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Direct sum decomposition and weak Krull-Schmidt Theorems

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Abstract

In this thesis we discuss the behaviour of direct sum decomposition in additive categories and in particular in categories of modules.

In the first part of the thesis, we investigate the ring theoretical properties that play a main role in the theory of factorization in additive categories, like the exchange property, semilocality and Goldie dimension. We stress the importance of the latter and we investigate with care the infinite case of the dual Goldie dimension of rings.

In the rest of the thesis, we use a more categorical approach, studying the behaviour of direct sum decomposition in additive categories. Given an additive category $\mathcal{C}$, its skeleton $V(\mathcal{C})$ has the structure of a commutative monoid under the operation of direct sum, and all the information about the regularity of the direct sum decomposition in the category $\mathcal{C}$ are traceable from the monoid $V(\mathcal{C})$. We study classes of categories where the direct sum decomposition behaves quite regularly; mainly we restrict to categories $\mathcal{C}$ whose monoid $V(\mathcal{C})$ is a Krull monoid, underlining the prominent role played by semilocal endomorphism rings. We analyze the peculiar behaviour of direct sum decomposition in some categories of modules, where the uniqueness of the decomposition is obtained up to two permutations, and we notice how this phenomenon is due to the presence of endomorphism rings of type two. In the last chapter we investigate what happens when we pass from finite direct sum of indecomposable objects to infinite direct sums, and we develop the setting for the phenomena we studied in the finite case to appear, both at a monoid theoretical and at a categorical level.

Sommario

In questa tesi discutiamo il comportamento della decomposizione in somma diretta in categorie additive e in particolare in categorie di moduli.

Nella prima parte della tesi, investighiamo le proprietà degli anelli che giocano un ruolo prominente nella teoria della fattorizzazione nelle categorie additive, come per esempio la proprietà di scambio, la semilocalità e la dimensione di Goldie. Vogliamo sottolineare l'importanza di quest'ultima e investighiamo con attenzione il caso infinito della dimensione duale di Goldie di un anello.

Nel resto della tesi, utilizziamo un approccio più categoriale, studiando il comportamento della decomposizione in somma diretta nelle categorie additive. Data una categoria additiva $\mathcal{C}$, il suo scheletro $V(\mathcal{C})$ ha la struttura di un monoide commutativo rispetto all'operazione di somma diretta, e tutte le informazioni riguardo la regolarità della decomposizione in somma diretta nella categoria $\mathcal{C}$ sono rintracciabili attraverso il monoide $V(\mathcal{C})$. Studiamo classi di categorie in cui la decomposizione in somma diretta assume un comportamento abbastanza regolare; principalemente ci restringiamo a categorie $\mathcal{C}$ il cui monoide

$V(\mathcal{C})$ è un monoide di Krull, evidenziando il ruolo prominente occupato da parte degli anelli degli endomorfismi semilocali. Analizziamo il comportamento peculiare della decomposizione in somma diretta in alcune categorie di moduli, dove l'unicità della decomposizione è garantita a meno di due permutazioni, e notiamo come questo fenomeno sia dovuto alla presenza di anelli degli endomorfismi di tipo due. Nell'ultimo capitolo investighiamo cosa succede quando passiamo da somme dirette finite di oggetti indecomponibili a somme dirette infinite, e sviluppiamo l'ambiente in cui i fenomeni studiati precedentemente nel caso finito si manifestano, sia ad un livello di teoria dei monodi sia ad un livello categoriale.

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Contents

Introduction

In the determination of a complex structure, there are two key steps: first it is necessary to identify the simplest elements, the basic constituents, the elementary pieces. Then one has to analyze how these elements interacts between them to give rise to more complicated structures. In various fields of knowledge, we have examples of this procedure.

- The first example that comes in mind is given by nature itself, since we understand matter as constituted by $\alpha\tau o\mu o\varsigma$. The idea that matter is composed of elementary particles dates back to ancient Greece, thanks to philosophers as Leucippus, Democritus and Epicurus. Only from the 19^{th} century, though, this approach was backed by some physical experiment. Nowadays, particle physics is the branch of physics that studies the elementary subatomic constituents of matter and radiation, and the interactions between them. Strictly speaking, in this context, the term particle is a misnomer because the dynamics of particle physics are governed by quantum mechanics and, as such, they exhibit wave-particle duality displaying particle-like behaviour under certain experimental conditions and wave-like behaviour in others (more technically they are described by state vectors in a Hilbert space).

- In the investigation of human kinds and their society, psychology studies the behaviour of individuals and their mental processes. Such a study regards the internal dynamics of a person, the links that intervene between him and what surrounds him, the human behaviour and the mental processes that exist between external stimulation and related answers. Sociology instead is the science that studies social structures, their organizations, the norms and the processes that unite, or separate, people not only as individuals but as exponents of associations, groups and institutions.

- In the LEGO construction game, the player is given a precise amount of building blocks and he has to follow closely the instructions to combine the pieces to construct a model. More generally, one can try, given a fair amount of bricks, to combine them appropriately to create a new model.

- A quite interesting example is provided by linguistics. On the one hand, this subject studies how phonemes, i.e. the smallest segmental units of sound, are used in the different languages to construct words and hence to communicate verbally. On the other hand, it investigates how different alphabets are used by different cultures to produce, by juxtaposition, words, and, composing them, sentences and written texts. Moreover, it analyzes also how one aspect translates into the other, that is how it is possible to pass from oral to written communication and viceversa.

Now we provide some mathematical examples.

- In mathematical logic, an atomic formula is a formula with no deeper propositional structure, that is, a formula that does not contain logical connectives or equivalently that has no strict sub-formulas. Using logical connectives and quantifiers it is possible to construct propositions starting from atomic formulas. On an higher level, the rules of inference allow then to deduce conclusions from assumed axioms.

- Every finite group G has a composition series, i.e. a subnormal series $1 = H_0 \lhd H_1 \lhd \dots \lhd H_n = G$ such that each H_i is a maximal normal subgroup of H_{i+1}. Equivalently H_{i+1}/H_i is a simple group for every $i = 0, \dots, n-1$. This allows us to decompose any finite group as a family of finite simple groups. The latter have been classified completely, providing a complete list of the building blocks that give rise to all finite groups through composition series. Moreover the Jordan-Hölder Theorem states that any two composition series of a given group are equivalent. That is, they have the same composition length and the same composition factors, up to permutation and isomorphism. Anyhow, this is not enough to conclude the classification of all finite groups, since a composition series does not determine the group itself.

- An integral domain R is atomic if every non-zero non-unit element can be written as the product of finitely many irreducible elements. If this can be done in a unique way, up to the multiplication by a unit and a reordering of the factors, R is said to be a unique factorization monoid.

Now we try to summarize in an abstract way what we are trying to illustrate with the above examples. Let M be an atomic monoid, i.e. every element is the sum of finitely many atoms. To study the behaviour of its operation, we can go through three steps:

1. identify the atoms of M;

2. determine when the sum of two finite families of atoms coincide;

3. determine all the possible factorizations for every element of the monoid.

In the easiest case, M is a free commutative monoid, i.e. $M \cong \mathbb{N}_0^{(I)}$ for some set I. In fact, the atoms of a free commutative monoid $\mathbb{N}_0^{(I)}$ are the elements with a unique non-zero entry equal to 1. Moreover, the sum of two finite families

$\{m_i\}_{i=1}^k$ and $\{n_j\}_{j=1}^l$ of atoms of $\mathbb{N}_0^{(I)}$ coincide if and only if $k = l$ and there exists a permutation σ of $\{1, \dots, k\}$ such that $m_i = n_{\sigma(i)}$ for every $i = 1, \dots, k$.

We are particularly interested in the monoids that describe the behaviour of the direct sum decomposition for some additive category $\mathcal{C}$. This monoids are defined as follows. Given an additive category $\mathcal{C}$, any skeleton of $\mathcal{C}$ has the structure of a commutative monoid $V(\mathcal{C})$, where the operation is the direct sum of objects.

In this case, the atoms of the monoid $V(\mathcal{C})$ are the indecomposable objects of the category $\mathcal{C}$. It is of clear interest to know for which additive categories $\mathcal{C}$ the monoid $V(\mathcal{C})$ is free. The theorems that prove that the monoid $V(\mathcal{C})$ is free for some additive category $\mathcal{C}$ are generally known as *Krull-Schmidt theorems.*

The Krull-Schmidt Theorem and its weaker versions

The starting point of the history of the Krull-Schmidt Theorem is the well known result of Frobenius and Stickelberger, that states that every finitely generated abelian group is isomorphic to a direct sum of primary cyclic groups and infinite cyclic groups, and this decomposition is unique [28].

The next step came in 1909, when Wedderburn proved that any two direct product decompositions of a finite group G into indecomposable factors $G = H_1 \times \dots \times H_r = K_1 \times \dots \times K_s$ are isomorphic [53]. Wedderburn's Theorem is stated as an exchange property between direct decompositions of maximum length. However, his proof makes no use of automorphisms.

Two years later, Remak [46] derived the same result showing also that the indecomposable factors are centrally isomorphic, i.e. there is an automorphism σ of G that is the identity modulo the center $Z(G)$ of G such that, after suitable relabeling of the indexes, $\sigma(H_i) = K_i$.

Subsequently, Krull [38] and Schmidt [49] extended this result to modules with finite length, obtaining the following Theorem.

Theorem 0.0.1 (Krull-Schmidt) *Let R be a ring and M_R a right R-module of finite length. Then there exists a decomposition*

$$M = M_1 \oplus \dots \oplus M_r$$

where each M_i, $i = 1, \dots, r$, is an indecomposable submodule of M. Moreover, if $M = N_1 \oplus \dots \oplus N_s$ is another decomposition of M into indecomposable modules, then $r = s$ and there exists a permutation σ of $\{1, \dots, r\}$ such that $M_i \cong N_{\sigma(i)}$ for every $i = 1, \dots, r$.

In 1950 Azumaya [6] extended the Theorem to the case of arbitrary direct sums of modules with local endomorphism ring. This result goes under the name of Krull-Schmidt-Azumaya Theorem (Theorem 1.2.8).

After proving the Theorem for the class of modules of finite length, Krull asked in 1932 whether a similar theorem holds also for the class of artinian modules. During the years, many partial results were obtained. For example, in 1969 Warfield [51] proved that such a theorem holds if the ring R is either commutative or noetherian. Anyhow, in 1995, in [21] it was proven that a Krull-Schmidt theorem cannot hold in general for the class of artinian modules over a ring R.

A problem similar to that of Krull was posed in 1975 by Warfield [52]. He proved that every finitely presented module over a serial ring is a direct sum of uniserial modules, and asked if such a decomposition is unique. In other words, Warfield asked if a Krull-Schmidt theorem holds for serial modules. The solution to this problem, a negative answer again, was provided in 1996 by Facchini [13]. Anyway, a certain regularity in the possible direct sum decompositions can still be observed. To be precise, there are two invariants under isomorphism $\sim_m$ and $\sim_e$ on the class of uniserial modules such that the following holds: given uniserial modules $U_1, \dots, U_n, V_1 \dots, V_m$, we have $U_1 \oplus \dots \oplus U_n \cong V_1 \oplus \dots \oplus V_m$ if and only if $m = n$ and there are *two* permutations σ, τ of $\{1, \dots, n\}$ such that $U_i \sim_m V_{\sigma(i)}$ and $U_i \sim_e V_{\tau(i)}$ for every $i = 1, \dots, n$. We can say then that the uniqueness of the decomposition is given not up to one permutation, but up to *two* permutations.

In the following years, other classes where such a weak Krull-Schmidt theorem holds were found. For instance the class of uniserial modules can be generalized to the class of biuniform modules, i.e. modules that are both uniform and couniform. Other classes are kernels of morphisms between indecomposable injective modules [11], cyclically presented modules over local rings [2], couniformly presented modules [17], artinian modules whose socle is isomorphic to the direct sum of two non isomorphic simple modules.

Following the evolution of the classical Krull-Schmidt Theorem into the Krull-Schmidt-Azumaya Theorem, a natural question to ask, investigating direct sum decompositions, is what happens when one considers arbitrary direct sums instead of finite ones. In the above mentioned examples, where a weak Krull-Schmidt theorem holds, one notices how some form of regularity is preserved in the infinite case. In the case of cyclically presented modules over a local ring, the behaviour is analogous to the finite case; in fact the uniqueness of the direct sum decomposition is still granted up to two bijections [3]. When we consider the case of uniserial modules, the situation becomes more complicated. We don't have anymore a completely symmetrical behaviour for the presence of the so-called non-quasi-small modules [42]. Anyway, we can still give a complete description of the direct sum decomposition. In fact, the following happens: given two families $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ of uniserial modules, we have $\oplus_{i \in I} U_i \cong \oplus_{j \in J} V_j$ if and only if there are two bijections $\sigma : I \to J$ and $\tau : I' = \{i \in I \mid U_i \text{ is quasi-small }\} \to J' = \{j \in J \mid V_j \text{ is quasi-small }\}$ such that $U_i \sim_m V_{\sigma(i)}$ for every $i \in I$ and $V_i \sim_e V_{\tau(i)}$ for every $i \in I'$ [45].

This thesis aims to present a collection, definitely not comprehensive, of recent results in the field obtained by the author and others. We combine together some well-known theorems and techniques, some results among the recent ones and some aside results which show how the literature is, as it is natural, full of material that could help spreading our comprehension and finding new approaches and ideas. Particular care is given, obviously, in pointing out the authors own contribution to the research in the field.

Organization of the thesis

The thesis is organized as follows.

In chapter 1, we collect the classical results that we need in ring theory. In particular we provide a proof of the Krull-Schmidt-Azumaya Theorem using the exchange property and we investigate semilocal rings. The latter would turn out to be a key concept for the regularity of the direct sum decomposition and, as a first evidence of this, we prove that a module with semilocal endomorphism ring cancels from direct sum. We also dedicate a whole section to Goldie dimension and dual Goldie dimension and we present their relation with the structure of rings.

In chapter 2, we generalize the Goldie dimension to the infinite case, trying to reprove in this setting the results we obtained in the finite case. We are particularly interested in the dual Goldie dimension, since its finiteness is strictly related to the cancellation of modules from direct sum. Making use of lattice theoretical techniques, we compute the dual Goldie dimension of some classes of rings, as Boolean rings, rings of continuous functions and abelian von Neumann regular rings.

Chapter 3 deals with the concepts of monoid theory that we need. This is very useful, since monoids are the proper setting to investigate factorization problems. In particular, the skeleton $V(\mathcal{C})$ of an additive category $\mathcal{C}$ has the structure of a monoid, when we consider the direct sum as operation, and the information about the regularity of the direct sum decomposition in the category $\mathcal{C}$ are traceable from the monoid $V(\mathcal{C})$. The correct family of monoids to consider to investigate the weak versions of the Krull-Schmidt theorem that we illustrated above are Krull monoids. By definition these are the monoids M that admit a divisor homomorphism into a free commutative monoid; this means that we can read the divisibility, and hence the factorization of the elements, of M looking at the divisibility in some free commutative monoid. Applying the theory of Krull monoids to the case of direct sum decomposition in additive categories, we provide a strategy to identify additive categories whose skeleton is a Krull monoid.

In chapter 4, we merge methods from category theory with the instruments developed in the previous chapter. With an high resemblance to what it is widely known for rings, we study maximal ideals of preadditive categories and the related simple categories. Maximal ideals do not exist in general for an arbitrary preadditive category $\mathcal{C}$, but they do always exist when $\mathcal{C}$ is semilocal, i.e. when $\mathcal{C}$ is a preadditive category with a non-zero object in which the endomorphism ring of every non-zero object is a semilocal ring. If $\mathcal{C}$ is a semilocal category, we get an isomorphism reflecting functor $F\colon \mathcal{C} \to \oplus_{\mathcal{M}\in \mathrm{Max}(\mathcal{C})}\mathcal{C}/\mathcal{M}$, where $\mathrm{Max}(\mathcal{C})$ is the class of all maximal ideals of $\mathcal{C}$, which allows us to get a good representation of the structure of semilocal categories. For an additive semilocal category $\mathcal{C}$, the functor F induces a monoid homomorphism $V(F)$ of the monoid $V(\mathcal{C})$ into the free commutative monoid $V(\oplus_{\mathcal{M}\in\mathrm{Max}(\mathcal{C})}\mathcal{C}/\mathcal{M}) \cong \mathbb{N}_0^{(\mathrm{Max}(\mathcal{C}))}$. If moreover idempotents split in $\mathcal{C}$, as $V(F)$ turns out to be a divisor homomorphism, the monoid $V(\mathcal{C})$ is necessarily a Krull monoid.

In chapter 5, we describe with accuracy the phenomenon of the weak Krull-Schmidt theorems. The key concepts here are those of ring and object of finite type. A ring R is said to be of finite type if $R/J(R)$ is isomorphic to a product of division rings and an object A of a preadditive category $\mathcal{C}$ is of finite type if its endomorphism ring $\mathrm{End}_{\mathcal{C}}(A)$ is so. Categories which have all objects of finite type are the natural setting where the so called weak Krull-Schmidt theorems hold. To underline this, we provide a fair number of examples. We conclude this chapter investigating categories $\mathcal{C}$ in which every object is of type ≤ 2. It turns out that the behaviour of direct sums of finitely many objects of type 2 is completely described by a graph whose connected components are either complete graphs or complete bipartite graphs. The vertices of the graphs are ideals in $\mathcal{C}$. The edges are isomorphism classes of objects. The complete bipartite graphs give rise to a behaviour described by a weak Krull-Schmidt theorem.

Chapter 6 is dedicated to generalize the weak Krull-Schmidt theorems to the infinite case. To follow a similar path to the one we took in the finite case, we first investigate the problem at a monoid theoretical level. Since usual monoids do not allow infinite sums, we introduce a new algebraic structure, that we call commutative infinitary monoid, where arbitrary infinite sums are possible. We look at the first properties of this structure, showing that there is a canonical way to pass from usual commutative monoids to infinitary ones. With this in hand, we define properly the Infinite 2-Krull-Schmidt Property, we give a complete description of the phenomenon and we apply our results to the skeleton $V(\mathcal{C})$ of a cocomplete category $\mathcal{C}$, endowed with the coproduct as operation. At this point we notice that we need some more generality to include the case of uniserial modules, where the existence of non-quasi-small modules ruins the symmetry of the Infinite 2-Krull-Schmidt Property. Therefore we need to go back to the monoid theoretical level and define a more general Infinite Quasi 2-Krull-Schmidt Property. Eventually we apply our results to the case of a preadditive category, obtaining a theorem that includes and generalizes the uniserial case.

For what concerns the fatherhood of the results, chapters 1, 3 and 5 are extracted from already known results, to give the appropriate setting for the author's own results. Generally, for each result, the reference from where it is taken is cited in the text. Chapter 2 is taken from [40] and chapter 4 from [22]. Chapter 6 is a personal work of the author not published yet.

Notations

For the reader convenience, we record here the assumptions we will taking for granted throughout the thesis and the use of the symbols that could be misinterpreted.

All rings we consider are associative rings R with $1_R \neq 0_R$. Anyhow, we will sometimes call ring also the endomorphism ring of the zero object of a category. We will try to be careful about this. Modules are, if not differently stated, unital right R-modules. All our monoids are commutative additive monoids, i.e. commutative additive semigroups with an identity element 0.

Subsets will be denoted by $\subseteq$ and proper subsets by $\subsetneq$. In general $\leq$ will denote a preorder and $a < b$ means $a \leq b$ and $a \neq b$.

The symbols $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ will denote respectively the set of positive integers, of the non-negative integers, of the integers, of the rationals and of the real numbers.

When writing Mod-R and mod-R we will mean the category of right R-modules and of finitely generated right R-modules, respectively.

We will use calligraphic letters $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots)$ to denote categories and ideals of categories, capital letters $(A, B, C, \dots)$ to denote rings, modules and objects of a category and small letters $(a, b, c, \dots)$ to denote elements. Moreover, to denote cardinal numbers we will use Hebrew letters $(\aleph, \beth, \gimel, \dots)$.

Foundations

In these years of research in subjects strictly related with category theory, I came across with several problems concerning the largeness of the structures I was working with. This is strictly related with the foundations of category theory. It is clear that the usual set theory, i.e. Zermelo-Fraenkel with the axiom of choice, which is denoted by ZFC, is not sufficient because category theory deals often with collections that are not sets but proper classes. Hence there is the need of a more powerful axiomatic set theory to work with. The requirements that one would ask for such a theory are the following.

- There should exist all the natural categories, as the category of all sets (with functions), the category of all groups (with group morphisms), the category of all topological spaces (with continuous maps) and even the category of all categories (with functors).

- If $\mathcal{A}$ and $\mathcal{B}$ are two categories, then there should be a category of all functors from $\mathcal{A}$ to $\mathcal{B}$, with natural transformations as functors.

- Standard set-theoretic operations used throughout mathematics, as taking unions and intersections of a set of sets or constructing powersets and order pairs, should be possible.

- The framework should be provably consistent from some standard set-theoretic background.

Many different approaches to the construction of such an axiomatic system exist, but none of them in fact satisfy all the above requirements. We just mention two paths that we do not follow: the first is the so called Grothendieck *axiom of* *universe*, that add to the usual ZFC the axiom that every set is contained in a universe. A universe is just a transitive set, which also contains the powerset of any of its members, and also contains the range of any definable function applied to any set inside the universe. The main negative aspect of this assumption is that the categories of all sets, all groups, etc. do not exist, but we are restricted to consider the categories of sets and groups inside a given universe.

The second approach is the one followed by Feferman, based on Quine's New Foundations (NF). NF is an axiomatic set theory that constitutes only of two axioms, extensionality and a comprehension schema, that solves the standard set-theoretical paradoxes using the concept of stratified formula. Doing so, there are no more problems in constructing big sets; in this theory, we have a set of all sets, a set of all groups, etc. The negative aspect of this theory is that some weird things happen at the basic set theoretical level. For example, it is no more true that the cardinality of the powerset must be bigger than the one of the set itself, and in fact it can be lower. From the point of view of a non-expert of the field as I am, this seems to be the most interesting way to find a satisfying axiomatization for category theory. Anyhow, due to the scarce understanding of it, I preferred to follow a most usual path.

We trail the most common road, following the approach of MacLane that distinguishes between small and large objects. This means that we have to consider an ontology that consists not only of sets, but also of classes. Lousily speaking, classes are the collections that are too big to be sets. Being more precise, sets are classes that are elements of other classes. The two most common axiomatic set theories, that are extensions of ZFC and add classes to their ontology, are Von Neumann-Bernays-Gödel set theory (NBG) and Morse-Kelley set theory (MK). The former is a conservative extension of ZFC and can be finitely axiomatized; by contrast, the latter is a proper extension of ZFC and cannot be finitely axiomatized. MK's strength stems from its axiom schema of Class Comprehension being impredicative, meaning that quantified variables can range over classes. This means that, for instance, we can have unions or intersections of classes of classes.

Anyway, there still are some problems with the use of MK. The difficulties arise from the fact that a class cannot be a member of another class. Hence, for examples, we cannot consider a class having as elements functors from a large category, or we cannot have classes with ideals of large categories as elements. Also, many times we had the need to consider the quotient class of a proper class with respect to an equivalence relation. It turns out that the quotient class, defined as the class of all the equivalence classes, cannot exist in MK, since an equivalence class can be a proper class and a class can have only sets as its elements. The solution to this problem is the use of a choice function, which chooses a representative element for every equivalence class, to construct a class of representatives of the equivalence classes.

Throughout the thesis we generally assume MK, whose axiomatization can be found well described in the appendix of the original book by Kelley [37]. The ad hoc solutions of the set-theoretical problems, that arose during the writing of this thesis, will be explained when they are needed.

Chapter 1

Semilocal endomorphism rings and Goldie dimension

1.1 Semisimple rings and modules

Let R be an associative ring with identity $1 \neq 0$. An R-module M is called a *simple* module if $M \neq 0$ and M has no other submodules than 0 and M itself. An R-module M is called *semisimple* if every submodule of M is a direct summand.

By these definitions, note that the zero module is semisimple but not simple. Clearly, every simple module is semisimple.

Lemma 1.1.1 *Every submodule and every quotient module of a semisimple module is semisimple.*

Proof. Let M be a semisimple module and N a submodule of M. If N' is a submodule of N, by the semisimplicity of M we have $M = N' \oplus N''$ for some submodule N'' of M. Hence

$$N = N \cap M = N \cap (N' \oplus N'') = N' \oplus (N \cap N'').$$

Now let $N = M/M_1$ be a factor module of the semisimple module M. Let M_2 be a submodule of M such that $M = M_1 \oplus M_2$. Then M_2 is semisimple and N is isomorphic to M_2. Hence N is semisimple. ■

There is a close relation between simple an semisimple modules. To understand it we first prove the following.

Lemma 1.1.2 *Any non-zero semisimple module contains a simple module.*

Proof. Let m be a non-zero element of M. By our previous Lemma, it suffices to prove our statement for the case $M = mR$. By Zorn's Lemma, there exists a submodule N of M maximal with respect to the property that $m \notin N$. Take a submodule N' of M such that $M = N \oplus N'$. To conclude we prove that

N' is simple. Indeed if N'' is a non-zero submodule of N', then $N \oplus N''$ must contain m and so $N \oplus N'' = M$, which implies that $N'' = N'$ as desired. ∎

Now we prove two other characterizations of semisimple modules, that are often used as definition of semisimplicity.

Proposition 1.1.3 *Let R be a ring and M an R-module. The following are equivalent:*

1. *M is semisimple;*
2. *M is the sum of a family of simple submodules;*
3. *M is the direct sum of a family of simple submodules.*

Proof.

(1)⇒(2) Let M_1 be the sum of all the simple submodules of M, and write $M = M_1 \oplus M_2$, where M_2 is a suitable submodule of M. If M_2 is different from the zero module, then it must contain a simple submodule. But this must be also in M_1. Hence $M_2 = 0$ and $M = M_1$.

(2)⇒(1) Let $\{M_i \mid i \in I\}$ be a family of simple modules such that $M = \sum_{i \in I} M_i$ an let N be any submodule of M. Consider the subsets $J \subseteq I$ satisfying the following conditions:

- $\sum_{j \in J} M_j$ is a direct sum;
- $N \cap \sum_{j \in J} M_j = 0$.

It is easy to check that Zorn's Lemma applies to the family of all such J's, with respect to ordinary inclusion. Thus we can pick a J to be maximal. For this J, let

$$M' = N + \sum_{j \in J} M_j = N \oplus \bigoplus_{j \in J} M_j.$$

To conclude the proof we need o show that $M' = M$. For this, it suffices to prove that $M' \supseteq M_i$ for every $i \in I$. If some $M_i \nsubseteq M'$, the simplicity of M_i implies that $M' \cap M_i = 0$. From this we have

$$M' + M_i = N \oplus \bigoplus_{j \in J} M_j \oplus M_i$$

that contradicts the maximality of J.

(2)⇒(3) Follows form the argument above applied to $N = 0$.

(3)⇒(2) Obvious. ∎

For a module M the *(Jacobson) radical* of M is defined to be the intersection of all maximal submodules of M, and is denoted by $\mathrm{Rad}(M)$. If M has no maximal submodules, we define $\mathrm{Rad}(M) = M$.

Lemma 1.1.4 *Let M and N be two R-modules. Then:*

1. *given an R-homomorphism $f: M \to N$, we have $f(\mathrm{Rad}(M)) \subseteq \mathrm{Rad}(N)$;*

2. *if* $f : M \to N$ *is an epimorphism and* $\ker(f) \subseteq \mathrm{Rad}(M)$*, then* $\mathrm{Rad}(N) = f(\mathrm{Rad}(M))$*. In particular* $\mathrm{Rad}(M/\mathrm{Rad}(M)) = 0$*.*

PROOF.

1. Let N' be a maximal submodule of N. It is enough to show that there exists a maximal submodule M' of M such that $f(M') \subseteq N'$. The homomorphism f composed with the canonical projection $\pi : N \to N/N'$ induces an homomorphism πf from M to the simple module N/N'. If $\pi f(M) = 0$, then $f(M)$ is contained in N'. If $\pi f(M) \neq 0$, it must be equal to N/N' since this module is simple. Hence $\ker(\pi f)$ is a maximal submodule of M such that $f(\ker(\pi f)) \subseteq N'$.

2. Since the submodules of the quotient module $M/Rad(M)$ correspond to the submodules of M containing $\mathrm{Rad}(M)$ it is clear that the maximal submodules of $M/\mathrm{Rad}(M)$ correspond to the maximal submodules of M and hence $\mathrm{Rad}(M/\mathrm{Rad}(M)) = 0$. If $f : M \to N$ is an epimorphism and $\ker(f) \subseteq \mathrm{Rad}(M)$, we have an isomorphism $M/\mathrm{Rad}(M) \cong N/f(\mathrm{Rad}(M))$. We obtain that $\mathrm{Rad}(N/f(\mathrm{Rad}(M))) = 0$ and this implies that $\mathrm{Rad}(N) \subseteq f(\mathrm{Rad}(M))$.

■

Lemma 1.1.5 *If* $\{M_i \mid i \in I\}$ *is a family of* R*-modules, then* $\mathrm{Rad}(\oplus_{i\in I} M_i) = \oplus_{i\in I}\mathrm{Rad}(M_i)$*. In particular, if* e *is an idempotent of* R*,* $\mathrm{Rad}(eR) = e\mathrm{Rad}(R)$*.*

PROOF. By our previous Lemma we have that $\mathrm{Rad}(\oplus_{i\in I} M_i) \supseteq \oplus_{i\in I}\mathrm{Rad}(M_i)$. Now let $m = (m_i)_{i\in I} \in \mathrm{Rad}(\oplus_{i\in I} M_i)$. Consider an index $j \in I$. For any maximal submodule N of M_j, $N \oplus \bigoplus_{i\in I\setminus\{j\}} M_i$ is a maximal submodule of $\oplus_{i\in I} M_i$, so $m \in N \oplus \bigoplus_{i\in I\setminus\{j\}} M_i$, which implies that $m_j \in N$. Therefore we have $m_j \in Rad(M_j)$ and similarly we can prove $m_i \in \mathrm{Rad}(M_i)$ for any $i \in I$. This shows that $\mathrm{Rad}(\oplus_{i\in I} M_i) \subseteq \oplus_{i\in I}\mathrm{Rad}(M_i)$.

The last sentence is justified from the fact that $R = eR \oplus (1-e)R$ for any idempotent $e \in R$. ■

For a ring R we define its *Jacobson radical* to be $\mathrm{Rad}(R_R)$ and we denote it by $J(R)$.

Lemma 1.1.6 *Let* y *be an element of a ring* R*. Then the following are equivalent:*

1. $y \in J(R)$*;*

2. $1 - yx$ *is right invertible for any* $x \in R$*;*

3. $My = 0$ *for any simple right* R*-module* M*.*

Proof. (1)⇒(2) Let $y \in \mathrm{Rad}(M)$. If $1 - yx$ is not right invertible for some $x \in R$, then $(1 - yx)R \subsetneq R$ is contained in a maximal right ideal M of R. But $1 - yx \in M$ and $y \in M$ implies that $1 \in M$, a contradiction.

(2)⇒(3) Assume $my \neq 0$ for an element $m \in M$. Then we must have $myR = M$. In particular $m = myx$ for some $x \in R$, so $m(1 - yx) = 0$. Using (2) we get that $m = 0$, a contradiction.

(3)⇒(1) For any maximal ideal M of R, the quotient module R/M is simple. By (3) we have $(R/M)y = 0$, which implies that $y \in M$. By the very definition we get $y \in J(R)$. ∎

For any R-module M, the *annihilator* of M is defined to be

$$\mathrm{Ann}(M) = \{r \in R \mid Mr = 0\}.$$

It is easy to see that it is a two-sided ideal of R. From (3) of Lemma 1.1.6 it follows immediately the following characterization of the Jacobson radical.

Corollary 1.1.7 *For any ring R, the Jacobson radical $J(R)$ equals the intersection of all the annihilators $\mathrm{Ann}(M)$, where M ranges among all the simple modules of R.*

The main consequence of this characterization is that in fact $J(R)$ is a two-sided ideal of R.

Lemma 1.1.8 *The following are equivalent for an element y of a ring R:*

1. $y \in J(R)$;
2. $1 - xy$ *is left invertible for any* $x \in R$;
3. $1 - xyz$ *is two-sided invertible for any* $x, z \in R$.

Proof. It is enough to prove that (1)⇔(3). The equivalence with (2) follows by symmetry. The implication (3)⇒(1) is obvious.

(1)⇒(3) Let $y \in J(R)$ and $x, z \in R$. Since $J(R)$ is a two-sided ideal, $xy \in J(R)$ and hence there exists $u \in R$ such that $(1 - xyz)u = 1$. Also $xyz \in J(R)$ and hence $u = 1 + (xyz)u$ is right invertible. Since it is also left invertible, we have that u is two-sided invertible and therefore $1 - xyz$ is two-sided invertible. ∎

Lemma 1.1.9 *Let I be a two-sided ideal of a ring R. Then:*

1. *if* $I \subseteq J(R)$, *then* $J(R/I) = J(R)/I$;
2. *if* $J(R/I) = 0$, *then* $J(R) \subseteq I$.

Proof.

1. The Jacobson radical of R/I is the intersection of all the maximal ideals of R containing I.

2. If $x \notin I$, there exists a maximal ideal M of R containing I such that $x \notin M$, so $x \notin J(R)$.

■

Now we prove Nakayama's Lemma, which will turn out quite helpful.

Lemma 1.1.10 *If M is a right module over a ring R, then $MJ(R) \subseteq \mathrm{Rad}(M)$.*

Proof. Let N be any maximal submodule of M. Since $J(R)$ annihilates all simple R-modules, $(M/N)J(R) = 0$. Therefore $MJ(R) \subseteq N$. But N is an arbitrary maximal submodule of M, hence $MJ(R) \subseteq \mathrm{Rad}(N)$. ■

Corollary 1.1.11 (Nakayama's Lemma) *Let M be a finitely generated R-module and let N be a submodule of M such that $N + MJ(R) = M$. Then $N = M$.*

Proof. Let M be a finitely generated R-module and let N be a proper submodule of M. Every non-zero finitely generated module has maximal submodules, so that $\mathrm{Rad}(M/N)$ is strictly contained in M/N. Lemma 1.1.10 implies that $(M/N)J(R) \subsetneq M/N$, so that $MJ(R) + N \subsetneq M$. ■

We say that a ring R is *J-semisimple* or *semiprimitive* if $J(R) = 0$. It is clear that for any ring R, the quotient ring $R/J(R)$ is semiprimitive. The rings R and $R/J(R)$ share some important properties.

Lemma 1.1.12 *The rings R and $R/J(R)$ have the same simple modules. An element $x \in R$ is left invertible (resp. right invertible, invertible) if and only if $x + J(R)$ is left invertible (resp. right invertible, invertible) in $R/J(R)$.*

Proof. Every simple module of a ring S is of the form S/M for some maximal ideal M of S. Since every maximal ideal of $R/J(R)$ is of the form $M/J(R)$ for some maximal ideal M of R, all the simple $R/J(R)$-modules are of the form $(R/J(R))/(M/J(R)) \cong R/M$.

For the second statement, let $y \in R$ such that $(y + J(R))(x + J(R)) = 1$ in $R/(J(R))$. Then $1 - yx \in J(R)$, so $1 - (1 - yx) = yx$ is left invertible. This clearly implies that x is left invertible in R. ■

Now we relate the Jacobson radical of a ring with semisimplicity.

Proposition 1.1.13 *Let R be a ring. The following conditions are equivalent:*

1. *R_R is a semisimple module;*
2. *every right R-module is semisimple;*
3. *every short exact sequence of right R-modules splits;*
4. *every right R-module is projective;*
5. *every right R-module is injective;*

6. the ring R is right artinian and $J(R) = 0$.

Proof. (1)⇔(2) It is clear that (2) implies (1). Assume R_R is semisimple module. Hence every cyclic module is semisimple and every module M is the sum of its cyclic submodules.

(2)⇔(3) Clear from the definition of semisimple module.

(3)⇔(4)⇔(5) Clear by the homological characterization of injective and projective modules.

(1)⇒ (6) Let $R_R = \sum_{i \in I} M_i$, where M_i are simple ideals. Since $1 \in R$, this sum must be finite. Hence R_R has finite composition length and therefore it is artinian (in fact it follows that it is also noetherian).

Since R_R is semisimple, there exist a right ideal I of R such that $R_R = J(R) \oplus I$ and idempotents $e, f \in R$ such that $J(R) = eR$, $I = fR$ and $e + f = 1$. Then $f = 1 - e$ is invertible since $e \in J(R)$. Since it is also idempotent, it follows $f = 1$ and hence $e = 0$. In particular $J(R) = eR = 0$.

(6)⇒ (1) Since R is right artinian it is clear that every non-zero right ideal contains a minimal right ideal. Moreover, every minimal right ideal I is a direct summand of R_R. Indeed, since $I \neq 0 = J(R)$, there exists a maximal right ideal M not containing I. Then $I \cap M = 0$ and $R_R = I \oplus M$.

Now suppose that R_R is not semisimple. Take a minimal right ideal I_1 of R, and write $R_R = I_1 \oplus J_1$. Since R_R is not semisimple $J_1 \neq 0$ and hence there exists a minimal right ideal $I_2 \subseteq J_1$. The ideal I_2 is a direct summand of R_R and hence also of J_1, so we can write $J_1 = I_2 \oplus J_2$. Continuing in this fashion, we get a descending chain of right ideals

$$J_1 \supseteq J_2 \supseteq \ldots$$

This contradicts the fact that R is right artinian and therefore R_R must be semisimple. ∎

A ring satisfying the equivalent conditions of Proposition 1.1.13 is said to be (right)*semisimple artinian*. It is possible to describe in a easy way the structure of semisimple artinian rings. We denote by $M_n(R)$ the ring of $n \times n$ matrices over a ring R.

Theorem 1.1.14 (Wedderburn-Artin) *A ring R is semisimple artinian if and only if there exist a finite number of division rings $D_1, \ldots, D_t$ and positive integers $n_1, \ldots, n_t$ such that $R \cong \prod_{i=1}^{t} M_{n_i}(D_i)$.*

Proof. First we prove that a ring of the form $R \cong \prod_{i=1}^{t} M_{n_i}(D_i)$ is semisimple artinian. To do this we show that $M_n(D)$ is a simple ring, right semisimple, for every division ring D and every positive integer n, and that the finite product of semisimple artinian rings is again a semisimple artinian ring.

Let D be a division ring. It is simple and hence, since every ideal of $M_n(D)$ is of the form $M_n(I)$ for an ideal I of D, also $M_n(D)$ is simple. Let V be the n-tuple row space D^n. The ring $M_n(D)$ acts on the right by matrix multiplication, so we can view V as a right $M_n(D)$-module. Elementary linear algebra shows that V is a simple right $M_n(D)$-module. Now consider the direct sum decomposition

$$M_n(D) = R_1 \oplus \ldots \oplus R_n$$

where R_i is the right ideal o $M_n(D)$ consisting of matrices all of whose rows are zero except for the i-th one. As a right $M_n(D)$-module, every R_i is isomorphic to V, hence $M_n(D) \cong nV$. This shows that the ring $M_n(D)$ is right semisimple.

Now let $R_1, \ldots, R_n$ be (right) semisimple artinian rings and let R be their direct product. We can write $R_i = I_{i1} \oplus \ldots \oplus I_{im_i}$ as a sum of simple right ideals. Viewing R_i as an ideal in R, every I_{ij} can be seen a simple right ideal of R. From

$$R_R = R_1 \oplus \ldots \oplus R_n = \oplus_{i,j} I_{i,j},$$

we conclude that R is right semisimple.

To prove the other implication, let R be a right semisimple artinian ring. Decompose R_R into a finite direct sum of simple right ideals. Grouping these according to their isomorphism type as right R-modules, we can write

$$R_R = n_1 V_1 \oplus \ldots \oplus n_t V_t,$$

where $V_1, \ldots, V_t$ are mutually non-isomorphic simple right R-modules. Let us now compute the endomorphism ring of the two R-modules in the above equality. On one side we have $\mathrm{End}_R(R) = R$. On the other side, let $D_i = \mathrm{End}_R(V_i)$. By Schur's Lemma, it is a division ring. Moreover we have that $\mathrm{End}_R(nV_i) = M_n(D_i)$ and that there are no non-zero morphisms between V_i and V_j if $i \neq j$. Hence we have

$$\begin{aligned}\mathrm{End}_R(n_1V_1 \oplus \ldots \oplus n_tV_t) &= \mathrm{End}_R(n_1V_1) \times \ldots \times \mathrm{End}_R(n_tV_t) \\ &= M_{n_1}(D_1) \times \ldots \times M_{n_t}(D_t).\end{aligned}$$

Thus, we get a ring isomorphism $R \cong M_{n_1}(D_1) \times \ldots \times M_{n_t}(D_t)$. ∎

Since the condition in the Wedderburn-Artin Theorem is right-left symmetric it follows that "right" can be replaced by "left" everywhere in Proposition 1.1.13 and we can remove the adjective "right" from the definition of semisimple artinian ring.

1.2 Local rings and the exchange property

Proposition 1.2.1 *The following conditions are equivalent for a ring R:*

1. *$R/J(R)$ is a division ring;*
2. *$J(R)$ is a maximal right ideal;*
3. *R has a unique maximal right ideal;*
4. *the sum of two non-invertible elements of R is non-invertible;*

5. $J(R)$ is the set of non-invertible elements of R.

PROOF.
(1)⇒(2) A ring is a division ring if and only if it has no non-trivial right ideals. Hence a quotient ring R/I is a division ring if and only if I is a maximal ideal.

(2)⇒(3) Every maximal ideal M of R contains $J(R)$. Hence if $J(R)$ is maximal, it must be the unique one.

(3)⇒(4) Every non-invertible element is contained in a maximal right ideal. Since there is only one maximal right ideal, every non-invertible element belongs to it.

(4)⇒(5) Our hypothesis says that the set of non-invertible elements is an ideal. It is clearly the unique maximal right ideal and henceforth equal to $J(R)$.

(5)⇒(1) Let $x + J(R)$ be a non-zero element of $R/J(R)$. Then there exist $y \in R$ such that $(x + J(R))(y + J(R)) = 1 + J(R)$ and $R/J(R)$ is a division ring.
■

Condition (1) is left/right symmetric and hence we can replace "right" with "left" in the other conditions of our proposition. A ring satisfying these equivalent conditions is called *local*.

The main role played by local rings in non-commutative algebra is as endomorphism ring of modules. The first easy observation is that a module with local endomorphism ring is indecomposable, i.e. it has no non-trivial direct sum decomposition. Now our aim is to prove the Krull-Schmidt-Azumaya Theorem, which shows the importance of local endomorphism rings in the decomposition of modules. We begin with a definition.

Definition 1.2.2 Given a cardinal $\aleph$, an R-module M is said to have the $\aleph$-*exchange property* if for any R-module G and any two direct sum decompositions

$$G = M' \oplus N = \oplus_{i \in I} A_i,$$

where $M' \cong M$ and $|I| \leq \aleph$, there are R-submodules B_i of A_i, $i \in I$, such that $G = M' \oplus \bigoplus_{i \in I} B_i$.

It is easy to prove that in fact the R-submodule B_i of A_i is a direct summand of A_i for every $i \in I$. A module has the *exchange property* if it has the $\aleph$-exchange property for every cardinal $\aleph$. It has the *finite exchange property* if it has the $\aleph$-exchange property for every finite cardinal $\aleph$. The class of modules with the $\aleph$-exchange property is closed under finite direct sums and direct summands.

Lemma 1.2.3 *Let M_1 and M_2 be two R-modules and $M = M_1 \oplus M_2$ their direct sum. Then, for any cardinal $\aleph$, the direct sum M has the $\aleph$-exchange property if and only if M_1 and M_2 have the $\aleph$-exchange property.*

PROOF. Suppose $M = M_1 \oplus M_2$ has the $\aleph$-exchange property,

$$G = M_1' \oplus N = \oplus_{i \in I} A_i,$$

$M_1' \cong M_1$ and $|I| \leq \aleph$. Then $G' = M_2 \oplus G = M' \oplus N = M_2 \oplus \bigoplus_{i \in I} A_i$, where $M' = M_1' \oplus M_2 \cong M$. Fix an element $k \in I$ and set $I' = I \setminus \{k\}$. Then $G' = M' \oplus N = (M_2 \oplus A_k) \oplus \bigoplus_{i \in I'} A_i$. Hence there exist submodules $B \subseteq M_2 \oplus A_k$ and $B_i \subseteq A_i$ for every $i \in I'$ such that

$$G' = M' \oplus B \oplus \bigoplus_{i \in I'} B_i.$$

Since $M_2 \subseteq M_2 \oplus B \subseteq M_2 \oplus A_k$, it follows that $M_2 \oplus B = M_2 \oplus B_k$, where $B_k = (M_2 \oplus B) \cap A_k$. Thus $M' \oplus B = M_1' \oplus M_2 \oplus B = M_1' \oplus M_2 \oplus B_k$. Substituting this into the above equality we obtain

$$G' = M_1' \oplus M_2 \oplus \bigoplus_{i \in I} B_i.$$

An application of the modular identity to the modules $M_1' \oplus \bigoplus_{i \in I} B_i \subseteq G$ and M_2 yields $G \cap (M_2 + (M_1' \oplus \bigoplus_{i \in I} B_i)) = (G \cap M_2) + (M_1' \oplus \bigoplus_{i \in I} B_i)$, that is, $G = M_1' \oplus \bigoplus_{i \in I} B_i$. Thus M_1 has the $\aleph$-exchange property.

Conversely suppose M_1 and M_2 have the $\aleph$ exchange property and

$$G = M_1' \oplus M_2' \oplus N = \oplus_{i \in I} A_i,$$

where $M_1' \cong M_1$, $M_2' \cong M_2$ and $|I| \leq \aleph$. Since M_1 has the $\aleph$-exchange property there are submodules $A_i' \subseteq A_i$ such that $G = M_1' \oplus M_2' \oplus N = M_1' \oplus \bigoplus_{i \in I} A_i'$. This implies that

$$G/M_1' = (M_2' + M_1'/M_1') \oplus (N + M_1'/M_1') = \oplus_{i \in I} (A_i' + M_1'/M_1').$$

By the $\aleph$-exchange property of $M_2 \cong M_2' + M_1'/M_1'$, we obtain that there exist submodules $B_i \subseteq A_i'$ such that

$$G/M_1' = (M_2' + M_1'/M_1') \oplus \bigoplus_{i \in I} (B_i + M_1'/M_1').$$

From this we deduce that $G = M_1' + M_2 + \sum_{i \in I} B_i$. In order to show that the sum is direct, suppose $m_1 + m_2 + \sum_{i \in I} b_i = 0$, where $m_1 \in M_1'$, $m_2 \in M_2'$ and $b_i \in B_i$ almost all zero. We have $(m_2 + M_1') + \sum_{i \in I} (b_i + M_1') = 0$ in G/M_1' so that $m_2 \in M_1'$ and $b_i \in M_1'$ for every $i \in I$. Then we get that $m_2 \in M_2' \cap M_1' = 0$ and $b_i \in B_i \cap M_1' = 0$. Therefore also $m_1 = 0$. This proves that

$$G = M_1' \oplus M_2' \oplus \bigoplus_{i \in I} B_i.$$

Thus $M = M_1 \oplus M_2$ has the $\aleph$-exchange property. ∎

Every module has the 1-exchange property. Now we show that the 2-exchange property is equivalent to the finite exchange property.

Lemma 1.2.4 *If an R-module M has the 2-exchange property, then it has also the finite exchange property.*

PROOF. It is sufficient to show, for an arbitrary integer $n \geq 2$, that if M has the n-exchange property, then it has also the $(n+1)$-exchange property. Hence, let M be a module with the n-exchange property ($n \geq 2$) and suppose

$$G = M' \oplus N = A_1 \oplus \ldots \oplus A_{n+1},$$

where $M' \cong M$. Set $P = A_1 \oplus \ldots \oplus A_n$, so that $G = M' \oplus N = P \oplus A_{n+1}$. Since M has the 2-exchange property, there exist submodules $P' \subseteq P$ and $B_{n+1} \subseteq A_{n+1}$ such that $G = M' \oplus P' \oplus B_{n+1}$. From $P' \subseteq P \subseteq P' \oplus (M' \oplus B_{n+1})$ and $B_{n+1} \subseteq A_{n+1} \subseteq B_{n+1} \oplus (M' \oplus P')$ we get $P = P' \oplus P''$ and $A_{n+1} = B_{n+1} \oplus A'_{n+1}$, where $P'' = (M' \oplus B_{n+1}) \cap P$ and $A'_{n+1} = (M' \oplus P') \oplus A_{n+1}$. From the decompositions

$$G = M' \oplus P' \oplus B_{n+1} = (P'' \oplus A'_{n+1}) \oplus (P' \oplus B_{n+1})$$

we infer that P'' is isomorphic to a direct summand of M'. Therefore P'' has the n-exchange property by Lemma 1.2.3. Since

$$P = P' \oplus P'' = A_1 \oplus \ldots \oplus A_n,$$

there exist submodules $B_i \subseteq A_i$, $i = 1, \ldots, n$, such that

$$P = P'' \oplus B_1 \oplus \ldots \oplus B_n.$$

From $P'' \subseteq M' \oplus B_{n+1} \subseteq G = P'' \oplus (P' \oplus A_{n+1})$ we deduce that $M' \oplus B_{n+1} = P'' \oplus P'''$, where $P''' = (M' \oplus B_{n+1}) \cap (P' \oplus A_{n+1})$. Therefore

$$\begin{aligned} G &= M' \oplus P' \oplus B_{n+1} = P' \oplus P'' \oplus P''' = P \oplus P''' \\ &= B_1 \oplus \ldots \oplus B_n \oplus P'' \oplus P''' = B_1 \oplus \ldots \oplus B_n \oplus B_{n-1} \oplus M' \end{aligned}$$

that is, M has the $(n+1)$ exchange property. ∎

Now we show a relation between the exchange property and local rings.

Proposition 1.2.5 *The following conditions are equivalent for an indecomposable R-module M:*

1. *M has local endomorphism ring;*
2. *M has the finite exchange property;*
3. *M has the exchange property.*

PROOF. (1)$\Rightarrow$(2) Suppose M is a module with local endomorphism ring $\mathrm{End}_R(M)$. By Lemma 1.2.4 it is enough to show that M has the 2-exchange property. Hence suppose that $G = M \oplus N = A_1 \oplus A_2$. We obtain

$$1_M = \pi_M \epsilon_M = \pi_M(\epsilon_{A_1}\pi_{A_1} + \epsilon_{A_2}\pi_{A_2})\epsilon_M = \pi_M \epsilon_{A_1}\pi_{A_1}\epsilon_M + \pi_M \epsilon_{A_2}\pi_{A_2}\epsilon_M,$$

where with ϵ and π we denote the canonical injections and projections, respectively. Since M has local endomorphism ring, one of the two summands, say $\pi_M \epsilon_{A_1} \pi_{A_1} \epsilon_M$, must be an automorphism. Let H be the image of the monomorphism $\epsilon_{A_1} \pi_{A_1} \epsilon_M$ so that $\epsilon_{A_1} \pi_{A_1} \epsilon_M$ induces an isomorphism $M \to H$ and $\pi_M|_H : H \to M$ is an isomorphism. This implies that $G = N \oplus H$ and the projection $G \to H$ with respect to this decomposition is $(\pi_M|_H)^{-1} \pi_M$. Since

$$H = \epsilon_{A_1} \pi_{A_1} \epsilon_M(M) \subseteq A_1 \subseteq H \oplus N = G$$

it follows that $A_1 = H \oplus B_1$, where $B_1 = N \cap A_1$ and the projection $A_1 \to H$ with respect to this decomposition is $(\pi_M|_H)^{-1} \pi_M|_{A_1}$. Therefore $G = A_1 \oplus A_2 = H \oplus (B_1 \oplus A_2)$. With respect to the last decomposition of G the projection $G \to H$ is $(\pi_M|_H)^{-1} \pi_M|_{A_1} \pi_{A_1} = (\pi_M|_H)^{-1} \pi_M \epsilon_{A_1} \pi_{A_1}$, and this mapping restricted to M is $(\pi_M|_H)^{-1} \pi_M \epsilon_{A_1} \pi_{A_1} \epsilon_M$. This is an isomorphism and this implies that $G = M \oplus B_1 \oplus A_2$.

(2)⇒(3) Let M be an indecomposable module with the finite exchange property and suppose $G = M \oplus N = \oplus_{i \in I} A_i$. Fix a non-zero element $x \in M$. There is a finite subset F of I such that $x \in \oplus_{i \in F} A_i$, so that $G = M \oplus N = \oplus_{i \in F} A_i \oplus A'$, where $A' = \oplus_{i \in I \setminus F} A_i$. Since M has the finite exchange property, there exist direct sum decompositions $A_i = B_i \oplus C_i$, $i = 1, \ldots, n$, and $A' = B' \oplus C'$ such that

$$G = M \oplus N = M \oplus B_1 \oplus \ldots \oplus B_n \oplus B'.$$

This implies that $M \cong C_1 \oplus \ldots \oplus C_n \oplus C$. Since M is indecomposable it must be isomorphic to one of the direct summands and all the other summands are zero. It is not possible that $M \cong C$ because this would imply that $M \cap \bigoplus_{i \in F} A_i = 0$. Hence there is an index $j \in F$ and a submodule B of A_j such that

$$G = M \oplus B \oplus \bigoplus_{j \neq i \in F} A_i \oplus A' = M \oplus B \oplus \bigoplus_{j \neq i \in I} A_i.$$

(3)⇒(1) Let M be an indecomposable module and suppose that $\mathrm{End}_R(M)$ is not a local ring. Then there exist two elements $\varphi, \psi \in \mathrm{End}_R(M)$ which are not automorphisms of M, such that $\varphi - \psi = 1_M$. Let $A = M_1 \oplus M_2$ be the external direct sum of two modules M_1, M_2, both equal to M, and let $\pi_i : A \to M_i$, $i = 1, 2$, be the canonical projections. The composition of the mappings

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} : M \to M_1 \oplus M_2 \text{ and } \begin{pmatrix} 1_M & -1_M \end{pmatrix} : M_1 \oplus M_2 \to M$$

is the identity mapping of M. Hence, if M' denotes the image of $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ and K the kernel of $\begin{pmatrix} 1_M & -1_M \end{pmatrix}$, we have $A = M' \oplus K$. If the exchange property were to hold for M, there would be direct summands B_1 of M_1 and B_2 o M_2 such that $A = M' \oplus K = M' \oplus B_1 \oplus B_2$. Since M_1 and M_2 are indecomposable, we would have either $A = M' \oplus M_1$ or $A = M' \oplus M_2$. If $A = M' \oplus M_1$, then $\pi_2|_{M'}$ is an isomorphism. Therefore, the composite mapping $\pi_2 \begin{pmatrix} \varphi \\ \psi \end{pmatrix} : M \to M_2$ is an isomorphism. But $\pi_2 \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \psi$, contradiction. Similarly if $A = M' \oplus M_2$. ■

Now we relate the exchange property with the existence of common refinements of direct sum decompositions. Let M be an R-module. Suppose that $\{M_i \mid i \in I\}$ and $\{N_j \mid j \in J\}$ are two families of R-submodules of M such that $M = \oplus_{i\in I} M_i = \oplus_{j\in J} N_j$. These two decompositions are said to be *isomorphic* if there is a bijection $\varphi: I \to J$ such that $M_i \cong N_{\varphi(i)}$ for every $i \in I$, and the second decomposition is a *refinement* of the first if there exists a surjective map $\varphi: J \to I$ such that $N_j \subseteq M_{\varphi(j)}$ for every $j \in J$.

Proposition 1.2.6 *Let $\aleph$ be a cardinal, let M be an R-module with the $\aleph$-exchange property and let $M = \oplus_{i\in I} M_i = \oplus_{j\in J} N_j$ be two direct sum decompositions of M with I finite and $|J| \leq \aleph$. Then these two direct sum decompositions of M have isomorphic refinements.*

Proof. We assume $I = \{0, 1, \dots, n\}$. We shall construct a chain $N_j \supseteq N'_{0,j} \supseteq N'_{1,j} \supseteq N'_{n,j}$ for every $j \in J$ such that

$$M = (\oplus_{i=0}^{k} M_i) \oplus (\oplus_{j\in J} N'_{k,j})$$

for every $k = 0, \dots, n$. We do this by induction on k. For $k = 0$, the module M_0 has the $\aleph$-exchange property and hence there exist submodules $N'_{0,j}$ of N_j such that $M = M_0 \oplus \bigoplus_{j\in J} N'_{0,j}$. Now suppose $1 \leq k \leq n$ and that the modules $N'_{k-1,j}$ such that $M = (\oplus_{i=0}^{k-1} M_i) \oplus (\oplus_{j\in J} N'_{k-1,j})$ have been constructed. Consider the direct sum decompositions

$$M = M_k \oplus (\oplus_{i=k+1}^{n} M_i) \oplus (\oplus_{i=0}^{k-1} M_i) = (\oplus_{j\in J} N'_{k-1,j}) \oplus (\oplus_{i=0}^{k-1} M_i).$$

Since M_k has the $\aleph$-exchange property, there exist submodules $N'_{k,j}$ of $N'_{k-1,j}$ such that $M = M_k \oplus (\oplus_{j\in J} N'_{k,j}) \oplus (\oplus_{i=1}^{k-1} M_i)$, which is what we had to prove.

For $k = n$, we have $M = (\oplus_{i=1}^{n} M_i) \oplus (\oplus_{j\in J} N'_{n,j})$ so that $N'_{n,j} = 0$ for every $j \in J$. Since the $N'_{k,j}$ are direct summands of M contained in $N'_{k-1,j}$ there is a direct sum decomposition $N'_{k-1,j} = N'_{k,j} \oplus N_{k,j}$ for every k and j. Similarly $N_j = N'_{0,j} \oplus N_{0,j}$. Hence $N_j = N_{0,j} \oplus N_{1,j} \oplus \dots \oplus N_{n,j}$ for every $j \in J$, so that $M = \oplus_{j\in J} \oplus_{i=0}^{n} N_{i,j}$ is a refinement of the decomposition $M = \oplus_{j\in J} N_j$.

As $M = (\oplus_{i=0}^{k-1} M_i) \oplus (\oplus_{j\in J} N'_{k-1,j}) = (\oplus_{i=0}^{k} M_i) \oplus (\oplus_{j\in J} N'_{k,j})$ for $k = 1, 2, \dots, n$, factorizing modulo $(\oplus_{i=0}^{k-1} M_i) \oplus (\oplus_{j\in J} N'_{k,j})$ we obtain that $M_k \cong \oplus_{j\in J} N_{k,j}$ for $k = 1, 2, \dots, n$. Similarly $M_0 \cong \oplus_{j\in J} N_{0,j}$. Hence for every $i = 0, \dots, n$ there is a decomposition $M_i = \oplus_{j\in J} N''_{i,j}$ with $N''_{i,j} \cong N_{i,j}$ for every i and j. Thus $\oplus_{i=0}^{n} \oplus_{j\in J} N''_{i,j}$ is a refinement of the decomposition $M = (\oplus_{i=0}^{k} M_i)$ isomorphic to $M = \oplus_{j\in J} \oplus_{i=0}^{n} N_{i,j}$. ∎

Now we are almost ready to prove the Krull-Schmidt-Azumaya Theorem. Before doing it, we need a last Lemma.

Lemma 1.2.7 *If a module M is a direct sum of modules with local endomorphism ring, then every indecomposable direct summand of M has local endomorphism ring.*

Proof. Suppose $M = A \oplus B = \oplus_{i \in I} M_i$, where A is indecomposable and all the modules M_i have local endomorphism ring. Let F be a finite subset of I such that $A \cap \oplus_{i \in F} M_i \neq 0$ and set $C = \oplus_{i \in F} M_i$. The module C has the exchange property, hence there exist direct sum decompositions $A = A' \oplus A''$ and $B = B' \oplus B''$ such that $M = C \oplus A' \oplus B'$. Note that A' is proper submodule of A, because $A \cap C \neq 0$ and $A' \cap C = 0$. Since A is indecomposable, it follows that $A' = 0$. Thus $M = C \oplus B'$ and $C \cong A \oplus B''$. Hence A is isomorphic to a direct summand of C. It follows that A has the exchange property by Lemma 1.2.3. Therefore A has local endomorphism ring by Proposition 1.2.5. ■

Theorem 1.2.8 (Krull-Schmidt-Azumaya) *Let M be a module that is a direct sum of modules with local endomorphism rings. Then any two direct sum decompositions of M into indecomposable direct summands are isomorphic.*

Proof. Suppose that $M = \oplus_{i \in I} M_i = \oplus_{j \in J} M_j$, where the modules M_i and N_j are indecomposable. By Lemma 1.2.7 all the modules M_i and N_j have local endomorphism rings. For $I' \subseteq I$ and $J' \subseteq J$ let

$$M(I') = \oplus_{i \in I'} M_i \text{ and } N(J') = \oplus_{j \in J'} N_j.$$

We know that $M(I')$ and $M(J')$ have the exchange property whenever I' and J' are finite. Since the summands N_j are indecomposable, for every finite subset $I' \subseteq I$ there exists a subset $J' \subseteq J$ such that $M = M(I') \oplus N(J \smallsetminus J')$ and hence $M(I') \cong N(J')$. By proposition 1.2.6 applied to the decompositions $M(I') \cong N(J')$, the two decompositions $M(I') = \oplus_{i \in I'} M_i$ and $N(J') = \oplus_{j \in J'} N_j$ have isomorphic refinements. From the indecomposability of the M_i and N_j we obtain that there is a bijection $\varphi\colon I' \to J'$ such that $M_i \cong N_{\varphi(i)}$ for every $i \in I'$. For every R-module A set

$$I(A) = \{i \in I \mid M_i \cong A\} \text{ and } J(A) = \{j \in J \mid N_j \cong A\}.$$

From what we have just seen it follows that $I(A)$ finite implies $|I(A)| \leq |J(A)|$ and if $I(A) \neq 0$ also $J(A) \neq 0$. Similarly $J(A)$ finite implies that $|J(A)| \leq |I(A)|$ and if $J(A) \neq 0$ also $I(A) \neq 0$. In order to prove the theorem it suffices to show that $|I(A)| = |J(A)|$ for every R-module A.

Suppose first that $I(A)$ is finite. In this case we argue by induction on $|I(A)|$. If $|I(A)| = 0$, then $|J(A)| = 0$. If $|I(A)| \geq 1$, fix an index $i_0 \in I(A)$. Then there is an index $j_0 \in J$ such that $M = M(\{i_0\}) \oplus N(J \smallsetminus \{j_0\})$. If we factorize modulo $M(\{i_0\})$ we obtain

$$N(J \smallsetminus \{j_0\}) \cong M(I \smallsetminus \{i_0\}).$$

From the inductive hypothesis we obtain $|I(A) \smallsetminus \{i_0\}| = |J(A) \smallsetminus \{j_0\}|$ and hence $|I(A)| = |J(A)|$.

By symmetry we can conclude that $J(A)$ finite implies $|I(A)| = |J(A)|$ as well.

Hence we can suppose that both $I(A)$ and $J(A)$ are infinite sets. By symmetry it is sufficient to show that $|J(A)| \leq |I(A)|$ for an arbitrary module A.

For each $i \in I(A)$ set $J_i = \{j \in J \mid M = M_i \oplus N(J \smallsetminus \{j\})\}$. Obviously $J_i \subseteq J(A)$. If x is a non-zero element of M_i, then there is a finite subset J'' of J such that $x \in N(J'')$. Hence $M_i \cap N(K) \neq 0$ for every $K \subseteq J$ that contains J''. Thus $J_i \subseteq J''$, so that J_i is finite.

We claim that $\cup_{i \in I(A)} J_i = J(A)$. In order to prove the claim, fix $j \in J(A)$. Then there exists a finite subset I' of I such that $N_j \cap M(I') \neq 0$. Hence there exists a finite subset J' of J such that $M = M(I') \oplus N(J \smallsetminus J')$. Note that $j \in J'$. Since $N(J' \smallsetminus \{j\})$ has the exchange property, we obtain that for every $i \in I'$ there exists a direct summand M_i' of M_i such that $M = N(J' \smallsetminus \{j\}) \oplus (\oplus_{i \in I'} M_i') \oplus N(J \smallsetminus J')$. Then $N_j \cong \oplus_{i \in I'} M_i'$, so that there exists an index $k \in I'$ with $M_k' = M_k$ and $M_i' = 0$ for every $i \in I'$, $i \neq k$. Note that $M_k \cong N_j \cong A$, so that $k \in I(A)$. Thus

$$M = N(J' \smallsetminus \{j\}) \oplus M_k \oplus N(J \smallsetminus J') = M_k \oplus N(J \smallsetminus \{j\}),$$

that is $j \in J_k$. Hence $j \in \cup_{i \in I(A)} J_i$, which proves the claim.

It follows that

$$|J(A)| = |\cup_{i \in I(A)} J_i| \leq |I(A)| \aleph_0 = |I(A)|.$$

■

Now we provide some examples of classes of modules with local endomorphism ring, providing the proper setting where to apply the Krull-Schmidt-Azumaya Theorem.

Lemma 1.2.9 *Let M be a right R-module and f an endomorphism of M.*

- *If n is a positive integer such that $f^n(M) = f^{n+1}(M)$, then $\ker(f^n) + f^n(M) = M$.*
- *If M is an artinian module, then f is an automorphism if and only if it is injective.*

Proof. If $f^n(M) = f^{n+1}(M)$, then $f^t(M) = f^{t+1}(M)$ for every $t \geq n$, so that $f^n(M) = f^{2n}(M)$. If $x \in M$, then $f^n(x) \in f^n(M) = f^{2n}(M)$, so that $f^n(x) = f^n(y)$ for some $y \in f^n(M)$. Therefore $z = x - y$ is in $\ker(f^n)$, and $x = y + z \in f^n(M) + \ker(f^n)$.

Now suppose that M is artinian. If f is injective endomorphism of M, the descending chain

$$M \supseteq f(M) \supseteq f^2(M) \supseteq \dots$$

is stationary, so that $\ker(f^n) + f^n(M) = M$ by the above. As f^n is injective, $\ker(f^n) = 0$, and therefore $f^n(M) = M$. In particular, f is surjective. ■

Proposition 1.2.10 *Every artinian module with simple socle has local endomorphism ring.*

Proof. Let M be an artinian module with simple socle. By previous lemma, we have that an endomorphism f of M is not automorphism if and only if it is not injective. Since the socle of an artinian module is essential, this is equivalent to $f(\mathrm{soc}(M)) = 0$. Since the set of all endomorphisms of M with this property form an ideal of $\mathrm{End}_R(M)$, we obtain that $\mathrm{End}_R(M)$ is a local ring with unique maximal ideal $J(\mathrm{End}_R(M)) = \{f \in \mathrm{End}_R(M) \mid f(\mathrm{soc}(M)) = 0\}$. ■

In a similar way one can prove that noetherian modules with a unique maximal submodule have local endomorphism ring. Now we provide another class of modules with local endomorphism ring. They are called Fitting modules due to this Lemma by Fitting.

Lemma 1.2.11 *If M is a module of finite length n and f is an endomorphism of M, then $M = \ker(f^n) \oplus f^n(M)$.*

Proof. Since M is of finite length n, both the chains

$$M \supseteq f(M) \supseteq F^2(M) \supseteq \dots$$

and

$$\ker(f) \subseteq \ker(f^2) \subseteq \ker(f^3) \subseteq \dots$$

are stationary at the n-th step. Applying Lemma 1.2.9 and its dual version, we obtain that $M = \ker(f^n) \oplus f^n(M)$. ■

We say that a right R-module is a *Fitting module* if for every endomorphism $f \in \mathrm{End}_R(M)$, there is a positive integer n such that $M = \ker(f^n) \oplus f^n(M)$. From Lemma 1.2.11 it is clear that modules of finite length are Fitting modules. It is easily seen that direct summands of Fitting modules are Fitting modules.

Proposition 1.2.12 *The endomorphism ring of any indecomposable Fitting module is local.*

Proof. If M is a Fitting module and f is an endomorphism of M, there exists a positive integer n such that $M = \ker(f^n) \oplus f^n(M)$. If M is indecomposable, two cases can occur. In the first case $f^n(M) = M$ and $\ker(f^n) = 0$. Then f^n is an automorphism of M, so that f itself is an automorphism of M. In the second case, $f^n(M) = 0$, that is, f is nilpotent. Hence every endomorphism of M is either nilpotent or an automorphism.

In order to show that $\mathrm{End}_R(M)$ is local, we must show that the sum of two non-invertible endomorphism is non-invertible. Suppose that f and g are two non-invertible endomorphisms of M such that $f+g$ is invertible. If $h = (f+g)^{-1}$ is the inverse of $f+g$, then $fh + gh = 1$. Since f and g are not automorphisms, neither fh nor gh are automorphisms. Therefore there exists a positive integer n such that $(gh)^n = 0$. Since

$$1 = (1 - gh)(1 + gh + (gh)^2 + \dots + (gh)^{n-1})$$

the endomorphism $1 - gh = fh$ is invertible. This contradiction proves the Lemma. ■

In particular, applying the Krull-Schmidt-Azumaya Theorem to the case of modules with finite length, we can recover the classical Krull-Schmidt Theorem.

Proposition 1.2.13 *Let M be an indecomposable injective R-module. Then:*

- *an endomorphism of M is an automorphism of M if and only if it is injective;*
- *the endomorphism ring of M is local.*

PROOF. If $f \in \mathrm{End}_R(M)$ is a monomorphism, then $f(M)$ is a submodule of M isomorphic to M. In particular $f(M)$ is a non-zero direct summand of M. Since M is indecomposable, $f(M) = M$ and f is an automorphism.

To prove that $\mathrm{End}_R(M)$ is a local ring,we have to show that the sum of two non-invertible endomorphisms f and g of M is non-invertible. By the above $\ker(f) \neq 0$ and $\ker(g) \neq 0$. Since an irreducible injective module is uniform, we have $\ker(f) \cap \ker(g) \neq 0$. Now

$$\ker(f) \cap \ker(g) \subseteq \ker(f+g),$$

so that $\ker(f+g) \neq 0$. Therefore $f+g$ is not invertible. ■

1.3 Goldie dimension

In this section we treat the concept of Goldie dimension, both for modular lattices and for modules, and we underline its connection with semilocal rings.

Throughout this section $(L, \vee, \wedge)$ will denote a *bounded modular lattice*, that is a lattice with a smallest element 0 and a greatest element 1 such that $a \wedge (b \vee c) = (a \wedge b) \vee c$ for every $a, b, c \in L$ with $c \leq a$. If $a, b \in L$, we call $[a, b] = \{x \in L \mid a \leq x \leq b\}$ the *interval* between a and b.

A finite subset $\{a_i \mid i \in I\}$ of $L \setminus \{0\}$ is said to be *join-independent* if $a_i \wedge (\bigvee_{i \neq j \in I} a_j) = 0$ for every $i \in I$. The empty subset of $L \setminus \{0\}$ is join-independent. An infinite subset of $L \setminus \{0\}$ is join-independent if all its finite subsets are join-independent.

Lemma 1.3.1 *Let $A \subseteq L \setminus \{0\}$ be a join-independent subset of a modular lattice L. For any non-zero element $a \in L$ such that $a \wedge (\bigvee_{b \in B} b) = 0$ for every finite subset $B \subseteq A$, we have that $A \cup \{a\}$ is join-independent.*

PROOF. We have to prove that every finite subset of $A \cup \{a\}$ is join-independent. It is clear for finite subsets of A. Hence it suffices to show that $B \cup \{a\}$ is join-independent for every finite subset $B \subseteq A$. Since $a \wedge (\bigvee_{b \in B} b) = 0$, it remains to prove that $b \wedge (a \vee \bigvee_{x \in B \setminus \{b\}} x) = 0$ for each $b \in B$. Now

$$\begin{aligned} (\bigvee_{y \in B} y) \wedge (a \vee \bigvee_{x \in B \setminus \{b\}} x) &= ((\bigvee_{y \in B} y) \wedge a) \vee (\bigvee_{x \in B \setminus \{b\}} x) \\ &= 0 \vee \bigvee_{x \in B \setminus \{b\}} x = \bigvee_{x \in B \setminus \{b\}} x, \end{aligned}$$

so that

$$\begin{aligned} b \wedge (a \vee \bigvee_{x \in B \setminus \{b\}} x) &= b \wedge (\bigvee_{y \in B} y) \wedge (a \vee \bigvee_{x \in B \setminus \{b\}} x) \\ &= b \wedge \bigvee_{x \in B \setminus \{b\}} x = 0. \end{aligned}$$

■

By Zorn's Lemma, every join-independent subset of $L \setminus \{0\}$ is contained in a maximal join-independent subset of $L \setminus \{0\}$.

An element $a \in L$ is *essential* if $a \wedge x = 0$ implies $x = 0$. Thus $0 \in L$ is essential if and only if $L = \{0\}$. If $a \leq b$ are elements of L, the element a is said to be *essential in* b if it is essential in $[0, b]$.

Lemma 1.3.2 *Let a, b and c be elements of L. If a is essential in b and b is essential in c, then a is essential in c.*

Proof. Let x be a non-zero element of $[0, c]$. Now $b \wedge x \neq 0$ since b is essential in c. Hence $a \wedge x = a \wedge (b \wedge x) \neq 0$ since a is essential in b. ■

Lemma 1.3.3 *Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be elements such that $\{b_1, \dots, b_n\}$ is join-independent. If a_i is essential in b_i for every $i = 1, \dots, n$, then $a_1 \vee \dots \vee a_n$ is essential in $b_1 \vee \dots \vee b_n$.*

Proof. Using induction, it is enough to prove the case with $n = 2$. Hence we can suppose that we have elements a_1, a_2 and b_1, b_2 such that $b_1 \wedge b_2 = 0$ and a_i is essential in b_i for $i = 1, 2$. If any of the four elements is zero, then the statement of the Lemma is trivial, hence we can assume they are all non-zero.

