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New construction methods for copulas and the multivariate case

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TO MY FAMILY
AND ALL PEOPLE
WHO HAVE SUPPORTED ME

Introduzione

In molti campi ci troviamo di fronte al problema di aggregare un insieme di risultati numerici, per ottenere un determinato valore; ciò avviene non soltanto in matematica ed in fisica, ma anche nell'ingegneria e nella maggior parte delle scienze economiche e sociali. Le funzioni di aggregazione, pertanto, sono usate per ottenere un risultato globale per ogni alternativa che consideri determinati criteri, anche se i problemi d'aggregazione sono molto vasti ed eterogenei. Ci sono, ad esempio, molti contributi sull'aggregazione di un numero finito od infinito di inputs reali [6, 7, 24, 31, 46, 61, 83], su argomenti che trattano input su scala ordinale [32] o sul problema di aggregare input complessi (come le distribuzioni di probabilità [81, 91] o i fuzzy sets [105]).
Le funzioni d'aggregazione, in particolare, giocano un ruolo importante nell'area dei processi decisionali, dove, infatti, i valori da aggregare sono tipicamente gradi di preferenza o di soddisfazione e, quindi, appartengono all'intervallo $[0,1]$.
Sia $n \in \mathbb{N}$, $n \geq 2$. Una *funzione d'aggregazione n–aria* è una mappa A da $[0,1]^n$ in $[0,1]$, che soddisfa alle seguenti proprietà:

(A1) $A(0,\dots,0) = 0$ e $A(1,\dots,1) = 1$;

(A2) A è crescente in ogni componente.

Una particolare classe di funzioni d'aggregazione è data dalle n-copule, la cui definizione è dovuta a A. Sklar nel 1959 [94]: una *n-copula* è la restrizione al cubo unitario $[0,1]^n$ di una funzione di distribuzione cumulativa multivariata, le cui marginali sono uniformi su $[0,1]$.
Più precisamente, una n-copula è una funzione $C : [0,1]^n \to [0,1]$ che soddisfa:

(C1) $C(\mathbf{u}) = 0$ se $u_i = 0$ per ogni $i = 1,\dots,n$;

(C2) $C(\mathbf{u}) = u_i$ se tutte le coordinate di $\mathbf{u}$ sono 1 eccetto u_i, cioè C ha *marginali monodimensionali uniformi*;

(C3) C è *n-crescente*, cioè $V_C(B) \geq 0$ per ogni n-box $B = [u_1,v_1] \times [u_2,v_2] \times \dots \times [u_n,v_n] \subseteq [0,1]^n$ con $u_i \leq v_i$, $i = 1,2,\dots,n$, dove il C-volume dell'n-box B è dato da

$$V_C(B) = \sum \varepsilon(z_1,\dots,z_n) \cdot C(z_1,\dots,z_n) \geq 0 \tag{1}$$

con

$$\varepsilon(z_1,\dots,z_n) = \begin{cases} 1 & \text{se } z_i = u_i \text{ per un numero pari di i,} \\ -1 & \text{se } z_i = u_i \text{ per un numero dispari di i.} \end{cases}$$

e la somma in (2.15) è estesa a tutti i vertici di B.

Le condizioni (C1) e (C2) sono note come *condizioni sui bordi*, mentre la condizione (C3) è nota come *monotonia*.

Nel caso bivariato la 2-crescenza è equivalente alla supermodularità e gli operatori d'aggregazione binari supermodulari, di cui le copule bivariate sono una sottoclasse, sono stati analizzati in dettaglio in [37, 39]. Ricordiamo che, considerata una funzione $A : [0,1]^n \to [0,1]$, A è supermodulare se, per ogni $\mathbf{x}$, $\mathbf{y} \in [0,1]^n$,

$$A(\mathbf{x} \wedge \mathbf{y}) + A(\mathbf{x} \vee \mathbf{y}) \geq A(\mathbf{x}) + A(\mathbf{y}),$$

dove

$$\begin{aligned} \mathbf{x} \vee \mathbf{y} &= (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \cdots, \max\{x_n, y_n\}), \\ \mathbf{x} \wedge \mathbf{y} &= (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \cdots, \min\{x_n, y_n\}). \end{aligned}$$

Una forma più forte di supermodularità è data dall'ultramodularità, che è stata discussa in generale in [67] e nell'applicazione alle copule in [56]. A nostro avviso l'ultramodularità è il modo più semplice per conservare l'assioma principale delle copule nel caso bivariato, perché, invece di essere definita sui rettangoli, lo è sui parallelogrammi (Fig. 3.1). Si può, quindi, verificare che una copula sia ultramodulare, controllando, soltanto, che le sezioni monodimensionali della copula siano convesse, essendo la supermodularità data per definizione.
Queste analisi ci consentiranno di costruire, ad esempio, nuove copule Archimediane, che meritano particolare attenzione per la convessità dei loro generatori additivi ed il legame stretto, che vedremo esserci tra convessità ed ultramodularità.
L'ultramodularità, infine, gode anche di un'altra proprietà interessante, ossia della chiusura per composizione, a differenza della supermodularità. Affinchè ciò avvenga in quest'ultimo caso, abbiamo bisogno di comporre la supermodularità con l'ultramodularità.

I particolari risultati trovati nel caso bidimensionale hanno indirizzato tale lavoro verso l'investigazione del caso multidimensionale. La supermodularità e l'assioma di n-crescenza non coincidono, sebbene tra essi vi sia, comunque, una stretta connessione (Def. 1.2 in [11]). La supermodularità è più forte della monotonia e nel caso multidimensionale è detta 2-*monotonia*, mentre l'ultramodularità *forte* 2-*monotonia*.
Nel caso multivariato si analizzano, dunque, concetti che generalizzano l'assioma di monotonia per le funzioni d'aggregazione con un duplice intento:

1. studiare le copule come particolari tipi di funzioni d'aggregazione,
2. vedere quando la proprietà di n-crescenza incontra la richiesta di monotonia per gli operatori d'aggregazione. Quest'approccio, infatti, porta alla definizione di k-monotonia, che per $k = n$ coincide con la n-crescenza.

Si ottengono, quindi, vari metodi di costruzione per copule multivariate e, di conseguenza, nella modellizzazione della dipendenza stocastica di vettori casuali con dimensione $n \geq 3$. Nell'Esempio B.1.7, in particolare, la chiusura delle funzioni d'aggregazione k-monotone

rispetto alla composizione con una funzione totalmente monotona, ci permetterebbe di ottenere una copula trivariata, anche se un problema aperto riguarda anche la possibilità di rilassare l'ipotesi di totale monotonia con quella più debole di forte k-monotonia. Con questo metodo otterremmo, comunque, una copula e non semplicemente un operatore d'aggregazione, perché per composizione si conserverebbe la proprietà di forte 3-monotonia, che coincide esattamente con la 3-crescenza nelle copule trivariate.
Nella prima parte (Esempio 4.3.10 (ii)) si considera, invece, un'applicazione al caso bivariato, utilizzando il Teorema 4.3.9.
Dal punto di vista probabilistico la supermodularità richiede semplicemente che una copula C sia una funzione di distribuzione valida. Considerandola, infatti, come funzione di distribuzione di due variabili casuali U_1 e U_2, osserviamo che, per ogni $u_{11}, u_{12}, u_{21}, u_{22}$ con $u_{11}, u_{12}, u_{21}, u_{22} \in [0,1]$, la disuguaglianza

$$C(u_{12}, u_{22}) - C(u_{12}, u_{21}) - C(u_{11}, u_{22}) + C(u_{11}, u_{21}) \geq 0$$

è equivalente alla seguente:

$$P(u_{11} \leq U_1 \leq u_{12}, u_{21} \leq U_2 \leq u_{22}) \geq 0.$$

L'interpretazione statistica dell'ultramodularità si collega, invece, allo studio delle proprietà di dipendenza delle variabili casuali. Le copule ultramodulari, in particolare, descrivono la struttura di dipendenza di vettori casuali stocasticamente decrescenti.
Le copule, quindi, in quanto funzioni di distribuzione congiunta, sono strettamente connesse con la misura di probabilità.
Questa tesi rappresenta, di conseguenza, un tipo di approccio unificante, poiché i concetti algebrici della teoria dei reticoli (supermodularità ed ultramodularità) si generalizzano con concetti tipici della teoria della misura (k-monotonia e forte k-monotonia) e le copule s'inseriscono esattamente in continuità fra i due differenti approcci.

Preface

Aggregation functions are mathematical objects that have the function of reducing a set of numbers into a unique representative number, combining several degrees of membership into one aggregated value. Particular kinds of aggregation functions are copulas which permit to represent joint distribution functions by splitting the marginal behaviour, embedded in the marginal distributions, from the dependence captured by the copula itself.
The word *copula* is a Latin noun that means a “link” and is used in grammar and logic to describe “that part of proposition which connects the subject and predicate” (*Oxford English Dictionary*). This word was first employed in a mathematical or statistical sense by Abe Sklar (1959) in the theorem (which now bears his name) describing the function that “joins together” one-dimensional distribution functions to form multivariate distribution functions. This theorem is a theorem of existence and, in the case of continuous marginal distribution functions is also a theorem of uniqueness, but it’s not a constructive theorem. Moreover, the concept of copula can be extended to n dimensions, but multivariate extensions are generally not easily to be done.
This thesis addresses and develops a new unified approach to copula-based modelling and characterizations of aggregation functions in the multivariate case. In fact the copula approach is particularly useful when we investigate the interaction between different arguments of aggregation functions. The problem of modelling interaction between attributes remains a difficult question in the theory of aggregation functions. The way to construct aggregation functions can be analyzed under various aspects: algebraic [23, 58, 60], analytical [12–16], probabilistic [37,39], and our aim is to propose some methods of construction connected with these points of view, in order to study n-increasing aggregation functions which are copulas. To cope with this problem, we have to understand the algebraic structure of lattice and supermodularity on a general lattice, because supermodularity is strictly connected to 2-increasingness (Def. 1.2 in [11]) and in the bivariate case copulas are a subclass of supermodular aggregation functions. Supermodularity is also connected to measure theory. For example it is known that a Choquet integral operator based on a fuzzy measure μ is superadditive if, and only if, the fuzzy measure μ is supermodular (see prop. 7.1.8).
Fuzzy measures are one of the most important areas in mathematics and so is the integral with respect to the fuzzy measure. The classical measure and the integral theory is based on the additivity of the set function. Additive property is sometimes important in some applications but sometimes becomes ineffective in the reasoning about the real world environment e.g., fuzzy logic, decision making, artificial intelligence, etc. In particular, with the introduction of fuzzy

set theory by Lofti A. Zadeh in 1965, which handles real life problems i.e. vagueness and ambiguity, the additive property of classical measures becomes a subject of controversy. During the seventies, M. Sugeno studied a common type of non-additive and monotonic set functions, called fuzzy measures.
So, a further aim of this thesis is to find out what is the connection between fuzzy measures and supermodular aggregation functions and in particular we want to investigate under which conditions Choquet integral yields supermodular aggregation functions.
Finally, our purpose is to open a new way for constructing supermodular aggregation functions in the multivariate case, which allows to extend several properties of copulas as well. In fact, in the multivariate case there are a lot of unsolved problems, in particular with regard to the multivariate decomposition of aggregation functions in a sum of copulas.

Organization

The thesis contains an introductory chapter with general definitions and a chapter presenting some other introductory topics with regard to the problems solved in the bivariate case. The others chapters discuss the multivariate case, by using some tools, such as aggregation evaluators.
In particular, we recall that the mathematical concept of supermodularity formalizes the idea of complementarity and in the literature it is defined for functions on a general lattice. So in the Introduction we are going to define some important definitions and properties for lattices, supermodularity and increasing differences.

Chapter 2 describes mainly supermodular and ultramodular functions at large, by starting to locate these concepts in the setting of aggregation functions and copulas.

In Chapter 3 we discuss supermodular and ultramodular aggregation functions in the multivariate case, with an interesting characterization of ultramodular functions which are also continuous.

Chapter 4 presents the copula approach to aggregation functions. The class of copulas has many distinct families, but we discuss few families of copulas suitable for our constructions, including a study of structural properties in connection with methods which yield ultramodular copulas in the bivariate case.

In Chapter 5 we will introduce triangular norms which are particular families of copulas, in order to find the solutions of a functional inequality which generalizes Frank's functional equation. Moreover, we will study aggregation evaluators and their connections with t-norms and t-conorms.

In Chapter 6 we present a new type of approach to aggregation functions, by using t-norms and in particular triangular norm-based measures, briefly called *T-measures*. So, we introduce TS-supermodularity like an extension of supermodular fuzzy measures.

Chapter 7 outlines some of analytical and algebraic methods connected with supermodular and ultramodular aggregation functions and that we use in the construction of copulas.
In this chapter we continue to deal with the multivariate case, by introducing two properties which are stronger than the monotonicity of aggregation functions, with some representation

results and with an application for constructing copulas.

Appendix A presents an application of multivariate copulas in the framework of multivariate dependence.

The thesis finishes with Appendix B where a number of the main open problems are listed after the detailed discussion treated in each section of this work. Of course the list is not exhaustive.

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Contents

Chapter 1

Introduction

This chapter introduces general concepts and relevant results for the present perspective on aggregation functions based on copulas. In particular, we discuss supermodular functions on a lattice and we explore some of their basic properties. Various common functional transformations maintain or generate supermodularity and, above all, there is an equivalence between supermodularity and a standard notion of complementarity, known also as "increasing differences". The concept of complementarity is well established in economics at least since Edgeworth and the basic idea of complementarity is that the marginal value of an action is increasing in the level of other actions avalaible. The mathematical concept of supermodularity formalizes the idea of complementarity and in the literature it is defined for functions on a general lattice, but our aim is to define supermodularity for aggregation functions.

1.1 Partially Ordered Sets and Lattices

This section introduces and develops concepts and properties involving order and lattices, by giving also characterizations of sublattice structure.
A lattice is a system of elements with two basic operations: formation of meet and formation of join, which are respectively denoted by $a \wedge b$ and $a \vee b$; this notation has been favoured by Birkhoff and MacLane.
To introduce lattices we define first the relation of **partial order** and then **partially ordered sets**, including chains. Whenever discussing a general partially ordered set, the associated ordering relation is denoted $\preceq$. Sometimes, the same symbol $\preceq$ may be used to denote different ordering relations on different partially ordered sets, where the particular context precludes any ambiguities. Any subset of $\mathbb{R}^n$ is taken to have $\leq$ as the associated ordering relation.

Let L be a set of elements; then a relation $\preceq$ of partial order over L is any dyadic relation over L which is:

(i) reflexive: for every $a \in L$, $a \preceq a$;

(ii) anti-symmetric: if $a \preceq b$ and $b \preceq a$, then $a = b$;

(iii) transitive: if $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

If $a \preceq b$ or $b \preceq a$, we say that a,b are comparable, otherwise a and b are incomparable, in notation $a \| b$.
A set L over which a relation $\preceq$ of partial order is defined is called a partially ordered set or **poset**.
Notice that elements of a partially ordered set need not be comparable with one another, though each must be comparable with itself.
If for every pair of elements a,b of a partially ordered set L we have either $a \preceq b$ or $b \preceq a$ or both, the set L is said to be simply or totally ordered and is called a **chain**. Since $a \preceq b$ and $b \preceq a$ imply $a = b$, hence we may define a chain as a partially ordered set in which for every pair of distinct elements a,b we have either $a \prec b$ or $b \prec a$ (*linearity* or *ordering property*). We note that any subset of a chain is itself a chain. A finite chain of n elements has a least and a greatest member, and is isomorphic with the sequence of natural numbers $(1,2,3,\ldots,n)$.

Suppose that L is a partially ordered set and A is a subset of L. If x' is in L and $x \preceq x'$ $(x' \preceq x)$ for each $x \in A$, then x' is an **upper (lower) bound** for A. If x' in A is an upper (lower) bound for A, then x' is the **greatest (least)** element of A. If x' is in A and there does not exist any x'' in A with $x' \prec x''$ $(x'' \prec x')$, then x' is a **maximal (minimal)** element of A. A greatest (least) element is a maximal (minimal) element. A partially ordered set can have at most one greatest (least) element, but it may have any number of maximal (minimal) elements. Distinct maximal (minimal) elements do not have ordering property, that is they are **unordered**. If the set of upper (lower) bounds of A has a least (greatest) element, then this **least upper bound (greatest lower bound)** of A is the **supremum (infimum)** of A and is denoted $\mathbf{sup_L}(\mathbf{A})$ $(\mathbf{inf_L}(\mathbf{A}))$ if the set L is not clear from context or **sup(A) (inf(A))** if the set L is clear from context.
If two elements, x' and x'', of a partially ordered set L have a least upper bound (greatest lower bound) in L, it is their **join (meet)** and is denoted $\mathbf{x}' \vee \mathbf{x}''$ $(\mathbf{x}' \wedge \mathbf{x}'')$. A partially ordered set that contains the join and the meet of each pair of its elements is a **lattice** $\langle L, \vee, \wedge \rangle$. It shall be convenient to lay down the convention $\bigvee \emptyset = \bigwedge \emptyset = \bot$.

A function $f(x)$ from a partially ordered set L to a partially ordered set Y is **increasing (decreasing)** if $x' \preceq x''$ in L implies $f(x') \preceq f(x'')$ $(f(x'') \preceq f(x'))$ in Y. A function is **monotone** if it is either increasing or decreasing. A function $f(x)$ from a partially ordered set L to a partially ordered set Y is **strictly increasing (strictly decreasing)** if $x' \prec x''$ in L implies $f(x') \prec f(x'')$ $(f(x'') \prec f(x'))$ in Y. It is common in the lattice theory literature [10, 97], to use the terms **isotone** and **antitone** rather than "increasing" and "decreasing", respectively, but the latter are used herein in order to maintain a more familiar terminology.

1.1.1 Sublattice Structure

If A is a subset of a lattice L and A contains the join and meet (with respect to L) of each pair of elements of A, then A is a **sublattice** of L. For a lattice L, let $\mathscr{L}(L)$ denote the set of all nonempty sublattices of L. If A is a sublattice of a lattice L, then A is itself a lattice and in A the join and meet of any two elements are the same as the join and meet of those same two elements in L. If L is a lattice, A is a sublattice of L, and A' is a sublattice of A, then A' is a sublattice of L. If A is a sublattice of L, L and A are lattices with the same ordering relation, i.e. $x \preceq_A y$ for $x, y \in A$ implies $x \preceq_L y$, A is not necessarily a sublattice of L.

If $f(x)$ is a function from a set L into a partially ordered set Y, then the **level sets** of $f(x)$ on L are the sets $\{x : x \in L, y \preceq f(x)\}$ for y in Y. A function $f(x)$ from a set L into a partially ordered set Y is a **generalized indicator function** for a subset A of L if

$$f(x) = \begin{cases} y'' & \text{for } x \in A, \\ y' & \text{for } x \in L \text{ and } x \notin A \end{cases}$$

where $y' \prec y''$ in Y; that is, if the only level sets of $f(x)$ on L are L, A and perhaps the empty set. An **indicator function** is a generalized indicator function with $Y = \mathbb{R}^1$, $y' = 0$ and $y'' = 1$.

If L is a partially ordered set, A is a subset of L and $L \cap [x, \infty)$ is a subset of A for each $x \in A$, then A is an **increasing set**. Equivalently, a subset A of a partially ordered set L is an increasing set if the indicator function of $A \cap [x, \infty)$ is an increasing function on L for each $x \in A$. Increasing sets are useful in characterizing properties of parameterized collections of distribution functions.

It can be shown that a nonempty finite lattice has a greatest element and a least element (Lemma 2.2.1 in [98]). A different proof of this result comes from combining the properties that any nonempty finite partially ordered set has a maximal element and a minimal element and that a maximal (minimal) element of a lattice is the greatest (least) element.

If L_i is a partially ordered set with binary relation $\preceq_i$ for each $i \in I$, then the **direct product** of these partially ordered sets is the partially ordered set consisting of the set $\times_{i \in I} L_i$ with the **product relation** $\preceq$ where $x' \preceq x''$ in $\times_{i \in I} L_i$ if $x'_i \preceq_i x''_i$ for each i in I. A special case of this direct product example is the partially ordered set $\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}^1 \ \forall\, i = 1, \ldots, n\}$ with the ordering relation $\leq$ where $x' \leq x''$ in $\mathbb{R}^n$ if $x'_i \leq x''_i$ in $\mathbb{R}^1$ for $i = 1, \ldots, n$. In fact $I = \{1, \ldots, n\}$, $L_i = \mathbb{R}^1$ with $\mathbb{R}^1$ having the usual ordering relation $\leq$ for each i in I and $\mathbb{R}^n = \times_{i \in I} L_i$.

If L and T are sets and S is a subset of $L \times T$, then the **section** of S at t in T is $\mathbf{S_t} = \{x : x \in L, (x,t) \in S\}$ and the **projection** of S on T is $\Pi_T S = \{t : t \in T, S_t \text{ is nonempty}\}$. Lemma 2.2.2 and Lemma 2.2.3 in [98] show that intersections, sections and projections of sublattices are also sublattices. Furthermore, the direct product of lattices is a lattice and of sublattices is a sublattice. Essential properties characterizing sublattices of the direct product of any finite collection of lattices can be expressed in terms of sublattices of the direct product of two lattices.

A lattice in which each nonempty subset A has a supremum $\vee A$ and an infimum $\wedge A$ is **complete**. The concept is self-dual and obviously half of the hypothesis is redundant. For a complete lattice L, the supremum of L is denoted by 1 and the infimum of L is denoted by 0. Thus L is a bounded lattice, with 1 as its greatest element and 0 as its least element.

By Lemma 2.2.1 in [98], any finite lattice is complete. A nonempty complete lattice has a greatest element and a least element. If A is a sublattice of a lattice L and if, for each nonempty subset A' of A, $\sup_L(A')$ and $\inf_L(A')$ exist and are contained in A, then A is a **subcomplete** sublattice of L. By Lemma 2.2.1, any finite sublattice of a lattice is subcomplete. Hence any sublattice of a finite lattice is subcomplete. Each closed interval in a complete lattice L is a subcomplete sublattice of L and the supremum and infimum with respect to the closed interval of any subset of the closed interval are the same as the supremum and infimum with respect to L of that same subset. If L is a lattice and A is a subcomplete sublattice of L, then

$\sup_L(A') = \sup_A(A')$ and $\inf_L(A') = \inf_A(A')$ for each nonempty subset A' of A, A itself is a complete lattice, and A has a greatest element and a least element if A is nonempty.

1.1.2 Lattice Homomorphisms

A monotone map $g : L \to M$ between lattices need not, in general, preserve meets and joins, but the following statements are equivalent:

- g is increasing;
- $g(x \wedge y) \leq (g(x) \wedge g(y)))$, $\forall x, y \in L$;
- $g(x \vee y) \geq (g(x) \vee g(y)))$, $\forall x, y \in L$.

The mapping g is called a *meet-morphism* if $g(x \wedge y) = (g(x) \wedge g(y)))$, $\forall x, y \in L$ and a *join-morphism* if $g(x \vee y) = (g(x) \vee g(y)))$, $\forall x, y \in L$. Obviously, meet- and join-morphisms are increasing mappings, but the converse does not hold. Similarly, the mapping g is called an *inf-morphism* if for any non-empty family $(x_i)_{i \in I}$ in L it holds that

$$g(\inf_{i \in I} x_i) = \inf_{i \in I} g(x_i)$$

and a *sup-morphism* if for any non-empty family $(x_i | i \in I)$ in L it holds that

$$g(\sup_{i \in I} x_i) = \sup_{i \in I} g(x_i).$$

If we consider the particular case of a function $\varphi : [0,1] \Rightarrow [0,1]$, the following statements are equivalent:

- φ is increasing;
- $(\varphi(\min(x,y)) = \min(\varphi(x), \varphi(y)))$, $\forall (x,y) \in [0,1]^2$;
- $(\varphi(\max(x,y)) = \max(\varphi(x), \varphi(y)))$, $\forall (x,y) \in [0,1]^2$.

Clearly, an increasing function $\varphi : [0,1] \to [0,1]$ is right-continuous if and only if it is an inf-morphism, and left-continuous if and only if it is a sup-morphism.
A function $g : L \to M$ that preserves finite meets and joins, that is, for which

- $g(x \wedge y) = (g(x) \wedge g(y)))$
- $g(x \vee y) = (g(x) \vee g(y)))$

is called a **lattice homomorphism**.

1. A lattice *monomorphism* or *lattice embedding* is an injective lattice homomorphism;
2. a lattice *epimorphism* is a surjective lattice homomorphism;

3. a lattice *endomorphism* of L is a lattice homomorphism from L to itself;

4. a lattice *isomorphism* is a bijective lattice homomorphism, or equivalently, an order isomorphism.

Definition 1.1.1 *Let* $\mathbf{L} = (L, \wedge_L, \vee_L, 0_L, 1_L)$ *and* $\mathbf{M} = (M, \wedge_M, \vee_M, 0_M, 1_M)$ *be bounded lattices. A function* $g : L \to M$ *for which* $g(0_L) = 0_M$ *and* $g(1_L) = 1_M$ *is called a* $\{0,1\}$-***homomorphism*** *if* $g(0_L) = 0_M$ *and* $g(1_L) = 1_M$.

If L and M are complete lattices, a *continuous homomorphism* from L to M is a function $h : L \to M$ such that :

$$h(\wedge A) = \wedge h(A) \qquad h(\vee A) = \vee h(A)$$

for every nonempty subset A of L.

1.2 Increasing differences and supermodular functions

Suppose that L and T are partially ordered sets and $f(x,t)$ is a real-valued function on a subset S of $L \times T$. For t in T, let S_t denote the section of S at t. If $f(x,t'') - f(x,t')$ is increasing, decreasing, strictly increasing, or strictly decreasing in x on $S_{t''} \cap S_{t'}$ for all $t' \prec t''$ in T, then $f(x,t)$ has, respectively, **increasing differences, decreasing differences, strictly increasing differences** or **strictly decreasing differences** in (x,t) on S. The conditions of these definitions do not distinguish between the first and second variables because $f(x',t'') - f(x',t') \leq f(x'',t'') - f(x'',t')$ if and only if $f(x'',t') - f(x',t') \leq f(x'',t'') - f(x',t'')$, and similarly for a strict inequality.
Suppose that L_i is a partially ordered set for each i in a set A, L is a subset of $\times_{i \in A} L_i$, an element x in L is expressed as $x = (x_i)_{i \in I}$, where x_i is in L_i for each i in I and $f(x)$ is a real-valued function on L. If, for all distinct i' and i'' in I and for all x'_i in L_i for all i in $I \setminus \{i', i''\}$, $f(x)$ has increasing differences, decreasing differences, strictly increasing differences, or strictly decreasing differences in $(x_{i'}, x_{i''})$ on the section of L at $\{x'_i : i \in A \setminus \{i', i''\}\}$, then $f(x)$ has, respectively, **increasing differences, decreasing differences, strictly increasing differences**, or **strictly decreasing differences** on L. If $f(x)$ is differentiable on $\mathbb{R}^n$, then $f(x)$ has increasing differences on $\mathbb{R}^n$ if and only if $\partial f(x)/\partial x_{i'}$ is increasing in $x_{i''}$ for all distinct i' and i'' and all x. If $f(x)$ is twice differentiable on $\mathbb{R}^n$, then $f(x)$ has increasing differences on $\mathbb{R}^n$ if and only if $\partial^2 f(x)/\partial x_{i'} \partial x_{i''} \geq 0$ for all distinct i' and i'' and all x.
Suppose that $f(x)$ is a real-valued function on a lattice L. If

$$f(x') + f(x'') \leq f(x' \vee x'') + f(x' \wedge x'')$$

for all x' and x'' in L, then $f(x)$ is **supermodular** on L. If

$$f(x') + f(x'') < f(x' \vee x'') + f(x' \wedge x'')$$

for all unordered x' and x'' in L, then $f(x)$ is **strictly supermodular** on L. If $-f(x)$ is (strictly) supermodular, then $f(x)$ is **(strictly) submodular**. A function that is both supermodular and

submodular is a **valuation**.
Theorem 2.6.1 and Corollary 2.6.1 in [98] show that a function has increasing differences on the direct product of a finite collection of chains if and only if the function is supermodular on that direct product, thereby characterizing supermodularity on the direct product of a finite collection of chains (and, in particular, on $\mathbb{R}^n$) in terms of the nonnegativity of all pairs of cross–differences. As in Lemma 2.2.4 in [98] the number 2 plays a fundamental role in characterizing sublattice structure, likewise it has a fundamental role for supermodular functions, since supermodularity on any finite product of chains is equivalent to supermodularity on the product of each pair of the chains. Thus supermodularity is a second–order property in the sense that for twice–differentiable functions on $\mathbb{R}^n$ each class of functions can be characterized by certain conditions on the matrix of second partial derivatives.
Theorem 2.6.1 in [98] shows that supermodularity implies increasing differences for a function on a sublattice of the direct product of lattices. Corollary 2.6.1 in [98] states that increasing differences imply supermodularity on the direct product of a finite collection of chains. This result is a consequence of Theorem 2.6.2 because any real–valued function on a chain is (strictly) supermodular. The result of Corollary 2.6.1, characterizing supermodularity in terms of increasing differences, is limited to domains that are the direct product of finitely many chains. Note that this corollary states that a function is supermodular on the direct product of finitely many chains if the function is supermodular on each 4-element sublattice $\{x', x'', x' \vee x'', x' \wedge x''\}$ such that x' and x'' each differ from $x' \wedge x''$ in exactly one component.
A function $f : \mathbb{R}^2 \to \mathbb{R}$ has increasing differences if, for any $t \geq t'$, $g(x) = f(x,t) - f(x,t')$ is an increasing function of x. A function $f : \mathbb{R}^S \to \mathbb{R}$ has increasing differences if for any s, t and x, the function $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$,

$$\hat{f}(\hat{x}_s, \hat{x}_t) = f(x_{-s,t}, \hat{x}_s, \hat{x}_t),$$

obtained by allowing only x_s and x_t to vary from x, has increasing differences.
There is an equivalence of supermodularity and increasing differences for a function $f : L \to \mathbb{R}$, where L is a finite dimensional product set $L = \times_{i \in I} L_i$, where each L_i is a chain in its order $\succeq_i$ and L ordered by the product order (see [98], Corollary 2.6.1).
Say that x and y are **disjoint**, written $x \perp y$, if the infimum of x and y is zero, i.e., if $x \wedge y = 0$ (see [89], Chapter 5, Section 1). So a mapping $f : \mathbb{R}^S \to \mathbb{R}$ is supermodular if and only if it displays increasing differences and so the following proposition characterizes supermodular functions.

Proposition 1.2.1 *A function $f : L \subseteq \mathbb{R}^n \to \mathbb{R}$ is supermodular if and only if*

$$f(x+h+k) - f(x+k) \geq f(x+h) - f(x) \tag{1.1}$$

when $x, y \in L$, for all h, k with $h, k \geq 0$, $h \perp k$ such that $x+h, x+k, x+h+k \in L$.

Increasing difference transfers the supermodularity condition to one involving the linear structure of $\mathbb{R}^n$.
It is worth noting that the supermodularity condition is only an “inter-attribute" relation. Intu-

itively increasing differences say that there must be “complementarity" between attributes.

Proposition 1.2.2 *An n-ary function* $f\colon [0,1]^n \to [0,1]$ *is supermodular if and only if each of its two-dimensional sections is supermodular, i.e., for each* $\mathbf{x} \in [0,1]^n$ *and all* $i, j \in \{1,2,\ldots,n\}$ *with* $i \neq j$*, the function* $f_{\mathbf{x},i,j}\colon [0,1]^2 \to [0,1]$ *given by* $f_{\mathbf{x},i,j}(u,v) = f(\mathbf{y})$*, where* $y_i = u$*,* $y_j = v$ *and* $y_k = x_k$ *for* $k \in \{1,2,\ldots,n\} \setminus \{i,j\}$*, is supermodular.*

Proof: On a generic lattice we have the similar result in the Definition 1.2 in [11]. The same result can be applied to our case as well, because $[0,1]^n$ is a lattice, □

Remark 1.2.3 (Supermodular lattices) *It is clear that with a supermodular indicator function we can define a supermodular sublattice in a lattice. This definition and in particular the description of supermodular sublattices in products of relatively complemented lattices can be found in [63].*

Definition 1.2.4 *A sublattice* L' *of a lattice* L *is called supermodular (in* L*) if, for any* $x \in L'$ *and* $y \in L$*, at least one of the elements* $x \wedge y, x \vee y$ *belongs to* L'*.*

Prototypical examples of supermodular sublattices in any lattice are its ideals, filters, and (set-theoretic) unions of an ideal and a filter.

In the next chapters we will apply all these concepts in the theory of aggregation functions and copulas, but, first of all, we are going to introduce the basic background, definitions and properties in this context.

Chapter 2

Aggregation functions and Copulas

Supermodular functions are extensively investigated in different research areas, both pure and applied. The supermodular property also goes by a variety of names such as L-superadditive (where L is mnemonic for lattice), superadditive and quasimonotone. Our aim is to apply this concept to aggregation functions, but, first of all, we recall the basic definitions and properties both for aggregation functions and copulas. Moreover, we will focus our attention to the main problem that we have when we want to deal with multivariate copulas as aggregation functions.

2.1 Aggregation Functions

Aggregation operators (also referred to as *means* or *mean operators*) correspond to particular mathematical functions used for information fusion, the broad area that studies methods to combine data or information supplied by multiple sources. Generally, we consider mathematical functions that combine a finite number of inputs, called arguments, into a single output. So, aggregation has for purpose the simultaneous use of different pieces of information provided by several sources, in order to come to a conclusion or a decision. They are applied in many different domains and in particular aggregation functions play important role in different approaches to decision making, where values to be aggregated are typically preference or satisfaction degrees and thus belong to the unit interval $[0,1]$. For more details, see [42].
A basic consideration for any aggregation operator is the type of data it is going to fuse. At present, there exists a large number of aggregation operators applicable to a broad range of data representation formalisms. For example, aggregation operators on the following formalisms have been considered in the literature: *numerical data, ordinal scales, fuzzy sets, belief functions*, among others. The construction of new functions on the basis of new properties or when considering new knowledge representation formalisms has been studied for a long time. For example, in the framework of aggregation of preferences, Llull (thirteenth century) and Nicholas Cusanus (fifteenth century) proposed methods that were later rediscovered by Condorcet and Borda (eighteenth century). They are the Condorcet rule (with the Copeland method for solving ties) and the Borda count.
Currently, in these early years of the 21st century, an important amount of literature is already

available, many significant results have been found (such as characterizations of various families of aggregation functions), and many connections have been done with other related fields or former works, such as triangular norms, conorms, uninorms, generalized means and ordered weighted aggregation (OWA) operators. Within these classes various important families have been identified, like the arithmetic mean, which is an aggregation function defined by

$$AM(x_1,\dots,x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

It is clear that any aggregation of the numbers $x_1,\dots,x_n$ cannot be made by means of usual arithmetic operations, unless these operations involve only order. For example, computing the arithmetic mean is forbidden, but the median or any order statistic is permitted.
To define general aggregation operators we use the definition given by Beliakov, Mesiar and Valášková (see page 220 in [5]). If we consider the behaviour of the aggregation in the best and in the worst case we expect that an aggregation satisfies the following boundary conditions:

$$A(0,\dots,0) = 0 \qquad \text{and} \qquad A(1,\dots,1) = 1$$

These conditions mean that if we observe only completely bad (or satisfactory) criteria the total aggregation has to bee completely bad (or satisfactory). We consider aggregation functions that satisfy the boundary conditions.
Increasingness is another property, which is often required for aggregation and commonly accepted for functions used to aggregate preferences.
So, as it has been shown in [75], we can define an aggregation operator as a function

$$A : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$$

that satisfies:

- (Idempotency) $A(x) = x \qquad \forall x \in [0,1]$;
- (Boundary conditions) $A(0,\dots,0) = 0$ and $A(1,\dots,1) = 1$;
- (Monotonicity) $A(x_1,\dots,x_n) \leq A(y_1,\dots,y_n)$ if $(x_1,\dots,x_n) \leq (y_1,\dots,y_n)$.

Idempotency and monotonicity imply that aggregation operators are functions that yield a value between the minimum and the maximum of the input values. Formally, they are operations that satisfy internality:

$$\min_i x_i \leq A(x_1,\dots,x_n) \leq \max_i x_i.$$

2.1.1 Lattice-Ordered Semigroups and the case $[0,1]^n$

Now we define basic building blocks of the theory to be developed here: groups, lattice-ordered groups [9, 27], even if most of these concepts is probably known to the reader.

Definition 2.1.1 *A set of elements* $g \in \mathcal{G}$ *forms a group with respect to a law of composition* $(a,b) \to a * b : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ *if is satisfies the following axioms:*

(a) (closure) *the set is closed with respect to multiplication; for any two elements* $a,b \in \mathcal{G}$*, the composition* $a * b \in \mathcal{G}$*;*

(b) (associative law) *the multiplication is associative* $(a * b) * c = a * (b * c)$ *for any three elements* $a,b,c \in \mathcal{G}$*;*

(c) (existence of an identity element) *there exists an identity element* $\mathbf{e} \in \mathcal{G}$ *such that*

$$\mathbf{e} * g = g * \mathbf{e} \quad \textit{for any} \quad g \in \mathcal{G}$$

(d) (existence of inverses) *for each* $g \in \mathcal{G}$*, there exists an* $g' \in \mathcal{G}$ *such that* $g * g' = \mathbf{e} = g' * g$*.*

If (a), (b) and (c) hold, but not necessarily (d), then G is called a *semigroup*. (Some authors don't require a semigroup to contain an identity element.)

We usually write $a * b$ and e as $a\,b$ and 1, or as $a + b$ and 0.

If the group is finite, the number of elements is called the order of the group and denoted $|G|$.
If the operation $a * b = b * a$ is commutative for all $a,b \in \mathcal{G}$, the group is *abelian*.
Two groups with the same multiplication table are said to be *isomorphic*.
The concept of a lattice ordered semi-group, or l-semigroup, arose naturally in the ideal theory.

