Introduzione

Nell'ambito del programma di dottorato dell'Università di Padova, l'autore ha sviluppato il suo progetto di ricerca sotto la supervisione del Prof. Andrei Zelevinsky della Northeastern University di Boston (MA). Andrei Zelevinsky, noto esperto di teoria delle rappresentazioni, geometria algebrica, combinatoria algebrica e poliedrale, ha negli ultimi anni incentrato la sua attività scientifica nello sviluppo della teoria delle *algebre cluster*, allo scopo di fornire una struttura algebrica all'interno della quale studiare i concetti di positività totale e di base canonica in gruppi algebrici semisemplici. La teoria delle algebre cluster si è sviluppata negli ultimi sette anni in diversi campi della matematica: teoria di Lie, rappresentazioni di quiver, Grassmanniane di quiver, geometria di Poisson e spazi di Teichmueller, sistemi dinamici discreti, geometria tropicale e altri ancora. Malgrado questi interessanti sviluppi, ancora manca nell'ambito della teoria una definizione generale del concetto di base canonica, una appunto tra le principali motivazioni per l'introduzione delle algebre cluster. Un primo passo in questo senso è stato fatto in [27] nel caso di algebre cluster di rango due. Il passo successivo è naturalmente lo studio delle algebre cluster di rango tre. Se ne richiama brevemente la definizione. Sia $\mathbb{P}$ un gruppo abeliano motiplicativo senza torsione ed $\mathcal{F}$ il campo $\mathbb{QP}(x_1, x_2, x_3)$ delle funzioni razionali in tre variabili sul campo $\mathbb{QP}$ delle frazioni dell'anello gruppale $\mathbb{ZP}$. Diciamo *cluster* ogni base di trascendenza di $\mathcal{F}$ su $\mathbb{QP}$, e variabile cluster ogni elemento di un cluster. Una *algebra cluster di rango tre* è una $\mathbb{ZP}$-sottoalgebra di $\mathcal{F}$ generata dalle variabili cluster appartenenti ai cluster ottenuti a partire da un cluster iniziale $\mathcal{C} = (s_1, s_2, s_3)$ tramite trasformazioni birazionali, dette *mutazioni*, governate da una matrice 3×3 antisimmetrizzabile $B_{\mathcal{C}}$ e da una terna di coefficienti $(p_1^{\mathcal{C}}, p_2^{\mathcal{C}}, p_3^{\mathcal{C}})$ appartenenti al gruppo $\mathbb{P}$, associati al cluster iniziale $\mathcal{C}$. La terna $\{\mathcal{C}, B_{\mathcal{C}}, (p_1^{\mathcal{C}}, p_2^{\mathcal{C}}, p_3^{\mathcal{C}}\}$ è detta *seme* dell'algebra cluster.

Questa classe di algebre è piuttosto vasta; se ne distinguono essenzialmente tre tipi: tipo finito, tipo affine e tipo indefinito. Le algebre cluster di tipo affine sono associate alle matrici di Cartan generalizzate di tipo affine di dimensione tre. Tali matrici sono classificate mediante i seguenti diagrammi di Dynkin:

$$
\begin{array}{llll}
A_2^{(1)}: & \begin{array}{c} \quad 2 \quad \\ 1 \;\diagup\!\!\!\!\!\!\quad\diagdown\; 3 \end{array}\!\!\text{(}1 - 2 - 3 - 1\text{)}; & A_4^{(2)}: & 1 \Leftarrow 2 \Leftarrow 3 \\
C_2^{(1)}: & 1 \Rightarrow 2 \Leftarrow 3; & D_3^{(2)}: & 1 \Leftarrow 2 \Rightarrow 3; \\
G_2^{(1)}: & 1 - 2 \Rrightarrow 3; & D_4^{(3)}: & 1 - 2 \Lleftarrow 3
\end{array}
$$

Ad ognuno dei sei diagrammi corrisponde una classe di algebre cluster parametrizzata dalla scelta del gruppo dei coefficienti $\mathbb{P}$.

Nell'ambito di questa tesi si affronta lo studio delle algebre cluster di tipo $A_2^{(1)}$, $C_2^{(1)}$ e $G_2^{(1)}$, ovvero *algebre cluster di rango tre di tipo affine non intrecciato ("untwisted").* Per ognuna delle algebre cluster di queste classi si sono ottenuti i seguenti risultati (alcuni dei quali ancora solo congetturati per i casi $C_2^{(1)}$ e $G_2^{(1)}$):

- descrizione dell'algebra per generatori e relazioni;
- descrizione del *grafo di scambio* che ha come vertici i clusters e come lati le loro mutazioni;
- studio dell'insieme degli elementi *"positivi"*;
- l'esistenza e l'esplicita costruzione di una base canonica;
- determinazione di tutte le possibili basi canoniche;
- parametrizzazione degli elementi di ogni base canonica mediante gli elementi del reticolo delle radici corrispondente alla matrice di Cartan associata all'algebra;
- si è introdotta una nuova descrizione delle basi canoniche come insiemi di generatori omogenei dell'algebra.

L' autore ha potuto sviluppare la sua ricerca nell'ambito di un' intensa attività seminariale all'Università di Padova. In questo ambito è cresciuto il suo interesse per gli aspetti della teoria delle algebre cluster connesse alle rappresentazioni dei quivers e alla teoria dei moduli Tilting. Si è cosi interessato agli sviluppi della teoria delle cosiddette *categorie cluster* e delle *algebre cluster tilted*. L'influenza di queste teorie sulla tesi è per lo più circoscritta al caso $A_2^{(1)}$. Si noti, infatti, che questo caso è l'unico in cui, nel diagramma di Dynkin associato, non compaiono lati multipli, ovvero l'unico in cui la matrice di Cartan associata è simmetrica. Un orientamento del diagramma di Dynkin di tipo $A_2^{(1)}$ fornisce così un quiver (negli altri cinque casi si ottiene un quiver valutato). Nel caso di algebre cluster associate a quivers (piuttosto che a quiver valutati), P. Caldero e F.Chapoton in [7] (nel caso A_n), poi P.Caldero e B.Keller in [8] e [9] (nel caso di quivers senza cicli orientati) e infine Y.Palu in [26] (per quivers con cicli orientati), hanno dimostrato l'esistenza di una mappa tra le rappresentazioni indecomponibili senza auto-estensioni del quiver e le variabili cluster che generano l'algebra, nel caso in cui il gruppo dei coefficienti sia il gruppo banale $\mathbb{P} = \{1\}$. Ad ogni rappresentazione del quiver la mappa associa una funzione razionale, i cui coefficienti sono caratteristiche di Eulero–Poincarè di varietà proiettive chiamate Grassmanniane di quiver, che, nel caso delle rappresentazioni indecomponibili senza auto-estensioni, fornisce l'espansione di Laurent delle variabili cluster che generano l'algebra in funzione delle variabili appartenenti al cluster iniziale.

L'autore ha considerato il quiver Q senza cicli orientati associato al diagramma di Dynkin di tipo $A_2^{(1)}$ ed ha ottenuto i seguenti risultati:

- studio delle Grassmanniane di quiver associate alle rappresentazioni indecomponibili senza auto-estensioni di Q e calcolo della loro caratteristica di Eulero-Poincarè;

- studio delle rappresentazioni regolari di Q in termini della loro Grassmanniana di quiver; in particolare classificazione di tali rappresentazioni a meno di *equivalenze destre* cosi come introdotto in [13];

- descrizione dell'immagine della mappa di Caldero–Chapoton–Keller: tale immagine è unione non disgiunta di due basi dell'algebra cluster diverse dalla base canonica. Una di queste basi risulta essere una naturale generalizzazione della base *semi–canonica* costruita in [11] per un'algebra cluster di rango due;

- generalizzazione dei precedenti risultati ad ogni scelta del gruppo dei coefficienti.

Introduction

The study of cluster algebras started in [16], [17], [27] and it has, by now, reached a remarkable stage of development in several directions. The main motivation for introducing this theory was to define an algebraic framework for understanding total positivity and canonical bases in semisimple algebraic groups (see [28] for details).

Here we study rank three cluster algebras of untwisted affine type with a particular attention to their *canonical basis*. Let us briefly recall their definition. Let $\mathbb{P}$ be a *semifield*, i.e. an abelian multiplicative group endowed with a binary operation of *(auxiliary) addition*, denoted by $\oplus$, which is commutative, associative and $a(b\oplus c) = ab\oplus ac$ for every $a, b, c \in \mathbb{P}$. An important example of semifield is the *tropical semifield*: let J be a finite set of indices, the tropical semifield $\mathrm{Trop}(u_j : j \in J)$ is an abelian multiplicative group freely generated by the elements u_j ($j \in J$). The addition $\oplus$ in $\mathrm{Trop}(u_j : j \in J)$ is defined by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} \doteq \prod_j u_j^{\min(a_j,b_j)}. \tag{0.0.1}$$

It can be shown that $\mathbb{P}$ is torsion–free so that its group ring $\mathbb{ZP}$ is a domain. We consider the ambient field $\mathcal{F} = \mathbb{QP}(x_1, x_2, x_3)$ of rational functions in three commuting variables over the field of fractions of $\mathbb{ZP}$. We call $\mathbb{P}$ the *coefficient group*. A *seed* in $\mathcal{F}$ (see Definition 1.1.2) is a triple $\Sigma = (B; \mathbf{x}, \mathbf{y})$ where B is a 3×3 skew–symmetrizable matrix, $\mathbf{x} = (x_1, x_2, x_3)$ is a free generating system of $\mathcal{F}$, i.e. $\mathcal{F} \simeq \mathbb{QP}(\mathbf{x})$; $\mathbf{y}$ is a triple of elements of the coefficient group $\mathbb{P}$. The set $\mathbf{x}$ is called the *cluster* of the seed Σ while its elements are called the *cluster variables* of Σ. The elements of $\mathbf{y}$ are called the *coefficients* of the seed Σ.

For every $k = 1, 2, 3$ and every seed Σ, there exists a seed Σ_k in $\mathcal{F}$ obtained from Σ by a *mutation in direction k* (see Definition 1.1.4). Seed mutations are involutive and hence define an equivalence relation on the set of seeds in $\mathcal{F}$: two seeds are equivalent if one is obtained from the other by a finite sequence of mutations. We denote by $\mathcal{O}(\Sigma)$ the equivalence class of a seed Σ. The cluster algebra $\mathcal{A}(\Sigma)$ with initial seed Σ is the $\mathbb{ZP}$–subalgebra of $\mathcal{F}$ generated by all the cluster variables of the seeds in $\mathcal{O}(\Sigma)$.

The *Cartan counterpart* of a skew–symmetrizable matrix B is a 3×3 generalized Cartan matrix (Definition 1.2.1) $C(B) = (c_{ij})$ such that $c_{ij} = -|b_{ij}|$ if $i \neq j$. If $C(B)$ is a generalized Cartan matrix of type $A_2^{(1)}$, $C_2^{(1)}$ or $G_2^{(1)}$, then $\mathcal{A}(\Sigma)$ is called a cluster algebra of type $A_2^{(1)}$, $C_2^{(1)}$ or $G_2^{(1)}$ respectively. These Cartan matrices are called 3×3 generalized Cartan matrices of untwisted affine type, so that we use the same terminology for the corresponding cluster algebras; their Dynkin diagrams are shown below:

$$A_2^{(1)}: \begin{tikzcd}[column sep=small] & 2 \arrow[dr, dash] & \\ 1 \arrow[ur, dash] \arrow[rr, dash] & & 3; \end{tikzcd} \qquad C_2^{(1)}: \quad 1 \Rightarrow 2 \Leftarrow 3; \qquad G_2^{(1)}: \quad 1 - 2 \Rrightarrow 3.$$

Every cluster $\mathbf{x}$ of $\mathcal{A}$ is a free generating system of $\mathcal{F}$ and hence every element of $\mathcal{A}$ is a rational function in the elements of $\mathbf{x}$. A famous result of S. Fomin and A. Zelevinsky found in [16] asserts that every element of a cluster algebra $\mathcal{A}$ inside $\mathcal{F}$ is actually a *Laurent* polynomial in *every* cluster of $\mathcal{A}$ rather than a rational function. This result is even known as the *Laurent phenomenon.*
We say that an element of $\mathcal{A}$ is *positive* if its Laurent expansion in *every* cluster of $\mathcal{A}$ has coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$. The set of positive elements is closed under addition and multiplications, and hence form a semiring. The initial problem of this thesis was *to describe the semiring of positive elements* in cluster algebras of type $A_2^{(1)}$ without coefficients, i.e. when $\mathbb{P} = \{1\}$. This problem has been solved in cluster algebras of rank 2 in [27] (for every choice of the coefficient group) and this thesis is the natural generalization of that work. A *canonical basis* $\mathbf{B}$ of $\mathcal{A}$ is a $\mathbb{ZP}$–basis of $\mathcal{A}$ such that the semiring of positive elements consists precisely of $\mathbb{Z}_{\geq 0}\mathbb{P}$–linear combinations of elements of $\mathbf{B}$. The problem of describing the semiring of positive elements is hence translated into the problem of *proving the existence of a canonical basis.* This problem is still open in general but there are some expectations about that: for example it is expected that "cluster monomials", i.e. monomials in cluster variables belonging to the same cluster, belong to the canonical basis, when such a basis exists. We now present the main results of the thesis.

Chapter 1: Background. The first chapter of this thesis is devoted to give a recollection of some known results.

Definition of cluster algebras. In Section 1.1 we recall the definition and some general properties about cluster algebras (the main reference for this section is [18]).

Root systems. In Section 1.2 we briefly recall the structure of rank three root systems of affine (untwisted) type, i.e. root systems associated with the Dynkin diagrams of type $A_2^{(1)}$, $C_2^{(1)}$ and $G_2^{(1)}$ shown before.

Bipartite cluster algebras. In Section 1.3 we collect some properties of rank three cluster algebras of *bipartite* type, i.e. in which there exists an exchange matrix B such that $b_{ij} = b_{ji} = 0$ for some $i \neq j$; or equivalently in which the Dynkin diagram of the corresponding Cartan counterpart of B is bipartite. In [18] bipartite cluster algebras of every rank (not necessarily 3) are studied in details. Note that the case $A_2^{(1)}$ is the only case of rank three cluster algebras of affine type that is *not* bipartite.

Quiver representations and the Caldero–Chapoton–Keller map. In Section 1.4 we recall some well–known facts about quiver representations and we introduce the Caldero–Chapoton–Keller map. We briefly recall its definition here. It does not cost too much to put ourselves in the general situation, even if we will use this map only in the rank three case. Let Q be an acyclic quiver with n vertices. With Q it remains naturally associated a skew–symmetric matrix $B_Q = (b_{ij})$ such that $b_{ij} = \mathrm{card}\{j \to i \in Q_1\} - \mathrm{card}\{i \to j \in Q_1\}$ where Q_1 is the set of arrows of Q. In particular with Q it remains associated a (coefficient–free) cluster algebra $\mathcal{A}(Q)$ inside the field $\mathbb{Q}(x_1, \cdots, x_n)$ with initial seed $\{B_Q, \{x_1, \cdots, x_n\}\}$. A famous result of F.Chapoton and P.Caldero (if Q is of type A), of P.Caldero and B.Keller (if Q is acyclic) (there is also a similar result for quivers with cycles due to Y.Palu) associates with a Q–representation M of dimension vector $\mathbf{d} = (d_1, \cdots, d_n)$, a Laurent polynomial X_M given by

$$X_M(x_1, \cdots, x_n) = x_1^{-d_1} \cdots x_n^{-d_n} \sum_{\mathbf{e}} \chi_{\mathbf{e}}(M) \prod_{i,j} (x_i^{d_j - e_j} x_j^{e_i})^{[b_{ij}]_+}$$

where $[b_{ij}]_+ \doteq \max(b_{ij}, 0)$ denotes the maximum between b_{ij} and zero. We call the map $M \mapsto X_M$ the *Caldero–Chapoton–Keller map.* In this formula $\chi_{\mathbf{e}}(M)$ denotes the Euler–Poincaré characteristic of the algebraic variety $Gr_{\mathbf{e}}(M)$ of the sub–representations of M of dimension vector $\mathbf{e} = (e_1, \cdots, e_n)$. $Gr_{\mathbf{e}}(M)$ is a projective variety called *quiver Grassmannian* (see [10] for more details about this variety). A representation M of Q is called *rigid* if it does not have self–extensions, i.e. $Ext^1(M, M) = 0$. The result due to P.Caldero and B.Keller in [9] says that the map $M \mapsto X_M$ is a bijection between indecomposable rigid Q–representations and cluster variables of $\mathcal{A}(Q)$.

Definition of canonical basis and its properties. Section 1.5 is an heuristic treatment of the problem of finding a canonical basis, i.e. given a subset $\mathbf{B}$ of a cluster algebra $\mathcal{A}$ we supply a list of techniques that one can use in order to show that $\mathbf{B}$ has properties of a canonical basis. For a cluster algebra $\mathcal{A}$ of *geometric type*, i.e. when $\mathbb{P}$ is a *tropical semifield*, we get an useful result in Theorem 1.5.7: it gives sufficient condition in order to show that a subset $\mathbf{B}$ of $\mathcal{A}$ (candidate to be a canonical basis) is a linearly independent set (over $\mathbb{ZP}$). These hypothesis imply at once the $\mathbf{g}$*–vector parametrization* of the elements of $\mathbf{B}$, an interesting parametrization of $\mathbf{B}$ by elements of $\mathbb{Z}^n$, where n is the cardinality of every cluster of $\mathcal{A}$. This property was proved in [18] for cluster monomials.

Chapter 2: Cluster algebras of type $A_2^{(1)}$. The second chapter contains the main original part of this thesis. In this chapter we solve the problem of finding a canonical basis in *every* cluster algebra of type $A_2^{(1)}$ of *geometric type*.

Principal Coefficients. We consider the cluster algebra $\mathcal{A}$ with *principal coefficients*

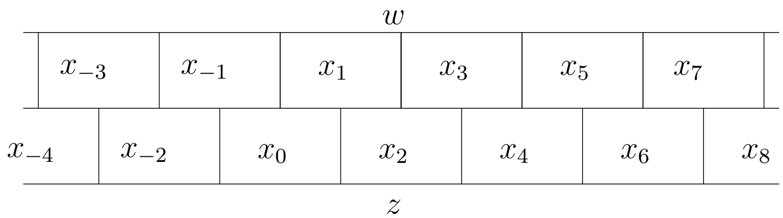

Figure 1: The exchange graph of $\mathcal{A}$

at the seed

$$\Sigma \doteq \{B = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \mathbf{x} = \{x_1, x_2, x_3\}, \mathbf{y} = \{y_1, y_2, y_3\}\}. \tag{0.0.2}$$

By definition this just means that $\mathcal{A} = \mathcal{A}(\Sigma)$ is the cluster algebra with initial seed Σ contained in the field $\mathcal{F} = \mathbb{QP}(x_1, x_2, x_3)$ where $\mathbb{P} = \mathrm{Trop}(y_1, y_2, y_3)$ is the tropical semifield generated by the coefficients of the seed Σ. The Cartan counterpart of B is a Cartan matrix of type $A_2^{(1)}$ and hence $\mathcal{A}(\Sigma)$ is a cluster algebra of type $A_2^{(1)}$. We prove (see Proposition 2.1.1) that $\mathcal{A}$ is the $\mathbb{ZP}$–subalgebra of $\mathcal{F}$ generated by infinitely many rational functions $\{x_m | \, m \in \mathbb{Z}\}$ recursively generated by a relation of the form:

$$x_m x_{m+3} = p_m^- x_{m+1} x_{m+2} + p_m^+$$

where $p_m^\pm \in \mathbb{P}$, together with the two more rational functions, that we call w and z, defined by

$$w \doteq \frac{y_2 x_1 + x_3}{x_2}, \qquad z \doteq \frac{y_1 y_3 \, x_1 x_2 + y_1 + x_2 x_3}{x_1 x_3}.$$

The elements $\{x_m\}$, w and z are all the cluster variables of $\mathcal{A}$ while the sets of the form $\{x_m, x_{m+1}, x_{m+2}\}$, $\{x_{2m+1}, w, x_{2m+3}\}$ and $\{x_{2m}, z, x_{2m+2}\}$ are the clusters of $\mathcal{A}$. Figure 1 shows the exchange graph of $\mathcal{A}$: by definition it has clusters as vertices and an edge between two clusters if they share exactly two cluster variables. In this figure cluster variables are associated with regions: there are infinitely many bounded regions labeled by the x_m's, and there are two unbounded regions labeled respectively by w and z. The cluster corresponding to a vertex common to three regions has the labeling cluster variables of these regions as elements. We refer to the exchange graph of $\mathcal{A}$ as the *brick wall*. Such a graph appeared in [16] as the exchange graph of a *coefficient–free* cluster algebra of type $A_2^{(1)}$ (see Section 2.3.11).

Canonical basis. We define elements $\{u_n | \, n \geq 0\}$ of $\mathcal{A}$ by the initial conditions:

$$u_0 = 1, \quad u_1 = zw - y_1 y_3 - y_2, \quad u_2 = u_1^2 - 2y_1 y_2 y_3$$

together with the recurrence relation for $n \geq 2$

$$u_{n+1} = u_1 u_n - (y_1 y_2 y_3)^n \, u_{n-1}.$$

Recall that a *cluster monomial* is a monomial in cluster variables belonging to the same cluster, i.e. in the algebra $\mathcal{A}$ cluster monomials are the monomials $x_m^a x_{m+1}^b x_{m+2}^c$, or $x_{2m+1}^a w^b x_{2m+3}^c$, or $x_{2m}^a z^b x_{2m+2}^c$ for every non–negative integers a, b, c and for every $m \in \mathbb{Z}$. The main result of the chapter is the following Theorem whose proof takes up the almost whole part of it.

Theorem 0.0.1. *The set* $\mathbf{B} = \{$*cluster monomials*$\} \cup \{u_n w^k,\ u_n z^k |\ n \geq 1,\ k \geq 0\}$ *is a canonical basis of* $\mathcal{A}$*. It is unique up to rescaling by elements of* $\mathbb{P}$*.*

$\mathbf{g}$*–vector parametrization of* $\mathbf{B}$. We prove (see Section 2.3.1) that every element b of $\mathbf{B}$ has the form:

$$b = F_b(y_1 \mathbf{x}^{\mathbf{b}_1}, y_2 \mathbf{x}^{\mathbf{b}_2}, y_3 \mathbf{x}^{\mathbf{b}_3}) \mathbf{x}^{\mathbf{g}_b} \qquad (0.0.3)$$

where $F_b \in \mathbb{Z}[y_1, y_2, y_3]$ is a primitive, i.e. not divisible by any y_i, polynomial in three variables, $\mathbf{g}_b \in \mathbb{Z}^3$ is an integer vector, $\mathbf{b}_i$ is the i–th column vector of the exchange matrix B and we use the notation $\mathbf{x}^{(g_1,g_2,g_3)^t} \doteq x_1^{g_1} x_2^{g_2} x_3^{g_3}$. F_b is called the *F–polynomial* of b, while $\mathbf{g}_b$ is called the $\mathbf{g}$*–vector* of b. Following [18], we choose the *principal $\mathbb{Z}^3$–grading of $\mathcal{A}$* given by for $i = 1, 2, 3$

$$deg(x_i) = \mathbf{e}_i, \quad deg(y_i) = -\mathbf{b}_i \qquad (0.0.4)$$

($\mathbf{e}_i$ is the i–th basis vector of $\mathbb{Z}^3$). It follows that the entries of F_b in (0.0.3) have degree zero; in other words the elements of $\mathbf{B}$ are *homogeneous* with degree $\mathbf{g}_b$ with respect to the principal $\mathbb{Z}^3$–grading of $\mathcal{A}$.

Denominator vectors and roots. By the Laurent phenomenon every element b of $\mathcal{A}$ is a Laurent polynomial in $\{x_1, x_2, x_3\}$ of the form

$$b = \frac{N_b(x_1, x_2, x_3)}{x_1^{d_1} x_2^{d_2} x_3^{d_3}}$$

for some primitive polynomial $N_b \in \mathbb{ZP}[x_1, x_2, x_3]$, and some non–negative integers d_1, d_2, d_3. We consider the root lattice Q of type $A_2^{(1)}$. We fix the basis of simple roots of Q, and we identify $\mathbb{Z}^3$ with Q. The map $b \mapsto \mathbf{d}(b) \doteq (d_1, d_2, d_3)$ is hence a map between $\mathcal{A}$ and Q; it is called the *denominator vector map* (in the cluster $\{x_1, x_2, x_3\}$).

Theorem 0.0.2. *The denominator vector map* $b \mapsto \mathbf{d}(b)$ *is a bijection between* $\mathbf{B}$ *and* Q*. Under this bijection positive real roots of the root system of type* $A_2^{(1)}$ *correspond to the set of cluster variables together with* $\{u_n w,\ u_n z |\ n \geq 1\}$*.*

$\mathbf{g}$*–vectors and denominator vectors.* In Proposition 2.3.7 we found an interesting description of the $\mathbf{g}$–vectors of the elements of $\mathbf{B}$ in terms of their denominator vectors. This can be seen as a generalization of a similar result for bipartite cluster algebras (Theorem 1.3.8): let Q_{In} be the acyclic quiver

$$Q_{In} \doteq \begin{tikzcd} & 2 \arrow[dl] & \\ 1 & & 3 \arrow[ul] \arrow[ll] \end{tikzcd} \qquad (0.0.5)$$

whose underlying graph is the Dynkin diagram of type $A_2^{(1)}$ (i.e. $B_{Q_{In}} = B$). We associate with the quiver $Q = Q_{In}$ its Euler matrix E_Q (recall that $(E_Q)_{ij} = 1$ if $i = j$, -1 if there is an arrow from i to j and 0 otherwise). We consider the piecewise–linear deformation $\mathcal{E}_Q$ of $-E_{Q_{In}}$ given by

$$\mathcal{E}_Q = \begin{pmatrix} -1 & 0 & 0 \\ [\ ?\ ]_+ & -1 & 0 \\ [\ ?\ ]_+ & [\ ?\ ]_+ & -1 \end{pmatrix} \tag{0.0.6}$$

$\mathcal{E}_Q$ acts on the root lattice $Q = \mathbb{Z}^3$ by the (piecewise–linear) action $*$ defined by

$$\mathcal{E}_Q * \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 + [a_1]_+ \\ -a_3 + [a_1]_+ + [a_2]_+ \end{pmatrix}$$

where $[a]_+ \doteq \max(a, 0)$.

Proposition 0.0.3. *Given $b \in \mathbf{B}$, its $\mathbf{g}$–vector $\mathbf{g}_b$ and its denominator vector $\mathbf{d}(b)$ are related by*

$$\mathbf{g}_b = \mathcal{E}_Q * \mathbf{d}(b) \tag{0.0.7}$$

Since the map $\mathcal{E}_Q$ is bijective we get the following corollary of this result:

Corollary 0.0.4. *The map $b \mapsto \mathbf{g}_b$ which associate to an element b of $\mathbf{B}$ its $\mathbf{g}$–vector $\mathbf{g}_b$, is a bijection between $\mathbf{B}$ and $\mathbb{Z}^3$.*

Explicit formulas We find the Laurent expansion of every element of $\mathbf{B}$ in *every* cluster of $\mathcal{A}$ (Theorem 2.1.9).

F–polynomials and quiver Grassmannians. In Section 2.5 we give an interpretation of the F–polynomials of the elements of $\mathbf{B}$ in terms of quiver representations. Let M be a Q_{In} representation, we define the polynomial $F_M \in \mathbb{Z}[y_1, y_2, y_3]$ given by

$$F_M = \sum_{\mathbf{e}=(e_1,e_2,e_3)} \chi_{\mathbf{e}}(M) y_1^{e_1} y_2^{e_2} y_3^{e_3}.$$

The map F has the following multiplicative property: $F_{M \oplus N} = F_M \cdot F_N$. If M is a rigid indecomposable Q–representation of dimension $\mathbf{d}$, by the result due to Caldero and Keller, there exists a unique cluster variable X_M with *denominator vector* $\mathbf{d}$. We get the following result.

Theorem 0.0.5. *If M is a rigid indecomposable Q_{In}–representation, F_M is the F–polynomial of the cluster variable X_M.*

In order to get the previous result we study the quiver Grassmannian $Gr_{\mathbf{e}}(M)$ associated with every indecomposable rigid representation M and we compute its Euler–Poincaré characteristic (Proposition 2.5.2). We note that, since of the muplti-pliative property of the map F, the previous Theorem gives also the F–polynomial of every cluster monomial of $\mathcal{A}$.

Semicanonical basis and non–rigid Q_{In}–representations. Once we have given an interpretation of cluster monomials in terms of quiver representations, we investigate an analogous interpretation for the other elements $\{u_n w^k, u_n z^k\}$ of $\mathbf{B}$. In order to do that we need to study *non*–rigid Q_{In}–representations. The indecomposable non–rigid Q_{In}–representations lie in infinitely many connected components of the Auslander–Reiten quiver of Q_{In} called tubes. There is one tube of *rank* two, i.e. the Auslander–Reiten translation τ has period two in this component, and infinitely many tubes of rank one parameterized by $k = \mathbb{C}$. The *regular homogeneous* representations are the indecomposable Q_{In}–representations given by, for every $n \geq 1$:

$$Reg_n^{\{3,2\}} \doteq \begin{tikzcd} & k^n \arrow[dl, "="'] & \\ k^n & & k^n \arrow[ll, "="] \arrow[ul, "J_n(0)"'] \end{tikzcd} ; \qquad Reg_n^{\{2,1\}} \doteq \begin{tikzcd} & k^n \arrow[dl, "J_n(0)"'] & \\ k^n & & k^n \arrow[ll, "="] \arrow[ul, "="'] \end{tikzcd}$$

$$Reg_n^{\{3,1\}}(\lambda) \doteq \begin{tikzcd} & k^n \arrow[dl, "="'] & \\ k^n & & k^n \arrow[ll, "J_n(\lambda)"] \arrow[ul, "="'] \end{tikzcd}.$$

where $J_n(\lambda)$ is the n-Jordan block of eigenvalue $\lambda \in k$. The arrows labeled by " = " are the identity map. The *regular non–homogeneous* representations are, for $n \geq 0$:

$$RN_n^w \doteq \begin{tikzcd} & k^{n+1} \arrow[dl, "\varphi_2^t"'] & \\ k^n & & k^n \arrow[ll, "="] \arrow[ul, "\varphi_1"'] \end{tikzcd} ; \qquad RN_n^z \doteq \begin{tikzcd} & k^n \arrow[dl, "\varphi_1"'] & \\ k^{n+1} & & k^{n+1} \arrow[ll, "="] \arrow[ul, "\varphi_2^t"'] \end{tikzcd}$$

where $\varphi_1, \varphi_2 :< u_1, \cdots, u_n > \to < v_1, \cdots, v_{n+1} >$, $\varphi_1(u_k) = v_k$ and $\varphi_2(u_k) = v_{k+1}$. One can see that for $n = 0$, RN_0^w (resp. RN_0^z) has dimension vector $(0,1,0)$ (resp. $(1,0,1)$) that is the denominator vector of w (resp. z) and hence this representation is rigid. We then concentrate on non–rigid regular representations, i.e. for $n \geq 1$. In section 2.6 we compute Euler–Poincaré characteristic of quiver Grassmannians associated with such Q_{In}–representations (Proposition 2.6.1), so that we have an explicit description of the image of F. In order to study the image of F it is natural to study the representations of Q_{In} up to *right–equivalence* (see Section 2.5.3). We show that $F_{Reg_n^{\{3,1\}}(\lambda)} = F_{Reg_n^{\{3,1\}}(0)}$ (indeed $Reg_n(\lambda)$ and $Reg_n(0)$ are *right–equivalent*), $F_{Reg_n^{\{2,1\}}} = F_{Reg_n^{\{3,2\}}}$, $F_{RN_n^w} = F_{Reg_n^{\{3,1\}}(0)} F_w$ and $F_{RN_n^z} = F_{Reg_n^{\{3,1\}}(0)} F_z$.

The natural question at this point is to see if this image is a set of "F–polynomials". In other words we ask if given such a module M there exists an element of $\mathcal{A}$ whose corresponding F–polynomial is F_M. The answer to this question is affirmative and it is given by the next theorem.

Theorem 0.0.6. *For every $n \geq 0$, the elements s_n and r_n defined by*

$$\begin{aligned} s_n &= u_n + \mathbf{y}^{\delta} u_{n-2} + \mathbf{y}^{2\delta} u_{n-4} + \cdots = \sum_{k \geq 0} \mathbf{y}^{k\delta} u_{n-2k}, \\ r_n &= s_n + y_2 s_{n-1}, \end{aligned}$$

have the form (0.0.3) *and* $F_{s_n} = F_{Reg_n^{\{3,1\}}(0)}$ *and* $F_{r_n} = F_{Reg_n^{\{3,2\}}}$. *In particular the set* $\mathcal{S} = \{$*cluster monomials*$\} \cup \{s_n w^k,\ s_n z^k \ :\ n \geq 1,\ k \geq 0\}$ *and the set* $\mathcal{R} =$

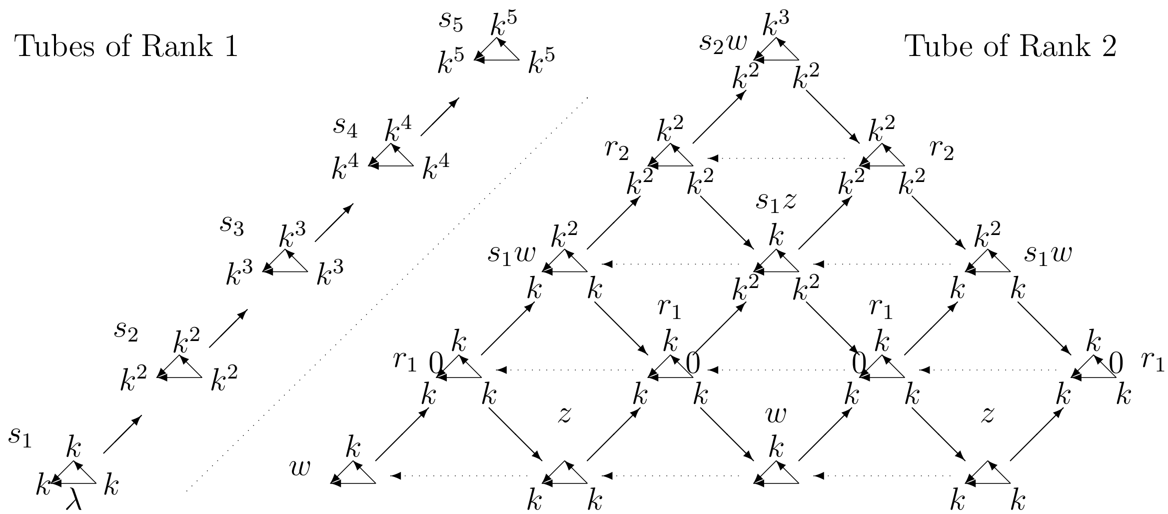

Figure 2: The shape of the tubes of the quiver Q_{In} and the image by the Caldero-Chapoton map.

$\{$cluster monomials$\}$ $\cup\{r_n w^k,\, r_n z^k : n \geq 1,\, k \geq 0\}$ *are $\mathbb{ZP}$–bases of $\mathcal{A}$ (that are not canonical basis).*

Figure 2 shows the image by the map F of the tubes. We call $\mathcal{S}$ a *"semicanonical" basis* of $\mathcal{A}$ in analogy with semicanonical basis found in [11] for a coefficient–free cluster algebra of type $A_1^{(1)}$. In [11] the semicanonical basis was parameterized by *Chebychev's polynomials* of the second kind, while the canonical basis by Chebychev's polynomials of the first kind. The same is true in $\mathcal{A}$ as it is shown in Corollary 2.6.7.

General coefficients In Section 2.4 we extend Theorem 0.0.1 to every choice of the coefficient *tropical* semifield. We consider a tropical semifield $\mathbb{P}$, the field $\mathcal{F} = \mathbb{QP}(x_1, x_2, x_3)$ and the cluster algebra $\mathcal{A}_{\mathbb{P}}$ inside $\mathcal{F}$ with initial seed given by (0.0.2). For every element b of $\mathbf{B}$ we define the element B in analogy with (0.0.3) by

$$B = F_b\left(\frac{y_1}{x_2x_3}, \frac{y_2x_1}{x_3}, y_3x_1x_2\right)\mathbf{x}^{\mathbf{g}_b} \tag{0.0.8}$$

If b is a cluster variable we call B a *principal* cluster variable and if b is a cluster monomial we call B a *principal* cluster monomial. We have the following result that is the main result of the Section.

Theorem 0.0.7. *The set $\mathbf{B}_{\mathbb{P}} = \{$principal cluster monomials$\} \cup \{U_nW^k, U_NZ^k | \, n \geq 1,\, k \geq 0\}$ is a canonical basis for $\mathcal{A}_{\mathbb{P}}$. This basis is unique up to rescaling by elements of $\mathbb{P}$.*

Chapter 3: Cluster algebras of type $C_2^{(1)}$ and $G_2^{(1)}$. The third chapter is devoted to the study of cluster algebras of type $C_2^{(1)}$ and $G_2^{(1)}$ of geometric type. We

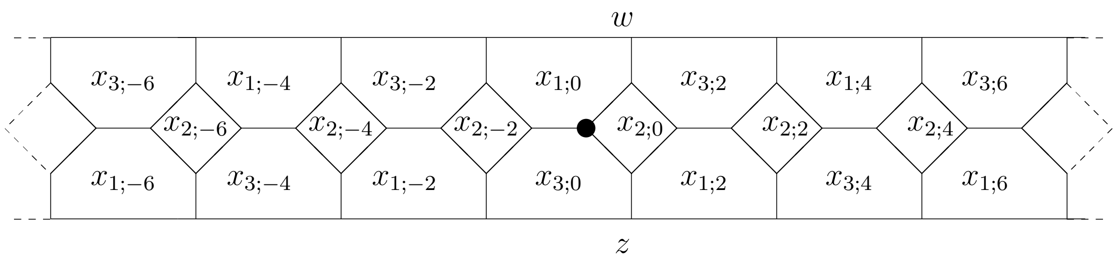

Figure 3: Exchange graph of a cluster algebra of type $C_2^{(1)}$ with principal coefficients at the labeled seed.

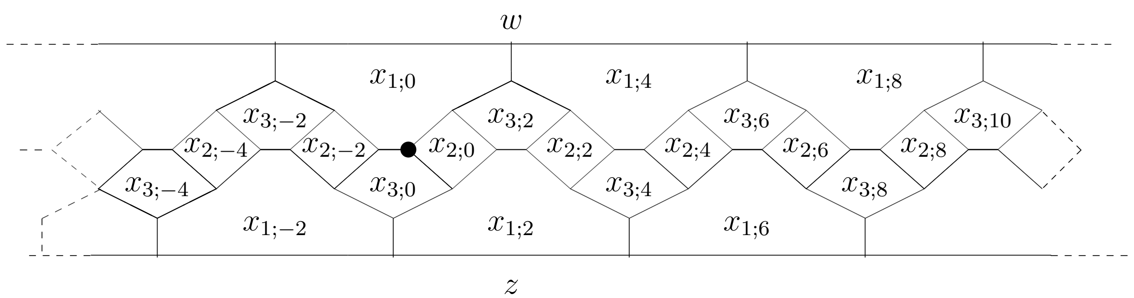

Figure 4: Exchange graph of a cluster algebra of type $G_2^{(1)}$ with principal coefficients at the labeled seed.

define such algebras by generators and relations. Figures 3 and 4 show their exchange graphs. We conjecture the existence of a canonical basis in every such algebras by exhibit explicit elements. In type $C_2^{(1)}$ this conjecture is motivated by Section 3.2 where we study the coefficient–free cluster algebra of type $C_2^{(1)}$: we find the canonical basis of such algebra; we prove this basis is parameterized by the root lattice of type $C_2^{(1)}$; finally we also get the explicit Laurent expansion of every element of such basis.

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Chapter 1

Background

1.1 Background on cluster algebras

In this section we recollect some results about cluster algebras that we will need in the next chapters. We are not going to give a self–contained treatment of the subject and we refer to [18] for all the details.

1.1.1 Definition of a cluster algebra

Definition 1.1.1 (semifields)**.** A *semifield* $(\mathbb{P}, \cdot, \oplus)$ is an abelian multiplicative group endowed with a binary operation of *(auxiliary) addition* $\oplus$ which is commutative, associative and $a(b \oplus c) = ab \oplus ac$ for every $a, b, c \in \mathbb{P}$.

An important example of a semifield is the *tropical semifield*: let J be a finite set of indices, the tropical semifield $\mathrm{Trop}(u_j : j \in J)$ is an abelian multiplicative group freely generated by the elements u_j $(j \in J)$. The addition $\oplus$ in $\mathrm{Trop}(u_j : j \in J)$ is defined by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} \doteq \prod_j u_j^{\min(a_j, b_j)}.$$

Another example of semifield is the *universal semifield* $\mathbb{Q}_{sf}(u_1, \cdots, u_\ell)$ introduced in [18, Definition 2.1]: by definition it is the set of all rational functions in ℓ independent variables $u_1, \cdots, u_\ell$ which can be written as a *subtraction–free* rational expressions in $u_1, \cdots, u_\ell$. For example $u^2 - u + 1 = \frac{(u+1)^3}{u+1} \in \mathbb{Q}_{sf}(u)$. $\mathbb{Q}_{sf}(u_1, \cdots, u_\ell)$ is a semifield with respect to the usual operations of multiplication and addition. This example is universal: any subtraction–free identity that holds in $\mathbb{Q}_{sf}(u_1, \cdots, u_\ell)$ remains valid for any elements $u_1, \cdots, u_\ell$ in an arbitrary semifield.

In [18, Section 5] it is shown that every semifield $\mathbb{P}$ is *torsion-free* as a multiplicative group , hence its group ring $\mathbb{ZP}$ is a domain.

As an ambient field for a cluster algebra $\mathcal{A}$ we consider the field $\mathcal{F}$ isomorphic to the field of rational functions in n independent variables with coefficients in $\mathbb{QP}$, the field of fractions of $\mathbb{ZP}$.

Definition 1.1.2 (seeds)**.** A *seed* in $\mathcal{F}$ is a triple $\Sigma = (B, \mathbf{x}, \mathbf{y})$ where:

- $B = (b_{ij})$ is an $n \times n$ integer matrix which is skew-symmetrizable, i.e. $d_i b_{ij} = -d_j b_{ji}$ for some positive integers $d_1, \cdots, d_n$. B is called the *exchange matrix* of Σ.

- $\mathbf{x} = \{x_1, \cdots, x_n\}$ is an n-tuple of elements of $\mathcal{F}$ forming a trascendence basis of $\mathcal{F}$, that is $\mathcal{F}$ is isomorphic to $\mathbb{QP}(x_1, \cdots, x_n)$. $\mathbf{x}$ is called the *cluster* of Σ and $x_1, \cdots x_n$ are called the *cluster variables* of Σ.

- $\mathbf{y} = \{y_1, \cdots, y_n\}$ is an n-tuple of elements of $\mathbb{P}$. $y_1, \cdots y_n$ are called the coefficients of Σ.

Sometimes it is useful to identify two seeds $(B, \mathbf{x}, \mathbf{y})$ and $(B', \mathbf{x}', \mathbf{y}')$ if there exists a permutation σ of the index set $I = \{1, \cdots, n\}$ such that $x'_i = x_{\sigma(i)}$, $y'_i = y_{\sigma(i)}$ and $b'_{ij} = b_{\sigma(i),\sigma(j)}$, for $i, j \in I$. The class of identified seeds is called an *unlabeled* seed, and its elements are called *labeled seeds* (see [18, Definition 2.3,]).

Example 1.1.3. The seed $\left(\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}\right)$ and the seed $\left(\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \{x_3, x_1, x_2\}, \{y_3, y_1, y_2\}\right)$ can be identified by $\sigma = (132)$.

Definition 1.1.4 (Seed mutations)**.** Let $(B, \mathbf{x}, \mathbf{y})$ be a seed in $\mathcal{F}$ and let $k \in I = \{1, \cdots, n\}$. The *seed mutation* μ_k *in direction* k transforms $(B, \mathbf{x}, \mathbf{y})$ into the seed $\mu_k(B, \mathbf{x}, \mathbf{y}) = (B', \mathbf{x}', \mathbf{y}')$ defined as follows:

- The entries of $B' = (b'_{ij})$ are given by
$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + sg(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise} \end{cases} \tag{1.1.1}$$
where $[x]_+ \doteq \max(x, 0)$ and $sg(x)$ is the sign function (we put $sg(0) = 0$).

- The coefficient tuple $\mathbf{y}' = (y'_1, \cdots, y'_n)$ is given by
$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k \\ y_j y_k^{[b_{kj}]_+}(y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k \end{cases} \tag{1.1.2}$$

- The cluster $\mathbf{x}' = \{x'_1, \cdots, x'_n\}$ is given by $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the *exchange relation*
$$x'_k = \frac{y_k \Pi_i x_i^{[b_{ik}]_+} + \Pi_i x_i^{[-b_{ik}]_+}}{(y_k \oplus 1)x_k}. \tag{1.1.3}$$

(The fact that $\mu_k(\Sigma)$ is again a seed follows immediately from the definition.)

Example 1.1.5. By using example 1.1.3, $\mu_1\left(\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}\right)$ is the (unlabeled) seed $\left(\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \{x_2, x_3, x_4\}, \{y_1^{(2)}, y_2^{(2)}, y_3^{(2)}\}\right)$ where $x_4 = \frac{x_2x_3+y_1}{x_1}$, $y_1^{(2)} = y_1y_2/(y_1 \oplus 1)$, $y_2^{(2)} = y_1y_3/(y_1 \oplus 1)$ and $y_3^{(2)} = 1/y_1$.

Definition 1.1.6 (cluster pattern)**.** Let $\mathbb{T}_n$ be the n–regular tree whose edges are labeled by the numbers $1, \cdots, n$ so that the n edges emanating from each vertex receive different labels. We write $t \overset{k}{\text{———}} t'$ to indicate that vertices $t, t' \in \mathbb{T}_n$ are joined by an edge labeled by k.
A *cluster pattern* is an assignement of a labeled seed Σ_t to every vertex $t \in \mathbb{T}_n$ such that the two seeds Σ_t and $\Sigma_{t'}$ are obtained from each other by a seed mutation in direction k whenever $t \overset{k}{\text{———}} t'$.

It is easy to see that μ_k is involutive and hence mutations define an equivalence relation in the class of seeds of $\mathcal{F}$: given two seeds Σ and Σ' we say that $\Sigma \sim \Sigma'$ if there exists a sequence $\{\mu_{k_1}, \mu_{k_2}, \cdots, \mu_{k_s}\}$ of mutations such that $\Sigma' = \mu_{k_s} \cdots \mu_{k_1} \Sigma$. We indicate by $\mathcal{O}(\Sigma)$ the equivalence class of Σ and with $\Xi(\Sigma)$ the set of cluster variables in $\mathcal{O}(\Sigma)$, that is the set of cluster variables of every seed in $\mathcal{O}(\Sigma)$.

The cluster algebra $\mathcal{A} = \mathcal{A}(\Sigma)$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by cluster variables in $\mathcal{O}(\Sigma)$: $\mathcal{A}(\Sigma) \doteq \mathbb{ZP}[\Xi(\Sigma)]$.

$\mathcal{A}(\Sigma)$ is called the cluster algebra *associated with* Σ or with *initial seed* Σ. By using this second name (with initial seed Σ) one wants to highlights that he is mainly considering the elements of $\mathcal{A}(\Sigma)$ as rational functions in the cluster variables of the cluster of Σ, more than rational functions in every other cluster (indeed clusters are free generating system for $\mathcal{F}$). On the other hand one should not forget that $\mathcal{A}(\Sigma) = \mathcal{A}(\Sigma')$ for every $\Sigma' \in \mathcal{O}(\Sigma)$.

1.1.2 Cluster algebras of geometric type

Let $\Sigma = (B, \mathbf{x}, \mathbf{y})$ be a seed in $\mathcal{F} = \mathbb{QP}(\mathbf{x})$. A cluster algebra $\mathcal{A}(\Sigma)$ is called of *geometric type* if $\mathbb{P}$ is a tropical semifield. In this case mutations are encoded into rectangular matrices ([18, Definition 2.12]) in the following way: let $\mathbb{P} = \mathrm{Trop}(x_{n+j} : j \in \{1, \cdots, r\})$ be a tropical semifield with generators $x_{n+1}, \cdots, x_m$ for some integer $m \geq n$. In this case every coefficient y'_i of every seed $\Sigma' \sim \Sigma$ is a (Laurent) monomial in the x_{n+j}'s of the form:

$$y'_i = \prod_j x_{n+j}^{b'_{n+j,i}}$$

for some integers $b_{n+j,i}$, $j = 1, \cdots, r$. Then we can extend the $n \times n$ exchange matrix B' of Σ' to a rectangular $m \times n$ matrix:

$$\widetilde{B}' = \begin{bmatrix} B' \\ \{b'_{ij}\} \end{bmatrix}_{i=n+1,\cdots,m;\; j=1,\cdots,n}.$$

We call the matrices $\widetilde{B}'$ rectangular exchange matrices. The advantage of working with rectangular exchange matrices is that mutations of coefficients (1.1.2) translate into matrix mutation (1.1.1) and the exchange relations take the simpler form

$$x'_k = \frac{\prod_{i=1}^{m} x_i^{[b_{ik}]_+} + \prod_{i=1}^{m} x_i^{[-b_{ik}]_+}}{x_k}. \tag{1.1.4}$$

Therefore in a cluster algebra of geometric type every seed Σ takes the form $\{\widetilde{B}, \mathbf{x}\}$, since both the coefficients and the exchange matrix of Σ are encoded into the rectangular matrix $\widetilde{B}$. The identification between such seeds is then naturally extended to rectangular exchange matrices by fixing the row indices $i = n+1, \cdots, m$ and by permuting the others.

A particularly important class of cluster algebras of geometric type are the algebras where the tropical semifield $\mathbb{P}$ is generated by the coefficient tuple of Σ, i.e. $\mathbb{P} = \mathrm{Trop}(y_1, \cdots, y_n)$. In this case $\mathcal{A}(\Sigma)$ is called a *cluster algebra with principal coefficients* at the seed Σ ([18, Definition 3.1]). In this case the rectangular exchange matrix of Σ is a $2n \times n$ matrix of the form

$$\widetilde{B} = \begin{bmatrix} B \\ \mathrm{Id_n} \end{bmatrix}$$

where Id_n is the $n \times n$ identity matrix. Moreover, in this case, suppose $\Sigma', \Sigma'' \in \mathcal{O}(\Sigma)$ and $\mu_k(\Sigma') = \Sigma''$, then the exchange relation (1.1.3) becomes

$$x''_k = \frac{\prod_{i=1}^{n} y_i^{[b'_{n+i,k}]_+} \prod_{i=1}^{n} x_i'^{[b'_{ik}]_+} + \prod_{i=1}^{n} y_i^{[-b'_{n+i,k}]_+} \prod_{i=1}^{n} x_i'^{[-b_{ik}]_+}}{x'_k}. \tag{1.1.5}$$

1.1.3 Laurent phenomenon

Theorem 1.1.7. *[16, Theorem 3.1] The cluster algebra $\mathcal{A}$ associated with a seed $(B, \mathbf{x} = \{x_1, \cdots, x_n\}, \mathbf{y} = \{y_1, \cdots, y_n\})$ is contained in the Laurent polynomial ring $\mathbb{ZP}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$, i.e. every element of $\mathcal{A}$ is a Laurent polynomial over $\mathbb{ZP}$ in the cluster variables $x_1, \cdots, x_n$, for every choice of the semifield $\mathbb{P}$.*

More explicitly, for every cluster $\mathcal{C} = \{s_1, \cdots, s_n\}$ of $\mathcal{A}$, an element z of $\mathcal{A}$ has the form

$$z = \frac{N_{\mathcal{C}}(s_1, \cdots, s_n)}{s_1^{d_1} \cdots s_n^{d_n}}$$

where $N_{\mathcal{C}}(s_1, \cdots, s_n) \in \mathbb{ZP}[s_1, \cdots, s_n]$ is a polynomial in n variables with coefficient in $\mathbb{ZP}$. The map $z \mapsto \mathbf{d}(z) = \mathbf{d}_{\mathcal{C}}(z) \doteq (d_1, \cdots, d_n)$ is called *denominator vector of z in the cluster $\mathcal{C}$*.

In view of [17, Proposition 11.2], for a cluster algebra with principal coefficients, Theorem 1.1.7 can be sharpened as follows.

Proposition 1.1.8. *Let $\mathcal{A}_\bullet(\Sigma)$ be a cluster algebra with principal coefficients at the seed $\Sigma = (B, \mathbf{x}, \mathbf{y})$. Then $\mathcal{A}_\bullet(\Sigma) \subseteq \mathbb{Z}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}; y_1, \cdots, y_n]$. That is every element of $\mathcal{A}$ is a Laurent polynomial in $x_1, \cdots, x_n$ whose coefficients are integer polynomials in $y_1, \cdots, y_n$. Thus we can associate to every cluster variable x of $\mathcal{A}_\bullet(\Sigma)$ both its expression in the seed Σ:*

$$x = X(x_1, \cdots, x_n, y_1, \cdots, y_n) \in \mathbb{Z}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}; y_1, \cdots, y_n] \tag{1.1.6}$$

and the polynomial:

$$F_x^\Sigma(y_1, \cdots, y_n) \doteq X(1, \cdots, 1, y_1, \cdots, y_n) \in \mathbb{Z}[y_1, \cdots, y_n] \tag{1.1.7}$$

called the F-polynomial associated with x in the seed Σ. We sometimes write F_s instead F_s^Σ when the seed is clear.

In view of the exchange relations the rational functions X in (1.1.6) and thus the F-polynomials are subtraction-free rational expressions. Then we can consider the evaluation of every F-polynomial in n elements of an *arbitrary* semifield $\mathbb{P}$, and we indicate it by $F|_\mathbb{P}$. To illustrate, let $F(u_1, u_2) = u_1^2 - u_1 u_2 + u_2^2$ and $\mathbb{P} = \mathrm{Trop}(y_1, y_2)$. Then $F|_\mathbb{P}(y_1, y_2) = \frac{y_1^3 \oplus y_2^3}{y_1 \oplus y_2} = 1$.

1.1.4 g–vectors and formulas for cluster variables

In a cluster algebra of rank n with principal coefficients, there exists a $\mathbb{Z}^n$–grading called *principal $\mathbb{Z}^n$-grading* that we are going to recall in this section.

Proposition 1.1.9. *[18, Proposition 6.1] Let $\mathcal{A} = \mathcal{A}_\bullet(\Sigma)$ be a cluster algebra with principal coefficients at the seed $\Sigma = (B, \mathbf{x}, \mathbf{y})$. Every cluster variable of $\mathcal{A}$ can be expressed as a Laurent polynomial X as in* (1.1.6)*. The Laurent polynomial X is homogeneous with respect to the $\mathbb{Z}^n$-grading of $\mathbb{Z}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}; y_1, \cdots, y_n]$ given by*

$$\deg(x_i) = \mathbf{e}_i \quad \deg(y_i) = -\mathbf{b}_j \tag{1.1.8}$$

where $\mathbf{e}_i$ is the i-th standard basis (column) vector in $\mathbb{Z}^n$, and $\mathbf{b}_j = \sum b_{ij} \mathbf{e}_i$ is the j-th column of the matrix B.

Definition 1.1.10. Let s be a cluster variable of $\mathcal{A}_\bullet(\Sigma)$. We define the $\mathbf{g}$*–vector* $\mathbf{g}_\Sigma(s)$ of s in the seed Σ, as the multi-degree of s with respect to (1.1.8), i.e.

$$\mathbf{g}_\Sigma(s) \doteq \deg(s) \in \mathbb{Z}^n \tag{1.1.9}$$

When the seed is clear we just write $\mathbf{g}(s)$ and we call it the $\mathbf{g}$–vector of s.