First we prove that $a_1 \vee b_2$ is essential in $b_1 \vee b_2$. Assume the contrary. Then there exists a non-zero element $x \in L$ such that $x \leq b_1 \vee b_2$ and $(a_1 \vee b_2) \wedge x = 0$. Since $\{a_1, b_2\}$ is join-independent, the set $\{a_1, b_2, x\}$ is join-independent by Lemma 1.3.1. In particular, $a_1 \wedge (b_2 \vee x) = 0$, so that $a_1 \wedge b_1 \wedge (b_2 \vee x) = 0$. Since a_1 is essential in b_1, this implies that $b_1 \wedge (b_2 \vee x) = 0$. Now $\{b_2, x\} \subseteq \{a_1, b_2, x\}$ is join-independent, and thus $b_1 \wedge (b_2 \vee x) = 0$ forces that $\{b_1, b_2, x\}$ is join-independent. In particular $x \wedge (b_1 \vee b_2) = 0$. But $x \leq b_1 \vee b_2$, so that $x = 0$. This contradiction proves the claim.

If we apply the claim to the four elements a_2, b_2, a_1, a_1 we obtain that $a_1 \vee a_2$ is essential in $a_1 \vee b_2$. The conclusion now follows from Lemma 1.3.2. ■

A lattice $L \neq \{0\}$ is *uniform* if all its non-zero elements are essential in L. An element a of L is called *uniform* if it is non-zero and the lattice $[0, a]$ is uniform.

Lemma 1.3.4 *If a modular lattice L does not contain infinite join-independent subsets, then for every non-zero element $a \in L$ there exists a uniform element $b \in L$ such that $b \leq a$.*

PROOF. Let a be a non-zero element of L and suppose that every element $b \leq a$ is not uniform. We shall define by induction a sequence $a_1, a_2, \dots$ of non-zero elements of $[0, a]$ such that, for every $n \geq 1$, the set $\{a_i \mid i = 1, \dots, n\}$ is join-independent and $\bigvee_{i=1}^n a_i$ is not essential in $[0, a]$. For $n = 1$, it is enough to notice that a is not uniform and hence there exist two non-zero elements $a_1, a_1' \in [0, a]$ such that $a_1 \wedge a_1' = 0$, i.e. a_1 has the required properties. Now suppose that $a_1, \dots, a_{n-1}$ have already been defined. Since $a_1 \vee \dots \vee a_{n-1}$ is not essential in $[0, a]$, there exists an elements $b \leq a$ such that $b \wedge (a_1 \vee \dots \vee a_{n_1}) = 0$. This element b is not uniform, hence there exist non-zero elements $a_n, a_n' \leq b$ such that $a_n \wedge a_n' = 0$. Then $a_n \wedge (a_1 \vee \dots \vee a_{n-1}) = 0$, so that $\{a_1, \dots, a_n\}$ is join-independent by Lemma 1.3.1. Moreover

$$\begin{aligned} a_n' \wedge (a_1 \vee \dots \vee a_n) &= a_n' \wedge b \wedge ((a_1 \vee \dots \vee a_{n-1}) \vee a_n) \\ &= a_n' \wedge ((b \wedge (a_1 \vee \dots \vee a_{n-1})) \vee a_n) \\ &= a_n' \wedge (0 \vee a_n) = 0. \end{aligned}$$

This completes the construction. Then we obtain an infinite join-independent set $\{a_n \mid n \geq 1\}$. ■

Theorem 1.3.5 *The following conditions are equivalent for a bounded modular lattice L:*

1. *L does not contain infinite join-independent subsets;*
2. *L contains a finite join-independent subset $\{a_1, \dots, a_n\}$ with a_i uniform for every $i = 1, \dots, n$ and $a_1 \vee \dots \vee a_n$ essential in L;*
3. *the cardinality of every join-independent subset of L is $\leq m$ for a non-negative integer m;*
4. *if $a_0 \leq a_1 \leq \dots$ is an ascending chain of elements of L, then there exists an index $i \geq 0$ such that a_i is essential in a_j for every $j \geq i$.*

Moreover, if these equivalent conditions hold and $\{a_1, \dots, a_n\}$ is a finite join-independent subset of L with a_i uniform for every $i = 1, \dots, n$ and $a_1 \vee \dots \vee a_n$ essential in L, then any other join-independent subset of L has cardinality $\leq n$.

PROOF. (1)⇒(2) Let $\mathcal{F}$ be the family of all join-independent subsets of L consisting only of uniform elements. The family $\mathcal{F}$ is non-empty by Lemma 1.3.4 and hence by Zorn's Lemma it has a maximal element X with respect to inclusion. By (1), the set X is finite, say $X = \{a_1, \dots, a_n\}$. The element $a_1 \vee \dots \vee a_n$ must be essential in L, otherwise there would exist a non-zero element $x \in L$ such that $(a_1 \vee \dots \vee a_n) \wedge x = 0$, and by Lemma 1.3.4 there would be a uniform element $b \in L$ such that $b \leq x$. Hence $(a_1 \vee \dots \vee a_n) \wedge b = 0$ and $\{a_1, \dots, a_n, b\}$ would be a join-independent subset of L strictly containing $\{a_1, \dots, a_n\}$, a contradiction.

(2)⇒(3) Suppose that (2) holds, so that there exists a finite join-independent subset $\{a_1, \dots, a_n\}$ with a_i uniform for every $i = 1, \dots, n$ and $a_1 \vee \dots \vee a_n$ essential in L. Assume that there exists a join-independent subset $\{b_1, \dots, b_k\}$ of L of cardinality $k > n$. For every $t = 0, 1, \dots, n$ we shall construct a subset X_t of $\{a_1, \dots, a_n\}$ of cardinality t and a subset Y_t of $\{b_1, \dots, b_k\}$ of cardinality $k - t$ such that $X_t \cap Y_t = \varnothing$ and $X_t \cup Y_t$ is a join-independent set. For $t = 0$ set $X_0 = \varnothing$ and $Y_0 = \{b_1, \dots, b_k\}$. Now suppose that X_t and Y_t have been constructed for some $0 \le t < n$. We shall construct X_{t+1} and Y_{t+1}. Since $|Y_t| = k - t > n - t > 0$, there exists $j = 1, \dots, k$ with $b_j \in Y_t$. Set

$$c = \bigvee_{y \in X_t \cup Y_t \smallsetminus \{b_j\}} y.$$

We claim that $c \wedge a_l = 0$ for some $l = 1, \dots, n$. Otherwise, if $c \wedge a_i \neq 0$ for every $i = 1, \dots, n$, then $c \wedge a_i$ is essential in a_i because a_i is uniform, so that $\bigvee_{i=1}^n c \wedge a_i$ is essential in $\bigvee_{i=1}^n a_i$ by Lemma 1.3.3. Since $\bigvee_{i=1}^n a_i$ is essential in 1, it follows that $\bigvee_{i=1}^n c \wedge a_i$ is essential in 1. Then $c \ge \bigvee_{i=1}^n c \wedge a_i$ is essential in 1, so that $c \wedge b_j \neq 0$. This contradicts the fact that $X_t \cup Y_t$ is join-independent and this contradiction proves the claim. From Lemma 1.3.1 and the claim it follows that $(X_t \cup \{a_l\}) \cup (Y_t \smallsetminus \{b_j\})$ is join-independent, so that $X_{t+1} = X_t \cup \{a_l\}$ and $Y_{t+1} = Y_t \smallsetminus \{b_j\}$ have the required properties. This completes the construction of the sets X_t and Y_t.

For $t = n$ we have a non-empty subset Y_n of $\{b_1, \dots, b_k\}$ such that

$$\{a_1, \dots, a_n\} \cup Y_n$$

is a join-independent subset of cardinality k, so that $(a_1 \vee \dots \vee a_n) \wedge y = 0$ for every $y \in Y_n$, and this contradicts the fact that $a_1 \vee \dots \vee a_n$ is essential in L. Hence every join-independent subset of L has cardinality $\le n$.

(3)⇒(4) If (4) does not hold, there is a chain $a_0 \le a_1 \le \dots$ of elements of L such that for every $i \ge 0$, the element a_i is not essential in a_{i+1}. Then for every $n \ge 0$, there exists a non-zero element $b_n \le a_{n+1}$ such that $a_n \wedge b_n = 0$. The set $\{b_n \mid n \ge 0\}$ is join-independent, so (3) does not hold.

(4)⇒(1) If (1) is not satisfied, then L contains a countable infinite join-independent subset $\{b_i \mid i \ge 0\}$. Set $a_n = \bigvee_{i=0}^n b_i$. Then $a_0 \le a_1 \le a_2 \le \dots$ and, for every $n \ge 0$, the element a_n is not essential in a_{n+1} since $a_n \wedge b_{n+1} = 0$. Hence (4) is not satisfied.

The last part of the statement has already been seen in the proof of (2)⇒(3).
■

Thus, for a modular lattice L, either there is a finite join-independent subset $\{a_1, \dots, a_n\}$ with a_i uniform for $i = 1, \dots, n$ and $a_1 \vee \dots \vee a_n$ essential in L, and in this case n is said to be the *Goldie dimension* of L, denoted by $\dim L$, or it contains an infinite join-independent subset, in which case it is said to have *infinite Goldie dimension*.

Now we apply the concepts we just introduced above to the modular lattice $\mathcal{L}(M)$ of all submodules of a given R-module M. The *Goldie dimension* of M, denoted by $\dim M$, is the Goldie dimension of the modular lattice $\mathcal{L}(M)$.

Since a module M is essential in its injective envelope $E(M)$, we have that $\dim M = \dim E(M)$.

We say that a module M is *uniform* if its lattice of submodules $\mathcal{L}(M)$ is a uniform lattice. By Theorem 1.3.5 it is clear that a module M has finite Goldie dimension if and only if contains an essential submodule that is the direct sum of uniform submodules $U_1, \dots, U_n$. In this case $E(M) = E(U_1) \oplus \dots \oplus E(U_n)$ is the finite direct sum of n indecomposable modules. Since any indecomposable injective module has local endomorphism ring, by the Krull-Schmidt-Azumaya Theorem, we have that the number of summands in any indecomposable decomposition of $E(M)$ does not depend on the decomposition. Hence a module M has Goldie dimension n if and only if its injective envelope is the direct sum of n indecomposable modules.

Now we collect the basis properties of the Goldie dimension for a module M. Their proof is elementary.

Proposition 1.3.6 *Let M be a module.*

1. $\dim M = 0$ *if and only if* $M = 0$*;*
2. $\dim M = 1$ *if and only if M is uniform;*
3. *if $N \subseteq M$ and M has finite Goldie dimension, then N has finite Goldie dimension and* $\dim N \leq \dim M$*;*
4. *if $N \subseteq M$ and M has finite Goldie dimension, then* $\dim N = \dim M$ *if and only if N is essential in M;*
5. *if M and M' are modules of finite Goldie dimension, then $M \oplus M'$ has finite Goldie dimension and* $\dim M \oplus M' = \dim M + \dim M'$.

Artinian modules and noetherian modules have finite Goldie dimension. For an artinian module M, the Goldie dimension of M equals the composition length of the socle $\mathrm{soc}(M)$.

We shall now apply our results to the dual lattice of the lattice $\mathcal{L}(M)$ of all submodules of a module M. The dual lattice of a modular lattice is also modular, so we can apply the results of this section to the dual of the lattice $\mathcal{L}(M)$ and translate them into the language of modules.

Let M be a right module. A finite set $\{N_i \mid i \in I\}$ of proper submodules of M is said to be *coindependent* if $N_i + \bigcap_{i \neq j \in I} N_j = M$ for every $i \in I$. An arbitrary set A of proper submodules of M is said to be *coindependent* if all its finite subsets are coindependent. A submodule N of M is said to be *superfluous* if it is essential in the dual of the lattice $\mathcal{L}(M)$, i.e. if $N + A \subsetneq M$ for every proper submodule $A \subsetneq M$. An R-module $M \neq 0$ is said to be *couniform* if the dual of the lattice $\mathcal{L}(M)$ is uniform. Every local module, that is, a module with a unique maximal submodule, is clearly couniform. From Theorem 1.3.5 we obtain the following.

Theorem 1.3.7 *The following conditions are equivalent for a module M:*

1. *there do not exist infinite coindependent sets of proper submodules of M;*

2. *there exists a finite coindependent set* $\{N_1, \ldots, N_n\}$ *of proper submodules of* M *with* M/N_i *couniform for every* $i = 1, \ldots, n$ *and* $N_1 \cap \ldots \cap N_n$ *superfluous in* M*;*

3. *the cardinality of the coindependent sets of proper submodules is* $\leq m$ *for a non-negative integer* m*;*

4. *if* $N_0 \supseteq N_1 \supseteq N_2 \supseteq \ldots$ *is a descending chain of submodules of* M*, then there exists* $i \geq 0$ *such that* N_i/N_j *is superfluous in* M/N_j *for every* $j \geq i$*.*

Moreover, if these equivalent conditions hold and $\{N_1, \ldots, N_n\}$ *is a finite coindependent set of proper submodules of* M *with* M/N_i *couniform for all* i *and* $N_1 \cap \ldots \cap N_n$ *superfluous in* M*, then every other coindependent set has cardinality* $\leq n$*.*

We define the *dual Goldie dimension* of a module M, denoted by $\operatorname{codim}(M)$, to be the Goldie dimension of the dual of the lattice $\mathcal{L}(M)$. It is clear, from (4) of previous theorem, that every artinian module has finite dual Goldie dimension. Dualizing Proposition 1.3.6, we obtain the following.

Proposition 1.3.8 *Let* M *be a module.*

1. $\operatorname{codim}(M) = 0$ *if and only if* $M = 0$*;*

2. $\operatorname{codim}(M) = 1$ *if and only if* M *is couniform;*

3. *if* $N \subseteq M$ *and* M *has finite dual Goldie dimension, then* M/N *has finite dual Goldie dimension and* $\operatorname{codim}(M/N) \leq \operatorname{codim}(M)$*;*

4. *if* $N \subseteq M$ *and* M *has finite dual Goldie dimension, then* $\operatorname{codim}(M) = \operatorname{codim}(M/N)$ *if and only if* N *is superfluous in* M*;*

5. *if* M *and* M' *are modules with finite dual Goldie dimension, then* $M \oplus M'$ *is a module with finite dual Goldie dimension and* $\operatorname{codim}(M \oplus M') = \operatorname{codim}(M) + \operatorname{codim}(M')$*.*

For a semisimple module, the dual Goldie dimension coincides with the composition length of the module. Hence for a semisimple artinian ring

$$\dim(R_R) = \dim({}_R R) = \operatorname{codim}(R_R) = \dim({}_R R).$$

We shall denote this finite dimension $\dim(R)$.

1.4 Semilocal rings

A ring R is a *semilocal ring* if $R/J(R)$ is a semisimple artinian ring. Since $J(R/J(R)) = 0$ for every ring R, it is clear that a ring R is semilocal if and only if $R/J(R)$ is a right artinian ring, if and only if $R/J(R)$ is a left artinian ring.

Proposition 1.4.1 *If a ring R has finitely many maximal right ideals, then it is semilocal. If $R/J(R)$ is commutative, also the converse holds.*

Proof. It is clear that for both conclusions, we may assume $J(R) = 0$. Assume that $M_1, \dots, M_n$ are all the maximal right ideals of R. Then $\cap_{i=1}^n M_i = 0$ and hence we have a injection of right R-modules

$$R \to \oplus_{i=1}^n R/M_i.$$

The latter has a composition series; thus, so does the former. This implies that the ring R is right artinian, and hence semilocal. Conversely, assume that R is commutative and artinian. Since we have assumed that $J(R) = 0$, the ring R is a direct product of a finite number of fields (for instance, by Theorem 1.1.14). Then the number of maximal ideals o R equals the number of factors in this decomposition. ∎

We remark that in general it is not true that a semilocal ring has finitely many right ideals. For example, a matrix algebra over a field is semilocal, but it may have infinitely many maximal right ideals.

Now we give some examples of semilocal rings.

- Any local ring is semilocal.
- Every right (or left) artinian ring is semilocal.
- If R is a semilocal ring, the ring $M_n(R)$ of $n \times n$ matrices with entries in R is semilocal. In fact $J(M_n(R)) = M_n(J(R))$ and hence $M_n(R)/J(M_n(R)) \cong M_n(R/J(R))$. If R is semilocal, its quotient $R/J(R)$ is semisimple artinian, and this implies that the matrix ring $M_n(R/J(R))$ is semisimple artinian.
- The direct product of two semilocal rings is semilocal.
- Every homomorphic image of a semilocal ring is a semilocal ring. In fact, let I be an ideal of a semilocal ring R. Since every simple R/I-module is a simple R-module, if $\pi: R \to R/I$ i the canonical projection, then $\pi(J(R)) \subseteq J(R/I)$. Hence π induces a surjective homomorphism $R/J(R) \to (R/I)/J(R/I)$. But every homomorphic image of a semisimple artinian ring is a semisimple artinian ring, and thus R/I is semilocal.

With the next two propositions we provide some more examples of semilocal rings.

Proposition 1.4.2 *Let k be a commutative semilocal ring and R be a k-algebra that is finitely generated as k-module. Then $J(R) \supseteq J(k)R$ and R is a semilocal ring.*

Proof. In order to show that $J(R) \supseteq J(k)R$, it is sufficient to prove that $MJ(k) = 0$ for every simple R-module M. Now $MJ(k)$ is a submodule of M, because $MJ(k)R = MRJ(k) = MJ(k)$. Since M is simple, either $MJ(K) = M$ or $MJ(k) = 0$. But R_k is finitely generated and M is an homomorphic image of R, so that M_k is finitely generated. By Nakayama's Lemma (1.1.11), $MJ(k) = M$ implies $M = 0$, a contradiction. Therefore $MJ(k) = 0$.

Then $R/J(k)R$ is a module-finite algebra over the artinian commutative ring $k/J(k)$. Since $k/J(k)$ is artinian, $R/J(k)R$ is an artinian module. In particular $R/J(k)R$ is an artinian ring, so R is semilocal. ∎

Proposition 1.4.3 *If R is a semilocal ring and e is a non-zero idempotent of R, then eRe is a semilocal ring.*

Proof. It is enough to show that if $R/J(R)$ is a right artinian ring, then also $eRe/J(eRe)$ is a right artinian ring. Suppose that there is a descending chain $J_1 \supseteq J_2 \supseteq \dots$ of right $eRe/J(eRe)$-ideals. It is clear that $J_1R+J(R) \supseteq J_2R+J(R) \supseteq \dots$ is a descending chain of right $R/J(R)$-ideals, hence it is stationary. If we show that $(IR+J(R))\cap eRe = I$ for every right eRe-ideal containing $J(eRe)$, it is clear that also the initial chain must be stationary. Let $\sum_k i_kr_k + j = ere$ be an element of $(IR + J(R)) \cap eRe$, with $i_k \in I$, $r, r_k \in R$ and $j \in J(R)$. Then $ere = e(\sum_k i_kr_k + j)e = \sum_k ei_ker_ke + eje \in IeRe + eJ(R)e \subseteq I$. ∎

Now we show that there is a strong connection between the semilocality of a ring and its dual Goldie dimension.

Proposition 1.4.4 *The following are equivalent for a ring R:*

1. *R is semilocal;*
2. *the right R-module R_R has finite dual Goldie dimension;*
3. *the left R-module ${}_RR$ has finite dual Goldie dimension.*

Moreover, if these conditions hold,

$$\operatorname{codim}(R_R) = \operatorname{codim}({}_RR) = \dim(R/J(R)).$$

Proof. (1)⇒(2) Let R be a semilocal ring and suppose that R_R has infinite dual Goldie dimension, i.e. there exists an infinite coindependent set $\{I_n \mid n \geq 1\}$ of proper right ideals of R. Then $R/\cap_{n=1}^k I_n$ is a direct sum of k non-zero cyclic modules for every $k \geq 1$. If C is a non-zero cyclic module, $C/CJ(R)$ is a non-zero module. Therefore $R/J(R) + \cap_{n=1}^k I_n$ is a direct sum of at least k non-zero modules for every $k \geq 1$. In particular $R/J(R)$ can not have finite length, so that R can not be semilocal.

(2)⇒(1) Suppose that R_R has finite dual Goldie dimension. Let $\mathcal{I}$ be the set of all right ideals of R that are finite intersection of maximal right ideals. Note that if $I, J \in \mathcal{I}$ and $I \subseteq J$ then R/I and R/J are semisimple modules of finite length and

$$\operatorname{codim}(R/J) \leq \operatorname{codim}(R/I).$$

Since $\operatorname{codim}(R/I) \leq \operatorname{codim}(R)$ for every I, it follows that every descending chain in $\mathcal{I}$ is finite, i.e. the partially ordered set $\mathcal{I}$ is artinian. In particular $\mathcal{I}$ has a minimal element. Since any intersection of two elements of $\mathcal{I}$ belongs to $\mathcal{I}$, the set $\mathcal{I}$ has a least element, which is the Jacobson radical $J(R)$. Hence $J(R) \in \mathcal{I}$ is a finite intersection of maximal right ideals. Therefore $R/J(R)$ is a semisimple artinian right R-module, hence R is semilocal.

Since (1) is left-right symmetric, (1), (2) and (3) are equivalent. Finally, $J(R)$ is a superfluous module of R_R(Lemma 1.1.11), so that if (2) holds, then $\operatorname{codim}(R_R) = \operatorname{codim}(R/J(R))$ by Proposition 1.3.8(4). ∎

Corollary 1.4.5 *Let P_R be a finitely generated projective module over a semilocal ring R. Then every surjective endomorphism of P_R is an automorphism. In particular, every right or left invertible element of a semilocal ring is invertible.*

Proof. Sice R is semilocal, the right module R_R has finite dual Goldie dimension, so that P_R has finite dual Goldie dimension. If $f: P_R \to P_R$ is a surjective endomorphism of P_R, then $\ker(f)$ is a direct summand of P_R, and $\ker(f) \oplus P_R \cong P_R$. Thus $\operatorname{codim}(\ker(f)) = 0$, so $\ker(f) = 0$.

For the second part of the statement, we show that if x and y are elements of R such that $xy = 1$, then also $yx = 1$. Since $xy = 1$, left multiplication by x is a surjective endomorphism $\mu_x: R_R \to R_R$. From $xy = 1$ it follows $yR \oplus \ker(\mu_x) = R$. Hence $yR = R$ and then y is also right invertible. Thus y is invertible and x is its two-sided inverse. ∎

Now we want to prove another characterization of semilocal rings, due to Camps and Dicks [9].

Lemma 1.4.6 *Let M be a right module over a ring R and let f and g be two endomorphisms of M. Then:*

1. $\ker(f - fgf) = \ker(f) \oplus \ker(1 - gf)$;
2. $\operatorname{coker}(f - fgf) \cong \operatorname{coker}(f) \oplus \operatorname{coker}(1 - fg)$.

Proof.

1. It is clear that $\ker(f) + \ker(1 - gf) \subseteq \ker(f - fgf)$. Conversely, if $x \in \ker(f - fgf)$, then $(1 - gf)(x) \in \ker(f)$, $gf(x) \in \ker(1 - gf)$ and $x = (1 - gf)(x) + gf(x)$, so that $\ker(f) + \ker(1 - gf) = \ker(f - fgf)$. It is easy to verify that $\ker(f) \cap \ker(1 - gf) = 0$.
2. Consider the mapping $\varphi: M \to \operatorname{coker}(f) \oplus \operatorname{coker}(1 - fg)$ defined by $\varphi(x) = (x + f(M), x + (1 - fg)(M))$, for every $x \in M$. We show that φ is a surjective mapping. Note that $M = fg(M) + (1 - fg)(M) \subseteq f(M) + (1 - fg)(M)$. Therefore for any $y, z \in M$, there exist $v \in f(M)$ and $w \in (1 - fg)(M)$ such that $y - z = v + w$. Set $x = y - v = z + w$. Then
$$\varphi(x) = (y + f(M), z + (1 - fg)(M)).$$

This shows that φ is surjective. The kernel of φ is $f(M) \cap (1-fg)(M)$ and thus we must show that $f(M) \cap (1-fg)(M) = (f-fgf)(M)$. Now if $f(x) = (1-fg)(y)$, with $x, y \in M$, then $y = f(x) + fg(y)$, so that

$$f(x) = (1-fg)(y) = (1-fg)(f(x)+fg(y)) = (f-fgf)(x+g(y)).$$

This proves that $f(M) \cap (1-fg)(M) \subseteq (f-fgf)(M)$. The opposite inclusion is easily verified.

∎

Theorem 1.4.7 (Camps and Dicks) *The following conditions are equivalent for a ring R:*

1. *R is semilocal;*
2. *There exist an integer $n \geq 0$ and a function $d: R \to \{1, \dots, n\}$ such that*
   - *(a) for every $a, b \in R$, $d(1-ab) + d(a) = d(a-aba)$;*
   - *(b) if $a \in R$ and $d(a) = 0$, then $a \in U(R)$.*
3. *There exists a partial order $\leq$ on the set R such that*
   - *(c) $(R, \leq)$ is an artinian poset;*
   - *(d) if $a, b \in R$ and $1 - ab \notin U(R)$, then $a - aba < a$.*

Proof. (1)⇒(2) If R is a semilocal ring, then R_R has finite dual Goldie dimension. Let $n = \operatorname{codim}(R_R)$ and $d: R \to \{1, \dots, n\}$ be defined by $d(a) = \operatorname{codim}(R/aR)$ for every $a \in R$.

In order to prove that d has property (a), consider two elements $a, b \in R$ and apply Lemma 1.4.6(2) to the two endomorphisms of the module R_R given by left multiplication by a and b respectively. Then $R/(a-aba)R \cong R/aR \oplus R/(1-ab)R$ implies that $d(1-ab) + d(a) = d(a-aba)$.

If $a \in R$ and $d(a) = 0$, then $R/aR = 0$, so that a is right invertible. Hence it is clear that $a \in U(R)$ by Corollary 1.4.5.

(2)⇒(3) If (2) holds, define a partial order $\leq$ on R by $a \leq b$ if and only if $a = b$ or $d(a) > d(b)$. Then (3) is easily verified.

(3)⇒(1) Let $\overline{R}$ denote $R/J(R)$ and let $\overline{r}$ denote $r + J(R) \in R/J(R)$ for every $r \in R$. Set

$$\mathcal{F} = \{ r \in R \mid \overline{r}^2 = \overline{r} \text{ and } (\overline{1-r})\overline{R} \text{ is a right ideal of finite length of } \overline{R} \}.$$

Note that $\mathcal{F} \neq 0$, because $1 \in \mathcal{F}$. Since $(R, \leq)$ is artinian, there exists an element $a \in \mathcal{F}$ minimal with respect to the order $\leq$.

Suppose $\overline{a} \neq \overline{0}$. Then $a \notin J(R)$, so that $aR \setminus J(R) \neq \varnothing$ and we can choose an element $ab \in aR \setminus J(R)$ that is minimal with respect to the order $\leq$. Since $ab \notin J(R)$, there exists $c \in R$ such that $1 - abc \notin U(R)$. Then by (d) we get $a - abca < a$. Set $a' = a - abca$, so that $a' < a$. We show that $a' \in \mathcal{F}$.

We claim that if $x \in R$ and $1 - abx \notin U(R)$, then $\overline{abxab} = \overline{ab}$. In order to prove the claim fix $x \in R$ with $1 - abx \notin U(R)$. From property (d) we get that $ab - abxab < ab$. Since ab is minimal in $aR \setminus J(R)$, it follows that $ab - abxab \in J(R)$. Hence $\overline{ab} = \overline{abxab}$.

Now apply the claim to $x = c$. Then $\overline{abcab} = \overline{ab}$, so that $\overline{abc}$ is idempotent. Then $\overline{a'}$ is also idempotent, because

$$\overline{a'}^2 = \overline{a - abca}^2 = \overline{a} - \overline{abca} - \overline{abca} + \overline{abcabca} = \overline{a} - \overline{abca} = \overline{a'}.$$

Note that $\overline{a - a'} = \overline{abca}$ also is idempotent.

It is easily verified that $\{\overline{1-a}, \overline{a-a'}, \overline{a'}\}$ is a complete set of orthogonal idempotents in $\overline{R}$. Therefore $\overline{R}_{\overline{R}} = (\overline{1-a})\overline{R} \oplus (\overline{a-a'})\overline{R} \oplus \overline{a'}\overline{R}$. We show that $(\overline{a-a'})\overline{R}$ is a simple $\overline{R}$-module. Since $\overline{a-a'} = \overline{abca}$, we get that

$$(\overline{a-a'})\overline{R} = \overline{abcaR} = \overline{abR}.$$

Moreover $\overline{abR} \neq \overline{0}$, otherwise $ab \in J(R)$. Now consider any $abd \in abR \setminus J(R)$. Since $abd \notin J(R)$, there exists $e \in R$ such that $1 - abde \notin U(R)$. Applying the claim with $x = de$, we see that $\overline{abdeab} = \overline{ab}$, so that $\overline{abdR} = \overline{abR}$. This shows that $(\overline{a-a'})\overline{R}$ is a simple $\overline{R}$-module.

Now $(\overline{1-a})\overline{R}$ is a module of finite length, so that

$$(\overline{1-a})\overline{R} \oplus (\overline{a-a'})\overline{R} = (\overline{1-a'})\overline{R}$$

has finite length. Thus $a' \in \mathcal{F}$.

But $a' < a$ and a was minimal in $\mathcal{F}$. This contradiction shows that $\overline{a} = \overline{0}$. Therefore $(\overline{1-a})\overline{R} = \overline{R}_{\overline{R}}$ has finite length, that is, $\overline{R}$ is right artinian and the proof of (1) is concluded.

Now suppose the equivalent conditions of the statement hold. We want to show that if $m = \dim(\overline{R}) = \operatorname{codim}(R)$ and n is any integer satisfying condition (2), then $m \leq n$. As $m = \dim(\overline{R})$, there are elements $e_1, \dots, e_m \in R$ such that $\{\overline{e_1}, \dots, \overline{e_m}\}$ is a complete set of non-zero orthogonal idempotents of $\overline{R}$. Define $a_0, \dots, a_m \in R$ by induction as follows: $a_0 = 1$ and $a_i = a_{i-1} - a_{i-1}e_i a_{a-i}$ for $i = 1, \dots, m$. Note that $\overline{a_i} = \overline{e_{i+1}} + \overline{e_{i+2}} + \dots + \overline{e_m}$ for every $i = 0, \dots, m$, so that $\overline{1 - a_{i-1}e_i} = \overline{1 - e_i} \notin U(\overline{R})$. It follows that $1 - a_{i-1}e_i \notin U(R)$. Applying property (b) we see that $d(1 - a_{i-1}e_i) > 0$, and applying property (a) we get $d(1 - a_{i-1}e_i) + d(a_{i-1}) = d(a_i)$, so that $d(a_{i-1}) < d(a_i)$ for every $i = 1, \dots, m$. from $d(a_0) < \dots < d(a_m)$ we obtain $m \leq n$. ∎

The last characterization that we give of semilocal ring uses the concept of local morphisms. Given two rings R and S, a ring morphism $\varphi: R \to S$ is said to be *local* if r is invertible whenever $\varphi(r)$ is invertible. For instance, if R is a ring and I is an ideal of R contained in $J(R)$, the canonical projection $R \to R/I$ is a local morphism. In the following Lemma we collect the first properties of local morphisms.

Lemma 1.4.8 *Let $\varphi: R \to S$ and $\psi: S \to T$ be two ring morphisms. Then:*

1. if φ is local, then $\ker(\varphi) \subseteq J(R)$*;*

2. if φ and ψ are local, then $\psi\varphi$ is local;

3. if $\psi\varphi$ is local, then φ is local.

PROOF.

1. Let $y \in \ker(\varphi)$. Since $\varphi(1_R - xy) = 1_S$ is invertible, also $1_R - xy$ is invertible, for any $x \in R$. This means that $y \in J(R)$.

2. Let $r \in R$ such that $\psi\varphi(r)$ is invertible in T. Since ψ is local, $\varphi(r)$ is invertible in S. Hence r is invertible in R since φ is local.

3. Let $r \in R$ such that $\varphi(r)$ is invertible. Then also $\psi\varphi(r)$ is invertible and hence r is invertible since $\psi\varphi$ is local.

■

Theorem 1.4.9 *If $\varphi: R \to S$ is a local morphism and* $\operatorname{codim}(S)$ *is finite, then* $\operatorname{codim}(R) \leq \operatorname{codim}(S)$*. In particular, a ring R is semilocal if and only if there exists a local morphism of R into a semilocal ring, if and only if there exists a local morphism of R into a semisimple artinian ring.*

PROOF. Let $\varphi: R \to S$ be a local morphism. By Proposition 1.4.4 and Theorem 1.4.7 there is a function $d: S \to \{0, \ldots, m\}$, where $m = \operatorname{codim}(S)$, satisfying (a) and (b) of (2) of Theorem 1.4.7. We want to show that the function $d\varphi: R \to \{0, \ldots, m\}$ satisfies (a) and (b) of (2) of Theorem 1.4.7 and hence, always by Theorem 1.4.7, we have that $\operatorname{codim}(R) \leq \operatorname{codim}(S)$. In fact we have

$$\begin{aligned} d\varphi(1 - xy) + d\varphi(x) &= d(1 - \varphi(x)\varphi(y)) + d(\varphi(x)) \\ &= d(\varphi(x) - \varphi(x)\varphi(y)\varphi(x)) \\ &= d\varphi(x - xyx) \end{aligned}$$

for any $x, y \in R$, and

$$d\varphi(x) = 0 \Rightarrow \varphi(x) \in U(S) \Rightarrow x \in U(R)$$

for any $x \in R$, since φ is local. ■

1.5 Semilocal endomorphism rings

In this section we prove some properties of objects that have a semilocal endomorphism ring. We say that a ring R has *left stable range 1* if, whenever $Ra + Rb = R$, there exists $r \in R$ such that $a + rb$ is invertible.

Proposition 1.5.1 (Bass) *Every semilocal ring R has left stable range 1.*

PROOF. Recalling that $u \in R$ is a unit if and only if $\overline{u} \in R/J(R)$ is a unit, we may replace R with $R/J(R)$ and assume that $R/J(R)$ is semisimple artinian. Using the Wedderburn-Artin Theorem we may further assume that $R = \mathrm{End}_D(V)$, where V is a finite dimensional right vector space over a division ring D. Now suppose that $Ra+Rb = R$. The left ideal Rb gives rise to a subspace $W = \{v \in V \mid Rbv = 0\}$ of V. In fact, we have that $Rb = \mathrm{Ann}(W) = \{f \in R \mid f(W) = 0\}$.

Note that the restriction of the action of a on W gives an isomorphism $W \to aW$. To see this, write $1 = ra+r'b$, where $r, r' \in R$. If $w \in W$ is such that $a(w) = 0$, then $w = (ra + r'b)w = r'b(w) = 0$, as desired. Now pick a D-automorphism f of V such that $f(w) = a(w)$ for every $w \in W$. Then $f - a \in \mathrm{Ann}(W) = Rb$, so $a + Rb$ contains the unit f of R. ∎

Using the above Proposition now we show that modules with semilocal endomorphism rings cancel from direct sums.

Proposition 1.5.2 *Let A, B and C be objects of a preadditive category $\mathcal{C}$. Suppose $E = \mathrm{End}_R(A)$ has left stable range 1. Then $A \oplus B \cong A \oplus C$ implies $B \cong C$.*

PROOF. Since $A \oplus B \cong A \oplus C$ there are two inverse morphisms

$$F = \begin{pmatrix} f_{A,A} & f_{B,A} \\ f_{A,C} & f_{B,C} \end{pmatrix} : A \oplus B \to A \oplus C$$

and

$$G = \begin{pmatrix} g_{A,A} & g_{C,A} \\ g_{A,B} & g_{C,B} \end{pmatrix} : A \oplus C \to A \oplus B.$$

Since GF is the identity on $A \oplus B$ we have

$$\begin{pmatrix} g_{A,A}f_{A,A} + g_{C,A}f_{A,C} & g_{A,A}f_{B,A} + g_{C,A}f_{B,C} \\ g_{A,B}f_{A,A} + g_{C,B}f_{A,C} & g_{A,B}f_{B,A} + g_{C,B}f_{B,C} \end{pmatrix} = \begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}.$$

From $g_{A,A}f_{A,A} + g_{C,A}f_{A,C} = 1_A$ it follows that $Ef_{A_A} + Eg_{C,A}f_{A,C} = E$. Hence there exists $t \in E$ such that $u = f_{A,A} + tg_{C,A}f_{A,C}$ is an automorphism of A. Consider the mapping

$$G' = \begin{pmatrix} 1_A & tg_{C,A} \\ g_{A,B} & g_{C,B} \end{pmatrix} : A \oplus B \to A \oplus C.$$

Then

$$G'F = \begin{pmatrix} u & v_{B,A} \\ 0 & 1_B \end{pmatrix}$$

is clearly an automorphism of $A \oplus B$. Since $F : A \oplus B \to A \oplus C$ is an automorphism, it follows that $G' : A \oplus B \to A \oplus C$ is an automorphism as well. But then

$$\begin{pmatrix} 1_A & 0 \\ -g_{A,B} & 1_B \end{pmatrix} G' \begin{pmatrix} 1_A & -tg_{C,A} \\ 0 & 1_C \end{pmatrix} = \begin{pmatrix} 1_A & 0 \\ 0 & g_{C,B} - g_{A,B}tg_{C,A} \end{pmatrix}$$

and hence the homomorphism $g_{C,B} - g_{A,B} t g_{C,A}: C \to B$ is an isomorphism of C into B.

■

If M_R is a right R-modules, let $\mathrm{add}(M_R)$ denote the full subcategory of Mod-R whose objects are all the modules isomorphic to direct summands of direct sums of finitely many copies of M_R. For example $\mathrm{add}(R_R) = \text{proj-}R$.

Lemma 1.5.3 *Let M_R be a non-zero right R-module and let $E = \mathrm{End}_R(M)$ be its endomorphism ring. The functors*

$$\mathrm{Hom}_R(M, -): \text{Mod-}R \to \text{Mod-}E \quad \textit{and} \quad - \otimes_E M: \text{Mod-}E \to \text{Mod-}R$$

induce an equivalence between the full subcategory $\mathrm{add}(M_R)$ *of* Mod-R *and the full subcategory* proj-E *of* Mod-E.

We use this Lemma to prove the following.

Proposition 1.5.4 *Let A and B two right R-modules. If A has semilocal endomorphism ring and there exists an integer n such that $A^n \cong B^n$, then $A \cong B$.*

Proof. We shall suppose that we have $M = \oplus_{i=1}^n A_i = \oplus_{i=1}^n B_i$, where each $A_i \cong A$ and $B_i \cong B$, A has local endomorphism ring, and prove that $A_1 \cong B_1$.

Note that M has semilocal endomorphism ring, since

$$\mathrm{End}_R(M) = M_n(\mathrm{End}_R(A)).$$

Let $\epsilon_i: A_i \to M$ and $\pi_i: M \to A_i$, $i = 1, \ldots, n$, be the canonical morphisms with respect to the decomposition $M = \oplus_{i=1}^n A_i$ and $\epsilon_i': B_i \to M$ and $\pi_i': M \to B_i$, $i = 1, \ldots, n$, be the canonical morphisms with respect to the decomposition $M = \oplus_{i=1}^n B_i$. Denote by $e_i = \epsilon_i \pi_i$ and $f_i = \epsilon_i' \pi_i'$. Applying Lemma 1.5.3 we obtain that there are monomorphism $\mathrm{Hom}_R(M, \epsilon_i): \mathrm{Hom}_R(M, A_i) \to E$, whose image is $e_i E$, and monomorphisms $\mathrm{Hom}_R(M, \epsilon_i'): \mathrm{Hom}_R(M, B_i) \to E$, whose image is $f_i E$. Therefore we have $e_i E \cong e_1 E$ and $f_i E \cong f_1 E$ for every i, so that $e_i E / e_i J(E) \cong e_1 E / e_1 J(E)$ and $f_i E / f_i J(E) \cong f_1 E / f_1 J(E)$. Hence

$$E/J(E) \cong \oplus_{i=1}^n e_i E / e_i J(E) \cong \oplus_{i=1}^n f_i E / f_i J(E).$$

Since E is semilocal, $E/J(E)$ is a semisimple right E-module of finite length, and therefore $e_1 E / e_1 J(E) \cong f_1 E / f_1 (E)$. By Proposition 3.3(b) of [14] $e_1 E \cong f_1 E$, so that $A_1 \cong B_1$ by Lemma 1.5.3, as desired. ■

Proposition 1.5.5 *Let A be a right R-module with semilocal endomorphism ring such that* $\mathrm{codim}(\mathrm{End}_R(A)) = n$. *Then A has at most 2^n isomorphic classes of direct summands.*

Proof. By Lemma 1.5.3 it is clear that direct summands of A corresponds to direct summands of its endomorphism ring $E = \mathrm{End}_R(A)$. Hence it is enough to show that E has at most 2^n non-isomorphic direct summands eE. By proposition 3.3(b) of [14] it suffices to show that $E/J(E)$ has at most 2^n non-isomorphic direct summands. This is obvious because $E/J(E)$ is semisimple artinian and $\dim(E/J(E)) = \operatorname{codim}(E) = n$. ■

Now we want to give some examples of modules with semilocal endomorphism ring. To do this we first need the following useful criterion.

Proposition 1.5.6 (Herbera and Shamsuddin) *Let M_R be a right module over a ring R.*

1. *If M_R has finite Goldie dimension and every injective endomorphism of M_R is bijective, then the endomorphism ring $\operatorname{End}_R(M)$ is semilocal and*

$$\operatorname{codim}(\operatorname{End}_R(M)) \leq \dim(M_R);$$

2. *if M_R has finite dual Goldie dimension and every surjective endomorphism of M_R is bijective, then the endomorphism ring $\operatorname{End}_R(M)$ is semilocal and*

$$\operatorname{codim}(\operatorname{End}_R(M)) \leq \operatorname{codim}(M_R);$$

3. *if M_R has finite Goldie dimension and finite dual Goldie dimension, then the endomorphism ring $\operatorname{End}_R(M)$ is semilocal and*

$$\operatorname{codim}(\operatorname{End}_R(M)) \leq \dim(M_R) + \operatorname{codim}(M_R).$$

Proof.

1. If $\dim(M_R)$ is finite, set $n = \dim(M_R)$ and define

$$\begin{array}{cccc} d_1\colon & \operatorname{End}_R(M) & \to & \{1,\dots,n\} \\ & f & \mapsto & \dim(\ker(f)). \end{array}$$

By Lemma 1.4.6(1), the mapping d_1 satisfies conditions (a) and (b) of Theorem 1.4.7. This proves (1).

2. If $\operatorname{codim}(M_R)$ is finite, set $m = \operatorname{codim}(M_R)$ and define

$$\begin{array}{cccc} d_2\colon & \operatorname{End}_R(M) & \to & \{1,\dots,m\} \\ & f & \mapsto & \operatorname{codim}(\operatorname{coker}(f)). \end{array}$$

By Lemma 1.4.6(2), the mapping d_2 satisfies conditions (a) and (b) of Theorem 1.4.7. This proves (2).

3. If $\dim(M_R)$ and $\operatorname{codim}(M_R)$ are both finite, set

$$d = d_1 + d_2\colon \operatorname{End}_R(M) \to \{1,\dots,m+n\}.$$

From this (3) follows.

■

Proposition 1.5.7 *Every artinian module has semilocal endomorphism ring.*

Proof. It is enough to apply (1) of previous Proposition, since every artinian module has finite Goldie dimension and every injective endomorphism is bijective. ■

Dually, we have the following.

Proposition 1.5.8 *Every noetherian module of finite dual Goldie dimension has semilocal endomorphism ring.*

Proof. It is enough to apply (2) of Proposition 1.5.6. ■

In the following chapters we will meet other relevant examples of objects (mainly modules) with semilocal endomorphism ring.

Chapter 2

Infinite dual Goldie dimension

We saw in the previous chapter that the Goldie dimension and the dual Goldie dimension play an important role in module theory. Till now we investigated only the finite case, and we saw that a ring has finite dual Goldie dimension if and only if it is semilocal. In this chapter we want to define and investigate the infinite case, focusing mainly on the dual Goldie dimension of the right R-module R_R. We start from the most general setting where the Goldie and the dual Goldie dimension makes sense, that is, the setting of bounded modular lattices. First, we analyze which properties of the finite Goldie dimension still hold in the infinite case and which do not. Then, we restrict our attention to the case of the dual Goldie dimension of the right R-module R_R. For this study, in which we are particularly interested, we can consider only the maximal right ideals of R instead of the whole lattice of right ideals of R. Eventually, we study in detail some relevant examples, computing their dual Goldie dimension. These examples show the difficulties that arise in passing from the finite case to the infinite one.

2.1 Goldie dimension on lattices

Let L be a bounded modular lattice, that is, a lattice L that satisfies the modular law $x \leq b \Rightarrow x \vee (a \wedge b) = (x \vee a) \wedge b$ and has a greatest element 1 and a smallest element 0. Recall that a subset $\{ a_i \mid i \in I \}$ of $L \setminus \{ 0 \}$ is said to be join-independent if $a_i \wedge (\bigvee_{i \neq j \in F} a_j) = 0$ for every $i \in I$ and every finite subset $F \subseteq I$ containing i.

Generalizing the definition that we gave in the previous chapter, we say that the *Goldie dimension* of L, denoted by $\dim(L)$, is defined as the supremum of all cardinals $\aleph$ such that L contains a join-independent subset of cardinality $\aleph$.

Remember that a subset $A = \{ a_i \mid i \in I \}$ of L is coindependent if for every finite subset $F \subseteq I$ and $i \in F$ we have $a_i \vee (\bigwedge_{j \neq i \in F} a_j) = 1$.

The *dual Goldie dimension* of L, denoted by $\mathrm{codim}(L)$, is the Goldie dimension of L^{op}, i.e. the supremum of all cardinals $\aleph$ such that L contains a coindependent subset of cardinality $\aleph$.

Given a cardinal number $\aleph$, say that $\aleph$ is *attained* in L if L contains a join-independent subset of cardinality $\aleph$. We recall that an infinite cardinal $\aleph$ is called *regular* if $\aleph_i < \aleph$ for $i \in I$ with $|I| < \aleph$ implies $\sum \aleph_i < \aleph$. Otherwise it is called *singular*. An uncountable, regular, limit cardinal is said to be *inaccessible*. We remind that the existence of inaccessible cardinals can not be proved in ZFC (Zermelo-Fraenkel with the axiom of choice) and that there are no such cardinals in the constructible universe (see for example [12]). In [48], Santa-Clara and Silva proved, generalizing results in [10] and [12], that the Goldie dimension of L can be not attained only if it is an inaccessible cardinal.

Definition 2.1.1 Let L be a bounded modular lattice and let $A = \{\, a_i \mid i \in I \,\}$ be a subset of L. A is called an *essential subset* if for every non-zero element $b \in L$, there exists a finite subset F of I such that $(\bigvee_{i\in F} a_i) \wedge b \neq 0$.

Similarly, A is a *superfluous subset* if for every $1 \neq b \in L$, there exists a finite subset F of I such that $(\bigwedge_{i\in F} a_i) \vee b \neq 1$. Obviously A is a superfluous subset in L if and only if A is an essential subset in L^{op}.

Let a be an element of L and $A = \{\, a_i \mid i \in I \,\}$ a subset of L such that $a_i \leq a$ for every $i \in I$. We say that A is *essential in a* if it is an essential subset of the lattice $[0, a]$.

A finite subset $A = \{\, a_i \mid i \in I \,\} \subseteq L$ is essential if and only if $\bigvee_{i\in I} a_i$ is essential in L. Similarly, A is superfluous if and only if $\bigwedge_{i\in I} a_i$ is superfluous in L.

Theorem 2.1.2 *Let $L \neq 0$ be a bounded modular lattice such that every non-zero element of L contains a uniform element. Let $\aleph$ be a cardinal number. Then the following are equivalent:*

1. *L does not contain join-independent subsets of cardinality $\geq \aleph$;*
2. *L contains an essential join-independent subset $\{\, a_i \mid i \in I \,\}$ of cardinality strictly less than $\aleph$, with a_i uniform for every $i \in I$;*
3. *there exists a cardinal $\beth < \aleph$ such that every join-independent subset of L has cardinality $\leq \beth$.*

Moreover, if these equivalent conditions hold, every essential join-independent subset $\{\, a_i \mid i \in I \,\}$, with a_i uniform for every $i \in I$, attains the Goldie dimension of L.

Proof.

(1)⇒(2) Let F be the set of all join-independent subsets of L consisting only of uniform elements. Since every element of L contains a uniform element, F is non-empty. By Zorn's lemma, F has a maximal element X with respect to inclusion. By (a), $\mathrm{card}(X) < \aleph$. At this point, we claim that X is an essential subset of L; otherwise there would exist a non-zero element $x \in L$ such that

$(\bigvee_{y \in F} y) \wedge x = 0$ for every finite subset $F \subseteq X$ and, by hypothesis, there would be a uniform element $b \in L$ such that $b \leq x$. Then $X \cup \{ b \}$ would be a join-independent set of uniform elements strictly containing X, a contradiction to the maximality of X.

(2)⇒(3) Suppose that there exists an essential join-independent subset $A = \{ a_i \mid i \in I \}$ with every a_i uniform, $\mathrm{card}(I) = \beth < \aleph$.
We claim that A is maximal between the join-independent subsets of L; otherwise there exists a non-zero element $b \in L$ such that $A \cup \{ b \}$ is join-independent. This means that for every finite subset F of I we have that $(\bigvee_{i \in F} a_i) \wedge b = 0$, but this clearly contradicts the hypothesis that A is an essential subset.
Then, by Theorem 1 of [33], we have

$$\mathrm{card}(J) \leq \mathrm{card}(I)$$

for every join-independent subset $\{ b_j \mid j \in J \}$ of L.

(3)⇒(1) Obvious.

The final remark is clear form the proof $(b) \Rightarrow (c)$. ■

The hypothesis that every non-zero element contains a uniform element is not only necessary, but also sufficient to claim that the lattice has an essential independent subset of uniform elements.

Proposition 2.1.3 *Let L be a bounded modular lattice. If L contains an essential join-independent subset $\{a_i \mid i \in I\}$ with a_i uniform for every $i \in I$, then every non-zero element of L contains a uniform element.*

Proof. Let x be a non-zero element of L. Since $\{a_i \mid i \in I\}$ is an essential set, there exists a finite subset $F \subseteq I$ such that $x \wedge (\bigvee_{i \in F} a_i) \neq 0$. Now

$$\dim([0, x \wedge (\bigvee_{i \in F} a_i)]) \leq \dim([0, \bigvee_{i \in F} a_i]) = |F|.$$

Hence, by Theorem 1.3.5, there is a uniform element $u \leq x \wedge (\bigvee_{i \in F} a_i) \leq x$. ■

In the finite case one has that also the following statement is equivalent to the ones in the theorem:

- if $a_0 \leq a_1 \leq a_2 \leq \ldots$ is an ascending chain of elements of L, then there exists $i \geq 0$ such that a_i is essential in a_j for every $j \geq i$.

One can try to generalize this to the infinite case and ask if the following condition is equivalent to the ones in the theorem:

4. there does not exist an ascending chain X of elements of L of cardinality $\aleph$ such that $\{ b \in X \mid b < a \}$ is not an essential set in a, for every a in the chain.

What happens is that just one implication continues to hold. We have that (3)⇒(4): if (4) does not hold, there exists an ascending chain X of elements of L of cardinality $\aleph$ such that $\{ b \in X \mid b < a \}$ is not an essential set in a, for every a

in the chain. Then, for every a in the chain, there exist a non-zero element $c_a \leq a$ in L such that $(\bigvee_F b) \wedge c_a = 0$, for any finite subset F of $\{ b \in X \mid b < a \}$. This implies that these elements c_a form a join-independent subset of L of cardinality $\aleph$. Thus (3) does not hold.

The problem is that the other implication is no longer true when we pass to the infinite case. Let us show that the implication (4)⇒(1) is false in general. Let X be a set of cardinality $\aleph$; consider L to be the sublattice of $(\wp(X), \subseteq)$ consisting of $\varnothing$, X itself and all the finite subsets of X. It is clear that L is modular, since it is a sublattice of a distributive lattice. Every singleton is an uniform element in L, then every non-zero element of L contains a uniform element.

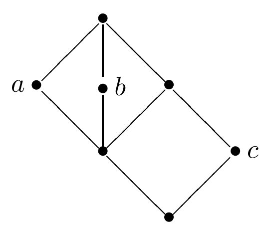
Since $x \wedge (\bigvee_{i=1}^n x_i) = 0$ for x, x_i singletons of X with $x \neq x_i$, $i = 1, \ldots, n$, we have that the set of the singletons is a join-independent subset of L.

On the other hand it is also obvious that every chain in L can have at most countable cardinality.

Proposition 2.1.4 *Let L be a bounded modular lattice and let $E \subseteq \wp(L)$ be the set of essential sets of L. If A is an independent essential set of L, then it is minimal in E.*

Proof. Let A be an independent essential set of L. If A' is an essential set strictly contained in A and $a \in A \smallsetminus A'$, there exists a finite set $F \subseteq A'$ such that $a \wedge (\bigvee_{i \in F} a_i) \neq 0$. This clearly contradicts the fact that A is independent. ∎

Conversely, it is not true in general that a minimal essential set is independent. To see this it is enough to look at the following example. Let L be the lattice



It is clear that the set $\{a, b\}$ is minimal essential but it is not independent. In fact, Puczyłowski proved in [41] that the one above is the only pathology that can appear considering minimality of essential subsets of uniform elements.

Before stating the next proposition, if L and L' are two bounded modular lattices, we denote by $L \oplus L'$ the direct sum of L and L', which, as a set, consists of the elements (l, l'), with $l \in L$ and $l' \in L'$ and have the operations defined componentwise.

Proposition 2.1.5 *Let L be a bounded modular lattice.*

1. $\dim(L) = 0$ *if and only if* $L = 0$*;*
2. $\dim(L) = 1$ *if and only if* L *is uniform;*

3. $\dim([0,a]) \leq \dim(L)$ *for every* $a \in L$*;*

4. $\dim([0,a]) = \dim(L)$ *if* a *is essential in* L*;*

5. *if* L' *is another modular lattice bounded, then* $\dim(L \oplus L') = \dim(L) + \dim(L')$.

PROOF. The proof of (1), (2), (3) and (4) are elementary (the original article of Alfred Goldie where these things were observed is [31]). To prove (5) it is enough to observe that if $\{ a_i \mid i \in I \}$ is an essential join-independent subset of uniform elements of L_1 and $\{ b_j \mid j \in J \}$ is an essential join-independent subset of uniform elements of L_2, then $\{ (a_i, 0) \mid i \in I \} \cup \{ (0, b_j) \mid j \in J \}$ is an essential join-independent subset of uniform elements of $L_1 \oplus L_2$. ∎

We notice that the converse implication of (4) holds only in the finite case.

Remark 2.1.6 The hypothesis that every element of the lattice contains a uniform element is always satisfied by the dual lattice of the right (left) ideals of a ring, since every right (left) ideal is contained in a maximal one. Therefore the dual Goldie dimension (left or right) of a ring is always attained.

2.2 Dual Goldie dimension of rings

In view of the previous remark, now we restrict to the case of the right dual Goldie dimension of a ring R. We can easily observe here a certain number of facts:

- the Jacobson radical is a superfluous ideal ([5], Prop. 9.18). This means that $\operatorname{codim}(R_R) = \operatorname{codim}(R_R/J(R_R))$ and so we can restrict our attention to semiprimitive rings;

- when we look for coindependent sets we can restrict to maximal right ideals. If $I_1, \dots, I_n$ are coindependent right ideals, i.e. $I_i + (\bigcap_{j \neq i} I_j) = R$, choosing maximal ideals $M_i \supseteq I_i$, we have that $M_i + (\bigcap_{j \neq i} M_j) = R$, which means that $M_1, \dots, M_n$ are coindependent maximal right ideals. Moreover $M_1, \dots, M_n$ are all distinct; in fact, if $M_i = M_j$, we have that $M_i = M_i + M_j \supseteq I_i + I_j$, and this contradicts the fact that I_i and I_j are coindependent.

Let us see now how the concepts that we introduced above translate in this particular case. Let $\{ M_i \mid i \in I \}$ be a set of maximal right ideals.

The set $\{ M_i \mid i \in I \}$ is coindependent if for every finite subset $F \subseteq I$ and $i \in F$, we have $M_i + (\bigcap_{i \neq j \in F} M_j) = R_R$, that is equivalent to saying that $\bigcap_{i \neq j \in F} M_j \nsubseteq M_i$; this, thanks to the Chinese Remainder Theorem, is also equivalent to

$$\frac{R}{\bigcap_{i \in F} M_i} \cong \bigoplus_{i \in F} \frac{R}{M_i}.$$

We have that $\bigcap_{i\in I} M_i$ is a superfluous ideal in R if and only if it is equal to the Jacobson radical $J(R)$. It is clear that $\bigcap_{i\in I} M_i \supseteq J(R)$, since $J(R)$ is the intersection of all maximal right ideals; on the other side the Jacobson radical of a ring R is the biggest superfluous right ideal ([5], Prop. 9.18), and therefore $\bigcap_{i\in I} M_i$ must be contained in it.

The set $\{ M_i \mid i \in I \}$ is superfluous, by definition, if for any proper right ideal $J \subseteq R_R$ there exists a finite subset $F \subseteq I$ such that $J + \bigcap_{i\in F} M_i \neq R_R$. Clearly, without loss of generality, we can take J to be a maximal ideal; therefore the condition we get is that for any maximal right ideal $M \subseteq R_R$ there exists a finite subset $F \subseteq I$ such that $M + \bigcap_{i\in F} M_i \neq R_R$; this is equivalent to saying that for every maximal right ideal there exists a finite subset $F \subseteq I$ such that $\bigcap_{i\in F} M_i \subseteq M$.

It is clear that if the set is superfluous, then $\bigcap_{i\in I} M_i = J(R)$ is superfluous. The converse implication is not true in general.

Example 2.2.1 Consider the polynomial ring $\mathbb{Q}[x]$ in one variable over the rational numbers. It is a principal ideal domain and $(p(x))$ is a maximal ideal if and only if $p(x)$ is an irreducible polynomial. Let $T = \{ (x-a) \mid a \in \mathbb{Z} \}$; we can notice that, since $(x-a) + \prod_{i=1}^{n}(x-b_i) = \mathbb{Q}[x]$ for any distinct $a, b_1, \dots, b_n$, the set T is coindependent. It is clear also that $\bigcap_T (x-a) = (0) = J(\mathbb{Q}[x])$. Anyway, if we take $p(x)$ to be an irreducible polynomial of degree bigger than 1, there does not exist a finite number of elements in T such that $(p(x)) \subseteq \bigcap_{i=1}^{n}(x-a_i)$.

Example 2.2.2 In this example we will show that the cardinalities of a superfluous coindependent set of maximal right ideals and of a coindependent set of maximal right ideals with superfluous intersection can be different. Let us consider the ring of continuous functions of the real numbers $C(\mathbb{R})$; for every $a \in \mathbb{R}$, the subset $M_a = \{ f \in C(\mathbb{R}) \mid f(a) = 0 \} \subseteq C(\mathbb{R})$ is a maximal ideal. The set $\{ M_a \mid a \in \mathbb{Q} \}$ is a coindependent set of cardinality $\aleph_0$, but it is not coindependent since none of the maximal ideals M_a, with $a \in \mathbb{R} \setminus \mathbb{Q}$, contains a finite intersection of maximal ideals in this set. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, the intersection of $\{ M_a \mid a \in \mathbb{Q} \}$ consists only of the zero function. On the other hand it is clear that to have a coindependent superfluous set it is necessary that it contains all the maximal ideals M_a, with $a \in \mathbb{R}$, and therefore it has cardinality at least $\mathfrak{c}$.

We saw that if $\{ M_i \mid i \in I \}$ is a coindependent set we have, for any finite subset $F \subseteq I$, an isomorphism $\frac{R}{\bigcap_{i\in F} M_i} \cong \bigoplus_{i\in F} \frac{R}{M_i}$. Therefore, in this situation, $\{ M_i \mid i \in I \}$ is a superfluous set if, for any maximal right ideal M of R_R, we have that there exists a finite subset $F \subseteq I$ such that $\frac{M}{\bigcap_{i\in F} M_i}$ is a maximal ideal of the semisimple ring $\bigoplus_{i\in F} \frac{R}{M_i}$.

Now we show that in the case of the dual Goldie dimension of a ring R, the converse of Proposition 2.1.4 holds.

Proposition 2.2.3 *Let $\mathcal{M}_R$ be the set of all maximal right ideals of a ring R and $\mathcal{F} \subseteq \wp(\mathcal{M}_R)$ the set of all superfluous sets of maximal ideals. If $\mathcal{F}$ has a minimal element T^*, then T^* is coindependent and hence $|T^*| = \operatorname{codim}(R_R)$.*

Proof. Let T^* be a minimal superfluous set of maximal ideals. To prove that it is coindependent, we have to show that for any $M_1, \dots, M_n \in T$, we have that $M_i + (\bigcap_{j \neq i} M_j) = R$ for any $i = 1, \dots, n$. Suppose not, i.e. there exists an i such that we have $M_i + (\bigcap_{j \neq i} M_j) \subsetneq R$; this means that $M_i + (\bigcap_{j \neq i} M_j) \supseteq M_i$ is contained in a maximal ideal, that clearly must be M_i itself. From this it follows that $\bigcap_{j \neq i} M_j \subseteq M_i$ contradicting the fact that T^* is minimal. Thanks to Theorem 2.1.2, it is clear that $|T^*| = \text{codim}(R_R)$. ∎

It clearly follows that

Corollary 2.2.4 *All minimal superfluous sets T of maximal right ideals of a ring R have the same cardinality.*

2.3 Examples and computations

In this section we want to analyze some concrete examples of lattices and rings and compute their dual Goldie dimension. We start with an example that will turn out to be very helpful. First we need some definitions.

Definition 2.3.1 Let L be a lattice. An *ideal* I of L is a non-empty subset of L such that:

- for every $x \in I$ and $y \in L$, $x \wedge y$ is in I;
- for every $x, y \in I$, their join $x \vee y$ is in I.

An ideal I is *maximal* if the only ideal that properly contains I is the whole lattice L.

Dually, a *filter* F of L is a non-empty subset of L such that:

- for every $x \in F$ and $y \in L$, $x \vee y$ is in F;
- for every $x, y \in F$, their meet $x \wedge y$ is in F.

A filter F is *maximal* if the only filter that properly contains F is the whole lattice L. In this case it is called *ultrafilter*.

Given an element a of a lattice L, the set of all the elements $b \in L$ such that $b \leq a$ forms an ideal of L, called the *principal ideal generated by a* and denoted by $(a]$. By Zorn's Lemma it is clear that every ideal of a bounded lattice is contained in a maximal ideal.

The set $I(L)$ of all ideals of L is a bounded lattice, considering the intersection of ideals as meet and the ideal generated by I and J, i.e. the intersection of all the ideals containing I and J, as the join of I and J. We have that if L is modular or distributive, then so is $I(L)$.

The same holds also for the set of all filters of L, since they are in fact the ideals of the dual lattice L^{op}.

An ideal I is *prime* if it is proper and for every $a, b \in L$ such that $a \wedge b \in I$, then either $a \in I$ or $b \in I$.

We recall that a *complement* for an element a of a bounded lattice L is an element b such that $a \wedge b = 0$ and $a \vee b = 1$. A bounded lattice L is *complemented* if every element in L has a complement.

Lemma 2.3.2 *1. If L is a distributive lattice, every maximal ideal is prime;*

2. if L is complemented, every prime ideal is maximal;

3. if P is a prime ideal and I and J are ideals of a lattice L, then $I \cap J \subseteq P$ implies that either $I \subseteq P$ or $J \subseteq P$.

PROOF.

1. Suppose $I \subseteq L$ is a maximal ideal and let $a, b \in L$ be such that $a \wedge b \in I$. If $a \notin I$, the ideal generated by I and $(a]$ is the whole lattice, so that $b \le i \vee a$ for some element $i \in I$. Then $b = b \wedge b \le (i \vee a) \wedge b = (i \wedge b) \vee (a \wedge b) \in I$.

2. Suppose I is a prime ideal of L. We show that for any element $a \in L \setminus I$, the ideal generated by I and $(a]$ is the whole lattice. Let b be a complement of a, then $b \in I$, since $a \wedge b = 0$ and I is prime. Now $1 = a \vee b$ is in the ideal generated by $(a]$ and I.

3. Let P be a prime ideal containing the intersection of two ideals I and J. If neither $I \subseteq P$ nor $J \subseteq P$ there exist two elements $i \in I \setminus P$ and $j \in J \setminus P$. Then $i \wedge j \in I \cap J \subseteq P$ and this contradicts the fact that P is prime.

■

The same observations we made about ideals of a ring hold for ideals of a lattice. Hence, the dual Goldie dimension of the lattice $I(L)$ of ideals of a lattice L is always attained and to compute it, it is enough to find a superfluous coindependent family of maximal ideals.

Proposition 2.3.3 *If L is a bounded distributive lattice, the family of all maximal ideals is coindependent. Hence the dual Goldie dimension of $I(L)$ equals the cardinality of the set of maximal ideals of L.*

PROOF. A family of maximal ideals $\{M_i\}_{i \in I}$ is coindependent if and only if for every $\bar{\imath} \in I$ and every finite subset $F \subseteq I$ we have that $\bigcap_{i \in F} M_i \subseteq M_{\bar{\imath}}$ implies that $\bar{\imath} \in F$. The maximal ideal $M_{\bar{\imath}}$ is prime and hence $\bigcap_{i \in F} M_i \subseteq M_{\bar{\imath}}$ implies, by (3) of Lemma 2.3.2, that there exists an $i \in I$ such that $M_i \subseteq M_{\bar{\imath}}$. Since they are both maximal ideals, they must coincide.

Since the set of all maximal ideals is clearly superfluous, it is clear that the dual Goldie dimension of $I(L)$ equals the cardinality of the set of maximal ideals of L. ■

If a lattice B is *Boolean*, i.e. distributive and complemented, it is usual to express the concepts in terms of filters and ultrafilters instead of ideals and maximal ideals. The situation does not change a lot since there is a one to one correspondence between maximal ideals and ultrafilters given by the complement. The set of ultrafilters is the underlying set of a topological space, called the *Stone space* and denoted by $S(B)$. The topology for $S(B)$ is generated by a basis consisting of all the sets of the form

$$\{x \in S(B) \mid b \in x\}$$

where b is an element of B.

Since the ring theoretical notion of ideal in Boolean rings corresponds to the lattice theoretical notion of ideal in Boolean lattices, in the correspondence that there is between the two categories, we have obviously the following.

Theorem 2.3.4 *Given a Boolean ring B, its dual Goldie dimension is given by the cardinality of its Stone space $S(B)$.*

As a particular case, using ([29],9.2), we get the following.

Corollary 2.3.5 *Let X be a set. The dual Goldie dimension of the Boolean ring $(\wp(X), \Delta, \cap)$ is equal to $2^{2^{|X|}}$.*

The boolean ring $(\wp(X), \Delta, \cap)$ can be seen also as the ring 2^X of functions from the set X to the field with two elements. If we take any field K, what we proved above holds also for the ring K^X of functions from the set X to the field K. To show this it is enough to establish an inclusion preserving bijection between the set of proper ideals of K^X and the set of proper filters of X.

Let I be a proper ideal of the ring K^X, we want to show that the collection $Z(I)$ of zero sets of elements of I form a filter of X. To do this we observe the following:

- the empty set is not a zero set, since the identity function does not belong to I;
- if $A \subseteq X$ is a zero set of an element $f \in I$ and $A \subseteq B \subseteq X$, then B is the zero set of the element $f \cdot \chi_{X \setminus B} \in I$, where with χ_Y we denote the characteristic function of the set $Y \subseteq X$;
- let A and B be the zero set of two elements in I. Multiplying them by the appropriate elements in the ring we find that also $\chi_{X \setminus A}$ and $\chi_{X \setminus B}$ are in the ideal I; therefore we have that $\chi_{X \setminus A} + \chi_{X \setminus B} - \chi_{X \setminus A} \cdot \chi_{X \setminus B}$ (the last term is needed only when the characteristic of K is 2) is an element of the ideal I, having as zero set $A \cap B$.

On the other hand, let F be a proper filter on X. We want to prove that the set $I(F) = \{ f : X \to K \mid Z(f) \in F \}$, where $Z(f)$ indicates the zero set of f, is a proper ideal of the ring K^X. To do this we notice the following:

- the identity of K^X is not in I since the empty set is not in F;
- if $f \in I$ and $g \in K^X$, we have that $fg \in I$ since $Z(f) \subseteq \mathbb{Z}(fg) \subseteq X$;

- if f and g are elements of I, then $f + g \in I$ since $Z(f + g) \supseteq Z(f) \cap Z(g)$.

To check that these two maps are each other's inverse, we observe that, for any proper ideal I of K^X

$$IZ(I) = \{f \in K^X \mid Z(f) \in Z(I)\} = \{f \in K^X \mid \exists g \in I \text{ such that } Z(f) = Z(g)\}.$$

From the equality $Z(f) = Z(g)$ we deduce that also $f \in I$ and so $IZ(I) = I$.

On the other hand, let F be a proper filter of X. We have

$$ZI(F) = \{Z(f) \subseteq X \mid f \in I(F)\} = \{Z(f) \subseteq X \mid Z(f) \in F\}.$$

It is clear that $ZI(F) \subseteq F$. To check that the equality holds it is enough to notice that every subset Y of X is of the form $Z(f)$ for some function $f \in K^X$. Since the two maps are clearly order preserving, in this way we have a lattice isomorphism between the lattice of ideals of K^X and the lattice of filters of the lattice $\wp(X)$. Hence we can conclude as follows.

Proposition 2.3.6 *Let X ba a set and K any field. Then the dual Goldie dimension of the ring K^X is equal to $2^{2^{|X|}}$.*

Now let X be a topological space and consider the ring $C(X)$ of continuous functions from X to the reals. Let $Z(f)$ be the zero set of an element $f \in C(X)$ and let $Z(X) = \{Z(f) \mid f \in C(X)\}$. Now we want to show that the set $Z(X)$ is in fact a sublattice of the distributive lattice $\wp(X)$, and hence a distributive lattice itself. Let $Z(f)$ and $Z(g)$ be the zero sets of two elements $f, g \in C(X)$. From the equalities

$$Z(f) \cup Z(g) = Z(fg)$$

and

$$Z(f) \cap Z(g) = Z(f^2 + g^2) = Z(|f| + |g|)$$

we easily deduce what we wanted. Now we want to show that there is a bijection between the set of ideals of the ring $C(X)$ and the set of filters of the distributive lattice $Z(X)$.

Let I be an ideal of the ring $C(X)$. The set $Z(I) = \{Z(f) \mid f \in I\}$ is an ideal of $Z(X)$ since

- if $f \in I$ and $g \in C(X)$, then $Z(f) \cup Z(g) = Z(fg) \in Z(I)$ since $fg \in I$;
- if $f, g \in I$, then $Z(f) \cap Z(g) = Z(f^2 + g^2) \in Z(I)$ since $f^2 + g^2 \in I$.

On the other hand, let F be a filter of $Z(X)$. The set $I(F) = \{f \in C(X) \mid Z(f) \in F\}$ is an ideal of $C(X)$, in fact

- if $Z(f) \in F$ and $g \in C(X)$, then $fg \in I(F)$ since $Z(fg) \supseteq Z(f) \in F$;
- if $Z(f)$ and $Z(g)$ are in F, then $f + g \in I(F)$ since $Z(f+g) \supseteq Z(f) \cap Z(g) \in F$.

Lemma 2.3.7 *Let X be a topological space.*

1. *For every filter F of $Z(X)$, we have $ZI(F) = F$;*

2. *for every ideal I if $C(X)$, we have $IZ(I) \supseteq I$.*

PROOF.

1. Let F be a filter of $Z(X)$. From

$$ZI(F) = \{Z(f) \mid f \in I(F)\} = \{Z(f) \mid f \text{ such that } Z(f) \in F\}$$

follows that $ZI(F) \subseteq F$. Since every element of $Z(X)$ is of the form $Z(f)$ for some $f \in C(X)$, it is clear that in fact the equality holds.

2. Let I be an ideal of $C(X)$. Form

$$\begin{aligned} IZ(I) &= \{f \in C(X) \mid Z(f) \in Z(I)\} \\ &= \{f \in C(X) \mid \exists g \in I \text{ such that } Z(f) = Z(g)\} \end{aligned}$$

it becomes clear that $IZ(I) \supseteq I$.

■

In (2) the inclusion may be proper, as the following example shows.

Example 2.3.8 Consider the principal ideal $I = (i)$ of $C(\mathbb{R})$, where i denotes the identity function. This consists of all the functions $f \in C(\mathbb{R})$ such that $f(x) = xg(x)$ for some function $g \in C(\mathbb{R})$. Since $Z(i) = \{0\}$ and $Z(I)$ is a filter of $Z(\mathbb{R})$, the filter $Z(I)$ is the set of all subsets of $\mathbb{R}$ containing 0. Hence the ideal $M = IZ(I)$ clearly consists of all the functions in $C(\mathbb{R})$ that vanish at 0. Hence M contains I. However $M \neq I$. For instance $i^{1/3} \in M \setminus I$. That $i^3 \in M$ is obvious. If $i^{1/3} \in I$, then $i^{1/3} = ig$ for some $g \in C(\mathbb{R})$. But then $g(x) = i^{-2/3}$ for $x \neq 0$, so that g can not be continuous at 0.