Definition 2.1.2 *By a multiplicative lattice or m-lattice, we mean a lattice L with a binary multiplication satisfying*

$$a\,(b \vee c) = a\,b \vee a\,c \quad \textit{and} \quad (a \vee b)\,c = a\,c \vee b\,c. \tag{2.1}$$

A zero of an m-lattice L is an element 0 *satisfying*

$$0 \wedge x = 0\,x = x\,0 = 0 \quad \forall x \in L. \tag{2.2}$$

A unity of an m-lattice L is an element e satisfying

$$e\,x = x\,e = x \quad \forall x \in L. \tag{2.3}$$

A infinity of L is an element I satisfying

$$I \vee x = I\,x = x\,I = I \quad \forall x \in L. \tag{2.4}$$

L is called commutative or associative if

$$x\,y = y\,x \quad \textit{or} \quad (x\,y)\,z = x\,(y\,z) \tag{2.5}$$

for all $x, y, z \in L$. If L is conditionally complete and satisfies the unrestricted distributive laws

$$a \vee b_\alpha = \vee(a\, b_\alpha) \quad and \quad (\vee a_\alpha)\, b = \vee(a_\alpha\, b), \tag{2.6}$$

it is called a complete m-lattice or cm-lattice. An associative m-lattice with unity is called a lattice ordered semigroup, or l-semigroup; if complete, it is called a cl-semigroup.

Now we recall the following theorem (theorem 1 in [9] on page 201):

Theorem 2.1.3 *In any m-lattice we have*

$$a \leq b \quad implies \quad x\,a \leq x\,b \quad and \quad a\,y \leq b\,y \quad \forall x, y; \tag{2.7}$$

$$(a \wedge b)(a \vee b) \leq b\,a \vee a\,b \quad \forall a, b. \tag{2.8}$$

If the m-lattice has a unity e, then

$$a \vee b = e \quad implies \quad a \wedge b = b\,a \vee a\,b, \quad and \tag{2.9}$$

$$a \vee b = a \vee c = e \quad implies \quad a \vee b\,c = a \vee (b \wedge c) = e. \tag{2.10}$$

If it has an element $z \leq e$ satisfying $z\,x = x\,z = z$ for all x, then this z is a zero.

A particular lattice ordered semigroup is $[0,1]^n$.
In fact we can easily check that $[0,1]^n$ is a semigroup with respect to the standard multiplication because it has the properties of closure, associativity and existence of an identity.
This structure of a semigroup, combined with the structure of a lattice, implies that $[0,1]^n$ is an m-lattice. Moreover, it is associative and complete, so $[0,1]^n$ is a cl-semigroup.
Concerning distributivity, $[0,1]^n$ is not distributive in general for $n > 1$, because it's not true that, given $a \leq x \leq b$, at most one y exists satisfying $x \wedge y = a$ and $x \vee y = b$ (corollary 1 on page 134 in [9]). We recall also that in a general lattice-ordered group L_V we have the following identity (see page 207 in [89]): $\forall \mathbf{x}, \mathbf{y} \in L_V$

$$\mathbf{x} + \mathbf{y} = \mathbf{x} \vee \mathbf{y} + \mathbf{x} \wedge \mathbf{y}. \tag{2.11}$$

This equality holds also for the lattice $[0,1]^n$, even if this one is not a lattice-ordered group.

2.1.2 Copulas

Particular aggregation functions are copulas and many of the basic results about copulas can be traced to the early work of Wassily Hoeffding. He also obtained best possible bound inequalities for these functions, characterized the distributions corresponding to those bounds and studied measures of dependence that are scale-invariant, that is invariant under strictly increasing transformations. In 1951 Fréchet obtained independently many of the same results of Hoeffding's work. In recognition of the shared responsibility for these important ideas we

will refer to "Fréchet-Hoeffding bounds" and "Fréchet-Hoeffding classes".
When Sklar wrote his paper in 1959 with the term "copula", he was collaborating with Bertold Schweizer in the development of the theory of probabilistic metric spaces. During the period from 1958 through 1976, most of the important results concerning copulas were obtained in the course of the study of probabilistic metric spaces, mainly in the study of binary operations in the space of the probability distribution functions. In 1942, Karl Menger (see [71]) proposed a probabilistic generalization of the theory of metric spaces, by replacing the number $d(p,q)$ by a distribution function F_{pq}, whose value $F_{pq}(x)$ for any real x is the probability that the distance between p and q is less than x. The first difficulty in the construction of probabilistic metric spaces comes when one tries to find a "probabilistic" analogue of the triangle inequality. Menger proposed $F_{pr}(x+y) \geq T(F_{pq}(x), F_{qr}(y))$, where T is a *triangle norm* or *t-norm*. Some t-norms are copulas, and conversely, some copulas are t-norms. For a history of the development of the theory of probabilistic metric spaces, see [90] and [91]. So, at the beginning, copulas were mainly used in the development of the theory of probabilistic metric spaces. Later, they were of interest to define nonparametric measures of dependence between random variables. In fact, with regard to the link between copulas and the study of dependence among random variables, it appears in the paper by Schweizer and Wolff (1981). In that paper they presented the basic invariance properties of copulas under strictly monotone transformations of random variables and introduced the measure of dependence now known as *Schweizer and Wolff's* σ. Since then, copulas began to play an important role in probability and mathematical statistics.

Definition 2.1.4 *A* 2*-copula, or simply, a copula is a function* $C : [0,1]^2 \to [0,1]$ *that satisfies:*

(a) $C(0,u) = C(u,0) = 0 \quad C(1,u) = C(u,1) = u \quad \forall u \in [0,1]$;

(b) C is a supermodular function.

It is easy to see that the function $\Pi(u,v) = uv$ satisfies conditions (i) and (ii) and hence is a copula. The copula Π, called the *product copula*, has an important statistical interpretation. The following Sklar's Theorem [94], which partially explains the importance of copulas in statistical modelling, justifies the role of copulas as dependence functions.

Theorem 2.1.5 (Sklar's theorem) . *Let H be a 2-dimensional distribution function with margins F and G. Then there exists a 2-copula C such that for all* (x,y) *in* $\overline{\mathbb{R}}^2$,

$$H(x,y) = C(F(x), G(y)). \tag{2.12}$$

If F and G are continuous, then C is unique; otherwise, C is uniquely determined on Ran F $\times$ *Ran G. Conversely, if C is a 2-copula and F and G are distribution functions, then the function H defined by* (2.12) *is a 2-dimensional distribution function with margins F and G.*

2.1.3 The Fréchet-Hoeffding Bounds for Joint Distribution

As a consequence of Sklar's Theorem, if X and Y are random variables with a joint distribution function H and margins F and G, respectively, then for all x, y in $\overline{\mathbf{R}}$,

$$\max(F(x)+G(y)-1,0) \leq H(x,y) \leq \min(F(x),G(y)) \tag{2.13}$$

or (since $H(x,y) = C(F(x),G(y))$)

$$W(u,v) = \max(u+v-1,0) \leq C(u,v) \leq \min(u,v) = M(u,v). \tag{2.14}$$

Since M and W are copulas, the above bounds are joint distribution functions, and are called the *Fréchet-Hoeffding bounds* for joint distribution functions H with margins F and G.
The copulas M, W and Π have important statistical interpretations. Let X and Y be continuous random variables, then:

(i) the copula of X and Y is $M(u,v)$ if and only if each of X and Y is almost surely an increasing function of the other;

(ii) the copula of X and Y is $W(u,v)$ if and only if each of X and Y is almost surely a decreasing function of the other;

(iii) the copula of X and Y is $\Pi(u,v) = uv$ if and only if X and Y are independent.

Among the most important results in probabilistic metric spaces - for the statistician - is the class of Archimedean t-norms, those t-norms T that satisfy $T(u,u) < u$ for all $u \in (0,1)$. Archimedean t-norms that are also copulas are called *Archimedean copulas*.

2.1.4 Archimedean Copulas

Here we focus on the class of *Archimedean copulas* because their properties fit the needs of the aggregation problem.
The family of Archimedean copulas is particularly interesting because they can be defined by means of a single function. Let ϕ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\phi(1) = 0$. The *pseudo-inverse* of ϕ is the function $\phi^{[-1]}$ with Dom $\phi^{[-1]} = [0,\infty]$ and Ran $\phi^{[-1]} = [0,1]$ given by

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t), & 0 \leq t \leq \phi(0), \\ 0, & \phi(0) \leq t \leq \infty. \end{cases}$$

Note that $\phi^{[-1]}$ is continuous and non-increasing on $[0,\infty]$, and strictly decreasing on $[0,\phi(0)]$.
The function C defined by

$$C(u,v) = \phi^{[-1]}(\phi(u)+\phi(v)) \quad u,v \in [0,1]^2$$

is a copula if and only if ϕ is convex.
Copulas of the form described above are called *Archimedean* copulas and the function ϕ is

called a *generator* of the copula.
Because of their simple forms, the ease with which they can be constructed and their many nice properties, Archimedean copulas frequently appear in discussions of multivariate distributions.

2.2 The main problem: the axiom of n-increasingness

The concept of copula can be extended to n dimensions, where an *n-copula* is the restriction to the unit n-cube $[0,1]^n$ of a multivariate cumulative distribution function, whose marginals are uniform on $[0,1]$. More precisely, an n-copula is a function $C : [0,1]^n \to [0,1]$ that satisfies:

(a) $C(\mathbf{u}) = 0$ if $u_i = 0$ for any $i = 1,\dots,n$, that is C is *grounded*;

(b) $C(\mathbf{u}) = u_i$ if all coordinates of $\mathbf{u}$ are 1 except u_i, that is C has *uniform one-dimensional marginals*;

(c) C is *n-increasing*, i.e. $V_C(B) \geq 0$ for any n-box $B = [u_1,v_1] \times [u_2,v_2] \times \dots \times [u_n,v_n] \subseteq [0,1]^n$ with $u_i \leq v_i$, $i = 1,2,\dots,n$, where the C-volume of the n-box B is given by

$$V_C(B) = \sum \varepsilon(z_1,\dots,z_n) \cdot C(z_1,\dots,z_n) \geq 0, \tag{2.15}$$

with

$$\varepsilon(z_1,\dots,z_n) = \begin{cases} 1 & \text{if } z_i = u_i \text{ for an even number of i's,} \\ -1 & \text{if } z_i = u_i \text{ for an odd number of i's} \end{cases}$$

and the sum in (2.15) is extended to all vertices of B.

Conditions (a) and (b) are known as *boundary conditions*, whereas condition (c) is known as *monotonicity*.
A copula $C : [0,1]^n \to [0,1]$ is called *absolutely continuous* if, when considered as a joint cdf, it has a joint density given by $\partial^n C^n(u_1,\dots,u_n)/\partial u_1,\dots,\partial u_n$.
If C has nth-order derivatives, n-increasing is equivalent to $\frac{\partial^n}{\partial u_1 \dots \partial u_n} C \geq 0$.
This definition is the multivariate extension of the concept of "increasing" for a univariate function when we interpret "increasing" as "increasing as a distribution function".
Various properties of copulas have been studied in literature, but most part of the research concentrates on the bivariate case, since multivariate extensions are generally not easily to be done. So we can begin with the construction of 3-copulas. We recall that 2-copulas join one-dimensional distribution functions to form bivariate distribution functions. The "naive" approach to constructing multidimensional distributions via copulas would be to use 2-copulas to join other 2-copulas. Unfortunately, this procedure can fail, i.e. a 3-place function C via 2-copulas is unnecessarily a 3-copula, as shown in [81]. If C_1 and C_2 are 2-copulas such that $C_2(C_1(u,v),w)$ is a 3-copula, we say that C_1 is *directly compatible* with C_2. The following theorem provides criteria for direct compatibility when one of C_1 or C_2 is M, W or Π. Its proof can be found in [86].

Theorem 2.2.1 *1. Every 2-copula C is directly compatible with Π;*

2. *The only 2-copula directly compatible with M is M.*
3. *The only 2-copula directly compatible with W is M.*
4. *M is directly compatible with every 2-copula C;*
5. *W is directly compatible only with Π; and*
6. *Π is directly compatible with a 2-copula C if and only if for all v_1, v_2, w_1, w_2 in $\mathbf{I}$ such that $v_1 \leq v_2$ and $w_1 \leq w_2$, the function*

$$u \to V_C([u\,v_1, u\,v_2] \times [w_1, w_2])$$

is nondecreasing on $\mathbf{I}$.

From Sklar's theorem, we know that if C is a 2-copula and F and G are univariate distribution functions, then $C(F(x), G(y))$ is always a two dimensional distribution function. Can we extend this procedure to higher dimensions by replacing F and G by multivariate distributions functions? That is, given $m+n \geq 3$, for what 2-copulas C is it true that if $F(\mathbf{x})$ is an $m-$ dimensional distribution function and $G(\mathbf{y})$ is an $n-$ dimensional distribution function, then $C(F(\mathbf{x}), G(\mathbf{y}))$ is an $(m+n)-$dimensional distribution function? The answer is provided in the following theorem [44]:

Theorem 2.2.2 *Let m and n be positive integers such that $m+n \geq 3$ and suppose that C is a 2-copula such that $H(\mathbf{x}, \mathbf{y}) = C(F(\mathbf{x}), G(\mathbf{y}))$ is an $(m+n)$-dimensional distribution function with margins $H(\mathbf{x}, \infty) = F(\mathbf{x})$ and $H(\infty, \mathbf{y}) = G(\mathbf{y})$ for all m-dimensional distribution functions $F(\mathbf{x})$ and n-dimensional distribution functions $G(\mathbf{y})$. Then $C = \Pi$.*

The following theorem [91] presents related results for the cases when the 2-copula C in the previous theorem is Π or M, and the multidimensional distribution functions F and G are copulas (or, if the dimension is 1, the identity function):

Theorem 2.2.3 *Let m and n be integers ≥ 2. Let C_1 be an m-copula and C_2 an n-copula.*

1. *Let C be the function from $\mathbf{I}^{m+n}$ to $\mathbf{I}$ given by*

$$C(u_1, u_2, \ldots, u_{m+n}) = M(C_1(u_1, u_2, \ldots, u_m), C_2(u_{m+1}, u_{m+2}, \ldots, u_{m+n})).$$

Then C is an $(m+n)$-copula if and only if $C_1 = M^m$ and $C_2 = M^n$.

2. *Let C', C'' and C''' be the functions defined by*

$$C'(u_1, u_2, \ldots, u_{m+1}) = \Pi(C_1(u_1, u_2, \ldots, u_m), u_{m+1}),$$

$$C''(u_1, u_2, \ldots, u_{n+1}) = \Pi(u_1, C_2(u_2, u_3, \ldots, u_{n+1})),$$

$$C'''(u_1, u_2, \ldots, u_{m+n}) = \Pi(C_1(u_1, u_2, \ldots, u_m), C_2(u_{m+1}, u_{m+2}, \ldots, u_{m+n})).$$

Then C' is always an $(m+1)$-copula, C'' is always an $(n+1)$-copula and C''' is always an $(m+n)$-copula.

For further information on this subject, we refer to [81] and [49].
So, in constructing copulas of higher dimensions, only few such methods are known in the literature. In particular, when we treat copulas as aggregation functions, the main problem is the crucial requirement of n-increasingness for aggregation functions.
We will see that this concept is strictly connected to k-monotonicity and to the bounded variation of a function. So, we will connect and apply both of them to the multivariate decomposition of aggregation functions in a sum of copulas.

2.3 Lattice-valued Aggregation Operators

Now, for completeness of information, we analyse the case when aggregation operators are defined on a general lattice. By definition, they need to have values in the same general lattice, endowed with a new operation, in order to introduce a more general concept of supermodularity.
Let $\mathbf{L} = (L, \leq, \mathbb{O}, \mathbb{1}, *)$ a lattice with the least element $\mathbb{O}$ and the greatest element $\mathbb{1}$. The ordering structure on L can be coordinate-wisely extended to L^n $(n \in \mathbb{N}^+)$, i.e. for the relation $\leq$ on L^n, defined by

$$(\alpha_1, \ldots, \alpha_n) \leq (\beta_1, \ldots, \beta_n) \Leftrightarrow \alpha_i \leq \beta_i \quad \forall i = 1, \ldots, n,$$

and for the elements $\mathbb{O}^n = (\mathbb{O}, \ldots, \mathbb{O})$, $\mathbb{1}^n = (\mathbb{1}, \ldots, \mathbb{1})$ of L^n, $\mathbf{L}_n = (L^n, \leq, \mathbb{O}^n, \mathbb{1}^n)$ forms a lattice with universal bounds $\mathbb{O}^n, \mathbb{1}^n$.
We consider a bounded lattice $\mathbf{L}$ and we denote by $* : \mathbf{L} \to \mathbf{L}$ an operation called **lattice operation**. Our basic requirements are the following, for any $x, y, z, w \in \mathbf{L}$:

1. Commutativity: $x * y \geq y * x$,
2. Monotonicity: $w * x \geq y * z$, if $w \geq y$ and $x \geq z$,
3. Associativity: $x * (y * z) = (x * y) * z$,
4. Neutral element: $\exists\, e \in L$, such that $x * e = e * x = x$.

Definition 2.3.1 *A mapping $A : \bigcup_{n \in \mathbb{N}^+} L^n \to L$ is called an aggregation operator on $\mathbf{L}_n$ if the following conditions are fulfilled:*

(AO1) *A preserves universal bounds, i.e.*

$$A(\mathbb{O}^n) = \mathbb{O} \quad \text{and} \quad A(\mathbb{1}^n) = \mathbb{1} \quad \text{for all} \quad n \in \mathbb{N}^+;$$

(AO2) *A preserves the order on L^n for all $n \in \mathbb{N}^+$, i.e.*

$$(\alpha_1, \ldots, \alpha_n) \leq (\beta_1, \ldots, \beta_n) \Rightarrow A(\alpha_1, \ldots, \alpha_n) \leq A(\beta_1, \ldots, \beta_n)$$

(AO3) *A is the identity mapping id_L on L, i.e. $A(\alpha) = \alpha$ for all $\alpha \in L$.*

For $n \geq 2$, a mapping $B : L^n \to L$ is called an n-ary aggregation operator on $\mathbf{L}_n = (L^n, \leq, \mathbb{O}, \mathbb{1})$ if and only if the conditions (AO1) and (AO2) are satisfied. A 1-ary aggregation operator $B : L \to L$ is the identity mapping id_L on L.

An aggregation operator A can be identified by a family of n-ary aggregation operators $\{A_n | n \in \mathbb{N}^+\}$. This means that for a given aggregation operator A on $\mathbf{L}_n$, we may associate a family of n-ary aggregation operators $\{A_n | n \in \mathbb{N}^+\}$ to A, which is defined by $A_n(\alpha_1, \dots, \alpha_n) = A(\alpha_1, \dots, \alpha_n)$. Conversely, if $\{A_n | n \in \mathbb{N}^+\}$ is a family of n-ary aggregation operators on $\mathbf{L}_n$, then we can define an aggregation operator A on $\mathbf{L}_n$ by $A(\alpha_1, \dots, \alpha_n) = A_n(\alpha_1, \dots, \alpha_n)$. It is obvious that the connection between the aggregation operators and the families of n-ary aggregation operators is bijective. Thus an aggregation operator and its associated family of n-ary aggregation operators $\{A_n | n \in \mathbb{N}^+\}$ can be conceived as the same thing.

2.3.1 Classification and general properties

We introduce some properties which could be desirable for the aggregation of criteria. Associativity is also an interesting property for aggregation operators. The associativity property concerns the "clustering" character of an aggregation operator.

Definition 2.3.2 *Let A be an aggregation operator on $\mathbf{L}_n = (L^n, \leq, \mathbb{O}, \mathbb{1})$.*

- *A is called to be associative iff*

$$A(\alpha_1, \dots, \alpha_k, \dots, \alpha_n) = A_2(A_k(\alpha_1, \dots, \alpha_k), A_{n-k}(\alpha_{k+1}, \dots, \alpha_n))$$

  for all $n \geq 2$, $k = 1, \dots, n-1$ and $\alpha_i \in L$, $\quad i = 1, \dots, n$.

- *A is called to be symmetric iff*

$$A(\alpha_1, \dots, \alpha_n) = A(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)})$$

  for all $n \in \mathbb{N}^+$, $\alpha_i \in L \quad i = 1, \dots, n$ and for all permutations σ of $\{1, \dots, n\}$.

- *A has the neutral element $e \in L$ iff for all $n \geq 2$ and $\alpha_i \in L \quad i = 1, \dots, n$, if $\alpha_k = e$ for some $k \in \{1, \dots, n\}$, then*

$$A(\alpha_1, \dots, \alpha_n) = A(\alpha_1, \dots, \alpha_{k-1}, \alpha_{k+1} \alpha_n).$$

2.3.2 Transformed aggregation operators

The idea of transformation of aggregation operators can be transparently illustrated on the well-known relation of the two basic arithmetic operations of addition and multiplication. Indeed, the addition Σ on $[-\infty, \infty]$ (respectively on $[0, \infty]$, $[-\infty, 0]$) and the multiplication Π on

$[0,\infty]$ (respectively on $[0,1]$, $[1,\infty]$) are related by the logarithmic transformation

$$\sum_{i=1}^{n}(-\log x_i) = -\log(\prod_{i=1}^{n})x_i. \tag{2.16}$$

Formally, (2.16) can be written into

$$\prod_{i=1}^{n} x_i = \varphi^{-1}(\sum_{i=1}^{n}\varphi(x_i)), \tag{2.17}$$

where $\varphi : [0,\infty] \to [-\infty,\infty]$ is given by $\varphi(x) = -\log x$. The relation (2.17) between $\prod$ and $\sum$ can be generalized to construct new supermodular aggregation functions from a given one. We present the following transformation construction method for n-ary aggregation functions only, because the extension to the construction of extended aggregation functions is obvious.

Proposition 2.3.3 *Consider two bounded lattices L and M with an isomorphism $\varphi : L \to M$. For $n \in \mathbb{N}$, let $A : M^n \to M$ be an n-ary aggregation function. Then the function $A_\varphi : L^n \to L$ defined in the following way*

$$A_\varphi(x_1,\dots,x_n) := \varphi^{-1}(A(\varphi(x_1),\dots,\varphi(x_n))) \tag{2.18}$$

is an n-ary aggregation function on L^n. A_φ is called the $\varphi-$transform of A.

Proof: Since φ is an isomorphism, it preserves the lattice operations, that is $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ and $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ and both φ and φ^{-1} preserve the order, i.e., $\varphi(x) \le \varphi(y)$ if and only if $x \le y$. At last the non decreasing monotonicity of A ensures the non decreasing monotonicity of A_φ. Moreover, the boundary conditions are satisfied because φ is an isomorphism. So, A_φ is an n-ary aggregation function on L^n. □

2.3.3 $*-$Supermodularity

Now we consider two bounded lattices $\mathbf{L}_n = (L^n, \le, \mathbb{0}, \mathbb{1}, *)$ and $\mathbf{M}_n = (M^n, \le, \mathbb{0}, \mathbb{1}, *_\varphi)$ with their different operations $*$ and $*_\varphi$, where $*_\varphi$ is called the $\varphi-$*transform* of $*$ and we have the following result:

Proposition 2.3.4 *Consider an n-ary aggregation function as defined by* (2.18).
A is $*$-supermodular if and only if A_φ is $*_\varphi$-supermodular.

Proof: The sufficient condition is obvious, by taking $\varphi = id$. With regard to the necessary one we have to prove that $A_\varphi(x \vee y) *_\varphi A_\varphi(x \wedge y) \ge A_\varphi(x) *_\varphi A_\varphi(y)$. But we know that A is $*-$supermodular, i.e.

$$A(\varphi(x_1 \vee y_1),\dots,\varphi(x_n \vee y_n)) * A(\varphi(x_1 \wedge y_1),\dots,\varphi(x_n \wedge y_n)) \ge$$
$$\ge A(\varphi(x_1),\dots,\varphi(x_n)) * A(\varphi(y_1),\dots,\varphi(y_n)).$$

If φ is an isomorphism, also φ^{-1} is an isomorphism and it preserves the order. So, we have

$$\varphi^{-1}\big(A(\varphi(x_1 \vee y_1),\dots,\varphi(x_n \vee y_n)) * A(\varphi(x_1 \wedge y_1),\dots,\varphi(x_n \wedge y_n))\big) \geq$$
$$\geq \varphi^{-1}\big(A(\varphi(x_1),\dots,\varphi(x_n)) * A(\varphi(y_1),\dots,\varphi(y_n))\big).$$

Clearly, the previous expression is equivalent to

$$\varphi^{-1}(A(\varphi(x_1 \vee y_1),\dots,\varphi(x_n \vee y_n))) *_\varphi \varphi^{-1}(A(\varphi(x_1 \wedge y_1),\dots,\varphi(x_n \wedge y_n))) \geq$$
$$\geq \varphi^{-1}(A(\varphi(x_1),\dots,\varphi(x_n))) *_\varphi \varphi^{-1}(A(\varphi(y_1),\dots,\varphi(y_n))), \quad \text{i.e.}$$

our thesis. □

For example, if we take $L = [0,1]$, $M = [-\infty,0]$ and $\varphi(x) = \log x$ with $* = +$ on M, we have that $*_\varphi = \cdot$ on L.
So, if we have $A : [-\infty,0]^n \to [-\infty,0]$, we obtain $A(\mathbf{x} \vee \mathbf{y}) + A(\mathbf{x} \wedge \mathbf{y}) \geq A(\mathbf{x}) + A(\mathbf{y})$ and $B := A_\varphi : [0,1]^n \to [0,1]$, so that $B(\mathbf{u} \vee \mathbf{v}) \cdot B(\mathbf{u} \wedge \mathbf{v}) \geq B(\mathbf{u}) \cdot B(\mathbf{v})$.
Note that we have introduced another kind of supermodularity, called $*(*_\varphi)$-*supermodularity*, because in a general lattice $\vee$-supermodularity holds trivially. In fact $A(x \vee y) \vee A(x \wedge y) = A(x \vee y) \geq A(x) \vee A(y)$ surely, because $x \vee y \geq x$, $x \vee y \geq y \Rightarrow A(x \vee y) \geq A(x)$ and $A(x \vee y) \geq A(y)$. A special subclass of $*(*_\varphi)$-supermodular aggregation functions is that formed by $*(*_\varphi)-$modular aggregation functions, i.e. those $A(A_\varphi)$'s for which

$$A(x \vee y) * A(x \wedge y) = A(x) * A(y)$$

(respectively, $A_\varphi(x \vee y) *_\varphi A_\varphi(x \wedge y) = A_\varphi(x) *_\varphi A_\varphi(y)$).

2.4 Stronger forms of supermodularity: ultramodularity

Now we consider the similar class of ultramodular functions that play an eminent role in different contexts. Ultramodular functions are also called "directionally convex functions" or "functions having increasing increments" and ultramodular functions are supermodular functions while the converse is in general false. For a detailed study of the properties of supermodular and ultramodular functions we refer to [67], [69], [93], [97] and [98] as well as the references therein contained.
We have said that, if we consider the special product set $\mathbb{R}^n$, we endow $\mathbb{R}^n$ with the usual *product order*, which says that $x > y$ if $x_i \geq y_i$ for $i = 1,2,\dots,n$. With this order $\mathbb{R}^n$ becomes a lattice, i.e., it is a partially ordered set where there is a supremum and an infimum to every pair of points in $\mathbb{R}^n$. We have denoted in the first section the supremum and infimum of x and y by $x \vee y$ and $x \wedge y$ respectively; it is not hard to see that

$$x \vee y = (\max\{x_1,y_1\},\max\{x_2,y_2\},\dots,\max\{x_n,y_n\}) \text{ and}$$
$$x \wedge y = (\min\{x_1,y_1\},\min\{x_2,y_2\},\dots,\min\{x_n,y_n\}).$$

A subset L of $\mathbb{R}^n$ is a sublattice (of $\mathbb{R}^n$) if for every pair of points x and y in L, both $x \vee y$ and $x \wedge y$ are also contained in L. Like supermodularity, ultramodular functions can be also defined on a generic lattice, ([48], [97] and [98]), but we focus our attention on functions f defined on a generic sublattice $L \subseteq \mathbb{R}^n$.

Definition 2.4.1 *A function $f : L \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be ultramodular iff*

$$f(x+h+k) - f(x+k) \geq f(x+h) - f(x)$$

for all $x \in L$ with $h, k \in L_+$.

2.4.1 Ultramodularity and convexity

Every subset of the lattice R^1 is a sublattice of R^1 and for real functions convexity in one variable is an interesting and fundamental analytical property, playing an important role in several mathematical fields and applications, especially when solving optimization problems [87, 88, 100, 103].

Definition 2.4.2 Let I be a subinterval of $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ be a real function.

(i) f is said to be *convex* if, for all $x, y \in I$ and for all $\lambda \in [0,1]$,

$$\lambda \cdot f(x) + (1-\lambda) \cdot f(y) \geq f(\lambda \cdot x + (1-\lambda) \cdot y); \tag{2.19}$$

(ii) f is said to be *Jensen convex* if, for all $x, y \in I$,

$$\frac{f(x)+f(y)}{2} \geq f\Big(\frac{x+y}{2}\Big). \tag{2.20}$$

Trivially, each convex function is also Jensen convex. There is a number of conditions which are equivalent to the convexity (2.19), as stated by the following remark.

Remark 2.4.3 Let I be a subinterval of $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ be a real function. Then we have:

(i) f is convex if and only if, for all $x, y \in I$ and for all $\varepsilon > 0$ such that $x < y$ and $y + \varepsilon \in I$,

$$f(y+\varepsilon) - f(y) \geq f(x+\varepsilon) - f(x). \tag{2.21}$$

(ii) If f is a continuous function then f is convex if and only if it is Jensen convex.

(iii) If f is a monotone function then f is convex if and only if it is Jensen convex.

(iv) If f is a bounded function then f is convex if and only if it is Jensen convex.

However, for real functions defined on subsets of $\mathbb{R}^n$ with $n > 1$, these definitions of convexity are no more equivalent, in general. For such functions, ultramodularity and convexity

are quite unrelated properties.
Ultramodular functions are supermodular, but the converse is in general false. For example, the function $(\prod_{i=1}^{n} x_i)^{1/n}$ is supermodular but it is not ultramodular because it is concave. This can be verified in several ways, such as directly verifying this inequality $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y) \quad \forall x, y \in [0,1]^n$ and $0 \leq \theta \leq 1$, verifying that the Hessian is positive semidefinite, or restricting the function to an arbitrary line and verifying concavity of the resulting function of one variable.

The following result (Corollary 4.1 of [67]) states the exact relationship between ultramodular and supermodular functions $A\colon [0,1]^n \to [0,1]$:

Proposition 2.4.4 *A function $A\colon [0,1]^n \to [0,1]$ is ultramodular if and only if A is supermodular and each of its one-dimensional sections is convex, i.e., for each $\mathbf{x} \in [0,1]^n$ and each $i \in \{1,\ldots,n\}$ the function $A_{\mathbf{x},i}\colon [0,1] \to [0,1]$ given by $A_{\mathbf{x},i}(u) = f(\mathbf{y})$, where $y_i = u$ and $y_j = x_j$ whenever $j \neq i$, is convex.*

Bivariate copulas are closely linked to the convexity of one-dimensional functions (e.g., additive generators of Archimedean copulas are convex). Copulas of higher dimensions describe the stochastic dependence structure of k-dimensional random vectors with $k > 2$, and they are linked to a stronger form of convexity of one-dimensional functions. For example, in the case of Archimedean copulas, the corresponding additive generator has a derivative of $(k-2)$-th order which is convex [70]. In the next chapter we will deeply analyse the idea of ultramodular functions and in the last chapter we will propose and study stronger versions of ultramodularity, leading to the stronger forms of convexity mentioned above in the case of functions in one variable.

Chapter 3

Supermodular and Ultramodular aggregation functions

In this chapter we will characterize the connections between the general class of all supermodular n–aggregation functions $\mathcal{A}_n^S$ and ultramodular ones $\mathcal{U}_n$. We will see that there are several modifications and constructions of aggregation functions which preserve the supermodularity, but only few of them preserve also ultramodularity.
Section 3.2.2 is devoted to some constructions of ultramodular aggregation functions, especially those based on the composition of appropriate functions. The structure of ultramodular functions is discussed in Section 3.3. We will also introduce modular aggregation functions and some basic results will be recalled.

3.1 Characterizations of some subclasses of supermodular aggregation functions

First results are related to the characterization of some subclasses of $\mathcal{A}_n^S$, following the statements considered in [37].
Specifically, we are proving the following results.

Proposition 3.1.1 *Let* $A \in \mathcal{A}_n^S$*. Then:*

(a) *the neutral element* $e \in [0,1]$ *of* A*, if it exists, is equal to* 1*;*

(b) *the annihilator* $a \in [0,1]$ *of* A*, if it exists, is equal to* 0*.*

(c) *if* A *is continuous on the border of* $[0,1]^n$*, then* A *is continuous on* $[0,1]^n$*.*

Proof: (a) Let $A \in \mathcal{A}_n^S$. Then the bivariate marginals are supermodular and have neutral element $e = 1$ by the proposition 3.1 in [37] and we have

$$\overline{A}(1,x_\beta) = A(a_1,\dots,a_{\alpha-1},1,a_{\alpha+1},\dots,a_{\beta-1},x_\beta,a_{\beta+1},\dots,a_n) = x_\beta,$$

for any couple of integers α, β such that $1 \le \alpha < \beta \le n$. So, by taking $a_i = 1\ \forall i = 1, \dots, n$, $i \ne \beta$, we can conclude that the neutral element e of $A \in \mathcal{A}_n^S$ is equal to 1.

(b) Similarly,

$$\overline{A}(0, x_\beta) = A(a_1, \dots, a_{\alpha-1}, 0, a_{\alpha+1}, \dots, a_{\beta-1}, x_\beta, a_{\beta+1}, \dots, a_n) = 0$$

and so $a = 0$ is the annihilator.

(c) Let A be continuous on the border of $[0,1]^n$ and let $\mathbf{x} = (x_1, \dots, x_n)$ be a point in $]0,1[^n$ such that A is not continuous in $\mathbf{x}$. Suppose, without loss of generality, that there exists a sequence $\{x_{1,\alpha}\}_{\alpha \in \mathbb{N}}$ in $[0,1]$, $x_{1,\alpha} \le x_1$ for every $\alpha \in \mathbb{N}$, such that $\{x_{1,\alpha}\}_{\alpha \in \mathbb{N}}$ tends to x_1 as $\alpha \to +\infty$ and

$$\lim_{\alpha \to +\infty} A(x_{1,\alpha}, x_2, \dots, x_n) < A(x_1, x_2, \dots, x_n).$$

Therefore, there exists $\varepsilon > 0$ and $\alpha_0 \in \mathbb{N}$ such that

$$A(x_1, \dots, x_n) - A(x_{1,\alpha}, \dots, x_{n,\alpha}) > \varepsilon$$

for every $\alpha \ge \alpha_0$. But, because A is continuous on the border of the unit square, there exists $\overline{\alpha} > \alpha_0$ such that $A(x_1, 1, \dots, 1) - A(x_{1,\overline{\alpha}}, 1, \dots, 1) < \varepsilon$. But this violets the supermodular property, because, in this case,

$$A(x_1, 1, \dots, 1) + A(x_{1,\overline{\alpha}}, x_2, \dots, x_n) - A(x_1, \dots, x_n) - A(x_{1,\overline{\alpha}}, 1, \dots, 1) < 0$$

Thus, the only possibility is that A is continuous on $[0,1]^n$. □

3.1.1 Bounds on arbitrary subsets of supermodular agops

Given a supermodular agop A, it is obvious that

$$A_S(x_1, \dots, x_n) \le A(x_1, \dots, x_n) \quad \forall x_i \in [0,1],$$

and A_S is the best-possible lower bound in the set $\mathcal{A}_n^S$, because it is supermodular.

Moreover, the best-possible upper bound in $\mathcal{A}_n^S$ is the greatest agop

$$A_G(x_1, \dots, x_n) = \begin{cases} 0 & \text{if } (x_1, \dots, x_n) = (0, \dots, 0), \\ 1 & \text{otherwise.} \end{cases}$$

Notice that A_G is not supermodular, e.g.

$$A(1, \dots, 1) + A(\mathbf{0}) - A(1, \dots, 1, 0, \dots, 0) - A(0, \dots, 0, 1, \dots, 1) = -1,$$

but it is the pointwise limit of the sequence A_n of supermodular agops, defined by

$$A_n(x_1,\dots,x_n) = \begin{cases} 1 & \text{if } (x_1,\dots,x_n) \in [1/n,1]^n; \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $(\mathcal{A}_n^S, \leq)$ is not a complete lattice. But the following result holds.

Proposition 3.1.2 *Every n-aggregation function is the supremum (wrt the pointwise order) of a suitable subset of $\mathcal{A}_n^S$.*

Proof: Let A be an agop; we may (and, in fact do) suppose that $A \neq A_G$, since this case has already been considered, and that A is not supermodular, this case being trivial. For every $x_0 \in [0,1]$, let $z_0 = A(\mathbf{x}_0)$ and consider the following supermodular agop:

$$\hat{A}_{\mathbf{x}_0} := \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{1}; \\ z_0 & \text{if } \mathbf{x} \in [x_0,1]^n \setminus \{\mathbf{1}\}; \\ 0 & \text{otherwise.} \end{cases}$$

In fact we have the following situation:

$\mathbf{x}+\mathbf{h}+\mathbf{k} \in$	$\mathbf{x}+\mathbf{k} \in$	$\mathbf{x}+\mathbf{h} \in$	$\mathbf{x} \in$	$\hat{A}_{\mathbf{x}_0} \in \mathcal{A}_n^S$
$\{\mathbf{1}\}$	$[x_0,1]^n \setminus \{\mathbf{1}\}$	$[x_0,1]^n \setminus \{\mathbf{1}\}$	$[x_0,1]^n \setminus \{\mathbf{1}\}$	$1 \geq z_0$
$[x_0,1]^n \setminus \{\mathbf{1}\}$	$[x_0,1]^n \setminus \{\mathbf{1}\}$	$([x_0,1]^n \setminus \{\mathbf{1}\})^C$	$([x_0,1]^n \setminus \{\mathbf{1}\})^C$	$0 \geq 0$
$([x_0,1]^n \setminus \{\mathbf{1}\})^C$	$([x_0,1]^n \setminus \{\mathbf{1}\})^C$	$([x_0,1]^n \setminus \{\mathbf{1}\})^C$	$([x_0,1]^n \setminus \{\mathbf{1}\})^C$	$0 \geq 0$

Then one has $A(x_1,\dots,x_n) = \sup\{\hat{A}_{\mathbf{x}_0} : \mathbf{x}_0 \in [0,1]^n\}$. □

3.1.2 Quasi–arithmetic means and Weighted quasi-arithmetic means

Now we see how to use convex functions in the construction of supermodular aggregation ones.