For $j = 1, \cdots, n$ we define

$$\hat{y}_j \doteq y_j \Pi_i x_i^{b_{ij}}. \tag{1.1.10}$$

We will see some interesting properties of these elements in section 1.1.6. For now we want just to point out that in view of Proposition 1.1.9, they have degree zero,

$$\deg(\hat{y}_j) = 0$$

for every $j = 1, \cdots, n$. In what follow we use the notation $\mathbf{a} = (a_1, \cdots, a_n)$, $\mathbf{s}^{\mathbf{a}} = s_1^{a_1} \cdots s_n^{a_n}$. The promised explicit formula for cluster variables in terms of the F-polynomials and $\mathbf{g}$-vectors is given by the following

Theorem 1.1.11. *[18, Corollary 6.3] Let $\mathcal{A}(\Sigma)$ be an arbitrary cluster algebra with coefficients in some semifield $\mathbb{P}$. Let s be a cluster variable of $\mathcal{A}(\Sigma)$. By definition s is obtained from a cluster variable x_k of the cluster of Σ by a sequence $\mu_{i_1} \circ \cdots \circ \mu_{i_t}$ of mutations* (1.1.3)*: $s = \mu_{i_1} \circ \cdots \circ \mu_{i_t}(x_k)$. We consider the cluster variable S of the cluster algebra $\mathcal{A}_\bullet(\Sigma)$ with principal coefficients at Σ given by the same sequence of mutations: $S = \mu_{i_1} \circ \cdots \circ \mu_{i_t}(x_k)$. To S are associated its F–polynomial F_S (given by* (1.1.7)*) and its $\mathbf{g}$–vector $\mathbf{g}_S$ (given by* (1.1.4)*) (in the seed Σ).*
Then the Laurent expansion of s in the initial seed $\Sigma = (B, \mathbf{x}, \mathbf{y})$ of $\mathcal{A}(\Sigma)$ is given by

$$s = \frac{F_S \mid_{\mathcal{F}} (\widehat{y}_1, \cdots, \widehat{y}_n)}{F_S \mid_{\mathbb{P}} (y_1, \cdots, y_n)} \mathbf{x}^{\mathbf{g}_S} \tag{1.1.11}$$

where we used the notation (1.1.10).

1.1.5 Computing F-polynomials and g-vectors

Let $\mathcal{A} = \mathcal{A}_\bullet(\Sigma)$ be a cluster algebra with principal coefficients at some seed Σ. Given an integer $k \in \{1, \cdots, n\}$, let $\Sigma' = \{\widetilde{B}' = (b'_{ij}), \{x'_1, \cdots, x'_n\}\}$ be a seed of $\mathcal{A}$, and $\Sigma'' = \mu_k(\Sigma') = \{\widetilde{B}'', \{x''_1, \cdots, x''_n\}\}$ be the mutation in direction k of the seed Σ'. We have

- given $i \in \{1, \cdots, n\}$, the F-polynomials F'_i and F''_i associated respectively with the cluster variable x''_i and x'_i are related to each other by the relations

$$\begin{aligned} F''_l &= F'_l \quad \text{for } l \neq k; &(1.1.12)\\ F'_k F''_k &= \prod_{i=1}^n y_i^{[b'_{n+i,k}]_+} \prod_{i=1}^n F_i'^{[b'_{ik}]_+} + \prod_{i=1}^n y_i^{[-b'_{n+i,k}]_+} \prod_{i=1}^n F_i'^{[-b_{ik}]_+}. &(1.1.13)\end{aligned}$$

  Moreover the F-polynomial F_i associated with the initial cluster variable x_i of Σ is 1. (This result is given in [18, Section 5]).

- Given $i \in \{1, \cdots, n\}$, the $\mathbf{g}$-vectors $\mathbf{g}'_i$ and $\mathbf{g}''_i$ associated respectively with the cluster variable x'_i and x''_i are related to each other by the relation

$$\begin{aligned} \mathbf{g}''_l &= \mathbf{g}'_l \quad \text{for } l \neq k; &(1.1.14)\\ \mathbf{g}''_k &= -\mathbf{g}'_k + \sum_{i=1}^n [b'_{ik}]_+ \mathbf{g}'_i - \sum_{i=1}^n [b'_{n+i,k}]_+ \mathbf{b}'_i. &(1.1.15)\end{aligned}$$

  where $\mathbf{b}'_i$ is the i–th column of B'. Moreover the $\mathbf{g}$-vector $\mathbf{g}_i$ associated with the initial cluster variable x_i of Σ is the i–th standard basis vector $\mathbf{e}_i$ in $\mathbb{Z}^n$. (This result is given in [18, Section 6]).

The previous formulas give a receipt for computing F–polynomials and $\mathbf{g}$–vectors of every cluster variable of $\mathcal{A}$ recursively.

1.1.6 g-vector parametrization

This section is quite useful in our treatment of canonical basis as it will be explained in Section 1.5. We follow [18, Section 7].

Let $\mathcal{A}$ be a cluster algebra of geometric type associated with the seed $\{\tilde{B}^0, \mathbf{x}\}$, (see (1.1.1)) and let $\{x_{n+1}, \cdots, x_m\}$ be the generators of the tropical semifield. We assume throughout this subsection that

$$\tilde{B}^0 \text{ has full rank } n. \tag{1.1.16}$$

In [4] it was shown that the same is true for every other matrix obtained from $\tilde{B}^0$ by sequence of mutations. Note that every cluster algebra with principal coefficients satisfy (1.1.16). For a seed $\Sigma_t = \{\mathbf{x}_t, \tilde{B}^t\}$ of $\mathcal{A}$, we define the elements $\hat{y}_{k;t}$ as follows:

$$\hat{y}_{k;t} = y_{k;t} \prod_{j=1}^{n} x_{j;t}^{b^t_{jk}} = \prod_{j=1}^{m} x_{j;t}^{b^t_{jk}} = \tilde{\mathbf{x}}_t^{\mathbf{b}^t_j} \tag{1.1.17}$$

where we use the short-hand notation $\tilde{\mathbf{x}}_\mathbf{t} = \{x_{1;t}, \cdots, x_{n;t}, x_{n+1}, \cdots, x_m\}$ (i.e. $x_{n+i;t} \doteq x_{n+i}$) and $\mathbf{b}^t_j$ is the j–th column of $\tilde{B}^t$. Condition (1.1.16) implies that the elements $\hat{y}_{k;t}$ are algebraically independent over $\mathbb{Z}$. Note that if $\mathcal{A}$ had principal coefficients at the seed Σ_t, (1.1.17) specializes to (1.1.10).
The assignment $t \mapsto (\{\hat{y}_{1;t}, \cdots, \hat{y}_{n;t}\}, B^t)$ is a Y-pattern, i.e. the following Lemma holds:

Lemma 1.1.12. *[18, Proposition 3.7] Whenever* $t \overset{k}{\text{—}} t'$ *then*

$$\hat{y}_{i;t'} = \begin{cases} \hat{y}_{i;t} & \text{if } i = k; \\ \hat{y}_{i;t}\, \hat{y}_{k;t}^{[b^t_{ki}]_+} (\hat{y}_{k;t} + 1)^{-b^t_{ki}} & \textit{otherwise.} \end{cases} \tag{1.1.18}$$

The meaning of the preceding Lemma is that the elements $\{\hat{y}_{k;t}\}$ satisfy the same mutation rule (1.1.2) satisfied by the coefficients of the seeds of a cluster algebra.

Definition 1.1.13. Let $\mathcal{M}$ be the set of all the elements z of $\mathcal{A}$ of the form:

$$z = R(\hat{y}_{1;t}, \cdots, \hat{y}_{n;t}) \prod_{i=1}^{m} x_{i;t}^{a_i} \tag{1.1.19}$$

where R is a rational function in n variables with coefficients in $\mathbb{Q}$ and the exponents are the entries of a vector $\mathbf{a} = (a_1, \cdots, a_m) \in \mathbb{Z}^m$ with integer coefficients.

The following Lemma says that if an element $z \in \mathcal{M}$ has the form (1.1.19) in a seed of $\mathcal{A}$ then it has the same form in every other seed of $\mathcal{A}$.

Lemma 1.1.14. *Suppose* $t \overset{k}{\text{—}} t'$ *. If z has the form* (1.1.19) *for the seed $\Sigma_{t'}$ in t', then it has the same form for the seed Σ_t in t.*

Proof. The proof follows by direct check, using Lemma 1.1.12. Indeed

$$\begin{aligned}
z &= R(\hat{y}_{1;t'}, \cdots, \hat{y}_{n;t'}) \prod_{i=1}^{m} x_{i;t'}^{a_i} \\
&= R(\hat{y}_{1;t}\, \hat{y}_{k;t}^{[b^t_{ki}]_+} (\hat{y}_{k;t}+1)^{-b^t_{ki}}, \cdots, \hat{y}_{k;t}^{-1}, \cdots, \hat{y}_{n;t}\, \hat{y}_{k;t}^{[b^t_{ki}]_+} (\hat{y}_{k;t}+1)^{-b^t_{ki}}) \prod_{i=1}^{m} x_{i;t'}^{a_i} \\
&= R'(\hat{y}_{1;t}, \cdots, \hat{y}_{n;t}) \prod_{i=1}^{m} x_{i;t'}^{a_i} \\
&= R'(\hat{y}_{1;t}, \cdots, \hat{y}_{n;t}) \prod_{i\neq k} x_{i;t}^{a_i} \left(\frac{\prod_{i=1}^m x_{i;t}^{[b_{ik}]_+} + \prod_{i=1}^m x_{i;t}^{[-b_{ik}]_+}}{x_{k;t}}\right)^{a_k} \\
&= R'(\hat{y}_{1;t}, \cdots, \hat{y}_{n;t}) \prod_{i\neq k} x_{i;t}^{a_i} \left(\frac{(\hat{y}_{k;t}+1) \prod_{i=1}^m x_{i;t}^{[-b_{ik}]_+}}{x_{k;t}}\right)^{a_k} \\
&= R'(\hat{y}_{1;t}, \cdots, \hat{y}_{n;t}) (\hat{y}_{k;t}+1)^{a_k} x_{k;t}^{-a_k} \prod_{i\neq k} x_{i;t}^{a_i + a_k[-b_{ik}]_+}
\end{aligned}$$

In the fourth equality we used formula (1.1.4) for the mutation of cluster variables in the geometric type and the definition of $\hat{y}_{k;t}$. □

In view of Theorem 1.1.11 cluster variables are elements of $\mathcal{M}$. Indeed the denominator of the formula (1.1.11) applied to a cluster variable of $\mathcal{A}$ (which is of geometric type), is a monomial in $\{x_{n+1}, \cdots, x_m\}$. Moreover $\mathcal{M}$ is clearly closed under multiplication, so every monomial in cluster variables lies in $\mathcal{M}$.

Definition 1.1.15. A rational function $R \in \mathbb{Q}(u_1, \cdots, u_n)$ is *primitive* if it is a quotient of two polynomials not divisible by any u_i, i.e. it has the form

$$R(u_1, \cdots, u_n) = q \cdot \frac{F(u_1, \cdots, u_n)}{G(u_1, \cdots, u_n)}$$

where $F, G \in \mathbb{Z}[u_1, \cdots, u_n]$, $u_i \nmid F, G$ for every $i = 1, \cdots, n$ and $q \in \mathbb{Q}$.

Lemma 1.1.16. *Under the hypothesis* (1.1.16)*, every element of $\mathcal{M}$ has a unique presentation of the form* (1.1.19) *where R is a primitive rational function.*

Proof. Once z has the form (1.1.19), in view of (1.1.17), z can be written in the required form, i.e. with R primitive. So it remains to prove that such a presentation is unique. Suppose it is not. Then there exist $F_1, F_2, G_1, G_2 \in \mathbb{Z}[u_1, \cdots, u_n]$ primitive and $q_1, q_2 \in \mathbb{Q}$ such that

$$z = q_1 \frac{F_1(\hat{y}_{1;t}, \cdots, \hat{y}_{n;t})}{F_2(\hat{y}_{1;t}, \cdots, \hat{y}_{n;t})} \prod_{i=1}^{m} x_{i;t}^{a_i} = q_2 \frac{G_1(\hat{y}_{1;t}, \cdots, \hat{y}_{n;t})}{G_2(\hat{y}_{1;t}, \cdots, \hat{y}_{n;t})} \prod_{i=1}^{m} x_{i;t}^{b_i}.$$

Then it follows immediately $q_1 = q_2$, $F_2 = G_2$ and

$$F_1(\hat{y}_{1;t}, \cdots, \hat{y}_{n;t}) = G_1(\hat{y}_{1;t}, \cdots, \hat{y}_{n;t}) \prod_{i=1}^{m} x_{i;t}^{c_i}, \tag{1.1.20}$$

where $c_i = b_i - a_i$. We want to show $c_1 = \cdots = c_n = 0$ and so $F_1 = G_1$. Let us write explicit formulas for F_1 and G_1 using multi-indices: $F_1(\hat{y}_{1;t}, \cdots, \hat{y}_{n;t}) = \sum_I \gamma_I \hat{y}_I^{\Gamma_I}$ with $\Gamma_I = (d_i : i \in I)$ and $G_1(\hat{y}_{1;t}, \cdots, \hat{y}_{n;t}) = \sum_I \delta_J \hat{y}_J^{\Delta_J}$ with $\Delta_J = (e_j : j \in J)$. So that $\hat{y}_J^{\Gamma_I} = \mathbf{x}^{\sum_{i \in I} d_i \mathbf{b}_i^t}$ and $\hat{y}_J^{\Delta_J} = \mathbf{x}^{\sum_{j \in J} e_j \mathbf{b}_j^t}$. By (1.1.20) there exist multi-indices I and J such that

$$\hat{y}_I^{\Gamma_I} = \hat{y}_J^{\Delta_J} \prod x_{i;t}^{c_i}$$

that implies $\mathbf{x}_t^{\sum d_i \mathbf{b}_i} = \mathbf{x}_t^{\mathbf{c} + \sum e_j \mathbf{b}_j}$. Since $x_{1;t}, \cdots, x_{m;t}$ are algebraically independent, it must be:

$$\sum d_i \mathbf{b}_i = \mathbf{c} + \sum e_j \mathbf{b}_j.$$

It follows that $\mathbf{c}$ lies in the $\mathbb{Z}$-span of the columns of $\tilde{B}^t$, and then $\prod x_{i;t}^{c_i}$ is a Laurent monomial in the $\hat{y}_{i;t}$. Since F_1 and G_1 are primitive, their ratio can be a Laurent monomial only if they are equal to each other. $\square$

We can now recall the definition of the $\mathbf{g}$–vector parametrization (of $\mathcal{M}$).

Definition 1.1.17. [18, Definition 7.9] For any $z \in \mathcal{M}$ and any $t \in \mathbb{T}_n$, the $\mathbf{g}$-vector of z with respect to t is the vector $\mathbf{g}_t(z) \in \mathbb{Z}^n$ defined as follows: if z is expressed (uniquely) in the form (1.1.19) with R primitive, then we set $\mathbf{g}_t(z) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$.

The following property of $\mathcal{M}$ follows immediately by the definition

Lemma 1.1.18. *The set of elements of $\mathcal{M}$ having the same $\mathbf{g}$–vector is closed under addition.*

Note that the previous definition is consistent with definition 1.1.9. Definition 1.1.17 implies at once that the $\mathbf{g}$-vector has the following multiplicative property:

$$\mathbf{g}_t(z_1 z_2) = \mathbf{g}_t(z_1) + \mathbf{g}_t(z_2).$$

1.1.7 Coefficient–specialization

Definition 1.1.19. [18, Definition 12.1] Let $\mathcal{A}$ and $\overline{\mathcal{A}}$ be two cluster algebras of rank n over the coefficient semifields $\mathbb{P}$ and $\overline{\mathbb{P}}$, respectively, with the respective families of cluster variables $\{x_{i;t}\}$ and $\{\overline{x_{i;t}}\}$, $i = 1, \cdots, n$ and $t \in \mathbb{T}_n$. We say that $\overline{\mathcal{A}}$ is obtained from $\mathcal{A}$ by a *coefficient specialization* if

1. $\mathcal{A}$ and $\overline{\mathcal{A}}$ have the same matrices $B_t = \overline{B}_t$ at every vertex $t \in \mathbb{T}_n$;

2. there is a homomorphism of multiplicative group $\varphi : \mathbb{P} \to \overline{\mathbb{P}}$ that extends to a unique ring homomorphism $\varphi : \mathcal{A} \to \overline{\mathcal{A}}$ such that $\varphi(x_{i;t}) = \overline{x}_{i;t}$ for all i and t.

1.2 Background on root systems

1.2.1 Symmetrizable Cartan matrices, roots and Dynkin diagrams

An $n \times n$ matrix $C = (c_{ij})$ is called a *generalized Caratan matrix* if it satisfies the following properties:

1. $c_{ii} = 2$ for every $i = 1, \cdots, n$;

2. c_{ij} are non positive *integers* for $i \neq j$;

3. $c_{ij} = 0$ implies $c_{ji} = 0$.

We recall the classification theorem of generalized Cartan matrix due to Vinberg:

Theorem 1.2.1. *[24, Theorem 4.3] Let C be a $n \times n$ generalized Cartan matrix, real and indecomposable. Then one and only one of the following three possibilities holds for both C and its transpose C^t:*

(Fin) $det(C) \neq 0$; there exists $u > 0$ such that $Cu > 0$; $Cv \geq 0$ implies $v > 0$ or $v = 0$;

(Aff) corank(C)=1; there exists $u > 0$ such that $Cu = 0$; $Cv \geq 0$ implies $Cv = 0$;

(Ind) there exists $u > 0$ such that $Cu < 0$; $Cv \geq 0$, $v \geq 0$ implies $v = 0$.

Cartan matrices satisfying (Fin) (resp. (Aff), (Ind)) are called of finite type (resp. affine and indefinite type). We have the following Corollary

Corollary 1.2.2. *Let C be a generalized Cartan matrix, real and indecomposable. Then C is of finite type (resp. affine or indefinite) if and only if there exists a vector $u > 0$ such that $Au > 0$ (resp. $= 0$ or < 0).*

Let $C = \{c_{ij}\}_{i,j=1}^{n}$ be a generalized Cartan matrix. Following [24] we associate with C a graph called the *Dynkin diagram* of C. If $c_{ij}c_{ji} \leq 4$ and $|c_{ij}| \geq |c_{ji}|$, the vertices i and j are joined by $|c_{ij}|$ edges and these lines are equipped with an arrow pointing *toward* i if $|c_{ij}| > 1$. If $c_{ij}c_{ji} > 4$, the vertices i and j are joined by a bold–faced line equipped with an ordered pair of integers $(|c_{ij}|, |c_{ji}|)$.

Figure 1.1 shows the Dynkin diagrams of the 3×3 Cartan matrices of affine type. The coordinates of the integer vector δ on the right of each such diagram are coefficients of a linear dependence between the columns of the corresponding generalized Cartan matrix.

$$
\begin{array}{llll}
A_2^{(1)}: & \begin{array}{c} \quad 2 \\ 1 \;\diagup\!\!\!\!-\!\!\!-\!\!\!-\!\!\!\diagdown\; 3 \end{array} & [\delta=(1,1,1)]; & A_4^{(2)}: 1 \Leftarrow 2 \Leftarrow 3 \quad [\delta=(2,2,1)]; \\
C_2^{(1)}: & 1 \Rightarrow 2 \Leftarrow 3 & [\delta=(1,2,1)]; & D_3^{(2)}: 1 \Leftarrow 2 \Rightarrow 3 \quad [\delta=(1,1,1)]; \\
G_2^{(1)}: & 1 - 2 \Rrightarrow 3 & [\delta=(1,2,3)]; & D_4^{(3)}: 1 - 2 \Lleftarrow 3 \quad [\delta=(1,2,1)].
\end{array}
$$

Figure 1.1: Affine Dynkin diagrams of rank three

1.2.2 Root system of type $A_2^{(1)}$

We briefly recall the structure of a root system of type $A_2^{(1)}$ (see [24, Chapter 6]): let

$$
C = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{1.2.1}
$$

be the Generalized Cartan matrix of type $A_2^{(1)}$ (its Dynkin diagram is shown in figure 1.1). C is a symmetric matrix of rank 2. The Kernel of $\mathcal{C}$ is generated by the element $\delta = (1,1,1)$. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a *realization* of C, i.e. $\mathfrak{h}$ is a four dimensional complex vector space, $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ (resp. $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$) is a linearly independent set in $\mathfrak{h}^*$ (resp. $\mathfrak{h}$), $\alpha_j(\alpha_i^\vee) = -1$ if $i \neq j$ and $\alpha_i(\alpha_i^\vee) = 2$. The set $\mathring{\Delta} = \{\pm\alpha_1, \pm\alpha_3, \pm(\alpha_1+\alpha_3)\}$ is then a root system of type A_2 in $\mathfrak{h}^*$ (it is shown in figure 1.2). Let $Q \doteq \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3$ be the *root lattice* and $\delta \doteq \alpha_1 + \alpha_2 + \alpha_3 \in Q$.

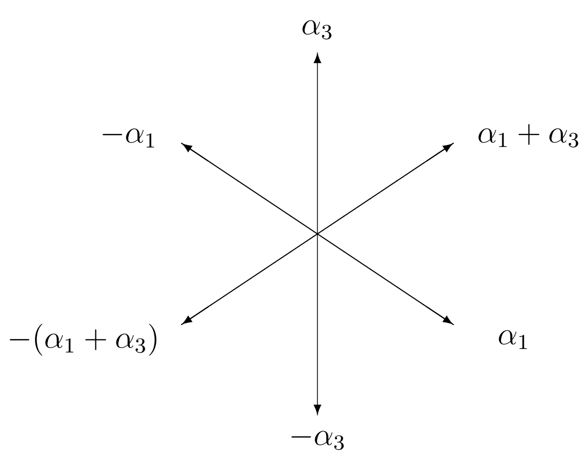

Figure 1.2: Root system of type A_2.

The root system Δ of type $A_2^{(1)}$ is the subset of Q given by the disjoint union $\Delta = \Delta^{Im} \cup \Delta^{re}$ where

$$
\Delta^{Im} = \{\pm n\delta \mid n \geq 1\} = \{(n,n,n) \mid n \in \mathbb{Z} \setminus \{0\}\},
$$

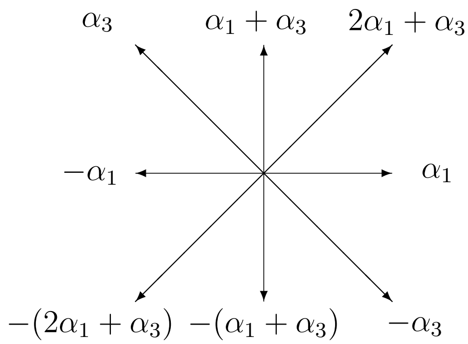

Figure 1.3: Root system of type C_2.

$$\Delta^{re} = \{\alpha + n\delta |\, \alpha \in \overset{\circ}{\Delta},\, n \in \mathbb{Z}\} = \{(n \pm 1, n, n), (n, n, n \pm 1), (n \pm 1, n, n \pm 1) |\, n \in \mathbb{Z}\}.$$

The elements of Δ^{Im} are the *imaginary roots* of Δ while Δ^{re} are the *real roots* of Δ. The positive roots are the roots with non-negative coordinates in the basis Π. Explicitly they are given by $\Delta_+ = \Delta_+^{Im} \cup \Delta_+^{re}$ where

$$\Delta_+^{Im} = \{n\delta |\, n \geq 1\} = \{(n, n, n) |\, n \geq 1\},$$

$$\begin{aligned} \Delta_+^{re} &= \{(n+1, n, n), (n, n, n+1), (n+1, n, n+1) |\, n \geq 0\} \cup \\ &\quad \{(n, n+1, n+1), (n+1, n+1, n), (n, n+1, n) |\, n \geq 0\}. \end{aligned}$$

1.2.3 Root system of type $C_2^{(1)}$

We briefly recall the structure of a root system of type $C_2^{(1)}$ (see [24, Chapter 6]): let

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \tag{1.2.2}$$

be the Generalized Cartan matrix of type $C_2^{(1)}$ (its Dynkin diagram is shown in figure 1.1). C is a *symmetrizable* matrix of rank 2 (i.e. DC is symmetric for some diagonal matrix with positive entries, e.g. $D = diag(2, 1, 2)$). The Kernel of $\mathcal{C}$ is generated by the element $\delta = (1, 2, 1)$. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a *realization* of C, i.e. $\mathfrak{h}$ is a four dimensional complex vector space, $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ (resp. $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$) is a linearly independent set in $\mathfrak{h}^*$ (resp. $\mathfrak{h}$), $\alpha_j(\alpha_i^\vee) = c_{ij}$ if $i \neq j$ and $\alpha_i(\alpha_i^\vee) = 2$. The set $\overset{\circ}{\Delta} = \{\pm\alpha_1, \pm\alpha_3, \pm(\alpha_1 + \alpha_3), \pm(2\alpha_1 + \alpha_3)\}$ is then a root system of type C_2 in $\mathfrak{h}^*$ (it is shown in figure 1.3). Let $Q \doteq \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3$ be the *root lattice* and $\delta \doteq \alpha_1 + 2\alpha_2 + \alpha_3 \in Q$. The root system Δ of type $C_2^{(1)}$ is the subset of Q given by the disjoint union $\Delta = \Delta^{Im} \cup \Delta^{re}$ where

$$\Delta^{Im} = \{\pm n\delta |\, n \geq 1\} = \{(n, 2n, n) |\, n \in \mathbb{Z} \setminus \{0\}\},$$

$$\Delta^{re} = \{\alpha + n\delta |\, \alpha \in \overset{\circ}{\Delta},\, n \in \mathbb{Z}\}.$$

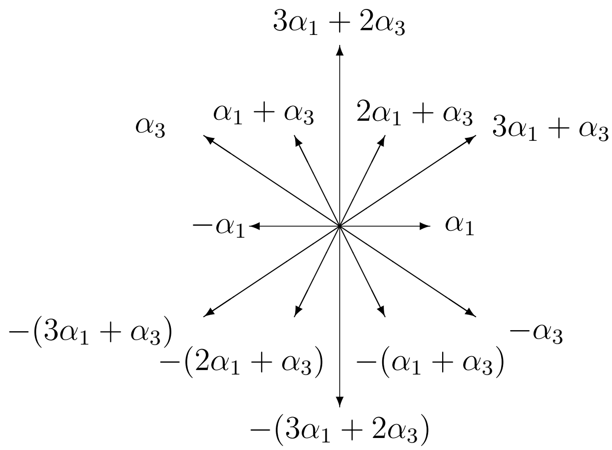

Figure 1.4: Root system of type G_2.

The elements of Δ^{Im} are the *imaginary roots* of Δ while Δ^{re} are the *real roots* of Δ. The positive roots are the roots with non-negative coordinates in the basis Π. Explicitly they are given by $\Delta_+ = \Delta_+^{Im} \cup \Delta_+^{re}$ where

$$\Delta_+^{Im} = \{n\delta |\, n \geq 1\} = \{(n, 2n, n) |\, n \geq 1\},$$

$$\begin{aligned} \Delta_+^{re} \;=\;& \{(n+1, 2n, n), (n+2, 2n, n+1), (n+1, 2n, n+1), (n, 2n, n+1) |\, n \geq 0\} \cup \\ & \{(n-1, 2n, n), (n-1, 2n, n-1), (n, 2n, n-1) |\, n \geq 1\} \cup \\ & \{(n-2, 2n, n-1) |\, n \geq 2\} \end{aligned}$$

1.2.4 Root system of type $G_2^{(1)}$

We briefly recall the structure of a root system of type $G_2^{(1)}$ (see [24, Chapter 6]): let

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix} \tag{1.2.3}$$

be the Generalized Cartan matrix of type $G_2^{(1)}$ (its Dynkin diagram is shown in figure 1.1). C is a *symmetrizable* matrix of rank 2 (i.e. DC is symmetric for some diagonal matrix with positive integer entries, e.g. $D = diag(3, 3, 1)$). The Kernel of $\mathcal{C}$ is generated by the element $\delta = (1, 2, 3)$. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of C, i.e. $\mathfrak{h}$ is a four dimensional complex vector space, $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ (resp. $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$) is a linearly independent set in $\mathfrak{h}^*$ (resp. $\mathfrak{h}$), $\alpha_j(\alpha_i^\vee) = c_{ij}$ if $i \neq j$ and $\alpha_i(\alpha_i^\vee) = 2$. The set $\mathring{\Delta} = \{\pm\alpha_1, \pm\alpha_3, \pm(3\alpha_1 + \alpha_3), \pm(2\alpha_1 + \alpha_3), \pm(3\alpha_1 + 2\alpha_3), \pm(\alpha_1 + \alpha_3)\}$ is then a root system of type G_2 in $\mathfrak{h}^*$ (it is shown in figure 1.4). Let $Q \doteq \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3$ be the root lattice and $\delta \doteq \alpha_1 + 2\alpha_2 + 3\alpha_3 \in Q$. The root system Δ of type $G_2^{(1)}$ is the subset of Q given by the disjoint union $\Delta = \Delta^{Im} \cup \Delta^{re}$ where

$$\Delta^{Im} = \{\pm n\delta |\, n \geq 1\} = \{(n, 2n, 3n) |\, n \in \mathbb{Z} \setminus \{0\}\},$$

$$\Delta^{re} = \{\alpha + n\delta |\, \alpha \in \mathring{\Delta},\, n \in \mathbb{Z}\}.$$

The elements of Δ^{Im} are the imaginary roots of Δ while Δ^{re} are the real roots of Δ. The positive roots are the roots with non-negative coordinates in the basis Π. Explicitly they are given by $\Delta_+ = \Delta_+^{Im} \cup \Delta_+^{re}$ where

$$\Delta_+^{Im} = \{n\delta \mid n \geq 1\} = \{(n, 2n, 3n) \mid n \geq 1\},$$

$$\begin{aligned}\Delta_+^{re} &= \{(n+1, 2n, 3n), (n+3, 2n, 3n+1), (n+2, 2n, 3n+1), (n+3, 2n, 3n+2),\\ &\quad (n+1, 2n, 3n+1), (n, 2n, 3n+1) \mid n \geq 0\} \cup \\ &\quad \{(n-1, 2n, 3n), (n-1, 2n, 3n-1) \mid n \geq 1\} \cup \\ &\quad \{(n-2, 2n, 3n-1) \mid n \geq 2\} \\ &\quad \cup \{(n-3, 2n, 3n-1), (n-3, 2n, 3n-2) \mid n \geq 3\}.\end{aligned}$$

1.3 Rank three bipartite cluster algebras

In figure 1.1 are listed all the Dynkin diagram of 3×3 generalized Cartan matrices of affine type. To every of them is associated a class of cluster algebras parameterized by the choice of the coefficient group $\mathbb{P}$. Since all of them are *bipartite* except one we want to recall the definition and the main properties of the cluster algebras associated with bipartite initial matrices discovered in [18, Sections 8,9,10]. We start with a general definition.

1.3.1 Definition

Definition 1.3.1. [18, Definition 8.1] A $n \times n$ skew–symmetrizable matrix $B = \{b_{ij}\}$ is *bipartite* if there exists a function $\varepsilon : \{1, \cdots, n\} \to \{1, -1\}$ such that

$$b_{ij} > 0 \Longrightarrow \begin{cases} \varepsilon(i) = 1, \\ \varepsilon(j) = -1. \end{cases} \tag{1.3.1}$$

A seed $(B, \mathbf{x}, \mathbf{y})$ in $\mathcal{F} = \mathbb{QP}(x_1, \cdots, x_n)$ is bipartite if B is bipartite.

In the rest of the section we restrict ourselves in rank three case: we consider a cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}_0, \mathbf{y}_0, B^0)$ associated with the bipartite seed:

$$\Sigma_0 = \{\mathbf{x}_0 \doteq \{x_{1;0}, x_{2;0}, x_{3;0}\}, \mathbf{y}_0 \doteq \{y_{1;0}, y_{2;0}, y_{3;0}\}, B = (b_{ij})_{i,j=1,2,3}\}. \tag{1.3.2}$$

Without lost of generality we choose the function ε in definition 1.3.1 as

$$\varepsilon(1) = \varepsilon(3) = -1, \quad \varepsilon(2) = +1.$$

In other words the initial exchange matrix is of the form

$$B^0 = \begin{pmatrix} 0 & - & 0 \\ + & 0 & + \\ 0 & - & 0 \end{pmatrix} \tag{1.3.3}$$

Recall that with every symmetrizable Cartan matrix $C = (c_{ij})_{i,j=1,\cdots,n}$ is uniquely associated a valued graph (its Dynkin diagram) $(\Gamma, \mathbf{d})$ whose vertices are numbered by $1, \cdots, n$, and two vertices i and j are joined by the valued edge $(|c_{ij}|, |c_{ji}|)$. Then the choice of ε correspond to the choice of the orientation of the graph associated with the Cartan counterpart $C(B^0) = 2 \cdot \mathrm{Id}_3 + \{-|b_{ij}|\} = \{c_{ij}\}$ of B given by

$$1 \xrightarrow[(|c_{12}|,|c_{21}|)]{} 2 \xleftarrow[(|c_{23}|,|c_{32}|)]{} 3\,. \tag{1.3.4}$$

1.3.2 Bipartite belt

Let μ_k be the seed mutation in direction $k \in \{1, 2, 3\}$. Since $b_{13} = b_{31} = 0$ it follows from Definition 1.1.4 that μ_1 and μ_3 commute. Then the following operators are well-defined:

$$\mu_+ = \mu_2, \quad \mu_- = \mu_1 \circ \mu_3.$$

It follows that $\mu_\pm(B) = -B$. We have a *bipartite belt* consisting of the seeds

$$\Sigma_r = \{\mathbf{x}_r, \mathbf{y}_r, (-1)^r B\} \qquad (r \in \mathbb{Z})$$

where $\mathbf{x}_r \doteq \{x_{1;r}, x_{2;r}, x_{3;r}\}$ is the cluster, $\mathbf{y}_r \doteq \{y_{1;r}, y_{2;r}, y_{3;r}\}$ are the coefficients of Σ_r, and Σ_r is defined by setting, for each $r > 0$:

$$\Sigma_r = \underbrace{\mu_\pm \cdots \mu_- \mu_+ \mu_-}_{r \text{ factors}}(\Sigma_0), \tag{1.3.5}$$

$$\Sigma_{-r} = \underbrace{\mu_\mp \cdots \mu_+ \mu_- \mu_+}_{r \text{ factors}}(\Sigma_0). \tag{1.3.6}$$

The diagram of mutations has the following shape:

$$\begin{tikzcd}
 & & \Sigma^1_{-2} \arrow[dl, "\mu_1"'] & & & \Sigma^3_0 \arrow[dr, "\mu_1"] & & & \\
\cdots & \boxed{\Sigma_{-2}} & & \boxed{\Sigma_{-1}} \arrow[ul, "\mu_3"'] \arrow[dl, "\mu_1"] & \boxed{\Sigma_0} \arrow[l, "\mu_2"'] \arrow[ur, "\mu_3"] \arrow[dr, "\mu_1"'] & & \boxed{\Sigma_1} \arrow[r, "\mu_2"] & \boxed{\Sigma_2} & \cdots \\
 & & \Sigma^3_{-2} \arrow[ul, "\mu_3"] & & & \Sigma^1_0 \arrow[ur, "\mu_3"'] & & &
\end{tikzcd} \tag{1.3.7}$$

Here boxes highlight the seeds of the bipartite belt, while the seed Σ^i_{2m} is the mutation in direction i of the seed Σ_{2m}, for $i = 1, 3$. We drew an arrow labeled by μ_k from a seed Σ to a seed Σ' whenever the mutation of Σ in direction k is Σ'.

1.3.3 Exchange relations in the bipartite belt

We want to recall some properties of the seeds in the bipartite belt: it is evident from the diagram that Σ_{2m-1} shares with Σ_{2m} the cluster variables $x_{1;2m}$ and $x_{3;2m}$.

Similarly the seed Σ_{2m+1} shares with Σ_{2m} the cluster variable $x_{2;2m}$. The coefficients mutations (1.1.2) take the form

$$\begin{aligned}
y_{1;2m}y_{1;2m+1} &= 1 && (1.3.8)\\
y_{2;2m+1} &= y_{2;2m}(y_{1;2m}\oplus 1)^{-b_{12}}(y_{3;2m}\oplus 1)^{-b_{32}} && (1.3.9)\\
y_{3;2m}y_{3;2m+1} &= 1 && (1.3.10)
\end{aligned}$$

$$\begin{aligned}
y_{1;2m-1} &= \frac{y_{1;2m}y_{2;2m}^{b_{21}}}{(y_{2;2m}\oplus 1)^{b_{21}}} && (1.3.11)\\
y_{2;2m-1}y_{2;2m} &= 1 && (1.3.12)\\
y_{3;2m-1} &= \frac{y_{3;2m}y_{2;2m}^{b_{23}}}{(y_{2;2m}\oplus 1)^{b_{23}}} && (1.3.13)
\end{aligned}$$

The exchange relations are

$$x_{i;2m}x_{i;2m+2} = \frac{y_{i;2m}x_{2;2m}^{b_{2i}}+1}{y_{i;2m}\oplus 1} \tag{1.3.14}$$

for $i = 1, 3$, and

$$x_{2;2m}x_{2;2m+2} = \frac{x_{1;2m+2}^{-b_{12}}x_{3;2m+2}^{-b_{32}}+y_{2;2m+1}}{y_{2;2m+1}\oplus 1} \tag{1.3.15}$$

We did not describe all the possible cluster variables and exchange relations of $\mathcal{A}$.

1.3.4 Denominator vectors and roots

Let Q be the root lattice associated with the Cartan counterpart $C = C(B) = \{c_{ij}\}$ of B. Let α_1, α_2 and α_3 be the simple roots of Q. We identify Q with $\mathbb{Z}^3$ and the simple roots with the standard basis of $\mathbb{Z}^3$. Let $W \subset GL(Q)$ be the corresponding Weyl group; it is generated by the simple reflections s_1, s_2 and s_3 which act on the simple roots by

$$s_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i.$$

It follows that $s_i(\alpha_i) = -\alpha_i$ and $s_i^2 = 1$. We define the elements $t_\pm \in W$ by setting

$$t_+ = s_2 \quad t_- = s_1s_3 = s_3s_1$$

Since $c_{13} = c_{31} = 0$, s_1 and s_3 commute. In particular, $t_\pm^2 = 1$. The action of t_- on the simple roots is given by

$$t_-(\alpha_j) = \begin{cases} -\alpha_j & \text{if } j = 1, 3\\ \alpha_2 - c_{12}\alpha_1 - c_{32}\alpha_3 & \text{if } j = 2\end{cases}$$

Let $\Phi^{\mathrm{Re}}_{\geq -1}$ be the union of the set of real positive roots of Q and the set of negative simple roots. We recall the involutive permutations τ_+ and τ_- of $\Phi^{\mathrm{Re}}_{\geq -1}$ are defined by

$$\tau_-(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_2\\ t_-(\alpha) & \text{otherwise}\end{cases} \tag{1.3.16}$$

$$\tau_+(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_1 \text{ or } \alpha = -\alpha_3 \\ t_+(\alpha) = s_2(\alpha) & \text{otherwise} \end{cases} \tag{1.3.17}$$

We can extend $\tau_\pm$ to Q by:

$$\tau_-(\sum_{i=1}^{3} a_i\alpha_i) = (-a_1 - c_{12}[a_2]_+)\alpha_1 + a_2\alpha_2 + (-a_3 - c_{32}[a_2]_+)\alpha_3 \tag{1.3.18}$$

$$\tau_+(\sum_{i=1}^{3} a_i\alpha_i) = a_1\alpha_1 + (-a_2 - c_{21}[a_1]_+ - c_{23}[a_3]_+)\alpha_2 + a_3\alpha_3$$

Definition 1.3.2. [18, Definition 10.2] We define the vectors $\mathbf{d}(i;m) \in Q$, for $i = 1,2,3$ and all $m \geq 0$ by setting:

$$\mathbf{d}(i;2m) = (\tau_-\tau_+)^m(-\alpha_i); \tag{1.3.19}$$

$$\mathbf{d}(i;-2m) = (\tau_+\tau_-)^m(-\alpha_i). \tag{1.3.20}$$

Theorem 1.3.3. *[18, Theorem 10.3] The denominator vector of a cluster variable $x_{i;2m}$ with respect to the initial cluster $\mathbf{x}_0$ is equal to $\mathbf{d}(i;2m)$, for any $i = 1,2,3$ and $m \in \mathbb{Z}$.*

Corollary 1.3.4. *[18, Corollary 10.6] Each cluster variable $x_{i;2m}$ can be written as*

$$x_{i;2m} = \frac{P_{i;2m}(x_{1;0}, x_{2;0}, x_{3_0})}{\mathbf{x}_0^{\mathbf{d}(i;2m)}}$$

where $P_{i;2m}$ is a polynomial with non-zero *constant term.*

Proposition 1.3.5. *[18, Proposition 10.7] For all $m \in \mathbb{Z}$ and $j \in \{1,2,3\}$ the elements $y_{j;2m}$ evaluated in the tropical semifield $\mathbb{P} = \mathrm{Trop}(y_{1;0}, y_{2;0}, y_{3;0})$ are given by*

$$\begin{array}{rcll} y_{j;2m}|_{\mathrm{Trop}(y_{1;0},y_{2;0},y_{3;0})} & = & \mathbf{y}_0^{-\mathbf{d}(j;2m)} & \text{if } j = 1,3, \\ y_{2;2m}|_{\mathrm{Trop}(y_{1;0},y_{2;0},y_{3;0})} & = & \mathbf{y}_0^{\mathbf{d}(2;2m)} & \end{array}$$

In the cluster algebra $\mathcal{A}_\bullet(\mathbf{x}_0, \mathbf{y}_0, B)$ with principal coefficients at Σ_0, Proposition 1.3.5 allow us to write the exchange relations between cluster variables in the bipartite belt in an easier way as shown by the following result.

Corollary 1.3.6. *In $\mathcal{A}_\bullet(\mathbf{x}_0, \mathbf{y}_0, B)$ the exchange relations* (1.3.14) *take the form*

$$x_{j;2m}x_{j;2m+2} = \mathbf{y}_0^{[-\mathbf{d}(j;2m)]_+} x_{2;2m}^{-c_{2j}} + \mathbf{y}^{[\mathbf{d}(j;2m)]_+}$$

$$x_{2;2m}x_{2;2m+2} = \mathbf{y}_0^{[\mathbf{d}(2;2m)]_+} x_{1;2m}^{-c_{12}} x_{3;2m}^{-c_{32}} + \mathbf{y}^{[-\mathbf{d}(2;2m)]_+}$$

Corollary 1.3.7. *For every cluster variable $x_{i;2m}$ belonging to the bipartite belt, the F–polynomial $F^0_{i;2m}$ with respect to the seed Σ_0:*

- *has constant term* 1*,*
- *has a a unique monomial, namely $\mathbf{y}^{[\mathbf{d}(i;2m)]_+}$, with coefficient* 1 *and divisible by all of the other occurring monomials.*

1.3.5 g–vectors

The next and last recall from [18] is about the $\mathbf{g}$–vectors of the cluster variables belonging to the bipartite belt.

Let E be the linear automorphism of the root lattice Q given by $E(a_1\alpha_1+a_2\alpha_2+a_3\alpha_3)=a_1\alpha_1-a_2\alpha_2+a_3\alpha_3$.

Theorem 1.3.8. *[18, Theorem 10.12] For every $m\in\mathbb{Z}$, the $\mathbf{g}$–vector $\mathbf{g}_{i;2m}$ and the denominator vector $\mathbf{d}(i;2m)$ of the cluster variable $x_{i;2m}$ with respect to the initial seed Σ_0 are related by:*

$$\mathbf{g}_{i;2m}=E\tau_-(\mathbf{d}(i;2m)).$$

Remark 1.3.9. For future purposes we want to give another interpretation of the piecewise linear operator $E\tau_-$. We choose the orientation 1.3.4 of the valued quiver Q_B associated with the Cartan counterpart of B^0. We associate to Q_B the matrix E_B defined by

$$(E_B)_{ij}=\begin{cases} 1 & \text{if } i=j,\\ b_{ij} & \text{if there exists an arrow from } i \text{ to } j,\\ 0 & \text{otherwise.}\end{cases}$$

Note that since of (1.3.3) if there exists an arrow from i to j, then b_{ij} is non–positive. We are interested in the *opposite* of E_B that in the basis of simple roots is given by

$$-E_B=\begin{pmatrix} -1 & -b_{12} & 0\\ 0 & -1 & 0\\ 0 & -b_{32} & -1\end{pmatrix}$$

We consider the piecewise–linear modification $\mathcal{E}_B$ of $-E_B$ given by

$$\mathcal{E}_B=\begin{pmatrix} -1 & -b_{12}[\ ?\ ]_+ & 0\\ 0 & -1 & 0\\ 0 & -b_{32}[\ ?\ ]_+ & -1\end{pmatrix}.$$

$\mathcal{E}_B$ acts on $Q=\mathbb{Z}^3$ by $*$ in the following way:

$$\mathcal{E}_B*(a_1\alpha_1+a_2\alpha_2+a_3\alpha_3)=\mathcal{E}_B*\begin{pmatrix} a_1\\ a_2\\ a_3\end{pmatrix}=\begin{pmatrix} -a_1-b_{12}[a_2]_+\\ -a_2\\ -b_{32}[a_2]_+-a_3\end{pmatrix}$$

$$=(-a_1-b_{12}[a_2]_+)\alpha_1-a_2\alpha_2+(-b_{32}[a_2]_+-a_3)\alpha_3 \qquad (1.3.21)$$

Finally we get

$$E\tau_-=\mathcal{E}_B. \qquad (1.3.22)$$

This follows immediately from (1.3.18).

Corollary 1.3.10. *For every cluster monomial of the form $s_1^{a_1}s_2^{a_2}s_3^{a_3}$, its $\mathbf{g}$–vector $\mathbf{g}$ and its denominator vector $\mathbf{d}$ with respect to the initial seed Σ_0 are related by:*

$$\mathbf{g}=E\tau_-(\mathbf{d}).$$

Proposition 1.3.11. *For any $m\in\mathbb{Z}$ let $\mathbf{g}_{j;2m}=\begin{pmatrix} g_1\\ g_2\\ g_3\end{pmatrix}$ and ${\mathbf{g}'}_{j;2m}=\begin{pmatrix} g'_1\\ g'_2\\ g'_3\end{pmatrix}$ be respectively the $\mathbf{g}$–vectors of the cluster variable $x_{j;2m}$ in Σ_{2m-1} and in $\Sigma_{2m}=\mu_2(\Sigma_{2m-1})$. Then they are related by the formula:*

$$g_i=\begin{cases} g'_i+[-b_{i2}]_+g'_2-b'_{i2}[-g'_2]_+ & \text{if } i=1,3\\ -g'_2 & \text{otherwise}\end{cases}$$

1.4 Background on quiver representations

After the article [5] the interplay between cluster algebras and quiver representations has been the subject of many other papers, e.g. [7], [8], [9] and [11]. Here we recall one of the results on it. In order to fix notations we first recall some well-known facts about quiver representations. For a detailed introduction see e.g [14], [3] or [1]. A quiver Q is an ordered pair $\{Q_0, Q_1\}$ where Q_0 is the set of *vertices* and Q_1 is the set of *arrows*, i.e. ordered pairs $\{i, j\}$ of vertices. For an arrow $\alpha = \{i, j\}$ we usually write $\alpha : i \to j$. $i = s(\alpha)$ is the *starting point* of α and $j = t(\alpha)$ is the *ending point* or *target* of α. A *path* in Q is either a concatenation $\alpha_n \cdots \alpha_1$ of arrows such that $t(\alpha_i) = s(\alpha_{i+1})$ or the symbol e_i for every $i \in Q_0$ called *trivial path.* Let k be an algebraically closed field of characteristic zero. The k-vector space with basis the paths of Q is called the *path algebra* kQ of Q. It is well-known that kQ is a k-algebra (see e.g. [2, III.1]). A *representation* V of a given quiver $Q = \{Q_0, Q_1\}$ over a field k is a collection $\{\{V_i\}_{i\in Q_0}, \{f_\alpha\}_{\alpha\in Q_1}\}$ of finite dimensional k-vector spaces V_i's together with linear maps $f_\alpha : V_i \to V_j$ whenever $\alpha : i \to j$. A *morphism* of two representations of Q, $V = \{\{V_i\}, \{f_\alpha\}\}$ and $W = \{\{W_i\}, \{g_\alpha\}\}$, is a collection of linear maps $\{h_i : V_i \to W_i\}_{i\in Q_0}$ such that the diagram

$$\begin{array}{ccc} V_i & \xrightarrow{f_\alpha} & V_j \\ h_i \downarrow & & \downarrow h_j \\ W_i & \underset{g_\alpha}{\longrightarrow} & W_j \end{array}$$

commutes, i.e. $h_j \circ f_\alpha = g_\alpha \circ h_i$, for every arrow $\alpha : i \to j$. A morphism $h = \{h_i\}$ is called a *monomorphism* (resp. *epimorphism*, *isomorphism*) if every h_i is injective (resp. surjective, bijective). A *sub-representation* W of a Q-representation V is a Q-representation endowed of a monomorphism (*inclusion*) into V. We write $W \leq V$ to indicate that W is a sub-representation of V. The *direct sum* of two Q-representations V and W is the representation $V \oplus W = \{\{V_i \oplus W_i\}_{i\in Q_0}, \{f_\alpha \oplus g_\alpha\}\}$. A representation is called *decomposable* if it is sum of two sub-representations of itself. Otherwise it is called *indecomposable*. The *dimension vector* of a Q-representation $V = \{V_i\}_{i\in Q_0}$ is the ordered collection $\mathbf{d} = \underline{dim}(V) = \{d_i\}$ of the positive numbers $d_i = dim_k(V_i)$. It is well-known that the category of Q-representations and the category of kQ-modules of finite k-dimension are equivalent. A Q-representation V of dimension $\mathbf{d}$ is called *rigid* if a generic representation of dimension $\mathbf{d}$ is isomorphic to V, or equivalently if it has no nontrivial self-extensions.

For a Q-representation V of dimension $\mathbf{d}$ and a dimension vector $\mathbf{e} \in \mathbb{Z}^n_{\geq 0}$ we define the quiver-Grassmannian

$$Gr_{\mathbf{e}}(V) \doteq \{W \leq V \,|\, \underline{dim}(W) = \mathbf{e}\}. \tag{1.4.1}$$

Quiver Grassmannians appears in many papers (see e.g. [7], [8] or [10]). By the definition, $Gr_{\mathbf{e}}(V)$ is a closed subvariety of the product of Grassmannians $\Pi_i Gr_{e_i}(V_i)$. We indicate by $\chi_{\mathbf{e}}(V)$ the Euler-Poincaré characteristic of $Gr_{\mathbf{e}}(V)$ (see e.g. [22, Section 4.5]). We associate to any representation M of dimension vector $\mathbf{d}$ the

Laurent polynomial $X_M(x_1, \cdots, x_n)$ in n variables $\{x_1, \cdots, x_n\}$ given by

$$X_M(x_1, \cdots, x_n) = x_1^{-d_1} \cdots x_n^{-d_n} \sum_{\mathbf{e}} \chi_{\mathbf{e}}(M) \prod_{i,j} (x_i^{d_j - e_j} x_j^{e_i})^{[b_{ij}]_+}$$

We call the map $M \mapsto X_M$ the *Caldero-Chapoton map*. It has the following property:

$$X_{M \bigoplus N} = X_M X_N$$

We associate to a quiver Q without loops, without oriented 2–cycles and with n vertices, a skew–symmetric matrix $B_Q = \{b_{ij}\}$ given by

$$b_{ij} = \mathrm{card}\{a \in Q_1 | s(a) = j, t(a) = i\} - \mathrm{card}\{a \in Q_1 | s(a) = j, t(a) = i\}. \qquad (1.4.2)$$

Note that the matrix B_Q defined here is the transpose of the matrix B_Q defined in [11]. Viceversa to every integer skew–symmetric $n \times n$ matrix B we associate a quiver Q_B having n vertices and b_{ij} arrows from j to i whenever $b_{ij} > 0$. We then associate to Q a skew–symmetric cluster algebra without coefficients $\mathcal{A}(Q)$ with initial seed $\{B_Q, \{x_1, \cdots, x_n\}\}$.

Theorem 1.4.1. *[9, Theorem 4] Let Q be an acyclic quiver with n vertices and let $\mathcal{A}(Q)$ be the coefficient–free cluster algebra with initial seed $\{B_Q, \{x_1, \cdots, x_n\}\}$. The correspondence $M \mapsto X_M(x_1, \cdots, x_n)$ is a bijection between the set of isomorphism classes of indecomposable rigid representations of the quiver Q, and the set of all cluster variables in $\mathcal{A}(Q)$ not belonging to the initial cluster $\{x_1, \cdots, x_n\}$.*

1.5 General techniques in finding canonical basis of a cluster algebra

This section is an heuristic treatment of the problem of finding a basis of a cluster algebra somehow related to the canonical basis of semisimple algebraic group (see [28] for more details). A general definition of such a basis does not exist. Here we give a definition (Definition 1.5.1), that works for the class of cluster algebras treated in this thesis together with the rank two cluster algebras of finite and affine type as shown in [27]. We collect the techniques that one can use for finding such a basis.

In the whole section $\mathcal{A}$ will be a cluster algebra in the field $\mathcal{F}$ of rational functions in n independent variables with coefficients in some semifield $\mathbb{P}$. When necessary we will restrict ourselves in a less generality. Recall that $\mathcal{A}$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by recursively defined rational functions called cluster variables. Usually the set of cluster variables is denoted by Ξ, so that we can write $\mathcal{A} \doteq \mathbb{ZP}[\Xi]$. Cluster variables are collected into clusters, that are maximal set of algebraically independent cluster variables. Moreover every cluster $\mathcal{C}$ of $\mathcal{A}$ generates the whole field $\mathcal{F}$, in other words $\mathcal{C}$ has cardinality n. The *Laurent phenomenon* asserts that every element of $\mathcal{A}$ is a Laurent polynomial with coefficients in $\mathbb{ZP}$ in every cluster of $\mathcal{A}$.

Let's start introducing the notion of positivity in $\mathcal{A}$.

1.5.1 Positivity and definition of canonical basis

An element z of $\mathcal{A}$ is called *positive* if the Laurent expansion of z in *every* cluster of $\mathcal{A}$ has coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$. In other words z is positive if for every cluster $\mathcal{C} = \{x_1, \cdots, x_n\}$ of $\mathcal{A}$, it has the form

$$z = \frac{N_{\mathcal{C}}(x_1, \cdots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}}$$

where $N_{\mathcal{C}}$ is a primitive, i.e. not divisible by any x_i, polynomial with coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$.

For the connection of cluster algebras with the problem of finding a Total positive basis for the coordinate ring of algebraic groups (a good introduction about this connection is given in [15]), it seems very interesting describe the set, actually the semiring, of positive elements. A semiring is a subset of a ring closed under addition and multiplication. We say that a positive element is *positive indecomposable* if it cannot be written as a sum of two non-zero positive elements.

Definition 1.5.1. If the set of positive indecomposable elements form a $\mathbb{Z}\mathbb{P}$-basis, then we call it a *canonical basis* of $\mathcal{A}$. In other words a set $\mathbf{B}$ of elements of $\mathcal{A}$ is a canonical basis of $\mathcal{A}$ if it satisfies the following properties:

CB1 $\mathbf{B}$ is a $\mathbb{Z}\mathbb{P}$-basis of $\mathcal{A}$, i.e.

- CB1a $\mathbf{B}$ is a linearly independent set over $\mathbb{Z}\mathbb{P}$;
- CB1b $\mathbf{B}$ spans $\mathcal{A}$ over $\mathbb{Z}\mathbb{P}$.

CB2 $\mathbf{B}$ coincides with the set of positive indecomposable elements, in particular they are positive.

Provided that CB1 holds, property CB2 is equivalent to the following

CB2' The semiring of positive elements coincides with the $\mathbb{Z}_{\geq 0}\mathbb{P}$-span of $\mathbf{B}$.

Provided that both CB1 and CB2 hold then clearly its $\mathbb{Z}_{\geq 0}\mathbb{P}$–linear combinations are positive elements and viceversa every positive elements is a $\mathbb{Z}_{\geq 0}\mathbb{P}$–linear combination of positive indecomposable elements. Viceversa if both CB1 and CB2' hold, if p is a positive indecomposable element then it must lie in $\mathbf{B}$ and every element of $\mathbf{B}$ is positive indecomposable.

We remark here that the fact that the set of positive indecomposable elements is linearly independent is not expected for every cluster algebra.

The canonical basis, when exists, is uniquely determined up to rescaling by elements of $\mathbb{P}$: given a subset $\mathbf{B}$ of $\mathcal{A}$ satisfying properties CB1 and CB2, every set $\mathbf{B}'$ whose elements are scalar multiples of the elements of $\mathbf{B}$ clearly satisfies the same properties.