This example tells us that, for a generic topological space X, there is not a bijection between the set of ideals of $C(X)$ and the set of filters of the distributive lattice $Z(X)$. Anyway from Lemma 2.3.7 we can deduce that there is a bijection between maximal ideals of $C(X)$ and ultrafilters of $Z(X)$. In fact, if M is a maximal ideal of $C(X)$, the ideal $IF(M)$ is proper and contains M, hence it needs to be equal to M. Moreover, if $M, M_1, \ldots, M_n$ are maximal ideals of $C(X)$, we have that

$$M \supseteq \bigcap_{i=1}^{n} M_i \iff Z(M) \supseteq \bigcap_{i=1}^{n} Z(M_i).$$

To show this we need to prove that $Z(\bigcap_{i=1}^n M_i) = \bigcap_{i=1}^n Z(M_i)$. It is clear that the inclusion $\subseteq$ holds. For the other inclusion, let $Y \in \bigcap_{i=1}^n Z(M_i)$; hence $Y = Z(f_i)$, with $f_i \in M_i$, for every $i = 1, \ldots, n$. But then $Y = Z(f_1 \cdot \ldots \cdot f_n)$, and $f_1 \cdot \ldots \cdot f_n \in M_1 \cdot \ldots \cdot M_n \subseteq \bigcap_{i=1}^n M_i$. Hence we have

$$
\begin{aligned}
M \supseteq \bigcap_{i=1}^{n} M_i \quad &\Rightarrow \quad Z(M) \supseteq Z(\bigcap_{i=1}^{n} M_i) = \bigcap_{i=1}^{n} Z(M_i) \\
&\Rightarrow \quad M = IZ(M) \supseteq I(\bigcap_{i=1}^{n} Z(M_i)) = IZ(\bigcap_{i=1}^{n} M_i) \supseteq \bigcap_{i=1}^{n} M_i.
\end{aligned}
$$

From this we deduce that a set of maximal ideals of $C(X)$ is coindependent if and only if the corresponding set of filters of X is coindependent. From this we can conclude the following.

Proposition 2.3.9 *Let X be a topological space. Then the dual Goldie dimension of the ring $C(X)$ is equal to the cardinality of the set of ultrafilters of the lattice $Z(X)$.*

PROOF. To what we said above, it is enough to add that every ultrafilter of $Z(X)$ is prime, since the lattice $Z(X)$ is distributive. Hence the set of all ultrafilters of $Z(X)$ is coindependent. ■

For any topological space X, the set of all ultrafilters of the lattice $Z(X)$ gives rise to a topological space, with the same topology that we introduced above for a general Stone space. This topological space is called the *Stone-Čech compactification* of X, which is denoted by βX. It is a compact Hausdorff topological space endowed with a continuous map from X to βX, having the following universal property: any continuous map $f: X \to K$, where K is a compact Hausdorff space, lifts uniquely to a continuous map $\beta f: \beta X \to K$.

Corollary 2.3.10 *Let X be a topological space. Then the dual Goldie dimension of the ring $C(X)$ is equal to the cardinality of the Stone-Čech compactification βX of X.*

Now we want to generalize what we did above to abelian von Neumann regular rings.

Proposition 2.3.11 *For a ring R, the following conditions are equivalent:*

1. *for every element $x \in R$ there exists $y \in R$ such that $xyx = x$;*
2. *every principal right (left) ideal of R is generated by an idempotent;*
3. *every finitely generated right (left) ideal of R is generated by an idempotent.*

PROOF. (1)⇒(2) Given an element $x \in R$, there exists $y \in R$ such that $xyx = x$. Then xy is an idempotent of R such that $xR = xyR$.

(2)⇒(3) It suffices to show that $xR + yR$ is principal for any $x, y \in R$. Now $xR = eR$ for some idempotent $e \in R$, and since $y - ey \in xR + yR$ we see that $xR + yR = eR + (y - ey)R$. There is an idempotent $f \in R$ such that $fR = (y - ey)R$ and we note that $ef = 0$. Consequently, $g = f - fe$ is an idempotent orthogonal to e. Observing that $fg = g$ and $gf = f$, we see that $gR = fR = (y - ey)R$, whence $xR + yR = eR + gR$. Inasmuch as e and g are orthogonal, we conclude that $xR + yR = (e + g)R$.

(3)⇒(1) Given $x \in R$, there exists an idempotent $e \in R$ such that $eR = xR$. Then $e = xy$ for some $y \in R$ and $x = ex = xyx$. ∎

A ring that satisfies these equivalent conditions is called a *von Neumann regular ring*. Such a ring is said to be *abelian* if all its idempotents are central. It is a general fact that the central idempotents of a ring R form a boolean lattice, where $e \wedge f = ef = fe$ and $e \vee f = e + f - ef$ for any two central idempotents $e, f \in R$. In the case of abelian von Neumann regular rings the lattice of central idempotents is isomorphic to the lattice of finitely generated ideals. In fact we have

$$eR \cap fR = efR \quad \text{and} \quad eR + fR = (e + f - ef)R.$$

Moreover we notice that in abelian von Neumann rings, every one-sided ideal is also two-sided, so we will talk just of ideals and not of right or left ideals.

As we did in our previous cases, we want to find a bijection between the set of ideals of R and the set of ideals of a Boolean lattice. For an abelian von Neumann regular ring R, the Boolean lattice of all the idempotents of R is what we are looking for.

Theorem 2.3.12 *Let R be an abelian von Neumann regular ring and let $B(R)$ be the Boolean lattice formed by all the idempotents of R. Then there is a bijection between the set of ideals of R and the set of ideals of $B(R)$. Hence the dual Goldie dimension of the ring R is equal to the cardinality of the Stone space of the lattice $B(R)$.*

Proof. Let R be an abelian von Neumann regular ring and let I be an ideal of R. Then the set of idempotents $\phi(I) = \{e \in B(R) \mid e \in I\}$ is an ideal of $B(R)$, in fact:

- $\phi(I)$ is not empty since every principal ideal of R is generated by an idempotent;
- if $e \in \phi(I)$ and $f \in B(R)$, then $e \wedge f = ef \in I$, so that $e \wedge f \in \phi(I)$;
- if $e, f \in \phi(I)$, then $e \vee f = e + f - ef \in I$, so that $e \vee f \in \phi(I)$.

Conversely, if J is an ideal of the Boolean lattice $B(R)$, we can associate to it the ideal $\psi(I)$ generated by I.

Now we need to prove that ϕ and ψ are inverse mappings. To show this, let I be an ideal of R. It is obvious that $\psi\phi(I) \subseteq I$. The other inclusion is clear since R is a von Neumann regular ring and hence every principal ideal is generated by an idempotent. Conversely, let J be an ideal of the Boolean lattice $B(R)$. It is obvious that $\phi\psi(J) \supseteq J$. To prove the other implication, let e be an idempotent in the ideal generated by J. Hence e belongs to the ideal generated by a finite number of idempotents $e_1, \dots, e_n$ in J. Since the ideal generated by $e_1, \dots, e_n$

is equal to the ideal generated by $e_1 \vee \ldots \vee e_n$, we have that $e \in (e_1 \vee \ldots \vee e_n)R$. Hence $e = e(e_1 \vee \ldots \vee e_n)$ belongs to the ideal J.

The last sentence clearly follows from Theorem 2.3.4. ∎

Chapter 3

Krull monoids

One of the most natural problem to consider in module theory is the behaviour of the direct sum decomposition of modules. We saw in chapter 1 the Krull-Schmidt-Azumaya Theorem, which describes completely the direct sum decompositions of any module that is a direct sum of modules with local endomorphism ring. If we consider the category of vector spaces over a division ring, it is well known that the linear dimension is an invariant that completely describes the behaviour of the direct sum of vector spaces up to isomorphism. In general the situation is not so clear and it is of great interest to have some more insight.

The proper setting to investigate problems about the behaviour of the direct sum in module categories, or more generally of additive categories, is the one of commutative monoids. In fact, every category $\mathcal{C}$ has a *skeleton* $V(\mathcal{C})$, that is, a full, isomorphism-dense subcategory in which no two distinct objects are isomorphic. It is well known that any two skeletons of $\mathcal{C}$ are isomorphic and they are equivalent to $\mathcal{C}$. If $\mathcal{C}$ is an additive category, or, in a particular case, a subcategory of the category Mod-R of all R-modules closed under isomorphism and direct sum, any skeleton $V(\mathcal{C})$ of $\mathcal{C}$ is endowed with the structure of commutative monoid, with respect to the operation defined by

$$\langle A\rangle + \langle B\rangle = \langle A \oplus B\rangle,$$

where with the angled brackets $\langle A\rangle$ we denote the element of the skeleton associated to A. We remark that, if the category $\mathcal{C}$ is not skeletally small, any skeleton $V(\mathcal{C})$ of $\mathcal{C}$ is not a set, but a proper class. To include this case, we do not require that a monoid is a set, but we allow it to be a proper class. It we need the underlying class to be a set, we will call it a proper monoid.

We denote by $U(M)$ the set of invertible elements of a monoid M. We call the monoid M *reduced* if $U(M) = \{0\}$, that is, if $a + b = 0$ implies $a = b = 0$. In any case, we denote by M_{red} the factor monoid $M/U(M)$ consisting of all cosets $x + U(M)$ with $x \in M$. Note that M_{red} is a reduced monoid. If a and b are two elements of the monoid M, define $a \le b$ if there exists an element $c \in M$ such that $a + c = b$. The relation $\le$ is reflexive, transitive and invariant under translation, i.e. for any $d \in M$, $a \leq b$ implies $a + d \leq b + d$. Thus $\leq$ defines a preorder on M, usually called the *algebraic preorder* of M.

Remark 3.0.1 Let $\mathcal{C}$ be an additive category. Then the monoid $V(\mathcal{C})$ is reduced and $\langle A \rangle \leq \langle B \rangle$ if and only if A is a direct summand of B in $\mathcal{C}$.

An *atom* of a monoid M is an element $a \in M$ such that $a = b + c$ implies $b = 0$ or $c = 0$. For an additive category $V(\mathcal{C})$, the atoms of the monoid $V(\mathcal{C})$ correspond exactly to the indecomposable elements of $\mathcal{C}$. We say that a monoid M is *atomic* if every element $a \in M$ is equal to the sum of finitely many atoms and, similarly, an additive category $\mathcal{C}$ is *atomic* if its monoid $V(\mathcal{C})$ is so.

Let k be a division ring. We denote by Vect$-k$ the category of all right k-vector spaces and by vect$-k$ the full subcategory of all finitely generated right k-vector spaces. It is easy to see that $V(\text{vect}-k) \cong \mathbb{N}_0$ and $V(\text{Vect}-k) \cong \text{Card}$, where Card is the class of cardinal numbers endowed with the operation given by the sum of cardinals. Both isomorphism are provided by the dimension function dim that associates to every vector space its linear dimension.

The Krull-Schmidt-Azumaya Theorem implies the following. Let R be a ring and $\mathcal{C}$ the subcategory of Mod-R of modules which are finite direct sums of modules with local endomorphism ring. Then the monoid $V(\mathcal{C})$ is free on the base given the class of modules with local endomorphism ring. Generally, an additive category $\mathcal{C}$ is said to be a *Krull-Schmidt category* if the commutative monoid $V(\mathcal{C})$ is free, that is, it is atomic and if $A_1, \dots, A_m$ and $B_1, \dots, B_n$ are indecomposable objects in $\mathcal{C}$ such that $\oplus_{i=1}^m A_i \cong \oplus_{j=1}^n B_j$, then $m = n$ and there exists a permutation $\sigma \in \mathcal{S}_n$ such that $A_i \cong B_{\sigma(i)}$ for every $i = 1, \dots, n$.

Now we consider the category proj-R of finitely generated projective modules over the ring R. Our aim is to understand the monoid $V(\text{proj-}R)$. We know that the monoid $V(\mathcal{C})$ is reduced and we notice that the module R_R is an order-unit of the monoid $V(\mathcal{C})$. A non-zero element u of a monoid M is an *order-unit* if for every element $a \in M$ there exists an integer $n \geq 0$ such that $a \leq nu$. We can define the category of monoids with an order-unit in the following way. Its objects are the commutative monoids M with a distinguished element $u \in M$, which is an order-unit. A morphism $f\colon M \to M'$ of commutative monoids with order-units is a morphism of commutative monoids that sends the order-unit $u \in M$ to the order-unit $u' \in M'$. The next Theorem, due to Bergman and Dicks ([7, Theorem 6.2 and Theorem 6.4] and [8, p. 315]), tells us that in fact the monoids of the form $V(\text{proj-}R)$ are exactly all reduced commutative monoids with order unit.

Theorem 3.0.2 *Let k be a field and let M be a reduced commutative proper monoid with order unit u. Then there exists a right and left hereditary k-algebra R such that $V(\text{proj-}R)$ and M are isomorphic as monoid with order unit.*

As a corollary of this result, we show that the theory of factorization in a commutative integral domain can be interpreted as direct sum decomposition in a suitable class of modules. Recall that if R is a commutative integral domain, Q is its field of fractions, Q^* is the multiplicative group of non-zero elements of Q and $U(R)$ is the group of units of R, then the factor group $G = Q^*/U(R)$ is a partially ordered abelian group, called the group of divisibility of R. Its positive cone G_+ is $R^*/U(R)$, where $R^* = R \setminus \{0\}$.

Corollary 3.0.3 *Let R be a commutative integral domain and G_+ the positive cone of the group of divisibility G of R. Then there exists a class $\mathcal{C}$ of finitely generated R-modules over a suitable ring R, closed for finite direct sums, direct summands and isomorphism, such that $V(\mathcal{C}) \cong G_+$.*

Proof. Set $M = G_+ \cup \{+\infty\}$. The addition on G_+ extends to an associative addition on M with $a + (+\infty) = (+\infty) + a = +\infty$ for every $a \in M$. The element $u = +\infty$ is an order-unit in the reduced commutative monoid M. By Theorem 3.0.2, there exists a ring S with $V(\text{proj-}S) \cong M$ as monoids with order-unit. Hence the class of all finitely generated projective right S-modules not isomorphic to S has the required properties. ∎

This corollary makes clear the connection between factorization in commutative integral domains and direct sum decomposition in additive categories and in fact show how the latter is a wider problem than the former.

3.1 Krull monoids

The easiest class of monoids to handle is the one of free monoids. For what concerns factorization problems, the next family to consider is the one of Krull monoids, where we can control divisibility by looking in a free monoid anyway.

Let M be a monoid. A non-zero monoid homomorphism $v: M \to \mathbb{N}_0$ is called a *valuation* of M, and $e(v) = \gcd(v(M))$ is called its *index*. If v is a valuation, then $e(v)^{-1}v(M)$ is a *numerical* monoid, that means, $\mathbb{N}_0 \setminus e(v)^{-1}v(M)$ is a finite set. A valuation $v: M \to \mathbb{N}_0$ is *essential* if for all $x, y \in M$ such that $v(x) \le v(y)$, there exists some $s \in M$ such that $x \le y + s$ and $v(s) = 0$. Obviously, v is essential if and only if $e(v)^{-1}v$ is essential.

A submonoid M' of M is *divisor-closed* if for any $x, y \in M$ such that $x \le y$, $y \in M'$ implies $x \in M'$. For any subset $U \subseteq M$, we denote by $[[U]]$ the smallest divisor-closed submonoid containing U. A *prime ideal* of M is a proper subset $P \subsetneq M$ such that $M \setminus P$ is a divisor-closed submonoid, that is, for any $x, y \in M$ we have $x + y \in P$ if and only if $x \in P$ or $y \in P$. If U is any subset of M, then $M \setminus [[M \setminus U]]$ is the largest prime ideal contained in U. A prime ideal P of a commutative monoid M is said to be a prime ideal *of height one* if it is minimal among non-empty prime ideals of M [15].

If P is a prime ideal of M, then the *localization M_P of M at P* is the monoid whose elements are all formal differences $x - s$, where $x, s \in M$, $s \notin P$ and, for all $x, x' \in M$ and $s, s' \in M \setminus P$, we have $x - s = x - s'$ in M_P if and only if there exists $t \in M \setminus P$ such that $x + s' + t = x' + s + t$ in M. In particular $G(M) = M_\varnothing$ is the *Grothendieck group* of M. The monoid $(M_P)_{\text{red}}$ is called the *reduced localization* of M at P. If $x, x' \in M$ and $s, s' \in M \setminus P$, then $x - s + U(M_P) = x' - s' + U(M_P)$ in $(M_P)_{\text{red}}$ if and only if there exist elements $t, t' \in M \setminus P$ such that $x + t = x' + t'$.

In particular, the homomorphism $M \to (M_P)_{\mathrm{red}}$, defined by $x \mapsto x - 0 + U(M_P)$, is surjective.

A monoid M is called *cancellative* if, for all $x, y, z \in M$, $x + z = y + z$ implies $x = y$. If M is cancellative, then it is contained in its Grothendieck group $G(M)$, usually called in this case the quotient group or the group of differences of M. If R is a ring, then the monoid $V(\text{proj-}R)$ is reduced, and $G(V(\text{proj-}R)) = K_0(R)$ is the Grothendieck group of the isomorphism classes of finitely generated projective R-modules.

A monoid M is called a *discrete valuation monoid* if M_{red} is isomorphic to $\mathbb{N}_0$. If M is a discrete valuation monoid, then there is a unique isomorphism $\theta: M_{\mathrm{red}} \to \mathbb{N}_0$ and the map $v_M: M \to \mathbb{N}_0$, defined by $v_M(x) = \theta(x + U(M))$, is an essential surjective valuation. If $v: M \to \mathbb{N}_0$ is any valuation of M, it is easy to see that $v = e(v)v_M$.

Lemma 3.1.1 *Let M be a monoid, $v: M \to \mathbb{N}_0$ a valuation and $P = \{x \in M \mid v(x) > 0\}$.*

1. *P is a prime ideal of M and v induces a surjective homomorphism*

$$\overline{v}: (M_P)_{\mathrm{red}} \to v(M)$$

defined by $\overline{v}(x - s + U(M_p)) = v(x)$ for all $x \in M$ and $s \in M \setminus P$;

2. *the following are equivalent:*
   - *(a) v is essential;*
   - *(b) $v(M) = e(v)\mathbb{N}_0$ and $\overline{v}$ is an isomorphism;*
   - *(c) M_P is a discrete valuation monoid.*

3. *If v is essential, then P is a prime ideal of height one of M.*

Proof.

1. It is easy to check that P is indeed a prime ideal. If $x, x' \in M$ and $s, s' \in M \setminus P$ are such that $x - s + U(M_P) = x' - s' + U(M_P)$, then there exist $t, t' \in M \setminus P$ such that $x + t = x' + t'$, whence $v(x) = v(x')$. Hence v induces a surjective homomorphism $\overline{v}$ as asserted.

2. (a)⇒(b) Set $e_0 = \min(v(M) \setminus \{0\}) = v(x_0)$ for some $x_0 \in M$. We shall prove that $v(M) = e_0\mathbb{N}_0$. Suppose that $x \in M$ and $v(x) = e_0 k + r$, where $k, r \in \mathbb{N}_0$ with $r < e_0$. Since $v(kx_0) = ke_0 \le v(x)$, there exists some $s \in M$ such that $kx_0 \le x + s$ and $v(s) = 0$. Thus $x + s = kx_0 + y$ for some $y \in M$, from which $v(y) = r$, and therefore $r = 0$.

   To prove that $\overline{v}$ is injective, let $x, x' \in M$ and $s, s' \in M \setminus P$ such that $\overline{v}(x - s + U(M_P)) = \overline{v}(x' - s' + U(M_P))$. Then $v(x) = v(x')$ and thus there exist elements $y \in M$ and $t \in M \setminus P$ such that $x + y = x' + t$. It follows that $v(y) = 0$, hence $y \in M \setminus P$ and $x - s + U(M_P) = x' - s' + U(M_P)$.

(b)⇒(a) Let x and y be elements of M such that $v(x) \leq v(y)$. Then there exists an element $a \in M$ such that $v(x) + v(a) = v(y)$, which means $\overline{v}(x + a + U(M_P)) = \overline{v}(y + U(M_P))$. Therefore there exist elements $s, t \in M \setminus P$ satisfying $x + a + s = y + t$ and consequently $x \leq y + t$ and $v(t) = 0$.

The equivalence of (b) and (c) follows directly from the definition of discrete valuation monoid observing that $v(M) = \overline{v}((M_P)_{\mathrm{red}})$ by (1).

3. By (2), M_P is a discrete valuation monoid. Hence it possesses exactly one non-empty prime ideal and therefore P is a prime ideal of height one of M.

■

A *divisor homomorphism* between two monoids M and N is a monoid homomorphism $\varphi: M \to N$ such that $\varphi(x) \leq \varphi(y)$ implies $x \leq y$ for every $x, y \in M$. This means that we can read the algebraic preorder of M by looking at the algebraic preorder of N using φ. A commutative monoid M is a *Krull monoid* if there exists a divisor homomorphism $\varphi: M \to F$ into a free monoid F. This means that there exists a family of valuations v_i, $i \in I$, given by the non-zero components of the divisor homomorphism, such that, for every $a, b \in M$:

- $v_i(a) = v_i(b)$ for every $i \in I$ if an only if $a + U(M) = b + U(M)$;
- $a \leq b$ if and only if $v_i(a) \leq v_i(b)$ for every $i \in I$;
- $v_i(a) = 0$ for almost all $i \in I$.

With the next Lemma we prove the first properties of Krull monoids.

Lemma 3.1.2 *Let M be a commutative monoid.*

1. *Every divisor homomorphism $\varphi: M \to F$ into a free monoid F induces an isomorphism $M_{\mathrm{red}} \to \varphi(M)$;*
2. *every reduced Krull monoid is cancellative;*
3. *M is a Krull monoid if and only if M_{red} is a Krull monoid.*

Proof.

1. Let $\varphi: M \to F$ be a divisor homomorphism of M into a free monoid F. Since $U(F) = \{0\}$ it is possible to define the induced homomorphism $\overline{\varphi}: M_{\mathrm{red}} \to F$. To prove that it is injective it is enough to notice that for an element $a \in M$, $\varphi(a) = 0_F = \varphi(0_M)$ implies $a \leq 0_M$, that is $a \in U(M)$.
2. If M is a reduced Krull monoid, from (1) we have that M is isomorphic to a submonoid of a free monoid F. Since F is cancellative, also M is so.

3. It is clear that if $\varphi: M \to F$ is a divisor homomorphism into a free monoid F, also the induced homomorphism $\overline{\varphi}: M_{\text{red}} \to F$ is a divisor homomorphism. Conversely, suppose that $\psi: M_{\text{red}} \to F$ is a divisor homomorphism of M_{red} into a free monoid F. We can define a monoid homomorphism $\varphi: M \to F$ by $\varphi(a) = \psi(a + U(M))$ for every $a \in M$, that turns out to be a divisor homomorphism.

■

Since we know that $V(\mathcal{C})$ is reduced for any additive category $\mathcal{C}$, Lemma 3.1.2(2) tells us that a necessary condition for $V(\mathcal{C})$ to be a Krull monoid is that it must be cancellative.

Proposition 3.1.3 *A monoid M is a reduced Krull monoid if and only if it is isomorphic to $\mathbb{N}^{(I)} \cap G$, where $\mathbb{N}^{(I)}$ is the free monoid on the basis I and G is a subgroup of $\mathbb{Z}^{(I)}$.*

Proof. Since M is a reduced Krull monoid, there is an injective divisor homomorphism $\varphi: M \to \mathbb{N}_0^{(I)}$, for some class I. Hence we can suppose that $M \subseteq \mathbb{N}_0^{(I)}$. The monoid M is cancellative and thus we can embed it in its Grothendieck group $G(M) \subseteq \mathbb{Z}^{(I)}$. Therefore $M \subseteq G(M) \cap \mathbb{N}_0^{(I)}$. To prove the opposite inclusion, let $x - s \in G(M) \cap \mathbb{N}_0^{(I)}$. This means that $\varphi(x) \geq \varphi(s)$ and hence, since φ is a divisor homomorphism, we have $x \geq s$. This implies that there exists an element $y \in M$ such that $x = y + s$, thus $y = x - s$ in $G(M)$. ■

Proposition 3.1.3 tells us that a reduced Krull monoid has a very nice geometrical behaviour. In fact it is the intersection of a lattice $G \subseteq \mathbb{Z}^{(I)}$ with the positive cone $\mathbb{N}^{(I)}$, so that the failure of the uniqueness of the factorization is minimal, due only to the presence of the border of $\mathbb{N}^{(I)} \cap G$.

Now we want to provide some examples of Krull monoids that are relevant in the study of factorizations.

Example 3.1.4 The example that gave rise to the study of Krull monoids is the following. An integral domain R is a Krull domain if and only if its multiplicative monoid $R^\bullet = R \setminus \{0\}$ is a Krull monoid.

Example 3.1.5 A nonzero element a of a ring R is said to be a *regular element* if it is neither a left nor a right zero divisor. An ideal a of R is called *regular* if it contains a regular element of R. A Marot ring is a non-zero commutative ring such that every regular ideal can be generated by regular elements. Then a Marot ring R is a Krull ring, in the sense of [36], if and only if the multiplicative monoid of regular elements of R is a Krull monoid.

Example 3.1.6 Let R be a Krull domain, $I \subseteq R$ a non-zero ideal and $G \subseteq (R/I)^\times$ a subgroup. Then the monoid

$$H_G = \{a \in R^\bullet \mid a + I \in G\}$$

is a Krull monoid, called the (regular) congruence monoid defined in R modulo I by G.

Example 3.1.7 Let G be an abelian group and let $G_0 \subseteq G$ be a non-empty subset. Denote by $F(G_0)$ the free monoid on the basis G_0. Then

$$\mathcal{B}(G_0) = \{ \prod_{g \in G_0} g^{n_g} \in F(G_0) \mid \sum_{g \in G} n_g g = 0 \} \subseteq F(G_0)$$

is called the *block monoid over* G_0. Clearly, the embedding $i: \mathcal{B}(G_0) \to F(G_0)$ is a divisor homomorphism and hence $\mathcal{B}(G_0)$ is a Krull monoid.

We now present an example that is not of purely algebraic nature and comes from analytic number theory. We say that a submonoid $H \subseteq D$ is *saturated* if $a, b \in H$, $c \in D$ and $a = b + c$ imply $c \in H$.

Example 3.1.8 A *quasi-formation* $[D, H, |\cdot|]$ consists of:

- a free monoid $D = F(P)$ on the basis P,
- a homomorphism $|\cdot|: D \to (\mathbb{N}, \cdot)$ such that $|a| = 1$ if and only if $a = 0$, and the Dirichlet series

$$\sum_{p \in P} |p|^{-s} \text{ converges for } \mathfrak{Re}(s) > 1,$$

- a saturated submonoid $H \subseteq D$ such that $G = D/H$ is finite, and for every $g \in G$ the function ψ_g, defined by

$$\psi_g(s) = \sum_{p \in P \cap g} |p|^{-s} - \frac{1}{|G|} \log \frac{1}{s-1} \qquad \text{for} \qquad \mathfrak{Re}(s) > 1,$$

  has a holomorphic extension to $s = 1$.

If $[D, H, |\cdot|]$ is a quasi-formation, then H is a Krull monoid.

For a concrete example of a quasi-formation, let R be the ring of integers of an algebraic number field or a holomorphy ring in an algebraic function field over a finite field and $H = \mathcal{H}(\mathcal{R})$ the multiplicative monoid of non-zero principal ideals of R. Let D be the multiplicative monoid of all non-zero ideals of R, and for $I \in D$, let $|I| = (D : I)$. Then $[D, H, |\cdot|]$ is a quasi-formation.

3.2 Divisor theories

Throughout this section we suppose that M is a cancellative monoid.

A divisor homomorphism $\varphi: M \to F$ into a free monoid F is a *divisor theory* if for every $u \in F$ there exist a finite family $\{x_1, \dots, x_n\}$ of elements of M such that $u = \min\{\varphi(x_1), \dots, \varphi(x_n)\}$, where the minimum is taken with respect to the algebraic preorder of F. Every cancellative Krull monoid possesses a divisor theory and if $\varphi: M \to F$ and $\varphi': M \to F'$ are two divisor theories, then there exists a unique isomorphism $\Phi: F \to F'$ such that $\Phi \circ \varphi = \varphi'$ [35, Theorems 23.4 and 20.4].

Let $\varphi: M \to F \cong \mathbb{N}_0^{(I)}$ be a divisor theory. Then φ has a unique extension to a group homomorphism $G(\varphi): G(M) \to G(F) \cong \mathbb{Z}^{(I)}$. The cokernel of $G(\varphi)$ is determined up to canonical isomorphism by φ and it is called the *class group* of M, denoted by $\mathrm{Cl}(M)$. If $\pi: G(F) \to \mathrm{Cl}(M)$ denotes the canonical projection, we have that $\mathrm{Cl}(M) = \pi(F)$. It follows that M is free if and only if the divisor theory φ is surjective, and $\mathrm{Cl}(M)$ is a torsion group if and only if for every $u \in F$ there exists some $q \geq 1$ such that $qu \in \varphi(M)$.

Given a divisor homomorphism $\varphi = (\varphi_i)_{i \in I}: M \to \mathbb{N}_0^{(I)}$ into a free monoid, there is a way to obtain from it a divisor theory. First, we observe that $\varphi' = (\lambda_i \varphi_i)_{i \in I'}: M \to \mathbb{N}_0^{(I')}$, where $i \in I \setminus I'$ implies $\varphi_i = 0$ and $\lambda_i \in e(\varphi_i)^{-1}\mathbb{N}_0 \setminus \{0\}$, is again a divisor homomorphism into a free monoid $\mathbb{N}_0^{(I')}$.

Proposition 3.2.1 *Let M be a Krull monoid, and let $\varphi = (\varphi_i)_{i \in I}: M \to \mathbb{N}_0^{(I)}$ be a divisor homomorphism such that $\varphi_i \neq 0$ for every $i \in I$ and $e(\varphi_i)^{-1}\varphi_i \neq e(\varphi_j)^{-1}\varphi_j$ whenever $i \neq j \in I$. Let J be the set of all the indices of $j \in I$ for which φ_j is essential.*

1. *The map $\varphi^* = (e(\varphi_j)^{-1}\varphi_j)_{j \in J}: M \to \mathbb{N}_0^{(J)}$ is a divisor theory, and φ is a divisor theory if and only if $I = J$ and $e(\varphi_i) = 1$ for every $i \in I$.*

2. *If $v: M \to \mathbb{N}_0$ is an essential valuation of M, then there exists some $j \in J$ such that $e(v)^{-1}v = e(\varphi_j)^{-1}\varphi_j$.*

Proof. See [34, Satz 1, Satz 2 and Korollar]. ∎

We say that two valuations $v_1, v_2: M \to \mathbb{N}_0$ of a monoid M are *equivalent* if $e(v_1)^{-1}v_1 = e(v_2)^{-1}v_2$. With the next proposition we prove that in Krull monoids there is a strong connection between essential valuations and prime ideals of height one.

Proposition 3.2.2 *Let M be a cancellative Krull monoid. Then:*

1. *every non-empty prime ideal of M contains a prime ideal of height one of the form $P_v = \{x \in M \mid v(x) > 0\}$ for some essential valuation v;*

2. *two essential valuations $v_1, v_2: M \to \mathbb{N}_0$ are equivalent if and only if $P_{v_1} = P_{v_2}$;*

3. *a valuation $v: M \to \mathbb{N}_0$ is essential if and only if $P_v = \{x \in M \mid v(x) > 0\}$ is a prime ideal of height one.*

Proof.

1. Let $\varphi = (\varphi_j)_{j \in J}: M \to \mathbb{N}_0^{(J)}$ be a divisor theory for M, such that every essential valuation of M is equivalent to some φ_j, $j \in J$. If P is a prime ideal of M and $P \not\supseteq P_{\varphi_j}$ for every $j \in J$, then for every $j \in J$ there exists $x_j \notin P$ such that $\varphi_j(x_j) > 0$. Let p be a fixed element of P, so that $\varphi_j(p) > 0$ if and only if j belongs to a finite subset F of J. Then
$$p \leq \sum_{j \in F} \varphi_j(p) x_j \notin P,$$

so that $p \notin P$. This contradiction shows that $P = \varnothing$. Thus every non-empty prime ideal of M contains a prime ideal of the form P_{φ_j}. In particular, every prime ideal of height one of M is of the type P_{φ_j}.

We already know from Lemma 3.1.1 that for any monoid, P_v is a prime ideal of height one for every essential valuation v.

2. It is clear that two equivalent valuations $v_1, v_2 : M \to \mathbb{N}_0$ give rise to the same ideal $P_{v_1} = P_{v_2}$. If v_1 and v_2 are two non-equivalent essential valuations, we can suppose that $v_1 = \varphi_i$ and $v_2 = \varphi_j$ for $i \neq j \in J$, where $\varphi = (\varphi_j)_{j\in J} : M \to \mathbb{N}_0^{(J)}$ is a divisor theory of M. Then there exists $x \in M$ with $\varphi_i(x) \neq 0$ and $\varphi_j(x) = 0$ because φ is a divisor theory. Hence $P_{\varphi_i} \neq P_{\varphi_j}$.

3. It is true in any monoid that the ideal P_v associated to an essential valuation v is minimal non-empty, as we saw in Lemma 3.1.1. If v is a valuation for which the ideal P_v is a prime ideal of height one, we know by (1) and (2) that v must be equivalent to an essential valuation, and hence it is essential itself.

■

When a Krull monoid M has a divisor theory $\varphi : M \to \mathbb{N}_0^m$, for some $m \geq 1$, we can describe prime ideals associated to non-essential valuations as the union of prime ideals of height one. We notice that this is the case if and only if M_{red} is finitely generated and non-zero.

Proposition 3.2.3 *Let M be a Krull monoid, $m \geq 1$, $\varphi = (\varphi_1, \ldots, \varphi_m) : M \to \mathbb{N}_0^m$ a divisor theory and $P_j = \{x \in M \mid \varphi_j(x) > 0\}$. Let $v : M \to \mathbb{N}_0$ be any valuation of M and $P_v = \{x \in M \mid v(x) > 0\}$ its associated ideal. Then there exist non-negative real numbers $c_1, \ldots, c_m$ such that $v = c_1\varphi_1 + \ldots + c_m\varphi_m$. If $v = c_1\varphi_1 + \ldots + c_m\varphi_m$ is any such representation of v, then $P_v = \bigcup_{c_j>0} P_j$.*

Proof. Since φ induces an isomorphism $M_{\mathrm{red}} \to \varphi(M)$ and the canonical homomorphism $M \to M_{\mathrm{red}}$, defined by $x \mapsto x + U(M)$, induces a bijection between the prime ideals of M and M_{red} and also between the valuations of M and M_{red}, we may assume that M is a submonoid of $\mathbb{N}_0^m$ and that the divisor theory $\varphi : M \to \mathbb{N}_0^m$ is the inclusion map.

Let $U \subseteq \mathbb{R}^m$ be the subspace generated by M, $C = \{\sum_{x\in M} \lambda_x x \mid \lambda_x \in \mathbb{R}_{\geq 0}, \lambda_x = 0$ for almost all $x\}$ the cone in U generated by M and $Q = G(M)$ the subgroup of $\mathbb{Z}^m$ generated by M. Let $\pi_j : \mathbb{R}^m \to \mathbb{R}$ be the projection onto the j-th coordinate and $\phi_j = \pi_{j|U} : U \to \mathbb{R}$ its restriction to U. Since $\varphi = (\phi_{1|M}, \ldots, \phi_{m|M}) : M \to \mathbb{N}_0^m$ is a divisor theory, it follows that $C = \bigcup_{j=1}^m \phi^{-1}\mathbb{R}_{\geq 0}$, and by the H. Weyl's Theorem [47, Theorem 17.3] the dual cone $\check{C}$ is the cone generated by $\varphi_1, \ldots, \varphi_m \in \mathbb{R}_{\geq 0}$ in the dual space $U^* = \mathrm{Hom}(U, \mathbb{R})$. If $v : M \to \mathbb{N}_0$ is any valuation, then there exists some $\varphi \in U^*$ such that $\varphi_{|M} = v$ and therefore $\varphi \in \check{C}$. Hence there exist $c_1, \ldots, c_m \in \mathbb{R}_{\geq 0}$ such that $\varphi = c_1\phi_1 + \ldots + c_m\phi_m$ and restriction to M implies $v = c_1\varphi_1 + \ldots + c_m\varphi_m$. The representation of P as a union of P_j's is now obvious.
■

With the next Theorem we give an equivalent characterization of Krull monoids as intersection of the localization at the prime ideals of height one.

Theorem 3.2.4 *The following conditions are equivalent for any cancellative monoid M:*

1. *M is a Krull monoid;*
2. *The localization M_P is a discrete valuation monoid for every prime ideal of height one P, every $x \in M$ is contained in at most finitely many prime ideals of height one, and $M = \cap_P M_P$, where P varies among prime ideals of height one.*

Proof. (1)⇒(2) Let M be a Krull monoid and let $\varphi = (\varphi_i)_{i \in I} : M \to \mathbb{N}_0^{(I)}$ be a divisor theory. If P is a prime ideal of height one, we know from Proposition 3.2.2 that $P = P_{\varphi_i}$ for some $i \in I$. Then $M_P = M_{P_{\varphi_i}}$ is a discrete valuation monoid by Lemma 3.1.1, and v induces an isomorphism $\overline{v}: (M_P)_{\text{red}} \to e(v)\mathbb{N}_0$. As M is cancellative, $M \subseteq M_P \subseteq M_\varnothing$ for every prime ideal of height one, so that $M \subseteq \cap_P M_P$, where P varies among prime ideals of height one.

For the opposite inclusion, suppose $x - y \in \cap_P M_P \subseteq M_\varnothing$ with $x, y \in M$. Then $\varphi_i(x-y) \geq 0$, so that $\varphi_i(x) \geq \varphi_i(y)$, for every $i \in I$. Thus $\varphi(x) \geq \varphi(y)$ and hence $x \geq y$. Therefore $x - y$ belong to the cancellative monoid M. Finally, for every $x \in M$ we have that $\varphi(x) \in \mathbb{N}_0^{(I)}$, so that $\varphi_i(x) \neq 0$ for at most finitely many $i \in I$. Thus x is contained in finitely many prime ideals of height one.

(2)⇒(1) Let M be a cancellative monoid satisfying the conditions stated in (2). Then the canonical homomorphisms $M \to (M_P)_{\text{red}}$, P ranging among prime ideals of height one, have the property that every $x \in M$ is mapped to zero for almost all minimal non-empty primes P, and each $(M_P)_{\text{red}}$ is isomorphic to $\mathbb{N}_0$. Thus these canonical homomorphisms define a monoid homomorphism $\varphi: M \to \mathbb{N}_0^{(I)}$, where I is a class of indexes for the prime ideals of height one of M. In order to show that φ is a divisor homomorphism, let $x, y \in M$ such that $\varphi(x) \leq \varphi(y)$. Then, for every prime ideal of height one P, there exists $s_P \in (M_P)_{\text{red}}$ such that $x + s_P = y$ in $(M_P)_{\text{red}}$. Thus for every prime ideal of height one P, there exist $u_P \in M$ and $t_P \in M \setminus P$ with $y + t_P = x + u_P$. Then $y - x = u_P - t_P \in M_P$ for every prime ideal of height one P, from which $y - x \in M$, that is, $x \leq y$ in M. ∎

3.3 Additive categories and Krull monoids

In this section we want to apply the theory about Krull monoids we developed in this chapter to the monoids of the form $V(\mathcal{C})$, where $\mathcal{C}$ is an additive category. We noticed already that such monoids are reduced and, if they are Krull monoids, they must also be cancellative. To avoid set theoretical complications, in this section we restrict to the case $\mathcal{C}$ skeletally small.

We say that *idempotents split* in a category $\mathcal{C}$, or that $\mathcal{C}$ has *splitting idempotents*, if every idempotent morphism in $\mathcal{C}$ has a kernel. If $\{\mathcal{C}_\lambda \mid \lambda \in \Lambda\}$ is a family of additive categories indexed in a set Λ, let $\prod_{\lambda\in\Lambda}\mathcal{C}_\lambda$ be the product category, whose objects are the sequences $S=(A_\lambda)_{\lambda\in\Lambda}$ with $A_\lambda\in\mathcal{C}_\lambda$, and whose morphisms are given by $\mathrm{Hom}_{\prod_{\lambda\in\Lambda}\mathcal{C}_\lambda}((A_\lambda)_{\lambda\in\Lambda},(A'_\lambda)_{\lambda\in\Lambda})=\prod_{\lambda\in\Lambda}\mathrm{Hom}_{\mathcal{C}_\lambda}(A_\lambda,A'_\lambda)$. Let $\coprod_{\lambda\in\Lambda}\mathcal{C}_\lambda$ be the full subcategory of $\prod_{\lambda\in\Lambda}\mathcal{C}_\lambda$ whose objects are the sequences $S=(A_\lambda)_{\lambda\in\Lambda}$ with almost all $A_\lambda=0$.

Since we want the monoid $V(\mathcal{C})$ to be cancellative, it is natural to look at the case when every endomorphism ring $\mathrm{End}_{\mathcal{C}}(A)$ is semilocal, for any object $A\in\mathcal{C}$. It is easier to start with the case when every endomorphism ring is semisimple artinian.

Proposition 3.3.1 *The following conditions are equivalent for any skeletally small additive category $\mathcal{C}$.*

1. *Idempotents splits in $\mathcal{C}$ and the endomorphism rings $\mathrm{End}_{\mathcal{C}}(A)$ of all objects A of $\mathcal{C}$ are semisimple artinian;*

2. *there exists a set $\{k_\lambda \mid \lambda\in\Lambda\}$ of division ring such that $\mathcal{A}$ is equivalent to $\coprod_{\lambda\in\Lambda}\mathrm{vect}\text{-}k_\lambda$.*

Proof. (1)⇒(2) Let $\mathcal{C}$ be a skeletally small additive category with splitting idempotents in which every endomorphism ring is semisimple artinian. As $\mathcal{C}$ is skeletally small, there exists a set $\{A_\lambda \mid \lambda\in\Lambda\}$ of representatives of the indecomposable objects of $\mathcal{C}$ up to isomorphism. Since idempotents split in $\mathcal{C}$, the endomorphism ring of every A_λ is a division ring k_λ. Every object A of $\mathcal{C}$ decomposes as a direct sum of finitely many objects whose endomorphism rings are division rings, hence they are necessary indecomposable objects. Assume that A_λ and $A_{\lambda'}$ are indecomposable objects and that $\mathrm{Hom}_{\mathcal{C}}(A_\lambda,A_{\lambda'})\neq 0$. We have already remarked that the endomorphism rings of A_λ and $A_{\lambda'}$ are division rings. In the additive category $\mathcal{C}$, the endomorphism ring of $A_\lambda\oplus A_{\lambda'}$ is the matrix ring

$$E=\begin{pmatrix}\mathrm{Hom}_{\mathcal{C}}(A_\lambda,A_\lambda) & \mathrm{Hom}_{\mathcal{C}}(A_{\lambda'},A_\lambda)\\ \mathrm{Hom}_{\mathcal{C}}(A_\lambda,A_{\lambda'}) & \mathrm{Hom}_{\mathcal{C}}(A_{\lambda'},A_{\lambda'})\end{pmatrix}$$

which is a semisimple artinian ring. Let $f\colon A_\lambda\to A_{\lambda'}$ be a non-zero morphism. Then the element $\begin{pmatrix}0&0\\ f&0\end{pmatrix}\in E$ is a non-zero element of E that induces by left multiplication a non-zero morphism of right E-modules from the indecomposable right ideal

$$\begin{pmatrix}1_{A_\lambda}&0\\0&0\end{pmatrix}E$$

into the indecomposable right ideal

$$\begin{pmatrix}0&0\\0&1_{A_{\lambda'}}\end{pmatrix}E.$$

As indecomposable right ideals are simple E-modules because E is semisimple artinian, the non-zero morphism induced by left multiplication is an isomorphism, and thus it has an inverse isomorphism. The inverse isomorphism is given by left multiplication by an element $\alpha \in E$, which is necessarily of the form $\alpha = \begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}$. So $g: A_{\lambda'} \to A_\lambda$ is a morphism in $\mathcal{C}$ such that $gf = 1_{A_\lambda}$ and $fg = 1_{A_{\lambda'}}$. We have thus proved that if A_λ and $A_{\lambda'}$ are two objects whose endomorphism rings are division rings and there is a non-zero morphism $A_\lambda \to A_{\lambda'}$, then $A_\lambda \cong A_{\lambda'}$.

As every object A of $\mathcal{C}$ decomposes as a direct sum of finitely many objects with local endomorphism rings, by the Krull-Schmidt-Azumaya Theorem there are only finitely many λ's such that $\mathrm{Hom}_{\mathcal{C}}(A_\lambda, A) \neq 0$. Thus $F = \prod_{\lambda \in \Lambda} \mathrm{Hom}_{\mathcal{C}}(A_\lambda, -): \mathcal{C} \to \coprod_{\lambda \in \Lambda} \mathrm{vect}\text{-}k_\lambda$ is an equivalence.

(2)⇒(1) is obvious. ∎

The skeletally small additive categories satisfying the equivalent conditions of Proposition 3.3.1 are called *amenable semisimple*. They are necessarily abelian.

For every additive category $\mathcal{C}$, there exists a functor $F: \mathcal{C} \to \hat{\mathcal{C}}$ into an additive category $\hat{\mathcal{C}}$ in which idempotents split, uniquely determined up to categorical equivalence, with the following universal property: for every functor $G: \mathcal{C} \to \mathcal{C}'$ of $\mathcal{C}$ into an additive category $\mathcal{C}'$ in which idempotents split, there exists a unique functor $H: \hat{\mathcal{C}} \to \mathcal{C}'$ such that $G = HF$. The category $\hat{\mathcal{C}}$ is called an *idempotent completion* of $\mathcal{C}$. To prove the existence of the idempotent completion of $\mathcal{C}$, take as objects of $\hat{\mathcal{C}}$ the pairs (A, e), where A is an object of $\mathcal{C}$ and e in an idempotent of $\mathrm{End}_{\mathcal{C}}(A)$, and as morphisms $(A, e) \to (B, f)$ the morphisms $\varphi: A \to B$ in $\mathcal{C}$ such that $f\varphi e = \varphi$. Define the functor $F: \mathcal{C} \to \hat{\mathcal{C}}$ by $F(A) = (A, 1_A)$ for every object A of $\mathcal{C}$.

Corollary 3.3.2 *The following conditions are equivalent for a skeletally small additive category $\mathcal{C}$:*

1. *The endomorphism ring* $\mathrm{End}_{\mathcal{C}}(A)$ *of every object* $A \in \mathcal{C}$ *is a semisimple artinian ring;*

2. *there exist a set* $\{k_\lambda \mid \lambda \in \Lambda\}$ *of division rings and a full and faithful functor* $H: \mathcal{C} \to \coprod_{\lambda \in \Lambda} \mathrm{vect}\text{-}k_\lambda$.

Proof. (1)⇒(2) Assume that (1) holds for the category $\mathcal{C}$. Then (1) also holds for the idempotent completion $\mathcal{C}$. Apply Proposition 3.3.1 to the skeletally small additive category $\hat{\mathcal{C}}$, so that there exist a set $\{k_\lambda \mid \lambda \in \Lambda\}$ of division rings and an equivalence $G: \hat{\mathcal{C}} \to \coprod_{\lambda \in \Lambda} \mathrm{vect}\text{-}k_\lambda$. The composite functor $H = GF$ of G and the canonical functor $F: \mathcal{C} \to \hat{\mathcal{C}}$ is full and faithful.

(2)⇒(1) is obvious. ∎

The *Jacobson radical* of a preadditive category $\mathcal{C}$ is the ideal $\mathcal{J}$ of $\mathcal{C}$ defined, for every pair A, B of objects of $\mathcal{C}$, by $\mathcal{J}(A, B) = \{ f \in \mathrm{Hom}_{\mathcal{C}}(A, B) \mid 1_A - gf$ has a left inverse for all $g \in \mathrm{Hom}_{\mathcal{C}}(B, A) \}$. The quotient category $\mathcal{C}/\mathcal{J}$ has zero Jacobson radical and there is a canonical functor $G: \mathcal{A} \to \mathcal{A}/\mathcal{J}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories and $F: \mathcal{A} \to \mathcal{B}$ an additive functor. We say that F is:

- *isomorphism reflecting* if for every pair A, A' of objects of $\mathcal{A}$, $F(A) \cong F(A')$ implies $A \cong A'$;
- *direct-summand reflecting* if for every pair A, A' of objects of $\mathcal{A}$ with $F(A)$ isomorphic to a direct summand of $F(A')$, A is isomorphic to a direct summand of A';
- *local* if, for every pair A, A' of objects of $\mathcal{A}$ and every morphism $f: A \to A'$ such that $F(f): F(A) \to F(A')$ is an isomorphism, f is an isomorphism.

Lemma 3.3.3 *Let $\mathcal{C}$ be an additive category and $\mathcal{J}$ its Jacobson radical. Then:*

- *the canonical functor $G: \mathcal{C} \to \mathcal{C}/\mathcal{J}$ is a full, isomorphism reflecting, local functor;*
- *if idempotents split in $\mathcal{C}$, then G is also direct-summand reflecting.*

Proof. It is clear that G is a full functor. If we prove that G is local, then it is automatically also isomorphism reflecting. Hence let A, A' be two objects of $\mathcal{A}$ and $f: A \to A'$ a morphism of $\mathcal{C}$ such that $F(f): F(A) \to F(A')$ is an isomorphism. This means that there exists a morphism $g: A' \to A$ in $\mathcal{C}$ such that $1_A - gf \in \mathcal{J}(A, A)$ and $1_{A'} - fg \in \mathcal{J}(A', A')$. Using the definition of the Jacobson radical it is straightforward to prove that in fact g is an inverse of f in $\mathcal{C}$.

Assume now that $\mathcal{C}$ is an additive category in which idempotents split. In order to show that G is direct-summand reflecting, take two objects A, A' of $\mathcal{C}$ with $G(A)$ isomorphic to a direct summand of $G(A')$. Then there exist morphisms $f: A \to A'$ and $g: A' \to A$ such that $1_A - gf \in \mathcal{J}(A, A)$. Thus $gf: A \to A$ has a two-sided inverse, so that A is isomorphic to a direct summand of A'. ■

Proposition 3.3.4 *Let $\mathcal{C}$ be an additive category with Jacobson radical $\mathcal{J}$. Let $G: \mathcal{C} \to \widehat{\mathcal{C}/\mathcal{J}}$ be the canonical functor of $\mathcal{C}$ into the idempotent completion $\widehat{\mathcal{C}/\mathcal{J}}$ of the factor category $\mathcal{C}/\mathcal{J}$. Then G is a full, isomorphism reflecting, local functor. If, moreover, idempotents split in $\mathcal{C}$, then G is also direct-summand reflecting.*

Proof. The objects of $\widehat{\mathcal{C}/\mathcal{J}}$ are the pairs $(A, \overline{\varphi})$, where A is an object of $\mathcal{C}$ and $\varphi: A \to A$ is an endomorphism of A in $\mathcal{C}$ such that $\overline{\varphi} = \varphi + \mathcal{J}(A, A)$ is an idempotent of $\mathrm{End}_{\mathcal{C}}(A)/J(\mathrm{End}_{\mathcal{C}}(A))$. The morphisms $(A, \overline{\varphi}) \to (B, \overline{\psi})$ in $\widehat{\mathcal{C}/\mathcal{J}}$ are the cosets $\overline{f} = f + \mathcal{J}(A, B)$, where $f: A \to B$ is a morphism in $\mathcal{C}$ such that $\psi f \varphi - f \in \mathcal{J}(A, B)$.

The canonical functor $G: \mathcal{C} \to \widehat{\mathcal{C}/\mathcal{J}}$ is the composite functor of:

- the functor $\mathcal{C} \to \mathcal{C}/\mathcal{J}$, which is full, isomorphism reflecting, local and, when idempotents split in $\mathcal{C}$, also direct-summand reflecting;
- the functor $\mathcal{C}/\mathcal{J} \to \widehat{\mathcal{C}/\mathcal{J}}$, which is full, faithful, isomorphism reflecting and local.

Then G is full, isomorphism reflecting and local.

Now assume that idempotents split in $\mathcal{C}$. Let A, A' be a pair of objects of $\mathcal{C}$ with $G(A)$ isomorphic to a direct summand of $G(A')$. Let $(K, \overline{\omega})$ be an object in $\widehat{\mathcal{C}/\mathcal{J}}$ such that $G(A) \oplus (K, \overline{\omega}) \cong G(A')$. Then there are morphisms $\overline{f} = f + \mathcal{J}(A, A') : (A, \overline{1_A}) \to (A', \overline{1_{A'}})$ and $\overline{g} = g + \mathcal{J}(A', A) : (A', \overline{1_{A'}}) \to (A, \overline{1_A})$ with $\overline{gf} = \overline{1_A}$ and $\ker(\overline{g}) = \ker(\overline{fg}) = (K, \overline{\omega})$. Then $1_a - gf \in \mathcal{J}(A, A)$, so that gf is invertible in the ring $\mathrm{End}_{\mathcal{C}}(A)$. Thus $f(gf)^{-1}g : A \to A'$ is idempotent. Write $1_{A'} - f(gf)^{-1}g = kl$ for some $l : A' \to B$ and some $k : B \to A'$ with $lk = 1_B$, so that k is the kernel of $f(gf)^{-1}g$. Applying the functor G we get that, for the idempotent $\overline{f(gf)^{-1}g} = \overline{fg}$, one has $\overline{1_{A'}} - \overline{fg} = \overline{kl}$ with $\overline{kl} = \overline{1_B}$, so that $\overline{k} : G(B) \to G(A')$ is the kernel of $\overline{fg}$. As kernels are unique up to isomorphism, we conclude that $G(B) \cong (K, \overline{\omega})$. In particular, this proves that G is direct-summand reflecting, because k kernel of the idempotent $f(gf)^{-1}g$ implies $A \oplus B \cong A'$. ∎

When the endomorphism rings $\mathrm{End}_{\mathcal{C}}(A)$ are all semilocal, the functor $G : \mathcal{C} \to \widehat{\mathcal{C}/\mathcal{J}}$ maps to the particularly good category $\widehat{\mathcal{C}/\mathcal{J}}$, as the next result shows.

Proposition 3.3.5 *Let $\mathcal{C}$ be a skeletally small additive category with Jacobson radical $\mathcal{J}$ and with the property that $\mathrm{End}_{\mathcal{C}}(A)$ is a semilocal ring for every object A of $\mathcal{C}$. Then the idempotent completion $\widehat{\mathcal{C}/\mathcal{J}}$ of the factor category $\mathcal{C}/\mathcal{J}$ is an amenable semisimple category.*

Proof. As the endomorphism ring of every object in $\mathcal{C}$ is semilocal, the endomorphism ring of every object in $\mathcal{C}/\mathcal{J}$ is semisimple artinian, so that $\widehat{\mathcal{C}/\mathcal{J}}$ is an amenable semisimple category by Proposition 3.3.1. ∎

Eventually, we prove that, if $\mathcal{C}$ is a skeletally small additive category with splitting idempotents has the property that $\mathrm{End}_{\mathcal{C}}(A)$ is a semilocal ring for every object $A \in \mathcal{C}$, then $V(\mathcal{C})$ is a Krull monoid.

Theorem 3.3.6 *Let $\mathcal{C}$ be a skeletally small additive category. Let F be an additive functor of $\mathcal{C}$ into an amenable semisimple category $\mathcal{D}$. If either*

- *F is direct-summand reflecting, or*
- *idempotents split in $\mathcal{C}$, and F is local,*

then $V(\mathcal{C})$ is a Krull monoid.

Proof. The functor $F : \mathcal{C} \to \mathcal{D}$ induces a monoid homomorphism

$$V(F) : V(\mathcal{C}) \to V(\mathcal{D}).$$

The monoid $V(\mathcal{D})$ is free because $\mathcal{D}$ is amenable semisimple.

The functor F is direct-summand reflecting if and only if $V(F)$ is a divisor homomorphism. If this is the case then $V(\mathcal{C})$ is a Krull monoid.

If idempotents split in $\mathcal{C}$ and F is local, then F induces a local ring homomorphism $\mathrm{End}_{\mathcal{C}}(A) \to \mathrm{End}_{\mathcal{D}}(F(A))$ for every object A of $\mathcal{C}$. Thus the endomorphism rings of all objects of $\mathcal{C}$ are semilocal rings. By Proposition 3.3.5, there is a direct-summand reflecting functor G of $\mathcal{C}$ into the amenable semisimple category $\widetilde{\mathcal{C}/\mathcal{J}}$. Thus $V(\mathcal{C})$ is a Krull monoid by the first case. ∎

To conclude the chapter we want to remark that, when one consider module categories, it is possible to invert the implication that states that a skeletally small full subcategory $\mathcal{C}$ of Mod-R closed under finite direct sums, direct summands and isomorphisms and such that $\mathrm{End}_R(A)$ is semilocal for every $A \in \mathcal{C}$, provides a reduced Krull monoid $V(\mathcal{C})$.

First, we notice that we can restrict to categories of finitely generated projective modules with semilocal endomorphism ring. Let $\mathcal{C}$ be a skeletally small full subcategory of Mod-R closed under finite direct sums, direct summands and isomorphisms and such that $\mathrm{End}_R(A)$ is semilocal for every $A \in \mathcal{C}$; let M_R be the direct sum of the modules in $V(\mathcal{C})$ and $E = \mathrm{End}_R(M)$ its endomorphism ring. For a ring R, denote with $\mathcal{S}_R$ the full subcategory of Mod-R consisting of all finitely generated projective modules with semilocal endomorphism ring. The categories $\mathcal{C}$ and $\mathcal{S}_E$ turn out to be equivalent via the functors $\mathrm{Hom}_R(M,-)\colon \mathcal{C} \to \mathcal{S}_E$ and $- \otimes_E M\colon \mathcal{S}_E \to \mathcal{C}$. In particular, the monoids $V(\mathcal{C})$ and $V(\mathcal{S}_E)$ are isomorphic.

Our next Theorem [26, Theorem 2.1] states that we can realize every reduced Krull monoid in the form $V(\mathcal{S}_R)$ for some ring R.

Theorem 3.3.7 *Let k be a field, M a reduced Krull monoid, I a set and $T\colon M \to \mathbb{N}_0^{(I)}$ a divisor homomorphism. Then there exist a k-algebra R and two monoid isomorphisms $M \to V(\mathcal{S}_R)$ and $\mathbb{N}_0^{(I)} \to V(\mathcal{S}_{R/J(R)})$ such that if $\tau\colon V(\mathcal{S}_R) \to V(\mathcal{S}_{R/J(R)})$ is the homomorphism induced by the canonical projection $\pi\colon R \to R/J(R)$, then the diagram of monoids and monoids homomorphisms*

$$\begin{tikzcd} M \arrow[r, "T"] \arrow[d, "\cong"] & \mathbb{N}^{(I)} \arrow[d, "\cong"] \\ V(\mathcal{S}_R) \arrow[r, "\tau"] & V(\mathcal{S}_{R/J(R)}) \end{tikzcd}$$

commutes.

Chapter 4

Maximal Ideals in Preadditive Categories and Semilocal Categories

In this chapter, our first aim is to study the maximal ideals of a preadditive category $\mathcal{C}$. To describe the maximal ideals of $\mathcal{C}$, we make use of the *ideal* $\mathcal{A}_I$ *of the category* $\mathcal{C}$ *associated to an ideal* I of the endomorphism ring $\mathrm{End}_{\mathcal{C}}(A)$ of an object A of $\mathcal{C}$. The ideal $\mathcal{A}_I$ consists of all morphisms $f\colon X \to Y$ in $\mathcal{C}$ such that $\beta f \alpha \in I$ for every pair of morphisms $\alpha\colon A \to X$ and $\beta\colon Y \to A$ in $\mathcal{C}$. These ideals $\mathcal{A}_I$ of the category $\mathcal{C}$ turn out to be an elementary, but useful and powerful, tool. They were already introduced in [23] and [24] in the case in which $\mathcal{C}$ is a category of modules.

When $\mathcal{C}$ is the category proj-R of all finitely generated projective right modules over a ring R, the maximal ideals of $\mathcal{C}$ are in one-to-one correspondence with the maximal two-sided ideals of the ring R. We give a complete description of *simple* preadditive categories, that is, the preadditive categories with exactly two ideals, necessarily the trivial ones. An additive category $\mathcal{C}$ with splitting idempotents is simple if and only if $\mathcal{C}$ is equivalent to the category proj-R for some simple ring R. Maximal ideals do not exist in general for an arbitrary preadditive category $\mathcal{C}$, but they do always exist when $\mathcal{C}$ is *semilocal*, i.e. when $\mathcal{C}$ is a preadditive category with a non-zero object in which the endomorphism ring of every non-zero object is a semilocal ring. If $\mathcal{C}$ is a semilocal category and $\mathcal{M}$ is a maximal ideal of $\mathcal{C}$, the factor category $\mathcal{C}/\mathcal{M}$ is not only simple, but also equivalent to a full subcategory of Mod-R whose objects are finitely generated semisimple right modules over a simple artinian ring R. Thus the objects B of $\mathcal{C}$ are completely described by a set of natural numbers indexed in the class $\mathrm{Max}(\mathcal{C})$ of all maximal ideals of $\mathcal{C}$. The natural number corresponding to a maximal ideal $\mathcal{M}$ of $\mathcal{C}$ is the Goldie dimension of B in the factor category $\mathcal{C}/\mathcal{M}$, hence coincides with the dual Goldie dimension of the ring $\mathrm{End}_{\mathcal{C}}(B)/\mathcal{M}(B,B)$. Thus we get an isomorphism reflecting functor $F\colon \mathcal{C} \to \oplus_{\mathcal{M}\in\mathrm{Max}(\mathcal{C})}\mathcal{C}/\mathcal{M}$, which allows us to get a good representation of the structure of semilocal categories.

When the category $\mathcal{C}$ is additive, any skeleton $V(\mathcal{C})$ of $\mathcal{C}$ has the structure of a large monoid, in which the operation is induced by coproduct. For an additive semilocal category $\mathcal{C}$ with splitting idempotents, the functor F induces a monoid homomorphism $V(F)$ of the monoid $V(\mathcal{C})$ into the free commutative monoid $V(\oplus_{\mathcal{M}\in\mathrm{Max}(\mathcal{C})}\mathcal{C}/\mathcal{M}) \cong \mathbb{N}_0^{(\mathrm{Max}(\mathcal{C}))}$. As $V(F)$ turns out to be a divisor homomorphism, one finds that the monoid $V(\mathcal{C})$ is necessarily a Krull monoid. For an additive semilocal category $\mathcal{C}$ with splitting idempotents, we can therefore characterize the essential valuations of the monoid $V(\mathcal{C})$ and give some natural divisor homomorphisms and divisor theories of $V(\mathcal{C})$.

The associated ideals allow us also to study when there are canonical one-to-one-correspondences between the two-sided ideals of the endomorphism rings $\mathrm{End}_{\mathcal{C}}(A)$ and $\mathrm{End}_{\mathcal{C}}(B)$ of two objects A and B of a preadditive category $\mathcal{C}$. This condition is strictly stronger than the Morita-equivalence of the two rings $\mathrm{End}_{\mathcal{C}}(A)$ and $\mathrm{End}_{\mathcal{C}}(B)$. For further information on modules with Morita-equivalent endomorphism rings, see [1].

4.1 Associated ideals and maximal ideals

Let $\mathcal{C}$ be a preadditive category. For any object A of $\mathcal{C}$ and any two-sided ideal I of $\mathrm{End}_{\mathcal{C}}(A)$, let $\mathcal{A}_I$ be the ideal of the category $\mathcal{C}$ defined as follows: a morphism $f\colon X \to Y$ in $\mathcal{C}$ is in $\mathcal{A}_I(X,Y)$ if and only if $\beta f\alpha \in I$ for every pair of morphisms $\alpha\colon A \to X$ and $\beta\colon Y \to A$ in $\mathcal{C}$. The ideal $\mathcal{A}_I$ is called the *ideal of $\mathcal{C}$ associated to I* [23, 24]. The ideal $\mathcal{A}_I$ is the greatest of the ideals $\mathcal{I}'$ of $\mathcal{C}$ with $\mathcal{I}'(A,A) \subseteq I$. It is easily seen that $\mathcal{A}_I(A,A) = I$. Clearly, the ideals of the category $\mathcal{C}$ associated to two distinct ideals of $\mathrm{End}_{\mathcal{C}}(A)$ are distinct.

Lemma 4.1.1 *Let A and B be non-zero objects of a preadditive category $\mathcal{C}$, let I be a maximal ideal of $\mathrm{End}_{\mathcal{C}}(A)$ and let $\mathcal{A}_I$ be the ideal of $\mathcal{C}$ associated to I. Set $I' = \mathcal{A}_I(B,B)$ and assume that $I' \neq \mathrm{End}_{\mathcal{C}}(B)$. Then:*

1. *the ideal of $\mathcal{C}$ associated to I' is equal to $\mathcal{A}_I$;*
2. *if $\mathrm{End}_{\mathcal{C}}(B)$ is a semilocal ring, I' is a maximal ideal of $\mathrm{End}_{\mathcal{C}}(B)$.*

Proof.

1. Clearly, I' is an ideal of $\mathrm{End}_{\mathcal{C}}(B)$. Let $\mathcal{A}_{I'}$ be the ideal of $\mathcal{C}$ associated to I'. We must show that $\mathcal{A}_I = \mathcal{A}_{I'}$. In order to prove that $\mathcal{A}_I \subseteq \mathcal{A}_{I'}$, fix a morphism $f \in \mathcal{A}_I(X,Y)$. Then, for every $\alpha\colon B \to X$ and $\beta\colon Y \to B$, one has that $\beta f\alpha \in \mathcal{A}_I(B,B) = I'$. Hence $f \in \mathcal{A}_{I'}(X,Y)$.

   For the inverse inclusion, suppose that $f\colon X \to Y$ is a morphism in the category $\mathcal{C}$ with $f \notin \mathcal{A}_I(X,Y)$. Then there are $\alpha\colon A \to X$ and $\beta\colon Y \to A$ with $g = \beta f\alpha \notin I$. Then $\mathrm{End}_{\mathcal{C}}(A)g\,\mathrm{End}_{\mathcal{C}}(A) + I = \mathrm{End}_{\mathcal{C}}(A)$. Suppose that $g \in \mathcal{A}_{I'}(A,A)$. Then

   $$\begin{aligned}&\mathrm{Hom}_{\mathcal{C}}(A,B)\,\mathrm{End}_{\mathcal{C}}(A)\mathrm{Hom}_{\mathcal{C}}(B,A)\\ &= \mathrm{Hom}_{\mathcal{C}}(A,B)(\mathrm{End}_{\mathcal{C}}(A)g\,\mathrm{End}_{\mathcal{C}}(A) + I)\mathrm{Hom}_{\mathcal{C}}(B,A) \subseteq I'.\end{aligned}$$

It follows that the whole ring $\operatorname{End}_{\mathcal{C}}(A)$ is contained in $\mathcal{A}_{I'}(A,A)$. As $I' \neq \operatorname{End}_{\mathcal{C}}(B)$, we have that $1_B \notin I' = \mathcal{A}_I(B,B)$, so that there exist $\varphi\colon A \to B$ and $\psi\colon B \to A$ with $\psi\varphi \notin I$. Then $\psi\varphi \operatorname{End}_{\mathcal{C}}(A)\psi\varphi \nsubseteq I$ because maximal ideals are prime. But $\varphi \operatorname{End}_{\mathcal{C}}(A)\psi \subseteq I' = \mathcal{A}_I(B,B)$, so that $\psi\varphi \operatorname{End}_{\mathcal{C}}(A)\psi\varphi \subseteq I$, a contradiction. This shows that $g \notin \mathcal{A}_{I'}(A,A)$. Hence there are homomorphisms $\alpha'\colon B \to A$ and $\beta'\colon A \to B$ with $\beta'\beta f \alpha\alpha' \notin I'$. In particular, $f \notin \mathcal{A}_{I'}(X,Y)$.

2. Assume $\operatorname{End}_{\mathcal{C}}(B)$ semilocal. The Jacobson radical $\mathcal{J}$ of the category $\mathcal{C}$ is the greatest ideal of $\mathcal{C}$ such that $\mathcal{J}(A,A)$ coincides with the Jacobson radical $J(\operatorname{End}_{\mathcal{C}}(A))$ of the ring $\operatorname{End}_{\mathcal{C}}(A)$ for every non-zero object A in $\mathcal{C}$. Since every maximal ideal is primitive and $J(\operatorname{End}_{\mathcal{C}}(A))$ is the intersection of all primitive ideals of $\operatorname{End}_{\mathcal{C}}(A)$, it follows that $\mathcal{J}(A,A) \subseteq I$. As $\mathcal{A}_I$ is the greatest of the ideals of $\mathcal{C}$ with this property, we get that $\mathcal{J} \subseteq \mathcal{A}_I$. Thus $\mathcal{J}(B,B) \subseteq \mathcal{A}_I(B,B)$, that is, $J(\operatorname{End}_{\mathcal{C}}(B)) \subseteq I'$. Now I' is a proper ideal of $\operatorname{End}_{\mathcal{C}}(B)$. As $\operatorname{End}_{\mathcal{C}}(B)$ is semilocal, it follows that $\operatorname{End}_{\mathcal{C}}(B)/I'$ is a semisimple artinian ring. In order to show that I' is maximal, we will prove that $\operatorname{End}_{\mathcal{C}}(B)/I'$ is a simple artinian ring.

   Assume the contrary, so that there exist elements f, g of $\operatorname{End}_{\mathcal{C}}(B)$ such that $f + I', g + I'$ are non-trivial orthogonal central idempotents of the ring $\operatorname{End}_{\mathcal{C}}(B)/I'$. Since these idempotents are non-zero, there exist $\alpha, \alpha'\colon A \to B$ and $\beta, \beta' \in B \to A$ with $\beta f \alpha \notin I$ and $\beta' g \alpha' \notin I$. Then we have $\beta' g \alpha' \operatorname{End}_{\mathcal{C}}(A)\beta f \alpha \nsubseteq I$. As $f + I', g + I'$ are orthogonal and central, we know that $g \operatorname{End}_{\mathcal{C}}(B) f \subseteq I'$. Hence, a fortiori, $g\alpha' \operatorname{End}_{\mathcal{C}}(A)\beta f \subseteq I'$. Now $I' = \mathcal{A}_I(B,B)$ implies $\beta' I' \alpha \subseteq I$. Thus $\beta' g \alpha' \operatorname{End}_{\mathcal{C}}(A)\beta f \alpha \subseteq I$, a contradiction that proves that I' is a maximal ideal.

■

Lemma 4.1.2 *Let A and A' be two fixed non-zero objects of a preadditive category $\mathcal{C}$, let M (resp. M') be a maximal two-sided ideal of $\operatorname{End}_{\mathcal{C}}(A)$ (resp. $\operatorname{End}_{\mathcal{C}}(A')$) and let $\mathcal{A}_M$ (resp. $\mathcal{A}_{M'}$) be the ideal of $\mathcal{C}$ associated to M (resp. M'). The following conditions are equivalent:*

1. *$\mathcal{A}_M \supseteq \mathcal{A}_{M'}$;*
2. *$\mathcal{A}_M(B,B) \supseteq \mathcal{A}_{M'}(B,B)$ for every object B in $\mathcal{C}$;*
3. *$M \supseteq \mathcal{A}_{M'}(A,A)$;*
4. *there exists an object $C \in \operatorname{Ob}(\mathcal{C})$ such that*

$$\operatorname{End}_{\mathcal{C}}(C) \neq \mathcal{A}_M(C,C) \supseteq \mathcal{A}_{M'}(C,C).$$

PROOF. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious. To prove (4) ⇒ (1), it is enough to observe that, by Lemma 4.1.1(1), the ideal of $\mathcal{C}$ associated to $\mathcal{A}_M(C,C)$ is equal to $\mathcal{A}_M$ and therefore $\mathcal{A}_M$ is the greatest of the ideals $\mathcal{I}$ of $\mathcal{C}$ such that $\mathcal{A}_M(C,C) \supseteq \mathcal{I}(C,C)$. Thus $\mathcal{A}_M \supseteq \mathcal{A}_{M'}$. ■

Let A be an object of a preadditive category $\mathcal{C}$. For every $X, Y \in \mathrm{Ob}(\mathcal{C})$, let $\mathrm{Hom}_{\mathcal{C}}(A, Y)\mathrm{Hom}_{\mathcal{C}}(X, A)$ be the subgroup of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ generated by all composite morphisms fg where f ranges in $\mathrm{Hom}_{\mathcal{C}}(A, Y)$ and g ranges in $\mathrm{Hom}_{\mathcal{C}}(X, A)$.

Proposition 4.1.3 *Let A and A' be two non-zero objects of a preadditive category $\mathcal{C}$, let M (resp. M') be a maximal two-sided ideal of $\mathrm{End}_{\mathcal{C}}(A)$ (resp. $\mathrm{End}_{\mathcal{C}}(A')$) and let $\mathcal{A}_M$ (resp. $\mathcal{A}_{M'}$) be the ideal of $\mathcal{C}$ associated to M (resp. M'). The following conditions are equivalent:*

1. $\mathcal{A}_M = \mathcal{A}_{M'}$;
2. $\mathcal{A}_M(B, B) = \mathcal{A}_{M'}(B, B)$ *for every* $B \in \mathrm{Ob}(\mathcal{C})$;
3. $M = \mathcal{A}_{M'}(A, A)$;
4. $M' = \mathcal{A}_M(A', A')$;
5. *there exists an object $C \in \mathrm{Ob}(\mathcal{C})$ such that*
$$\mathrm{End}_{\mathcal{C}}(C) \neq \mathcal{A}_M(C, C) = \mathcal{A}_{M'}(C, C);$$
6. $\mathrm{Hom}_{\mathcal{C}}(A, A')M\mathrm{Hom}_{\mathcal{C}}(A', A) \subseteq M'$ *and* $\mathrm{Hom}_{\mathcal{C}}(A, A')\mathrm{Hom}_{\mathcal{C}}(A', A) \nsubseteq M'$;
7. $\mathrm{Hom}_{\mathcal{C}}(A', A)M'\mathrm{Hom}_{\mathcal{C}}(A, A') \subseteq M$ *and* $\mathrm{Hom}_{\mathcal{C}}(A', A)\mathrm{Hom}_{\mathcal{C}}(A, A') \nsubseteq M$;
8. *there exist two morphisms $\varphi: A \to A', \psi: A' \to A$ such that*
$$\psi\,\mathrm{End}_{\mathcal{C}}(A')\varphi \nsubseteq M,\ \varphi\,\mathrm{End}_{\mathcal{C}}(A)\psi \nsubseteq M',\ \psi M'\varphi \subseteq M \text{ and } \varphi M\psi \subseteq M'.$$

PROOF. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (5)$ and $(2) \Rightarrow (4) \Rightarrow (5)$ are obvious. The implication $(5) \Rightarrow (1)$ follows from the previous lemma.

