Gap phenomena and controllability in free end-time problems with active state constraints ☆

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Abstract

This paper is concerned with gap phenomena and controllability conditions for free end-time optimal control problems with endpoint and state constraints, in which the data are permitted to be measurable with respect to the time variable. In particular, we prove sufficient conditions to avoid a gap between the infimum of the original minimum problem and an extended problem, obtained by first enlarging the set of original controls and then convexifying the extended velocities set. These conditions, which also guarantee controllability of the original system to an extended solution, are given in terms of normality of multipliers for the Maximum Principle, involving an extended minimizer with possibly active state constraint at the endpoints. In the free time case, links between absence of a gap and normality have only recently been studied, for the relaxed problem without state constraints. This paper establishes such links for a more general extension admitting active state constraints. Furthermore, under additional constraint qualification conditions we improve the normality test for no gap, by considering nondegenerate multipliers only.

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1. Introduction

For any pair $s_1,\, s_2 \in \mathbb{R}$, $s_1 < s_2$, consider the *original control system*

$$\begin{cases} \dot{y}(s) = \mathcal{F}(s, y(s), \omega(s), \alpha(s)), \\ \omega(s) \in V(s), \quad \alpha(s) \in A(s), \end{cases} \qquad \text{a.e. } s \in [s_1, s_2] \tag{1.1}$$

and the state and endpoint constraints

$$h(s, y(s)) \le 0 \quad \forall s \in [s_1, s_2], \qquad (s_1, y(s_1), s_2, y(s_2)) \in \mathcal{T}. \tag{1.2}$$

The data comprise the functions $\mathcal{F} : \mathbb{R}^{1+n+m+q} \to \mathbb{R}^n$, $h : \mathbb{R}^{1+n} \to \mathbb{R}$, the closed set $\mathcal{T} \subset \mathbb{R}^{1+n+1+n}$, and the set-valued maps $A : \mathbb{R} \rightsquigarrow \mathbb{R}^q$, $V : \mathbb{R} \rightsquigarrow \mathbb{R}^m$, where A takes as values compact sets while the values of V are bounded but not necessarily closed sets.

We first embed the set of control-trajectory pairs $(s_1, s_2, \omega, \alpha, y)$ of (1.1), referred to as *strict sense processes*, into the set of *extended processes*, where the control component $\omega(s)$ takes values in the closure $\overline{V(s)}$ of $V(s)$. Afterwards, the set of extended processes is embedded into the set of *relaxed extended processes,* given by the elements $(s_1, s_2, \underline{\omega}, \underline{\alpha}, \lambda, y)$ satisfying the following *relaxed extended control system,* in which the extended velocity sets are convexified:

$$\begin{cases} \dot{y}(s) = \sum_{k=0}^{n} \lambda^k(s) \mathcal{F}(s, y(s), \omega^k(s), \alpha^k(s)), \\ \underline{\omega}(s) = (\omega^0, \dots, \omega^n)(s) \in \overline{V(s)}^{1+n}, \\ \underline{\alpha}(s) = (\alpha^0, \dots, \alpha^n)(s) \in A(s)^{1+n}, \quad \lambda(s) \in \Delta_n, \quad \text{a.e. } s \in [s_1, s_2]. \end{cases} \tag{1.3}$$

Here, Δ_n is the n-dimensional simplex:

$$\Delta_n := \left\{ \lambda = (\lambda^0, \dots, \lambda^n) : \quad \lambda^k \ge 0,\ k = 0, \dots, n,\ \sum_{k=0}^{n} \lambda^k = 1 \right\}.$$

Any process is called *feasible* when the associated trajectory y satisfies the constraints (1.2) and we will use Γ, Γ_e, and Γ_r to denote the subsets of feasible strict sense, feasible extended, and feasible relaxed extended processes, respectively (see Section 2 below for the precise assumptions and definitions). Notice that we can identify a strict sense or an extended process $(s_1, s_2, \omega, \alpha, y)$ with any relaxed extended process $(s_1, s_2, \underline{\omega}, \underline{\alpha}, \lambda, y)$ with $\underline{\omega} = (\omega, \dots, \omega)$ and $\underline{\alpha} = (\alpha, \dots, \alpha)$, so that we have $\Gamma \subseteq \Gamma_e \subseteq \Gamma_r$.

Since we are interested in local properties, we introduce a concept of *distance* between trajectories, including left and right endpoints. Precisely, for all (s_1, s_2, y), (s_1', s_2', y') with $s_1 < s_2$, $s_1' < s_2'$, and $y : [s_1, s_2] \to \mathbb{R}^n$, $y' : [s_1', s_2'] \to \mathbb{R}^n$ continuous functions, we define the distance

$$d_\infty\big((s_1, s_2, y), (s_1', s_2', y')\big) := |s_1 - s_1'| + |s_2 - s_2'| + \|\tilde{y} - \tilde{y}'\|_{L^\infty}, \tag{1.4}$$

where $\tilde{y} : \mathbb{R} \to \mathbb{R}^n$ denotes the extension of the function y obtained by setting $\tilde{y}(s) := y(s_1)$ for all $s < s_1$ and $\tilde{y}(s) := y(s_2)$ for all $s > s_2$.

In this paper, we consider two related problems: (i) given a cost function $\Psi : \mathbb{R}^{1+n+1+n} \to \mathbb{R}$, find necessary conditions to have at a feasible relaxed extended process $\bar{z} := (\bar{s}_1, \bar{s}_2, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y})$ a *local infimum gap*, namely, the existence of some $\delta > 0$ such that

$$\Psi(\bar{s}_1, \bar{y}(\bar{s}_1), \bar{s}_2, \bar{y}(\bar{s}_1)) < \inf_{(s_1,s_2,\omega,\alpha,y)\in\Gamma,\ d_\infty((s_1,s_2,y),(\bar{s}_1,\bar{s}_2,\bar{y}))<\delta} \Psi(s_1, y(s_1), s_2, y(s_1));$$

(ii) determine sufficient *controllability* conditions for the original constrained control system to a feasible relaxed extended process $\bar{z}$, that is, for any $\varepsilon > 0$ there exists some feasible strict sense process $(s_1, s_2, \omega, \alpha, y) \in \Gamma$, such that

$$d_\infty\big((s_1, s_2, y), (\bar{s}_1, \bar{s}_2, \bar{y})\big) < \varepsilon.$$

Note that, even when the function Ψ is continuous and the set of strict sense processes is d_∞-dense in the set of relaxed processes, a local infimum gap may occur. Indeed, the presence of constraints might imply that there does not exist any *feasible* approximating strict sense process.

As a first main result, we prove that, if a local infimum gap occurs at a feasible relaxed extended process $\bar{z}$, then the free end-time, constrained, nonsmooth version of the Maximum Principle established in [1] is valid in abnormal form (i.e., with zero cost multiplier) at $\bar{z}$. We derive as corollaries that: *normality* of multipliers (i.e., all sets of multipliers with cost multiplier $\neq 0$) guarantees the absence of gap, and *non-existence of non trivial* (i.e., not identically zero) *abnormal multipliers* implies controllability.

However, when the state constraint is active at the initial point, a situation which is difficult to exclude a priori, it is well known that there can always be degenerate multipliers, with zero cost multiplier. In this case, a normality test for gap avoidance becomes useless, unless only nondegenerate multipliers can be considered. This is the question we address in the second part of the work. Here, under some additional constraint qualification conditions, we prove that, if there is a local infimum gap at a feasible relaxed extended process, then $\bar{z}$ is *nondegenerate abnormal*, that is, abnormal for a nondegenerate version of the Maximum Principle considered above.

Controllability of a control system to a reference trajectory, which might not solve the original system, and occurrence of infimum gaps, when the original class of processes is extended in order to achieve existence of minimizers, are largely investigated issues. In particular, links between these properties and normality of multipliers in the Maximum Principle have been established since the early works [2, 3, 4, 5], up to the more recent results [6, 7, 8, 9, 10, 11, 12].

The novelties of this paper lie, on the one hand, into the generality of the extension, which includes as particular cases both the convex relaxation investigated in [6, 7] and the impulsive extension treated in [8, 10], allowing for measurable time dependence of the data and (active) state constraints. On the other hand, we relate nondegeneracy with the conditions for no gap occurence.

Apart from the recent paper [13], which, however, only deals with the relaxed problem without state constraints, all previous work has addressed, exclusively, either fixed end-time optimal control problems (see e.g., [2, 4, 5, 9, 6, 7]) or free end-time problems with Lipschitz continuous time dependence and control constraint set independent of time ([8, 10, 11, 12]). We point out that the Lipschitz case differs substantially from the case with measurable time dependence of the data, in that the former can be reduced to a fixed end-time problem by a change of independent variable. Free end-time problems with measurable time dependence and state constraints have received considerable attention since the late '80s, especially in relation to the study of optimality conditions (see e.g. [14, 15] and references therein). In particular, a motivation to investigate situations with active state constraint at the optimal free end-times came from the observation that a minimizing trajectory evolving on the boundary of the constraint set and terminating at a discontinuity point of the dynamics was a frequently encountered phenomenon in a variety of threshold problems (associated, for instance, with abrupt changes in a tariff or rate of return on investment at prespecified times, as described in [16, 1] and references therein).

The question of determining sufficient conditions to avoid the gap in the form of nondegenerate normality conditions, has been addressed for the first time only recently, in [10, 11]. In particular, in [10] we introduced, just for the impulsive extension, sufficient conditions for each set of multipliers to be nondegenerate. These conditions, however, did not cover the case of fixed initial point, for which we provided sufficient nondegeneracy conditions in [11]. In the present paper, we unify and extend all the previous results to the general free end-time problem with time-dependent control constraint sets considered here. It is worth mentioning that, although our conditions are partially inspired by well-known conditions for the nondegeneracy of the Maximum Principle (see for instance [17, 18, 19, 20, 21] and references therein), the techniques of the proofs utilized in Section 5 below and in [11] are original. In particular, by means of perturbation and penalization techniques and by Ekeland's variational principle, we construct a sequence of approximating problems with strict sense optimal processes, whose multipliers are shown to converge to an abnormal nondegenerate multiplier for the given relaxed extended process $\bar{z}$.

The paper is organized as follows. In Section 2 we introduce notation, main definitions, and precise assumptions. In Section 3 we relate the presence of gap with abnormality, and derive some consequences. In Section 4 we provide constraint qualification conditions, under which we obtain a refinement of the previous results, involving nondegenerate multipliers only, and give some examples to illustrate the theoretical results. Section 5 contains the main proofs.

2. Notation, main definitions, and assumptions

2.1. Notation

Given an interval $I \subseteq \mathbb{R}$ and a set $X \subseteq \mathbb{R}^k$, we write $C^0(I;X)$, $W^{1,1}(I;X)$, $\mathfrak{M}(I;X)$, $L^1(I;X)$, $L^\infty(I;X)$, for the set of continuous, absolutely continuous, Lebesgue measurable, Lebesgue integrable, and essentially bounded functions defined on I and with values in X, respectively. We will not specify domain and

codomain when the meaning is clear and we will use $\|\cdot\|_{L^1(I)}$, $\|\cdot\|_{L^\infty(I)}$, or also $\|\cdot\|_{L^1}$, $\|\cdot\|_{L^\infty}$ to denote the L^1 and the ess-sup norm, respectively. Furthermore, we denote by $\ell(X)$, $\mathrm{co}(X)$, $\mathrm{Int}(X)$, $\overline{X}$, ∂X the Lebesgue measure, the convex hull, the interior, the closure, and the boundary of X, respectively. As customary, χ_X is the characteristic function of X, namely $\chi_X(x) = 1$ if $x \in X$ and $\chi_X(x) = 0$ if $x \in \mathbb{R}^k \setminus X$. Given two nonempty subsets X_1, X_2 of $\mathbb{R}^k$, we denote by $X_1 + X_2$ the set $\{x_1 + x_2 \,|\, x_1 \in X_1,\, x_2 \in X_2\}$ and by $d_H(X_1, X_2)$ the Hausdorff distance between X_1 and X_2. Let $X \subseteq \mathbb{R}^{k_1+k_2}$ for some natural numbers k_1, k_2, and write $x = (x_1, x_2) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ for any $x \in X$. Then, $\mathrm{proj}_{x_i} X$ will denote the projection of X on $\mathbb{R}^{k_i}$, for $i = 1, 2$. We denote the closed unit ball in $\mathbb{R}^k$ by $\mathbb{B}_k$, omitting the dimension when it is clear from the context. Given a closed set $\mathcal{O} \subseteq \mathbb{R}^k$, we define the distance of a point $z \in \mathbb{R}^k$ from $\mathcal{O}$ as $d_{\mathcal{O}}(z) := \min_{y\in\mathcal{O}} |z - y|$. Given a set-valued map $F : I \rightsquigarrow X$ with closed images and a function $f : I \to X$, we write $\Pi_{F(s)} f(s)$ to denote the projection of f on F, namely $\Pi_{F(s)} f(s) := \big\{x \in F(s) :\ |x - f(s)| = d_{F(s)}(f(s))\big\}$. We set $\mathbb{R}_{\geq 0} := [0, +\infty[$, $\mathbb{R}_{\leq 0} :=]-\infty, 0]$, and $\mathbb{R}_{>0} :=]0, +\infty[$. For any $a, b \in \mathbb{R}$, we write $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.
Given $s_1 < s_2$, we use $NBV^+([s_1, s_2]; \mathbb{R})$ to denote the space of increasing, real valued functions μ on $[s_1, s_2]$ of bounded variation, vanishing at the point s_1 and right continuous on $]s_1, s_2[$. Each $\mu \in NBV^+([s_1, s_2]; \mathbb{R})$ defines a Borel measure on $[s_1, s_2]$, still denoted by μ, its total variation function is indicated by $\mu([s_1, s_2])$, and its support is $\mathrm{spt}\{\mu\}$.
The *limiting normal cone* $N_C(\bar{x})$ to a closed set $C \subseteq \mathbb{R}^k$ at $\bar{x} \in \mathbb{R}^k$ is

$$N_C(\bar{x}) := \left\{ \eta :\ \exists x_i \overset{C}{\to} \bar{x},\ \eta_i \to \eta \ \text{ s.t. } \ \limsup_{x \to x_i} \frac{\eta_i \cdot (x - x_i)}{|x - x_i|} \leq 0 \ \ \forall i \right\},$$

in which the notation $x_i \xrightarrow{C} \bar{x}$ means that $(x_i)_i \subset C$. Let $G : \mathbb{R}^k \to \mathbb{R}$ be a lower semicontinuous function, the *limiting subdifferential* of G at $\bar{x} \in \mathbb{R}^k$ is

$$\partial G(\bar{x}) := \left\{ \xi\colon\ \exists \xi_i \to \xi,\ x_i \to \bar{x} \text{ s.t. } \limsup_{x \to x_i} \frac{\xi_i \cdot (x - x_i) - G(x) + G(x_i)}{|x - x_i|} \leq 0 \ \ \forall i \right\}.$$

Let $G : \mathbb{R}^k \to \mathbb{R}$ be a locally Lipschitz continuous function and let $\mathrm{diff}(G)$, ∇G denote the set of differentiability points and the usual gradient operator of G, respectively. The *hybrid subdifferential* of G at $\bar{x} \in \mathbb{R}^k$ is

$$\partial^> G(\bar{x}) := \mathrm{co} \left\{ \xi\colon\ \exists (x_i)_i \text{ s.t. } \ x_i \overset{\mathrm{diff}(G)\setminus\{\bar{x}\}}{\to} \bar{x},\ G(x_i) > 0 \ \forall i,\ \nabla G(x_i) \to \xi \right\}.$$

Let $G : \mathbb{R}^k \to \mathbb{R}^l$ be a locally Lipschitz continuous function, the *Clarke's generalized Jacobian* of G at $\bar{x}$ is

$$DG(\bar{x}) := \mathrm{co} \left\{ \xi\colon\ \exists (x_i)_i \text{ s.t. } \ x_i \overset{\mathrm{diff}(G)\setminus\{\bar{x}\}}{\to} \bar{x} \text{ and } \nabla G(x_i) \to \xi \right\}.$$

Recall that, when $l = 1$, $DG = \mathrm{co}\, \partial G$. If $G : \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R}^l$ and $x = (x_1, x_2) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$, we use $D_{x_i} G$, $\nabla_{x_i} G$, and, if $l = 1$, $\partial_{x_i} G$, to denote partial Clarke's generalized Jacobian, partial gradient operator, and partial limiting subdifferential of G w.r.t. x_i, for $i = 1, 2$. Our main sources on nonsmooth analysis are [22, 23].