Viceversa if two sets $\mathbf{B}$ and $\mathbf{B}'$ satisfy the previous properties, then every element b' of $\mathbf{B}'$ is a combination $b' = \sum p_i b_i$ of elements of $\mathbf{B}$ with coefficients in $p_i \in \mathbb{Z}\mathbb{P}$; since b' is positive, $p_i \in \mathbb{Z}_{\geq 0}\mathbb{P}$; since b' is positive indecomposable every $p_i = 0$ except one p_0, i.e. $b' = p_0 b_0$. Then the elements of $\mathbf{B}'$ are scalar multiples of the elements of $\mathbf{B}$.

There are some special cases in which the canonical basis is unique. For example when $\mathbb{P} = \{1\}$, i.e. in the so called coefficient-free case, if the canonical basis exists it is unique.

There exists a natural coefficient specialization between a canonical basis of $\mathcal{A}$ and the canonical basis of the corresponding coefficient-free cluster algebra, sending $\mathbb{P}$ onto $\{1\}$. In some cases, somewhat surprisingly, the viceversa also holds: there exists a map between a coefficient-free cluster algebra and a generic cluster algebra of the same type mapping the canonical basis of the former into a particular canonical basis of the latter one, that determines all the others. This is the case, for example, of the rank two cluster algebras of finite and affine type, as it is shown in [27, Section 6].

In the next subsections we are going to collect the main techniques that one can use to prove that a given subset $\mathbf{B}$ of $\mathcal{A}$, candidate to be a canonical basis, satisfies properties CB1 and CB2 (or CB2').

1.5.2 Straightening relations

Let $\mathbf{B}$ be a subset of $\mathcal{A}$ candidate to be a canonical basis. We want to illustrate how one can prove that $\mathbf{B}$ spans $\mathcal{A}$ over $\mathbb{Z}\mathbb{P}$ and that its elements are positive. We assume that $\mathbf{B}$ has the following property

$$\text{Monomials in the elements of } \mathbf{B} \text{ span } \mathcal{A} \tag{1.5.1}$$

Indeed it is expected that $\mathbf{B}$ contains all the cluster monomials, that are monomials in cluster variables belonging to the same cluster. Let $\mathfrak{M} = \{M = b_1^{a_1} \cdots b_n^{a_n} | b_i \in \mathbf{B},\ a_i \in \mathbb{Z}_{\geq 0}\}$ be the set of monomials in the elements of $\mathbf{B}$. We assume the following condition

$$\mathfrak{M} \text{ is a well–ordered set.} \tag{1.5.2}$$

The positivity will be given by induction on the order (1.5.2).

Once we have property (1.5.1), in order to prove the span property of $\mathbf{B}$ it is sufficient to show that the generic monomial $M \in \mathfrak{M}$ is a linear combination of elements of $\mathbf{B}$ over $\mathbb{Z}\mathbb{P}$. Let $\mathbf{B}_{\mathfrak{M}}$ be the set of all the elements of $\mathbf{B}$ that are not divisible by elements of $\mathbf{B}$. We consider all the *minimal forbidden monomial*:

$$\mathfrak{M}\mathfrak{F} \doteq \{m = b_1 \cdots b_n | b_i \in \mathbf{B}_{\mathfrak{M}},\ m/b_i \in \mathbf{B},\ i = 1, \cdots, n,\ m \notin \mathbf{B}\}. \tag{1.5.3}$$

The expansions of the elements of $\mathfrak{M}\mathfrak{F}$ in $\mathbf{B}$ are called *straightening relations*. Then it suffices to show that every monomial $M \in \mathfrak{M}$ which has at least one of the forbidden monomial as a factor, can be written as a linear combination of monomials of smaller degree. We will show that this can be done by replacing some forbidden factor of M with its expression given by the appropriate straightening relation.

1.5.3 Newton polytopes

The Laurent phenomenon asserts that every element of $\mathcal{A}$ is a Laurent polynomial in every cluster of $\mathcal{A}$. It means that for every cluster $\mathcal{C} = \{x_1, \cdots, x_n\}$ of $\mathcal{A}$, every element z is a $\mathbb{ZP}$-linear combination of Laurent monomials in $\{x_1, \cdots, x_n\}$:

$$z = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha \mathbf{x}^\alpha \qquad a_\alpha \in \mathbb{ZP},\ \mathbf{x} = (x_1, \cdots, x_n) \tag{1.5.4}$$

the a_α's not all non-zero. The *Newton polytope of z in the cluster $\mathcal{C}$*, denoted by $\mathrm{Newt}_{\mathcal{C}}(z)$, is the convex hull in $\mathbb{Z}^n$ of all the α in (1.5.4) such that a_α is non-zero. We say that a vertex γ of $\mathrm{Newt}_{\mathcal{C}}(z)$ is *monic* if the corresponding Laurent monomial appears in (1.5.4) with coefficient in $\mathbb{P}$ (instead of $\mathbb{ZP}$). We also say that z is monic in $\mathcal{C}$ if all the vertices of $\mathrm{Newt}_{\mathcal{C}}(z)$ are monic.

Newton polytopes are a powerful tool in the study of canonical basis. We remark that the power of this tool offsets its intrinsic combinatorial nature that makes them hard to control. On the other hand having a good control of them, can make the life simpler: for example in the case of a cluster algebra of type $A_2^{(1)}$ (Chapter 2), we recognize this algebra to be graded after we noticed that all the Newton polytopes of the cluster variables were actually polygons (see Remark2.3.35).

The following is a Lemma that one can try to prove in order to have Theorem 1.5.3 below.

Lemma 1.5.2 (Key Lemma). *For every element b of $\mathbf{B}$ there exists a cluster $\mathcal{C} = \mathcal{C}_b$ and a* monic *vertex γ_b of $\mathrm{Newt}_{\mathcal{C}}(b)$ such that γ_b doesn't lie in $\mathrm{Newt}_{\mathcal{C}}(b')$ if $b' \neq b$ is another element of $\mathbf{B}$.*

An immediate consequence of Lemma 1.5.2 is the following result.

Theorem 1.5.3. *If a subset $\mathbf{B}$ of $\mathcal{A}$ satisfies Lemma 1.5.2, then $\mathbf{B}$ is a $\mathbb{ZP}$–linearly independent set. Moreover if also $\mathbf{B}$ spans $\mathcal{A}$ over $\mathbb{ZP}$ and its elements are positive, then they are positive indecomposable.*

Proof. We want to prove that if $\mathbf{B}$ satisfies the Key Lemma then it is a linearly independent set over $\mathbb{ZP}$, i.e. it has property CB1a of Definition 1.5.1. Consider an expression π of zero as a $\mathbb{ZP}$-linear combination of elements of $\mathbf{B}$. Suppose that an element b of $\mathbf{B}$ appears with coefficient $a_b \in \mathbb{ZP}$ in π. We can expand π in the cluster $\mathcal{C}_b = \{s_1, \cdots, s_n\}$ of Lemma 1.5.2 so that π becomes a sum of Laurent monomials in $\{s_1, \cdots, s_n\}$. Since of the Key Lemma, the Laurent monomial $\mathbf{s}^{\gamma_b}$ appears with coefficient $a_b y_b$, where y_b is the coefficient of $\mathbf{s}^{\gamma_b}$ in the expansion of b in the cluster $\mathcal{C}_b$. Then we have $a_b y_b = 0$. Since γ_b is monic, $y_b \in \mathbb{P}$ and we get $a_b = 0$. By repeating this argument for all the elements of $\mathbf{B}$ in π, we get that all the coefficients must be zero. Since π is an arbitrary expression, $\mathbf{B}$ is a linearly independent set over $\mathbb{ZP}$.

Now suppose that $\mathbf{B}$ is a $\mathbb{ZP}$–basis of $\mathcal{A}$, its elements are positive and that $\mathbf{B}$ satisfies the Key Lemma. We want to prove that every positive element p of $\mathcal{A}$ is a $\mathbb{Z}_{\geq 0}\mathbb{P}$–linear combination of elements of $\mathbf{B}$, i.e. property CB2' of Definition 1.5.1.

Since $\mathbf{B}$ spans $\mathcal{A}$ over $\mathbb{ZP}$, we express p as a $\mathbb{ZP}$–linear combination π of elements of $\mathbf{B}$ with coefficients in $\mathbb{ZP}$. Suppose that $b \in \mathbf{B}$ appears in π with non–zero coefficient a_b. We expand p in the cluster $\mathcal{C}_b = \{s_1, \cdots, s_n\}$ of the Key Lemma. Then the Laurent monomial $\mathbf{s}^{\gamma_b}$ appears with coefficient $a_b y_b$, where y_b is the coefficient of $\mathbf{s}^{\gamma_b}$ in the expansion of b in the cluster $\mathcal{C}_b$. Since p is positive, $a_b y_b \in \mathbb{Z}_{\geq 0}\mathbb{P}$; since γ is monic, $y_b \in \mathbb{P}$; then $a_b \in \mathbb{Z}_{\geq 0}\mathbb{P}$. By repeating the argument for all the elements of $\mathbf{B}$ appearing in π we get that all the coefficients of π lie in $\mathbb{Z}_{\geq 0}\mathbb{P}$. $\square$

Remark 1.5.4. We want to remark here that if an element p of $\mathcal{A}$ is *positive*, its Newton polytope $Newt_{\mathcal{C}}(p)$ in every cluster $\mathcal{C}$ of $\mathcal{A}$, is invariant under coefficient specializations (see Definition 1.1.19). In particular, one can study $Newt_{\mathcal{C}}(p)$ in the *coefficient–free* cluster algebra obtained from $\mathcal{A}$ by the coefficient specialization $\mathbb{P} \twoheadrightarrow \{1\}$ sending every element of the coefficient semifield $\mathbb{P}$ onto $\{1\}$.

1.5.4 g–vector parametrization of the canonical basis

Let us restrict ourselves to the hypothesis of section 1.1.6, i.e. let $\mathcal{A} = \mathcal{A}(\Sigma^0)$ be a cluster algebra of geometric type of rank n associated with the seed

$$\Sigma^0 = \{\mathbf{x}_0, \widetilde{B}^0\}$$

with coefficients in the semifield $\mathbb{P} = \mathrm{Trop}(x_{n+1}, \cdots, x_m)$. Recall that $\tilde{B}^0 = \{b_{ij}\}$ is a $m \times n$ matrix, whose principal part $B^0 = \{b_{ij}\}_{i,j=1,\cdots,n}$ is skew-symmetrizable (see section 1.1.1) and $\mathbf{x}_0 = \{x_{1;0}, \cdots, x_{n;0}\}$. We assume that

$$\widetilde{B}^0 \text{ has full rank } n. \tag{1.5.5}$$

This condition is satisfied in a cluster algebra with principal coefficients. Let $\mathbf{B}$ be a subset of $\mathcal{A}$ (candidate to be a canonical basis) having the following distinguished properties:

[*hom*] Every element $b \in \mathbf{B}$ has the form

$$b = F_b(\hat{y}_{1;0}, \cdots, \hat{y}_{n;0}) \prod_{i=1}^{m} x_{i;0}^{a_i} = F_b(\hat{y}_{1;0}, \cdots, \hat{y}_{n;0}) \tilde{\mathbf{x}}_0^{\tilde{\mathbf{g}}_b} \tag{1.5.6}$$

where F_b is a primitive polynomial (see definition 1.1.15) in n variables, $\tilde{\mathbf{x}}_0 = (x_{1;0} \cdots, x_{n;0}, x_{n+1}, \cdots, x_m)$ and

$$\hat{y}_{k;0} = y_{k;0} \prod_{j=1}^{n} x_{k;0}^{b^0_{jk}} = \prod_{j=1}^{m} x_{j;0}^{b^0_{jk}} = \tilde{\mathbf{x}}_0^{\mathbf{b}^0_j}$$

for $k = 1, \cdots, n$ (defined in (1.1.17)) and $y_k = \prod_i x_{n+i}^{b_{n+i,k}}$. The elements of $\mathbf{B}$ are hence elements of the set $\mathcal{M}$ defined in section 1.1.6. We denote the $\mathbf{g}$–vector of b by $\mathbf{g}^0_b \doteq (a_1, \cdots, a_n)^t$ in the seed Σ^0. (By Lemma 1.1.14, every element of $\mathbf{B}$ has the same expression (1.5.6) in every seed of $\mathcal{A}$).

[$\mathbf{g}$] The map $b \mapsto \mathbf{g}_b^0$ between $\mathbf{B}$ and $Q = \mathbb{Z}^n$ which associates with an element b of $\mathbf{B}$ its $\mathbf{g}$–vector $\mathbf{g}_b^0$ in Σ^0 is injective.

[F] For every $b \in \mathbf{B}$, F_b has constant term 1.

[B^0] The $\mathbb{Z}_{\geq 0}$–span of the columns $\mathbf{b}_1^0, \cdots, \mathbf{b}_n^0$ of the matrix $\tilde{B}^0$ does not contain lines. In other words every expression of zero as a positive linear combination of them has zero coefficients:

$$a_1 \mathbf{b}_1^0 + \cdots + a_n \mathbf{b}_n^0 = 0,\ a_1, \cdots, a_n \in \mathbb{Z}_{\geq 0} \Rightarrow a_1 = \cdots = a_n = 0$$

The previous properties imply that an element b of $\mathbf{B}$ has the form

$$b = \tilde{\mathbf{x}}_0^{\tilde{\mathbf{g}}_b} + \sum_{\mathbf{c}=(c_1,\cdots,c_n)\in\mathbb{Z}_{\geq 0}^n\setminus\{\mathbf{0}\}} \alpha_{\mathbf{c}}\, \tilde{\mathbf{x}}_0^{(\tilde{\mathbf{g}}_b + \sum c_i \mathbf{b}_i^0)} \tag{1.5.7}$$

where $\mathbf{b}_1^0, \cdots, \mathbf{b}_n^0$ are the columns of $\widetilde{B}^0$ and the sum is over *non–negative* and non–zero integer vectors $\mathbf{c} = (c_1, \cdots, c_n) \in \mathbb{Z}_{\geq 0}^n \setminus \{\mathbf{0}\}$.

We introduce on $\mathbb{Z}^m$ the following binary relation:

$$\mathbf{a} \leq_{B^0} \mathbf{b} \Longleftrightarrow \exists\, \alpha_1, \cdots, \alpha_n \in \mathbb{Z}_{\geq 0} : \mathbf{a} = \mathbf{b} + \sum_{i=1}^n \alpha_i \mathbf{b}_i^0$$

Since of property [B^0], $\leq_{B^0}$ is a partial order in $\mathbb{Z}^m$. It induces a partial order on the monomials $\widetilde{\mathbf{x}}^{\mathbf{a}}$ in $x_1, \cdots, x_n, x_{n+1}, \cdots, x_m$ given by

$$\widetilde{\mathbf{x}}^{\mathbf{a}} \leq \widetilde{\mathbf{x}}^{\mathbf{b}} \Longleftrightarrow \mathbf{a} \leq_{B^0} \mathbf{b}. \tag{1.5.8}$$

Moreover, since of (1.5.7), it induces a partial order on $\mathbf{B}$ given by:

$$b \leq b' \Longleftrightarrow \tilde{\mathbf{g}}_b \leq_{B^0} \tilde{\mathbf{g}}_{b'} \tag{1.5.9}$$

In particular every finite subset of $\mathbf{B}$ has a minimal element. The following Lemma is a refinement of the Key Lemma 1.5.2:

Lemma 1.5.5 (Key Lemma2)**.** *Suppose that a subset $\mathbf{B}$ of $\mathcal{A}$ satisfy properties [hom], [$\mathbf{g}$], [F] and [B^0]. Then in every finite subset $\mathbf{B}'$ of $\mathbf{B}$ there exists an element $b \in \mathbf{B}'$ such that the Laurent monomial $\tilde{\mathbf{x}}_0^{\tilde{\mathbf{g}}_b}$ is a summand of the Laurent expansion of $b' \in \mathbf{B}'$ in the cluster $\mathbf{x}_0$ if and only if $b' = b$.*

Proof. We pick a minimal element b of $\mathbf{B}'$ with respect to (1.5.9). Now if the Laurent monomial $\tilde{\mathbf{x}}_0^{\widetilde{\mathbf{g}}_b}$ is a summand of the Laurent expansion of an element b' of $\mathbf{B}'$ in the cluster $\mathbf{x}_0$, then $\widetilde{\mathbf{g}}_{b'} \leq \widetilde{\mathbf{g}}_b$. For the minimality of b it must be $\widetilde{\mathbf{g}}_{b'} = \widetilde{\mathbf{g}}_b$; then in particular $\mathbf{g}_{b'}^0 = \mathbf{g}_b^0$. For the property [$\mathbf{g}$] we get $b' = b$. $\square$

Remark 1.5.6. The fact that $\widetilde{x}^{\mathbf{g}_b}$ is not a summan of the Laurent expansion of every other element of B' doesn't imply that $p\widetilde{x}^{\mathbf{g}_b}$ for some $p \in \mathbb{P}$ satisfies the same property.

Clearly if the Key lemma 1.5.5 holds, then also Key Lemma 1.5.2 holds. Key Lemma2 is a refinement of Key Lemma 1.5.2 since here the cluster is fixed. The following Theorem is the analogous of Theorem 1.5.3.

Theorem 1.5.7. *Provided that the elements of a subset* $\mathbf{B}$ *of* $\mathcal{A}$ *satisfy properties [hom], [*$\mathbf{g}$*], [F] and [*B^0*], then* $\mathbf{B}$ *is a* $\mathbb{ZP}$*–linearly independent set. Moreover if* $\mathbf{B}$ *spans* $\mathcal{A}$ *over* $\mathbb{ZP}$*, its elements are positive, and the following condition holds:*

[Ind] For every $b \in \mathbf{B}$*, the (Laurent) monomial* $\mathbf{x}_b^{\mathbf{g}}$ *in variables* $x_1, \cdots, x_n$ *is not a summand of the Larent expansion of every other element* $b' \in \mathbf{B}$*,* $b' \neq b$*,*

then they are positive indecomposable.

Proof. Let $\pi = \sum_{b \in \mathbf{B}'} a_b b$ be a finite $\mathbb{ZP}$–linear combination of elements of $\mathbf{B}$. Without lost of generality we assume that $a_b \in \mathbb{P}$ for every $b \in \mathbf{B}'$. Suppose that $\pi = 0$. We expand π in the cluster $\mathbf{x}_0$. We consider the set $L = \{a_b \tilde{\mathbf{x}_0}^{\tilde{\mathbf{g}}_b}\}$ of leading terms of π. We note that if two elements $a_b \tilde{\mathbf{x}_0}^{\tilde{\mathbf{g}}_b}$ and $a_b \tilde{\mathbf{x}_0}^{\tilde{\mathbf{g}}'_b}$ of this set are equal, then $\mathbf{g}_b = \mathbf{g}'_b$; since of property [$\mathbf{g}$], $b = b'$ and hence $\tilde{\mathbf{g}}_b = \tilde{\mathbf{g}}_{b'}$ and finally $a_b = a_{b'}$.

Since L is finite there exists a minimal element $a_{b_0} \tilde{\mathbf{x}}_0^{\tilde{\mathbf{g}}_{b_0}}$ (note that b_0 is *not* necessarily minimal in $\mathbf{B}'$). In particular this element does not appear in the Laurent expansion of every other element of $\mathbf{B}$. We conclude $a_{b_0} = 0$. We now consider the set $\mathbf{B}' \setminus \{b_0\}$ and proceed by induction on its cardinality in order to get $a_b = 0$ for every $b \in \mathbf{B}'$.

Condition [Ind] is much stronger than the fact that $\tilde{\mathbf{x}}_0^{\tilde{\mathbf{g}}_b}$ is not a summand of every other element of $\mathbf{B}$ (see Example). Now suppose that $\mathbf{B}$ spans $\mathcal{A}$ over $\mathbb{ZP}$, its elements are positive and that $\mathbf{B}$ satisfies property [Ind]. We want to prove that every positive element p of $\mathcal{A}$ is a $\mathbb{Z}_{\geq 0}\mathbb{P}$–linear combination of elements of $\mathbf{B}$, i.e. property CB2' of Definition 1.5.1 holds. We consider the expansion $\pi = \sum_{b \in \mathbf{B}'} a_b b = p$ of p in $\mathbf{B}$, where $\mathbf{B}'$ is the finite subset of $\mathbf{B}$ of the elements b such that $a_b \neq 0$. We expand p in the cluster $\mathbf{x}_0$. We pick a minimal element b of $\mathbf{B}'$. Then the Laurent monomial $\tilde{\mathbf{x}}_0^{\tilde{\mathbf{g}}_b}$ corresponding to this minimal element b, appears in π with coefficient a_b. Since p is positive, $a_b \in \mathbb{Z}_{\geq 0}\mathbb{P}$. We now consider the element $\pi' = \pi - a_b b$. Note that at this point it is not clear that π' is positive. We pick a minimal element b' of the new set $\mathbf{B}'' = \mathbf{B}' \setminus \{b\}$. Then the Laurent monomial $\tilde{\mathbf{x}}_0^{\tilde{\mathbf{g}}_{b'}}$ corresponding to this minimal element b', is *not* a summand of every other element of $\mathbf{B}''$, hence, in particular, it appears in π' with coefficient $a_{b'}$. If $a_{b'}$ is in $\mathbb{Z}_{\geq 0}\mathbb{P}$ we continue with $\pi'' = \pi' - b'$, so we can assume without lost of generality that $a_{b'}$ is *not* in $\mathbb{Z}_{\geq 0}\mathbb{P}$. Since π is positive, the coefficient of the Laurent monomial $\tilde{\mathbf{x}}_0^{\tilde{\mathbf{g}}_{b'}}$ must be in $\mathbb{Z}_{\geq 0}\mathbb{P}$ and hence $a_{b'} \tilde{\mathbf{x}}_0^{\tilde{\mathbf{g}}_{b'}}$ must be a summand of b. But this means that the Laurent monomial $\mathbf{x}_{b'}^{\mathbf{g}}$ is a summand of b, against the hypothesis [ind]. Then $a_{b'} \in \mathbb{Z}_{\geq 0}\mathbb{P}$. $\square$

Chapter 2

Cluster algebras of type $A_2^{(1)}$

This chapter is devoted to the study of cluster algebras of type $A_2^{(1)}$. A coefficient-free cluster algebra of this type first appeared in [16, Example 7.8]. It is the only class of rank three cluster algebras of affine type that are not bipartite. In particular we do not have the results of section 1.3 at our disposal but we still find similar statements, e.g. we find an interesting connection (Proposition 2.3.7) between denominator vectors and **g**–vectors that can be viewed as a generalization of Theorem 1.3.8. On the other hand we are able to produce explicit Laurent expansions in all the clusters of all the elements of the canonical basis using quiver representations (Section 2.5).

We will mainly concentrate ourselves in the principal coefficients setting. We find canonical basis, its parametrization in term of the root lattice and explicit formulas for its elements in this setting. In section 2.4 we then generalize all these results to any *tropical* semi–field.

2.1 Algebraic structure of a cluster algebra of type $A_2^{(1)}$ with principal coefficients

Let $\mathbb{P} = \mathrm{Trop}(y_1, y_2, y_3)$ be the tropical semi-field (see section 1.1.1) with generators $\{y_1, y_2, y_3\}$. Let $\mathcal{F} = \mathbb{QP}(x_1, x_2, x_3)$ be the field of rational functions in 3 commuting variables over $\mathbb{QP}$, the field of fractions of $\mathbb{ZP}$. We consider inside $\mathbb{P}$ the sequence $\{y_{1;m} : m \in \mathbb{Z}\}$ defined by the initial data $y_{1;1} = y_1$, $y_{1;i} = y_{i+3}^{-1}$ for $i = 0, -1, -2$ and $y_{1;-3} = y_3$, together with the recurrence relations

$$y_{1;m} = \begin{cases} y_2\, y_{1;\,\mathrm{sg}(m)(|m|-1)} & \text{if } m \leq -4, 2 \leq m,\ m \text{ even} \\ y_1 y_3\, y_{1;\,\mathrm{sg}(m)(|m|-1)} & \text{if } m \leq -5, 3 \leq m,\ m \text{ odd} \end{cases} \tag{2.1.1}$$

Here $\mathrm{sg}(m)$ is the sign of m. The explicit solution of this recursion will be given in (2.1.23).

Recursively define elements $x_m \in \mathcal{F}$, $m \in \mathbb{Z}$ by

$$x_m x_{m+3} = \frac{x_{m+1}x_{m+2} + y_{1;m}}{y_{1;m} \oplus 1} = \frac{y_{3;m+1}x_{m+1}x_{m+2} + 1}{y_{3;m+1} \oplus 1} \tag{2.1.2}$$

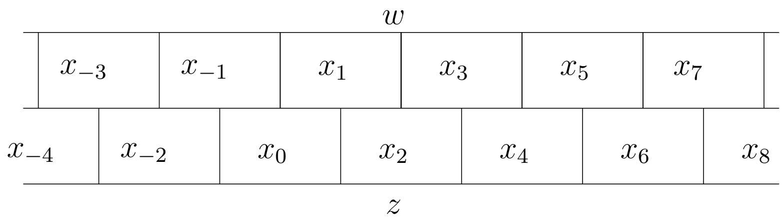

Figure 2.1: The exchange graph of $\mathcal{A}$

where $y_{3;m+1} \doteq y_{1;m}^{-1}$ (this choice makes the second equality obvious). In particular $y_{3;1} = y_3$ which explains the terminology. Note that, in view of (2.1.1), the denominators in the middle term of (2.1.2) are all 1 except when $m = 0, -1, -2$.

We also define

$$w \doteq \frac{y_2 x_1 + x_3}{x_2}, \tag{2.1.3}$$

$$z \doteq \frac{y_1 y_3\, x_1 x_2 + y_1 + x_2 x_3}{x_1 x_3}. \tag{2.1.4}$$

Let $\mathcal{A}$ be the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by the elements $\{x_m, w, z : m \in \mathbb{Z}\}$. In Proposition 2.1.1 below we will show that $\mathcal{A}$ is a cluster algebra of type $A_2^{(1)}$ with principal coefficients at the initial seed

$$\Sigma_{In} \doteq \{B = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}\}. \tag{2.1.5}$$

The generators of $\mathcal{A}$ are its *cluster variables*, while the *clusters* of $\mathcal{A}$ are the sets $\{x_m, x_{m+1}, x_{m+2}\}$, $\{x_{2m-1}, w, x_{2m+1}\}$ and $\{x_{2m-2}, z, x_{2m}\}$ for every $m \in \mathbb{Z}$. The *exchange graph* of $\mathcal{A}$ is shown in figure 2.1: it has clusters as vertices and an edge between two clusters $\mathcal{C}$ and $\mathcal{C}'$ whenever $|\mathcal{C} \cap \mathcal{C}'| = 2$. In this figure cluster variables are associated with regions. The generic cluster $\{s_1, s_2, s_3\}$ corresponds to the (unique) vertex common to the three regions labeled by s_1, s_2 and s_3.

Using (2.1.1) and (2.1.2) we obtain more relations between cluster variables: for every $m \in \mathbb{Z}$

$$w x_{2m} = \frac{y_{2;2m-1} x_{2m-1} + x_{2m+1}}{y_{2;2m-1} \oplus 1} = \frac{x_{2m-1} + y^{(w)}_{2;2m-1} x_{2m+1}}{y^{(w)}_{2;2m-1} \oplus 1} \tag{2.1.6}$$

where

$$y_{2;2m-1} \doteq \begin{cases} y_2 & \text{if } m \geq 1, \\ 1/y_3 & \text{if } m = 0, \\ 1/y_1 y_3 & \text{if } m \leq -1, \end{cases} \tag{2.1.7}$$

and $y^{(w)}_{2;2m-1} \doteq y^{-1}_{2;2m-1}$. Indeed for $m = 1$ (2.1.6) is nothing but the definition of w; for $m \geq 1$ (resp. $m \leq 1$) one can proceed by induction on m expanding the element $w x_{2m} x_{2m+2}$ (resp. $w x_{2m} x_{2m-2}$). Similarly: for every $m \in \mathbb{Z}$ we have

$$z x_{2m+1} = \frac{y_{2;2m} x_{2m} + x_{2m+2}}{y_{2;2m} \oplus 1} = \frac{x_{2m} + y^{(z)}_{2;2m} x_{2m+2}}{y^{(1)}_{2;2m} \oplus 1} \tag{2.1.8}$$

where

$$y_{2;2m} \doteq \begin{cases} y_1y_3 & \text{if } m \geq 1, \\ y_1 & \text{if } m = 0, \\ 1/y_2 & \text{if } m \leq -1. \end{cases} \tag{2.1.9}$$

and $y^{(z)}_{2;2m} \doteq y^{-1}_{2;2m}$. Moreover from (2.1.2), using (2.1.6) and (2.1.8), we obtain for every $m \in \mathbb{Z}$:

$$x_{2m-2}x_{2m+2} = \frac{x^2_{2m} + y^{(z)}_{1;2m-2}z}{y^{(z)}_{1;2m-2} \oplus 1} = \frac{y^{(z)}_{3;2m}x^2_{2m} + z}{y^{(z)}_{3;2m} \oplus 1} \tag{2.1.10}$$

and

$$x_{2m-1}x_{2m+3} = \frac{x^2_{2m+1} + y^{(w)}_{1;2m-1}w}{y^{(w)}_{1;2m-1} \oplus 1} = \frac{y^{(w)}_{3;2m+1}x^2_{2m+1} + w}{y^{(w)}_{3;2m+1} \oplus 1} \tag{2.1.11}$$

where

$$y^{(c)}_{1;m} = y_{1;m}(y_{2;m} \oplus 1), \tag{2.1.12}$$

$y^{(c)}_{3;m+2} \doteq (y^{(c)}_{1;m})^{-1}$ and $c = w$ if m is odd and $c = z$ if m is even. Indeed one can expand $x_{2m-2}x_{2m+2}$ (resp. $x_{2m-1}x_{2m+3}$) in the cluster $\{x_{2m-1}, x_{2m}, x_{2m+1}\}$ (resp. $\{x_{2m}, x_{2m+1}, x_{2m+2}\}$) using (2.1.2). Then using (2.1.1) and (2.1.8) (resp. (2.1.6)) one get the result.

We refer to the relations (2.1.2), (2.1.6), (2.1.8), (2.1.10) and (2.1.11) as *exchange relations* of $\mathcal{A}$.

Proposition 2.1.1. *$\mathcal{A}$ is a cluster algebra of type $A^{(1)}_2$ with principal coefficients at the initial seed Σ_{In} defined in* (2.1.5)*. The exchange graph of $\mathcal{A}$ is given by Figure 2.1.*

Proof. The proof is based on the following Lemma:

Lemma 2.1.2. *The (unlabeled)* seeds *of a cluster algebra of type $A^{(1)}_2$ with principal coefficients at the initial seed Σ_{In} are*

$$\begin{aligned} \Sigma_m &\doteq \{\tilde{B}_m, \{x_m, x_{m+1}, x_{m+2}\}\}, & (2.1.13)\\ \Sigma^w_{2m-1} &\doteq \{\tilde{B}^{cyclic}_{2m-1}, \{x_{2m-1}, w, x_{2m+1}\}\}, & (2.1.14)\\ \Sigma^z_{2m-1} &\doteq \{\tilde{B}^{cyclic}_{2m}, \{x_{2m}, z, x_{2m+2}\}\} & (2.1.15) \end{aligned}$$

for every $m \in \mathbb{Z}$. For every m, they are mutually related by the following diagram of mutations:

$$\begin{tikzcd} \Sigma^w_{2m-1} \arrow[rr, shift left] \arrow[d, leftrightarrow] & & \Sigma^w_{2m+1} \arrow[ll, shift left] \arrow[d, leftrightarrow] & \\ \Sigma_{2m-1} \arrow[r, shift left] & \Sigma_{2m} \arrow[l, shift left] \arrow[r, shift left] \arrow[d, leftrightarrow] & \Sigma_{2m+1} \arrow[l, shift left] \arrow[r, shift left] & \Sigma_{2m+2} \arrow[l, shift left] \arrow[d, leftrightarrow] \\ & \Sigma^z_{2m} \arrow[rr, shift left] & & \Sigma^z_{2m+2} \arrow[ll, shift left] \end{tikzcd} \tag{2.1.16}$$

where the right (resp. left) arrows $\to$ (resp. $\leftarrow$) stand for mutations in direction 1 (resp. 3) and vertical arrows for mutations in direction 2. Σ_m is not equivalent to Σ_n if $m \neq n$, in particular the exchange graph of $\mathcal{A}$ is given by figure 2.1.

$\tilde{B}_m$ and $\tilde{B}_m^{cyclic}$ are 6×3 rectangular matrices of the form

$$\tilde{B}_m \doteq \begin{bmatrix} B \\ B_m \end{bmatrix} \quad \tilde{B}_m^{cyclic} \doteq \begin{bmatrix} B^{cyclic} \\ B_m^{cyclic} \end{bmatrix}$$

where $B^{cyclic} \doteq \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$. The square matrices at the bottom are given by: for every $n \geq 1$

$$B_m = \begin{cases} \begin{pmatrix} n & 1 & -n \\ n & 0 & -n+1 \\ n-1 & 1 & -n+1 \end{pmatrix} & m = 2n \\[2ex] \begin{pmatrix} n & 0 & -n+1 \\ n-1 & 1 & -n+1 \\ n-1 & 0 & -n+2 \end{pmatrix} & m = 2n-1 \\[2ex] \begin{pmatrix} n-2 & 0 & -n+1 \\ n-1 & -1 & -n+1 \\ n-1 & 0 & -n \end{pmatrix} & m = -2n \\[2ex] \begin{pmatrix} n-1 & -1 & -n+1 \\ n-1 & 0 & -n \\ n & -1 & -n \end{pmatrix} & m = -2n-1 \end{cases}$$

$B_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$, $B_{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$.

$$B_m^{cyclic} = \begin{cases} \begin{pmatrix} n & -1 & -n+1 \\ n & 0 & -n+1 \\ n-1 & -1 & -n+2 \end{pmatrix} & m = 2n \\[2ex] \begin{pmatrix} n & 0 & -n+1 \\ n-1 & -1 & -n+2 \\ n-1 & 0 & -n+2 \end{pmatrix} & m = 2n-1 \\[2ex] \begin{pmatrix} n-2 & 0 & -n+1 \\ n-2 & 1 & -n+1 \\ n-1 & 0 & -n \end{pmatrix} & m = -2n \\[2ex] \begin{pmatrix} n-2 & 1 & -n+1 \\ n-1 & 0 & -n \\ n-1 & 1 & -n \end{pmatrix} & m = -2n-1 \end{cases}$$

$B_0^{cyclic} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$, $B_{-1}^{cyclic} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$.

Proof of Lemma 2.1.2. We need to prove the diagram (2.1.16) for every $m \in \mathbb{Z}$. For $m = 1$ Σ_1 is the initial seed Σ. For $m \geq 1$ (resp. $m \leq 1$) a direct check shows that $\tilde{B}_{m+1}$ (resp. $\tilde{B}_{m-1}$) is obtained from $\tilde{B}_m$ by a matrix mutation in direction 1 (resp. 3) and by reordering the index set with the permutation (132) (resp. (123)). Then it follows $\tilde{B}_m = \mu_3(\tilde{B}_{m+1})$ for $m \geq 1$ (resp. $\tilde{B}_{m+1} = \mu_1(\tilde{B}_m)$ for $m \leq 1$). We define x_{m+3} (resp. x_m) to be the cluster variable obtained by the mutation of the cluster variable x_m (resp. x_{m+2}) in the (unlabeled) seed Σ_m. The central line of the diagram is proved.

We define w (resp. z) to be the cluster variable obtained by mutation of the cluster variable x_2 (resp. x_3) in the (unlabeled) seed Σ_1 (resp. Σ_2). Explicitly w (resp. z) is defined by (2.1.3) (resp. (2.1.4)). Let $\{y_{1;m}, y_{2;m}, y_{3;m}\}$ the triple of coefficients of the seed Σ_m. Since $\mathcal{A}$ has principal coefficients, in particular it is of geometric type, then, by definition,

$$y_{i;m} = \prod_{j=1}^{3} y_j^{b^m_{i,3+j}}$$

where $B_m = \{b^m_{ij}\}$. Note that $\{y_{1;m}\}$ (resp. $\{y_{2;m}\}$) satisfies the recurrence relation (2.1.1) (resp. (2.1.7)-(2.1.9)). Moreover given $\delta \doteq (1,1,1)$, for every $n \geq 1$ we have

$$y_{1;2n} = \mathbf{y}^{n\delta} y_3^{-1}; \quad y_{1;2n} = \mathbf{y}^{n\delta} y_2^{-1} y_3^{-1}; \quad y_{1;-2n} = \mathbf{y}^{(n-1)\delta} y_1^{-1} y_2^{-1}; \quad y_{1;2n} = \mathbf{y}^{(n-1)\delta} y_1^{-1}.$$

By induction on m one can prove the equality

$$c = \frac{y_{2;m} x_m + x_{m+2}}{(y_{2;m} \oplus 1) x_{m+1}}. \tag{2.1.17}$$

where $c = w$ if m is odd and $c = z$ if m is even. By (2.1.17) we conclude that w (resp. z) is the cluster variable obtained by the mutation of the cluster variable x_{2m} (resp. x_{2m+1}) in the seed Σ_{2m-1} (resp. Σ_{2m}) for every $m \in \mathbb{Z}$. Moreover it is straightforward to check that the matrix $\tilde{B}^{cyclic}_m$ is obtained from $\tilde{B}_m$ by a matrix mutation in direction 2. Then the diagram (2.1.16) is proved. The fact that the diagram is not finite, follows observing that the first column of B_m has at least one positive coordinate while the third one at least one negative coordinate. Hence if Σ_m is equivalent to Σ_n then either $y_{1;m} = y_{1;n}$ or $y_{1;m} = y_{2;n}$. Since of the previous formulas this is the case only if $m = n$. This finishes the proof of Lemma 2.1.2 $\square$

It remains to prove relations (2.1.2), (2.1.6), (2.1.8), (2.1.10) and (2.1.11) are the *exchange relations* of a cluster algebra of type $A_2^{(1)}$. Let $\{y_{1;m}, y_{2;m}, y_{3;m}\}$ (resp. $\{y^{(c)}_{1;m}, y^{(c)}_{2;m}, y^{(c)}_{3;m}\}$) the triple of coefficients of the seed Σ_m (resp. Σ^c_m for $c = w$ or z). Then using Lemma 2.1.2 and the definition 1.1.3 of exchange relations we get the desired result. Moreover we also get formulas (2.1.1), (2.1.7), (2.1.12). $\square$

2.1.1 Canonical basis

We construct the canonical basis of $\mathcal{A}$.

Definition 2.1.3. Let us define

$$u \doteq zw - y_1 y_3 - y_2. \tag{2.1.18}$$

In view of (2.1.2) $\mathbb{P}$ is contained in $\mathcal{A}$ and then $u \in \mathcal{A}$. Let $u_1, u_2, \dots$ be the sequence of polynomials defined by the initial condition

$$\begin{aligned} u_0 &= 1, \\ u_1 &= u, \\ u_2 &= u_1^2 - 2y_1y_2y_3 \end{aligned} \tag{2.1.19}$$

together with the recurrence relation for $n \geq 2$

$$u_{n+1} = u_1 u_n - \mathbf{y}^{\delta}\, u_{n-1}. \tag{2.1.20}$$

where $\delta = (1,1,1)^t$.

By definition u_n is a polynomial in u and since u lies in $\mathcal{A}$, $u_n \in \mathcal{A}$ for $n \geq 1$. The definition of the u_n's is a generalization of analogous definition in rank two cluster algebras of affine type given in [27]. In loc.cit. the group of coefficients was just $\{1\}$ and the definition was given in terms of Chebychev's polynomials of the first kind. Here if $\mathbb{P}$ was $\{1\}$, i.e. $\mathcal{A}$ was a coefficient–free cluster algebra, then u_n would be the n–th Chebychev's polynomials of the first kind T_n evaluated in u.

Recall that a monomial in cluster variables belonging to the same cluster is called a *cluster monomial*.

Theorem 2.1.4. *The set* $\mathbf{B}$ *of cluster monomials and of the elements* $\{u_n w^k, u_n z^k : n \geq 1, k \geq 0\}$ *is a canonical basis of* $\mathcal{A}$ *(see definition 1.5.1).*

The proof of the Theorem will be given in section 2.3. We point out here that, as mentioned in the first chapter, all the canonical basis of $\mathcal{A}$ can be obtained from $\mathbf{B}$ by scalar multiplication.

2.1.2 Parametrization of the canonical basis

Our next result provides a parametrization of $\mathbf{B}$. Let $Q = \mathbb{Z}^3$ be a lattice of rank 3 with a fixed basis $\{\alpha_1, \alpha_2, \alpha_3\}$. We sometimes identify $\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 \in Q$ with its coordinate vector (a_1, a_2, a_3) with respect to the chosen basis $\{\alpha_1, \alpha_2, \alpha_3\}$.

Theorem 2.1.5. *For every* $\alpha = (a_1, a_2, a_3) \in Q$*, there is a unique basis element* $X[\alpha] \in \mathbf{B}$ *of the form*

$$X[\alpha] = \frac{N_\alpha(x_1, x_2, x_3)}{x_1^{a_1} x_2^{a_2} x_3^{a_3}}, \tag{2.1.21}$$

where N_α *is a polynomial with coefficients in* $\mathbb{ZP}$ *not divisible by any* x_i*. The correspondence* $\alpha \mapsto X[\alpha]$ *is a bijection between* Q *and* $\mathbf{B}$*. In particular the denominator vector map* $x \mapsto \mathbf{d}(x)$ *in the cluster* $\{x_1, x_2, x_3\}$*, restricts to a bijection between* $\mathbf{B}$ *and* Q*.*

Following [16], we identify Q with the root lattice of the affine root system of type $A_2^{(1)}$ (see section 1.2.2) so that α_1, α_2 and α_3 become simple roots. In section 1.2.2 we recall the structure of a root system of type $A_2^{(1)}$. For the initial cluster variables x_1, x_2 and x_3, the correspondence (2.1.21) takes the form

$$x_1 = \frac{1}{x_1^{-1}} = X[-\alpha_1], \quad x_2 = \frac{1}{x_2^{-1}} = X[-\alpha_2], \quad x_3 = \frac{1}{x_3^{-1}} = X[-\alpha_3].$$

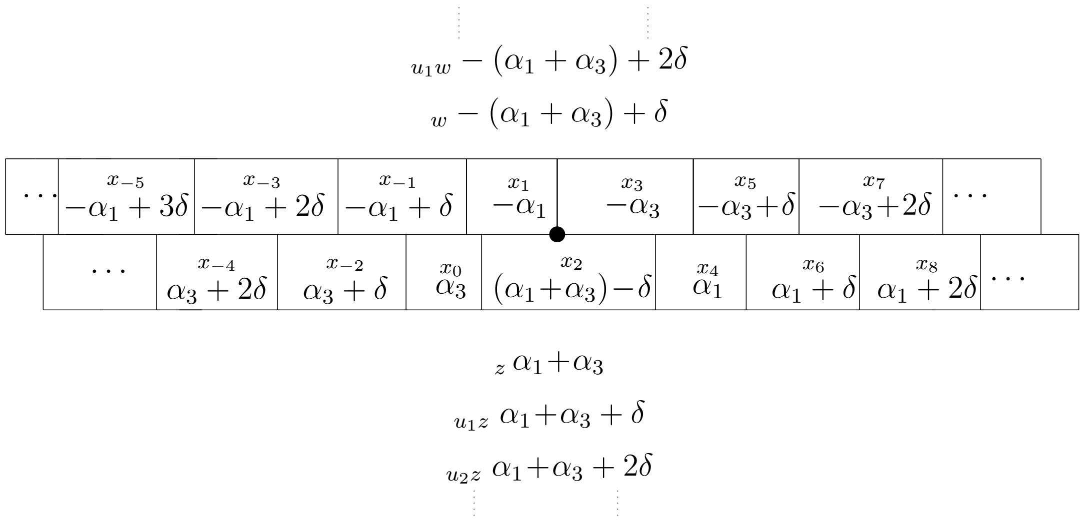

Figure 2.2: Denominator vectors and real roots in type $A_2^{(1)}$

Proposition 2.1.6. *The cluster variables x_m different from x_1, x_2 and x_3, and the elements u_n have the form*

$$x_{-(2n+1)} = X[-\alpha_1+(n+1)\delta], \qquad x_{2(n+2)+1} = X[-\alpha_3+(n+1)\delta]$$

$$x_{-2n} = X[\alpha_3+n\delta], \qquad x_{2(n+2)} = X[\alpha_1+n\delta]$$

$$w = X[-(\alpha_1+\alpha_3)+\delta], \qquad u_n = X[n\delta], \quad z = X[\alpha_1+\alpha_3] \tag{2.1.22}$$

for $n \geq 0$. In particular $u_n w = X[-(\alpha_1+\alpha_3)+(n+1)\delta]$ and $u_n z = X[(\alpha_1+\alpha_3)+n\delta]$. The correspondence $\alpha \mapsto X[\alpha]$ is hence a bijection between the positive real roots and $\{x_m : m \in \mathbb{Z} \setminus \{1,2,3\}\} \cup \{u_n z, u_n w : n \geq 0\}$. Moreover cluster variables are in bijection with the set of positive real Schur roots. Figure 2.2 describes the situation.

Using Proposition 2.1.6 one can recognize that the element $\mathbf{y}^{\mathbf{d}(x_{m+3})}$ satisfies (2.1.1) for every $m \in \mathbb{Z}$ (here we use the standard notation $\mathbf{s}^{(a_1,a_2,a_3)} \doteq s_1^{a_1} s_2^{a_2} s_3^{a_3}$). In particular we get

$$y_{1;m} = \mathbf{y}^{\mathbf{d}(x_{m+3})} \tag{2.1.23}$$

We conclude this section by pointing out an important property of denominator vectors.

Definition 2.1.7. [18, Definition 6.12] A collection of vectors in $\mathbb{Z}^n$ (or in $\mathbb{R}^n$) are *sign–coherent* (to each other) if, for any $i \in \{1, \cdots, n\}$, the ith coordinates of all of these vectors are either all non–negative or all non–positive.

Corollary 2.1.8. *Denominator vectors of cluster variables belonging to the same cluster are sign–coherent.*

Proof. It follows after a glance at figure 2.2 or figure 2.4. $\square$

2.1.3 Explicit formulas for the elements of the canonical basis

Our next result provides explicit formulas for the elements of $\mathbf{B}$ in every cluster of $\mathcal{A}$. In view of the exchange relations, it is sufficient to consider only two clusters, namely $\{x_1, x_2, x_3\}$ and $\{x_1, w, x_3\}$ and only cluster variables x_m with $m \geq 1$. Indeed the expansion of a cluster variable x_{m+n} (resp. x_{2m+n}) in the cluster $\{x_m, c, x_{m+2}\}$, where $c = w$ or $c = x_{m+1}$, (resp. $\{x_{2m}, z, x_{2m+2}\}$) is obtained by the expansion of x_{1+n} (resp. x_{2m+1+n}) in the cluster $\{x_1, c, x_3\}$ (resp. $\{x_{2m+1}, w, x_{2m+3}\}$) by replacing x_1 with x_m, c with x_2 when $c \neq w$, x_3 with x_{m+2} and y_i with $y_{i;m}$ (resp. x_{2m+1} with x_{2m}, w with z, x_{2m+3} with x_{2m+2} and $y^{(w)}_{i;2m+1}$ with $y^{(z)}_{i;2m}$), $i = 1, 2, 3$, $n, m \in \mathbb{Z}$. Moreover the expansion of x_{-m} is obtained from the expansion of x_{m+2} by replacing x_1 with x_3, x_3 with x_1 and y_1 with y_3^{-1}, y_2 with y_2^{-1} and y_3 with y_1^{-1}.

Theorem 2.1.9. *For every $m \geq 1$ the following formulas hold. In the cluster $\{x_1, x_2, x_3\}$*

$$x_{2m+1} = \frac{\sum_{\mathbf{e}} \chi_{2m+1}(\mathbf{e})\mathbf{y}^{\mathbf{e}}\mathbf{x}^{(e_2+e_3, m-1-e_1+e_3, 2m-e_2-e_1-2)} + x_2^{m-1}x_3^{2m-2}}{x_1^{m-1}x_2^{m-1}x_3^{m-2}} \tag{2.1.24}$$

where $\chi_{2m+1}(e_1, e_2, e_3) \doteq \binom{e_1-e_3}{e_2-e_3}\binom{m-1-e_3}{m-1-e_1}\binom{e_1-1}{e_3}$.

$$x_{2m} = \frac{\sum_{\mathbf{e}} \chi_{2m}(\mathbf{e})\mathbf{y}^{\mathbf{e}}\mathbf{x}^{(e_2+e_3, m-1-e_1+e_3, 2m-3-e_1-e_2)} + x_2^{m-1}x_3^{2m-3}}{x_1^{m-1}x_2^{m-2}x_3^{m-2}} \tag{2.1.25}$$

where $\chi_{2m}(e_1, e_2, e_3) \doteq \binom{e_1-1}{e_3}[\binom{e_1-e_3}{e_2-e_3}\binom{m-2-e_3}{m-2-e_1}+\binom{e_1-e_3-1}{e_2-e_3}\binom{m-2-e_3}{m-1-e_1}]$.

In the cluster $\{x_1, w, x_3\}$

$$x_{2m+1} = \frac{\sum_{e_1,e_3} \binom{m-1-e_3}{m-1-e_1}\binom{e_1-1}{e_3} y_1^{e_1}(y_2y_3)^{e_3}\, x_1^{2e_3} w^{e_1-e_3} x_3^{2m-2e_1-2} + x_3^{2m-2}}{x_1^{m-1}x_3^{m-2}} \tag{2.1.26}$$

$$x_{2m} = \frac{\sum_{\mathbf{e}} \chi^w_{2m}(\mathbf{e}) y_1^{e_1}(y_2^{e_2}y_3)^{e_3} x_1^{2e_3+1-e_2} w^{e_1-e_3} x_3^{2m-3-2e_1+e_2} + (y_2x_1+x_3)x_3^{2m-3}}{x_1^{m-1}wx_3^{m-2}} \tag{2.1.27}$$

where $\chi^w_{2m}(e_1, e_2, e_3) \doteq \binom{e_1-1}{e_3}\binom{m-1-e_2-e_3}{m-1-e_1-e_2}\binom{1}{e_2}$.

The Laurent expansion of u_n, $n \geq 1$, in the cluster $\{x_1, x_2, x_3\}$ is given by

$$u_n = \frac{\mathbf{y}^{n\delta}x_1^{2n}x_2^n + x_2^nx_3^{2n} + \sum_{\mathbf{e}} \chi_{u_n}(e_1, e_2, e_3)\mathbf{y}^{\mathbf{e}}\mathbf{x}^{(e_2+e_3, n-e_1+e_3, 2n-e_1-e_2)}}{x_1^nx_2^nx_3^n} \tag{2.1.28}$$

where $\chi_{u_n}(e_1, e_2, e_3) \doteq \binom{e_1-e_3}{e_1-e_2}[\binom{n-e_3}{n-e_1}\binom{e_1-1}{e_3}+\binom{n-e_3-1}{n-e_1}\binom{e_1-1}{e_3-1}]$. *In the cluster $\{x_1, w, x_3\}$*

$$u_n = \frac{\mathbf{y}^{n\delta}\, x_1^{2n}+x_3^{2n} + \sum_{e_1,e_3} \chi^w_{u_n}(e_1, e_3) y_1^{e_1}(y_2y_3)^{e_3}\, x_1^{2e_3} w^{e_1-e_3} x_3^{2n-2e_1}}{x_1^nx_3^n} \tag{2.1.29}$$

where $\chi^w_{u_n}(e_1, e_3) \doteq \binom{n-e_3}{n-e_1}\binom{e_1-1}{e_3} + \binom{n-e_3-1}{n-e_1}\binom{e_1-1}{e_3-1}$. *In the cluster $\{x_1, w, x_3\}$ the expansion of z is given by*

$$z = \frac{y_1\, y_2\, y_3\, x_1^2 + y_1\, y_3\, x_1\, x_3 + y_1\; w + y_2\, x_1x_3 + x_3^2}{x_1wx_3} \tag{2.1.30}$$

2.2 F–polynomials and quiver Grassmannians

The main result of this section is a description of the F–polynomial associated with every element of the canonical basis $\mathbf{B}$ of $\mathcal{A}$ in terms of quiver–Grassmannians defined in (1.4.1). We associate with the initial exchange matrix B of the seed Σ_{In} given in (2.1.5), the quiver Q_{In} having b_{ij} arrows from j to i if $b_{ij} \geq 0$:

$$Q_{In} \doteq \begin{tikzcd} & 2 \arrow[dl] & \\ 1 & & 3 \arrow[ll] \arrow[ul] \end{tikzcd} \tag{2.2.1}$$

In view of Theorem 1.4.1 the denominator vector of a cluster variable s is the dimension vector of a (unique) *rigid* indecomposable module M_s.

Theorem 2.2.1. *The F–polynomial F_s associated with a cluster variable s in the seed Σ_{In}, is given by*

$$F_s = \sum_{\mathbf{e}} \chi_{\mathbf{e}}(M_s) y_1^{e_1} y_2^{e_2} y_3^{e_3} \tag{2.2.2}$$

The proof of Theorem 2.2.1 will be given in section 2.5.

Once we have given a representation theoretic interpretation of cluster variables and hence of cluster monomials, we investigate an analogous interpretation for the other elements $\{u_n w^k, u_n z^k\}$ of $\mathbf{B}$. In order to do that we need to study *non*–rigid Q_{In}–representations. The indecomposable non–rigid Q_{In}–representations form infinitely many connected components of the Auslander–Reiten quiver of Q_{In} called tubes. There is one tube of *rank* two, i.e. the Auslander–Reiten translation τ has period two in this component, and infinitely many tubes of rank one parameterized by the choice of elements of $\lambda \in \mathbb{C} = k$. We define the *regular homogeneous* representations to be

$$Reg_n^{\{3,2\}} \doteq \begin{tikzcd} & k^n \arrow[dl, "="'] & \\ k^n & & k^n \arrow[ll, "="] \arrow[ul, "J_n(0)"'] \end{tikzcd} ; \qquad Reg_n^{\{2,1\}} \doteq \begin{tikzcd} & k^n \arrow[dl, "J_n(0)"'] & \\ k^{n+1} & & k^n \arrow[ll, "="] \arrow[ul, "="'] \end{tikzcd}$$

$$Reg_n^{\{3,1\}}(\lambda) \doteq \begin{tikzcd} & k^n \arrow[dl, "="'] & \\ k^n & & k^n \arrow[ll, "J_n(\lambda)"] \arrow[ul, "="'] \end{tikzcd} .$$

where $J_n(\lambda)$ is the n-Jordan block of eigenvalue $\lambda \in k$. The arrows labeled by $=$ are the identity map. We also define the *regular non–homogeneous* to be

$$RN_n^w \doteq \begin{tikzcd} & k^{n+1} \arrow[dl, "="'] & \\ k^n & & k^n \arrow[ll, "="] \arrow[ul, "J_n(0)"'] \end{tikzcd} ; \qquad RN_n^z \doteq \begin{tikzcd} & k^n \arrow[dl, "J_n(0)"'] & \\ k^{n+1} & & k^{n+1} \arrow[ll, "="] \arrow[ul, "="'] \end{tikzcd}$$

Figure 2.3 shows the shape of such components.

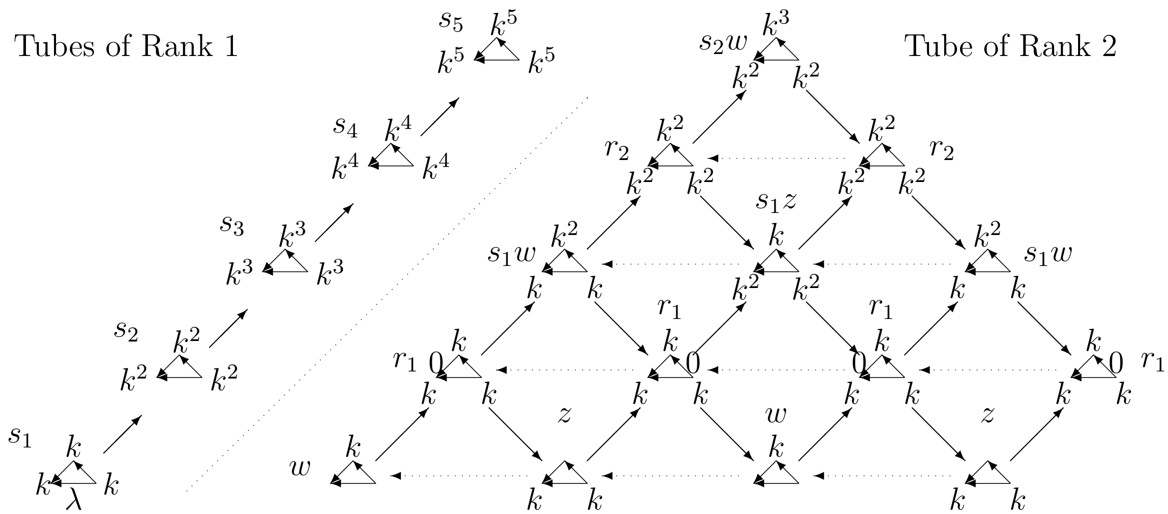

Figure 2.3: The shape of the tubes of the quiver Q_{In} and the image by the Caldero-Chapoton map.