$(3) \Rightarrow (6)$ Assume (3), so that $\mathrm{Hom}_{\mathcal{C}}(A, A')M\mathrm{Hom}_{\mathcal{C}}(A', A) \subseteq M'$.
If $\mathrm{Hom}_{\mathcal{C}}(A, A')\mathrm{Hom}_{\mathcal{C}}(A', A) \subseteq M'$, then $\mathrm{End}_{\mathcal{C}}(A) \subseteq \mathcal{A}_{M'}(A, A) = M$, a contradiction.

$(6) \Rightarrow (8)$ Suppose that (6) holds, so that there exist two morphisms $\varphi: A \to A', \psi: A' \to A$ with $\varphi\psi \notin M'$, but $\varphi M\psi \subseteq M'$. From $\varphi\psi \notin M'$, using the fact that M' is a prime ideal, we get that $\varphi\psi\,\mathrm{End}_{\mathcal{C}}(A')\varphi\psi \nsubseteq M'$. From (6), it follows that $\psi\,\mathrm{End}_{\mathcal{C}}(A')\varphi \nsubseteq M$. Finally, $\varphi\,\mathrm{End}_{\mathcal{C}}(A)\psi M'\varphi\,\mathrm{End}_{\mathcal{C}}(A)\psi \subseteq M'$ because M' is an ideal of $\mathrm{End}_{\mathcal{C}}(A')$, so that, $\varphi(\mathrm{End}_{\mathcal{C}}(A)\psi M'\varphi\,\mathrm{End}_{\mathcal{C}}(A) + M)\psi \subseteq M'$. Using (6) again, we get that $\mathrm{End}_{\mathcal{C}}(A)\psi M'\varphi\,\mathrm{End}_{\mathcal{C}}(A) \subseteq M$, from which $\psi M'\varphi \subseteq M$.

$(8) \Rightarrow (1)$ Assume that (8) holds. Let X and Y be two objects of $\mathcal{C}$ and $f: X \to Y$ be a morphism in $\mathcal{A}_M(X, Y)$. We want to show that f belongs to $\mathcal{A}_{M'}(X, Y)$. Fix arbitrary morphisms $\alpha: A' \to X$ and $\beta: Y \to A'$. We must prove that $\beta f\alpha \in M'$. Now $f \in \mathcal{A}_M(X, Y)$ implies that

$$(\mathrm{End}_{\mathcal{C}}(A)\psi\,\mathrm{End}_{\mathcal{C}}(A')\beta)f(\alpha\,\mathrm{End}_{\mathcal{C}}(A')\varphi\,\mathrm{End}_{\mathcal{C}}(A)) \subseteq M.$$

By (8), we get that

$$\varphi(\mathrm{End}_{\mathcal{C}}(A)\psi\,\mathrm{End}_{\mathcal{C}}(A')\beta f\alpha\,\mathrm{End}_{\mathcal{C}}(A')\varphi\,\mathrm{End}_{\mathcal{C}}(A))\psi \subseteq M',$$

so that

$$\mathrm{End}_{\mathcal{C}}(A')(\varphi\,\mathrm{End}_{\mathcal{C}}(A)\psi\,\mathrm{End}_{\mathcal{C}}(A')\beta f\alpha\,\mathrm{End}_{\mathcal{C}}(A')\varphi\,\mathrm{End}_{\mathcal{C}}(A)\psi)\,\mathrm{End}_{\mathcal{C}}(A') \subseteq M'.$$

This is the product of the three ideals

$$\mathrm{End}_{\mathcal{C}}(A')\varphi\,\mathrm{End}_{\mathcal{C}}(A)\psi\,\mathrm{End}_{\mathcal{C}}(A'), \qquad \mathrm{End}_{\mathcal{C}}(A')\beta f\alpha\,\mathrm{End}_{\mathcal{C}}(A')$$

and

$$\mathrm{End}_{\mathcal{C}}(A')\varphi\,\mathrm{End}_{\mathcal{C}}(A)\psi\,\mathrm{End}_{\mathcal{C}}(A')$$

of the ring $\mathrm{End}_{\mathcal{C}}(A')$ with $\mathrm{End}_{\mathcal{C}}(A')\varphi\,\mathrm{End}_{\mathcal{C}}(A)\psi\,\mathrm{End}_{\mathcal{C}}(A') \notin M'$ by (8). As M' is prime, we get $\beta f\alpha \in M'$. This proves that $\mathcal{A}_M \subseteq \mathcal{A}_{M'}$.

The proofs that $\mathcal{A}_{M'} \subseteq \mathcal{A}_M$ and (4) ⇒ (7) ⇒ (8) are similar. ∎

We say that an ideal $\mathcal{M}$ of a preadditive category $\mathcal{C}$ is *maximal* if the improper ideal of $\mathcal{C}$ is the unique ideal of the category $\mathcal{C}$ properly containing $\mathcal{M}$. Clearly, if all objects of $\mathcal{C}$ are zero objects, maximal ideals do not exist in $\mathcal{C}$. The next lemma characterizes maximal ideals.

Lemma 4.1.4 *Let $\mathcal{C}$ be a preadditive category and $\mathcal{M}$ be a proper ideal of $\mathcal{C}$. Then $\mathcal{M}$ is a maximal ideal if and only if, for every object A of $\mathcal{C}$ with $\mathcal{M}(A,A) \neq \mathrm{End}_{\mathcal{C}}(A)$, one has that:* (1) *$\mathcal{M}(A,A)$ is a maximal ideal of* $\mathrm{End}_{\mathcal{C}}(A)$*, and* (2) *$\mathcal{M}$ is the ideal of $\mathcal{C}$ associated to $\mathcal{M}(A,A)$.*

Proof. Let $\mathcal{M}$ be a maximal ideal of $\mathcal{C}$, and let A be an object of $\mathcal{C}$ with $\mathcal{M}(A,A) \neq \mathrm{End}_{\mathcal{C}}(A)$. Clearly, $\mathcal{M}(A,A)$ is an ideal of $\mathrm{End}_{\mathcal{C}}(A)$. If $\mathcal{M}(A,A)$ is not a maximal ideal of $\mathrm{End}_{\mathcal{C}}(A)$, let I be a maximal ideal of $\mathrm{End}_{\mathcal{C}}(A)$ properly containing it, and let $\mathcal{A}_I$ be the ideal associated to I. Then $\mathcal{M}(A,A) \subsetneq I$, so that $\mathcal{M} \subsetneq \mathcal{A}_I$. Thus $\mathcal{A}_I$ is the improper ideal, which is a contradiction because $\mathcal{A}_I(A,A) = I$.

Now let $\mathcal{A}_M$ be the ideal of $\mathcal{C}$ associated to the maximal ideal $M = \mathcal{M}(A,A)$ of $\mathrm{End}_{\mathcal{C}}(A)$. Then $\mathcal{M} \subseteq \mathcal{A}_M$. Since $\mathcal{A}_M$ is proper, it follows that $\mathcal{M} = \mathcal{A}_M$ is the ideal associated to $\mathcal{M}(A,A)$.

Conversely, let $\mathcal{M}$ be a proper ideal of $\mathcal{C}$ with the property that, for every object A of $\mathcal{C}$ with $\mathcal{M}(A,A) \neq \mathrm{End}_{\mathcal{C}}(A)$, (1) and (2) hold. Let $\mathcal{I}$ be an ideal properly containing $\mathcal{M}$. Then $\mathcal{I}(A,A) \supseteq \mathcal{M}(A,A)$ for every object $A \in \mathrm{Ob}(\mathcal{C})$. If $\mathcal{I}(A,A) = \mathcal{M}(A,A)) \neq \mathrm{End}_{\mathcal{C}}(A)$ for some $A \in \mathrm{Ob}(\mathcal{C})$, then $\mathcal{I} \subseteq \mathcal{M}$, because $\mathcal{M}$ is the ideal associated to $\mathcal{M}(A,A)$, that is, the greatest of the ideals $\mathcal{I}'$ with $\mathcal{I}'(A,A) \subseteq \mathcal{M}(A,A)$. This is a contradiction, because $\mathcal{M}$ is properly contained in $\mathcal{I}$. Therefore, for every $A \in \mathrm{Ob}(\mathcal{C})$, either $\mathcal{I}(A,A) \supset \mathcal{M}(A,A)$ or $\mathcal{I}(A,A) = \mathcal{M}(A,A) = \mathrm{End}_{\mathcal{C}}(A)$. In both cases, $\mathcal{I}(A,A) = \mathrm{End}_{\mathcal{C}}(A)$, so $\mathcal{I}$ is the improper ideal, as we wanted to prove. ∎

For a ring R, let proj-R denote the full subcategory of Mod-R whose objects are all finitely generated projective modules.

Proposition 4.1.5 *For any ring R, the maximal ideals of the category* proj-R *are exactly the ideals of* proj-R *associated to the maximal two-sided ideals of the ring R.*

Proof. Let $\mathcal{M}$ be a maximal ideal in proj-R. Then $\mathcal{M}(R_R, R_R)$ is either equal to $\mathrm{End}_R(R) \cong R$ or is a maximal ideal of $\mathrm{End}_R(R) \cong R$ (Lemma 4.1.4). If $\mathcal{M}(R_R, R_R) = \mathrm{End}_R(R)$, then $\mathcal{M}(P_R, P_R) = \mathrm{End}_R(P)$ for every finitely generated projective module P_R, and $\mathcal{M}$ is the improper ideal, contradiction. Hence $\mathcal{M}(R_R, R_R)$ is a maximal ideal M of $\mathrm{End}_R(R) \cong R$, so that $\mathcal{M}$ is associated to M by Lemma 4.1.4(2).

Conversely, let M be any maximal two-sided ideal of $\mathrm{End}_R(R)$ and let $\mathcal{A}_M$ be the ideal of proj-R associated to M. Let $\mathcal{I}$ be an ideal of proj-R containing $\mathcal{A}_M$. Then $\mathcal{I}(R_R, R_R)$ is an ideal of $R \cong \mathrm{End}_R(R)$ containing $\mathcal{A}_M(R_R, R_R)$. If $\mathcal{I}(R_R, R_R) = \mathrm{End}_R(R)$, then $\mathcal{I}$ is the improper ideal. If $\mathcal{I}(R_R, R_R) = \mathcal{A}_M(R_R, R_R)$, then $\mathcal{A}_M = \mathcal{I}$ by the maximality of the associated ideal $\mathcal{A}_M$ among the ideals $\mathcal{I}'$ with $\mathcal{I}'(R_R, R_R) \subseteq \mathcal{A}_M(R_R, R_R)$. ■

4.2 Simple additive categories

Let A be an object of a preadditive category $\mathcal{C}$. Let $\mathrm{add}(A)$ denote the subclass of $\mathrm{Ob}(\mathcal{C})$ consisting of all objects $B \in \mathrm{Ob}(\mathcal{C})$ for which there exist $n > 0$ and morphisms $f_1, \dots, f_n: A \to B$ and $g_1, \dots, g_n: B \to A$ with $\sum_{i=1}^n f_i g_i = 1_B$. When $\mathcal{C}$ is additive with splitting idempotents, then $\mathrm{add}(A)$ is the class of objects of the smallest additive full subcategory of $\mathcal{C}$ with splitting idempotents containing A and closed under isomorphism. For example, if $\mathcal{C} = $ Mod-R, then $\mathrm{add}(R_R)$ coincides with the class proj-R of all finitely generated projective right R-modules. We will denote as $\mathrm{add}(A)$ not only the subclass of $\mathrm{Ob}(\mathcal{C})$, but also the full subcategory of $\mathcal{C}$ whose class of objects is $\mathrm{add}(A)$.

Lemma 4.2.1 *Let A be a non-zero object of a preadditive category $\mathcal{C}$, set $R = \mathrm{End}_{\mathcal{C}}(A)$ and consider the additive functor $F = \mathrm{Hom}_{\mathcal{C}}(A, -): \mathcal{C} \to$ Mod-R. The following properties hold:*

1. *the functor F induces a full and faithful functor $\mathrm{add}(A) \to$ proj-R;*
2. *if $\mathcal{C}$ is an additive category with splitting idempotents, then F induces an equivalence $\mathrm{add}(A) \to$ proj-R.*

Proof. Let B be an object of $\mathrm{add}(A)$, so that there exist $f_1, \dots, f_n: A \to B$ and $g_1, \dots, g_n: B \to A$ with $\sum_{i=1}^n f_i g_i = 1_B$. Applying F, we obtain that $\sum_{i=1}^n F(f_i)F(g_i) = 1_{F(B)}$, where $F(f_i): F(A) \to F(B)$ and $F(g_i): F(B) \to F(A)$. Thus the module $F(B)$ is a direct summand of $F(A)^n \cong R_R^n$, hence a finitely generated projective right R-module. To see that the restriction of F to $\mathrm{add}(A)$ is faithful, let $f: B \to B'$ be a morphism of $\mathrm{add}(A)$ such that $F(f) = 0$, i.e. $fh = 0$ for every $h \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Since $1_B = \sum_{i=1}^n f_i g_i$, we have $f = f1_B = \sum_{i=1}^n (ff_i)g_i = 0$. In order to prove that the restriction of F is full, let B, B' be a pair of objects in $\mathrm{add}(A)$ and let $\varphi: \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, B')$ be a right R-module morphism. Define $f: B \to B'$ by $f = \sum_{i=1}^n \varphi(f_i) g_i$. We want to show that $F(f) = \varphi$. The right R-module $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is generated by the n elements f_i, because every $g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ can be written as $1_B g = \sum_{i=1}^n f_i g_i g$. Therefore it suffices to show that $F(f)(f_k) = \varphi(f_k)$ for every $k = 1, \dots, n$. This is true, because, by the $\mathrm{End}_{\mathcal{C}}(A)$-linearity of φ, $F(f)(f_k) = ff_k = \sum_{i=1}^n \varphi(f_i)g_i f_k = \varphi(\sum_{i=1}^n f_i g_i f_k) = \varphi(f_k)$. This proves (1).

Now assume $\mathcal{C}$ additive and with splitting idempotents. Let P be a finitely generated projective right R-module. There are morphisms $\alpha_i: P \to R_R$ and $\beta_i: R_R \to P$ such that $1_P = \sum_{i=1}^n \beta_i \alpha_i$. Therefore the endomorphism of R_R^n defined by the matrix $(\alpha_i \beta_j)$ is an idempotent endomorphism with image P. As $\mathcal{C}$ is additive and the restriction of F to $\mathrm{add}(A)$ is full, there is an endomorphism f of A^n in $\mathcal{C}$ such that $F(f) = (\alpha_i \beta_j)$. Since the restriction of F is faithful, f must be idempotent, hence splits. Let $g: A^n \to B$ and $h: B \to A^n$ be morphisms in $\mathcal{C}$ with $hg = f$ and $gh = 1_B$. Then $F(g): F(A^n) \to F(B)$ and $F(h): F(B) \to F(A^n)$ are right R-module morphisms with $F(h)F(g) = F(f)$ and $F(g)F(h) = 1_{F(B)}$. Hence $F(g)$ is onto, so that $F(h)$ and $F(f)$ have the same image. Now the image of $F(f) = (\alpha_i \beta_j)$ is P, and $F(g)F(h) = 1_{F(B)}$ implies that the image of $F(h)$ is isomorphic to $F(B)$. Thus $P \cong F(B)$, as desired. ∎

We say that a preadditive category is *simple* if it has exactly two ideals, necessarily the trivial ones. Hence, a simple category has non-zero objects. Clearly, the dual of a simple category is a simple category.

Theorem 4.2.2 *The following conditions are equivalent for a preadditive category $\mathcal{C}$:*

1. *$\mathcal{C}$ is a simple category;*
2. *$\mathcal{C}$ has a non-zero object, the endomorphism ring of every non-zero object of $\mathcal{C}$ is a simple ring, and, for every $A, B, C \in \mathrm{Ob}(\mathcal{C})$ with $A \neq 0$ and every $f: B \to C$, if $\beta f \alpha = 0$ for every $\alpha: A \to B$ and every $\beta: C \to A$, then $f = 0$;*
3. *$\mathcal{C}$ has a non-zero object, and every non-zero object of $\mathcal{C}$ is a generator and a cogenerator for $\mathcal{C}$ and has a simple endomorphism ring;*
4. *$\mathcal{C}$ has a non-zero object and there exists a simple ring R such that $\mathcal{C}$ is equivalent to a full subcategory of the category* proj-R *of all finitely generated projective right R-modules.*

Proof. (1) ⇒ (2) follows immediately from Lemma 4.1.4, because if $\mathcal{C}$ is simple, then, for every non-zero object A, the zero ideal of $\mathcal{C}$ is the ideal associated to the zero ideal of $\mathrm{End}_{\mathcal{C}}(A)$.

(2) ⇒ (3) Assume that (2) holds. Let A be a non-zero object. We must show that A is a generator, that is, if $f: B \to C$ is a non-zero morphism in $\mathcal{C}$, then there exists a morphism $\alpha: A \to B$ such that $f\alpha \neq 0$. Now, by Condition (2), $f \neq 0$ implies that there exist $\alpha: A \to B$ and $\beta: C \to A$ such that $\beta f \alpha \neq 0$. In particular, $f\alpha \neq 0$. Thus A is a generator. Similarly, A is a cogenerator.

(3) ⇒ (4) Suppose that (3) holds. Let A be a fixed non-zero object of $\mathcal{C}$, so that the endomorphism ring $R = \mathrm{End}_{\mathcal{C}}(A)$ of A is a simple ring. Let us show that $\mathrm{add}(A) = \mathcal{C}$. Let $B \neq 0$ be an object of $\mathcal{C}$. As B is a generator and $1_A \neq 0_A$, there exists $\alpha: B \to A$ with $1_A \alpha \neq 0_A \alpha$, that is, $\alpha \neq 0$. As B is a

cogenerator, there exists $\beta: A \to B$ with $\beta\alpha \neq 0$. As $\mathrm{End}_{\mathcal{C}}(B)$ is simple, the two-sided ideal of $\mathrm{End}_{\mathcal{C}}(B)$ generated by $\beta\alpha$ is the whole ring, that is, there exist $\gamma_1, \dots, \gamma_n, \gamma'_1, \dots, \gamma'_n \in \mathrm{End}_{\mathcal{C}}(B)$ with $1_B = \sum_{i=1}^n \gamma_i\beta\alpha\gamma'_i$. Hence B is an object of $\mathrm{add}(A)$, and $\mathrm{add}(A) = \mathcal{C}$. By Lemma 4.2.1(1), the additive functor $F = \mathrm{Hom}_{\mathcal{C}}(A, -): \mathcal{C} \to \text{Mod-}R$ is full and faithful.

(4) $\Rightarrow$ (1) Let $\mathcal{C}$ be a full subcategory of proj-R for some simple ring R and let $\mathcal{I}$ be a non-zero ideal of $\mathcal{C}$. We must show that $\mathcal{I}$ is the improper ideal. Fix a non-zero morphism $f: A \to B$ in $\mathcal{I}$. We must prove that every morphism $g: X \to Y$ in $\mathcal{C}$ is in $\mathcal{I}$. There exist an epimorphism $\pi_A: R^n \to A$ and a monomorphism $\varepsilon_B: B \to R^m$. Hence the morphism $\varepsilon_B f \pi_A: R^n \to R^m$ is a non-zero morphism in Mod-R. Thus there exist morphisms $\varepsilon_R: R \to R^n$ and $\pi_R: R^m \to R$ with $\pi_R\varepsilon_B f\pi_A\varepsilon_R: R \to R$ non-zero. As R is simple, there exist endomorphisms $f_1, \dots, f_n, g_1, \dots, g_n$ of R_R with $\sum_{i=1}^n f_i\pi_R\varepsilon_B f\pi_A\varepsilon_R g_i = 1_R$. Now Y is a direct summand of R_R^t, so that there exist $\alpha: Y \to R_R^t$ and $\beta: R_R^t \to Y$ with $\beta\alpha = 1_Y$. Let $\pi_1, \dots, \pi_t: R^t \to R$ and $\varepsilon_1, \dots, \varepsilon_t: R \to R^t$ be such that $\sum_{j=1}^t \varepsilon_j\pi_j = 1_{R^t}$. Then $g = \beta\alpha g = \beta 1_{R^t}\alpha g = \sum_{j=1}^t \beta\varepsilon_j\pi_j\alpha g = \sum_{j=1}^t \beta\varepsilon_j 1_R\pi_j\alpha g = \sum_{i=1}^n \sum_{j=1}^t (\beta\varepsilon_j f_i\pi_R\varepsilon_B) f (\pi_A\varepsilon_R g_i\pi_j\alpha g)$ is in $\mathcal{I}(X, Y)$. ■

Remark 4.2.3 By Condition (4) of Theorem 4.2.2, every simple preadditive category is necessarily skeletally small.

By the same Condition (4), every full subcategory of a simple preadditive category containing a non-zero object is a simple category.

Proposition 4.2.4 *An additive category $\mathcal{C}$ with splitting idempotents is simple if and only if it is equivalent to the category* proj-R *for some simple ring R.*

Proof. If R is a simple ring, the category proj-R is simple by Theorem 4.2.2, (4) $\Rightarrow$ (1).

Conversely, let $\mathcal{C}$ be a simple additive category with splitting idempotents. By Theorem 4.2.2, every non-zero object of $\mathcal{C}$ has a simple endomorphism ring. In the proof of Theorem 4.2.2, (3) $\Rightarrow$ (4), we have seen that if A is a fixed non-zero object, then $\mathrm{add}(A) = \mathcal{C}$. By Lemma 4.2.1(2), the functor $F = \mathrm{Hom}_{\mathcal{C}}(A, -): \mathcal{C} \to \text{proj-}R$ is an equivalence. ■

Remark 4.2.5 Maximal ideals of a preadditive category $\mathcal{C}$ coincide with kernels of non-zero functors $F: \mathcal{C} \to \text{proj-}R$, where R ranges in the class of simple rings. Let us show that it suffices to consider the simple rings R of the type $\mathrm{End}_{\mathcal{C}}(A)/M$, where A is a non-zero object of $\mathcal{C}$ and M is a maximal ideal of $\mathrm{End}_{\mathcal{C}}(A)$. If $\mathcal{M}$ is a maximal ideal of a category $\mathcal{C}$, then $\mathcal{M}$ is associated to a maximal ideal M of the endomorphism ring $\mathrm{End}_{\mathcal{C}}(A)$ of a non-zero object A of $\mathcal{C}$ (Lemma 4.1.4). The image of A in the factor category $\mathcal{C}/\mathcal{M}$ is a non-zero object of $\mathcal{C}/\mathcal{M}$ whose endomorphism ring is the simple ring $\mathrm{End}_{\mathcal{C}}(A)/M$. We have seen in the proof of (3) $\Rightarrow$ (4) in Theorem 4.2.2 that the functor

$$F = \mathrm{Hom}_{\mathcal{C}/\mathcal{M}}(A, -): \mathcal{C}/\mathcal{M} \to \text{proj-}(\mathrm{End}_{\mathcal{C}}(A)/M)$$

is full and faithful. Hence the kernel of the functor

$$\mathrm{Hom}_{\mathcal{C}}(A,-)/\mathcal{M}(A,-) = \mathrm{Hom}_{\mathcal{C}/\mathcal{M}}(A,-): \mathcal{C} \to \text{proj-}(\mathrm{End}_{\mathcal{C}}(A)/M)$$

is $\mathcal{M}$.

4.3 Existence of maximal ideals in a preadditive category. Examples. The Jacobson radical.

We begin this section showing that maximal ideals do not necessarily exist in a preadditive category $\mathcal{C}$ with a non-zero object, even in the case in which $\mathcal{C}$ is a small abelian category.

Example 4.3.1 Let k be a division ring and $\overline{V}_k$ a right vector space of infinite dimension d. Let $\mathcal{C}$ be either the whole category Vect$-k$ or the full subcategory of Vect$-k$ whose objects are all vector subspaces of $\overline{V}_k$ of dimension strictly less than d. Recall that for any k-linear mapping $f: V_k \to W_k$, the *rank* $\rho(f)$ of f is the dimension of the image $\mathrm{im}(f)$. Let $\aleph$ be an infinite cardinal, and consider the ideal $\mathcal{I}_\aleph$ of $\mathcal{C}$ defined, for every $V_k, W_k \in \mathrm{Ob}(\mathcal{C})$, by $\mathcal{I}_\aleph(V_k, W_k) = \{ f \in \mathrm{Hom}_k(V, W) \mid \rho(f) < \aleph \}$. We leave to the reader the verification that, for $f: V_k \to W_k$ and $f': V'_k \to W'_k$, there exist $\alpha: V_k \to V'_k$ and $\beta: W'_k \to W_k$ such that $f = \beta f' \alpha$ if and only if $\rho(f) \le \rho(f')$. It follows that the ideals of $\mathcal{C}$ are the zero ideal, the improper ideal and the ideals $\mathcal{I}_\aleph$ for every infinite cardinal $\aleph$ (clearly, in the second case, in which $\mathcal{C} \ne$ Vect$-k$, the improper ideal and the ideals $\mathcal{I}_\aleph$ with $\aleph \ge d$ coincide). An object V_k of $\mathcal{C}$ becomes the zero object in the factor category $\mathcal{C}/\mathcal{I}_\aleph$ if and only if $\dim(V_k) < \aleph$. For the category Vect$-k$, maximal ideals do not exist, and in the second case, in which $\mathcal{C} \ne$ Vect$-k$, maximal ideals exist in $\mathcal{C}$ if and only if d is the successor of a cardinal d' (and in this case $\mathcal{C}$ has a unique maximal ideal, which is the ideal $\mathcal{I}_{d'}$). For instance, if $d = \aleph_\omega$, then d is not the successor of a cardinal, the ideals of the small abelian category $\mathcal{C}$ are the two trivial ideals and the ideals $\mathcal{I}_{\aleph_n}$ with n finite ordinal, and maximal ideals do not exist in $\mathcal{C}$. If $d = \aleph_1$, then d is the successor of the cardinal $\aleph_0$, and $\mathcal{C}$ has $\mathcal{I}_{\aleph_0}$ as its unique maximal ideal. In this case, the canonical functor $\mathcal{C} \to \mathcal{C}/\mathcal{I}_{\aleph_0}$ is not isomorphism-reflecting, because all finitely-dimensional objects of $\mathcal{C}$ become zero objects in the factor category $\mathcal{C}/\mathcal{I}_{\aleph_0}$.

This example also shows that, though every maximal ideal of a category $\mathcal{C}$ is the ideal associated to a maximal ideal of the endomorphism ring of a non-zero object of $\mathcal{C}$ (Lemma 4.1.4), the converse is not always true, even if the category is small. In the previous example, if V_k is a vector space of infinite dimension $\aleph$ that is an object of $\mathcal{C}$, the ideal $\mathcal{I}_\aleph$ is the ideal associated to the maximal ideal of $\mathrm{End}_k(V)$ that consists of all the endomorphisms of V_k of rank strictly less than $\aleph$. If $V_k \ne 0$ is a vector space of finite dimension that is an object of $\mathcal{C}$, the zero ideal of $\mathcal{C}$ is the ideal associated to the maximal ideal of $\mathrm{End}_k(V)$, which is the zero ideal of $\mathrm{End}_k(V)$.

Definition 4.3.2 A *semilocal* category is a preadditive category with a non-zero object such that the endomorphism ring of every non-zero object is a semilocal ring.

The rest of this chapter will be mainly devoted to describing the structure of semilocal categories.

Proposition 4.3.3 *Let $\mathcal{C}$ be a semilocal category. Then:*

1. *every ideal of $\mathcal{C}$ associated to a maximal ideal of the endomorphism ring of a non-zero object of $\mathcal{C}$ is a maximal ideal of $\mathcal{C}$;*
2. *in $\mathcal{C}$, every proper ideal is contained in a maximal ideal;*
3. *maximal ideals exist in $\mathcal{C}$.*

Proof.

1. Let M be a maximal two-sided ideal of the endomorphism ring $\mathrm{End}_{\mathcal{C}}(A)$ for some non-zero object $A \in \mathrm{Ob}(\mathcal{C})$. We will prove that the ideal $\mathcal{A}_M$ associated to M is maximal. Let $\mathcal{I}$ be an ideal in $\mathcal{C}$ properly containing $\mathcal{A}_M$. By Lemma 4.1.1(2), for any non-zero object B in the semilocal category $\mathcal{C}$, $\mathcal{A}_M(B,B)$ is always either $\mathrm{End}_{\mathcal{C}}(B)$ or a maximal ideal of $\mathrm{End}_{\mathcal{C}}(B)$. Since $\mathcal{I}$ properly contains $\mathcal{A}_M$, the ideal $\mathcal{I}(B,B)$ also must be either $\mathrm{End}_{\mathcal{C}}(B)$ or the maximal ideal $\mathcal{A}_M(B,B)$ of $\mathrm{End}_{\mathcal{C}}(B)$. If we suppose that $\mathcal{I}(B,B) = \mathcal{A}_M(B,B)$ is a maximal ideal of $\mathrm{End}_{\mathcal{C}}(B)$, by Lemma 4.1.1(1) we obtain that $\mathcal{I} \subseteq \mathcal{A}_M$. This is not possible because $\mathcal{I}$ properly contains $\mathcal{A}_M$. Therefore $\mathcal{I}$ is the improper ideal and so $\mathcal{A}_M$ is maximal.
2. Let $\mathcal{I}$ be a proper ideal of $\mathcal{C}$, so that there exists a non-zero object A of $\mathcal{C}$ with $\mathcal{I}(A,A) \neq \mathrm{End}_{\mathcal{C}}(A)$. If M is a maximal ideal of $\mathrm{End}_{\mathcal{C}}(A)$ containing $\mathcal{I}(A,A)$, Part (1) yields that the ideal associated to M is a maximal ideal of $\mathcal{C}$ containing $\mathcal{I}$.
3. Follows from (2) applied to the zero ideal of $\mathcal{C}$.

■

Recall that if R is a simple artinian ring, then R has a unique simple right module S up to isomorphism, and all finitely generated right R-modules M are semisimple and isomorphic to S^n, where n is the Goldie dimension of M.

Corollary 4.3.4 *Let $\mathcal{C}$ be a semilocal category and $\mathcal{M}$ a maximal ideal of $\mathcal{C}$. Then there exist a simple artinian ring R and a full and faithful functor $F\colon \mathcal{C}/\mathcal{M} \to \text{fgss-}R$ of the factor category $\mathcal{C}/\mathcal{M}$ into the full subcategory* fgss-R *of* Mod-R *whose objects are all finitely generated semisimple right R-modules. Moreover, for every object B of $\mathcal{C}$, the Goldie dimension of the semisimple right R-module $F(B)$ is equal to* $\mathrm{codim}(\mathrm{End}_{\mathcal{C}}(B)/\mathcal{M}(B,B))$.

Proof. Argue as in the proof of Theorem 4.2.2((3) ⇒ (4)) applying Lemma 4.2.1(1). Let A be a non-zero object of the factor category $\mathcal{C}/\mathcal{M}$. The ring R is the ring $\mathrm{End}_{\mathcal{C}}(A)/\mathcal{M}(A,A)$, which is simple artinian because $\mathcal{M}(A,A)$ is a maximal ideal in the semilocal ring $\mathrm{End}_{\mathcal{C}}(A)$. Over such a ring R, every module is semisimple and projective. For every finitely generated semisimple module M, the Goldie dimension of M coincides with the dual Goldie dimension of its endomorphism ring, so that for every object B of $\mathcal{C}$, $\dim(F(B)) = \mathrm{codim}(\mathrm{End}_R(F(B))) = \mathrm{codim}(\mathrm{End}_{\mathcal{C}/\mathcal{M}}(B)) = \mathrm{codim}(\mathrm{End}_{\mathcal{C}}(B)/\mathcal{M}(B,B))$. ∎

If $\mathcal{C}$ is a semilocal category, we can consider the class of all pairs (A,M), where A ranges in the class of all non-zero objects of $\mathcal{C}$ and M is a maximal ideal in the endomorphism ring $\mathrm{End}_{\mathcal{C}}(A)$. Define an equivalence relation $\sim$ on this class by $(A,M) \sim (A',M')$ if $\mathcal{A}_M = \mathcal{A}_{M'}$. Cf. Proposition 4.1.3. Let $\mathrm{Max}(\mathcal{C})$ be a class of representatives modulo $\sim$. We call $\mathrm{Max}(\mathcal{C})$ the *maximal spectrum* of $\mathcal{C}$. For a semilocal category, the class $\mathrm{Max}(\mathcal{C})$ collects all maximal ideals of $\mathcal{C}$ but, since an ideal of a large category is not a set, we cannot define the maximal spectrum as the class of all maximal ideal of $\mathcal{C}$; we have then to give this alternative definition, which is correct in MK.

Example 4.3.5 Let $\mathcal{C}$ be a preadditive category in which $\mathrm{End}_{\mathcal{C}}(A)$ is a local ring for every $A \in \mathrm{Ob}(\mathcal{C})$. In particular, $\mathcal{C}$ has no zero objects. We will prove that there is a bijection f between $\mathrm{Max}(\mathcal{C})$ and $V(\mathcal{C})$, defined by $f(A,M) = \langle A \rangle$ for every $(A,M) \in \mathrm{Max}(\mathcal{C})$. To prove that f is injective, let $(A,M),(A',M') \in \mathrm{Max}(\mathcal{C})$ be such that $A \cong A'$. If $g\colon A \to A'$ is an isomorphism, then, for every morphism $h\colon X \to Y$ in $\mathcal{C}$, $h \in \mathcal{A}_M(X,Y)$ if and only if the endomorphism $\beta h \alpha$ of A is not an automorphism of A for every $\alpha\colon X \to A$ and every $\beta\colon Y \to A$, if and only if the endomorphism $g\beta h\alpha g^{-1}$ of A' is not an automorphism of A' for every $\alpha\colon X \to A$ and every $\beta\colon Y \to A'$, if and only $h \in \mathcal{A}_{M'}(X,Y)$. Thus $\mathcal{A}_M = \mathcal{A}_{M'}$ and $(A,M) \sim (A',M')$, so that $(A,M) = (A',M')$ in $\mathrm{Max}(\mathcal{C})$.

In order to show that f is onto, fix $A \in V(\mathcal{C})$. Then A is non-zero, so that the local ring $\mathrm{End}_{\mathcal{C}}(A)$ has a maximal ideal M. Let $(A',M') \in \mathrm{Max}(\mathcal{C})$ be such that $(A,M) \sim (A',M')$. Then $\mathcal{A}_M = \mathcal{A}_{M'}$ implies that there exist $g\colon A \to A'$, $h\colon A' \to A$ and $\alpha\colon A' \to A'$ with $h\alpha g$ an automorphism of A by Proposition 4.1.3, (1) ⇒ (6). Thus there exists $g'\colon A \to A$ with $h\alpha g g' = 1_A$. Hence h is right invertible and $\alpha g g' h$ is a non-zero idempotent of $\mathrm{End}_{\mathcal{C}}(B)$. As $\mathrm{End}_{\mathcal{C}}(B)$ is a local ring, its only non-zero idempotent is the identity. Thus h is also left invertible, hence an isomorphism. Thus $\langle A' \rangle = \langle A \rangle$ and $f(A',M') = \langle A \rangle$.

Example 4.3.6 There are two standard operations that can be performed on a preadditive category $\mathcal{C}$. We can construct the category $\mathrm{sum}(\mathcal{C})$ whose objects are formal direct sums of finitely many objects of $\mathcal{C}$, so that $\mathrm{sum}(\mathcal{C})$ is an additive category containing $\mathcal{C}$, and we can construct the category $\hat{\mathcal{C}}$ whose objects are all pairs (A,e), with $A \in \mathrm{Ob}(\mathcal{C})$ and e an idempotent in $\mathrm{End}_{\mathcal{C}}(A)$, so that $\hat{\mathcal{C}}$ turns out to be a category with splitting idempotents containing $\mathcal{C}$. As morphisms between finite direct sums are matrices of morphisms, and a matrix is in an ideal if and only if all entries of the matrix are in the ideal, it is clear that both operations do not change the ideals of the category. Thus the maximal ideals are essentially the same for the three categories $\mathcal{C}$, $\mathrm{sum}(\mathcal{C})$ and $\hat{\mathcal{C}}$. For instance, if $\mathcal{C}$ is the category of Example 4.3.5, then $\mathrm{sum}(\mathcal{C})$ is a category in which every object is a direct sum of finitely many objects with a local endomorphism ring, and the maximal ideals of $\mathrm{sum}(\mathcal{C})$ correspond to the objects of $V(\mathcal{C})$.

For a further example of maximal spectrum of a semilocal category, let R be a ring and $\mathcal{S}_R$ be the full subcategory of proj-R whose objects are the finitely generated projective R-modules with a semilocal endomorphism ring. Let us briefly describe the structure of the objects of $\mathcal{S}_R$. If A_R is a finitely generated projective module and $\mathrm{End}_R(A)$ is semilocal, then $A_R/A_RJ(R)$ is a finitely generated projective $R/J(R)$-module and $\mathrm{End}_{R/J(R)}(A/AJ(R)) \cong \mathrm{End}_R(A)/J(\mathrm{End}_R(A))$ is semisimple artinian [5, Corollary 17.12], so that $A_R/A_RJ(R)$ is a direct sum of finitely generated projective indecomposable $R/J(R)$-modules, that is, $A_R/A_RJ(R)$ is a direct sum of finitely many simple R-modules (cf. the remark after the statement of Theorem 2.1 in [4]). Let $\mathcal{H}$ be the full subcategory of Mod-R whose objects are all simple homomorphic images of finitely generated projective modules with a semilocal endomorphism ring, and let $V(\mathcal{H})$ be a skeleton of $\mathcal{H}$. Thus $A_R/A_RJ(R) \cong \oplus_{S\in V(\mathcal{H})} S^{n_S}$ for suitable integers $n_S \geq 0$, almost all zero. For every $S \in V(\mathcal{H})$, there is a unique R-submodule A_S of A_R, with $A_S \supseteq A_RJ(R)$ and $A_S/A_RJ(R)$ the S-socle of $A_R/A_RJ(R)$, that is, $A_S/A_RJ(R) \cong S^{n_S}$, so that $A_R/A_RJ(R) = \oplus_{S\in V(\mathcal{H})} A_S/A_RJ(R)$. Now $\mathrm{End}_R(A)/J(\mathrm{End}_R(A)) \cong \mathrm{End}_{R/J(R)}(A/AJ(R)) \cong \prod_{S\in V(\mathcal{H})} \mathrm{End}_R(A_S/AJ(R))$, where each $\mathrm{End}_R(A_S/AJ(R))$ is the zero ring if $n_S = 0$ and is a simple artinian ring otherwise. It follows that the maximal ideals of $\mathrm{End}_R(A)$ are the ideals $M_S = \{\, f \in \mathrm{End}_R(A) \mid f(A_S) \subseteq A_RJ(R)\,\}$, where S ranges in the objects of $V(\mathcal{H})$ with $n_S > 0$.

Theorem 4.3.7 *Let R be a ring, $\mathcal{S}_R$ be the full subcategory of* proj-R *whose objects are the finitely generated projective R-modules with semilocal endomorphism rings, and $\mathcal{H}$ be the full subcategory of* Mod-R *whose objects are all simple homomorphic images of finitely generated projective modules with a semilocal endomorphism ring. Then there is a bijection* $\mathrm{Max}(\mathcal{S}_R) \to V(\mathcal{H})$.

Proof. We use the notation introduced in the paragraph immediately before the statement of the Theorem. Let $(A_R, M) \in \mathrm{Max}(\mathcal{S}_R)$, so that M is a maximal ideal of the semilocal endomorphism ring of the finitely generated projective module A_R. Define a correspondence $\Phi\colon \mathrm{Max}(\mathcal{S}_R) \to V(\mathcal{H})$ associating to (A_R, M) the unique $S \in V(\mathcal{H})$ with $n_S > 0$ and $M = M_S$.

In order to show that the correspondence Φ is onto, fix $S \in V(\mathcal{H})$. The module S is a homomorphic image of an object A_R of $\mathcal{S}_R$. Let (A'_R, M') be the element of $\mathrm{Max}(\mathcal{S}_R)$ with $(A'_R, M') \sim (A_R, M_S)$. We will prove that $\Phi(A'_R, M') = S$. For this, we must show that $M' = \{\, f \in \mathrm{End}_R(A') \mid f(A'_S) \subseteq A'_RJ(R)\,\}$. Let N be the ideal in the right term of this equality. We know that N is always either a maximal ideal or the improper ideal of $\mathrm{End}_R(A')$. Hence it suffices to prove that $N \subseteq M'$. By Proposition 4.1.3((1) $\Leftrightarrow$ (3)), the condition $(A'_R, M') \sim (A_R, M_S)$ implies that $M' = \mathcal{A}_{M_S}(A'_R, A'_R) = \{\, f \in \mathrm{End}_R(A') \mid \beta f \alpha \in M_S$ for every $\alpha\colon A_R \to A'_R$ and $\beta\colon A'_R \to A_R\,\} = \{\, f \in \mathrm{End}_R(A') \mid \beta f \alpha(A_S) \subseteq$

$A_R J(R)$ for every $\alpha\colon A_R \to A'_R$ and $\beta\colon A'_R \to A_R\}$. To show that $N \subseteq M'$, assume $f \in N$, $\alpha\colon A_R \to A'_R$ and $\beta\colon A'_R \to A_R$. Then $\alpha(A_S) \subseteq A'_S$, so that $\beta f\alpha(A_S) \subseteq \beta f(A'_S) \subseteq \beta(A'_R J(R)) \subseteq A_R J(R)$. This proves that $N \subseteq M'$, and Φ is onto.

In order to prove that Φ is injective, fix two finitely generated projective modules A_R and A'_R with semilocal endomorphism rings, let M and M' be two maximal ideals of $\mathrm{End}_R(A)$ and $\mathrm{End}_R(A')$ respectively, and suppose $\Phi(A_R, M) = \Phi(A'_R, M')$. Then there exists $S \in V(\mathcal{H})$ such that $M = M_S = \{ f \in \mathrm{End}_R(A) \mid f(A_S) \subseteq A_R J(R) \}$ and $M' = M'_S = \{ f' \in \mathrm{End}_R(A') \mid f'(A'_S) \subseteq A'_R J(R) \}$. We must prove that $\mathcal{A}_M = \mathcal{A}_{M'}$. By Propositions 4.1.3 and 4.3.3, it suffices to show that $\mathcal{A}_M(A'_R, A'_R) = \{ f' \in \mathrm{End}_R(A') \mid \beta f'\alpha(A_S) \subseteq A_R J(R)$ for every $\alpha\colon A_R \to A'_R, \beta\colon A'_R \to A_R \}$ is contained in M'. Let f' be a morphism in $\mathrm{End}_R(A') \setminus M'$. Then $f'(A'_S) \nsubseteq A'_R J(R)$, so that f' induces a non-zero endomorphism $\overline{f'}$ of $A'_S/A'_R J(R)$. Now $A_S/A_R J(R)$ and $A'_S/A'_R J(R)$ are direct sums of a non-zero finite number of copies of S, so that there exist morphisms $\overline{\alpha}\colon A_S/A_R J(R) \to A'_S/A'_R J(R)$ and $\overline{\beta}\colon A'_S/A'_R J(R) \to A_S/A_R J(R)$ with $\overline{\beta f'\alpha} \neq 0$. As A_R and A'_R are projective, $\overline{\alpha}$ and $\overline{\beta}$ can be lifted to morphisms $\alpha\colon A_R \to A'_R$ and $\beta\colon A'_R \to A_R$, so that $\beta f'\alpha(A_S) \nsubseteq A_R J(R)$. Thus $f' \notin \mathcal{A}_M(A'_R, A'_R)$. ■

Let $\mathcal{C}_\lambda$ be a preadditive category for every index λ ranging in a class Λ. We define the *weak direct sum* $\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$ of the categories $\mathcal{C}_\lambda$ as follows. The objects of $\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$ are the finite sets $\{ (\lambda_1, A_1), (\lambda_2, A_2), \dots, (\lambda_n, A_n) \}$, where $n \geq 0$ is an integer, $\lambda_1, \dots, \lambda_n$ are distinct elements of Λ and A_i is a non-zero object of $\mathcal{C}_{\lambda_i}$ for every $i = 1, 2, \dots, n$. The set of all morphisms between two objects $\{ (\lambda_1, A_1), (\lambda_2, A_2), \dots, (\lambda_n, A_n) \}$ and $\{ (\mu_1, B_1), (\mu_2, B_2), \dots, (\mu_m, B_m) \}$ of the category $\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$ is

$$\bigoplus_{\substack{i=1,\dots,n\\ j=1,\dots,m\\ \lambda_i=\mu_j}} \mathrm{Hom}_{\mathcal{C}_{\lambda_i}}(A_i, B_j).$$

The following remarks are probably redundant, but help clarify the notion of weak direct sum, which we have just introduced.

(a) The weak direct sum $\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$ of preadditive categories $\mathcal{C}_\lambda$ is a preadditive category.

(b) The category $\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$ has always a unique zero object, given by the empty set.

(c) Any two objects

$$\{ (\lambda_1, A_1), (\lambda_2, A_2), \dots, (\lambda_n, A_n) \} \quad \text{and} \quad \{ (\mu_1, B_1), (\mu_2, B_2), \dots, (\mu_m, B_m) \}$$

of $\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$ are isomorphic if and only if $\{ \lambda_1, \dots, \lambda_n \} = \{ \mu_1, \dots, \mu_m \}$ and $A_i \cong B_j$ in $\mathcal{C}_{\lambda_i}$ for every i and j with $\lambda_i = \mu_j$.

(d) Suppose that the class Λ is a set, so that we can construct the product category $\prod_{\lambda\in\Lambda}\mathcal{C}_\lambda$. Assume that all the preadditive categories $\mathcal{C}_\lambda$ have a zero object. Then the weak direct sum $\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$ is equivalent to the full subcategory of the product category $\prod_{\lambda\in\Lambda}\mathcal{C}_\lambda$ whose objects are the sequences $(A_\lambda)_{\lambda\in\Lambda}$ of objects $A_\lambda \in \mathrm{Ob}(\mathcal{C}_\lambda)$ with A_λ a zero object of $\mathcal{C}_\lambda$ for almost all $\lambda \in \Lambda$.

In (e) and (f) below, for any preadditive category $\mathcal{C}$, let $\mathcal{C}^+$ denote the category obtained from $\mathcal{C}$ adjoining a further zero object (so that $\mathcal{C}^+$ and $\mathcal{C}$ are equivalent if $\mathcal{C}$ already has a zero object), and let $\mathcal{C}^*$ denote the full subcategory of $\mathcal{C}$ whose objects are all non-zero objects of $\mathcal{C}$.

(e) Let M be a subclass of Λ. Then the categories $\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$ and $(\oplus_{\lambda\in M}\mathcal{C}^*_\lambda)\oplus(\oplus_{\lambda\in\Lambda\setminus M}\mathcal{C}_\lambda)$ are isomorphic.

(f) Let $\mathcal{C}_1,\dots,\mathcal{C}_n$ be $n\geq 1$ preadditive categories. Then the category $\oplus_{i=1}^n\mathcal{C}_i$ is isomorphic to the category $(\prod_{i=1}^n\mathcal{C}^*_i)^+$.

For every $\lambda_0\in\Lambda$, there is a canonical functor $E_{\lambda_0}\colon\mathcal{C}_{\lambda_0}\to\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$, which is full and faithful. Moreover, let $\mathcal{D}$ be an additive category with a zero object and $G_\lambda\colon\mathcal{C}_\lambda\to\mathcal{D}$ be an additive functor for every $\lambda\in\Lambda$. Then there exists an additive functor $G\colon\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda\to\mathcal{D}$ such that GE_λ is naturally isomorphic to G_λ for every $\lambda\in\Lambda$. The additive functor G with this property is unique up to natural isomorphism.

Let $\mathcal{D}$ be a preadditive category and $F_\lambda\colon\mathcal{D}\to\mathcal{C}_\lambda$, where λ ranges in the class Λ, be an additive functor. Assume that, for every object $A\in\mathrm{Ob}(\mathcal{D})$, the object $F_\lambda(A)$ is a zero object of $\mathcal{C}_\lambda$ for almost all $\lambda\in\Lambda$. Define an additive functor $F\colon\mathcal{D}\to\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$ in the following way. Consider the composite functors $E_\lambda F_\lambda\colon\mathcal{D}\to\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$, $\lambda\in\Lambda$. Let A be an object of $\mathcal{D}$ and let $\lambda_1,\dots,\lambda_n$ be the elements $\lambda\in\Lambda$ such that $F_\lambda(A)$ is a non-zero object of $\mathcal{C}_\lambda$. Let $F(A)$ be the coproduct in $\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$ of the objects $E_{\lambda_1}F_{\lambda_1}(A),\dots,E_{\lambda_n}F_{\lambda_n}(A)$. Now let $f\colon A\to B$ be a morphism in $\mathcal{D}$ and let $\mu_1,\dots,\mu_m$ be the elements $\mu\in\Lambda$ such that $F_\mu(B)$ is a non-zero object of $\mathcal{C}_\mu$. Then F maps f to the $m\times n$ matrix having (i,j)-entry equal to $E_{\mu_i}F_{\mu_i}(f)$ for $\mu_i=\lambda_j$, and all the other entries equal to zero.

We say that the functor $F\colon\mathcal{D}\to\oplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$ is induced by the collection of functors $F_\lambda\colon\mathcal{D}\to\mathcal{C}_\lambda$, $\lambda\in\Lambda$.

Theorem 4.3.8 *Let $\mathcal{C}$ be a semilocal category. Then:*

1. *the Jacobson radical of $\mathcal{C}$ is the intersection of all maximal ideals of $\mathcal{C}$ and, for every object A in $\mathcal{C}$, there exist finitely many maximal ideals $\mathcal{M}_1,\dots,\mathcal{M}_n$ $(n\geq 0)$ such that, for every maximal ideal $\mathcal{M}$ in $\mathcal{C}$, A is a non-zero object in $\mathcal{C}/\mathcal{M}$ if and only if $\mathcal{M}=\mathcal{M}_i$ for some $i\in\{1,\dots,n\}$;*

2. *the functor $F\colon\mathcal{C}\to\oplus_{\mathcal{M}\in\mathrm{Max}(\mathcal{C})}\mathcal{C}/\mathcal{M}$, induced by the collection of canonical functors $\mathcal{C}\to\mathcal{C}/\mathcal{M}$, $\mathcal{M}\in\mathrm{Max}(\mathcal{C})$, is isomorphism reflecting;*

3. *if $\mathcal{C}$ is additive with splitting idempotents, the functor F is direct-summand reflecting.*

Proof.

1. Let A,B be objects of $\mathcal{C}$. Let f be a morphism in $\mathcal{J}(A,B)$. Let $\mathcal{M}$ be a maximal ideal of $\mathcal{C}$. We want to prove that $f\in\mathcal{M}(A,B)$. By Lemma 4.1.4,

either $\mathcal{M}(A,A) = \mathrm{End}_{\mathcal{C}}(A)$ or $\mathcal{M}(A,A)$ is a maximal ideal of $\mathrm{End}_{\mathcal{C}}(A)$ and $\mathcal{M}$ is the ideal of $\mathcal{C}$ associated to $\mathcal{M}(A,A)$. If $\mathcal{M}(A,A) = \mathrm{End}_{\mathcal{C}}(A)$, then $\mathcal{M}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$, so that $f \in \mathcal{M}(A,B)$ and we are done. Hence we can assume that $M = \mathcal{M}(A,A)$ is a maximal ideal of $\mathrm{End}_{\mathcal{C}}(A)$ and $\mathcal{M}$ is the ideal of $\mathcal{C}$ associated to M. Now $f \in \mathcal{J}(A,B)$, so that $\beta f \alpha \in \mathcal{J}(A,A) = J(\mathrm{End}_{\mathcal{C}}(A)) \subseteq M$ for every $\alpha\colon A \to A$ and every $\beta\colon B \to A$. Thus f is in the maximal ideal $\mathcal{M}$ associated to M.

Conversely, assume that $f \in \mathcal{M}(A,B)$ for every maximal ideal $\mathcal{M}$ of $\mathcal{C}$. If A is a zero object in $\mathcal{C}$, then $f = 0 \in \mathcal{J}(A,B)$. If A is not a zero object in $\mathcal{C}$, let M be any maximal ideal of the ring $\mathrm{End}_{\mathcal{C}}(A)$. Let $\mathcal{A}_M$ be the ideal in $\mathcal{C}$ associated to M, so that $\mathcal{A}_M$ is maximal by Proposition 4.3.3(1). Now $f \in \mathcal{A}_M(A,B)$ implies that $gf \in \mathcal{A}_M(A,A) = M$ for every $g\colon B \to A$. Since this is true for every maximal ideal M of $\mathrm{End}_{\mathcal{C}}(A)$, which is a semilocal ring, it follows that $gf \in \mathcal{J}(A,A)$ for every $g\colon B \to A$. Thus $1_A - gf$ has a left inverse, and $f \in \mathcal{J}(A,B)$. Thus $\mathcal{J}$ is the intersection of all the maximal ideals of $\mathcal{C}$.

Now let A be an object of $\mathcal{C}$. If $A = 0$ in $\mathcal{C}$, then $A = 0$ in $\mathcal{C}/\mathcal{M}$ for all maximal ideals $\mathcal{M}$. Assume A non-zero. Then $\mathrm{End}_{\mathcal{C}}(A)$ is a semilocal ring. Let $M_1, \dots, M_n$ be the maximal ideals of $\mathrm{End}_{\mathcal{C}}(A)$, and $\mathcal{A}_{M_1}, \dots, \mathcal{A}_{M_n}$ be the corresponding associated ideals. Let $\mathcal{M}$ be a maximal ideal of $\mathcal{C}$ different from $\mathcal{A}_{M_i}$ for every $i = 1, \dots, n$. As $\mathcal{M}$ is not associated to $M_1, \dots, M_n$, we get that $\mathcal{M}(A,A) = \mathrm{End}_{\mathcal{C}}(A)$ by Lemma 4.1.4. Thus A is the zero object in $\mathcal{C}/\mathcal{M}$.

This concludes the proof of (1). Notice that (1) allows us to say that the collection of canonical functors $\mathcal{C} \to \mathcal{C}/\mathcal{M}$, $\mathcal{M} \in \mathrm{Max}(\mathcal{C})$ induces a functor F into the weak direct sum category. It sends the object A of $\mathcal{C}$ into the finite set $\{(\mathcal{M}_1, A_1), \dots, (\mathcal{M}_n, A_n)\}$, where A_i is the image of A in $\mathcal{C}/\mathcal{M}_i$.

2. Let B be a non-zero object of $\mathcal{C}$. By (1), there are only finitely many maximal ideals $\mathcal{M}$ with B non-zero in $\mathcal{C}/\mathcal{M}$. Let $\mathcal{M}_1, \dots, \mathcal{M}_n$ be these finitely many distinct maximal ideals. Thus $\mathcal{M}_i(B,B) \neq \mathrm{End}_{\mathcal{C}}(B)$, so that $\mathcal{M}_i$ is the ideal associated to the maximal ideal $\mathcal{M}_i(B,B)$ of $\mathrm{End}_{\mathcal{C}}(B)$ by Lemma 4.1.4. By Proposition 4.3.3(1), the maximal ideals of the ring $\mathrm{End}_{\mathcal{C}}(B)$ are exactly the n ideals $\mathcal{M}_i(B,B)$. Since $\mathrm{End}_{\mathcal{C}}(B)$ is semilocal, we have a canonical isomorphism

$$\mathrm{End}_{\mathcal{C}}(B)/J(\mathrm{End}_{\mathcal{C}}(B)) \cong \prod_{i=1}^{n} \mathrm{End}_{\mathcal{C}}(B)/\mathcal{M}_i(B,B).$$

Hence there exists, for every $i = 1, \dots, n$, a $\delta_i \in \mathrm{End}_{\mathcal{C}}(B)$ such that $\delta_i \equiv 1_B \pmod{\mathcal{M}_i(B,B)}$, $\delta_i \equiv 0_B \pmod{\mathcal{M}_j(B,B)}$ for every $j \neq i$, and $\delta_i\delta_j \in J(\mathrm{End}_{\mathcal{C}}(B))$ for every $i \neq j$.

In order to prove (2), let A be an object of $\mathcal{C}$ with $A \cong B$ in $\mathcal{C}/\mathcal{M}$ for every maximal ideal $\mathcal{M}$ of $\mathcal{C}$. Then A also is non-zero in $\mathcal{C}$, otherwise $B = 0$ in $\mathcal{C}/\mathcal{M}$ for every maximal ideal $\mathcal{M}$ of $\mathcal{C}$, so that $B = 0$ in $\mathcal{C}/\mathcal{J}$, hence $B = 0$ in $\mathcal{C}$, contradiction. Thus A and B are both non-zero in $\mathcal{C}$. Moreover, for every $\mathcal{M}$, $A = 0$ in $\mathcal{C}/\mathcal{M}$ if and only if $B = 0$ in $\mathcal{C}/\mathcal{M}$. Hence we can apply the argument of the previous paragraph to A also, and we find that there exist endomorphisms $\delta'_1, \dots, \delta'_n \in \mathrm{End}_{\mathcal{C}}(A)$ such that $\delta'_i \equiv 1_A \pmod{\mathcal{M}_i(A,A)}$, $\delta'_i \equiv 0_A \pmod{\mathcal{M}_j(A,A)}$ for $j \neq i$, and $\delta'_i\delta'_j \in J(\mathrm{End}_{\mathcal{C}}(A))$ for $i \neq j$.

For every $i = 1, \dots, n$, let $f_i \colon A \to B$ be a morphism in $\mathcal{C}$ that becomes an isomorphism in $\mathcal{C}/\mathcal{M}_i$ and $g_i \colon B \to A$ be a morphism that lifts to $\mathcal{C}$ the inverse of f_i in $\mathcal{C}/\mathcal{M}_i$. Set $f = \sum_{i=1}^n \delta'_i f_i \delta_i$ and $g = \sum_{i=1}^n \delta_i g_i \delta'_i$. Then

$$gf = \sum_{i,j} \delta_i g_i \delta'_i \delta'_j f_j \delta_j \equiv \delta_i g_i \delta'_i f_i \delta_i \equiv g_i f_i \equiv 1_A \pmod{\mathcal{M}_i}$$

for all i, hence modulo $\mathcal{J}$, so that f is left invertible in $\mathcal{C}$. Similarly, the composite morphism fg is invertible in $\mathcal{C}$, so that f is right invertible. Thus f is an isomorphism in $\mathcal{C}$.

3. In order to prove (3), assume $\mathcal{C}$ additive with splitting idempotents and A an object of $\mathcal{C}$ with $F(A)$ a direct summand of $F(B)$. Now we have morphisms $f_i \colon A \to B$ and $g_i \colon B \to A$ such that $g_i f_i \equiv 1_A \pmod{\mathcal{M}_i}$, $i = 1, \dots, n$. Set $f = \sum_{i=1}^n \delta_i f_i$ and $g = \sum_{i=1}^n g_i \delta_i$. Then

$$gf = \sum_{i,j} g_i \delta_i \delta_j f_j \equiv \sum_{i=1}^n g_i \delta_i f_i \pmod{\mathcal{J}},$$

so that $gf \equiv g_i f_i \pmod{\mathcal{M}_i}$, that is, $gf \equiv 1_A \pmod{\mathcal{M}_i}$ for every $i = 1, \dots, n$. As $F(A)$ is a direct summand of $F(B)$ and $\mathcal{M}(B,B) = \mathrm{End}_{\mathcal{C}}(B)$ for every maximal ideal $\mathcal{M}$ of $\mathcal{C}$ different from $\mathcal{M}_1, \dots, \mathcal{M}_n$, it follows that $\mathcal{M}(A,A) = \mathrm{End}_{\mathcal{C}}(A)$ for every maximal ideal $\mathcal{M}$ of $\mathcal{C}$ different from $\mathcal{M}_1, \dots, \mathcal{M}_n$, so that $gf \equiv 1_A \pmod{\mathcal{M}}$ for every $\mathcal{M} \in \mathrm{Max}(\mathcal{C})$. By (1), $gf \equiv 1_A \pmod{\mathcal{J}}$, so that gf is left invertible in $\mathrm{End}_{\mathcal{C}}(A)$. Thus $g'f = 1_A$ for a suitable g'. Hence A is isomorphic to a direct summand of B.

∎

This theorem does not hold omitting the hypothesis that the category be semilocal. For instance, if $\mathcal{C}$ is the category of all vector spaces of dimension $\leq \aleph_1$, then $\mathcal{C}$ has a unique maximal ideal $\mathcal{M}$ consisting of all morphisms of rank $\leq \aleph_0$, and all vector spaces of dimension $\leq \aleph_0$ turn out to be isomorphic modulo $\mathcal{M}$.

4.4 The monoid $V(\mathcal{C})$

Let M_λ be a small monoid for every index λ ranging in a class Λ. The *direct sum* of the monoids M_λ is the large monoid $\oplus_{\lambda \in \Lambda} M_\lambda$ defined as follows. Let $\oplus_{\lambda \in \Lambda} M_\lambda$ be the class having as elements the finite sets $\{ (\lambda_1, a_1), (\lambda_2, a_2), \dots, (\lambda_n, a_n) \}$, where $n \geq 0$ is an integer, $\lambda_1, \dots, \lambda_n$ are distinct elements of Λ and a_i is a non-zero element of M_{λ_i} for every $i = 1, \dots, n$. If $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m, \nu_1, \dots, \nu_p$ are distinct elements of Λ, define

$$\begin{aligned}
&\{(\lambda_1, a_1), \dots, (\lambda_n, a_n), (\mu_1, b_1), \dots, (\mu_m, b_m)\} \\
&\quad + \{(\lambda_1, c_1), \dots, (\lambda_n, c_n), (\nu_1, d_1), \dots, (\nu_p, d_p)\} \\
&= \{(\lambda_1, a_1 + c_1), \dots, (\lambda_n, a_n + c_n), (\mu_1, b_1), \dots, (\mu_m, b_m), (\nu_1, d_1), \dots, (\nu_p, d_p)\}.
\end{aligned}$$

Then $\oplus_{\lambda \in \Lambda} M_\lambda$ with this operation becomes a large commutative monoid. If every monoid M_λ coincide with a single monoid M, we denote the direct sum $\oplus_{\lambda \in \Lambda} M$ by $M^{(\Lambda)}$.

Let $\mathcal{C}$ be an additive semilocal category in which idempotents split. The direct-summand reflecting functor $F: \mathcal{C} \to \oplus_{\mathcal{M} \in \mathrm{Max}(\mathcal{C})} \mathcal{C}/\mathcal{M}$ of Theorem 4.3.8(3) induces a unique monoid homomorphism $V(F): V(\mathcal{C}) \to V(\oplus_{\mathcal{M} \in \mathrm{Max}(\mathcal{C})} \mathcal{C}/\mathcal{M})$, which is a divisor homomorphism. Moreover,

$$V(\oplus_{\mathcal{M} \in \mathrm{Max}(\mathcal{C})} \mathcal{C}/\mathcal{M}) \cong \oplus_{\mathcal{M} \in \mathrm{Max}(\mathcal{C})} V(\mathcal{C}/\mathcal{M}) \cong \mathbb{N}_0^{(\mathrm{Max}(\mathcal{C}))}$$

by Corollary 4.3.4. Thus there is a divisor homomorphism of $V(\mathcal{C})$ into a free commutative monoid, and the monoid $V(\mathcal{C})$ turns out to be a Krull monoid.

Let us show that this argument can be inverted. Let X be a set, $\mathbb{N}_0^{(X)}$ the free commutative monoid with free set X of generators and $\mathbb{Z}^{(X)}$ the free abelian group with free set X of generators. The elements of $\mathbb{N}_0^{(X)}$ will be denoted as functions $s: X \to \mathbb{N}_0$ with $s(x) = 0$ for almost all x. The *support* of an element $s \in \mathbb{N}_0^{(X)}$ is the finite set $\mathrm{supp}(s) = \{ x \in X \mid s(x) \neq 0 \}$. For every preadditive category $\mathcal{C}$ let $\mathrm{sum}(\mathcal{C})$ and $\mathcal{A}dd(\mathcal{C}) = \widehat{\mathrm{sum}(\mathcal{C})}$ denote the additive category generated by $\mathcal{C}$ and the additive category with splitting idempotents generated by $\mathcal{C}$ respectively. Notice that the maximal ideals of $\mathcal{C}$, $\mathrm{sum}(\mathcal{C})$ and $\mathcal{A}dd(\mathcal{C})$ coincide as we saw in Example 4.3.6.

Theorem 4.4.1 *Let X be a set and S be a subset of the monoid $\mathbb{N}_0^{(X)}$ such that $\bigcup_{s \in S} \mathrm{supp}(s) = X$. Let $\mathbb{N}_0 S$ be the submonoid of $\mathbb{N}_0^{(X)}$ generated by S and $\mathbb{Z}S$ be the subgroup of $\mathbb{Z}^{(X)}$ generated by S. Then there exists a preadditive category $\mathcal{C}$ such that the full and faithful embeddings $\mathcal{C} \hookrightarrow \mathrm{sum}(\mathcal{C}) \hookrightarrow \mathcal{A}dd(\mathcal{C})$ induce a commutative diagram of sets and mappings*

$$\begin{array}{ccccccc}
S & \hookrightarrow & \mathbb{N}_0 S & \hookrightarrow & \mathbb{Z}S \cap \mathbb{N}_0^{(X)} & \hookrightarrow & \mathbb{N}_0^{(X)} \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
V(\mathcal{C}) & \hookrightarrow & V(\mathrm{sum}(\mathcal{C})) & \hookrightarrow & V(\mathcal{A}dd(\mathcal{C})) & \hookrightarrow & \mathbb{N}_0^{(\mathrm{Max}(\mathcal{C}))}.
\end{array}$$

Here the vertical arrows represent bijections, and the squares in the middle and on the right are commutative squares of monoids and monoid homomorphisms.

Proof. The embedding $\mathbb{Z}S \cap \mathbb{N}_0^{(X)} \hookrightarrow \mathbb{N}_0^{(X)}$ is a divisor homomorphism, so that $\mathbb{Z}S \cap \mathbb{N}_0^{(X)}$ is a reduced Krull monoid. For any ring R, let $\mathcal{S}_R$ denote the full subcategory of Mod-R consisting of all finitely generated projective right R-modules with semilocal endomorphism rings. Let k be a field. By [26, Theorem 2.1], there exist a k-algebra R and two monoid isomorphisms $g\colon \mathbb{Z}S \cap \mathbb{N}_0^{(X)} \to V(\mathcal{S}_R)$ and $h\colon \mathbb{N}_0^{(X)} \to V(\mathcal{S}_{R/J(R)})$ such that if $\tau\colon V(\mathcal{S}_R) \to V(\mathcal{S}_{R/J(R)})$ is the homomorphism induced by the natural surjection $\pi\colon R \to R/J(R)$, then the diagram of monoids and monoid homomorphisms

$$\begin{array}{ccc} \mathbb{Z}S \cap \mathbb{N}_0^{(X)} & \hookrightarrow & \mathbb{N}_0^{(X)} \\ {\scriptstyle g}\downarrow{\scriptstyle \cong} & & {\scriptstyle h}\downarrow{\scriptstyle \cong} \\ V(\mathcal{S}_R) & \xrightarrow{\tau} & V(\mathcal{S}_{R/J(R)}) \end{array}$$

commutes.

We claim that $V(\mathcal{S}_{R/J(R)}) \cong \mathbb{N}_0^{(\mathrm{Max}(\mathcal{S}_R))}$. In order to prove the claim, notice that $V(\mathcal{S}_{R/J(R)}) \cong \mathbb{N}_0^{(X)}$. Hence it suffices to show that there is a one-to-one correspondence between $\mathrm{Max}(\mathcal{S}_R)$ and the class of atoms of $V(\mathcal{S}_{R/J(R)})$. By Theorem 4.3.7, $\mathrm{Max}(\mathcal{S}_R) \cong V(\mathcal{H})$ (notation as in Theorem 4.3.7). By [26, Proposition 2.5], the class of atoms of $V(\mathcal{S}_{R/J(R)})$ consists of a class of representatives of the simple projective $R/J(R)$-modules. Thus it suffices to prove that a simple R-module M is a homomorphic image of a finitely generated projective R-module with a semilocal endomorphism ring if and only if it is a projective $R/J(R)$-module. Now if the simple R-module M is a homomorphic image of a finitely generated projective R-module A_R with $\mathrm{End}_R(A)$ semilocal, then $M \otimes R/J(R) \cong M$ is a homomorphic image of $A_R \otimes R/J(R) \cong A_R/A_RJ(R)$, which is a direct sum of finitely many simple modules, as we have remarked in the paragraph before the statement of Theorem 4.7. Hence, since $A_R/A_RJ(R)$ is a projective $R/J(R)$-module, the $R/J(R)$-module M also is projective. Conversely, if M is a simple projective $R/J(R)$-module, then M corresponds to an element $x \in X$ via the isomorphism h, so that $x \in \mathrm{supp}(s)$ for some $s \in S$. This element s corresponds to a projective module $A_R \in V(\mathcal{S}_R)$ via g and M is a homomorphic image of A_R. Thus $M \in \mathcal{H}$. This concludes the proof of the claim.

Define the category $\mathcal{C}$ as the full subcategory of $\mathcal{S}_R$ whose class of objects is $g(S)$. Then $\mathcal{A}dd(\mathcal{C})$ is equivalent to $\mathcal{S}_R$ and there is a monoid isomorphism $\mathbb{N}_0S \cong V(\mathrm{sum}(\mathcal{C}))$. ■

4.5 Comparing ideals of endomorphism rings of distinct objects

This section is devoted to comparing the ideals of the endomorphism ring of two objects A and B of a preadditive category $\mathcal{C}$. The most natural way of associating to every ideal of $\mathrm{End}_{\mathcal{C}}(A)$ an ideal of $\mathrm{End}_{\mathcal{C}}(B)$ is to associate to the ideal I of $\mathrm{End}_{\mathcal{C}}(A)$ the ideal $\mathcal{A}_I(B,B)$ of $\mathrm{End}_{\mathcal{C}}(B)$, where $\mathcal{A}_I$ denotes the ideal in the category $\mathcal{C}$ associated to I. Similarly, we can associate to each ideal K of $\mathrm{End}_{\mathcal{C}}(B)$ the ideal $\mathcal{A}_K(A,A)$ of $\mathrm{End}_{\mathcal{C}}(A)$. We will compare the ideals of $\mathrm{End}_{\mathcal{C}}(A)$ and those of $\mathrm{End}_{\mathcal{C}}(B)$ via these correspondences. The lattice of all two-sided ideals of the endomorphism ring $\mathrm{End}_{\mathcal{C}}(A)$ of an object A will be denoted by $\mathcal{L}(\mathrm{End}_{\mathcal{C}}(A))$.

Lemma 4.5.1 *The following conditions are equivalent for two non-zero objects A and B of a preadditive category $\mathcal{C}$:*

1. *the mappings*
$$\alpha: \mathcal{L}(\mathrm{End}_{\mathcal{C}}(A)) \to \mathcal{L}(\mathrm{End}_{\mathcal{C}}(B)),$$
$$I \in \mathcal{L}(\mathrm{End}_{\mathcal{C}}(A)) \mapsto \mathcal{A}_I(B,B) \in \mathcal{L}(\mathrm{End}_{\mathcal{C}}(B)),$$
and
$$\beta: \mathcal{L}(\mathrm{End}_{\mathcal{C}}(B)) \to \mathcal{L}(\mathrm{End}_{\mathcal{C}}(A)),$$
$$K \in \mathcal{L}(\mathrm{End}_{\mathcal{C}}(B)) \mapsto \mathcal{A}_K(A,A) \in \mathcal{L}(\mathrm{End}_{\mathcal{C}}(A)),$$
where $\mathcal{A}_I$ and $\mathcal{A}_K$ are the ideals associated to I and K respectively, are such that $\alpha\beta = 1_{\mathcal{L}(\mathrm{End}_{\mathcal{C}}(B))}$;

2. $\mathrm{Hom}_{\mathcal{C}}(A,B)\mathrm{Hom}_{\mathcal{C}}(B,A) = \mathrm{End}_{\mathcal{C}}(B)$*;*

3. $\mathrm{add}(B) \subseteq \mathrm{add}(A)$*.*

Moreover, if the category $\mathcal{C}$ is additive with splitting idempotents, the previous conditions are also equivalent to:

4. *there exists a non-negative integer n such that B is isomorphic to a direct summand of A^n.*

Proof. (1) ⇒ (2) Set $K = \mathrm{Hom}_{\mathcal{C}}(A,B)\mathrm{Hom}_{\mathcal{C}}(B,A)$. The correspondence β sends K to $\beta(K) = \mathrm{End}_{\mathcal{C}}(A)$ and α sends $\mathrm{End}_{\mathcal{C}}(A)$ to $\alpha(\mathrm{End}_{\mathcal{C}}(A)) = \mathrm{End}_{\mathcal{C}}(B)$. Thus (1) implies that $\mathrm{Hom}_{\mathcal{C}}(A,B)\mathrm{Hom}_{\mathcal{C}}(B,A) = \mathrm{End}_{\mathcal{C}}(B)$.