Proposition 3.1.3 *Let M_f be a quasi–arithmetic mean, viz. let a continuous strictly monotone function $f : [0,1] \to \mathbb{R}$ exist such that*

$$M_f(\mathbf{x}) := f^{-1}\left(\frac{f(x_1) + \dots + f(x_n)}{n}\right).$$

Then $M_f \in \mathcal{A}_n^S$ if, and only if, f^{-1} is convex.

Proof: $M_f \in \mathcal{A}_n^S$ if, and only if, for any couple of integers α, β, such that $1 \leq \alpha < \beta \leq n$, $\overline{M}_f(x_\alpha, x_\beta) = M_f(a_1,\dots,a_{\alpha-1},x_\alpha,a_{\alpha+1},\dots,a_{\beta-1},x_\beta,a_{\beta+1},\dots,a_n) \in \mathcal{A}_2^S$, if and only if f^{-1} is convex. □

Due to well-known characterization of quasi-arithmetic means M_f bounded from above by the arithmetic mean M, that is thanks to lemma 1 in [24], we have the next result.

Corollary 3.1.4 *$M_f \in \mathcal{A}_n^S$ if and only if $M_f \leq M$.*

A similar result holds for weighted quasi-arithmetic means, that is with n-dimensional weighting vector $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$ such that $\sum_{i=1}^n w_i = 1$ and $W_f := f^{-1}\left(\sum_{i=1}^n w_i f(x_i)\right)$ we have

Corollary 3.1.5 *$W_f \in \mathcal{A}_n^S$ if and only if $W_f \leq W$.*

Proof: Thanks to convexity of f^{-1}, by using Jensen's inequality we have:

$$W_f = f^{-1}\left(\sum_{i=1}^n w_i f(x_i)\right) \leq \sum_{i=1}^n w_i f^{-1}(f(x_i)) = \sum_{i=1}^n w_i x_i = W.$$

□

A special class of aggregation operators are the so called OWA operators (ordered weighted averaging operators) introduced in [104] and related to the Choquet integral [45]. We recall that the OWA operator is given by

$$OWA(x_1, \ldots, x_n) = \sum_{i=1}^n w_i \cdot x_{\sigma(i)},$$

where $(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ is a nondecreasing permutation of the n-tuple $(x_1, \ldots, x_n)$. Following [31], it is known that a Choquet integral operator based on a fuzzy measure m is supermodular if, and only if, the fuzzy measure m is supermodular. So, OWA operators are supermodular if and only if their weighting vector $(w_1, \ldots, w_n)$ is decreasing.

3.1.3 Modular aggregation functions

A special subclass of $\mathcal{A}_n^S$ is that formed by modular aggregation functions, i.e. those A's for which

$$A(\mathbf{x} \wedge \mathbf{y}) + A(\mathbf{x} \vee \mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y}),$$

for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$. For these operators the following characterization holds.

Proposition 3.1.6 *For an agop A the following statements are equivalent:*

(a) *A is modular;*

(b) *there exist increasing functions f_i from $[0,1]$ into $[0,1]$, such that*

$$A(x_1, \ldots, x_n) = \sum_{i=1}^n f_i(x_i), \tag{3.1}$$

with $f_i(0) = 0$, $\forall i = 1, \ldots, n$ *and* $\sum_{i=1}^{n} f_i(1) = 1$;

(c) *A is strongly additive, i.e., if* $\mathbf{x} \wedge \mathbf{y} = \mathbf{0}$ *and* $\mathbf{x} + \mathbf{y} \in [0,1]^n$, *then* $A(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y})$.

Proof: $(a) \Rightarrow (b)$ If A is modular, set $f_i(x_i) := A(0, \ldots, x_i, \ldots, 0)$, $\forall i = 1, \ldots, n$.
From modularity of A,

$$A(\mathbf{x}) + A(\mathbf{0}) = A(x_1, 0 \ldots, 0) + A(0, x_2, \ldots, x_n) = f_1(x_1) + A(0, x_2, \ldots, x_n).$$

But $A(0, x_2, \ldots, x_n)$ is modular and so we have:

$$\begin{aligned} A(0, x_2, \ldots, x_n) + A(\mathbf{0}) &= A(0, x_2, 0 \ldots, 0) + A(0, 0, x_3 \ldots, x_n) = \\ &= f_2(x_2) + A(0, 0, x_3, \ldots, x_n), \end{aligned}$$

which implies (b) recursively.
$(b) \Rightarrow (c)$ $A(\mathbf{x}+\mathbf{y}) = A(x_1 + y_1, \ldots, x_n + y_n) = \sum_{i=1}^{n} f_i(x_i + y_i)$.
But $\mathbf{x} \wedge \mathbf{y} = \mathbf{0}$ and so $\sum_{i=1}^{n} f_i(x_i + y_i) = \sum_{i=1}^{n} f_i(x_i) + f_i(y_i)$, that is our thesis.
$(c) \Rightarrow (a)$ We note that $\mathbf{x} + \mathbf{y} = \mathbf{x} \wedge \mathbf{y} + \mathbf{x} \vee \mathbf{y}$. So,

$$A(\mathbf{x} \wedge \mathbf{y}) + A(\mathbf{x} \vee \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y}).$$

□

3.2 Ultramodular multivariate aggregation functions

Definition 3.2.1 An n-ary aggregation function $A\colon [0,1]^n \to [0,1]$ is called *ultramodular* if, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in [0,1]^n$ with $\mathbf{x} + \mathbf{y} + \mathbf{z} \in [0,1]^n$,

$$A(\mathbf{x} + \mathbf{y} + \mathbf{z}) - A(\mathbf{x} + \mathbf{y}) \geq A(\mathbf{x} + \mathbf{z}) - A(\mathbf{x}). \tag{3.2}$$

Ultramodularity implies supermodularity of aggregation functions. To see this, for arbitrary $\mathbf{x}, \mathbf{y} \in [0,1]^n$ put first $\mathbf{u} = \mathbf{y} - \mathbf{x} \wedge \mathbf{y}$ and $\mathbf{v} = \mathbf{x} - \mathbf{x} \wedge \mathbf{y}$. Then we get

$$\mathbf{x} \vee \mathbf{y} = \mathbf{x} + \mathbf{y} - \mathbf{x} \wedge \mathbf{y} = \mathbf{x} \wedge \mathbf{y} + \mathbf{u} + \mathbf{v}$$

and, because of (3.2),

$$\begin{aligned} A(\mathbf{x} \vee \mathbf{y}) + A(\mathbf{x} \wedge \mathbf{y}) &= A(\mathbf{x} \wedge \mathbf{y} + \mathbf{u} + \mathbf{v}) + A(\mathbf{x} \wedge \mathbf{y}) \\ &\geq A(\mathbf{x} \wedge \mathbf{y} + \mathbf{v}) + A(\mathbf{x} \wedge \mathbf{y} + \mathbf{u}) \\ &= A(\mathbf{x}) + A(\mathbf{y}). \end{aligned}$$

In the case of one-dimensional aggregation functions, ultramodularity (3.2) is just standard convexity. Therefore, ultramodularity can also be seen as an extension of one-dimensional convexity.

Remark 3.2.2 (i) *Because of Proposition 1.2.2 and Proposition 2.4.4, for an n-ary aggregation function* $A\colon [0,1]^n \to [0,1]$ *the following are equivalent:*

(a) *A is ultramodular;*

(b) *each two-dimensional section of A is ultramodular;*

(c) *each two-dimensional section of A is supermodular and each one-dimensional section of A is convex.*

(ii) *Another equivalent condition to the ultramodularity (3.2) of an n-ary aggregation function* $A\colon [0,1]^n \to [0,1]$ *is the validity of*

$$A(\mathbf{x}+\mathbf{u})+A(\mathbf{x}-\mathbf{u}) \geq A(\mathbf{x}+\mathbf{v})+A(\mathbf{x}-\mathbf{v}) \tag{3.3}$$

for all $\mathbf{x},\mathbf{u} \in [0,1]^n$, $\mathbf{v} \in \mathbb{R}^n$ *with* $|\mathbf{v}| \leq \mathbf{u}$ *and* $\mathbf{x}+\mathbf{u}, \mathbf{x}-\mathbf{u}, \mathbf{x}+\mathbf{v}, \mathbf{x}-\mathbf{v} \in [0,1]^n$ *(indeed, it is sufficient to put* $\mathbf{y} = \mathbf{u}+\mathbf{v}$ *and* $\mathbf{z} = \mathbf{u}-\mathbf{v}$*). Relaxing the requirement* $\mathbf{u} \in [0,1]^n$ *and* $|\mathbf{v}| \leq \mathbf{u}$ *into* $\mathbf{u} \in \mathbb{R}^n$ *and* $|\mathbf{v}| \leq |\mathbf{u}|$ *we get the definition of* symmetrically monotone functions *given in [96]. Note that symmetrically monotone aggregation functions* $A\colon [0,1]^n \to [0,1]$ *are exactly ultramodular aggregation functions which are modular, i.e.,* $A(\mathbf{x}) = \sum_{i=1}^n f_i(x_i)$ *with* $f_i\colon [0,1] \to [0,1]$ *being convex for each* $i \in \{1,\ldots,n\}$ *(compare Propositions 3.1.6 and 2.4.4).*

(iii) *For* $n = 2$*, the ultramodularity (3.2) of an aggregation function* $A\colon [0,1]^2 \to [0,1]$ *is equivalent to A being* P-increasing *(see [41]), i.e., to*

$$A(u_1,v_1)+A(u_4,v_4) \geq \max\big(A(u_2,v_2)+A(u_3,v_3), A(u_3,v_2)+A(u_2,v_3)\big)$$

for all $u_1,u_2,u_3,u_4,v_1,v_2,v_3,v_4 \in [0,1]$ *satisfying* $u_1 \leq u_2 \wedge u_3 \leq u_2 \vee u_3 \leq u_4$, $v_1 \leq v_2 \wedge v_3 \leq v_2 \vee v_3 \leq v_4$, $u_1+u_4 \geq u_2+u_3$, *and* $v_1+v_4 \geq v_2+v_3$.

(iv) *Given a copula* $C\colon [0,1]^2 \to [0,1]$, *for each* $c \in [0,1]$ *the horizontal section* $h_c\colon [0,1] \to [0,1]$ *given by* $h_c(x) = C(x,c)$ *obviously satisfies* $h_c(0) = 0$ *and* $h_c(1) = c$*. Then the strongest convex horizontal section* h_c *is given by* $h_c(u) = c \cdot u$*, corresponding to the product copula* Π*. It is easy to verify that* Π *is an ultramodular copula, and hence* Π *is the strongest ultramodular copula. From a statistical point of view this means that each ultramodular copula is* Negative Quadrant Dependent *(NQD, for more details see [82]).*

3.2.1 Connections with supermodular aggregation functions and copulas

As in [37] for the case $n = 2$ we consider the notion of P-increasing functions where a function $\psi\colon [0,1]^n \to [0,1]$ is called P-increasing if it is increasing and ultramodular.
A function $\mathbf{K}\colon [0,1]^n \to [0,1]^m$ is said to be supermodular if the coordinate functions $K_1,\ldots,K_m$, $i = 1,\ldots,m$ defined by $\mathbf{K}(\mathbf{x}) = (K_1(\mathbf{x}),\ldots,K_m(\mathbf{x}))$ are supermodular.

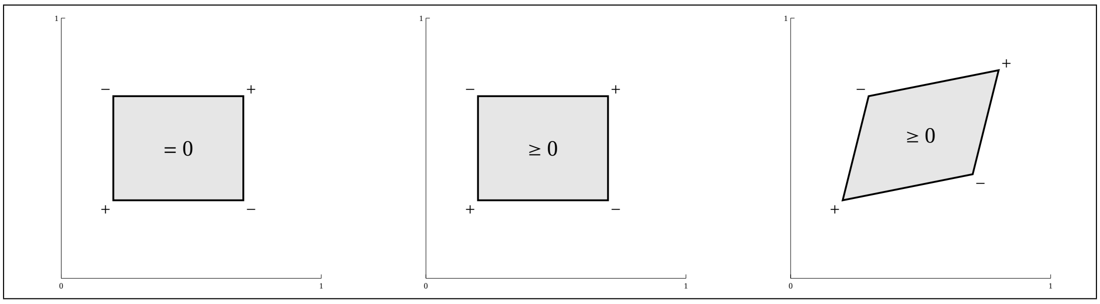

Figure 3.1: Modularity (left), supermodularity (center), and ultramodularity of $f\colon [0,1]^2 \to [0,1]$

Proposition 3.2.3 *If* $\psi\colon [0,1]^m \to [0,1]$ *is a P-increasing function and* $\mathbf{K}\colon [0,1]^n \to [0,1]^m$ *is a supermodular increasing function then the function* $H\colon [0,1]^n \to [0,1]$ *given by*

$$H(x_1,\ldots,x_n) = \psi(\mathbf{K})(\mathbf{x}) = \psi(K_1(\mathbf{x}),\ldots,K_m(\mathbf{x}))$$

is supermodular.

Proof: We consider 3 vectors $\mathbf{x},\mathbf{h},\mathbf{k}$ such that $\mathbf{h},\mathbf{k} \geq 0$ and $\mathbf{h} \perp \mathbf{k}$.
For all $i = 1,\ldots,m$, $K_i(\mathbf{x}+\mathbf{h}+\mathbf{k}) - K_i(\mathbf{x}+\mathbf{k}) \geq K_i(\mathbf{x}+\mathbf{h}) - K_i(\mathbf{x})$ and then there exist s_i, t_i with $t_i \geq s_i \geq 0$ such that

$$K_i(\mathbf{x}+\mathbf{h}+\mathbf{k}) = K_i(\mathbf{x}+\mathbf{k}) + t_i \qquad K_i(\mathbf{x}+\mathbf{h}) = K_i(\mathbf{x}) + s_i$$

So there exist $\mathbf{s},\mathbf{t}$ vectors in $\mathbb{R}^m$ such that $\mathbf{t} \geq \mathbf{s} \geq \mathbf{0}$ and

$$\mathbf{K}(\mathbf{x}+\mathbf{h}+\mathbf{k}) = \mathbf{K}(\mathbf{x}+\mathbf{k}) + \mathbf{t} \qquad \mathbf{K}(\mathbf{x}+\mathbf{h}) = \mathbf{K}(\mathbf{x}) + \mathbf{s}.$$

Since ψ is a P-increasing function and K is increasing in each variable one has:

$$\psi(\mathbf{K})(\mathbf{x}+\mathbf{h}+\mathbf{k}) - \psi(\mathbf{K})(\mathbf{x}+\mathbf{k}) = \psi(\mathbf{K}(\mathbf{x}+\mathbf{k})) + \mathbf{t}) - \psi(\mathbf{K}(\mathbf{x}+\mathbf{k})) \geq$$

$$\psi(\mathbf{K}(\mathbf{x}+\mathbf{k}) + \mathbf{s}) - \psi(\mathbf{K}(\mathbf{x}+\mathbf{k})) \geq \psi(\mathbf{K}(\mathbf{x}) + \mathbf{s})) - \psi(\mathbf{K}(\mathbf{x})) =$$

$$\psi(\mathbf{K}(\mathbf{x}+\mathbf{h})) - \psi(\mathbf{K}(\mathbf{x})) = \psi(\mathbf{K})(\mathbf{x}+\mathbf{h}) - \psi(\mathbf{K})(\mathbf{x}).$$

□

Corollary 3.2.4 *Let* $A \in \mathcal{A}_n^S$. *Let* $\varphi : [0,1] \to [0,1]$ *be a continuous increasing and convex function with* $\varphi(0) = 0$ *and* $\varphi(1) = 1$ *then the function*

$$A_\varphi(\mathbf{x}) := \varphi(A(x_1,\ldots,x_n))$$

is in $\mathcal{A}_n^S$.

Proof: It is obvious that $A_\varphi(\mathbf{0}) = 0$ and $A_\varphi(\mathbf{1}) = 1$. Then, it suffices to apply the above theorem to the function $H(x_1,\dots,x_n) = \psi(K_1(\mathbf{x}))$, with $\psi = \varphi$ and $K_1 = A$. In fact scalar convex functions are ultramodular and so ψ is P-increasing. □

Corollary 3.2.5 *Let f_i be increasing functions from $[0,1]$ into $[0,1]$ such that $f_i(0) = 0$ and $f_i(1) = 1$ for every $i = 1,2\dots,n$. Let A be in $\mathcal{A}_n^S$. Then, the function defined by*

$$A_{f_1,\dots,f_n}(x_1,\dots,x_n) := A(f_1(x_1),\dots,f_n(x_n))$$

is in $\mathcal{A}_n^S$.

Proof: It is obvious that $A_{f_1,\dots,f_n}(0,\dots,0) = 0$, $A_{f_1,\dots,f_n}(1,\dots,1) = 1$ and $A_{f_1,\dots,f_n}$ is increasing in each place, since it is the composition of increasing functions. Moreover, given $x_1^j \le x_2^j$, $\forall j = 1,\dots,n$, one obtains

$$\begin{aligned}&A_{f_1,\dots,f_n}(x_1^1,x_1^2,\dots,x_1^n) + A_{f_1,\dots,f_n}(x_2^1,x_2^2,\dots,x_2^n) \ge \\ &A_{f_1,\dots,f_n}(x_2^1,\dots,x_2^h,x_1^{h+1},\dots,x_1^n) + A_{f_1,\dots,f_n}(x_1^1,\dots,x_1^h,x_2^{h+1},\dots,x_2^n),\end{aligned}$$

because of the supermodularity of A and the increasingness of f_i. □

Corollary 3.2.6 *Let $A \in \mathcal{A}_n^S$. Let $\varphi : [0,1] \to [0,1]$ be a continuous and strictly monotone function with $\varphi(0) = 0$ and $\varphi(1) = 1$. The following statements are equivalent:*

(a) *φ is concave;*

(b) *for every $A \in \mathcal{A}_n^S$, the function*

$$A_\varphi(\mathbf{x}) := \varphi^{-1}\left(A(\varphi(x_1),\dots,\varphi(x_n))\right)$$

is in $\mathcal{A}_n^S$.

Proof: $(a) \Rightarrow (b)$ If φ is concave and positive, then φ^{-1} is convex. So, by using the results of the corollaries 3.2.4 and 3.2.5 we have our thesis.
$(b) \Rightarrow (a)$ If $A_\varphi(\mathbf{x})$ is in $\mathcal{A}_n^S$, we can consider

$$M_\varphi(\mathbf{x}) := \varphi^{-1}\left(\frac{\varphi(x_1) + \dots + \varphi(x_n)}{n}\right).$$

Thanks to proposition 3.1.3, $M_\varphi(\mathbf{x}) \in \mathcal{A}_n^S$ if and only if φ^{-1} is convex. So, φ is concave. □

Corollary 3.2.7 *Let A,B be two n–dimensional copulas and let C be a P-increasing bivariate copula. Then the function $H\colon [0,1]^n \to [0,1]$ given by*

$$H(x_1,\dots,x_n) = C(A(x_1,\dots,x_n),B(x_1,\dots,x_n))$$

is an element of $\mathcal{A}_n^S$.

Proof: By applying the proposition 3.2.3 with $\psi = C$, $A = K_1$ and $B = K_2$ we have our thesis. □

Corollary 3.2.8 *Let C_1, C_2, C_3 be 3 bivariate copulas and $\psi\colon [0,1]^3 \to [0,1]$ a P-increasing function. Then the function $H\colon [0,1]^3 \to [0,1]$ given by*

$$H(x_1, x_2, x_3) = \psi(C_1(x_1, x_2), C_2(x_1, x_3), C_3(x_2, x_3))$$

is a supermodular aggregation function.

Proof: As in the previous proof, by posing $K_1 = C_1$, $K_2 = C_2$ and $K_3 = C_3$. □

3.2.2 Other constructions

Ultramodularity is preserved by the composition of aggregation functions (here the monotonicity of the aggregation functions is crucial). First of all, we give the following result:

Theorem 3.2.9 *Let $A\colon [0,1]^n \to [0,1]$ be an aggregation function and $k \geq 2$. Then the following are equivalent:*

(i) *A is ultramodular.*

(ii) *If $B_1, \dots, B_n\colon [0,1]^k \to [0,1]$ are nondecreasing supermodular functions then the composite $D\colon [0,1]^k \to [0,1]$ given by $D(\mathbf{x}) = A(B_1(\mathbf{x}), \dots, B_n(\mathbf{x}))$ is a supermodular function.*

Proof: To show that (i) implies (ii), let A be an ultramodular aggregation function and $B_1, \dots, B_n$ be nondecreasing supermodular functions. Evidently, D is an aggregation function. Choose $\mathbf{x}, \mathbf{y} \in [0,1]^k$ and denote, for each $i \in \{1, \dots, n\}$, $a_i = B_i(\mathbf{x}) - B_i(\mathbf{x} \wedge \mathbf{y})$ and $b_i = B_i(\mathbf{y}) - B_i(\mathbf{x} \wedge \mathbf{y})$, $\mathbf{u} = (a_1, \dots, a_n)$, $\mathbf{v} = (b_1, \dots, b_n)$, and $\mathbf{z} = (B_1(\mathbf{x} \wedge \mathbf{y}), \dots, B_n(\mathbf{x} \wedge \mathbf{y}))$. The monotonicity of the B_i's implies $\mathbf{u}, \mathbf{v} \in [0,1]^n$, and their supermodularity $(B_1(\mathbf{x} \vee \mathbf{y}), \dots, B_n(\mathbf{x} \vee \mathbf{y})) \geq \mathbf{u} + \mathbf{v} + \mathbf{z}$. Now, the monotonicity and the ultramodularity of A yield

$$\begin{aligned} D(\mathbf{x} \vee \mathbf{y}) &\geq A(\mathbf{z} + \mathbf{u} + \mathbf{v}) \\ &\geq A(\mathbf{z} + \mathbf{u}) + A(\mathbf{z} + \mathbf{v}) - A(\mathbf{z}) \\ &= D(\mathbf{x}) + D(\mathbf{y}) - D(\mathbf{x} \wedge \mathbf{y}), \end{aligned}$$

i.e., D is supermodular.

Now suppose that (ii) holds. To show that the one-dimensional sections of A are convex, consider, without loss of generality, the function $f\colon [0,1] \to [0,1]$ given by $f(x) = A(x, u_2, \dots, u_n)$, where $u_2, \dots, u_n \in [0,1]$ are fixed. Define the functions $B_1, \dots, B_n\colon [0,1]^k \to [0,1]$ by $B_1(\mathbf{x}) = \frac{x_1 + x_2}{2}$ and $B_i(\mathbf{x}) = u_i$ for $i > 1$. If, for arbitrary $x, y \in [0,1]$, we put $\mathbf{x} = (x, y, 0, \dots, 0)$ and $\mathbf{y} = (y, x, 0, \dots, 0)$ then we obtain $D(\mathbf{x}) = D(\mathbf{y}) = f\left(\frac{x+y}{2}\right)$, $D(\mathbf{x} \wedge \mathbf{y}) = f(x \wedge y)$, and $D(\mathbf{x} \vee \mathbf{y}) = f(x \vee y)$. Since $B_1, \dots, B_n$ are nondecreasing supermodular functions, also D is supermodular, proving $\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right)$, i.e., the convexity of f. Note that, in the case $n = 1$, this means that A is ultramodular. If $n > 1$, because of Proposition 1.2.2 it suffices to show the supermodularity of the two-dimensional sections of A. This can be seen by defining $g\colon [0,1]^2 \to [0,1]$ by $g(x) = A(x, y, u_3, \ldots, u_n)$, $B_1(\mathbf{x}) = x_1$, $B_2(\mathbf{x}) = x_2$, and $B_i(\mathbf{x}) = u_i$ for $i > 2$, and by using similar arguments as above. □

Now we are ready to show that the class of ultramodular aggregation functions is closed under composition.

Theorem 3.2.10 *Let $A\colon [0,1]^n \to [0,1]$ and $B_1, \ldots, B_n\colon [0,1]^k \to [0,1]$ be ultramodular aggregation functions. Then the composite function $D\colon [0,1]^k \to [0,1]$ given by*

$$D(\mathbf{x}) = A(B_1(\mathbf{x}), \ldots, B_n(\mathbf{x}))$$

is also an ultramodular aggregation function.

Proof: Because of Theorem 3.2.9, D is supermodular (this holds also if $k = 1$; indeed, the first part of the proof of Theorem 3.2.9 works also in the case $k = 1$), and thus only the convexity of its one-dimensional sections needs to be shown.
Let $g\colon [0,1] \to [0,1]$ be a one-dimensional section of the composite function D, i.e., there are one-dimensional sections $f_1, \ldots, f_n\colon [0,1] \to [0,1]$ of $B_1, \ldots, B_n$, respectively, (which are convex because of Proposition 2.4.4) such that $g(x) = A(f_1(x), \ldots, f_n(x))$. Of course, g is nondecreasing and its convexity is equivalent to the validity of the Jensen inequality

$$g(x+a) - g(x) \leq g(x+2a) - g(x+a) \tag{3.4}$$

for all $x, a \in [0,1]$ with $x + 2a \leq 1$. From the convexity of $f_1, \ldots, f_n$ we obtain

$$0 \leq f_i(x+a) - f_i(x) \leq f_i(x+2a) - f_i(x+a),$$

for each $i \in \{1, \ldots, n\}$. Putting $a_i = f_i(x+a) - f_i(x)$ and $b_i = f_i(x+2a) - f_i(x+a)$, we have

$$\begin{aligned} g(x+2a) &= A(f_1(x+2a), \ldots, f_n(x+2a)) = \\ &= A(f_1(x) + a_1 + b_1, \ldots, f_n(x) + a_n + b_n) \geq \\ &\geq A(f_1(x) + a_1, \ldots, f_n(x) + a_n) + A(f_1(x) + b_1, \ldots, f_n(x) + b_n) + \\ &\quad - A(f_1(x), \ldots, f_n(x)) \geq \\ &\geq 2g(x+a) - g(x), \end{aligned}$$

which proves (3.4). Here the first inequality follows from the ultramodularity and the second one from the monotonicity of A. □

Theorem 3.2.10 has several important consequences (some of them can be found in [67, Proposition 4.1].

Corollary 3.2.11 *Let $A_1, \dots, A_j \colon [0,1]^n \to [0,1]$ be n-ary ultramodular aggregation functions and $f\colon [0,1] \to [0,1]$ a nondecreasing function with $f(0) = 0$ and $f(1) = 1$. Then we have:*

(i) *each convex combination of $A_1, \dots, A_j$ is an n-ary ultramodular aggregation function;*

(ii) *the product of $A_1, \dots, A_j$ is an n-ary ultramodular aggregation function;*

(iii) *if $A\colon [0,1]^n \to [0,1]$ is an n-ary ultramodular aggregation function and f is convex then the composition $f \circ A$ is an n-ary ultramodular aggregation function;*

(iv) *if $A\colon [0,1]^2 \to [0,1]$ is a binary associative ultramodular aggregation function then, for each $k > 2$, the k-ary extension of A is an k-ary ultramodular aggregation function.*

Proof: Statements (i)–(iii) follow from Theorem 3.2.10 taking into account that the weighted arithmetic mean, the product Π (which is a copula with linear, i.e., convex one-dimensional sections) and the function f in (iii) (for nondecreasing functions in one variable convexity means ultramodularity) are ultramodular aggregation functions.
The proof of (iv) is done by induction: if the k-ary extension $A^{(k)}$ of A is ultramodular then also $A^{(k+1)}\colon [0,1]^{k+1} \to [0,1]$ given by $A^{(k+1)}(x_1, \dots, x_k, x_{k+1}) = A(A^{(k)}(x_1, \dots, x_k), x_{k+1})$ is also ultramodular as a consequence of the ultramodularity of the functions $B_1, B_2\colon [0,1]^{k+1} \to [0,1]$ given by $B_1(x_1, \dots, x_k, x_{k+1}) = A^{(k)}(x_1, \dots, x_k)$ and $B_2(x_1, \dots, x_k, x_{k+1}) = x_{k+1}$, respectively. □

3.3 Structure of ultramodular aggregation functions

We denote, for $n \in \mathbf{N}$, by $\mathcal{U}_n$ the set of n-ary ultramodular aggregation functions and we put $\mathcal{U} = \bigcup_{n \in \mathbf{N}} \mathcal{U}_n$. Because of Theorem 3.2.10, the set $\mathcal{U}$ is closed under composition of functions. Moreover, this means that each $\mathcal{U}_n$ is a convex set (it is even a compact subset of the set of all functions from $[0,1]^n$ to $[0,1]$, equipped with the topology of pointwise convergence). The set $\mathcal{U}_1$ consists of all convex, nondecreasing functions $f\colon [0,1] \to [0,1]$ satisfying $f(0) = 0$ and $f(1) = 1$, and its smallest and greatest elements are $\mathbf{1}_{\{1\}}$ and $\mathrm{id}_{[0,1]}$, respectively. Since for each $f \in \mathcal{U}_1$ its restriction $f|_{[0,1[}$ is continuous, it is possible to write f as a convex combination of a continuous element g of $\mathcal{U}_1$ and $\mathbf{1}_{\{1\}}$: indeed, $f = \lambda g + (1-\lambda)\mathbf{1}_{\{1\}}$ where $\lambda = f(1^-)$ and λg is the continuous extension of $f|_{[0,1[}$.

We start with showing that for an $A \in \mathcal{U}_n$ to be continuous it is sufficient to show that it is continuous at the point $\mathbf{1}$:

Lemma 3.3.1 *An n-ary ultramodular aggregation function $A\colon [0,1]^n \to [0,1]$ is continuous if and only if* $\sup\{A(\mathbf{x}) \mid \mathbf{x} \in [0,1[^n\} = 1$.

Proof: Suppose that $A \in \mathcal{U}_n$ is non-continuous, but continuous at the point $\mathbf{1}$. Because of the monotonicity of A, there is some non-continuous one-dimensional section of A. From the convexity of this section we know that this non-continuity can occur only in its right endpoint. This means that there is some $i \in \{1, \dots, n\}$ and some $\mathbf{x} \in [0,1]^n$ such that for all $s \in [0,1[$

$$A(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) - A(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_n) \geq \varepsilon > 0.$$

Since A is continuous in $\mathbf{1}$ there is an $\alpha \in [0,1[$ such that

$$A(\mathbf{1}) - A(1,\dots,1,\alpha,1,\dots,1) < \varepsilon.$$

Putting $\mathbf{u} = (x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_n)$ and $\mathbf{v} = (1,\dots,1,\alpha,1,\dots,1)$ we obtain $A(\mathbf{u} \vee \mathbf{v}) - A(\mathbf{v}) < \varepsilon$ and $A(\mathbf{u}) - A(\mathbf{u} \wedge \mathbf{v}) \geq \varepsilon$, contradicting the supermodularity of A. The converse implication is obvious. □

Based on that, we have the following decomposition of elements of $\mathcal{U}_n$ for $n > 1$:

Proposition 3.3.2 *Each function $A \in \mathcal{U}_n$ can be written as a convex combination $A = \lambda A^* + (1-\lambda)A^{**}$ where $\lambda = \sup\{A(\mathbf{x}) \mid \mathbf{x} \in [0,1[^n\}$, $A^* \in \mathcal{U}_n$ is continuous and A^{**} is an n-ary aggregation function with $A^{**}(\mathbf{x}) = 0$ for all $\mathbf{x} \in [0,1[^n$.*

Proof: The monotonicity of A and the continuity of each of its one-dimensional sections imply that $A|_{[0,1[^n}$ is continuous (compare Remark 1.3(ii) in [59] for the case $n = 2$). Let $B\colon [0,1]^n \to [0,1]$ be the (unique) continuous extension of $A|_{[0,1[^n}$. If $\lambda = B(\mathbf{1}) = 0$ then $B = 0$, and A^* can be chosen arbitrarily and $A^{**} = A$. If $\lambda > 0$ then $A^* = \frac{1}{\lambda}B$ is a continuous element of $\mathcal{U}_n$. Now, if $\lambda = 1$ then $A^* = B = A$, and A^{**} can be chosen arbitrarily. If $0 < \lambda < 1$ then $A^{**} = \frac{A-B}{1-\lambda}$, and we have $A^{**}(\mathbf{1}) = 1$. Because of $A|_{[0,1[^n} = B|_{[0,1[^n}$ we get $A^{**}(\mathbf{x}) = 0$ for all $\mathbf{x} \in [0,1[^n$. The monotonicity of A^{**} is equivalent to the monotonicity of its one-dimensional sections which is non-trivial only if, for some fixed $\mathbf{a} \in [0,1]^n$, one of its coordinates equals 1. Without loss of generality, consider the section $h\colon [0,1] \to [0,1]$ given by $h(x) = A(x,1,a_3,\dots,a_n)$. For each $\varepsilon \in]0,1[$, the ultramodularity of A implies

$$\begin{aligned} A(y,1,a_3,\dots,a_n) &- A(x,1,a_3,\dots,a_n) \\ &\geq A(y,1-\varepsilon,a_3,\dots,a_n) - A(x,1-\varepsilon,a_3,\dots,a_n) \end{aligned} \tag{3.5}$$

for all $x,y \in [0,1]$ with $x < y$. Taking the limit $\varepsilon \to 0$, (3.5) turns into

$$A(y,1,a_3,\dots,a_n) - A(x,1,a_3,\dots,a_n) \geq B(y,1,a_3,\dots,a_n) - B(x,1,a_3,\dots,a_n),$$

implying $h(x) \leq h(y)$. □

Remark 3.3.3 (i) The aggregation function A^{**} mentioned in Proposition 3.3.2 is not ultramodular, in general. Indeed, define $A \in \mathcal{U}_2$ by

$$A(x,y) = \begin{cases} \max(5xy - \frac{9}{2}, 0) & \text{if } (x,y) \in [0,1[^2, \\ \max(\frac{5}{9}y, 5y-4) & \text{if } x = 1, \\ \max(\frac{5}{9}x, 5x-4) & \text{if } y = 1. \end{cases}$$

Then $\lambda = \frac{1}{2}$, and A^* and A^{**} are given by $A^*(x,y) = \max(10xy - 9, 0)$ and

$$A^{**}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \min(\frac{10}{9}y, 1) & \text{if } x = 1, \\ \min(\frac{10}{9}x, 1) & \text{if } y = 1. \end{cases}$$

The section $h\colon [0,1] \to [0,1]$ given by $h(x) = A^{**}(x,1) = \min(\frac{10}{9}x, 1)$ is not convex, i.e., A^{**} is not ultramodular.

(ii) Because of Proposition 3.3.2, the ultramodularity of an n-ary aggregation function implies its continuity up to the right boundary of $[0,1]^n$, extending a similar fact for nondecreasing functions to dimension n.

Given a fixed $\mathbf{v} \in [0,1]^n$ and $\mathbf{t}_1, \ldots, \mathbf{t}_k \in [0,\infty[^n$ we define

$$E_{\mathbf{v};\mathbf{t}_1,\ldots,\mathbf{t}_k} = \left\{ \mathbf{x} \in [0,1]^n \,\middle|\, \mathbf{x} = \mathbf{v} + \sum_{j=1}^{k} \alpha_j \cdot \mathbf{t}_j \text{ and } \alpha_1, \ldots, \alpha_k \in [0,1] \right\}.$$

Evidently, to cover all possible k-dimensional sections of $[0,1]^n$, it is enough to consider $k \leq n$ and independent vectors $\mathbf{t}_1, \ldots, \mathbf{t}_k$. As special cases (with $k = 1$) we mention the i-th one-dimensional section with arbitrary $\mathbf{v}$ and $\mathbf{t} = \mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, and the diagonal section with $\mathbf{v} = (0, \ldots, 0)$ and $\mathbf{t} = (1, \ldots, 1)$. As a consequence of Definition 3.2.1, for a given ultramodular aggregation function $A\colon [0,1]^n \to [0,1]$, also the restriction $A|_{E_{\mathbf{v};\mathbf{t}_1,\ldots,\mathbf{t}_k}}$ of A to $E_{\mathbf{v};\mathbf{t}_1,\ldots,\mathbf{t}_k}$ is ultramodular.

Theorem 3.3.4 *Let $A\colon [0,1]^n \to [0,1]$ be an aggregation function with $A(\mathbf{x}) = 0$ for all $\mathbf{x} \in [0,1[^n$. Then A is ultramodular if and only if the following hold:*

(i) *all $(n-1)$-dimensional sections $B_i = A|_{E_i}$ of A, $i \in \{1, \ldots, n\}$, are ultramodular, where $E_i = E_{\mathbf{e}_i;\mathbf{e}_1,\ldots,\mathbf{e}_{i-1},\mathbf{e}_{i+1},\ldots,\mathbf{e}_n}$.*

(ii) *for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and all $\mathbf{x} \in E_i \cap E_j$ we have*

$$A(\mathbf{x}) \geq \sup\{B_i(\mathbf{y}) \mid \mathbf{y} \in E_i, \mathbf{y} < \mathbf{x}\} + \sup\{B_j(\mathbf{z}) \mid \mathbf{z} \in E_j, \mathbf{z} < \mathbf{x}\}.$$

Proof: Since the last inequality follows from the ultramodularity of A, the necessity is obvious. Conversely, evidently each one-dimensional section of A is either constant zero up to the endpoint (and thus convex) or it coincides with some one-dimensional section of some B_i, again showing its convexity. The validity of (ii) is trivial if $\mathbf{x} \in [0,1[^n$ or $\mathbf{y} \in [0,1[^n$. Suppose that $\mathbf{x}, \mathbf{y} \in [0,1]^n \setminus [0,1[^n$. Then there are $i, j \in \{1, \ldots, n\}$ with $\mathbf{x} \in E_i$ and $\mathbf{y} \in E_j$. If $\mathbf{x} \wedge \mathbf{y} \in [0,1]^n \setminus [0,1[^n$ then we may suppose $i = j$, and (ii) follows from the supermodularity of B_i. If $\mathbf{x} \wedge \mathbf{y} \in [0,1[^n$ then $i \neq j$ and $\mathbf{x} \vee \mathbf{y} \in E_i \cap E_j$, in which case (ii) follows from the fact that $\mathbf{x} < \mathbf{x} \wedge \mathbf{y}$ and $\mathbf{y} < \mathbf{x} \wedge \mathbf{y}$. □

Example 3.3.5 An aggregation function $A\colon [0,1]^2 \to [0,1]$ is non-continuous, ultramodular and satisfies $A(\mathbf{x}) = 0$ for all $\mathbf{x} \in [0,1[^2$ if and only if there are numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0,1]$ with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1$ and continuous, nondecreasing convex functions $f, g\colon [0,1] \to [0,1]$ such that

$$A(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \lambda_1 + \lambda_2 \cdot f(x) & \text{if } (x,y) \in [0,1[\times \{1\}, \\ \lambda_3 + \lambda_4 \cdot g(y) & \text{if } (x,y) \in \{1\} \times [0,1[, \\ 1 & \text{otherwise.} \end{cases}$$

The smallest non-continuous ultramodular binary aggregation function vanishing on $[0,1[^2$ is $\mathbf{1}_{\{(1,1)\}}$, and there is no greatest aggregation function of this type. However, for each $\alpha \in [0,1]$, the function A_α given by

$$A_\alpha(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \alpha & \text{if } (x,y) \in [0,1[\times \{1\}, \\ 1-\alpha & \text{if } (x,y) \in \{1\} \times [0,1[, \\ 1 & \text{otherwise.} \end{cases}$$

is a maximal non-continuous ultramodular binary aggregation function vanishing on $[0,1[^2$.