2.3 Proofs

We are going to show that the elements of $\mathbf{B}$ satisfy all the hypothesis of Theorem 1.5.7. In this process we will need Theorem 2.1.5. Even if its proof is completely independent, for the convenience of the reader and for a higher rigorousness we include it in this section. Explicit formulas will yield positivity and straightening relations the span property.

2.3.1 Homogeneity of the elements of B

Proposition 2.3.1. *The elements of* $\mathbf{B}$ *belong to the set* $\mathcal{M}$ *defined in Section 1.1.6. In other words the Laurent expansion of every element* b *of* $\mathbf{B}$ *in the initial seed* (2.1.5) *has the form*

$$b = F_b(\widehat{y}_1, \widehat{y}_2, \widehat{y}_3)\mathbf{x}^{\mathbf{g}_b} \tag{2.3.1}$$

where F_b *is a primitive polynomial in three variables,* $\mathbf{g}_b = (g_1, g_2, g_3)^t$ *is an integer vector and*

$$\widehat{y}_1 = \tfrac{y_1}{x_2x_3}, \quad \widehat{y}_2 = \tfrac{y_2x_1}{x_3}, \quad \widehat{y}_3 = y_3x_1x_2 \tag{2.3.2}$$

(see (1.1.17)*). In particular property [hom] of Theorem 1.5.7 is satisfied by* $\mathbf{B}$.

Proof. By Proposition 1.1.9 cluster variables and hence cluster monomials belong to $\mathcal{M}$. By using Lemma 1.1.18 we are going to show that u_n belongs to $\mathcal{M}$ as well, for $n \geq 1$. By their definitions (2.1.3)–(2.1.4), it follows that the $\mathbf{g}$–vector i.e. the principal $\mathbb{Z}^3$–degree given by (1.1.8), of w and z is respectively $(0, -1, 1)^t$ and $(-1, 1, 0)^t$. Then $\deg(zw) = \deg(z) + \deg(w) = (-1, 0, 1)^t$. We note that

$\deg(y_1y_3) = -\underline{b}_1 - \underline{b}_3 = (-1,0,1)^t = -\underline{b}_2 = \deg(y_2)$. Moreover

$$y_1y_3 = (\hat{y}_1\hat{y}_3)\frac{x_3}{x_1}, \qquad , y_2 = \hat{y}_2\frac{x_3}{x_1},$$

hence they belong to $\mathcal{M}$ and have the same $\mathbf{g}$–vector as zw. By Lemma 1.1.18 we conclude that $u = zw - y_1y_3 - y_2$ belongs to $\mathcal{M}$ and its $\mathbf{g}$–vector is given by $\mathbf{g}_{u_1} = (-1,0,1)^t$. Similarly $\deg(y_1y_2y_3) = -\underline{b}_1 - \underline{b}_2 - \underline{b}_3 = (-2,0,2)^t$ and $y_1y_2y_3 = \hat{y}_1\hat{y}_2\hat{y}_3x_1^{-2}x_3^2$, hence u_2 (defined in (2.1.19)) belongs to $\mathcal{M}$ and its $\mathbf{g}$–vector is $2\mathbf{g}_{u_1}$. Proceeding by induction on $n \geq 2$ and using its definition (2.1.20), u_n belongs to $\mathcal{M}$ and it is homogeneous of degree $n\mathbf{g}_{u_1}$. □

Definition 2.3.2. For every $b \in \mathbf{B}$, we denote by $\mathbf{g}_b$ and F_b respectively the $\mathbf{g}$–vector and the F–polynomial of b in the initial seed (2.1.5). We denote by $\mathbf{g}_b^w$ and by F_b^w respectively the $\mathbf{g}$-vector and the F–polynomial of b in the cluster $\{x_1, w, x_3\}$.

2.3.2 g–vectors

Proposition 2.3.3. *With the notations of Definition 2.3.2, for every $m \geq 0$:*

$$\mathbf{g}_{2m+1} = \begin{pmatrix} -m+1 \\ 0 \\ m \end{pmatrix} \quad \mathbf{g}_{2m+2} = \begin{pmatrix} -m \\ 1 \\ m \end{pmatrix} \tag{2.3.3}$$

$$\mathbf{g}_{-(2m+1)} = \begin{pmatrix} -m \\ -1 \\ m \end{pmatrix} \quad \mathbf{g}_{-2m} = \begin{pmatrix} -m \\ 0 \\ m-1 \end{pmatrix} \tag{2.3.4}$$

Moreover for every $n \geq 0$

$$\mathbf{g}_w = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \mathbf{g}_z = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{g}_{u_n} = \begin{pmatrix} -n \\ 0 \\ n \end{pmatrix} \tag{2.3.5}$$

Proof. In section 1.1.5 explicit formulas for the $\mathbf{g}$-vectors are given in terms of the rectangular matrices $\tilde{B}_m$: using Lemma 2.1.2, the $\mathbf{g}$-vectors are uniquely determined by the initial conditions

$$\mathbf{g}_i = \mathbf{e}_i \quad (i = 1,2,3)$$

where $\mathbf{e}_i$ is the i-th standard basis vector of $\mathbb{Z}^3$, together with the recurrence relations

$$\begin{aligned}
\mathbf{g}_{2m+3} &= -\mathbf{g}_{2m} + \begin{pmatrix} -2m+1 \\ 1 \\ 2m \end{pmatrix} \\
\mathbf{g}_{2m+2} &= -\mathbf{g}_{2m-1} + \begin{pmatrix} -2m+2 \\ 1 \\ 2m-1 \end{pmatrix} \\
\mathbf{g}_{-2m-2} &= -\mathbf{g}_{-2m+1} + \mathbf{g}_{-2m-1} + \mathbf{g}_{-2m} \\
\mathbf{g}_{-2m-1} &= -\mathbf{g}_{-2m+2} + \mathbf{g}_{-2m} + \mathbf{g}_{-2m+1}
\end{aligned}$$

for every $m \geq 1$, together with $\mathbf{g}_0 = -\mathbf{g}_3$, $\mathbf{g}_{-1} = -\mathbf{g}_2$ and $\mathbf{g}_{-2} = -\mathbf{g}_1$. The claim follows by checking that the given vectors satisfy these recurrence relations.

The $\mathbf{g}$–vector of w and z can be obtained directly from the definition (2.1.3) and (2.1.4) respectively. Then from (2.1.18) we get $\mathbf{g}_{u_1} = \mathbf{g}_w + \mathbf{g}_z$ and from (2.1.19) and (2.1.20) we get $\mathbf{g}_{u_n} = n\,\mathbf{g}_{u_1}$ and we are done. □

2.3.3 F–polynomials

We denote the F–polynomial in the initial seed Σ_{In} associated with the cluster variable x_m by F_m. The following Proposition provides explicit formulas for these polynomials.

Proposition 2.3.4. *For $m \geq 0$*

$$F_{2m+1} = \sum_{e_1,e_2,e_3} \chi_{2m+1}(e_1,e_2,e_3) y_1^{e_1} y_2^{e_2} y_3^{e_3} + 1 \tag{2.3.6}$$

where $\chi_{2m+1}(e_1,e_2,e_3) \doteq \binom{e_1-e_3}{e_2-e_3}\binom{m-1-e_3}{m-1-e_1}\binom{e_1-1}{e_3}$.

For $m \geq 1$

$$F_{2m} = \sum_{e_1,e_2,e_3} \chi_{2m}(e_1,e_2,e_3) y_1^{e_1} y_2^{e_2} y_3^{e_3} + 1 \tag{2.3.7}$$

where $\chi_{2m}(e_1,e_2,e_3) \doteq \binom{e_1-1}{e_3}[\binom{e_1-e_3}{e_2-e_3}\binom{m-2-e_3}{m-2-e_1} + \binom{e_1-e_3-1}{e_2-e_3}\binom{m-2-e_3}{m-1-e_1}]$.

For $m \geq 0$

$$F_{-(2m+1)} = \sum_{e_1,e_2,e_3} \chi_{-(2m+1)}(e_1,e_2,e_3) y_1^{e_1} y_2^{e_2} y_3^{e_3} + y_1^m y_2^{m+1} y_3^{m+1} \tag{2.3.8}$$

where $\chi_{-(2m+1)}(e_1,e_2,e_3) \doteq \binom{e_1+1-e_3}{e_2-e_3}\binom{m-e_3}{m-e_1}\binom{e_1+1}{e_3}$;

$$F_{-2m} = \sum_{e_1,e_2,e_3} \chi_{-2m}(e_1,e_2,e_3) y_1^{e_1} y_2^{e_2} y_3^{e_3} + y_1^m y_2^m y_3^{m+1} \tag{2.3.9}$$

where $\chi_{-2m}(e_1,e_2,e_3) \doteq \binom{m-e_3}{m-e_1}[\binom{e_1+1-e_3}{e_1-e_2}\binom{e_1}{e_3-1} + \binom{e_1-e_3}{e_1-e_2}\binom{e_1}{e_3}]$.

For $n \geq 1$

$$F_{u_n} = y_1^n y_2^n y_3^n + \sum_{e_1,e_2,e_3} \chi_{u_n}(e_1,e_2,e_3) y_1^{e_1} y_2^{e_2} y_3^{e_3} + 1 \tag{2.3.10}$$

where $\chi_{u_n}(e_1,e_2,e_3) \doteq \binom{e_1-e_3}{e_2-e_3}[\binom{n-e_3}{n-e_1}\binom{e_1-1}{e_3} + \binom{n-e_3-1}{n-e_1}\binom{e_1-1}{e_3-1}]$ *and* $F_{u_0} = 1$.

Proof. By (2.1.3) and (2.1.4), the F–polynomial of w and z is respectively

$$\begin{aligned} F_w(y_1,y_2,y_3) &= y_2 + 1, & (2.3.11)\\ F_z(y_1,y_2,y_3) &= y_1y_3 + y_1 + 1. & (2.3.12) \end{aligned}$$

Using Lemma 2.1.2, the recurrence relations given in section 1.1.5 between the F-polynomials become the following: the initial conditions are

$$\begin{aligned} F_1 = F_2 = F_3 &= 1 \\ F_0 &= y_3 + 1 \\ F_{-1} &= y_2F_0 + 1 \end{aligned}$$

and for $m \geq 1$

$$\begin{aligned}
F_z F_{2m+1} &= y_1 y_3 F_{2m} + F_{2m+2}, && (2.3.13)\\
F_w F_{2m} &= y_2 F_{2m-1} + F_{2m+1}, && (2.3.14)\\
F_z F_{-2m-1} &= y_2 F_{-2m} + F_{-2m-2}, && (2.3.15)\\
F_w F_{-2m} &= y_1 y_3 F_{-2m+1} + F_{-2m-1}. && (2.3.16)
\end{aligned}$$

From its definition (2.1.18) the F–polynomial of u_1 is given by

$$F_{u_1}(y_1, y_2, y_3) = F_z F_w(y_1, y_2, y_3) - y_1 y_3 - y_2 = y_1 y_2 y_3 + y_1 y_2 + y_1 + 1. \qquad (2.3.17)$$

Moreover from (2.1.19) and (2.1.20) we have for $n \geq 2$

$$\begin{aligned}
F_{u_2}(y_1, y_2, y_3) &= F_{u_1}(y_1, y_2, y_3)^2 - 2 y_1 y_2 y_3, && (2.3.18)\\
F_{u_{n+1}}(y_1, y_2, y_3) &= F_{u_1} F_{u_n}(y_1, y_2, y_3) - y_1 y_2 y_3 F_{u_{n-1}}(y_1, y_2, y_3). && (2.3.19)
\end{aligned}$$

The proof follows now by induction on m and n. $\square$

Corollary 2.3.5. **B** *satisfies property [F] of Theorem 1.5.7, i.e. for every $b \in \mathcal{B}$ the corresponding F–polynomial F_b has constant term* 1.

2.3.4 Proof of Theorem 2.1.9

Proposition 2.3.1 provides explicit formulas of the elements of **B** in terms of their **g**–vectors and F–polynomials. Hence formulas (2.1.24), (2.1.25) and (2.1.28) follow by the corresponding formulas (2.3.49), (2.3.5) for their **g**–vectors and from the corresponding formulas (2.3.6), (2.3.7) and (2.3.10) for their F–polynomials.

We have the following formulas

$$x_2 = \frac{y_2 x_1 + x_3}{w}. \qquad (2.3.20)$$

By applying formulas (2.3.20) to the formulas (2.1.24) , (2.1.25), (2.1.28) and (2.1.4) one gets respectively formulas (2.1.26), (2.3.103), (2.1.29) and (2.1.30).

2.3.5 Proof of Theorem 2.1.5

Clearly every cluster monomial $s_1^a s_2^b s_3^c$ also has the form (2.1.21), i.e. can be written as $X[\alpha]$ with $\alpha = a\mathbf{d}(s_1) + b\mathbf{d}(s_2) + c\mathbf{d}(s_3)$. We also note that

$$\mathbf{d}(u_1) = \mathbf{d}(w) + \mathbf{d}(z) = \delta \doteq \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix} \right).$$

Since of its definition (2.1.20), $\mathbf{d}(u_n) = n\mathbf{d}(u_1)$. Moreover $\mathbf{d}(u_n w^k) = n\mathbf{d}(u_1) + k\mathbf{d}(w)$ and $\mathbf{d}(u_n z^k) = n\mathbf{d}(u_1) + k\mathbf{d}(z)$. Hence the cone in Q defined by

$$\mathcal{C}_{\mathrm{Im}} = \mathbb{Z}_{\geq 0}\mathbf{d}(w) + \mathbb{Z}_{\geq 0}\mathbf{d}(z)$$

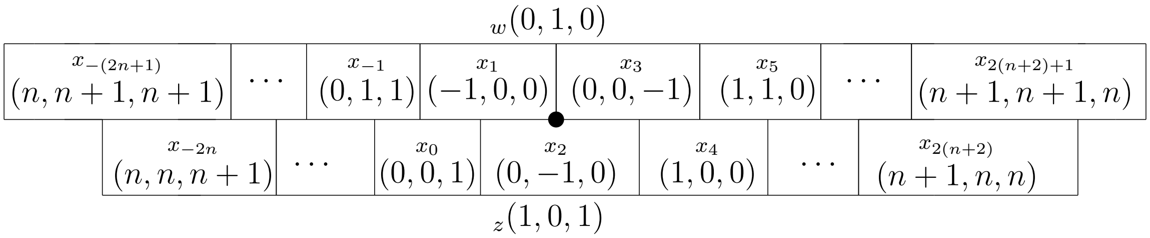

Figure 2.4: Denominator vectors of cluster variables in the cluster $\mathcal{C}_{In}$

is in bijection with the set $\{\mathbf{d}(u_nw^k), \mathbf{d}(u_nw^k) | \, n, k \geq 0\}$.

To complete the proof it's enough to show the following:

$$\text{For every cluster } \{s_1, s_2, s_3\}, \text{ the vectors } \mathbf{d}(s_1), \mathbf{d}(s_2) \text{ and } \mathbf{d}(s_3) \text{ form a } \mathbb{Z}\text{-basis of } Q. \tag{2.3.21}$$

$$\text{For every cluster } \{s_1, s_2, s_3\}, \text{ the vectors } \mathbf{d}(s_1), \mathbf{d}(s_2) \text{ and } \mathbf{d}(s_3) \text{ are the only positive real roots contained in the additive semigroup } \mathcal{C}_{\{s_1,s_2,s_3\}} := \mathbb{Z}_{\geq 0}\mathbf{d}(s_1) + \mathbb{Z}_{\geq 0}\mathbf{d}(s_2) + \mathbb{Z}_{\geq 0}\mathbf{d}(s_3). \tag{2.3.22}$$

$$\text{The union } \bigcup \mathcal{C}_{\{s_1,s_2,s_3\}} \text{ is equal to } Q - \mathcal{C}_{\mathrm{Im}}. \tag{2.3.23}$$

The exchange relation (2.1.2) implies at once the following relation between denominator vectors of cluster variables

$$\mathbf{d}(x_m) + \mathbf{d}(x_{m+3}) = [\mathbf{d}(x_{m+1}) + \mathbf{d}(x_{m+2})]_+ \tag{2.3.24}$$

where the operation $b \mapsto [b]_+$ is understood component–wise. By induction on m we get formulas (2.1.22), i.e. for $m \geq 1$

$$\mathbf{d}(x_{2m+1}) = \begin{pmatrix} m-1 \\ m-1 \\ m-2 \end{pmatrix}, \quad \mathbf{d}(x_{2m+2}) = \begin{pmatrix} m \\ m-1 \\ m-1 \end{pmatrix}, \tag{2.3.25}$$

$$\mathbf{d}(x_{-2m+1}) = \begin{pmatrix} m-1 \\ m \\ m \end{pmatrix}, \quad \mathbf{d}(x_{-2m+2}) = \begin{pmatrix} m-1 \\ m-1 \\ m \end{pmatrix} \tag{2.3.26}$$

and also for $n \geq 1$

$$\mathbf{d}(w) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{d}(z) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{d}(u_n) = \begin{pmatrix} n \\ n \\ n \end{pmatrix}. \tag{2.3.27}$$

In particular (2.3.24) becomes

$$\mathbf{d}(x_m) + \mathbf{d}(x_{m+3}) = [\mathbf{d}(x_{m+1})]_+ + [\mathbf{d}(x_{m+2})]_+. \tag{2.3.28}$$

We observe that

$$[\mathbf{d}(x_m)]_+ = \begin{cases} 0 & \text{if } m = 1, 2, 3, \\ \mathbf{d}(x_m) & \text{otherwise }. \end{cases} \tag{2.3.29}$$

Clearly $\{\mathbf{d}(x_1), \mathbf{d}(x_2), \mathbf{d}(x_3)\}$ is a $\mathbb{Z}$–basis of $Q = \mathbb{Z}^3$. Then using (2.3.28) and (2.3.29) we get by induction on m

$$|\det(\mathbf{d}(x_m), \mathbf{d}(x_{m+1}), \mathbf{d}(x_{m+2}))| = |\det(\mathbf{d}(x_{m+1}), \mathbf{d}(x_{m+2}), \mathbf{d}(x_{m+3}))| = 1$$

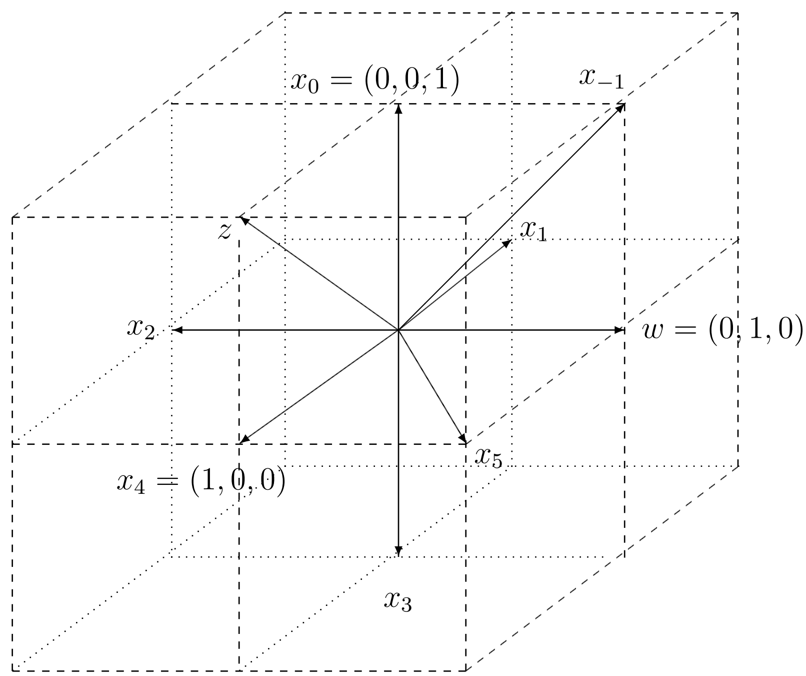

Figure 2.5: Denominator vectors of the cluster variable having at least one coordinate equal to zero. We wrote x_m for $\mathbf{d}(x_m)$. Note the clusters involving here form a fan whose union is $Q \setminus Q_+$

for every $m \in \mathbb{Z}$. Moreover it is straightforward to check directly that

$$|\det(\mathbf{d}(x_{2m-1}), \mathbf{d}(w), \mathbf{d}(x_{2m+1}))| = 1$$

and

$$|\det(\mathbf{d}(x_{2m-2}), \mathbf{d}(z), \mathbf{d}(x_{2m}))| = 1$$

for every $m \in \mathbb{Z}$ and (2.3.21) is proved.

We consider the basis of simple roots $\alpha_1, \alpha_2, \alpha_3$ of Q and the corresponding coordinate system (g_1, g_2, g_3). By using (2.3.52) and (2.3.53), or figure 2.4, we observe that there are four lines in the affine space containing the points corresponding to all the x_m's different from x_2. They contain respectively the "negative odd", "positive odd", the "negative even" and the "positive even" cluster variables: they are

$$\ell^-_{\text{odd}} = \begin{cases} g_2 = g_3 \\ g_1 = g_2 - 1 \end{cases} ; \quad \ell^+_{\text{odd}} = \begin{cases} g_1 = g_2 \\ g_3 = g_2 - 1 \end{cases} .$$
$$\ell^-_{\text{even}} = \begin{cases} g_1 = g_2 \\ g_3 = g_2 + 1 \end{cases} ; \quad \ell^+_{\text{even}} = \begin{cases} g_2 = g_3 \\ g_1 = g_2 + 1 \end{cases} .$$

We define the two-dimensional subspace of $Q_{\mathbb{R}}$ containing respectively both ℓ^+_{odd} and ℓ^-_{even} and both ℓ^-_{odd} and ℓ^+_{even}:

$$P := \{g_1 = g_2\}; \quad T := \{g_2 = g_3\}.$$

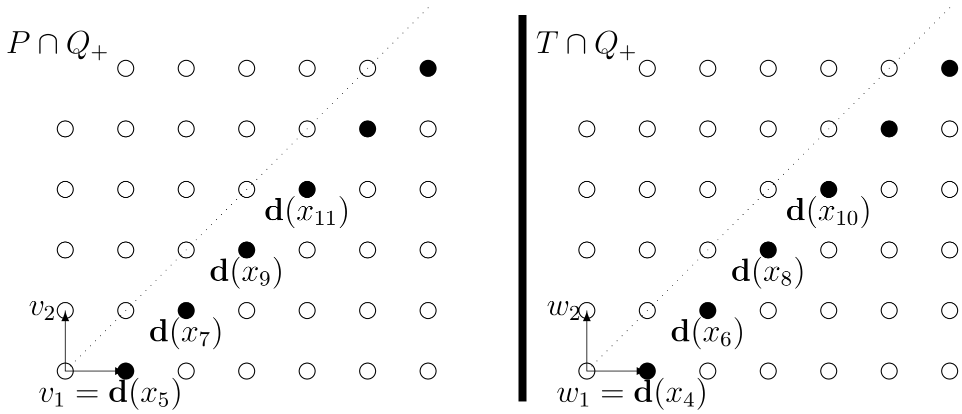

Figure 2.6: $P \cap Q_+$ and $T \cap Q_+$: they intersect themselves into the dotted line of equation $g_1 = g_2 = g_3$.

Formulas (2.3.52) and (2.3.53) have the following interesting and expected (see [18, Conjecture 7.5]) property: given a cluster $\mathcal{C} = \{s_1, s_2, s_3\}$ the corresponding denominator vectors $\{\mathbf{d}(s_1), \mathbf{d}(s_2), \mathbf{d}(s_3)\}$ are *sign–coherent*, i.e. the ith coordinates of all of them are either all non–negative or all non–positive. It means that if the initial cluster variable x_i lies in the cluster $\mathcal{C}$ then the ith coordinates of the other two elements of $\mathcal{C}$ are 0. Figure 2.5 shows denominator vectors of the cluster variables having at least one coordinate equal to zero. Let us analyze this figure: the cones $\mathcal{C}_{\{s_1,s_2,s_3\}}$ involved here satisfies property 2.3.22, i.e. they do not overlap themselves. Moreover their union is the entire lattice except the interior of the positive octant $Q_+ = \mathbb{Z}_{\geq 0}\mathbf{d}(x_4) + \mathbb{Z}_{\geq 0}\mathbf{d}(w) + \mathbb{Z}_{\geq 0}\mathbf{d}(x_0)$.

We now concentrate our attention on the other cones. Let $\mathcal{C}_P$ be the (open) cone inside $P \cap Q_+$ defined by $\mathcal{C}_P = \{0 < g_3 < g_1\} \cup \{0, 0, 0\}$. By (2.3.52), $\mathbf{d}(x_{2n+1}) \in \mathcal{C}_P$ for $n \geq 2$. The vectors $v_1 = (1, 1, 0)^t = \mathbf{d}(x_5)$ and $v_2 = (0, 0, 1)^t = \mathbf{d}(x_0)$ form a $\mathbb{Z}$–basis of P such that $\mathcal{C}_P$ is given by $\mathbb{Z}_{\geq 0}v_1 + \mathbb{Z}_{\geq 0}(v_1 + v_2)$. (This can be seen for example observing that $\det(v_1, v_2, (1, 0, 0)^t) = 1$ and that v_1 and v_2 have at least one coordinate equal to zero). $\mathbf{d}(x_{2n+1}) = a_{n1}v_1 + a_{n2}v_2$ where $a_{n1} \doteq n - 1$ and $a_{n2} \doteq n - 2$. The sequence a_{n2}/a_{n1} is strictly increasing. It has limit

$$\lim_{n\to\infty} \frac{a_{n2}}{a_{n1}} = 1$$

we conclude that the set of cones $\{\{\mathbb{Z}_{\geq 0}\mathbf{d}(x_{2n+1}) + \mathbb{Z}_{\geq 0}\mathbf{d}(x_{2n+3})\} : n \geq 2\}$ is a fan in P whose union is $\mathcal{C}_P$. Hence we have

$$\bigcup_{n\geq 2} \mathcal{C}_{\{x_{2n+1}, w, x_{2n+3}\}} = \mathbb{Z}_{\geq 0}\mathbf{d}(w) + \mathcal{C}_P$$

and these cones have no common interior points. Similarly let $\mathcal{C}_T$ be the (open) cone inside $T \cup Q_+$ defined by $\mathcal{C}_T = \{0 < g_2 < g_1\} \cup \{0, 0, 0\}$. $\mathbf{d}(x_{2n}) \in \mathcal{C}_T$ for $n \geq 2$. $w_1 = (1, 0, 0)^t = \mathbf{d}(x_4)$ and $w_2 = (0, 1, 1)^t$ form a $\mathbb{Z}$–basis of T such that $\mathcal{C}_T = \mathbb{Z}_{\geq 0}w_1 + \mathbb{Z}_{\geq 0}(w_1 + w_2)$. In this basis $\mathbf{d}(x_{2n}) = b_{n1}w_1 + b_{n2}w_2$ with $b_{n1} = n-1$ and

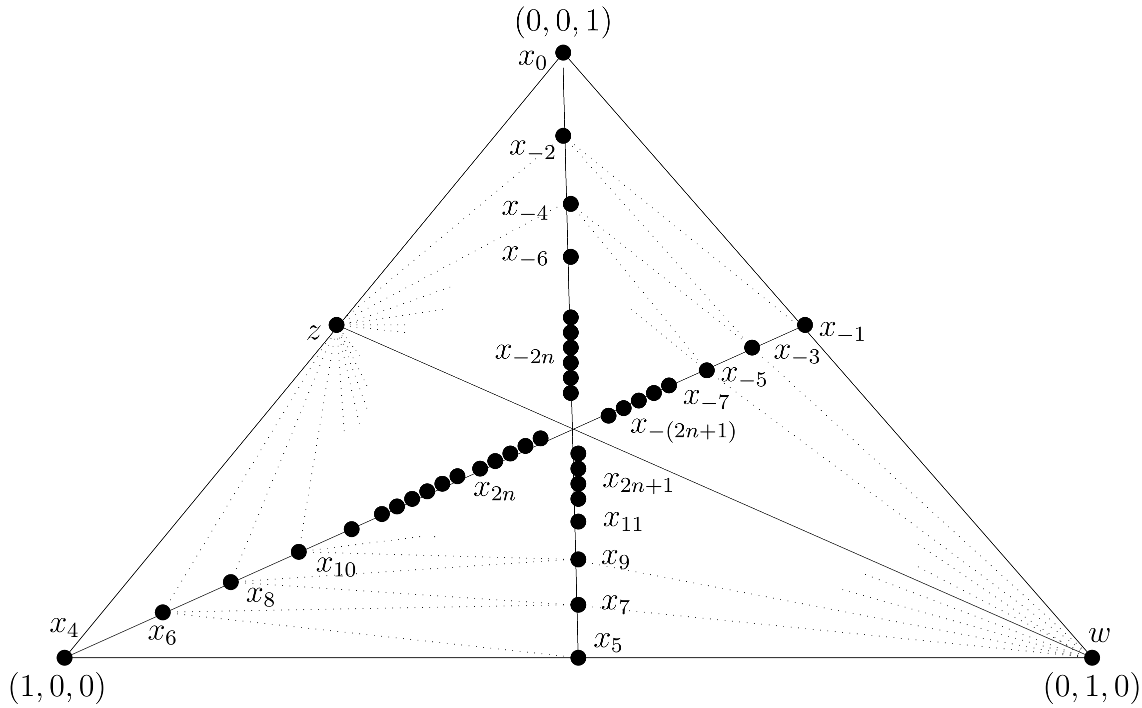

Figure 2.7: "cluster triangulation" of the intersection between $Q^+_{\mathbb{R}}$ and the plane $\mathcal{P} = \{g_1 + g_2 + g_3 = 1\}$. With abuse of language we wrote x_m to denote $\mathbb{R}\alpha(x_m) \cap \mathcal{P}$. The line between w and z is $\mathcal{C}_{Im} \cap \mathcal{P}$. The line between x_0 and x_5 is $P \cap \mathcal{P}$. The line between x_4 and x_{-1} is $T \cap \mathcal{P}$.

$b_{n2} = n-2$. Figure 2.6 shows the cones $P \cap Q_+$ and $T \cap Q_+$ in the chosen basis $\{v_1, v_2\}$ and $\{w_1, w_2\}$ respectively. The strictly increasing sequence $\{b_{n2}/b_{n1}\}$ has limit 1 for $n \to \infty$. We conclude that the set of cones $\{\{\mathbb{Z}_{\geq 0}\mathbf{d}(x_{2n}) + \mathbb{Z}_{\geq 0}\mathbf{d}(x_{2n+2})\} : n \geq 2\}$ is a fan in T whose union is the closure of $\mathcal{C}_T$. Hence we have

$$\bigcup_{n \geq 2} \mathcal{C}_{\{x_{2n}, z, x_{2n+2}\}} = \mathbb{Z}_{\geq 0}\mathbf{d}(z) + \mathcal{C}_T.$$

It follows from the previous arguments that $\mathcal{C}_P + \mathcal{C}_T = \bigcup_{m \geq 4} \mathcal{C}_{\{x_m, x_{m+1}, x_{m+2}\}}$ and that the interior of two different cones $\mathcal{C}_{\{x_m, x_{m+1}, x_{m+2}\}}$ and $\mathcal{C}_{\{x_n, x_{n+1}, x_{n+2}\}}$ are disjoint.

Figure 2.7 shows the intersection of the plane $\mathcal{P} = \{g_1 + g_2 + g_3 = 1\}$ with the positive octant Q_+. The marked points illustrate the mutual position of $\mathbb{R}\mathbf{d}(x_m) \cap \mathcal{P}$. Dotted lines join two points corresponding to cluster variables belonging to the same cluster. We have obtained all the points in the triangle between z, w and x_4. We are going to obtain the others by reflecting through the line between z and w: we consider the orthogonal reflection r_{Im} with respect to the imaginary cone $\mathcal{C}_{Im}$. It acts on vectors by exchanging the first coordinate with the third one: $r_{Im}(a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3) = a_3\alpha_1 + a_2\alpha_2 + a_1\alpha_3$. In particular it fixes $\mathcal{C}_{Im}$. It sends $\mathbf{d}(x_m)$ to $\mathbf{d}(x_{4-m})$ for $m \geq 3$.

We have just obtained $\mathcal{C}_{Im} + \mathbb{Z}_{\geq 0}\mathbf{d}(x_4)$ as union of mutually disjoint cones of the form $\mathcal{C}_{\{s_1, s_2, s_3\}}$ with $s_i = x_m, w$ or z with $m \geq 4$. By applying r_{Im} we obtain

$\mathcal{C}_{Im} + \mathbb{Z}_{\geq 0}\mathbf{d}(x_0)$ as mutually disjoint union of cones of the form $\mathcal{C}_{\{s_1,s_2,s_3\}}$ with $s_i = x_m, w$ or z with $m \leq 0$. Since $Q_+ = \mathcal{C}_{Im} + \mathbb{Z}_{\geq 0}\mathbf{d}(x_4) + \mathbb{Z}_{\geq 0}\mathbf{d}(x_0)$ we are done.

Corollary 2.3.6. *For every $m \geq 2$, $\mathbf{d}(x_{4-m}) = (13)\mathbf{d}(x_m)$, where (13) is the automorphism of Q that exchanges the third entry with the first one in the basis of simple roots.*

2.3.6 g–vector parametrization of B

We associate to the quiver $Q = Q_{In}$ defined in (2.2.1) its Euler matrix E_Q (recall that $(E_Q)_{ij} = 1$ if $i = j$, -1 if there is an arrow from i to j and 0 otherwise). We consider the piecewise–linear deformation $\mathcal{E}_Q$ of $-E_{Q_{In}}$ given by

$$\mathcal{E}_Q = \begin{pmatrix} -1 & 0 & 0 \\ [\ ?\ ]_+ & -1 & 0 \\ [\ ?\ ]_+ & [\ ?\ ]_+ & -1 \end{pmatrix} \tag{2.3.30}$$

Since of Remark 1.3.9 the following result can be seen as a generalization of Theorem 1.3.8 to the non–bipartite case.

Proposition 2.3.7. *Given $b \in \mathbf{B}$, its $\mathbf{g}$–vector $\mathbf{g}_b$ and its denominator vector $\mathbf{d}(b)$ are related by*

$$\mathbf{g}_b = \mathcal{E}_Q * \mathbf{d}(b) \tag{2.3.31}$$

where $*$ is understood as in (1.3.21).

Proof. It follows from the explicit formulas for the $\mathbf{g}$–vectors given in Proposition 2.3.3 and from the explicit formulas for the denominator vectors given in (2.3.52), (2.3.53) and (3.2.17) that formula (2.3.31) holds for cluster variables and for the u_n's. By Corollary 2.1.8 denominator vectors of cluster variables belonging to the same cluster are sign–coherent. $\mathcal{E}_Q$ is hence linear in every cone generated by such vectors. Then the claim follows for cluster monomials. By (3.2.17) denominator vectors of the u_n's, w and z lie in the positive octant Q_+ in which $\mathcal{E}_Q$ is linear. The claim is hence true for $u_n w^k$ and $u_n z^k$, $n, k \geq 0$. □

Corollary 2.3.8. *The map $b \mapsto \mathbf{g}_b$ which associates to an element $b \in \mathbf{B}$ its $\mathbf{g}$–vector in the initial cluster $\mathbf{g}_b$ is a bijection between $\mathbf{B}$ and Q.*

Proof. $\mathcal{E}_Q$ is injective. Then the claim follows combining Theorem 2.1.5 with Proposition 2.3.7. □

2.3.7 Linear independence of B

Proposition 2.3.9. $\mathbf{B}$ *is a linearly independent set over $\mathbb{ZP}$.*

Proof. We verify that the set $\mathbf{B}$ satisfies hypothesis of Theorem 1.5.7. By Proposition 2.3.1, every element of $\mathbf{B}$ is homogeneous and satisfies property [hom]. The columns of the matrix $B = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$ clearly satisfy hypothesis $[B^0]$, i.e. $a\mathbf{b}_1 + b\mathbf{b}_2 + c\mathbf{b}_3 = \mathbf{0}$, $a, b, c \geq 0$ implies $a = b = c = 0$, where $\mathbf{b}_i$ is the i–th column of B. F–polynomials have constant term 1 by Corollary 2.3.5. The map $b \mapsto \mathbf{g}_b$ is injective by Corollary 2.3.8. □

2.3.8 F–polynomials, g–vectors and denominator vectors in the cluster $\{x_1, w, x_3\}$

In this section we study another cluster algebra $\mathcal{A}^{Cycl}$ of type $A_2^{(1)}$ closely related to $\mathcal{A}$. We need some results on $\mathcal{A}^{Cycl}$ in order to get the positivity of the elements of the canonical basis $\mathbf{B}$ of $\mathcal{A}$.

Let $\mathbb{P}$ be the tropical semi–field (see Definition 1.1.1) $\mathrm{Trop}(\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3)$ generated by three elements $\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3$ and let $\mathcal{F} = \mathbb{QP}(x_1, w, x_3)$ be the field of rational functions in three commuting variables x_1, w and x_3 with coefficients in $\mathbb{QP}$. We define $\mathcal{A}^{Cycl}$ to be the cluster algebra inside $\mathcal{F}$ with principal coefficients at the initial "cyclic" seed:

$$\Sigma_{In}^{Cycl} = \{B^{cyclic} \doteq \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}, \{x_1, w, x_3\}, \{\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3\}\}.$$

Here, as the same as in Lemma 2.1.2, the name "cyclic" comes from the fact that the quiver associated with B^{cyclic} (see Section 2.2) has a cycle.

B^{cyclic} is the principal part of one of the exchange matrices of the algebra $\mathcal{A}$ shown in Lemma 2.1.2, so that *$\mathcal{A}^{Cycl}$ is a cluster algebra of type $A_2^{(1)}$*.

We denote the generators of $\mathcal{A}^{Cycl}$ inside $\mathcal{F}$, i.e. its cluster variables, with the same symbols as in $\mathcal{A}$. More precisely if a cluster variable $s = \mu(x)$ in $\mathcal{A}$ is obtained from a cluster variable x of the cluster of the initial seed Σ_{In} by a *sequence* μ of mutations, we denote by the same letter s the cluster variable $\mu \circ \mu_2(x)$ of $\mathcal{A}^{Cycl}$ obtained after the sequence $\mu \circ \mu_2$ of mutations from the initial cyclic seed Σ_{In}^{Cycl} (here μ_2 denotes mutation in direction two). This is consistent with the following fact: the mutation in direction two of the seed Σ_{In}^{Cycl} (see Definition 1.1.4) is the seed

$$\Sigma_1 \doteq \{B \doteq \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \{x_1, x_2, x_3\}, \{y_{1;1}, y_{2;1}, y_{3;1}\}\}$$

where the coefficients and the cluster variable x_2 are given by

$$y_{1;1} = \mathrm{y}_1\mathrm{y}_2, \quad y_{2;1} = \frac{1}{\mathrm{y}_2}, \quad y_{3;1} = \mathrm{y}_3, \quad x_2 = \frac{x_1 + \mathrm{y}_2 x_3}{w}. \tag{2.3.32}$$

Since of Lemma 2.1.2 all the cluster variables of $\mathcal{A}^{Cycl}$ are obtained in this way. We now introduce the corresponding of the elements u_n of $\mathcal{A}$, in $\mathcal{A}^{Cycl}$. In view of Theorem 1.1.11 the Laurent expansion in the cluster $\{x_1, x_2, x_3\}$ of every cluster variable s of $\mathcal{A}^{Cycl}$ is given by:

$$s = \frac{F_s\mid_{\mathcal{F}}(\widehat{y}_{1;1}, \widehat{y}_{2;1}, \widehat{y}_{3;1})}{F_s\mid_{\mathbb{P}}(y_{1;1}, y_{2;1}, y_{3;1})}\mathbf{x}^{\mathbf{g}_s} \tag{2.3.33}$$

where $\widehat{y}_{i;1}$, $i = 1, 2, 3$, is given by (2.3.2), i.e. in this case, using (2.3.32):

$$\widehat{y}_{1;1} = \frac{y_{1;1}}{x_2 x_3} = \frac{\mathrm{y}_1\mathrm{y}_2}{x_2 x_3}, \quad \widehat{y}_{2;1} = \frac{y_{2;1}x_1}{x_3} = \frac{x_1}{\mathrm{y}_2 x_3}, \quad \widehat{y}_{3;1} = y_{3;1}x_1x_2 = \mathrm{y}_3 x_1 x_2.$$

The Laurent expansion of s in the initial cluster $\{x_1, w, x_3\}$ is obtained from (2.3.33) by mutating the cluster variable x_2 as in (2.3.32):

$$s = \frac{F_s\mid_{\mathcal{F}}\left(\frac{\mathrm{y}_1\mathrm{y}_2 w}{(x_1+\mathrm{y}_2x_3)x_3}, \frac{x_1}{\mathrm{y}_2x_3}, \frac{\mathrm{y}_3x_1(x_1+\mathrm{y}_2x_3)}{w}\right)}{F_s\mid_{\mathbb{P}}(\mathrm{y}_1\mathrm{y}_2, \frac{1}{\mathrm{y}_2}, \mathrm{y}_3)} \cdot (x_1, \frac{x_1 + \mathrm{y}_2x_3}{w}, x_3)^{\mathbf{g}_s}. \tag{2.3.34}$$

Having this in mind we give the following definition:

Definition 2.3.10. For every $n \geq 0$ we define:

$$u_n \doteq F_{u_n} |_{\mathcal{F}} \left(\frac{\mathrm{y}_1\mathrm{y}_2 w}{(x_1+\mathrm{y}_2x_3)x_3}, \frac{x_1}{\mathrm{y}_2x_3}, \frac{\mathrm{y}_3x_1(x_1+\mathrm{y}_2x_3)}{w}\right) \cdot \left(x_1, \frac{x_1+\mathrm{y}_2x_3}{w}, x_3\right)^{\mathbf{g}_{u_n}} \quad (2.3.35)$$

where F_{u_n} is the polynomial given by (2.3.10) and $\mathbf{g}_{u_n}$ is the vector given in (2.3.5).

The following Proposition shows that the elements $\{u_n\}$ are elements of $\mathcal{A}^{Cycl}$ and satisfy relations similar to the relations satisfied by the corresponding elements $\{u_n\}$ of $\mathcal{A}$.

Proposition 2.3.11. *For every $n \geq 1$ the rational function u_n defined in* (2.3.35) *satisfy the initial conditions:*

$$u_0 = 1, \quad u_1 = \tfrac{1}{\mathrm{y}_2} z w - \mathrm{y}_1\mathrm{y}_2\mathrm{y}_3 - \tfrac{1}{\mathrm{y}_2}, \quad u_2 = u_1^2 - 2\mathrm{y}_1\mathrm{y}_3, \quad (2.3.36)$$

together with the recurrence relations

$$u_{n+1} = u_1 u_n - \mathrm{y}_1\mathrm{y}_3\, u_{n-1}. \quad (2.3.37)$$

In particular they are elements of $\mathcal{A}^{Cycl}$.

Proof. Recall from (2.3.17) that

$$F_{u_1}(y_1, y_2, y_3) = y_1y_2y_3 + y_1y_2 + y_1 + 1.$$

and from (2.3.5), $\mathbf{g}_{u_1} = \left[\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right]$. Then (2.3.35) becomes for $n = 1$:

$$u_1 = \frac{\mathrm{y}_1\mathrm{y}_3x_1^2 + \mathrm{y}_1 w + x_3^2}{x_1x_3}. \quad (2.3.38)$$

Recall from (2.3.12) that

$$F_z(y_1, y_2, y_3) = y_1y_3 + y_1 + 1$$

and from (2.3.5) that $\mathbf{g}_z = \left[\begin{smallmatrix} -1 \\ 1 \\ 0 \end{smallmatrix}\right]$. We then conclude from (2.3.33) that the expansion of the cluster variable z in the initial cluster of $\mathcal{A}^{Cycl}$ is

$$z = \frac{\mathrm{y}_1\mathrm{y}_2^2\mathrm{y}_3x_1x_3 + \mathrm{y}_1\mathrm{y}_2\mathrm{y}_3x_1^2 + \mathrm{y}_1\mathrm{y}_2 w + x_1x_3 + \mathrm{y}_2x_3^2}{x_1 w x_3} \quad (2.3.39)$$

from which the second equality of (2.3.36) follows.
From (2.3.18) and (2.3.19) we know that

$$\begin{aligned} F_{u_2}(y_1, y_2, y_3) &= F_{u_1}(y_1, y_2, y_3)^2 - 2y_1y_2y_3, \\ F_{u_{n+1}}(y_1, y_2, y_3) &= F_{u_1}F_{u_n}(y_1, y_2, y_3) - y_1y_2y_3F_{u_{n-1}}(y_1, y_2, y_3). \end{aligned}$$

By applying this equalities to (2.3.35) and using the fact that $\mathbf{g}_{u_n} = \left[\begin{smallmatrix} -n \\ 0 \\ n \end{smallmatrix}\right]$ we get the desired relations. □

Definition 2.3.12. Let $\mathbf{B}$ be the set of cluster monomials of $\mathcal{A}^{Cycl}$ together with the set of elements $\{u_n w^k\, u_n z^k :\ n \geq 1,\ k \geq 0\}$.

We denote the F–polynomial, the $\mathbf{g}$–vector and the denominator vector with respect to the initial seed of $\mathcal{A}^{Cycl}$ of an element b of $\mathbf{B}$ by respectively F_b^w, $\mathbf{g}_b^w$ and $\mathbf{d}^w(b)$; we denote by F_b, $\mathbf{g}_b$ and $\mathbf{d}(b)$ respectively the F–polynomial, the $\mathbf{g}$–vector and the denominator vector of the corresponding element b in the initial seed of $\mathcal{A}$. For the cluster variable x_m, $m \in \mathbb{Z}$, we abbreviate $F_{x_m} = F_m$ and $\mathbf{g}_{x_m} = \mathbf{g}_m$.

In order to get formulas for F–polynomials, $\mathbf{g}$–vectors and denominator vectors of the elements of $\mathbf{B}$, it is sufficient to find them for the following subset of $\mathbf{B}$:

$$\mathcal{B} \doteq \{\text{cluster variables}\} \sqcup \{u_n :\ n \geq 1\}.$$

Proposition 2.3.13. *For every $m \geq 0$:*

$$F^w_{2m+1}(\mathrm{y}_1,\mathrm{y}_2,\mathrm{y}_3) = \sum_{e_1,e_3} \binom{m-1-e_3}{m-1-e_1}\binom{e_1-1}{e_3}\mathrm{y}_1^{e_1}\mathrm{y}_3^{e_3} + 1; \tag{2.3.40}$$

$$F^w_{2m+2}(\mathrm{y}_1,\mathrm{y}_2,\mathrm{y}_3) = \sum_{\mathbf{e}} \binom{e_1-1}{e_3}\binom{m-1-e_3+e_2}{m-1-e_1+e_2}\binom{1}{e_2}\mathrm{y}_1^{e_1}\mathrm{y}_2^{e_2}\mathrm{y}_3^{e_3} + \mathrm{y}_2 + 1; \tag{2.3.41}$$

$$F^w_{-(2m+1)}(\mathrm{y}_1,\mathrm{y}_2,\mathrm{y}_3) = \sum_{e_1,e_3} \binom{m-e_3}{m-e_1}\binom{e_1+1}{e_3}\mathrm{y}_1^{e_1}\mathrm{y}_3^{e_3} + \mathrm{y}_1^m\mathrm{y}_3^{m+1}; \tag{2.3.42}$$

$$F^w_{-2m}(\mathrm{y}_1,\mathrm{y}_2,\mathrm{y}_3) = \sum_{\mathbf{e}} \binom{e_1+1-e_2}{e_3-e_2}\binom{m-e_3}{m-e_1}\binom{1}{e_2}\mathrm{y}^{\mathbf{e}} + \mathrm{y}_1^m\mathrm{y}_3^{m+1}(\mathrm{y}_2+1). \tag{2.3.43}$$

$$F^w_z = \mathrm{y}_1\mathrm{y}_2^2\mathrm{y}_3 + \mathrm{y}_1\mathrm{y}_2\mathrm{y}_3 + \mathrm{y}_1\mathrm{y}_2 + \mathrm{y}_2 + 1. \tag{2.3.44}$$

For every $n \geq 1$:

$$F^w_{u_n}(\mathrm{y}_1,\mathrm{y}_2,\mathrm{y}_3) = \mathrm{y}_1^n\mathrm{y}_3^n + \sum_{e_1,e_3}\left[\binom{n-e_3}{n-e_1}\binom{e_1-1}{e_3} + \binom{n-e_3-1}{n-e_1}\binom{e_1-1}{e_3-1}\right]\mathrm{y}_1^{e_1}\mathrm{y}_3^{e_3} + 1. \tag{2.3.45}$$

Proof. The Proof is based on the following Lemma.

Lemma 2.3.14. *For every element b of $\mathcal{B}$ different from w in $\mathcal{A}^{Cycl}$, the F–polynomials F_b and F_b^w are related by the following formula:*

$$F^w_b(\mathrm{y}_1,\mathrm{y}_2,\mathrm{y}_3) = \frac{F_b(\frac{\mathrm{y}_1\mathrm{y}_2}{1+\mathrm{y}_2}, \frac{1}{\mathrm{y}_2}, \mathrm{y}_3(1+\mathrm{y}_2))}{F_b\mid_{\mathbb{P}}(\mathrm{y}_1\mathrm{y}_2, \frac{1}{\mathrm{y}_2}, \mathrm{y}_3)} \cdot (1+\mathrm{y}_2)^{\mathbf{g}_{2;b}} \tag{2.3.46}$$

where $g_{2;b}$ denotes the second component of the vector $\mathbf{g}_b$.

Proof of Lemma 2.3.14. By equating in (2.3.34) $x_1 = w = x_3 = 1$ we get the desired relation. $\square$

We can give a refinement of Lemma 2.3.14:

Lemma 2.3.15. *For every element b of $\mathcal{B}$ different from w, (2.3.46) takes the form:*

$$F_b^w(\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3) = \begin{cases} F_b(\frac{\mathrm{y}_1\mathrm{y}_2}{1+\mathrm{y}_2}, \frac{1}{\mathrm{y}_2}, \mathrm{y}_3(1+\mathrm{y}_2)) \cdot \frac{\mathrm{y}_2}{(1+\mathrm{y}_2)} & \text{if } b = x_{-(2m+1)},\ m \geq 0, \\ \\ F_b(\frac{\mathrm{y}_1\mathrm{y}_2}{1+\mathrm{y}_2}, \frac{1}{\mathrm{y}_2}, \mathrm{y}_3(1+\mathrm{y}_2)) \cdot (1+\mathrm{y}_2)^{g_{2;b}} & \text{otherwise} \end{cases} \tag{2.3.47}$$

Proof of Lemma 2.3.15. By direct check in formulas (2.3.40)-(2.3.44) one can easily see that

$$F_b \mid_{\mathbb{P}} (\mathrm{y}_1\mathrm{y}_2, \frac{1}{\mathrm{y}_2}, \mathrm{y}_3) = \begin{cases} \frac{1}{\mathrm{y}_2} & \text{if } b = x_{-(2m+1)} \text{ for some } m \geq 0 \\ 1 & \text{otherwise} \end{cases} \tag{2.3.48}$$

Moreover by (2.3.50), $g_{2;-(2m+1)} = -1$ for every $m \geq 0$. $\square$

To conclude the proof of Proposition 2.3.13 we apply (2.3.46) using the explicit formula for the F–polynomial F_b given in Proposition 2.3.4 and the explicit formula for $\mathbf{g}_b$ given in Proposition 2.3.3. $\square$

Corollary 2.3.16. *For every element b of $\mathbf{B}$, F_b^w has constant term 1.*

Proof. For b in $\mathcal{B}$, this follows directly by formulas (2.3.40)-(2.3.44). When $b = b_1 \cdots b_n$ is a product of elements of $\mathbf{B}$, $F_b^w = F_{b_1}^w \cdot \cdots \cdot F_{b_n}^w$. Since every F_{b_i} has constant term 1, the same is true for F_b $\square$

Proposition 2.3.17. *The $\mathbf{g}$–vectors of the elements of $\mathcal{B}$ are given by: for every $m \geq 0$:*

$$\mathbf{g}_{2m+1}^w = \begin{pmatrix} -m+1 \\ 0 \\ m \end{pmatrix} \quad \mathbf{g}_{2m+2}^w = \begin{pmatrix} -m+1 \\ -1 \\ m \end{pmatrix} \tag{2.3.49}$$

$$\mathbf{g}_{-(2m+1)}^w = \begin{pmatrix} -m \\ 1 \\ m-1 \end{pmatrix} \quad \mathbf{g}_{-2m}^w = \begin{pmatrix} -m \\ 0 \\ m-1 \end{pmatrix} \tag{2.3.50}$$

where $\mathbf{g}_m^w = \mathbf{g}_{x_m}^w$. Moreover for every $n \geq 1$

$$\mathbf{g}_w^w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{g}_z^w = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \quad \mathbf{g}_{u_n}^w = \begin{pmatrix} -n \\ 0 \\ n \end{pmatrix} \tag{2.3.51}$$

Proof. The proof is based on the following Lemma.

Lemma 2.3.18. *For every element b of $\mathcal{B}$, the $\mathbf{g}$–vectors $\mathbf{g}_b = \begin{pmatrix} g_{1;b} \\ g_{2;b} \\ g_{3;b} \end{pmatrix}$ and $\mathbf{g}_s^w = \begin{pmatrix} g_{1;b}^w \\ g_{2;b}^w \\ g_{3;b}^w \end{pmatrix}$ are related to each other by the following formula:*

$$g_{j;b}^w = \begin{cases} g_{1;b} + g_{2;b} - \min(g_{2;b}, 0) & \text{if } j = 1, \\ -g_{2;b} & \text{if } j = 2, \\ g_{3;b} + \min(g_{2;b}, 0) & \text{if } j = 3. \end{cases}$$

Proof of Lemma 2.3.18. By Theorem 1.1.11, Corollary 2.3.16 and Definition 2.3.10, the expansion of the element b of $\mathcal{B}$ in the initial cluster Σ_{In}^{Cycl} of $\mathcal{A}^{Cycl}$ is given by:

$$b = F_b^w(\widehat{\mathrm{y}}_1, \widehat{\mathrm{y}}_2, \widehat{\mathrm{y}}_3) \cdot (x_1, w, x_3)^{\mathbf{g}_b^w}$$

where, by Definition 1.1.10,

$$\widehat{\mathrm{y}}_1 = \mathrm{y}_1 \tfrac{w}{x_3^2} \quad \widehat{\mathrm{y}}_2 = \mathrm{y}_2 \tfrac{x_3}{x_1} \quad \widehat{\mathrm{y}}_3 = \mathrm{y}_3 \tfrac{x_1^2}{w}.$$

On the other hand the same expansion is given by (2.3.34). By equating this two expressions we get:

$$\frac{F_b\,|_{\mathcal{F}}\,(\frac{\mathrm{y}_1\mathrm{y}_2 w}{(x_1+\mathrm{y}_2x_3)x_3}, \frac{x_1}{\mathrm{y}_2x_3}, \frac{\mathrm{y}_3x_1(x_1+\mathrm{y}_2x_3)}{w})}{F_b\,|_{\mathbb{P}}\,(\mathrm{y}_1\mathrm{y}_2, \frac{1}{\mathrm{y}_2}, \mathrm{y}_3)}\mathbf{x}^{\mathbf{g}_b} = F_b^w(\frac{\mathrm{y}_1 w}{x_3^2}, \frac{\mathrm{y}_2x_3}{x_1}, \frac{\mathrm{y}_3x_1^2}{w}) \cdot (x_1, w, x_3)^{\mathbf{g}_b^w}$$

By Lemma 2.3.14 and (2.3.48) we have

$$(x_1, \frac{x_1+\mathrm{y}_2x_3}{w}, x_3)^{\mathbf{g}_b} = \begin{cases} \frac{x_3}{x_1} \cdot (\frac{x_1+\mathrm{y}_2x_3}{x_1})^{g_{2;s}} \cdot (x_1, w, x_3)^{\mathbf{g}_b^w} & \text{if } b = x_{-(2m+1)},\ m \geq 0, \\ (\frac{x_1+\mathrm{y}_2x_3}{x_1})^{g_{2;b}} \cdot (x_1, w, x_3)^{\mathbf{g}_b^w} & \text{otherwise.} \end{cases},$$

from which the desired result follows by using the explicit description of $\mathbf{g}_s$ given in Proposition 2.3.3. This concludes the proof of Lemma 2.3.18. □

Using Proposition 2.3.3 we apply Lemma 2.3.18 to the explicit formulas given there and we conclude the proof of Proposition 2.3.17. □

As a corollary of the previous results we get explicit formulas for the elements of $\mathcal{B}$ in the initial cluster of $\mathcal{A}^{Cycl}$.