(2) ⇒ (3) Assume $D \in \mathrm{add}(B)$, so that there exist morphisms $f_1,\dots,f_n: B \to D$ and $g_1,\dots,g_n: D \to B$ with $1_D = \sum_{i=1}^n f_i g_i$. If (2) holds, there exist morphisms $h_1,\dots,h_m: A \to B$ and $l_1,\dots,l_m: B \to A$ such that $1_B = \sum_{j=1}^m h_j l_j$. Hence $1_D = \sum_{i=1}^n f_i 1_B g_i = \sum_{i,j} f_i h_j l_j g_i$. Therefore $D \in \mathrm{add}(A)$.

(3) ⇒ (1) The composition $\alpha\beta$ sends an ideal $K \in \mathcal{L}(\mathrm{End}_{\mathcal{C}}(B))$ to the ideal $\alpha\beta(K) = \{\, g \in \mathrm{End}_{\mathcal{C}}(B) \mid \alpha\delta g\gamma\beta \in K$ for every $\alpha: A \to B, \gamma: A \to B, \beta: B \to A, \delta: B \to A \,\}$. Obviously, $K \subseteq \alpha\beta(K)$. If (3) holds, then $1_B = \sum_{i=1}^n f_i h_i$ for suitable $f_i \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ and $h_i \in \mathrm{Hom}_{\mathcal{C}}(B,A)$. Therefore $g \in \alpha\beta(K)$ implies $g = \sum_{i,j} f_i h_i g f_j h_j \in K$.

(2) ⇔ (4) is trivial. ∎

Let A and B be non-zero objects of a preadditive category $\mathcal{C}$. Consider the bimodules

$$ {}_{\mathrm{End}_{\mathcal{C}}(B)}P_{\mathrm{End}_{\mathcal{C}}(A)} = \mathrm{Hom}_{\mathcal{C}}(A,B) \quad \text{and} \quad {}_{\mathrm{End}_{\mathcal{C}}(A)}Q_{\mathrm{End}_{\mathcal{C}}(B)} = \mathrm{Hom}_{\mathcal{C}}(B,A)$$

and the bimodule homomorphisms $\theta: P \otimes Q \to \mathrm{End}_{\mathcal{C}}(B)$, defined by $\theta(f \otimes g) = fg$, and $\phi: Q \otimes P \to \mathrm{End}_{\mathcal{C}}(A)$, defined by $\phi(g \otimes f) = gf$ for every $f \in P$ and $g \in Q$. Since $\theta(f \otimes g)f' = f\phi(g \otimes f')$ and $g\theta(f \otimes g') = \phi(g \otimes f)g'$ for $f, f' \in P, g, g' \in Q$, the couple (θ,ϕ) defines a Morita pair for (P,Q) [5, Exercise 22.5].

When θ and ϕ are both epic, the rings $\mathrm{End}_{\mathcal{C}}(A)$ and $\mathrm{End}_{\mathcal{C}}(B)$ are Morita equivalent [5, Exercise 22.7].

Theorem 4.5.2 *The following conditions are equivalent for two non-zero objects A and B of a preadditive category $\mathcal{C}$:*

1. *the mappings*

$$\begin{aligned}\alpha\colon \mathcal{L}(\mathrm{End}_{\mathcal{C}}(A)) &\to \mathcal{L}(\mathrm{End}_{\mathcal{C}}(B)),\\ I \in \mathcal{L}(\mathrm{End}_{\mathcal{C}}(A)) &\mapsto \mathcal{A}_I(B,B) \in \mathcal{L}(\mathrm{End}_{\mathcal{C}}(B)),\end{aligned}$$

and

$$\begin{aligned}\beta\colon \mathcal{L}(\mathrm{End}_{\mathcal{C}}(B)) &\to \mathcal{L}(\mathrm{End}_{\mathcal{C}}(A)),\\ K \in \mathcal{L}(\mathrm{End}_{\mathcal{C}}(B)) &\mapsto \mathcal{A}_K(A,A) \in \mathcal{L}(\mathrm{End}_{\mathcal{C}}(A)),\end{aligned}$$

are mutually inverse one-to-one correspondences;

2. *the ideal* $\mathrm{Hom}_{\mathcal{C}}(A,B)\mathrm{Hom}_{\mathcal{C}}(B,A)$ *equals the whole ring* $\mathrm{End}_{\mathcal{C}}(B)$ *and similarly* $\mathrm{Hom}_{\mathcal{C}}(B,A)\mathrm{Hom}_{\mathcal{C}}(A,B) = \mathrm{End}_{\mathcal{C}}(A)$*;*

3. $\mathrm{add}(A) = \mathrm{add}(B)$*;*

4. *in the Morita pair (θ, ϕ) for the bimodules*

$${}_{\mathrm{End}_{\mathcal{C}}(B)}P_{\mathrm{End}_{\mathcal{C}}(A)} = \mathrm{Hom}_{\mathcal{C}}(A,B) \quad \textit{and} \quad {}_{\mathrm{End}_{\mathcal{C}}(A)}Q_{\mathrm{End}_{\mathcal{C}}(B)} = \mathrm{Hom}_{\mathcal{C}}(B,A),$$

both θ and ϕ are epic.

Moreover, if the category $\mathcal{C}$ is additive with splitting idempotents, the previous are equivalent also to:

5. *there exist two non-negative integers n and m such that A is isomorphic to a direct summand of B^n and B is isomorphic to a direct summand of A^m.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) follow immediately from Lemma 4.5.1. (2) $\Leftrightarrow$ (4) is obvious. ■

Remark 4.5.3 Condition (4) is strictly stronger than the condition "The rings $\mathrm{End}_{\mathcal{C}}(A)$ and $\mathrm{End}_{\mathcal{C}}(B)$ are Morita equivalent." For instance, it is easy to construct examples of abelian groups G that are not free, but whose endomorphism ring is isomorphic to $\mathbb{Z}$. The simplest example is probably the subgroup G of $\mathbb{Q}$ generated by all p^{-1}, where p ranges in the set of all prime numbers. The group G contains $\mathbb{Z}$. As G is torsion-free of rank 1, its endomorphism ring $\mathrm{End}_{\mathbb{Z}}(G)$ is a subring of $\mathbb{Q}$, that is, consists of multiplications by rational numbers q. More precisely, $\mathrm{End}_{\mathbb{Z}}(G)$ consists of all $q \in \mathbb{Q}$ with $qp^{-1} \in G$. Thus $\mathrm{End}_{\mathbb{Z}}(G) = \{\, q \in \mathbb{Q} \mid q \in \bigcap_p pG \,\}$. Now $G/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}/p\mathbb{Z}$, so that $\bigcap_p pG \subseteq \mathbb{Z}$. It follows that $\mathrm{End}_{\mathbb{Z}}(G) \cong \mathbb{Z}$. In particular, $\mathrm{End}_{\mathbb{Z}}(G)$ and $\mathbb{Z}$ are Morita equivalent, but $\mathrm{add}(G_{\mathbb{Z}}) \neq \mathrm{add}(\mathbb{Z}_{\mathbb{Z}})$.

4.6 Trace ideals

Let $\mathcal{C}$ be a preadditive category. For any subclass $\mathcal{U}$ of $\mathrm{Ob}(\mathcal{C})$, define the *trace* $\mathrm{Tr}(\mathcal{U})$ *of* $\mathcal{U}$ *in* $\mathcal{C}$ as the ideal of $\mathcal{C}$ given by

$$\mathrm{Tr}(\mathcal{U})(B,C) = \sum_{A\in\mathcal{U}} \mathrm{Hom}_{\mathcal{C}}(A,C)\mathrm{Hom}_{\mathcal{C}}(B,A)$$

for any pair B,C of objects of $\mathcal{C}$. An ideal $\mathcal{I}$ of $\mathcal{C}$ is called a *trace ideal* if it is equal to $\mathrm{Tr}(\mathcal{U})$ for some subclass $\mathcal{U}$ of $\mathrm{Ob}(\mathcal{C})$. In particular, the improper ideal is $\mathrm{Tr}(\mathrm{Ob}(\mathcal{C}))$ and the zero ideal is $\mathrm{Tr}(\varnothing)$, so that they are both trace ideals. By a *maximal trace ideal*, we mean a trace ideal that is maximal in the class of all proper trace ideals. Notice that the trace of $\mathcal{U}$, when $\mathcal{U}$ consists of a unique object, has already appeared in Proposition 4.1.3, Conditions (5) and (5') (cf. the paragraph before the statement of Proposition 4.1.3) and in the previous section.

Remark 4.6.1 We call these ideals of the category $\mathcal{C}$ *trace* ideals, because when $\mathcal{C}$ is a full subcategory of Mod-R and R_R is an object of $\mathcal{C}$, the ideal $\mathrm{Tr}(\mathcal{U})(R_R,R_R)$ of $\mathrm{End}_R(R)\cong R$ is what is usually called the *trace* of the class $\mathcal{U}$ of modules. More generally, Anderson and Fuller define on Page 109 of [5] the trace of a class $\mathcal{U}$ in a module M_R as the submodule of M_R generated by $\mathcal{U}$. This corresponds to the submodule $\mathrm{Tr}(\mathcal{U})(R_R,M_R)$ of $\mathrm{Hom}_R(R,M)\cong M_R$.

Lemma 4.6.2 *For any subclass* $\mathcal{U}$ *of* $\mathrm{Ob}(\mathcal{C})$*, the trace* $\mathrm{Tr}(\mathcal{U})$ *is the smallest ideal* $\mathcal{I}$ *of* $\mathcal{C}$ *such that* $\mathcal{I}(B,B)=\mathrm{End}_{\mathcal{C}}(B)$ *for every object* $B\in\mathcal{U}$*. That is, it is the ideal of* $\mathcal{C}$ *generated by the class* $\{1_B \mid B\in\mathcal{U}\}$*.*

Proof. Let $\mathcal{U}$ be a subclass of $\mathrm{Ob}(\mathcal{C})$ and let $\mathcal{I}$ be an ideal of $\mathcal{C}$ such that $\mathcal{I}(B,B)=\mathrm{End}_{\mathcal{C}}(B)$ for every object $B\in\mathcal{U}$. If a morphism $f\colon C\to D$ is in $\mathrm{Tr}(\mathcal{U})$, then $f=f_1+\dots+f_n$, where every f_i is a morphism that factors through an object A_i of $\mathcal{U}$. Since $\mathcal{I}(A_i,A_i)=\mathrm{End}_{\mathcal{C}}(A_i)$ for every $i=1,\dots,n$, it follows that $f_i\in\mathcal{I}$ for every $i=1,\dots,n$. Hence $f\in\mathcal{I}$, and therefore $\mathrm{Tr}(\mathcal{U})\subseteq\mathcal{I}$. ■

Let $\mathcal{C}$ be a preadditive category and $\mathcal{U}$ be a subclass of $\mathrm{Ob}(\mathcal{C})$. Let $\mathrm{add}(\mathcal{U})$ denote the subclass of $\mathrm{Ob}(\mathcal{C})$ consisting of all objects $B\in\mathrm{Ob}(\mathcal{C})$ for which there exist $n\geq 0$, $A_1,\dots,A_n\in\mathcal{U}$ and morphisms $f_i\colon A_i\to B$ and $g_i\colon B\to A_i$ with $\sum_{i=1}^n f_ig_i=1_B$. Clearly, $\mathcal{U}\subseteq\mathrm{add}(\mathcal{U})$. We say that a subclass $\mathcal{U}$ of $\mathrm{Ob}(\mathcal{C})$ is *additively closed* if $\mathcal{U}=\mathrm{add}(\mathcal{U})$. The class $\mathrm{add}(\mathcal{U})$ when $\mathcal{U}$ consists of a unique object A has already appeared at the beginning of Section 4.2 and in Section 4.5.

Let $\mathcal{I}$ be an ideal of a preadditive category $\mathcal{C}$. We will denote by $\mathcal{Z}(\mathcal{I})$ the class of all objects $A\in\mathrm{Ob}(\mathcal{C})$ that become the zero object in the factor category $\mathcal{C}/\mathcal{I}$, that is, the objects $A\in\mathrm{Ob}(\mathcal{C})$ with $1_A\in\mathcal{I}(A,A)$.

Proposition 4.6.3 *Let* $\mathcal{C}$ *be a preadditive category.*

1. *If* $\mathcal{I}$ *is an ideal of* $\mathcal{C}$*, then the subclass* $\mathcal{Z}(\mathcal{I})$ *of* $\mathrm{Ob}(\mathcal{C})$ *is additively closed;*

2. $\mathcal{Z}(\mathrm{Tr}(\mathcal{U}))\supseteq\mathcal{U}$ *for every subclass* $\mathcal{U}$ *of* $\mathrm{Ob}(\mathcal{C})$*;*

3. *a subclass $\mathcal{U}$ of* $\mathrm{Ob}(\mathcal{C})$ *is additively closed if and only if* $\mathcal{Z}(\mathrm{Tr}(\mathcal{U})) = \mathcal{U}$*;*

4. $\mathrm{Tr}(\mathcal{Z}(\mathcal{I})) \subseteq \mathcal{I}$ *for every ideal $\mathcal{I}$ of $\mathcal{C}$;*

5. *an ideal $\mathcal{I}$ of $\mathcal{C}$ is a trace ideal if and only if* $\mathrm{Tr}(\mathcal{Z}(\mathcal{I})) = \mathcal{I}$*;*

6. *there is an inclusion-preserving one-to-one correspondence between the trace ideals of $\mathcal{C}$ and the additively closed subclasses of* $\mathrm{Ob}(\mathcal{C})$*.*

PROOF.

1. We must prove that $\mathrm{add}(\mathcal{Z}(\mathcal{I})) \subseteq \mathcal{Z}(\mathcal{I})$. This is easily seen.

2. is trivial.

3. The implication ($\Leftarrow$) follows from (1). In order to prove the implication ($\Rightarrow$), it suffices to prove that $\mathcal{Z}(\mathrm{Tr}(\mathcal{U})) \subseteq \mathcal{U}$ by (2). Hence, assume $\mathcal{U}$ additively closed and $A \in \mathcal{Z}(\mathrm{Tr}(\mathcal{U}))$. As $\mathrm{Tr}(\mathcal{U})$ is the ideal of $\mathcal{C}$ generated by $\{\, 1_B \mid B \in \mathcal{U}\,\}$, it follows that, for every $X, Y \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Tr}(\mathcal{U})(X,Y) = \{\, \sum_{i=1}^n f_i g_i \mid n \geq 0,\ A_1, \dots, A_n \in \mathcal{U},\ f_i\colon A_i \to Y,\ g_i\colon X \to A_i \,\}$. Thus $A \in \mathcal{Z}(\mathrm{Tr}(\mathcal{U}))$ implies $1_A \in \mathrm{Tr}(\mathcal{U})(A,A)$, so that $A \in \mathrm{add}(\mathcal{U}) = \mathcal{U}$.

4. If $\mathcal{I}$ is an ideal, then $\mathcal{Z}(\mathcal{I})$ consists of the objects A with $1_A \in \mathcal{I}(A,A)$, and the trace $\mathrm{Tr}(\mathcal{Z}(\mathcal{I}))$ is the ideal of $\mathcal{C}$ generated by these 1_A's. Hence $\mathrm{Tr}(\mathcal{Z}(\mathcal{I})) \subseteq \mathcal{I}$.

5. The implication ($\Leftarrow$) is trivial. For the reverse implication, let $\mathcal{I} = \mathrm{Tr}(\mathcal{U})$ be a trace ideal. Here $\mathcal{U}$ is a subclass of $\mathrm{Ob}(\mathcal{C})$. The correspondences $\mathrm{Tr}(-)$ and $\mathcal{Z}(-)$ are clearly inclusion-preserving, that is, $\mathcal{U} \subseteq \mathcal{V}$ implies $\mathrm{Tr}(\mathcal{U}) \subseteq \mathrm{Tr}(\mathcal{V})$ and $\mathcal{I} \subseteq \mathcal{J}$ implies $\mathcal{Z}(\mathcal{I}) \subseteq \mathcal{Z}(\mathcal{J})$. By (2), $\mathcal{Z}(\mathrm{Tr}(\mathcal{U})) \supseteq \mathcal{U}$. Hence $\mathrm{Tr}(\mathcal{Z}(\mathrm{Tr}(\mathcal{U}))) \supseteq \mathrm{Tr}(\mathcal{U})$, that is, $\mathrm{Tr}(\mathcal{Z}(\mathcal{I})) \supseteq \mathcal{I}$.

6. is now clear.

■

If $\mathcal{C}$ is any category, not necessarily preadditive, we can consider the class $V(\mathcal{C})$ of objects of a skeleton of the category $\mathcal{C}$. The class $V(\mathcal{C})$ is pre-ordered by the pre-order $\preceq$ defined, for every $A, B \in V(\mathcal{C})$, by $B \preceq A$ if there exist morphisms $f\colon A \to B$ and $g\colon B \to A$ with $fg = 1_B$. If the category $\mathcal{C}$ is additive, then $V(\mathcal{C})$ is a large monoid via the operation induced by direct sum. Hence not only has $V(\mathcal{C})$ the pre-order $\preceq$, but also the algebraic pre-order $\leq$. Clearly, $B \leq A$ implies $B \preceq A$ for every $A, B \in V(\mathcal{C})$. When the category $\mathcal{C}$ is additive and idempotents split, the two pre-orders coincide. If, moreover, $\mathcal{C}$ is directly finite, then the two equal pre-orders on $V(\mathcal{C})$ are partial orders.

Let M be any monoid. Recall that a submonoid M' of M is divisor-closed if $x, y \in M$, $x \leq y$ in M and $y \in M'$ implies $x \in M'$. Clearly, if $\mathcal{C}$ is an additive category with splitting idempotents, there is a one-to-one correspondence between additively closed subclasses of $\mathrm{Ob}(\mathcal{C})$ and divisor-closed submonoids of $V(\mathcal{C})$. The complements of the divisor-closed submonoids of a monoid M are the prime ideals of M. It follows that there is an order-reversing one-to-one correspondence between trace ideals of $\mathcal{C}$ and prime ideals of $V(\mathcal{C})$. Clearly, there is a one-to-one correspondence between maximal trace ideals of $\mathcal{C}$ and prime ideals of $V(\mathcal{C})$ of height one.

For the rest of the chapter, assume that $\mathcal{C}$ is an additive semilocal category with splitting idempotents. Then $V(\mathcal{C})$ is a reduced Krull monoid, hence a cancellative monoid. If $v: V(\mathcal{C}) \to \mathbb{N}_0$ is a valuation of $V(\mathcal{C})$ and $P_v = \{ B \in V(\mathcal{C}) \mid v(B) > 0 \}$, then P_v is a prime ideal of $V(\mathcal{C})$, and v is essential if and only if the prime ideal P_v has height one, by Lemma 3.2.2. In a Krull monoid M, every non-empty prime ideal M contains a prime ideal of height one. Thus every proper trace ideal of an additive semilocal category $\mathcal{C}$ with splitting idempotents is contained in a maximal trace ideal.

In the next Proposition, we describe the valuations associated to the maximal ideals of the category $\mathcal{C}$. If $\mathcal{M}$ is a maximal ideal of an additive semilocal category $\mathcal{C}$ with splitting idempotents, the associated valuation is $w_{\mathcal{M}}: V(\mathcal{C}) \to V(\mathcal{C}/\mathcal{M}) \cong \mathbb{N}_0$. Thus $w_{\mathcal{M}}(A)$ is the Goldie dimension of the semisimple object A of $\mathcal{C}/\mathcal{M}$. Equivalently, $w_{\mathcal{M}}(A)$ is the dual Goldie dimension of the semilocal ring $\mathrm{End}_{\mathcal{C}}(A)/\mathcal{M}(A,A)$ (Corollary 4.3.4).

Proposition 4.6.4 *Let $\mathcal{M}$ be a maximal ideal of an additive semilocal category $\mathcal{C}$ with splitting idempotents. Let $w_{\mathcal{M}}: V(\mathcal{C}) \to V(\mathcal{C}/\mathcal{M}) \cong \mathbb{N}_0$ be the valuation of the Krull monoid $V(\mathcal{C})$ induced by the canonical projection $\mathcal{C} \to \mathcal{C}/\mathcal{M}$. Let A be a non-zero object of $\mathcal{C}$ and M be a maximal ideal of the endomorphism ring of A such that the maximal ideal $\mathcal{M}$ is associated to M. Then:*

1. *the prime ideal $P_{w_{\mathcal{M}}}$ of $V(\mathcal{C})$ associated to the valuation $w_{\mathcal{M}}$ is*

$$V(\mathcal{C}) \setminus \mathcal{Z}(\mathcal{M}) = \{ B \in V(\mathcal{C}) \mid \mathrm{Hom}_{\mathcal{C}}(B,A)\mathrm{Hom}_{\mathcal{C}}(A,B) \notin M \};$$

2. *the following conditions are equivalent:*
   - (a) *the valuation $w_{\mathcal{M}}$ is essential;*
   - (b) *the prime ideal $P_{w_{\mathcal{M}}}$ is of height one;*
   - (c) *the divisor-closed submonoid*

     $$D_{\mathcal{M}} = \{ B \in V(\mathcal{C}) \mid \mathrm{Hom}_{\mathcal{C}}(B,A)\mathrm{Hom}_{\mathcal{C}}(A,B) \subseteq M \}$$

     of $V(\mathcal{C})$ is maximal among the proper divisor-closed submonoids of $V(\mathcal{C})$;
   - (d) *the additively closed subclass $\{ B \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(B,A)\mathrm{Hom}_{\mathcal{C}}(A,B) \subseteq M \}$ of $\mathrm{Ob}(\mathcal{C})$ is maximal among the additively closed proper subclasses of $\mathrm{Ob}(\mathcal{C})$.*

Proof.

1. An object B of $V(\mathcal{C})$ is in $P_{w_{\mathcal{M}}}$ if and only if $w_{\mathcal{M}}(B) > 0$, that is, if and only if B is not a zero object in $\mathcal{C}/\mathcal{M}$, i.e. if and only if $B \notin \mathcal{Z}(\mathcal{M})$. Moreover, B is not a zero object in $\mathcal{C}/\mathcal{M}$ if and only if $1_B \notin \mathcal{M}(B,B)$, that is, if and only if there exist $\alpha: A \to B$ and $\beta: B \to A$ with $\beta\alpha \notin M$, i.e. if and only if $\mathrm{Hom}_{\mathcal{C}}(B,A)\mathrm{Hom}_{\mathcal{C}}(A,B) \nsubseteq M$.

2. The equivalence of (a) and (b) is proved in Lemma 3.2.2. The equivalence of (b), (c) and (d) follows from the one-to-one correspondences between prime ideals of $V(\mathcal{C})$, divisor-closed submonoids of $V(\mathcal{C})$ and additively closed subclasses of $\mathcal{C}$.

■

In the next two results, we consider the valuations associated to the prime ideals of the Krull monoid $V(\mathcal{C})$. If P is a prime ideal of $V(\mathcal{C})$, then $\mathcal{I}_P = \mathrm{Tr}(V(\mathcal{C}) \setminus P)$ is a trace ideal of $\mathcal{C}$, and we can define a valuation $v_P: V(\mathcal{C}) \to \mathbb{N}_0$ by $v_P(A) = \mathrm{codim}(\mathrm{End}_{\mathcal{C}}(A)/\mathcal{I}_P(A,A))$ for every $A \in V(\mathcal{C})$. Recall that two valuations v, v' of a monoid M are said to be equivalent if $e(v)^{-1}v = e(v')^{-1}v'$. For a reduced Krull monoid M, there is a one-to-one correspondence between prime ideals of height one and essential valuations modulo equivalence.

Proposition 4.6.5 *Let $\mathcal{C}$ be an additive semilocal category with splitting idempotents.*

1. *If P is a prime ideal of $V(\mathcal{C})$, then $P_{v_P} = P$. In particular, P is of height one if and only if the valuation v_P is essential.*

2. *If $v: V(\mathcal{C}) \to \mathbb{N}_0$ is any essential valuation, then P_v is a prime ideal of height one and v_{P_v} is equivalent to v.*

Proof.

1. Let P be a prime ideal of $V(\mathcal{C})$. The prime ideal P_{v_P} consists of all the objects $A \in V(\mathcal{C})$ such that $\mathcal{I}_P(A,A) \neq \mathrm{End}_{\mathcal{C}}(A)$, i.e. P_{v_P} consists of the elements of $V(\mathcal{C}) \setminus \mathcal{Z}(\mathcal{I}_P)$. Thus $P_{v_P} = P$. The second part follows by Lemma 3.2.2.

2. follows from (1) and Lemma 3.2.2.

■

Thus, if $\mathcal{C}$ is an additive semilocal category with splitting idempotents, the monoid $V(\mathcal{C})$ is a Krull monoid, and therefore we can equivalently study essential valuations of $V(\mathcal{C})$, prime ideals of height one in $V(\mathcal{C})$ or maximal trace ideals of $\mathcal{C}$. In this correspondence, to every maximal trace ideal $\mathcal{I}$, we can associate an essential valuation $w_{\mathcal{I}}$ of $V(\mathcal{C})$ defined by $w_{\mathcal{I}}(A) = v_{(V(\mathcal{C}) \setminus \mathcal{Z}(\mathcal{I}))}(A)$. Notice that, if $\mathcal{I}$ is a maximal $\mathcal{M}$ ideal of $\mathcal{C}$, this last $w_{\mathcal{I}}$ notation is consistent with the valuations $w_{\mathcal{M}}$, which we had previously introduced.

In the next lemma, we consider small categories to avoid set-theoretical problems.

Theorem 4.6.6 *If $\mathcal{C}$ is an additive semilocal small category with splitting idempotents, then the maximal trace ideals of $\mathcal{C}$ form a set* $\operatorname{Max}(\operatorname{Tr})$ *and the mapping* $\overline{w} = (\overline{w_{\mathcal{I}}})_{\mathcal{I}\in\operatorname{Max}(\operatorname{Tr})}: V(\mathcal{C}) \to \mathbb{N}_0^{(\operatorname{Max}(\operatorname{Tr}))}$*, defined by* $\overline{w_{\mathcal{I}}} = e(w_{\mathcal{I}})^{-1}w_{\mathcal{I}}$ *for every* $\mathcal{I} \in \operatorname{Max}(\operatorname{Tr})$*, is a divisor theory.*

Proof. We have $\operatorname{Max}(\operatorname{Tr}) = \{\mathcal{I}_P \mid P$ is a prime ideal of height one in the monoid $V(\mathcal{C})\}$. By Proposition 4.6.5, every essential valuation of $V(\mathcal{C})$ is equivalent to a valuation of the form $w_{\mathcal{I}}$ for some maximal trace ideal $\mathcal{I}$. Now conclude by Proposition 3.2.1. ∎

Finally, we will consider the valuations associated to any ideal $\mathcal{I}$ of an additive semilocal category $\mathcal{C}$ with splitting idempotents. For any such ideal $\mathcal{I}$, we can define a valuation $w_{\mathcal{I}}: V(\mathcal{C}) \to \mathbb{N}_0$ by $w_{\mathcal{I}}(A) = \operatorname{codim}(\operatorname{End}_{\mathcal{C}}(A)/\mathcal{I}(A,A))$. Notice that this notation is consistent with the valuations $w_{\mathcal{M}}$, which we had previously introduced only in the case of $\mathcal{M}$ a maximal ideal of $\mathcal{C}$, and with the valuations $w_{\mathcal{I}}$, where $\mathcal{I}$ is a trace ideal. Moreover, we have $v_P = w_{\mathcal{I}_P}$. For any ideal $\mathcal{I}$ of $\mathcal{C}$, the corresponding prime ideal $V(\mathcal{C}) \setminus \mathcal{Z}(\mathcal{I})$ will be denoted by $\Phi(\mathcal{I})$. In the next Proposition, we characterize for which ideals $\mathcal{I}$ of $\mathcal{C}$ the valuation $w_{\mathcal{I}}$ is essential.

Proposition 4.6.7 *Let $\mathcal{C}$ be an additive semilocal category with splitting idempotents. The following conditions are equivalent for a proper ideal $\mathcal{I}$ of $\mathcal{C}$:*

1. *the valuation $w_{\mathcal{I}}$ is essential;*
2. *the prime ideal $\Phi(\mathcal{I})$ of $V(\mathcal{C})$ has height one;*
3. *the ideal $\mathcal{I}$ contains a maximal trace ideal.*

Proof. (1) $\Leftrightarrow$ (2) The prime ideal $\Phi(\mathcal{I}) = V(\mathcal{C}) \setminus \mathcal{Z}(\mathcal{I})$ is equal to the prime ideal $P_{w_{\mathcal{I}}}$. We conclude by Lemma 3.2.2.

(2) $\Rightarrow$ (3) If $\Phi(\mathcal{I}) = V(\mathcal{C}) \setminus \mathcal{Z}(\mathcal{I})$ is a prime ideal of height one, then $V(\mathcal{C}) \cap \mathcal{Z}(\mathcal{I})$ is a maximal divisor-closed submonoid of $V(\mathcal{C})$, so that $\mathcal{Z}(\mathcal{I})$ is a maximal additively closed proper subclass of $\operatorname{Ob}(\mathcal{C})$. Thus the corresponding trace ideal $\operatorname{Tr}(\mathcal{Z}(\mathcal{I}))$ is a maximal trace ideal of $\mathcal{C}$. But $\operatorname{Tr}(\mathcal{Z}(\mathcal{I})) \subseteq \mathcal{I}$ by Proposition 4.6.3(4).

(3) $\Rightarrow$ (2) Let $\mathcal{I}$ be an ideal of $\mathcal{C}$ containing a maximal trace ideal $\mathcal{J}$. Then $\mathcal{Z}(\mathcal{I}) \supseteq \mathcal{Z}(\mathcal{J})$, so that $\Phi(\mathcal{I}) = V(\mathcal{C}) \setminus \mathcal{Z}(\mathcal{I}) \subseteq V(\mathcal{C}) \setminus \mathcal{Z}(\mathcal{J}) = \Phi(\mathcal{J})$. As $\mathcal{J}$ is a maximal trace ideal, its corresponding prime ideal $\Phi(\mathcal{J})$ has height one. Since $\mathcal{I}$ is a proper ideal, $\mathcal{Z}(\mathcal{I})$ must be a proper subclass of $\operatorname{Ob}(\mathcal{C})$, so that $\Phi(\mathcal{I}) = V(\mathcal{C}) \setminus \mathcal{Z}(\mathcal{I})$ is not the empty ideal. Thus $\Phi(\mathcal{I}) = \Phi(\mathcal{J})$. ∎

Example 4.6.8 Let R be the localization of the ring $\mathbb{Z}$ of integers at the multiplicatively closed subset of $\mathbb{Z}$ consisting of the integers prime with 6, so that R is a commutative semilocal principal ideal domain with two maximal ideals generated by 2 and 3 respectively. Let $\mathcal{C}$ be the full subcategory proj-R of Mod-R. Then $\mathcal{C}$ is an additive semilocal category with splitting idempotents. Since every finitely generated projective R-module is free, the monoid $V(\mathcal{C})$ is isomorphic to $\mathbb{N}_0$. By Proposition 4.1.5, $\mathcal{C}$ has exactly two maximal ideals, $\mathcal{M}_2$ and $\mathcal{M}_3$. The canonical monoid homomorphism $V(\mathcal{C}) \to V(\mathcal{C}/\mathcal{M}_2) \oplus V(\mathcal{C}/\mathcal{M}_3)$ corresponds to the monoid homomorphism $\mathbb{N}_0 \to \mathbb{N}_0 \oplus \mathbb{N}_0$, $n \mapsto (n,n)$. The two valuations $w_{\mathcal{M}_2}$ and $w_{\mathcal{M}_3}$ are essential and coincide. The monoid $\mathbb{N}_0$ has exactly two prime ideals, which are the empty ideal and the ideal $\mathbb{N}$ of positive integers. Correspondingly, $\mathcal{C}$ has exactly two trace ideals, necessarily the zero ideal and the improper ideal. Thus $\mathcal{C}$ has only one maximal trace ideal, which is the zero ideal, corresponding to the prime ideal $\mathbb{N}$ of $\mathbb{N}_0$. The corresponding valuation $v_{\mathbb{N}}: V(\mathcal{C}) \to \mathbb{N}_0$, defined by $v_{\mathbb{N}}(A) = \operatorname{codim}(\operatorname{End}_{\mathcal{C}}(A))$ for every $A \in V(\mathcal{C})$, sends the free module R^n to $2n$.

We conclude giving a further divisor homomorphism of the monoid $V(\mathcal{C})$ into a free commutative monoid. Let $\mathcal{C}$ be a small additive semilocal category with splitting idempotents. For every finite subset S of $\operatorname{Ob}(\mathcal{C})$, set

$$D_S = \bigcup_{\substack{\mathcal{I} \in \operatorname{Max}(\operatorname{Tr}) \\ S \cap \mathcal{Z}(\mathcal{I}) = \varnothing}} \mathcal{Z}(\mathcal{I}).$$

Let $\nu_S: V(\mathcal{C}) \to \mathbb{N}_0$ be the valuation defined, for every $A \in V(\mathcal{C})$, by $\nu_S(A) = \operatorname{codim}(\operatorname{End}_{\mathcal{C}}(A)/\operatorname{Tr}(D_S)(A,A))$. Notice that $\nu_S = w_{\operatorname{Tr}(D_S)}$.

Theorem 4.6.9 *Let $\mathcal{C}$ be a small additive semilocal category with splitting idempotents. Let $\wp_f(\operatorname{Ob}(\mathcal{C}))$ be the set of all finite subsets of the set $\operatorname{Ob}(\mathcal{C})$. Then the mapping $\nu = (\nu_S)_{S \in \wp_f(\operatorname{Ob}(\mathcal{C}))}$ is a divisor homomorphism of the monoid $V(\mathcal{C})$ into the free commutative monoid $\mathbb{N}_0^{(\wp_f(\operatorname{Ob}(\mathcal{C})))}$.*

Proof. We have seen in Theorem 4.6.6 that $\overline{w} = (\overline{w_{\mathcal{I}}})_{\mathcal{I} \in \operatorname{Max}(\operatorname{Tr})}: V(\mathcal{C}) \to \mathbb{N}_0^{(\operatorname{Max}(\operatorname{Tr}))}$ is a divisor theory. In order to prove the theorem, it suffices to show that, for every $\mathcal{I} \in \operatorname{Max}(\operatorname{Tr})$, there exists $S \in \wp_f(\operatorname{Ob}(\mathcal{C}))$ with $w_{\mathcal{I}}$ and ν_S equivalent valuations. Fix an ideal $\mathcal{I} \in \operatorname{Max}(\operatorname{Tr})$. Let $\delta_{\mathcal{I}}$ be the element of $\mathbb{N}_0^{(\operatorname{Max}(\operatorname{Tr}))}$ that is one in the coordinate indexed by $\mathcal{I}$ and zero in all the other coordinates. By the definition of divisor theory, there exists a finite set $T = \{A_1, \dots, A_m\}$ of objects of $V(\mathcal{C})$ such that $\delta_{\mathcal{I}} = \min\{\overline{w}(A_1), \dots, \overline{w}(A_m)\}$. We claim that $D_T = \mathcal{Z}(\mathcal{I})$. To prove the claim notice that an object B of $\mathcal{C}$ is in D_T if and only if there exists $\mathcal{K} \in \operatorname{Max}(\operatorname{Tr})$ with $T \cap \mathcal{Z}(\mathcal{K}) = \varnothing$ and $B \in \mathcal{Z}(\mathcal{K})$. Now $T \cap \mathcal{Z}(\mathcal{K}) = \varnothing$ if and only if $w_{\mathcal{K}}(A_i) > 0$ for every $i = 1, \dots, m$, if and only if $\overline{w_{\mathcal{K}}}(A_i) > 0$ for every $i = 1, \dots, m$. Now $\delta_{\mathcal{I}} = \min\{\overline{w}(A_1), \dots, \overline{w}(A_m)\}$ implies that $\overline{w_{\mathcal{K}}}(A_i) > 0$ for every $i = 1, \dots, m$ if and only if $\mathcal{K} = \mathcal{I}$. Thus B belongs to D_T if and only if $B \in \mathcal{Z}(\mathcal{I})$. This proves our claim. In particular, D_T is an additively closed subclass of $\operatorname{Ob}(\mathcal{C})$. In order to conclude the proof of the theorem, it suffices to show that $w_{\mathcal{I}}$ and ν_T are equivalent valuations. For this, it is enough to show that $P_{w_{\mathcal{I}}} = \{A \in V(\mathcal{C}) \mid w_{\mathcal{I}}(A) > 0\}$ is equal to $P_{\nu_T} = \{A \in V(\mathcal{C}) \mid \nu_T(A) > 0\}$. (Notice that $w_{\mathcal{I}}$ essential implies that ν_T also is essential by Lemma 3.2.2 Now $P_{w_{\mathcal{I}}} = \{A \in V(\mathcal{C}) \mid 1_A \notin \mathcal{I}(A,A)\} = V(\mathcal{C}) \setminus \mathcal{Z}(\mathcal{I})$, and $P_{\nu_T} = \{A \in V(\mathcal{C}) \mid \operatorname{End}_{\mathcal{C}}(A) \neq \operatorname{Tr}(D_T)(A,A)\} = V(\mathcal{C}) \setminus \mathcal{Z}(\operatorname{Tr}(D_T))$. Hence it remains to show that $\mathcal{Z}(\mathcal{I}) = \mathcal{Z}(\mathrm{Tr}(D_T))$. But $\mathcal{Z}(\mathrm{Tr}(D_T)) = D_T$ because D_T is an additively closed subclass of $\mathrm{Ob}(\mathcal{C})$, and $D_T = \mathcal{Z}(\mathcal{I})$ as we have seen in the claim. Thus $P_{w_\mathcal{I}} = P_{\nu_T}$ and $\overline{w}_\mathcal{I}$ is equivalent to ν_T by Lemma 3.2.2. ∎

Chapter 5

Weak Krull-Schmidt Theorems

5.1 Rings and objects of finite type

In our previous chapters we investigated Krull monoids and issues concerning factorizations in them, applying it mainly to the case of the monoid $V(\mathcal{C})$ associated to an additive category $\mathcal{C}$. Though the Krull-Schmidt Theorem does not hold in general for Krull monoids, we saw that they still preserve a certain regularity in the factorization. In this chapter we want to present a particular class of categories $\mathcal{C}$ that provide examples of Krull monoids. In these categories the uniqueness of the direct sum decomposition is not controlled by a single permutation, as it is in the usual Krull-Schmidt Theorem, but by a finite number of permutations. The meaning of this will be clarified in what follows.

We say that a ring R has *type* n if the factor ring $R/J(R)$ is a direct product of n division rings and we say that R has *finite type* if it has type n for some integer $n \geq 1$. If a ring R has finite type, then the type n of R coincides with the dual Goldie dimension $\operatorname{codim}(R_R)$ of R_R according to Proposition 1.4.4.

A ring R has type 1 if and only if it is a local ring, if and only if there is a local morphism of R into a division ring. With the next proposition, we generalize this fact.

Before stating the proposition, we need to recall that a *completely prime ideal* P of a ring R is a proper ideal P of R such that for every $x, y \in R$, $xy \in P$ implies that either $x \in P$ or $y \in P$. Recall that if R is a ring, $P_1, \dots, P_n$ are completely prime two-sided ideals of R, and I is a right ideal of R contained in $\bigcup_{i=1}^{n} P_i$, then $I \subseteq P_i$ for some i.

Proposition 5.1.1 *The following conditions are equivalent for a ring R with Jacobson radical $J(R)$ and a positive integer n:*

1. *the ring R has type n;*

2. *there exists a local morphism of the ring R into a direct product of m division rings for some positive integer m, and n is the smallest of such positive integers m;*

3. *R has exactly n distinct maximal right ideals, and they are all two-sided ideals in R.*

PROOF. (1)⇒(2) The canonical projection $R \to R/J(R)$ is always a local morphism. From $R/J(R) \cong D_1 \times \ldots \times D_n$, with D_i a division ring for every $i = 1, \ldots, n$ we have that there is an onto local morphism $R \to R/J(R) \cong D_1 \times \ldots \times D_n$ with kernel $J(R)$. For an arbitrary local morphism $\varphi: R \to D'_1 \times \ldots \times D'_m$ of R into the direct product of m division rings $D'_1, \ldots, D'_m$, the dual Goldie dimension $\operatorname{codim}(R) = n$ must be less or equal than $\operatorname{codim}(D'_1 \times \ldots \times D'_m) = m$ by Theorem 1.4.9.

(2)⇒(3) Assume that (2) holds. Let $\varphi: R \to D_1 \times \ldots \times D_n$ be a local morphism with $D_1, \ldots, D_n$ division rings. Set $P_i = \ker(\pi_i \varphi)$, where π_i is the canonical projection of $D_1 \times \ldots \times D_n$ onto D_i. Then P_i is a completely prime two-sided ideal of R, and $U(R) = R \setminus (\cup_{i=1}^n P_i)$ because φ is a local morphism.

Any proper right ideal of R is contained in $\bigcup_{i=1}^n P_i$, hence in one of the P_i's. In particular, the unique maximal right ideals of R are at most $P_1, \ldots, P_n$ and they are all two-sided ideals. Assume that the P_i's are not all distinct, or that one of them is not maximal. In both cases there exist two indices $i, j = 1, \ldots, n$ with $i \neq j$ and $P_i \subseteq P_j$. It is then easy to check that $(\pi_1 \varphi, \ldots, \widehat{\pi_i} \varphi, \ldots, \pi_n \varphi): R \to D_1 \times \ldots \times \hat{D}_i \times \ldots \times D_n$ is a local morphism, which contradicts the minimality of n.

(3)⇒(2) If $Q_1, \ldots, Q_n$ are all the maximal right ideals of R, then they are pairwise comaximal, so that the canonical projection $\pi: R \to \oplus_{i=1}^n R/Q_i$ is a right R-module morphism, which is onto by the Chinese Remainder Theorem. Since they are all two-sided, π is a ring morphism, with kernel $J(R)$. Hence $R/J(R) \cong \prod_{i=1}^n R/Q_i$, and the rings R/Q_i do not have non-trivial right ideals. Therefore the R/Q_i's are division rings. ∎

Notice that the definition of having type n is a left/right symmetric condition, so that it is equivalent also to: R has exactly n distinct maximal left ideals, and they are all two-sided in R. Hence, for rings of finite type, the set of all maximal right ideals, the set of all maximal left ideals and the set of all maximal two-sided ideals coincide, and we will talk simply of maximal ideals, without mentioning the side.

We say that an object A of a preadditive category $\mathcal{C}$ has *type n* if its endomorphism ring $\operatorname{End}_{\mathcal{C}}(M)$ is a ring of type n. It may be convenient to consider the zero object of a category the unique object of type 0. We will say that an object A has *finite type* if it has type n for some positive integer n. Objects of type one are exactly the objects with a local endomorphism ring.

Lemma 5.1.2 *Let A and B be two objects of a preadditive category $\mathcal{C}$ with splitting idempotents, with A of finite type. If A is isomorphic to a direct summand of B^k for some $k \geq 1$, then A is isomorphic to a direct summand of B.*

Proof. Let $P_1, \dots, P_n$ be all the maximal ideals of $\mathrm{End}_{\mathcal{C}}(A)$. Suppose there are morphisms $f: A \to B^k$ and $g: B^k \to A$ such that $gf = 1_A$. Then $1_A = \sum_{i=1}^k g\epsilon_i\pi_i f$, where $\epsilon_i: B \to B^k$ and $\pi_i: B^k \to B$ are the canonical injections and projections, respectively. Since $\mathrm{End}_{\mathcal{C}}(A)/J(\mathrm{End}_{\mathcal{C}}(A))$ is canonically isomorphic to $\prod_{j=1}^n \mathrm{End}_{\mathcal{C}} A/P_j$, there exists, for every $j = 1, \dots, n$ an endomorphism $q_j \in \mathrm{End}_{\mathcal{C}}(A)$ such that $q_j \notin P_j$ but $q_j \in \bigcap_{l \neq j} P_l$. Then $\sum_{i=1}^k (q_j g\epsilon_i)(\pi_i f q_j) = q_j^2 \notin P_j$. Hence for every $j = 1, \dots, n$ there exists an index i with $(q_j g\epsilon_i)(\pi_i f q_j) \notin P_j$. For this index i, set $f_j = \pi_i f q_j$ and $g_j = q_j g\epsilon_i$, so that $f_j: A \to B$ and $g_j: B \to A$ are such that $g_j f_j \notin P_j$ but $g_j f_j \in \bigcap_{l \neq j} P_l$. Now consider the homomorphisms $f': A \to B$ and $g': B \to A$ defined by $f' = \sum_{j=1}^n f_j q_j$ and $g' = \sum_{j=1}^n q_j g_j$. Obviously $g'f'$ is not contained in any maximal ideal of $\mathrm{End}_{\mathcal{C}}(A)$, hence is left invertible. If h is a left inverse of $g'f'$, then $(hg')f' = 1_A$. Therefore A is isomorphic to a direct summand of B, since idempotents split in $\mathcal{C}$. ∎

Let $\mathcal{C}$ be a preadditive subcategory and A an object of $\mathcal{C}$ of type n. If $\mathcal{I}$ is an ideal of $\mathcal{C}$, we say that $\mathcal{I} \in V(A)$ is $\mathcal{I}$ is equal to the ideal $\mathcal{A}_P$ of $\mathcal{C}$ associated to a maximal ideal P of the endomorphism ring $\mathrm{End}_{\mathcal{C}}(A)$.

Theorem 5.1.3 *Let $\mathcal{C}$ be a preadditive category and A, B objects of $\mathcal{C}$ of type m and n, respectively. Then $A \cong B$ if and only if, for any ideal $\mathcal{I}$ of $\mathcal{C}$, we have $\mathcal{I} \in V(A) \iff \mathcal{I} \in V(B)$.*

Proof. Assume that $\mathcal{I} \in V(A) \iff \mathcal{I} \in V(B)$, for an ideal $\mathcal{I}$ of $\mathcal{C}$. Then A and B have the same type n, and the n maximal ideals of $\mathrm{End}_{\mathcal{C}}(A)$ and $\mathrm{End}_{\mathcal{C}}(B)$ can be labeled in such a way that if $P_1, \dots, P_n$ are the maximal ideal of $\mathrm{End}_{\mathcal{C}}(A)$ and $P_1', \dots, P_n'$ are the maximal ideal of $\mathrm{End}_{\mathcal{C}}(B)$, then $\mathcal{A}_{P_i} = \mathcal{A}_{P_i'}$ for every $i = 1, \dots, n$.

Suppose that we can find homomorphisms $\alpha_1, \dots, \alpha_n: A \to B$ and homomorphisms $\beta_1, \dots, \beta_n: B \to A$ such that $\alpha_i\beta_i, \beta_i\alpha_i \notin \mathcal{A}_{P_i} = \mathcal{A}_{P_i'}$ but $\alpha_i, \beta_i \in \mathcal{A}_{P_j}$ for any $i, j = 1, \dots, n$ with $i \neq j$. Consider $\alpha = \sum_{i=1}^n \alpha_i$ and $\beta = \sum_{i=1}^n \beta_i$. Then $\beta\alpha$ is not contained in any maximal ideal of $\mathrm{End}_{\mathcal{C}}(A)$ and $\alpha\beta$ is not contained in any maximal ideal of $\mathrm{End}_{\mathcal{C}}(B)$. Therefore $\beta\alpha$ and $\alpha\beta$ are automorphisms and $A \cong B$.

It remains to explain how one can find those $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$. Let $h \in \mathrm{End}_{\mathcal{C}}(A)$ be such that $h \notin P_1$ and $h \in \bigcap_{j=2}^n P_j$. Then the same relations hold for h^2. In particular $h^2 \notin \mathcal{A}_{P_1} = \mathcal{A}_{P_1'}$, therefore there are $f: B \to A$ and $g: A \to B$ such that $gh^2 f \notin P_1'$. Similarly, there are $f': A \to B$ and $g': B \to A$ such that $g'gh^2ff' \notin P_1$. Put $\alpha_1 = f'g'gh$ and $\beta_1 = hf$. Since $h \in \bigcap_{j=2}^n P_j$, we have that $\alpha_1, \beta_1 \in \bigcap_{j=2}^n \mathcal{A}_{P_j}$. On the other hand neither $\alpha_1\beta_1 = f'g'gh^2f$ nor $\beta_1\alpha_1 = hff'g'gh$ are in $\mathcal{A}_{P_1} = \mathcal{A}_{P_1'}$ because P_1 is completely prime and $(g'gh^2ff')^2 = g'gh^2ff'g'gh^2ff' = g'gh^2f(f'g'gh^2f)f' = g'gh(hff'g'gh)hff' \notin P_1$. Similarly we can find $\alpha_2, \dots, \alpha_n$ and $\beta_2, \dots, \beta_n$. ∎

Proposition 5.1.4 *Let $\mathcal{C}$ be an additive category whose objects are direct sums of finitely many objects of finite type. For every object A of finite type, let P be a maximal ideal of $\mathrm{End}_{\mathcal{C}}(A)$ and let $\mathcal{A}_P$ the ideal of $\mathcal{C}$ associated to P. Then the categories $\mathcal{C}/\mathcal{A}_P$ and $\mathrm{vect}\text{-}\mathrm{End}_{\mathcal{C}}(A)/P$ are equivalent.*

Proof. Let F be the canonical functor $F:\mathcal{C} \to \mathcal{C}/\mathcal{A}_P$. We verify that $G = \mathrm{Hom}_{\mathcal{C}/\mathcal{A}_P}(F(A), -)$ is a category equivalence of the category $\mathcal{C}/\mathcal{A}_P$ into the category of all right vector spaces over the division ring $\mathrm{End}_R(A)/P$. From Lemma 4.1.1 we deduce that every object of $\mathcal{C}/\mathcal{A}_P$ is isomorphism to an object of the form $F(A^t)$. Since G induces an isomorphism between the endomorphism ring of $F(A)$ in $\mathcal{C}/\mathcal{A}_P$ and the endomorphism ring of the vector space $G(F(A))$, it is clear that G is a categorical equivalence. ∎

Now, following the approach of chapter 3, we prove our next Proposition.

Proposition 5.1.5 *Let $\mathcal{C}$ be an additive category whose objects are direct sums of finitely many objects of finite type. The canonical functor*

$$U:\mathcal{C} \to \oplus_{\mathcal{I}}\mathcal{C}/\mathcal{I},$$

where $\mathcal{I}$ varies among the ideals of $\mathcal{C}$ of the form $\mathcal{A}_P$ for some maximal ideal P of the endomorphism ring $\mathrm{End}_{\mathcal{C}}(A)$ of some object A of $\mathcal{C}$ of finite type, is full. The ideal $\mathcal{K}er(U)$ is the Jacobson radical of the category $\mathcal{C}$ and therefore the functor U is local and isomorphism reflecting.

Moreover, if idempotents split in $\mathcal{C}$, then U is direct-summand reflecting.

Proof. Let us prove that U is full. Let A, B be objects of $\mathcal{C}$. There exist objects $A_1, \dots, A_k$ and $B_1, \dots, B_l$ of finite type such that $A = \oplus_{i=1}^k A_i$ and $B = \oplus_{j=1}^l B_j$. Since U preserves finite direct sums, it is enough to consider the case $k = l = 1$. Thus suppose that A and B are modules of finite type. Let $\overline{f}: U(A) \to U(B)$ be a morphism in $\oplus_{\mathcal{I}}\mathcal{C}/\mathcal{I}$. We have to find a morphism $f: A \to B$ such that $U(f) = \overline{f}$. Recall that there are only finitely many ideals $\mathcal{I}_i$, $i = 1, \dots, n$ of $\mathcal{C}$ of the form $\mathcal{A}_P$ for some maximal ideal P of the endomorphism ring $\mathrm{End}_{\mathcal{C}}(C)$ of some object C of $\mathcal{C}$ of finite type, such that A and B can be non-zero objects in the quotient category $\mathcal{C}/\mathcal{I}_i$. For every such ideal $\mathcal{I}_i$, let $f_i: A \to B$ be a morphism such that its image in the category $\mathcal{C}/\mathcal{I}_i$ is the corresponding component of $\overline{f}$.

Since we have that $\mathrm{End}_{\mathcal{C}}(A)/J(\mathrm{End}_{\mathcal{C}}(A))$ is canonically isomorphic to the direct product of $\mathrm{End}_{\mathcal{C}}(A)$ modulo its maximal ideals, we have that, for every ideal $\mathcal{I}_i$, $i = 1, \dots, n$, there exists $\delta_i \in \mathrm{End}_{\mathcal{C}}(A)$ such that $\delta_i \equiv 1_A$ modulo $\mathcal{I}_i(A, A)$, and $\delta_i \equiv 0_A$ modulo every other maximal ideal of $\mathrm{End}_{\mathcal{C}}(A)$. Then put $f = \sum_{i=1}^n f_i \delta_i$.

To prove that $\mathcal{K}er(U)$ is equal to the Jacobson radical $\mathcal{J}$ of $\mathcal{C}$, let $f: B \to C$ be a morphism in $\mathcal{J}(B, C)$. Since $\beta f \alpha \in \mathcal{J}(A, A) = J(\mathrm{End}_{\mathcal{C}}(A))$ for every morphism $\alpha: A \to B$ and $\beta: C \to A$, it clear that $U(f) = 0$. Conversely, suppose that $f \notin \mathcal{J}(B, C)$. This implies that there exist a morphism $g: C \to B$ such that $1_B - gf$ is not right invertible. Hence there exists a maximal ideal P of $\mathrm{End}_{\mathcal{C}}(B)$ with $1_B - gf \in P$. Thus $gf \notin P$ and this implies that $f \notin \mathcal{A}_P$ and hence $f \notin \mathcal{K}er(U)$.

Let us prove now that U is a local functor. Let A and B be objects of $\mathcal{C}$ and let $f: A \to B$ be a morphism such that $U(f)$ is an isomorphism. Since U is full, there exists $g: B \to A$ such that $U(1_A - gf) = 0$ and $U(1_B - fg) = 0$. That is $1_A - gf \in J(\mathrm{End}_{\mathcal{C}}(A))$ and $1_B - fg \in J(\mathrm{End}_{\mathcal{C}}(B))$. So fg and gf are both automorphisms and then also f is an isomorphism. Any local and full functor is isomorphism reflecting, so U must be so.

To prove the last sentence of the Proposition, suppose that $\mathcal{C}$ has splitting idempotents. Let A and B be objects of $\mathcal{C}$ such that $U(A)$ is isomorphic to a direct summand of $U(B)$. In particular, there are morphisms $\overline{f}: U(A) \to U(B)$ and $\overline{g}: U(B) \to U(A)$ such that $\overline{g}\overline{f} = 1_{U(A)}$. Since U is full and local, there exists $f: A \to B$ and $g: B \to A$ such that gf is an isomorphism. Since idempotents split in $\mathcal{C}$, it follows that A is a direct summand of B in $\mathcal{C}$. ■

It is a direct consequence of our last Proposition and Theorem 3.3.6 that the monoid $V(\mathcal{C})$ is a Krull monoid, for any additive category $\mathcal{C}$ with splitting idempotents whose objects are direct sums of finitely many objects of finite type.

5.2 Examples of objects of finite type

With this section we want to provide various examples of modules of finite type, that will provide the setting for the next section and the starting point for our next, and last, chapter.

We will say that a module M_R over a ring R is *heterogenous* if for every direct-sum decomposition $M_R = M_1 \oplus M_2 \oplus M_3$, $M_1 \cong M_2$ implies $M_1 = M_2 = 0$; that is, if M_R does not have two distinct non-zero isomorphic direct summands. For instance a semisimple module is heterogeneous if and only if it is a direct sum of a family of pairwise non-isomorphic simple modules. Clearly, a heterogeneous finitely generated semisimple module is a module of finite type.

For a projective module P_R we have the following.

Lemma 5.2.1 *The following conditions are equivalent for a projective right modules P_R over an arbitrary ring R:*

1. *P_R is couniform;*
2. *P_R is the projective cover of a simple module;*
3. *P_R has type 1;*
4. *there exists an idempotent $e \in R$ such that $P_R \cong eR$ and eRe a local ring;*
5. *P_R is a local finitely generated module;*
6. *P_R is finitely generated, non-zero and all its proper submodules are superfluous.*

Proof. (1)⇒(2) By [5, Proposition 17.14] we know that P_R has a maximal submodule Q_R. Hence the module P_R/Q_R is simple and, since P_R is couniform, Q_R is superfluous in P_R.

(2)⇒(1) Clearly P_R is the projective cover of a simple module if and only if P_R contains a superfluous maximal module Q_R. Hence every submodule of P_R must be contained in Q_R and so P_R is uniform.

(2)⇒(3) Let $J = J(R)$ be the Jacobson radical of R. Since PJ is contained in every maximal submodule of P and contains every superfluous submodule of P, it turns out that PJ is a superfluous maximal submodule of P. By [5, Corollary 17.12] we have

$$\mathrm{End}_R(P)/J(\mathrm{End}_R(P)) \cong \mathrm{End}_R(P/PJ),$$

that is a division ring by Schur's Lemma. Thus $\mathrm{End}_R(P)$ is local.

(3)⇒(2) Suppose that $\mathrm{End}_R(P)$ is a local ring. Thus P_R is non-zero and by [5, Proposition 17.14] there is a maximal submodule $K \subsetneq P$. We claim that the epimorphism $p: P \to P/K$ is a projective cover. To show this we have to prove that K is superfluous in P. Suppose that $K + L = P$ for some $L \subseteq P$. Then

$$P/K \cong (L+K)/K \cong L/(L \cap K);$$

so there is a non-zero homomorphism $f: P \to L/(L \cap K)$. Thus, since P is projective, there is an endomorphism $s: P \to L \subseteq P$ such that $ps = f$. Since $f \neq 0$, $\mathrm{im}(s) \nsubseteq K$, from which it follows that $\mathrm{im}(s)$ is not superfluous in P. Therefore $s \notin J(\mathrm{End}_R(P))$ and, by [5, Proposition 17.11], s is an invertible endomorphism of P. Then $L = P$. We showed that K is superfluous in P.

(2)⇒(4) Every simple module is an epimorphic image of R so, by [5, Lemma 17.17], a projective cover of P_R must be isomorphic to a direct summand of R_R. That is $P_R \cong eR$ for some idempotent $e \in R$. By (3) the endomorphism ring $\mathrm{End}_R(eR) = eRe$ is a local ring.

(4)⇒(5) It is clear that $P_R \cong eR$ is finitely generated and it is local since it has a superfluous maximal submodule.

(5)⇒(6) In a finitely generated module, every proper submodule is contained in a maximal submodule. Hence P_R local implies that P_R has a greatest submodule, necessarily superfluous.

(6)⇒(2) Trivial. ■

More generally, we can characterize finitely generated projective modules of type n.

Proposition 5.2.2 *The following conditions are equivalent for a finitely generated projective module P over a ring R:*

1. *P_R has type n;*
2. *$P/PJ(R)$ is a heterogeneous semisimple module of Goldie dimension n;*
3. *P_R is the projective cover of a heterogeneous semisimple module of Goldie dimension n.*

PROOF. (1)⇒(2) Let $E = \mathrm{End}_R(P)$. Since E is of type n, P has at most n maximal submodules. Let $M_1, \dots, M_m$ be the maximal submodules of P_R. Then

$P/PJ(R)$ is isomorphic to a submodule of $P/M_1 \oplus \ldots \oplus P/M_m$. So $P/PJ(R)$ is isomorphic to $S_1 \oplus \ldots \oplus S_k$, where $S_1, \ldots, S_k$ are simple modules. Suppose $f: S_1 \to S_2$ and $g: S_2 \to S_1$ are mutually inverse isomorphisms. We know that $I_1 = \{f \in E \mid \overline{f}(P/PJ(R)) \subseteq S_2 \oplus S_3 \oplus \ldots \oplus S_k\}$ and $I_2 = \{f \in E \mid \overline{f}(P/PJ(R)) \subseteq S_1 \oplus S_3 \oplus \ldots \oplus S_k\}$ cannot be contained in the same maximal right ideal of E because $I_1 + I_2 = E$. Let $h \in E$ be such that $\overline{h} = f \oplus g \oplus 1_{S_3} \oplus \ldots \oplus 1_{S_k}$. The homomorphism $\overline{h}$ is an isomorphism and hence h is an invertible element of E. Clearly $h(I_1) \subseteq I_2$ and so $I_1 \subseteq h^{-1}(I_2)$, and we have a contradiction, because all the maximal ideals of E are two-sided. Thus $S_1, \ldots, S_k$ are pairwise non-isomorphic.

It remains to prove that $k = n$. Clearly $\mathrm{End}_R(S_1 \oplus \ldots \oplus S_k)$ has exactly k maximal right ideals. But this ring is factor of E modulo the superfluous ideal $I = \{f \in E \mid f(P) \subseteq PJ(R)\}$.

(2)⇒(1) Suppose that $P/PJ(R) = S_1 \oplus \ldots \oplus S_n$, where the S_i are pairwise non-isomorphic simple modules. Then the ring $\mathrm{End}_R(P/PJ(R))$ has exactly n maximal right ideals and each maximal right ideal is two-sided. This ring is a factor of E modulo a superfluous ideals. Thus (1) holds.

(2)⇔(3) is analogous to the proof of (2)⇔(5) of Lemma 5.2.1. ∎

Dually, an injective module of finite Goldie dimension is of finite type if and only if it is a heterogeneous module, if and only if it is the direct sum of finitely many pairwise non-isomorphic indecomposable injective modules.

Proposition 5.2.3 *Let M_R be a module of finite Goldie dimension n with injective envelope $E(M_R)$ heterogeneous and with the property that every injective endomorphism of M_R is bijective. Then M_R has type $\leq n$.*

Proof. Consider the ring homomorphism

$$\varphi: \mathrm{End}_R(M) \to \mathrm{End}_R(E(M))/J(\mathrm{End}_R(E(M)))$$

defined, for every $f \in \mathrm{End}_R(M)$, by $\varphi(f) = f_0 + J(\mathrm{End}_R(E(M)))$, where $f_0 \in \mathrm{End}_R(E(M))$ denotes any extension of f to $E(M_R)$. Notice that φ is well defined, that is, it does not depend on the choice of the extension f_0, because if f_0' is any other extension of f, then $f_0 - f_0'$ is zero on the essential submodule M_R of $E(M_R)$, hence belongs to $J(\mathrm{End}_R(E(M)))$. The ring morphism φ is a local morphism, because if $f \in \mathrm{End}_R(M)$ and $\varphi(f)$ is invertible in $\mathrm{End}_R(E(M))/J(\mathrm{End}_R(E(M)))$, then f_0 is invertible in $\mathrm{End}_R(E(M))$, hence f is injective. Thus f is invertible in $\mathrm{End}_R(M)$ by hypothesis. Finally, $E(M_R)$ is heterogenous, that is $E(M_R) = E_1 \oplus \ldots \oplus E_n$ with $E_1, \ldots, E_n$ pairwise non-isomorphic indecomposable injective modules. Hence every homomorphism $E_i \to E_j$ with $i \neq j$ has non-zero kernel. Thus $\mathrm{End}_R(E(M))/J(\mathrm{End}_R(E(M))) \cong \prod_{i=1}^n \mathrm{End}_R(E_i)/J(\mathrm{End}_R(E_i))$ is a product of division rings. ∎

Corollary 5.2.4 *If M_R is an artinian module with an heterogeneous socle, then M_R is a module of type n, where $n \leq \dim(M_R)$.*

Proof. Heterogeneous socle implies $E(M_R)$ heterogeneous. Moreover every injective endomorphism of M_R is bijective. ∎

Consider the category $(\text{Mod-}R)'$ with the same objects as Mod-R and, for right R-modules A and B, with

$$\operatorname{Hom}_{(\text{Mod-}R)'}(A,B) = \varinjlim \operatorname{Hom}_R(A, B/B'),$$

where the direct limit is taken over the upward directed family of superfluous submodules B' of B. Let $F\colon \text{Mod-}R \to (\text{Mod-}R)'$ be the canonical functor. We shall denote the image $F(A)$ in $(\text{Mod-}R)'$ of a right R-module A by $\overline{A}$, and the image $F(f)$ of a morphism f by $\overline{f}$.

Proposition 5.2.5 *Let M_R a module of finite dual Goldie dimension n with the property that every surjective endomorphism of M_R is bijective. If every homomorphic image of M_R is heterogeneous, then M_R has type $\leq n$.*

Proof. The canonical functor $F\colon \text{Mod-}R \to (\text{Mod-}R)'$ induces a ring morphism $\varphi\colon \operatorname{End}_R(M) \to \operatorname{End}_{(\text{Mod-}R)'}(F(M))$, defined, for every $f \in \operatorname{End}_R(M)$, by $\varphi(f) = \overline{f}$. The ring morphism φ is a local morphism, because if $f \in \operatorname{End}_R(M)$ and $\varphi(f)$ is invertible in $\operatorname{End}_{(\text{Mod-}R)'}(F(M))$, then f is a surjective endomorphism of M_R, hence f is invertible in $\operatorname{End}_R(M)$ by hypothesis.

Since M_R has finite dual Goldie dimension n, there exist n submodules $N_1, \dots, N_n$ of M_R such that every quotient M/N_i is a couniform module, the submodule $N = \cap_{i=1}^n N_i$ is superfluous in M_R and the canonical injective mapping

$$M_R/N \to M/N_1 \oplus \dots \oplus M/N_n$$

is bijective. Hence $F(M_R) \cong F(M_R/N) \cong F(M/N_1) \oplus \dots \oplus F(M/N_n)$ in $(\text{Mod-}R)'$, so that

$$\operatorname{End}_{(\text{Mod-}R)'}(F(M)) \cong \operatorname{End}_{(\text{Mod-}R)'}(F(M/N_1) \oplus \dots \oplus F(M/N_n)).$$

Now the endomorphism rings $\operatorname{End}_{(\text{Mod-}R)'}(F(M/N_i))$ are division rings, and

$$\operatorname{Hom}_{(\text{Mod-}R)'}(F(M/N_i), F(M/N_j)) = 0$$

for $i \neq j$ because every homomorphic image of M_R is heterogeneous. Thus $\operatorname{End}_{(\text{Mod-}R)'}(F(M)) \cong \prod_{i=1}^n \operatorname{End}_{(\text{Mod-}R)'}(F(M/N_i))$ is a direct product of division rings. ∎

Corollary 5.2.6 *If M_R is a noetherian module with $M_R/M_RJ(R)$ heterogeneous semisimple, then M_R is a module of type n, where $n \leq \dim(M_R/M_RJ(R))$.*

Proof. It suffices to apply the previous Proposition. Notice that:

- if M_R is a noetherian module, then $M_RJ(R)$ is superfluous in M_R by Nakayama's Lemma, so that

$$\operatorname{codim}(M_R) = \operatorname{codim}(M_R/M_RJ(R)) = \dim(M_R/M_RJ(R));$$

- every surjective endomorphism of M_R is bijective;
- if $M_R/M_RJ(R)$ is a heterogeneous semisimple module, then every homomorphic image of M_R is heterogeneous. To prove this, notice that if $M_R/N = S \oplus S' \oplus C$ with $S \cong S'$, then $(M_R/N)/(M_R/N)J(R) = S/SJ(R) \oplus S'/S'J(R) \oplus C/CJ(R)$, and $(M_R/N)/(M_R/N)J(R) \cong M_R/(M_RJ(R)+N)$ is heterogeneous because it is a homomorphic image of the heterogeneous semisimple module $M_R/M_RJ(R)$. Hence $S/SJ(R) = S'/S'J(R) = 0$. But S, S' are finitely generated because M_R is noetherian, so that $S = S' = 0$ by Nakayama's Lemma.

■

Proposition 5.2.7 *Let E and E' be injective heterogeneous modules of finite Goldie dimension n, m respectively, and let $\varphi: E \to E'$ be a module morphism. Then $\ker(\varphi)$ has type $\leq m + n$.*

Proof. By [20, Lemma 5.2], there is a local morphism

$$\chi: \mathrm{End}_R(\ker(\varphi)) \to \frac{\mathrm{End}_R(E(\ker(\varphi)))}{J(\mathrm{End}_R(E(\ker(\varphi))))} \times \frac{\mathrm{End}_R(E(\varphi(E)))}{J(\mathrm{End}_R(E(\varphi(E))))}.$$

Now E, E' heterogeneous implies $E(\ker(\varphi))$, $E(\varphi(E))$ heterogeneous, so that

$$\frac{\mathrm{End}_R(E(\ker(\varphi)))}{J(\mathrm{End}_R(E(\ker(\varphi))))} \text{ and } \frac{\mathrm{End}_R(E(\varphi(E)))}{J(\mathrm{End}_R(E(\varphi(E))))}$$

are direct products of $\leq n$ and $\leq m$ division rings, respectively. ■

Proposition 5.2.8 *Let M_R be a module of finite Goldie dimension n, of finite dual Goldie dimension m, with injective envelope $E(M_R)$ heterogeneous and with the property that every homomorphic image of M_R is heterogeneous. Then M_R has type $\leq n + m$.*

Proof. By [20, Proposition 6.4], there is a local morphism

$$\varphi: \mathrm{End}_R(M) \to \mathrm{End}_R(E(M))/J(\mathrm{End}_R(E(M))) \times \mathrm{End}_{(\text{Mod-}R)'}(F(M)).$$

The ring $\mathrm{End}_R(E(M))/J(\mathrm{End}_R(E(M)))$ is a direct product of $\leq n$ division rings and $\mathrm{End}_{(\text{Mod-}R)'}(F(M))$ is a direct product of $\leq m$ division rings. ■

Proposition 5.2.9 *If there is an exact sequence*

$$0 \to K_R \xrightarrow{\epsilon} P_R \xrightarrow{\pi} M_R \to 0,$$

where P_R is a projective module of type m and K is a superfluous submodule of P_R of type n, then M_R is a module of type $\leq m + n$.

Proof. Since P_R and K_R are of finite type, their endomorphism rings $\mathrm{End}_R(P)$ and $\mathrm{End}_R(K)$ have finite dual Goldie dimension, respectively m and n. Given any endomorphism $f \in \mathrm{End}_R(M)$, there are endomorphisms f_0, f_1 of P_R and K_R, respectively, making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_R & \longrightarrow & P_R & \longrightarrow & M_R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & K_R & \longrightarrow & P_R & \longrightarrow & M_R & \longrightarrow & 0. \end{array}$$

Let us prove that the position $\Psi: f \to (f_0 + J(\mathrm{End}_R(P)), f_1 + J(\mathrm{End}_R(M)))$ well defines a local morphism

$$\Psi: \mathrm{End}_R(M) \to \mathrm{End}_R(P)/J(\mathrm{End}_R(P)) \times \mathrm{End}_R(K)/J(\mathrm{End}_R(K)).$$

In order to prove that Ψ is well defined, assume that both f_0 and f_0' lift f to P_R, then $f_0 - f_0'$ maps P_R into K_R, so that the image of $f_0 - f_0'$ is superfluous in P_R. This implies $f_0 - f_0' \in J(\mathrm{End}_R(P))$. Let us prove that $f_1 + J(\mathrm{End}_R(K))$ depends only on f and not on the choice of the lifting f_0 of f. Let f_0' be another lifting of f and f_1' the corresponding restriction to K_R. We must show that $f_1 - f_1' \in J(\mathrm{End}_R(K))$. Since both f_0 and f_0' lift f, we have $(f_0 - f_0')(P_R) \subseteq K_R$, and hence the difference $f_0 - f_0': P_R \to P_R$ factors through the kernel $\epsilon: K_R \to P_R$ of π, i.e. $f_0 - f_0' = \epsilon g$ for a suitable morphism $g: P_R \to K_R$. In order to prove that $f_1 - f_1' \in J(\mathrm{End}_R(K))$, we must prove that for any endomorphism h of K_R, $1_K - h(f_1 - f_1')$ is an automorphism of K_R. Now $h(f_1 - f_1'): K_R \to K_R$ is the restriction to K_R of $hg: P_R \to K_R$. Hence $1_K - h(f_1 - f_1')$ is the endomorphism of K_R obtained by restriction of the endomorphism $1_P - \epsilon hg$ of P_R. As K is superfluous in P, the image $\epsilon hg(P_R) \subseteq K_R$ is superfluous in P_R. Therefore $1_P - \epsilon hg$ is a surjective endomorphism of P_R, hence a splitting epimorphism. Since modules with a semilocal endomorphism ring are directly finite, $1_P - \epsilon hg$ is an automorphism of P_R. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_R & \longrightarrow & P_R & \longrightarrow & M_R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_K - h(f_1 - f_1')} & & \downarrow{\scriptstyle 1_P - \epsilon hg} & & \downarrow{\scriptstyle 1_M} & & \\ 0 & \longrightarrow & K_R & \longrightarrow & P_R & \longrightarrow & M_R & \longrightarrow & 0. \end{array}$$

By the Snake Lemma, $1_P - \epsilon hg$ and 1_M automorphisms imply $1_K - h(f_1 - f_1')$ automorphism, as desired. This proves that Ψ is a well defined ring morphism. It is a local morphism, because if $f \in \mathrm{End}_R(M)$, $f_0 + J(\mathrm{End}_R(P))$ is invertible in $\mathrm{End}_R(P)/J(\mathrm{End}_R(P))$ and $f_1 + J(\mathrm{End}_R(K))$ is invertible in $\mathrm{End}_R(K)/J(\mathrm{End}_R(K))$, then f_0 is an automorphism of P_R and f_1 is an automorphism of K_R. By the Snake Lemma, f is an automorphism of M_R. ∎

5.3 The Krull-Schmidt Theorem in the case 2

In this section we restrict to objects of type 2. There is nothing really special about the number 2, except that everything can be expressed in easier terms.

Anyway, it is possible to generalize what we present in this section to the case n.