We have the following characterization of maximal continuous binary ultramodular aggregation functions:

Proposition 3.3.6 *A function $A\colon [0,1]^2 \to [0,1]$ is a maximal continuous ultramodular aggregation function (i.e., there is no continuous ultramodular aggregation function $B\colon [0,1]^2 \to [0,1]$ with $B(x,y) \geq A(x,y)$ for all $(x,y) \in [0,1]^2$ and $B(x_0,y_0) > A(x_0,y_0)$ for some $(x_0,y_0) \in [0,1]^2$) if and only if A is a weighted arithmetic mean, i.e., $A(x,y) = \lambda \cdot x + (1-\lambda) \cdot y$ for some $\lambda \in [0,1]$.*

Proof: Note that for each supermodular aggregation function $A\colon [0,1]^2 \to [0,1]$ we necessarily have $A(1,0) + A(0,1) \leq 1$. Moreover, if $A(1,0) + A(0,1) = 1$, i.e., $A(1,1) - A(1,0) - A(0,1) + A(0,0) = 0$, then for each rectangle $[x,x^*] \times [y,y^*]$ necessarily $A(x^*,y^*) - A(x^*,0) - A(x,y^*) + A(x,y) = 0$, and the aggregation function is additive, i.e., $A(x,y) = A(x,0) + A(0,y)$. On the other hand, since the one-dimensional sections of the ultramodular aggregation function A are convex, we get $A(x,0) \leq x \cdot A(1,0) = \lambda \cdot x$ and $A(0,y) \leq y \cdot A(0,1) = (1-\lambda) \cdot y$, where $\lambda = A(1,0)$. Then clearly each weighted arithmetic mean given by $A(x,y) = \lambda \cdot x + (1-\lambda) \cdot y$ is a maximal element of the set of all continuous elements of $\mathcal{U}_2$. Moreover, these facts also prove that each element of $\mathcal{U}_2$ satisfying $A(1,0) + A(0,1) = 1$ which is different from the weighted arithmetic mean is bounded from above by a corresponding weighted arithmetic mean (with coinciding values at the corner points of the unit square). On the other hand, if for each continuous element A of $\mathcal{U}_2$ we put $a = A(1,0)$ and $b = A(0,1)$ and if $a + b < 1$ then, as already mentioned, $A(x,0) \leq x \cdot a$, and evidently $A(0,y) \leq b \cdot y < (1-a) \cdot y$. Moreover, due to the convexity of the one-dimensional sections of A, we have $A(1,y) \leq a + (1-a) \cdot y$ and

$A(x,1) \leq (1-a)+a\cdot x$, implying $A(x,y) \leq a\cdot x+(1-a)\cdot y$ for all $(x,y)\in[0,1]^2$. Hence the weighted arithmetic mean B given by $B(x,y)=a\cdot x+(1-a)\cdot y$ satisfies $B\geq A$ and $B(0,1)>A(0,1)$, i.e., A is not a maximal element of the set of all continuous members of $\mathcal{U}_2$. $\square$

Chapter 4

2-Increasing agops

In the case of aggregation functions we have denoted supermodular n–ary aggregation ones by $\mathcal{A}_n^S$. Now we want to discuss the interesting connections between $\mathcal{A}_n^S$, $\mathcal{A}_2^S$ and copulas. Section 4.3 deals with some special ultramodular aggregation functions, especially with ultramodular copulas and in subsection 4.3.1 a method for constructing bivariate copulas based ultramodular aggregation functions is proposed and exemplified. We will continue the discussion about construction procedures which yield supermodular and ultramodular copulas known as ordinal sums, flipping, and cycle shiftings.

4.1 A copula-based approach to aggregation functions

A binary aggregation function, briefly called agop, is said to be 2-increasing if, for all x_1, x_2, y_1 and $y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$, one has

$$V_A([x_1,x_2] \times [y_1,y_2]) := A(x_1,y_1) + A(x_2,y_2) - A(x_1,y_2) - A(x_2,y_1) \geq 0. \tag{4.1}$$

A deep analysis of this class of functions can be found in [37, 39]. Moreover, inequality (4.1) is called *rectangular inequality* and $V_A([x_1,x_2] \times [y_1,y_2])$ is said to be the A-*volume* of $[x_1,x_2] \times [y_1,y_2]$. Notice that inequality (4.1) is equivalent to the fact that both the functions

$$t \to A(x_2,t) - A(x_1,t) \quad \text{and} s \to A(s,y_2) - A(s,y_1)$$

are increasing for $x_1 \leq x_2$ and $y_1 \leq y_2$, respectively. This property is also known as *moderate growth* (see [73]).
A 2-increasing aggregation function is also *supermodular* with respect to the pointwise order on $[0,1]^2$. In fact, if $\mathbf{x} \wedge \mathbf{y}$ and $\mathbf{x} \vee \mathbf{y}$ denote, respectively, the componentwise minimum and maximum of two points $\mathbf{x}$ and $\mathbf{y}$ of $[0,1]^2$, then inequality (4.1) can be rewritten in the form $A(\mathbf{x} \wedge \mathbf{y}) + A(\mathbf{x} \vee \mathbf{y}) \geq A(\mathbf{x}) + A(\mathbf{y})$. The class of 2-increasing agops will be denoted by $\mathcal{A}_2$ and the most important characterization between $\mathcal{A}_n^S$ and $\mathcal{A}_2^S$ is Proposition 1.2.2.
The supermodularity of a function $f\colon [0,1]^n \to [0,1]$ is preserved if the arguments are distorted,

i.e., if $g_1,\dots,g_n\colon [0,1]\to[0,1]$ are nondecreasing functions, then the function $h\colon [0,1]^n\to[0,1]$ given by $h(\mathbf{x}) = f(g_1(x_1),\dots,g_n(x_n))$ is supermodular (if f is a supermodular aggregation function with $f(g_1(0),\dots,g_n(0)) = 0$ and $f(g_1(1),\dots,g_n(1)) = 1$ then h is also a supermodular aggregation function).

Some important examples of supermodular aggregation functions are the restrictions to $[0,1]^2$ of bivariate distribution functions F such that $F(0,0) = 0$ and $F(1,1) = 1$, in particular copulas, which correspond to the case when the marginal distributions are uniform on the unit interval (and in such a case, they possess a neutral element 1) (see [81, 82]), the *smallest agop*, defined by

$$A_S(x,y) = \begin{cases} 1 & \text{if } (x,y) = (1,1), \\ 0 & \text{otherwise.} \end{cases}$$

and the weighted arithmetic means (see [22]).
Copulas play an important role in the representation of supermodular binary aggregation functions. The following result is taken from [39], Theorem 17:

Proposition 4.1.1 *An aggregation function $A\colon [0,1]^2\to[0,1]$ is supermodular if and only if there are nondecreasing functions $g_1,g_2,g_3,g_4\colon [0,1]\to[0,1]$ with $g_i(1) = 1$ for $i\in\{1,2,3,4\}$ and $g_1(0) = g_2(0) = 0$, a copula $C\colon [0,1]^2\to[0,1]$ with $C(g_3(0),g_4(0)) = 0$, and numbers $a,b,c\in[0,1]$ with $a+b+c = 1$ such that, for all $(x,y)\in[0,1]^2$,*

$$A(x,y) = a\cdot g_1(x) + b\cdot g_2(y) + c\cdot C(g_3(x),g_4(y)). \tag{4.2}$$

If 0 is an annihilator of the aggregation function $A\colon [0,1]^2\to[0,1]$, i.e., if $A(x,0) = A(0,x) = 0$ for all $x\in[0,1]$, then (4.2) reduces to

$$A(x,y) = C(f(x),g(y)), \tag{4.3}$$

where $f,g\colon [0,1]\to[0,1]$ are nondecreasing functions with $f(1) = g(1) = 1$ and C satisfies $C(f(0),g(0)) = 0$. Note that then we have $f(x) = A(x,1)$ and $g(x) = A(1,x)$ for all $x\in[0,1]$.

Proposition 4.1.1 can be read also in this way: each binary supermodular aggregation function is a convex combination of a modular aggregation function and a distorted copula.

In general, the composition of (super-)modular functions is not necessarily (super-)modular: the functions $A,B\colon [0,1]^2\to[0,1]$ given by $A(x,y) = \sqrt{x}$ and $B(x,y) = \frac{x+y}{2}$ are both modular and, therefore, supermodular. However, the composition $A(B,B)\colon [0,1]^2\to[0,1]$ given by $A(B,B)(x,y) = \sqrt{\frac{x+y}{2}}$ is not supermodular.

4.2 Binary ultramodular aggregation function

Propositions 4.1.1 and 3.2.2 imply the following representation for binary supermodular aggregation functions:

Corollary 4.2.1 *If $A \in \mathcal{U}_2$ then we have*

$$A = \lambda \cdot A_1 + (1 - \lambda) \cdot A_2, \tag{4.4}$$

where A_1 is a modular element of $\mathcal{U}_2$, A_2 is a supermodular binary aggregation function with annihilator 0*, and $\lambda = 1 - A(1,0) - A(0,1) \in [0,1]$.*

Proof: If $A \in \mathcal{U}_2$ then, because of Proposition 4.1.1, there exist nondecreasing functions $g_1, g_2, g_3, g_4 \colon [0,1] \to [0,1]$ with $g_i(1) = 1$ for $i \in \{1,2,3,4\}$ and $g_1(0) = g_2(0) = 0$, a copula C with $C(g_3(0), g_4(0)) = 0$, and numbers $a, b, c \in [0,1]$ with $a + b + c = 1$ such that, for all $(x,y) \in [0,1]^2$,

$$A(x,y) = a \cdot g_1(x) + b \cdot g_2(y) + c \cdot C(g_3(x), g_4(y)) = \lambda \cdot A_1 + (1 - \lambda) \cdot A_2,$$

where $\lambda = a + b$, $A_1(x,y) = \frac{1}{a+b} \cdot (a \cdot g_1(x) + b \cdot g_2(y))$ whenever $\lambda > 0$ (and A_1 an arbitrary modular element of $\mathcal{U}_2$ if $\lambda = 0$), and $A_2(x,y) = C(g_3(x), g_4(y))$ (if $\lambda = 1$ then A_2 can be chosen arbitrarily). The supermodularity of A_2 is a consequence of the supermodularity of C. □

A full characterization of the elements of the set $\mathcal{U}_2$ is still missing. Obviously, if $A_2 \in \mathcal{U}_2$ has annihilator 0 then (4.4) yields an element $A \in \mathcal{U}_2$ for each modular $A_1 \in \mathcal{U}_2$ and each $\lambda \in [0,1]$.

Remark 4.2.2 (i) The supermodular aggregation function A_2 mentioned in Corollary 4.2.1 is not ultramodular, in general. Take, e.g., $A \in \mathcal{U}_2$ given by

$$A(x,y) = \begin{cases} \frac{4}{3} xy & \text{if } (x,y) \in \left[0, \frac{1}{2}\right]^2, \\ \frac{2x+2y-1}{3} & \text{otherwise.} \end{cases}$$

Then $\lambda = \frac{2}{3}$, $A_1(x,y) = f(x) + f(y)$ with $f(x) = \max(2x - 1, 0)$ and $A_2(x,y) = g(x) \cdot g(y)$ with $g(x) = \min(2x, 1)$, and A_2 is supermodular. However, A_2 is not ultramodular, since the section $h \colon [0,1] \to [0,1]$ given by $h(x) = A(x,1) = g(x)$ is not convex.

(ii) If A_2 is a supermodular binary aggregation function with annihilator 0 which is not ultramodular then the set $[\lambda_0, 1]$ of all λ such that, for some modular $A_1 \in \mathcal{U}_2$, the convex combination $\lambda \cdot A_1 + (1 - \lambda) \cdot A_2$ is ultramodular, is a proper subset of $[0,1]$. In other words, $\lambda_0 = 0$ if and only A_2 is ultramodular. It is not difficult to show that, for A_1 and A_2 considered in (i), we have $\lambda_0 = \frac{2}{3}$.

(iii) There are supermodular binary aggregation functions A_2 with annihilator 0 such that the set $[\lambda_0, 1]$ in (ii) is trivial, i.e., $\lambda_0 = 1$ (in which case A_2 is necessarily non-ultramodular). An example for such an A_2 is the geometric mean, i.e., $A_2(x,y) = \sqrt{x \cdot y}$ (note that it has unbounded partial derivatives).

Proposition 4.2.3 *A binary aggregation function $A \in \mathcal{U}_2$ can be written as in* (4.4) *with $A_1 \in \mathcal{U}_2$ being modular and $A_2 \in \mathcal{U}_2$ having annihilator* 0 *if and only if, for all $r, s \in [0,1]$, the* *functions* $h_r, v_s\colon [0,1] \to [0,1]$ *given by* $h_r(x) = A(x,r) - A(x,0)$ *and* $v_s(y) = A(s,y) - A(0,y)$*, respectively, are convex.*

Proof: Assume that A can be written as in (4.4). If $\lambda < 1$ then $h_r(x) = (1-\lambda)\cdot A_2(x,y)$ and $v_s(y) = (1-\lambda)\cdot A_2(x,y)$, i.e., h_r and v_s are multiples of sections of $A_2 \in \mathcal{U}_2$ and, therefore, convex. If $\lambda = 1$ then $A_1 = A$, implying that $h_r = A(0,r)$ and $v_s = A(s,0)$ are constant and, therefore, convex.

Conversely, suppose that all the functions h_r and v_s are convex. Because of Corollary 4.2.1, $A = \lambda \cdot A_1 + (1-\lambda)\cdot A_2$ with $A_1 \in \mathcal{U}_2$ being modular and A_2 being supermodular with annihilator 0. Then $v_s(y) = \lambda \cdot A_1(s,0) + (1-\lambda)\cdot A_2(s,y)$, and the convexity of v_s implies that either $\lambda = 1$ (in which case $A = A_1$ and A_2 can be chosen arbitrarily) or $g\colon [0,1] \to [0,1]$ given by $g(y) = A_2(x,y)$ is a convex section of A_2. Thus, if A is not modular then A_2 is necessarily ultramodular. □

Now we present a way to construct continuous ultramodular aggregation operators from (possibly) non-ultramodular ones.

Proposition 4.2.4 *Let* $C\colon [0,1]^2 \to [0,1]$ *be a copula,* $f, g\colon [0,1] \to [0,1]$ *nondecreasing surjections, and assume that all sections of* $A_2\colon [0,1]^2 \to [0,1]$ *given by* $A_2(x,y) = C(f(x), g(y))$ *have bounded second derivatives. Then there is a* $\lambda \in [0,1[$ *and a modular aggregation function* $A_1 \in \mathcal{U}_2$ *such that* $A\colon [0,1]^2 \to [0,1]$ *given by* $A = \lambda \cdot A_1 + (1-\lambda)\cdot A_2$ *is an ultramodular aggregation function.*

Proof: Define the functions $\alpha, \beta\colon [0,1] \to \mathbb{R}$ by

$$\alpha(x) = \inf\left\{ \tfrac{\partial^2}{\partial x^2} A_2(x,y) \;\middle|\; y \in [0,1] \right\},$$
$$\beta(y) = \inf\left\{ \tfrac{\partial^2}{\partial y^2} A_2(x,y) \;\middle|\; x \in [0,1] \right\},$$

respectively, and the functions $\gamma, \delta, \varepsilon, \zeta\colon [0,1] \to \mathbb{R}$ by

$$\gamma(x) = \int_0^x \max(-\alpha(u), 0)\, du, \qquad \varepsilon(x) = \int_0^x \max(-\beta(u), 0)\, du,$$
$$\delta(x) = \int_0^x \gamma(u)\, du \qquad \zeta(x) = \int_0^x \varepsilon(u)\, du,$$

respectively. Then it is not difficult to check that, for $\lambda_0 = \frac{\delta(1)+\zeta(1)}{\delta(1)+\zeta(1)+1}$, for each $\lambda \in [\lambda_0, 1]$ and for all $a, b \in [0,1]$ with $a + b = \frac{\lambda - \lambda_0}{\lambda\cdot(1-\lambda_0)}$, the function $A_1\colon [0,1]^2 \to [0,1]$ given by

$$A_1(x,y) = a\cdot x + \tfrac{1-\lambda}{\lambda}\cdot \delta(x) + b\cdot y + \tfrac{1-\lambda}{\lambda}\cdot \zeta(y)$$

is modular, implying that $A = \lambda \cdot A_1 + (1-\lambda)\cdot A_2$ is an ultramodular aggregation operator. □

Example 4.2.5 Using the notations of Proposition 4.2.4 and of its proof, put $C = \Pi$ and define the functions f and g by $f(x) = 2x - x^2$ and $g(x) = \frac{3x-x^3}{2}$. Note that the function A_2 is not ultramodular. Then we get $\alpha(x) = -2$, $\beta(x) = -3x$, $\delta(x) = x^2$, $\zeta(x) = \frac{x^3}{2}$, and $\lambda_0 = \frac{3}{5}$. Finally, we obtain the two-parametric family $(A_{\lambda,a})_{\lambda \in [\frac{3}{5},1], a \in [0, \frac{5\lambda-3}{2\lambda}]}$ of ultramodular aggregation functions given by

$$A_{\lambda,a}(x,y) = \lambda\big(ax + \tfrac{1-\lambda}{\lambda}x^2 + (\tfrac{5\lambda-3}{2\lambda} - a)y + \tfrac{1-\lambda}{\lambda}\tfrac{y^3}{2}\big) + (1-\lambda)\tfrac{(2x-x^2)\cdot(3y-y^3)}{2}.$$

4.2.1 Modifications and constructions

Now we recall some of the modifications and constructions in the framework of copulas (i.e., which preserve supermodularity).

Remark 4.2.6 For each binary copula $C\colon [0,1]^2 \to [0,1]$ we have $W \le C \le M$, where the lower and upper Fréchet-Hoeffding bounds W and M are given by $W(x,y) = \max(x+y-1, 0)$ and $M(x,y) = \min(x,y)$, respectively.

(i) *Ordinal sum*: If $(C_k)_{k\in K}$ is a family of copulas and if $(]a_k,b_k[)_{k\in K}$ is a family of pairwise disjoint open subintervals of $[0,1]$ then the ordinal sum $C = (\langle a_k, b_k, C_k\rangle)_{k\in K}$ is given by

$$C(x,y) = \begin{cases} a_k + (b_k - a_k)C_k\big(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\big) & \text{if } (x,y) \in]a_k,b_k[^2, \\ M(x,y) & \text{otherwise.} \end{cases}$$

(ii) *W-ordinal sum*: If $(C_k)_{k\in K}$ is a family of copulas and if $(]a_k,b_k[)_{k\in K}$ is a family of pairwise disjoint open subintervals of $[0,1]$ then the W-ordinal sum $C = W\text{-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$ is given by (see [40, 54])

$$C(x,y) = \begin{cases} (b_k - a_k)C_k\big(\frac{x-a_k}{b_k-a_k}, \frac{y-1+b_k}{b_k-a_k}\big) & \text{if } (x,y) \in]a_k,b_k[\times]1-b_k, 1-a_k[, \\ W(x,y) & \text{otherwise.} \end{cases}$$

(iii) *g-ordinal sum*: If $(C_k)_{k\in K}$ is a family of copulas and if $(]a_k,b_k[)_{k\in K}$ is a family of pairwise disjoint open subintervals of $[0,1]$ then the g-ordinal sum $C = \text{g-}(\langle a_k, b_k, C_k\rangle)_{k\in K}$ is given by (see [74])

$$C(x,y) = \begin{cases} a_k y + (b_k - a_k)\cdot C_k\big(\frac{x-a_k}{b_k-a_k}, y\big) & \text{if } x \in]a_k,b_k[, \\ xy & \text{otherwise.} \end{cases}$$

(iv) *Flipping*: If C is a copula then the flippings C^- and C_- are given by (see [82])

$$C^-(x,y) = x - C(x, 1-y), \tag{4.5}$$

$$C_-(x,y) = y - C(1-x, y), \tag{4.6}$$

(v) *Survival copula*: If C is a copula then the survival copula $\widehat{C}$ is given by (see [82])

$$\widehat{C}(x,y) = x + y - 1 + C(1-x, 1-y).$$

(vi) *Cycle shifting*: If C is a copula and $s \in]0,1[$ then the cycle shiftings $C_s^{x\text{-cshift}}$ and $C_s^{y\text{-cshift}}$ are given by (see [29])

$$C_s^{x\text{-cshift}}(u,v) = \begin{cases} C(u+s,v) - C(s,v) & \text{if } u \in [0, 1-s], \\ C(u+s-1,v) + v - C(s,v) & \text{otherwise,} \end{cases}$$

$$C_s^{y\text{-cshift}}(u,v) = \begin{cases} C(u,v+s) - C(u,s) & \text{if } v \in [0, 1-s], \\ C(u,v+s-1) + u - C(u,s) & \text{otherwise.} \end{cases}$$

So, we see that there are several modifications and constructions of aggregation functions which preserve the supermodularity. However, only few of them preserve also ultramodularity. Without reference to distribution functions or random variables, we construct grounded 2-increasing functions on $\mathbf{I}^2$ with uniform margins, utilizing some information of a geometric nature, such as a description of the support or the shape of the graphs of horizontal, vertical, or diagonal sections. In the "ordinal sum" construction the members of a set of copulas are scaled and translated in order to construct a new copula.
The only copula whose ultramodularity is preserved by flipping or cycle shifting is the product copula Π. From Remark 3.2.2(iv) we know that a necessary condition for a copula C to be an ultramodular $C \leq \Pi$, i.e., C is Negative Quadrant Dependent (NQD). However, flipping changes the property NQD into PQD (Positive Quadrant Dependent, see [82]), i.e., if $C \leq \Pi$ then $C^- \geq \Pi$ and $C_- \geq \Pi$, implying that the only ultramodular copula remaining ultramodular after flipping is Π. If C is a copula and h_c a horizontal section thereof, then the corresponding horizontal section h_c^* of the copula $C_s^{x\text{-cshift}}$ is given by

$$h_c^*(u) = \begin{cases} h_c(s+u) - h_c(s) & \text{if } u \in [0, 1-s], \\ h_c(u+s-1) + c - h_c(s) & \text{otherwise.} \end{cases}$$

Then, if h_c is convex, h_c^* is convex only if h_c is linear, i.e., if $h_c(u) = c \cdot u$, implying $C = \Pi$. Analogous reasoning based on vertical sections can be used for y-cycle shifting. Moreover, no non-trivial ordinal sum or g-ordinal sum of copulas can be ultramodular. On the other hand, a W-ordinal sum of copulas is ultramodular if and only if each summand copula is ultramodular. Similarly, a survival copula $\widehat{C}$ is ultramodular if and only if C is ultramodular.

4.3 Special ultramodular aggregation functions

If an ultramodular binary aggregation function with annihilator 0 has also neutral element 1 then it necessarily is an ultramodular copula, i.e., a copula with convex sections. In statistics, where a copula C describes the dependence structure of a random vector (X,Y), the ultramodularity of C is equivalent to X being stochastically decreasing with respect to Y (and Y being stochastically decreasing with respect to X). Clearly, the set $\mathcal{C}_u$ of all ultramodular binary copulas is convex. The greatest element of $\mathcal{C}_u$ is the product Π, and the smallest element of $\mathcal{C}_u$ is the lower Fréchet-Hoeffding bound W.

Because of [82, 92], each associative copula is an ordinal sum [28, 82, 92] of Archimedean copulas. However, if for a copula C we have $C(a,a) = a$ for some $a \in]0,1[$, then $C(x,a) = \min(x,a)$ implies that there are non-convex sections, i.e., C cannot be ultramodular. Therefore, each associative ultramodular copula C is a trivial ordinal sum of Archimedean copulas, i.e., C itself must be Archimedean. Recall that a binary aggregation function $C\colon [0,1]^2 \to [0,1]$ is an Archimedean copula if and only if there is a continuous, strictly decreasing convex function $t\colon [0,1] \to [0,\infty]$ with $t(1) = 0$ such that for all $(x,y) \in [0,1]$ (see [79])

$$C(x,y) = t^{-1}(\min(t(x)+t(y),t(0))). \tag{4.7}$$

The function t is called an additive generator of C, and it is unique up to a positive multiplicative constant.

If we want to see whether an Archimedean copula is ultramodular, i.e., has convex horizontal and vertical sections, its symmetry (as a consequence of (4.7)) and boundary conditions tell us that it suffices to check the convexity of all horizontal sections for $a \in]0,1[$.

Theorem 4.3.1 *Let $C\colon [0,1]^2 \to [0,1]$ be an Archimedean copula with a two times differentiable additive generator $t\colon [0,1] \to [0,\infty]$. Then C is ultramodular if and only if $\frac{1}{t'}$ is a convex function.*

Proof: Suppose that C is an Archimedean copula with a two times differentiable additive generator t. Then C is ultramodular if and only if, for each $a \in]0,1[$ and $c = t(a) \in]0,t(0)[$, the section $f\colon [0,1] \to [0,1]$ given by

$$f(x) = C(x,a) = t^{-1}(\min(t(x)+c,t(0))),$$

is convex, which is equivalent to $f''(x) \geq 0$ whenever $f(x) > 0$, i.e., for each $x \in [0,1]$ with $f(x)+c < t(0)$

$$f''(x) = \frac{t''(x)\cdot(t'(t^{-1}(t(x)+c)))^2 - (t'(x))^2\cdot t''(t^{-1}(t(x)+c))}{(t'(t^{-1}(t(x)+c)))^3} \geq 0.$$

Since t' is negative on $]0,1[$, this means

$$\frac{t''(x)}{(t'(x))^2} \leq \frac{t''(t^{-1}(t(x)+c))}{(t'(t^{-1}(t(x)+c)))^2}. \tag{4.8}$$

Put $u = t^{-1}(t(x)+c) < x$. Because of $\frac{t''(x)}{(t'(x))^2} = \left(-\frac{1}{t'(x)}\right)'$, (4.8) can be rewritten as

$$\left(\frac{1}{t'(x)}\right)' \geq \left(\frac{1}{t'(u)}\right)'. \tag{4.9}$$

Since a can be chosen arbitrarily, (4.9) holds for all $u,x \in]0,1[$ with $u < x$, i.e., the derivative of the function $\frac{1}{t'}$ is nondecreasing implying that $\frac{1}{t'}$ is convex. □

Remark 4.3.2 (i) The requirement that the additive generator t of C be two times differentiable cannot be dropped in Theorem 4.3.1: define $t\colon [0,1] \to [0,\infty]$ by

$$t(x) = \begin{cases} -\log(\frac{8}{9}x) & \text{if } x \in \left[0, \frac{1}{2}\right], \\ -2\log\frac{2x+1}{3} & \text{otherwise.} \end{cases}$$

Then $\frac{1}{t'}$ is given by $\frac{1}{t'(x)} = \max(-x, -\frac{2x+1}{4})$, and it is convex, but the horizontal section at $\frac{3}{4}$ of the Archimedean copula C generated by t is not convex, i.e., C is not ultramodular.

(ii) Theorem 4.3.1 gives also a hint how to construct ultramodular Archimedean copulas: if $g\colon [0,1] \to [0,\infty]$ is a differentiable, convex, non-increasing function with $g(1) < 0$ then $t\colon [0,1] \to [0,\infty]$ given by

$$t(x) = -\int_x^1 \frac{1}{g(u)}\,du \tag{4.10}$$

is an additive generator of an ultramodular Archimedean copula.

Example 4.3.3 For each $\lambda \in]0,1[$, the function $g_\lambda\colon [0,1] \to [0,\infty]$ given by $g_\lambda(x) = -x^{1-\lambda}$ is differentiable, convex, non-increasing and satisfies $g_\lambda(1) < 0$. Then (4.10) yields the additive generator $t_\lambda\colon [0,1] \to [0,\infty]$ given by $t_\lambda(x) = \frac{1-x^\lambda}{\lambda}$, and the corresponding copula C_λ is given by

$$C_\lambda(x,y) = \left(\max(x^\lambda + y^\lambda - 1, 0)\right)^{\frac{1}{\lambda}}.$$

Note that the limit cases for λ going to 0 and 1 are Π and W, respectively, and that $(C_\lambda)_{\lambda \in]0,1]}$ is the family of non-strict Clayton copulas (see [82]).

As already mentioned in Section 3.2.2, some constructions involving copulas preserve the convexity of their horizontal and vertical sections. First of all, for each ultramodular copula C, also the corresponding survival copula $\widehat{C}$ (see [82]) given by $\widehat{C}(x,y) = x+y-1+C(1-x,1-y)$ is ultramodular. Also, the W-ordinal sum (see [40,54]) of ultramodular copulas is ultramodular.

Example 4.3.4 (i) For $\lambda = 0.5$, the survival copula $\widehat{C}_{0.5}$ of the corresponding Clayton copula is given by

$$\widehat{C}_{0.5}(x,y) = x+y-1+\left(\max(\sqrt{1-x}+\sqrt{1-y}-1, 0)\right)^2,$$

it is ultramodular (but not associative).

(ii) The copula $C = W\text{-}(\langle 0, 0.5, \Pi \rangle)$ is given by

$$C(x,y) = \begin{cases} x \cdot (2y-1) & \text{if } (x,y) \in [0,0.5] \times [0.5,1], \\ W(x,y) & \text{otherwise;} \end{cases}$$

it is also ultramodular (but not associative).

Two types of flipping [29, 82] of a copula C, leading to C^- and C_- given by (4.5) and (4.6) turn convex sections into concave ones, and viceversa. Thus, starting from a copula C with concave horizontal and vertical sections, the flipped copulas C^- and C_- are ultramodular. Concerning copulas with concave sections we recall the following result [55, Theorem 3]:

Theorem 4.3.5 *Let C be an Archimedean copula with additive generator t, let t' be the left derivative of t on $]0,1]$ and $t'(0) = t'(0^+)$. Then all the one-dimensional sections of C are concave if and only if $t'(0) = \infty$, t' is finite on $]0,1]$, and $\frac{1}{t'}$ is concave.*

Example 4.3.6 (i) Similarly as in Example 4.3.3, for each $\lambda \in]0,\infty[$, consider the concave function $g_\lambda \colon [0,1] \to [0,\infty]$ given by $g_\lambda(x) = -x^{\lambda-1}$. It is related via $g_\lambda = \frac{1}{t'_\lambda}$ to the additive generator $t_\lambda \colon [0,1] \to [0,\infty]$ given by $t_\lambda(x) = \frac{x^{-\lambda}-1}{\lambda}$, and the corresponding copula C_λ is given by

$$C_\lambda(x,y) = \left(\max(x^{-\lambda} + y^{-\lambda} - 1, 0)\right)^{-\frac{1}{\lambda}},$$

which is exactly a strict Clayton copula (see [82]). Then both $(C_\lambda)^-$ and $(C_\lambda)_-$ are ultramodular (and not associatve):

$$(C_\lambda)^-(x,y) = x - \left(x^{-\lambda} + (1-y)^{-\lambda} - 1\right)^{-\frac{1}{\lambda}},$$
$$(C_\lambda)_-(x,y) = y - \left((1-x)^{-\lambda} + y^{-\lambda} - 1\right)^{-\frac{1}{\lambda}};$$

observe that $(C_\lambda)_-(x,y) = (C_\lambda)^-(y,x)$.

(ii) The only Archimedean copulas with the property that also the flipped copulas given by (4.5) and (4.6) are Archimedean are the Frank copulas $(F_\lambda)_{\lambda \in]0,\infty[}$ given by

$$F_\lambda(x,y) = \begin{cases} x \cdot y & \text{if } \lambda = 1, \\ \log_\lambda \left(1 + \frac{(\lambda^x - 1)\cdot(\lambda^y - 1)}{\lambda - 1}\right) & \text{otherwise.} \end{cases}$$

Observe that $(F_\lambda)^- = (F_\lambda)_- = F_{\frac{1}{\lambda}}$, and that F_λ is ultramodular if and only if $\lambda \in [1,\infty[$.

Recently, an interesting class of copulas which are invariant under univariate conditioning was introduced in [36], compare also [74]. For each continuous, convex and strictly decreasing function $f \colon [0,1] \to [0,\infty]$ with $f(1) = 0$ (i.e., for each additive generator of an Archimedean copula, see (4.7)) define the function $C_f \colon [0,1]^2 \to [0,1]$ by

$$C_f(x,y) = \begin{cases} 0 & \text{if } x = 0, \\ x \cdot f^{-1}\left(\min\left(\frac{f(y)}{x}, f(0)\right)\right) & \text{otherwise.} \end{cases} \tag{4.11}$$

It was shown in [36, Proposition 3.1] that C_f is a copula, and f was called a *horizontal generator* of C_f. In full analogy, we can use f as a *vertical generator* to construct the copula

$C^f\colon [0,1]^2 \to [0,1]$ via

$$C^f(x,y) = \begin{cases} 0 & \text{if } y=0, \\ y \cdot f^{-1}\big(\min\big(\frac{f(x)}{y}, f(0)\big)\big) & \text{otherwise.} \end{cases} \tag{4.12}$$

Note also that we have $C_{c\cdot f} = C_f$ and $C^{c\cdot f} = C^f$ for each positive constant c. Using similar reasoning as in the proof of Theorem 4.3.1 we obtain the following result:

Theorem 4.3.7 *Let $f\colon [0,1] \to [0,\infty]$ be a two times differentiable horizontal or vertical generator. If $\frac{1}{f'}$ is a convex function then C_f and C^f are ultramodular.*

Because of the similarity between the Theorems 4.3.1 and 4.3.7 we can give some examples based on the same functions as those in Example 4.3.3.

Example 4.3.8 For each $\lambda \in\,]0,1[$, define the function $f_\lambda\colon [0,1] \to [0,\infty]$ by $f_\lambda(x) = 1 - x^\lambda$. Then all the requirements of Theorem 4.3.7 are fulfilled and, therefore, the (asymmetric) copulas C_{f_λ} and C^{f_λ} given by

$$C_{f_\lambda}(x,y) = \max\big(x^{\lambda-1}\cdot y^\lambda + x^\lambda - x^{\lambda-1}, 0\big)^{\frac{1}{\lambda}},$$
$$C^{f_\lambda}(x,y) = \max\big(x^{\lambda}\cdot y^{\lambda-1} + y^\lambda - y^{\lambda-1}, 0\big)^{\frac{1}{\lambda}},$$

are ultramodular.
Observe that $C_{f_1} = C^{f_1} = W$ and that, taking into account $f_0 = \lim_{\lambda\searrow 0} f_\lambda = -\log$, we have $C_{f_0} = \lim_{\lambda\searrow 0} C_{f_\lambda}$, where C_{f_0} is given by

$$C_{f_0}(x,y) = \begin{cases} 0 & \text{if } x=0, \\ x\cdot y^{\frac{1}{x}} & \text{otherwise.} \end{cases}$$

However, Theorem 4.3.7 provides only a sufficient condition for the ultramodularity of copulas (in contrast to Theorem 4.3.1 where a necessary and sufficient condition is given): indeed, if $f\colon [0,1] \to [0,1]$ is given by $f(x) = \frac{1}{x} - 1$, then the copulas C_f and C^f, given by

$$C_f(x,y) = \frac{x^2 y}{1-y+xy}, \qquad C^f(x,y) = \frac{xy^2}{1-x+xy},$$

are both ultramodular, but $\frac{1}{f'}$ is not convex (in fact, it is concave).

4.3.1 Constructing copulas by means of ultramodular aggregation functions

Theorem 3.2.10 has important implications for the construction of bivariate copulas.

Theorem 4.3.9 *Let $A\colon [0,1]^n \to [0,1]$ be a continuous ultramodular aggregation function, $C_1,\ldots,C_n\colon [0,1]^2 \to [0,1]$ copulas and the functions $f_1,\ldots,f_n, g_1,\ldots,g_n\colon [0,1] \to [0,1]$ satisfy*

$f_i(1) = g_i(1) = 1$, *for each* $i \in \{1,\ldots,n\}$ *with* $A(f_1(0),\ldots,f_n(0)) = A(g_1(0),\ldots,g_n(0)) = 0$. *Define* $\xi,\eta\colon [0,1] \to [0,1]$ *by*

$$\xi(x) = \sup\{u \in [0,1] \mid A(f_1(u),\ldots,f_n(u)) \le x\},$$
$$\eta(x) = \sup\{u \in [0,1] \mid A(g_1(u),\ldots,g_n(u)) \le x\}.$$

Then the function $C\colon [0,1]^2 \to [0,1]$ *given by*

$$C(x,y) = A\big(C_1(f_1\circ\xi(x), g_1\circ\eta(x)),\ldots,C_n(f_n\circ\xi(x), g_n\circ\eta(x))\big) \tag{4.13}$$

is a copula.

Proof: Defining $f,g\colon [0,1]\to[0,1]$ by $f(x) = A(f_1(x),\ldots,f_n(x))$ and $g(x) = A(g_1(x),\ldots,g_n(x))$ it is clear that f and g are surjective nondecreasing functions (and, therefore, continuous), while ξ and η are the (upper) pseudo-inverses of f and g (see [58, 92]). Consequently, proving that 1 is the neutral element of C, $f\circ\xi = g\circ\eta = \mathrm{id}_{[0,1]}$. Because of Theorem 3.2.10, C is also ultramodular and, therefore, a copula. □

Theorem 4.3.9 can be used to construct non-symmetric copulas from symmetric ones, some of which have already been considered in the literature.

Example 4.3.10 (i) If we put $n = 2$, $A = C_1 = \Pi$ and define, for $\alpha,\beta \in [0,1]$, the functions f_1, f_2, g_1, g_2 by $f_1(x) = x^{1-\alpha}$, $f_2(x) = x^{\alpha}$, $g_1(x) = x^{1-\beta}$, and $g_2(x) = x^{\beta}$, then for each copula C_2 the construction in (4.13) yields the copula $C_{\alpha,\beta}$ given by

$$C_{\alpha,\beta} = x^{1-\alpha}\cdot y^{1-\beta}\cdot C_2(x^{\alpha}, y^{\beta}). \tag{4.14}$$

Note that $C_{\alpha,\beta}$ was shown to be a copula in [50], and that a generalization of (4.14) based on the n-ary product Π was given in [64].