2.2. Main definitions

In order to introduce the precise concepts of strict sense, extended and relaxed extended process, for any pair s_1, $s_2 \in \mathbb{R}$, $s_1 < s_2$, we set

$$\begin{aligned}
&\mathcal{A}([s_1, s_2]) := \{\alpha \in \mathfrak{M}([s_1, s_2]; \mathbb{R}^q) : \ \alpha(s) \in A(s) \text{ a.e. } s \in [s_1, s_2]\},\\
&\mathcal{V}([s_1, s_2]) := \{\omega \in \mathfrak{M}([s_1, s_2]; \mathbb{R}^m) : \ \omega(s) \in V(s) \text{ a.e. } s \in [s_1, s_2]\},\\
&\mathcal{W}([s_1, s_2]) := \{\omega \in \mathfrak{M}([s_1, s_2]; \mathbb{R}^m) : \ \omega(s) \in \overline{V(s)} \text{ a.e. } s \in [s_1, s_2]\},\\
&\Lambda([s_1, s_2]) := \mathfrak{M}([s_1, s_2]; \Delta_n).
\end{aligned}$$

Definition 2.1 (Processes and feasible processes)**.** We refer to any element $(s_1, s_2, \omega, \alpha, y)$ with $s_1 < s_2$, controls $\omega \in \mathcal{W}([s_1, s_2])$, $\alpha \in \mathcal{A}([s_1, s_2])$, and trajectory $y \in W^{1,1}([s_1, s_2]; \mathbb{R}^n)$ that satisfies

$$\dot{y}(s) = \mathcal{F}(s, y(s), \omega(s), \alpha(s)) \quad \text{a.e. } s \in [s_1, s_2], \tag{2.1}$$

as *extended process*. An extended process $(s_1, s_2, \omega, \alpha, y)$ is called a *strict sense process* if $\omega \in \mathcal{V}([s_1, s_2])$. A strict sense or extended process is *feasible* when it satisfies the constraints (1.2), namely, if $h(s, y(s)) \le 0$ for all $s \in [s_1, s_2]$ and $(s_1, y(s_1), s_2, y(s_2)) \in \mathcal{T}$.

We define *relaxed (extended) process* any element $(s_1, s_2, \underline{\omega}, \underline{\alpha}, \lambda, y)$, where $s_1 < s_2$, $\underline{\omega} \in \mathcal{W}^{1+n}([s_1, s_2])$, $\underline{\alpha} \in \mathcal{A}^{1+n}([s_1, s_2])$, $\lambda \in \Lambda([s_1, s_2])$, and $y \in W^{1,1}([s_1, s_2]; \mathbb{R}^n)$ satisfies

$$\dot{y}(s) = \sum_{k=0}^{n} \lambda^k(s) \mathcal{F}(s, y(s), \omega^k(s), \alpha^k(s)) \ \text{ a.e. } s \in [s_1, s_2]. \tag{2.2}$$

A relaxed process is *feasible* when it satisfies (1.2).

As already observed, we identify an extended process $(s_1, s_2, \omega, \alpha, y)$ with any relaxed process $(s_1, s_2, \underline{\omega}, \underline{\alpha}, \lambda, y) = (s_1, s_2, \omega, \dots, \omega, \alpha, \dots, \alpha, \lambda, y)$ and the sets Γ, Γ_e of feasible strict sense and feasible extended processes, respectively, with subsets of the set of feasible relaxed processes Γ_r. Hence, we will often simply call *process* any relaxed, extended, or strict sense process.

We consider *local* notions of *minimum, infimum gap*, and *controllability.*

Definition 2.2 (Minimizer)**.** Let $\tilde{\Gamma} \in \{\Gamma, \Gamma_e, \Gamma_r\}$. Given a continuous function $\Psi : \mathbb{R}^{1+n+1+n} \to \mathbb{R}$, a process $\bar{z} := (\bar{s}_1, \bar{s}_2, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y}) \in \tilde{\Gamma}$ is called a *local Ψ-minimizer for problem* $(P_{\tilde{\Gamma}})$ if, for some $\delta > 0$, one has

$$\begin{aligned}
\Psi(\bar{s}_1, \bar{y}(\bar{s}_1), \bar{s}_2, \bar{y}(\bar{s}_2)) = \min \Big\{ \Psi(s_1, y(s_1), s_2, y(s_2)) : \quad (s_1, s_2, \underline{\omega}, \underline{\alpha}, \lambda, y) \in \tilde{\Gamma},\\
d_\infty((s_1, s_2, y), (\bar{s}_1, \bar{s}_2, \bar{y})) < \delta \Big\}.
\end{aligned}$$

The process $\bar z \in \tilde\Gamma$ is a (*global*) Ψ*-minimizer for problem* $(P_{\tilde\Gamma})$ if

$$\Psi(\bar s_1, \bar y(\bar s_1), \bar s_2, \bar y(\bar s_2)) = \min_{\tilde\Gamma} \Psi(s_1, y(s_1), s_2, y(s_2)).$$

Definition 2.3 (Infimum gap)**.** Let $\Psi : \mathbb{R}^{1+n+1+n} \to \mathbb{R}$ be a continuous function. Fix $\bar z := (\bar s_1, \bar s_2, \underline{\bar\omega}, \underline{\bar\alpha}, \bar\lambda, \bar y) \in \Gamma_r$. We say that *at $\bar z$ there is a local Ψ-infimum gap* if, for some $\delta > 0$,

$$\Psi(\bar s_1, \bar y(\bar s_1), \bar s_2, \bar y(\bar s_2)) < \inf \Big\{ \Psi(s_1, y(s_1), s_2, y(s_2)) : \\ (s_1, s_2, \omega, \alpha, y) \in \Gamma, \quad d_\infty\big((s_1, s_2, y), (\bar s_1, \bar s_2, \bar y)\big) < \delta \Big\}.^{1} \tag{2.3}$$

The notion of local Ψ-infimum gap is related to the following topological properties.

Definition 2.4 (Isolated process and controllability)**.** Let us fix a process $\bar z := (\bar s_1, \bar s_2, \underline{\bar\omega}, \underline{\bar\alpha}, \bar\lambda, \bar y) \in \Gamma_r$. We call $\bar z$ *isolated* if, for some $\delta > 0$,

$$\big\{(s_1, s_2, \omega, \alpha, y) \in \Gamma : \quad d_\infty\big((s_1, s_2, y), (\bar s_1, \bar s_2, \bar y)\big) < \delta\big\} = \emptyset.$$

We say that *the constrained control system* (1.1)-(1.2) *is controllable to* $\bar z$ if $\bar z$ is not isolated, that is, for every $\varepsilon > 0$ there is some process $(s_1, s_2, \omega, \alpha, y) \in \Gamma$ such that $d_\infty\big((s_1, s_2, y), (\bar s_1, \bar s_2, \bar y)\big) < \varepsilon$.

In fact, the following equivalences are valid.

Proposition 2.5. *Given a feasible relaxed process $\bar z := (\bar s_1, \bar s_2, \underline{\bar\omega}, \underline{\bar\alpha}, \bar\lambda, \bar y)$, the following properties are equivalent:*

(i) *$\bar z$ is isolated;*

(ii) *for every continuous function Ψ, at $\bar z$ there is a Ψ-local infimum gap;*

(iii) *given a continuous function Ψ, at $\bar z$ there is a local Ψ-infimum gap.*

Proof. The implication (i)$\Rightarrow$(ii) is immediate, since if $\bar z$ is isolated, then the right-hand-side in (2.3) is equal to $+\infty$. Also the fact that (ii)$\Rightarrow$(iii) is obvious. It remains only to show that (iii)$\Rightarrow$(i). Assume by contradiction that (iii) holds true but $\bar z$ is not isolated. Then, for some $\delta > 0$ as in Definition 2.3 and any sequence $(\varepsilon_i)_i \subset]0, \delta[$, $\varepsilon_i \downarrow 0$, there exists a sequence of feasible strict sense processes $(s_{1_i}, s_{2_i}, \omega_i, \alpha_i, y_i) \in \Gamma$ such that $d_\infty\big((s_{1_i}, s_{2_i}, y_i), (\bar s_1, \bar s_2, \bar y)\big) < \varepsilon_i < \delta$, so that

$$\Psi(\bar s_1, \bar y(\bar s_1), \bar s_2, \bar y(\bar s_2)) < \inf \Big\{ \Psi(s_1, y(s_1), s_2, y(s_2)) : \quad (s_1, s_2, \omega, \alpha, y) \in \Gamma, \\ d_\infty\big((s_1, s_2, y), (\bar s_1, \bar s_2, \bar y)\big) < \delta \Big\} \le \Psi(s_{1_i}, y_i(s_{1_i}), s_{2_i}, y_i(s_{2_i})).$$

As $i \to +\infty$, we get the desired contradiction and the proof is complete. □

[1] As customary, when the set is empty we set the infimum $= +\infty$.

From Proposition 2.5 it follows that having a local Ψ-infimum gap at $\bar{z}$ is independent of the choice of Ψ. For this reason, in the following we simply say that *at $\bar{z}$ there is a local infimum gap.*

2.3. Assumptions

The hypotheses we invoke are of local nature: they relate to a reference feasible relaxed process $(\bar{s}_1, \bar{s}_2, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y})$ and a parameter $\theta > 0$. Let $\theta' > 0$ and define the θ'*-tube* of the process $(\bar{s}_1, \bar{s}_2, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y})$ as

$$\Sigma_{\theta'} := \{(s,x) \in \mathbb{R} \times \mathbb{R}^n : \quad s \in [\bar{s}_1 - \theta', \bar{s}_2 + \theta'], \ |x - \bar{y}(s)| \le \theta'\},$$

where $\bar{y}$ is extended by constant extrapolation.

(H1) *The set-valued map $A : \mathbb{R} \rightsquigarrow \mathbb{R}^q$ is Borel measurable and takes compact sets as values. The set-valued map $V : \mathbb{R} \rightsquigarrow U$ is Borel measurable and $U \subseteq \mathbb{R}^m$ is a compact set. Moreover, for every $i \in \mathbb{N}$ there exists a closed Borel measurable set-valued map $V_i : \mathbb{R} \rightsquigarrow U$ such that $V_i(s) \subseteq V(s)$ for a.e. $s \in [\bar{s}_1 - \theta, \bar{s}_2 + \theta]$, and*

$$d_H(V_i(s), \overline{V(s)}) \le \psi_i(s) \qquad a.e. \ s \in [\bar{s}_1 - \theta, \bar{s}_2 + \theta],$$

where $\psi_i \in (L^1 \cap L^\infty)([\bar{s}_1 - \theta, \bar{s}_2 + \theta]; \mathbb{R}_{\ge 0})$ and

$$\|\psi_i\|_{L^1([\bar{s}_1 - \theta, \bar{s}_2 + \theta])} \to 0, \qquad \|\psi_i\|_{L^\infty([\bar{s}_1 - \theta, \bar{s}_1 + \theta] \cup [\bar{s}_2 - \theta, \bar{s}_2 + \theta])} \to 0.$$

(H2) *The constraint function h is Lipschitz continuous on Σ_θ, i.e. there is some constant $K_h > 0$ such that*

$$|h(s,x) - h(s',x')| \le K_h |(s,x) - (s',x')| \qquad \forall (s,x), \ (s',x') \in \Sigma_\theta.$$

(H3) (i) *For all $(x,w,a) \in \mathbb{R}^n \times U \times \mathbb{R}^q$, $\mathcal{F}(\cdot, x, w, a)$ is Lebesgue measurable on $\mathbb{R}$ and for any $(s,x) \in \Sigma_\theta$, $\mathcal{F}(s,x,\cdot,\cdot)$ is continuous on $U \times \mathbb{R}^q$. Moreover, there exists $K_{\mathcal{F}} > 0$ such that*

$$|\mathcal{F}(s,x,w,a)| \le K_{\mathcal{F}}, \qquad |\mathcal{F}(s,x',w,a) - \mathcal{F}(s,x,w,a)| \le K_{\mathcal{F}} |x' - x|,$$

for all (s,x,w,a), $(s,x',w,a) \in \Sigma_\theta \times U \times A(s)$.

(ii) *There exists some continuous increasing function $\varphi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ with $\varphi(0) = 0$ such that for any $(s,x,a) \in \Sigma_\theta \times A(s)$, we have*

$$\begin{aligned} &|\mathcal{F}(s,x,w',a) - \mathcal{F}(s,x,w,a)| \le \varphi(|w' - w|) \qquad \forall w', \ w \in U, \\ &D_x\mathcal{F}(s,x,w',a) \subseteq D_x\mathcal{F}(s,x,w,a) + \varphi(|w' - w|)\,\mathbb{B} \qquad \forall w', \ w \in U. \end{aligned}$$

Remark 2.6. Some comments on the hypotheses are in order.

(i) By hypothesis **(H1)**, for any $\delta > 0$ there exists some $\iota_\delta \in \mathbb{N}$ such that, for every $i \ge \iota_\delta$, one has

$$\int_{\bar{s}_1 - \theta}^{\bar{s}_2 + \theta} \psi_i(s)\, ds \le \delta, \quad \|\psi_i\|_{L^\infty([\bar{s}_1 - \theta, \bar{s}_1 + \theta] \cup [\bar{s}_1 - \theta, \bar{s}_1 + \theta])} \le \delta.$$

Thus, given an arbitrary measurable function $\omega(s) \in \overline{V(s)}$ for a.e. $s \in [\bar{s}_1 - \theta, \bar{s}_2 + \theta]$, from [24, Cor. 8.2.13] it follows that there is some measurable selection $\omega_\delta(s) \in \Pi_{V_{\iota_\delta}(s)}(\omega(s))$ for a.e. $s \in [\bar{s}_1 - \theta, \bar{s}_2 + \theta]$ such that

$$\begin{cases} \|\omega_\delta - \omega\|_{L^1([\bar{s}_1-\theta,\bar{s}_2+\theta])} \le \displaystyle\int_{\bar{s}_1-\theta}^{\bar{s}_2+\theta} \psi_{\iota_\delta}(s)\, ds \le \delta, \\ \|\omega_\delta - \omega\|_{L^\infty([\bar{s}_1-\theta,\bar{s}_1+\theta]\cup[\bar{s}_1-\theta,\bar{s}_1+\theta])} \le \delta. \end{cases} \tag{2.4}$$

As a consequence, **(H1)** implies in particular the density of the control set $\mathcal{V}([s_1, s_2])$ in $\mathcal{W}([s_1, s_2])$ in the L^1-norm, for every $s_1,\ s_2 \in \mathbb{R}$ such that $\bar{s}_1 - \theta \le s_1 < s_2 \le \bar{s}_2 + \theta$. Hypothesis **(H1)** is satisfied, for instance, when $V(\cdot) \equiv V$ is bounded and there exists a sequence $(V_i)_i$ of closed subsets of V such that $V_i \subseteq V_{i+1}$ for every i and $\bigcup_{i=1}^{+\infty} V_i = V$, as assumed in [11].