Proposition 2.3.19. *For every $m \geq 0$:*

$$x_{2m+1} = \frac{\sum_{e_1,e_3} \binom{m-1-e_3}{m-1-e_1}\binom{e_1-1}{e_3} \mathrm{y}_1^{e_1}\mathrm{y}_3^{e_3}\, x_1^{2e_3} w^{e_1-e_3} x_3^{2m-2e_1-2} + x_3^{2m-2}}{x_1^{m-1}x_3^{m-2}};$$

$$x_{2m+2} = \frac{\sum_{\mathbf{e}} \binom{e_1-1}{e_3}\binom{m-1-e_3+e_2}{m-1-e_1+e_2}\binom{1}{e_2} \mathrm{y}^{\mathbf{e}} x_1^{2e_3+1-e_2} w^{e_1-e_3} x_3^{2m-1-2e_1+e_2} + \mathrm{y}_2 x_3^{2m} + x_1 x_3^{2m-1}}{x_1^m w x_3^{m-1}};$$

$$x_{-(2m+1)} = \frac{\sum_{e_1,e_3} \binom{m-e_3}{m-e_1}\binom{e_1+1}{e_3} \mathrm{y}_1^{e_1}\mathrm{y}_3^{e_3}\, x_1^{2e_3} w^{e_1-e_3+1} x_3^{2m-2e_1} + \mathrm{y}_1^m \mathrm{y}_3^{m+1} x_1^{2m+2}}{x_1^m x_3^{m+1}};$$

$$x_{-2m} = \frac{\sum_{\mathbf{e}} \binom{e_1+1-e_2}{e_3-e_2}\binom{m-e_3}{m-e_1}\binom{1}{e_2} \mathrm{y}^{\mathbf{e}} x_1^{2e_3-e_2} w^{e_1-e_3} x_3^{2m-2e_1+e_2} + \mathrm{y}_1^m \mathrm{y}_3^{m+1} x_1^{2m+1}(\mathrm{y}_2x_3 + x_1)}{x_1^m w x_3^{m+1}}.$$

For every $n \geq 1$:

$$u_n = \frac{\mathrm{y}_1^n \mathrm{y}_3^n\, x_1^{2n} + x_3^{2n} + \sum_{e_1,e_3} [\binom{n-e_3}{n-e_1}\binom{e_1-1}{e_3} + \binom{n-e_3-1}{n-e_1}\binom{e_1-1}{e_3-1}] \mathrm{y}_1^{e_1}\mathrm{y}_3^{e_3}\, x_1^{2e_3} w^{e_1-e_3} x_3^{2n-2e_1}}{x_1^n x_3^n}.$$

Corollary 2.3.20. *For $m \geq 1$*

$$\mathbf{d}^w(x_{2m+1}) = \left(\begin{smallmatrix} m-1 \\ 0 \\ m-2 \end{smallmatrix} \right), \quad \mathbf{d}^w(x_{2m+2}) = \left(\begin{smallmatrix} m-1 \\ 1 \\ m-2 \end{smallmatrix} \right), \tag{2.3.52}$$

$$\mathbf{d}^w(x_{-2m+3}) = \left(\begin{smallmatrix} m-2 \\ 0 \\ m-1 \end{smallmatrix} \right), \quad \mathbf{d}^w(x_{-2m+2}) = \left(\begin{smallmatrix} m-2 \\ 1 \\ m-1 \end{smallmatrix} \right) \tag{2.3.53}$$

and also for $n \geq 1$

$$\mathbf{d}^w(w) = \left(\begin{smallmatrix} 0 \\ -1 \\ 0 \end{smallmatrix} \right), \quad \mathbf{d}^w(z) = \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix} \right), \quad \mathbf{d}^w(u_n) = \left(\begin{smallmatrix} n \\ 0 \\ n \end{smallmatrix} \right). \tag{2.3.54}$$

together with $\mathbf{d}^w(x_2) = \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix} \right]$.

Remark 2.3.21. The F–polynomial F_z^w of the cluster variable z in $\mathcal{A}^{Cycl}$ is given in (2.3.44) while its denominator vector $\mathbf{d}^w(z)$ is given in (3.2.17). In particular the following equality holds:

$$F_z^w|_{\mathrm{Trop}(\mathrm{y}_1,\mathrm{y}_2,\mathrm{y}_3)}(\frac{1}{\mathrm{y}_1}, \frac{1}{\mathrm{y}_2}, \frac{1}{\mathrm{y}_3}) = \frac{1}{\mathrm{y}_1\mathrm{y}_2^2\mathrm{y}_3}.$$

In [18, Conjecture 7.17] it was expected the right-hand side to be $\mathrm{y}^{-\mathbf{d}^w(z)} = \frac{1}{\mathrm{y}_1\mathrm{y}_2\mathrm{y}_3}$. This counterexample appears also in [6] and in [21].

2.3.9 Positivity of the elements of the canonical basis B

As a corollary of the previous results we get the positivity of the elements of the canonical basis $\mathcal{B}$ as shown by the following proposition.

Proposition 2.3.22. *The Laurent expansion of every element of the canonical basis* $\mathbf{B}$ *in every cluster of $\mathcal{A}$ has coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$.*

Proof. Given an element b of $\mathcal{B} \doteq \{\text{cluster variables}\} \sqcup \{u_n : \ n \geq 1\}$, its Laurent expansion in a cluster $\mathcal{C} = (s_1, s_2, s_3)$ of $\mathcal{A}$ is given by:

$$b = \frac{F_b^{\mathcal{C}}(\widehat{y}_{1;\mathcal{C}}, \widehat{y}_{2;\mathcal{C}}, \widehat{y}_{3;\mathcal{C}})}{F_b^{\mathcal{C}}|_{\mathbb{P}}(y_{1;\mathcal{C}}, y_{2;\mathcal{C}}, y_{3;\mathcal{C}})} \cdot \mathbf{s}^{\mathbf{g}_b^{\mathcal{C}}}$$

where $F_b^{\mathcal{C}}$ and $\mathbf{g}_b^{\mathcal{C}}$ are respectively the F–polynomial and the $\mathbf{g}$–vectors of b in the cluster $\mathcal{C}$, and $\{y_{1;\mathcal{C}}, y_{2;\mathcal{C}}, y_{3;\mathcal{C}}\}$ are the coefficients of the (unique) seed of $\mathcal{A}$ with cluster $\mathcal{C}$. By the symmetries of the exchange relations, it is sufficient to consider only the two clusters $\{x_1, x_2, x_3\}$ and $\{x_1, w, x_3\}$. By Proposition 2.3.4 and Proposition 2.3.13 we know F–polynomials in these two clusters have coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$. We conclude the elements of $\mathcal{B}$ are positive. The claim follows by the fact the all the other elements of $\mathbf{B}$ are products of elements of $\mathcal{B}$. $\square$

2.3.10 Straightening relations and span property

The main result of this section is the following:

Proposition 2.3.23. *The set* $\mathbf{B}$ *of cluster monomials and of the elements* $\{u_n w^k, u_n z^k : n \geq 1, k \geq 0\}$ *defined in Theorem 2.1.4 spans* $\mathcal{A}$ *over* $\mathbb{ZP}$.

The strategy for proving Proposition 2.3.23 is described in Section 1.5.2. We briefly remind here the arguments. Since $\mathbf{B}$ contains cluster variables, monomials in its elements span $\mathcal{A}$ over $\mathbb{ZP}$. It is then sufficient to express every such monomial as a $\mathbb{ZP}$–linear combination of elements of $\mathbf{B}$. In order to do that we will need "straightening relations", i.e. explicit expressions for the expansion in $\mathbf{B}$ of the "minimal" monomials that are *not* elements of $\mathbf{B}$. These monomials are minimal with respect to the following ordering: the generic monomial M has the form $M = u_{n1}^{a_1} \cdots u_{n_s}^{a_s} x_{m_1}^{b_1} \cdots x_{m_t}^{b_t} w^c z^d$ where $0 < n_1 < \cdots < n_s$, $m_1 < \cdots < m_t$ and the exponents are positive integers. We define the *multi-degree* $\mu(M) = (\mu_1(M), \mu_2(M), \mu_3(M)) \in \mathbb{Z}^3_{\geq 0}$ by setting

$$\begin{cases} \mu_1(M) &= \sum_{i=1}^s a_i + \sum_{j=1}^t b_j + c + d \\ \mu_2(M) &= m_t - m_1; \\ \mu_3(M) &= b_1 + b_t. \end{cases} \tag{2.3.55}$$

One can immediately see that the minimal monomials in the elements of $\mathbf{B}$ with respect to the multi–degree (2.3.55) are the following:

$$u_n u_p; \quad u_n x_m; \quad x_m x_{m+2+n},$$

for every $n, p \geq 1$ and $m \in \mathbb{Z}$. Indeed $\mu_1(u_n x_m) = \mu_1(u_n u_p) = \mu_1(x_m x_{m+2+2n}) = 2$ and hence they are minimal ($\mu_1(M) = 1$ if and only if M is either a cluster variable or u_n). Moreover they are the only monomials not belonging to $\mathbf{B}$ having this property.

Propositions 2.3.24 and 2.3.27 give the desired straightening relations. The proof of Proposition 2.3.23 will be given at the end of the present section.

Proposition 2.3.24. *For every* $n, p \geq 1$

$$u_n u_p = \begin{cases} u_{n+p} + \mathbf{y}^{p\delta} u_{n-p} & n > p \\ u_{2n} + 2\mathbf{y}^{n\delta} & n = p \end{cases} \tag{2.3.56}$$

where $\delta = (1,1,1)^t$ *is the denominator vector of* u_1 *in the initial cluster of* $\mathcal{A}$.

Proof. We use the definition of the u_n's given in Definition 2.1.3. We assume now that $u_0 = 2$, so that the relation $u_1 u_n = u_{n+1} + \mathbf{y}^\delta u_{n-1}$ holds for every $n \geq 1$ (instead of holding only for $n \geq 2$ as in Definition 2.1.3). Moreover, with this convention, we have to prove that for every $p : \;\; 1 \leq p \leq n$ we have

$$u_n u_p = u_{n+p} + \mathbf{y}^{p\delta} u_{n-p} \tag{2.3.57}$$

If $n = p = 1$ then (2.3.57) is the definition (2.1.19) of u_2; we assume $n \geq 2$ and we proceed by induction on $p \geq 1$: if $p = 1$, then (2.3.57) is just the definition (2.1.20) of u_{n+1}, we then assume $2 \leq p+1 \leq n$

$$u_n u_{p+1} = u_n[u_1 u_p - \mathbf{y}^{\delta} u_{p-1}] =$$

$$= u_1[u_{n+p} + \mathbf{y}^{p\delta} u_{n-p}] - \mathbf{y}^{\delta}[u_{n+p-1} + \mathbf{y}^{(p-1)\delta} u_{n-p+1}] =$$

$$= u_{n+1+p} + \mathbf{y}^{\delta} u_{n+p-1} + \mathbf{y}^{p\delta}[u_{n+1-p} + \mathbf{y}^{\delta} u_{n-p-1}] - \mathbf{y}^{\delta}[u_{n+p-1} + \mathbf{y}^{(p-1)\delta} u_{n-p+1}] =$$

$$= u_{n+p+1} + \mathbf{y}^{(p+1)\delta} u_{n-(p+1)}.$$

$\square$

In order to get the other straightening relations we will need the following notations.

Definition 2.3.25. We introduce the following deformation of the coefficients: for every $m \in \mathbb{Z}$ we define

$$\xi_m = \begin{cases} \mathbf{y}^{\mathbf{d}(x_{m+3})} = y_{1;m} & \text{if } m \geq 1 \\ \mathbf{y}^{\mathbf{d}(x_m)} = y_{1;m-3} & \text{if } m \leq 0 \end{cases} \tag{2.3.58}$$

and also

$$\zeta_n^-(m) \doteq \begin{cases} \xi_m \oplus \mathbf{y}^{n\delta} & \text{if } m \geq 1 \\ 1 & \text{if } m \leq 0 \end{cases} \quad ; \quad \zeta_n^+(m) \doteq \begin{cases} 1 & \text{if } m \geq 1 \\ \xi_m \oplus \mathbf{y}^{n\delta} & \text{if } m \leq 0 \end{cases} \tag{2.3.59}$$

where $n\delta = (n, n, n)^t$ is the denominator vector of u_n in the initial cluster of $\mathcal{A}$. Moreover for every integer $k \geq 0$ we define

$$\gamma_1(k) \doteq y_1^{\lceil \frac{k}{2} \rceil} y_2^{\lfloor \frac{k}{2} \rfloor} y_3^{\lceil \frac{k}{2} \rceil}; \quad \gamma_2(k) \doteq y_1^{\lfloor \frac{k}{2} \rfloor} y_2^{\lceil \frac{k}{2} \rceil} y_3^{\lfloor \frac{k}{2} \rfloor}; \quad \gamma_3(k) \doteq \begin{cases} \mathbf{y}^{\frac{k}{2}\delta} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

and we define for $i = 1, 2, 3$ the corresponding elements of $\mathcal{A}$:

$$\Gamma_i(n) = \sum_{k \geq 0} (\lfloor \frac{k}{2} \rfloor + 1) \cdot \gamma_i(k) \cdot u_{n-k}.$$

We also define for every $m \in \mathbb{Z}$ and $m_1 \geq 0$:

$$\eta^-_{m;\, m_1} = \begin{cases} \xi_m \oplus \xi_{m+m_1} & \text{if } m \leq 0 < m + m_1 \\ 1 & \text{otherwise} \end{cases}$$

and

$$\eta^+_{m;\, m_1} = \begin{cases} 1 & \text{if } m \leq 0 < m + m_1 \\ \xi_m \oplus \xi_{m+m_1} & \text{otherwise} \end{cases}$$

Here we collect some properties of the elements introduced in Definition 2.3.25 that we will need later.

Lemma 2.3.26. *With notations of Definition 2.3.25 we have the following results:*

1. *For every $m \in \mathbb{Z}$ and $k \geq 0$:*

$$\xi_m \oplus \xi_{m+k} = \begin{cases} \xi_m & \text{if } m \geq 1, \\ \xi_{m+k} & \text{if } m+k \leq 0; \end{cases} \tag{2.3.60}$$

moreover

$$\xi_m \oplus \mathbf{y}^\delta = \begin{cases} \xi_m & \text{if } -1 \leq m \leq 2, \\ \mathbf{y}^\delta & \text{otherwise.} \end{cases} \tag{2.3.61}$$

If $m \geq 0$ and $n \geq 1$ we get

$$\xi_{-m} \oplus \xi_n = \begin{cases} \xi_{-m} & \text{if } m < n-1, \\ \mathbf{y}^{k\delta} & \text{if } m = n-1 = 2k, \\ y_2\mathbf{y}^{k\delta} & \text{if } m = n-1 = 2k+1, \\ \xi_n & \text{if } m > n-1. \end{cases} \tag{2.3.62}$$

2. *For every $m \in \mathbb{Z}$ and $n \geq 1$ the following relation holds*

$$\zeta_n^+(m) = (13)\zeta_n^-(1-m); \quad \zeta_n^-(m) = (13)\zeta_n^+(1-m) \tag{2.3.63}$$

where (13) is the automorphism of $\mathbb{P}$ that exchanges y_1 with y_3.

3. *For every $n \geq 1$ and $i \in \{1,2,3\}$ we have:*

$$u_1\Gamma_i(n) = \Gamma_i(n+1) + \mathbf{y}^\delta\Gamma_i(n-1) - \gamma_i(n+1) \tag{2.3.64}$$

Proof of Lemma 2.3.26. (2.3.60) and (2.3.61) follow directly by the definition of ξ_m and ξ_{m+k} by using figure 2.4: indeed one can see that $\mathbf{d}(x_m) \leq \mathbf{d}(x_{m+k})$ (resp. $\mathbf{d}(x_m) \geq \mathbf{d}(x_{m+k})$) if $m \geq 1$ (resp. $m+k \leq 0$). (Here $\leq$ is understood term by term).

We now want to compute $\xi_{-m} \oplus \xi_n \doteq \mathbf{y}^{\mathbf{d}(x_{-m})} \oplus \mathbf{y}^{\mathbf{d}(x_{n+3})}$. By Corollary 2.3.6, $\mathbf{d}(x_{-m}) = (13)\mathbf{d}(x_{m+4})$, where (13) is the linear operator on $\mathbb{Z}^3$ that exchanges the first entry with the third one. We now consider all the possible cases:

If $m+4 < n+3$ then $\mathbf{d}(x_{m+4}) \leq \mathbf{d}(x_{n+3})$; since $m+4$ and $n+3$ are positive integers, $\mathbf{d}(x_{m+4})$ and $\mathbf{d}(x_{-m})$ have respectively the form (d_3+1, d_2, d_3) and $(d_3'+1, d_2', d_3')$ for some $d_2, d_3, d_2', d_3' \geq 0$; in particular $\mathbf{d}(x_{-m}) = (d_3, d_2, d_3+1)$. Since by hypothesis $d_3 < d_3'$ and $d_2 < d_2'$, we conclude $\mathbf{d}(x_{-m}) \leq \mathbf{d}(x_{n+3})$ so that $\mathbf{y}^{\mathbf{d}(x_{-m})} \oplus \mathbf{y}^{\mathbf{d}(x_{n+3})} = \mathbf{y}^{\mathbf{d}(x_{-m})}$.

If $m+4 = n+3 = 2k+4$ for some $k \geq 0$, then by (2.3.52) (or figure 2.4), $\mathbf{d}(x_{m+4}) = (k+1, k, k)^t$ so that $(13)\mathbf{d}(x_{m+4}) = (k, k, k+1)^t$; then $\mathbf{y}^{\mathbf{d}(x_{-m})} \oplus \mathbf{y}^{\mathbf{d}(x_{n+3})} = \mathbf{y}^{k\delta}$.

If $m+4 = n+3 = 2k+5$ for some $k \geq 0$, then by (2.3.52) (or figure 2.4), $\mathbf{d}(x_{m+4}) = (k+1, k+1, k)^t$ so that $(13)\mathbf{d}(x_{m+4}) = (k, k+1, k+1)^t$; then $\mathbf{y}^{\mathbf{d}(x_{-m})} \oplus \mathbf{y}^{\mathbf{d}(x_{n+3})} = y_2\mathbf{y}^{k\delta}$.

If $m+4 > n+3$ then $\mathbf{d}(x_{m+4}) \geq \mathbf{d}(x_{n+3})$, then also (13)$\mathbf{d}(x_{m+4}) \geq \mathbf{d}(x_{n+3})$.

and (2.3.62) is proved.

Formula (2.3.64) follows by (2.3.56). We give the details here:

$$u_1 \cdot \Gamma_i(n) = \sum_{k=0}^{n} (\lfloor \frac{k}{2} \rfloor + 1)\gamma_i(k)(u_1 u_{n-k}) =$$

$$\sum_{k=0}^{n-2} (\lfloor \frac{k}{2} \rfloor + 1)\gamma_i(k) u_{n+1-k} + (\lfloor \frac{n-1}{2} \rfloor + 1)\gamma_i(n-1) \cdot u_2 +$$

$$+(\lfloor \frac{n}{2} \rfloor + 1)\gamma_i(n) u_1 + (\lfloor \frac{n-1}{2} \rfloor + 1)\gamma_i(n-1)\mathbf{y}^{\delta} +$$

$$+\mathbf{y}^{\delta} \cdot [\sum_{k=0}^{n-2} (\lfloor \frac{k}{2} \rfloor + 1)\gamma_i(k) u_{n-1-k} + (\lfloor \frac{n-1}{2} \rfloor + 1)\gamma_i(n-1)] =$$

$$= \Gamma_i(n+1) - \gamma_i(n+1) + \mathbf{y}^{\delta} \cdot \Gamma_i(n-1)$$

The last equality follows from the fact that $\lfloor \frac{n-1}{2} \rfloor + 1 = \lfloor \frac{n+1}{2} \rfloor$ and $\gamma_i(n-1)\mathbf{y}^{\delta} = \gamma_i(n+1)$. $\square$

Proposition 2.3.27. *With notations of Definition 2.3.25 the following "straightening relations" hold.*

(i) *For every $m \in \mathbb{Z}$ and $n \geq 1$:*

$$u_n x_m = \zeta_n^-(m)\, x_{m-2n} + \zeta_n^+(m)\, x_{m+2n} \tag{2.3.65}$$

(ii) *For every $m \in \mathbb{Z}$* even *and $n \geq 0$:*

$$x_m x_{m+2n+3} = \eta^-_{m;\,2n+3}\, x_{m+n+1} x_{m+n+2} + \eta^+_{m;\,2n+3}\, \Gamma_1(n) \tag{2.3.66}$$

(iii) *For every $m \in \mathbb{Z}$* odd *and $n \geq 0$:*

$$x_m x_{m+2n+3} = \eta^-_{m;\,2n+3}\, x_{m+n+1} x_{m+n+2} + \eta^+_{m;\,2n+3}\, \Gamma_2(n) \tag{2.3.67}$$

(iv) *For every $m \in \mathbb{Z}$* even *and $n \geq 2$:*

$$x_m x_{m+2n} = \eta^-_{m;\,2n}\, x_{m+2\lfloor \frac{n}{2} \rfloor} x_{m+2\lceil \frac{n}{2} \rceil} + \eta^+_{m;\,2n}\, \Gamma_3(n-2) z \tag{2.3.68}$$

(v) *For every $m \in \mathbb{Z}$* odd *and $n \geq 2$:*

$$x_m x_{m+2n} = \eta^-_{m;\,2n}\, x_{m+2\lfloor \frac{n}{2} \rfloor} x_{m+2\lceil \frac{n}{2} \rceil} + \eta^+_{m;\,2n}\, \Gamma_3(n-2) w \tag{2.3.69}$$

Proof. We prove part (i) by induction on $n \geq 1$. We prove it for $n = 1$. By using exchange relations (2.1.6) and (2.1.8) it is easy to see that for every $m \in \mathbb{Z}$ we have:

$$u_1 x_m = (zw - y_1 y_3 - y_2)x_m =$$
$$= x_{m-2}[\frac{y_{2;m-1} \cdot y_{2;m-2}}{(y_{2;m-1} \oplus 1)(y_{2;m-2} \oplus 1)}] + x_{m+2}[\frac{1}{(y_{2;m-1} \oplus 1)(y_{2;m-2} \oplus 1)}] +$$
$$+x_m[\frac{y_{2;m-1} \cdot y_{2;m-2}}{(y_{2;m-1} \oplus 1)(y_{2;m-2} \oplus 1)} + \frac{y_{2;m}}{(y_{2;m-1} \oplus 1)(y_{2;m} \oplus 1)} - y_1 y_3 - y_2]$$

where $y_{2;k}$ is defined in (2.1.7) for k odd and in (2.1.9) for k even. The following Lemma, whose proof is by direct check, gives the desired result.

Lemma 2.3.28. *For every $m \in \mathbb{Z}$ the following relations hold:*

1. $\frac{y_{2;m-1} y_{2;m-2}}{(y_{2;m-1} \oplus 1)(y_{2;m-2} \oplus 1)} = \zeta_1^-(m)$
2. $\frac{1}{(y_{2;m-1} \oplus 1)(y_{2;m-2} \oplus 1)} = \zeta_1^+(m)$
3. $\frac{y_{2;m-1} y_{2;m-2}}{(y_{2;m-1} \oplus 1)(y_{2;m-2} \oplus 1)} + \frac{y_{2;m}}{(y_{2;m-1} \oplus 1)(y_{2;m} \oplus 1)} - y_1 y_3 - y_2 = 0$

We now proceed by induction on $n \geq 1$. We use the convention that $u_0 = 2$ so that the relation:

$$u_{n+1} = u_1 u_n - \mathbf{y}^\delta u_{n-1}$$

(given in Definition 2.1.3) holds for every $n \geq 1$. Moreover, with this convention, since $\zeta_0^\pm(m) = 1$, (2.3.65) still holds for $n = 0$. We have

$$u_{n+1} x_m = u_1 u_n x_m - \mathbf{y}^\delta u_{n-1} x_m =$$

$$= u_n[\zeta_1^-(m) x_{m-2} + \zeta_1^+(m) x_{m+2}] +$$
$$-\mathbf{y}^\delta[\zeta_{n-1}^-(m) x_{m-2n+2} + \zeta_{n-1}^+(m) x_{m+2n-2}] =$$

$$= \zeta_1^-(m)[\zeta_n^-(m-2) x_{m-2-2n} + \zeta_n^+(m-2) x_{m-2+2n}] +$$
$$+\zeta_1^+(m)[\zeta_n^-(m+2) x_{m+2-2n} + \zeta_n^+(m+2) x_{m+2+2n}] +$$
$$-\mathbf{y}^\delta \zeta_{n-1}^-(m) x_{m+2-2n} - \mathbf{y}^\delta \zeta_{n-1}^+(m) x_{m-2+2n} =$$

$$= x_{m-2-2n}[\zeta_1^-(m)\zeta_n^-(m-2)] + x_{m-2+2n}[\zeta_1^-(m)\zeta_n^+(m-2) - \mathbf{y}^\delta \zeta_{n-1}^+(m)] +$$
$$+x_{m+2-2n}[\zeta_1^+(m)\zeta_n^-(m+2) - \mathbf{y}^\delta \zeta_{n-1}^-(m)] + x_{m+2+2n}[\zeta_1^+(m)\zeta_n^+(m+2)]$$

The claim follows by Lemma 2.3.29 below.

Lemma 2.3.29. *For every $m \in \mathbb{Z}$ and $n \geq 1$ we have*

1. $\zeta_1^-(m)\zeta_n^-(m-2) = \zeta_{n+1}^-(m)$;
2. $\zeta_1^-(m)\zeta_n^+(m-2) - \mathbf{y}^\delta \zeta_{n-1}^+(m) = 0$;

3. $\zeta_1^+(m)\zeta_n^-(m+2) - \mathbf{y}^\delta \zeta_{n-1}^-(m) = 0$*;*

4. $\zeta_1^+(m)\zeta_n^+(m+2) = \zeta_{n+1}^+(m)$

The proof of Lemma 2.3.29 is by direct check.

We prove part *(ii)* and *(iii)* together. It is convenient to prove that the following relation holds for every $m \in \mathbb{Z}$, $n \geq 0$ and $i = 1$ if m is even and 2 if m is odd:

$$\eta^+_{m;2n+3}\Gamma_i(n) = x_m x_{m+2n+3} - \eta^-_{m;2n+3} x_{m+n+1} x_{m+n+2} \tag{2.3.70}$$

We proceed by induction on $n \geq 0$. We first prove (2.3.70) for $n = 0$. In this case $\Gamma_1(0) = \Gamma_2(0) = 1$. By the exchange relation (2.1.2) we know that for every $m \in \mathbb{Z}$ the following relation holds:

$$x_m x_{m+3} = \begin{cases} y_{m+3} x_{m+1} x_{m+2} + 1 & \text{if } m = 0, -1, -2 \\ x_{m+1} x_{m+2} + y_{1;m} & \text{otherwise} \end{cases} \tag{2.3.71}$$

By part 1 of Lemma 2.3.26, it is immediate to verify that

$$\eta^-_{m;3} = \begin{cases} y_{m+3} & \text{if } m = 0, -1, -2 \\ 1 & \text{otherwise} \end{cases} \quad ; \quad \eta^+_{m;3} = \begin{cases} 1 & \text{if } m = 0, -1, -2 \\ y_{1;m} & \text{otherwise} \end{cases}$$

so that (2.3.70) specializes to (2.3.71) when $n = 0$, i.e. for every $m \in \mathbb{Z}$ the following relation holds

$$x_m x_{m+3} = \eta^-_{m;3} x_{m+1} x_{m+2} + \eta^+_{m;3}. \tag{2.3.72}$$

We now assume $n \geq 1$. In this case, using the inductive hypothesis we have:

$$\Gamma_i(n+1) = u_1 \Gamma_i(n) - \mathbf{y}^\delta \Gamma_i(n-1) + \gamma_i(n+1) =$$

$$= \frac{u_1}{\eta^+_{m;2n+3}} \cdot [x_m x_{m+2n+3} - \eta^-_{m;2n+3} x_{m+n+1} x_{m+n+2}] +$$
$$- \frac{\mathbf{y}^\delta}{\eta^+_{m;2n+1}} \cdot [x_m x_{m+2n+1} - \eta^-_{m;2n+1} x_{m+n} x_{m+n+1}] + \gamma_i(n+1) =$$

$$= \frac{x_m}{\eta^+_{m,2n+3}} \cdot [\zeta_1^-(m+2n+3) x_{m+2n+1} + \zeta_1^+(m+2n+3) x_{m+2n+5}] +$$
$$- \frac{\eta^-_{m;2n+3}}{\eta^+_{m;2n+3}} \cdot [\zeta_1^-(m+n+2) x_{m+n} x_{m+n+1} + \zeta_1^+(m+n+2) x_{m+n+1} x_{m+n+4}] +$$
$$- \frac{\mathbf{y}^\delta}{\eta^+_{m;2n+1}} \cdot [x_m x_{m+2n+1} - \eta^-_{m;2n+1} x_{m+n} x_{m+n+1}] + \gamma_i(n+1) =$$

$$= x_m x_{m+2n+1} [\tfrac{\zeta_1^-(m+2n+3)\eta^+_{m;2n+1} - \mathbf{y}^\delta \eta^+_{m;2n+3}}{\eta^+_{m;2n+1}\eta^+_{m;2n+3}}] + x_m x_{m+2n+5} [\tfrac{\zeta_1^+(m+2n+3)}{\eta^+_{m;2n+3}}] +$$
$$+ x_{m+n} x_{m+n+1} [\tfrac{\mathbf{y}^\delta \eta^-_{m;2n+1}\eta^+_{m;2n+3} - \eta^-_{m;2n+3}\zeta_1^-(m+n+2)\eta^+_{m;2n+1}}{\eta^+_{m;2n+1}\eta^+_{m;2n+3}}] +$$
$$- \frac{\zeta_1^+(m+n+2)\eta^-_{m;2n+3}}{\eta^+_{m;2n+3}} x_{m+n+1} x_{m+n+4} + \gamma_i(n+1) =$$

$$= x_m x_{m+2n+1}[\frac{\zeta_1^-(m+2n+3)\eta^+_{m;2n+1} - \mathbf{y}^\delta \eta^+_{m;2n+3}}{\eta^+_{m;2n+1}\eta^+_{m;2n+3}}] + x_m x_{m+2n+5}[\frac{\zeta_1^+(m+2n+3)}{\eta^+_{m;2n+3}}] +$$
$$+ x_{m+n} x_{m+n+1}[\frac{\mathbf{y}^\delta \eta^-_{m;2n+1}\eta^+_{m;2n+3} - \eta^-_{m;2n+3}\zeta_1^-(m+n+2)\eta^+_{m;2n+1}}{\eta^+_{m;2n+1}\eta^+_{m;2n+3}}] +$$
$$- \frac{\zeta_1^+(m+n+2)\eta^-_{m;2n+3}}{\eta^+_{m;2n+3}}[\eta^-_{m+n+1;3} x_{m+n+2} x_{m+n+3} + \eta^+_{m+n+1;3}] + \gamma_i(n+1)$$

Lemma 2.3.30 below shows that this polynomial is equal to

$$\frac{1}{\eta^+_{m;2n+5}}[x_m x_{m+2n+5} - \eta^-_{m;2n+5} x_{m+n+2} x_{m+n+3}]$$

and we are done.

We prove (iii) and (iv) together. In order to do that we introduce the variable c depending on $m \in \mathbb{Z}$ in the following way: c is w if m is odd and c is z if m is even. With this convention, both (2.3.68) and (2.3.69) are equivalent to the following:

$$c\Gamma_3(n-2) = \frac{1}{\eta^+_{m;2n}}\left[x_m x_{m+2n} - \eta^-_{m;2n} x_{m+2\lfloor \frac{n}{2} \rfloor} x_{m+2\lceil \frac{n}{2} \rceil}\right]. \qquad (2.3.73)$$

In order to prove (2.3.73) we proceed by induction on $n \geq 2$. We verify directly the formula for $n = 2$ and $n = 3$. We then assume $n \geq 4$. By using (2.3.64) and the inductive hypothesis we get the following equality:

$$c\Gamma_3(n-2) =$$

$$x_m x_{m+2n+4}[\frac{\zeta_1^-(m+2n-2)\eta^+_{m;\,2n-4} - \mathbf{y}^\delta \eta^+_{m;\,2n-2}}{\eta^+_{m;\,2n-2}\eta^+_{m;\,2n-4}}] +$$
$$x_m x_{m+2n}[\frac{\zeta_1^+(m+2n-2)}{\eta^+_{m;\,2n-2}}] +$$
$$x_{m+2\lfloor \frac{n-2}{2} \rfloor} x_{m+2\lceil \frac{n-2}{2} \rceil}[\frac{\mathbf{y}^\delta \eta^-_{m;\,2n-4}\eta^+_{m;\,2n-2} - \eta^-_{m;\,2n-2}\eta^+_{m;\,2n-4}\zeta_1^-(m+2\lceil \frac{n-1}{2} \rceil)}{\eta^+_{m;\,2n-2}\eta^+_{m;\,2n-4}}] +$$
$$-\frac{\eta^-_{m;\,2n-2}\zeta_1^+(m+2\lceil \frac{n-1}{2} \rceil)}{\eta^+_{m;\,2n-2}} x_{m+2\lfloor \frac{n-1}{2} \rfloor} x_{m+2\lceil \frac{n-1}{2} \rceil} + c\gamma_3(n-2)$$

Lemma 2.3.30 concludes the proof. □

Lemma 2.3.30. *For every $n \geq 1$, $m_1 \geq 3$ and $m \in \mathbb{Z}$ the following equalities hold in $\mathbb{ZP}$:*

1. $\zeta_1^-(m+m_1+2)\eta^+_{m;m_1} - \mathbf{y}^\delta \eta^+_{m;m_1+2} = 0$
2. $\zeta_1^+(m+m_1) = \eta^+_{m;m_1}/\eta^+_{m;m_1+2}$
3. $\mathbf{y}^\delta \eta^-_{m;m_1} \cdot \eta^+_{m;m_1+2} - \eta^-_{m;m_1+2}\eta^+_{m;m_1}\zeta_1^-(m+\lceil \frac{m_1+2}{2} \rceil) = 0$

4. $\zeta_1^+(m+n+2)\eta^-_{m;2n+3}\eta^-_{m+n+1;3} = \eta^+_{m;2n+3}\eta^-_{m;2n+5}/\eta^+_{m;2n+5}$

5. For $i = 1$ if m is even and $i = 2$ if m is odd we have for every $n \geq 1$:
$\gamma_i(n+1) \cdot \eta^+_{m;2n+3} - \zeta_1^+(m+n+2) \cdot \eta^-_{m;2n+3} \cdot \eta^+_{m+n+1;3} = 0$

6. $\frac{\eta^-_{m;\,2n-2}\zeta_1^+(m+2\lceil\frac{n-1}{2}\rceil)}{\eta^+_{m;\,2n-2}} x_{m+2\lfloor\frac{n-1}{2}\rfloor} x_{m+2\lceil\frac{n-1}{2}\rceil} - c\gamma_3(n-2) = \frac{\eta^-_{m;2n}}{\eta^+_{m;2n}} x_{m+2\lfloor\frac{n}{2}\rfloor} x_{m+2\lceil\frac{n}{2}\rceil}$

The proof of Lemma (2.3.30) follows by direct check.

Proof of Proposition 2.3.23

In order to prove that $\mathbf{B}$ spans $\mathcal{A}$, we need to show that every monomial M in the variables u_n and in the cluster variables is a linear combination of elements of $\mathbf{B}$. The generic M has the form $M = u_{n1}^{a_1} \cdots u_{n_s}^{a_s} x_{m_1}^{b_1} \cdots x_{m_t}^{b_t} w^c z^d$ where $0 < n_1 < \cdots < n_s$, $m_1 < \cdots < m_t$ and the exponents are positive integers. We will use the multi–degree defined in (2.3.55). Therefore, to complete the proof, we proceed by induction on $\mu(M)$. If $\mu_1(M) = 1$ then M is a cluster variable or one of the u_n's. Then it suffices to show that every monomial M which has at least one of the "forbidden" products as a factor, can be written as a linear combination of monomials of (lexicographically) smaller multi-degree. We will show that this can be done by replacing some "forbidden" factor of M with its expression given by the appropriate relation in Propositions 2.3.24 and 2.3.27. Indeed, if $\sum_{i=1}^s a_i \geq 2$ (resp. $\sum_{i=1}^s a_i = 1$) then one can apply (2.3.56) (resp. (2.3.65)), expressing M as a linear combination of monomials with smaller value of μ_1. So we can assume that $M = x_{m_1}^{b_1} \cdots x_{m_t}^{b_t} w^c z^d$. If both c and d are positive, by using the fact that $zw = u_1 + 2$, one obtains again a sum of two monomials with smaller value of μ_1. So we can assume that $d = 0$ (resp. $c = 0$) and that we can apply the exchange relation (2.1.6) (resp. (2.1.8)), i.e. some m_i is odd (resp. even). We again obtain a sum of two monomials having smaller value of μ_1 than the initial one. So we can assume that M has one of the following forms: $M_1 = (\Pi_{m_i \text{ odd}} x_{m_i}^{b_i})w^c$ or $M_2 = (\Pi_{m_i \text{ even}} x_{m_i}^{b_i})z^d$ or $M_3 = x_{m_1}^{b_1} \cdots x_{m_t}^{b_t}$ with $m_t - m_1 \geq 3$. We apply either (2.3.72) or (2.3.68) or (2.3.69) to the product $x_{m_1}x_{m_t}$. By inspection, in the resulting expression for both M_1 and M_2, all the monomials except at most one that has smaller value of μ_1 have the same value of μ_1. By further inspection, for every such monomial M', if $\min(b_1, b_t) = 1$ (resp. $\min(b_1, b_t) \geq 2$) then $\mu_2(M') < \mu_2(M)$ (resp. $\mu_2(M') = \mu_2(M)$ and $\mu_3(M') = \mu_3(M) - 2$) . Analogously in the resulting expression for M_3, there is precisely one monomial M' with $\mu_1(M') = \mu_1(M)$, while the rest of the terms have smaller value of μ_1. Moreover if $\min(b_1, b_t) = 1$ (resp. $\min(b_1, b_t) \geq 2$) then $\mu_2(M') < \mu_2(M)$ (resp. $\mu_2(M') = \mu_2(M)$ and $\mu_3(M') = \mu_3(M) - 2$) .

2.3.11 Coefficient–free cluster algebra of type $A_2^{(1)}$

Let $\mathcal{F} = \mathbb{Q}(x_1, x_2, x_3)$ be the field of rational functions in three (commuting) independent variables x_1, x_2 and x_3 with rational coefficients. Recursively define elements

$\overline{x}_m \in \mathcal{F}$ by the initial conditions $\overline{x}_i = x_i$ for $i = 1, 2, 3$, together with the recurrence relations for $m \in \mathbb{Z}$:

$$\overline{x}_m \overline{x}_{m+3} = \overline{x}_{m+1} \overline{x}_{m+2} + 1. \tag{2.3.74}$$

Define also the elements $\overline{w}, \overline{z} \in \mathcal{F}$ by

$$\overline{w} = \frac{\overline{x}_1 + \overline{x}_3}{\overline{x}_2}, \tag{2.3.75}$$

$$\overline{z} = \frac{\overline{x}_1 \overline{x}_2 + \overline{x}_2 \overline{x}_3 + 1}{\overline{x}_1 \overline{x}_3}. \tag{2.3.76}$$

Let $\mathcal{A}_{\{1\}}$ be the $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by all the $\overline{x}_m$, $\overline{w}$ and $\overline{z}$. An easy induction shows that the following relations hold for every $m \in \mathbb{Z}$

$$\overline{w}\overline{x}_{2m} = \overline{x}_{2m-1} + \overline{x}_{2m+1}, \tag{2.3.77}$$
$$\overline{z}\overline{x}_{2m+1} = \overline{x}_{2m} + \overline{x}_{2m+2}. \tag{2.3.78}$$

Moreover from (2.3.74), (2.3.75) and (2.3.76) we obtain

$$\overline{x}_{2m-1}\overline{x}_{2m+3} = \overline{x}_{2m+1}^2 + \overline{w}, \tag{2.3.79}$$
$$\overline{x}_{2m-2}\overline{x}_{2m+2} = \overline{x}_{2m}^2 + \overline{z}. \tag{2.3.80}$$

for all $m \in \mathbb{Z}$. We refer to the generators of $\mathcal{A}_{\{1\}}$ as *cluster variables* and to the relations (2.3.74), (2.3.75), (2.3.76), (2.3.79) and (2.3.80) as *exchange relations*. The sets of the form

$$\{\overline{x}_m, \overline{x}_{m+1}, \overline{x}_{m+2}\}, \{\overline{x}_{2m+1}, \overline{w}, \overline{x}_{2m+3}\}, \{\overline{x}_{2m}, \overline{z}, \overline{x}_{2m+2}\}$$

for $m \in \mathbb{Z}$ are the *clusters* of $\mathcal{A}_{\{1\}}$. Note that clusters are algebraically independent sets. In particular $\mathcal{A}_{\{1\}} \subset \mathbb{Q}(\mathcal{C})$ for every cluster $\mathcal{C}$. We set $\mathcal{C}_{In} \doteq \{x_1, x_2, x_3\}$.

In [16, Example 7.8] it is shown that $\mathcal{A}_{\{1\}}$ is the *coefficient-free* cluster algebra with initial seed

$$\{B = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \{x_1, x_2, x_3\}\}, \tag{2.3.81}$$

and that the previous terminology is consistent with the theory of cluster algebras. Moreover the *exchange graph* of this cluster algebra is the two-layer brick wall shown in figure 2.1.

We now describe the canonical basis $\mathbf{B}$ of $\mathcal{A}_{\{1\}}$ explicitly. For every cluster $\mathcal{C} = \{s_1, s_2, s_3\}$ an element of the form $s_1^p s_2^q s_3^r$ for some $p, q, r \in \mathbb{Z}_{\geq 0}$ is called a *cluster monomial* (or more precisely a *cluster monomial in* $\mathcal{C}$ when we want to emphasize the cluster $\mathcal{C}$). We introduce an element $\overline{u} \in \mathcal{A}$ by setting

$$\overline{u} \doteq \overline{z}\overline{w} - 2. \tag{2.3.82}$$

Let $T_0, T_1, \dots$ be the sequence of Chebyshev polynomials of the first kind given by $T_{-n} = 0$, $T_0 = 1$, and $T_n(t + t^{-1}) = t^n + t^{-n}$ for $n > 0$. We define a sequence $\overline{u}_1, \overline{u}_2, \dots$ of elements of $\mathcal{A}$ by setting $\overline{u}_n = T_n(\overline{u})$, i.e.

$$\overline{u}_0 = 1, \quad \overline{u}_1 = \overline{u}, \quad \overline{u}_2 = \overline{u}_1^2 - 2 \tag{2.3.83}$$

together with the recurrence relation for $n \geq 2$

$$\overline{u}_{n+1} = \overline{u}_1 \overline{u}_n - \overline{u}_{n-1}. \tag{2.3.84}$$

It follows immediately from the definition that $\overline{u}\overline{x}_m = \overline{x}_{m-2} + \overline{x}_{m+2}$: indeed we can consider the automorphism t of $\mathcal{A}_{\{1\}}$ that sends $x_m \mapsto x_{m+2}$, so that $\overline{u} = t + t^{-1}$. Then by definition we get: $\overline{u}_n \overline{x}_m = (t^n + t^{-n})\overline{x}_m = \overline{x}_{m-2n} + \overline{x}_{m+2n}$.

Lemma 2.3.31. *The map $\varphi : \mathbb{P} \to \{1\}$ which sends every element of $\mathbb{P}$ onto 1, extends uniquely to a ring epimorphism $\varphi : \mathcal{A} \to \mathcal{A}_{\{1\}}$ that is a coefficient specialization (defined in Section (1.1.7)), i.e. $\varphi(s) = \overline{s}$ for every cluster variable s of $\mathcal{A}$. Moreover φ restrict to a bijection between $\mathbf{B}$ and $\overline{\mathbf{B}}$ such that $\varphi(b) = \overline{b}$ for every $b \in \mathbf{B}$.*

Proof. The exchange relations (2.3.74) are obtained from the exchange relations (2.1.2) by specializing every element of $\mathbb{P}$ to $\{1\}$. It implies that $\varphi(x_m) = \overline{x}_m$. Moreover the elements w and z of $\mathcal{A}$ defined in (2.1.3) and (2.1.4) respectively are mapped by φ onto the elements $\overline{w}$ and $\overline{z}$ of $\mathcal{A}_{\{1\}}$.

The fact that $\varphi(u_n) = \overline{u}_n$ follows by observing that the defining relations for the elements $\{u_n\}$, specialize to the defining relations for the elements $\{\overline{u}_n\}$ when $\mathbb{P}$ is mapped onto $\{1\}$. $\square$

Proposition 2.3.32. *The set $\overline{\mathbf{B}} = \{$cluster monomials$\} \cup \{\overline{u}_n \overline{w}^k, \overline{u}_n \overline{z}^k \,|\, n \geq 1,\, k \geq 0\}$ is a $\mathbb{Z}$–basis of $\mathcal{A}$ whose elements are positive.*

Proof. Since $\overline{B}$ is the image by φ of the $\mathbb{ZP}$–basis $\mathbf{B}$ of $\mathcal{A}$, and φ is surjective, it follows that $\overline{\mathbf{B}}$ spans $\mathcal{A}_{\{1\}}$ over $\mathbb{Z}$. Moreover since $\mathbf{B}$ satisfies hypotheses of Theorem 1.5.7, the same is true for $\overline{\mathbf{B}}$ and hence $\overline{\mathbf{B}}$ is a linearly independent set. The elements of $\overline{\mathbf{B}}$ are positive since they are image by φ of positive elements. $\square$

Newton polygons in the cluster $\{x_1, x_2, x_3\}$

In this section we study Newton polytopes in the initial cluster $\mathcal{C}_{In} = \{x_1, x_2, x_3\}$ of the elements of the basis $\mathbf{B}$ of the coefficient–free cluster algebra $\mathcal{A}_{\{1\}}$ of type $A_2^{(1)}$ introduced in Section 2.3.11. Recall that the Newton polytope $\mathrm{Newt}_{\mathcal{C}}(x)$ of a Laurent polynomial $x \in \mathbb{Z}[s_1^{\pm 1}, s_2^{\pm 1}, s_3^{\pm 1}]$ with respect to the ordered set $\mathcal{C} = \{s_1, s_2, s_3\}$ is the convex hull in $Q_{\mathbb{R}} = \mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_2 \oplus \mathbb{R}\alpha_3$ of all lattice points $g = (g_1, g_2, g_3)$ such that the monomial $s^g := s_1^{g_1} s_2^{g_2} s_3^{g_3}$ appears with a non-zero coefficient in x. We say that a vertex γ of $Newt_{\mathcal{C}}(x)$ is *monic* if the corresponding monomial $\mathbf{s}^\gamma$ has coefficient 1 in the expansion of x in $\mathcal{C}$. With some abuse of language we say that an element x of $\mathcal{A}$ is *monic* in the cluster $\mathcal{C}$ if all the vertices of $\mathrm{Newt}_{\mathcal{C}}(x)$ are monic.

Example 2.3.33. By the definitions 2.3.75 and 2.3.76

$$\mathrm{Newt}_{\{x_1,x_2,x_3\}}(\overline{w}) = \mathrm{Conv}\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\} \tag{2.3.85}$$

$$\mathrm{Newt}_{\{x_1,x_2,x_3\}}(\overline{z}) = \mathrm{Conv}\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}\} \tag{2.3.86}$$

where Conv means convex hull in $Q_{\mathbb{R}} = \mathbb{R}^3$. In particular $\overline{w}$ and $\overline{z}$ are monic in the initial cluster $\{x_1, x_2, x_3\}$ and their Newton polytopes are polygons contained respectively in the plane $P_2 = \{(g_1, g_2, g_3) : g_1 - g_2 + g_3 = 2\}$ and $P_{-2} = \{(g_1, g_2, g_3) : g_1 - g_2 + g_3 = -2\}$.

By the symmetries in the exchange relations (2.3.74), the Laurent expansion of the cluster variable x_{-m} is obtained from the Laurent expansion of the cluster variable x_{m+4} by exchanging the variable x_1 and the variable x_3. In particular the corresponding Newton polytopes are related to each other by:

$$Newt_{\{x_1,x_2,x_3\}}(\overline{x}_{-m}) = (13)Newt_{\{x_1,x_2,x_3\}}(\overline{x}_{m+4}). \tag{2.3.87}$$

where (13) is the automorphism of $Q = \mathbb{Z}^3$ that exchanges the first coordinate with the third one. Th efollowing Proposition gives the explicit description of Newton polytopes in the initial cluster $\mathcal{C}_{In} = \{x_1, x_2, x_3\}$ of the elements of $\overline{\mathbf{B}}$.

Proposition 2.3.34. *The elements $\{\overline{x}_m : m \in \mathbb{Z}\}$ and $\{\overline{u}_n : n \geq 1\}$ of $\mathcal{A}_{\{1\}}$ are monic in the initial cluster $\{x_1, x_2, x_3\}$. Moreover, for $m \geq 2$ and $n \geq 1$, the following explicit formulas hold:*

$$\mathrm{Newt}_{\{x_1,x_2,x_3\}}(\overline{x}_{2m+1}) = Conv\{\begin{pmatrix} 1-m \\ 0 \\ m \end{pmatrix}, \begin{pmatrix} 1-m \\ 1-m \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1-m \\ 2-m \end{pmatrix}, \begin{pmatrix} m-2 \\ -1 \\ 2-m \end{pmatrix}\} \tag{2.3.88}$$

$$\mathrm{Newt}_{\{x_1,x_2,x_3\}}(\overline{x}_{2m}) = Conv\{\begin{pmatrix} 1-m \\ 1 \\ m-1 \end{pmatrix}, \begin{pmatrix} 1-m \\ 2-m \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2-m \\ 2-m \end{pmatrix}, \begin{pmatrix} m-3 \\ 0 \\ 2-m \end{pmatrix}\} \tag{2.3.89}$$

$$\mathrm{Newt}_{\{x_1,x_2,x_3\}}(\overline{u}_n) = Conv\{\begin{pmatrix} -n \\ 0 \\ n \end{pmatrix}, \begin{pmatrix} -n \\ -n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -n \\ -n \end{pmatrix}, \begin{pmatrix} n \\ 0 \\ -n \end{pmatrix}\} \tag{2.3.90}$$

Proof. In Theorem 2.1.9, formula (2.1.24), we have found the Laurent expansion of $\overline{x}_{2m+1}$ in the cluster $\mathcal{C}_{In} = \{x_1, x_2, x_3\}$. By specializing all the coefficients to 1, we get the following formula (written in the form predicted in Proposition 2.3.1):

$$\overline{x}_{2m+1} = \mathbf{x}^{\begin{pmatrix} 1-m \\ 0 \\ m \end{pmatrix}} \left[\sum_{\mathbf{e}} \binom{e_1-e_3}{e_2-e_3}\binom{m-1-e_3}{m-1-e_1}\binom{e_1-1}{e_3} x_1^{e_2+e_3} x_2^{e_3-e_1} x_3^{-e_2-e_1} + 1\right]$$

We deduce that the Newton polytope $Newt_{\mathcal{C}_{In}}(x_{2m+1})$ of x_{2m+1} in the cluster $\mathcal{C}_{In}$ is the convex hull of the set $\{\begin{pmatrix} e_2+e_3+1-m \\ e_3-e_1 \\ -e_2-e_1+m \end{pmatrix} | (e_1, e_2, e_3)^t \in E\}$ where $E = \{\mathbf{e} = (e_1, e_2, e_3)^t | \ 0 \leq e_3 \leq e_1 - 1; \ e_3 \leq e_2 \leq e_1; \ 1 \leq e_1 \leq m-1\} \cup \{(0,0,0)^t\}$. We consider the affine map

$$f : \mathbb{A}^3 \to \mathbb{A}^3 : \ \mathbf{e} \mapsto B\mathbf{e} + \begin{pmatrix} 1-m \\ 0 \\ m \end{pmatrix}.$$

where B is the initial exchange matrix given in (2.3.81). The map f sends convex sets in convex sets and $Newt_{\mathcal{C}_{In}}(x_{2m+1}) = f(Conv(E))$ where $Conv(E)$ is the smallest convex set containing E. It is easy to see that the convex set $Conv(E)$ is the convex hull of the following points of the affine space:

$$Conv(E) = Conv\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} m-1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} m-1 \\ m-1 \\ 0 \end{pmatrix}, \begin{pmatrix} m-1 \\ m-1 \\ m-2 \end{pmatrix}\}$$

We hence apply the map f to every generator of $Conv(E)$ and we find (2.3.88). The fact that $\overline{x}_{2m+1}$ is monic follows directly by (2.3.88) and by the explicit formula.

The proof of (2.3.89) is quite similar. In Theorem 2.1.9, formula (2.1.25), we have found the Laurent expansion of $\overline{x}_{2m}$ in the cluster $\mathcal{C}_{In} = \{x_1, x_2, x_3\}$. We then specialize all the coefficients to 1 in that formula and we get:

$$\overline{x}_{2m} = \mathbf{x}^{\begin{pmatrix} 1-m \\ 1 \\ m-1 \end{pmatrix}} \Big\{ \sum_{\mathbf{e}} \binom{e_1-1}{e_3} [\binom{e_1-e_3}{e_2-e_3} \binom{m-2-e_3}{m-2-e_1} + \binom{e_1-e_3-1}{e_2-e_3} \binom{m-2-e_3}{m-1-e_1}] x_1^{e_2+e_3} x_2^{e_3-e_1} x_3^{-e_2-e_1} + 1 \Big\}$$

We deduce that the Newton polytope $Newt_{\mathcal{C}_{In}}(\overline{x}_{2m+2})$ of $\overline{x}_{2m+2}$ in the cluster $\mathcal{C}_{In}$ is the convex hull of the set $\{\begin{pmatrix} e_2+e_3+1-m \\ e_3-e_1+1 \\ m-e_2-e_1 \end{pmatrix} | (e_1, e_2, e_3)^t \in E\}$ where $E = E_1 \cup E_2 \cup (0,0,0)^t$ where $E_1 = \{\mathbf{e} = (e_1, e_2, e_3)^t | \ 0 \leq e_3 \leq e_1 - 1; \ e_3 \leq e_2 \leq e_1; \ 1 \leq e_1 \leq m-2\}$ and $E_2 = \{\mathbf{e} = (e_1, e_2, e_3)^t | \ 0 \leq e_3 \leq e_1 - 1; \ e_3 \leq e_2 \leq e_1 - 1; \ 1 \leq e_1 \leq m-1\}$. We consider the affine map

$$f : \mathbb{A}^3 \to \mathbb{A}^3 : \ \mathbf{e} \mapsto B\mathbf{e} + \begin{pmatrix} 1-m \\ 1 \\ m-1 \end{pmatrix}.$$

where B is the initial exchange matrix. The map f sends convex sets in convex sets and $Newt_{\mathcal{C}_{In}}(\overline{x}_{2m+1}) = f(Conv(E))$. If $m = 2$, $E = \{(0,0,0)^t, (1,0,0)^t\}$ and hence

$$Newt_{\mathcal{C}_{In}}(\overline{x}_4) = \ Conv\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\}$$

which is (2.3.89) for $m = 2$.