(ii) If we put $n = 2$, $A = W$, $C_1 = C_2 = M$ and define the functions f_1, f_2, g_1, g_2 by $f_1(x) = g_2(x) = \frac{x+2}{3}$ and $f_2(x) = g_1(x) = \frac{2x+1}{3}$, then the construction in (4.13) yields the copula C given by

$$C(x,y) = \frac{1}{3}\cdot\max(\min(x+1, 2y) + \min(2x, y+1) - 1, 0).$$

(iii) If C_1 and C_2 are arbitrary copulas and $f,g\colon [0,1]\to[0,1]$ are continuous nondecreasing functions with $f(1) = g(1) = 1$ then $A\colon [0,1]^2 \to [0,1]$ given by

$$A(x,y) = W(C_1(f(x),g(y)), C_2(x+1-f(x), y+1-g(y)))$$

is a copula. If, e.g., $C_1 = C_2 = \Pi$ and $f(x) = \frac{1+4x}{5}$ and $g(x) = \frac{2+x}{3}$ then $A(x,y) = \max\big(\frac{(2x+3)(2y+3)}{10}, 0\big)$. Observe that A is an Archimedean copula, its additive generator t being given by $t(x) = -\log\frac{2x+3}{5}$. In fact, each Archimedean copula whose additive generator t, for some $c \in]0,1]$, is given by $t(x) = -\log(c \cdot x + 1 - c)$ can be obtained in this way, and all these copulas are ultramodular.

(iv) Let $A\colon [0,1]^n \to [0,1]$ be the n-ary extension of a binary ultramodular Archimedean copula with additive generator $t\colon [0,1] \to [0,\infty]$, i.e.,

$$A(\mathbf{x}) = t^{-1}\Big(\min\Big(t(0), \sum_{i=1}^{n} t(x_i)\Big)\Big).$$

Assume that $C_1 = \cdots = C_n = D$, D being an Archimedean copula with additive generator $\psi\colon [0,1] \to [0,\infty]$, and $f_1 = \cdots = f_n = g_1 = \cdots = g_n = \mathrm{id}_{[0,1]}$. Then the construction in (4.13) yields an Archimedean copula C whose additive generator $\rho\colon [0,1] \to [0,\infty]$ is given by $\rho(x) = \psi\big(t^{-1}\big(\frac{t(x)}{n}\big)\big)$. If the additive generator t is two times differentiable (implying that $\frac{1}{t'}$ is convex because of Theorem 4.3.1) then it is not difficult to check that the function $\zeta\colon [0,1] \to [0,1]$ given by $\zeta(x) = t^{-1}\big(\frac{t(x)}{n}\big)$ is concave, strictly increasing and satisfies $\zeta(1) = 1$. Therefore, C is a ζ-transform of D, i.e.,

$$C(x,y) = \zeta^{-1}(\max(\zeta(0), D(\zeta(x), \zeta(y))));$$

note that the concavity of ζ is sufficient to show that C is a copula [60].

4.4 Extensions of 2–increasing agops

In the literature, there is a variety of construction methods for aggregation functions (see [22] and the references therein).
Some of these methods are used to obtain a supermodular aggregation function. An *ordinal sum* for n–ary aggregation functions is a function defined in the following way:

$$D : [0,1]^n \to [0,1]$$

is a symmetric continuous aggregation function which is strictly monotone (cancellative) on $]0,1[^n$. Then the D-ordinal sum $A^D : [0,1]^n \to [0,1]$ of idempotent aggregation functions $A_i : [a_{i-1}, a_i]^n \to [a_{i-1}, a_i]$, $i = 1, \ldots, k$, $0 = a_0 < a_1 < \ldots < a_k = 1$, is given as a (unique) solution of the equation

$$D(A^D(x_1, \ldots, x_n), a_1, \ldots, a_{k-1}) = D(A_1(x_1^{(1)}, \ldots, x_n^{(1)}), \ldots, A_k(x_1^{(k)}, \ldots, x_n^{(k)})).$$

Some results have been recently obtained on copulas in [76].
Let an agop A be defined in the following way:

$$A(\mathbf{x}) := \sum_{i=1}^{k} A_i(\mathbf{x}^{(i)}) - \sum_{i=0}^{k-1} a_i, \tag{4.15}$$

where $A_i : [a_{i-1}, a_i]^n \to [a_{i-1}, a_i]$, $i = 1, \ldots, k$, $x^{(i)} = \min(a_i, \max(a_{i-1}, x))$, $\cup_{i=1}^{k} [a_{i-1}, a_i] = [0,1]$, with $a_0 = 0$ and $a_k = 1$. So, we have the following results.

Proposition 4.4.1 *Let A be a bivariate aggregation function defined by* (4.15)*. Then A is 2-increasing if and only if every A_i is 2-increasing.*

Proof: If A is 2-increasing $\forall x_1, x_2, y_1, y_2 \in [0,1]$, with $x_1 \leq x_2$ and $y_1 \leq y_2$, we have

$$V_A([x_1,x_2]\times[y_1,y_2]) := A(x_1,y_1)+A(x_2,y_2)-A(x_1,y_2)-A(x_2,y_1) \geq 0$$

In particular by taking $x_1^{(i)}, x_2^{(i)}, y_1^{(i)}, y_2^{(i)} \in [a_{i-1}, a_i]$, we have

$$V_A\left([x_1^{(i)},x_2^{(i)}]\times[y_1^{(i)},y_2^{(i)}]\right) \geq 0,$$

that is A_i is 2-increasing.
Viceversa, if A_i is 2-increasing, we have $A(x_1^{(i)},y_1^{(i)})+A(x_2^{(i)},y_2^{(i)})-A(x_1^{(i)},y_2^{(i)})-A(x_2^{(i)},y_1^{(i)}) \geq 0$. So,

$$\sum_{i=1}^{k} A(x_1^{(i)},y_1^{(i)})+A(x_2^{(i)},y_2^{(i)})-A(x_1^{(i)},y_2^{(i)})-A(x_2^{(i)},y_1^{(i)}) \geq 0$$

that is A is $2-$increasing. □

Proposition 4.4.2 *Let A be a bivariate agop defined by* (4.15)*. Then A is the maximal 2-increasing agop extending $A_1,\ldots,A_n$, that is $A_{|I_i} = A_i$.*

Proof: We can prove this by induction. For $k = 2$ we have two summands with points $0, a, 1$ on the intervals $I_1 = [0,a]$ and $I_2 = [a,1]$ and so $A_{|I_1} = A_1$ and $A_{|I_2} = A_2$. If $x \leq a \leq y$ then $A(x,y) = A(x,a)+A(a,y)-a$; however, due to the nonnegativity of volume of box $[x,a]\times[a,y]$ of any 2-increasing agop B extending A_1 and A_2, it follows that $B(a,y)+B(x,a) = A_2(a,y)+A_1(x,a) = A(a,y)+A(x,a) \geq B(x,y)+B(a,a) = B(x,y)+a$ and thus $B(x,y) \leq A(a,y)+A(x,a)-A(a,a) = A(x,y)$. Now we suppose this maximal extension for $k-1$ summands is the strongest one and we want to verify the same for k summands. So we suppose $A(x,y) = \sum_{i=1}^{k-1} A_i(x,y) - \sum_{i=0}^{k-2} a_i$. Thus $A(x,y) = \sum_{i=1}^{k} A_i(x,y) - \sum_{i=0}^{k-1} a_i = \sum_{i=1}^{k-1} A_i(x,y) + A_k(x,y) - \sum_{i=0}^{k-2} a_i - a_{k-1} \leq \sum_{i=1}^{k-1} B_i(x,y) + B_k(x,y) - \sum_{i=0}^{k-2} a_i - a_{k-1} \leq \sum_{i=1}^{k} B_i(x,y) - \sum_{i=0}^{k-1} a_i = B(x,y)$. We have applied 2-summands extension to the last summand A_k on $[a_{k-1}, a_k]$ and any extension B of the first $k-1$ summands on $[0, a_{k-1}]$. □

Proposition 4.4.3 *Let A be defined by* (4.15)*. Then A is maximal 2-increasing agop extending $A_1,\ldots,A_n$, that is $A_{|I_i} = A_i$ in the multivariate case as well.*

Proof: We recall that in the multivariate case A is 2-increasing if the functions obtained by fixing $n-2$ variables are 2-increasing. For example in the trivariate case we need that $A(x_1,x_2,b_3)$, $A(b_1,x_2,x_3)$ and $A(x_1,b_2,x_3)$ are 2-increasing.
In general $A : [0,1]^n \to [0,1]$ is 2-increasing if for any couple of integers α, β such that $1 \leq \alpha <$

$\beta \leq n$ and all $x_i \in [0,1]$ with $i \in \{1,\dots,n\} \setminus \{\alpha,\beta\}$, the function $\overline{A} : [0,1]_\alpha \times [0,1]_\beta \to [0,1]$, given by

$$\overline{A}(x_\alpha, x_\beta) = A(a_1,\dots,a_{\alpha-1},x_\alpha,a_{\alpha+1},\dots,a_{\beta-1},x_\beta,a_{\beta+1},\dots,a_n)$$

is 2-increasing. The previous proposition says that they extend $A_1,\dots,A_n$ and they are the maximal. So, we can conclude that A is the maximal. □

Proposition 4.4.4 *Let A be a bivariate aggregation function defined by (4.15). Then, for strictly increasing continuous concave (respectively decreasing, convex) functions $f : [0,1] \to [-\infty,\infty]$, we have that A_f is the 2-increasing agop extension, where*

$$A_f : [0,1]^2 \to [0,1] \qquad \textit{and}$$

$$A_f(\mathbf{x}) := f^{-1}\left(\sum_{i=1}^{k} f(A_i(\mathbf{x}^{(i)}) - \sum_{i=0}^{k-1} f(a_i)\right) \tag{4.16}$$

Proof: Let $t_4^{(i)} = A_i(x_2^{(i)}, y_2^{(i)})$, $t_3^{(i)} = A_i(x_1^{(i)}, y_2^{(i)})$, $t_2^{(i)} = A_i(x_2^{(i)}, y_1^{(i)})$ and $t_1^{(i)} = A_i(x_1^{(i)}, y_1^{(i)})$. We recall that A_i is 2-increasing. Thus, for all f strictly monotone $f \circ A_i$ is 2-increasing again (corollary 3.2.6) and we have

$$f(t_4^{(i)}) - f(t_3^{(i)}) - f(t_2^{(i)}) + f(t_1^{(i)}) \geq 0.$$

So,

$$\sum_{i=1}^{k} f(t_4^{(i)}) - f(t_3^{(i)}) - f(t_2^{(i)}) + f(t_1^{(i)}) \geq 0.$$

Thus, applying lemma 2.5 in [78] and using that f^{-1} is convex and A_i is increasing in each variable ($t_1 \leq t_2 \leq t_4$ and $t_1 \leq t_3 \leq t_4$), we have $f^{-1}(\sum_{i=1}^{k} f(t_4^{(i)}) - f(t_3^{(i)}) - f(t_2^{(i)}) + f(t_1^{(i)})) \geq 0$, that is our thesis. □

Note that $x^{(i)} = \min(a_i, \max(a_{i-1}, x))$ is a point from $[a_i, b_i]$ closest to x.
So, e.g., if $f(x) = id(x) = x$, we get

$$A^{(id)}(\mathbf{x}) := \left(\sum_{i=1}^{k} A_i(\mathbf{x}^{(i)}) - \sum_{i=0}^{k-1} a_i\right),$$

that is (4.15).

Remark 4.4.5 *The theorem 5 in [24] says that there need not exists neither an upper nor a lower bound of M_f in the class of quasi-arithmetic means. Similarly the extension in the previous proposition is not maximal neither minimal.*

Remark 4.4.6 *Thanks to Proposition 1.2.2, we can conclude that aggregation functions defined by (4.15) and (4.16) are supermodular ordinal sum if and only if every A_i is 2-increasing.*

Another way to construct a 2-increasing agop is the following one: let us consider F, $G : [0,1]^d \to [0,1]$ and $H : [0,1]^2 \to [0,1]$, like in the construction introduced by [25]. An aggregation function $\bar{A} \in \mathcal{A}^S_{2d}$ is a function $\bar{A} : [0,1]^d \times [0,1]^d \to [0,1]$, such that

(A1) $\bar{A}(0,\ldots,0) = 0$ and $\bar{A}(1,\ldots,1) = 1$;

(A2) $\bar{A}(\mathbf{x}_{(d)}, \mathbf{y}_{(d)})$ is 2-increasing.

In order to discuss 2-increasingness in the case of double aggregation operators, we use the Cartesian partial ordering defined in the section 2.1 in [25].
So, let $\mathbf{z} = (\mathbf{x}_{(n)}, \mathbf{y}_{(m)})$ and $\mathbf{z}' = (\mathbf{x}'_{(r)}, \mathbf{y}'_{(s)})$ be elements of $[0,1]^d \times [0,1]^d$. The relation $\leq_{2\pi}$ is defined as follows: $\mathbf{z} \leq_{2\pi} \mathbf{z}'$ if and only if $\mathbf{x}_{(n)} \leq_\pi \mathbf{x}'_{(r)}$ and $\mathbf{y}_{(m)} \leq_\pi \mathbf{y}'_{(s)}$. This implies that $n = r$, $m = s$, $x_1 \leq x'_1, \ldots, x_n \leq x'_n$ and $y_1 \leq y'_1, \ldots, y_m \leq y'_m$.
So, in our case, $\bar{A}$ is 2-increasing if

$$\bar{A}(\mathbf{x}_{(d)}, \mathbf{y}_{(d)}) + \bar{A}(\mathbf{x}'_{(d)}, \mathbf{y}'_{(d)}) \geq \bar{A}(\mathbf{x}_{(d)}, \mathbf{y}'_{(d)}) + \bar{A}(\mathbf{x}'_{(d)}, \mathbf{y}_{(d)}), \tag{4.17}$$

$\forall\, \mathbf{x}_{(d)} \leq_\pi \mathbf{x}'_{(d)}$ and $\mathbf{y}_{(d)} \leq_\pi \mathbf{y}'_{(d)}$, $d \in \mathbb{N}$.
Like in the bivariate case, the inequality (4.17) is equivalent to the fact that both the functions $\mathbf{t}_{(d)} \to \bar{A}(\mathbf{x}_{(d)}, \mathbf{t}_{(d)}) - \bar{A}(\mathbf{x}'_{(d)}, \mathbf{t}_{(d)})$ and $\mathbf{s}_{(d)} \to \bar{A}(\mathbf{s}_{(d)}, \mathbf{y}_{(d)}) - \bar{A}(\mathbf{s}_{(d)}, \mathbf{y}'_{(d)})$ are increasing for all $\mathbf{x}_{(d)} \leq \mathbf{x}'_{(d)}$ and for $\mathbf{y}_{(d)} \leq \mathbf{y}'_{(d)}$, respectively.

Remark 4.4.7 *In general $\mathcal{A}^S_2 = \mathcal{A}^S_n$, but in particular we have the following relation between $\mathcal{A}^S_{2d}$ and $\mathcal{A}^S_n$, for any d, such that $2d = n$.*

Proposition 4.4.8 *$\mathcal{A}^S_{2d} \subset \mathcal{A}^S_n$, for any d, such that $2d = n$.*

Proof: This result is a consequence of the general one given by [85], because supermodular aggregation functions are a particular class of that one considered in the definition 2.4 [85]. Moreover the function defined in the counterexample of that proof is an aggregation function, because it satisfies (A1) and (A2). □

Now we want to characterize the concept of 2-increasingness of double aggregation functions in terms of H.

Theorem 4.4.9 *Let $\bar{A} = A_{F,G,H}$ be a double aggregation function and $\leq_{2\pi}$ the ordering relation introduced previously. Then $\bar{A} \in \mathcal{A}^S_{2d}$ with respect to $\leq_{2\pi}$ if and only if H is 2-increasing.*

Proof: F and G are increasing, because they are aggregation functions and so $\bar{A} = A_{F,G,H}$ is 2-increasing if and only if the requirement (4.17) is satisfied, that is

$$\begin{aligned} H(F(\mathbf{x}_{(d)}), G(\mathbf{y}_{(d)})) + H(F(\mathbf{x}'_{(d)}), G(\mathbf{y}'_{(d)})) \geq H(F(\mathbf{x}_{(d)}), G(\mathbf{y}'_{(d)})) + \\ + H(F(\mathbf{x}'_{(d)}), G(\mathbf{y}_{(d)})), \end{aligned}$$

$\forall\, F(\mathbf{x}_{(d)}) \leq F(\mathbf{x}'_{(d)})$ and $G(\mathbf{y}_{(d)}) \leq G(\mathbf{y}'_{(d)})$. So $\bar{A}$ is 2-increasing if and only if H is 2-increasing. □

Remark 4.4.10 *If $\bar{A} \in \mathcal{A}_{2d}^{S}$ and if it has a neutral element, then, thanks to the Proposition* (3.1.1)*, the neutral element is equal to* 1*. So, if H is* 2*-increasing and it has the neutral element, then it has uniform one-dimensional marginals and H is a copula.*

Chapter 5

Triangular norms and Aggregation Evaluators

In the Introduction, we presented ultramodular and supermodular properties on a generic sublattice $L \subset \mathbb{R}^n$. But on an arbitrary bounded lattice we can introduce also the general definition of an evaluator and combine this concept with supermodularity and ultramodularity. Moreover, we will see also interesting connections between evaluators and triangular norms and conorms.

5.1 Triangular norms

A triangular norm (briefly t-norm) T is defined to be a two-place function

$$T : [0,1] \times [0,1] \to [0,1]$$

fulfilling the following properties:

$T(1,y) = y$, for each $y \in [0,1]$	*Boundary Condition*
$T(x,y_1) \leq T(x,y_2)$, for all $x,y_1,y_2 \in [0,1]$, if $y_1 \leq y_2$	*Monotonicity*
$T(x,y) = T(y,x)$, for all $x,y \in [0,1]$	*Commutativity*
$T(x,T(y,z)) = T(T(x,y),z)$, for all $x,y,z \in [0,1]$.	*Associativity*

Note that a t-norm defines a semigroup on $[0,1]$ with unit 1 and annihilator 0 and where the semigroup operation is order-preserving and commutative.
Given a t-norm T, the two-place function

$$S : [0,1] \times [0,1] \to [0,1],$$

defined by

$$S(x,y) = 1 - T(1-x, 1-y)$$

is called a t-conorm (or the dual of T). Obviously S fulfills monotonicity, commutativity, associativity and

$$S(x,0) = x \qquad\qquad Boundary \quad Condition$$

Here we deal with the Frank family of t-norms T_s, $s \in [0,\infty]$.
For each $s \in [0,\infty]$, the *Frank t-norms* are defined by the formulas

- the minimum t-norm, $T_M(x,y) := \min\{x,y\}$,
- the product t-norm, $T_P(x,y) := x \cdot y$,
- the Łukasiewicz t-norm $T_L(x,y) := \max\{0, x+y-1\}$,
- if $s \in (0,\infty) \setminus \{1\}$,

$$T_s(x,y) := \log_s \left[1 + \frac{(s^x-1)(s^y-1)}{s-1}\right] \tag{5.1}$$

The basic t-conorms (dual of four basic t-norms) are:

- the maximum t-conorm, $S_M(x,y) := \max\{x,y\}$,
- the probabilistic sum, $S_P(x,y) := x + y - x \cdot y$,
- the Łukasiewicz t-conorm $S_L(x,y) := \min\{1, x+y\}$,
- if $s \in (0,\infty) \setminus \{1\}$, $\quad S_s(x,y) := 1 - \log_s \left[1 + \frac{(s^{1-x}-1)(s^{1-y}-1)}{s-1}\right]$.

Moreover, we have also the following kinds:

- the drastic product, $T_D(x,y) = \begin{cases} 0, & \text{if} \quad \max\{x,y\} < 1, \\ \min\{x,y\}, & \text{if} \quad \max\{x,y\} = 1, \end{cases}$
- the drastic sum, $S_D(x,y) = \begin{cases} 1, & \text{if} \quad \min\{x,y\} > 0, \\ \max\{x,y\}, & \text{if} \quad \min\{x,y\} = 0, \end{cases}$
- if $\gamma \geq 0$, $\quad T^{\gamma}(x,y) = \dfrac{x \cdot y}{\gamma + (1-\gamma)(x+y-x\cdot y)}$,
- if $\gamma \geq 0$, $\quad S^{\gamma}(x,y) = \dfrac{x+y+(\gamma-2)x\cdot y}{1+(\gamma-1)\cdot x \cdot y}$.

The family of t-norms $\{T^{\gamma} | \gamma \geq 0\}$ was studied by Hamacher [47], while the family $\{T_s | s \in [0,\infty]\}$ appeared first in Frank's [43] investigation of the functional equation

$$x + y = T(x,y) + S(x,y), \quad \forall x,y \in [0,1], \tag{5.2}$$

where T is a triangular norm and S is an associative function on the unit square. Note that the only strict solutions of (5.2) are just t-norms T_s for $s \in [0,\infty]$ (and $T_0 = T_M$ with $T_\infty = T_L$ are the limits of these T_s) and the corresponding S_s are just the dual t-conorms, i.e., $S_s(x,y) =$

$1-T_s(1-x,1-y)$. In particular Frank [43] showed that the t-norms T_s, $0 \le s \le \infty$, form a single family in the sense that T_M, T_P and T_L are the limits of T_s corresponding to their subscripts. We also have, for each t-norm T,

$$T_D \le T \le T_M. \tag{5.3}$$

5.1.1 Solutions of a functional inequality

If we define a function $\hat{C}$ from $[0,1]^2$ into $[0,1]$ by

$$\hat{C}(x,y) = x+y-1+C(1-x,1-y),$$

the function $\hat{C}$ is a copula and we refer to $\hat{C}$ as the *survival copula* of X and Y. Moreover the operator $\hat{}$ is involutive, i. e., $(\hat{C})\hat{} = C$. Notice that $\hat{C}$ "couples" the joint survival function to its univariate margins in a manner completely analogous to the way in which a copula connects the joint distribution function to its margins. Moreover the joint survival function $\overline{C}$ for two uniform $(0,1)$ random variables whose joint distribution function is the copula C is $\overline{C}(x,y) = 1-x-y+C(x,y) = \hat{C}(1-x,1-y)$.
Two other functions closely related to copulas and survival copulas are the dual of a copula and the co-copula [91]. The dual of a copula C is the function $\tilde{C}$ defined by $\tilde{C}(x,y) = x+y-C(x,y)$ and the co-copula is the function C^d defined by $C^d(x,y) = 1-C(1-x,1-y)$. Neither of these is a copula, but when C is the copula of a pair of random variables X and Y, the dual of the copula and the co-copula each express a probability of an event involving X and Y.
A t-norm that satisfies supermodularity condition is a copula, and in view of Lemma 1.4.2 in [1] and the paragraph immediately preceding Definition 1.3.2 in [1], an associative copula is a t-norm. Many of the important copulas and families of copulas are associative. However, there are commutative copulas that are not associative, and hence not t-norms; and there are t-norms that satisfy (2.14) but not supermodularity, and hence are not copulas. As observed in [1], the first thing to note about (5.2) is that, since S is nondecreasing, T satisfies the 1-Lipschitz condition. Hence T must be a copula, say C, and S must be $\tilde{C}$, the associated dual copula of C which is given by $\tilde{C}(x,y) = x+y-C(x,y)$.
Thus, in view of Lemma 1.4.6 in [1], our aim is to determine all solution pairs $(C,\tilde{C})$ for which C is an associative copula and the dual copula $\tilde{C}$ is also (simultaneously) associative.
A straightforward calculation yields that

$$x+y-C(x,y) = 1-C(1-x,1-y)$$

i.e., that

$$\tilde{C} = C^d$$

and since the associativity of C_α implies that C_α^d is associative, it follows that, for each $\alpha \in [0,+\infty]$, the pair (C_α, C_α^d) is a solution of (5.2). The remarkable and surprising fact is that these pairs (C_α, C_α^d) and pairs $(C,\tilde{C})$ for which C is an ordinal sum of C_α's are the only solutions. In fact this is the main subject of [43] in the following theorem:

Theorem 5.1.1 *The pair (T,S) is a solution of (5.2) if and only if*

(a) *$T = C_\alpha$, where C_α is given by (5.1) for some $\alpha \in [0, +\infty]$ and $S = C_\alpha^d$; or*

(b) *T is an ordinal sum of C_α's and $S = \tilde{T}$.*

So, Frank answered in [43] to the question when C and $\tilde{C}$ are simultaneously associative, that is if and only if C belongs to the important family of copulas that now bears his name.
Similarly to Frank in [43] we want to study the functional inequality

$$T(x,y) + S(x,y) \leq x + y, \quad \forall x, y \in [0,1] \tag{5.4}$$

Of course all the solutions of the functional equation in (5.2) are the solutions of functional inequality (5.4). Now we have to find other solutions, but we observe that this means exactly an associative copula.
In fact for theorem 5.1.1 the functional inequality (5.4) is equivalent to solve

$$C \leq \hat{C}(x,y),$$

which of course is not valid for each copula, but survival operation on copulas preserves their ordering and thus $C \leq \hat{C}$ implies $\hat{C} \leq \hat{\hat{C}} = C$, hence then necessarily $C = \hat{C}$. So in copulas the inequality is valid exactly for copulas stable under survival operation. For the previous observation the same happens to the opposite inequality, that it is for

$$T(x,y) + S(x,y) \geq x + y, \quad \forall x, y \in [0,1], \tag{5.5}$$

and inequality leads to the same results as equality.

5.1.2 Additive and multiplicative generators

It is straightforward that, given a t-norm T and a strictly increasing bijection $\varphi : [0,1] \to [0,1]$, the function $T_\varphi : [0,1]^2 \to [0,1]$ given by

$$T_\varphi(x,y) = \varphi^{-1}(T(\varphi(x), \varphi(y))) \tag{5.6}$$

is again a t-norm.
In other words, the t-norms T and T_φ are isomorphic in the sense that for all $(x,y) \in [0,1]^2$

$$\varphi(T_\varphi(x,y)) = T(\varphi(x), \varphi(y)).$$

If we want to construct t-norms as transformations of the additive semigroup $([0,\infty], +)$ and the multiplicative semigroup $([0,1], \cdot)$, respectively, monotone (but not necessarily bijective) functions are used and a generalized inverse, the pseudo-inverse [58] (see also [59], Section 3.1) is needed.
The following result ([59], Theorem 3.23) is more general in the sense that the continuity of the one-place function is not needed, that the requirement of the closedness of the range under addition can be relaxed, and that the inverse function is replaced by the pseudo-inverse. On

the other hand, it is slightly more special since we want to construct an operation on the unit interval with neutral element 1.

Theorem 5.1.2 *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function with $f(1) = 0$ such that f is right-continuous at 0 and*

$$f(x) + f(y) \in Ran(f) \cup [f(0),\infty] \tag{5.7}$$

for all $(x,y) \in [0,1]^2$. The following function $T : [0,1]^2 \to [0,1]$ is a t-norm:

$$T(x,y) = f^{(-1)}(f(x) + f(y)). \tag{5.8}$$

In theorem 5.1.2, the pseudo-inverse $f^{[-1]}$ may be replaced by any monotone function $g : [0,\infty] \to [0,1]$ with $g|_{\text{Ran}(f)} = f^{[-1]}|_{\text{Ran}(f)}$. In some very abstract settings (see, e.g., [91]), such a function g (which may be non-monotone) is called a quasi-inverse of f.
It is obvious that a multiplication of f in Theorem 5.1.2 by a positive constant does not change the resulting t-norm T.

Definition 5.1.3 *An additive generator $t : [0,1] \to [0,\infty]$ of a t-norm T is a strictly decreasing function which is right-continuous at 0 and satisfies $t(1) = 0$, such that for all $(x,y) \in [0,1]^2$ we have*

$$\begin{aligned} t(x) + t(y) &\in Ran(t) \cup [t(0),\infty] \\ T(x,y) &= t^{(-1)}(t(x) + t(y)). \end{aligned} \tag{5.9}$$

For example, starting with the function $t : [0,1] \to [0,\infty]$ given by $t(x) = 1 - x$ we get the Łukasiewicz t-norm T_L and $t(x) = -\ln x$ produces the product T_P.
Combining the continuity with some algebraic properties, we obtain two extremely important classes of t-norms.

Definition 5.1.4 **(i)** *A t-norm T is called strict if it is continuous and strictly monotone, i.e. if $0 \le a < b \le 1; 0 < c \le 1$ then $T(a,c) < T(b,c)$.*

(ii) *A t-norm T is called nilpotent if it is continuous and if each $a \in]0;1[$ is a nilpotent element of T.*

A consequence of Proposition 3.31 and 5.6 in [59] is that the product T_P and the Łukasiewicz t-norm T_L are not only prototypical examples of strict and nilpotent t-norms, respectively, but that each continuous Archimedean t-norm is isomorphic either to T_P or to T_L.

5.2 Uniform approximation of a continuous t-norm by means of generators

Since Dombi [34] it is known that the strongest t-norm T_M can be approximated by means of generated t-norms (either strict or nilpotent ones).

Theorem 5.2.1 *Let f be an additive generator of a continuous Archimedean t-norm T. For $\lambda \in]0,\infty[$, define $f_\lambda : [0,1] \to [0,\infty]$ by $f_\lambda(x) = (f(x))^\lambda$. Then also f_λ is an additive generator of a continuous Archimedean t-norm T_λ (which is strict if and only if T is strict) for any $\lambda \in]0,\infty[$. More, for all $x,y \in [0,1]$,*

$$\lim_{\lambda \to \infty} T_\lambda(x,y) = T_M(x,y) = \min(x,y).$$

Moreover in [72] the author shows that each continuous t-norm T can be approximated with an arbitrary small given accuracy by some strict t-norm.

Theorem 5.2.2 *Let T be a continuous t-norm and let $\delta \in]0,1[$ be given. Then there exist a strict t-norm $T^{<\delta>}$ which is a δ-approximation of T, i.e., for all $x,y \in [0,1]$ it is*

$$|T(x,y) - T^{<\delta>}(x,y)| < \delta.$$

5.3 Aggregation Evaluators

Let $X \neq \emptyset$ be a given at most countable set. Then by $L = [0,1]^X$ we denote the system of all functions $f : X \to [0,1]$. Hence $(L,\wedge,\vee,\top,\bot)$ is a lattice with top and bottom elements $\top$ and $\bot$, equal to constants 1 and 0, respectively. In [14], so-called evaluators have been defined on the system L. For the purpose of our work we define an evaluator as follows:

Definition 5.3.1 *A function $\phi : L \to [0,1]$ is said to be a normalized evaluator on L iff it satisfies the following properties:*

- $\phi(0) = 0$, $\phi(1) = 1$,
- $\phi(f) \leq \phi(g)$ *for all* $f,g \in L$ *such that* $f \leq g$.

In fact, evaluators can be defined for an arbitrary bounded lattice. However, we restrict our considerations to the lattice L.
An evaluator ϕ is called *existential* if for arbitrary $x \in L$,

$$\phi(x) = 0 \Rightarrow x = 0.$$

An evaluator ϕ is called *universal* if for arbitrary $x \in L$,

$$\phi(x) = 1 \Rightarrow x = 1.$$

The standard comparison of real numbers allows us to compare evaluators on the same system of objects. In this case we utilize the usual pointwise ordering of functions. This means that if ϕ_1 and ϕ_2 are two evaluators on L, we say that ϕ_1 is greater than ϕ_2, with notation $\phi_2 \leq \phi_1$ if for all $x \in L$, $\phi_2(x) \leq \phi_1(x)$. The greatest evaluator is the existential evaluator ϕ_E defined for all $x \in L$ by

$$\phi_E(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ 1 & \text{otherwise.} \end{cases}$$

The smallest evaluator is the universal evaluator ϕ_U defined for all $x \in L$ by

$$\phi_U(x) = \begin{cases} 1 & \text{if} \quad x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We consider finite distributive lattices $(L_1, \leq_1), \ldots, (L_n, \leq_n)$ and their product $L := L_1 \times \ldots \times L_n$ endowed with the product order $\leq$. Elements x of L can be written in their vector form $(x_1, \ldots, x_n)$. L is also a distributive lattice whose join-irreducible elements are of the form $(\bot_1, \ldots, \bot_{i-1}, j_i, \bot_{i+1}, \ldots, \bot_n)$, for some i and some join-irreducible element j_i of L_i. With some abuse of language, we shall also call j_i this element of L.
Lattice functions are real-valued mappings defined over product lattices of the above form. We denote by $\mathbb{R}^L$ the set of lattice functions over L.

Now we introduce the modular, supermodular and ultramodular evaluators on the general bounded lattice $\langle L, \leq, 0, 1 \rangle$.

Definition 5.3.2 *An operation $SM : L \to [0,1]$ is said to be a supermodular evaluator iff it is an evaluator and it satisfies the following property:*

$$SM(x \wedge y) + SM(x \vee y) \geq SM(x) + SM(y).$$

In the case of equality in the above equation, we have the *modular* evaluator.

Definition 5.3.3 *An operation $U : L \to [0,1]$ is said to be a ultramodular evaluator (UM evaluator for short) iff U is an evaluator satisfying the property:*

$$U(x_1) + U(x_4) \geq U(x_2) + U(x_3)$$

for all collections $\{x_1, x_2, x_3, x_4\}$ of elements in L such that $x_1 \leq x_2 \leq x_4$ and $x_1 \vee x_4 = x_2 \vee x_3$.

One of the most important consequences is that the t-norms are SM evaluators, while the t-conorms are not.
It is known that aggregation of evaluators yields an evaluator (see Proposition 1 in [13]). Now we continue the discussion about aggregation of several kinds of evaluators and we will focus on aggregation of supermodular evaluators. The following propositions are respectively the extensions of two important results which hold for aggregation functions, i.e. Proposition 3.2.3 and Corollary 3.2.4.

Let $K_1, \ldots, K_m : L \to [0,1]$, $i = 1, 2, \ldots, m$ be SM evaluators. A function $K : L \to [0,1]^m$ given by $K(x) = (K_1(x), \ldots, K_m(x))$ is said to be an SM evaluator.

Proposition 5.3.4 *If $\psi \colon [0,1]^m \to [0,1]$ is an increasing UM evaluator and $K \colon L \to [0,1]^m$ is an increasing SM evaluator, then the function $H \colon L' \to [0,1]$ given by*

$$H(x_1, \ldots, x_n) = \psi(K)(x) = \psi(K_1(x), \ldots, K_m(x))$$

is an SM evaluator.

Corollary 5.3.5 *Let A be an SM evaluator. If* $\varphi : [0,1] \to [0,1]$ *is a continuous increasing and convex function with* $\varphi(0) = 0$ *and* $\varphi(1) = 1$ *then the function*

$$A_\varphi(\mathbf{x}) := \varphi(A(x_1,\dots,x_n))$$

is an SM evaluator.

Proof: It is obvious that $A_\varphi(0) = 0$ and $A_\varphi(1) = 1$. Then, it suffices to apply the above theorem to the function $H(x_1,\dots,x_n) = \psi(K_1(\mathbf{x}))$, with $\psi = \varphi$ and $K_1 = A$. In fact scalar convex functions are ultramodular and so ψ is increasing and ultramodular. □

5.4 Aggregation of evaluators

Aggregation functions are special kinds of evaluators, because a particular bounded lattice is $[0,1]$ and as a consequence $[0,1]^n$. Let $n \in \mathbb{N}$, $n \geq 2$. We recall that an *n–ary aggregation function* is a mapping $A : \bigcup_{n\in\mathbb{N}^+} [0,1]^n \to [0,1]$ that satisfies the following properties:

(A1) $A(0,\dots,0) = 0$ and $A(1,\dots,1) = 1$;

(A2) A is increasing in each component.

Each aggregation function A can be canonically represented by a family $(A_{(n)})_{n\in\mathbb{N}}$ of n-ary operations, e.g., functions $A_{(n)} : [0,1]^n \to [0,1]$, given by

$$A_{(n)}(x_1,\dots,x_n) = A(x_1,\dots,x_n).$$

Each function $A_{(n)}$ is an evaluator on the bounded lattice $([0,1]^n, \leq, 0, 1)$. If $A(x_1,\dots,x_n) = 0$ implies that $x_i = 0$ for $i = 1,\dots,n$, we say that aggregation operator A does not have zero divisors. In this case, function $A_{(n)}$ is an existentional evaluator and A is an existentional aggregator. If $A(x_1,\dots,x_n) = 1$ implies that $x_i = 1$ for $i = 1,\dots,n$, function $A_{(n)}$ is a universal evaluator and A is a universal aggregator. Moreover, we have the following result (Proposition 6 in [14]).

Proposition 5.4.1 *Let* $\Phi = \{\phi_i\}_{i=1}^n$ *be an ordered system of evaluators on a bounded lattice* $(L, \leq, \perp, \top)$ *and let A be an aggregation function. Then the function* $A_\Phi : L \to [0,1]$ *defined for all* $a \in L$ *by*

$$A_\Phi(a) = A(\phi_1(a),\dots,\phi_n(a))$$

is an evaluator on L.

Aggregation of existential (universal) evaluators by an existential (universal) aggregation function yields an existential (universal) evaluator.
It is known that an aggregation function A dominates an aggregation function B, denoted by $A \gg B$, if for all $(x_{i1,\dots,x_{in}}) \in [0,1]^n$, $i = 1,\dots,m$, and $m,n \in \mathbb{N}$

$$A(B(x_{11},\dots,x_{1n}), B(x_{21},\dots,x_{2n}),\dots,B(x_{m1},\dots,x_{mn})) \geq$$

$$\geq B(A(x_{11},\ldots,x_{m1}),A(x_{12},\ldots,x_{m2}),\ldots,A(x_{1n},\ldots,x_{nm}))$$

5.4.1 TS-evaluators

Now we recall the following definitions introduced in [14].

Definition 5.4.2 *Consider a bounded lattice $(L,\leq,\bot,\top)$, a t-norm T and a t-conorm S. An evaluator ϕ on L is called a T-evaluator iff for all $a,b\in L$*

$$T(\phi(a),\phi(b))\leq\phi(a\wedge b),$$

and it is called an S-evaluator iff

$$S(\phi(a),\phi(b))\geq\phi(a\vee b).$$

Each T-evaluator is also a T_1-evaluator for any t-norm T_1 weaker than T. Each S-evaluator is also an S_1 evaluator for any t-conorm S_1 stronger than S.
We recall also the following propositions proved in [14].