(ii) Hypotheses **(H2)** and **(H3)**(i) are quite standard assumptions, while condition **(H3)**(ii), which prescribes additional regularity properties of the dynamics $\mathcal{F}$ in the w-variable, reflects the different roles played by the controls α and ω, as only the set of w-control values is extended by replacing $V(s)$ with $\overline{V(s)}$ for a.e. s. Condition **(H3)**(ii) is fulfilled, for instance, when $\mathcal{F}(s, x, w, a) = \mathcal{F}_1(s, x, a) + \mathcal{F}_2(s, x, w, a)$, where $\mathcal{F}_1$, $\mathcal{F}_2$ satisfy **(H3)**(i), $\mathcal{F}_2(s, \cdot, w, a)$ is C^1, and $\nabla_x \mathcal{F}_2$ is continuous on the compact set $\Sigma_\theta \times U \times A(s)$. It is also verified when the dynamics function has a polynomial dependence on the control variable w, with coefficients satisfying **(H3)**(i) in the remaining variables, as in Examples 4.9, 4.10 below.

3. Abnormality and local infimum gap

In this section we state a theorem relating the existence of a gap and the validity of a constrained Maximum Principle in abnormal form for a free-time optimal control problem, in which both end-times are choice variables. From this result we deduce that normality is a sufficient condition for gap-avoidance and a local controllability condition.

Given an essentially bounded function $g : \mathbb{R} \to \mathbb{R}$ and a point $\bar{s} \in \mathbb{R}$, the essential value of g at $\bar{s}$ is the set

$$\operatorname*{ess}_{s \to \bar{s}} g(s) := \left[\lim_{\delta \downarrow 0} \left(\operatorname*{ess\ inf}_{s \in [\bar{s}-\delta, \bar{s}+\bar{\delta}]} g(s) \right),\ \lim_{\delta \downarrow 0} \left(\operatorname*{ess\ sup}_{s \in [\bar{s}-\delta, \bar{s}+\bar{\delta}]} g(s) \right) \right].$$

For the properties of the essential value we refer to [23, Sect. 8].

Definition 3.1 (Normal and abnormal extremal)**.** Let $\bar{z} := (\bar{s}_1, \bar{s}_2, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y})$ be a feasible relaxed process for which hypotheses **(H1)**–**(H3)** are verified. Given a function $\Psi : \mathbb{R}^{1+n+1+n} \to \mathbb{R}$ which is Lipschitz continuous on a neighborhood of $(\bar{s}_1, \bar{y}(\bar{s}_1), \bar{s}_2, \bar{y}(\bar{s}_2))$, we say that $\bar{z}$ is a Ψ-*extremal* if there exist a path $p \in W^{1,1}([\bar{s}_1, \bar{s}_2]; \mathbb{R}^n)$, numbers $k_1,\ k_2 \in \mathbb{R}$, $\gamma \ge 0$, $\beta_1 \ge 0$, $\beta_2 \ge 0$, a measure

$\mu \in NBV^+([\bar s_1,\bar s_2];\mathbb{R})$, and a Borel measurable and μ-integrable function $m : [\bar s_1,\bar s_2] \to \mathbb{R}^n$, such that:

$$\|p\|_{L^\infty([\bar s_1,\bar s_2])} + \mu([\bar s_1,\bar s_2]) + \beta_1 + \beta_2 + \gamma \neq 0; \tag{3.1}$$

$$-\dot p(s) \in \sum_{k=0}^{n} \bar\lambda^k(s)\,\mathrm{co}\,\partial_x \left\{ q(s)\cdot \mathcal{F}(s,(\bar y,\bar\omega^k,\bar\alpha^k)(s))\right\} \quad \text{a.e. } s\in[\bar s_1,\bar s_2]; \tag{3.2}$$

$$\begin{aligned}(-k_1, p(\bar s_1), k_2, -q(\bar s_2)) &\in \gamma\partial\Psi(\bar s_1,\bar y(\bar s_1),\bar s_2,\bar y(\bar s_2)) \\ &\quad + N_{\mathcal{T}}(\bar s_1,\bar y(\bar s_1),\bar s_2,\bar y(\bar s_2)) + \beta_1\partial h(\bar s_1,\bar y(\bar s_1))\times\beta_2\partial h(\bar s_2,\bar y(\bar s_2));\end{aligned} \tag{3.3}$$

$$k_1 \in \operatorname*{ess}_{s\to\bar s_1}\Big(\max_{(w,a)\in\overline{V(s)}\times A(s)} p(\bar s_1)\cdot\mathcal{F}\big(s,\bar y(\bar s_1),w,a\big)\Big), \tag{3.4}$$

$$k_2 \in \operatorname*{ess}_{s\to\bar s_2}\Big(\max_{(w,a)\in\overline{V(s)}\times A(s)} q(\bar s_2)\cdot\mathcal{F}\big(s,\bar y(\bar s_2),w,a\big)\Big); \tag{3.5}$$

for every $k=0,\dots,n$, for a.e. $s\in[\bar s_1,\bar s_2]$, one has

$$q(s)\cdot\mathcal{F}\big(s,\bar y(s),\bar\omega^k(s),\bar\alpha^k(s)\big) = \max_{(w,a)\in\overline{V(s)}\times A(s)} q(s)\cdot\mathcal{F}\big(s,\bar y(s),w,a\big); \tag{3.6}$$

$$m(s)\in\partial_x^{>} h\,(s,\bar y(s)) \qquad \mu\text{-a.e.}; \tag{3.7}$$

$$\mathrm{spt}(\mu)\subseteq\{s\in[\bar s_1,\bar s_2]:\ h\,(s,\bar y(s))=0\}, \tag{3.8}$$

where

$$q(s) := \begin{cases} p(s) + \int_{[\bar s_1,s[} m(\sigma)\mu(d\sigma) & s\in[\bar s_1,\bar s_2[,\\ p(\bar s_2) + \int_{[\bar s_1,\bar s_2]} m(\sigma)\mu(d\sigma) & s=\bar s_2.\end{cases}$$

Furthermore, for $j\in\{1,2\}$, $\beta_j=0$ if either $h(\bar s_j,\bar y(\bar s_j))<0$ or the s_j-component of the endpoint constraint set $\mathcal{T}$ is the single point $\{\bar s_j\}$.

We will call a Ψ-extremal *normal* if all multipliers $(p,k_1,k_2,\beta_1,\beta_2,\gamma,\mu,m)$ as above have $\gamma>0$, and *abnormal* when it is not normal. Clearly, an abnormal Ψ-extremal is abnormal for every Ψ, thus in the following it will be simply called an *abnormal extremal*.

Theorem 3.2. *Let $\bar z := (\bar s_1,\bar s_2,\underline{\bar\omega},\underline{\bar\alpha},\bar\lambda,\bar y)$ be a feasible relaxed process for which hypotheses* **(H1)**–**(H3)** *are verified. If at $\bar z$ there is a local infimum gap, then $\bar z$ is an abnormal extremal.*

We prove this result in Section 5. As corollaries of Theorem 3.2, we get the following sufficient conditions for gap-avoidance and controllability.

Theorem 3.3. *Let $\bar z := (\bar s_1,\bar s_2,\underline{\bar\omega},\underline{\bar\alpha},\bar\lambda,\bar y)$ be a feasible relaxed process for which hypotheses* **(H1)**–**(H3)** *are verified. Let $\Psi:\mathbb{R}^{1+n+1+n}\to\mathbb{R}$ be a Lipschitz continuous function on a neighborhood of $(\bar s_1,\bar y(\bar s_1),\bar s_2,\bar y(\bar s_2))$. When $\bar z$ is a local Ψ-minimizer for (P_{Γ_e}) or (P_{Γ_r}) which is a normal Ψ-extremal, then*

$$\Psi(\bar s_1,\bar y(\bar s_1),\bar s_2,\bar y(\bar s_1)) = \inf\Psi(s_1,y(s_1),s_2,y(s_1)),$$

over all processes $(s_1,s_2,\omega,\alpha,y)\in\Gamma$ with $d_\infty((s_1,s_2,y),(\bar s_1,\bar s_2,\bar y))<\delta$. Similarly, if $\bar z$ is Ψ-minimizer for (P_{Γ_e}) or (P_{Γ_r}) which is a Ψ-normal extremal, then the above equality holds for the infimum over the whole set Γ.

Proof. Since $\Gamma \subseteq \Gamma_e \subseteq \Gamma_r$, when $\bar{z}$ is a local Ψ-minimizer for (P_{Γ_e}) or (P_{Γ_r}) there exists some $\delta > 0$ such that

$$\Psi(\bar{s}_1, \bar{y}(\bar{s}_1), \bar{s}_2, \bar{y}(\bar{s}_2)) \leq \inf \Big\{ \Psi(s_1, y(s_1), s_2, y(s_2)) : \\ (s_1, s_2, \omega, \alpha, y) \in \Gamma, \quad d_\infty\big((s_1, s_2, y), (\bar{s}_1, \bar{s}_2, \bar{y})\big) < \delta \Big\}. \tag{3.9}$$

At this point, the proof of the first statement is trivial: indeed, if $\bar{z}$ satisfies (3.9) as a strict inequality, then at $\bar{z}$ there is a local Ψ-infimum gap. But in this case $\bar{z}$ could not be a normal Ψ-extremal, in view of Theorem 3.2. Hence, the inequality in (3.9) is in fact an equality. Let now $\bar{z}$ be a Ψ-minimizer for (P_{Γ_e}) or (P_{Γ_r}). Then, it satisfies the relation

$$\Psi(\bar{s}_1, \bar{y}(\bar{s}_1), \bar{s}_2, \bar{y}(\bar{s}_2)) \leq \inf_{(s_1, s_2, \omega, \alpha, y) \in \Gamma} \Psi(s_1, y(s_1), s_2, y(s_2))$$

and, if we suppose that the inequality is strict, this implies again that at $\bar{z}$ there is a local infimum gap. Thus, arguing as above we still get a contradiction. □

Remark 3.4. By the free-time constrained maximum principle [1, Thm. 5.2], local Ψ-minimizers of (P_r) are Ψ-extremals in a stronger form than in Definition 3.1, in which the costate differential inclusion (3.2) is replaced by

$$-\dot{p}(s) \in \operatorname{co} \partial_x \left\{ \sum_{k=0}^{n} \bar{\lambda}^k(s)\, q(s) \cdot \mathcal{F}(s, (\bar{y}, \bar{\omega}^k, \bar{\alpha}^k)(s)) \right\} \text{ a.e. } s \in [\bar{s}_1, \bar{s}_2]. \tag{3.10}$$

The need to consider (3.2) derives from the perturbation technique used in the proof of Theorem 3.2 (see also [7, 11]). In fact, (3.2) may differ from (3.10) only in case of nonsmooth dynamics. Precisely, if $\mathcal{F}(s, \cdot, \bar{\omega}^k(s), \bar{\alpha}^k(s))$ is continuously differentiable at $\bar{y}(s)$, for all $k = 0, \dots, n$ and a.e. $s \in [\bar{s}_1, \bar{s}_2]$, then both differential inclusions reduce to the adjoint equation

$$-\dot{p}(s) = \sum_{k=0}^{n} \bar{\lambda}^k(s)\, q(s) \cdot \nabla_x \mathcal{F}(s, (\bar{y}, \bar{\omega}^k, \bar{\alpha}^k)(s)) \text{ a.e. } s \in [\bar{s}_1, \bar{s}_2].$$

In order to establish sufficient controllability conditions, given a reference process $\bar{z} := (\bar{s}_1, \bar{s}_2, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y}) \in \Gamma_r$ for which hypotheses **(H1)**–**(H3)** are verified, we introduce the set $\mathcal{M}(\bar{z})$ of multipliers $(p, k_1, k_2, \beta_1, \beta_2, \mu, m)$, where $p \in W^{1,1}([\bar{s}_1, \bar{s}_2]; \mathbb{R}^n)$, $k_1, k_2 \in \mathbb{R}$, $\beta_1, \beta_2 \geq 0$, $\mu \in NBV^+([\bar{s}_1, \bar{s}_2]; \mathbb{R})$, $m : [\bar{s}_1, \bar{s}_2] \to \mathbb{R}^n$ is a Borel measurable and μ-integrable function, that meet conditions (3.2), (3.4)–(3.8) (for q as in Definition 3.1), and such that

$$\|p\|_{L^\infty([\bar{s}_1, \bar{s}_2])} + \mu([\bar{s}_1, \bar{s}_2]) + \beta_1 + \beta_2 \neq 0,$$
$$(-k_1, p(\bar{s}_1), k_2, -q(\bar{s}_2)) \in N_{\mathcal{T}}(\bar{s}_1, \bar{y}(\bar{s}_1), \bar{s}_2, \bar{y}(\bar{s}_2)) \\ + \beta_1 \partial h(\bar{s}_1, \bar{y}(\bar{s}_1)) \times \beta_2 \partial h(\bar{s}_2, \bar{y}(\bar{s}_2)).$$

Theorem 3.5. *Let $\bar{z} := (\bar{s}_1, \bar{s}_2, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y})$ be a feasible relaxed process and assume that hypotheses* **(H1)**–**(H3)** *are verified. If $\mathcal{M}(\bar{z}) = \emptyset$, then the constrained control system* (1.1)-(1.2) *is controllable to $\bar{z}$.*

Proof. Theorem 3.5 is simply the contrapositive statement of Theorem 3.2. Indeed, if the constrained control system (1.1)-(1.2) is not controllable to $\bar{z}$, then $\bar{z}$ is an isolated process, which means that at $\bar{z}$ there is a local infimum gap, in view of Proposition 2.5. Now, Theorem 3.2 implies that $\bar{z}$ is an abnormal extremal, and this guarantees that $\mathcal{M}(\bar{z}) \neq \emptyset$. □

Remark 3.6. In order to simplify the exposition, we considered a Mayer problem with a single inequality state constraint. Actually, from quite standard arguments (see e.g. [10, 23]) all the results of this paper could be extended: (i) to a Bolza problem, with cost of the form

$$J(s_1, s_2, \omega, \alpha, y) := \Psi(s_1, y(s_1), s_2, y(s_2)) + \int_{s_1}^{s_2} \mathcal{L}(s, y(s), \omega(s), \alpha(s))\, ds,$$

with $\mathcal{L} : \mathbb{R}^{1+n+m+q} \to \mathbb{R}$ which satisfies the same regularity assumptions as the dynamics $\mathcal{F}$; (ii) to $N \geq 1$ inequality state constraints $h_j(s, y(s)) \leq 0$ for all $s \in [s_1, s_2]$ ($j = 1, \dots, N$), where each h_j satisfies hypothesis **(H2)**; (iii) to implicit time-dependent state constraints of the form $y(s) \in Y(s)$ for all $s \in [s_1, s_2]$, where $Y : \mathbb{R} \rightsquigarrow \mathbb{R}^n$ is a Lipschitz continuous set-valued map.