During this section we will denote by $\mathcal{D}$ a preadditive category of indecomposable objects of type ≤ 2 and by $\mathcal{C} = \mathrm{sum}(\mathcal{D})$ the additive category generated by $\mathcal{D}$. For such a category $\mathcal{C}$, we already know that the monoid $V(\mathcal{C})$ is cancellative. Also, the monoid $V(\mathcal{C})$ is *half-factorial*, i.e. if $a_1, \dots, a_n, b_1, \dots, b_m$ are atoms such that $\sum_{i=1}^n a_i = \sum_{j=1}^m b_j$, then $m = n$. More precisely, if $U_1, \dots, U_n$ and $U'_1, \dots, U'_m$ are objects of type 1, and $V_1, \dots, V_r$ and $V'_1, \dots, V'_s$ are indecomposable objects of the type 2, and

$$U_1 \oplus \ldots \oplus U_n \oplus V_1 \oplus \ldots \oplus V_r \cong U'_1 \oplus \ldots \oplus U'_m \oplus V'_1 \oplus \ldots \oplus V'_s,$$

then $m = n$, $r = s$, there is a permutation σ of $\{1, \dots, n\}$ such that $U_i \cong U'_{\sigma(i)}$ for every $i = 1, \dots, n$, and $V_1 \oplus \ldots \oplus V_r \cong V'_1 \oplus \ldots \oplus V'_s$. To see this, notice that if an object with local endomorphism ring is a direct summand of the direct sum of two objects, then it is isomorphic to a direct summand of one of the two objects, and proceed by induction using the fact that objects with semilocal endomorphism rings cancel from direct sum. This proves that $n = m$, that there is a permutation σ of $\{1, \dots, n\}$ such that $U_i \cong U'_{\sigma(i)}$, for every $i = 1, \dots, n$, and that $V_1 \oplus \ldots \oplus V_r \cong V'_1 \oplus \ldots \oplus V'_s$. In order to prove that $r = s$, notice that the dual Goldie dimension of the endomorphism ring $\mathrm{End}_R(V_1 \oplus \ldots \oplus V_r)$ is $2r$.

From this it follows that $V(\mathcal{C})$ decomposes as the direct product of a free commutative monoid and a monoid $V(\mathcal{C}_2)$, where $\mathcal{C}_2 = \mathrm{sum}(\mathcal{D}_2)$ and $\mathcal{D}_2$ is the full subcategory of $\mathcal{D}$ whose objects are indecomposable objects of type 2.

The property that we look for is the following. We say that *2-Krull-Schmidt Property* holds for a category $\mathcal{D}$ of indecomposable objects if there exist two equivalence relations $\sim$ and $\equiv$ on the class of objects of $\mathcal{D}$ such that, for objects $U_1, \dots, U_n, V_1, \dots, V_m$, we have $U_1 \oplus \ldots \oplus U_n \cong V_1 \oplus \ldots \oplus V_m$ in $\mathrm{sum}(\mathcal{D})$ if and only if $m = n$ and there are two permutations σ, τ of $\{1, \dots, n\}$ such that $U_i \sim V_{\sigma(i)}$ and $U_i \equiv V_{\tau(i)}$ for every $i = 1, \dots, n$.

Assume that the Krull-Schmidt Theorem holds for a category $\mathcal{D}$ of indecomposable objects of type ≤ 2. If ρ is an equivalence relation on the class $V(\mathcal{D})$ with the property that if $U_1, \dots, U_n, V_1, \dots, V_m \in V(\mathcal{D})$ and $U_1 \oplus \ldots \oplus U_n \cong V_1 \oplus \ldots \oplus V_m$ in $\mathrm{sum}(\mathcal{D})$, then $m = n$ and there exists a permutation σ on $\{1, \dots, n\}$ such that $U_i \rho V_{\sigma(i)}$ for every $i = 1, \dots, n$, it follows that the restriction of ρ to the class $V(\mathcal{D}_2)$, where $\mathcal{D}_2$ denoted the full subcategory of objects of type 2, has the same property. Conversely, if the 2-Krull-Schmidt holds for $\mathcal{D}_2$ with respect to the equivalence relations ρ_2 and ρ'_2, then it holds for $\mathcal{D}$ as well. To see this, extend the equivalence relation ρ_2 on $V(\mathcal{D}_2)$ to the equivalence relation $\Delta_{V(\mathcal{D}_1)} \dot{\cup} \rho_2$, where $\mathcal{D}_1$ is the full subcategory of $\mathcal{D}$ of objects of type 1 and $\Delta_{V(\mathcal{D}_1)} = \{(U, U) \mid U \in V(\mathcal{D}_1)\}$ is the diagonal of $V(\mathcal{D}_1)$. Similarly for ρ'_2. This proves the following Lemma.

Lemma 5.3.1 *Let $\mathcal{D}$ be a preadditive category of indecomposable objects of type ≤ 2 and $\mathcal{D}_2$ the full subcategory of indecomposable objects of type 2. Then the 2-Krull-Schmidt Property holds for $\mathcal{D}$ if and only if it holds for $\mathcal{D}_2$.*

In order to study the behaviour of the direct-sum decomposition in an additive category $\mathcal{C}$ whose objects are finite direct sums of indecomposable objects of type 2, we will now associate a graph $G(\mathcal{D})$ to the full subcategory $\mathcal{D}$ of indecomposable objects. The graph $G(\mathcal{D}) = (V, E)$ associated to the class $\mathcal{D}$ has as its class V of vertices the class $\mathrm{Max}(\mathcal{D})$. For every object A of $\mathcal{D}$, set $V(A) = \{(A, P), (A, Q)\} \subseteq \mathrm{Max}(\mathcal{D})$, where P and Q are the two maximal ideals of the endomorphism ring $\mathrm{End}_{\mathcal{D}}(A)$. The class of edges of G is $E = \{V(A) \mid A \in \mathcal{D}\}$. The edge $V(A) = \{(A, P), (A, Q)\}$ joins the two vertices (A, P) and (A, Q) in $\mathrm{Max}(\mathcal{D})$. The graph $G(\mathcal{D})$ has no multiple edges and no loops, by Theorem 5.1.3.

We can associate a commutative monoid $V(G)$ to any graph $G = (V, E)$. Given the graph $G = (V, E)$, where the elements of E are subsets of V of cardinality 2, consider the free commutative monoid $\mathbb{N}_0^{(V)}$ having as free class of generators the class $\{\delta_v \mid v \in V\}$. If $l = \{v, w\} \in E$ is an edge of G, define $\delta_l = \delta_v + \delta_w \in \mathbb{N}_0^{(V)}$. Let $V(G)$ be the submonoid of $\mathbb{N}_0^{(V)}$ generated by all the elements $\delta_l \in \mathbb{N}_0^{(V)}$, where l ranges in E. By Proposition 5.1.5, for any category $\mathcal{D}$ of indecomposable objects of type 2, the monoids $V(\mathrm{sum}(\mathcal{D}))$ and $V(G(\mathcal{D}))$ are isomorphic.

All the previous results about categories can be stated in the language of graphs. For instance, we can say that the *2-Krull-Schmidt Property* holds for the graph $G = (V, E)$ if there exist two equivalences $\sim$ and $\equiv$ on the class E of edges of G with the following property. Let $l_1, \dots, l_n, e_1, \dots, e_m \in E$ be edges of G; then $\delta_{l_1} + \dots + \delta_{l_n} = \delta_{e_1} + \dots + \delta_{e_m}$ in $V(G)$ if and only if $m = n$ and there exist two permutations σ, τ of $\{1, \dots, n\}$ such that $l_i \sim e_{\sigma(i)}$ and $l_i \equiv e_{\tau(i)}$ for every $i = 1, \dots, n$.

Recall that a graph if *bipartite* it there exists a partition $X \dot\cup Y$ of its class of vertices for which $X \neq \varnothing$, $Y \neq \varnothing$ and every edge connects a vertex in X to a vertex in Y. A graph is bipartite if and only if it does not contain cycles of odd length. A graph is called a *complete bipartite graph* if there is a partition $X \dot\cup Y$ of its set of vertices for which $X \neq \varnothing$, $Y \neq \varnothing$, and there are no edges between any two vertices of X, no edges between any two vertices of Y, and there is exactly one edge between any vertex in X and any vertex in Y. For every pair of disjoint set X and Y, let $B(X \dot\cup Y)$ be the complete bipartite graph with the disjoint union $X \dot\cup Y$ as set of vertices and one edge $e_{(x,y)}$ connecting any vertex $x \in X$ and any vertex $y \in Y$.

Proposition 5.3.2 *The following conditions are equivalent for a graph $G = (V, E)$:*

1. *the 2-Krull-Schmidt Property holds for the graph G;*
2. *there exist a complete bipartite graph $B(X \dot\cup Y)$ and an injective monoid homomorphism $V(G) \to V(B(X \dot\cup Y))$ that sends atoms to atoms.*

Proof. (1)⇒(2) Let $\sim$ and $\equiv$ be two equivalence relations on E such that, for every $l_1, \dots, l_n, e_1, \dots, e_m \in E$, $\delta_{l_1} + \dots + \delta_{l_n} = \delta_{e_1} + \dots + \delta_{e_m}$ in $V(G)$ if and only if $m = n$ and there exist two permutations σ, τ of $\{1, \dots, n\}$ such that $l_i \sim e_{\sigma(i)}$ and $l_i \equiv e_{\tau(i)}$ for every $i = 1, \dots, n$. Let $E/\sim$ be the quotient set of E modulo $\sim$. The canonical projection $\pi_\sim \colon E \to E/\sim$ induces a monoid homomorphism $\hat{\pi_\sim}$ of $V(G)$ into the free commutative monoid $\mathbb{N}_0^{(E/\sim)}$. It is defined as follows. If we denote by $[l]_\sim$ the image of $l \in E$ in $E/\sim$, then $\hat{\pi_\sim}$ maps an arbitrary element $\delta_{l_1} + \dots + \delta_{l_n}$ of $V(G)$ to $\delta_{[l_1]_\sim} + \dots + \delta_{[l_n]_\sim}$. Similarly, for the other equivalence relation $\equiv$, we get a monoid homomorphism $\hat{\pi_\equiv} \colon V(G) \to \mathbb{N}_0^{(E/\equiv)}$, and the product morphism $\hat{\pi_\sim} \times \hat{\pi_\equiv} \colon V(G) \to \mathbb{N}_0^{(E/\sim)} \times \mathbb{N}_0^{(E/\equiv)}$ is injective.

Now consider the bipartite graph $B(X \dot\cup Y)$ with $X = E/\sim$ and $Y = E/\equiv$. The monoid $V(B(X \dot\cup Y))$ is the submonoid of the free commutative monoid $\mathbb{N}_0^{(X \dot\cup Y)}$, which has $X \dot\cup Y = E/\sim \dot\cup\, E/\equiv$ as free class of generators, generated by the elements $\delta_{[l]_\sim} + \delta_{[e]_\equiv} \in \mathbb{N}_0^{(X \dot\cup Y)}$. Since the image of $\hat{\pi_\sim} \times \hat{\pi_\equiv}$ is generated by the elements $\delta_{[l]_\sim} + \delta_{[l]_\equiv}$ with $l \in E$, it follows that the image of the injective monoid morphism $\hat{\pi_\sim} \times \hat{\pi_\equiv}$ is contained in $V(B(X \dot\cup Y))$. Finally, the atoms δ_l of $V(G)$ are mapped by $\hat{\pi_\sim} \times \hat{\pi_\equiv}$ to the atoms $\delta_{[l]_\sim} + \delta_{[l]_\equiv}$ of $V(B(X \dot\cup Y))$.

(2)$\Rightarrow$(1) Suppose that there exist a complete bipartite graph $B(X \dot\cup Y)$ and an injective monoid homomorphism $\varphi \colon V(G) \to V(B(X \dot\cup Y))$ that sends atoms to atoms. For every edge $e \in E$, the atom $\delta_e \in V(G)$ is sent by φ to an atom $\varphi(\delta_e) = \delta_{e_X} + \delta_{e_Y} \in V(B(X \dot\cup Y))$, where e_X is a vertex in X and e_Y is a vertex in Y. Define an equivalence relation $\sim$ on E as follows. Given two edges $e, l \in E$, we have $e \sim l$ if $e_X = l_X$. Similarly define an equivalence relation $\equiv$ on E by $e \equiv l$ if and only if $e_Y = l_Y$. Since φ is injective, for edges $e_1, \dots, e_n, l_1, \dots, l_m$, we have that $\sum_{i=1}^n \delta_{e_i} = \sum_{j=1}^m \delta_{l_j}$ if and only if $\varphi(\sum_{i=1}^n \delta_{e_i}) = \sum_{i=1}^n \varphi(\delta_{e_i}) = \sum_{i=1}^n \delta_{e_{i_X}} + \delta_{e_{i_Y}}$ equals $\varphi(\sum_{j=1}^m \delta_{l_i}) = \sum_{j=1}^m \varphi(\delta_{l_i}) = \sum_{j=1}^m \delta_{l_{i_X}} + \delta_{l_{i_Y}}$. This clearly happens if and only if $m = n$ and there are two bijections σ, τ of $\{1, \dots, n\}$ such that $\delta_{e_{i_X}} = \delta_{l_{\sigma(i)_X}}$ and $\delta_{e_{i_Y}} = \delta_{l_{\tau(i)_Y}}$ for every $i = 1, \dots, n$. Hence $\sum_{i=1}^n \delta_{e_i} = \sum_{j=1}^m \delta_{l_j}$ if and only if $m = n$ and $e_i \sim l_{\sigma(i)}$ and $e_i \equiv l_{\tau(i)}$ for every $i = 1, \dots, n$. ■

Corollary 5.3.3 *The 2-Krull-Schmidt Property holds for a graph G if and only if it holds for all the connected components of G.*

Condition (2) of Proposition 5.3.2 is hereditary in the sense that if it holds for a graph G, it holds for any subgraph of G.

Corollary 5.3.4 *If the 2-Krull-Schmidt Property holds for a graph G, it also holds for any subgraph of G.*

Corollary 5.3.5 *The 2-Krull-Schmidt Property holds for any bipartite graph G.*

Proof. A bipartite graph G is contained in a complete bipartite graph $B(X \dot\cup Y)$ and the inclusion $V(G) \to V(B(X \dot\cup Y))$ sends atoms to atoms. ■

If $\aleph$ is a cardinal number, the complete graph on a set of vertices of cardinality $\aleph$ will be denoted by $K_\aleph$. If $\aleph \geq \beth \geq 1$ are cardinal numbers, the complete bipartite graph $B(X \dot\cup Y)$ with $|X| = \aleph$ and $|Y| = \beth$ will be denoted by $K_{\aleph,\beth}$.

(The graphs we deal with are sometimes large, in the sense that vertices and edges can form classes that are not sets. It is clear that we can define the complete graph K_α and the complete bipartite graph $K_{\alpha,\beta}$ for arbitrary classes α and β also).

Proposition 5.3.6 *If the 2-Krull-Schmidt Property holds for a graph G, then G does not contain copies of the complete graph K_4.*

Proof. By corollary 5.3.4 it suffices to show that the 2-Krull-Schmidt Property does not hold for the complete graph $G = K_4$. Assume the contrary. Then $G = K_4$ has six edges $l_1, \dots, l_6$ such that $\delta_{l_1} + \delta_{l_2} = \delta_{l_3} + \delta_{l_4} = \delta_{l_5} + \delta_{l_6}$, and, by Proposition 5.3.2, there are a complete bipartite graph $B(X \dot\cup Y)$ and an injective monoid homomorphism $\psi\colon V(G) \to V(B(X \dot\cup Y))$ that sends atoms to atoms. Let l'_i be the edge of $B(X \dot\cup Y)$ such that $\delta_{l'_i} = \psi(\delta_{l_i})$. Then $l'_1, \dots, l'_6$ are six distinct edges of $B(X \dot\cup Y)$ and $\delta_{l'_1} + \delta_{l'_2} = \delta_{l'_3} + \delta_{l'_4} = \delta_{l'_5} + \delta_{l'_6}$. This implies that any vertex of l'_1 is incident both to l'_3 or l'_4 and to l'_5 or l'_6, hence has degree at least three in the subgraph of $B(X \dot\cup Y)$ with the six edges l'_i. Since the sum of the degrees of the vertices is equal to twice the number of edges, that is, is equal to twelve, it follows that the subgraph of $B(X \dot\cup Y)$ with the six edges l'_i has four vertices of degree three. Hence this subgraph of $B(X \dot\cup Y)$ is isomorphic to K_4. In particular, K_4 would be a bipartite graph, which is a contradiction. ∎

Proposition 5.3.7 *If a graph G has at most one cycle of odd length in each connected component, then the 2-Krull-Schmidt Property holds for the graph G.*

Proof. By Corollary 5.3.3, we can assume the graph $G = (V, E)$ connected. By Corollary 5.3.5 we can assume the connected graph G has exactly one cycle $l_1, \dots, l_n$ of odd length $n \geq 3$. The edge l_1 is not on any cycle of even length, because if $l_1, e_1, \dots, e_m$ is a cycle of even length $m+1$, then $e_1, \dots, e_m, l_n, l_{n-1}, \dots, l_2$ would be another cycle of odd length $m + n - 1$. Thus the graph G and the graph $G' = (V, E \setminus \{l_1\})$ have the same cycles of even length. It follows that $V(G) \cong V(G') \oplus \mathbb{N}_0$. Now apply Proposition 5.3.2 and Corollary 5.3.5. ∎

Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$. We will say that $\mathcal{D}$ *satisfies condition (DSP)* if whenever we have $A \oplus B \cong C \oplus D$ in $\mathcal{C}$ with $A, B, C \in \mathcal{D}$, then also $D \in \mathcal{D}$.

Every full subcategory $\mathcal{D}$ of an additive category $\mathcal{C}$ with splitting idempotents has a *(DSP)-closure*, that is, there is a smallest full subcategory $\mathcal{D}'$ of $\mathcal{C}$, containing $\mathcal{D}$ and satisfying condition (DSP). Define $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}'_{n+1}$ as the full subcategory of $\mathcal{C}$ whose class of objects is the class of object of $\mathcal{D}'_n$ together with the objects $D \in \mathcal{C}$ for which there exist $A, B, C \in \mathcal{D}'_n$ with $A \oplus B \cong C \oplus D$, for every $n \geq 0$. Then the full subcategory of $\mathcal{C}$ whose class of objects is the union of the classes of objects of the categories $\mathcal{D}'_n$ is the (DSP)-closure of $\mathcal{D}$.

Recall that if codim denotes the dual Goldie dimension and A, B are arbitrary objects of an additive category $\mathcal{C}$, then

$$\operatorname{codim}(\operatorname{End}_{\mathcal{C}}(A \oplus B)) = \operatorname{codim}(\operatorname{End}_{\mathcal{C}}(A)) + \operatorname{codim}(\operatorname{End}_{\mathcal{C}}(B)).$$

If $\mathcal{D}$ is a full subcategory of indecomposable objects of type ≤ 2 of an additive category $\mathcal{C}$, then its (DSP)-closure $\mathcal{D}'$ is a full subcategory of $\mathcal{C}$ of indecomposable objects with semilocal endomorphism ring of dual Goldie dimension ≤ 2. To see this, argue by induction on n. If $A, B, C \in \mathcal{D}'_n$ have semilocal endomorphism ring of dual Goldie dimension ≤ 2 and $A \oplus B \cong C \oplus D$, then the endomorphism ring of D cannot have dual Goldie dimension ≥ 3, otherwise C has local endomorphism ring, hence $C \cong A$ and $B \cong D$ or $C \cong B$ and $A \cong D$, contradiction. Hence $\operatorname{codim}(\operatorname{End}_{\mathcal{C}}(D)) \leq 2$. If D is not indecomposable, then it must be a direct sum of two objects with local endomorphism ring, hence isomorphic to either A or B, and we get a contradiction again. Similarly, one sees that if $\mathcal{D}$ is a full subcategory of indecomposable objects of type 2 of an additive category $\mathcal{C}$, then its (DSP)-closure $\mathcal{D}'$ is a full subcategory of $\mathcal{C}$ of indecomposable objects with semilocal endomorphism ring of dual Goldie dimension = 2.

We say that a full subcategory $\mathcal{D}$ of indecomposable objects of type 2 of an additive category $\mathcal{C}$ satisfies *weak (DSP)* if for every $U, U', W \in \mathcal{D}$ such that the edges $V(U)$ and $V(U')$ are not incident, and for every object $X \in \mathcal{C}$, $U \oplus U' \cong W \oplus X$ implies $X \in \mathcal{D}$. By [25, Lemma 5.1], any full subcategory $\mathcal{D}$ of indecomposable objects of type 2 of an additive category $\mathcal{C}$ with splitting idempotents satisfies weak (DSP). If a full subcategory $\mathcal{D}$ of indecomposable objects of type 2 of an additive category $\mathcal{C}$ satisfies weak (DSP), then in the graph $V(\mathcal{D})$ any two distinct vertices connected by a path of length 3 are adjacent.

Lemma 5.3.8 *Let G be a connected graph with the property that any two distinct vertices connected by a path of length 3 are adjacent. Then G is either a complete graph or a complete bipartite graph.*

Proof. Assume that $G = (V, E)$ satisfies the hypotheses. The statement is trivial for $|V| \leq 2$, so we can suppose $|V| \geq 3$. Fix a vertex $v_0 \in V$. Let $X = \{v \in V \mid v \text{ is adjacent to } v_0\}$ and $Y = V \setminus X$, so that in particular $v_0 \in Y$ and $X \neq \varnothing$. Let G' be the subgraph of G defined by $G' = (V, E')$, where $E' = \{\{v, w\} \in E \mid \{v, w\} \cap X \neq \varnothing \text{ and } \{v, w\} \cap Y \neq \varnothing\}$. Then G' is a bipartite graph. Let us prove that it is a complete bipartite graph. If $x \in X$ and $y \in Y$, then x is adjacent to v_0 and y is not. If $y = v_0$, there is an edge between x and y. If $y \neq v_0$, then there is a path in G between y and v_0, which can be shortened to a path of length ≤ 2. Since $y \neq v_0$ is not adjacent to v_0, there is a path of length 2 in G between y and v_0. Hence there is a path of length 3 in G between y and x, so that x and y are adjacent in G.

Now if $E = E'$, then G is a complete bipartite graph and we are done. Assume $E' \subsetneq E$. There exists an edge in E with both vertices either in X or in Y. Suppose that x_0, x_0' are two vertices in X with $\{x_0, x_0'\} \in E$. Fix any two distinct vertices $x, x' \in X$. Then $\{x, v_0\}, \{v_0, x_0\}, \{x_0, x_0'\}, \{x_0', v_0\}, \{v_0, x'\}$ is a path in E of length 5, possibly with equal consecutive edges. In any case, this path can be shortened to a path of length 1 in G between the two distinct vertices x, x'. Thus all vertices of X are adjacent. Now fix any two distinct vertices $y, y' \in Y$. Then $\{y, x_0\}, \{x_0, x_0'\}, \{x_0', y'\} \in E$ is a path of length 3 in G between the two distinct vertices y, y' of G. Thus y and y' are adjacent in G. ∎

Proposition 5.3.9 *Let $\mathcal{C}$ be an additive category. The following conditions are equivalent for a full subcategory $\mathcal{D}$ of $\mathcal{C}$ of indecomposable objects of type 2 satisfying weak (DSP):*

1. *the 2-Krull-Schmidt Property holds for $\mathcal{D}$;*

2. *the graph $G(\mathcal{D})$ does not contain subgraphs isomorphic to K_4;*

3. *every connected component of $G(\mathcal{D})$ is either a complete bipartite graph or isomorphic to K_3.*

Proof. (1)⇒(2) has been proved in 5.3.6.

(2)⇒(3) By Lemma 5.3.8, the graph $G(\mathcal{D})$ is a disjoint union of complete bipartite graphs and complete graphs. The only complete graphs that do not contain subgraphs isomorphic to K_4 are K_1, K_2 and K_3. Now K_1 cannot appear as a connected component of $G(\mathcal{D})$ and $K_2 \cong K_{1,1}$.

(3)⇒(1) follows from Corollaries 5.3.3 and 5.3.5 and Proposition 5.3.7. ∎

The full subcategory $\mathcal{D}_2$ of all indecomposable objects of type 2 of an additive category $\mathcal{C}$ with splitting idempotents satisfies weak (DSP), hence we have the following.

Theorem 5.3.10 *Let $\mathcal{D}_2$ be the full subcategory of all indecomposable objects of type 2 of an additive category $\mathcal{C}$ with splitting idempotents. The 2-Krull-Schmidt Property holds for $\mathcal{D}_2$ if and only if $G(\mathcal{D}_2)$ does not contain subgraphs isomorphic to K_4, if and only if every connected component of $G(\mathcal{D}_2)$ is either isomorphic to K_3 or a complete bipartite graph.*

Theorem 5.3.11 *Let $\mathcal{C}$ be an additive category. Then exactly one of the following two conditions holds for a full subcategory $\mathcal{D}$ of $\mathcal{C}$ of indecomposable objects of type 2 satisfying weak (DSP).*

- *Either there exist two objects $U_1, U_2 \in \mathcal{D}$ such that $U_1 \oplus U_2$ has three non-isomorphic direct-sum decompositions.*

- *Or there exist two ideals $\mathcal{I}, \mathcal{K}$ of the full subcategory $\mathcal{S} = \mathrm{sum}(\mathcal{D})$ of $\mathcal{C}$, whose objects are all finite direct sums of objects in $\mathcal{D}$, with $\mathcal{S}/\mathcal{I}$ and $\mathcal{S}/\mathcal{K}$ amenable semisimple categories and the canonical functor $F\colon \mathcal{S} \to \mathcal{S}/\mathcal{I} \times \mathcal{S}/\mathcal{K}$ isomorphism reflecting.*

Proof. The dichotomy corresponds to the fact if $G(\mathcal{D})$ contains or not a subgraph isomorphic to K_4. For the first point, it suffices to notice that a graph $G = (V, E)$ contains a copy of the graph K_4 if and only if there exist six distinct edges $l_1, \dots, l_6 \in E$ such that $\delta_{l_1} + \delta_{l_2} = \delta_{l_3} + \delta_{l_4} = \delta_{l_5} + \delta_{l_6}$ in $V(G)$.

Hence, suppose that $G(\mathcal{D})$ does not contain subgraphs isomorphic to K_4, or, equivalently, that every connected component of $G(\mathcal{D})$ is either isomorphic to K_3 or bipartite. For every connected component of $G(\mathcal{D})$ isomorphic to K_3, fix an object A with $V(A)$ in the connected component. Let $\mathcal{D}'$ be the full subcategory of $\mathcal{D}$ whose objects are all objects in $\mathcal{D}$ not isomorphic to any of the fixed objects A. Hence the graph $G(\mathcal{D}')$ is now bipartite, because we have interrupted all triangles in $G(\mathcal{D})$, and the graphs $G(\mathcal{D})$ and $G(\mathcal{D}')$ have the same class V of vertices. Let $\mathcal{S}' = \mathrm{sum}(\mathcal{D}')$ be the full subcategory of $\mathcal{S}$ whose objects are all direct sums of finitely many objects in $\mathcal{D}'$. Let $\mathcal{C}_{FT}$ be the full subcategory of $\mathcal{C}$ whose objects are finite direct sums of objects of finite type. Thus we have full subcategories $\mathcal{S}' \subseteq \mathcal{S} \subseteq \mathcal{C}_{FT} \subseteq \mathcal{C}$. Correspondingly, we have a commutative diagram of canonical functors

$$\begin{array}{ccccc} \mathcal{S}' & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{C}_{FT} \\ \Big\downarrow{\scriptstyle F'} & & \Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle U} \\ \bigoplus_{\mathcal{I}} \mathcal{S}'/\mathcal{I} & \xrightarrow{E} & \bigoplus_{\mathcal{I}} \mathcal{S}/\mathcal{I} & \longrightarrow & \bigoplus_{\mathcal{I}} \mathcal{C}_{FT}/\mathcal{I}, \end{array}$$

where $\mathcal{I}$ varies among the ideals of the category of the form $\mathcal{A}_P$ for some maximal ideal P of the endomorphism ring $\mathrm{End}_{\mathcal{C}}(A)$ of some object A of the category. Since the canonical functor U is full and isomorphism reflecting by Proposition 5.1.5, also F must be isomorphism reflecting. Let $V = X_1 \dot\cup X_2$ be a bipartition corresponding to the bipartite graph $G(\mathcal{D}')$. The square on the left in the previous diagram becomes

$$\begin{array}{ccc} \mathcal{S}' & \longrightarrow & \mathcal{S} \\ \Big\downarrow{\scriptstyle F_1'\times F_2'} & & \Big\downarrow{\scriptstyle F_1\times F_2} \\ \bigoplus_{\mathcal{I}\in X_1} \mathcal{S}'/\mathcal{I} \oplus \bigoplus_{\mathcal{I}\in X_2} \mathcal{S}'/\mathcal{I} & \xrightarrow{E_1\times E_2} & \bigoplus_{\mathcal{I}\in X_1} \mathcal{S}/\mathcal{I} \oplus \bigoplus_{\mathcal{I}\in X_2} \mathcal{S}/\mathcal{I}. \end{array}$$

Consider the ideals $\mathcal{K}er(F_i)$ of the category $\mathcal{S}$, $i = 1,2$. The functor $F_i \colon \mathcal{S} \to \oplus_{\mathcal{I}\in X_i}\mathcal{S}/\mathcal{I}$ induces a faithful functor $G_i \colon \mathcal{S}/\mathcal{K}er(F_i) \to \oplus_{\mathcal{I}\in X_i}\mathcal{S}/\mathcal{I}$. In order to conclude, it suffices to prove that the faithful functor G_i is an equivalence. Now U full implies F full, hence F_i full, so that G_i is full. Similarly for $\mathcal{S}'$. Hence we find a faithful functor $G_i' \colon \mathcal{S}'/\mathcal{K}er(F_i) \to \oplus_{\mathcal{I}\in X_i}\mathcal{S}'/\mathcal{I}$, which now is not only full, but clearly also dense, hence an equivalence. We claim that $\mathcal{S}'/\mathcal{I} \cong \mathcal{S}/\mathcal{I}$ for every ideal $\mathcal{I}$ of the form $\mathcal{A}_P$ for some maximal ideal P of the endomorphism ring $\mathrm{End}_{\mathcal{C}}(A)$ of some object A of the category. From the claim, it will follow that E_i is an equivalence. Now F_i' and E_i dense, imply F_i dense. Thus G_i is an equivalence.

Hence it suffices to prove the claim. Clearly, the inclusion $\mathcal{S}' \to \mathcal{S}$ induces a full and faithful functor $\mathcal{S}'/\mathcal{I} \to \mathcal{S}/\mathcal{I}$. We must prove that this functor is dense, and for this it suffices to show that for any of the originally fixed objects A there exists an object A' of $\mathcal{S}'$ with $A' \cong A$ in the category $\mathcal{S}/\mathcal{I}$. If $\mathcal{I}$ is a vertex of $\mathcal{D}$ that is not one of the two vertices of the edge $V(A)$, we can take $A' = 0$. Assume that $\mathcal{I}$ is one of the two vertices of $V(A)$. We can assume that the connected component of $G(\mathcal{D})$ is the triangle with three edges $V(A_1), V(A_2), V(A_3)$, with three vertices $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$, that each edge is $V(A_i) = \{\mathcal{I}_k, \mathcal{I}_l\}$ for $\{i,k,l\} = \{1,2,3\}$, that $V(A) = V(A_1)$ and $\mathcal{I} = \mathcal{I}_2$. Then $A_1 \cong A_3$ in the category $\mathcal{S}/\mathcal{I}$, so that A_3 has the required property. This proves the claim. ■

Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$ whose class of objects is a class of indecomposable objects of $\mathcal{C}$. An ideal of the category $\mathcal{D}$

is said to be *completely prime* if, for every A, B, C object of $\mathcal{D}$ and morphisms $f: A \to B$ and $g: B \to C$, one has $gf \in \mathcal{I}$ if and only if $f \in \mathcal{I}$ or $g \in \mathcal{I}$. For the rest of the section we will suppose also that a completely prime ideal $\mathcal{I}$ of $\mathcal{D}$ satisfies $\mathcal{I}(A, A) \neq \mathrm{End}_{\mathcal{D}}(A)$ for every object A of $\mathcal{D}$. Notice that if $\mathcal{I}$ is a completely prime ideal of $\mathcal{D}$, in the quotient category $\mathcal{C}/\mathcal{I}$ the endomorphism ring of every object is an integral domain, not necessarily commutative. If A, B are objects of $\mathcal{D}$, we will say that A and B have the same $\mathcal{I}$-class, and write $[A]_{\mathcal{I}} = [B]_{\mathcal{I}}$, if there exist morphisms $f: A \to B$ and $g: B \to A$ in $\mathcal{D}$ that are not in $\mathcal{I}$. Having the same $\mathcal{I}$-class turn out to be an equivalence relation on the class of objects of $\mathcal{D}$.

Proposition 5.3.12 *Let $\mathcal{C}$ be an additive category, $\mathcal{D}$ a full subcategory of $\mathcal{C}$ of indecomposable objects and $\mathcal{D}_2$ its full subcategory consisting of all the objects of type 2. The following conditions are equivalent:*

1. *there exist two completely prime ideals $\mathcal{P}$, $\mathcal{Q}$ of $\mathcal{D}$ such that, for every object $U \in \mathcal{D}$, the set $\mathrm{End}_{\mathcal{D}}(U) \setminus (\mathcal{P}(U,U) \cup \mathcal{Q}(U,U))$ is the set of all automorphisms of U;*

2. *all objects of $\mathcal{D}$ have type ≤ 2 and the graph $G(\mathcal{D}_2)$ is bipartite;*

3. *there exist two additive functors $F_i: \mathcal{D} \to \mathcal{A}_i$, $i = 1, 2$, of the full subcategory $\mathcal{D}$ of $\mathcal{C}$ into two amenable semisimple categories $\mathcal{A}_i$, such that, for every object $U \in \mathcal{D}$, $F_i(U)$ is a simple object of $\mathcal{A}_i$ and for every $f \in \mathrm{End}_{\mathcal{D}}(U)$, f is an automorphism of U if and only if $F_1(f)$ and $F_2(f)$ are automorphisms of $F_1(U)$ and $F_2(U)$ respectively.*

Proof. (1)⇒(2) Assume that there exist two ideals $\mathcal{P}, \mathcal{Q}$ satisfying condition (1) and let U be an object in $\mathcal{D}$. One of the two following conditions hold: either the ideals $\mathcal{P}(U,U)$ and $\mathcal{Q}(U,U)$ are comparable, in which case $\mathrm{End}_{\mathcal{D}}(U)$ is a local ring with unique maximal ideal the biggest ideal among $\mathcal{P}(U,U)$ and $\mathcal{Q}(U,U)$, or $\mathcal{P}(U,U)$ and $\mathcal{Q}(U,U)$ are not comparable, they are the two distinct maximal ideals of $\mathrm{End}_{\mathcal{D}}(U)$, $\mathrm{End}_{\mathcal{D}}(U)$ is a ring of type two, $J(\mathrm{End}_{\mathcal{D}}(U)) = \mathcal{P}(U,U) \cap \mathcal{Q}(U,U)$ and $\mathrm{End}_{\mathcal{D}}(U)/J(\mathrm{End}_{\mathcal{D}}(U))$ is canonically isomorphic to the direct product of the two division rings $\mathrm{End}_{\mathcal{D}}(U)/\mathcal{P}(U,U)$ and $\mathrm{End}_{\mathcal{D}}(U)/\mathcal{Q}(U,U)$. In particular $\mathrm{End}_{\mathcal{D}}(U)$ is a ring of type ≤ 2 for every object $U \in \mathcal{D}$.

Let $\mathcal{I}$ be a vertex of $G(\mathcal{D}_2)$. Then $\mathcal{I}$ is the ideal associated to a maximal ideal I of $\mathrm{End}_{\mathcal{D}}(U)$ for some object $U \in \mathcal{D}_2$. Now $\mathrm{End}_{\mathcal{D}}(U) \setminus (\mathcal{P}(U,U) \cup \mathcal{Q}(U,U))$ is the set of all automorphisms of U, and $\mathrm{End}_{\mathcal{D}}(U)$ is a ring of type 2, so that either $I = \mathcal{P}(U,U)$ or $I = \mathcal{Q}(U,U)$, but not both. Therefore either $\mathcal{P} \subseteq \mathcal{I}$ or $\mathcal{Q} \subseteq \mathcal{I}$ but not both. Let X be the class of the vertices $\mathcal{I}$ with $\mathcal{P} \subseteq \mathcal{I}$ and Y be the class of the vertices $\mathcal{I}$ with $\mathcal{Q} \subseteq \mathcal{I}$, so that the set of vertices of $G(\mathcal{D}_2)$ is $X \,\dot\cup\, Y$. For every object $U \in \mathcal{D}$, the edge $V(U)$ connects the ideal of $\mathcal{D}$ associated to $\mathcal{P}(U,U)$ and the ideal of $\mathcal{D}$ associated to $\mathcal{Q}(U,U)$. That is, a vertex of X and a vertex of Y.

(2)⇒(3) Assume that condition (2) holds. The idea of the proof of Theorem 5.3.11 yields a functor $F': \mathcal{D}_2 \to \oplus_{\mathcal{I}} \mathcal{D}_2/\mathcal{I}$, with $\mathcal{I}$ ranging in the class V of vertices of $G(\mathcal{D}_2)$. The bipartition $V = X_1 \dot\cup X_2$ of $G(\mathcal{D}_2)$ induces two functors $F_i'\colon \mathcal{D}_2 \to \oplus_{\mathcal{I}\in X_i}\mathcal{D}_2/\mathcal{I}$, $i = 1, 2$. Notice that $\oplus_{\mathcal{I}\in X_i}\mathcal{D}_2/\mathcal{I}$ is an amenable semisimple category. For every morphism f in $\mathcal{D}_2$, either $F_1'(f)$ is an isomorphism or $F_1'(f) = 0$. Similarly for $F_2'(f)$. Therefore $\mathcal{P} = \mathcal{K}er(F_1')$ and $\mathcal{Q} = \mathcal{K}er(F_2')$ are completely prime ideals of $\mathcal{D}_2$, obtained as intersections of ideals associated to maximal ideals of the rings $\mathrm{End}_{\mathcal{D}}(U)$, with U ranging in the objects of $\mathcal{D}_2$, and $\mathcal{D}_2/\mathcal{P} \cong \oplus_{\mathcal{I}\in X_1}\mathcal{D}_2/\mathcal{I}$ and $\mathcal{D}_2/\mathcal{Q} \cong \oplus_{\mathcal{I}\in X_2}\mathcal{D}_2/\mathcal{I}$ are amenable semisimple categories. For every object U of $\mathcal{D}_2$, the ideals $\mathcal{P}(U,U)$ and $\mathcal{Q}(U,U)$ are the two maximal ideals of $\mathrm{End}_{\mathcal{D}}(U)$, so that $\mathrm{End}_{\mathcal{D}}(U) \setminus (\mathcal{P}(U,U) \cup \mathcal{Q}(U,U))$ is the set of all automorphisms of U.

We will now extend $\mathcal{P}$ and $\mathcal{Q}$ to completely prime ideals of the category $\mathcal{D}$. Let $\mathcal{J}$ be the Jacobson radical of the category $\mathcal{D}$. Since the ideals $\mathcal{P}$, $\mathcal{Q}$ are intersections of ideals associated to maximal ideals of endomorphism rings of objects of modules of type 2, the restriction of $\mathcal{J}$ to $\mathcal{D}_2$ is contained both in $\mathcal{P}$ and $\mathcal{Q}$. Let $\mathcal{D}_1$ be the full subcategory of $\mathcal{D}$ whose objects are all the objects of $\mathcal{D}$ of type 1. Define $\mathcal{P}(A,B) = \mathcal{J}(A,B)$ if $A \in \mathcal{D}_1$ or $B \in \mathcal{D}_1$. Observe that $\mathcal{J}(A,B) = \mathrm{Hom}_{\mathcal{D}}(A,B)$ provided $A \in \mathcal{D}_1$ and $B \in \mathcal{D}_2$, or $A \in \mathcal{D}_2$ and $B \in \mathcal{D}_1$. Moreover, $\mathcal{J}(A,B)$ is the set of all non-isomorphisms of $\mathrm{Hom}_{\mathcal{D}}(A,B)$ if $A, B \in \mathcal{C}_1$. Now it is straightforward to check that $\mathcal{P}$ is a completely prime ideal in $\mathcal{D}$ and that $\mathcal{D}/\mathcal{P} \cong \mathcal{D}_2/\mathcal{P} \times \mathcal{D}_1/\mathcal{J}$ is an amenable semisimple category. The ideal $\mathcal{Q}$ can be extended to $\mathcal{D}$ in a similar way. The canonical functors $F_1\colon \mathcal{D} \to \mathcal{D}/\mathcal{P}$ and $F_2\colon \mathcal{D} \to \mathcal{D}/\mathcal{Q}$ are the required functors.

(3)⇒(1) It suffices to define $\mathcal{P} = \mathcal{K}er(F_1)$ and $\mathcal{Q} = \mathcal{K}er(F_2)$. ∎

Theorem 5.3.13 *Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$ of indecomposable objects. Let $\mathcal{P}, \mathcal{Q}$ be a pair of completely prime ideals of $\mathcal{D}$ with the property that, for every object $A \in \mathcal{D}$, $f\colon A \to A$ is an automorphism if and only if $f \notin \mathcal{P}(A,A) \cup \mathcal{Q}(A,A)$. Let $A_1, \dots, A_n, B_1, \dots, B_m$ be objects of $\mathcal{D}$. Then the objects $A_1 \oplus \dots \oplus A_n$ and $B_1 \oplus \dots \oplus B_m$ of $\mathcal{C}$ are isomorphic if and only if $n = m$ and there are two permutations σ, τ of $\{1, \dots, n\}$ with $[A_i]_{\mathcal{P}} = [B_{\sigma(i)}]_{\mathcal{P}}$ and $[A_i]_{\mathcal{Q}} = [B_{\tau(i)}]_{\mathcal{Q}}$ for every $i = 1, \dots, n$.*

Proof. (⇒) By Proposition 5.3.12, the objects of $\mathcal{D}$ are of type 1 or 2. Assume $A_1 \oplus \dots \oplus A_n \cong B_1 \oplus \dots \oplus B_m$. We have seen that $m = n$ and that there are two permutations α, β of $\{1, \dots, n\}$ and a positive integer $k \le n$ such that:

- $A_{\alpha(1)}, \dots, A_{\alpha(k)}, B_{\beta(1)}, \dots, B_{\beta(k)}$ have type 2,
- $A_{\alpha(1)} \oplus \dots \oplus A_{\alpha(k)} \cong B_{\beta(1)} \oplus \dots \oplus B_{\beta(k)}$ and
- $A_{\alpha(k+1)} \cong B_{\beta(k+1)}, \dots, A_{\alpha(n)} \cong B_{\beta(n)}$ are objects of type 1.

Assume for the moment, for simplicity of notation, that α and β are the identity permutations. Let $\mathcal{D}_2$ be the full subcategory of $\mathcal{D}$ of objects of type 2. Let $\mathrm{Max}(\mathcal{D}_2)$ be the class of all the ideals $\mathcal{I}$ of $\mathcal{D}_2$ associated to a maximal ideal I of $\mathrm{End}_{\mathcal{D}_2}(A)$ for some object A of $\mathcal{D}_2$. By Proposition 5.1.4, there is an injective homomorphism $\oplus_{\mathcal{I}\in\mathrm{Max}(\mathcal{D}_2)} d_{\mathcal{I}}\colon V(\mathcal{D}_2) \to \mathbb{N}_0^{(\mathrm{Max}(\mathcal{D}_2))}$. In the proof of previous proposition we have seen that, for every $\mathcal{I} \in \mathrm{Max}(\mathcal{D}_2)$, either $\mathcal{P} \subseteq \mathcal{I}$ or $\mathcal{Q} \subseteq \mathcal{I}$, but not both. That is, there is a partition $S_{\mathcal{P}} \dot\cup S_{\mathcal{Q}}$ of $\mathrm{Max}(\mathcal{D}_2)$, where $S_{\mathcal{P}} = \{\mathcal{I} \in \mathrm{Max}(\mathcal{D}_2) \mid \mathcal{P} \subseteq \mathcal{I}\}$ and $S_{\mathcal{Q}} = \{\mathcal{I} \in \mathrm{Max}(\mathcal{D}_2) \mid \mathcal{Q} \subseteq \mathcal{I}\}$. Thus we have an injective morphism $\bigoplus_{\mathcal{I} \in S_{\mathcal{P}}} d_{\mathcal{I}} \oplus \bigoplus_{\mathcal{I} \in S_{\mathcal{Q}}} d_{\mathcal{I}} \colon V(\mathcal{D}_2) \to \mathbb{N}_0^{(S_{\mathcal{P}})} \oplus \mathbb{N}_0^{(S_{\mathcal{Q}})}$. For every A in $\mathcal{D}_2$, there is exactly one $\mathcal{I} \in S_{\mathcal{P}}$ with $d_{\mathcal{I}}(A) = 1$, and $d_{\mathcal{I}}(A) = 0$ for all the other $\mathcal{I}$'s in $S_{\mathcal{P}}$. Similarly for $\mathcal{Q}$. Hence $A_1 \oplus \ldots \oplus A_n \cong B_1 \oplus \ldots \oplus B_m$ implies that there exists a permutation σ of $\{1, \ldots, k\}$ such that, for every $\mathcal{I} \in S_{\mathcal{P}}$, one has $A_i \cong B_{\sigma(i)}$ in the factor category $\mathcal{D}_2/\mathcal{I}$. Similarly, there exists a permutation τ of $\{1, \ldots, k\}$ such that, for every $\mathcal{I} \in S_{\mathcal{Q}}$, one has $A_i \cong B_{\tau(i)}$ in the factor category $\mathcal{D}_2/\mathcal{I}$. Hence, for every $\mathcal{I} \in S_{\mathcal{P}}$ and every $i = 1, \ldots, k$, there exist morphisms $\varphi_{i,\mathcal{I}} \colon A_i \to B_\sigma(i)$ and $\psi_{i,\mathcal{I}} \colon B_{\sigma(i)} \to A_i$ with $\psi_{i,\mathcal{I}}\varphi_{i,\mathcal{I}} - 1_{A_i} \in \mathcal{I}(A_i, A_i)$ and $\varphi_{i,\mathcal{I}}\psi_{i,\mathcal{I}} - 1_{B_{\sigma(i)}} \in \mathcal{I}(B_{\sigma(i)}, B_{\sigma(i)})$. Now, for every $i = 1, \ldots, k$, $\mathcal{P}(A_i, A_i)$ is a maximal ideal of $\mathrm{End}_{\mathcal{D}_2}(A_i)$, so that its associated ideal is an ideal $\mathcal{I} \in \mathrm{Max}(\mathcal{D}_2)$ necessarily contained in $\mathcal{P}$, that is, an ideal $\mathcal{I} \in S_{\mathcal{P}}$. Thus $\psi_{i,\mathcal{I}}\varphi_{i,\mathcal{I}} - 1_{A_i} \in \mathcal{I}(A_i, A_i)$ for this $\mathcal{I}$ implies $\psi_{i,\mathcal{I}}\varphi_{i,\mathcal{I}} \notin \mathcal{P}(A_i, A_i)$. Hence $\varphi_{i,\mathcal{I}} \notin \mathcal{P}(A_i, B_{\sigma(i)})$ and $\psi_{i,\mathcal{I}} \notin \mathcal{P}(B_{\sigma(i)}, A_i)$. Thus $[A_i]_{\mathcal{P}} = [B_{\sigma(i)}]_{\mathcal{P}}$. Similarly, $[A_i]_{\mathcal{Q}} = [B_{\tau(i)}]_{\mathcal{Q}}$ for every $i = 1, \ldots, k$.

Now that we have found the permutations σ and τ, we go back to the notations of the first paragraph of this proof and see that $\beta\sigma\alpha^{-1}$ is a bijection between the A_i's and the B_i's of type 2 that preserves the $\mathcal{P}$ classes. Similarly, $\beta\tau\alpha^{-1}$ is bijection between the A_i's and the B_i's of type 2 that preserves the $\mathcal{Q}$ classes. As $A_{\alpha(i)} \cong B_{\beta(i)}$ for every $i = k+1, \ldots, n$ one sees that $\beta\alpha^{-1}$ is a bijection between the A_i's and the B_i's of type 1 that preserves the isomorphism classes, hence the $\mathcal{P}$ classes. Combining these two bijections, one find a permutation of $\{1, \ldots, n\}$ that preserves the $\mathcal{P}$ classes, as desired. Similarly for the $\mathcal{Q}$ classes.

($\Leftarrow$) Let $A_1, \ldots, A_n, B_1, \ldots, B_n$ be objects in $\mathcal{D}$ and let σ, τ be permutations of $\{1, \ldots, n\}$ with $[A_i]_{\mathcal{P}} = [B_{\sigma(i)}]_{\mathcal{P}}$ and $[A_i]_{\mathcal{Q}} = [B_{\tau(i)}]_{\mathcal{Q}}$ for every $i = 1, \ldots, n$. Let $\mathcal{S}$ denote the full subcategory of $\mathcal{C}$ whose objects are all finite direct sums of finitely many objects in $\mathcal{D}$. In order to show that $A_1 \oplus \ldots \oplus A_n \cong B_1 \oplus \ldots \oplus B_n$, it suffices to show that their images in the category $\mathcal{S}/\mathcal{I}$ are isomorphic objects of $\mathcal{S}/\mathcal{I}$ for every ideal $\mathcal{I}$ of $\mathcal{S}$ associated to a maximal ideal I of the endomorphism ring of some object A in $\mathcal{D}$. Assume $A \in \mathcal{D}$ and let I be a maximal ideal of the ring $\mathrm{End}_{\mathcal{D}}(A)$. As we have seen in the proof of Proposition 5.3.12((1)⇒(2)), either $I = \mathcal{P}(A, A)$ or $I = \mathcal{Q}(A, A)$. Suppose, for instance, that $I = \mathcal{P}(A, A)$. Then $\mathcal{I} \supseteq \mathcal{P}$. Let us prove that $A_i \cong B_{\sigma(i)}$ in $\mathcal{S}/\mathcal{I}$ for every $i = 1, \ldots, n$. In the factor category $\mathcal{S}/\mathcal{I}$, the images of the objects of $\mathcal{D}$ are either zero or isomorphic to the image of A by Lemma 4.1.1. Hence it suffices to show that A_i is zero in $\mathcal{S}/\mathcal{I}$ if and only if $B_{\sigma(i)}$ is zero in $\mathcal{S}/\mathcal{I}$. If A_i is zero in $\mathcal{S}/\mathcal{I}$, then $\mathcal{I}(A_i, A_i) = \mathrm{End}_{\mathcal{D}}(A_i)$, so that $hl \in I = \mathcal{P}(A, A)$ for every $h \colon A_i \to A$ and $l \colon A \to A_i$. In order to prove that $B_{\sigma(i)}$ is zero in $\mathcal{S}/\mathcal{I}$, fix $\alpha \colon A \to B_{\sigma(i)}$ and $\beta \colon B_{\sigma(i)} \to A$. We must prove that $\beta\alpha \in I$. From $[A_i]_{\mathcal{P}} = [B_{\sigma(i)}]_{\mathcal{P}}$, we get that there are morphisms $f \colon A_i \to B_{\sigma(i)}$ and $g \colon B_{\sigma(i)} \to A_i$ with $f \notin \mathcal{P}(A_i, B_{\sigma(i)})$ and $g \notin \mathcal{P}(B_{\sigma(i)}, A_i)$. If $\beta\alpha \notin I = \mathcal{P}(A, A)$, then $\beta, \alpha \notin \mathcal{P}$, so that $\beta f \notin \mathcal{P}(A_i, A)$ and $g\alpha \notin \mathcal{P}(A, A_i)$ because $\mathcal{P}$ is completely prime, contradicting the fact that $hl \in I = \mathcal{P}(A, A)$ for every $h \colon A_i \to A$ and $l \colon A \to A_i$. This proves that $\beta\alpha \in I$, so that $B_{\sigma(i)}$ is zero in $\mathcal{S}/\mathcal{I}$. Similarly $B_{\sigma(i)}$ zero in $\mathcal{S}/\mathcal{I}$ implies A_i zero in $\mathcal{S}/\mathcal{I}$. Thus $A_i \cong B_{\sigma(i)}$ in $\mathcal{S}/\mathcal{I}$ for every $i = 1, \dots, n$, so that $A_1 \oplus \dots \oplus A_n \cong B_1 \oplus \dots \oplus B_n$ in $\mathcal{S}/\mathcal{I}$ for every associated ideal $\mathcal{I}$. This concludes the proof. ∎

We conclude the chapter with some examples of completely prime ideals and applications of the theory we have developed above.

Example 5.3.14 If R is a ring, $\mathcal{U}$ is the full subcategory of Mod-R whose objects are all uniform right R-modules, and $\mathcal{P}$ is defined by $\mathcal{P}(A,B) = \{f: A \to B \mid f \text{ non-injective}\}$, then $\mathcal{P}$ is a completely prime ideal of $\mathcal{U}$.

Example 5.3.15 Dually, if $\mathcal{C}$ is the category of all couniform right R-modules and $\mathcal{Q}(A,B)$ consists of all non-surjective morphisms from A to B, then $\mathcal{Q}$ is a completely prime ideal of $\mathcal{C}$.

Example 5.3.16 If $\mathcal{B}$ is the category of all biuniform right R-modules and $\mathcal{P}$, $\mathcal{Q}$ are the restrictions to $\mathcal{B}$ of the previous completely prime ideals, then the pair $\mathcal{P}$, $\mathcal{Q}$ satisfies the hypotheses of Theorem 5.3.13. The class $\mathcal{B}$ satisfies (DSP).

Example 5.3.17 If $\mathcal{C}$ is the full subcategory of Mod-R whose objects are all right R-modules whose endomorphism is local, the Jacobson radical is a completely prime ideal of $\mathcal{C}$. The class $\mathcal{C}$ satisfies (DSP).

Example 5.3.18 Let $\mathcal{K}$ be the full subcategory of Mod-R whose objects are all kernels of morphisms $f: E_1 \to E_2$, where E_1 and E_2 range in the class of all uniform injective modules. If E_1, E_2, E_1', E_2' are uniform injective modules and $\varphi: E_1 \to E_2$ and $\varphi': E_1' \to E_2'$ are two non-injective morphisms, any morphisms $f: \ker(\varphi) \to \ker(\varphi')$ extend to a morphism $f_1: E_1 \to E_2$. Moreover f_1 induces a morphism $\tilde{f}_1: E_1/\ker(\varphi) \to E_2/\ker(\varphi')$, which extends to a morphism $f_2: E_2 \to E_2'$. If $\mathcal{Q}$ is defined by $\mathcal{Q}(\ker(\varphi), \ker(\varphi')) = \{f: \ker(\varphi) \to \ker(\varphi') \mid f_2 \text{ is not injective}\}$, then $\mathcal{Q}$ is a completely prime ideal of $\mathcal{K}$. If $\mathcal{P}$ is the restriction of the ideal in Example 5.3.14 to the category $\mathcal{K}$, then the pair $\mathcal{P}$, $\mathcal{Q}$ satisfies the hypotheses of Theorem 5.3.13. The class $\mathcal{K}$ satisfies condition (DSP). Cf. [11].

Example 5.3.19 Dualizing our previous example, we say that a module M_R is *couniformly presented* if it is non-zero and there exists an exact sequence

$$0 \to C_R \to P_R \to M_R \to 0$$

with P_R projective and both C_R and P_R couniform. Given any two couniformly presented modules M_R and M_R' with their couniform presentations $0 \to C_R \to P_R \to M_R \to 0$ and $0 \to C_R' \to P_R' \to M_R' \to 0$, every morphism $f: M_R \to M_R'$ lift to a morphism $f_0: P_R \to P_R'$, that induces a morphism $f_1: C_R \to C_R'$ by restriction. If $\mathcal{P}$ is defined by $\mathcal{P}(M_R, M_R') = \{f: M_R \to M_R' \mid f_1 \text{ is not surjective}\}$, then $\mathcal{P}$ is a completely prime ideal of the full subcategory $\mathcal{S}$ of Mod-R whose objects are all couniformly presented modules. If $\mathcal{Q}$ is the restriction of the ideal in Example 5.3.15 to the category $\mathcal{S}$, then the pair $\mathcal{P}$, $\mathcal{Q}$ satisfies the hypotheses of Theorem 5.3.13. The class $\mathcal{S}$ satisfies condition (DSP). Cf. [17].

Example 5.3.20 Let R be a ring and let S_1, S_2 be two fixed non-isomorphic simple right R-modules. Let $\mathcal{C}$ be the full subcategory of Mod-R whose objects are all artinian right R-modules A_R with $\mathrm{soc}(A_R) \cong S_1 \oplus S_2$. Set $\mathcal{P}_i(A, B) = \{f\colon A \to B \mid f(\mathrm{soc}_{S_i}(A_R)) = 0\}$. The pair of completely prime ideals $\mathcal{P}_1$, $\mathcal{P}_2$ satisfies the hypotheses of Theorem 5.3.13.

Chapter 6

The infinite Krull-Schmidt Property in the case 2

We saw in the previous chapter some examples of categories where the 2-Krull-Schmidt Property holds. A natural question is to ask what happens when one considers arbitrary direct sums instead of finite ones. The purpose of this chapter is to study, in an abstract setting, the Infinite 2-Krull-Schmidt Property. To achieve the most possible generality, we first investigate the problem at a monoid theoretical level. Since usual monoids do not allow infinite sums, we introduce a new algebraic structure, that we call commutative infinitary monoid, where arbitrary infinite sums are possible. In section 6.2 we define this new structure and look at its first properties, showing that there is a canonical way to pass from usual commutative monoids to infinitary ones. Then we define properly the Infinite 2-Krull-Schmidt Property and we give a complete description of the phenomenon (Theorem 6.3.6). Eventually, we apply our results to the main example of commutative infinitary monoid, that is the skeleton $V(\mathcal{C})$ of a cocomplete category $\mathcal{C}$, endowed with the coproduct as operation.

6.1 Completely prime ideals and associated ideals

Given an ideal $\mathcal{I}$ of a preadditive category $\mathcal{C}$ and two objects $A, B \in \mathcal{C}$, we write $A \sim_{\mathcal{I}} B$ when there exist morphisms $\alpha: A \to B$ and $\beta: B \to A$ such that $\alpha \notin \mathcal{I}$ and $\beta \notin \mathcal{I}$. In general, the relation $A \sim_{\mathcal{I}} B$ is only symmetric. Recall that $\mathcal{I}$ is a completely prime ideal of $\mathcal{C}$ if the composition $gf: A \to C$ of two morphisms $f: A \to B$ and $g: B \to C$ is in $\mathcal{I}$ if and only if $f \in \mathcal{I}$ or $g \in \mathcal{I}$ and every object of $\mathcal{C}$ is non-zero in $\mathcal{C}/\mathcal{I}$. If this is the case, then the relation $\sim_{\mathcal{I}}$ is also reflexive and transitive. We denote by $[A]_{\mathcal{I}}$ the equivalence class of $\sim_{\mathcal{I}}$ containing A.

Lemma 6.1.1 *Let $\mathcal{C}$ be a preadditive category and let $\mathcal{I}$ be a completely prime ideal of $\mathcal{C}$. Let A be an object of $\mathcal{C}$. Let $I = \mathcal{I}(A, A)$ and $\mathcal{A}_I$ be the ideal of $\mathcal{C}$*

associated to I. Then the following are equivalent for an object B of $\mathcal{C}$:

1. *B is a non-zero object in $\mathcal{C}/\mathcal{A}_I$;*
2. *$[A]_{\mathcal{I}} = [B]_{\mathcal{I}}$;*
3. *$\mathcal{A}_I(B,B) = \mathcal{I}(B,B) \neq \mathrm{End}_{\mathcal{C}}(B)$.*

If $\mathrm{End}_{\mathcal{C}}(A)/I$ is a division ring, then the previous are equivalent also to:

4. *$\mathrm{End}_{\mathcal{C}}(B)/\mathcal{A}_I(B,B)$ is a division ring.*

Proof.

(1)⇔(2) B is a non-zero object of $\mathcal{C}/\mathcal{A}_I$ if and only if $1_B \notin \mathcal{A}_I(B,B)$. By definition of associated ideal, this means that there exist a morphism $\alpha\colon A \to B$ and a morphism $\beta\colon B \to A$ such that $\beta 1_B \alpha = \beta\alpha \notin I = \mathcal{I}(A,A)$. Since $\mathcal{I}$ is completely prime, this is equivalent to saying that α and β are not in $\mathcal{I}$.

(2)⇒(3) It is enough to prove that $\mathcal{A}_I(B,B) = \mathcal{I}(B,B)$. If we suppose that an endomorphism f of B is in $\mathcal{I}(B,B)$, then $\beta f\alpha \in I = \mathcal{I}(A,A)$ for every $\alpha\colon A \to B$ and $\beta\colon B \to A$. Then $f \in \mathcal{A}_I(B,B)$. Conversely, suppose $f \notin \mathcal{I}(B,B)$. By (2) we know that there exist morphisms $\alpha\colon A \to B$ and $\beta\colon B \to A$ such that $\alpha,\beta \notin \mathcal{I}$. Therefore, since $\mathcal{I}$ is completely prime, $\beta f\alpha \notin I = \mathcal{I}(A,A)$ and this means that $f \notin \mathcal{A}_I(B,B)$.

(3)⇒(1) Obvious.

Assume now that $\mathrm{End}_{\mathcal{C}}(A)/I$ is a division ring.

(2)⇒(4) If (2) holds, there exist a morphism $\alpha\colon A \to B$ and a morphism $\beta\colon B \to A$ such that $\alpha,\beta \notin \mathcal{I}$. Since $\mathrm{End}_{\mathcal{C}}(A)/I$ is a division ring, we have that, for any $f \in \mathrm{End}_{\mathcal{C}}(B)$ that is not in $\mathcal{A}_I(B,B) = \mathcal{I}(B,B)$, there exists $g \in \mathrm{End}_{\mathcal{C}}(A)$ such that $1_A - \beta f\alpha g \in I$. Then also $\alpha(1_A - \beta f\alpha g)\beta = \alpha\beta(1_B - f\alpha g\beta)$ is in $\mathcal{I}(B,B)$. Since $\mathcal{I}$ is completely prime and α,β are not in $\mathcal{I}$, we have that $1_B - f\alpha g\beta \in \mathcal{I}(B,B)$. In other words, $\alpha g\beta + \mathcal{I}(B,B)$ is a right inverse for $f + \mathcal{I}(B,B)$ in $\mathrm{End}_{\mathcal{C}}(B)/\mathcal{I}(B,B) = \mathrm{End}_{\mathcal{C}}(B)/\mathcal{A}_I(B,B)$.

(4)⇒(1) Obvious. ■

Remark 6.1.2 Let $\mathcal{C}$ be a preadditive category and let $\mathcal{I}$ be a completely prime ideal of $\mathcal{C}$. Let A and B be two objects of $\mathcal{C}$. Then in $\mathcal{C}$ there is not a biproduct of A and B. To see this, suppose that in $\mathcal{C}$ there exists a biproduct $A \oplus B$ and consider the canonical embeddings ϵ_A, ϵ_B and the canonical projections π_A, π_B. We have that $1_A = \pi_A\epsilon_A$ and $1_B = \pi_B\epsilon_B$ imply that $\epsilon_A, \epsilon_B, \pi_A, \pi_B$ are not in $\mathcal{I}$. But, since $\mathcal{I}$ is completely prime, $\pi_B\epsilon_A = 0$ implies that either π_B or ϵ_A is in $\mathcal{I}$. This is a contradiction.

Following [23], if A is an object of a preadditive category $\mathcal{C}$ and I is an ideal in $\mathrm{End}_{\mathcal{C}}(A)$, we say that A is *I-small* if for every family of objects M_λ, $\lambda \in \Lambda$, and morphisms $\alpha\colon A \to \oplus_{\lambda\in\Lambda} M_\lambda$ and $\beta\colon \oplus_{\lambda\in\Lambda} M_\lambda \to A$ with $\beta\alpha \notin I$, there exists $\mu \in \Lambda$ such that $\beta\epsilon_\mu\pi_\mu\alpha \notin I$.

If R is a ring and $\mathcal{C}$ is a preadditive full subcategory of Mod-R, we denote by $\mathrm{Sum}(\mathcal{C})$ the category whose objects are direct sums of (possibly infinitely many) objects in $\mathcal{C}$.

At this point, we can generalize Proposition 5.1.4 to the infinite case.

Proposition 6.1.3 *Let R be a ring, $\mathcal{C}$ be a full preadditive subcategory of* Mod-R *and let $\mathcal{I}$ be a completely prime ideal of $\mathcal{C}$. Let A be an object of $\mathcal{C}$ such that* $\mathrm{End}_{\mathcal{C}}(A)/\mathcal{I}(A,A)$ *is a division ring. Let $I = \mathcal{I}(A,A)$ and $\mathcal{A}_I$ be the ideal of* $\mathrm{Sum}(\mathcal{C})$ *associated to I. If A is I-small, then the category* $\mathrm{Sum}(\mathcal{C})/\mathcal{A}_I$ *is equivalent to the category of all right vector spaces over the division ring* $\mathrm{End}_{\mathcal{C}}(A)/\mathcal{I}(A,A)$.

Proof. By Lemma 6.1.1 we have that, for every object B of $\mathcal{C}$, when we pass to the factor category $\mathcal{C}/\mathcal{A}_I$, either $B = 0$ or $B \cong A$. Then it is enough to use [23, Lemma 3.2] ∎

Therefore there is a direct-summand preserving functor of $\mathrm{Sum}(\mathcal{C})$ into the category of right vector spaces over the division ring $\mathrm{End}_{\mathcal{C}}(A)/I$ with the property that, for every object $X = \oplus_{\lambda \in \Lambda} M_\lambda$ of $\mathrm{Sum}(\mathcal{C})$, with the M_λ's in $\mathcal{C}$, the dimension of the vector space corresponding to X is equal to the cardinality of the set $\{\, \lambda \in \Lambda \mid [M_\lambda]_{\mathcal{I}} = [A]_{\mathcal{I}} \,\}$. Hence this cardinality depends only on X and not on the direct-sum representation $X = \oplus_{\lambda\in\Lambda} M_\lambda$ of X as a direct sum of elements of $\mathcal{C}$.

6.2 Commutative infinitary monoids

Let M be a class. If $\aleph$ is a cardinal number, we can define the class $M^{\aleph} = \{ f\colon \aleph \to M \mid f \text{ is a function } \}$. An $\aleph$-*operation* on M is a function $p_\aleph\colon M^\aleph \to M$.

We define a *commutative infinitary monoid* to be a class M together with an operation $p_\aleph$ for every cardinal number $\aleph$ such that:

- $p_1\colon M^1 \to M$ is the canonical bijection that sends the map $f\colon 1 \to M$, defined by $f(1) = m$, to the element $m \in M$;

- if $\aleph_i, i \in I$, and $\aleph$ are cardinal numbers, $\gamma_i\colon \aleph_i \to \aleph$, $i \in I$, are injective maps such that $\aleph = \dot{\bigcup}_{i\in I}\gamma_i(\aleph_i)$ and $\aleph_I = |I|$, then, for any $f \in M^\aleph$, $p_\aleph(f) = p_{\aleph_I}(\Gamma)$, where $\Gamma \in M^I$ is the function from I to M defined by $\Gamma(i) = p_{\aleph_i}(f\gamma_i)$, for any $i \in I$.

Remark 6.2.1 In a similar way to what we did above, we can define classes of monoids where one can perform sums only up to a given regular cardinal. In fact, if $\beth$ is a regular cardinal, one defines a *commutative* $\beth$-*infinitary monoids* to be a class M together with operations $p_\aleph$ for every cardinal number $\aleph < \beth$ satisfying the above axioms. In this fashion, it is easy to notice that commutative $\aleph_0$-infinitary monoids are exactly the usual commutative monoids.

Denote by e the image of the empty function, that is the only element of M^0, through p_0. The element e behaves like an identity element for M, i.e. if we sum infinitely many times e to an element $m \in M$ we obtain m again. To show this, consider any pair of cardinals $\aleph$ and $\beth$ and any function $f\colon \aleph \to M$; by our second axiom, choosing $\aleph_1 = \aleph$ and $\aleph_i = 0$ for every $i \in \beth$, we obtain the equality $p_\aleph(f) = p_{\beth+1}(\Gamma_f)$, where $\Gamma_f \in M^{\beth+1}$ is the function from $\beth + 1$ to M that sends 1 to $p_\aleph(f)$ and $i \in \beth$ to $p_0(\varnothing) = e$ for every $i \in \beth$.

If we choose $|I| = 1$ in our second axiom, we obtain that $p_\aleph(f) = p_\aleph(f\sigma)$ for any cardinal $\aleph$, any $f \in M^\aleph$ and any permutation $\sigma\colon \aleph \to \aleph$. This means that the order of the summands does not influence the result. Hence it makes sense to call such a structure commutative.

Similarly, it is easy to show that the associative property holds.

Example 6.2.2 Let Card be the class of all cardinal numbers. Endow Card with the operations $p_\aleph\colon \mathrm{Card}^\aleph \to \mathrm{Card}$ defined by $p_\aleph(f) = \sum_{i\in\aleph} f(i)$. Then Card becomes a commutative infinitary monoid. In fact $p_1(f_\beth) = \beth$ for every $\beth \in \mathrm{Card}$, where $f_\beth$ is the map that sends 1 to $\beth$. Moreover, given a cardinal $\aleph$ and injective maps $\gamma_i\colon \aleph_i \to \aleph$, $i \in I$ such that $\aleph = \dot{\bigcup}_{i\in I}\gamma_i(\aleph_i)$, we have that $p_\aleph(f) = \sum_{i\in\aleph} f(i)$ is equal to $p_{\aleph_I}(\Gamma) = \sum_{i\in I} p_{\aleph_i}(f\gamma_i) = \sum_{i\in I}\sum_{j\in\aleph_i} f\gamma_i(j)$.

If I is a class and for any $i \in I$ we have a commutative infinitary monoid M_i, we define in the following way their *direct sum* $\oplus_{i\in I} M_i$. The elements of $\oplus_{i\in I} M_i$ are the sets $\{(i_\lambda, m_{i_\lambda})\}_{\lambda\in\Lambda}$, where Λ is a set, the i_λ are distinct elements of I and m_{i_λ} is an element of M_{i_λ} for every $\lambda \in \Lambda$. We identify two elements $\{(i_\lambda, m_{i_\lambda})\}_{\lambda\in\Lambda}$ and $\{(j_\mu, m_{j_\mu})\}_{\mu\in\Lambda'}$ of $\oplus_{i\in I} M_i$ when $m_{i_\lambda} = m_{j_\mu}$ if $i_\lambda = j_\mu$, $m_{i_\lambda} = e_{i_\lambda}$ for every i_λ that is different from any j_μ, $\mu \in \Lambda'$, and $m_{j_\mu} = e_{j_\mu}$ for every j_μ that is different from any i_λ, $\lambda \in \Lambda$.

Define the operations $p^\oplus_\aleph$ on $\oplus_{i\in I} M_i$ in the following way. If $f \in (\oplus_{i\in I} M_i)^\aleph$ is a function from $\aleph$ to $\oplus_{i\in I} M_i$ that sends $a \in \aleph$ to $\{(i^a_\lambda, m^a_{i_\lambda})\}_{\lambda\in\Lambda_a}$, set $p^\oplus_\aleph(f) = \{(i^f_\lambda, p^i_\aleph(f_{i_\lambda}))\}_{\lambda\in\Lambda}$ where $\Lambda = \bigcup_{a\in\aleph}\Lambda_a$ and $f_{i_\lambda} \in M_i^\aleph$ is the function from $\aleph$ to M_i that sends $a \in \aleph$ to $m^a_{i_\lambda} \in M_i$, where without loss of generality we can consider $m^a_{i_\lambda} = e$ if $\lambda \notin \Lambda_a$.

We can consider the category $\widehat{Mon}$ of all commutative infinitary monoids. The morphisms between two commutative infinitary monoids M_1 and M_2 are the functions α between the classes M_1 and M_2 such that for every cardinal $\aleph$ and every $f \in M_1^\aleph$, we have $\alpha p^1_\aleph(f) = p^2_\aleph(\alpha f)$.

Proposition 6.2.3 *Let I be a class and let M_i be a commutative infinitary monoid for every $i \in I$. Then the direct sum $\oplus_{i\in I} M_i$ is the coproduct of the M_i's in the category of commutative infinitary monoids.*

Proof. For any index $\bar{\imath} \in I$, there is a morphism of commutative infinitary monoids $f_{\bar{\imath}}$ from $M_{\bar{\imath}}$ to $\oplus_{i\in I} M_i$ that sends every element $m \in M_{\bar{\imath}}$ to $\{(\bar{\imath}, m)\}$. If there is another commutative infinitary monoid N together with morphisms of commutative infinitary monoids $g_i\colon M_i \to N$ for every $i \in I$, then there exists a unique morphism of commutative infinitary monoids $f\colon \oplus_{i\in I} M_i \to N$ such that

$g_i = f \circ f_i$ for every $i \in I$, defined in the following way: an element $\{(i_\lambda, m_{i_\lambda})\}_{\lambda\in\Lambda}$ of $\oplus_{i\in I} M_i$ is sent by f to the element $p^N_\aleph(\iota)$ where $\aleph = |\Lambda|$ and $\iota \in N^\aleph$ is the function from Λ to N that sends $\lambda \in \Lambda$ to $g_{i_\lambda}(m_{i_\lambda})$. ∎

We notice that, if I is a set, then $\oplus_{i\in I} M_i$ is the product of the monoids M_i. If I is not a set, but a proper class, the product of the commutative infinitary monoids M_i, $i \in I$, does not exist in MK.