Proposition 5.4.3 *Let T^* and S^* be mutually dual t-norm and t-conorm, respectively. Assume a De Morgan bounded lattice $(L,\leq,',\bot,\top)$. An evaluator ϕ on L is a T^*-evaluator iff $\bar{\phi}$ is an S^*-evaluator.*

Proposition 5.4.4 *Let T^* be a Frank t-norm and S^* be its dual t-conorm. Assume a De Morgan bounded lattice and complemented lattice $(L,\leq,',\bot,\top)$. An evaluator ϕ on L which is a T^* and at the same time an S^*-evaluator is a self-dual evaluator.*

Now we can define a $TS-$evaluator in the following way:

Definition 5.4.5 *Consider a bounded lattice $(L,\leq,\bot,\top)$, a t-norm T and a t-conorm S. An evaluator ϕ on L is called a TS-evaluator iff it is a T-evaluator and at the same time an S-evaluator.*

5.4.2 Triangular norms and supermodular evaluators

For a selected t-norm T^* (t-conorm S^*) from the four basic t-norms (t-conorms), we will focus on the following issues:

- when supermodular evaluators are related to T^*-evaluators (S^*-evaluators),
- when supermodular evaluators are related to other T-evaluators (S-evaluators).

Triangular norms are special types of aggregation functions. T-norms T_M,T_P,T_L and T_D are universal aggregation functions, while t-conorms S_M,S_P,S_L and S_D are existential aggregation functions. Concerning their dominance we have

$$T_M\gg T_P\gg T_L\gg T_D,\qquad S_D\gg S_L\gg S_P\gg S_M.$$

So, now it is interesting to prove the dominance between *SM*-evaluators and the minimum t-norm T_M. In fact we have the following result:

Proposition 5.4.6 *For any $n \in \mathbb{N}$, the strongest t-norm T_M dominates the class of SM-evaluators.*

Proof: We need to prove that $\forall \mathbf{x}, \mathbf{y} \in [0,1]^n$, we have that

$$T_M(SM(\mathbf{x}), SM(\mathbf{y})) \geq SM(T_M(x_1,y_1), \ldots, T_M(x_n,y_n)).$$

In fact $SM(\mathbf{y}) \geq SM(\mathbf{x} \wedge \mathbf{y})$ and also $SM(\mathbf{x}) \geq SM(\mathbf{x} \wedge \mathbf{y})$. Then,

$$T_M(SM(\mathbf{x}), SM(\mathbf{y})) = \min\{SM(\mathbf{x}), SM(\mathbf{y})\} \geq SM(\mathbf{x} \wedge \mathbf{y})$$

and so we have

$$T_M(SM(\mathbf{x}), SM(\mathbf{y})) \geq SM(\mathbf{x} \wedge \mathbf{y}) = SM(T_M(x_1,y_1), \ldots, T_M(x_n,y_n)),$$

that is our thesis: $T_M \gg SM$. □

Concerning t-conorms, we have the following result:

Proposition 5.4.7 *For any $n \in \mathbb{N}$, the class of SM-evaluators dominates the maximum t-conorm S_M.*

Proof: We need to prove that $\forall \mathbf{x}, \mathbf{y} \in [0,1]^n$, we have that

$$S_M(SM(\mathbf{x}), SM(\mathbf{y})) \leq SM(S_M(x_1,y_1), \ldots, S_M(x_n,y_n)).$$

In fact $SM(\mathbf{y}) \leq SM(\mathbf{x} \vee \mathbf{y})$ and also $SM(\mathbf{x}) \leq SM(\mathbf{x} \vee \mathbf{y})$. Then,

$$S_M(SM(\mathbf{x}), SM(\mathbf{y})) = \max\{SM(\mathbf{x}), SM(\mathbf{y})\} \leq SM(\mathbf{x} \vee \mathbf{y})$$

and so we have

$$S_M(SM(\mathbf{x}), SM(\mathbf{y})) \leq SM(\mathbf{x} \vee \mathbf{y}) = SM(S_M(x_1,y_1), \ldots, S_M(x_n,y_n)),$$

that is our thesis: $SM \gg S_M$. □

So, thanks to propositions 5.4.6 and 5.4.7 we can conclude that an *SM*-evaluator is not a T_M evaluator neither an S_M evaluator, as we can see in the following example.

Example 5.4.8 *Let $(2^X, \subseteq, ', \emptyset, X)$ be the complemented bounded lattice with intersection and union as the lattice operations, where $X = \{a,b,c\}$. Let $\phi : 2^X \to [0,1]$ be given for all $A \in 2^X$ by*

$$\phi(A) = \begin{cases} \frac{1}{4-|A|} & \text{if } \quad a \in A, \\ 0 & \text{otherwise}. \end{cases}$$

Then $\phi(X) = 1$, $\phi(a) = \frac{1}{3}$, $\phi(\{a,b\}) = \phi(\{a,c\}) = \frac{1}{2}$, *while* $\phi(c) = \phi(b) = \phi(\{c,b\}) = \phi(\emptyset) = 0$.
The authors in [14] have proved that ϕ *is a* T_P*-evaluator but not a* T_M*-evaluator neither an* S_P*-evaluator. But each* S_M*-evaluator is an* S_P*-evaluator. So, if* ϕ *is not an* S_P*-evaluator, it is not an* S_M*-evaluator either. Now we prove that* ϕ *is supermodular, that is an SM-evaluator. So, we need to prove that* $\forall A, B \subset 2^X$*, we have that*

$$\phi(A \cup B) + \phi(A \cap B) \geq \phi(A) + \phi(B).$$

In fact we have the following situation, by considering that we can exchange A with B and also b with c. Moreover the only non trivial case is $A \nsubseteq B$*, with* $a \in A$*, but* $a \notin B$ *as we can see in the following scheme.*

A	B	$A \cup B$	$A \cap B$	$\phi(A \cup B) + \phi(A \cap B) \geq \phi(A) + \phi(B)$
X	$\forall B$	A	B	$\phi(A) + \phi(B) \geq \phi(A) + \phi(B)$
$A \subseteq B$	B	B	A	$\phi(A) + \phi(B) \geq \phi(A) + \phi(B)$
$a \notin A$	$a \notin B$			$0 \geq 0$
$\{a\}$	$\{b\}$	$\{a,b\}$	$\emptyset$	$\frac{1}{2} \geq \frac{1}{3}$
$\{a\}$	$\{b,c\}$	X	$\emptyset$	$1 \geq \frac{1}{3}$
$\{a,b\}$	$\{c\}$	X	$\emptyset$	$1 \geq \frac{1}{2}$

Remark 5.4.9 *In general we cannot establish a dominance relationship in the following cases:*

1. *the dominance between SM-evaluators and the product t-norm* T_P*. In fact we cannot say that : for any* $n \in \mathbb{N}$*, the product t-norm* T_P *dominates the class of SM-evaluators, i.e. we cannot say that* $\forall \mathbf{x}, \mathbf{y} \in [0,1]^n$*, we have*

   $$T_P(SM(\mathbf{x}), SM(\mathbf{y})) \geq SM(T_P(x_1,y_1), \ldots, T_P(x_n,y_n)),$$

   or equivalently In fact for $n = 1$*, if we take* $SM(x) = 2x - x^2$ *we have that* $SM(\frac{1}{2}) = \frac{3}{4}$ *and so* $SM(\frac{1}{2})SM(\frac{1}{2}) = \frac{9}{16}$*. Then if we take* $SM(\frac{1}{2} \times \frac{1}{2}) = \frac{7}{16} \leq \frac{9}{16}$*. So in this case* $SM(x \times x) \leq SM(x) \times SM(x)$*.*
   On the other side if we take $SM(x) = 1 - \sqrt{1-x}$ *and* $x = 0.64$ *we have* $SM(0.64) = 0.4$*. So* $SM(0.64) \times SM(0.64) = 0.16$*, while* $SM(0.64 \times 0.64) = 1 - \sqrt{0.36 \times 1.64} \geq 1 - 0.6 \times 1.3 = 0.22$*. So, in this case we have the opposite inequality* $SM(x \times x) \geq SM(x) \times SM(x)$*.*

2. *the dominance between the Łukasiewicz t-norm and SM-evaluators. In fact it cannot be*

   $$T_L(SM(\mathbf{x}), SM(\mathbf{y})) \geq SM(T_L(x_1,y_1), \ldots, T_L(x_n,y_n))$$

   because, if $\max\{0, SM(\mathbf{x}) + SM(\mathbf{y}) - 1\} = 0$*, then* $SM(T_L(x_1,y_1), \ldots, T_L(x_n,y_n)) \leq 0$*,* *which violets the boundary conditions. On the other side it cannot be*

$$T_L(SM(\mathbf{x}), SM(\mathbf{y})) \leq SM(T_L(x_1, y_1), \ldots, T_L(x_n, y_n))$$

because, if
$\max\{0, SM(\mathbf{x}) + SM(\mathbf{y}) - 1\} = SM(\mathbf{x}) + SM(\mathbf{y}) - 1$*, then* $SM(\mathbf{x}) + SM(\mathbf{y}) - 1 \geq 0$*, and, at the same time, if*
$SM(T_L(x_1, y_1), \ldots, T_L(x_n, y_n)) = 0$*, then* $SM(\mathbf{x}) + SM(\mathbf{y}) - 1 \leq 0$*,*
which violets the hypothesis.

So, of course we have the following dominance relationship:

$$T_M \gg SM \gg S_M.$$

Chapter 6

T-evaluators and supermodular fuzzy measures

After combining the concept of evaluator with supermodularity and ultramodularity, we will devote this chapter to know under which conditions aggregation of supermodular evaluators yields a supermodular evaluator.
First of all, we will introduce a new kind of measures on crisp sets, called T-evaluator measure and then, by introducing fuzzy sets theory, we will study in particular under which conditions the Choquet integral of a fuzzy set with respect to a fuzzy measure is supermodular.

6.1 Choquet Integral

Now we are focusing our attention to the concept of the integral with respect to *nonadditive set functions*, known as the *Choquet integral*. This concept is useful in many fields such as *mathematical economics* (e.g., Marinacci and Montrucchio [72]) and *multicriteria decision making* (e.g., Grabisch et al. [42]), where the problem of ranking of alternatives with respect to a set of criteria is the following one. Let $X = \{1,\dots,n\}$ be the set of criteria under consideration in some decision problem. Let us consider a function $\mu : 2^X \to \mathbb{R}$ such that $\mu(\emptyset) = 0$.
When μ is monotonic, that is $\mu(A) \le \mu(B)$ whenever $A \subseteq B$, then it is called a *capacity* (Choquet, 1953) or *fuzzy measure* (Sugeno, 1974). The capacity is normalized if in addition $\mu(X) = 1$.

6.1.1 Fuzzy measures

For applications, several distinguished classes of fuzzy measures are important. We list some of them in the next definitions for the sake of selfcontainedness, though these well known properties can be found, e. g., in [31, 83, 101].

Definition 6.1.1 *Let $X = \{1,2,\dots,n\}, n \in \mathbb{N}$ be a fixed set of criteria. A mapping $\mu : 2^X \to [0,1]$ is called a fuzzy measure whenever $\mu(\emptyset) = 0$, $\mu(X) = 1$ and for all $A \subseteq B \subseteq X$, it holds $\mu(A) \le \mu(B)$.*

Distinguished classes of fuzzy measures are determined by their respective properties.

Definition 6.1.2 *A fuzzy measure μ on X is called:*

1. *additive if*
$$\forall A, B \in 2^X, \quad \mu(A \cup B) = \mu(A) + \mu(B),$$
2. *subadditive (submeasure) whenever*
$$\forall A, B \in 2^X, \quad \mu(A \cup B) \leq \mu(A) + \mu(B),$$
3. *superadditive (supermeasure) whenever*
$$\forall A, B \in 2^X, \quad A \cap B = \emptyset, \quad \mu(A \cup B) \geq \mu(A) + \mu(B),$$
4. *submodular whenever*
$$\forall A, B \in 2^X, \quad \mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B),$$
5. *supermodular whenever*
$$\forall A, B \in 2^X, \quad \mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B),$$
6. *symmetric if for any subsets A, B, $|A| = |B|$ implies $\mu(A) = \mu(B)$.*

The conjugate or dual of a capacity μ is a capacity $\overline{\mu}$ defined by
$$\overline{\mu}(A) := \mu(X) - \mu(A), \quad A \subseteq X.$$

For other kinds of measures, like belief, plausibility, possibility and necessity, there is a deep description in [99]. Observe that if a fuzzy measure is both submodular and supermodular, it is modular and thus a probability measure on X. Evidently, each supermodular fuzzy measure is also superadditive and similarly, each submodular fuzzy measure is subadditive.

Definition 6.1.3 *Let S be a t-conorm and $X = \{x_1, \ldots, x_n\}$. The fuzzy measure $\mu : 2^X \to [0,1]$ is called S-measure if for all $A, B \in 2^X$ such that $A \cap B = \emptyset$*
$$\mu(A \cup B) = S(\mu(A), \mu(B)).$$
Let T be a t-norm. The fuzzy measure $\mu^* : 2^X \to [0,1]$ is called T-measure if for all $A, B \in 2^X$ such that $A \cup B = X$
$$\mu^*(A \cap B) = T(\mu^*(A), \mu^*(B)).$$

Now we recall some results obtained in [95].

Theorem 6.1.4 *Let S be a t-conorm. Then the following are equivalent:*

i) *each S-measure μ is subadditive*

ii) $S \leq S_\infty$.

However, for superadditive measures we have only the following weaker result.

Theorem 6.1.5 *Let S be a t-conorm, such that each S-measure μ is superadditive. Then $S \geq S_\infty$.*

For submodular fuzzy measures we have the next result.

Theorem 6.1.6 *Let S be a continuous t-conorm. Then the following are equivalent:*

i) *each S-measure μ is submodular*

ii) *S is* 1*-Lipschitz t-conorm.*

Note that there are non-continuous t-conorms S such that each S-measure μ is necessarily submodular.

Proposition 6.1.7 *An S-measure $\mu : 2^X \to [0,1]$ is an S-evaluator on 2^X and a T-measure $\mu^* : 2^X \to [0,1]$ is a T-evaluator on 2^X.*

6.1.2 T-evaluator measure

Every strict (i.e., strictly increasing on $(0,1] \times (0,1]$ and continuous) t-norm T admits the representation

$$T(a,b) = t^{-1}(t(a)+t(b)) \tag{6.1}$$

for all $a, b \in I$, where the function $f : I \to [0,\infty]$ is continuous and strictly decreasing with $t(0) = +\infty$ and $t(1) = 0$, and t^{-1} is the inverse of t. In the definition 5.1.3, we called t an additive generator of T. Let T be a strict t-norm with an additive generator $t : [0,1] \to [0,\infty]$ and μ be a fuzzy measure on a σ-algebra $\mathcal{A}$ of subsets of a set X, i.e. $\mu : \mathcal{A} \to [0,1]$.
Moreover, let μ be a T-evaluator, i.e. $T(\mu(A),\mu(B)) \leq \mu(A \cap B)$, then $t^{-1}(t(\mu(A)) + t(\mu(B))) \leq \mu(A \cap B)$. Now, by posing $t \circ \mu = \nu$ we have

$$\nu(A \cap B) \leq \nu(A) + \nu(B) \tag{6.2}$$

We observe that if $A = E^c$ and $B = F^c$, then with $C = (E \cup F)^c$ we have

$$\nu((E \cup F)^c) = \nu(E^c \cap F^c) \leq \nu(E^c) + \nu(F^c)$$

Now, by considering the following measure $M : \mathcal{A} \to [0,\infty]$, such that $M(A) = \nu(A^c) = t(\mu(A^c))$, we have:

- $M(\emptyset) = 0,\ M(X) = +\infty,$ i.e. M is an infinite measure;
- if $A \subset B$ then $M(A) \leq M(B)$, i.e. M is monotone;

- $M(A \cup B) \leq M(A) + M(B)$, i.e. M is subadditive.

Finally we take the set of all monotone infinite subadditive measures $\mathcal{M} = \{M | M : \mathcal{A} \to [0, \infty]\}$ and we consider the set $\mathcal{M}_T$ of all capacities which are T-evaluators, i.e.,

$$\mathcal{M}_T = \{\mu | \mu : \mathcal{A} \to [0,1], \mu \text{ is a T-evaluator}\}.$$

Thanks to the previous observations we have that $\mu \in \mathcal{M}_T$ if and only if $M \in \mathcal{M}$ and $M(A) = t(\mu(A^c))$ (or also $\mu(A) = t^{-1}(M(A^c))$).
Let us consider two strict t-norms T_1 and T_2, we see that $\mu_1 \in \mathcal{M}_{T_1}$ if and only if $\mu_2 \in \mathcal{M}_{T_2}$, such that $t_1(\mu_1(A)) = t_2(\mu_2(A))$ (i.e. $\mu_2 = (t_2^{-1} \circ t_1) \circ \mu_1$).
Finally we want to prove the following result:

Theorem 6.1.8 *Let X be a nonempty set and μ a capacity on X. Then the following are equivalent:*

- *μ is a T_M-evaluator;*
- *μ is a T-evaluator for any strict t-norm T.*

Proof: Of course for any $T \neq T_M$ we have $\mathcal{M}_{T_M} \subset \mathcal{M}_T$ and so $\mathcal{M}_{T_M} \subset \bigcap_{\text{strict t-norms } T} \mathcal{M}_T$.
Now we want to prove the converse, i.e. $\mathcal{M}_T \subset \mathcal{M}_{T_M}$, $\forall T \neq T_M$. In fact, $\forall T \neq T_M$, and $\forall \mu \in \mathcal{M}_T$, we have that μ is a T-evaluator, i.e. $T(\mu(A), \mu(B)) \leq \mu(A \cap B)$ with A and B subsets of X.
This means that $\mu(A \cap B) = T(\mu(A \cap B), 1) = T(\mu(A \cap B), \mu(X)) \geq T(\mu(A), \mu(B))$.
As a consequence of Theorem 5.2.1, the strongest t-norm T_M can be approximated by strict t-norms, i.e., for any $\delta > 0$ there is a strict t-norm T_δ, such that $|T_M(x,y) - T_\delta(x,y)| < \delta$, for all $x, y \in [0,1]^2$.
Recall that starting from an arbitrary additive generator f of a strict t-norm T, the power $t_\lambda = t^\lambda$, $\lambda \in]0, \infty[$, is again an additive generator of a strict t-norm, which we denote T_λ (see also Theorem 5.2.1).
If μ is a T_λ-evaluator, then $T_\lambda(\mu(A), \mu(B)) \leq T_\lambda(\mu(A \cap B), 1)$.
So,

$$\lim_{\lambda \to \infty} T_\lambda(\mu(A), \mu(B)) = T_M(\mu(A), \mu(B)) \leq \lim_{\lambda \to \infty} T_\lambda(\mu(A \cap B), 1) = T_M(\mu(A \cap B), 1) = \mu(A \cap B).$$

This means that $T_M(\mu(A), \mu(B)) \leq \mu(A \cap B)$, i.e. μ is a T_M-evaluator, that is our thesis. □

6.1.3 Choquet Integrals for Nonnegative Functions

Let X be a nonempty set, $\mathcal{A}$ a σ-algebra of subsets of X and $\mu : \mathcal{A} \to [0, \infty]$ a monotone measure, such that $(X, \mathcal{A}, \mu)$ is a monotone measure space.
Also, let $A \in \mathcal{A}$ and f be a nonnegative measurable function on $(X, \mathcal{A})$. The Lebesgue integral of f with respect to μ may be not well defined due to the nonadditivity of μ.
Fortunately, there are some equivalent definitions of the Lebesgue integral that may yet be valid with respect to monotone measures. One of them is the Riemann integral, as shown in

Section 8.1 of [102]. When it is used in this way, the integral is usually referred to as a Choquet integral.

Definition 6.1.9 *The Choquet integral of a nonnegative measurable function f with respect to monotone measure μ on measurable set A , denoted by $(C)\int_A f\,d\mu$, is defined by the formula*

$$(C)\int_A f\,d\mu = (C)\int_0^\infty \mu(F_\alpha \cap A)\,d\alpha,$$

where $F_\alpha = \{x|f(x) \geq \alpha\}$ for $\alpha \in [0,\infty)$. When $A = X$, $(C)\int_X f\,d\mu$ is usually written as $(C)\int f\,d\mu$.

Since f in Definition 6.1.9 is measurable, we know that $F_\alpha = \{x|f(x) \geq \alpha\} \in \mathcal{A}$ for $\alpha \in [0,\infty)$ and, therefore, $F_\alpha \cap A \in \mathcal{A}$. So, $\mu(F_\alpha \cap A)$ is well defined for all $\alpha \in [0,\infty)$. Furthermore, F_α is a class of sets that are nonincreasing with respect to α and so are sets in $F_\alpha \cap A$. Since monotone measure μ is a nondecreasing set function, we know that $\mu(F_\alpha \cap A)$ is a nondecreasing function of α and, therefore, the above Riemann integral makes sense. Thus, the Choquet integral of a nonnegative measurable function with respect to a monotone measure on a measurable set is well defined.

6.1.4 Properties of the Choquet Integral

Unlike the Lebesgue integral, the Choquet integral is generally nonlinear with respect to its integrand due to the nonadditivity of μ. That is, we may have

$$(C)\int (f+g)\,d\mu \neq (C)\int f\,d\mu + (C)\int g\,d\mu$$

for some nonnegative measurable functions f and g.
However, the Choquet integral has some properties of the Lebesgue integral. These properties are listed in the following theorem, which can be found in Section 11.2 of [102].

Theorem 6.1.10 *Let f and g be nonnegative measurable functions on $(X,\mathcal{A},\mu)$, A and B be measurable sets and a be a nonnegative real constant. Then,*

1. *$(C)\int_A 1\,d\mu = \mu(A)$;*
2. *$(C)\int_A f\,d\mu = (C)\int f \cdot \chi_A\,d\mu$;*
3. *If $f \leq g$ on A, then $(C)\int_A f\,d\mu \leq (C)\int_A g\,d\mu$;*
4. *If $A \subset B$ then $(C)\int_A f\,d\mu \leq (C)\int_B f\,d\mu$;*
5. *$(C)\int_A a f\,d\mu = a \cdot (C)\int_A f\,d\mu$.*

For a given fuzzy measure μ, an aggregation operator can be built by means of any fuzzy integral and we restrict our attention to the Choquet integral.

6.1.5 Choquet integral-based aggregation functions

An axiomatic characterization of the Choquet integral as an aggregation operator was proposed by Marichal in [66]. In particular, the discrete Choquet integral is an adequate aggregation operator that extends the weighted arithmetic mean by taking into consideration of the interaction among criteria. Moreover, the Choquet integral identifies with the weighted arithmetic mean (discrete Lebesgue integral) as soon as the fuzzy measure is additive.

If $n \in \mathbf{N}$ and $X = \{1, \ldots, n\}$ then, for a fuzzy measure μ on X, i.e., a nondecreasing function $\mu\colon 2^X \to [0,1]$ with $\mu(\emptyset) = 0$ and $\mu(X) = 1$, and $\mathbf{x} \in [0,1]^n$ the Choquet integral [26] is given by

$$\begin{aligned}\mathrm{Ch}(\mu, \mathbf{x}) &= \int_0^1 \mu(\{x_i \geq u\})\, du \\ &= \sum_{i=1}^{n} x_{\pi(i)} \big(\mu(\{\pi(i), \ldots, \pi(n)\}) - \mu(\{\pi(i+1), \ldots, \pi(n)\})\big),\end{aligned}$$

where $\pi\colon X \to X$ is a permutation of X with $x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(n)}$ and, by convention, $\{\pi(n+1), \pi(n)\} = \emptyset$.

For a fixed fuzzy measure μ, the function $\mathrm{Ch}_\mu\colon [0,1]^n \to [0,1]$ given by $\mathrm{Ch}_\mu(\mathbf{x}) = \mathrm{Ch}(\mu, \mathbf{x})$ is an aggregation function, a so-called *Choquet integral-based aggregation function*.

Two particular cases are of interest.

- If μ is additive, then the Choquet integral reduces to a weighted arithmetic mean:

$$\mathrm{Ch}(\mu, \mathbf{x}) = \sum_{i=1}^{n} x_{\pi(i)} \mu(\pi(i)).$$

- If μ is symmetric, the Choquet integral reduces to the so-called Ordered Weighted Average (OWA) introduced by Yager (Yager, 1988):

$$\mathrm{Ch}(\mu, \mathbf{x}) = \sum_{i=1}^{n} (\mu_{n-i+1} - \mu_{n-i}) x_{\pi(i)}$$

  with $\mu_i := \mu(P)$, such that $|P| = i$, $P \subseteq X$ and π is defined as before.

It is possible to construct *symmetric* universal fuzzy measure with the help of a one-dimensional generator $g : [0,1] \to [0,1]$, a nondecreasing function with $g(0) = 0$, $g(1) = 1$, namely

$$\mu^g(P) = g\Big(\frac{|P|}{n}\Big), \quad \text{with} \quad n \in \mathbb{N}. \tag{6.3}$$

The corresponding Choquet integral based aggregation operator is given by

$$A^g(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_{\pi(i)} \Big(g\Big(\frac{i}{n}\Big) - g\Big(\frac{i-1}{n}\Big)\Big) = \sum_{i=1}^{n} w_{i,n} x_{\pi(i)},$$

i.e., it is the OWA operator with the weights

$$w_{i,n} = g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right), \tag{6.4}$$

which are non-negative and sum up to one.
An *additive* universal fuzzy measure can be generated using

$$\mu^g(P) = \sum_{i \in P}\left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right) \tag{6.5}$$

and the corresponding Choquet integral based operator is a weighted mean

$$A^g(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i\left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right) = \sum_{i=1}^{n} w_{i,n}x_i.$$

Evidently, the only universal fuzzy measure which is both symmetric and additive is linked to the identity generator $g(x) = x$, in which case A^g the arithmetic mean.
Summarizing, one can build Choquet integral based general aggregation operators with the help of a one-dimensional generator g, by defining universal fuzzy measures using (6.3) (symmetric measure) and (6.5) (additive measure). The corresponding operators are an OWA operator and a weighted mean, with the weights defined by (6.4). Following [31], we will see later in Proposition 7.1.8 that a Choquet integral operator based on a fuzzy measure μ is superadditive if, and only if, the fuzzy measure μ is supermodular.

6.2 Fuzzy sets

Throughout this paper X will denote a nonempty set and, following [102, 106], we recall the definition of a fuzzy event in $\mathbb{R}^n$, through the use of the concept of a fuzzy set.
Specifically, a fuzzy set $f \in \mathbb{R}^n$ is defined by a characteristic function $m_f : \mathbb{R}^n \to [0,1]$ which associates with each x in $\mathbb{R}^n$ its "grade of membership", $m_f(x)$ in f. To distinguish between the characteristic function of a nonfuzzy set and the characteristic function of a fuzzy set, the latter will be referred to as a *membership* function.
A standard fuzzy set is called *normalized* if

$$\sup_{x \in X} m(x) = 1$$

Since any ordinary set F can be defined by its characteristic function $\chi_F : X \to \{0,1\}$, it is a special standard fuzzy set.

Definition 6.2.1 *If $m_f(x) \le m_g(x)$ for any $x \in X$, we say that fuzzy set f is included in fuzzy set g and we write $f \subset g$. If $f \subset g$ and $g \subset f$, we say that f and g are equal, which we write as $f = g$.*

Definition 6.2.2 *Let f and g be fuzzy sets. The standard union of f and g, $f \cup g$, is defined by*

$$m_{f \cup g}(x) = m_f(x) \vee m_g(x), \qquad \forall x \in X,$$

where $\vee$ denotes the maximum operator.

Definition 6.2.3 *Let f and g be fuzzy sets. The standard intersection of f and g, $f \cap g$, is defined by*

$$m_{f \cap g}(x) = m_f(x) \wedge m_g(x), \qquad \forall x \in X,$$

where $\wedge$ denotes the minimum operator.

Similar to the way operations on ordinary sets are treated, we can generalize the standard union and the standard intersection for an arbitrary class of fuzzy sets: if $\{f_r | r \in R\}$ is a class of fuzzy sets, where R is an arbitrary index set, then $\cup_{r \in R} f_r$ is the fuzzy set having membership function $\sup_{r \in R} m_{f_r}(x)$, $x \in X$, and $\cap_{r \in R} f_r$ is the fuzzy set having membership function $\inf_{r \in R} m_{f_r}(x)$, $x \in X$.

Definition 6.2.4 *Let f be a fuzzy set. The standard complement of f, $\bar{f}$, is defined by*

$$m_{\bar{f}}(x) = 1 - m_f(x), \qquad \forall x \in X.$$

Two or more of the three basic operations can also be combined. For example, the difference $f - g$ of fuzzy sets f and g can be expressed as $f \cap \bar{g}$, so that

$$m_{f-g}(x) = \min[m_f(x), 1 - m_g(x)]$$

for all $x \in X$.

6.2.1 Operations on Fuzzy Sets

Operations on fuzzy sets are performed using triangular norms. As a natural generalization of a measure space, Butnariu and Klement introduced T-tribes of fuzzy sets with T-measures. They made the first steps towards a characterization of monotonic real-valued T-measures for a Frank triangular norm T. Later on, Mesiar, Barbieri, Navara and Weber found independently two generalizations, one for vector-valued T-measures with respect to Frank t-norms (in particular for nonmonotonic ones) [4], the other for monotonic real-valued T-measures with respect to general strict t-norms [80].

The concept of a T-measure serves not only as a fuzzification of classical measure theory; if was successfully applied in [21] to find solutions of games with fuzzy coalitions.

Being an associative operation, a t-norm T may be extended to an arbitrary finite number of elements, then we denote it by $T_{k=1}^{n} a_k$. We may use this notation also for countably many arguments,

$$T_{n \in \mathbb{N}} a_n = \lim_{n \to \infty} T_{k=1}^{n} a_k;$$

as the limit of a monotonic bounded sequence, it is well defined.
Let X be a set and $\mathscr{B}$ be a $\sigma-$algebra of subsets of X. The *$\mathscr{B}$-generated tribe* is the collection $\mathscr{I}$ of all functions $f : X \to [0,1]$ (fuzzy subsets of X) which are *$\mathscr{B}$-measurable*. In order to define measures on $\mathscr{I}$, we fix a $t-$norm T (fuzzy conjunction), i.e., a binary operation $T : [0,1]^2 \to [0,1]$ which is commutative, associative, nondecreasing and satisfies the boundary condition $T(a,1) = a$ for all $a \in [0,1]$ (see [91]). For the other necessary fuzzy logical operations, we take the *standard fuzzy negation* $' : [0,1] \to [0,1]$ defined by $a' := 1-a$, and the *t-conorm* $S : [0,1]^2 \to [0,1]$ *dual* to T, i.e., $S(a,b) := T(a',b')'$.
We extend the operations $T,'$ and S to operations T, ^{c}, S (fuzzy intersection, fuzzy complement and fuzzy union) on $\mathscr{I}$ pointwise:

$$T(f,g)(x) := T(f(x),g(x)),$$

$$f^c(x) := f(x)',$$

$$S(f,g)(x) := S(f(x),g(x)).$$

6.2.2 T-measures

Triangular norm-based measures *(T-measures)* appear under various names, and in specific analytical forms, in fields ranging from *Mathematical Statistics* to *Capacity Theory* ([43]), *Probability and Measure Theory* ([18,51–53]), *Pattern Recognition* and *Game Theory* ([19]). In [20] the authors study the triangular norm-based measures, namely T-measures defined on subsets of the unit cube $[0,1]^X$, which are triangular norm-based tribes ($T-tribes$) and they find out under which conditions T-measures can be represented as integrals of specific Markov kernels. The main result in [20] shows that any fundamental triangular norm based T-tribe $\mathscr{I}$ consists of functions, which are measurable with respect to the intrinsic σ-algebra $\mathscr{I}^\vee$ corresponding to $\mathscr{I}$ (i.e. with respect to the σ-algebra of those sets whose characteristic functions belong to $\mathscr{I}$).

T-measures were introduced in [21] as a natural generalization of $\sigma-$additive measures on $\sigma-$algebras.

Definition 6.2.5 *A function $\mu : \mathscr{I} \to [0,1]$ is a monotone T-measure if it satisfies the following axioms:*

(M1) $\mu(0) = 0$,

(M2) $T(\mu(f),\mu(g)) + S(\mu(f),\mu(g)) = \mu(f) + \mu(g)$,

(M3) $f_n \nearrow f \Rightarrow \mu(f_n) \nearrow \mu(f)$,

where the symbol $\nearrow$ denotes monotone increasing convergence. If, moreover, $\mu(1) = 1$, then μ is called a normalized T-measure.

If, moreover, $\mu(1_X) < \infty$ and $f \le g$ implies $\mu(f) \le \mu(g)$, then μ is called a finite monotone T-measure. Note that if $\mathscr{I}$ consists of crisp elements only then a T-measure μ may be considered as an ordinary $\sigma-$additive measure.

The definition of T-measures does not relate T-measures to lattice operations; this is possible for strict Frank t-norms:

Proposition 6.2.6 *[6, Lemma 5.5] Let T be a Frank t-norm. Then each T-measure μ is a modular function, i.e., it satisfies*

$$\mu(f \wedge g) + \mu(f \vee g) = \mu(f) + \mu(g)$$

for all f, g.

The extension of the operations intersection, union and complementation in ordinary set theory to fuzzy sets was always done pointwise: one considered two two-place functions $T : [0,1] \times [0,1] \to [0,1]$, $S : [0,1] \times [0,1] \to [0,1]$ and one-place function $N : [0,1] \to [0,1]$ and extended them in the usual way: if f, g are two fuzzy sets, then

$$T(f,g)(x) = T(f(x), g(x)), \tag{6.6}$$

$$S(f,g)(x) = S(f(x), g(x)), \tag{6.7}$$

$$N(f)(x) = N(f(x)). \tag{6.8}$$

In his first paper Zadeh suggested to use $T(x,y) = T_M(x,y) = \min(x,y)$ for intersection, $S(x,y) = S_M(x,y) = \max(x,y)$ for union and $N(x) = 1 - x$ for complementation.
Alsina et al. [2] and Prade [84] suggested to use a t-norm for intersection and its t-conorm for union of fuzzy sets.
For the following two lemmas proved by Klement in [53], we consider a measurable space $(X, \mathscr{A})$, i.e., a non-empty set X and a $\sigma-$algebra $\mathscr{A}$ on X. As usual, the unit interval $[0,1]$ is equipped with the σ-algebra $\mathscr{B}$ of all Borel subsets of $[0,1]$.

Lemma 6.2.7 *Let T be a $\mathcal{B}^2$-measurable t-norm, S its dual, $f, g : (X, \mathscr{A}) \to ([0,1], \mathscr{B})$ measurable functions and $(f)_{n\in\mathbb{N}}$ a sequence of measurable functions from $(X, \mathscr{A})$ into $([0,1], \mathscr{B})$. Then the following functions are also $\mathcal{B}^2$-measurable:*

$$T(f,g), \qquad S(f,g), \qquad \top_{n\in\mathbb{N}} \mu_n, \qquad \underset{n\in\mathbb{N}}{S} f_n.$$

Lemma 6.2.8 *Let T be a continuous t-norm, S its dual and $\mu : (X, \mathscr{A}) \to ([0,1], \mathscr{B})$ a measurable function. Then there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of measurable step functions from $(X, \mathscr{A})$ into $([0,1], \mathscr{B})$ such that*

$$\mu = \underset{n\in\mathbb{N}}{S} s_n.$$

6.3 TS-supermodular fuzzy measures

Now we want to study TS-supermodularity like an extension of supermodular measures. First of all, we recall that the *support* of a fuzzy set $f \in [0,1]^X$ is $\mathit{Supp} f := \{x \in X : f(x) > 0\}$.

So we introduce the following measure. A function $\mu : \mathscr{I} \to \mathbb{R}_+$ (where $\mathbb{R}_+$ denotes the set of all nonnegative real numbers) is a *monotone supermodular T-measure*, briefly called *TS-measure*, if it satisfies the following axioms:

(M1) $\mu(0) = 0$

(M2) $\mu(T(f,g)) + \mu(S(f,g)) \geq \mu(f) + \mu(g)$

(M3) $f_n \nearrow f \Rightarrow \mu(f_n) \nearrow \mu(f)$,

where the symbol $\nearrow$ denotes monotone increasing convergence. If, moreover, $\mu(1) = 1$, then μ is called a *normalized supermodular TS-measure*.

Remark 6.3.1 *If f, g are constants, also $S(f,g) = const$ and similarly $T(f,g) = const$ and so we can conclude that*

$$\mu(S(f,g)) = \mu(T(f,g)) = T(\mu(f), \mu(g)) = S(\mu(f), \mu(g)) = 1$$

Proof: If $f = \text{const}$, then $\mu(f) = \mu(\{x \in X : f(x) \geq \text{const}\}) = \mu(X) = 1$. A similar observation holds also for g and so $T(\mu(f), \mu(g)) = T(1,1)) = 1 = S(\mu(f), \mu(g)) = S(1,1)$.
On the other side $\mu(S(f,g)) = \mu(\{x \in X : (S(f,g))(x) \geq \text{const}\}) = \mu(X) = 1$ and so we have our thesis. □

Proposition 6.3.2 *A TS-measure $\mu : 2^X \to [0,1]$ is a T-evaluator on 2^X.*
Proof.

It's a particular case of proposition 7 in [14], because from this one a T-measure is a T-evaluator, so a TS-measure is a T-evaluator as well.

We recall also a result which can be found in [14] and which shows how T_L- and S_L-evaluators of crisp sets can be extended to T_L- and S_L-evaluators of fuzzy sets.

Proposition 6.3.3 *Consider $X \neq \emptyset$. Let P be a fuzzy measure which is a T_L- (S_L)-evaluator on 2^X. Then a mapping $\varphi : \mathcal{F}(X) \to [0,1]$ defined for all $f \in \mathcal{F}(X)$ by*

$$\varphi(f) = (C) \int f dP,$$

is a T_L-(S_L)- evaluator on $\mathcal{F}(X)$, where $(C) \int f dP$ stands for the Choquet integral of the fuzzy set f with respect to P.