4. Nondegenerate abnormality and infimum gap

The normality test to avoid a local infimum gap at some process $\bar{z} := (\bar{s}_1, \bar{s}_2, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y}) \in \Gamma_r$ established in Theorem 3.3 might be useless when the initial point $(\bar{s}_1, \bar{y}(\bar{s}_1)) \in \partial\Omega$, where Ω is the state constraint set, defined by

$$\Omega := \{(t, x) \in \mathbb{R}^{1+n} \,:\, h(t, x) \leq 0\}. \tag{4.1}$$

Indeed, in this case $\bar{z}$ is very often an abnormal extremal, since, at least disregarding the endpoint constraints, there may be *degenerate* sets of multipliers $(p, k_1, k_2, \beta_1, \beta_2, \gamma, \mu, m)$ that meet all the conditions of Definition 3.1 with

$$\mu \equiv \mu(\{\bar{s}_1\}) \neq 0, \qquad p \equiv -m(\bar{s}_1)\mu(\{\bar{s}_1\}), \qquad \gamma = \beta_1 = \beta_2 = 0. \tag{4.2}$$

This section is devoted to provide some sufficient conditions to refine the results of Section 3, in order to exclude degenerate multipliers. We will conclude with some examples.

Let us point out that we cannot simply consider any of the conditions of nondegeneracy known in the literature to prove that a process $\bar{z}$ at which there is a local infimum gap is abnormal *and nondegenerate*. In particular, our strategy to prove that $\bar{z}$ is an abnormal extremal is to apply the Ekeland Principle to a sequence of optimization problems over strict sense processes, so that the sequence of Ekeland minimizers approximate the reference relaxed process $\bar{z}$. By applying the Maximum Principle to these minimizers we derive, in the limit, a maximum principle in abnormal form for $\bar{z}$. Hence, on the one hand, we would need a condition of nondegeneracy for each of these minimizers, which remains so by passing to the limit. On the other hand, for the approximating problems we cannot invoke, for instance, controllability conditions of the kind introduced in [25, 26] (see also [17], [23, Sec. 10.6]), since they require Hamiltonians which are Lipschitz continuous in time, while the Hamiltonians of our Ekeland optimization problems are at most measurable in time (see problems (P_i) in the proof of Theorem 4.6 below). Let us recall also [27], where this kind of nondegeneracy conditions are extended to differential inclusions with bounded variation in time.

4.1. Nondegeneracy conditions for general endpoint constraints

In the case of general endpoint constraints, we consider the following condition, which ensures that a multiplier as in (4.2) cannot exist.

Condition for nondegeneracy (CN). *A process* $(\bar{s}_1, \bar{s}_2, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y}) \in \Gamma_r$ *is said to satisfy condition* **(CN)** *if*

$$\partial_x^{>} h(\bar{s}_1, \bar{y}(\bar{s}_1)) \cap \left(-\mathrm{proj}_{x_1} N_{\mathcal{T}}(\bar{s}_1, \bar{y}(\bar{s}_1), \bar{s}_2, \bar{y}(\bar{s}_2))\right) = \emptyset. \tag{4.3}$$

Condition **(CN)** extends a condition first introduced in [10], for the impulsive extension with Lipschitz continuous data in the time variable. It is *a posteriori* requirement, that ensures the nondegeneracy of *every* (Ψ-)extremal, similarly to the strengthened nontriviality conditions derived in [19, Cor. 3.1].

Proposition 4.1. *Let* $(\bar{s}_1, \bar{s}_2, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y})$ *be a* Ψ*-extremal for* $\Psi : \mathbb{R}^{1+n+1+n} \to \mathbb{R}$ *Lipschitz continuous on a neighborhood of* $(\bar{s}_1, \bar{y}(\bar{s}_1), \bar{s}_2, \bar{y}(\bar{s}_2))$*. Assume that condition* **(CN)** *is satisfied. Then any multiplier* $(p, k_1, k_2, \beta_1, \beta_2, \gamma, \mu, m)$ *that meets all the condition of Definition 3.1, is a* nondegenerate multiplier*, that is, it satisfies the following strengthened nontriviality condition*

$$\|q\|_{L^\infty([\bar{s}_1, \bar{s}_2])} + \gamma + \mu(]\bar{s}_1, \bar{s}_2]) + \beta_1 + \beta_2 \neq 0, \tag{4.4}$$

where q *is as in Definition 3.1.*

Proof. Let $(p, k_1, k_2, \beta_1, \beta_2, \gamma, \mu, m)$ be a multiplier associated to the process $(\bar{s}_1, \bar{s}_2, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y})$ as in Definition 3.1. Assume by contradiction that (4.4) is not satisfied. Then by conditions (3.1)–(3.8), $(p, k_1, k_2, \beta_1, \beta_2, \gamma, \mu, m)$ satisfies (4.2) and one has

$$m(\bar{s}_1) \in \partial_x^{>} h(\bar{s}_1, \bar{y}(\bar{s}_1)), \qquad p(\bar{s}_1) \in \mathrm{proj}_{x_1} N_{\mathcal{T}}(\bar{s}_1, \bar{y}(\bar{s}_1), \bar{s}_2, \bar{y}(\bar{s}_2)).$$

Since $N_{\mathcal{T}}(\bar{s}_1, \bar{y}(\bar{s}_1), \bar{s}_2, \bar{y}(\bar{s}_2))$ is a cone, this implies

$$m(\bar{s}_1) \in \partial_x^{>} h(\bar{s}_1, \bar{y}(\bar{s}_1)) \cap \left(-\mathrm{proj}_{x_1} N_{\mathcal{T}}(\bar{s}_1, \bar{y}(\bar{s}_1), \bar{s}_2, \bar{y}(\bar{s}_2))\right),$$

in contradiction with (4.3). □

As a consequence of Proposition 4.1, when **(CN)** is valid, in Theorems 3.2, 3.3, and 3.5 we can equivalently consider nondegenerate multipliers only.

Condition **(CN)** is trivially satisfied when, for instance, $(\bar{s}_1, \bar{y}(\bar{s}_1)) \in \mathrm{Int}(\Omega)$, as in this case $\partial_x^{>} h(\bar{s}_1, \bar{y}(\bar{s}_1)) = \emptyset$. For less trivial situations in which condition **(CN)** is satisfied, we refer the reader to [10, Rem. 3.1, 3.2].

4.2. Nondegeneracy conditions in the case of fixed initial point

We analyze the case with fixed initial point, for which it is immediate to see that condition **(CN)** is never verified if the point lies on $\partial\Omega$. Given some value $\check{x}_0 \in \mathbb{R}^n$ and a closed set $\tilde{\mathcal{T}} \subseteq \mathbb{R}^{1+n}$, the endpoint constraint set $\mathcal{T}$ takes now the form

$$\mathcal{T} = \{(0, \check{x}_0)\} \times \tilde{\mathcal{T}}. \tag{4.5}$$

Since the initial time is always zero, in this subsection for any $S > 0$ we simply write (S, ω, α, y), $(S, \underline{\omega}, \underline{\alpha}, \lambda, y)$, $\mathcal{V}(S)$, $\mathcal{W}(S)$, $\mathcal{A}(S)$, $\Lambda(S)$ in place of $(0, S, \omega, \alpha, y)$, $(0, S, \underline{\omega}, \underline{\alpha}, \lambda, y)$, $\mathcal{V}([0, S])$, $\mathcal{W}([0, S])$, $\mathcal{A}([0, S])$, $\Lambda([0, S])$, respectively. Furthermore, we imply that all processes satisfy $y(0) = \check{x}_0$.

Definition 4.2 (Nondegenerate normal and abnormal extremal)**.** Let $\bar{z} := (\bar{S}, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{y})$ be a feasible relaxed process for which **(H1)**–**(H3)** are verified. Given a function $\Psi : \mathbb{R}^{1+n} \to \mathbb{R}$ which is Lipschitz continuous on a neighborhood of $(\bar{S}, \bar{y}(S))$, we call *nondegenerate multiplier* any element $(p, k, \beta, \gamma, \mu, m)$ that meets conditions (3.2), (3.6), (3.7) and (3.8) of Definition 3.1 on $[0, \bar{S}]$, and satisfies the strengthened nontriviality condition

$$\mu(]0, \bar{S}]) + \|q\|_{L^\infty([0,\bar{S}])} + \gamma + \beta \neq 0, \tag{4.6}$$

the transversality conditions

$$(k, -q(\bar{S})) \in \gamma \partial \Psi(\bar{S}, \bar{y}(\bar{S})) + N_{\tilde{\mathcal{T}}}(\bar{S}, \bar{y}(\bar{S})) + \beta\, \partial h(\bar{S}, \bar{y}(\bar{S})), \tag{4.7}$$

and

$$k \in \operatorname*{ess}_{s \to \bar{S}} \Big(\max_{(w,a) \in \overline{V(s)} \times A(s)} q(\bar{S}) \cdot \mathcal{F}\big(s, \bar{y}(\bar{S}), w, a\big) \Big) \tag{4.8}$$

(q is as in Definition 3.1), and $\beta = 0$ if either $h(\bar{S}, \bar{y}(\bar{S})) < 0$ or $\tilde{\mathcal{T}} \subseteq \{\bar{S}\} \times \mathbb{R}^n$. We call $\bar{z}$ a *nondegenerate* Ψ*-extremal* if nondegenerate multipliers exist. Then, we say that $\bar{z}$ is *nondegenerate normal* if all possible choices of nondegenerate multipliers have $\gamma > 0$, and *nondegenerate abnormal* if there is at least one nondegenerate multiplier with $\gamma = 0$.

Clearly, a nondegenerate abnormal (Ψ-)extremal is an abnormal extremal and any normal Ψ-extremal is nondegenerate normal. However, we have examples where a Ψ-minimizer of the extended problem is a nondegenerate normal Ψ-extremal, but an abnormal extremal (see example 4.10 below, or [11, Ex. 5.1]). In these situations, the nondegenerate normality test established in Theorem 4.7 below detects the absence of gap, while Theorem 3.3 gives no information.

To introduce sufficient nondegeneracy conditions, we first extend the relaxed control system by introducing a new variable, ξ. Precisely, with a small abuse of notation, in the following we call relaxed process any element $(S, \underline{\omega}, \underline{\alpha}, \lambda, \xi, y)$ with $S > 0$ and $(\underline{\omega}, \underline{\alpha}, \lambda, \xi, y) \in \mathcal{W}^{1+n}(S) \times \mathcal{A}^{1+n}(S) \times \Lambda(S) \times W^{1,1}([0, S]; \mathbb{R}^{1+n} \times \mathbb{R}^n)$, which satisfies the Cauchy problem

$$\begin{cases} (\dot{\xi}, \dot{y})(s) = \Big(\lambda(s), \displaystyle\sum_{k=0}^{n} \lambda^k(s) \mathcal{F}(s, (y, \omega^k, \alpha^k)(s))\Big) \text{ a.e. } s \in [0, S], \\ (\xi, y)(0) = (0, \check{x}_0). \end{cases} \tag{4.9}$$

Furthermore, we define the subset $\Lambda^1(S) := \mathfrak{M}([0,S];\Delta^1_n) \subset \Lambda(S)$, where

$$\Delta^1_n := \bigcup_{k=0}^{n} \{e_k\} \quad (e_0,\dots,e_n \text{ canonical basis of } \mathbb{R}^{1+n}), \tag{4.10}$$

and observe that a relaxed process $(S,\underline{\omega},\underline{\alpha},\lambda,\xi,y)$ with $\lambda \in \Lambda^1(S)$ corresponds to the extended process (S,ω,α,ξ,y),[2] where

$$(\omega,\alpha) := \sum_{k=0}^{n} (\omega_k,\alpha_k)\chi_{\{s\in[0,S]:\ \lambda(s)=e_k\}}.$$

Let $(\bar S,\underline{\bar\omega},\underline{\bar\alpha},\bar\lambda,\bar\xi,\bar y)$ be the reference feasible relaxed process. We consider the following hypothesis.

(H4) *If $(0,\check x_0) \in \partial\Omega$, there exist $\tilde\delta > 0$, $C > 1$, $(\varepsilon_i)_i \subseteq]0,\bar S]$ with $\varepsilon_i \downarrow 0$, $(\tilde r_i)_i \subseteq L^1([0,\bar S];\mathbb{R}_{\ge 0})$ with $\lim_{i\to+\infty} \|\tilde r_i\|_{L^1([0,\bar S])} \to 0$, a sequence of Lebesgue measurable subsets $\tilde E_i$ of $[0,\bar S]$ with $\lim_{i\to+\infty} \ell(\tilde E_i) = \bar S$, a sequence of extended processes $(\bar S,\tilde\omega_i,\tilde\alpha_i,\tilde\lambda_i,\tilde\xi_i,\tilde y_i)$ with $(\tilde\omega_i,\tilde\alpha_i,\tilde\lambda_i) \in (\mathcal{W}(\bar S)\cap\mathcal{V}(C\sqrt[4]{\varepsilon_i}))\times\mathcal{A}(\bar S)\times\Lambda^1(\bar S)$ for every i, and a sequence of extended controls $(\hat\omega_i,\hat\alpha_i) \in \mathcal{W}(C\sqrt[4]{\varepsilon_i})\times\mathcal{A}(C\sqrt[4]{\varepsilon_i})$, enjoying for any i the following properties:*

(i) *one has*
$$\|(\tilde\xi_i,\tilde y_i) - (\bar\xi,\bar y)\|_{L^\infty([0,\bar S])} \le \varepsilon_i \tag{4.11}$$

(ii) *one has*
$$h(s,\tilde y_i(s)) \le 0 \qquad \forall s \in [0, C\sqrt[4]{\varepsilon_i}]\,; \tag{4.12}$$

(iii) *for a.e. $s \in \tilde E_i$, one has*
$$(\tilde\omega_i,\tilde\alpha_i,\tilde\lambda_i)(s) \in \bigcup_{k=0}^{n}\{(\bar\omega^k(s),\bar\alpha^k(s),e^k)\} + (\tilde r_i(s)\mathbb{B}_m)\times\{0\}\times\{0\} \tag{4.13}$$

(iv) *for all $(\zeta_0,\zeta) \in \partial^* h(0,\check x_0)$, for a.e. $s \in [0, C\sqrt[4]{\varepsilon_i}]$ one has*
$$\zeta\cdot\big[\mathcal{F}(s,\check x_0,(\hat\omega_i,\hat\alpha_i)(s)) - \mathcal{F}(s,\check x_0,(\tilde\omega_i,\tilde\alpha_i)(s))\big] \le -\tilde\delta. \tag{4.14}$$

Remark 4.3. Hypothesis **(H4)** is effective only when $(0,\check x_0) \in \partial\Omega$ and, disregarding the state constraint (4.12), the existence of approximating controls that satisfy the remaining conditions (4.11), (4.13) is a straightforward consequence of the Relaxation Theorem together with assumption **(H1)**, as we will see in the proof of Theorem 3.2 below. At the same time, relation (4.14) is a suitable modification of known constraint qualification conditions (see for instance [18, 19]) that will be crucial in order to show that the reference process is a nondegenerate extremal, as well as abnormal.

[2] According to the above convention, we include in the string also the new variable ξ.