Lemma 6.2.4 *The direct sum of copies of* Card*, indexed in a class I, is a free object on a basis I in the category of commutative infinitary monoids.*

Proof. Let I be a class and $F \cong \oplus_{i\in I}\mathrm{Card}$ be the direct sum of $|I|$ copies of Card. The canonical injection is the map of classes $\iota: I \to F$ that sends an element $i \in I$ to the element $\{(i,1)\} \in F$. Let M be any commutative infinitary monoid with a map of classes $\kappa: I \to M$. We need to prove that there is a unique morphism of commutative infinitary monoids $\tilde\kappa: F \to M$ such that $\kappa = \tilde\kappa\iota$. It is clear that we need to define $\tilde\kappa(\{(i_\lambda, \aleph_\lambda)\}_{\lambda\in\Lambda}) = p^M_{|\Lambda|}(p)$ where $p: |\Lambda| \to M$ is the map that associates to every $\lambda \in \Lambda$ the element $p^M_{\aleph_\lambda}(\kappa_{i\lambda})$, and $k_{i_\lambda}: \aleph_\lambda \to M$ sends any $s \in \aleph_\lambda$ to $\kappa(i_\lambda)$. ∎

An element m of M is an *atom* if $m = p_{\aleph_2}(m_1, m_2)$ implies $m_1 = e$ or $m_2 = e$. We say that M is *atomic* if every element of M is a sum of possibly infinitely many atoms. Unless otherwise explicitly mentioned, lowercase letter will always denote atoms of an atomic commutative infinitary monoid and we will write generic elements of an atomic commutative infinitary monoid as sum of atoms.

We will identify every function $f \in M^\aleph$ with the image of the elements of $\aleph$ and denote any $\aleph$-operation $p_\aleph$ of M by the symbol $\sum$, since this will not create any confusion.

It is clear that every commutative infinitary monoid M can be seen as a usual commutative monoid. Indeed, this gives rise to a forgetful functor ^{f} from the category $\widehat{Mon}$ of commutative infinitary monoids to the category Mon of usual commutative monoids. The functor ^{f} forgets all the $\aleph$-operations $p_\aleph$ for $\aleph \geq \aleph_0$.

There is also a functor from the category of commutative monoids to the category of commutative infinitary monoids. We can define it in the following way. Let M be a commutative monoid. Consider the class $\{\aleph \to M \mid \aleph \in \mathrm{Card}\}$ of all the functions from a cardinal to M and quotient it by the equivalence relation $\sim$ defined by $(f: \aleph_1 \to M) \sim (g: \aleph_2 \to M)$ if and only if $\aleph_1 = \aleph_2$ and there exists a bijection $\sigma: \aleph_1 \to \aleph_2$ such that $f = g\sigma$. On this class, that we will denote by M', we define the $\aleph$-operations $p_\aleph$. Given a cardinal $\aleph$, we define the $\aleph$-operation $p_\aleph: M'^\aleph \to M'$ by juxtaposition of functions, i.e. if $f \in M'^\aleph$ sends $s \in \aleph$ to $f_s: \beth_s \to M$, the $\aleph$-operation $p_\aleph$ sends f to

$$\begin{array}{rccc} p_\aleph(f): & \dot\bigcup_{s\in\aleph}\beth_s & \to & M \\ & x \in \beth_s & \mapsto & f_s(x). \end{array}$$

In this way the class M' becomes a commutative infinitary monoid. Our functor from the category of commutative monoids to the category of commutative infinitary monoids is defined sending a commutative monoid M to the commutative infinitary monoid $\hat{M}$ obtained by taking the quotient of the commutative infinitary monoid M' with respect to the congruence generated by the relation $\alpha+\beta \equiv \gamma$, where $\alpha: 1 \to M'$ sends 1 to m_1, $\beta: 1 \to M'$ sends 1 to m_2 and $\gamma: 1 \to M'$ sends 1 to $m_1 + m_2$.

If we are given a morphism of commutative monoids $f: M_1 \to M_2$, the morphism of commutative infinitary monoids $\hat{f}: \hat{M}_1 \to \hat{M}_2$ is defined as follows. If $\alpha: \aleph \to M_1$ is an element of $\hat{M}_1$, its image through $\hat{f}$ is defined to be $\hat{f}(\alpha) = f\alpha: \aleph \to M_2$.

Proposition 6.2.5 *The functors* ${}^f: \widehat{Mon} \to Mon$ *is right adjoint to the functor* $\hat{}: Mon \to \widehat{Mon}$.

Proof. We have to prove that for every commutative monoid M and every commutative infinitary monoid N, there is a bijection $\mathrm{Hom}_{Mon}(M, N^f) \leftrightarrow \mathrm{Hom}_{\widehat{Mon}}(\hat{M}, N)$. Let α be a morphism of commutative monoids between M and N^f. To it we associate the morphism of commutative infinitary monoids $p\hat{\alpha}$, where p is the morphism of commutative infinitary monoids defined by

$$\begin{array}{rccc} p: & \widehat{N^f} & \to & N \\ & (h: \aleph \to N^f) & \mapsto & p_\aleph(h). \end{array}$$

Conversely, if $\beta: \hat{M} \to N$ is a morphism of commutative infinitary monoids, we associate to it he morphism of commutative monoids $\beta^f q$, where q is the morphism of commutative monoids defined by

$$\begin{array}{rccc} q: & M & \to & \hat{M}^f \\ & m & \mapsto & (q(m): 1 \mapsto m). \end{array}$$

To prove the proposition we need to prove that the composition of the two maps that we defined is the identity in both ways. To show this, let α be a morphism of commutative monoids between M and N^f. We have

$$\begin{array}{rcl} (p\hat{\alpha})^f q(m) & = & (p\hat{\alpha})^f (q(m): 1 \mapsto m) = (p\hat{\alpha})(q(m): 1 \mapsto m) \\ & = & p_1(\alpha q(m): 1 \mapsto \alpha(m)) = \alpha(m), \end{array}$$

for any $m \in M$. Hence $(p\hat{\alpha})^f q = \alpha$. On the other hand, if $\beta: \hat{M} \to N$ is a morphism of commutative infinitary monoids and $\kappa: \aleph \to M$ is an element of $\hat{M}$, we have

$$\begin{array}{rcl} p\widehat{(\beta^f q)}(\kappa) & = & p_\aleph\left(\begin{array}{ccc} \aleph & \to & N^f \\ i & \mapsto & \beta^f q\kappa(i) = \beta(q(\kappa(i)): 1 \mapsto \kappa(i)) \end{array} \right) \\ & = & \beta(p_\aleph(q(\kappa(i)): 1 \mapsto \kappa(i))) = \beta(\kappa) \end{array}$$

and this proves $p\widehat{(\beta^f q)} = \beta$. ■

6.3 Infinite 2-Krull-Schmidt Property

In the setting of commutative infinitary monoids, we can generally define when a monoid satisfies an infinite Krull-Schmidt property. We say that the *Infinite Krull-Schmidt Property* holds for an atomic commutative infinitary monoid M if, given two families $\{ a_i \mid i \in I \}$ and $\{ b_j \mid j \in J \}$ of atoms of M, we have $\sum_{i\in I} a_i = \sum_{j\in J} b_j$ if and only if there exists a bijection $\sigma: I \to J$ such that $a_i = b_{\sigma(i)}$ for every $i \in I$. It is clear that a commutative infinitary monoid satisfies the Infinite Krull-Schmidt Property if and only if it is free.

We say that the *Infinite 2-Krull-Schmidt Property* holds for an atomic commutative infinitary monoid if there exist two equivalence relations $\sim$ and $\equiv$ on the class A of atoms of M such that, given two families $\{ a_i \mid i \in I \}$ and $\{ b_j \mid j \in J \}$ of atoms of M, we have $\sum_{i\in I} a_i = \sum_{j\in J} b_j$ if and only if there exist two bijections $\sigma, \tau: I \to J$ such that $a_i \sim b_{\sigma(i)}$ and $a_i \equiv b_{\tau(i)}$ for every $i \in I$.

In this chapter, by a *graph* we mean a class V of vertices together with a class E of edges, which are 2-element subsets of V. We can associate an atomic commutative infinitary monoid $M(G)$ to any graph $G = (V, E)$. Given a graph $G = (V, E)$, where the elements of E are subsets of V of cardinality 2, consider the free commutative infinitary monoid $F(V)$ on the basis $\{ \delta_v \mid v \in V \}$. If $l = \{v, w\} \in E$ is an edge of G, define $\delta_l = \delta_v + \delta_w \in F(V)$. Let $M(G)$ be the commutative infinitary submonoid of $F(V)$ generated by the elements δ_l, where l ranges in E.

More generally, given a class A we denote by the symbol $F(A)$ the free commutative infinitary monoid on the basis A.

Proposition 6.3.1 *Let M be an atomic commutative infinitary monoid. Then the following are equivalent:*

1. *the Infinite 2-Krull-Schmidt Property holds for M;*
2. *there exist a complete bipartite graph $B(X \dot\cup Y)$ and an injective morphism of commutative infinitary monoids $M \to M(B(X \dot\cup Y))$ that sends atoms to atoms.*

Proof. Suppose that the Infinite 2-Krull-Schmidt Property holds for M. Let A be the class of atoms of M and let $A/\sim$ and $A/\equiv$ be the quotient classes of A modulo the equivalence relations $\sim$ and $\equiv$, respectively. The canonical projection $\pi_\sim: A \to A/\sim$ induces a morphism of commutative infinitary monoids $\widehat{\pi_\sim}: M \to F(A/\sim)$ defined as follows, $\widehat{\pi_\sim}(\sum_{i\in I} a_i) = \sum_{i\in I} \pi_\sim(a_i)$. Similarly, the other canonical projection $\pi_\equiv: A \to A/\equiv$ induces a morphism of commutative infinitary monoids $\widehat{\pi_\equiv}: M \to F(A/\equiv)$ defined by $\widehat{\pi_\equiv}(\sum_{i\in I} a_i) = \sum_{i\in I} \pi_\equiv(a_i)$.

Since the Infinite 2-Krull-Schmidt Property holds, the product morphism $\widehat{\pi_\sim} \times \widehat{\pi_\equiv}: M \to F(A/\sim) \times F(A/\equiv)$ is injective.

Now consider the complete bipartite graph $B(X \dot\cup Y)$, where $X = A/\sim$ and $Y = A/\equiv$. The monoid $M(B(X \dot\cup Y))$ is the submonoid of the free commutative infinitary monoid $F(X \dot\cup Y)$ on the basis $X \dot\cup Y = A/\sim \dot\cup A/\equiv$, generated by the elements $[l]_\sim + [e]_\equiv$, with l and e ranging in A. Since the image of $\widehat{\pi_\sim} \times \widehat{\pi_\equiv}$ is generated by the elements $[l]_\sim + [l]_\equiv$, with $l \in A$, it follows that the image of the injective morphism of commutative infinitary monoids $\widehat{\pi_\sim} \times \widehat{\pi_\equiv}$ is contained in $M(B(X \dot\cup Y))$.

Finally, the atoms a of M are mapped by $\widehat{\pi_\sim} \times \widehat{\pi_\equiv}$ to the atoms $[a]_\sim + [a]_\equiv$ of $M(B(X \dot\cup Y))$. This completes the proof of (1) ⇒ (2).

Assume now that there exist a complete bipartite graph $B(X \dot\cup Y)$ and an injective morphism of commutative infinitary monoids $\varphi: M \to M(B(X \dot\cup Y))$ that respects infinite sums and sends atoms to atoms. Therefore every atom a of M is sent by φ to an atom $\varphi(a) = x_a + y_a$ of $M(B(X \dot\cup Y))$, with $x_a \in X$ and $y_a \in Y$. Since φ preserves infinite sums, an element $\sum_{i \in I} a_i$ of M is sent to $\sum_{i \in I}(x_{a_i} + y_{a_i})$. Therefore two elements $a = \sum_{i \in I} a_i$ and $b = \sum_{j \in J} b_j$ of M are equal if and only if $\varphi(a) = \sum_{i \in I} x_{a_i} + y_{a_i}$ is equal to $\varphi(b) = \sum_{j \in J} x_{b_j} + y_{b_j}$. This is equivalent to say that there exist two bijections $\sigma, \tau: I \to J$ such that $x_{a_i} = x_{b_{\sigma(i)}}$ and $y_{a_i} = y_{b_{\tau(i)}}$ for every $i \in I$. Defining $a \sim b$ if and only if $x_a = x_b$ and $a \equiv b$ if and only if $y_a = y_b$ it becomes clear that the Infinite 2-Krull-Schmidt Property holds. ∎

We can interpret the commutative infinitary monoid associate to a graph from another point of view.

Example 6.3.2 Let K and L be two classes. Consider the class D of all pairs $\begin{pmatrix} a \\ b \end{pmatrix}$ with $a \in K$ and $b \in L$. Let F be the free commutative infinitary monoid on the basis D. We can look to the elements of F as $2 \times \aleph$ matrices, with $\aleph$ any cardinal number, such that all the entries of the first row are in K and all the entries of the second row are in L; with this interpretation, the operation of F is just the juxtaposition of matrices. We have to be careful since two $2 \times \aleph$ matrices M and N are equal in F if there exist a bijection $\rho: \aleph \to \aleph$ such that the i-th column of M is equal to the $\rho(i)$-th column of N, for every $i \in \aleph$. On F we consider the congruence $\sim$ defined by the following: given two $2 \times \aleph$ matrices M and N we have $M \sim N$ if and only if there are two bijections $\sigma, \tau: \aleph \to \aleph$ such that $m_{1,i} = n_{1,\sigma(i)}$ and $m_{2,i} = n_{2,\tau(i)}$ for every $i \in \aleph$. Lousily speaking, $M \sim N$ if and only if the first row of M has the same entries of the first row of N, counting with multiplicity, and the second row of M has the same entries of the second row of N, counting with multiplicity.

Let C be the quotient monoid of F by the congruence $\sim$. Then C is a commutative infinitary monoid and it is clear that the Infinite 2-Krull-Schmidt Property holds for C.

It easy to see that the commutative infinitary monoids constructed in this way are exactly the monoids $M(B(X \dot\cup Y))$. In fact it is enough to take $X = K$ and $Y = L$. Hence, given any atomic commutative infinitary monoid M for which the Infinite 2-Krull-Schmidt Property holds with respect to the equivalence relations $\sim$ and $\equiv$, there exist an atomic commutative infinitary monoid C constructed as in Example 6.3.2 and an injective morphism of commutative infinitary monoids $\varphi: M \to C$. It is enough to take K equal to the class of equivalence classes of the atoms of M with respect to the equivalence relation $\sim$

and L equal to the class of equivalence classes of the atoms of M with respect to the equivalence relation $\equiv$. Then we can define $\varphi\colon M \to C$ as the morphism of commutative infinitary monoids that sends an atom $a \in M$ to the matrix $\begin{pmatrix} [a]_\sim \\ [a]_\equiv \end{pmatrix}$. Since the Infinite 2-Krull-Schmidt Property holds both for M and C, it is clear that φ is well-defined and injective.

Lemma 6.3.3 *Let $\sim$ and $\equiv$ be two equivalence relations on a class S. Then the following are equivalent:*

1. *$a \sim b$ and $a \equiv c$ implies that there exists $d \in S$ such that $d \sim c$ and $d \equiv b$;*
2. *$[b]_\sim \cap [c]_\equiv \neq \varnothing$ implies $[b]_\equiv \cap [c]_\sim \neq \varnothing$;*
3. *the composite relation $\sim \circ \equiv$ is symmetric;*
4. *$\sim \circ \equiv$ is equal to $\equiv \circ \sim$.*

We say that two relations on a class S are *permutable* if they satisfy the equivalent conditions of Lemma 6.3.3. It is clear that if $\sigma = \tau$, then σ and τ are permutable. We will see more interesting examples of permutable relations in the following chapters.

We say that the *Strong Infinite 2-Krull-Schmidt Property* holds for an atomic commutative infinitary monoid M if the Infinite 2-Krull-Schmidt Property holds with respect to two permutable equivalence relations, i.e. if there exist two permutable equivalence relations $\sim$ and $\equiv$ on the class A of atoms of M such that, given two families $\{\, a_i \mid i \in I \,\}$ and $\{\, b_j \mid j \in J \,\}$ of atoms of M, we have $\sum_{i \in I} a_i = \sum_{j \in J} b_j$ if and only if there exist two bijections $\sigma, \tau\colon I \to J$ such that $a_i \sim b_{\sigma(i)}$ and $a_i \equiv b_{\tau(i)}$ for every $i \in I$.

Theorem 6.3.4 *Let M be an atomic commutative infinitary monoid. Suppose that there are two permutable equivalence relations $\sim$ and $\equiv$ on the class A of atoms of M such that:*

1. *$a = b$ if and only if $a \sim b$ and $a \equiv b$;*
2. *if $a \sim b$ and $a \equiv c$, there exists an element $d \in A$ such that $a + d = b + c$, $d \sim c$ and $d \equiv b$.*

Let $\{\, a_i \mid i \in I \,\}$ and $\{\, b_j \mid j \in J \,\}$ be two families of atoms of M. Then $\sum_{i \in I} a_i = \sum_{j \in J} b_j$ if there exist two bijections $\sigma, \tau\colon I \to J$ such that $a_i \sim b_{\sigma(i)}$ and $a_i \equiv b_{\tau(i)}$ for every $i \in I$.

Proof. Consider two families $\{\, a_i \mid i \in I \,\}$ and $\{\, b_j \mid j \in J \,\}$ of atoms of M and suppose that there are two bijections $\sigma, \tau\colon I \to J$ such that $a_i \sim b_{\sigma(i)}$ and $a_i \equiv b_{\tau(i)}$ for every $i \in I$. We have to show that $\sum_{i \in I} a_i = \sum_{j \in J} b_j$.

The symmetric group $\mathcal{S}_I$ consisting of all bijections $I \to I$ acts on the family I in a natural way. Let C be the cyclic subgroup of $\mathcal{S}_I$ generated by $\tau^{-1}\sigma \in S_I$. Then C acts on the family I. For every element $i \in I$ let $[i] = \{\, (\tau^{-1}\sigma)^z(i) \mid$

$z \in \mathbb{Z}\}$ denote the C-orbit of i. Let $\sigma([i])$ be the image of the orbit i via the bijection σ.

Fix $i \in I$. We claim that $\sum_{k\in[i]} a_k = \sum_{l\in\sigma([i])} b_l$.

Set $i_z = (\tau^{-1}\sigma)^z(i)$, $j_z = \sigma(i_z)$, $a_z = a_{i_z}$ and $b_z = b_{j_z}$. In this notation the equality $a_i \sim b_{\sigma(i)}$ for every $i \in I$ implies that $a_z \sim b_z$ for every $z \in \mathbb{Z}$ and similarly the equality $a_i \equiv b_{\tau(i)}$ implies that $a_z \equiv b_{z-1}$ for every $z \in \mathbb{Z}$.

We now prove that there are elements x_n and y_n in A satisfying the following properties for every $n \geq 1$:

- $x_{n-1} + y_n = b_{n-1} + b_{-n}$ and $x_n + y_n = a_n + a_{-n}$;
- $x_n \sim a_n$ and $x_n \equiv a_{-n}$;
- $y_n \sim b_{-n}$ and $y_n \equiv b_{n-1}$.

Set $x_0 = a_0$. Since $a_0 \sim b_0$ and $a_0 \equiv b_{-1}$, there is an element $y_1 \in A$ such that $b_0 + b_{-1} = x_0 + y_1$ with $y_1 \sim b_{-1}$ and $y_1 \equiv b_0$. Therefore $a_1 \equiv b_0 \equiv y_1$ and $a_{-1} \sim b_{-1} \sim y_1$. Hence there is an element $x_1 \in A$ such that $a_1 + a_{-1} = x_1 + y_1$ with $x_1 \sim a_1$ and $x_1 \equiv a_{-1}$. Thus x_0, x_1 and y_1 have the required properties.

Now let $n > 1$ and suppose that x_t and y_t satisfying the required properties have already been constructed for $t < n$. Since $x_{n-1} \sim a_{n-1} \sim b_{n-1}$ and $x_{n-1} \equiv a_{-n+1} \equiv b_{-n}$, there exists an element $y_n \in A$ such that $x_{n-1} + y_n = b_{n-1} + b_{-n}$ with $y_n \sim b_{-n}$ and $y_n \equiv b_{n-1}$. From $a_{-n} \sim b_{-n}$ and $a_n \equiv b_{n-1}$ it follows that $y_n \sim a_{-n}$ and $y_n \equiv a_n$. Again, there exists an element $x_n \in A$ such that $x_n + y_n = a_n + a_{-n}$ with $x_n \sim a_n$ and $x_n \equiv a_{-n}$.

Now suppose that the orbit $[i]$ is an infinite set. Then

$$\begin{aligned}
\sum_{k\in[i]} a_k &= \sum_{n\in\mathbb{Z}} a_n = a_0 + \sum_{n\geq 1}(a_n + a_{-n}) = x_0 + \sum_{n\geq 1}(x_n + y_n) = \\
&= \sum_{n\geq 1}(x_{n-1} + y_n) = \sum_{n\geq 1}(b_{n-1} + b_{-n}) = \sum_{n\in\mathbb{Z}} b_n = \\
&= \sum_{l\in\sigma([i])} b_l.
\end{aligned}$$

Now suppose that the orbit $[i]$ is a finite set with $q = 2n+1$ elements. Then $x_n \sim a_n \sim b_n$ and $x_n \equiv a_{-n} \equiv b_{-n-1} = b_n$ imply $x_n = b_n$ and

$$\begin{aligned}
\sum_{k\in[i]} a_k &= a_0 + \sum_{k=1}^{n}(a_k + a_{-k}) = x_0 + \sum_{k=1}^{n}(x_k + y_k) = \\
&= \sum_{k=1}^{n}(x_{k-1} + y_k) + x_n = \sum_{k=1}^{n}(b_{k-1} + b_k) + b_n = \\
&= \sum_{l\in\sigma([i])} b_l.
\end{aligned}$$

For the case $q = 2n$, $n \geq 1$, we have $a_n = a_{-n}$, hence $y_n \sim a_{-n}$ and $y_n \equiv a_n$ imply that $y_n = a_n$. Then

$$\begin{aligned}\sum_{k\in[i]} a_k &= a_0 + \sum_{k=1}^{n-1}(a_k + a_{-k}) + a_n = x_0 + \sum_{k=1}^{n-1}(x_{k-1} + y_k) + y_n = \\ &= \sum_{k=1}^{n}(x_{k-1} + y_k) = \sum_{k=1}^{n}(b_{n-1} + b_{-n}) = \\ &= \sum_{l\in\sigma([i])} b_l.\end{aligned}$$

When the index i runs over all the indices in I, we get that the orbits $[i]$ form a partition of I into disjoint countable subsets $I = \bigcup_{i\in I}[i]$ and their images form a partition of J into disjoint countable subsets $J = \bigcup_{i\in I}\sigma([i])$. By the claim $\sum_{k\in[i]} a_k = \sum_{l\in\sigma([i])} b_l$ for every orbit $[i]$, so that $\sum_{i\in I} a_i = \sum_{j\in J} b_j$. ∎

We say that an atomic monoid M is *a-cancellative* if for any atom b and any other elements $c, d \in M$, we have $b + c = b + d \Rightarrow c = d$.

We say that an equivalence relation $\sim$ on the class A of atoms of an atomic commutative infinitary monoid *controls the infinite* if the following is true: consider two sets $\{a_i \mid i \in I\}$ and $\{b_j \mid j \in J\}$ of atoms of M and the sets $I_\sim(k) = \{\, i \in I \mid a_i \sim a_k \,\}$ and $J_\sim(k) = \{\, j \in J \mid b_j \sim a_k \,\}$ for $k \in I$. Whenever $\sum_{i\in I} a_i = \sum_{j\in J} b_j$ and the sets $I_\sim(k)$ and $J_\sim(k)$ are both infinite, for every $t \in I_\sim(k)$ there exists a subset $A(t) \subseteq J$ with $|A(t)| \le |I_\sim(k)|$ such that $J_\sim(k) \subseteq \bigcup_{t\in I_\sim(k)} A(t)$ and, similarly, for every $u \in J_\sim(k)$ there exists a subset $B(u) \subseteq I$ with $|B(u)| \le |J_\sim(k)|$ such that $I_\sim(k) \subseteq \bigcup_{u\in J_\sim(k)} B(u)$.

Theorem 6.3.5 *Let M be an atomic a-cancellative commutative infinitary monoid. Suppose that there are two permutable equivalence relations $\sim$ and $\equiv$ on the class A of atoms of M such that:*

1. *$a = b$ if and only if $a \sim b$ and $a \equiv b$;*
2. *if a is a summand of $\sum_{j\in J} b_j$, then there exist $j_1, j_2 \in J$ such that $a \sim b_{j_1}$ and $a \equiv b_{j_2}$;*
3. *if $a \sim b$ and $a \equiv c$, there exists an element $d \in A$ such that $a + d = b + c$, $d \sim c$ and $d \equiv b$.*

If $\sim$ controls the infinite, the equality $\sum_{i\in I} a_i = \sum_{j\in J} b_j$ implies that there is a bijection $\sigma\colon I \to J$ such that $a_i \sim b_{\sigma(i)}$ for every $i \in I$.

Proof. Fix an index $k \in I$ and consider the two subclasses $I_\sim(k) = \{\, i \in I \mid a_i \sim a_k \,\}$ of I and $J_\sim(k) = \{\, j \in J \mid b_j \sim a_k \,\}$ of J. It is obvious that the $I_\sim(k), k \in I$, form a partition of I. Note that the $J_\sim(k), k \in I$, also form a partition of J because for every $j \in J$ there is a $k \in I$ with $b_j \sim a_k$ and for every $k \in I$ there is a $j \in J$ with $b_j \sim a_k$ by (2).

In order to establish the existence of the bijection $\sigma\colon I \to J$ preserving the $\sim$-classes of $\{\, a_i \mid i \in I \,\}$ and $\{\, b_j \mid j \in J \,\}$, it is sufficient to prove that the cardinalities $|I_\sim(k)|$ and $|J_\sim(k)|$ are equal for every $k \in I$.

Suppose first that either $I_\sim(k)$ or $J_\sim(k)$ is a finite set. Without loss of generality we may assume $|I_\sim(k)| \le |J_\sim(k)|$. Suppose that $|I_\sim(k)| < |J_\sim(k)|$. Take $\bar\imath \in I_\sim(k)$; then, by (2), there exist $j_1, j_2 \in J$ such that $a_{\bar\imath} \sim b_{j_1}$ and $a_{\bar\imath} \equiv b_{j_2}$.

If $j_2 \in J_\sim(k)$, then $a_{\bar\imath} = b_{j_2}$ by (1) and, since M is a-cancellative, we get $\sum_{i \in I \smallsetminus \{\bar\imath\}} a_i = \sum_{j \in J \smallsetminus \{j_2\}} b_j$.

On the other hand, if $j_2 \notin J_\sim(k)$ we have that, by (3), there exists $d \in A$ such that $d \sim b_{j_2} \not\sim a_{\bar\imath}$, $d \equiv b_{j_1}$ and $b_{j_1} + b_{j_2} = a_{\bar\imath} + d$. Then $\sum_{i \in I} a_i = \sum_{j \in J} b_j = b_{j_1} + b_{j_2} + \sum_{j \in J \smallsetminus \{j_1, j_2\}} b_j = a_{\bar\imath} + d + \sum_{j \in J \smallsetminus \{j_1, j_2\}} b_j$ implies $\sum_{i \in I \smallsetminus \{\bar\imath\}} a_i = d + \sum_{j \in J \smallsetminus \{j_1, j_2\}} b_j$.

An easy induction shows that after $|I_\sim(k)|$ steps we get the required contradiction.

Now suppose that $I_\sim(k)$ and $J_\sim(k)$ are both infinite. By symmetry it is sufficient to prove that $|J_\sim(k)| \le |I_\sim(k)|$. By hypothesis, for every $t \in I_\sim(k)$ there exists a subclass $A(t) \subseteq J$ with $|A(t)| \le I_\sim(k)$ such that $J_\sim(k) \subseteq \bigcup_{t \in I_\sim(k)} A(t)$. Looking at the cardinalities, we obtain that $|J_\sim(k)| \le |I_\sim(k)||I_\sim(k)| = |I_\sim(k)|$. Hence $|J_\sim(k)| = |I_\sim(k)|$ if $I_\sim(k)$ and $J_\sim(k)$ are both infinite. ∎

Combining the results of Theorem 6.3.4 and Theorem 6.3.5, we obtain the following.

Theorem 6.3.6 *Let M be an atomic commutative infinitary monoid and let $\sim$ and $\equiv$ be two permutable equivalence relations on the class A of atoms of M. Then the following are equivalent:*

1. *the Strong Infinite 2-Krull-Schmidt Property holds for $\sim$ and $\equiv$;*
2. *the following hypotheses hold for M:*
   - *(a) M is a-cancellative;*
   - *(b) $a = b$ if and only if $a \sim b$ and $a \equiv b$;*
   - *(c) if a is a summand of $\sum_{j \in J} b_j$ there exist $j_1, j_2 \in J$ such that $a \sim b_{j_1}$ and $a \equiv b_{j_2}$;*
   - *(d) if $a \sim b$ and $a \equiv c$, there exists an element $d \in A$ such that $a + d = b + c$, $d \sim c$ and $d \equiv b$;*
   - *(e) $\sim$ controls the infinite;*
   - *(f) $\equiv$ controls the infinite.*
3. *There exist a complete bipartite graph $B(X \dot\cup Y)$ and an injective morphism of commutative infinitary monoids $\varphi\colon M \to M(B(X \dot\cup Y))$ that sends atoms to atoms and such that, for any $z_1 \in X$ and $z_2 \in Y$, we have $d(z_1, z_2) \le 3$ in $\varphi(M)$ implies $d(z_1, z_2) = 1$ in $\varphi(M)$.*

Proof. (1)⇒(2) Suppose that the Strong Infinite 2-Krull-Schmidt Property holds for $\sim$ and $\equiv$. Then it is clear that $a = b$ if and only if $a \sim b$ and $a \equiv b$. If a is a summand of $\sum_{j \in J} b_j$, there exist atoms $c_k, k \in K$, such that $a + \sum_{k \in K} c_k = \sum_{j \in J} b_j$; then by the Infinite 2-Krull-Schmidt Property, there exist $j_1, j_2 \in J$ such that $a \sim b_{j_1}$ and $a \equiv b_{j_2}$. Conversely, if $a \sim b_{j_1}$ and $a \equiv b_{j_2}$, by the permutability of $\sim$ and $\equiv$, there exists a $d \in M$ such that $d \sim b_{j_2}$ and $d \equiv b_{j_1}$; then it is clear that $a + d = b_{j_1} + b_{j_2}$. To prove that M is a-cancellative, let a be an atom and $\sum_{j\in J} b_j$ and $\sum_{k\in K} c_k$ be generic elements of M; then, if $a + \sum_{j\in J} b_j = a + \sum_{k\in K} c_k$, by the Infinite 2-Krull-Schmidt Property we get that there exist two bijections $\sigma, \tau \colon J' \to K'$, with $|J'| = |J| + 1$ and $|K'| = |K| + 1$, such that the conclusion of the Theorem holds; σ and τ clearly induces bijections between J and K for which the conclusion of the Theorem holds, but then $\sum_{j\in J} b_j = \sum_{k\in K} c_k$ and M is a-cancellative.

To prove (e) and (f), suppose $\sum_{i\in I} a_i = \sum_{j\in J} b_j$. By the Infinite 2-Krull-Schmidt Property, there exist two bijections $\sigma, \tau \colon I \to J$ such that $a_i \sim b_{\sigma(i)}$ and $a_i \equiv b_{\tau(i)}$ for every $i \in I$. Then we can take $A(t) = \{\sigma(t)\}$ and we get $J_\sim(K) \subseteq \bigcup_{t\in I_\sim(K)} \{\sigma(t)\}$. Similarly, defining $B(u) = \{\sigma^{-1}(u)\}$, we get $I_\sim(K) \subseteq \bigcup_{u\in J_\sim(K)} \{\sigma^{-1}(u)\}$. We can do the same for $\equiv$ using τ instead of σ.

(2)$\Rightarrow$(1) The hypothesis of Theorem 6.3.4 are satisfied and then we have $\sum_{i\in I} a_i = \sum_{j\in J} b_j$ if there exist two bijections $\sigma, \tau \colon I \to J$ such that $a_i \sim b_{\sigma(i)}$ and $a_i \equiv b_{\tau(i)}$ for every $i \in I$.

It is clear that if we assume (2) all the hypothesis of Theorem 6.3.5 are satisfied both for $\sim$ and $\equiv$. Then $\sum_{i\in I} a_i = \sum_{j\in J} b_j$ implies that there exist two bijections $\sigma, \tau \colon I \to J$ such that $a_i \sim b_{\sigma(i)}$ and $a_i \equiv b_{\tau(i)}$ for every $i \in I$.

We just proved that the Strong Infinite 2-Krull-Schmidt Property holds.

(1)$\Leftrightarrow$(3) It is enough to use Proposition 6.3.1 and notice that the permutability of the relations $\sim$ and τ translates into the condition that, for any z_1, z_2 both in X or in Y, we have $d(z_1, z_2) \le 3 \Rightarrow d(z_1, z_2) = 1$.

■

6.4 Infinite 2-Krull-Schmidt Property in cocomplete categories

Let $\mathcal{C}$ be an additive category. Any skeleton $V(\mathcal{C})$ of $\mathcal{C}$ has the structure of a large commutative monoid, in which the operation is induced by coproduct. If the category $\mathcal{C}$ is *cocomplete*, i.e. any set of objects of $\mathcal{C}$ admits a coproduct, then $V(\mathcal{C})$ is a commutative infinitary monoid. We will always assume that every element of our category $\mathcal{C}$ is a (possibly infinite) coproduct of indecomposable objects of $\mathcal{C}$, so that the commutative infinitary monoid $V(\mathcal{C})$ is always atomic.

We say that the *Infinite 2-Krull-Schmidt Property* holds for a cocomplete category $\mathcal{C}$ if it holds for the monoid $V(\mathcal{C})$. Similarly, we say that the *Strong Infinite 2-Krull-Schmidt Property* holds for $\mathcal{C}$ if it holds for $V(\mathcal{C})$.

Now we want to investigate how we can obtain the conditions of Theorem 6.3.6(2) when we are considering the commutative infinitary monoid $V(\mathcal{C})$ associated to a cocomplete category $\mathcal{C}$.

We will say that two ideals I and J of a ring R realize all maximal ideals, if for every maximal one-sided ideal M of R, we have $M = I$ or $M = J$. It is clear, by Proposition 5.1.1, that if the ideals I and J realize all the maximal ideals of a ring R, then R is a ring of type ≤ 2.

Lemma 6.4.1 *Let $\mathcal{I}$ and $\mathcal{J}$ be two completely prime ideals of a preadditive category $\mathcal{D}$ and let A and B be two objects of $\mathcal{D}$ such that $\mathcal{I}(A,A)$ and $\mathcal{J}(A,A)$ are proper ideals of $\mathrm{End}_{\mathcal{D}}(A)$ realizing all maximal ideals of $\mathrm{End}_{\mathcal{D}}(A)$, and $\mathcal{I}(B,B)$ and $\mathcal{J}(B,B)$ are proper ideals of $\mathrm{End}_{\mathcal{D}}(B)$ realizing all maximal ideals of $\mathrm{End}_{\mathcal{D}}(B)$. Then $A \cong B$ in $\mathcal{D}$ if and only if $[A]_{\mathcal{I}} = [B]_{\mathcal{I}}$ and $[A]_{\mathcal{J}} = [B]_{\mathcal{J}}$.*

Proof. Since $\mathcal{I}(A,A)$, $\mathcal{J}(A,A)$, $\mathcal{I}(B,B)$ and $\mathcal{J}(B,B)$ are all proper ideals, it is clear that $A \cong B$ implies $[A]_{\mathcal{I}} = [B]_{\mathcal{I}}$ and $[A]_{\mathcal{J}} = [B]_{\mathcal{J}}$.

On the other hand, suppose that $[A]_{\mathcal{I}} = [B]_{\mathcal{I}}$ and $[A]_{\mathcal{J}} = [B]_{\mathcal{J}}$. This means that there exist morphisms $\alpha, \gamma : A \to B$ and $\beta, \delta : B \to A$ such that α and β are not in $\mathcal{I}$ and γ and δ are not in $\mathcal{J}$. Among the three morphisms α, γ and $\alpha + \gamma$ we can find a morphism $f : A \to B$ that is not in $\mathcal{I}$ and not in $\mathcal{J}$. Similarly, among the three morphisms β, δ and $\beta + \delta$ we can find a morphism $g : B \to A$ that is not in $\mathcal{I}$ and not in $\mathcal{J}$. The composite morphisms fg and gf are not in $\mathcal{I}$ and not in $\mathcal{J}$, and therefore are automorphisms. ∎

From now on, suppose that $\mathcal{C}$ is a cocomplete category and let $\mathcal{D}$ be the full subcategory of indecomposable objects of $\mathcal{C}$.

Suppose that $\mathcal{I}$ and $\mathcal{J}$ are two completely prime ideals of $\mathcal{D}$. We say that the category $\mathcal{C}$ is *$\mathcal{D}$-splitting* if, for any $A, B, C \in \mathcal{D}$ with $[A]_{\mathcal{I}} = [B]_{\mathcal{I}}$ and $[A]_{\mathcal{J}} = [C]_{\mathcal{J}}$, there exists an object $D \in \mathcal{D}$ such that $A \oplus D = B \oplus C$. In the expression $\mathcal{D}$-splitting, there is no direct reference to the ideals $\mathcal{I}$ and $\mathcal{J}$, since they will always be clear from the context.

Lemma 6.4.2 *Let $\mathcal{C}$ be a cocomplete category. Let $\mathcal{I}$ and $\mathcal{J}$ be two completely prime ideals of $\mathcal{D}$ and let $A \neq 0$ and $U_1, \dots, U_n$, $n \geq 2$ be objects of $\mathcal{D}$ such that $\mathcal{I}(A,A)$ and $\mathcal{J}(A,A)$ are proper ideals of $\mathrm{End}_{\mathcal{D}}(A)$ realizing all maximal ideals of $\mathrm{End}_{\mathcal{D}}(A)$. Suppose that A is isomorphic to a direct summand of $U_1 \oplus \dots \oplus U_n$ and $A \ncong U_i$ for every i. Then there are two distinct indices $i, j = 1, \dots, n$ such that $[A]_{\mathcal{I}} = [U_i]_{\mathcal{I}}$ and $[A]_{\mathcal{J}} = [U_j]_{\mathcal{J}}$.*

Proof. Let $A \neq 0$ and $U_1, \dots, U_n$, $n \geq 2$, be objects of $\mathcal{D}$ such that $\mathcal{I}(A,A)$ and $\mathcal{J}(A,A)$ are proper ideals of $\mathrm{End}_{\mathcal{D}}(A)$ realizing all maximal ideals of $\mathrm{End}_{\mathcal{D}}(A)$. Suppose that A is isomorphic to a non-zero direct summand of $U_1 \oplus \dots \oplus U_n$ and $A \ncong U_i$ for every $i = 1, \dots, n$. Hence there are morphisms $f = (f_k)_{k=1}^n : A \to U_1 \oplus \dots \oplus U_n$ and $g = (g_k)_{k=1}^n : U_1 \oplus \dots \oplus U_n \to A$ such that $gf = \sum_{k=1}^n g_k f_k = 1_A$ and none of the $g_k f_k$ is an isomorphism. Therefore there exist two distinct indices i and j in $1, \dots, n$ such that $g_i f_i \notin \mathcal{I}(A,A)$ and $g_j f_j \notin \mathcal{J}(A,A)$. It follows that $[A]_{\mathcal{I}} = [U_i]_{\mathcal{I}}$ and $[A]_{\mathcal{J}} = [U_j]_{\mathcal{J}}$.
∎

Corollary 6.4.3 *Let $\mathcal{C}$ be a $\mathcal{D}$-splitting additive category. Let $\mathcal{I}$ and $\mathcal{J}$ be two completely prime ideals of $\mathcal{D}$ and let A, U_1 and U_2 objects of $\mathcal{D}$ such that the ideals $\mathcal{I}(A,A)$ and $\mathcal{J}(A,A)$ are proper ideals of $\mathrm{End}_{\mathcal{D}}(A)$ realizing all maximal ideals of $\mathrm{End}_{\mathcal{D}}(A)$, and $\mathcal{I}(U_i,U_i)$ and $\mathcal{J}(U_i,U_i)$ are proper ideals of $\mathrm{End}_{\mathcal{D}}(U_i)$ realizing all maximal ideals of $\mathrm{End}_{\mathcal{D}}(U_i)$, $i = 1,2$. Then $[A]_{\mathcal{I}} = [U_1]_{\mathcal{I}}$ and $[A]_{\mathcal{J}} = [U_2]_{\mathcal{J}}$ implies that there exists an object D of $\mathcal{C}$ such that $[D]_{\mathcal{I}} = [U_2]_{\mathcal{I}}$ and $[D]_{\mathcal{J}} = [U_1]_{\mathcal{J}}$.*

PROOF. Since the category $\mathcal{C}$ is $\mathcal{D}$-splitting and $[A]_{\mathcal{I}} = [U_1]_{\mathcal{I}}$ and $[A]_{\mathcal{J}} = [U_2]_{\mathcal{J}}$, there exists an object $D \in \mathcal{D}$ such that $A \oplus D = U_1 \oplus U_2$. Now, using the fact that U_1 and U_2 are direct summands of $A \oplus D$, by Lemma 6.4.2 we get that either A or D is in the same $\sim_{\mathcal{I}}$-class of U_2 and similarly either A or D is in the same $\sim_{\mathcal{J}}$-class of U_1. If $[U_1]_{\mathcal{J}} = [A]_{\mathcal{J}}$ we get by Lemma 6.4.1 that $A \cong U_1$. Using again Lemma 6.4.2 we have that either U_1 or U_2 is in the same $\sim_{\mathcal{J}}$-class of D. Since $[U_1]_{\mathcal{J}} = [A]_{\mathcal{J}} = [U_2]_{\mathcal{J}}$, we have also $[D]_{\mathcal{J}} = [U_1]_{\mathcal{J}}$. Similarly, if $[U_2]_{\mathcal{I}} = [A]_{\mathcal{I}}$ we get that $A \cong U_2$. Using Lemma 6.4.2 we have that either U_1 or U_2 is in the same $\sim_{\mathcal{I}}$-class of D. Hence $[U_1]_{\mathcal{I}} = [A]_{\mathcal{I}} = [U_2]_{\mathcal{I}}$ implies that $[D]_{\mathcal{I}} = [U_2]_{\mathcal{I}}$. ■

Remark that this corollary states that the equivalence relations $\sim_{\mathcal{I}}$ and $\sim_{\mathcal{J}}$ are permutable.

Definition 6.4.4 Let $\mathcal{C}$ be a cocomplete category and let $\mathcal{I}$ be an ideal of $\mathcal{C}$. We say that an object U in $\mathcal{D}$ is *$\mathcal{I}$-quasi-small* if for every family of objects $M_\lambda \in \mathcal{D}, \lambda \in \Lambda$, and homomorphisms $\alpha: U \to \oplus_{\lambda\in\Lambda} M_\lambda$ and $\beta: \oplus_{\lambda\in\Lambda} M_\lambda \to U$ with $\beta\alpha \notin \mathcal{I}$, the class $\{\, \mu \in \Lambda \mid \beta\epsilon_\mu\pi_\mu\alpha \notin \mathcal{I} \,\}$ is non-empty.

With the following Lemma we remark that being $\mathcal{I}$-quasi-small is invariant under $\sim_{\mathcal{I}}$-equivalence.

Lemma 6.4.5 *Let $\mathcal{C}$ be a cocomplete category and let $\mathcal{I}$ be a completely prime ideal of $\mathcal{D}$. Let A and B be two objects of $\mathcal{D}$ such that $[A]_{\mathcal{I}} = [B]_{\mathcal{I}}$. Then A is $\mathcal{I}$-quasi-small if and only if B is $\mathcal{I}$-quasi-small.*

PROOF. Let $f: A \to B$ and $g: B \to A$ morphisms that are not in $\mathcal{I}$ and suppose A is $\mathcal{I}$-quasi-small. To prove that also B is $\mathcal{I}$-quasi-small, suppose that we have a family $\{\, M_\lambda \mid \lambda \in \Lambda \,\}$ of objects of $\mathcal{D}$ and morphisms $\alpha: B \to \oplus_{\lambda\in\Lambda} M_\lambda$ and $\beta: \oplus_{\lambda\in\Lambda} M_\lambda \to B$ with $\beta\alpha \notin \mathcal{I}$. Then also $g\beta\alpha f \notin \mathcal{I}$ and this implies that the class $\{\, \mu \in \Lambda \mid g\beta\epsilon_\mu\pi_\mu\alpha f \notin \mathcal{I} \,\}$ is non-empty. Since $\mathcal{I}$ is completely prime, this is equivalent to say that the class $\{\, \mu \in \Lambda \mid \beta\epsilon_\mu\pi_\mu\alpha \notin \mathcal{I} \,\}$ is non-empty. ■

Proposition 6.4.6 *Let $\mathcal{C}$ be a cocomplete category and let $\mathcal{I}$ be a completely prime ideal of $\mathcal{D}$. Let A be an $\mathcal{I}$-quasi-small object of $\mathcal{D}$ that is non-zero in $\mathcal{C}/\mathcal{I}$. If A is a direct summand of $\oplus_{\lambda\in\Lambda} M_\lambda$ for a family of objects $M_\lambda \in \mathcal{D}, \lambda \in \Lambda$, then there exists an index $\mu \in \Lambda$ such that $[A]_{\mathcal{I}} = [M_\mu]_{\mathcal{I}}$.*

PROOF. Since A is a direct summand of $\oplus_{\lambda\in\Lambda} M_\lambda$, there are morphisms $\epsilon_A: A \to \oplus_{\lambda\in\Lambda} M_\lambda$ and $\pi_A: \oplus_{\lambda\in\Lambda} M_\lambda \to A$ such that $\pi_A\epsilon_A = 1_A \notin \mathcal{I}$. Then, since A is $\mathcal{I}$-quasi-small, the class $\{\, \mu \in \Lambda \mid \pi_A\epsilon_\mu\pi_\mu\epsilon_A \notin \mathcal{I} \,\}$ is non-empty. Hence, for every μ in this class, we have that $[A]_{\mathcal{I}} = [M_\mu]_{\mathcal{I}}$. ■

Definition 6.4.7 Let $\mathcal{C}$ be a cocomplete category and let $\mathcal{I}$ be an ideal of $\mathcal{D}$. We say that an object U in $\mathcal{D}$ is *$\mathcal{I}$-small* if for every family of objects $M_\lambda \in \mathcal{D}$, $\lambda \in \Lambda$, and homomorphisms $\alpha: U \to \oplus_{\lambda\in\Lambda} M_\lambda$ and $\beta: \oplus_{\lambda\in\Lambda} M_\lambda \to U$ with $\beta\alpha \notin \mathcal{I}$, the set $\{\, \mu \in \Lambda \mid \beta\epsilon_\mu\pi_\mu\alpha \notin \mathcal{I} \,\}$ is finite non-empty.

It is clear that every $\mathcal{I}$-small object is also $\mathcal{I}$-quasi small, for any ideal $\mathcal{I}$ of the category $\mathcal{D}$.

Remark 6.4.8 If $\mathcal{C}$ is a cocomplete category of right R-modules, $\mathcal{I}$ is an ideal of $\mathcal{D}$ and U is an object of $\mathcal{D}$ such that $\mathrm{End}_{\mathcal{D}}(U)/\mathcal{I}(U,U)$ is a division ring, then U is $\mathcal{I}$-small if and only if it is $\mathcal{I}$-quasi-small. In fact, suppose that for every family of R-modules $M_\lambda \in \mathcal{D}$, $\lambda \in \Lambda$ and homomorphisms $\alpha: U \to \oplus_{\lambda\in\Lambda} M_\lambda$ and $\beta: \oplus_{\lambda\in\Lambda} M_\lambda \to U$ with $\beta\alpha \notin \mathcal{I}$, the class $\{\, \mu \in \Lambda \mid \beta\epsilon_\mu\pi_\mu\alpha \notin \mathcal{I} \,\}$ is non-empty. Suppose also that there exist a family of R-modules $N_\lambda \in \mathcal{D}$, $\lambda \in \Lambda$ and homomorphisms $\gamma: U \to \oplus_{\lambda\in\Lambda} N_\lambda$ and $\delta: \oplus_{\lambda\in\Lambda} N_\lambda \to U$ such that there are infinitely many indices $\mu_i \in \Lambda$, $i \in \mathbb{N}$, with $f_i = \delta\epsilon_{\mu_i}\pi_{\mu_i}\gamma \notin \mathcal{I}$ for every $i \in \mathbb{N}$. Since $\mathrm{End}_{\mathcal{D}}(U)/\mathcal{I}(U,U)$ is a division ring, we have that there exists a morphism $g_i \in \mathrm{End}_{\mathcal{D}}(U)/\mathcal{I}(U,U)$ with $h_i = 1_U - g_i f_i \in \mathcal{I}(U,U)$ for every $i \in \mathbb{N}$. Consider the morphism $H: U \to U^{(\mathbb{N})}$ whose i-th component is defined as

$$H_i = h_i - h_{i-1} = 1_U - g_i f_i - 1_U + g_{i-1} f_{i-1} = g_{i-1} f_{i-1} - g_i f_i$$

(define h_0 to be 0). If we denote by $\Sigma: U^{(\mathbb{N})} \to U$ the morphism that sends an element of $U^{(\mathbb{N})}$ to the sum of its components, we obtain that $\Sigma H = 1_U \notin \mathcal{I}(U,U)$ but $\Sigma\epsilon_i\pi_i H = H_i \in \mathcal{I}$ for every $i \in \mathbb{N}$. This contradicts our hypothesis.

Now we use together our two previous Theorems to obtain a sufficient condition for an additive category $\mathcal{C}$ to satisfy the Strong Infinite 2-Krull-Schmidt Property.

Theorem 6.4.9 *Let $\mathcal{C}$ be a $\mathcal{D}$-splitting cocomplete category and let $\mathcal{I}$ and $\mathcal{J}$ be completely prime ideals of $\mathcal{D}$. Suppose that, for every object A in $\mathcal{D}$, the ideals $\mathcal{I}(A,A)$ and $\mathcal{J}(A,A)$ are proper ideals realizing all maximal ideals of $\mathrm{End}_{\mathcal{D}}(A)$. Suppose also that every object in $\mathcal{D}$ is $\mathcal{I}$-small and $\mathcal{J}$-small.*

Let $\{\, U_i \mid i \in I \,\}$ and $\{\, V_j \mid j \in J \,\}$ be two families of objects of $\mathcal{D}$. Then $\oplus_{i\in I} U_i \cong \oplus_{j\in J} V_j$ if and only if there are two bijections $\sigma, \tau: I \to J$ such that $[U_i]_{\mathcal{I}} = [V_{\sigma(i)}]_{\mathcal{I}}$ and $[U_i]_{\mathcal{J}} = [V_{\tau(i)}]_{\mathcal{J}}$ for every $i \in I$.

Proof. Observe that the conclusion of the Theorem is that the Strong Infinite 2-Krull-Schmidt Property holds for the monoid $V(\mathcal{C})$ with respect to the equivalence relations $\sim_{\mathcal{I}}$ and $\sim_{\mathcal{J}}$.

Then, to prove the theorem, it is enough to verify that all the conditions of (2) in Proposition 6.3.6 hold. Since all the endomorphism rings of U_i and V_j are semilocal, it is clear that $V(\mathcal{C})$ is a-cancellative. Condition (b) is given by Lemma 6.4.1. Condition (c) by Proposition 6.4.6. Condition (d) comes from the $\mathcal{D}$-splitting property and Corollary 6.4.3. To prove condition (e), suppose $\oplus_{i\in I} U_i \cong \oplus_{j\in J} V_j$ and consider the classes $I_{\sim_{\mathcal{I}}}(k) = \{i \in I \mid U_i \sim_{\mathcal{I}} U_k\}$ and $J_{\sim_{\mathcal{I}}}(k) = \{j \in J \mid V_j \sim_{\mathcal{I}} U_k\}$ for an element $k \in I$. For any $t \in I_{\sim_{\mathcal{I}}}(k)$, the canonical injection $\epsilon_t: U_t \to \oplus_{j\in J} V_j$ and the canonical projection $\pi_t: \oplus_{j\in J} V_j \to U_t$ satisfy $\pi_t\epsilon_t = 1_U \notin \mathcal{I}$ and hence, since U_t is $\mathcal{I}$-small, the set $A(t) = \{j \in J \mid \pi_t\epsilon_j\pi_j\epsilon_t \notin J\}$ is finite non-empty. We are left to prove that $J_{\sim_{\mathcal{I}}}(k) \subseteq \bigcup_{t\in I_{\sim_{\mathcal{I}}}(k)} A(t)$. To show this consider $V_{\bar{j}} \in J_{\sim_{\mathcal{I}}}(k)$. Since $V_{\bar{j}}$ is a direct summand of $\oplus_{i\in I} U_i$, as above we get that the set $B(\bar{\jmath}) = \{i \in I \mid \pi_{\bar{\jmath}}\epsilon_i\pi_i\epsilon_{\bar{\jmath}} \notin \mathcal{I}\}$ is finite non-empty. Since $\mathcal{I}$ is completely prime, we have that $\pi_{\bar{\jmath}}\epsilon_i\pi_i\epsilon_{\bar{\jmath}} \notin \mathcal{I}$ if and only if $\pi_i\epsilon_{\bar{\jmath}}\pi_{\bar{\jmath}}\epsilon_i \notin \mathcal{I}$. Hence we have that $\bar{\jmath} \in A(i)$ for every $i \in B(\bar{\jmath})$. We can prove the same for $\mathcal{J}$ to conclude the proof for (e). Similarly we prove also condition (f). ■

6.5 Artinian modules with heterogeneous socle of length 2

In this section we want to apply what we did in our previous section to a concrete category. Let R be a ring, S_1 and S_2 two non-isomorphic simple R-modules and consider the category $\mathcal{D} = \{\, M \in \text{Mod-}R \mid M \text{ artinian, } \operatorname{soc}(M) \cong S_1 \oplus S_2 \,\}$. For any module $M_A \in \mathcal{D}$, we settle $\operatorname{soc}(M_A) = M_A^1 \oplus M_A^2$ with $M_A^1 \cong S_1$ and $M_A^2 \cong S_2$.

Let $\mathcal{C} = \operatorname{Sum}(\mathcal{D})$ be the category whose objects are direct sums of objects in $\mathcal{D}$. In $\mathcal{D}$ we define the ideals $\mathcal{I}^1$ and $\mathcal{I}^2$ in the following way: for every pair of objects M_A and M_B of $\mathcal{D}$ let $\mathcal{I}^1(M_A, M_B) = \{\, f\colon M_A \to M_B \mid f(M_A^1) = 0 \,\}$ and $\mathcal{I}^2(M_A, M_B) = \{\, f\colon M_A \to M_B \mid f(M_A^2) = 0 \,\}$.

Proposition 6.5.1 *Let $\mathcal{C} = \operatorname{Sum}(\mathcal{D})$ the category whose objects are direct sums of objects in $\mathcal{D}$. Then $\mathcal{I}^1$ and $\mathcal{I}^2$ are completely prime ideals of $\mathcal{D}$ and every (right, left, two-sided) maximal ideal of the endomorphism ring of an objects M_A of $\mathcal{D}$ is equal either to $\mathcal{I}^1(M_A, M_A)$ or to $\mathcal{I}^2(M_A, M_A)$.*

Proof. First, we want to prove that $\mathcal{I}^1$ and $\mathcal{I}^2$ are in fact ideals of $\mathcal{D}$. Since the situation is completely symmetric, we prove it only for $\mathcal{I}^1$. It is clear that the zero morphism is in $\mathcal{I}^1$ and that $f, g \in \mathcal{I}^1$ implies $f + g \in \mathcal{I}^1$. Now suppose $f \in \mathcal{I}^1(M_A, M_B)$. If $g\colon M_B \to M_C$ is any morphism in $\mathcal{C}$, then it is easy to see that $gf \in \mathcal{I}^1(M_A, M_C)$. If $g\colon M_C \to M_A$, to prove that $fg \in \mathcal{I}^1(M_C, M_B)$ it is enough to show that $g(M_C^1) \subseteq M_A^1$. Since every morphism sends the socle to the socle, we know that $g(M_C^1) \subseteq M_A^1 \oplus M_A^2$. The simple component M_C^1 is sent by g to zero or to a module isomorphic to S_1. Then it is easy to deduce that $g(M_C^1) \subseteq M_A^1$.

Next we show that the ideals $\mathcal{I}^1$ and $\mathcal{I}^2$ are completely prime. Again, we show this only for $\mathcal{I}^1$. To prove it, consider $f\colon M_A \to M_B$ and $g\colon M_B \to M_C$ such that $gf \in \mathcal{I}^1$. Suppose that $f \notin \mathcal{I}^1$ so that $f(M_A^1) \neq 0$. This implies that $f(M_A^1) = M_B^1$ and then $g(M_B^1) = gf(M_A^1) = 0$ means that $g \in \mathcal{I}^1$.

Eventually, we show that $\mathcal{I}^1$ and $\mathcal{I}^2$ realize all maximal ideals of the endomorphism ring of an object M_A in $\mathcal{D}$. Suppose first that $\mathcal{I}^1(M_A, M_A)$ and $\mathcal{I}^2(M_A, M_A)$ are comparable. Without loss of generality we can assume $\mathcal{I}^1(M_A, M_A) \supseteq \mathcal{I}^2(M_A, M_A)$. In this case, for any morphism $f\colon M_A \to M_A$ that is not in $\mathcal{I}^1(M_A, M_A)$, we have that $\ker(f) \cap \operatorname{soc}(M_A) = 0$. Since the socle is essential in an artinian module, f must be injective, and hence an automorphism of M_A. This means that $\mathcal{I}^1(M_A, M_A)$ is the unique maximal ideal of $\operatorname{End}_{\mathcal{D}}(M_A)$. On the other hand, suppose that $\mathcal{I}^1(M_A, M_A)$ and $\mathcal{I}^2(M_A, M_A)$ are not comparable. To prove that $\mathcal{I}^1(M_A, M_A)$ is a maximal ideal, consider

$g \in \mathrm{End}_{\mathcal{D}}(M_A)$ that is not in $\mathcal{I}^1(M_A, M_A)$. If $g(M_A^2) \neq 0$, then g is an automorphism. If $g(M_A^2) = 0$, take an element $f \in \mathcal{I}^1(M_A, M_A) \setminus \mathcal{I}^2(M_A, M_A)$; then $\ker(f+g) = 0$ and so $f+g$ is an automorphism. Similarly we can prove that $\mathcal{I}^2$ is a maximal ideal.

With the same argument as above we can prove that if an ideal of $\mathrm{End}_{\mathcal{D}}(M_A)$ is not contained in $\mathcal{I}^1(M_A, M_A)$ or in $\mathcal{I}^2(M_A, M_A)$, then it contains an automorphism and so it is the whole endomorphism ring. ∎

To apply Theorem 6.4.9 we need to show that $\mathcal{C}$ is $\mathcal{D}$-splitting. To show this, let M_A, M_B and M_C be three objects in $\mathcal{D}$ such that $[M_A]_{\mathcal{I}^1} = [M_B]_{\mathcal{I}^1}$ and $[M_A]_{\mathcal{I}^2} = [M_C]_{\mathcal{I}^2}$. This means that there exist morphisms $f: M_A \to M_B$, $g: M_B \to M_A$, $h: M_A \to M_C$ and $l: M_C \to M_A$ with $f, g \notin \mathcal{I}^1$ and $h, l \notin \mathcal{I}^2$. If gf is an automorphism, $M_A \cong M_B$ and then $M_A \oplus M_C \cong M_B \oplus M_C$. Similarly, if lh is an automorphism, we have $M_A \cong M_C$ and hence $M_A \oplus M_B \cong M_B \oplus M_C$. If gf and lh are not automorphism, then $gf \in \mathcal{I}^2 \setminus \mathcal{I}^1$ and $lh \in \mathcal{I}^1 \setminus \mathcal{I}^2$. Therefore $gf + lh$ is an automorphism of M_A factoring through $M_B \oplus M_C$ and so there exists an R-module D such that $M_A \oplus D \cong M_B \oplus M_C$. To conclude, we need to show that $D \in \mathcal{D}$, i.e. D is artinian and $\mathrm{soc}(D) \cong S_1 \oplus S_2$. It is clear that D is artinian. Moreover, we have that $\mathrm{soc}(M_A \oplus D) \cong \mathrm{soc}(M_A) \oplus \mathrm{soc}(D) \cong M_A^1 \oplus M_A^2 \oplus \mathrm{soc}(D)$ is isomorphic to $\mathrm{soc}(M_B \oplus M_C) \cong \mathrm{soc}(M_B) \oplus \mathrm{soc}(M_C) \cong M_B^1 \oplus M_B^2 \oplus M_C^1 \oplus M_C^2$; by cancellation we get $\mathrm{soc}(D) \cong S_1 \oplus S_2$.

Lemma 6.5.2 *Every object in $\mathcal{D}$ is $\mathcal{I}^1$-small and $\mathcal{I}^2$-small.*

Proof. We prove the Lemma only for $\mathcal{I}^1$, then by symmetry it works also for $\mathcal{I}^2$. By Remark 6.4.8 it is enough to prove that any object M_A in $\mathcal{D}$ is $\mathcal{I}^1$-quasi-small. To prove this we have to show that for any family of objects $M_\lambda, \lambda \in \Lambda$, in $\mathcal{D}$ and homomorphisms $\alpha: M_A \to \oplus_{\lambda\in\Lambda} M_\lambda$ and $\beta: \oplus_{\lambda\in\Lambda} M_\lambda \to M_A$ with $\beta\alpha \notin \mathcal{I}^1$, the class $\{\, \mu \in \Lambda \mid \beta\epsilon_\mu\pi_\mu\alpha \notin \mathcal{I}^1 \,\}$ is not empty. Suppose then that $\beta\alpha \notin \mathcal{I}^1$, i.e. $\beta\alpha(M_A^1) \neq 0$. The image $\alpha(M_A^1)$ is a submodule of $\oplus_{\lambda\in F} M_\lambda^1$ for some finite $F \subseteq \Lambda$. Since $\beta(\oplus_{\lambda\in F} M_\lambda^1) \neq 0$, there exists an element $\mu \in F$ such that $\beta(M_\mu^1) \neq 0$. But this means that $\beta\epsilon_\mu\pi_\mu\alpha(M_A^1) \neq 0$ and hence $\beta\epsilon_\mu\pi_\mu\alpha$ is not in $\mathcal{I}^1$. ∎

Theorem 6.5.3 *Let R be a ring, S_1 and S_2 two non-isomorphic simple R-modules and consider the category $\mathcal{D} = \{\, M \in \text{Mod-}R \mid M \text{ artinian}, \mathrm{soc}(M) \cong S_1 \oplus S_2 \,\}$. Let $\mathcal{C} = \mathrm{Sum}(\mathcal{D})$ be the category whose objects are direct sums of objects in $\mathcal{D}$ and $\mathcal{I}^1 = \{\, f: M_A \to M_B \mid f(M_A^1) = 0 \,\}$ and $\mathcal{I}^2 = \{\, f: M_A \to M_B \mid f(M_A^2) = 0 \,\}$. Then the Strong Infinite 2-Krull-Schmidt Property holds for $\mathcal{C}$, with respect to the equivalence relations $\sim_{\mathcal{I}^1}$ and $\sim_{\mathcal{I}^2}$.*

6.6 Noetherian modules of dimension two over their radical

We can dualize everything we did in our previous section. Let R be a ring and $S_1 \ncong S_2$ be two non-isomorphic simple R-modules. Consider the category

$\mathcal{D} = \{ M \in \text{Mod-}R \mid M \text{ noetherian}, M/\mathrm{Rad}(M) \cong S_1 \oplus S_2 \}$ and its closure under small coproducts $\mathcal{C} = \mathrm{Sum}(\mathcal{D})$. For every object $M_A \in \mathcal{D}$, denote by M_A^1 the submodule of M_A containing $\mathrm{Rad}(M_A)$ such that $M_A^1/\mathrm{Rad}(M_A) \cong S_1$ and by M_A^2 the submodule of M_A containing $\mathrm{Rad}(M_A)$ such that $M_A^2/\mathrm{Rad}(M_A) \cong S_2$. For every pair of objects M_A and M_B of $\mathcal{D}$ define $\mathcal{I}_1(M_A, M_B) = \{ f\colon M_A \to M_B \mid f(M_A) \subseteq M_B^1 \}$ and $\mathcal{I}_2(M_A, M_B) = \{ f\colon M_A \to M_B \mid f(M_A) \subseteq M_B^2 \}$.

Proposition 6.6.1 *Let $\mathcal{C} = \mathrm{Sum}(\mathcal{D})$ the category whose objects are direct sums of objects in $\mathcal{D}$. Then $\mathcal{I}_1$ and $\mathcal{I}_2$ are completely prime ideals of $\mathcal{D}$ and every (right, left, two-sided) maximal ideal of the endomorphism ring of an object M_A of $\mathcal{D}$ is equal either to $\mathcal{I}_1(M_A, M_A)$ or to $\mathcal{I}_2(M_A, M_A)$.*

Proof. First, we want to prove that $\mathcal{I}_1$ and $\mathcal{I}_2$ are ideals of $\mathcal{D}$. It is clear that the zero morphism is in $\mathcal{I}_1$ and that $f, g \in \mathcal{I}_1$ implies $f + g \in \mathcal{I}_1$. Now suppose $f \in \mathcal{I}_1(M_A, M_B)$. If $g\colon M_B \to M_C$ is a morphism in $\mathcal{C}$, we have that $g(M_B^1)$ is contained in M_C^1 and hence $gf(M_A) \subseteq g(M_B^1) \subseteq M_C^1$. If $g\colon M_C \to M_B$, it is easy to see that $fg \in \mathcal{I}_1(M_C, M_B)$ since $fg(M_C) \subseteq f(M_A) \subseteq M_B^1$. Similarly for $\mathcal{I}_2$.

Next we show that the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ are completely prime. By symmetry, we show it only for $\mathcal{I}_1$. To prove this, consider $f\colon M_A \to M_B$ and $g\colon M_B \to M_C$ such that $gf \in \mathcal{I}_1$. Suppose that $g \notin \mathcal{I}_1$ so that $g(M_B) \nsubseteq M_C^1$. This implies that $f(M_A)$ is contained in M_B^1 and then $f \in \mathcal{I}_1$.

Eventually, we show that $\mathcal{I}_1$ and $\mathcal{I}_2$ realize all the maximal ideals of the endomorphism ring of an object M_A in $\mathcal{D}$. Suppose first that $\mathcal{I}_1(M_A, M_A)$ and $\mathcal{I}_2(M_A, M_A)$ are comparable. Without loss of generality we can assume $\mathcal{I}_1(M_A, M_A) \supseteq \mathcal{I}_2(M_A, M_A)$. In this case, for any morphism $f \in \mathrm{End}_{\mathcal{D}}(M_A) \setminus \mathcal{I}_1(M_A, M_A)$, we have $\mathrm{im}(f) + \mathrm{Rad}(M_A) = M_A$. Since the radical of a noetherian module is superfluous, we get that f must be surjective, and hence an automorphism of M_A. This means that $\mathcal{I}_1(M_A, M_A)$ is the unique maximal ideal of $\mathrm{End}_{\mathcal{D}}(M_A)$. On the other hand, suppose that $\mathcal{I}_1(M_A, M_A)$ and $\mathcal{I}_2(M_A, M_A)$ are not comparable. To prove that $\mathcal{I}_1(M_A, M_A)$ is a maximal ideal, consider $g \in \mathrm{End}_{\mathcal{D}}(M_A)$ that is not in $\mathcal{I}_1(M_A, M_A)$. If $g(M_A) \nsubseteq M_A^2$, then g is an automorphism. If $g(M_A) \subseteq M_A^2$, take an element $f \in \mathcal{I}_1(M_A, M_A) \setminus \mathcal{I}_2(M_A, M_A)$; then $(f + g)(M_A) = M_A$ and so $f + g$ is an automorphism. In the same way we can prove that $\mathcal{I}_2$ is a maximal ideal.

With the same argument as above we can prove that if an ideal of $\mathrm{End}_{\mathcal{D}}(M_A)$ is not contained in $\mathcal{I}_1(M_A, M_A)$ or in $\mathcal{I}_2(M_A, M_A)$, then it contains an automorphism and so it is the whole endomorphism ring. ∎

To apply Theorem 6.4.9 we need to show that $\mathcal{C}$ is $\mathcal{D}$-splitting. To show this, let M_A, M_B and M_C be three objects in $\mathcal{D}$ such that $[M_A]_{\mathcal{I}_1} = [M_B]_{\mathcal{I}_1}$ and $[M_A]_{\mathcal{I}_2} = [M_C]_{\mathcal{I}_2}$. This means that there exist morphisms $f\colon M_A \to M_B$, $g\colon M_B \to M_A$, $h\colon M_A \to M_C$ and $l\colon M_C \to M_A$ with $f, g \notin \mathcal{I}_1$ and $h, l \notin \mathcal{I}_2$. If gf is an automorphism, $M_A \cong M_B$ and then $M_A \oplus M_C \cong M_B \oplus M_C$. Similarly, if lh is an automorphism, we have $M_A \cong M_C$ and hence $M_A \oplus M_B \cong M_B \oplus M_C$. If gf and lh are not automorphism, then $gf \in \mathcal{I}_2 \setminus \mathcal{I}_1$ and $lh \in \mathcal{I}_1 \setminus \mathcal{I}_2$. Therefore $gf + lh$ is an automorphism of M_A factoring through $M_B \oplus M_C$ and so there exists an R-module D such that $M_A \oplus D \cong M_B \oplus M_C$. To conclude, we need to show that $D \in \mathcal{D}$, i.e. D is noetherian and $M/\mathrm{Rad}(M) \cong S_1 \oplus S_2$. It is clear that D is noetherian. We have that $M_A \oplus D/\mathrm{Rad}(M_A \oplus D) \cong M_A/\mathrm{Rad}(M_A) \oplus D/\mathrm{Rad}(D) \cong S_1 \oplus S_2 \oplus D/\mathrm{Rad}(D)$ is isomorphic to $M_B \oplus M_C/\mathrm{Rad}(M_B \oplus M_C) \cong M_B/\mathrm{Rad}(M_B) \oplus M_C/\mathrm{Rad}(M_C) \cong S_1 \oplus S_2 \oplus S_1 \oplus S_2$; by cancellation we get $D/\mathrm{Rad}(D) \cong S_1 \oplus S_2$.

Lemma 6.6.2 *Every object in $\mathcal{D}$ is $\mathcal{I}_1$-small and $\mathcal{I}_2$-small.*

Proof. We prove the Lemma only for $\mathcal{I}_1$, by symmetry it holds also for $\mathcal{I}_2$. By Remark 6.4.8 it is enough to prove that any object M_A in $\mathcal{D}$ is $\mathcal{I}_1$-quasi-small. To prove this we have to show that for any family of objects $M_\lambda \in \mathcal{D}$, $\lambda \in \Lambda$, and homomorphisms $\alpha: M_A \to \oplus_{\lambda\in\Lambda} M_\lambda$ and $\beta: \oplus_{\lambda\in\Lambda} M_\lambda \to M_A$ with $\beta\alpha \notin \mathcal{I}_1$, the class $\{ \mu \in \Lambda \mid \beta\epsilon_\mu\pi_\mu\alpha \notin \mathcal{I}_1 \}$ is not empty. Suppose then that $\beta\alpha \notin \mathcal{I}_1$, i.e. $\beta\alpha(M_A) \nsubseteq M_A^1$. This means that there exists an element $m \in M_A$ such that $\beta\alpha(m) \notin M_A^1$. Let F be a finite subset of Λ such that $\alpha(m) \in \oplus_{\lambda\in F} M_\lambda$. Then $\beta\alpha(m) = \sum_{\lambda\in F} \beta\epsilon_\lambda\pi_\lambda\alpha(m) \notin M_A^1$. This implies that there exists an index $\mu \in \Lambda$ such that $\beta\epsilon_\mu\pi_\mu\alpha \notin \mathcal{I}_1$. ■

Theorem 6.6.3 *Let R be a ring, S_1 and S_2 two non-isomorphic simple R-modules and consider the category*

$$\mathcal{D} = \{ M \in \text{Mod-}R \mid M \text{ noetherian}, \ M/\mathrm{Rad}(M) \cong S_1 \oplus S_2 \}.$$

Let $\mathcal{C} = \mathrm{Sum}(\mathcal{D})$ be the category whose objects are direct sums of objects in $\mathcal{D}$ and $\mathcal{I}_1 = \{ f: M_A \to M_B \mid f(M_A) \subseteq M_B^1 \}$ and $\mathcal{I}_2 = \{ f: M_A \to M_B \mid f(M_A) \subseteq M_B^2 \}$. Then the Strong Infinite 2-Krull-Schmidt Property holds for $\mathcal{C}$, with respect to the equivalence relations $\sim_{\mathcal{I}_1}$ and $\sim_{\mathcal{I}_2}$.

6.7 Representations of type 1 pointwise of quivers with 2 vertices

A *quiver* $Q = (Q_0, Q_1)$ is a directed graph, whose set of vertices is Q_0 and whose set of arrows is Q_1. For any arrow $a \in Q_1$, we denote by $i(a)$ the *initial vertex* and by $t(a)$ the *terminal vertex* of the arrow a. Thus $a: i(a) \to t(a)$. A *path* p is a juxtaposition of arrows $a_1, \dots, a_n$ such that $i(a_{i+1}) = t(a_i)$ for every $i = 1, \dots, n-1$. We define also $i(p) = i(a_1)$ and $t(p) = t(a_n)$ and we say that n is the *length* of the path. If p, q are paths such that $t(p) = i(q)$, then pq is the juxtaposition of p and q. We will consider only quivers with two vertices and a finite number of arrows.