Now concerning *SM* evaluators we want to prove the following result:

Proposition 6.3.4 *Consider $X = \{x_1, \cdots, x_n\}$. Let P be a fuzzy measure which is both a SM-evaluator on 2^X. Then a mapping $\varphi : \mathcal{F}(X) \to [0,1]$ defined for all $f \in \mathcal{F}(X)$ by*

$$\varphi(f) = (C)\int f dP,$$

is an SM-evaluator on $\mathcal{F}(X)$, where $(C)\int f dP$ stands for the Choquet integral of the fuzzy-valued function f with respect to P and so the following relation holds:

$$(C)\int (f \vee g) dP + (C)\int (f \wedge g) dP \geq (C)\int f dP + (C)\int g dP, \tag{6.9}$$

Proof: We skip the proof that φ is an evaluator. The Choquet integral can be expressed as

$$(C)\int f dP = \int_0^1 P(\{x \in X : f(x) \geq \alpha\}) d\alpha,$$

where the right-hand side integral is the standard Riemann integral.
By hypothesis we know that

$$P(\{x \in X : (f \vee g)(x) \geq \alpha\}) + P(\{x \in X : (f \wedge g)(x) \geq \alpha\}) \geq$$
$$\geq P(\{x \in X : f(x) \geq \alpha\}) + P(\{x \in X : g(x) \geq \alpha\})$$

and hence also

$$\int_0^1 P(\{x \in X : (f \vee g)(x) \geq \alpha\}) d\alpha + \int_0^1 P(\{x \in X : (f \wedge g)(x) \geq \alpha\}) d\alpha \geq$$
$$\int_0^1 P(\{x \in X : f(x) \geq \alpha\}) d\alpha + \int_0^1 P(\{x \in X : g(x) \geq \alpha\}) d\alpha,$$

and so we have proved the inequality 6.9, i.e.

$$\varphi(f \vee g) + \varphi(f \wedge g) \geq \varphi(f) + \varphi(g)$$

□

Example 6.3.5 *Consider $X = \{x_1, x_2\}$ and a supermodular measure P, such that $P(x_1) = \omega_1$ with $0 < \omega_1 < 1$ and $P(x_2) = 0$.*
In the following table we consider the fuzzy sets f and g.

	x_1	x_2
f	0.4	0.7
g	0.8	0.5

So we have

1. *(C)* $\int f dP = \int_0^1 P(\{x \in X : f(x) \geq t\}) dt = \int_0^{0.4} P(\{X\}) dt = 0.4;$

2. (C) $\int g dP = \int_0^1 P(\{x \in X : g(x) \geq t\})dt = \int_0^{0.5} P(\{X\})dt + \int_{0.5}^{0.8} P(\{x_1\})dt = 0.5 + 0.3\omega_1$;

Now we consider the fuzzy sets $f \wedge g$ and $f \vee g$ and their respective Choquet integrals:

	x_1	x_2
$f \wedge g$	0.4	0.5
$f \vee g$	0.8	0.7

1. (C) $\int (f \wedge g) dP = \int_0^1 P(\{x \in X : (f \wedge g)(x) \geq t\})dt = \int_0^{0.4} P(\{X\})dt = 0.4$;
2. (C) $\int (f \vee g) dP = \int_0^1 P(\{x \in X : (f \vee g)(x) \geq t\})dt = \int_0^{0.7} P(\{X\})dt + \int_{0.7}^{0.8} P(\{x_1\})dt = 0.7 + 0.1\omega_1$;

and we see
$1.1 + 0.1\omega_1 \geq 0.9 + 0.3\omega_1$, *i.e.* $0.2 - 0.2\omega_1 \geq 0$. *So the Choquet integral is an SM– evaluator on* $\mathcal{F}(X)$, *i.e. our thesis.*

Example 6.3.6 *Let* $X = \{x_1, x_2, x_3\}$ *and an SM–evaluator like in the example 5.4.8, i.e.* $P : 2^X \to [0,1]$ *be given for all* $A \in 2^X$ *by*

$$P(A) = \begin{cases} \frac{1}{4-|A|} & \text{if} \quad x_1 \in A, \\ 0 & \text{otherwise}. \end{cases}$$

In the following table we consider the fuzzy sets f and g.

	x_1	x_2	x_3
f	0.1	0.2	0.5
g	0.8	0.3	0.4

and we have

1. (C) $\int f dP = \int_0^1 P(\{x \in X : f(x) \geq t\})dt = \int_0^{0.1} P(\{X\})dt = 0.1$;
2. (C) $\int g dP = \int_0^1 P(\{x \in X : g(x) \geq t\})dt = \int_0^{0.3} P(\{X\})dt + \int_{0.3}^{0.4} P(\{x_1, x_3\})dt + {} + \int_{0.4}^{0.8} P(\{x_1\})dt = 0.3 + 0.05 + 0.13 = 0.48$;

Now we consider the fuzzy sets $f \wedge g$ and $f \vee g$ and their respective Choquet integrals: and we

	x_1	x_2	x_3
$f \wedge g$	0.1	0.2	0.4
$f \vee g$	0.8	0.3	0.5

have

1. (C) $\int (f \wedge g) dP = \int_0^1 P(\{x \in X : (f \wedge g)(x) \geq t\})dt = \int_0^{0.1} P(\{X\})dt = 0.1$;

2. *(C)* $\int (f \vee g) dP = \int_0^1 P(\{x \in X : (f \vee g)(x) \geq t\}) dt = \int_0^{0.3} P(\{X\}) dt + \int_{0.3}^{0.5} P(\{x_1, x_3\}) dt + \int_{0.5}^{0.8} P(\{x_1\}) dt = 0.3 + 0.2 \cdot 0.5 + 0.3 \cdot \frac{1}{3} = 0.6$;

Finally we can see $0.7 \geq 0.58$ *and so the Choquet integral is an SM- evaluator on* $\mathcal{F}(X)$.

Chapter 7

The multivariate case

Following the ideas of stronger forms of monotonicity for unary real functions and for capacities, this chapter aims at discussing aggregation functions $A\colon [0,1]^n \to [0,1]$ which are k-monotone or strongly k-monotone (see [57]). In Section 7.2, k-monotone and strongly k-monotone aggregation functions are discussed in general, i.e., for $k = 2,3,\ldots,\infty$. Under some specific requirements, well-known aggregation functions are recovered. Section 7.2.1 is devoted to the particular cases $k = 2$ and $k = \infty$ and in Section 7.3 we investigate the copula decomposition of n-monotone aggregation functions with interesting results for $n \geq 3$.

7.1 Stronger types of monotonicity

The monotonicity of a real function $f\colon I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is some real interval, can be strengthened into the total monotonicity. Recall that a real function f is *totally monotone* if it is smooth and all its derivatives are nonnegative. In particular, a real function $f\colon [0,1] \to [0,1]$ is totally monotone if and only if $f(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$ with $a_i \geq 0$ for all $i \in \mathbb{N} \cup \{0\}$ and $\sum_{i=0}^{\infty} a_i \leq 1$. Observe that if $f(0) = 0$ and $f(1) = 1$ are required then necessarily $a_0 = 0$ and $\sum_{i=0}^{\infty} a_i = 1$. Similarly, the monotonicity of capacities can be strengthened into the k-monotonicity, $k = 2,3,\ldots,\infty$. Recall that, for a measurable space $(X,\mathcal{A})$, a mapping $m\colon \mathcal{A} \to [0,1]$ is called a *capacity* if $m(\emptyset) = 0$, $m(X) = 1$ and m is monotone, i.e., $m(E) \leq m(F)$ whenever $E \subseteq F$. For a fixed $k \in \mathbb{N} \setminus \{1\}$, m is called *k-monotone* if for all $E_1,\ldots,E_k \in \mathcal{A}$ we have

$$m\left(\bigcup_{i=1}^{k} E_i\right) \geq \sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} m\left(\bigcap_{j \in J} E_j\right) \tag{7.1}$$

Moreover, if a capacity m satisfies (7.1) for all $k \in \mathbb{N} \setminus \{1\}$ then m is called an *∞-monotone capacity* (or, equivalently, a *belief measure*). For more details see [83, 102].

The k-monotonicity (7.1) of a capacity m can be formulated in an equivalent way: m is

k-monotone if for all $r \in \{2,\ldots,k\}$ and for all pairwise disjoint $E, E_1, \ldots, E_r \in \mathcal{A}$,

$$\sum_{J \subseteq \{1,\ldots,r\}} (-1)^{r-|J|} m\Big(E \cup \bigcup_{j \in J} E_j\Big) \geq 0. \tag{7.2}$$

Inequality (7.2) can be generalized to an arbitrary bounded lattice $(L, \vee, \wedge, \mathbf{0}, \mathbf{1})$. Indeed, let $g\colon L \to \mathbb{R}$ be a nondecreasing mapping, i.e., $g(a) \leq g(b)$ whenever $a \leq b$. Then g is *k-monotone*, $k \in \mathbb{N} \setminus \{1\}$, if for all $r \in \{2,\ldots,k\}$, for all $a \in L$, and for all pairwise disjoint $a_1, \ldots, a_r \in L$ (i.e., $a_1 \wedge a_2 = \mathbf{0}$, etc.) we have

$$\sum_{J \subseteq \{1,\ldots,r\}} (-1)^{r-|J|} g\Big(a \vee \bigvee_{j \in J} a_j\Big) \geq 0. \tag{7.3}$$

If the lattice L under consideration is a sublattice of some vector lattice (and if $\mathbf{0}$ is the neutral element of the addition on that vector space) then another condition equivalent to (7.3) can be given: a nondecreasing mapping $g\colon L \to \mathbb{R}$ is k-monotone if for all $r \in \{2,\ldots,k\}$ and for all $a, a_1, \ldots, a_r \in L$ with

$$a = a + \bigvee a_i = a + a_1 + \cdots + a_r \in L$$

we have

$$\sum_{J \subseteq \{1,\ldots,r\}} (-1)^{r-|J|} g\Big(a + \sum_{j \in J} a_j\Big) \geq 0. \tag{7.4}$$

(observe that $\bigvee a_i = a_1 + \cdots + a_r$ is equivalent to $a_1, \ldots, a_r$ being pairwise disjoint).

Moreover, in this case the following strong k-monotonicity related to (7.4) can be introduced: a nondecreasing mapping $g\colon L \to \mathbb{R}$ is called *strongly k-monotone* if for all $r \in \{2,\ldots,k\}$ and for all $a, a_1, \ldots, a_r \in L$ with $a + a_1 + \cdots + a_r \in L$ we have

$$\sum_{J \subseteq \{1,\ldots,r\}} (-1)^{r-|J|} g\Big(a + \sum_{j \in J} a_j\Big) \geq 0. \tag{7.5}$$

Observe that if $(L, \vee, \wedge, \mathbf{0}, \mathbf{1}) = (\mathcal{A}, \cup, \cap, \emptyset, X)$ then conditions (7.2) and (7.3) coincide (if we put $m = g$). Moreover, taking into account that each set $E \in \mathcal{A}$ is represented by the corresponding characteristic function $\mathbf{1}_E$, then $\iota\colon \mathcal{A} \to \mathbb{R}^X$ defined by $\iota(E) = \mathbf{1}_E$ provides an embedding of $(\mathcal{A}, \cup, \cap, \emptyset, X)$ into the vector lattice $(\mathbb{R}^X, \sup, \inf, \mathbf{0}, \mathbf{1})$, where $\mathbf{0}$ and $\mathbf{1}$ are the constant functions assuming only the value 0 and 1, respectively. Then $\iota(\mathcal{A})$ is a bounded sublattice of $\mathbb{R}^X$ (and even a sublattice of $\{0,1\}^X$). Putting $g(\mathbf{1}_E) = m(E)$, we see the equivalence of (7.2), (7.4) and (7.5).

7.1.1 Exact evaluators

Consider a bounded real-valued mapping f defined on a non-empty subset X. The set of all functions f on X is denoted by L and it is a lattice. In what follows, we use the term *functional* to refer to a real-valued map defined on some subset A of a lattice L. If $\underline{\Gamma}$ denotes a functional, then $\overline{\Gamma}$ denotes its *conjugate*, defined by

$$\overline{\Gamma}(f) := -\underline{\Gamma}(-f),$$

for any function f in $-\mathrm{dom}\underline{\Gamma} := \{-f : f \in \mathrm{dom}\underline{\Gamma}\}$. So, $\mathrm{dom}\overline{\Gamma} = -\mathrm{dom}\underline{\Gamma}$.
So, evaluators are particular kinds of functionals and it is interesting to introduce the family of *exact evaluators*.
Exact functionals are real-valued functionals that are monotone, super-additive, positively homogenous and translation invariant (or constant additive). They were introduced and studied by Maaß [65] in an attempt to unify and generalise a number of notions in the literature, such as coherent lower previsions, exact cooperative games and coherent risk measures.
A special subclass of exact functionals are *n-monotone* exact functionals, for $n \geq 1$. They have been studied in [30] and represented in terms of Choquet integral. Hence we will represent also this subclass in terms of evaluators, but first of all we give the definition of *exact evaluators*.
An evaluator ϕ on L is called *exact* whenever for any functions f and g on X, any non-negative real number λ, and any real number μ, it holds that

E1 $\phi(\lambda f) = \lambda\phi(f)$ (positive homogeneity);

E2 $\phi(f+g) \geq \phi(f)+\phi(g)$ (superadditivity);

E3 $\phi(f+c) = \phi(f)+\phi(c)$ (constant additivity).

An evaluator defined on an arbitrary subset of L is called *exact* if it can be extended to an exact evaluator on all of L.
The following definition is a special case of Choquet's general definition of n-monotonicity [26].

Definition 7.1.1 *Let $n \in \mathbb{N}^*$ and let ϕ be an evaluator whose domain domϕ is a lattice of bounded real-valued mapping f on X. Then we call ϕ n-monotone if for all $p \in \mathbb{N}$, $p \leq n$ and all $f, f_1, \ldots, f_p$ in dom ϕ:*

$$\sum_{I \subseteq \{1,\ldots,p\}} (-1)^{|I|} \phi\left(f \wedge \bigwedge_{i \in I} f_i\right) \geq 0.$$

The conjugate of an n-monotone evaluator is called *n-alternating*. An ∞-monotone evaluator (i.e, an evaluator which is n-monotone for all $n \in \mathbb{N}$) is also called *completely monotone*, and its conjugate *completely alternating*.
In this definition, and further on, we use the convention that for $I = \emptyset$, $\bigwedge_{i \in I} f_i$ simply drops out of the expressions (we could let it be equal to $+\infty$). Clearly, if an evaluator ϕ is n-monotone, it is also p-monotone for $1 \leq p \leq n$. The following proposition gives an immediate alternative characterization for the n-monotonicity on evaluators.

Proposition 7.1.2 *Let $n \in \mathbb{N}^*$ and consider an evaluator ϕ whose domain domϕ is a lattice of bounded real-valued mapping f on X. Then ϕ is n-monotone if and only if for all $p \in \mathbb{N}$,*

$2 \leq p \leq n$ *and all* $f_1, \ldots, f_p$ *in dom* ϕ*:*

$$\phi\left(\bigvee_{i=1}^{p} f_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,p\}} (-1)^{|I|+1} \phi\left(\bigwedge_{i \in I} f_i\right).$$

Exactness guarantees n-monotonicity only if $n = 1$: any exact evaluator on a lattice of bounded real-valued mapping f on X is monotone by definition, but not necessarily 2-monotone.

7.1.2 Superadditive evaluators

Concavity and supermodularity are two independent properties, but in a classic article Choquet [26] (Theorem 54.1) claimed that supermodularity implies concavity for the important class of the positively homogeneous functionals defined on ordered vector spaces. Unfortunately his argument was incomplete and thus his claim remained open. Anyway it turned out that Choquet's claim is true in the special case of $\mathbb{R}^n$ with coordinate-wise order, but beyond that it need not be true even for finite-dimensional Riesz spaces. The investigation to what extent Choquet's claim holds in general Riesz spaces can be found in [68].
For positively homogeneous evaluators (and so for Choquet integral as an aggregation operator) concavity and superadditivity are equivalent properties. Moreover there is another interesting result about supermodularity which implies superadditivity (the converse is trivial): we need C^2 for a positively homogeneous evaluator.

Definition 7.1.3 *An evaluator* $\phi : \mathbb{R}^n_+ \to [0,1]$ *is positively homogeneous if* $\phi(\alpha x) = \alpha\phi(x)$ *for all* $\alpha \geq 0$ *and all* $x \in \mathbb{R}^n_+$.

Definition 7.1.4 *An evaluator* $\phi : \mathbb{R}^n_+ \to [0,1]$ *is superlinear if it is positively homogeneous and superadditive.*

Theorem 7.1.5 *Let* $\phi : \mathbb{R}^n_+ \to [0,1]$ *be an ultramodular evaluator. Then,* ϕ *is positively homogeneous if and only if it is linear.*

Proof: This is a particular case of Theorem 5.1 in [67], where the authors show that ultramodular functions are never positively homogeneous, unless they are linear. □

Theorem 7.1.6 *Let* X *be a nonempty set and consider a function* $f : X \to [0,1]^n$*. Then the Choquet integral* $Ch_m(f)$ *is a positively homogeneous evaluator.*

Proof: This is a particular case of the property 1.1 in [62]. □

When constructing ultramodular aggregation functions, we can focus on special types of aggregation functions. However, in some cases the ultramodularity can be a contradictory or rather restrictive requirement. For instance, *disjunctive aggregation functions* (such as *triangular conorms* [59]) cannot be ultramodular. As an example of the second type we have the *Choquet integral* [26, 31] presented in 6.1.5.

Theorem 7.1.7 *An aggregation operator is ultramodular if and only if it is linear.*

Proof: With regard to aggregation function we can write an aggregation function in the following way:

$$A(x_1,x_2,\dots,x_n)=\sum_{i=1}^{n}x_{\pi(i)}[m(\{\pi(i),\dots,\pi(n)\})-m(\{\pi(i+1),\dots,\pi(n)\})],$$

where μ is a given fuzzy measure. It follows from Denneberg that A is superadditive if and only if μ is supermodular. Moreover, from Theorem 7.1.5 and Theorem 7.1.6, A is ultramodular if and only if it is linear. □

Now we give an alternative proof of this result in the following proposition:

Proposition 7.1.8 *Let* $\mathrm{Ch}_\mu\colon [0,1]^n \to [0,1]$ *be a Choquet integral-based aggregation function based on a fuzzy measure* μ *on* $X=\{1,\dots,n\}$*. Then we have:*

(i) Ch_μ *is superadditive, i.e., for all* $\mathbf{x},\mathbf{y}\in[0,1]^n$ *with* $\mathbf{x}+\mathbf{y}\in[0,1]^n$ *we have*

$$\mathrm{Ch}_\mu(\mathbf{x}+\mathbf{y})\geq \mathrm{Ch}_\mu(\mathbf{x})+\mathrm{Ch}_\mu(\mathbf{y}),$$

if and only if the fuzzy measure μ *is supermodular.*

(ii) Ch_μ *is ultramodular if and only if the fuzzy measure* μ *is modular, i.e.,* Ch_μ *is a weighted arithmetic mean.*

Proof: Statement (i) follows from [31]. If μ is modular (i.e., a probability measure) then Ch_μ is a weighted arithmetic mean and, thus, ultramodular. Conversely, if Ch_μ is ultramodular, then Ch_μ is also superadditive (indeed, it suffices to put $\mathbf{x}=\mathbf{0}$ in (3.2)), and thus μ is supermodular because of (i), and each one-dimensional section of Ch_μ is convex. This means in particular that, for an arbitrary permutation σ of $X\setminus\{1\}$, the function $f\colon [0,1]\to[0,1]$ given by

$$f(x)=\mathrm{Ch}_\mu\left(x,\tfrac{\sigma(2)-1}{n},\dots,\tfrac{\sigma(n)-1}{n}\right)$$

is convex. Clearly, f is a continuous piecewise linear function which is linear on each interval $\left[\frac{i-1}{n},\frac{i}{n}\right]$, $i\in\{1,\dots,n\}$. If τ denotes the inverse permutation of σ then the corresponding slopes of the restrictions $f|_{\left[\frac{i-1}{n},\frac{i}{n}\right]}$, $i\in\{1,\dots,n\}$, are given by $\mu(\{1,\tau(2),\dots,\tau(i)\})-\mu(\{\tau(2),\dots,\tau(i)\})$ whenever $i<n$, and by $\mu(\{1\})$ for $i=n$. Therefore, the convexity of f is equivalent to $h\colon X\to[0,1]$, where $h(i)$ is the slope of f on the interval $\left[\frac{i-1}{n},\frac{i}{n}\right]$, being non-decreasing. A similar claim holds for each other coordinate $j\in X$, i.e., for all $i,j\in X$ with $i\neq j$ and all $B\subseteq X\setminus\{i,j\}$ we have

$$\mu(B\cup\{i,j\})-\mu(B\cup\{i\})\leq\mu(B\cup\{j\})-\mu(B). \tag{7.6}$$

Because of the supermodularity of μ, the converse inequality of 7.6 holds, too, i.e., we have $\mu(B\cup\{i,j\})-\mu(B\cup\{i\})=\mu(B\cup\{j\})-\mu(B)$. For $B=\emptyset$ this means that $\mu(\{i,j\})=\mu(\{i\})+$

$\mu(\{j\})$, and for $B = \{k\}$ with $k \in X \setminus \{i, j\}$ we obtain

$$\mu(\{i,j,k\}) + \mu(\{j\}) = \mu(\{i,k\}) + \mu(\{j,k\})$$
$$= \mu(\{i\}) + \mu(\{k\}) + \mu(\{j\}) + \mu(\{k\}),$$

i.e., $\mu(\{i,j,k\}) = \mu(\{i\}) + \mu(\{j\}) + \mu(\{k\})$. By induction, $\mu(B) = \sum_{i \in B} \mu(\{i\})$ for each $B \subseteq X$, i.e., μ is modular. □

Remark 7.1.9 Observe that each $\{0,1\}$-valued supermodular fuzzy measure on X has the form μ_B, B being some non-empty subset of X, where $\mu_B(A) = 1$ if $B \subseteq A$, and $\mu_B(A) = 0$ otherwise. Then $\mathrm{Ch}_{\mu_B}(\mathbf{x}) = \min\{x_i \mid i \in B\}$ for each $\mathbf{x} \in [0,1]^n$. Moreover, a general supermodular fuzzy measure on X is a convex combination of $\{0,1\}$-valued supermodular capacities on X, and thus each superadditive Choquet integral-based aggregation function $\mathrm{Ch}_\mu \colon [0,1]^n \to [0,1]$ has the form

$$\mathrm{Ch}_\mu(\mathbf{x}) = \sum_{j=1}^{k} \lambda_j \cdot \min\{x_i \mid i \in B_j\},$$

where $k \in \mathbf{N}$, $\lambda_j > 0$ and $\emptyset \subset B_j \subseteq X$ for $j \in \{1,..,k\}$, and $\sum_{j=1}^{k} \lambda_j = 1$.

We shall identify the function $\min \colon [0,1]^k \to [0,1]$ and the greatest lower bound $\min \colon 2^{[0,1]} \to [0,1]$, i.e., both $\min(x_1,\ldots,x_n)$ and $\min\{x_1,\ldots,x_n\}$ mean the same, namely, the smallest of the numbers $x_1,\ldots,x_n \in [0,1]$. Since min is supermodular for each arity, this implies that each superadditive Choquet integral-based aggregation function $\mathrm{Ch}_\mu \colon [0,1]^n \to [0,1]$ is supermodular (this result can be found in [83, Theorem 7.17]).

It is also interesting to see the equivalence between superadditivity and supermodularity in the case of positively homogeneous evaluators.

Proposition 7.1.10 *A positively homogeneous evaluator $\phi : \mathbb{R}_+^2 \to [0,1]$ is superadditive if and only if it is supermodular.*

The example of Choquet [26] (p.288) shows that Proposition 7.1.10 does not hold in general in $\mathbb{R}^2$ when $n > 2$. In particular in [68] there is the following key definition:

Definition 7.1.11 *A class of evaluators $\phi : \mathbb{R}_+^n \to [0,1]$ has the Choquet property if its members are concave whenever they are supermodular.*

In particular the class of positively homogeneous evaluators and the class of translation invariant evaluators have the Choquet property. Observe that for positively homogeneous evaluators concavity and superadditivity are equivalent properties and so for this case Definition 7.1.11 can be equivalently stated in terms of supermodularity and superadditivity.

Theorem 7.1.12 *The positively homogeneous evaluators $\phi : \mathbb{R}_+^n \to [0,1]$ have the Choquet property.*

Proof: This is a particular case of Theorem 3 in [68]. □

Proposition 7.1.13 *Assume that an evaluator $\phi :]0,\infty[^n \to [0,1]$ is positively homogeneous and C^2. If ϕ is (sub/super) modular then it is (sub/super) additive.*

Proof: This is a particular case of Proposition 2.2 in [62]. □

7.2 (Strongly) k-monotone aggregation functions

Based on (7.4) and (7.5), we introduce the following stronger forms of monotonicity for aggregation functions.

Definition 7.2.1 Let $A\colon [0,1]^n \to [0,1]$ be an aggregation function and $k \in \mathbb{N} \setminus \{1\}$.

(i) The aggregation function A is called *k-monotone* if for each $r \in \{2,\dots,k\}$ and for all $\mathbf{x},\mathbf{x}_1,\dots,\mathbf{x}_r \in [0,1]^n$ with $\mathbf{x}+\mathbf{x}_1+\cdots+\mathbf{x}_r = \mathbf{x}+\bigvee \mathbf{x}_i \in [0,1]^n$ we have

$$\sum_{J\subseteq\{1,\dots,r\}} (-1)^{r-|J|} A\Big(\mathbf{x}+\bigvee_{j\in J}\mathbf{x}_j\Big) \geq 0. \tag{7.7}$$

(ii) The aggregation function A is said to be *strongly k-monotone* if for each $r \in \{2,\dots,k\}$ and for all $\mathbf{x},\mathbf{x}_1,\dots,\mathbf{x}_r \in [0,1]^n$ with $\mathbf{x}+\mathbf{x}_1+\cdots+\mathbf{x}_r \in [0,1]^n$ we have

$$\sum_{J\subseteq\{1,\dots,r\}} (-1)^{r-|J|} A\Big(\mathbf{x}+\sum_{j\in J}\mathbf{x}_j\Big) \geq 0. \tag{7.8}$$

(iii) The aggregation function A is called *strongly ∞-monotone* (*totally monotone*) if it is strongly k-monotone for each $k \in \mathbb{N}\setminus\{1\}$.

Note that if $\mathbf{x}+\mathbf{x}_1+\cdots+\mathbf{x}_r = \mathbf{x}+\bigvee \mathbf{x}_i \in [0,1]^n$ then formulae (7.7) and (7.8) coincide (and then $\mathbf{x}_1,\dots,\mathbf{x}_r$ have pairwise disjoint supports, i.e., $\min(\mathbf{x}_i,\mathbf{x}_j) = \mathbf{0}$ for all $i \neq j$). Clearly, for an n-ary aggregation function A, its k-monotonicity for $k > n$ is equivalent to the n-monotonicity of A, which is not true for strong monotonicity. For example, for a unary aggregation function $f\colon [0,1] \to [0,1]$, k-monotonicity is just the nondecreasingness of f, while strong 2-monotonicity of f is equivalent to its convexity.

The following results can be found in [17].

Proposition 7.2.2 *Let $f\colon [0,1] \to [0,1]$ be an aggregation function. Then we have:*

(i) *f is strongly k-monotone for some $k \in \mathbb{N}\setminus\{1\}$ if and only if all derivatives of f of order $1,\dots,k-2$ are nonnegative and $f^{(k-2)}$ is a nondecreasing convex function.*

(ii) *f is strongly ∞-monotone if and only if f is a totally monotone real function, i.e., it has non-negative derivatives of all orders on $[0,1[$.*

Proposition 7.2.3 *Let $A\colon [0,1]^n \to [0,1]$ be an aggregation function. Then A is totally monotone if and only if all partial derivatives of A are nonnegative. In particular, this means that*

$$A(u_1,\ldots,u_n) = \sum a_{i_1,\ldots,i_n} \cdot {u_1}^{i_1} \cdots {u_n}^{i_n},$$

where $i_1,\ldots,i_n$ run from 0 to ∞, $a_{0,\ldots,0} = 0$, all $a_{i_1,\ldots,i_n} \geq 0$, and $\sum a_{i_1,\ldots,i_n} = 1$.

As a particular consequence of Proposition 7.2.3 we see that, for each $n \in \mathbb{N}$, the product $\Pi\colon [0,1]^n \to [0,1]$ is a totally monotone aggregation function. Also, each weighted arithmetic mean $W\colon [0,1]^n \to [0,1]$ given by $W(u_1,\ldots,u_n) = \sum w_i \cdot u_i$ is totally monotone.

Proposition 7.2.4 *Fix $k \in \{2,3,\ldots,\infty\}$. Then for all $n,m \in \mathbb{N}$ and for all strongly k-monotone n-ary aggregation functions $A\colon [0,1]^n \to [0,1]$ and for all strongly k-monotone m-ary aggregation functions $B_1,\ldots,B_n\colon [0,1]^m \to [0,1]$ also the composite function $D\colon [0,1]^m \to [0,1]$ given by*

$$D(\mathbf{x}) = A(B_1(\mathbf{x}),\ldots,B_n(\mathbf{x}))$$

is strongly k-monotone.

It is possible to show that for each fixed $n \in \mathbb{N}$ and $k \in \{2,3,\ldots,\infty\}$, the class of all (strongly) k-monotone n-ary aggregation functions is convex and compact (with respect to the topology of pointwise convergence).

For $n \in \mathbb{N} \setminus \{1\}$ and for n-ary aggregation functions $A\colon [0,1]^n \to [0,1]$, the notion of n-increasingness was introduced in the framework of copulas [81, 94]:

Definition 7.2.5 Let $n \geq 2$. An aggregation function $A\colon [0,1]^n \to [0,1]$ is called *n-increasing* if for all $\mathbf{x},\mathbf{y} \in [0,1]^n$ with $\mathbf{x} \leq \mathbf{y}$ we have

$$\sum_{J \subseteq \{1,\ldots,n\}} (-1)^{n-|J|} A(\mathbf{z}_J) \geq 0, \tag{7.9}$$

where $\mathbf{z}_J \in [0,1]^n$ is given by $z_j = y_j$ if $j \in J$, and $z_j = x_j$ otherwise.

It is not difficult to check that, under the hypotheses of Definition 7.2.5, formulae (7.9) and (7.7) coincide, i.e., n-monotonicity and n-increasingness for n-ary aggregation functions mean the same. Hence, k-monotonicity extends the notion of n-increasingness to higher dimensions.

Remark 7.2.6 (i) Because of [17], strong k-monotone aggregation functions are important in the theory of non-additive measures: for k-monotone capacities $m_1,\ldots,m_n$ acting on a fixed measurable space $(X,\mathcal{A})$ and for a strongly k-monotone n-ary aggregation function A, the set function $A(m_1,\ldots,m_n)\colon \mathcal{A} \to [0,1]$ given by

$$A(m_1,\ldots,m_n)(E) = A(m_1(E),\ldots,m_n(E))$$

is a k-monotone capacity whenever A is strongly k-monotone (if $|X| \geq k$, this is also necessary condition if the claim should be valid for arbitrary k-monotone capacities $m_1,\ldots,m_n$).

(ii) k-monotonicity is an axiom for k-dimensional copulas [94].

(iii) Strong 2-monotonicity is known also as *ultramodularity*, and it was discussed in general in [67] (see also [56]). Another name for 2-monotonicity is *supermodularity*, a widely used concept in the theory of non-additive measures and of aggregation functions.

7.2.1 (Strongly) 2-monotone aggregation functions

Recall that an aggregation function $C\colon [0,1]^2 \to [0,1]$ which is 2-monotone and satisfies $C(x,1) = C(1,x) = x$ for all $x \in [0,1]$ is called a 2-*copula* (or, shortly, a *copula*). Copulas play a key role in the description of the stochastic dependence of two-dimensional random vectors and they are substantially exploited in several applications in finance, hydrology, etc. The construction of new types of copulas is one of the important theoretical tasks allowing a better modelling of real problems involving stochastic uncertainty. So, the result from [39], i.e. proposition 4.1.1 can be written in the following way:

Proposition 7.2.7 *An aggregation function $A\colon [0,1]^2 \to [0,1]$ is 2-monotone if and only if there are nondecreasing functions $g_1, g_2, g_3, g_4\colon [0,1] \to [0,1]$ with $g_i(0) = 0$ and $g_i(1) = 1$ for each $i \in \{1,2,3,4\}$, a binary copula $C\colon [0,1]^2 \to [0,1]$, and numbers $a,b,c \in [0,1]$ with $a+b+c = 1$ such that, for all $(x,y) \in [0,1]^2$,*

$$A(x,y) = a \cdot g_1(x) + b \cdot g_2(y) + c \cdot C(g_3(x), g_4(y)). \tag{7.10}$$

Similarly, we can formulate respectively Remark 3.2.2(c) and Propositions 3.2.9 in the following way:

Proposition 7.2.8 *An aggregation function $A\colon [0,1]^2 \to [0,1]$ is strongly 2-monotone if and only if A is 2-monotone and each horizontal and each vertical section of A is a convex function.*

In the class of copulas, the greatest strongly 2-monotone copula is the product copula Π, while the smallest strongly 2-monotone copula is the Fréchet-Hoeffding lower bound W given by $W(x,y) = \max(x+y-1,0)$. Note that the only totally monotone 2-copula is the product copula Π.

Theorem 7.2.9 *Let $A\colon [0,1]^n \to [0,1]$ be an aggregation function and $k \geq 2$. Then the following are equivalent:*

(i) *A is strongly 2-monotone.*

(ii) *If $B_1, \ldots, B_n\colon [0,1]^k \to [0,1]$ are nondecreasing 2-monotone functions then the composite $D\colon [0,1]^k \to [0,1]$ given by $D(\mathbf{x}) = A(B_1(\mathbf{x}), \ldots, B_n(\mathbf{x}))$ is a 2-monotone function.*

7.3 Construction of multivariate copulas

The subject of assessing probabilistic dependence between one-dimensional distribution functions to construct a joint distribution function is an important task in probability theory and statistics. As we have already told, copula function captures the dependence relationships among the individual random variables as each multivariate distribution can be represented in terms of its marginals through a given copula structure.
The aim of this section is to present the copula approach for studying aggregation problems. It can be extended to n dimensions and for the sake of simplicity first of all we are considering the case $n = 2$. Specifically, binary aggregation operators satisfying the 2-increasing property are analysed in details in [37] and [39], with the most important result recalled in proposition 4.1.1. So, the proofs in the following section about the bivariate case will be provided just for completeness of information.
We are studying a class of aggregation functions that can be expressed in terms of marginal functions by using the method of copulas. Elimination of marginals through copulas helps to model and understand dependence structure between variables more effectively, as the dependence has nothing to do with the marginal behaviour.
Like for the bivariate case, for any n-copula:

$$W(u_1,\dots,u_n) \le C(u_1,\dots,u_n) \le M(u_1,\dots,u_n),$$

but, even if the upper function M is an n-copula for any $n \in \mathbb{N}$, the lower function W is not an n-copula for any $n > 2$.
Moreover, Sklar's theorem holds in the multivariate case as well.

Theorem 7.3.1 (Sklar [72]) *Let H be an n-dimensional distribution function with margins $F_1, F_2, \dots, F_n$. Then there exists an n-copula C such that for all $\mathbf{x} \in \overline{\mathbb{R}}^n$,*

$$H(x_1,x_2,\dots,x_n) = C(F_1(x_1),F_2(x_2),\dots,F_n(x_n)) \tag{7.11}$$

If $F_1, F_2, \dots, F_n$ are all continuous, then C is unique; otherwise, C is uniquely determined on $Ran\ F_1 \times Ran\ F_2 \times \dots \times Ran\ F_n$. Conversely, if C is an n-copula and $F_1, F_2, \dots, F_n$ are distribution functions, then the function H defined by (7.11) is an n-dimensional distribution function with margins $F_1, F_2, \dots, F_n$.

The following proposition gives an interesting characterization concerning multivariate copulas and supermodular aggregation functions.

Proposition 7.3.2 *The set of all n–dimensional copulas $\mathcal{C}_n$ is contained in $\mathcal{A}_n^S$.*

Proof: In fact all n–dimensional copulas are n-increasing and so they are 2-increasing by lemma 2.1 in [77] and by proposition 1.2.2 they belong to $\mathcal{A}_n^S$. □

Remark 7.3.3 *The converse is not true: in fact, if we consider the smallest aggregation function A^{small}, defined by $A^{small}(\mathbf{x}) = 1$ if $\mathbf{x} = (1,\dots,1)$, and $A^{small}(\mathbf{x}) = 0$, otherwise, we see that A^{small} is an element of $\mathcal{A}_n^S$, but it is not a copula.*

Let f be a real valued function on $[a_1,b_1]\times\ldots\times[a_n,b_n] = [a,b] \subset \mathbb{R}^n$, $-\infty < a < b < \infty$. Given a set of indexes $I \subseteq N = \{1,\dots,n\}$, let $f(x_I;y_{-I})$ denote the value of f at the point in $[a,b]$ whose ith element is equal to x_i if $i \in I$ and is equal to y_i otherwise. For any n-dimensional rectangle $R = [c,d] \subseteq [a,b]$, let $\Delta_R f = \sum_{I\subseteq\{1,\dots,n\}}(-1)^{|I|} f(c_I;d_{-I})$. For a nonempty $I \subseteq N$, let f_I denote the function on $\Pi_{i\in I}[a_i,b_i]$ obtained by fixing the jth argument of f equal to b_j whenever $j \notin I$, and letting the other arguments vary (see [8]).
Now we are going to apply this observation to n-monotone aggregation functions, for $n \geq 2$, and so we are going to see another approach to the bivariate case, which has been deeply analysed in the previous chapters and in particular in Propositions 3.2.2 and 4.1.1 and Corollary 4.2.1.

7.3.1 The bivariate case

We define the following one-dimensional marginals of a binary aggregation function A:

$$F_1(x_1) = A(x_1,1) \qquad F_2(x_2) = A(1,x_2)$$

Proposition 7.3.4 *A is a 2-increasing continuous binary aggregation function with*

$$A(t,0) = 0 \quad \text{and} \quad A(0,t) = 0 \qquad \forall\, t \in [0,1] \tag{7.12}$$

if and only if there exist a copula C and two one-dimensional increasing and uniformly continuous functions F_1 and F_2 from $[0,1]$ to $[0,1]$ with $F_1(0) = 0 = F_2(0)$ and $F_1(1) = 1 = F_2(1)$, such that

$$A(x_1,x_2) = C(F_1(x_1),F_2(x_2)) \qquad \forall\, x_1,x_2 \in [0,1].$$

Proof.