We hence assume $m \geq 3$. In this case it is easy to see that $Conv(E)$ is the convex hull of the following points of the affine space:

$$Conv(E) = Conv\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} m-1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} m-2 \\ m-2 \\ 0 \end{pmatrix}, \begin{pmatrix} m-1 \\ m-2 \\ 0 \end{pmatrix}, \begin{pmatrix} m-1 \\ m-2 \\ m-2 \end{pmatrix}, \begin{pmatrix} m-2 \\ m-2 \\ m-3 \end{pmatrix}\}$$

We hence apply the map f and we get

$$Newt_{\mathcal{C}_{In}}(\overline{x}_{2m}) = \ Conv\{\begin{pmatrix} 1-m \\ 1 \\ m-1 \end{pmatrix}, \begin{pmatrix} 1-m \\ 2-m \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3-m \\ 3-m \end{pmatrix}, \begin{pmatrix} -1 \\ 2-m \\ 2-m \end{pmatrix}, \begin{pmatrix} m-3 \\ 0 \\ 2-m \end{pmatrix}, \begin{pmatrix} m-4 \\ 0 \\ 3-m \end{pmatrix}\}$$

In order to get (2.3.89) we need to show that the third and the last generator of $Newt_{\mathcal{C}_{In}}(\overline{x}_{2m})$ are convex combinations of the others. This can be done by direct check. The fact that $\overline{x}_{2m}$ is monic follows directly by (2.3.89) and by the explicit formula.

In Theorem 2.1.9, formula (2.1.28), we have found the Laurent expansion of the element $\overline{u}_n$, $n \geq 1$, in the cluster $\mathcal{C}_{In} = \{x_1, x_2, x_3\}$. We then specialize all the coefficients to 1 and we get the following formula (written in the spirit of Proposition 2.3.1):

$$\overline{u}_n = \mathbf{x}^{\begin{pmatrix} -n \\ 0 \\ n \end{pmatrix}} \Big[x_1^{2n} x_3^{-2n} + \sum_{\mathbf{e}} \binom{e_1-e_3}{e_1-e_2} [\binom{n-e_3}{n-e_1} \binom{e_1-1}{e_3} + \\ + \binom{n-e_3-1}{n-e_1} \binom{e_1-1}{e_3-1}] x_1^{e_2+e_3} x_2^{e_3-e_1} x_3^{-e_2-e_1} + 1 \Big]$$

We deduce that the Newton polytope $Newt_{\mathcal{C}_{In}}(\overline{u}_n)$ of $\overline{u}_n$ in the cluster $\mathcal{C}_{In}$ is the convex hull of the set $\{\begin{pmatrix} e_2+e_3-n \\ e_3-e_1 \\ -e_2-e_1+n \end{pmatrix} | (e_1,e_2,e_3)^t \in E\}$ where $E = \{(e_1,e_2,e_3)^t \in \mathbb{Z}^3 |\ 0 \le e_3 \le e_1-1;\ 0 \le e_2 \le e_1;\ 1 \le e_1 \le n\} \cup \{(0,0,0)^t\} \cup \{(n,n,n)^t\}$. We consider the affine map

$$f : \mathbb{A}^3 \to \mathbb{A}^3 :\ \mathbf{e} \mapsto B\mathbf{e} + \begin{pmatrix} -n \\ 0 \\ n \end{pmatrix}.$$

where B is the initial exchange matrix. The map f sends convex sets in convex sets and $Newt_{\mathcal{C}_{In}}(x_{2m+1}) = f(Conv(E))$. $Conv(E)$ is the convex hull of the following points of the affine space:

$$Conv(E) = Conv\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ n \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ n \\ n-1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ n \\ n \end{pmatrix}\} =$$
$$Conv\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ n \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ n \\ n-1 \end{pmatrix}, \begin{pmatrix} n \\ n \\ n \end{pmatrix}\}.$$

We hence apply the map f to every generator of $Conv(E)$ and we find:

$$Conv\{\begin{pmatrix} -n \\ 0 \\ n \end{pmatrix}, \begin{pmatrix} -n \\ -n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -n \\ -n \end{pmatrix}, \begin{pmatrix} n-1 \\ -1 \\ -n \end{pmatrix}, \begin{pmatrix} n \\ 0 \\ -n \end{pmatrix}\}.$$

Since $\begin{pmatrix} n-1 \\ -1 \\ -n \end{pmatrix} = \frac{1}{n}\begin{pmatrix} 0 \\ -n \\ -n \end{pmatrix} + \frac{n-1}{n}\begin{pmatrix} n \\ 0 \\ -n \end{pmatrix}$ we find the desired (2.3.90). The fact that $\overline{u}_n$ is monic follows directly by (2.3.90) and by the explicit formula. □

Remark 2.3.35. We note that Newton polytopes of the elements of cluster variables are actually *polygons*. This is equivalent to the fact that the algebra $\mathcal{A}$ is graded. Indeed if we choose the grade $g(x_1) = g(x_3) = 1$, $g(y) = 0$ for every $y \in \mathbb{P}$ and $g(x_2) = -1$ from the exchange relations it follows that

$$g(w) = 2;\quad g(x_{2m+1}) = 1;\quad g(u_n) = 0;\quad g(x_{2m}) = -1;\quad g(z) = -2. \tag{2.3.91}$$

In view of that Newton polygons are contained in affine planes: more explicitly let $P_i = \{(e_1,e_2,e_3) \in Q|\ e_1 - e_2 + e_3 = i\}$. Then for every $m \in \mathbb{Z}$, we have $Newt_{\mathcal{C}_{In}}(x_{2m+1}) \subset P_1$, $Newt_{\mathcal{C}_{In}}(u_n) \subset P_0$ and $Newt_{\mathcal{C}_{In}}(x_{2m}) \subset P_{-1}$. Figure 2.8 represents the polygons Π_m and Υ_n in the subspace P_0 such that $Newt_{\mathcal{C}_{In}}(x_m) = \Pi_m + (i,0,0)$, where $i=-1$ if m is odd and $i=1$ if m is even, $\Upsilon_n = Newt_{\mathcal{C}_{In}}(u_n)$.

The following result is a corollary of Proposition 2.3.34.

Proposition 2.3.36. *a) If b is a cluster monomial containing at least one cluster variable different from x_1, x_2 and x_3, then there exists a non-zero linear form on $Q_{\mathbb{R}}$*

$$\varphi_b(g_1,g_2,g_3) = \alpha_b g_1 + \beta_b g_2 + \gamma_b g_3, \qquad \alpha_b, \beta_b, \gamma_b \ge 0$$

such that $\mathrm{Newt}_{\{x_1,x_2,x_3\}}(b) \subset \{\varphi_b < 0\}$. *In particular* $\mathrm{Newt}_{\{x_1,x_2,x_3\}}(b)$ *has empty intersection with the positive cone $Q_+ = \mathbb{Z}_{\ge 0}\alpha_1 + \mathbb{Z}_{\ge 0}\alpha_2 + \mathbb{Z}_{\ge 0}\alpha_3$. Table 2.1 shows the linear form φ_b for every choice of the cluster monomial b.*

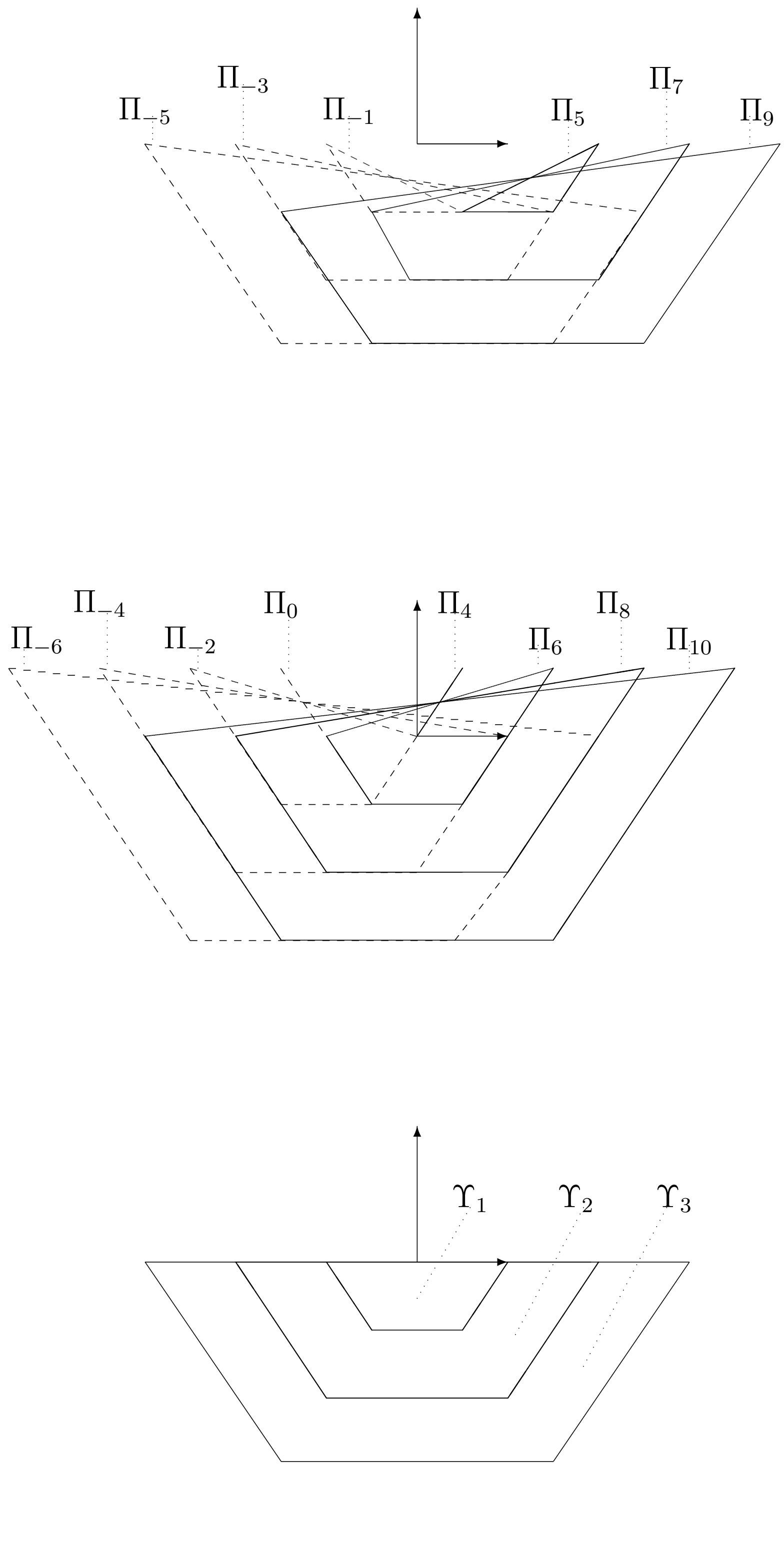

Figure 2.8: Polygons in the subspace $P_0 = \{e_1 - e - 2 + e_3 = 0\}$ corresponding to Newton polygons of the elements of $\mathbf{B}$ by, from above to below: $\mathrm{Newt}_{\{x,x_2,x_3\}}(x_{2m+1}) = (1,0,0)^t + \Pi_{2m+1}$, $Newt_{\{x_1,x_2,x_3\}}(x_{2m}) = (-1,0,0)^t + \Pi_{2m}$, $\mathrm{Newt}_{\{x_1,x_2,x_3\}}(u_n) = \Upsilon_n$. The orthogonal vectors in each figure are the basis $v_1 = (-1,0,1)^t$, $v_2 = (1,2,1)^t$ of P_0.

b) For $k \geq 0$ and $n \geq 1$

$$\mathrm{Newt}_{\{x_1,x_2,x_3\}}(u_n w^k) \subset \{g_1 + 2g_2 + g_3 \leq 0\}$$

c) For $k > 0$ and $n \geq 1$

$$\mathrm{Newt}_{\{x_1,x_2,x_3\}}(u_n z^k) \subset \{2g_1 + g_2 + 2g_3 \leq 0\}$$

.

In particular the monomial $x_1^a x_2^b x_3^c$, for every non-negative integers a, b and c, doesn't appear in the Laurent expansion of any other element of $\mathbf{B}$ with respect to the initial cluster $\{x_1, x_2, x_3\}$.

Proof. Since $\mathrm{Newt}(s_1^p s_2^q s_3^r) = p\mathrm{Newt}(s_1) + q\mathrm{Newt}(s_2) + r\mathrm{Newt}(s_3)$, it is sufficient to find a linear form assuming negative values on the vertices of $\mathrm{Newt}(s_1)$, $\mathrm{Newt}(s_2)$ and $\mathrm{Newt}(s_3)$. Using the previous formulas we are going to give an explicit solution of the corresponding systems of linear inequalities. Let's prove a). We'll distinguish the different kinds of cluster monomials.

- $b = x_{2m+1}^p x_{2m+2}^q x_{2m+3}^r$. We put $\alpha_{2m+1} := \alpha_b$, $\beta_{2m+1} := \beta_b$ and $\gamma_{2m+1} := \gamma_b$. For $m \geq 2$, using (2.3.89) and (2.3.88), it is sufficient to solve the following system of linear inequalities:

$$\begin{cases}
\alpha(1-m) + \gamma(m) < 0 \\
\alpha(1-m) + \beta(1-m) + \gamma < 0 \\
\beta(1-m) + \gamma(2-m) < 0 \\
\alpha(m-2) + \beta(-1) + \gamma(2-m) < 0 \\
\alpha(-m) + \beta(1) + \gamma(m) < 0 \\
\alpha(-m) + \beta(1-m) < 0 \\
\alpha(-1) + \beta(1-m) + \gamma(1-m) < 0 \\
\alpha(m-2) + \gamma(1-m) < 0 \\
\alpha(-m) + \gamma(m+1) < 0 \\
\alpha(-m) + \beta(-m) + \gamma < 0 \\
\beta(-m) + \gamma(1-m) < 0 \\
\alpha(m-1) + \beta(-1) + \gamma(1-m) < 0 \\
\alpha > 0 \\
\beta > 0 \\
\gamma > 0
\end{cases}$$

  it is equivalent to the following one

$$\begin{cases}
\frac{m-2}{m-1}\alpha < \gamma < \frac{m-1}{m}\alpha \\
(m-1)(\alpha-\gamma) < \beta < m(\alpha-\gamma) \\
\alpha > 0 \\
\beta > 0 \\
\gamma > 0
\end{cases}$$

An explicit solution could be the following:

$$\alpha_{2m+1} = m(m-1), \quad \beta_{2m+1} = m(m-1), \quad \gamma_{2m+1} = m^2 - 2m + 1/2. \tag{2.3.92}$$

In the same way for $m = 1$ we obtain

$$\alpha_3 = 1, \quad \beta_3 = \begin{cases} 0 & \text{if } q \neq 0 \\ 1 & \text{if } q = 0 \end{cases}, \quad \gamma_3 = 0.$$

Using the symmetries of $\mathcal{A}$ we have for $m \geq 0$

$$\alpha_{-(2m+1)} = \gamma_{2(m+1)+1}, \quad \beta_{-(2m+1)} = \beta_{2(m+1)+1}, \quad \gamma_{-(2m+1)} = \alpha_{2(m+1)+1}. \tag{2.3.93}$$

- $b = x_{2m}^p x_{2m+1}^q x_{2m+2}^r$. We put $\alpha_{2m} := \alpha_b$, $\beta_{2m} := \beta_b$ and $\gamma_{2m} := \gamma_b$. If $m \geq 4$ the corresponding system is equivalent to

$$\begin{cases} (m-2)(\alpha - \gamma) < \beta < (m-1)(\alpha - \gamma) \\ \frac{m-2}{m-1}\alpha < \gamma < \frac{m-1}{m}\alpha \end{cases}$$

An esplicit solution could be

$$\alpha_{2m} = m(m-1), \, \beta_{2m} = (m-1/2)(m-3/2), \gamma_{2m} = m^2 - 2m + 1/2. \tag{2.3.94}$$

In the other "positive" cases it is easy to verify that the following are solutions:

$$\begin{array}{lll} \alpha_2 = 1, & \beta_2 = 0, & \gamma_2 = 0. \\ \alpha_4 = 6, & \beta_4 = 3, & \gamma_4 = 2. \\ \alpha_6 = 9, & \beta_6 = 5, & \gamma_6 = 5. \end{array}$$

As before we have

$$\alpha_{-(2m)} = \gamma_{2(m+1)}, \quad \beta_{-(2m)} = \beta_{2(m+1)}, \quad \gamma_{-(2m)} = \alpha_{2(m+1)}. \tag{2.3.95}$$

- $b = x_{2m+1}^p w^q x_{2m+3}^r$. In this case we put $\alpha_{2m+1}^w := \alpha_b$, $\beta_{2m+1}^w := \beta_b$ and $\gamma_{2m+1}^w := \gamma_b$. For $m \geq 3$ the corresponding system becomes

$$\begin{cases} \gamma < \frac{m-1}{m}\alpha \\ \beta > (m-1)(\alpha - \gamma) \\ \beta > \alpha \end{cases}$$

A solution could be the following:

$$\alpha_{2m+1}^w = m, \quad \beta_{2m+1}^w = 2m, \quad \gamma_{2m+1}^w = m - 2. \tag{2.3.96}$$

In the same way we have

$$\begin{array}{lll} \alpha_1^w = 0, & \beta_1^w = 1, & \gamma_1^w = 0. \\ \alpha_3^w = \{ \begin{array}{ll} 1 & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{array}, & \beta_3^w = 1, & \gamma_3^w = 0. \\ \alpha_5^w = 4, & \beta_5^w = 5, & \gamma_5^w = 1. \end{array}$$

If $m \geq 0$

$$\alpha^w_{-(2m+1)} = \gamma^w_{2(m+1)+1}, \quad \beta^w_{-(2m+1)} = \beta^w_{2(m+1)+1}, \quad \gamma^w_{-(2m+1)} = \alpha^w_{2(m+1)+1}. \tag{2.3.97}$$

- $b = x^p_{2m} z^q x^r_{2m+2}$. In this case we put $\alpha^z_{2m} := \alpha_b$, $\beta^z_{2m} := \beta_b$ and $\gamma^z_{2m} := \gamma_b$. The corresponding system for $m \geq 4$ is equivalent to

$$\begin{cases} \frac{m-2}{m-1}\alpha < \gamma < \alpha \\ \beta < \gamma \\ \beta < (m-1)(\alpha - \gamma) \end{cases}$$

A solution could be

$$\alpha^z_{2m} = m(m-1), \quad \beta^z_{2m} = \frac{m}{4}(m-1), \quad \gamma^z_{2m} = m(m - \tfrac{3}{2}). \tag{2.3.98}$$

For the other "positive" cases an easily to verify solution could be

$$\begin{array}{lll} \alpha^z_2 = 1, & \beta^z_2 = 0 & \gamma^z_2 = \begin{cases} 0 & \text{if } r \neq 0 \\ 1 & \text{if } r = 0 \end{cases}, \\ \alpha^z_4 = 4, & \beta^z_4 = 1, & \gamma^z_4 = 2. \\ \alpha^z_6 = 3, & \beta^z_6 = 1, & \gamma^z_6 = 2. \end{array}$$

For the other cases

$$\alpha_{-(2m)} = \gamma_{2(m+1)}, \quad \beta_{-(2m)} = \beta_{2(m+1)}, \quad \gamma_{-(2m)} = \alpha_{2(m+1)}. \tag{2.3.99}$$

From (2.3.90), (2.3.85) and (2.3.86) and from the lemma 2.3.38 below, b) and c) follow by direct check. Then the only monomial that could appear in the Laurent expansion of $u_n w^k$ or in $u_n z^k$ is 1. For $k > 0$ the conclusion in these two cases follows observing that $g(u_n w^k) = 2k$ and $g(u_n z^k) = -2k$, so 1 cannot appear in their Laurent expansion. For $k = 0$ we observe that by definition $u_n x_3 = x_{3-2n} + x_{3+2n}$. So 1 appears in the Laurent expansion of u_n if and only if x_3 appears in the Laurent expansion of x_{3-2n} and of x_{2n+3}. By part a) this cannot be the case since their Newton polygons do not intersect the positive octant Q_+ (in particular they cannot contain the point $(0,0,1)$). $\square$

Lemma 2.3.37.

$$Conv\{p_i : i = 1, \cdots, n\} + Conv\{q_j : j = 1, \cdots, m\} = Conv\{p_i + q_j : i, j\}$$

Proof. Let $x = \sum \lambda_i p_i$ and $y = \sum \mu_j q_j$ with $\sum \lambda_i = \sum \mu_j = 1$ and $\lambda_i, \mu_j \geq 0$. Then

$$x + y = \sum_i \lambda_i p_i + \sum_j \mu_j q_j = \sum_i (\sum_j \mu_j) \lambda_i p_i + \sum_j (\sum_i \lambda_i) \mu_j q_j = \sum_{i,j} \mu_j \lambda_i (p_i + q_j)$$

and $\sum_{i,j} \mu_j \lambda_i = \lambda_1(\sum \mu_j) + \cdots + \lambda_n(\sum \mu_j) = 1$. On the other hand let $x = \sum \lambda_{ij}(p_i + q_j)$ with $\sum_{i,j} \lambda_{ij} = 1$, $\lambda_{ij} \geq 0$. Then

$$x = \sum_{i,j} (\lambda_{ij} p_i + \lambda_{ij} q_j) = \sum_i (\sum_j \lambda_{ij}) p_i + \sum_j (\sum_i \lambda_{ij}) q_j$$

$\square$

b	$\varphi_b(g_1, g_2, g_3)$	
$x_{2m+1}^p x_{2m+2}^q x_{2m+3}^r$	$m(m-1)g_1 + m(m-1)g_2 + (m^2 - 2m + \frac{1}{2})g_3$	$m \geq 2$
$x_3^p x_5^r$	g_1	
$x_3^p x_4^q x_5^r$	$g_1 + g_2$	$q > 0$
$x_{2m}^p x_{2m+1}^q x_{2m+2}^q$	$m(m-1)g_1 + (m - \frac{1}{2})(m - \frac{3}{2})g_2 + (m^2 - 2m + \frac{1}{2})g_3$	$m \geq 4$
$x_2^p x_3^q x_4^q$	g_1	
$x_4^p x_5^q x_6^q$	$6g_1 + 3g_2 + 2g_3$	
$x_6^p x_7^q x_8^q$	$9g_1 + 5g_2 + 5g_3$	
$x_{2m+1}^p w^q x_{2m+3}^r$	$mg_1 + 2mg_2 + (m-2)g_3$	$m \geq 3$
$x_1^p w^q x_3^r$	g_2	
$x_3^p x_5^r$	$g_1 + g_2$	
$x_3^p w^q x_5^r$	g_2	$q > 0$
$x_5^p w^q x_7^r$	$4g_1 + 5g_2 + g_3$	$q > 0$
$x_{2m}^p z^q x_{2m+2}^r$	$m(m-1)g_1 + \frac{m}{4}(m-1)g_2 + m(m - \frac{3}{2})g_3$	$m \geq 4$
$x_2^p z^q x_4^r$	g_1	$r > 0$
$x_2^p z^q$	$g_1 + g_3$	
$x_4^p z^q x_6^r$	$4g_1 + g_1 + 2g_3$	
$x_6^p z^q x_8^r$	$3g_1 + g_1 + 2g_3$	
$u_n w^k$	$g_1 + g_2 + g_3$	$n \geq 1,\ k \geq 0$
$u_n z^k$	$g_1 + g_2 + g_3$	$n \geq 1,\ k \geq 0$

Table 2.1: Every Laurent monomial y appearing in the Laurent expansion of b in the cluster $\mathcal{C}_{In}$ must satisfy the equation $\varphi_b(y) \leq 0$. The linear form of the other elements of $\mathbf{B}$ (involving cluster variables x_{-m}, $m \geq 0$) are obtained from these by (2.3.93), (2.3.95), (2.3.97) and (2.3.99).

Lemma 2.3.38. *Let α and β two non-zero positive scalars. Then*

$$Conv\{\alpha p_i : i = 1, \cdots, n\} + Conv\{\beta p_i : i = 1, \cdots, n\} = Conv\{(\alpha+\beta)p_i : i = 1, \cdots, n\}$$

In particular $k Conv\{p_i : i = 1, \cdots, n\} = Conv\{kp_i : i = 1, \cdots, n\}$ for all positive integer k.

Proof. $\mathrm{Conv}\{\alpha p_i\} + \mathrm{Conv}\{\beta p_i\} = \mathrm{Conv}\{\alpha p_i + \beta p_j\}$ by Lemma 2.3.37. In particular $\mathrm{Conv}\{(\alpha + \beta)p_i\} \subseteq \mathrm{Conv}\{\alpha p_i\} + \mathrm{Conv}\{\beta p_i\}$. On the other hand we first note that

$$\alpha p_i + \beta p_j = \frac{\alpha}{\alpha+\beta}[(\alpha+\beta)p_i] + \frac{\beta}{\alpha+\beta}[(\alpha+\beta)p_j] \in \mathrm{Conv}\{(\alpha+\beta)p_i\}.$$

Then

$$\begin{aligned}\sum_i \lambda_i(\alpha p_i) + \sum_j \mu_j(\beta p_j) &= \sum_i(\sum_j \mu_j)\lambda_i(\alpha p_i) + \sum_j(\sum_i \lambda_i)\mu_j(\beta p_j) \\ &= \sum_{i,j} \lambda_i\mu_j(\alpha p_i + \beta p_j) \in \mathrm{Conv}\{(\alpha+\beta)p_i\}\end{aligned}$$

since $\sum_{i,j} \lambda_i\mu_j = 1$ and $\lambda_i\mu_j > 0$ $\square$

Newton polygons in the cluster $\{x_1, w, x_3\}$

In this section we find explicit formulas for the Newton polytopes of the elements of $\overline{\mathbf{B}}$ in the cluster $\{x_1, \overline{w}, x_3\}$. In order to simplify notations we put $w \doteq \overline{w}$.

Proposition 2.3.39. *For $m \geq 2$ and $n \geq 1$ the following explicit formulas hold:*

$$\mathrm{Newt}_{\{x_1,w,x_3\}}(\overline{x}_{2m+1}) = Conv\{\begin{pmatrix} 1-m \\ 0 \\ m \end{pmatrix}, \begin{pmatrix} m-3 \\ 1 \\ 2-m \end{pmatrix}, \begin{pmatrix} 1-m \\ m-1 \\ 2-m \end{pmatrix}\} \tag{2.3.100}$$

$$\mathrm{Newt}_{\{x_1,w,x_3\}}(\overline{x}_{2m}) = Conv\{\begin{pmatrix} 2-m \\ -1 \\ m-1 \end{pmatrix}, \begin{pmatrix} m-3 \\ 0 \\ 2-m \end{pmatrix}, \begin{pmatrix} 1-m \\ -1 \\ m \end{pmatrix}, \begin{pmatrix} 1-m \\ m-2 \\ 2-m \end{pmatrix}\} \tag{2.3.101}$$

$$\mathrm{Newt}_{\{x_1,w,x_3\}}(\overline{u}_n) = Conv\{\begin{pmatrix} n \\ 0 \\ -n \end{pmatrix}, \begin{pmatrix} -n \\ 0 \\ n \end{pmatrix}, \begin{pmatrix} -n \\ n \\ -n \end{pmatrix}\} \tag{2.3.102}$$

Moreover u_n is monic in the cluster $\{x_1, w, x_3\}$.

Proof. By Theorem2.1.9, formula 2.1.26, we have the explicit expression of x_{2m+1} in the cluster $\{x_1, w, x_3\}$. We then specialize all the coefficients to 1 and we get the following formula (written in the spirit of Proposition 2.3.1)

$$\overline{x}_{2m+1} = \quad (x_1, w, x_3)^{\begin{pmatrix} 1-m \\ 0 \\ m \end{pmatrix}} \left[\sum_{e_1,e_3} \binom{m-1-e_3}{m-1-e_1}\binom{e_1-1}{e_3} x_1^{2e_3} w^{e_1-e_3} x_3^{-2e_1} + 1\right]$$

We deduce that the Newton polytope $Newt_{\{x_1,w,x_3\}}(x_{2m+1})$ of x_{2m+1} in the cluster $\mathcal{C}_{Cyclic}$ is the convex hull of the set $\{\begin{pmatrix} 2e_3+1-m \\ e_1-e_3 \\ -2e_1+m \end{pmatrix} | (e_1, e_2, e_3)^t \in E\}$ where $E = \{(e_1, 0, e_3)^t \in \mathbb{Z}^3 | \ 0 \leq e_3 \leq e_1 - 1; \ 1 \leq e_1 \leq m-1\} \cup \{(0, 0, 0)^t\}$. We consider the convex hull $Conv(E)$ of E, i.e. the smallest convex set containing E. It is the convex hull of the following points of the affine space:

$$Conv(E) = Conv\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} m-1 \\ 0 \\ m-2 \end{pmatrix}, \begin{pmatrix} m-1 \\ 0 \\ 0 \end{pmatrix}\}$$

Let us consider the affine map

$$f : \mathbb{A}^3 \to \mathbb{A}^3 : \ \mathbf{e} \mapsto B^{Cyclic}\mathbf{e} + \begin{pmatrix} 1-m \\ 0 \\ m \end{pmatrix} = \begin{pmatrix} 2e_3 - e_2 \\ e_1 - e_3 \\ -2e_1 + e_2 \end{pmatrix} + \begin{pmatrix} 1-m \\ 0 \\ m \end{pmatrix}.$$

where $B^{cyclic} \doteq \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$ is the exchange matrix of the seed containing the cluster $\{x_1, \overline{w}, x_3\}$. The map f sends convex sets in convex sets and $Newt_{\mathcal{C}_{In}}(x_{2m+1}) = f(Conv(E))$. We hence apply the map f to every generator of $Conv(E)$ and we find (2.3.100). The fact that $\overline{x}_{2m+1}$ is monic follows directly by (2.3.100) and by the explicit formula.

The proof of (2.3.101) follows the same strategy. By specilizing all the coefficients to 1 in formula 2.3.103 we get:

$$\overline{x}_{2m} = (x_1, w, x_3)^{\begin{pmatrix} 2-m \\ -1 \\ m-1 \end{pmatrix}} \left[\sum_{\mathbf{e}} \binom{m-2-e_3+e_2}{m-2-e_1+e_2}\binom{e_1-1}{e_3}\binom{1}{e_2} x_1^{2e_3-e_2} w^{e_1-e_3} x_3^{-2e_1+e_2} + \frac{x_3}{x_1} + 1\right]$$

For $m = 2$ we get

$$\overline{x}_4 = \frac{1}{x_1} + \frac{x_3^2}{wx_1} + \frac{x_3}{w}$$

and (2.3.101) holds in this case. We hence assume $m \geq 3$. We deduce that the Newton polytope $Newt_{\{x_1,w,x_3\}}(\overline{x}_{2m})$ of $\overline{x}_{2m}$ in the cluster $\mathcal{C}_{Cyclic}$ is the convex hull of the set $\{\begin{pmatrix} 2e_3 - e_2 + 2 - m \\ e_1 - e_3 - 1 \\ -2e_1 + e_2 + m - 1 \end{pmatrix} | (e_1, e_2, e_3)^t \in E\}$ where $E = \{(e_1, e_2, e_3)^t \in \mathbb{Z}^3 |\ 0 \leq e_3 \leq e_1 - 1;\ 1 \leq e_1 \leq m - 2 + e_2,\ 0 \leq e_2 \leq 1\} \cup \{(0,1,0)^t\} \cup \{(0,0,0)^t\}$. We consider the convex hull $Conv(E)$ of E. It is the convex hull of the following points of the affine space:

$$Conv(E) = Conv\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} m-2 \\ 0 \\ m-3 \end{pmatrix}, \begin{pmatrix} m-2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} m-1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} m-1 \\ 1 \\ m-2 \end{pmatrix}\}$$

Let us consider the affine map

$$f : \mathbb{A}^3 \to \mathbb{A}^3 : \ \mathbf{e} \mapsto B^{Cyclic}\mathbf{e} + \begin{pmatrix} 2-m \\ -1 \\ m-1 \end{pmatrix}$$

The map f sends convex sets in convex sets and $Newt_{\mathcal{C}_{In}}(\overline{x}_{2m}) = f(Conv(E))$. We hence apply the map f to every generator of $Conv(E)$ and we find

$$Newt_{\{x_1,w,x_3\}}(\overline{x}_{2m}) = Conv\{\begin{pmatrix} 2-m \\ -1 \\ m-1 \end{pmatrix}, \begin{pmatrix} m-4 \\ 0 \\ 3-m \end{pmatrix}, \begin{pmatrix} 2-m \\ m-3 \\ 3-m \end{pmatrix}, \begin{pmatrix} 1-m \\ -1 \\ m \end{pmatrix}, \begin{pmatrix} 1-m \\ m-2 \\ 2-m \end{pmatrix}, \begin{pmatrix} m-3 \\ 0 \\ 2-m \end{pmatrix}\}$$

The second and the third vectors are convex combination of the others:

$$\begin{pmatrix} m-4 \\ 0 \\ 3-m \end{pmatrix} = \frac{1}{2m-2}\begin{pmatrix} 1-m \\ -1 \\ m \end{pmatrix} + \frac{1}{2(m-1)(m-2)}\begin{pmatrix} 1-m \\ m-2 \\ 2-m \end{pmatrix} + \frac{2m-5}{2m-4}\begin{pmatrix} m-3 \\ 0 \\ 2-m \end{pmatrix}$$

and

$$\begin{pmatrix} 2-m \\ m-3 \\ 3-m \end{pmatrix} = \frac{1}{2m-3}\begin{pmatrix} 2-m \\ -1 \\ m-1 \end{pmatrix} + \frac{2m-5}{2m-3}\begin{pmatrix} 1-m \\ m-2 \\ 2-m \end{pmatrix} + \frac{1}{2m-3}\begin{pmatrix} m-3 \\ 0 \\ 2-m \end{pmatrix}.$$

The fact that $\overline{x}_{2m+1}$ is monic follows directly by (2.3.101) and by the explicit formula.

We now prove (2.3.102). In formula 2.1.29 we put all the coefficients to 1 and we get:

$$\overline{u}_n = \mathbf{x}_w^{\begin{pmatrix} -n \\ 0 \\ n \end{pmatrix}} \left\{ x_1^{2n} x_3^{-2n} + \sum\nolimits_{\mathbf{e}} \left[\binom{n-e_3}{n-e_1}\binom{e_1-1}{e_3} + \binom{n-e_3-1}{n-e_1}\binom{e_1-1}{e_3-1} \right] x_1^{2e_3} w^{e_1-e_3} x_3^{-2e_1} + 1 \right\}$$

where $\mathbf{x}_w \doteq (x_1, w, x_3)$.

We deduce that the Newton polytope $Newt_{\{x_1,w,x_3\}}(\overline{u}_n)$ of $\overline{u}_n$ in the cluster $\mathcal{C}_{Cyclic}$ is the convex hull of the set $\{\begin{pmatrix} 2e_3-n \\ e_1-e_3 \\ -2e_1+n \end{pmatrix} | \, (e_1, e_2, e_3)^t \in E\}$ where $E = \{(e_1, 0, e_3)^t \in \mathbb{Z}^3 | \ 1 \leq e_3 \leq e_1 - 1; \ 1 \leq e_1 \leq n\} \cup \{(e_1, 0, 0)^t \in \mathbb{Z}^3 | \ \ 1 \leq e_1 \leq n\} \cup \{(n, 0, n)^t\} \cup \{(0, 0, 0)^t\}$. We consider the convex hull $Conv(E)$ of E. It is the convex hull of the following points of the affine space:

$$Conv(E) = Conv\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ 0 \\ n \end{pmatrix}\}$$

Let us consider the affine map

$$f : \mathbb{A}^3 \to \mathbb{A}^3 : \ \mathbf{e} \mapsto B^{Cyclic}\mathbf{e} + \begin{pmatrix} -n \\ 0 \\ n \end{pmatrix}$$

The map f sends convex sets in convex sets and $Newt_{\mathcal{C}_{In}}(\overline{u}_n) = f(Conv(E))$. We hence apply the map f to every generator of $Conv(E)$ and we find the desired (2.3.102). The fact that $\overline{u}_n$ is monic follows directly by (2.3.102) and by the explicit formula. □

Corollary 2.3.40. *For $n \geq 1$ and $k > 0$*

$$\mathrm{Newt}_{\{x_1,w,x_3\}}(u_n z^k) = Conv\{\begin{pmatrix} n+k \\ -k \\ -n-k \end{pmatrix}, \begin{pmatrix} -n-k \\ -k \\ n+k \end{pmatrix}, \begin{pmatrix} -n-k \\ n \\ -n-k \end{pmatrix}\} \tag{2.3.103}$$

$$\mathrm{Newt}_{\{x_1,w,x_3\}}(u_n w^k) = Conv\{\begin{pmatrix} n \\ k \\ -n \end{pmatrix}, \begin{pmatrix} -n \\ k \\ n \end{pmatrix}, \begin{pmatrix} -n \\ n+k \\ -n \end{pmatrix}\} \tag{2.3.104}$$

In particular $\mathrm{Newt}_{\{x_1,w,x_3\}}(u_n z^k) \subset \{g_1 + 2g_2 + g_3 = -2k\}$ *and* $\mathrm{Newt}_{\{x_1,w,x_3\}}(u_n w^k) \subset \{g_1 + 2g_2 + g_3 = 2k\} \cap \{g_1 + g_3 \leq 0\}$.

Proof. By definition, $x_2 = \frac{x_1+x_3}{w}$ and $zx_3 = x_2 + x_4$. From (2.3.101),

$$\begin{aligned} \mathrm{Newt}_{\{x_1,w,x_3\}}(x_2 + x_4) \ &= \ \mathrm{Conv}\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\} \ &= \ \mathrm{Conv}\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}\}. \end{aligned}$$

Then

$$\mathrm{Newt}_{\{x_1,w,x_3\}}(z) = \mathrm{Conv}\left\{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\} \tag{2.3.105}$$

By the lemma 2.3.38 and the formula (2.3.102), $\mathrm{Newt}_{\{x_1,w,x_3\}}(u_n z^k) = \mathrm{Conv}\{(n,0,-n)^t, (-n,0,n)^t, (-n,n,-n)^t\} + \mathrm{Conv}\{(k,-k,-k)^t, (-k,0,-k)^t, (-k,-k,k)^t\} = \{(n+k,-k,-n-k)^t, (n-k,0,-n-k)^t, (n-k,-k,-n+k)^t, (-n+k,-k,n-k)^t\ (-n-k,0,n-k)^t, (-n-k,-k,n+k)^t, (-n+k,n-k,-n-k)^t\ (-n-k,n,-n-k)^t, (-n-k,n-k,-n+k)^t\}$. So (2.3.103) and (2.3.104) follow by direct check. $\square$

The next result is an analogous of the proposition 2.3.36 for the cluster $\{x_1, w, x_3\}$.

Proposition 2.3.41. *If b is a cluster monomial containing at least one cluster variable different from x_1, w and x_3, then there exists a non-zero linear form on $Q_{\mathbb{R}}$*

$$\varphi_b(g_1, g_2, g_3) = \alpha_b g_1 + \beta_b g_2 + \gamma_b g_3, \qquad \alpha_b, \beta_b, \gamma_b \geq 0$$

such that $\mathrm{Newt}_{\{x_1,w,x_3\}}(b) \subset \{\varphi_b < 0\}$. In particular $\mathrm{Newt}_{\{x_1,w,x_3\}}(b)$ has empty intersection with the positive cone $Q_+ = \mathbb{Z}_{\geq 0}\alpha_1 + \mathbb{Z}_{\geq 0}\alpha_2 + \mathbb{Z}_{\geq 0}\alpha_3$.
Moreover the monomial $x_1^a w^b x_3^c$, for every non-negative integers a, b and c, does not appear in the Laurent expansion of any other element of $\mathbf{B}$ with respect to the cluster $\{x_1, w, x_3\}$.

Proof. We're going to follow the proof of the proposition 2.3.36.

- $b = x_{2m+1}^p x_{2m+2}^q x_{2m+3}^r$. We put $\alpha_{2m+1} := \alpha_b$, $\beta_{2m+1} := \beta_b$ and $\gamma_{2m+1} := \gamma_b$. Using 2.3.101 and 2.3.100, it's sufficient to solve the following system of linear inequalities:

$$\begin{cases} \alpha(1-m) + \gamma(m) < 0 \\ \alpha(m-3) + \beta + \gamma(2-m) < 0 \\ \alpha(1-m) + \beta(m-1) + \gamma(2-m) < 0 \\ \alpha(1-m) + \beta(-1) + \gamma(m) < 0 \\ \alpha(m-2) + \gamma(1-m) < 0 \\ \alpha(-m) + \beta(-1) + \gamma(m+1) < 0 \\ \alpha(-m) + \beta(m-1) + \gamma(1-m) < 0 \\ \alpha(-m) + \gamma(m+1) < 0 \\ \alpha(m-2) + \beta + \gamma(1-m) < 0 \\ \alpha(-m) + \beta(m) + \gamma(1-m) < 0 \\ \alpha > 0 \\ \beta > 0 \\ \gamma > 0 \end{cases}$$

  For $m \geq 2$ this is equivalent to the following one

$$\begin{cases} \frac{m-2}{m-1}\alpha < \gamma < \frac{m-1}{m}\alpha \\ \gamma < (m-1)\beta - (m-2)\alpha \\ \alpha > 0 \\ \beta > 0 \\ \gamma > 0 \end{cases}$$

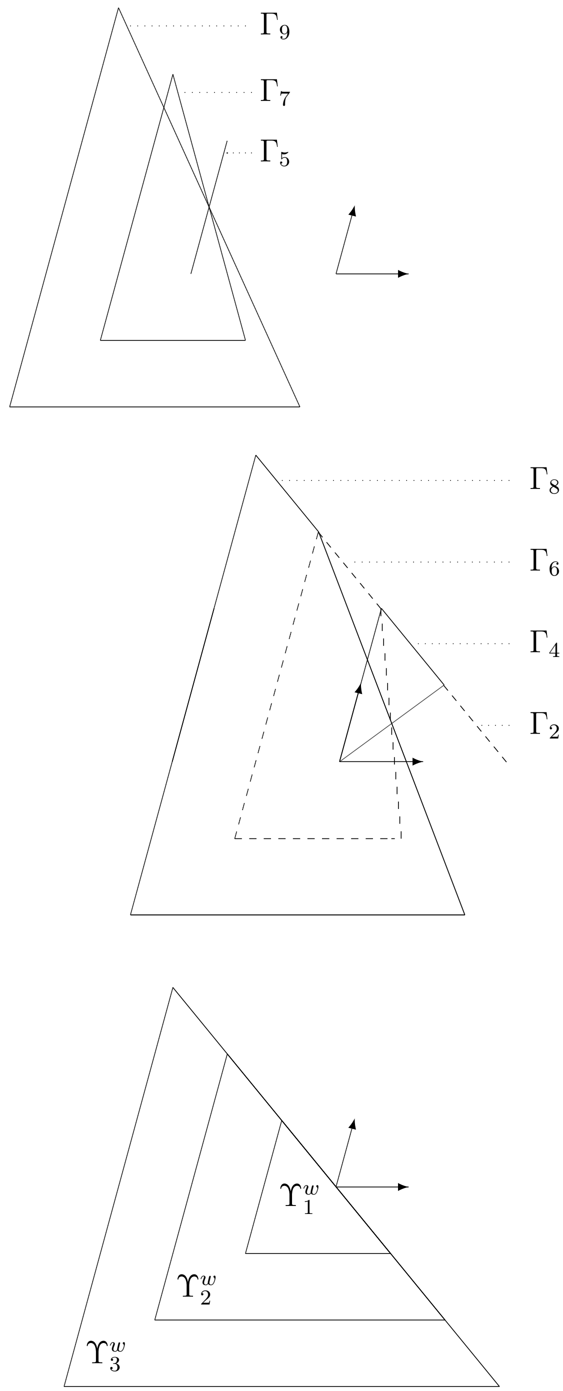

Figure 2.9: The triangles Γ_{2m+1} in the subspace $Q_0 = \{g_1 + 2g_2 + g_3 = 0\}$: $\mathrm{Newt}_{\{x_1,w,x_3\}}(x_{2m+1}) = (1,0,0) + \Gamma_{2m+1}$The quadrlilaterals Γ_{2m} in the subspace $Q_0 = \{g_1 + 2g_2 + g_3 = 0\}$: $\mathrm{Newt}_{\{x_1,w,x_3\}}(x_{2m}) = (-1,0,0) + \Gamma_{2m}$The triangles Υ_n^w in the subspace $Q_0 = \{g_1 + 2g_2 + g_3 = 0\}$: $\mathrm{Newt}_{\{x_1,w,x_3\}}(u_n) = \Upsilon_n^w$. We use the basis vector of Q_0 given by $\{w_1 = (1, -1/2, 0)^t, w_2 = (0, -1/2, 1)^t\}$

An explicit solution could be:

$$\alpha_{2m+1} = m(m-1), \quad \beta_{2m+1} = \tfrac{1}{4}(m-1), \quad \gamma_{2m+1} = m^2 - 2m + 1/2. \tag{2.3.106}$$

In the same way we obtain

$$\begin{array}{lll} \alpha_1 = 0 & \beta_1 = 1 & \gamma_1 = 0 \\ \alpha_3 = 1, & \beta_3 = \begin{cases} 0 & \text{if } r \neq 0 \\ 1 & \text{if } r = 0 \end{cases}, & \gamma_3 = 0 \end{array}$$

By using the symmetries of $\mathcal{A}$ and the definition of b, we have for $m \geq 0$

$$\alpha_{-(2m+1)} = \gamma_{2(m+1)+1}, \quad \beta_{-(2m+1)} = \beta_{2(m+1)+1}, \quad \gamma_{-(2m+1)} = \alpha_{2(m+1)+1}. \tag{2.3.107}$$

- $b = x_{2m}^p x_{2m+1}^q x_{2m+2}^r$. We put $\alpha_{2m} := \alpha_b$, $\beta_{2m} := \beta_b$ and $\gamma_{2m} := \gamma_b$. If $m \geq 4$ the corresponding system is equivalent to

$$\begin{cases} \frac{m-2}{m-1}\alpha < \gamma < \frac{m-1}{m}\alpha \\ (m-1)\gamma - (m-2)\alpha < \beta < (m-2)\gamma - (m-3)\alpha \end{cases}$$

An explicit solution could be

$$\alpha_{2m} = m(m-1), \quad \beta_{2m} = m-1, \quad \gamma_{2m} = m^2 - 2m + 1/2. \tag{2.3.108}$$

In the other "positive" cases it is easy to verify that the following are solutions:

$$\begin{array}{lll} \alpha_2 = \begin{cases} 1 & \text{if } p = 0 \\ 0 & \text{if } p \neq 0 \end{cases}, & \beta_2 = 1, & \gamma_2 = 0. \\ \alpha_4 = 2, & \beta_4 = 1, & \gamma_4 = 1/2. \\ \alpha_6 = 12, & \beta_6 = 3, & \gamma_6 = 7. \end{array}$$

As before we have

$$\alpha_{-(2m)} = \gamma_{2(m+1)}, \quad \beta_{-(2m)} = \beta_{2(m+1)}, \quad \gamma_{-(2m)} = \alpha_{2(m+1)}. \tag{2.3.109}$$

- $b = x_{2m+1}^p w^q x_{2m+3}^r$. In this case we put $\alpha_{2m+1}^w := \alpha_b$, $\beta_{2m+1}^w := \beta_b$ and $\gamma_{2m+1}^w := \gamma_b$. For $m \geq 4$ the corresponding system becomes

$$\begin{cases} \frac{m-2}{m-1}\alpha < \gamma < \frac{m-1}{m}\alpha \\ \beta = 0 \end{cases}$$

For $m \geq 3$ a solution could be:

$$\alpha_{2m+1}^w = m(m-1), \quad \beta_{2m+1}^w = 0, \quad \gamma_{2m+1}^w = m^2 - 2m + 1/2. \tag{2.3.110}$$

In the same way we have

$$\begin{array}{lll} \alpha_3^w = 1, & \beta_3^w = 0, & \gamma_3^w = 0. \\ \alpha_5^w = 4, & \beta_5^w = 0, & \gamma_5^w = 1. \end{array}$$

If $m \geq 0$

$$\alpha_{-(2m+1)}^w = \gamma_{2(m+1)+1}^w, \quad \beta_{-(2m+1)}^w = \beta_{2(m+1)+1}^w, \quad \gamma_{-(2m+1)}^w = \alpha_{2(m+1)+1}^w. \tag{2.3.111}$$

b	$\varphi_b(g_1, g_2, g_3)$	
$x_{2m+1}^p x_{2m+2}^q x_{2m+3}^r$	$m(m-1)g_1 + \frac{1}{4}(m-1)g_2 + (m^2 - 2m + \frac{1}{2})g_3$	$m \geq 2$
$x_1^p x_2^q x_3^r$	g_2	
$x_3^p x_4^q$	$g_1 + g_2$	
$x_3^p x_4^q x_5^r$	g_1	$r > 0$
$x_{2m}^p x_{2m+1}^q x_{2m+2}^q$	$m(m-1)g_1 + (m-1)g_2 + (m^2 - 2m + \frac{1}{2})g_3$	$m \geq 4$
$x_2^p x_3^q x_4^q$	g_2	
$x_4^p x_5^q x_6^q$	$2g_1 + g_2 + \frac{1}{2}g_3$	
$x_6^p x_7^q x_8^q$	$12g_1 + 3g_2 + 7g_3$	
$x_{2m+1}^p w^q x_{2m+3}^r$	$m(m-1)g_1 + (m^2 - 2m + \frac{1}{2})g_3$	$m \geq 3$
$x_3^p w^q x_5^r$	g_1	
$x_5^p w^q x_7^r$	$4g_1 + g_3$	
$x_{2m}^p z^q x_{2m+2}^r$	$g_1 + 2g_2 + g_3$	$m \geq 4$
$u_n w^k$	$g_1 + g_3$	$n \geq 1,\ k \geq 0$
$u_n z^k$	$g_1 + g_3$	$n \geq 1,\ k \geq 0$

Table 2.2: Every Laurent monomial y appearing in the Laurent expansion of b in the cluster $\mathcal{C}_{In}$ must satisfy the equation $\varphi_b(y) \leq 0$. The linear form of the other elements of $\mathbf{B}$ (involving cluster variables x_{-m}, $m \geq 0$) are obtained from these by (2.3.93), (2.3.95), (2.3.97) and (2.3.99).

- $b = x_{2m}^p z^q x_{2m+2}^r$. In this case we observe that $g(b) = -p - 2q - r$. Then

$$\mathrm{Newt}_{\{x_1,w,x_3\}}(b) \subset Q_{-p-2q-r} = \{g_1 + 2g_2 + g_3 = -p - 2q - r < 0\}. \quad (2.3.112)$$

It remains only to prove that the monomial $x_1^a w^b x_3^c$ does not appear in the $u_n w^k$ and $u_n z^k$'s Laurent expansion in $\{x_1, w, x_3\}$. By corollary 2.3.40, $u_n w^k$ could have only w^k with this property. But this happens if and only if 1 appears in the Laurent expansion (in $\{x_1, w, x_3\}$) of u_n. We observe that by definition $u_n x_3 = x_{3-2n} + x_{3+2n}$. So 1 appears in the Laurent expansion of u_n if and only if x_3 appears in the Laurent expansion of either x_{3-2n} or x_{2n+3}. This cannot be the case because their Newton polygons do not intersect the positive octant Q_+ (in particular they cannot contain the point $(0,0,1)^t$). Still by the corollary 2.3.40, for $k > 0$, the Laurent expansion of $u_n z^k$ cannot contain any such monomial, since its Newton polygon does not intersect the positive octant. □

The following Lemma is the analogous of the Key Lemma 1.5.2.

Lemma 2.3.42. *For every element b of $\overline{\mathbf{B}}$ there exists a cluster $\mathcal{C} = \mathcal{C}_b$ and a* monic *vertex γ_b of* $\mathrm{Newt}_{\mathcal{C}}(\overline{b})$ *such that $\gamma_{\overline{b}}$ does not lie in* $\mathrm{Newt}_{\mathcal{C}}(b')$ *if $b' \neq b$ is another element of $\overline{\mathbf{B}}$.*

Proof. If b is a cluster monomial in the elements of a cluster $\mathcal{C}$, by Propositions 2.3.36 and 2.3.41 we can choose $\mathcal{C}_b = \mathcal{C}$. If b is an element of $\{\overline{u}_n \overline{w}^k, \overline{u}_n \overline{z}^k \,|\, n \geq 1,\ k \geq 0\}$

we claim that the following couples have all the desired properties:

$$\begin{array}{lll} \text{if } b = \overline{u}_n & \text{then} & \mathcal{C} = \{x_1, x_2, x_3\} \text{ and } \gamma = (-n, 0, n); \\ \text{if } b = \overline{u}_n w^k & \text{then} & \mathcal{C} = \{x_1, w, x_3\} \text{ and } \gamma = (-n, k, n); \\ \text{if } b = \overline{u}_n \overline{z}^k & \text{then} & \mathcal{C} = \{x_2, z, x_4\} \text{ and } \gamma = (-n, k, n). \end{array}$$

Indeed let us consider first the case $b = \overline{u}_n$. By Proposition 2.3.34, c^γ occurs in the Laurent expansion of $\overline{u}_n$ with coefficient 1. We show that c^γ does not occur in $\overline{u}_p$ for $p \neq n$ as well: by definition $\overline{u}_p = T_p(u_1)$. By using the Taylor expansion of T_p in $t + t^{-1}$ we have

$$\overline{u}_p = \sum_{k=0}^{p} \frac{1}{k!} T_p^{(k)}(t + t^{-1}) l^k$$

Since the derivative of T_p is a positive linear combinations of T_q with $q < p$, the only monomials $x_1^a x_2^b x_3^c$ with $a - c = 0$ are such that $(a, b, c) = (\pm p, 0, \mp p)$. Moreover s^γ does not occur in any cluster monomial of $\overline{\mathbf{B}}$ since γ does *not* satisfy the linear inequalities $\varphi_b(\gamma) \leq 0$ where φ_b is given by Table **??**. Finally s^γ cannot appear in the Laurent expansion of $\overline{u}_p w^k$ and $\overline{u}_p z^k$ if $k > 0$, because $g(s^\gamma) = 0$ whereas $g(\overline{u}_n w^k) = 2k$ and $g(\overline{u}_n z^k) = -2k$. The other two cases follow by the same arguments using Table 2.2 instead of table **??**. □

Theorem 2.3.43. *The set* $\overline{\mathbf{B}} = \{\textit{cluster monomials}\} \cup \{\overline{u}_n \overline{w}^k, \overline{u}_n \overline{z}^k \,|\, n \geq 1,\, k \geq 0\}$ *is the canonical basis of* $\mathcal{A}_{\{1\}}$.

Proof. Since of Proposition 2.3.32 it remains to prove that the elements of $\overline{\mathbf{B}}$ are positive indecomposable. This is done as in the proof of Theorem 1.5.3. □

2.3.12 The elements of B are positive indecomposable

In Propositions 2.3.9 and 2.3.23 we have proved the set $\mathbf{B} = \{\text{cluster monomial}\} \cup \{u_n \overline{w}^k, u_n z^k :\ n \geq 1,\, k \geq 0\}$ is a $\mathbb{ZP}$–basis of the cluster algebra $\mathcal{A}$ with principal coefficients at the initial seed

$$\Sigma_{In} \doteq \{B = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}\}.$$

Moreover the elements of $\mathbf{B}$ are positive (Proposition 2.3.22).

Proposition 2.3.44. *The elements of* $\mathbf{B}$ *in* $\mathcal{A}$ *are positive indecomposable.*

Proof. It follows by Lemma 2.3.31 and Theorem 2.3.43. □

This conclude the proof of Theorem 2.1.4.

2.4 General coefficients

In this section we allow $\mathbb{P}$ to be a generic *tropical* semifield (see Definition 1.1.1). Let $\mathcal{A}_{\mathbb{P}}$ be a cluster algebra (of type $A_2^{(1)}$) inside the field $\mathbb{QP}(x_1, x_2, x_3)$ with initial seed

$$\Sigma_{In} \doteq \{B = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}\}.$$

By Theorem 1.1.11 we know that the cluster variables of $\mathcal{A}_{\mathbb{P}}$ are $\{x_m\}_{m\in\mathbb{Z}} \cup \{z, w\}$ and every cluster variable s has the form

$$s = \frac{F_s \mid_{\mathcal{F}} (\widehat{y}_1, \widehat{y}_2, \widehat{y}_n)}{F_s \mid_{\mathbb{P}} (y_1, y_2, y_3)} \mathbf{x}^{\mathbf{g}_s}$$

where

$$\widehat{y}_1 = \tfrac{y_1}{x_2 x_3}, \quad \widehat{y}_2 = \tfrac{y_2 x_1}{x_3}, \quad \widehat{y}_3 = y_3 x_1 x_2 \tag{2.4.1}$$

(defined in (2.3.2)), F_s and $\mathbf{g}_s$ are respectively the corresponding F–polynomial given by Proposition 2.3.4 and the corresponding $\mathbf{g}$–vector given by Proposition 2.3.3.