A quiver can be seen as a category in a natural way, by taking Q_0 as the set of objects and paths between vertices as the morphisms with the juxtaposition as composition. In order to obtain a category, we need to allow for every vertex $v \in Q_0$ the *trivial path* e_v with initial and terminal vertex both equal to v, such that $pe_v = p$ and $e_v q = q$ for every path p with $t(p) = v$ and every path q with $i(q) = v$.

Let R be a ring. The category of functors $(Q, \text{Mod-}R)$ from the category associated to the quiver Q described above to the category Mod-R is the category of representations of Q by right R-modules and R-modules homomorphisms. Since Mod-R is abelian, it follows that $(Q, \text{Mod-}R)$ is abelian and limits are computed pointwise.

More explicitly, a *representation* M of the quiver $Q = (Q_0, Q_1)$ in the category $(Q, \text{Mod-}R)$ is a family $\{M_i\}_{i \in Q_0}$ of right R-modules together with a family of R-modules homomorphisms $\{M_a : M_{i(a)} \to M_{t(a)}\}_{a \in Q_1}$. A *morphism of representations* $f: M \to M'$ is a collection of morphisms $\{f_i : M_i \to M'_i\}_{i \in Q_0}$ such that $M'_a f_{i(a)} = f_{t(a)} M_a$ for all arrows $a \in Q_1$.

The *path ring* is the ring obtained by considering the free right R-module $R[Q]$ on the set of paths P of Q. The product of two paths x and y is defined to be their juxtaposition xy if $t(x) = i(y)$ and to be $0_{R[Q]}$ if $t(x) \neq i(y)$. For arbitrary elements of $R[Q]$ the product is defined as

$$\sum_{x \in P} x r_x \cdot \sum_{y \in P} y r'_y = \sum_{x, y \in P} x y r_x r'_y.$$

Note that in general the ring $R[Q]$ may not have an identity. Nevertheless, if Q_0 is finite, $R[Q]$ is an associative ring with identity $e = \sum_{i \in Q_0} e_i$ and R embeds in $R[Q]$ via the injective ring morphism $r \mapsto er$.

There is an equivalence of categories between Mod-$R[Q]$ and $(Q, \text{Mod-}R)$ thus representations of quivers can be seen as modules over a suitable ring.

Let Q be a quiver with two vertices. We consider the category $\mathcal{D}$ of representations of Q pointwise of type 1, i.e of representations $M = \{M_1, M_2\}$ of Q such that M_i is of type 1 for $i = 1, 2$. Morphisms between two representations $M = \{M_1, M_2\}$ and $N = \{N_1, N_2\}$ are couples of arrows $\{f_1 : M_1 \to N_1, f_2 : M_2 \to N_2\}$ such that $f_2 M_a = N_a f_1$ for any arrow $a: 1 \to 2$ in Q and $f_1 M_b = N_b f_2$ for every arrow $b: 2 \to 1$ in Q. Let $\mathcal{C} = \text{Sum}(\mathcal{D})$ be the category whose objects are direct sums of objects in $\mathcal{D}$.

Recall that, for any ring R, the Jacobson radical

$$\begin{aligned} \mathcal{J}(M, N) &= \{f : M \to N \mid 1_M - gf \text{ has a left inverse for every } g : N \to M\} \\ &= \{f : M \to N \mid 1_M - gf \text{ has an inverse for every } g : N \to M\} \end{aligned}$$

is a two-sided ideal of the category Mod-R.

Lemma 6.7.1 *Let R be any ring and $\mathcal{L}$ be the full subcategory of* Mod-R *of objects with local endomorphism ring. Then $\mathcal{J}$ is a completely prime ideal in $\mathcal{L}$.*

Proof. Let M, M_1 and M_2 be right R-modules with local endomorphism ring. Consider two morphisms $f: M_1 \to M$ and $g: M \to M_2$ such that $gf \in \mathcal{J}(M_1, M_2)$. This means that $1_{M_1} - hgf$ is left invertible for every $h: M_2 \to M_1$. If we suppose that $g \notin \mathcal{J}(M, M_2)$ we have that there exists $l: M_2 \to M$ such that $1_M - lg$ is not invertible in $\text{End}_R(M)$. Since M has local endomorphism ring, it needs to be $1_M - lg \in \mathcal{J}(M, M) = J(\text{End}_R(M))$ and hence lg is invertible. This implies that g is left invertible and therefore we must have $f \in \mathcal{J}(M_1, M)$. ■

Given objects M and N in $\mathcal{D}$, we define the ideals $\mathcal{J}_1(M,N) = \{f\colon M \to N \mid f_1 \in \mathcal{J}(M_1,N_1)\}$ and $\mathcal{J}_2(M,N) = \{f\colon M \to N \mid f_2 \in \mathcal{J}(M_2,N_2)\}$.

Proposition 6.7.2 *The ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ are completely prime ideals of $\mathcal{D}$ such that they realize all (right, left, two-sided) maximal ideals of the endomorphism rings of the objects of $\mathcal{D}$.*

Proof. Since the situation is completely symmetric, we prove the proposition only for $\mathcal{J}_1$. First, we want to prove that $\mathcal{J}_1$ is in fact an ideal of $\mathcal{D}$. It is clear that the zero morphism is in $\mathcal{J}_1$. If we are given two morphisms $f,g\colon M \to N$ in $\mathcal{J}_1(M,N)$, we have that $f_1, g_1 \in \mathcal{J}(M_1,N_1)$ and this implies $f+g \in \mathcal{J}_1(M,N)$, i.e. $(f+g)_1 = f_1 + g_1 \in \mathcal{J}(M_1,N_1)$. It is clear that if we compose $f \in \mathcal{J}_1(M,N)$ with any morphism g in $\mathcal{C}$, we remain in $\mathcal{J}_1$, since $\mathcal{J}$ itself is an ideal.

To show that the ideal $\mathcal{J}_1$ is completely prime in $\mathcal{D}$ it is enough to apply Lemma 6.7.1.

Now we show that, given an object M of $\mathcal{D}$, the ideals $\mathcal{J}_1(M,M)$ and $\mathcal{J}_2(M,M)$ realize all maximal ideals of the endomorphism ring $\mathrm{End}_{\mathcal{D}}(M)$. Suppose first that $\mathcal{J}_1(M,M)$ and $\mathcal{J}_2(M,M)$ are comparable. Without loss of generality we can assume $\mathcal{J}_1(M,M) \supseteq \mathcal{J}_2(M,M)$. In this case, for any morphism $f \notin \mathcal{J}_1(M,M)$ we have that both f_1 and f_2 are isomorphisms and hence f itself must be an isomorphism of M. Therefore $\mathcal{J}_1(M,M)$ is the unique maximal ideal of $\mathrm{End}_{\mathcal{D}}(M)$. Now suppose that $\mathcal{J}_1(M,M)$ and $\mathcal{J}_2(M,M)$ are not comparable. To prove that $\mathcal{J}_1(M,M)$ is a maximal ideal, consider $g \in \mathrm{End}_{\mathcal{D}}(M) \setminus \mathcal{J}_1(M,M)$. This means that g_1 is an isomorphism. If also g_2 is an isomorphism then g itself must be an isomorphism. If $g_2 \in \mathcal{J}(M_2,M_2)$, choose an element $f \in \mathcal{J}_1(M,M) \setminus \mathcal{J}_2(M,M)$; then $f+g$ is neither in $\mathcal{J}_1(M,M)$ nor $\mathcal{J}_2(M,M)$ and hence it must be an isomorphism. Similarly we can prove that $\mathcal{J}_2(M,M)$ is a maximal ideal of $\mathrm{End}_{\mathcal{D}}(M)$.

With the same argument as above we can prove that if an ideal of $\mathrm{End}_{\mathcal{D}}(M)$ is not contained in $\mathcal{J}_1(M,M)$ or in $\mathcal{J}_2(M,M)$, then it contains an automorphism and so it must be the whole endomorphism ring. ∎

To apply Theorem 6.4.9 we need to show that $\mathcal{C}$ is $\mathcal{D}$-splitting. To show this, let M,N and P be three objects in $\mathcal{D}$ such that $[M]_{\mathcal{J}_1} = [N]_{\mathcal{J}_1}$ and $[M]_{\mathcal{J}_2} = [P]_{\mathcal{J}_2}$. This means that there exist morphisms $f\colon M \to N$, $g\colon N \to M$, $h\colon M \to P$ and $l\colon P \to M$ with $f,g \notin \mathcal{J}_1$ and $h,l \notin \mathcal{J}_2$. Hence the homomorphisms of R-modules $f_1\colon M_1 \to N_1$, $g_1\colon N_1 \to M_1$, $h_2\colon M_2 \to P_2$ and $l_2\colon P_2 \to M_2$ are in fact isomorphisms. Consider the representation T of our quiver Q having $T_1 = P_1$, $T_2 = N_2$ and such that for every arrow $a\colon Q_1 \to Q_2$ we have $T_a = N_a g_1^{-1} l_1$ and for every arrow $a'\colon Q_2 \to Q_1$ we have $T_{a'} = g_2 l_2^{-1} P_{a'}$. To prove that $M \oplus T \cong N \oplus P$ we consider the following morphism of representations e, defined by:

$$\begin{array}{ccc} M_1 \oplus P_1 & & M_2 \oplus N_2 \\ \begin{pmatrix} f_1 & g_1^{-1} l_1 \\ h_1 & 1 \end{pmatrix}\Big\downarrow & & \Big\downarrow\begin{pmatrix} f_2 & 1 \\ h_2 & l_2^{-1} g_2 \end{pmatrix} \\ N_1 \oplus P_1 & & N_2 \oplus P2. \end{array}$$

To be morphisms of representations of the quiver Q they need to commute with the morphisms $M_a \oplus T_a$ and $N_a \oplus T_a$ for every arrow $a{:}\, Q_1 \to Q_2$ and they need to commute with the morphisms $M_{a'} \oplus T_{a'}$ and $N_{a'} \oplus T_{a'}$ for every arrow $a'{:}\, Q_2 \to Q_1$. To check this we verify that

$$\begin{pmatrix} f_2 & 1 \\ h_2 & l_2^{-1} g_2 \end{pmatrix} \begin{pmatrix} M_a & 0 \\ 0 & T_a \end{pmatrix} = \begin{pmatrix} N_a & 0 \\ 0 & P_a \end{pmatrix} \begin{pmatrix} f_1 & g_1^{-1} l_1 \\ h_1 & 1 \end{pmatrix}$$

for every arrow $a{:}\, Q_1 \to Q_2$ and

$$\begin{pmatrix} f_1 & g_1^{-1} l_1 \\ h_1 & 1 \end{pmatrix} \begin{pmatrix} M_{a'} & 0 \\ 0 & T_{a'} \end{pmatrix} = \begin{pmatrix} N_{a'} & 0 \\ 0 & P_{a'} \end{pmatrix} \begin{pmatrix} f_2 & 1 \\ h_2 & l_2^{-1} g_2 \end{pmatrix}$$

for every arrow $a'{:}\, Q_2 \to Q_1$. To verify the equalities it is enough to use the commutativity of f, g, h and l with the morphisms of the representations.

Lemma 6.7.3 *Every object in $\mathcal{D}$ is $\mathcal{J}_1$-small and $\mathcal{J}_2$-small.*

Proof. We prove the Lemma only for $\mathcal{J}_1$, then by symmetry it works also for $\mathcal{J}_2$. By Remark 6.4.8 it is enough to prove that any object $M \in \mathcal{D}$ is $\mathcal{J}_1$-quasi-small. To prove this we have to show that for any family of objects $M^\lambda \in \mathcal{D}$, $\lambda \in \Lambda$, and homomorphisms $\alpha{:}\, M \to \oplus_{\lambda\in\Lambda} M^\lambda$ and $\beta{:}\, \oplus_{\lambda\in\Lambda} M^\lambda \to M$ with $\beta\alpha \notin \mathcal{J}_1$, the class $\{\mu \in \Lambda \mid \beta\epsilon_\mu\pi_\mu\alpha \notin \mathcal{J}_1\}$ is not empty. Suppose then that $\beta\alpha \notin \mathcal{J}_1$, i.e. $\beta_1\alpha_1 \notin \mathcal{J}(M_1, M_1)$. Fix an element $m \in M_1$; then there exists a finite subset $F \subseteq \Lambda$ such that $\alpha_1(m) \in \oplus_{\lambda\in\Lambda} M_1^\lambda$. Let $M' = \oplus_{\lambda\in\Lambda\setminus F} M_1^\lambda$. Then we have $\beta_1\alpha_1 = \beta_1 1_{\oplus_{\lambda\in\Lambda} M_1^\lambda}\alpha_1 = \beta_1(\epsilon_{M'}\pi_{M'} + \sum_{\lambda\in F}\epsilon_\lambda\pi_\lambda)\alpha_1$. Since $\beta_1\alpha_1$ is not in $\mathcal{J}$, also one of the summands must not belong to $\mathcal{J}$. It can not be that $\beta_1\epsilon_{M'}\pi_{M'}\alpha_1$ is not in $\mathcal{J}(M_1, M_1)$, because this would mean that $\beta_1\epsilon_{M'}\pi_{M'}\alpha_1$ is an isomorphism and this contradicts $\pi_{M'}\alpha_1(m) = 0$. Hence there exists and index $\mu \in F$ such that $\beta_1\epsilon_\mu\pi_\mu\alpha_1 \notin \mathcal{J}(M_1, M_1)$. ∎

Theorem 6.7.4 *Let R be a ring and Q a quiver with two vertices. Consider the category $\mathcal{D}$ of representations of Q pointwise of type 1. Let $\mathcal{C} = \mathrm{Sum}(\mathcal{D})$ be the category whose objects are direct sums of objects in $\mathcal{D}$. In $\mathcal{D}$ we have the ideals $\mathcal{J}_1(M,N) = \{f{:}\, M \to N \mid f_1 \in \mathcal{J}(M_1, N_1)\}$ and $\mathcal{J}_2(M,N) = \{f{:}\, M \to N \mid f_2 \in \mathcal{J}(M_2, N_2)\}$. Then the Strong Infinite 2-Krull-Schmidt Property holds for $\mathcal{C}$ with respect to the equivalence relations $\sim_{\mathcal{J}_1}$ and $\sim_{\mathcal{J}_2}$.*

6.8 Infinite Quasi 2-Krull-Schmidt Property

In this section we want to generalize our section 6.3. Let M be an atomic commutative infinitary monoid and let A be the class of atoms of M. Suppose we are given two subclasses A' and A'' of A such that $A' \cup A'' = A$, an equivalence relation $\sim$ on A' and an equivalence relation $\equiv$ on A''. We say that the *Infinite Quasi 2-Krull-Schmidt Property* holds for M with respect to the equivalence relations $\sim$ and $\equiv$ if for any couple of families $\{\, a_i \mid i \in I \,\}$ and $\{\, b_j \mid j \in J \,\}$

of atoms of M, we have that $\sum_{i\in I} a_i = \sum_{j\in J} b_j$ if and only if there exist two bijections $\sigma: I' = \{ i \in I \mid a_i \in A' \} \to J' = \{ j \in J \mid b_j \in A' \}$ and $\tau: I'' = \{ i \in I \mid a_i \in A'' \} \to J'' = \{ j \in J \mid b_j \in A'' \}$ such that $a_i \sim b_{\sigma(i)}$ for every $i \in I'$ and $a_i \equiv b_{\tau(i)}$ for every $i \in I''$.

Proposition 6.8.1 *Let M be an atomic commutative infinitary monoid and let A be the class of atoms of M. Suppose we are given two subclasses A' and A'' of A such that $A' \cup A'' = A$, an equivalence relation $\sim$ on A' and an equivalence relation $\equiv$ on A''. Then the following are equivalent:*

1. *the Infinite Quasi 2-Krull-Schmidt Property holds for M;*
2. *there exist a complete bipartite graph $B(X \dot\cup Y)$ and an injective morphism of commutative infinitary monoids $\varphi: M \to F(X \dot\cup Y)$ that sends atoms of M either to vertices of $B(X \dot\cup Y)$ or to edges of $B(X \dot\cup Y)$ such that at least one of the two vertices is not in the image.*

Proof. Suppose that the Infinite Quasi 2-Krull-Schmidt Property holds for M. Let $A'/\sim$ and $A''/\equiv$ be the quotient classes of A' and A'' modulo $\sim$ and $\equiv$, respectively. The canonical morphism

$$\begin{array}{rcccl} \pi_\sim: & A & \to & A'/\sim & \\ & a & \mapsto & [a]_\sim & \text{if } a \in A' \\ & a & \mapsto & 0 & \text{if } a \notin A' \end{array}$$

induces a canonical morphism $\widehat{\pi_\sim}: M \to F(A'/\sim)$ defined by $\widehat{\pi_\sim}(\sum_{i\in I} a_i) = \sum_{i\in I} \pi_\sim(a_i)$. Similarly, the canonical morphism

$$\begin{array}{rcccl} \pi_\equiv: & A & \to & A''/\equiv & \\ & a & \mapsto & [a]_\equiv & \text{if } a \in A'' \\ & a & \mapsto & 0 & \text{if } a \notin A'' \end{array}$$

induces a canonical morphism $\widehat{\pi_\equiv}: M \to F(A''/\equiv)$ defined by $\widehat{\pi_\equiv}(\sum_{i\in I} a_i) = \sum_{i\in I} \pi_\equiv(a_i)$. Since the Infinite Quasi 2-Krull-Schmidt Property holds for M, the product morphism $\widehat{\pi_\sim} \times \widehat{\pi_\equiv}: M \to F(A'/\sim) \times F(A''/\equiv) \cong F(A'/\sim \dot\cup A''/\equiv)$ is injective. For every atom $a \in A$ of M, we have that

$$\widehat{\pi_\sim} \times \widehat{\pi_\equiv}(a) = \begin{cases} [a]_\sim & \text{if } a \in A' \setminus A'' \\ [a]_\equiv & \text{if } a \in A'' \setminus A' \\ [a]_\sim + [a]_\equiv & \text{if } a \in A' \cap A'' \end{cases}$$

Hence $\widehat{\pi_\sim} \times \widehat{\pi_\equiv}$ is a morphism of commutative infinitary monoids from M to the free commutative infinitary monoid $F(A'/\sim \dot\cup A''/\equiv)$, that has as basis the vertices of the bipartite graph $B(A'/\sim \dot\cup A''/\equiv)$, that sends atoms of M to vertices or edges of $B(A'/\sim \dot\cup A''/\equiv)$. It is not possible that, given three atoms a, a_1 and a_2 of M, we have $\widehat{\pi_\sim} \times \widehat{\pi_\equiv}(a) = [a]_\sim + [a]_\equiv$, $\widehat{\pi_\sim} \times \widehat{\pi_\equiv}(a_1) = [a]_\sim$ and $\widehat{\pi_\sim} \times \widehat{\pi_\equiv}(a_2) = [a]_\equiv$. In fact, this means that

$$\widehat{\pi_\sim} \times \widehat{\pi_\equiv}(a) = \widehat{\pi_\sim} \times \widehat{\pi_\equiv}(a_1) + \widehat{\pi_\sim} \times \widehat{\pi_\equiv}(a_2) = \widehat{\pi_\sim} \times \widehat{\pi_\equiv}(a_1 + a_2).$$

Since $\widehat{\pi_\sim} \times \widehat{\pi_\equiv}$ is injective, we get $a = a_1 + a_2$ but this contradicts the fact that a is an atom.

Now suppose that there exist a complete bipartite graph $B(X\dot{\cup}Y)$ and an injective morphism of commutative infinitary monoids $\varphi: M \to F(X\dot{\cup}Y)$ that sends atoms of M to vertices or edges of $B(X\dot{\cup}Y)$. Let A' be the subclass of A consisting of the atoms such that $\varphi(a)$ contains a vertex in X and let A'' be the subclass of A consisting of the atoms such that $\varphi(a)$ contains a vertex in Y. Given an atom $a \in A$, we set $\varphi(a) = x_a + y_a$, with $x_a \in X$ and $y_a \in Y$, where one of the two summands can be zero. Hence we get $A' = \{ a \in A \mid x_a \neq 0 \}$ and $A'' = \{ a \in A \mid y_a \neq 0 \}$. Since φ preserves infinite sums, we have that $\varphi(\sum_{i\in I} a_i) = \sum_{i\in I}(x_{a_i} + y_{a_i})$, where, for every $i \in I$, one among x_{a_i} and y_{a_i} can be zero. Since φ is injective, we have that $\sum_{i\in I} a_i$ and $\sum_{j\in J} b_j$ are equal if and only if $\varphi(\sum_{i\in I} a_i) = \sum_{i\in I}(x_{a_i} + y_{a_i})$ is equal to $\varphi(\sum_{j\in J} b_j) = \sum_{j\in J}(x_{b_j} + y_{b_j})$. This happens if and only if there exist two bijections $\sigma: I' = \{ i \in I \mid a_i \in A' \} \to J' = \{ j \in J \mid b_j \in A' \}$ and $\tau: I'' = \{ i \in I \mid a_i \in A'' \} \to J'' = \{ j \in J \mid b_j \in A'' \}$ such that $x_{a_i} = x_{b_{\sigma(i)}}$ for every $i \in I'$ and $y_{a_i} = y_{b_{\tau(i)}}$ for every $i \in I''$. If we define $a \sim b$ if $x_a = x_b \neq 0$ and $a \equiv b$ if $y_a = y_b \neq 0$ it becomes clear that the Infinite Quasi 2-Krull-Schmidt Property holds for M. ■

Similarly to what we had in section 6.3, also here we can interpret the above Proposition from another point of view.

Example 6.8.2 Let K and L be two classes and let $K' \subseteq K$ and $L' \subseteq L$. Consider the class D of all the couples $\begin{pmatrix} a \\ b \end{pmatrix}$ with $a \in K$ and $b \in L$ such that $a \in K'$ or $b \in L'$. Let F be the class containing all $2 \times \aleph$ matrices, with $\aleph$ any cardinal number, such that all the columns are elements of D. On the class F we consider the following equivalence relation: given a $2 \times \aleph$ matrix M and a $2 \times \daleth$ matrix N we say that $M \sim N$ if and only if there exist two bijections $\sigma: I' = \{i \in \aleph \mid m_{1,i} \in K'\} \to J' = \{j \in \daleth \mid n_{1,j} \in K'\}$ and $\tau: I'' = \{i \in \aleph \mid m_{2,i} \in L'\} \to J'' = \{j \in \daleth \mid n_{2,j} \in L'\}$ such that $m_{1,i} = n_{1,\sigma(i)}$ for every $i \in I'$ and $m_{2,i} = n_{2,\tau(i)}$ for every $i \in I''$.

We define an atomic commutative infinitary monoid C considering the class $F/\sim$ together with the operation induced by the juxtaposition of matrices. It is clear that the Infinite Quasi 2-Krull-Schmidt Property holds for C.

Given any atomic commutative infinitary monoid M for which the Infinite Quasi 2-Krull-Schmidt Property holds with respect to the equivalence relations $\sim$ on A' and $\equiv$ on A'', there exist an atomic commutative infinitary monoid C constructed as in Example 6.8.2 and an injective morphism of commutative infinitary monoids $\varphi: M \to C$. It is enough to take K' equal to the class of equivalence classes of A' with respect to the equivalence relation $\sim$, $|K| = |K'|+1$, L' equal to the class of the equivalence classes A'' with respect to the equivalence relation $\equiv$ and $|L| = |L'| + 1$. Then we can define $\varphi: M \to C$ as the morphism of commutative infinitary monoids that sends an atom $a \in M$ to the matrix $\begin{pmatrix} [a]_\sim \\ [a]_\equiv \end{pmatrix}$, where we define $[a]_\sim = *$ if $a \notin A'$ and $[a]_\equiv = \star$ if $a \notin A''$. Since the

Infinite Quasi 2-Krull-Schmidt Property holds both for M and C, it is clear that φ is well-defined and injective.

Given the equivalence relation $\sim$ on A' we can extend it to an equivalence relation $\sim'$ on the whole A in the following way: we say that $a \sim' b$ if either $a, b \in A'$ and $a \sim b$, or both a and b are not in A'. Similarly we can extend the equivalence relation $\equiv$ on A'' to an equivalence relation $\equiv'$ on A.

If $\sim$ is an equivalence relation on A' and $\equiv$ is an equivalence relation on A'', we say that $\sim$ and $\equiv$ are *permutable* if, given $a \in A' \cap A''$, $b \in A'$ and $c \in A''$ such that $a \sim b$ and $a \equiv c$, there exists $d \in A$ such that $d \sim' c$ and $d \equiv' b$.

Example 6.8.3 In the setting of Example 6.8.2, the induced equivalence relations of D are permutable if it is not to possible to have the following: given two elements $a, a' \in K$ such that $a' \in K'$ and $a \in K \setminus K'$ and two elements $b, b' \in L$ such that $b' \in L'$ and $b \in L \setminus L'$, the matrices $\begin{pmatrix} a \\ b' \end{pmatrix}$, $\begin{pmatrix} a' \\ b' \end{pmatrix}$ and $\begin{pmatrix} a' \\ b \end{pmatrix}$ belong to D. One way to avoid such problems is to ask that $K = K'$ or $L = L'$.

Theorem 6.8.4 *Let M be an atomic commutative infinitary monoid. Suppose that there are two permutable equivalence relations $\sim$ on A' and $\equiv$ on A'' such that:*

1. *$a = b$ if and only if $a \sim' b$ and $a \equiv' b$;*
2. *if $a \sim b$ and $a \equiv c$, then there exists an element $d \in A$ such that $a + d = b + c$, $d \sim' c$ and $d \equiv' b$;*
3. *if we have two families $\{ a_i \mid i = 0, 1, \dots \}$ and $\{ b_j \mid j = 0, 1, \dots \}$ such that $a_0 \notin A'$, all the other elements are in $A' \cap A''$ and $b_i \equiv a_i$ and $b_i \sim a_{i+1}$, then $\sum_{i=0}^{\infty} a_i = \sum_{j=0}^{\infty} b_j$;*
4. *if we have two families $\{ a_i \mid i = 0, 1, \dots \}$ and $\{ b_j \mid j = 0, 1, \dots \}$ such that $a_0 \notin A''$, all the other elements are in $A' \cap A''$ and $b_i \sim a_i$ and $b_i \equiv a_{i+1}$, then $\sum_{i=0}^{\infty} a_i = \sum_{j=0}^{\infty} b_j$.*

Let $\{ a_i \mid i \in I \}$ and $\{ b_j \mid j \in J \}$ be two families of elements of A. Then $\sum_{i \in I} a_i = \sum_{j \in J} b_j$ if there are two bijections $\sigma \colon I' = \{ i \in I \mid a_i \in A' \} \to J' = \{ j \in J \mid b_j \in A' \}$ and $\tau \colon I'' = \{ i \in I \mid a_i \in A'' \} \to J'' = \{ j \in J \mid b_j \in A'' \}$ such that $a_i \sim b_{\sigma(i)}$ for every $i \in I'$ and $a_i \equiv b_{\tau(i)}$ for every $i \in I''$.

Proof. Suppose that there are two bijections $\sigma \colon I' = \{ i \in I \mid a_i \in A' \} \to J' = \{ j \in J \mid b_j \in A' \}$ and $\tau \colon I'' = \{ i \in I \mid a_i \in A'' \} \to J'' = \{ j \in J \mid b_j \in A'' \}$ such that $a_i \sim b_{\sigma(i)}$ for every $i \in I'$ and $a_i \equiv b_{\tau(i)}$ for every $i \in I''$.

For every element $i \in I$ we construct inductively two sets, $[i] \subseteq I$ and $[[i]] \subseteq J$. We start from imposing $i \in [i]$. Then $k \in [i] \cap I'$ implies that $\sigma(k) \in [[i]]$ and $k \in [i] \cap I''$ implies that $\tau(k) \in [[i]]$. Similarly, $k \in [[i]] \cap J'$ implies that $\sigma^{-1}(k) \in [i]$ and $k \in [[i]] \cap J''$ implies that $\tau^{-1}(k) \in [i]$. We claim that $\sum_{k \in [i]} a_i = \sum_{l \in [[i]]} b_l$.

To simplify the notation we set $i_0 = i$ and, if they exist, for $z \in \mathbb{Z}$, $j_z = \sigma(i_z)$ if $z \geq 0$ and $j_z = \tau(i_{z+1})$ if $z < 0$, and $i_z = \tau^{-1}(j_{z-1})$ if $z > 0$ and $i_z = \sigma^{-1}(j_z)$ if $z < 0$. We set also $a_z = a_{i_z}$ and $b_z = b_{j_z}$ whenever they exist. Then, when they exist, we have $a_z \sim b_z$ and $a_z \equiv b_{z-1}$. We can distinguish, up to symmetry and reindexing, three cases:

(a) every element a_z and b_z exists and is in $A' \cap A''$. In this case we can prove the claim exactly as in the proof of Theorem 6.3.4.

(b) $a_0 \notin A''$, $a_z \in A' \cap A''$ for every $z \geq 1$ and $b_z \in A' \cap A''$ for every $z \geq 0$. In this case the claim is equivalent to our hypothesis (4).

(c)There exists an integer $n \geq 0$ such that $a_0, b_n \notin A''$ and $a_z, b_z \in A' \cap A''$ for every other z. If $n = 0$, by (1) we have that $a_0 = b_0$ and the claim is proved. To prove the claim in the case $n > 0$ we will show by induction that for every $0 < k \leq n$ there exists $d_k \in A' \setminus A''$ such that $d_k \sim a_k$ and $\sum_{i=0}^{k} a_i = (\sum_{j=0}^{k-1} b_j) + d_k$. We know that $b_0 \sim a_0$ and $b_0 \equiv a_1$. Hence by (2) there exists an element d_1 such that $d_1 \sim a_1$, $d_1 \notin A''$ and $b_0 + d_1 = a_0 + a_1$. Now suppose that d_k has been defined for every $k < t$. We have that $b_{t-1} \sim a_{t-1} \sim d_{t-1}$ and $b_{t-1} \equiv a_t$. By (2) we get that there exists d_t such that $d_t \sim a_t$, $d_t \equiv' d_{t-1}$, i.e. $d_t \notin A''$, and $b_{t-1} + d_t = d_{t-1} + a_t$. Since by inductive hypothesis we have $\sum_{i=0}^{t-1} a_i = (\sum_{j=0}^{t-2} b_j) + d_{t-1}$ we obtain that $\sum_{i=0}^{t} a_i = (\sum_{j=0}^{t-2} b_j) + d_{t-1} + a_t = (\sum_{j=0}^{t-1} b_j) + d_t$. Since d_n and b_n are not in A'' and $d_n \sim a_n \sim b_n$, by (1) we obtain that $d_n = b_n$ concluding the proof of the claim.

It can not happen that there is an integer $n \geq 1$ such that $a_0 \notin A''$, $a_n \notin A'$ and $a_z, b_z \in A' \cap A''$ for every other z. In fact, if $n = 1$ we have that $a_0 \sim b_0$ and $a_1 \equiv b_0$ imply that there exists $d \in A$ such that $d \sim' a_1$ and $d \equiv' a_0$. Since $a_0 \notin A''$ and $a_1 \notin A'$ such an element d does not exist. If $n > 1$, we prove by induction that for every $1 \leq k < n$ there exists $c_k \in A' \cap A''$ such that $c_k \sim a_0$, $c_k \equiv b_k$ and $\sum_{i=1}^{k} a_i + c_k = \sum_{j=0}^{k} b_j$. From the relations $a_1 \sim b_1$ and $a_1 \equiv b_0$ we deduce that there exists $c_1 \in A' \cap A''$ such that $c_1 \sim b_0 \sim a_0$, $c_1 \equiv b_1$ and $a_1 + c_1 = b_0 + b_1$. Now suppose we constructed c_{k-1}. From the relations $a_k \sim b_k$ and $a_k \equiv b_{k-1} \equiv c_{k-1}$ we obtain that there exists $c_k \in A' \cap A''$ such that $c_k \sim c_{k-1} \sim a_0$, $c_k \equiv b_k$ and $a_k + c_k = c_{k-1} + b_k$. Since by inductive hypothesis we have $\sum_{i=1}^{k-1} a_i + c_{k-1} = \sum_{j=0}^{k-1} b_j$ we obtain that $\sum_{i=1}^{k} a_i + c_k = \sum_{j=0}^{k} b_j$. Since $c_{n-1} \sim a_0$ and $c_{n-1} \equiv b_{n-1} \equiv a_n$, by (2) there should exist $d \in A$ such that $d \sim' a_n$ and $d \equiv' a_0$. Since $a_0 \notin A''$ and $a_n \notin A'$ such an element d can not exist.

When the index i runs all over the indices in I, we get that the sets $[i]$ form a partition of I and the sets $[[i]]$ form a partition of J. By the claim $\sum_{k \in [i]} a_i = \sum_{l \in [[i]]} b_l$ for every $i \in I$ and therefore $\sum_{i \in I} a_i = \sum_{j \in J} b_j$. ■

Similarly to the concept defined in section 6.3, we say that the equivalence relation $\sim$ on A' *controls the infinite on* A' if the following happens: let $\{ a_i \mid i \in I \}$ and $\{ b_j \mid j \in J \}$ be two families of elements of A such that $\sum_{i \in I} a_i = \sum_{j \in J} b_j$. Considering, for $k \in I'$, the classes $I_\sim(k) = \{ i \in I \mid a_i \sim a_k \}$ and $J_\sim(k) = \{ j \in J \mid b_j \sim a_k \}$ we suppose that, whenever $I_\sim(k)$ and $J_\sim(k)$ are infinite, for every $t \in I_\sim(k)$ there exists a subset $A(t) \subseteq J$ with $|A(t)| \leq |I_\sim(k)|$ such that $J_\sim(k) \subseteq \bigcup_{t \in I_\sim(k)} A(t)$ and, similarly, for every $u \in J_\sim(t)$ there exists a subset $B(u) \subseteq I$ with $|B(u)| \leq |J_\sim(k)|$ such that $I_\sim(k) \subseteq \bigcup_{u \in J_\sim(k)} B(u)$.

Theorem 6.8.5 *Let M be an atomic a-cancellative commutative infinitary monoid. Let A' and A'' be two subclasses of the class A of atoms of M such that $A' \cup A'' = A$. Suppose that there are two permutable equivalence relations $\sim$ on A' and $\equiv$ on A'' such that:*

1. *$a = b$ if and only if $a \sim' b$ and $a \equiv' b$;*
2. *let $a \in A$ be a summand of $\sum_{j \in J} b_j$. If $a \in A'$ there exists $j_1 \in J$ such that $a \sim b_{j_1}$; if $a \in A''$ there exists $j_2 \in J$ such that $a \equiv b_{j_2}$;*
3. *given atoms $a \in A' \cap A''$, $b \in A'$ and $c \in A''$ such that $a \sim b$ and $a \equiv c$, then there exists an element $d \in A$ such that $a + d = b + c$, $d \sim' c$ and $d \equiv' b$.*

If $\sim$ controls the infinite on A', then $\sum_{i \in I} a_i = \sum_{j \in J} b_j$ implies that there is a bijection $\sigma: I' \to J'$ such that $a_i \sim b_{\sigma(i)}$ for every $i \in I'$.

Proof. Let $\{ a_i \mid i \in I \}$ and $\{ b_j \mid j \in J \}$ be two families of elements of A such that $\sum_{i \in I} a_i = \sum_{j \in J} b_j$. Fix an index $k \in I'$ and consider the two subclasses $I_\sim(k) = \{ i \in I \mid a_i \sim a_k \}$ of I' and $J_\sim(k) = \{ j \in J \mid b_j \sim a_k \}$ of J'. As in proof of Theorem 6.3.5, the classes $I_\sim(k), k \in I'$ form a partition of I' and the classes $J_\sim(k), k \in I'$ form a partition of J'.

In order to establish the existence of the bijection between the $\sim$-classes of $\{ a_i \mid i \in I' \}$ and $\{ b_j \mid j \in J' \}$, it is sufficient to prove that the cardinalities $|I_\sim(k)|$ and $|J_\sim(k)|$ are equal for every $k \in I'$.

Suppose first that either $I_\sim(k)$ or $J_\sim(k)$ is a finite set. Without loss of generality we may assume $|I_\sim(k)| \le |J_\sim(k)|$. Suppose that $|I_\sim(k)| < |J_\sim(k)|$. Take $\bar{\imath} \in I_\sim(k)$; then, by (2), there exist $j_1 \in J'$ such that $a_{\bar{\imath}} \sim b_{j_1}$. Suppose first that $\bar{\imath} \notin I''$. If $b_{j_1} \notin A''$ then $a_{\bar{\imath}} = b_{j_1}$ and, since M is a-cancellative, we get $\sum_{i \in I \smallsetminus \{\bar{\imath}\}} a_i = \sum_{j \in J \smallsetminus \{j_1\}} b_j$. If $b_{j_1} \in A''$ then there exists a_{i_1} such that $b_{j_1} \equiv a_{i_1}$. If $a_{i_1} \sim b_{j_1}$ we obtain $a_{i_1} = b_{j_1}$ and hence $\sum_{i \in I \smallsetminus \{i_1\}} a_i = \sum_{j \in J \smallsetminus \{j_1\}} b_j$. If $a_{i_1} \not\sim' b_{j_1}$, we can exclude the case $a_{i_1} \notin A'$, since if this is true we obtain by (3) that there should exist an atom d that is neither in A' nor in A''. Hence we must have $a_{i_1} \in A'$; then by (3) there exists $d \in A$ such that $d \sim a_{i_1}$, $d \notin A''$ and $b_{j_1} + d = a_{\bar{\imath}} + a_{i_1}$. Therefore

$$\sum_{i \in I} a_i = a_{\bar{\imath}} + a_{i_1} + \sum_{i \in I \smallsetminus \{\bar{\imath}, i_1\}} a_i = b_{j_1} + d + \sum_{i \in I \smallsetminus \{\bar{\imath}, i_1\}} a_i = \sum_{j \in J} b_j$$

implies $d + \sum_{i \in I \smallsetminus \{\bar{\imath}, i_1\}} a_i = \sum_{j \in J \smallsetminus \{j_1\}} b_j$.

If $\bar{\imath} \in I''$ there exists also $j_2 \in J''$ such that $a_{\bar{\imath}} \equiv b_{j_2}$. We can exclude that $b_{j_1} \notin A''$ and $b_{j_2} \notin A'$ since this would imply, by (3), that there exists an atom d that is neither in A' nor in A''. If $a_{\bar{\imath}} \equiv b_{j_1}$ we obtain $a_{\bar{\imath}} = b_{j_1}$ and hence $\sum_{i \in I \smallsetminus \{\bar{\imath}\}} a_i = \sum_{j \in J \smallsetminus \{j_1\}} b_j$ since M is a-cancellative. If $j_2 \in J_\sim(k)$, then $a_{\bar{\imath}} = b_{j_2}$ by (1) and we get $\sum_{i \in I \smallsetminus \{\bar{\imath}\}} a_i = \sum_{j \in J \smallsetminus \{j_2\}} b_j$. In the remaining case, where $j_2 \notin J_\sim(K)$, $a_{\bar{\imath}} \not\equiv' b_{j_1}$ and at least one among b_{j_1} and b_{j_2} is in $A' \cap A''$ we have, by (3), that there exists d such that $d \sim' b_{j_2}$, $d \equiv' b_{j_1}$ and $b_{j_1} + b_{j_2} = a_{\bar{\imath}} + d$. Then

$$\sum_{i \in I} a_i = \sum_{j \in J} b_j = b_{j_1} + b_{j_2} + \sum_{j \in J \smallsetminus \{j_1, j_2\}} b_j = a_{\bar{\imath}} + d + \sum_{j \in J \smallsetminus \{j_1, j_2\}} b_j$$

implies $\sum_{i\in I\setminus\{\bar{\imath}\}} a_i = d + \sum_{j\in J\setminus\{j_1,j_2\}} b_j$.

An easy induction shows that after $|I_\sim(K)|$ steps we get the required contradiction.

If $I_\sim(k)$ and $J_\sim(k)$ are both infinite, the proof goes exactly as in the last paragraph of the proof of Theorem 6.3.5.

Therefore we get that there exists a bijection $\sigma\colon I' \to J'$ such that $a_i \sim b_{\sigma(i)}$ for every $i \in I'$. ■

Combining the results of Theorem 6.8.4 and Theorem 6.8.5, we obtain the following.

Theorem 6.8.6 *Let M be an atomic commutative infinitary monoid. Let A' and A'' be two subclasses of the class A of atoms of M such that $A' \cup A'' = A$. Suppose that there are two permutable equivalence relations $\sim$ on A' and $\equiv$ on A''. Then the following are equivalent:*

1. *the Infinite Quasi 2-Krull-Schmidt Property holds for $\sim$ and $\equiv$;*
2. *the following hypotheses hold for M:*
   - (a) *M is a-cancellative;*
   - (b) *$a = b$ if and only if $a \sim' b$ and $a \equiv' b$;*
   - (c) *let $a \in A$ be a summand of $\sum_{j\in J} b_j$. If $a \in A'$ there exists $j_1 \in J$ such that $a \sim b_{j_1}$; if $a \in A''$ there exists $j_2 \in J$ such that $a \equiv b_{j_2}$;*
   - (d) *given atoms $a \in A' \cap A''$, $b \in A'$ and $c \in A''$ such that $a \sim b$ and $a \equiv c$, then there exists an element $d \in A$ such that $a + d = b + c$, $d \sim' c$ and $d \equiv' b$;*
   - (e) *if we have two families $\{ a_i \mid i = 0, 1, \dots \}$ and $\{ b_j \mid j = 0, 1, \dots \}$ such that $a_0 \notin A'$, all the other elements are in $A' \cap A''$ and $b_i \equiv a_i$ and $b_i \sim a_{i+1}$, then $\sum_{i=0}^{\infty} a_i = \sum_{j=0}^{\infty} b_j$;*
   - (f) *if we have two families $\{ a_i \mid i = 0, 1, \dots \}$ and $\{ b_j \mid j = 0, 1, \dots \}$ such that $a_0 \notin A''$, all the other elements are in $A' \cap A''$ and $b_i \sim a_i$ and $b_i \equiv a_{i+1}$, then $\sum_{i=0}^{\infty} a_i = \sum_{j=0}^{\infty} b_j$;*
   - (g) *$\sim$ controls the infinite on A';*
   - (h) *$\equiv$ controls the infinite on A''.*

PROOF. The implication (2) ⇒ (1) comes directly from Theorem 6.8.4 and 6.8.5.

The implication (1) ⇒ (2) is obtained applying the Infinite Quasi 2-Krull-Schmidt Property in some particular cases. It is clear that $a = b$ if and only if $a \sim' b$ and $a \equiv' b$. If $a \in A$ is a summand of $\sum_{j\in J} b_j$ it is clear from the Infinite Quasi 2-Krull-Schmidt Property that, if $a \in A'$, there exists $j_1 \in J$ such that $a \sim b_{j_1}$ and, if $a \in A''$, there exists $j_2 \in J$ such that $a \equiv b_{j_2}$. If we are given atoms $a \in A' \cap A''$, $b \in A'$ and $c \in A''$ such that $a \sim b$ and $a \equiv c$, the Property would tell us that $a = b + c$ if $b \notin A''$ and $c \notin A'$; this is clearly not possible, since a is an atom. Hence at least one among b and c is in $A' \cap A''$, the permutability of $\sim$ and $\equiv$ provides us an element $d \in A$ such that $d \sim' c$ and $d \equiv' b$ and the Property assures us that $a + d = b + c$. If a is an atom of M and $\sum_{i \in I} a_i$ and $\sum_{j \in J} b_j$ are two any other elements of M such that $a + \sum_{i \in I} a_i = a + \sum_{j \in J} b_j$, by the Property we obtain that there exist a bijection $\sigma \colon I' \cup \{*\} \to J' \cup \{*\}$ such that $a_i \sim' b_{\sigma(i)}$ for every $i \in I' \cup \{*\}$ and a bijection $\tau \colon I'' \cup \{*\} \to J'' \cup \{*\}$ such that $a_i \equiv' b_{\tau(i)}$ for every $i \in I'' \cup \{*\}$. Then it is clear that we have also bijections $\sigma' \colon I' \to J'$ such that $a_i \sim b_{\sigma'(i)}$ for every $i \in I'$ and $\tau' \colon I'' \to J''$ such that $a_i \sim b_{\tau'(i)}$ for every $i \in I''$ and hence $\sum_{i \in I} a_i = \sum_{j \in J} b_j$.

To prove (g) and (h), suppose $\sum_{i \in I} a_i = \sum_{j \in J} b_j$. By the Infinite Quasi 2-Krull-Schmidt Property, there exist two bijections $\sigma \colon I' \to J'$ such that $a_i \sim b_{\sigma(i)}$ for every $i \in I'$ and $\tau \colon I'' \to J''$ such that $a_i \equiv b_{\tau(i)}$ for every $i \in I''$. Then we can take, for every $t \in I'$, $A(t) = \{\sigma(t)\}$ and we get $J_\sim(k) \subseteq \bigcup_{t \in I_\sim(k)} \{\sigma(t)\}$ for every $k \in I'$. Similarly, defining $B(u) = \{\sigma^{-1}(u)\}$ for every $u \in J'$, we get $I_\sim(k) \subseteq \bigcup_{u \in J_\sim(k)} \{\sigma^{-1}(u)\}$ for every $k \in I'$. We can do the same for $\equiv$ using τ instead of σ. ■

6.9 Infinite quasi 2-Krull-Schmidt Property in cocomplete categories

Let $\mathcal{C}$ be a $\mathcal{D}$-splitting cocomplete category and let $\mathcal{I}$ and $\mathcal{J}$ be completely prime ideals of $\mathcal{D}$. Suppose that for every object A in $\mathcal{D}$ the ideals $\mathcal{I}(A, A)$ and $\mathcal{J}(A, A)$ are proper realizing all maximal ideals of $\mathrm{End}_{\mathcal{D}}(A)$. In this setting we would like to find conditions that make the Infinite Quasi 2-Krull-Schmidt Property hold for $\sim_{\mathcal{I}}$ and $\sim_{\mathcal{J}}$.

We know from section 6.4 that the Strong Infinite 2-Krull-Schmidt Property holds for the class of objects of $\mathcal{C}$ that are direct sum of $\mathcal{I}$-small and $\mathcal{J}$-small objects of $\mathcal{D}$.

The first step that we need to do, to create the setting for a Infinite Quasi 2-Krull-Schmidt Property for the commutative infinitary monoid $V(\mathcal{C})$, is to determine the subclasses $\mathcal{D}'$ and $\mathcal{D}''$ of $\mathcal{D}$ that will play the role of the classes A' and A'' of the previous section. To define $\mathcal{D}'$ and $\mathcal{D}''$ we look at Theorem 6.8.6 and we extrapolate the hypotheses that A' and A'' need to satisfy. Since the situation is completely symmetric, we will deal only with $\mathcal{D}'$ and A'. We get the following:

- *$\sim$ controls the infinite on A'.*
  Similarly to what we did in section 6.4 we can assume that every object in $\mathcal{D}'$ is $\mathcal{I}$-small to assure the validity of this hypothesis.

- *let $a \in A$ be a summand of $\sum_{j \in J} b_j$. If $a \in A'$ there exist $j_1 \in J$ such that $a \sim b_{j_1}$.*
  Since we assume that every element of $\mathcal{D}'$ is $\mathcal{I}$-small, it is $\mathcal{I}$-quasi small and so by Proposition 6.4.6 we get that also this hypothesis is satisfied.

- *if we have two families* $\{ a_i \mid i = 0, 1, \dots \}$ *and* $\{ b_j \mid j = 0, 1, \dots \}$ *such that* $a_0 \notin A'$, *all the other elements are in* $A' \cap A''$ *and* $b_i \equiv a_i$ *and* $b_i \sim a_{i+1}$, *then* $\sum_{i=0}^{\infty} a_i = \sum_{j=0}^{\infty} b_j$.
  We say that the couple of ideals $\mathcal{I}$ and $\mathcal{J}$ is $\mathcal{I}$-*starting* if, given two families $\{ B_i \mid i = 0, 1, \dots \}$ and $\{ C_j \mid j = 0, 1, \dots \}$ such that B_0 is not $\mathcal{I}$-small, all the other elements are both $\mathcal{I}$-small and $\mathcal{J}$-small and $[C_i]_{\mathcal{J}} = [B_i]_{\mathcal{J}}$ and $[C_i]_{\mathcal{I}} = [B_{i+1}]_{\mathcal{I}}$, then $\oplus_{i=0}^{\infty} B_i = \oplus_{j=0}^{\infty} C_j$.
- *given* $a, b \notin A'$, *we have* $a = b$ *if and only if* $a \equiv b$.
  We say that the ideals $\mathcal{I}$ and $\mathcal{J}$ are *complementary* if, given two non-$\mathcal{I}$-small objects B and C in $\mathcal{D}$, they are equal if and only if $[B]_{\mathcal{J}} = [C]_{\mathcal{J}}$, and similarly if B and C are non-$\mathcal{J}$-small, they are equal if and only if $[B]_{\mathcal{I}} = [C]_{\mathcal{I}}$.

With these hypotheses we get the following.

Theorem 6.9.1 *Let $\mathcal{C}$ be a $\mathcal{D}$-splitting cocomplete category and let $\mathcal{I}$ and $\mathcal{J}$ be completely prime ideals of $\mathcal{D}$. Suppose that for every object $A \in \mathcal{D}$, the ideals $\mathcal{I}(A, A)$ and $\mathcal{J}(A, A)$ are proper ideals of* $\mathrm{End}_{\mathcal{D}}(A)$ *realizing all maximal ideals of* $\mathrm{End}_{\mathcal{D}}(A)$. *Suppose that the couple of ideals $\mathcal{I}$ and $\mathcal{J}$ is both $\mathcal{I}$-starting and $\mathcal{J}$-starting and that $\mathcal{I}$ and $\mathcal{J}$ are complementary.*

Let $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ be two families of objects of $\mathcal{D}$. Then $\oplus_{i \in I} U_i \cong \oplus_{j \in J} V_j$ if and only if there is a bijection $\sigma\colon I' = \{i \in I \mid U_i \text{ is } \mathcal{I}\text{-small}\} \to J' = \{j \in J \mid V_j \text{ is } \mathcal{I}\text{-small}\}$ and a bijection $\tau\colon I'' = \{i \in I \mid U_i \text{ is } \mathcal{J}\text{-small}\} \to J'' = \{j \in J \mid V_j \text{ is } \mathcal{J}\text{-small}\}$ such that $[U_i]_{\mathcal{I}} = [V_{\sigma(i)}]_{\mathcal{I}}$ for every $i \in I'$ and $[U_i]_{\mathcal{J}} = [V_{\tau(i)}]_{\mathcal{J}}$ for every $i \in I''$.

Proof. It is enough to check that all the conditions of the second point of Theorem 6.8.6 are satisfied.

The endomorphism ring of every object $A \in \mathcal{D}$ is semilocal and hence A cancels from direct sums.

By Lemma 6.4.1 and the hypothesis that the ideals $\mathcal{I}$ and $\mathcal{J}$ are complementary, we get that $B \cong C$ if and only if $[B]'_{\mathcal{I}} = [C]'_{\mathcal{I}}$ and $[B]'_{\mathcal{J}} = [C]'_{\mathcal{J}}$, where with the $'$ we denote the extended equivalence relation as we defined it before Theorem 6.8.4.

Suppose that $B \in \mathcal{D}$ is a direct summand of $\oplus_{j \in J} C_j$. If B is $\mathcal{I}$-small, there is a $j \in J$ such that $[B]_{\mathcal{I}} = [C_j]_{\mathcal{I}}$. Similarly, if B is $\mathcal{J}$-small, there is a $j \in J$ such that $[B]_{\mathcal{J}} = [C_j]_{\mathcal{J}}$.

If we are given objects B, C and D in $\mathcal{D}$ such that $[B]_{\mathcal{I}} = [C]_{\mathcal{I}}$ and $[B]_{\mathcal{J}} = [D]_{\mathcal{J}}$, by Corollary 6.4.3 there exists an object $E \in \mathcal{D}$ such that $B \oplus E \cong C \oplus D$, $[E]_{\mathcal{I}} = [D]_{\mathcal{I}}$ and $[E]_{\mathcal{J}} = [C]_{\mathcal{J}}$; a fortiori it must be $[E]'_{\mathcal{I}} = [D]'_{\mathcal{I}}$ and $[E]'_{\mathcal{J}} = [C]'_{\mathcal{J}}$.

Since the couple of ideals $\mathcal{I}$ and $\mathcal{J}$ is both $\mathcal{I}$-starting and $\mathcal{J}$-starting, it is easy to see that all the entries of (2) in Theorem 6.8.6 are satisfied and hence the Infinite Quasi 2-Krull-Schmidt Property holds for the commutative infinitary monoid $V(\mathcal{C})$ with respect to the equivalence relations $\sim_{\mathcal{I}}$ and $\sim_{\mathcal{J}}$ considered respectively on the classes $\mathcal{D}'$ of $\mathcal{I}$-small objects and $\mathcal{D}''$ of $\mathcal{J}$-small objects. ∎

6.10 Uniserial modules

In this section we want to present the unique (up to now) known example of cocomplete category where the Infinite Quasi 2-Krull-Schmidt Property holds but the Infinite 2-Krull Schmidt Property does not.

Throughout this section we will consider unital right modules over an associative ring R with $1 \neq 0$. We say that an R-module M is *uniserial* if its lattice of submodules is linearly ordered under inclusion. Let $\mathcal{D}$ be the full subcategory of Mod-R consisting of all non-zero uniserial R-modules and let $\mathcal{C} = \mathrm{Sum}(\mathcal{D})$ be its closure under infinite direct sum. In the category $\mathcal{D}$ we define the ideals $\mathcal{I}$ and $\mathcal{J}$ in the following way: given any two uniserial modules $M, N \in \mathcal{D}$ let

$$\mathcal{I}(M,N) = \{f : M \to N \mid f \text{ is not a monomorphism }\},$$

and

$$\mathcal{J}(M,N) = \{f : M \to N \mid f \text{ is not an epimorphism }\}.$$

Lemma 6.10.1 *The ideals $\mathcal{I}$ and $\mathcal{J}$ are completely prime ideals of $\mathcal{D}$.*

Proof. Let M, N and P be three non-zero objects of $\mathcal{D}$.

To prove that $\mathcal{I}$ is a completely prime ideal of $\mathcal{D}$ we show that for any morphisms $\alpha : N \to M$ and $\beta : M \to P$ such that $\beta\alpha$ is a monomorphism, we have that both α and β are monomorphisms. It is clear that α must be a monomorphism since $\ker(\alpha) \subseteq \ker(\beta\alpha)$. To prove that β is a monomorphism notice that $\beta\alpha$ monomorphism implies that $\alpha(N) \cap \ker(\beta) = 0$; since M is uniserial we have that either $\alpha(N) = 0$ or $\ker(\beta) = 0$. It can not happen that $\alpha(N) = 0$ since we proved α to be a monomorphism, hence it must be $\ker(\beta) = 0$.

To prove that $\mathcal{J}$ is a completely prime ideal of $\mathcal{D}$ we show that for any morphism $\gamma : N \to M$ and $\delta : M \to P$ such that $\delta\gamma$ is an epimorphism, we have that both γ and δ are epimorphisms. It is clear that δ is an epimorphism since $\mathrm{im}(\delta) \supseteq \mathrm{im}(\delta\gamma)$. Now suppose that γ is not an epimorphism, so that $\gamma(N) \subsetneq M$. Since $P \neq 0$, we have also $\ker(\delta) \subsetneq M$. Then, since M is uniserial, we have that $\gamma(N) + ker(\delta) \subsetneq M$. Now δ induce a one-to-one order preserving correspondence between the submodules of M containing $\ker(\delta)$ and the submodules of P. Hence $\gamma(N) + ker(\delta) \subsetneq M$ implies $\delta(\gamma(N) + ker(\delta)) = \delta\gamma(N) \subsetneq \delta(M)$. This contradicts the fact that $\delta\gamma$ is an epimorphism, hence γ must be an epimorphism. ∎

Proposition 6.10.2 *Let M be a non-zero uniserial R-module and E its endomorphism ring $\mathrm{End}_R(M)$. Then $\mathcal{I}(M,M)$ and $\mathcal{J}(M,M)$ are proper ideals of E, every proper ideal of E is contained either in $\mathcal{I}(M,M)$ or in $\mathcal{J}(M,M)$ and either*

- *$\mathcal{I}(M,M)$ and $\mathcal{J}(M,M)$ are comparable so that E is a local ring with maximal ideal $\mathcal{I}(M,M) \cup \mathcal{J}(M,M)$, or*

- *$\mathcal{I}(M,M)$ and $\mathcal{J}(M,M)$ are not comparable, $\mathcal{I}(M,M)\cap\mathcal{J}(M,M)$ is the Jacobson radical $J(E)$ of E and $E/J(E)$ is canonically isomorphic to the direct product $E/\mathcal{I}(M,M)\times E/\mathcal{J}(M,M)$ of the division rings $E/\mathcal{I}(M,M)$ and $E/\mathcal{J}(M,M)$.*

PROOF. Let K be an arbitrary proper left or right ideal of E. Since $\mathcal{I}(M,M)\cup\mathcal{J}(M,M)$ is the set of non-invertible elements of $\mathrm{End}_R(M)$, it must be $K\subseteq\mathcal{I}(M,M)\cup\mathcal{J}(M,M)$. If we suppose $K\nsubseteq\mathcal{I}(M,M)$ and $K\nsubseteq\mathcal{J}(M,M)$, there must be elements $x\in K\smallsetminus\mathcal{I}(M,M)$ and $y\in K\smallsetminus\mathcal{J}(M,M)$. Then x must be in $\mathcal{J}(M,M)$ and $y\in\mathcal{I}(M,M)$. Hence we get that $x+y\in K\smallsetminus(\mathcal{I}(M,M)\cup\mathcal{J}(M,M))$. This contradicts the fact that $K\subseteq\mathcal{I}(M,M)\cup\mathcal{J}(M,M)$.

Thus every right or left ideal of E is contained either in $\mathcal{I}(M,M)$ or in $\mathcal{J}(M,M)$. Therefore the unique maximal ideals (right, left and two-sided) can be only $\mathcal{I}(M,M)$ and $\mathcal{J}(M,M)$. If these two ideals are comparable it is the clear that E is a local ring with maximal ideal $\mathcal{I}(M,M)\cup\mathcal{J}(M,M)$. Otherwise $\mathcal{I}(M,M)$ and $\mathcal{J}(M,M)$ are the only two distinct maximal ideals of E. Therefore $\mathcal{I}(M,M)\cap\mathcal{J}(M,M)$ is the Jacobson radical $J(E)$ of E and there is a canonical injective ring morphism $E/J(E)\to E/\mathcal{I}(M,M)\times E/\mathcal{J}(M,M)$. Since $E=\mathcal{I}(M,M)+\mathcal{J}(M,M)$ this morphism is onto by the Chinese Remainder Theorem. ■

From this it follows that every object in $\mathcal{D}$ has semilocal endomorphism ring and hence it cancels from direct sums. This means that the monoid $V(\mathcal{C})$ is a-cancellative.

With our next Lemma we prove that the category $\mathcal{C}$ is $\mathcal{D}$-splitting.

Lemma 6.10.3 *Let M,N and P be uniserial R-modules such that $[M]_{\mathcal{I}}=[N]_{\mathcal{I}}$ and $[M]_{\mathcal{J}}=[P]_{\mathcal{J}}$. There is a uniserial module Q such that $M\oplus Q\cong N\oplus P$.*

PROOF. If we have three non-zero uniserial modules M,N and P such that $[M]_{\mathcal{I}}=[N]_{\mathcal{I}}$ and $[M]_{\mathcal{J}}=[P]_{\mathcal{J}}$, we have morphisms $f\colon M\to N$, $g\colon N\to M$, $h\colon M\to P$ and $l\colon P\to M$ such that $f,g\notin\mathcal{I}$ and $h,l\notin\mathcal{J}$. Consider the morphisms

$$\begin{pmatrix} f \\ h \end{pmatrix}\colon M\to N\oplus P$$

and

$$\begin{pmatrix} g & l \end{pmatrix}\colon N\oplus P\to M$$

whose composite map is

$$\begin{pmatrix} g & l \end{pmatrix}\begin{pmatrix} f \\ h \end{pmatrix}=gf+lh\colon M\to M.$$

If gf is an isomorphism, then also f and g are isomorphisms by Lemma 6.10.1. This means that $M\cong N$ and we can choose Q to be P. Similarly, if lh is an isomorphism, also h and l are isomorphisms, M is isomorphic to P and we can choose Q to be isomorphic to N.

Hence we can suppose that neither gf nor lh are isomorphisms. Then $gf + lh$ is neither in $\mathcal{I}(M, M)$ nor in $\mathcal{J}(M, M)$ and therefore it must be an isomorphism. Hence M is a direct summand of $N \oplus P$ and, more precisely, we have that

$$N \oplus P \cong M \oplus \ker(\begin{array}{cc} g & l \end{array}).$$

Moreover

$$\begin{array}{rcl} \ker(\begin{array}{cc} g & l \end{array}) & = & \{n \oplus p \in N \oplus P \mid g(n) + l(p) = 0\} \\ & = & \{(g^{\leftarrow}(l(-p)), p) \mid l(p) \in g(N)\} \\ & \cong & l^{\leftarrow}(g(N)), \end{array}$$

and $l^{\leftarrow}(g(N))$ is a uniserial module because it is a submodule of P. Hence we can choose $Q = \ker(\begin{array}{cc} g & l \end{array})$. ∎

Now we want to investigate the classes of $\mathcal{I}$-small and $\mathcal{J}$-small uniserial modules.

Lemma 6.10.4 *Every non-zero uniserial module is $\mathcal{I}$-small.*

Proof. By Remark 6.4.8 it is enough to prove that every uniserial module is $\mathcal{I}$-quasi-small. Let $\oplus_{i \in I} A_i$ be a direct sum of uniserial modules and suppose $U \oplus B = \oplus_{i \in I} A_i$ for a non-zero uniserial module U. We denote by $\epsilon_U : U \to \oplus_{i \in I} A_i$ and $\pi_U : \oplus_{i \in I} A_i \to U$ the canonical morphisms with respect o the direct summand U and by $\epsilon_j : A_j \to \oplus_{i \in I} A_i$ and $\pi_j : \oplus_{i \in I} A_i \to A_j$ the canonical morphisms with respect to the direct summand A_j. We need to prove that there exists an index $k \in I$ such that $\pi_U \epsilon_k \pi_k \epsilon_U$ is an injective endomorphism of U.

To do this, take a non-zero element $x \in U$. Then $x \in A_{i_1} \oplus \ldots \oplus A_{i_n}$ for some $i_1, \ldots, i_n \in I$. Set $C = \oplus_{i \neq i_1, \ldots, i_n} A_i$ and let $\epsilon_C : C \to \oplus_{i \in I} A_i$ and $\pi_C : \oplus_{i \in I} A_i \to C$ be the canonical morphisms with respect to the direct summand C. Then

$$\begin{array}{rcl} 1_U & = & \pi_U \epsilon_U = \pi_U (\epsilon_{i_1} \pi_{i_1} + \ldots + \epsilon_{i_n} \pi_{i_n} + \epsilon_C \pi_C) \epsilon_U \\ & = & \pi_U \epsilon_{i_1} \pi_{i_1} \epsilon_U + \ldots + \pi_U \epsilon_{i_n} \pi_{i_n} \epsilon_U + \pi_U \epsilon_C \pi_C \epsilon_U . \end{array}$$

Since $1_U \notin \mathcal{I}$, at least one of the summands must be a monomorphism. It can not be the last one since $\pi_U \epsilon_C \pi_C \epsilon_U(x) = 0$. Therefore there is an index $t = 1, \ldots, n$ such that $\pi_U \epsilon_{i_t} \pi_{i_t} \epsilon_U$ is a monomorphism. ∎

On the other hand, it is not true that every non-zero uniserial module is $\mathcal{J}$-small. An example of a non-zero uniserial module over a uniserial domain that is not $\mathcal{J}$-small was given by Puninski in [42].

From our previous Lemma we deduce that the couple of ideals $\mathcal{I}$ and $\mathcal{J}$ is $\mathcal{I}$-starting since there are no non-$\mathcal{I}$-small non-zero uniserial modules. Hence, to be able to use Theorem 6.9.1, we are left to prove that the couple of ideals is $\mathcal{J}$-starting and complementary. To prove these conditions we refer to [44] and [45].

For any uniserial module U, define the submodule $U_m \subseteq U$ as the intersection of all the submodules of U that are isomorphic to U. If $S \subseteq \mathrm{End}_R(U)$ is the set of all monic endomorphisms of U, then $U_m = \bigcap_{f\in S} \mathrm{im}(f)$. Dually, let $T \subseteq \mathrm{End}_R(U)$ be the set of all epic endomorphisms of U and define $U_e = \sum_{f\in T} \ker(f)$.

Recall that an R-module M is said to be *small* if for every family $\{N_i \mid i \in I\}$ of R-modules and any homomorphism $\varphi: M \to \oplus_{i\in I} N_i$ there is a finite subset $F \subseteq I$ such that $\pi_i f = 0$ for every $i \in I \setminus F$. Clearly any small module is $\mathcal{I}$-small with respect to any ideal $\mathcal{I}$. For uniserial modules we have the following.

Lemma 6.10.5 ([14, Proposition 2.45]) *Every uniserial module that is not small is countably generated.*

Recall that a family of morphisms $f_\lambda: U \to M_\lambda$, $\lambda \in \Lambda$ is *summable* if for every $x \in U$ there is a finite subset $\Lambda' \subseteq \Lambda$ such that $f_\lambda(x) = 0$ for every $\lambda \in \Lambda \setminus \Lambda'$.

Proposition 6.10.6 *A uniserial module is non-$\mathcal{J}$-small if and only if $U_m \subsetneq U = U_e$ and U is a countably generated module.*

Proof. Let U be a uniserial module that is not $\mathcal{J}$-small. Since it is not small, it has to be countably generated. Any uniserial module with local endomorphism ring is $\mathcal{J}$-small since it has the exchange property [14, Example 9.29], so U has not local endomorphism ring and hence $U_m \subsetneq U$.

Now we need to prove that for any element $x \in U$ there is an endomorphism $f \in \mathrm{End}_R(U)$ such that $f(x) = x$ but f is not an automorphism. By contradiction, suppose that there exists an element $u \in U$ such that for every endomorphism f of U, $f(u) = u$ implies that f is an automorphism. Let M_λ, $\lambda \in \Lambda$ be a family of uniserial modules and let $\alpha: U \to \oplus_{\lambda\in\Lambda} M_\lambda$ and $\beta: \oplus_{\lambda\in\Lambda} M_\lambda \to U$ be morphisms such that $\beta\alpha = 1_U$. Let Λ' be a finite subset of Λ such that $\beta\epsilon_\lambda\pi_\lambda\alpha(u) = 0$ for every $\lambda \in \Lambda \setminus \Lambda'$. Then $\sum_{\lambda\in\Lambda'} \beta\epsilon_\lambda\pi_\lambda\alpha(u) = u$, hence $\sum_{\lambda\in\Lambda'} \beta\epsilon_\lambda\pi_\lambda\alpha$ is an automorphism. It is clear that one of the summands must be an epimorphism. This contradicts our hypothesis that U is non-$\mathcal{J}$-small. Hence, for any $u \in U$, there is a morphism $f: U \to U$ such that f is not an automorphism and $f(u) = u$. Then $1 - f$ is an epimorphism having u in its kernel, thus $U_e = U$.

Now let U be a countably generated uniserial module satisfying $U_m \subsetneq U = U_e$. To prove that U is not finitely generated, let $0 \neq u \in U_e$. By definition of U_e there exists an epimorphism f of U such that $f(u) = 0$. Let $v \in U$ such that $f(v) = u$. Then $f^2(v) = 0$ and $v \in U_e$. Since $u \neq 0$, $vR \nsubseteq uR$.

Next we prove that for any element $u \in U$ there is a monomorphism of U such that $f(u) = u$ but f is not an automorphism. Let $0 \neq u \in U = U_e$. Then there is an epimorphism $f: U \to U$ such that $f(u) = 0$. Let $g: U \to U$ be any monomorphism that is not an automorphism. Then $f + g$ is an automorphism and $(f+g)(u) = g(u)$. Now $(f+g)^{-1}g: U \to U$ is a monomorphism that is not an automorphism and $(f+g)^{-1}g(u) = u$.

Let x_n, $n \geq 0$ be a countable set of generators of U with $0 \subsetneq x_0R \subsetneq x_1R \subsetneq \dots$. For every n there is a monomorphism f of U such that $f(x_n) = x_n$ but f is not an automorphism. Define a family of endomorphisms as follows: $g_0 = f_0$ and $g_n = f_n - f_{n-1}$ for every $n \geq 1$. It is easy to see that $\{g_n \mid n \geq 0\}$ is a summable family of endomorphisms of U, $\sum_{n\geq 0} g_n = 1_U$ and every g_n is not an epimorphism. Hence U is not $\mathcal{J}$-small. ∎

Lemma 6.10.7 *Let U be a uniserial module satisfying $U_m \subsetneq U_e$. Then for any uniserial module V such that $[U]_\mathcal{I} = [V]_\mathcal{I}$ we have $V_m \subsetneq V_e$. Moreover, U_e is the union of its proper submodules that are isomorphic to V.*

Proof. We can suppose that V is a submodule of U such that $U_m \subsetneq V \subsetneq U_e$ because $[V]_\mathcal{I} = [U_e]_\mathcal{I}$. Now there is an epimorphism $f{:}\, U \to U$ such that $f(V) = 0$. The submodule $W = f^{-1}(V)$ is isomorphic to V, let $g{:}\, V \to W$ be an isomorphism. Then $W_m = U_m$ and $gf_{|W}{:}\, W \to W$ is an epimorphism having V, and thus also W_m, in its kernel. Hence $W_m \subsetneq W_e$.

Let X be a submodule of U_e which is a union of proper submodules of U_e that are isomorphic to V. Suppose $X \neq U_e$. Then there is an epimorphism $f{:}\, U \to U$ such that $f(X) = 0$. Now $f^{-1}(V)$ is a proper submodule of U_e isomorphic to V. Since $X \subsetneq f^{-1}(V)$, we have a contradiction and $X = U_e$. ∎

With our next Proposition we prove that the couple of ideals $\mathcal{I}$ and $\mathcal{J}$ is complementary.

Proposition 6.10.8 *Let U and V be non-zero uniserial modules that are not $\mathcal{J}$-small. Then $[U]_\mathcal{I} = [V]_\mathcal{I}$ if and only if $U \cong V$.*

Proof. Suppose that $[U]_\mathcal{I} = [V]_\mathcal{I}$. To prove the claim it is enough to find an epimorphism $f{:}\, V \to U$. By Proposition 6.10.6 we can apply Lemma 6.10.7 and hence U is a union of submodules isomorphic to V. As U is countably generated, there is a chain $X_1 \subseteq X_2 \subseteq \ldots \subseteq U$ such that for any $i \in \mathbb{N}$ there is an epimorphism $f{:}\, V \to X_i$. The sum of these epimorphisms induces an epimorphism $\varphi{:}\, \oplus_{i\in\mathbb{N}} V_i \to U$, where $V_i = V$ and $\varphi(V_i) = X_i$ for any $i \in \mathbb{N}$. Since $V = V_e$ and V is countably generated, it is possible to construct by induction elements $v_1, v_2, \ldots \in V$ and homomorphisms $h_1, h_2, \ldots$ such that the following conditions are satisfied:

- $v_1, v_2, \ldots$ generate V;
- for any $i \in \mathbb{N}$, the homomorphism $h_i{:}\, V \to V_i$ is an epimorphism and $h_{i+1}(v_i) = 0$;
- for any $i \geq 2$, $\varphi(h_i(v_i)) \notin X_{i-1}$.

The family $\{h_i \mid i \in \mathbb{N}\}$ is a summable family of homomorphisms $V \to \oplus_{i\in\mathbb{N}} V_i$, since $h_j(v_i) = 0$ whenever $j > i$. Let $f = \varphi h$, where $h = \sum_{i\in\mathbb{N}} h_i$. By the properties of the h_i, $f(v_i) \notin X_{i-1}$ for any $i \geq 2$. Thus f is an epimorphism and we are done. ∎

The last condition that we need is that the couple of ideals $\mathcal{I}$ and $\mathcal{J}$ is $\mathcal{J}$-starting. We just state it without proving it.

Proposition 6.10.9 ([45, Lemma 2.5]) *Let* $\{ B_i \mid i = 0, 1, \dots \}$ *and* $\{ C_j \mid j = 0, 1, \dots \}$ *be two families of uniserial modules such that* B_0 *is not* $\mathcal{J}$*-small, all the other modules are both* $\mathcal{I}$*-small and* $\mathcal{J}$*-small and* $[C_i]_{\mathcal{I}} = [B_i]_{\mathcal{I}}$ *and* $[C_i]_{\mathcal{J}} = [B_{i+1}]_{\mathcal{J}}$*, then* $\oplus_{i=0}^{\infty} B_i = \oplus_{j=0}^{\infty} C_j$.

At this point we can apply Theorem 6.9.1 and obtain the following.

Theorem 6.10.10 *Let* $\{U_i \mid i \in I\}$ *and* $\{V_j \mid j \in J\}$ *be non-empty families of non-zero uniserial modules. Let* $I' = \{i \in I \mid U_i$ *is* $\mathcal{J}$*-small*$\}$ *and* $J' = \{j \in J \mid V_j$ *is* $\mathcal{J}$*-small*$\}$. *Then* $\oplus_{i \in I} U_i \cong \oplus_{j \in J} V_j$ *if and only if there exist a bijection* $\sigma: I \to J$ *and a bijection* $\tau: I' \to J'$ *such that* $[U_i]_{\mathcal{I}} = [V_{\sigma(i)}]_{\mathcal{J}}$ *for any* $i \in I$ *and* $[U_i]_{\mathcal{J}} = [V_{\tau(i)}]_{\mathcal{J}}$ *for any* $i \in I'$.

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