Remark 4.4. Hypothesis **(H4)** is a weaker version of a condition first introduced in [11]. Indeed, in [11] we required the existence of some $\tilde{\delta} > 0$, $\bar{s} \in]0, \bar{S}]$, $(\bar{S}, \tilde{\omega}_i, \tilde{\alpha}_i, \tilde{\lambda}_i, \tilde{\xi}_i, \tilde{y}_i)$ with $(\tilde{\omega}_i, \tilde{\alpha}_i, \tilde{\lambda}_i) \in (\mathcal{W}(\bar{S}) \cap \mathcal{V}(\bar{s})) \times \mathcal{A}(\bar{S}) \times \Lambda^1(\bar{S})$, and $(\hat{\omega}_i, \hat{\alpha}_i) \in \mathcal{V}(\bar{s}) \times \mathcal{A}(\bar{s})$, such that $\|(\tilde{\xi}_i, \tilde{y}_i) - (\bar{\xi}, \bar{y})\|_{L^\infty([0,\bar{S}])} \to 0$ and enjoying for every i the properties (ii), (iv) above, with $\bar{s}$ in place of $C\sqrt[4]{\varepsilon_i}$.

Hypothesis **(H4)** is trivially satisfied when the reference process is in fact a strict sense process on some interval $[0, \bar{s}]$ and satisfies a classical constraint qualification condition introduced in [18], namely, if there exist $\bar{s}$, $\tilde{\delta} > 0$, $(\tilde{\omega}, \tilde{\alpha})$, $(\hat{\omega}, \hat{\alpha}) \in \mathcal{W}(\bar{s}) \times \mathcal{A}(\bar{s})$ such that for a.e. $s \in [0, \bar{s}]$, $\dot{\bar{y}} = \mathcal{F}(s, \bar{y}, \tilde{\omega}, \tilde{\alpha})$ and

$$\sup_{(\zeta_0,\zeta) \in \partial^* h(0,\check{x}_0)} \zeta \cdot \left[\mathcal{F}(s, \check{x}_0, (\hat{\omega}, \hat{\alpha})(s)) - \mathcal{F}(s, \check{x}_0, (\tilde{\omega}, \tilde{\alpha})(s))\right] \leq -\tilde{\delta}.$$

In the special case of an impulsive extension, assumptions implying **(H4)** weaker than the last one can be found e.g. in [11, Rem. 5.2].

Remark 4.5. Let hypotheses **(H1)**–**(H4)** be satisfied with reference to some process $(\bar{S}, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{\xi}, \bar{y})$. First of all, arguing as in [11, Rem. 3.1], on can assume without loss of generality that, for any i, the extended control $(\hat{\omega}_i, \hat{\alpha}_i)$ in **(H4)** is in fact a strict sense control, which belongs to $\mathcal{V}(C\sqrt[4]{\varepsilon_i}) \times \mathcal{A}(C\sqrt[4]{\varepsilon_i})$. Furthermore, **(H4)** implies that there exist $\hat{\delta}$, $\varepsilon > 0$ such that, for any i, $(\zeta_0, \zeta) \in \operatorname{co} \partial h(\sigma, x)$ with $\sigma \in [0, \varepsilon]$, $x \in \{\check{x}_0\} + \varepsilon\mathbb{B}$, $s \leq C\sqrt[4]{\varepsilon_i}$, any continuous path $y : [0, s] \to \{\check{x}_0\} + \varepsilon\mathbb{B}$, and any measurable map $\eta : [0, s] \to \{0, 1\}$, the following integral condition holds:

$$\int_0^s \eta(\sigma)\zeta \cdot \left[\mathcal{F}(\sigma, (y, \hat{\omega}_i, \hat{\alpha}_i)(\sigma)) - \mathcal{F}(\sigma, (y, \tilde{\omega}_i, \tilde{\alpha}_i)(\sigma))\right] d\sigma \leq -\hat{\delta}\, l(s, \eta(\cdot)), \quad (4.15)$$

where

$$l(s, \eta(\cdot)) := \ell(\{\sigma \in [0, s] \,:\, \eta(\sigma) = 1\}). \quad (4.16)$$

In particular, (4.15) holds for all $\zeta \in \partial_x^{>} h(\sigma, x)$, with (σ, x) as above.

In the case with fixed initial point, Theorem 3.2 can be refined as follows:

Theorem 4.6. *Let $\mathcal{T}$ be as in (4.5). Let $\bar{z} := (\bar{S}, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{\xi}, \bar{y})$ be a feasible relaxed process for which hypotheses* **(H1)**–**(H4)** *are verified. Then, if at $\bar{z}$ there is a local infimum gap, $\bar{z}$ is a* nondegenerate *abnormal extremal.*

The proof of this result will be given in Section 5. Arguing as in the previous section, from Theorem 4.6 we can derive the following results.

Theorem 4.7. *Let $\mathcal{T}$ be as in (4.5). Let $\bar{z} := (\bar{S}, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{\xi}, \bar{y})$ be a feasible relaxed process for which hypotheses* **(H1)**–**(H4)** *are verified. Let $\Psi : \mathbb{R}^{1+n} \to \mathbb{R}$ be a Lipschitz continuous function on a neighborhood of $(\bar{S}, \bar{y}(\bar{S}))$. When $\bar{z}$ is a local Ψ-minimizer for (P_{Γ_e}) or (P_{Γ_r}) which is a* nondegenerate *normal Ψ-extremal, then*

$$\Psi(\bar{S}, \bar{y}(\bar{S})) = \inf\, \Psi(S, y(S))$$

over all processes $(0, S, \omega, \alpha, y) \in \Gamma$ with $d_\infty\left((0, S, y), (0, \bar{S}, \bar{y})\right) < \delta$.
Similarly, if $\bar{z}$ is Ψ-minimizer for (P_{Γ_e}) or (P_{Γ_r}) which is a nondegenerate *normal Ψ-extremal, then $\bar{z}$ realizes the infimum of Ψ over Γ.*

Let $\bar{z} := (\bar{S}, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{\xi}, \bar{y})$ be a feasible relaxed process for which hypotheses **(H1)**–**(H4)** are verified. Define the set $\mathcal{M}^*(\bar{z})$ of multipliers (p, k, β, μ, m), where $p \in W^{1,1}([0, \bar{S}]; \mathbb{R}^n)$, $k \in \mathbb{R}$, $\beta \geq 0$, $\mu \in NBV^+([0, \bar{S}]; \mathbb{R})$, $m : [0, \bar{S}] \to \mathbb{R}^n$ is a Borel measurable and μ-integrable function, that meet conditions (3.2), (3.6), (3.7) and (3.8) of Definition 3.1 on $[0, \bar{S}]$, and (4.8) (for q as in Definition 3.1), and such that

$$\|q\|_{L^\infty([0,\bar{S}])} + \mu(]0, \bar{S}]) + \beta \neq 0, \quad (k, -q(\bar{S})) \in N_{\tilde{\mathcal{T}}}(\bar{S}, \bar{y}(\bar{S})) + \beta \partial h(\bar{S}, \bar{y}(\bar{S})).$$

Theorem 4.8. *Let $\mathcal{T}$ be as in* (4.5). *Let $\bar{z} := (\bar{S}, \underline{\bar{\omega}}, \underline{\bar{\alpha}}, \bar{\lambda}, \bar{\xi}, \bar{y})$ be a feasible relaxed process for which hypotheses* **(H1)**–**(H4)** *are verified. If $\mathcal{M}^*(\bar{z}) = \emptyset$, then the constrained control system* (1.1)-(1.2) *is controllable to $\bar{z}$.*

4.3. Some examples

The following example shows that the minimum of a constrained optimal control problem, of its extension, and of the relaxed extended problem can all be different from each other. Accordingly with the results in the previous sections, the extended and the relaxed extended minimizer are abnormal extremals (actually, nondegenerate abnormal extremals, as condition **(CN)** is verified).

Example 4.9. Consider the optimal control problem

$$\begin{cases} \qquad\qquad \text{minimize} \quad -y^1(S) \\ \text{over } S > 0, \ (\omega, \alpha, y) \in \mathcal{V}(S) \times \mathcal{A}(S) \times W^{1,1}([0, S]; \mathbb{R}^3), \text{ satisfying} \\ \dot{y}(s) = \mathcal{F}(s, y(s), \omega(s), \alpha(s)) \quad \text{a.e. } s \in [0, S], \\ h(y(s)) = y^1(s) - 1 \leq 0 \ \ \forall s \in [0, S], \\ y(0) \in \mathbb{R} \times \{0\} \times \{0\}, \quad (S, y(S)) \in \left[\frac{3}{4}, +\infty\right[\times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{\leq 0}, \end{cases} \tag{4.17}$$

where $\mathcal{V}(S) := \mathfrak{M}([0, S],]0, 1])$, $\mathcal{A}(S) := \mathfrak{M}([0, S], \{-1, 1\})$, and the function $\mathcal{F} : \mathbb{R}^6 \to \mathbb{R}^3$, $\mathcal{F}(s, x, w, a) = \mathcal{F}(s, x^1, x^2, x^3, w, a)$, is given by

$$\mathcal{F}(s, x, w, a) := \begin{cases} \left((x^2)^2, x^1 a, (x^2)^2\right) & s \in \left[0, \frac{1}{2}\right], \\ \left(0, x^1 w - x^3 a, w\right) & s \in \left]\frac{1}{2}, +\infty\right[. \end{cases}$$

First of all, we note that, adopting the terminology of the previous sections, there are no feasible strict sense processes for problem (4.17), so the infimum cost for (4.17) is $+\infty$. Indeed, if (S, ω, α, y) was a feasible strict sense process, we should have $\omega(s) > 0$ for a.e. $s \in [0, S]$ and $S \geq 3/4$, from which the contradiction follows:

$$0 \geq y^3(S) - y^3(0) = \int_0^{1/2} (y^2(s))^2 \, ds + \int_{1/2}^S \omega(s) \, ds > 0. \tag{4.18}$$

Let us now consider the corresponding extended problem, where for any $S > 0$ the extended controls ω belong to the set $\mathfrak{M}([0, S], [0, 1])$. In this case, the feasible extended process $\bar{z} = (\bar{S}, \bar{\omega}, \bar{\alpha}, \bar{y})$ given by

$$\bar{S} = 1, \quad \bar{\omega} \equiv 0, \quad \bar{\alpha} \equiv 1, \quad \bar{y} \equiv (0, 0, 0),$$

is a minimizer of the extended problem, with cost = 0. In fact, for any feasible extended process (S,ω,α,y) arguing similarly to (4.18), now we have $y^2\equiv 0$ on $[0,1/2]$ and $\omega=0$ a.e. on $[1/2,S]$, so that $y^1(s)=y^1(0)$ for every $s\in[0,S]$. Recalling that $\alpha(s)\in\{-1,1\}$, the equalities $0=\dot y^2(s)=\alpha(s)y^1(0)$ for a.e. $s\in[0,1/2]$ imply that $y^1\equiv 0$.

Finally, we consider the relaxed extended problem which, given the linearity of the dynamics in the control variables, is equivalent to considering problem (4.17), with $\mathfrak{M}([0,S],[0,1])$ and $\mathfrak{M}([0,S],[-1,1])$ that replace the control sets $\mathcal{V}(S)$, $\mathcal{A}(S)$, respectively. As it is easy to see, a feasible relaxed minimizer is now given by the process $\tilde z=(\tilde S,\tilde\omega,\tilde\alpha,\tilde y)$, where

$$\tilde S=1,\quad \tilde\omega\equiv 0,\quad \tilde\alpha\equiv 0,\quad \tilde y\equiv(1,0,0).$$

(Observe that, because of the state constraint, any feasible relaxed trajectory must satisfy $y^1(s)\le 1$ for every $s\in[0,S]$.) Thus, the minimum cost of the relaxed problem is $=-1$. In conclusion, the minimum cost is $+\infty$ on feasible strict sense processes, 0 on feasible extended processes, and -1 on feasible relaxed processes.

Note that both the minimizing processes $\bar z$ and $\tilde z$ are abnormal extremals, actually, nondegenerate abnormal extremals: just choose in Definition 3.1 the set of nondegenerate multipliers $(p,k_1,k_2,\beta_1,\beta_2,\gamma,\mu,m)$, where $p=(p_1,p_2,p_3)\equiv(0,0,-1)$, $k_1=k_2=\beta_1=\beta_2=0$, $\gamma=0$, and μ, $m\equiv 0$. Furthermore, the nondegeneracy condition **(CN)** is trivially satisfied for $\bar z$, as $h(\bar y(0))=-1<0$, so that $\partial_x^{>}h(\bar y(0))=\emptyset$, but also for $\tilde z$, since $\partial_x^{>}h(\tilde y(0))=(1,0,0)$ and the normal cone $N_{\mathbb{R}\times\{0\}\times\{0\}}(\tilde y(0))=\{0\}\times\mathbb{R}\times\mathbb{R}$, so that $\partial_x^{>}h(\tilde y(0))\cap N_{\mathbb{R}\times\{0\}\times\{0\}}(\tilde y(0))=\emptyset$.

In the following example there is no infimum gap but this fact cannot be deduced from the normality criterion in Theorem 3.3, since the extended minimizer is abnormal. Instead, the absence of gap is detected by Theorem 4.7, as hypothesis **(H4)** is satisfied and the minimizer is nondegenerate normal.

Example 4.10. Let us consider the constrained optimal control problem

$$\begin{cases} \qquad\qquad \text{minimize} \quad -y^2(S) \\ \text{over } S>0,\ (\omega,\alpha,y)\in\mathcal{V}(S)\times\mathcal{A}(S)\times W^{1,1}([0,S];\mathbb{R}^4),\ \text{satisfying} \\ \dot y(s)=\mathcal{F}(s,y(s),\omega(s),\alpha(s))\quad \text{a.e. } s\in[0,S], \\ y(0)=(0,1,0,0) \\ y(s)\in\Omega\ \ \forall s\in[0,S],\quad y(S)\in\mathcal{T}, \end{cases} \tag{4.19}$$

where the function $\mathcal{F}:\mathbb{R}^9\to\mathbb{R}^4$, $\mathcal{F}(s,x,w,a)=\mathcal{F}(s,x^1,\dots,x^4,w^1,w^2,w^3,a)$, is given by

$$\mathcal{F}(s,x,w,a):=\begin{cases} \big(w^1,w^2,(x^3+x^4)w^3,-w^3\big) & s\in[0,1[, \\ \big(w^1,w^2,x^3x^4w^1-w^3,-x^2w^3\big) & s\in[1,3[, \\ \big(0,0,x^2\alpha,(x^3)^2\big) & s\in[3,+\infty[, \end{cases}$$

and $\Omega:=\mathbb{R}\times[-1,1]^3$, $\mathcal{T}:=\{1\}\times[-1,0]\times[0,1]^2$, $V:=\{w=(w^1,w^2,w^3)\in\mathbb{R}_{>0}\times\mathbb{R}^2:\ \ |w|=1\}$, $A:=\{-1,1\}$.

Hence, for any $S > 0$ the set of strict sense controls is $\mathcal{V}(S) \times \mathcal{A}(S)$ with $\mathcal{V}(S) := \mathfrak{M}([0,S],V)$, $\mathcal{A}(S) := \mathfrak{M}([0,S],A)$, while the set of extended controls is $\mathcal{W}(S) \times \mathcal{A}(S)$, where $\mathcal{W}(S) := \mathfrak{M}([0,S],\overline{V})$.