Recall from Section 1.5.1 that if a canonical basis of $\mathcal{A}_{\mathbb{P}}$ exists, it is determined up to rescaling by elements of $\mathbb{P}$. Having this in mind we give the following definition.

Definition 2.4.1. For every cluster variable s of $\mathcal{A}_{\mathbb{P}}$ we define

$$S \doteq F_s|_{\mathbb{P}}(y_1, y_2, y_3) \cdot s. \tag{2.4.2}$$

We call these elements of $\mathcal{A}_{\mathbb{P}}$ *principal cluster variables*. Similarly given a cluster monomial b of $\mathcal{A}_{\mathbb{P}}$ we call the element $B \doteq F_b|_{\mathbb{P}}(y_1, y_2, y_3) \cdot b$ a *principal cluster monomial* of $\mathcal{A}_{\mathbb{P}}$.
For every $n \geq 0$ we define

$$U_n \doteq F_{u_n}(\widehat{y}_1, \widehat{y}_2, \widehat{y}_3) \cdot \frac{x_3^n}{x_1^n} \tag{2.4.3}$$

Principal cluster monomials are elements of the universal semifield in six variables $\mathbb{Q}_{sf}(y_1, y_2, y_3, x_1, x_2, x_3)$ (see Definition 1.1.1) since they are rational functions with positive coefficients in these six variables; moreover in this semifield a principal cluster variable S of $\mathcal{A}_{\mathbb{P}}$ coincide with the rational expression of the cluster variable s of the cluster algebra $\mathcal{A}$ with principal coefficients at the initial cluster Σ_{In}, which explains the terminology.

Theorem 2.4.2. *The set* $\mathbf{B} \doteq \{$*principal cluster monomials*$\} \cup \{U_n Z^k,\ U_n W^k :\ n \geq 1,\ k \geq 0\}$ *is a canonical basis of* $\mathcal{A}_{\mathbb{P}}$ *(see Definition1.5.1), i.e.* $\mathbf{B}$ *is a* $\mathbb{ZP}$*–basis of* $\mathcal{A}_{\mathbb{P}}$ *whose elements are positive indecomposable.*

Proof. The elements of $\mathbf{B}$ are linearly independent over $\mathbb{ZP}$ since $\mathbf{B}$ satisfies hypothesis of Theorem 1.5.7. The straightening relations given by Proposition 2.3.27 hold in $\mathbb{Q}_{sf}(y_1, y_2, y_3, x_1, x_2, x_3)$. So we can use the same argument as in the proof of Proposition 2.3.23 in Section 2.3.10 in order to conclude that $\mathbf{B}$ spans $\mathcal{A}_{\mathbb{P}}$ over $\mathbb{ZP}$.

So $\mathbf{B}$ is a $\mathbb{ZP}$–basis of $\mathcal{A}_{\mathbb{P}}$. We now prove that the elements of $\mathbf{B}$ are positive. The expansion of an element $B \in \mathbf{B}$ in the cluster $\{x_1, w, x_3\}$ is given by

$$B = F_b|_{\mathbb{P}}(y_1, y_2, y_3)\frac{F_b^w(\widehat{y}_{1;w}, \widehat{y}_{2;w}, \widehat{y}_{3;w})}{F_b^w|_{\mathbb{P}}(y_{1;w}, y_{2;w}, y_{3;w})}(x_1, w, x_3)^{\mathbf{g}_b^w}$$

where F_b^w is the F–polynomial of b given by Proposition 2.3.13 and $\mathbf{g}_b^w$ is the $\mathbf{g}$–vector of b given by Proposition 2.3.17 and $\{y_{1;w}, y_{2;w}, y_{3;w}\}$ are the coefficients of the (unique) seed of $\mathcal{A}_{\mathbb{P}}$ with cluster $\{x_1, w, x_3\}$. Then the Laurent expansion of every element of $\mathbf{B}$ in the two clusters $\{x_1, x_2, x_3\}$ and $\{x_1, w, x_3\}$ has coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$. By the symmetry of the exchange graph we conclude they are positive.

The fact that the elements of $\mathbf{B}$ are positive indecomposable follows by Lemma 2.3.31 and Theorem 2.3.43. □

2.5 F–polynomials and quiver Grassmannians

This section is somehow independent on the previous ones. We consider the acyclic quiver Q_{In} of type $A_2^{(1)}$:

$$Q_{In} \doteq \quad \begin{tikzcd} & 2 \arrow[dl, "a"'] & \\ 1 & & 3 \arrow[ll, "c"] \arrow[ul, "b"'] \end{tikzcd},$$

and we study the map $F : Rep(Q_{In}) \to \mathbb{Z}[y_1, y_2, y_3]$ which associates with every Q_{In}–representation M the polynomial $F_M(y_1, y_2, y_3)$ defined by:

$$F_M(y_1, y_2, y_3) = \sum_{\mathbf{e}=(e_1,e_2,e_3)} \chi_{\mathbf{e}}(M) y_1^{e_1} y_2^{e_2} y_3^{e_3} \tag{2.5.1}$$

where $\chi_{\mathbf{e}}(M)$ denotes the Euler–Poincaré characteristic of the quiver Grassmannian $Gr_{\mathbf{e}}(M)$, the projective variety of the sub–representations of M with dimension vector $\mathbf{e} = (e_1, e_2, e_3)$. The map F is the natural generalization of the Caldero–Chapoton–Keller map in the coefficients–setting.
The map F has the following multiplicative property:

$$F_{M\oplus N}(y_1, y_2, y_3) = F_M(y_1, y_2, y_3) \cdot F_N(y_1, y_2, y_3). \tag{2.5.2}$$

The proof of this fact follows from [10, Proposition 1], where it is shown that $\chi_{\mathbf{e}}(M \oplus N) = \sum_{\mathbf{f}+\mathbf{g}=\mathbf{e}} \chi_{\mathbf{f}}(M)\chi_{\mathbf{g}}(N)$.
Let us collect the main results of the present section that also justifies the interest of the map F: Proposition 2.1.6 shows that the denominator vector in the initial cluster $\{x_1, x_2, x_3\}$ of the cluster algebra $\mathcal{A}$ restricts to a bijection between cluster variables of $\mathcal{A}$ and a proper subset (the real *Schur* roots) of the real roots of the root system of type $A_2^{(1)}$ (see section 1.2.2 for more informations about the structure of a root system of type $A_2^{(1)}$). By definition every Schur root $\mathbf{d}$ is the dimension vector of a unique (up to isomorphisms) *rigid* module M, i.e. a module without nontrivial self–extensions. This terminology comes from the well known results due to Kac

[23]. Real Schur roots depends on the orientation of the quiver (see [12]) while real roots do not. It is well–known (see [23, Theorem 3]) that every real root determines a unique (up to isomorphisms) Q_{In}–representation. Briefly we have the following bijections:

$$\begin{array}{ccccc} \text{cluster variables} & \xleftrightarrow[\text{vector}]{\text{den.}} & \text{Schur roots} & \xleftrightarrow[\text{vector}]{\text{dim.}} & \text{Rigid } Q_{In}\text{–representations} \\ s & & \mathbf{d}(s) & & M_s \end{array}$$

One of the main results of the present section (Proposition 2.5.2) shows that for every cluster variable s the polynomial $F_{M_s}(y_1, y_2, y_3)$ is the *F–polynomial* of s (see Section 2.3.3). Moreover, it follows from (2.5.2) that for every rigid Q_{In}–representation $M = \oplus M_i$ (sum of indecomposable rigid representations M_i), there exists a unique cluster monomial $s_1^a s_2^b s_3^c$ such that its F–polynomial is exactly F_M. In other words the image of the rigid Q_{In} representations by F is a set of "F–polynomials".
In section 2.6 we will compute Euler–Poincaré characteristic of quiver Grassmannians associated with *non–rigid* indecomposable Q_{In}–representations (Proposition 2.6.1), so that we have an explicit description of the image of F. The natural question at this point is to see if this image is a set of "F–polynomials". In other words we asked if there exist elements of $\mathcal{A}$ whose corresponding F–polynomials are F_M where M is an indecomposable *non–rigid* Q_{In}–representation (the image of the rigid representations has just been described). The answer to this question is affermative and the set of such elements is divided into two families $\{s_n : \ n \geq 0\}$ and $\{r_n : \ n \geq 0\}$ (Definition 2.6.5). Moreover if we complete one of these families by the set of cluster monomials we get the two sets $\mathcal{S} = \{\text{cluster monomials}\} \cup \{s_n w^k, s_n z^k : \ n \geq 1, \ k \geq 0\}$ and $\mathcal{R} = \{\text{cluster monomials}\} \cup \{r_n w^k, r_n z^k : \ n \geq 1, \ k \geq 0\}$. Proposition 2.6.6 shows that both $\mathcal{S}$ and $\mathcal{R}$ are two $\mathbb{ZP}$–basis of $\mathcal{A}$ different from every canonical basis. We call $\mathcal{S}$ a *"semicanonical" basis* of $\mathcal{A}$ in analogy with semicanonical basis found in [11] for a coefficient–free cluster algebra of type $A_1^{(1)}$. In loc.cit. the semicanonical basis was parameterized by *Chebychev's polynomials* of the second kind, while the canonical basis by Chebychev's polynomials of the first kind. The same is true in $\mathcal{A}$ as it is shown in Corollary 2.6.7.
We begin by recalling the well–known classification of the indecomposable rigid Q_{In}-representations. We assume that all the representations are over the field $k = \mathbb{C}$ of complex numbers.

2.5.1 Indecomposable rigid Q_{In}–representations

Here we recall the classification of the representations of the quiver Q_{In} that are *rigid*, i.e. without non–trivial self–extensions. The classification of the indecomposable representations of Q_{In} will be completed in Section 2.6 where Proposition 2.6.1 will show the indecomposable Q_{In}–representations that are not rigid. The quiver Q_{In} is the acyclic quiver of type $A_2^{(1)}$ and its classification is well–known in literature (see e.g. [14] or [23]). In this thesis Q_{In} is the quiver associated with the exchange matrix of a cluster algebra of type $A_2^{(1)}$, and it was introduced in Section 2.2.

Proposition 2.5.1. *The indecomposable rigid Q_{In}-representations have dimension vector the real (Schur) roots: $(n{+}1, n{+}1, n)$, $(n{+}1, n, n)$, $(n, n{+}1, n{+}1)$, $(n, n, n{+}1)$, $(0, 1, 0)$ and $(1, 0, 1)$ for every $n \geq 0$.*

We denote by S_2 the simple representation of dimension $(0, 1, 0)$ and by S_{13} the representation of dimension $(1, 0, 1)$. They correspond respectively to the cluster variable w and z and they are sometimes called *simple regulars*. They are at the bottom of the tube of rank two in the AR–quiver of Q_{In} (see figure 2.3).

For $n \geq 0$, let $\mathbf{M}_{2(n+2)+1}$ and $\mathbf{M}_{2(n+2)}$ be the Q_{In}-representations of dimension vector respectively the real root $(n+1, n+1, n)$ and $(n+1, n, n)$. They correspond respectively to the cluster variable $x_{2(n+2)+1}$ and $x_{2(n+2)}$ (see figure 2.4) which explains the terminology. By using Kac's Theorem ([23, Theorem 3]) we can assume there exist basis $\{v_1, \cdots, v_{n+1}\}$ of $\mathbf{M}_{2(n+2)+1;\,1}$ and $\mathbf{M}_{2(n+2)+1;\,2}$ (resp. $\mathbf{M}_{2(n+2);\,1}$), and basis $\{u_1, \cdots, u_n\}$ of $\mathbf{M}_{2(n+2)+1;\,3}$ (resp. $\mathbf{M}_{2(n+2);\,2}$ and $\mathbf{M}_{2(n+2);\,3}$) such that

$$\mathbf{M}_{2(n+2)+1} = \begin{tikzcd} & k^{n+1} \arrow[dl, "="'] & \\ k^{n+1} & & k^n \arrow[ll, "\varphi_2"] \arrow[ul, "\varphi_1"'] \end{tikzcd} ; \qquad \mathbf{M}_{2(n+2)} = \begin{tikzcd} & k^{n} \arrow[dl, "\varphi_1"'] & \\ k^{n+1} & & k^n \arrow[ll, "\varphi_2"] \arrow[ul, "="'] \end{tikzcd}$$

where $\varphi_1(u_k) = v_k$ and $\varphi_2(u_k) = v_{k+1}$ for $k = 1, \cdots, n$ and the maps labeled by "$=$" are the identity map. Indeed it is not hard to prove that their endomorphism ring is a local ring and thus they are indecomposable.
The *duality functor* D sends the module $M = (M_1, M_2, M_3, f_a, f_b, f_c)$ to the module $DM = (M_3^*, M_2^*, M_1^*, f_a^*, f_b^*, f_c^*)$ where M_i^* is the dual vector space of M_i, $i = 1, 2, 3$. We define $M_{-(2n+1)} \doteq DM_{2(n+2)+1}$ and $M_{-2n} = DM_{2(n+2)}$ for every $n \geq 0$. It follows that $M_{-(2n+1)}$ (resp. M_{-2n}) is the unique (up to isomorphisms) module of dimension vector $(n, n+1, n+1)$ (resp. $(n, n, n+1)$). Moreover given a Q_{In}–representation M of dimension (d_1, d_2, d_3), the duality functor D induces a map $N \mapsto (M/N)^*$ between $Gr_{(e_1,e_2,e_3)}(M)$ and $Gr_{(d_3-e_3,\, d_2-e_2,\, d_1-e_1)}(DM)$ so that we have in particular

$$\chi_{(e_1,e_2,e_3)}(DM) = \chi_{(d_3-e_3, d_2-e_2, d_1-e_1)}(M). \tag{2.5.3}$$

We will use this fact later in the proofs.

2.5.2 F–polynomials of cluster variables

Here we compute Euler–Poincaré characteristic of quiver Grassmannians associated with the indecomposable *rigid* Q_{In}–representations.

Proposition 2.5.2. *Using the previous terminology the following formulas hold:*

$$\chi_{(e_1,e_2,e_3)}(\mathbf{M}_{2(n+2)+1}) = \binom{e_1 - e_3}{e_2 - e_3}\binom{n+1-e_3}{n+1-e_1}\binom{e_1 - 1}{e_3} \tag{2.5.4}$$

(with the convention that the right hand side is equal to 1 if $e_1 = e_2 = e_3 = 0$);

$$\chi_{\mathbf{e}}(\mathbf{M}_{2(n+2)}) = \binom{e_1 - 1}{e_3}\left[\binom{e_1 - e_3}{e_2 - e_3}\binom{n - e_3}{n - e_1} + \binom{e_1 - e_3 - 1}{e_2 - e_3}\binom{n - e_3}{n - e_1 + 1}\right] \tag{2.5.5}$$

(with the convention that the right hand side is equal to 1 *if* $e_1 = e_2 = e_3 = 0$*);*

$$\chi_{\mathbf{e}}(S_2) = \begin{cases} 1 & \textit{if} \;\; \mathbf{e} = \mathbf{0} \;\textit{or}\; \mathbf{e} = (0,1,0) \\ 0 & \textit{otherwise} \end{cases} \tag{2.5.6}$$

$$\chi_{\mathbf{e}}(S_{13}) = \begin{cases} 1 & \textit{if} \;\; \mathbf{e} = \mathbf{0} \;\textit{or}\; \mathbf{e} = (1,0,0) \;\textit{or}\; \mathbf{e} = (1,0,1) \\ 0 & \textit{otherwise} \end{cases} \tag{2.5.7}$$

In particular for every rigid Q_{In}*–representation* M*,* F_M *is the* F*–polynomial of the cluster variable with denominator vector (in the cluster* $\{x_1, x_2, x_3\}$*) the dimension vector of* M*.*

The proof is based on a suitable fiber bundle between the quiver Grassmannians associated with Q_{In} and quiver Grassmannians associated with the Kronecker quiver $\mathcal{K}$. We hence recall some facts about $\mathcal{K}$ and its quiver Grassmannians. Before doing that we recall the useful notion of "right equivalence".

2.5.3 Right–equivalence

In this section we recall the concept of "right–equivalence" given in [13] specialized to the case treated in this thesis. This is useful in order to compute Euler–Poincaré characteristic of quiver Grassmannians.

Definition 2.5.3. Let $Q = (Q_0, Q_1)$ be a finite quiver. Two finite dimensional kQ–modules $M = \oplus_{i \in Q_0} M_i$ and $N = \oplus_{i \in Q_0} N_i$ are called *right–equivalent* if there exists an automorphism $\psi : kQ \to kQ$ of the path algebra and an isomorphism $\phi : M \to N$ of Q_0–graded k–vector spaces such that

$$\phi(\alpha \cdot m) = \psi(\alpha) \circ \phi(m) \tag{2.5.8}$$

for every $\alpha \in kQ$ and $m \in M$ (here we have denoted by $\cdot$ and $\circ$ respectively the action of kQ on M and N).

The following Lemma justifies the introduction of the previous definition in this section.

Lemma 2.5.4. *Let* Q *be a finite acyclic quiver and* $(M, \cdot)$ *and* $(N, \circ)$ *two finite dimensional* kQ*–modules. If* M *and* N *are right–equivalent, then, for every dimension vector* $\mathbf{e}$*,* $Gr_{\mathbf{e}}(M) = Gr_{\mathbf{e}}(N)$*. In particular* $\chi_{\mathbf{e}}(M) = \chi_{\mathbf{e}}(N)$*.*

Proof. By hypothesis there exists an isomorphism of k–vector spaces $\phi : M \to N$ and an automorphism ψ of the path algebra kQ such that (2.5.8) holds for every $\alpha \in kQ$ and $m \in M$. We introduce another structure $\star$ of kQ–module on N by $\alpha \star n \doteq \psi(\alpha) \circ n$. Since ψ is a kQ–automorphism, $(N, \star)$ is a kQ–module. By (2.5.8), $(M, \cdot)$ and $(N, \star)$ are isomorphic by ϕ as kQ–modules. In particular we have $Gr_{\mathbf{e}}((M, \cdot)) = Gr_{\mathbf{e}}((N, \star))$. Now it is easy to see that every submodule of $(N, \circ)$ is a submodule of $(N, \star)$ and viceversa since ψ is an automorphism of kQ. We conclude that $Gr_{\mathbf{e}}((N, \circ)) = Gr_{\mathbf{e}}((N, \star))$ and we are done. $\square$

2.5.4 Indecomposable representations of the Kronecker quiver

Here we recall the well–known classification of the indecomposable representations of the Kronecker quiver

$$\mathcal{K}: \qquad 1 \underset{b}{\overset{a}{\leftleftarrows}} 2$$

We assume that all the representations are over the field $k = \mathbb{C}$ of complex numbers.

Proposition 2.5.5. *1. The indecomposable rigid $\mathcal{K}$-representations' dimension vector are the real roots $(n, n+1)$ and $(n+1, n)$, $n \geq 0$.*

2. For $n \geq 0$, let $\mathbf{m}_{n+3} = \{(m_{n+3;i}), (\varphi_i)\}_{i=1,2}$ and $\mathbf{m}_{-n} = \{(m_{-n;i}), (\varphi_i^*)\}_{i=1,2}$ be respectively the (up to isomorphisms) indecomposable representation of dimension vector $(n+1, n)$ and $(n, n+1)$. We assume that $m_{-n,1} = m^*_{n+3;2}$ and $m_{-n;2} = m^*_{n+3;1}$. Then there exists a basis $\{u_1, \dots, u_n\}$ in $m_{n+3;2}$ and a basis $\{v_1, \dots, v_{n+1}\}$ in $m_{n+3;1}$ such that $\varphi_1(u_k) = v_k$ and $\varphi_2(u_k) = v_{k+1}$ for $k \in [1, n]$. In these bases

$$\mathbf{m}_{n+3}: \qquad k^{n+1} \underset{\varphi_2}{\overset{\varphi_1}{\leftleftarrows}} k^n \; ; \qquad \mathbf{m}_{-n}: \qquad k^n \underset{\varphi_2^t}{\overset{\varphi_1^t}{\leftleftarrows}} k^{n+1}$$

where φ_i^t is the transpose of the matrix φ_i, $i = 1, 2$.

3. The indecomposable regular $\mathcal{K}$-representations have dimension vector (n, n) for every $n \geq 1$. For every $\lambda \in k$, they are, up to isomorphisms, the following

$$\mathbf{m}_n^{Reg}(\lambda): \qquad k^n \underset{J_n(\lambda)}{\overset{=}{\leftleftarrows}} k^n \; ; \qquad \mathbf{m}_n^{Reg}(\infty): \qquad k^n \underset{=}{\overset{J_n(0)}{\leftleftarrows}} k^n$$

where $J_n(\lambda)$ is the n-Jordan block of eigenvalue $\lambda \in k$ and the maps labeled by "$=$" are the identity map.

4. $\mathbf{m}_{n+3}$, $\mathbf{m}_{-n}$ and $\mathbf{m}_n^{Reg}$ for $n \geq 0$ are the all indecomposable $\mathcal{K}$–representations.

This result is due to L. Kronecker [25]; for a modern treatment see [20, Section 5.4] or [1].
Next result provides the Euler–Poincaré characteristic of the quiver Grassmannians associated with the indecomposable $\mathcal{K}$–representations. Recall from section 2.2, that given a $\mathcal{K}$–representation $\mathbf{m}$, $\chi_{(e_1,e_2)}(\mathbf{m})$ denotes the Euler–Poincaré characteristic of the variety of all sub–representations of $\mathbf{m}$ with dimension vector (e_1, e_2). In order to do that it is sufficient to consider only a small class of indecomposable $\mathcal{K}$–representations as it is shown by the following Lemma.

Lemma 2.5.6. *With notations of Proposition 2.5.5, we have*

- *$\mathbf{m}_n^{Reg}(\lambda)$ and $\mathbf{m}_n^{Reg}(\infty)$ are right equivalent for every $\lambda \in k$. In particular $\chi_{\mathbf{e}}(\mathbf{m}_n^{Reg}(\lambda)) = \chi_{\mathbf{e}}(\mathbf{m}_n^{Reg}(\infty))$.*

- $\chi_{(e_1,e_2)}(\mathbf{m}_{-n}) = \chi_{(n+1-e_2,n-e_1)}(\mathbf{m}_{n+3})$

Proof. In Definition 2.5.3 we choose $\phi : \mathbf{m}_n^{Reg}(\lambda) \to \mathbf{m}_n^{Reg}(0)$ to be the identity map and the automorphism ψ defined on the generators of $k\mathcal{K}$ by $\psi : a \mapsto b;\ b \mapsto a + \lambda b$ and identity on the idempotents.
Second part follows from the fact that $\mathbf{m}_{-n} = D\mathbf{m}_{n+3}$ where D is the duality functor that sends a module $m = (m_1, m_2, f_a, f_b)$ to the module $Dm = (m_2^*, m_1^*, f_a^*, f_b^*)$ being m_i^* the dual vector space of m_i. D induces an isomorphism of algebraic variety $n \mapsto D(m/n)$ between $Gr_{(e_1,e_2)}(m)$ and $Gr_{(d_2-e_2,d_1-e_1)}(Dm)$ where $d_i \doteq dim(M_i)$. $\square$

Proposition 2.5.7. *[11, Propositions 4.3 and 5.3] With notations of Proposition 2.5.5*

$$\chi_{(e_1,e_2)}(\mathbf{m}_{n+3}) = \binom{n+1-e_2}{n+1-e_1}\binom{e_1-1}{e_2} + \delta_{e_1,0}\delta_{e_2,0} \tag{2.5.9}$$

where $\delta_{a,b}$ is the Kronecker delta;

$$\chi_{(e_1,e_2)}(\mathbf{m}_n^{Reg}) = \binom{n-e_2}{n-e_1}\binom{e_1}{e_2}. \tag{2.5.10}$$

2.5.5 Proof of Proposition 2.5.2

Let us recall some properties of the Euler-Poincaré characteristic. We follow the treatment in [22, Section 4.5], where the Euler-Poincaré characteristic $\chi(X)$ is defined for any complex algebraic variety X (not necessarily smooth, projective or irreducible). The following facts are shown in loc.cit.

If $\mathbb{A}$ is a finite dimensional affine space, then $\chi(\mathbb{A}) = 1$. (2.5.11)

If a variety X is a disjoint union of finitely many locally closed subvarieties X_i, then $\chi(X) = \sum \chi(X_i)$. (2.5.12)

If $X \to Z$ is a fiber bundle (locally trivial in the Zariski topology) with fiber Y, then $\chi(X) = \chi(Y)\chi(Z)$. (2.5.13)

In our situation $X = Gr_{\mathbf{e}}(M)$ is a projective variety; in particular every Zariski–open subset of X is a locally closed subvariety. As a consequence of (2.5.11) and (2.5.12), the Schubert cell decomposition of the Grassmannian implies that

$$\chi(\mathrm{Gr}_r(V)) = \binom{\dim V}{r}. \tag{2.5.14}$$

Let us prove (2.5.4). The surjective morphism of algebraic variety

$$Gr_{(e_1,e_2,e_3)}\left(\begin{tikzcd}[row sep=small] & k^{n+1} \arrow[dl, "="'] & \\ k^{n+1} & & k^n \arrow[ul, "\varphi_1"'] \arrow[ll, "\varphi_2"] \end{tikzcd}\right) \twoheadrightarrow Gr_{(e_1,e_3)}\left(k^{n+1} \overset{\varphi_1}{\underset{\varphi_2}{\leftleftarrows}} k^n\right) \quad : (N_1,N_2,N_3) \mapsto (N_1,N_3)$$

sending the tern (N_1, N_2, N_3) onto the pair (N_1, N_3), is a (locally trivial) fiber bundle with fiber $Gr_{(e_2-e_3)}(e_1-e_3)$. From (2.5.12), using (2.5.9) and (2.5.14), (2.5.4) follows. We now prove (2.5.5). We need the following Lemma.

Lemma 2.5.8. *Given a Q_{In}–representation of the form*

$$\begin{tikzcd}[column sep=small] & M_2 \arrow[dl, "f_a"'] & \\ M = M_1 & & M_3 \arrow[ul, "f_b"'] \arrow[ll, "f_c"] \end{tikzcd}$$

where $M_i = K \oplus I$ for some $i = 1, 2, 3$, then $\chi_{\mathbf{e}}(M) = \chi_{\mathbf{e}}(M_K) + \chi_{\mathbf{e}}(M_I)$ where

$$M_K \doteq \{N = (N_1, N_2, N_3) \in Gr_{\mathbf{e}}(M)|\ N_i \supseteq K\}$$

and

$$M_I \doteq \{N = (N_1, N_2, N_3) \in Gr_{\mathbf{e}}(M)|\ N_i \subseteq I\}.$$

Proof. Clearly $Gr_{\mathbf{e}}(M)$ is the disjoint union of the closed subset M_K and its complement (locally closed) $M_K^c = \{N | N_i \not\supseteq K\}$ in $Gr_{\mathbf{e}}(M)$. Then $\chi_{\mathbf{e}}(M) = \chi(M_K) + \chi(M_K^c)$. The projection onto I in M_i induces a surjective morphism of algebraic variety $\pi : M_K^c \twoheadrightarrow M_I$ with affine fiber. We then have $\chi(M_K^c) = \chi(M_I)$. □

Let us prove (2.5.5). We consider the decomposition $M_1 = Im(\varphi_1) \oplus \mathbb{C}v_{n+1}$ of the vector space at the vertex 1 of the module $\mathbf{M}_{2(n+2)} = (M_1, M_2, M_3)$. By Lemma 2.5.8 we have

$$\chi_{(e_1,e_2,e_3)} \left(\begin{tikzcd}[column sep=small] & k^n \arrow[dl, "\varphi_1"'] & \\ k^{n+1} & & k^n \arrow[ul, "="'] \arrow[ll, "\varphi_2"] \end{tikzcd} \right) = \chi(G_1) + \chi(G_2)$$

where $G_1 \doteq \{(N_1, N_2, N_3) \in Gr_{\mathbf{e}}(\mathbf{M}_{2(n+2)}) | N_1 \subseteq Im(\varphi_1)\}$ and $G_2 \doteq \{(N_1, N_2, N_3) \in Gr_{\mathbf{e}}(\mathbf{M}_{2(n+2)}) | N_1 \supseteq \mathbb{C}v_{n+1}\}$. We now compute these Euler–Poincaré characteristics. Let us start with $\chi(G_1)$: it is easy to realize that

$$G_1 \simeq Gr_{(e_1,e_2,e_3)} \left(\begin{tikzcd}[column sep=small] & k^n \arrow[dl, "="'] & \\ k^n & & k^{n-1} \arrow[ul, "\varphi_1"'] \arrow[ll, "\varphi_2"] \end{tikzcd} \right)$$

i.e. G_1 is nothing but $Gr_{\mathbf{e}}(\mathbf{M}_{2(n+1)+1})$. By using (2.5.4) we get

$$\chi(G_1) = \binom{e_1 - e_3}{e_2 - e_3} \binom{n - e_3}{n - e_1} \binom{e_1 - 1}{e_3} + \delta_{e_1,0}\delta_{e_2,0}\delta_{e_3,0}. \tag{2.5.15}$$

Similarly one can easily realize that

$$G_2 \simeq Gr_{(e_1-1,\, e_2,\, e_3)} \left(\begin{tikzcd}[column sep=small] & k^n \arrow[dl, "="'] & \\ k^n & & k^n \arrow[ul, "="'] \arrow[ll, "J_n(0)"] \end{tikzcd} \right)$$

where $J_n(0)$ is the n–th Jordan block with eigenvalue zero. In Proposition 2.6.1 below, we will see that the representation on the right–hand side is an indecomposable non–rigid Q_{In}–representation denoted by Reg_n. In (2.6.1) the Euler–Poincaré characteristic of $Gr_{\mathbf{e}}(Reg_n)$ is computed. We recall here the (easy) proof in order to make the treatment of the proof of Proposition 2.5.2 completely self–contained. The surjective morphism of algebraic variety

$$Gr_{(e_1,e_2,e_3)}\left(\begin{tikzcd}[ampersand replacement=\&] \& k^n \arrow[dl, "="'] \& \\ k^n \& \& k^n \arrow[ul, "="'] \arrow[ll, "J_n(0)"] \end{tikzcd}\right) \twoheadrightarrow Gr_{(e_1,e_3)}\left(\begin{tikzcd}[ampersand replacement=\&] k^n \& k^n \arrow[l, shift right, "="'] \arrow[l, shift left, "J_n(0)"] \end{tikzcd}\right) \quad : (N_1,N_2,N_3) \mapsto (N_1,N_3)$$

sending the tern (N_1,N_2,N_3) onto the pair (N_1,N_3), is a (locally trivial) fiber bundle with fiber $Gr_{(e_2-e_3)}(e_1-e_3)$. By using (2.5.10) we get

$$\chi(G_2) = \chi_{(e_1-1,\,e_2,\,e_3)}(Reg_n) = \binom{e_1-e_3-1}{e_2-e_3}\binom{n-e_3}{n-e_1+1}\binom{e_1-1}{e_3}. \qquad (2.5.16)$$

From (2.5.15) and (2.5.16) we get the desired (2.5.5).

2.6 Regular representations and semicanonical basis

In this section we study the Euler–Poincaré characteristic of the quiver Grassmannians associated with *non–rigid* indecomposable Q_{In}–representations. We begin by recalling the classification of such modules.

Proposition 2.6.1. *1. The non–rigid indecomposable Q_{In}-representations have dimension vectors: (n,n,n), $(n,n+1,n)$ and $(n+1,n,n+1)$ for every $n \geq 1$.*

2. The Q_{In}-representations of dimension the imaginary root $n\delta = (n,n,n)$ for every $n \geq 1$ are called regular homogeneous. *They are, up to isomorphisms, the following*

$$Reg_n^{\{3,2\}} \doteq \begin{tikzcd}[ampersand replacement=\&] \& k^n \arrow[dl, "="'] \& \\ k^n \& \& k^n \arrow[ul, "J_n(0)"'] \arrow[ll, "="] \end{tikzcd} ; \qquad Reg_n^{\{2,1\}} \doteq \begin{tikzcd}[ampersand replacement=\&] \& k^n \arrow[dl, "J_n(0)"'] \& \\ k^n \& \& k^n \arrow[ul, "="'] \arrow[ll, "="] \end{tikzcd}$$

$$Reg_n^{\{3,1\}}(\lambda) \doteq \begin{tikzcd}[ampersand replacement=\&] \& k^n \arrow[dl, "="'] \& \\ k^n \& \& k^n \arrow[ul, "="'] \arrow[ll, "J_n(\lambda)"] \end{tikzcd}.$$

where $J_n(\lambda)$ is the n-Jordan block of eigenvalue $\lambda \in k$. The arrows labeled by " $=$ " are the identity map.

3. The Q_{In}-representations of dimension $(n,n+1,n)$ and $(n+1,n,n+1)$ for every $n \geq 0$ are called regular non-homogeneous *(for $n=0$ we recognize the simple regular* rigid *representations introduced in Section 2.5.1). For $n \geq 1$ they have non-trivial self-extensions but their dimension vector is a real root.*

By the Kac's theorem [23, Theorem 3], they are uniquely, up to isomorphisms, determined by their dimensions. We can hence assume they are the following

$$RN_n^w \doteq \begin{tikzcd} & k^{n+1} \arrow[dl, "\varphi_2^t"'] & \\ k^n & & k^n \arrow[ul, "\varphi_1"'] \arrow[ll, "="] \end{tikzcd} ; \qquad RN_n^z \doteq \begin{tikzcd} & k^{n} \arrow[dl, "\varphi_1"'] & \\ k^{n+1} & & k^{n+1} \arrow[ul, "\varphi_2^t"'] \arrow[ll, "="] \end{tikzcd}$$

where $\varphi_1, \varphi_2 :< u_1, \cdots, u_n > \to < v_1, \cdots, v_{n+1} >$, $\varphi_1(u_k) = v_k$ *and* $\varphi_2(u_k) = v_{k+1}$ *have been introduced in Section 2.5.1.*

RN_n^w (resp. RN_n^z) is a regular non–homogeneous Q_{In}–representation that contains (the module corresponding to) w (resp. z) as a submodule; which explains the terminology. The notation also determined uniquely their position in the Auslander–Reiten quiver (see figure 2.3). One can also prove part 3 by a case-by-case inspection assuming Proposition 2.5.5 below. Indeed the representations defined there are indecomposable since their endomorphism ring is local (see [1, III.Example 1.8]).

Lemma 2.6.2.
- *For every non–zero* $\lambda \in k$, $Reg_n^{\{3,1\}}(\lambda)$ *and* $Reg_n^{\{3,1\}}(0))$ *are right–equivalent. In particular*

$$\chi_{\mathbf{e}}(Reg_n^{\{3,1\}}(\lambda)) = \chi_{\mathbf{e}}(Reg_n^{\{3,1\}}(0)).$$

- *For every* Q_{In}*–representation* M *of dimension vector* (d_1, d_2, d_3) *we have*

$$\chi_{(e_1,e_2,e_3)}(DM) = \chi_{(d_3-e_3,d_2-e_2,d_1-e_1)}(M).$$

where D *is the duality functor defined in Section 2.5.1.*

Proof. With notations of Definition 2.5.3, we choose $\phi : Reg_n^{\{3,1\}}(\lambda) \to Reg_n^{\{3,1\}}(0)$ to be the identity, and for the automorphism ψ of kQ we choose the automorphism $c \mapsto \lambda ab + c$.

The duality functor D induces a map $^\perp : Gr_{(e_1,e_2,e_3)}(M) \to Gr_{(d_3-e_3,d_2-e_2,d_1-e_1)}(DM)$ which send $N \mapsto N^\perp \doteq (M/N)^*$ that is an isomorphism of algebraic varieties. $\square$

We simply denote by Reg_n the representation $Reg_n^{\{3,1\}}(0)$.

Proposition 2.6.3.

$$\chi_{(e_1,e_2,e_3)}(Reg_n) = \binom{e_1-e_3}{e_2-e_3}\binom{n-e_3}{n-e_1}\binom{e_1}{e_3} \tag{2.6.1}$$

$$\chi_{\mathbf{e}}(Reg_n^{\{2,1\}}) = \binom{e_1}{e_3}\left[\binom{e_1-e_3}{e_2-e_3}\binom{n-e_3}{n-e_1} + \binom{e_1-e_3}{e_2-e_1-1}\binom{n-1-e_1}{n-1-e_3}\right] \tag{2.6.2}$$

$$\chi_{\mathbf{e}}(Reg_n^{\{3,2\}}) = \binom{n-e_3}{n-e_1}\left[\binom{e_1-e_3}{e_1-e_2}\binom{e_1}{e_3} + \binom{e_1-e_3}{e_3-e_2-1}\binom{e_3-1}{e_1-1}\right] \tag{2.6.3}$$

$$\chi_{(e_1,e_2,e_3)}(RN_n^w) = \binom{n-e_3}{n-e_1}\left[\binom{e_1-e_3}{e_1-e_2+1}\binom{e_1}{e_3} + \binom{e_1-e_3}{e_1-e_2}\binom{e_1}{e_3}\right] \qquad (2.6.4)$$

$$\begin{aligned}\chi_{(e_1,e_2,e_3)}(RN_n^z) = \binom{n-e_3+1}{n-e_1+1}\binom{e_1-e_3}{e_1-e_2-1}\binom{e_1-1}{e_3-1} + \\ +\binom{n-e_3}{n-e_1+1}\binom{e_1-e_3-1}{e_1-e_2-1}\binom{e_1-1}{e_3} + \binom{n-e_3}{n-e_1}\binom{e_1-e_3}{e_1-e_2}\binom{e_1}{e_3} \qquad (2.6.5)\end{aligned}$$

Proof. Clearly $Reg_n^{\{3,2\}} = DReg_n^{\{2,1\}}$ and hence (2.6.3) follows from (2.6.2) by using (2.5.3).
Let us prove (2.6.1) (this proof already appeared in the proof of Proposition 2.5.2). The surjective morphism of algebraic variety

$$\begin{tikzcd} Gr_{(e_1,e_2,e_3)}\big(& & k^n \arrow[dl, "="'] & & & & \\ & k^n & & k^n\big) \arrow[ll, "J_n(0)"] \arrow[ul, "="'] \arrow[r, two heads] & Gr_{(e_1,e_3)}\big(k^n & k^n\big) \arrow[l, shift left, "="'] \arrow[l, shift right, "J_n(0)"] & : (N_1,N_2,N_3) \mapsto (N_1,N_3) \end{tikzcd}$$

sending the tern (N_1, N_2, N_3) onto the pair (N_1, N_3), is a (locally trivial) fiber bundle with fiber $Gr_{(e_2-e_3)}(e_1-e_3)$. By using (2.5.10) we get (2.6.1).
We prove both (2.6.2) and (2.6.4) together by induction on n. For $n = 1$ the surjective morphism of algebraic varieties:

$$\begin{tikzcd} Gr_{(e_1,e_2,e_3)}\big(& & k \arrow[dl, "0"'] & & & & \\ & k & & k\big) \arrow[ll, "="] \arrow[ul, "="'] \arrow[r, two heads] & Gr_{(e_1,e_3)}\big(k & k\big) \arrow[l, shift left, "="'] \arrow[l, shift right, "0"] & : (N_1,N_2,N_3) \mapsto (N_1,N_3) \end{tikzcd}$$

is a fiber bundle with fiber $Gr_{(e_2-e_3)}(k)$; then, by using (2.5.10), we get $\chi_{\mathbf{e}}(Reg_1^{\{2,1\}}) = \binom{1-e_3}{1-e_1}\binom{e_1}{e_3}\binom{1}{e_2-e_3}$ which coincides with (2.6.2) for $n = 1$. For $n = 0$, $RN_0^w = S_2$ and (2.6.4) holds. We now proceed by induction on n. The image of the Jordan block $J_n(0)$ is the subspace of $Reg_{n;1}^{\{2,1\}}$ generated by $\{v_1, \cdots, v_{n-1}\}$; by Lemma 2.5.8 we get $\chi_{\mathbf{e}}(Reg_n^{\{2,1\}}) = \chi_{\mathbf{e}}(G_1) + \chi_{\mathbf{e}}(G_2)$ where

$$\begin{aligned} G_1 &= \{(N_1, N_2, N_3) \in Gr_{\mathbf{e}}(Reg_n^{\{2,1\}}) | N_1 \supseteq kv_n\}, \\ G_2 &= \{(N_1, N_2, N_3) \in Gr_{\mathbf{e}}(Reg_n^{\{2,1\}}) | N_1 \subseteq Im(J_n(0))\}. \end{aligned}$$

It is easy to see that

$$\begin{tikzcd} G_1 \simeq Gr_{(e_1-1,\, e_2-1,\, e_3-1)}\big(& & k^{n-1} \arrow[dl, "J_n(0)"'] & \\ & k^{n-1} & & k^{n-1}\big) \arrow[ll, "="] \arrow[ul, "="'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G_2 \simeq Gr_{\mathbf{e}}\big(& & k^{n} \arrow[dl, "\varphi_2^t"'] & \\ & k^{n-1} & & k^{n-1}\big) \arrow[ll, "="] \arrow[ul, "\varphi_1"'] \end{tikzcd}$$

We hence have:

$$\chi_{\mathbf{e}}(Reg_n^{\{2,1\}}) = \chi_{\mathbf{e-1}}(Reg_{n-1}^{\{2,1\}}) + \chi_{\mathbf{e}}(RN_{n-1}^w).$$

Now (2.6.2) follows by the inductive hypothesis. In order to prove (2.6.4) we again use Lemma 2.5.8. We consider the decomposition of $RN^w_{n;2}$ by the kernel and the image of φ_2^t. We get $\chi_{\mathbf{e}}(RN^w_n) = \chi_{\mathbf{e}}(H_1) + \chi_{\mathbf{e}}(H_2)$ where

$$\begin{aligned} H_1 &= \{(N_1, N_2, N_3) \in Gr_{\mathbf{e}}(RN^w_n) | N_2 \supseteq Ker(\varphi_2^t) = kv_1\}, \\ H_2 &= \{(N_1, N_2, N_3) \in Gr_{\mathbf{e}}(RN^w_n) | N_1 \subseteq Im(\varphi_2^t) = k^n\}. \end{aligned}$$

It is easy to see that

$$H_1 \simeq Gr_{(e_1,\,e_2-1,\,e_3)}\left(\begin{tikzcd} & k^n \arrow[dl, "="'] & \\ k^n & & k^n \arrow[ll, "="] \arrow[ul, "J_n(0)"'] \end{tikzcd}\right) \quad \text{and} \quad H_2 \simeq Gr_{(e_1,\,e_2,\,e_3)}\left(\begin{tikzcd} & k^n \arrow[dl, "="'] & \\ k^n & & k^{n-1} \arrow[ll, "\varphi_2"] \arrow[ul, "\varphi_1"'] \end{tikzcd}\right)$$

We hence get:

$$\chi_{\mathbf{e}}(RN^w_n) = \chi_{(e_1,\,e_2-1,e_3)}(Reg_n^{\{3,2\}}) + \chi_{\mathbf{e}}(\mathbf{M}_{2(n+1)+1}),$$

from which (2.6.4) follows by using (2.6.3) and (2.5.4).
Let us prove (2.6.5). We use Lemma (2.5.8). The image of φ_1 in RN^z_n is generated by $\{v_1, \cdots, v_n\}$. We have $\chi_{\mathbf{e}}(RN^z_n) = \chi_{\mathbf{e}}(G_1) + \mathbf{G_1}$ where

$$\begin{aligned} G_1 &= \{(N_1, N_2, N_3) \in Gr_{\mathbf{e}}(RN^z_n) | N_1 \supseteq kv_{n+1}\}, \\ G_2 &= \{(N_1, N_2, N_3) \in Gr_{\mathbf{e}}(Reg_n^{\{2,1\}}) | N_1 \subseteq Im(\varphi_1)\}. \end{aligned}$$

One can easily recognize that

$$G_1 \simeq Gr_{(e_1-1,\,e_2,\,e_3)}\left(\begin{tikzcd} & k^n \arrow[dl, "="'] & \\ k^n & & k^{n+1} \arrow[ll, "\varphi_1^t"] \arrow[ul, "\varphi_2^t"'] \end{tikzcd}\right) \simeq Gr_{(n+1-e_3,n-e_2,n-e_1+1)}\left(\begin{tikzcd} & k^n \arrow[dl, "\varphi_2"'] & \\ k^{n+1} & & k^n \arrow[ll, "\varphi_1"] \arrow[ul, "="'] \end{tikzcd}\right)$$

and by (2.5.5) we have

$$\begin{aligned} \chi_{\mathbf{e}}(G_1) &= \chi_{(n+1-e_3,n-e_2,n-e_1+1)}(\mathbf{M}_{2(n+2)}) \\ &= \binom{n-e_3}{n-e_1+1}\left[\binom{e_1-e_3}{e_1-e_2-1}\binom{e_1-1}{e_3-1} + \binom{e_1-e_3-1}{e_1-e_2-1}\binom{e_1-1}{e_3}\right]. \end{aligned} \tag{2.6.6}$$

One can also recognize that

$$G_2 \simeq Gr_{(e_1,\,e_2,\,e_3)}\left(\begin{tikzcd} & k^n \arrow[dl, "="'] & \\ k^n & & k^n \arrow[ll, "\varphi_1^t"] \arrow[ul, "J_n(0)"'] \end{tikzcd}\right)$$

and from (2.6.3) we get

$$\begin{aligned} \chi_{\mathbf{e}}(G_2) &= \chi_{\mathbf{e}}(Reg_n^{\{3,2\}}) \\ &= \binom{n-e_3}{n-e_1}\left[\binom{e_1-e_3}{e_1-e_2}\binom{e_1}{e_3} + \binom{e_1-e_3}{e_1-e_2-1}\binom{e_1-1}{e_3-1}\right]. \end{aligned} \tag{2.6.7}$$

By summing up (2.6.6) and (2.6.7) we get the desired (2.6.5). □

The following result gives the relations between polynomials associated with regular Q_{In}–representations.

Corollary 2.6.4.

$$F_{Reg_n^{\{2,1\}}}(y_1, y_2, y_3) = F_{Reg_n}(y_1, y_2, y_3) + y_2 F_{Reg_{n-1}}(y_1, y_2, y_3) \tag{2.6.8}$$

$$F_{Reg_n^{\{2,1\}}} = F_{Reg_n^{\{3,2\}}} \tag{2.6.9}$$

$$F_{RN_n^w} = F_{Reg_n} \cdot F_w = F_{Reg_n \oplus S_2}. \tag{2.6.10}$$

$$F_{RN_n^z} = F_{Reg_n} \cdot F_z = F_{Reg_n \oplus S_{13}}. \tag{2.6.11}$$

Proof. From (2.6.2) we note that

$$\chi_{\{e_1,e_2,e_3\}}(Reg_n^{\{2,1\}}) = \chi_{\{e_1,e_2,e_3\}}(Reg_n) + \chi_{\{e_1,e_2-1,e_3\}}(Reg_{n-1}^{\{2,1\}}).$$

Then (2.6.8) follows from the definition.

Since $Reg_n^{\{3,2\}} = DReg_n^{\{2,1\}}$ (both of dimension vector $n\delta$), by (2.5.3) we have: $\chi_{\mathbf{e}}(Reg_n^{\{2,1\}}) = \chi_{n\delta-\mathbf{e}}(Reg_n^{\{2,1\}})$.

It follows by direct check by using the obvious equalities:

$$\chi_{(e_1,e_2,e_3)}(RN_n^w) = \chi_{(e_1,\,e_2-1,\,e_3)}(Reg_n) + \chi_{(e_1,e_2,e_3)}(Reg_n)$$

and

$$\chi_{\mathbf{e}}(RN_n^z) = \chi_{(e_1-1,\,e_2,\,e_3-1)}(Reg_n) + \chi_{(e_1-1,\,e_2,\,e_3)}(Reg_n) + \chi_{(e_1,e_2,e_3)}(Reg_n).$$

□

Now everything is in place for the introduction of the *semicanonical* basis of $\mathcal{A}$. Recall that $\mathcal{A}$ is a subalgebra of $\mathcal{F} = \mathbb{QP}(x_1, x_2, x_3)$ where $\mathbb{P}$ is the free abelian (multiplicative) group generated by $\{y_1, y_2, y_3\}$; in $\mathcal{F}$ we have already considered the monomials (see (2.4.1))

$$\widehat{y}_1 = \tfrac{y_1}{x_2 x_3}, \quad \widehat{y}_2 = \tfrac{y_2 x_1}{x_3}, \quad \widehat{y}_3 = y_3 x_1 x_2$$

Definition 2.6.5. For every $n \geq 1$ we define

$$s_n = F_{Reg_n}(\widehat{y}_1, \widehat{y}_2, \widehat{y}_3)\mathbf{x}^{(-n,0,n)^t}$$

We also define

$$r_n = F_{Reg_n^{\{2,1\}}}(\widehat{y}_1, \widehat{y}_2, \widehat{y}_3)\mathbf{x}^{(-n,0,n)^t}$$

Note that $(-n, 0, n)^t$ is the $\mathbf{g}$–vector of u_n. The following Proposition gives another two $\mathbb{ZP}$–basis of $\mathcal{A}$.

Proposition 2.6.6.

$$s_n = u_n + \mathbf{y}^{\delta} u_{n-2} + \mathbf{y}^{2\delta} u_{n-4} + \cdots = \sum_{k \geq 0} \mathbf{y}^{k\delta} u_{n-2k}, \tag{2.6.12}$$

$$r_n = s_n + y_2 s_{n-1}. \tag{2.6.13}$$

In particular the set $\mathcal{S} = \{$cluster monomials$\} \cup \{s_n w^k,\ s_n z^k : n \geq 1,\ k \geq 0\}$ and $\mathcal{R} = \{$cluster monomials$\} \cup \{r_n w^k,\ r_n z^k : n \geq 1,\ k \geq 0\}$ are $\mathbb{ZP}$–basis of $\mathcal{A}$. We call $\mathcal{S}$ a semicanonical basis of $\mathcal{A}$.

Proof. Recall from (2.3.10) that the F–polynomial of u_n is given by

$$F_{u_n} = y_1^n y_2^n y_3^n + \sum_{\mathbf{e}=(e_1,e_2,e_3)} \chi_{u_n}(\mathbf{e}) y_1^{e_1} y_2^{e_2} y_3^{e_3} + 1.$$

where $\chi_{u_n}(e_1, e_2, e_3) \doteq \binom{e_1-e_3}{e_2-e_3}[\binom{n-e_3}{n-e_1}\binom{e_1-1}{e_3} + \binom{n-e_3-1}{n-e_1}\binom{e_1-1}{e_3-1}]$ and $F_{u_0} = 1$. By using the identity $\binom{a-1}{b-1} + \binom{a-1}{b} = \binom{a}{b} - \delta_{a,0}\delta_{b,0}$, we have

$$\chi_{\mathbf{e}}(Reg_n) = \chi_{u_n}(\mathbf{e}) + \chi_{\mathbf{e}-\mathbf{1}}(Reg_{n-2}) + \delta_{e_1,0}\delta_{e_2,0}\delta_{e_3,0} + \delta_{e_1,n}\delta_{e_2,n}\delta_{e_3,n}$$

where $\mathbf{1} = (1, 1, 1)$ and δ is the Kronecker delta. We then have:

$$F_{u_n}(y_1, y_2, y_3) = F_{Reg_n}(y_1, y_2, y_3) - y_1 y_2 y_3 F_{Reg_{n-2}}(y_1, y_2, y_3) \tag{2.6.14}$$

from which one can easily prove that $F_{Reg_n} = F_{u_n} + \mathbf{y}^{\delta} F_{u_{n-2}} + \mathbf{y}^{2\delta} F_{u_{n-4}} + \cdots$ by induction on n. Since $\widehat{\mathbf{y}}^{\delta} =$Then (2.6.12) follows from the definition.

The equation (2.6.13) follows from (2.6.8). □

Corollary 2.6.7. *For every $n \geq 2$ we have*

$$u_n = s_n - y_1 y_2 y_3 s_{n-2}. \tag{2.6.15}$$

Moreover $u_1 = s_1$ and $u_0 = s_0 = 1$.

Remark 2.6.8. When all the coeficients y_1, y_2 and y_3 equals 1, the relation (2.6.15) becomes the well known relation between Chebychev's polynomials of the first kind and Chebychev's polynomials of the second kind. Moreover in this setting the straightening relation (2.1.20) becomes $u_{n+1} = u_1 u_n - u_{n-1}$; we can hence see that $u_n = T_n(u_1)$ is the n-th Chebychev's polynomial $T_n(u_1)$ of the first kind computed at $u_1 = zw - 2$ and $s_n = U_n(u_1)$ where U_n is the n–th Chebychev's polynomial of the second kind.

Example 2.6.9. The family $\{s_n\}$ satisfies property [hom], [F], [$\mathbf{g}$] and [B^0] of Theorem 1.5.7, and hence they are linearly independent over $\mathbb{ZP}$. They do not satisfy property [ind], since the Laurent monomial $x_3^{n-2}/x_1^{n-2} = \mathbf{x}^{\mathbf{g}_{s_{n-2}}}$ appears with coefficient 1 in the Laurent expansion of s_{n-2} and with coefficient $y_1 y_2 y_3$ in the Laurent expansion of s_n (as one can easily see from its definition 2.6.5).

Chapter 3

Cluster algebras of type $C_2^{(1)}$ and $G_2^{(1)}$

Let $\mathbb{P} = \mathrm{Trop}(y_1, y_2, y_3)$ be the tropical semifield generated by the elements y_1, y_2 and y_3. Let $\mathcal{F} = \mathbb{QP}(x_1, x_2, x_3)$ be the field of rational functions in three commuting variables x_1, x_2 and x_3 over the field of fractions of the group ring $\mathbb{ZP}$. We study two particular cluster algebras inside $\mathcal{F}$ of type $C_2^{(1)}$ and $G_2^{(1)}$.

3.1 Type $C_2^{(1)}$

Let $\mathcal{C}$ be the cluster algebra inside $\mathcal{F}$ with (principal coefficients at the) initial seed

$$\Sigma_0 = \{B = B(t_0) = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}, \{x_{1;0}, x_{2;0}, x_{3;0}\}, \{y_1, y_2, y_3\}\}.$$

The Cartan counterpart of B is the matrix

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \tag{3.1.1}$$

of type $C_2^{(1)}$. Then $\mathcal{C}$ is a cluster algebra of type $C_2^{(1)}$ with principal coefficients at the initial seed Σ_0. $\mathcal{C}$ is a cluster algebra of bipartite type, hence we have all the results of section 1.3 at our disposal. $\mathcal{C}$ is generated inside $\mathcal{F}$ by the *distinct* elements $\{x_{i;2m} : \ i = 1, 2, 3; \ m \in \mathbb{Z}\}$ defined by the exchange relations in the Corollary 1.3.6, i.e. denoted by $\mathbf{d}(x_{i;2m})$ the denominator vector of $x_{i;2m}$ in the initial cluster, we have the initial conditions

$$x_{1;0}x_{1;2} = y_1 x_{2;0}^2 + 1 \tag{3.1.2}$$

$$x_{2;0}x_{2;-2} = y_2 x_{1;0}x_{3;0} + 1 \tag{3.1.3}$$

$$x_{3;0}x_{3;2} = y_3 x_{2;0}^2 + 1 \tag{3.1.4}$$

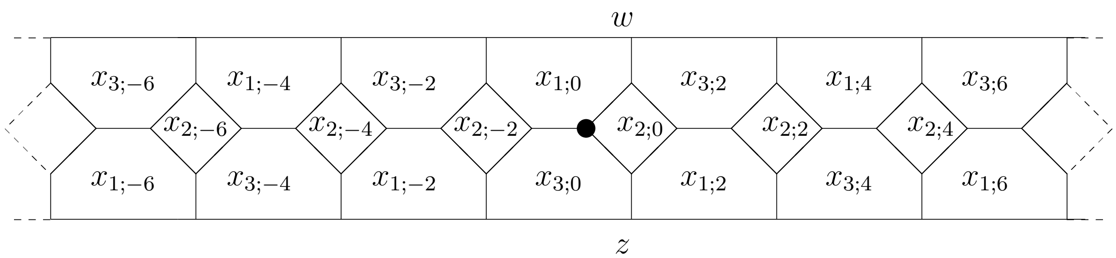

Figure 3.1: Exchange Graph of the cluster algebra $\mathcal{C}$

together with the recursive relations:

$$x_{1;2m}x_{1;2m+2} = x_{2;2m}^2 + \mathbf{y}^{\mathbf{d}(x_{1;2m})} \tag{3.1.5}$$
$$x_{2;2m}x_{2;2m+2} = \mathbf{y}^{\mathbf{d}(x_{2;2m})}x_{1;2m}x_{3;2m} + 1 \tag{3.1.6}$$
$$x_{3;2m}x_{3;2m+2} = x_{2;2m}^2 + \mathbf{y}^{\mathbf{d}(x_{3;2m})} \tag{3.1.7}$$

and by the elements

$$w \doteq \frac{x_{1;0}x_{3;0} + y_2y_3x_{2;0}^2 + y_2}{x_{2;0}x_{3;0}} \tag{3.1.8}$$
$$z \doteq \frac{x_{1;0}x_{3;0} + y_1y_2x_{2;0}^2 + y_2}{x_{1;0}x_{2;0}} \tag{3.1.9}$$
$$\tag{3.1.10}$$

The exchange graph of $\mathcal{C}$ is given in figure 3.1.