Since for any $x \in \mathcal{T}$ one has $x^2 \in [-1,0]$ and the cost function is $\Psi(x) = -x^2$, for the relaxed extended problem associated to problem (4.19) every feasible process such that $y^2(S) = 0$ is a minimizer. In particular, the following process $\bar{z} := (\bar{S}, \bar{\omega}^0, \bar{\omega}, \bar{y}^0, \bar{y}, \bar{\nu})$, where

$$\bar{S} = 2, \qquad \bar{\omega} = (\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3) = (1,0,0)\chi_{[0,1]} + (0,-1,0)\chi_{]1,2]},$$
$$\bar{y} = (\bar{y}^1, \bar{y}^2, \bar{y}^3, \bar{y}^4) = (s,1,0,0)\chi_{[0,1]} + (1,2-s,0,0)\chi_{[1,2]},$$

is a feasible extended process which is a minimizer of the (relaxed) extended problem. Notice that $\bar{z}$ is not a strict sense process, since $\bar{\omega}^1 \equiv 0$ on $]1,2]$.

From the free-time constrained maximum principle [1, Thm. 5.2], $\bar{z}$ is a Ψ-extremal accordingly to Definition 3.1. Hence, there exists a set of multipliers (p,k,β,γ,μ,m), that meets the conditions of Definition 3.1 on $[0,\bar{S}]$. In particular, $p = (p_1,\dots,p_4) \in W^{1,1}([0,2];\mathbb{R}^4)$ solves the adjoint system, so that $p \equiv (\bar{p}_1,\dots,\bar{p}_4)$ is constant on $[0,2]$. Furthermore, $\beta = 0$ since $\bar{y}(2) \in \operatorname{Int}\Omega$, and $\mu([0,s]) = \mu([0,1])$ for every $s \in [1,2]$ as $\bar{y}(s) \in \operatorname{Int}\Omega$ for every $s \in]1,2]$. Notice that, for $s \in [0,1]$, $\bar{y}(s) \in \Omega$ is equivalent to $h(\bar{y}(s)) \le 0$ for $h(x) := x^2 - 1$. Thus, $m(s) \in \partial_x^{>} h(\bar{y}(s)) = (0,1,0,0)$ μ-a.e. in $[0,2]$ and the function $q = (q_1,\dots,q_4)$ (as in Definition 3.1) is given by

$$q_2(s) = \begin{cases} \bar{p}_2 + \mu([0,s[) & \text{if } s \in [0,1] \\ \bar{p}_2 + \mu([0,1]) & \text{if } s \in]1,2] \end{cases}, \quad (q_1,q_3,q_4) \equiv (\bar{p}_1,\bar{p}_3,\bar{p}_4).$$

From the transversality condition (4.7) it follows that $k = 0$, $q_1 = \bar{p}_1 \in \mathbb{R}$, $q_2(2) = \bar{p}_2 + \mu([0,1]) = \gamma - \lambda_1$ for some $\lambda_1 \ge 0$, $q_3 = \bar{p}_3 \ge 0$, and $q_4 = \bar{p}_4 \ge 0$. The second transversality condition (4.8) implies that

$$\max_{(w^1,w^2,w^3)\in\overline{V}} \left\{\bar{p}_1 w^1 + q_2(2) w^2 - \bar{p}_3 w^3\right\} = k = 0,$$

from which, considering the controls $(1,0,0)$, $(0,\pm1,0)$, and $(0,0,\pm1)$, it follows that $\bar{p}_1 \le 0$, $q_2(2) = 0$, and $\bar{p}_3 = 0$. Notice that $q_2(s) = \bar{p}_2 + \mu([0,s[) \le \bar{p}_2 + \mu([0,1]) = 0$ for every $s \in [0,1[$. Therefore, the maximality condition in $[0,1[$, that reads

$$\max_{(w^1,w^2,w^3)\in\overline{V}} \left\{\bar{p}_1 w^1 + q_2(s) w^2 - \bar{p}_4 w^3\right\} = \bar{p}_1 \le 0,$$

implies that $\bar{p}_1 = 0$, $q_2(s) = \bar{p}_2 + \mu([0,s[) = 0$ for a.e. $s \in [0,1[$, and $\bar{p}_4 = 0$. In particular, $q(s) = 0$ for a.e. $s \in [0,2]$, $\mu([0,s[) = -\bar{p}_2$ for a.e. $s \in [0,1]$, so that $\mu([0,s[) = \mu(\{0\}) = -\bar{p}_2$ for all $s \in [0,2]$, and $\gamma = \lambda_1$.

At this point, by choosing $\gamma = \lambda_1 = 0$ and $\mu = \delta_{\{0\}}$ we obtain a set of degenerate multipliers that meets all the conditions of the maximum principle, so proving that $\bar{z}$ is an abnormal extremal.

However, from the above analysis we can deduce that $\bar{z}$ is nondegenerate normal. Indeed, for each choice of admissible multipliers one has $\beta = k = 0$, $\|q\|_{L^\infty([0,2])} = 0$, and $\mu(]0,2]) = 0$, so that $\gamma \neq 0$ as soon as they verify the strengthened nontriviality condition (4.6). Furthermore, as it is easy to check, condition **(H4)** is verified if we set $\tilde{\omega}_i := \bar{\omega}$ and $\hat{\omega}_i :\equiv (0,-1,0)$ for every $i \in \mathbb{N}$ (see also Remark 4.4). Consequently, Theorem 4.7 guarantees that at $\bar{z}$ there is no infimum gap.

5. Proof of Theorem 3.2, 4.6

Since the proofs involve only processes with trajectories close to the reference trajectory, using standard cut-off techniques we can assume that hypotheses **(H2)-(H3)** are satisfied in the whole space $\mathbb{R}^{1+n}$.

5.1. Proof of Theorem 4.6

In view of the above remark, for any $S > 0$ and any $(\underline{\omega}, \underline{\alpha}, \lambda) \in \mathcal{W}^{1+n}(S) \times \mathcal{A}^{1+n}(S) \times \Lambda(S)$ there exists a unique solution (ξ, y) to the Cauchy problem (4.9), and this solution is defined on the whole interval $[0, S]$. In the following, such solution will be denoted by $(\xi, y)[\underline{\omega}, \underline{\alpha}, \lambda]$. Similarly, for any $(\omega, \alpha) \in \mathcal{W}(S) \times \mathcal{A}(S)$, we will write $y[\omega, \alpha]$ to denote the corresponding solution to (2.1) on $[0, S]$, with initial condition $y(0) = \check{x}_0$.

Step 1. Define the function $\Phi : \mathbb{R}^{1+n+1} \to \mathbb{R}$, given by $\Phi(t, x, z) := d_{\tilde{\mathcal{T}}}(t, x) \vee z$, and for any $S > 0$ and $y \in W^{1,1}([0, S]; \mathbb{R}^n)$, introduce the payoff

$$\mathcal{J}(S, y) := \Phi\Big(S, y(S), \max_{s \in [0,S]} \frac{h(s, y(s))}{1 \vee K_h}\Big).$$

Let $(\varepsilon_i)_i$, $C > 1$, and $((\tilde{\omega}_i, \tilde{\alpha}_i, \tilde{\lambda}_i, \tilde{\xi}_i, \tilde{y}_i))_i$ be as in hypothesis **(H4)**, so that (4.11) holds. For every i, let $\rho_i \geq 0$ satisfy

$$\rho_i^4 = \sup\left\{\mathcal{J}(S, y) : \ (S, \omega, \alpha, \xi, y) \in \Gamma, \ d_\infty((0, S, (\xi, y)), (0, \bar{S}, (\bar{\xi}, \bar{y}))) \leq C^4 \varepsilon_i\right\},$$

By the Lipschitz continuity of Φ and since $\bar{z}$ is an isolated process by Proposition 2.5, for $\delta > 0$ as in Definition 2.4, for i large enough, we have

$$0 < \rho_i \leq C\sqrt[4]{\varepsilon_i} < \sqrt[4]{\delta}, \qquad \lim_{i \to +\infty} \rho_i = 0. \tag{5.1}$$

By well-known continuity properties of the input-output map $(\omega, \alpha) \mapsto y[\omega, \alpha]$, for every ε_i there exists $\delta_i > 0$ such that for any $\omega \in \mathfrak{M}([0, \bar{S}]; U)$ with $\|\omega - \tilde{\omega}_i\|_{L^1([0,\bar{S}])} \leq \delta_i$, one has $\|y[\omega, \tilde{\alpha}_i] - \tilde{y}_i\|_{L^\infty([0,\bar{S}])} \leq (C^4 - 1)\varepsilon_i$. According to Remark 2.6,(i), for any i let us choose a measurable control $\mathring{\omega}_i(s) \in V_{\iota_{\delta_i}}(s)$ for a.e. $s \in [0, \bar{S}]$, such that $\|\mathring{\omega}_i - \tilde{\omega}_i\|_{L^1([0,\bar{S}])} \leq \delta_i$.

For every i, set

$$\mathcal{V}_{\delta_i}(\bar{S}) := \{\omega \in \mathfrak{M}([0, \bar{S}]; U) : \ \omega(s) \in V_{\iota_{\delta_i}}(s) \text{ a.e. } s \in [0, \bar{S}]\}, \tag{5.2}$$

and consider the optimal control problem

$$
(\hat{P}_i)\begin{cases}\text{Minimize } \ \mathcal{J}(S,y)\\ \text{over } S>0,\ (\omega,\alpha,\lambda,\eta)\in\mathcal{V}_{\delta_i}(S)\times\mathcal{A}(S)\times\Lambda^1(S)\times\mathfrak{M}([0,S];\{0,1\}),\\ \text{and trajectories } \ (\xi,y)\in W^{1,1}([0,S];\mathbb{R}^{1+n}\times\mathbb{R}^n),\ \text{satisfying}\\ \dot{\xi}(s)=\lambda(s) \qquad \text{a.e. } s\in[0,S]\\ \dot{y}(s)=\mathcal{F}(s,y,\tilde{\omega}_i,\tilde{\alpha}_i)+\eta(s)[\mathcal{F}(s,y,\hat{\omega}_i,\hat{\alpha}_i)-\mathcal{F}(s,y,\tilde{\omega}_i,\tilde{\alpha}_i)] \text{ a.e. } s\in[0,\rho_i]\\ \dot{y}(s)=\mathcal{F}(s,y(s),\omega(s),\alpha(s)) \quad \text{ a.e. } s\in]\rho_i,S]\\ (\xi,y)(0)=(0,\check{x}_0), \qquad d_\infty((0,S,y),(0,\bar{S},\bar{y}))\le\delta,\end{cases}
$$

where $(\hat{\omega}_i,\hat{\alpha}_i)$ is as in hypothesis **(H4)** and is assumed to belong to $\mathcal{V}(C\sqrt[4]{\varepsilon_i})\times\mathcal{A}(C\sqrt[4]{\varepsilon_i})$ in view of Remark 4.5. We call an element $(S,\omega,\alpha,\lambda,\eta,\xi,y)$ verifying the constraints in $(\hat{P}_i)$ a *process for problem* $(\hat{P}_i)$ and use $\mathcal{S}_i$ to denote the set of such processes. By introducing, for every $(S',\omega',\alpha',\lambda',\eta',\xi',y')$, $(S,\omega,\alpha,\lambda,\eta,\xi,y)\in\mathcal{S}_i$, the distance

$$
\begin{aligned}\mathbf{d}((S',\omega',\alpha',\lambda',\eta',\xi',y'),(S,\omega,\alpha,\lambda,\eta,\xi,y)) &:= |S-S'|+\|\omega'-\omega\|_{L^1([0,S\wedge S'])}\\ &+\ell\left(\{s\in[0,S\wedge S']:\ (\alpha',\lambda',\eta')(s)\neq(\alpha,\lambda,\eta)(s)\}\right),\end{aligned} \tag{5.3}
$$

we can make $(\mathcal{S}_i,\mathbf{d})$ a complete metric space. Notice that, by the very definition of ρ_i, the process $\check{z}_i:=(\bar{S},\check{\omega}_i,\check{\alpha}_i,\check{\lambda}_i,\check{\eta}_i,\check{\xi}_i,\check{y}_i)$ with $\check{\omega}_i:=\mathring{\omega}_i$, $\check{\alpha}_i:=\tilde{\alpha}_i$, $\check{\lambda}_i:=\tilde{\lambda}_i$, $\check{\eta}_i\equiv 0$, and $(\check{\xi}_i,\check{y}_i)$ the corresponding trajectory in $(\hat{P}_i)$ (belongs to $\mathcal{S}_i$ and), is a ρ_i^4-minimizer for problem $(\hat{P}_i)$.[3] In particular, $\check{\xi}_i=\tilde{\xi}_i$ and one has $\|(\check{\xi}_i,\check{y}_i)-(\tilde{\xi}_i,\tilde{y}_i)\|_{L^\infty([0,\bar{S}])}\le(C^4-1)\varepsilon_i$. Hence, from (4.11) it follows that

$$
d_\infty\big((0,\bar{S},(\check{\xi}_i,\check{y}_i)),(0,\bar{S},(\bar{\xi},\bar{y}))\big)=\|(\check{\xi}_i,\check{y}_i)-(\bar{\xi},\bar{y})\|_{L^\infty([0,\bar{S}])}\le C^4\varepsilon_i. \tag{5.4}
$$

Then, from Ekeland's Principle one can deduce that there exists a process $z_i:=(S_i,\omega_i,\alpha_i,\lambda_i,\eta_i,\xi_i,y_i)\in\mathcal{S}_i$ which is a minimizer of the optimization problem

$$
(P_i)\begin{cases}\text{Minimize } \ \mathcal{J}(S,y)+\rho_i^2\ \Big[|S-S_i|+\int_0^S\nu_i(s,\omega(s),\alpha(s),\lambda(s),\eta(s))\,ds\Big]\\ \text{over } \ (S,\omega,\alpha,\lambda,\eta,\xi,y)\in\mathcal{S}_i,\end{cases}
$$

where the function $\nu_i:[0,\bar{S}+\delta]\times U\times\mathbb{R}^q\times\Delta_n^1\times\{0,1\}\to\mathbb{R}$ is defined as

$$
\nu_i(s,w,a,\lambda,\eta):=\begin{cases}|w-\omega_i(s)|+\chi_{\{(a,\lambda,\eta)\neq(\alpha_i(s),\lambda_i(s),\eta_i(s))\}}(s,a,\lambda,\eta), & s\in[0,S_i],\\ 0 & s\in]S_i,\ \bar{S}+\delta].\end{cases}
$$

Furthermore, z_i satisfies

$$
\mathbf{d}\big((S_i,\omega_i,\alpha_i,\lambda_i,\eta_i,\xi_i,y_i),(\bar{S},\check{\omega}_i,\check{\alpha}_i,\check{\lambda}_i,\check{\eta}_i,\check{\xi}_i,\check{y}_i)\big)\le\rho_i^2. \tag{5.5}
$$

[3]Notice that, for any $S>0$ and any control, the Cauchy problem in $(\hat{P}_i)$ is a special case of (4.9), hence it admits a unique solution, which is defined on $[0,S]$. In particular, all processes in $\mathcal{S}_i$ are strict sense processes.

In order to apply Ekeland's variational principle, the domain of minimization must be a complete metric space. For this reason, unlike usual, we apply Ekeland's principle to the sequence of problems $(\hat{P}_i)$ on different domains, in each of which strict sense controls ω must belong to a closed subset $\mathcal{V}_{\delta_i}(S)$ of the set of strict sense controls $\mathcal{V}(S)$, which is generally not closed in the L^1-norm (it is in fact dense in the set of extended controls, $\mathcal{W}(S)$). By (5.4), (5.5), and the continuity of the input-output map associated to (4.9), it follows that, eventually for a subsequence, on $[0, \bar{S}+\delta]$ one has

$$\left\|(\xi_i, y_i) - \left(\bar{\xi}, \bar{y}\right)\right\|_{L^\infty} \to 0, \quad \left(\dot{\xi}_i, \dot{y}_i\right) \rightharpoonup \left(\dot{\bar{\xi}}, \dot{\bar{y}}\right) \quad \text{weakly in } L^1. \tag{5.6}$$

Here (we do not rename) the functions ξ_i, y_i, $\bar{\xi}$, $\bar{y}$ are extended to $[0, \bar{S}+\delta]$ by constant extrapolation and the derivatives are set equal to 0 accordingly. Furthermore, hypothesis **(H4)** and (5.5) imply that there exist a sequence of measurable subsets $E_i \subseteq [0, \bar{S}] \cap [\rho_i, S_i]$ and $(r_i)_i \subset L^1([0, \bar{S}]; \mathbb{R}_{\geq 0})$ such that $\ell(E_i) \to \bar{S}$, $\|r_i\|_{L^1([0,\bar{S}])} \to 0$ as $i \to +\infty$, and, for every i and for a.e. $s \in E_i$:

$$(\omega_i, \alpha_i, \lambda_i)(s) \in \bigcup_{k=0}^{n} \{(\bar{\omega}^k(s), \bar{\alpha}^k(s), e^k)\} + (r_i(s)\mathbb{B}_m) \times \{0\} \times \{0\}. \tag{5.7}$$

Step 2. For each $i \in \mathbb{N}$, set

$$c_i := \max_{s \in [0, S_i]} h(s, y_i(s)), \qquad \tilde{h}(s, x, c) := h(s, x) - c \quad \forall (s, x, c) \in \mathbb{R}^{1+n+1}.$$

As it is easy to verify, the process $(z_i, c_i) = (S_i, \omega_i, \alpha_i, \lambda_i, \eta_i, \xi_i, y_i, c_i)$ turns out to be a minimizer for the optimization problem

$$(Q_i) \begin{cases} \text{Minimize } \Phi\left(S, y(S), \frac{c(S)}{1 \vee K_h}\right) \\ \qquad\qquad\qquad + \rho_i^2 \left\{|S - S_i| + \int_0^S \nu_i(s, \omega(s), \alpha(s), \lambda(s), \eta(s))\, ds\right\} \\ \text{over } (S, \omega, \alpha, \lambda, \eta, \xi, y) \in \mathcal{S}_i, \ c \in W^{1,1}([0, S]; \mathbb{R}), \text{ verifying} \\ \dot{c}(s) = 0, \qquad \tilde{h}(s, y(s), c(s)) \leq 0 \quad \forall s \in [0, S]. \end{cases}$$

Since $\bar{z}$ is isolated, from (5.6) it follows that $\Phi\left(S_i, y_i(S_i), \frac{c_i}{1 \vee K_h}\right) > 0$ for all i, namely, at least one of the following inequalities holds true:

$$d_{\tilde{\mathcal{T}}}(S_i, y_i(S_i)) > 0, \qquad c_i > 0. \tag{5.8}$$

Passing eventually to a subsequence, we may suppose $c_i > 0$ for every i. Indeed, if this is not the case, condition (5.8) implies $d_{\mathcal{T}}(S_i, y_i(S_i)) > 0$. Thus, the process $(S_i, \omega_i, \alpha_i, \lambda_i, \xi_i, y_i, c_i)$ can be replaced by $(S_i, \omega_i, \alpha_i, \lambda_i, \xi_i, y_i, \hat{c}_i)$ with $\hat{c}_i := d_{\tilde{\mathcal{T}}}(S_i, y_i(S_i))/2 > 0$, which is still a minimizer of problem (Q_i). Since, for every i, $\rho_i \leq C\sqrt[4]{\varepsilon_i}$ by (5.1) and the constraint qualification condition (4.15) is valid on the interval $[0, C\sqrt[4]{\varepsilon_i}]$, [11, Lemma 6.1] is applicable, which states that

$$h(s, y_i(s)) \leq 0 < c_i \qquad \forall s \in [0, \rho_i],$$

i.e. the constraint is inactive on $[0, \rho_i]$.

Our aim is now to apply a free end-time constrained Pontryagin Maximum Principle to problem (Q_i) with reference to the minimizer (z_i, c_i). By well known properties of subdifferentials (see [23]) and the conditions in (5.8), we deduce that $\overset{>}{\partial}_{t,x,c}\tilde{h}(s, y_i(s), c_i) = \overset{>}{\partial}_{t,x}h(s, y_i(s)) \times \{-1\}$ and that the relation $(\zeta_i, \zeta_{c_i}) \in \partial\Phi\left(S_i, y_i(S_i), \frac{c_i}{1\vee K_h}\right)$ implies the existence of some $\sigma_i^1, \sigma_i^2 \geq 0$ with $\sigma_i^1 + \sigma_i^2 = 1$, verifying

$$\zeta_i = (\zeta_{s_i}, \zeta_{y_i}) \in \sigma_i^1 \left(\partial d_{\tilde{\mathcal{T}}}(S_i, y_i(S_i)) \cap \partial\mathbb{B}_{1+n}\right), \qquad \zeta_{c_i} = \frac{\sigma_i^2}{1 \vee K_h}.$$

Furthermore, $\sigma_i^k = 0$ for $k \in \{1, 2\}$, when the maximum in $d_{\tilde{\mathcal{T}}}(S_i, y_i(S_i)) \vee \frac{c_i}{1\vee K_h}$ is strictly greater than the k-th term in the maximization. Thus, the Maximum Principle [1, Thm. 5.2] yields the existence of a path $(p_i, \pi_i) \in W^{1,1}([0, S_i]; \mathbb{R}^{n+1})$, numbers $k_i \in \mathbb{R}$, $\gamma_i \geq 0$, $\beta_i \geq 0$, $\sigma_i^1 \geq 0$, $\sigma_i^2 \geq 0$ with $\sum_{k=1}^2 \sigma_i^k = 1$, a measure $\mu_i \in NBV^+([0, S_i]; \mathbb{R})$, and a Borel measurable and μ_i-integrable function $m_i : [0, S_i] \to \mathbb{R}^n$, verifying the following conditions:

(i)′ $\gamma_i + \|p_i\|_{L^\infty([0,S_i])} + \|\pi_i\|_{L^\infty([0,S_i])} + \mu_i([0, S_i]) + \beta_i = 1$;

(ii)′ $-\dot{p}_i(s) \in \text{co}\ \partial_x\Big\{q_i(s) \cdot \mathcal{F}(s, (y_i, \omega_i, \alpha_i)(s))\Big\}$ for a.e. $s \in [\rho_i, S_i]$, and $\dot{\pi}_i(s) = 0$ for a.e. $s \in [0, S_i]$;

(iii)′ $(k_i, -q_i(S_i)) \in (\gamma_i\rho_i^2\, \mathbb{B}_1 \times \{0_n\}) + \beta_i\partial h(S_i, y_i(S_i)) + \gamma_i\sigma_i^1\left(\partial d_{\tilde{\mathcal{T}}}(S_i, y_i(S_i)) \cap \partial\mathbb{B}_{1+n}\right)$,
$\pi_i(0) = 0, \;\; -\pi_i(S_i) + \int_{[0,S_i]} \mu_i(d\sigma) = \gamma_i \dfrac{\sigma_i^2}{1 \vee K_h} - \beta_i$;

(iv)′ $k_i \in \underset{s\to S_i}{\text{ess}}\Big(\max_{(w,a)\in V_{\iota_{\delta_i}}(s)\times A(s)} q_i(S_i) \cdot \mathcal{F}\big(s, y_i(S_i), w, a\big)\Big) + M\gamma_i\rho_i^2\, \mathbb{B}_1$;

(v)′ $m_i(s) \in \overset{>}{\partial}_x h\,(s, y_i(s)), \qquad \mu_i$-a.e. $s \in [0, S_i]$,

(vi)′ $\text{spt}(\mu_i) \subseteq \{s \in [0, S_i] : h\,(s, y_i(s)) - c_i = 0\} \subset [\rho_i, S_i]$,

(vii)′$_1$ $\int_0^{\rho_i} \eta_i\, p_i \cdot [\mathcal{F}(s, y_i, \hat{\omega}_i, \hat{\alpha}_i) - \mathcal{F}(s, y_i, \tilde{\omega}_i, \tilde{\alpha}_i)]\, ds$
$\geq \int_0^{\rho_i} \Big\{(1 - \eta_i)\, p_i \cdot [\mathcal{F}(s, y_i, \hat{\omega}_i, \hat{\alpha}_i) - \mathcal{F}(s, y_i, \tilde{\omega}_i, \tilde{\alpha}_i)] - M\gamma_i\rho_i^2\Big\} ds$;

(vii)′$_2$ $\int_{\rho_i}^{S_i} q_i \cdot \mathcal{F}(s, y_i, \omega_i, \alpha_i) ds \geq \int_{\rho_i}^{S_i} \left[q_i \cdot \mathcal{F}(s, y_i, w, \alpha) - M\gamma_i\rho_i^2\right] ds$
for all $(\omega, \alpha, \lambda) \in \mathcal{V}_{\iota_{\delta_i}}(S_i) \times \mathcal{A}(S_i) \times \Lambda^1(S_i)$,

for some $M > 1$ depending on the diameter of the bounded set U, where

$$q_i(s) := \begin{cases} p_i(s) + \int_{[0,s[} m_i(\sigma)\mu_i(d\sigma) & s \in [0, S_i[, \\ p_i(S_i) + \int_{[0,S_i]} m_i(\sigma)\mu_i(d\sigma) & s = S_i \end{cases}$$

and $\beta_i = 0$ if $h(S_i, y_i(S_i)) < c_i$.

Observe that, for each i, by (ii)$'$ and (iii)$'$ we derive

$$\mu_i([0,S_i]) = \int_{[0,S_i]} \mu_i(ds) = \gamma_i \frac{\sigma_i^2}{1 \vee K_h} - \beta_i \qquad \text{and} \qquad \pi_i \equiv 0. \tag{5.9}$$

Furthermore, by (iv)$'$ we deduce that

$$k_i \leq K_{\mathcal{F}} \|q_i\|_{L^\infty([0,S_i])} + M\gamma_i\rho_i^2.$$

Accordingly, since $\|m_i\|_{L^\infty([0,S_i])} \leq K_h$ and $\partial h(\cdot,\cdot) \subseteq K_h\mathbb{B}_{1+n}$, by (iii)$'$,

$$\begin{aligned} \gamma_i\sigma_i^1 - K_h\beta_i - \gamma_i\rho_i^2 &\leq |(k_i, -q_i(S_i))| \\ &\leq (K_{\mathcal{F}}+1)\|p_i\|_{L^\infty([0,S_i])} + K_h(K_{\mathcal{F}}+1)\mu_i([0,S_i]) + M\gamma_i\rho_i^2. \end{aligned} \tag{5.10}$$

By adding up the non-triviality condition (i)$'$, (5.9), and (5.10), for i sufficiently large we get

$$\begin{aligned} (K_{\mathcal{F}}+2)\|p_i\|_{L^\infty([0,S_i])} &+ (1 + 1 \vee K_h + K_h(1+K_{\mathcal{F}}))\mu_i([0,S_i]) \\ &+ (1 + K_h + 1 \vee K_h)\beta_i \geq 1 - \gamma_i + \gamma_i(\sigma_i^1 + \sigma_i^2) - (M+1)\gamma_i\rho_i^2 \geq \frac{1}{2}, \end{aligned}$$

since $\rho_i \downarrow 0$ and $\sigma_i^1 + \sigma_i^2 = 1$. Hence, scaling the multipliers, we obtain

$$\|p_i\|_{L^\infty([0,S_i])} + \mu_i([0,S_i]) + \beta_i = 1, \qquad \gamma_i \leq \tilde{L}, \tag{5.11}$$

where $\tilde{L} := 2[(K_{\mathcal{F}}+2) \vee (1 + 1 \vee K_h + K_h(1+K_{\mathcal{F}}))]$.

Step 3. Now, we pass to the limit in the relations obtained in *Step 2.* As for the trajectories, consider the functions p_i extended to $[\bar{s}_1 - \delta, \bar{s}_2 + \delta]$ by constant extrapolation to the left and to the right. Extend also the measures μ_i and the functions m_i to $[0, \bar{S}+\delta]$ by setting them identically zero outside $[0, S_i]$. Then, by Banach-Alaoglu's Theorem, there exist a subsequence of $(\mu_i)_i$, $\mu \in NBV^+([0,\bar{S}+\delta];\mathbb{R})$, $m : [0,\bar{S}+\delta] \to \mathbb{R}^n$ Borel measurable and μ-integrable, such that (we do not relabel) $\mu_i \overset{*}{\rightharpoonup} \mu$ weakly* in the dual space $C^*([0,\bar{S}+\delta])$ and $m_i\mu_i(ds) \overset{*}{\rightharpoonup} m\mu(ds)$ (see [23, Proposition 9.2.1]). Furthermore, the $p_i \in W^{1,1}([0,\bar{S}+\delta];\mathbb{R}^n)$ are uniformly bounded and have uniformly bounded derivatives, so that there is some path $p \in W^{1,1}([0,\bar{S}+\delta];\mathbb{R}^n)$ such that, along a suitable subsequence, on $[0,\bar{S}+\delta]$ one has

$$p_i \to p \ \text{ in } L^\infty, \qquad \dot{p}_i \rightharpoonup \dot{p} \ \text{ weakly in } L^1, \qquad q_i \to q \ \text{ in } L^1, \tag{5.12}$$

where

$$q(s) := \begin{cases} p(s) + \int_{[0,s[} m(\sigma)\mu(d\sigma) & s \in [0,\bar{S}+\delta[\\ p(\bar{S}+\delta) + \int_{[0,\bar{S}+\delta]} m(\sigma)\mu(d\sigma) & s = \bar{S}+\delta. \end{cases}$$

By (i)$'$ and (iii)$'$, the real sequences $(k_i)_i$, $(\beta_i)_i$ are bounded. Hence, eventually for a further subsequence, there exist $k \in \mathbb{R}$ and $\beta \geq 0$ such that $k_i \to k$ and $\beta_i \to \beta$, as $i \to +\infty$. In the limit, condition (5.11) yields

$$\|p\|_{L^\infty([0,\bar{S}])} + \mu([0,\bar{S}]) + \beta = 1, \tag{5.13}$$

while (iii)$'$, (v)$'$, and (vi)$'$ imply the transversality conditions (4.7), and the properties (3.7), (3.8) of m and μ, respectively. Notice that if $h(\bar S, \bar y(\bar S)) < 0$, then $h(S_i, y_i(S_i)) < 0 < c_i$ for i sufficiently large by (5.5) and (5.6), hence $\beta_i = 0$ for any i large, so that $\beta = 0$. Similarly, if $\tilde{\mathcal{T}} \subset \{\bar S\} \times \mathbb{R}^n$, then $S_i \equiv \bar S$, hence $\beta_i \equiv 0$ and consequently $\beta = 0$. Condition (4.8) on k follows from the stability properties of essential values (see [23, Chapter 8]), once we observe that **(H1)** (see Remark 2.6,(i)) and **(H3)**,(ii) imply, for a.e. $s \in [S_i - \delta, S_i + \delta]$ ($\subseteq [\bar S - \theta, \bar S + \theta]$),

$$0 \le \max_{(w,a) \in \overline{V(s)} \times A(s)} q_i(S_i) \cdot \mathcal{F}\big(s, y_i(S_i), w, a\big) - \max_{(w,a) \in V_{\iota_{\delta_i}}(s) \times A(s)} q_i(S_i) \cdot \mathcal{F}\big(s, y_i(S_i), w, a\big) \le (1 + K_h)\varphi(\delta_i).$$