Proposition 3.1.1. *For every $m \neq 0$ the denominator vector of $x_{1;2m}$, $x_{2;2m}$ and $x_{3;2m}$ in the initial cluster $\{x_{1;0}, x_{2;0}, x_{3;0}\}$ is given by: For every $m \geq 1$*

$$\mathbf{d}(x_{1;2m}) = \begin{cases} \begin{pmatrix} m \\ 2(m-1) \\ m-1 \end{pmatrix} = \alpha_1 + (m-1)\delta & \text{if } m \text{ is odd,} \\ \begin{pmatrix} m-1 \\ 2(m-1) \\ m \end{pmatrix} = \alpha_3 + (m-1)\delta & \text{if } m \text{ is even} \end{cases} \tag{3.1.11}$$

$$\mathbf{d}(x_{2;2m}) = \begin{pmatrix} m \\ 2m-1 \\ m \end{pmatrix} = -\alpha_2 + m\delta \tag{3.1.12}$$

$$\mathbf{d}(x_{3;2m}) = \begin{cases} \begin{pmatrix} m \\ 2(m-1) \\ m-1 \end{pmatrix} = \alpha_1 + (m-1)\delta & \text{if } m \text{ is even} \\ \begin{pmatrix} m-1 \\ 2(m-1) \\ m \end{pmatrix} = \alpha_3 + (m-1)\delta & \text{if } m \text{ is odd} \end{cases} \tag{3.1.13}$$

$$\mathbf{d}(x_{1;-2m}) = \begin{cases} \begin{pmatrix} m \\ 2m \\ m-1 \end{pmatrix} = -\alpha_3 + m\delta & \textit{if } m \textit{ is odd,} \\ \begin{pmatrix} m-1 \\ 2m \\ m \end{pmatrix} = -\alpha_1 + m\delta & \textit{if } m \textit{ is even} \end{cases} \tag{3.1.14}$$

$$\mathbf{d}(x_{2;-2m}) = \begin{pmatrix} m-1 \\ 2(m-1)+1 \\ m-1 \end{pmatrix} = \alpha_2 + (m-1)\delta \tag{3.1.15}$$

$$\mathbf{d}(x_{3;-2m}) = \begin{cases} \begin{pmatrix} m \\ 2m \\ m-1 \end{pmatrix} = -\alpha_3 + m\delta & \textit{if } m \textit{ is even} \\ \begin{pmatrix} m-1 \\ 2m \\ m \end{pmatrix} = -\alpha_1 + m\delta & \textit{if } m \textit{ is odd} \end{cases} \tag{3.1.16}$$

where $\delta = (1,2,1)^t$ (cf. Section 1.2.3). For every $n \geq 1$:

$$\mathbf{d}(w) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{d}(z) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{d}(u_n) = \begin{pmatrix} n \\ 2n \\ n \end{pmatrix} = n\delta. \tag{3.1.17}$$

Proof of Proposition 3.1.1. By the echange relations we have for every non–zero integer m:

$$\mathbf{d}(x_{i;2m}) + \mathbf{d}(x_{i;2m+2}) = 2\mathbf{d}(x_{2;2m})$$

from which all the formulas (3.1.11)–(3.1.16) follow by induction on m by using the obvious initial condition given by the exchange relations (3.1.2)–(3.1.4).

The denominator vector of w and z is recognized by their definition (3.1.8) and (3.1.9). The denominator vector of u_n is recognized directly by its definition (3.1.20) since $\mathbf{d}(u_1) = \mathbf{d}(z) + \mathbf{d}(w) = \delta$ and

$$\mathbf{d}(u_{n+1}) = \mathbf{d}(u_1) + \mathbf{d}(u_n).$$

□

We want to find a canonical basis of $\mathcal{C}$. In order to do that the following definition is fundamental.

Definition 3.1.2. For every $n \geq 2$ we define the elements of $\mathcal{F}$ by

$$u_1 = zw - y_2y_3 - y_1y_2 \tag{3.1.18}$$
$$u_2 = u_1^2 - 2\mathbf{y}^\delta \tag{3.1.19}$$
$$u_{n+1} = u_1u_n - \mathbf{y}^\delta u_{n-1} \tag{3.1.20}$$

where $\delta \doteq (1,2,1)^t$.

Conjecture 3.1.3. *The set $\mathbf{B} = \{$cluster monomials$\} \cup \{u_nw^k, u_nz^k :\ n \geq 1,\ k \geq 0\}$ is a canonical basis of the cluster algebra $\mathcal{C}$, i.e. $\mathbf{B}$ is a $\mathbb{ZP}$–basis of $\mathcal{C}$ and its elements are positive indecomposable.*

This conjecture is motivating by the results obtained in the coefficient–free setting that we are going to give in the next section.

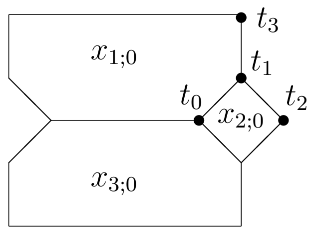

Figure 3.2: Exchange Graph around the initial vertex t_0

3.2 Coefficient–free cluster algebra of type $C_2^{(1)}$

In this section we study the coefficient–free cluster algebra of type $C_2^{(1)}$. We find the canonical basis for this algebra and we find explicit formulas for its elements. This section is independent on the others and completely self–contained.

3.2.1 Algebraic structure of a (coefficient-free) cluster algebra of type $C_2^{(1)}$

We study of the (coefficient-free) cluster algebra $\mathcal{C}$ with initial seed

$$\{B = B(t_0) = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}, \{r_1, s_1, v_1\}\}.$$

We are changing notations with respect to the general case since it is convenient. The corresponding Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \tag{3.2.1}$$

of type $C_2^{(1)}$. Let $B(i)$ be the principal submatrix of $B(\mathcal{C})$ obtained by removing the i-th row and column. Since $B(\mathcal{C})$ is of affine type, $B(1)$, $B(2)$ and $B(3)$ are all of finite type. More precisely $B(1)$ is of type C_2, $B(2)$ is of type $A_1 \times A_1$ and $B(3)$ of type B_2. The corresponding Coxeter numbers are respectively $h = 4$, $h = 2$ and $h = 4$. By the useful result [16, Theorem 7.7], we know the cluster variable r_1 appears in exactly $h + 2 = 6$ different seeds, the variable s_1 in exactly $h + 2 = 4$ different seeds and the variable v_1 in exactly $h + 2 = 6$ different seeds. So in the exchange graph they must appear in the way showed by the figure 3.2. In this figure vertices correspond to seeds and a variable inside a region touching a vertex t, corresponds to a cluster variable of the seed in t. For example t_0 is the initial vertex (i.e. the vertex corresponding to the initial seed), t_1 corresponds to the seed obtained after a mutation in direction 3 from the initial one and t_2 is obtained from t_1 by mutation in direction 1. The matrices in t_1 and t_2 are respectively

$$B(t_1) = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } B(t_2) = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix}.$$

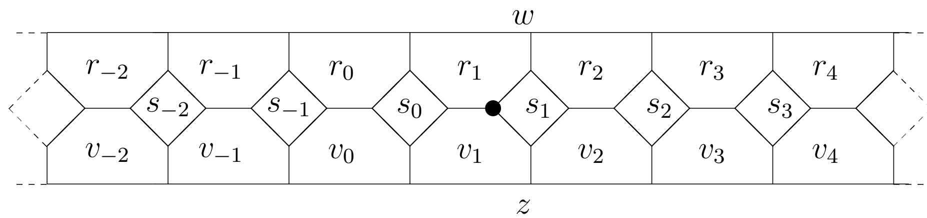

Figure 3.3: The Exchange Graph of $\mathcal{C}$.

We note that B is invariant under the permutation (13) of the index set. Clearly for every permutation σ of the index set and every index i the following diagram

$$\begin{array}{ccc} B & \xrightarrow{\mu_i} & B^i \\ \sigma\downarrow & & \downarrow\sigma \\ B' & \xrightarrow{\mu_{\sigma(i)}} & B'^{(i)} \end{array} \tag{3.2.2}$$

commutes. In particular the mutation in direction 1 of B gives the matrix $(13)B(t_1)$ obtained from $B(t_1)$ after the permutation (13) of the index set. We've found all the matrices in the square. The mutation of $B(t_1)$ in direction 2 gives the matrix

$$B(t_3) = \begin{pmatrix} 0 & 1 & -2 \\ -2 & 0 & 2 \\ 2 & -1 & 0 \end{pmatrix}.$$

By (3.2.2) the matrix on the opposite vertex to t_3 with respect to s_1 is $(13)B(t_3)$. Moreover we note that the mutation of $B(t_2)$ in direction 2 is nothing but B. Then we have obtained both all the exchange matrices for the cluster algebra $\mathcal{C}$ and a surjective map from the graph in the figure 3.3 and the *exchange graph* of $\mathcal{C}$. Consequently we've obtained all the *exchange relations* of $\mathcal{C}$:

$$r_m v_{m+1} = s_m^2 + 1 \tag{3.2.3}$$

$$v_m r_{m+1} = s_m^2 + 1 \tag{3.2.4}$$

$$w s_m = r_m + r_{m+1} \tag{3.2.5}$$

$$s_{m-1} s_m = r_m v_m + 1 \tag{3.2.6}$$

$$z s_m = v_m + v_{m+1} \tag{3.2.7}$$

$$r_{m-1} r_{m+1} = r_m^2 + w^2 \tag{3.2.8}$$

$$v_{m-1} v_{m+1} = v_m^2 + z^2 \tag{3.2.9}$$

We note that w and z appear in infinitely many clusters otherwise r_m and v_m wouldn't appear in exactly 6 clusters. To see that the graph in the figure 3.3 is the exchange graph of $\mathcal{C}$ we have only to prove that the cluster variables are all distinct (i.e. the map above is also injective). In the initial cluster $\{r_1, v_1, s_1\}$ we have the *denominator map*

$$\mathbf{d} : \mathcal{C} \to Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3 \ : \ y = \frac{F_y(r_1, s_1, v_1)}{r_1^{d_1} s_1^{d_2} v_1^{d_3}} \mapsto d_1\alpha_1 + d_2\alpha_2 + d_3\alpha_3 \tag{3.2.10}$$

Clearly $\mathbf{d}(r_1) = -\alpha_1$, $\mathbf{d}(s_1) = -\alpha_2$ and $\mathbf{d}(v_1) = -\alpha_3$.

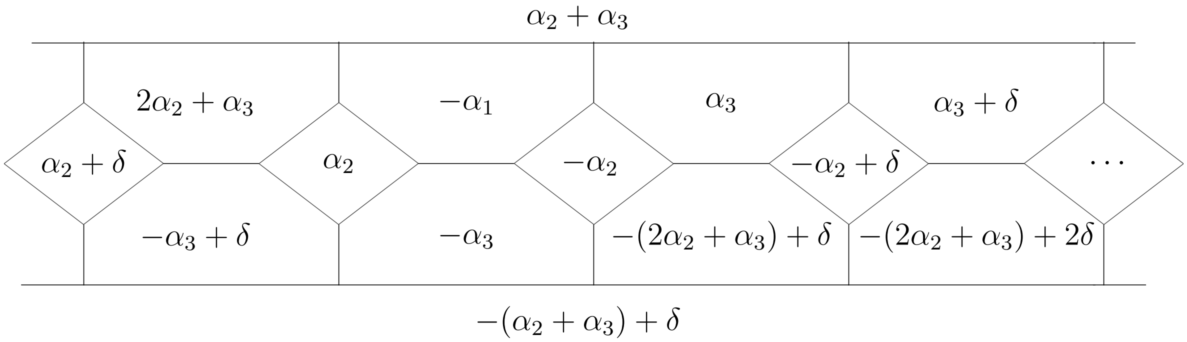

Figure 3.4: Denominator vectors in the initial cluster

Proposition 3.2.1. *For every $n \geq 1$:*

$$\mathbf{d}(r_{n+1}) = \begin{pmatrix} n-1 \\ 2(n-1) \\ n \end{pmatrix} = \alpha_3 + (n-1)\delta \tag{3.2.11}$$

$$\mathbf{d}(s_n) = \begin{pmatrix} n-1 \\ 2(n-1)-1 \\ n-1 \end{pmatrix} = -\alpha_2 + (n-1)\delta \tag{3.2.12}$$

$$\mathbf{d}(v_{n+1}) = \begin{pmatrix} n \\ 2(n-1) \\ n-1 \end{pmatrix} = \alpha_1 + (n-1)\delta \tag{3.2.13}$$

For every $n \leq 0$

$$\mathbf{d}(r_{n+1}) = \begin{pmatrix} n-1 \\ 2n \\ n \end{pmatrix} = -\alpha_1 + n\delta \tag{3.2.14}$$

$$\mathbf{d}(s_n) = \begin{pmatrix} n \\ 2n+1 \\ n \end{pmatrix} = \alpha_2 + n\delta \tag{3.2.15}$$

$$\mathbf{d}(v_{n+1}) = \begin{pmatrix} n \\ 2n \\ n-1 \end{pmatrix} = -\alpha_3 + n\delta \tag{3.2.16}$$

where $\delta = (1,2,1)^t$ (cf. Section 1.2.3). For every $n \geq 1$:

$$\mathbf{d}(w) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{d}(z) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \tag{3.2.17}$$

Proof. It follows from Proposition 3.1.1. $\square$

We conclude that the cluster variables are all distinct. The figure 3.4 shows the denominator vectors of the cluster variables. In this figure we put out the correspondence with the real roots of type $C_2^{(1)}$ once we choose $\mathring{\Pi} = \{\alpha_2, \alpha_3\}$ (see Section 1.2.3).

By the exchange relations the following useful formulas hold

$$wv_m = zr_m = s_{m-1} + s_m \tag{3.2.18}$$

Indeed

$$wv_m s_m = (r_m + r_{m+1})v_m = r_m v_m + s_m^2 + 1 = zr_m s_m = s_{m-1}s_m + s_m^2$$

We introduce an element $u \in \mathcal{A}$ by setting

$$u = zw - 2. \tag{3.2.19}$$

Let $T_0, T_1, \ldots$ be the sequence of Chebyshev polynomials of the first kind given by $T_0 = 1$, and $T_n(t + t^{-1}) = t^n + t^{-n}$ for $n > 0$. We define the sequence $u_1, u_2, \ldots$ of elements of $\mathcal{C}$ by setting $u_n = T_n(u)$. From (3.2.19) we obtain

$$\mathbf{d}(u) = \mathbf{d}(z) + \mathbf{d}(w) = \delta.$$

If we consider the automorphism t of $\mathcal{C}$ which sends x_m to x_{m+2}, for $x = r, s$ or v, we have by definition

$$u_n x_m = x_{m-n} + x_{m+n}. \tag{3.2.20}$$

In particular by induction on n, it can be shown that:

$$\mathbf{d}(u_n) = \begin{pmatrix} n \\ 2n \\ n \end{pmatrix} = n\delta. \tag{3.2.21}$$

3.2.2 Explicit Laurent expansions

In this section we give the explicit Laurent expansion of the elements of the set $\mathbf{B} = \{\text{cluster monomials}\} \cup \{u_n w^k, u_n z^k\}$ in the initial cluster $\{r_1, s_2, v_1\}$ of $\mathcal{C}$.

Theorem 3.2.2. • *For every $n \geq 2$ the following formulas hold:*

$$r_n = \frac{s_1^{4n-6} + \sum_{q+r\leq 2n-4} \binom{2n-4-r}{q}\binom{2n-3-q}{r} r_1^q s_1^{2r} v_1^q}{r_1^{n-2} s_1^{2n-4} v_1^{n-1}} \tag{3.2.22}$$

$$s_n = \frac{s_1^{4n-4} + \sum_{q+r\leq 2n-3} \binom{2n-3-r}{q}\binom{2n-2-q}{r} r_1^q s_1^{2r} v_1^q}{r_1^{n-1} s_1^{2n-3} v_1^{n-1}} \tag{3.2.23}$$

$$v_n = \frac{s_1^{4n-6} + \sum_{q+r\leq 2n-4} \binom{2n-4-r}{q}\binom{2n-3-q}{r} r_1^q s_1^{2r} v_1^q}{r_1^{n-1} s_1^{2n-4} v_1^{n-2}} \tag{3.2.24}$$

• *For every $n \geq 0$ the following formulas hold*

$$r_{-n} = \frac{(r_1 v_1)^{2n+2} + \sum_{q+r\leq 2n+1} \binom{2n+2-r}{q}\binom{2n+1-q}{r} r_1^q s_1^{2r} v_1^q}{r_1^{n} s_1^{2n+2} v_1^{n+1}} \tag{3.2.25}$$

$$s_{-n} = \frac{(r_1 v_1)^{2n+1} + \sum_{q+r\leq 2n} \binom{2n+1-r}{q}\binom{2n-q}{r} r_1^q s_1^{2r} v_1^q}{r_1^{n} s_1^{2n+1} v_1^{n}} \tag{3.2.26}$$

$$v_{-n} = \frac{(r_1 v_1)^{2n+2} + \sum_{q+r\leq 2n+1} \binom{2n+2-r}{q}\binom{2n+1-q}{r} r_1^q s_1^{2r} v_1^q}{r_1^{n+1} s_1^{2n+2} v_1^{n}} \tag{3.2.27}$$

• *For every $n \geq 1$ the following formula holds*

$$u_n = \frac{(r_1 v_1)^{2n} + s_1^{4n} + \sum_{q+r\leq 2n-1} \frac{2n}{2n-q-r}\binom{2n-1-r}{q}\binom{2n-1-q}{r} r_1^q s_1^{2r} v_1^q}{r_1^{n} s_1^{2n} v_1^{n}} \tag{3.2.28}$$

Proof. By the symmetries in the exchange relations, the Laurent expansion of r_{-n} $(n \geq 0)$(resp. $s_{-n}(n \geq 1)$, $v_{-n}(n \geq 0)$) in the cluster $\{r_1, s_0, v_1\}$ is obtained from the Laurent expansion of r_{n+2} (resp. s_{n+1}, v_{n+2}) in the cluster $\{r_1, s_1, v_1\}$ by exchanging s_1 with s_0. Then, applying the relation $s_0 s_1 = r_1 v_1 + 1$, the formula (3.2.25) (resp. (3.2.26), (3.2.27)) follows from (3.2.22) (resp. (3.2.23), (3.2.24)).

Moreover the Laurent expansion of v_n in the cluster $\{r_1, s_1, v_1\}$ is obtained from the Laurent expansion of r_n by interchanging the variables r_1 and v_1. Then (3.2.24) follows from (3.2.22).

It remains to prove both (3.2.22) and (3.2.23).

For $n = 2$, (3.2.22) is nothing but the exchange relation (3.2.3), while (3.2.23) becomes

$$s_2 = \frac{s_1^4 + \sum_{q+r \leq 1} \binom{1-r}{q}\binom{2-q}{r} r_1^q s_1^{2r} v_1^q}{r_1 s_1 v_1} = \frac{s_1^4 + 1 + r_1 v_1 + 2s_1^2}{r_1 s_1} = \frac{r_2 v_2 + 1}{s_1}$$

that is (3.2.6). By induction on n a direct calculation shows that the right hand side of (3.2.22) (resp. (3.2.23)) satisfies (3.2.5) (resp. (3.2.18)). $\square$

3.2.3 Canonical basis

Theorem 3.2.3. *The set* $\mathbf{B} = \{$*cluster monomials*$\} \cup \{u_n w^k, u_n z^k \,|\, n \geq 1,\ k \geq 0\}$ *is the canonical basis of* $\mathcal{C}$.

Proof. The proof is organized as follows: in Corollary 3.2.5 we prove that $\mathbf{B}$ is a linearly independent set over $\mathbb{Z}$; by Theorem 3.2.2 it follows that the elements of $\mathbf{B}$ are positive; in Section 3.2.4 we prove the elements of $\mathbf{B}$ are positive indecomposable and that $\mathbf{B}$ spans $\mathcal{C}$ over $\mathbb{Z}$. $\square$

Our next result provides a parametrization of $\mathbf{B}$. Let $Q = \mathbb{Z}^3$ be the root lattice of type $C_2^{(1)}$. We fix the basis $\{\alpha_1, \alpha_2, \alpha_3\}$ of simple roots. We will sometimes write a point $\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 \in Q$ simply as $\alpha = (a_1, a_2, a_3)$.

Theorem 3.2.4. *In the situation of Theorem* 3.2.3*, for every* $\alpha = (a_1, a_2, a_3) \in Q$*, there is a unique basis element* $b = b[\alpha] \in \mathbf{B}$ *of the form*

$$b[\alpha] = \frac{N_\alpha(x_1, x_2, x_3)}{x_1^{a_1} x_2^{a_2} x_3^{a_3}}, \tag{3.2.29}$$

where N_α *is a polynomial with constant term* 1*. The correspondence* $\alpha \mapsto b[\alpha]$ *is a bijection between* Q *and* $\mathbf{B}$*. Under this bijection the cluster variables different from* $\{r_1, s_1, v_1\}$ *and the elements* $\{u_n w, u_n z\}$ *correspond to all the positive real roots of the Kac-Moody algebra of type* $C_2^{(1)}$*.*

Proof. The second part follows by Proposition 3.2.1 using the results of Section 1.2.3.

We now prove the bijection with Q. Since $\mathbf{d}(u_n) = n\mathbf{d}(u_1)$, $\mathbf{d}(u_1) = \mathbf{d}(z) + \mathbf{d}(w)$ and $\mathbf{d}(w)$ and $\mathbf{d}(z)$ lie in the positive octant, the image of $\{u_n z,\ u_n w : n \geq 0\}$ is the cone

$$\mathcal{C}_{Im} := \mathbb{Z}_{\geq 0}\mathbf{d}(z) + \mathbb{Z}_{\geq 0}\mathbf{d}(w) = \mathbb{Z}_{\geq 0}(\alpha_1 + \alpha_2) + \mathbb{Z}_{\geq 0}(\alpha_2 + \alpha_3)$$

Then it remains to show the following:

$$\text{For every cluster } \{t_1, t_2, t_3\}, \text{ the vectors } \mathbf{d}(t_1),\ \mathbf{d}(t_2) \text{ and } \mathbf{d}(t_3) \text{ form a } \mathbb{Z}\text{-basis of } Q. \tag{3.2.30}$$

$$\text{For every cluster } \{t_1, t_2, t_3\}, \text{ the vectors } \mathbf{d}(t_1),\ \mathbf{d}(t_2) \text{ and } \mathbf{d}(t_3) \text{ are the only positive real roots contained in the additive semigroup } \mathcal{C}_{\{t_1,t_2,t_3\}} := \mathbb{Z}_{\geq 0}\mathbf{d}(t_1) + \mathbb{Z}_{\geq 0}\mathbf{d}(t_2) + \mathbb{Z}_{\geq 0}\mathbf{d}(t_3). \tag{3.2.31}$$

$$\text{The union } \bigcup \mathcal{C}_{\{t_1,t_2,t_3\}} \text{ is equal to } Q - \mathcal{C}_{\mathrm{Im}}. \tag{3.2.32}$$

From Proposition 3.2.1 it follows by direct check that given a cluster $\{s_1, s_2, s_3\}$,

$$\det(\mathbf{d}(s_1), \mathbf{d}(s_2), \mathbf{d}(s_3)) = \pm 1 \tag{3.2.33}$$

(here *det* is the determinant view as a multi-linear function on the column of matrices) and (3.2.30) is proved.

In order to prove (3.2.31) we first observe that the clusters containing at least one initial cluster variable r_1, s_1 or v_1 (showed in figure3.5) satisfy both (3.2.31) and (3.2.32). Indeed their union equals $Q \setminus \overset{\circ}{Q}_+$ where $\overset{\circ}{Q}_+$ is the strictly positive octant

$$\overset{\circ}{Q}_+ = \{\lambda = \sum_{i=1}^{3} k_i \alpha_i \in Q : k_i > 0\}.$$

Moreover the rest of the denominator vectors $\{\mathbf{d}(s) : s \neq r_1, s_1, v_1\}$ are contained in $Q_+ \setminus \overset{\circ}{\mathcal{C}}_{Im}$ where $\overset{\circ}{\mathcal{C}}_{Im}$ is the interior of $\mathcal{C}_{Im}$. Then they satisfy (3.2.31). Now we can consider the other clusters. We are going to show that whenever $\mathbf{d}(t)$ lies in $\mathcal{C}_{\{t_1,t_2,t_3\}}$ for a cluster variable t, then t is either t_1 or t_2 or t_3. Let us consider the operator

$$\sigma_2(\alpha) = \begin{cases} -\alpha & \text{if } \alpha = \alpha_2 \\ 2\alpha_2 + \alpha & \text{if } \alpha = \alpha_1, \alpha_3 \end{cases}$$

defined on the generators and extended by linearity to Q. σ_2 fixes $\mathcal{C}_{Im}$ pointwise, then the subspace generated by $\mathcal{C}_{Im}$ is the maximal subspace with this property. In particular $\sigma_2(\alpha_i + \delta) = \sigma_2(\alpha_i) + \delta$ for $i = 1, 2, 3$. It follows σ_2 induces an involution in the set of the clusters not containing r_1 or v_1 which sends s_m into s_{1-m}. Then it is sufficient to consider only the case in which all t, t_1, t_2 and t_3 are r_m, s_m, v_m with $m \geq 2$, s_1, z or w. By Proposition3.2.1, the corresponding denominator vectors are all contained in the cone $\mathcal{D}$ in Q spanned by $-\alpha_2$ and $\mathcal{C}_{Im}$. Now we can reduce further our case by case proof by considering the involution of Q defined on the generators by

$$(13) : \alpha_1 \mapsto \alpha_3 \text{ and } \alpha_2 \mapsto \alpha_2.$$

It induces an automorphism of the exchange graph sending r_1 into v_1. It sends clusters in clusters. The maximal subspace fixed by (13) contains α_2 and $\alpha_1 + \alpha_3$. It cuts $\mathcal{D}$ into two cones. Now we only have to treat the corresponding two cases.

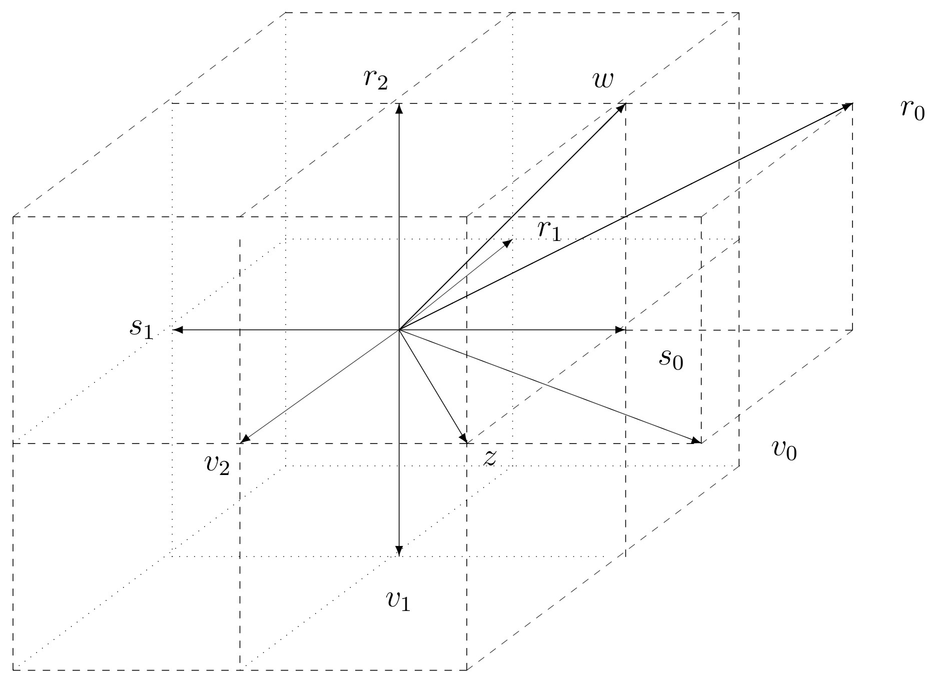

Figure 3.5: Denominator vectors having at least a zero coordinate

First of all let us assume $s = \{r_m, w, r_{m+1}\}$ and $t \in \{r_{m+n+1}, w, r_{m+n+2}\}$ for some $n \geq 0$. $\mathbf{d}(r_{m+n+1})$ and $\mathbf{d}(r_{m+n+2})$ lie on the opposite side of the plane through $\mathbf{d}(w)$ and $\mathbf{d}(r_{m+1})$ with respect to $\mathbf{d}(r_m)$ (indeed $\mathbf{d}(r_{m+n+1}) = \mathbf{d}(r_{m+1}) + (n+1)\delta$ and $\mathbf{d}(r_m) = \mathbf{d}(r_{m+1}) - \delta$). It follows they can't be in $\mathcal{C}(s)$. Moreover the other cluster variables lie on the (closed) half-space of Q not containing w. So let us consider the second case. Let us first assume

$$s = \{r_m, s_m, r_{m+1}\}_{m \geq 2}$$

We have

$$\begin{array}{rcll} \mathbf{d}(r_{m+n+1}) & = & \mathbf{d}(r_{m+1}) + n\delta & \\ \mathbf{d}(v_{m+n}) & = & \mathbf{d}(v_{m+1}) + (n-1)\delta & = -\mathbf{d}(r_{m+1}) + 2\mathbf{d}(s_m) + n\delta \\ \mathbf{d}(s_{m+n}) & = & \mathbf{d}(s_m) + n\delta & \end{array}$$

then they are contained in the half-space of Q opposite to $\mathbf{d}(r_m)$ with respect to the plane through $\mathbf{d}(s_m)$ and $\mathbf{d}(r_{m+1})$ for every $n \geq 1$. Moreover

$$\mathbf{d}(v_m) = 2\mathbf{d}(s) - \mathbf{d}(r_m) - \delta$$

so $\mathbf{d}(v_m)$ is opposite to $\mathbf{d}(r_{m+1}) = \mathbf{d}(r_m) + \delta$ with respect to the 2-dimensional subspace through $\mathbf{d}(s_m)$ and $\mathbf{d}(r_m)$. We conclude that $\mathbf{d}(r_k)$, $\mathbf{d}(s_k)$, $\mathbf{d}(v_k)$ are not contained in $\mathring{\mathcal{C}}(s)$ for $k \geq m$. Analogously when

$$s = \{r_m, s_m, v_m\}$$

we have

$$\begin{array}{rcl}\mathbf{d}(r_{m+n}) & = & \mathbf{d}(r_m)+n\delta \;\; = -\mathbf{d}(v_m)+2\mathbf{d}(s_m)+(n-1)\delta \\ \mathbf{d}(s_{m+n}) & = & \mathbf{d}(s_m)+n\delta\end{array}$$

for $n \geq 1$ they are opposite to $\mathbf{d}(r_m)$ with respect to the plane through $\mathbf{d}(s_m)$ and $\mathbf{d}(v_m)$. Moreover

$$\mathbf{d}(v_{m+n}) = \mathbf{d}(v_m)+n\delta = 2\mathbf{d}(s_m)-\mathbf{d}(r_m)+(n-1)\delta$$

for $n \geq 1$ are opposite to $\mathbf{d}(v_m)$ with respect to the plane through $\mathbf{d}(r_m)$ and $\mathbf{d}(s_m)$. Finally when

$$s = \{r_{m+1}, s_m, v_{m+1}\}$$

we have

$$\begin{array}{rcl}\mathbf{d}(s_m) & = & \frac{1}{2}\mathbf{d}(r_{m+1})+\frac{1}{2}\mathbf{d}(v_{m+1})-\frac{1}{2}\delta \\ \mathbf{d}(s_{m+n}) & = & \frac{1}{2}\mathbf{d}(r_{m+1})+\frac{1}{2}\mathbf{d}(v_{m+1})+(\frac{1}{2}+n)\delta \\ \mathbf{d}(r_{m+1+n}) & = & \mathbf{d}(r_{m+1})+n\delta \\ \mathbf{d}(v_{m+1+n}) & = & \mathbf{d}(v_{m+1})+n\delta\end{array}$$

so they are opposite to $\mathbf{d}(s_m)$ with respect to the plane through $\mathbf{d}(r_{m+1})$ and $\mathbf{d}(v_{m+1})$. This shows (3.2.31).

In order to show (3.2.32) it is sufficient to show

$$\begin{array}{c}\mathbb{Z}_{\geq 0}\alpha_1+\mathbb{Z}_{\geq 0}\alpha_3+\mathbb{Z}_{\geq 0}\delta = \bigcup_{m\geq 2}\mathcal{C}(r_m,s_m,v_m)\cup \\ \cup\bigcup_{m\geq 2}\mathcal{C}(r_m,s_m,r_{m+1})\cup\bigcup_{m\geq 3}\mathcal{C}(r_m,s_{m-1},v_m)\cup\bigcup_{m\geq 2}\mathcal{C}(v_m,s_m,v_{m+1}).\end{array} \tag{3.2.34}$$

together with

$$\mathbb{Z}_{\geq 0}(\alpha_2+\alpha_3)+\mathbb{Z}_{\geq 0}\alpha_3+\mathbb{Z}_{\geq 0}\delta = \bigcup_{m\geq 2}\mathcal{C}(r_m,w,r_{m+1}) \tag{3.2.35}$$

Indeed applying the linear operators s_2 and (13) we cover the entire Q_+ which we think as the following union

$$Q_+ =<\alpha_1,\alpha_3,\delta>\cup<\alpha_3,\alpha_2+\alpha_3,\delta>\cup<\alpha_2+\alpha_3,2\alpha_2+\alpha_3,\delta>\cup$$

$$\cup<2\alpha_2+\alpha_3,2\alpha_2+\alpha_1,\delta>\cup<2\alpha_2+\alpha_1,\alpha_1+\alpha_2,\delta>\cup<\alpha_1+\alpha_2,\alpha_1,\delta>$$

where $<\alpha,\beta,\gamma>$ is the additive semigroup generated in Q by α, β and γ. We observe that for every linearly independent vectors α and β we have

$$\mathbb{Z}_{\geq 0}\alpha+\mathbb{Z}_{\geq 0}\beta = \bigcup_{n\geq 0}\left[\mathbb{Z}_{\geq 0}(\alpha+n\beta)+\mathbb{Z}_{\geq 0}(\alpha+(n+1)\beta)\right].$$

Indeed for every $\gamma = a\alpha + b\beta$ there exists a unique n such that $\frac{1}{n+1} \leq \frac{a}{b} \leq \frac{1}{n}$. Then (3.2.35) follows. Moreover

$$\mathbb{Z}_{\geq 0}\alpha_1 + \mathbb{Z}_{\geq 0}\alpha_3 + \mathbb{Z}_{\geq 0}\delta = \bigcup_{n\geq 0} [\mathbb{Z}_{\geq 0}(\alpha_3 + n\delta) + \mathbb{Z}_{\geq 0}(\alpha_3 + (n+1)\delta)]$$

$$+\bigcup_{n\geq 0} [\mathbb{Z}_{\geq 0}(\alpha_1 + n\delta) + \mathbb{Z}_{\geq 0}(\alpha_1 + (n+1)\delta)] =$$

$$\bigcup_{n\geq 0} [\mathbb{Z}_{\geq 0}(\alpha_3 + n\delta) + \mathbb{Z}_{\geq 0}(\alpha_1 + n\delta) + \mathbb{Z}_{\geq 0}(\alpha_3 + (n+1)\delta) + \mathbb{Z}_{\geq 0}(\alpha_1 + (n+1)\delta)]$$

Now for every $n \geq 0$ we have

$$\mathbb{Z}_{\geq 0}(\alpha_3 + n\delta) + \mathbb{Z}_{\geq 0}(\alpha_1 + n\delta) + \mathbb{Z}_{\geq 0}(\alpha_3 + (n+1)\delta) + \mathbb{Z}_{\geq 0}(\alpha_1 + (n+1)\delta) =$$

$$\begin{array}{ccccc} = [\mathbb{Z}_{\geq 0}(\alpha_3 + n\delta) & + & \mathbb{Z}_{\geq 0}(\alpha_1 + \alpha_2 + \alpha_3 + n\delta) & + & \mathbb{Z}_{\geq 0}(\alpha_1 + n\delta)] + \\ [\mathbb{Z}_{\geq 0}(\alpha_3 + n\delta) & + & \mathbb{Z}_{\geq 0}(\alpha_1 + \alpha_2 + \alpha_3 + n\delta) & + & \mathbb{Z}_{\geq 0}(\alpha_3 + (n+1)\delta)] + \\ [\mathbb{Z}_{\geq 0}(\alpha_1 + n\delta) & + & \mathbb{Z}_{\geq 0}(\alpha_1 + \alpha_2 + \alpha_3 + n\delta) & + & \mathbb{Z}_{\geq 0}(\alpha_1 + (n+1)\delta)] + \\ [\mathbb{Z}_{\geq 0}(\alpha_3 + (n+1)\delta) & + & \mathbb{Z}_{\geq 0}(\alpha_1 + \alpha_2 + \alpha_3 + n\delta) & + & \mathbb{Z}_{\geq 0}(\alpha_1 + (n+1)\delta)] \end{array}$$

$$= \mathcal{C}(r_{n+2}, s_{n+2}, v_{n+2}) \cup \mathcal{C}(r_{n+2}, s_{n+2}, r_{n+3}) \cup$$

$$\cup \mathcal{C}(v_{n+2}, s_{n+2}, v_{n+3}) \cup \mathcal{C}(r_{n+3}, s_{n+2}, v_{n+3})$$

□

Corollary 3.2.5. **B** *is a linearly independent set and its elements are positive indecomposable.*

Proof. For $\gamma = g_1\alpha_1 + g_2\alpha_2 + g_3\alpha_3 \in Q$, we abbreviate $t^\gamma = r_1^{g_1} s_1^{g_2} v_1^{g_3}$. We will use the product partial order on $Q = \mathbb{Z}^3$:

$$\gamma_1 \geq \gamma_2 \;\Leftrightarrow\; \gamma_1 - \gamma_2 \in Q_+ = \mathbb{Z}_{\geq 0}\alpha_1 + \mathbb{Z}_{\geq 0}\alpha_2 + \mathbb{Z}_{\geq 0}\alpha_3. \tag{3.2.36}$$

By Theorem 3.2.4, **B** can be parameterized by Q so that the element $b[\alpha]$ corresponding to $\alpha \in Q$ has the form

$$b[\alpha] = t^{-\alpha} + \sum_{\gamma > -\alpha} c_\gamma t^\gamma \ . \tag{3.2.37}$$

Now suppose that a (finite) integer linear combination of elements $b[\alpha] \in \mathbf{B}$ is equal to 0. Let $S \subset Q$ be the set of all α such that $b[\alpha]$ occurs with a non-zero coefficient in this linear combination. If S is non-empty, pick a maximal element $\alpha \in S$; in view of (3.2.37), the (Laurent) monomial $t^{-\alpha}$ does not occur in any $b[\beta]$ for $\beta \in S - \{\alpha\}$, which gives a desired contradiction. □

3.2.4 The set B spans $\mathcal{C}$ and its elements are positive indecomposable

We use straightening relations (see Section1.5.2).

Proposition 3.2.6. • *The following relation holds for all $p \geq n \geq 1$:*

$$u_n u_p = \begin{cases} u_{p-n} + u_{p+n} & \text{if} \quad p > n; \\ 2 + u_{2n} & \text{if} \quad p = n. \end{cases} \tag{3.2.38}$$

- *For all $m \in \mathbb{Z}$ and $n \geq 1$, we have*

$$r_m r_{m+n} = r_{m+\lfloor \frac{n}{2} \rfloor} r_{m+\lceil \frac{n}{2} \rceil} + (\sum_{k>0} k u_{n-2k}) w^2. \tag{3.2.39}$$

- *For all $m \in \mathbb{Z}$ and $n \geq 1$*

$$v_m v_{m+n} = v_{m+\lfloor \frac{n}{2} \rfloor} v_{m+\lceil \frac{n}{2} \rceil} + (\sum_{k>0} k u_{n-2k}) z^2. \tag{3.2.40}$$

- *For all $m \in \mathbb{Z}$ and $n \geq 1$*

$$s_m s_{m+2n} = s_{m+n}^2 + \sum_{k=0}^{2n-1} (k+1) u_{2n-1-k}. \tag{3.2.41}$$

- *For all $m \in \mathbb{Z}$ and $n \geq 1$*

$$s_m s_{m+2n+1} = r_{m+n} v_{m+n} + \sum_{k=0}^{2n} (k+1) u_{2n-k}. \tag{3.2.42}$$

- *For all $m \in \mathbb{Z}$ and $n \geq 1$*

$$r_m v_{m+2n} = v_m r_{m+2n} = r_{m+n} v_{m+n} + \sum_{k=1}^{2n} k u_{2n-k}. \tag{3.2.43}$$

- *For all $m \in \mathbb{Z}$ and $n \geq 1$*

$$r_m s_{m+n} = s_m r_{m+n+1} = r_{m+\lceil \frac{n}{2} \rceil} s_{m+\lfloor \frac{n}{2} \rfloor} + (\sum_{k=1}^{n} \lceil \frac{k}{2} \rceil u_{n-k}) w. \tag{3.2.44}$$

- *For all $m \in \mathbb{Z}$ and $n \geq 1$*

$$r_m v_{m+2n+1} = v_m r_{m+2n+1} = s_{m+n}^2 + \sum_{k=1}^{2n+1} k u_{2n+1-k}. \tag{3.2.45}$$

Proof. (3.2.38) follows immediately by the definition of the Chebyshev's polynomials. It is convenient to prove the following reformulation of (3.2.39):

$$t^{-\lceil \frac{n}{2} \rceil} r_{m+\lceil \frac{n}{2} \rceil} t^{\lceil \frac{n}{2} \rceil} r_{m+\lfloor \frac{n}{2} \rfloor} = t^{-\lfloor \frac{n}{2} \rfloor} r_{m+\lfloor \frac{n}{2} \rfloor} t^{\lfloor \frac{n}{2} \rfloor} r_{m+\lceil \frac{n}{2} \rceil} \tag{3.2.46}$$

$$t^{-\lfloor\frac{n}{2}\rfloor}r_{m+\lfloor\frac{n}{2}\rfloor}t^{\lfloor\frac{n}{2}\rfloor}r_{m+\lceil\frac{n}{2}\rceil}=r_{m+\lfloor\frac{n}{2}\rfloor}r_{m+\lceil\frac{n}{2}\rceil}+(\sum_{k>0}ku_{n-2k})w^2 \tag{3.2.47}$$

Since $\lfloor\frac{n}{2}\rfloor+\lceil\frac{n}{2}\rceil=n$ they are equivalent to (3.2.39).

For n even (3.2.46) is obvious. If n is odd $\lceil\frac{n}{2}\rceil=\lfloor\frac{n}{2}\rfloor+1$ and it follows by direct check.

To prove (3.2.47) we proceed by induction on n. If $n=0$ or 1 it is obvious. For $n=2$ it is nothing else than the exchange relation (3.2.8). let us assume (3.2.47) true for n and $n-1$ and let us prove it for $n+1$.

$$t^{-\lfloor\frac{n+1}{2}\rfloor}r_{m+\lfloor\frac{n+1}{2}\rfloor}t^{\lfloor\frac{n+1}{2}\rfloor}r_{m+\lceil\frac{n+1}{2}\rceil}=t^{-\lfloor\frac{n}{2}\rfloor}r_{m+\lfloor\frac{n}{2}\rfloor}t^{\lfloor\frac{n}{2}\rfloor+1}r_{m+\lceil\frac{n}{2}\rceil}$$

$$=u_1t^{-\lfloor\frac{n}{2}\rfloor}r_{m+\lfloor\frac{n}{2}\rfloor}t^{\lfloor\frac{n}{2}\rfloor}r_{m+\lceil\frac{n}{2}\rceil}-t^{-\lfloor\frac{n}{2}\rfloor}r_{m+\lfloor\frac{n}{2}\rfloor}t^{\lfloor\frac{n}{2}\rfloor-1}r_{m+\lceil\frac{n}{2}\rceil}$$

$$=u_1t^{-\lfloor\frac{n}{2}\rfloor}r_{m+\lfloor\frac{n}{2}\rfloor}t^{\lfloor\frac{n}{2}\rfloor}r_{m+\lceil\frac{n}{2}\rceil}-t^{-\lceil\frac{n-1}{2}\rceil}r_{m+\lceil\frac{n-1}{2}\rceil}t^{\lceil\frac{n-1}{2}\rceil}r_{m+\lfloor\frac{n-1}{2}\rfloor}$$

$$\begin{aligned}&=u_1[r_{m+\lfloor\frac{n}{2}\rfloor}r_{m+\lceil\frac{n}{2}\rceil}+(\textstyle\sum_{k>0}ku_{n-2k})w^2]+\\&\quad-[r_{m+\lfloor\frac{n-1}{2}\rfloor}r_{m+\lceil\frac{n-1}{2}\rceil}+(\textstyle\sum_{k>0}ku_{n-1-2k})w^2]\end{aligned}$$

$$\begin{aligned}&=r_{m+\lfloor\frac{n}{2}\rfloor+1}r_{m+\lceil\frac{n}{2}\rceil}+r_{m+\lfloor\frac{n}{2}\rfloor-1}r_{m+\lceil\frac{n}{2}\rceil}+(\textstyle\sum_{k>0}ku_1u_{n-2k})w^2+\\&\quad-[r_{m+\lfloor\frac{n-1}{2}\rfloor}r_{m+\lceil\frac{n-1}{2}\rceil}+(\textstyle\sum_{k>0}ku_{n-1-2k})w^2].\end{aligned}$$

Now we must distinguish the case even from the case odd.

If n is even

$$\begin{aligned}&=r_{m+\lceil\frac{n+1}{2}\rceil}r_{m+\lfloor\frac{n+1}{2}\rfloor}+[\sum_{0<k<\frac{n-1}{2}}k(u_{n-1-2k}+u_{n+1-2k})]w^2+\frac{n}{2}u_1w^2\\&\quad-(\textstyle\sum_{k>0}ku_{n-1-2k})w^2\end{aligned}$$

$$=r_{m+\lfloor\frac{n+1}{2}\rfloor}r_{m+\lceil\frac{n+1}{2}\rceil}+(\sum_{0<k<\frac{n-1}{2}}ku_{n+1-2k})w^2+\frac{n}{2}u_1w^2$$

$$=r_{m+\lfloor\frac{n+1}{2}\rfloor}r_{m+\lceil\frac{n+1}{2}\rceil}+(\textstyle\sum_{k>0}ku_{n+1-2k})w^2$$

If n is odd

$$\begin{aligned}&=r^2_{m+\lceil\frac{n}{2}\rceil}+r^2_{m+\lfloor\frac{n}{2}\rfloor}+w^2+(\textstyle\sum_{k>0}ku_1u_{n-2k})w^2+\\&\quad-r^2_{m+\lfloor\frac{n-1}{2}\rfloor}-(\textstyle\sum_{k>0}ku_{n-1-2k})w^2\end{aligned}$$

$$=r^2_{m+\lceil\frac{n+1}{2}\rceil}+w^2+(\sum_{0<k<\frac{n-1}{2}}ku_{n+1-2k})w^2+\frac{n-1}{2}(u_2+2)w^2-\frac{n-1}{2}w^2$$

$$=r_{m+\lfloor\frac{n+1}{2}\rfloor}r_{m+\lceil\frac{n+1}{2}\rceil}+(\textstyle\sum_{k>0}ku_{n+1-2k})w^2$$

The proof of (3.2.40) is completely equivalent using (3.2.9) instead of (3.2.8). (3.2.41)-(3.2.45) follow after a similar calculation. The first equalities of (3.2.44)-(3.2.45) can be shown using for example the following argument: for $n = 1$ we have

$$v_{m+2}r_m = v_{m+1}r_{m+1} + u_1 + 2 = v_{m+n}r_m.$$

let us proceed by induction on n:

$$v_m r_{m+2} r_m = r_m v_{m+2n-2} r_{m+2n} = r_m v_{m+2n} r_{m+2n-2}$$

and similarly for the others. □

Proposition 3.2.7. *The set* $\mathbf{B} = \{$*cluster monomials*$\} \cup \{u_n z^k, u_n w^k | n \geq 1,\ k \geq 0\}$ *spans the cluster algebra* $\mathcal{C}$ *over* $\mathbb{Z}$ *and its elements are positive indecomposable.*

Proof. We define the *multi-degree*

$$\begin{aligned} \mu_1(M) &= \sum_{i=1}^{s} a_i + \sum_{j=1}^{t} b_j + c + d \\ \mu_2(M) &= m_t - m_1; \\ \mu_3(M) &= b_1 + b_t. \end{aligned}$$

defined on a generic monomial $M = u_{n1}^{a_1} \cdots u_{n_s}^{a_s} x_{m_1}^{b_1} \cdots x_{m_t}^{b_t} w^c z^d$ where x is either r, s or v. We proceed by induction on $\mu(M)$. If $\mu_1(M) = 1$ then M is a cluster variable or one of the u_n's. Then it suffices to show that every monomial M which has at least one of the "forbidden" products as a factor, can be written as a linear combination of monomials of (lexicographically) smaller multi-degree. We will show that this can be done by replacing some "forbidden" factor of M with its expression given by the appropriate relation in Proposition 3.2.6. Indeed, if $\sum_{i=1}^{s} a_i \geq 2$ (resp. $\sum_{i=1}^{s} a_i = 1$) then one can apply (3.2.38) (resp. (3.2.20)), expressing M as a linear combination of monomials with smaller value of μ_1. So we can assume that $M = x_{m_1}^{b_1} \cdots x_{m_t}^{b_t} w^c z^d$. If both c and d are positive, using (3.2.19) one obtains again a sum of two monomials with smaller value of μ_1. So we can assume that $c = 0$ (resp. $d = 0$) and that we can apply (3.2.7) (resp. (3.2.5)) or (3.2.18). We again obtain a sum of two monomials having smaller value of μ_1 than the initial one. So we can assume that M has one of the following forms: $M_1 = (\Pi_i r_{m_i}^{b_i}) w^c$ or $M_2 = (\Pi_i v_{m_i}^{b_i}) z^d$ or $M_3 = (\Pi_i s_{m_i}^{b_i})$ with $m_t - m_1 \geq 3$. We apply the appropriate formula in the Proposition 3.2.6 to the product $x_{m_1} x_{m_t}$. By inspection, in the resulting expression for both M_1 and M_2, all the monomials except at most one that has smaller value of μ_1 have the same value of μ_1. By further inspection, for every such monomial M', if $\min(b_1, b_t) = 1$ (resp. $\min(b_1, b_t) \geq 2$) then $\mu_2(M') < \mu_2(M)$ (resp. $\mu_2(M') = \mu_2(M)$ and $\mu_3(M') = \mu_3(M) - 2$) . Analogously in the resulting expression for M_3, there is precisely one monomial M' with $\mu_1(M') = \mu_1(M)$, while the rest of the terms have smaller value of μ_1. Moreover if $\min(b_1, b_t) = 1$ (resp. $\min(b_1, b_t) \geq 2$) then $\mu_2(M') < \mu_2(M)$ (resp. $\mu_2(M') = \mu_2(M)$ and $\mu_3(M') = \mu_3(M) - 2$).

Once we have that the elements of $\mathbf{B}$ are positive and they span $\mathcal{C}$ over $\mathbb{Z}$ we can prove they are positive indecomposable by using the same argument as in Corollary 3.2.5: we prove that the expansion of a *positive* element of $\mathcal{C}$ in the $\mathbb{Z}$–basis $\mathbf{B}$ has positive coefficients. Suppose that a (finite) integer linear combination of elements $b[\alpha] \in \mathbf{B}$ is equal to p. Let $S \subset Q$ be the set of all α such that $b[\alpha]$ occurs with a non-zero coefficient in this linear combination. If S is non-empty, pick a maximal element $\alpha \in S$ with respect to (3.2.36); in view of (3.2.37), the (Laurent) monomial $t^{-\alpha}$ does not occur in any $b[\beta]$ for $\beta \in S-\{\alpha\}$, we hence conclude that the coefficient of $t^{-\alpha}$ in this linear combination is positive. Since $b[\alpha]$ is positive, its coefficient in this linear combination is positive. By repeating the same argument for every element of S we get the claim. $\square$

3.3 Type $G_2^{(1)}$

Let $\mathcal{G}$ be the cluster algebra inside $\mathcal{F}$ with initial seed

$$\Sigma_0 = \{B = B(t_0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -3 & 0 \end{pmatrix}, \{x_{1;0}, x_{2;0}, x_{3;0}\}, \{y_1, y_2, y_3\}\}.$$

The Cartan counterpart of B is the matrix

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}$$

of type $G_2^{(1)}$. Then $\mathcal{G}$ is a cluster algebra of type $G_2^{(1)}$ with principal coefficients at the initial seed Σ_0. $\mathcal{G}$ is a cluster algebra of bipartite type, hence we have all the results of section 1.3 at our disposal. $\mathcal{G}$ is generated inside $\mathcal{F}$ by the elements $\{x_{i;2m} : i = 1, 2, 3,\ m \in \mathbb{Z}\}$ defined by the exchange relations in the Corollary 1.3.6 and by the elements:

$$w \doteq \frac{x_{1;0}x_{3;0}^3 + y_2(y_3x_{2;0} + 1)^2}{x_{2;0}x_{3;0}^2} \tag{3.3.1}$$

$$z \doteq \frac{y_2(y_1x_{2;0} + 1)(y_3x_{2;0} + 1) + x_{1;0}x_{3;0}^3}{x_{1;0}x_{2;0}x_{3;0}}. \tag{3.3.2}$$

The exchange graph of $\mathcal{G}$ is given by Figure 3.6

The following definition is fundamental in order to get a canonical basis of $\mathcal{G}$.

Definition 3.3.1.

$$u_1 = zw - y_2y_3^2 - y_1y_2y_3 \tag{3.3.3}$$

$$u_2 = u_1^2 - 2\mathbf{y}^\delta \tag{3.3.4}$$

$$u_{n+1} = u_1u_n - \mathbf{y}^\delta u_{n-1} \tag{3.3.5}$$

where $\delta \doteq (1, 2, 3)^t$.

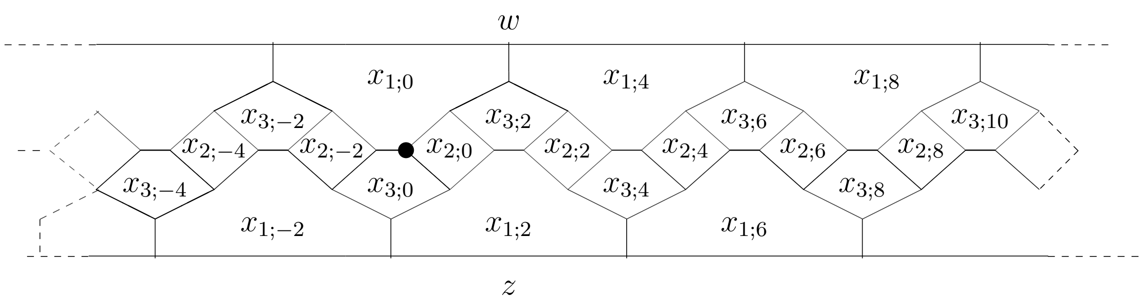

Figure 3.6: Exchange graph of the cluster algebra $\mathcal{G}$

We want to note here the analogy between Definition 3.1.2, Definition 3.3.1 and Definition 2.1.3: in all these cases

$$u_1 = zw - \mathbf{y}^{\mathbf{d}(w)} - \mathbf{y}^{\mathbf{d}(z)}$$

and the u_n is a modification of the Cebychev's polynomial $T_n(u_1)$ of the first kind evaluated in u_1.

Conjecture 3.3.2. *The set* $\mathbf{B} = \{$*cluster monomials*$\} \cup \{u_n w^k, u_n z^k : \ n \geq 1, \ k \geq 0\}$ *is a canonical basis of the cluster algebra* $\mathcal{G}$*, i.e.* $\mathbf{B}$ *is a* $\mathbb{ZP}$*–basis of* $\mathcal{G}$ *and its elements are positive indecomposable.*

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