As for the costate differential inclusion, recalling that $\partial_x(q \cdot \mathcal{F}) = q \cdot D_x\mathcal{F}$ for any $q \in \mathbb{R}^n$, by (ii)$'$, (5.7) and **(H3)**,(ii) we deduce that, for a.e. $s \in E_i$,

$$\big(- \dot p_i, \dot\xi_i, \dot y_i\big)(s) \in \bigcup_{k=0}^{n} \Big(\text{co } \partial_x \big\{ q(s) \cdot \mathcal{F}(s, (y_i, \bar\omega^k, \bar\alpha^k)(s)) \big\}, e^k, \mathcal{F}(s, (y_i, \bar\omega^k, \bar\alpha^k)(s)) \Big) + \big([(1 + K_{_h})\varphi(r_i(s)) + K_{_\mathcal{F}} |q_i(s) - q(s)|] \mathbb{B}_n \big) \times \{0\} \times \big(\varphi(r_i(s))\mathbb{B}_n\big).$$

Passing to the limit as $i \to +\infty$ (eventually, for a subsequence), which is possible by [23, Th. 2.5.3], we get that, for a.e. $s \in [0, \bar S]$ and for some function $\lambda \in \Lambda(\bar S)$,

$$\Big(- \dot p, \dot{\bar\xi}, \dot{\bar y} \Big)(s) \in \sum_{k=0}^{n} \lambda^k(s) \Big(\text{co } \partial_x \left\{ q(s) \cdot \mathcal{F}(s, (\bar y, \bar\omega^k, \bar\alpha^k)(s)) \right\}, e^k, \mathcal{F}(s, (\bar y, \bar\omega^k, \bar\alpha^k)(s)) \Big).$$

This implies that p satisfies (3.2), since $\lambda = \dot{\bar\xi} = \bar\lambda$ almost everywhere.
To obtain the maximality condition (3.6), take an arbitrary $(\omega, \alpha) \in \mathcal{W}(\bar S + \delta) \times \mathcal{A}(\bar S + \delta)$. In view of Remark 2.6,(i), **(H1)** implies that, for any i, there exists some $v_i \in \mathcal{V}_{\delta_i}(\bar S + \delta)$ such that $\|\omega - v_i\|_{L^1} \le \delta_i \downarrow 0$. By (vii)$'$, we deduce that, for any i, one has

$$\int_0^{\bar S + \delta} q_i \cdot \dot y_i \chi_{[\rho_i, S_i]} \, ds \ge \int_0^{\bar S + \delta} \left\{ q_i \cdot \mathcal{F}(s, y_i, v_i, \alpha) - M\gamma_i\rho_i^2 \right\} \chi_{[\rho_i, S_i]} \, ds.$$

Passing to the limit and using (5.6), (5.12) in the left hand side and the Dominated Convergence Theorem in the right hand side, we obtain

$$\int_0^{\bar S} q(s) \cdot \dot{\bar y}(s) \, ds \ge \int_0^{\bar S} q(s) \cdot \mathcal{F}(s, (\bar y, \omega, \alpha)(s)) \, ds.$$

Since this is true for any (ω, α) as above, by a measurable selection theorem we can conclude that $q(s) \cdot \dot{\bar y}(s) = \max_{(w,a) \in \overline{V(s)} \times A(s)} q(s) \cdot \mathcal{F}(s, \bar y(s), w, a)$ for a.e. $s \in [0, \bar S]$, which implies (3.6). Thus $\bar z$ is an abnormal extremal. To prove that it is in fact a nondegenerate abnormal extremal, it remains to show that the above multipliers fulfill the strengthened non-triviality condition

$$\|q\|_{L^\infty([0,\bar S])} + \mu(]0,\bar S]) + \beta \neq 0. \tag{5.14}$$

Indeed, assume by contradiction that $\|q\|_{L^\infty([0,\bar S])} + \mu(]0,\bar S]) + \beta = 0$. Then, the non-triviality condition (5.13) yields that $\mu(\{0\}) \neq 0$ and $p \equiv -\mu(\{0\})\zeta$ for some $\zeta \in \partial_x^{>} h(0,\check x_0)$. For every i, by the maximality condition (vii)$'_1$ and relation (4.15) (recalling that $\rho_i \le C\sqrt[4]{\varepsilon_i}$ by (5.1)) it follows that

$$\begin{aligned}
0 \ge& \int_0^{\rho_i} (1-2\eta_i)\, p\cdot[\mathcal F(s,y_i,\hat\omega_i,\hat\alpha_i) - \mathcal F(s,y_i,\tilde\omega_i,\tilde\alpha_i)]\, ds \\
&+ \int_0^{\rho_i} \left\{(1-2\eta_i)\,(p_i-p)\cdot[\mathcal F(s,y_i,\hat\omega_i,\hat\alpha_i) - \mathcal F(s,y_i,\tilde\omega_i,\tilde\alpha_i)] - M\gamma_i\rho_i^2\right\} ds \\
\ge& \int_0^{\rho_i} p\cdot[\mathcal F(s,y_i,\hat\omega_i,\hat\alpha_i) - \mathcal F(s,y_i,\tilde\omega_i,\tilde\alpha_i)]\,\chi_{\{\sigma:\ \eta_i(\sigma)=0\}}(s)\, ds \\
&- \int_0^{\rho_i} p\cdot[\mathcal F(s,y_i,\hat\omega_i,\hat\alpha_i) - \mathcal F(s,y_i,\tilde\omega_i,\tilde\alpha_i)]\,\chi_{\{\sigma:\ \eta_i(\sigma)=1\}}(s)\, ds \\
&\quad -\rho_i(2K_{\mathcal F}\|p_i-p\|_{L^\infty} + M\tilde L\rho_i^2) \\
\ge& \mu(\{0\})\,\hat\delta\, l(\rho_i,1-\eta_i(\cdot)) - 2K_{\mathcal F}K_h\, l(\rho_i,\eta_i(\cdot)) - \rho_i(2K_{\mathcal F}\|p_i-p\|_{L^\infty} + M\tilde L\rho_i^2) \\
\ge& \rho_i\left[\mu(\{0\})\,\hat\delta - \mu(\{0\})\,\hat\delta\,\rho_i - 2K_{\mathcal F}K_h\rho_i - 2K_{\mathcal F}\|p_i-p\|_{L^\infty} - M\tilde L\rho_i^2\right] > 0,
\end{aligned}$$

where we use the facts that $l(\rho_i,\eta_i(\cdot)) \le \rho_i^2$ and consequently $l(\rho_i,1-\eta_i(\cdot)) \ge \rho_i - \rho_i^2$, which follow from (5.5). Thus, we get a contradiction and the proof is complete.

□

5.2. Proof of Theorem 3.2

For any $s_1 < s_2$, set $\Lambda^1([s_1,s_2]) := \mathfrak M([s_1,s_2];\Delta_n^1)$, where Δ_n^1 is as in (4.10). Preliminarily, notice that, for any initial condition $\check x_0 \in \mathbb R^n$, the input-output map $\mathcal W([s_1,s_2]) \times \mathcal A([s_1,s_2]) \times \Lambda^1([s_1,s_2]) \ni (\omega,\alpha,\lambda) \mapsto (\xi,y)$, where $(\xi,y) = (\xi,y)[s_1,s_2,\check x_0,\omega,\alpha,\lambda]$ denotes the unique solution to

$$(\dot\xi,\dot y)(s) = (\lambda(s),\mathcal F(s,y(s),\omega(s),\alpha(s))) \quad \text{for a.e. } s\in[s_1,s_2] \tag{5.15}$$

with initial condition $(\xi,y)(s_1) = (0,\check x_0)$, is well defined and continuous, in view of the considerations at the beginning of this section. Let $\bar z := (\bar s_1,\bar s_2,\underline{\bar\omega},\underline{\bar\alpha},\bar\lambda,\bar y)$ be as in the theorem's statement, set $\bar\xi(s) := \int_{\bar s_1}^s \bar\lambda(s')\,ds'$ for all $s\in[\bar s_1,\bar s_2]$ and observe that $(\bar\xi,\bar y)$ solves the differential inclusion

$$\big(\dot\xi,\dot y\big)(s) \in \text{co}\bigcup_{k=0}^n\left\{\big(e^k,\mathcal F(s,y(s),\bar\omega^k(s),\bar\alpha^k(s))\big)\right\} \qquad \text{a.e. } s\in[\bar s_1,\bar s_2].$$

Since $\bar z$ is isolated in view of Proposition 2.5, there is some $\delta>0$ as in Definition 2.4. Fixed a sequence $\varepsilon_i \downarrow 0$, $\varepsilon_i < \delta/2$, by the Relaxation Theorem [23, Th. 2.7.2] for every i there is a measurable control $(\bar\omega_i,\bar\alpha_i,\bar\lambda_i)(s) \in$

$\bigcup_{k=0}^n\{(\bar\omega^k(s),\bar\alpha^k(s),e^k)\}$ for a.e. $s\in[\bar s_1,\bar s_2]$, such that the pair $(\bar\xi_i,\bar y_i)$, where $(\bar\xi_i,\bar y_i):=(\xi,y)[\bar s_1,\bar s_2,\bar y(\bar s_1),\bar\omega_i,\bar\alpha_i,\bar\lambda_i]$, satisfies

$$\|(\bar\xi_i,\bar y_i)-(\bar\xi,\bar y)\|_{L^\infty([\bar s_1,\bar s_2])}\le\varepsilon_i. \tag{5.16}$$

Choose $\delta_i\in]0,\varepsilon_i[$ such that, for any $\omega\in\mathcal{W}([\bar s_1,\bar s_2])$, $\|\omega-\bar\omega_i\|_{L^1([\bar s_1,\bar s_2])}\le\delta_i$, one has $\|(\xi,y)[\bar s_1,\bar s_2,\bar y(\bar s_1),\omega,\bar\alpha_i,\bar\lambda_i]-(\bar\xi_i,\bar y_i)\|_{L^\infty([\bar s_1,\bar s_2])}\le\varepsilon_i$. Let $\check\omega_i(s)\in V_{\iota_{\delta_i}}(s)$ for a.e. $s\in[\bar s_1,\bar s_2]$ ($V_{\iota_{\delta_i}}(s)$ as in Remark 2.6,(i)) be a strict sense control satisfying $\|\check\omega_i-\bar\omega_i\|_{L^1([\bar s_1,\bar s_2])}\le\delta_i$, which exists owing to hypothesis **(H1)**. Hence, setting $\check\alpha_i:=\bar\alpha_i$, $\check\lambda_i:=\bar\lambda_i$, and $(\check\xi_i,\check y_i):=(\xi,y)[\bar s_1,\bar s_2,\bar y(\bar s_1),\check\omega_i,\check\alpha_i,\check\lambda_i]$, we get a strict sense process $(\bar s_1,\bar s_2,\check\omega_i,\check\alpha_i,\check\lambda_i,\check\xi_i,\check y_i)$[4] enjoying the properties

$$\|(\check\xi_i,\check y_i)-(\bar\xi_i,\bar y_i)\|_{L^\infty([\bar s_1,\bar s_2])}\le\varepsilon_i, \tag{5.17}$$

and, for some sequence $(\check r_i)_i\subset L^1([\bar s_1,\bar s_2];\mathbb{R}_{\ge0})$ converging to 0 in L^1,

$$(\check\omega_i,\check\alpha_i,\check\lambda_i)(s)\in\bigcup_{k=0}^n\{(\bar\omega^k(s),\bar\alpha^k(s),e^k)\}+(\check r_i(s)\mathbb{B}_m)\times\{0\}\times\{0\}\ \text{ a.e. } s\in[\bar s_1,\bar s_2].$$

Now, set $\Phi(s_1,x_1,s_2,x_2,z):=d_{\mathcal{T}}(s_1,x_1,s_2,x_2)\vee z$ for all (s_1,x_1,s_2,x_2,z) in $\mathbb{R}^{1+n+1+n+1}$ and, for any $s_1<s_2$, $y\in W^{1,1}([s_1,s_2];\mathbb{R}^n)$, consider the payoff

$$\mathcal{J}(s_1,s_2,y):=\Phi\Big(s_1,y(s_1),s_2,y(s_2),\ \max_{s\in[s_1,s_2]}h(s,y(s))\Big).$$

For every i, let $\rho_i\ge0$ satisfy

$$\rho_i^4=\sup\big\{\mathcal{J}(s_1,s_2,y):\ (s_1,s_2,w,\alpha,y)\in\Gamma,\ d_\infty\big((s_1,s_2,y),(\bar s_1,\bar s_2,\bar y)\big)\le2\varepsilon_i\big\}$$

and introduce the set $\mathcal{S}_i$ of processes $(s_1,s_2,\omega,\alpha,\lambda,\xi,y)$, where $s_1<s_2$, the control $(\omega,\alpha,\lambda)\in\mathcal{V}_{\delta_i}([s_1,s_2])\times\mathcal{A}([s_1,s_2])\times\Lambda^1([s_1,s_2])$ with $\mathcal{V}_{\delta_i}([s_1,s_2]):=\{\omega\in\mathfrak{M}([s_1,s_2];U):\ \omega(s)\in V_{\iota_{\delta_i}}\ \text{a.e. } s\in[s_1,s_2]\}$, and (ξ,y) satisfies (5.15) and has $d_\infty\big((s_1,s_2,y),(\bar s_1,\bar s_2,\bar y)\big)\le\delta$. This set is a complete metric space if endowed with the distance

$$\begin{aligned}\mathbf{d}((s_1',s_2',\omega',\alpha',\lambda',\xi',y'),(s_1,s_2,\omega,\alpha,\lambda,\xi,y))&:=|s_1'-s_1|+|s_2'-s_2|\\&+|y'(s_1')-y(s_1)|+\|\omega'-\omega\|_{L^1(I)}+\ell\{s\in I:\ (\alpha',\lambda')(s)\ne(\alpha,\lambda)(s)\},\end{aligned} \tag{5.18}$$

where $I:=[s_1'\vee s_1,s_2'\wedge s_2]$. Notice that by (5.16), (5.17) it follows that $d_\infty((\bar s_1,\bar s_2,\check y_i),(\bar s_1,\bar s_2,\bar y))\le2\varepsilon_i$, so that the process $(\bar s_1,\bar s_2,\check\omega_i,\check\alpha_i,\check\lambda_i,\check\xi_i,\check y_i)$ is a ρ_i^4-minimizer for the optimal control problem

$$\begin{cases}\text{minimize}\quad\mathcal{J}(s_1,s_2,y)\\ \text{over processes } (s_1,s_2,\omega,\alpha,\lambda,\xi,y)\in\mathcal{S}_i.\end{cases}$$

From now on, except for minor obvious changes, the proof proceeds similarly to the proof of Theorem 4.6 and is actually simpler, since we disregard the nondegeneracy issue. Hence, we omit it. □

[4]As in the proof of Theorem 4.6, with a small abuse of notation we call *process* also an originally defined process, where the variable ξ is added.

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