Of course, if $A(x_1,x_2) = C(F_1(x_1),F_2(x_2))$, A is 2-increasing with $A(0,0) = 0$ and $A(1,1) = 1$. So we must prove the necessary condition. We can observe thanks to the lemma 2.1.5 in [81] that

$$|A(x_1,x_2) - A(x_1',x_2')| \leq |F_1(x_1) - F_1(x_1')| + |F_2(x_2) - F_2(x_2')|$$

for all $x_1,x_1',x_2,x_2' \in [0,1]$. Then, if $F_1(x_1) = F_1(x_1')$ and $F_2(x_2) = F_2(x_2')$, it follows that $A(x_1,x_2) = A(x_1',x_2')$. So, we can define a function C whose domain is $[0,1]^2$ with range $[0,1]$ defined by $C(u_1,u_2) = A(x_1,x_2)$ where $u_1 = F_1(x_1), u_2 = F_2(x_2)$.
We can prove that $A(x_1,x_2) = C(F_1(x_1),F_2(x_2))$ is a copula. It follows directly from the properties of A that the function C is a 2-increasing function, such that $C(0,u_2) = 0 = C(u_1,0)$. For the other boundary condition we have:

$$u_2 = C(1,u_2) = C(1,F_2(x_2)) = A(1,u_2),$$

for each u_2. Moreover, $F_1(x_1)$ and $F_2(x_2)$ are increasing and uniformly continuous thanks to the definition 2.2.5 and the corollary 2.2.6 in [81].

Aggregation operators that satisfy equation (7.12) frequently appear in many applications. For example, we can consider a multi-attribute decision problem where the attribute may be health state and consumption level or decisions involving trade-offs between quality and quantity.
Now we generalize the previous proposition with the following one.

Proposition 7.3.5 *A is a 2-monotone binary aggregation function if and only if there exist a constant k, a copula C, two increasing functions A_1, A_2 and two one-dimensional increasing functions $G_1(x_1)$, $G_2(x_2)$, such that*

$$A(x_1,x_2) = A_1(x_1) + A_2(x_2) + kC(G_1(x_1), G_2(x_2)) \tag{7.13}$$

Proof: Let us start with the sufficient condition. Of course, $A(x_1,x_2) = A_1(x_1) + A_2(x_2) + kC(G_1(x_1), G_2(x_2))$ is 2-monotone. Moreover $A(0,0) = 0$ and $A(1,1) = A_1(1) + A_2(1) + k$. So, by taking $k = 1 - A_1(1) - A_2(1)$, we have that $A(x_1,x_2)$ is an aggregation function.
Now we prove the necessary condition. First of all, by using the previous observation (see [8]), we observe that $R = [0,x_1] \times [0,x_2]$, $N = \{1,2\}$ and so $I \in \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.
Therefore, we have the following decomposition

$$\Delta_R(A) \triangleq A(x_1,x_2) - A(x_1,0) - A(0,x_2) + A(0,0).$$

So, rearranging the previous equation, we have

$$A(x_1,x_2) = A(x_1,0) + A(0,x_2) + \Delta_R(A).$$

We observe that

$$\begin{aligned} V_{\Delta_R(A)}([0,x_1] \times [0,x_2]) &= \Delta_R(A)(x_1,x_2) - \Delta_R(A)(x_1,0) - \Delta_R(A)(0,x_2) + \Delta_R(A)(0,0) \\ &= V_A([0,x_1] \times [0,x_2]) \geq 0, \end{aligned}$$

and so $\Delta_R(A)$ is 2-monotone and satisfies the hypothesis of Proposition 7.3.4 if $\Delta_R(A)(1,1) = 1$. If, otherwise, $\Delta_R(A)(1,1) = k$, with $0 < k < 1$, the hypothesis of Proposition 7.3.4 are satisfied by using the function $\Delta_R(A)(x_1,x_2)/\Delta_R(A)(1,1)$ and so there exists a copula C and two increasing functions G_1 and G_2 such that $\Delta_R(A)(x_1,x_2) = \Delta_R(A)(1,1)C(G_1(x_1), G_2(x_2))$. □

7.3.2 The trivariate case

The next proposition introduces an analogy between probability distributions and this class of aggregation functions, but first of all we define the following one-dimensional marginal functions

$$F_1(x_1) = A(x_1,1,1), \quad F_2(x_2) = A(1,x_2,1), \quad F_3(x_3) = A(1,1,x_3)$$

Proposition 7.3.6 (Particular case) *A is a* 3*-monotone trivariate aggregation function with*

$$A(x_1,x_2,0)=A(x_1,0,x_3)=A(0,x_2,x_3)=0 \tag{7.14}$$

if and only if there exists a copula C and three one-dimensional increasing and uniformly continuous marginals, such that

$$A(x_1,x_2,x_3)=C(F_1(x_1),F_2(x_2),F_3(x_3))$$

Proof: Like for the bivariate case, the sufficient condition is trivial. So, we prove the necessary condition only. Thanks to the previous observations, we have again the following decomposition with $R=[0,x_1]\times[0,x_2]\times[0,x_3]$:

$$\begin{aligned}\Delta_R(A) &\triangleq A(x_1,x_2,x_3)-A(x_1,x_2,0)-A(x_1,0,x_3)+A(x_1,0,0)+\\ &- A(0,x_2,x_3)+A(0,x_2,0)+A(0,0,x_3)-A(0,0,0)\\ &= A(x_1,x_2,x_3)-A(x_1,x_2,0)-A(x_1,0,x_3)-A(0,x_2,x_3)+\\ &+ A(x_1,0,0)+A(0,x_2,0)+A(0,0,x_3),\end{aligned}$$

because $A(0,0,0)=0$ for the property of aggregation function. So, for our hypothesis we obtain

$$A(x_1,x_2,x_3)=\Delta_R(A).$$

It remains to prove that $\Delta_R(A)$ is a copula. Surely it is 3-monotone, because this is our hypothesis for A. So, we must prove that $\Delta_R(A)$ is grounded and it has one-dimensional marginals. We can observe thanks to the Lemma 2.10.4 in [81] that

$$|A(x_1,x_2,x_3)-A(y_1,y_2,y_3)|\le|F_1(x_1)-F_1(y_1)|+|F_2(x_2)-F_2(y_2)|+|F_3(x_3)-F_3(y_3)|$$

for all $\mathbf{x},\mathbf{y}\in[0,1]$. Then, if $F_1(x_1)=F_1(y_1)$, $F_2(x_2)=F_2(y_2)$ and $F_3(x_3)=F_3(y_3)$ it follows that $A(\mathbf{x})=A(\mathbf{y})$. So, it is well-defined a function C whose domain is $[0,1]^3$ with range $[0,1]$, such that $C(F_1(x_1),F_2(x_2),F_3(x_3))=A(x_1,x_2,x_3)$. Therefore, we can prove that $A(\mathbf{x})=C(F_1(x_1),F_2(x_2),F_3(x_3))$ has one-dimensional marginals. In fact we have

$$C(1,1,F_3(x_3))=A(1,1,x_3)=F_3(x_3).$$

Verifications of the other conditions are similar. Moreover, $F_1(x_1)$, $F_2(x_2)$ and $F_3(x_3)$ are increasing and uniformly continuous thanks to the Lemma 2.10.3 and the Theorem 2.10.7 in [81]. □

Now we give the more general proposition.

Proposition 7.3.7 (General case) *A is a* 3*-monotone trivariate aggregation function if and only if there exist three increasing functions* A_1*,* A_2*,* A_3*, three 2-monotone functions* $A_{1,2}$*,* $A_{2,3}$*,*

$A_{1,3}$, *a constant* k_0 *and a copula* C_0, *such that*

$$\begin{aligned} A(x_1,x_2,x_3) &= A_{1,2}(x_1,x_2)+A_{1,3}(x_1,x_3)+A_{2,3}(x_2,x_3)+ \\ &- A_1(x_1)-A_2(x_2)-A_3(x_3)+k_0C_0(F_1^0(x_1),F_2^0(x_2),F_3^0(x_3)) \end{aligned} \tag{7.15}$$

Proof: Thanks to the previous observations, it is enough to define

$$A_1(x_1) := A(x_1,0,0), \qquad A_2(x_2) := A(0,x_2,0), \qquad A_3(x_3) := A(0,0,x_3) \tag{7.16}$$

and

$$A_{1,2}(x_1,x_2) := A(x_1,x_2,0), \quad A_{1,3}(x_1,x_3) := A(x_1,0,x_3), \quad A_{2,3}(x_2,x_3) := A(0,x_2,x_3). \tag{7.17}$$

With respect to the sufficient condition we observe that

$$\Delta_R(A)(1,1,1)C_0(F_1^0(x_1),F_2^0(x_2),F_3^0(x_3))$$

is 3-increasing and $A_{1,2}(x_1,x_2)$ is 2-increasing, that is

$$\begin{aligned} V_{A_{1,2}}([0,x_1]\times[0,x_2]) &= A_{1,2}(0,0)-A_{1,2}(0,x_2)+A_{1,2}(x_1,x_2)-A_{1,2}(x_1,0) = \\ &= A_{1,2}(x_1,x_2)-A_2(x_2)-A_1(x_1)\geq 0. \end{aligned} \tag{7.18}$$

So $A_{1,2}(x_1,x_2)-A_1(x_1)\geq A_{1,2}(x_1,x_2)-A_1(x_1)-A_2(x_2)\geq 0$.
Similarly $A_{1,3}(x_1,x_3)-A_3(x_3)\geq 0$ and $A_{2,3}(x_2,x_3)-A_2(x_2)\geq 0$. Then we can conclude that $A(x_1,x_2,x_3)$ is 3-monotone.
With regard to the necessary condition, we must prove that (7.16) and (7.17) are increasing and 2-monotone, respectively.
We have already said that increasingness is a property required to aggregation preferences. It remains to prove 2-monotonicity. We observe that $A_{1,2}(x_1,x_2) := A(x_1,x_2,0) = A(x_1,x_2,x_3) - A(0,0,x_3)$ and so it's 2-monotone thanks to the lemma 2.1 in [77].
At last, we must prove that

$$C_0(F_1^0(x_1),F_2^0(x_2),F_3^0(x_3)) = \frac{\Delta_R(A)}{k_0}$$

is a copula, with $k_0 = \Delta_R(A)(1,1,1)\neq 0$. Indeed, $\Delta_R(A)$ satisfies the hypothesis of proposition (7.3.6) for its construction. So, there exists a copula C_0 and three one-dimensional marginals F_1^0, F_2^0 and F_3^0, such that

$$\Delta_R(A) = k_0C_0(F_1^0(x_1),F_2^0(x_2),F_3^0(x_3)).$$

This ends the proof. □

As a consequence, we have the following result.

Corollary 7.3.8 *A is a trivariate* 3*-monotone aggregation function if and only if there exist* *three increasing functions A_1, A_2, A_3, four constants k_i, for $i = 0, 1, \dots, 3$, three copulas and a trivariate one, such that*

$$\begin{aligned} A(x_1,x_2,x_3) &= A_1(x_1)+A_2(x_2)+A_3(x_3)+k_0C_0(F_1^0(x_1),F_2^0(x_2),F_3^0(x_3))+ \\ &+ k_1C_1(F_2^1(x_2),F_3^1(x_3))+k_2C_2(F_1^2(x_1),F_3^2(x_3))+k_3C_3(F_1^3(x_1),F_2^3(x_2)) \end{aligned}$$

Proof: The sufficient condition is obvious. As regards the necessary one, we use the bivariate case, that is we have studied that

$$A(x_1,x_2,0) = A(x_1,0,0)+A(0,x_2,0)+\Delta_R(A)(1,1,0)C_3(F_1^3(x_1),F_2^3(x_2)).$$

Of course $A(1,1,0) < 1$ because $A(1,1,1) = 1$ and we must multiply $A(x_1,x_2,0)$ by $A(1,1,0)^{-1}$ to have an aggregation function. Anyway, we have the following decomposition:

$$A(x_1,0,x_3) = A_1(x_1)+A_3(x_3)+\Delta_R(A)(1,0,1)C_2(F_1^2(x_1),F_3^2(x_3))$$

and

$$A(0,x_2,x_3) = A_2(x_2)+A_3(x_3)+\Delta_R(A)(0,1,1)C_1(F_2^1(x_2),F_3^1(x_3)).$$

So,

$$\begin{aligned} A(x_1,x_2,x_3) &= A(x_1,x_2,0)+A(x_1,0,x_3)+A(0,x_2,x_3)-A(x_1,0,0)-A(0,x_2,0)+ \\ &- A(0,0,x_3)+\Delta_R(A) = \\ &= 2A(x_1,0,0)+2A(0,x_2,0)+2A(0,0,x_3)+\Delta_R(A)(0,1,1)C_1(F_2^1(x_2),F_3^1(x_3))+ \\ &+ \Delta_R(A)(1,0,1)C_2(F_1^2(x_1),F_3^2(x_3))+\Delta_R(A)(1,1,0)C_3(F_1^3(x_1),F_2^3(x_2))+ \\ &- A(x_1,0,0)-A(0,x_2,0)-A(0,0,x_3)+\Delta_R(A) = \\ &= A(x_1,0,0)+A(0,x_2,0)+A(0,0,x_3)+\Delta_R(A)(0,1,1)C_1(F_2^1(x_2),F_3^1(x_3))+ \\ &+ \Delta_R(A)(1,0,1)C_2(F_1^2(x_1),F_3^2(x_3))+\Delta_R(A)(1,1,0)C_3(F_1^3(x_1),F_2^3(x_2))+ \\ &+ \Delta_R(A)(1,1,1)C_0(F_1^0(x_1),F_2^0(x_2),F_3^0(x_3)) \end{aligned}$$

At last, we must prove that

$$\frac{\Delta_R(A)}{\Delta_R(A)(1,1,1)}$$

is a copula. In fact, it satisfies the hypothesis of proposition (7.3.6) for its construction. So, there exists a copula C_0 and three one-dimensional marginals $F_1^0(x_1)$, $F_2^0(x_2)$ and $F_3^0(x_3)$, such that $\Delta_R(A) = \Delta_R(A)(1,1,1)C_0(F_1^0(x_1),F_2^0(x_2),F_3^0(x_3))$.
As a consequence, by posing $k_0 = \Delta_R(A)(1,1,1)$, $k_1 = \Delta_R(A)(0,1,1)$, $k_2 = \Delta_R(A)(1,0,1)$ and at last $k_3 = \Delta_R(A)(1,1,0)$, we have our thesis. □

7.3.3 The general case

Let us consider $A(x_1,x_2,x_3,x_4)$, with $R=[0,x_1]\times[0,x_2]\times[0,x_3]\times[0,x_4]$. By using the previous observation, we note that

$$\begin{aligned} A(x_1,x_2,x_3,x_4) &= A(x_1,x_2,x_3,0)+A(x_1,0,x_3,x_4)+A(x_1,x_2,0,x_4)+A(0,x_2,x_3,x_4)+\\ &- A(x_1,0,0,x_4)-A(x_1,x_2,0,0)-A(x_1,0,x_3,0)-A(0,x_2,x_3,0)+\\ &- A(0,x_2,0,x_4)-A(0,0,x_3,x_4)+A(x_1,0,0,0)+A(0,x_2,0,0)+\\ &+ A(0,0,x_3,0)+A(0,0,0,x_4)+\Delta_R(A). \end{aligned}$$

By using and iterating this decomposition, we obtain at the end:

1. Three bivariate copulas $C(F_i,F_j)$ $\forall i=1,2,3 \quad j>i$, with $F_i=F_i(x_i)$.
2. One trivariate copula $C(F_i,F_j,F_l)$ $\forall i=1,2,3 \quad i<j<l$.

In fact the trivariate copulas come out only from the respective trivariate aggregation functions, while, about the bivariate ones, we have that, for the same i and j only one goes out from the respective bivariate aggregation function. We have the other two thanks to the trivariate aggregation functions. For example, we have three $C(F_1,F_2)$: one from $A(x_1,x_2,0,0)$, that is $k^{34}C^{34}(F_1^{34},F_2^{34})$, another from $A(x_1,x_2,x_3,0)$, that is $k^4C^4(F_1^4,F_2^4)$ and the last from $A(x_1,x_2,0,x_4)$, that is $k^3C^3(F_1^3,F_2^3)$.
So we can write the following decomposition:

$$A(x_1,x_2,x_3,x_4)=\sum_{i=1}^{4}A_i(x_i)+\sum_{\substack{\{h,k\}\\h=1}}^{4}\alpha^{\{\overline{\{h,k\}}=0\}}C^{\{\overline{\{h,k\}}=0\}}(F_h^{\{\overline{\{h,k\}}=0\}},F_k^{\{\overline{\{h,k\}}=0\}})+$$

$$+\sum_{\substack{\{i,j,l\}\\i=1}}^{4}\alpha^{\{\overline{\{i,j,l\}}=0\}}C^{\{\overline{\{i,j,l\}}=0\}}(F_i^{\{\overline{\{i,j,l\}}=0\}},F_j^{\{\overline{\{i,j,l\}}=0\}},F_l^{\{\overline{\{i,j,l\}}=0\}})+\alpha C(F_1,F_2,F_3,F_4).$$

where $\{\overline{\{h,k\}}=0\}$ is the set of complementary attributes of $\{x_h,x_k\}$, when, at least one of them is equal to zero and $\{\overline{\{i,j,l\}}=0\}$ is the set of complementary attributes of $\{x_i,x_j,x_l\}$, when, at least one of them is equal to zero too.
Now we can prove the general result.

Proposition 7.3.9 (General case) *A is an n-monotone aggregation function if and only if there exist n increasing functions $A_i(x_i)$, $\binom{n}{2}$ bivariate copulas C, $\binom{n}{k}$ k-copulas.... and an n-copula, such that*

$$\begin{aligned} A(x_1,\ldots,x_n) &= \sum_{i=1}^{n}A_i(x_i)+\sum_{\substack{\{r,s\}\\r=1}}^{n}\overline{\alpha}_{r,s}C(\overline{F}_{r,s})+\ldots\\ \ldots &+ \sum_{\substack{\{r,s,\ldots,t\}\\r=1}}^{n}\overline{\alpha}_{r,s,\ldots,t}C(\overline{F}_{r,s,\ldots,t})+\ldots+\alpha C(F_1,\ldots,F_n), \end{aligned} \tag{7.19}$$

with

$$\begin{aligned} A_i(x_i) &= A(0,\ldots,x_i,\ldots,0) \\ \overline{F}_{r,s,\ldots,t} &= (F_r^{\{\overline{\{r,s,\ldots,t\}}=0\}}, F_s^{\{\overline{\{r,s,\ldots,t\}}=0\}}, \ldots, F_t^{\{\overline{\{r,s,\ldots,t\}}=0\}}) \end{aligned}$$

and {r,s,...,t} denotes the set of attributes that are not equal to zero.

Proof: The sufficient condition is obvious, so we can prove the necessary one by induction: $n=2$ and $n=3$ are right and $n-1 \Rightarrow n$. So our inductive hypothesis is:

$$\begin{aligned} A(x_1,\ldots,x_{n-1}) &= \sum_{i=1}^{n-1} A_i(x_i) + \sum_{\substack{\{r,s\} \\ r=1}}^{n-1} \overline{\alpha}_{r,s} C(\overline{F}_{r,s}) + \ldots \\ \ldots &+ \sum_{\substack{\{r,s,\ldots,t\} \\ r=1}}^{n-1} \overline{\alpha}_{r,s,\ldots,t} C(\overline{F}_{r,s,\ldots,t}) + \ldots + \alpha C(F_1,\ldots,F_{n-1}) \end{aligned}$$

and we can apply this to the previous formula, but first we define $A(x_1,\ldots,x_j,\ldots,x_n)_{|x_j=0} = A(\mathbf{x}_\circ^j)$,
where $\mathbf{x}_\circ^j$ is the set of complementary attributes of x_j, with $x_j = 0$ and similarly $\mathbf{x}_\circ^{\{j,l,\ldots,m\}}$ as the set of complementary attributes of $x_j = x_l = \ldots = x_m = 0$.
So, we have

$$A(x_1,\ldots,x_n) = \underbrace{\sum_{j=1}^{n} A(\mathbf{x}_\circ^j)}_{n} - \underbrace{\sum_{\substack{j=1 \\ l>j}}^{n} A(\mathbf{x}_\circ^{j,l})}_{\binom{n}{2}} + \ldots + (-1)^{k-1} \underbrace{\sum_{\substack{j=1 \\ l>j \\ \ldots \\ m>l}}^{n} A(\mathbf{x}_\circ^{j,l,\ldots,m})}_{\binom{n}{k}} + \ldots + \Delta_R(A)(F_1,\ldots,F_n). \tag{7.20}$$

Then, by using our hypothesis, we obtain

$$\begin{aligned} A(\mathbf{x}_\circ^j) &= A(x_1,\ldots,x_j,\ldots,x_n)_{|x_j=0} = A(x_1,\ldots,x_{n-1}) = \sum_{\substack{i=1 \\ i\neq j}}^{n-1} A_i(x_i) + \\ &+ \sum_{\substack{\{r,s\} \\ r=1}}^{n-1} \overline{\alpha}_{r,s} C(\overline{F}_{r,s}) + \ldots + \sum_{\substack{\{r,s,\ldots,t\} \\ r=1}}^{n-1} \overline{\alpha}_{r,s,\ldots,t} C(\overline{F}_{r,s,\ldots,t}) + \ldots \\ &\ldots + \alpha^j C(F_1,\ldots,F_{n-1}) \;\; \forall j \end{aligned}$$

Similarly we get

$$
\begin{aligned}
A(\mathbf{x}_{\circ}^{j,l}) &= \sum_{\substack{i=1\\ i\neq(j,l)\\ l>j}}^{n-2} A_i(x_i) + \sum_{\substack{\{r,s\}\\ r=1}}^{n-2} \overline{\alpha}_{r,s} C(\overline{F}_{r,s}) + \ldots \\
&\ldots + \sum_{\substack{\{r,s,\ldots,t\}\\ r=1}}^{n-2} \overline{\alpha}_{r,s,\ldots,t} C(\overline{F}_{r,s,\ldots,t}) + \ldots + \alpha^{j,l} C(F_1,\ldots,F_{n-2}) \;\; \forall j,l
\end{aligned}
$$

and

$$
\begin{aligned}
A(\mathbf{x}_{\circ}^{j,l,\ldots,m}) &= \sum_{\substack{i=1\\ i\neq(j,l,\ldots,m)\\ l>j\\ \ldots\\ m>l}}^{n-k} A_i(x_i) + \sum_{\substack{\{r,s\}\\ r=1}}^{n-k} \overline{\alpha}_{r,s} C(\overline{F}_{r,s}) + \ldots \\
&\ldots + \sum_{\substack{\{r,s,\ldots,t\}\\ r=1}}^{n-k} \overline{\alpha}_{r,s,\ldots,t} C(\overline{F}_{r,s,\ldots,t}) + \ldots + \alpha^{j,l,\ldots,m} C(F_1,\ldots,F_{n-k}) \;\; \forall j,l,m.
\end{aligned}
$$

So, we replace these latter equations in (7.20) with

$$
\begin{aligned}
\sum_{j=1}^{n} A(\mathbf{x}_{\circ}^{j}) &= \sum_{j=1}^{n} \sum_{\substack{i=1\\ i\neq j}}^{n-1} A_i(x_i) + \sum_{j=1}^{n} \sum_{\substack{\{r,s\}\\ r=1}}^{n-1} \overline{\alpha}_{r,s} C(\overline{F}_{r,s}) + \ldots \\
&\ldots + \sum_{j=1}^{n} \sum_{\substack{\{r,s,\ldots,t\}\\ r=1}}^{n-1} \overline{\alpha}_{r,s,\ldots,t} C(\overline{F}_{r,s,\ldots,t}) + \ldots + \sum_{j=1}^{n} \alpha^{j} C(F_1,\ldots,F_{n-1});
\end{aligned}
$$

$$
\begin{aligned}
\sum_{\substack{j=1\\ l>j}}^{n} A(\mathbf{x}_{\circ}^{j,l}) &= \sum_{\substack{j=1\\ l>j}}^{n} \sum_{\substack{i=1\\ i\neq(j,l)\\ l>j}}^{n-2} A_i(x_i) + \sum_{\substack{j=1\\ l>j}}^{n} \sum_{\substack{\{r,s\}\\ r=1}}^{n-2} \overline{\alpha}_{r,s} C(\overline{F}_{r,s}) + \ldots \\
&\ldots + \sum_{\substack{j=1\\ l>j}}^{n} \sum_{\substack{\{r,s,\ldots,t\}\\ r=1}}^{n-2} \overline{\alpha}_{r,s,\ldots,t} C(\overline{F}_{r,s,\ldots,t}) + \ldots + \sum_{\substack{j=1\\ l>j}}^{n} \alpha^{j,l} C(F_1,\ldots,F_{n-2})
\end{aligned}
$$

and

$$\begin{aligned}\sum_{\substack{j=1\\l>j\\m>l}}^{n} A(\mathbf{x}_{\circ}^{j,l,\dots,m}) &= \sum_{\substack{j=1\\l>j\\ \dots\\m>l}}^{n} \sum_{\substack{i=1\\i\neq(j,l,\dots,m)\\l>j\\ \dots\\m>l}}^{n-k} A_i(x_i) + \sum_{\substack{j=1\\l>j\\ \dots\\m>l}}^{n} \sum_{\substack{\{r,s\}\\r=1}}^{n-k} \overline{\alpha}_{r,s} C(\overline{F}_{r,s}) + \dots \\ &\dots + \sum_{\substack{j=1\\l>j\\ \dots\\m>l}}^{n} \sum_{\substack{\{r,s,\dots,t\}\\r=1}}^{n-k} \overline{\alpha}_{r,s,\dots,t} C(\overline{F}_{r,s,\dots,t}) + \dots \\ &\dots + \sum_{\substack{j=1\\l>j\\ \dots\\m>l}}^{n} \alpha^{j,l,\dots,m} C(F_1, \dots, F_{n-k}).\end{aligned}$$

Therefore, we have our thesis. □

Appendix A

Example: the multivariate dependence

A.1 Copulas and Random Variables

We will use capital letters, such as X and Y, to represent random variables and lower case letters x, y to represent their values. We will say that F is the *distribution function of the random variable* X when for all x in $\overline{\mathbb{R}}$, $F(x) = P[X \leq x]$. We are defining distribution functions of random variables to be right-continuous, even if this is simply a matter of custom and convenience. Left-continuous distribution functions would serve equally as well. A random variable is continuous if its distribution function is continuous.
We will let I denote the unit interval $[0,1]$.
The following theorem shows that the product copula $\Pi(u,v) = uv$ characterizes independent random variables when the distribution functions are continuous. Its proof follows from Sklar's theorem and the observation that X and Y are independent if and only if $H(x,y) = F(x)G(y)$ for all (x,y) in $\overline{\mathbb{R}}^2$.

Theorem A.1.1 *Let X and Y be continuous random variables. Then X and Y are independent if and only if* $C_{XY} = \Pi$.

Much of the usefulness of copulas in the study of non parametric statistics derives from the fact that for strictly monotone transformations of the random variables copulas are either invariant, or change in predictable ways. Recall that if the distribution function of a random variable X is continuous, and if α is a strictly monotone function whose domain contains Ran X, then the distribution function of the random variable $\alpha(X)$ is also continuous. First we recall the theorem 2.4.3 in [81], where the case of strictly increasing transformations has been treated.

Theorem A.1.2 *Let X and Y be continuous random variables with copula* C_{XY}. *If* α *and* β *are strictly increasing on Ran X and Ran Y respectively, then* $C_{\alpha(X)\beta(Y)} = C_{XY}$. *Thus* C_{XY} *is invariant under strictly increasing transformations of X and Y.*

When at least one of α and β is strictly decreasing, we obtain results in which the copula of the random variables $\alpha(X)$ and $\beta(Y)$ is a simple transformation of C_{XY}. Specifically, we have:

Theorem A.1.3 *Let X and Y be continuous random variables with copula C_{XY}. Let α and β be strictly monotone on Ran X and Ran Y respectively.*

1. *If α is strictly increasing and β is strictly decreasing, then*

$$C_{\alpha(X)\beta(Y)}(u,v) = u - C_{XY}(u,1-v).$$

2. *If α is strictly decreasing and β is strictly increasing, then*

$$C_{\alpha(X)\beta(Y)}(u,v) = v - C_{XY}(1-u,v).$$

3. *If α and β are both strictly decreasing, then*

$$C_{\alpha(X)\beta(Y)}(u,v) = u+v-1+C_{XY}(1-u,1-v).$$

A.1.1 Symmetry properties

Let $(a,b) \in \mathbb{R}^2$ and (X,Y) a random pair. We say that X is symmetric about a if the cumulative distribution functions of $(X-a)$ and $(a-X)$ are identical. The following definitions generalize this symmetry concept to the bivariate case:

- X and Y are exchangeable if (X,Y) and (Y,X) are identically distributed;
- (X,Y) is marginally symmetric about (a,b) if X and Y are symmetric about a and b respectively;
- (X,Y) is radially symmetric about (a,b) if $(X-a,Y-b)$ and $(a-X,b-Y)$ follow the same joint cumulative distribution function;
- (X,Y) is jointly symmetric about (a,b) if the pairs of random variables

$$(X-a,Y-b), \quad (a-X,b-Y), \quad (X-a,b-Y) \quad \text{and} \quad (a-X,Y-b)$$

  have a common joint cumulative distribution function.

The following theorem provides conditions on ϕ to ensure that the couple (X,Y) with associated copula C_θ is radially (or jointly) symmetric.

Theorem A.1.4 **(i)** *If X and Y are identically distributed then X and Y are exchangeable. Besides, if (X,Y) is marginally symmetric about (a,b) then:*

(ii) *(X,Y) is radially symmetric about (a,b) if and only if either $\forall u \in I$, $\phi(u) = \phi(1-u)$ or $\forall u \in I$, $\phi(u) = -\phi(1-u)$;*

(iii) *(X,Y) is jointly symmetric about (a,b) if and only if $\forall u \in I$, $\phi(u) = -\phi(1-u)$.*

A.1.2 Survival Copulas

In many applications, the random variables of interest represent the lifetimes of individuals or objects in some population. The probability of an individual living or surviving beyond time x is given by $\overline{F}(x) = P[X > x] = 1 - F(x)$, called *survival function* (or *survivor function*, or *reliability function*), where, as before, F denotes the distribution function of X. When dealing with lifetimes, the natural range of a random variable is often $[0, +\infty)$; however, we will use the term "survival function" for $P[X > x]$ even when the range is $\overline{\mathbb{R}}$.

For a pair (X,Y) of random variables with joint distribution function H, the *joint survival function* is given by $\overline{H}(x,y) = P[X > x, Y > y]$. The margins of $\overline{H}$ are the functions $\overline{H}(x,+\infty)$ and $\overline{H}(-\infty,y)$, which are the univariate survival functions $\overline{F}$ and $\overline{G}$, respectively.

A.2 Concepts of dependence

In this section we note (X,Y) a random pair with joint cdf H, copula C and margins F and G. For the sake of simplicity, we assume that X and Y are exchangeable. Several concepts of dependence have been introduced and characterized in terms of copulas. X and Y are

- Positive Function Dependent (PFD) if for any integrable real-valued function g

$$\mathbb{E}_h[g(X)g(Y)] - \mathbb{E}_h[g(X)]\mathbb{E}_h[g(Y)] \geq 0,$$

  where $\mathbb{E}_h$ is the expectation symbol relative to the density h.

- Positively Quadrant Dependent (PDQ) if $P(X \leq x, Y \leq y) \geq P(X \leq x)P(Y \leq y)$, for all $(x,y) \in \mathbb{R}^2$ or equivalently

$$\forall (u,v) \in I^2, \quad C(u,v) \geq uv. \tag{A.1}$$

- Left Tail Decreasing $(LTD(Y|X))$ if $P(Y \leq y | X \leq x)$ is non-increasing in x for all y, or equivalently, see Theorem 5.2.5 in Nelsen (2006), $u \to C(u,v)/u$ is non-increasing for all $v \in I$.

- Right Tail Increasing $(RTI(Y|X))$ if $P(Y > y | X > x)$ is nondecreasing in x for all y or, equivalently, $u \to (v - C(u,v))/(1-u)$ is non-increasing for all $v \in I$.

- Stochastically Increasing $(SI(Y|X))$ if $P(Y > y | X = x)$ is nondecreasing in x for all y .

- Left Corner Set Decreasing (LCSD) if $P(X \leq x, Y \leq y | X \leq x', Y \leq y')$ is non-increasing in x' and y' for all x and y, or equivalently, see Corollary 5.2.17 in Nelsen (2006), C is a totally positive function of order 2 (TP_2), *i.e.* for all $(u_1,u_2,v_1,v_2) \in I^4$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$, one has

$$C(u_1,v_1)C(u_2,v_2) - C(u_1,v_2)C(u_2,v_1) \geq 0. \tag{A.2}$$

This property is equivalent to Positively Likelihood Ratio Dependent (PLR), which is defined if and only if C is absolutely continuous and its density c satisfies (A.2), with C replaced by c.

- Right Corner Set Increasing (RCSI) if $P(X > x, Y > y | X > x', Y > y')$ is nondecreasing in x' and y' for all x and y, or equivalently, the survival copula $\hat{C}$ associated to C is a totally positive function of order 2.

More broadly, one has the following definition:

Definition A.2.1 *Let A and B be subsets of $[0,1]$. A function C defined on $A \times B$ is said to be totally positive of order k, denoted TP_k, if for all m, $1 \le m \le k$ and all $u_1 < \ldots < u_m$, $v_1 < \ldots < v_m$ ($u_i \in A$, $v_j \in B$)*

$$C\begin{pmatrix} u_1, \ldots, u_m \\ v_1, \ldots, v_m \end{pmatrix} \equiv det \begin{bmatrix} C(u_1, v_1), \ldots, C(u_1, v_m) \\ C(u_m, v_1), \ldots, C(u_m, v_m) \end{bmatrix} \ge 0. \tag{A.3}$$

When the inequalities (A.3) are strict for $m = 1, \ldots, k$, C is called *strictly totally positive of order k (STP_k).*
There are several obvious consequences of the definition.

1. If a and b are nonnegative functions defined, respectively, on A and B and if K is TP_k then $a(u)b(v)C(u,v)$ is TP_k.

2. If g and h are defined on A and B, respectively, and monotone in the same direction, and if C is TP_k on $g(A) \times h(B)$, then $C(g(u), h(v))$ is TP_k on $A \times B$.

The following Corollary 5.2.6 in Nelsen [81] gives us the criteria for tail monotonicity in terms of the partial derivatives of C.

Corollary A.2.2 *Let X and Y be continuous random variables with copula C. Then*

1. *$LTD(Y|X)$ if and only if for any v in I, $\frac{\partial C(u,v)}{\partial u} \le \frac{C(u,v)}{u}$ for almost all u;*

2. *$LTD(X|Y)$ if and only if for any u in I, $\frac{\partial C(u,v)}{\partial v} \le \frac{C(u,v)}{v}$ for almost all v;*

3. *$RTI(Y|X)$ if and only if for any v in I, $\frac{\partial C(u,v)}{\partial u} \ge \frac{v - C(u,v)}{(1-u)}$ for almost all u;*

4. *$RTI(X|Y)$ if and only if for any u in I, $\frac{\partial C(u,v)}{\partial v} \ge \frac{u - C(u,v)}{(1-v)}$ for almost all v.*

When X and Y are exchangeable, there is no reason to distinguish $SI(Y|X)$ and $SI(X|Y)$, which will be both noted SI. Similarly, we will denote LTD the equivalent properties $LTD(Y|X)$ and $LTD(X|Y)$, and RTI, $RTI(Y|X)$ or $RTI(X|Y)$. The following theorem in [3] is devoted to the study of properties of positive dependence of any pair (X,Y) associated with the copula C_θ defined by (A.5). Similar results can be established for the corresponding concepts of negative dependence.

Theorem A.2.3 *Let* $\theta > 0$ *and* (X,Y) *a random pair with copula* C_θ.

- *X and Y are PFD.*
- *X and Y are PQD if and only if either* $\forall u \in I, \phi(u) \geq 0$ *or* $\forall u \in I, \phi(u) \leq 0$.
- *X and Y are LTD if and only if* $\phi(u)/u$ *is monotone.*
- *X and Y are RTI if and only if* $\phi(u)/(u-1)$ *is monotone.*
- *X and Y are LCSD if and only if they are LTD.*
- *X and Y are RCSI if and only if they are RTI.*
- *X and Y are SI if and only if* $\phi(u)$ *is either concave or convex.*
- *X and Y have the* TP_2 *density property if and only if they are SI.*

A.3 Multivariate dependence modeling using copulas

Analyzing the dependence between the components $X_1,\ldots,X_n$ of a random vector $\mathbf{X}$ is subject to various lines of statistical research. For this purpose, copula functions (or simply copulas) have been introduced by Sklar (1959) which allow for a separation between the marginal distributions and the dependence structure. Moreover, construction principles for copulas based on certain functions ("generator functions") have gained in importance. For example, Archimedean copulas are constructed by (a possibly rather complicated) composition of a specific generator function and its corresponding pseudo inverse. In contrast to that, Amblard and Girard (2002) discuss a very simple construction principle of copulas on the basis of certain generator functions and a "dependence parameter" θ. Specific generalized Farlie - Gumbel (or Sarmanov) copulas are generated by a single function (so-called generator or generator function) defined on the unit interval. An alternative approach to generalize the FGM family of copulas is to consider the semi-parametric family of symmetric copulas. This family is generated by a univariate function, determining the symmetry (radial symmetry, joint symmetry) and dependence property (quadrant dependence, total positivity) of copulas.
A multivariate data set, which exhibit complex patterns of dependence, particularly in the tails, can be modelled using a cascade of lower-dimensional copulas. Moreover, these copulas allow for a direct characterization of symmetry properties, ordering properties and association measures. Recently, Amblard and Girard (2004) also state a semiparametric estimation method for the underlying generator function. However, the parameter θ is not identified in the semiparametric context.
One of the most popular parametric families of copulas is the Farlie-Gumbel-Morgenstern (FGM) family defined when $\theta \in [-1,1]$ by

$$C_\theta^{FGM}(u,v) = uv + \theta u(1-u)v(1-v) \tag{A.4}$$

and studied in Farlie (1960), Gumbel (1960) and Morgenstern (1956).
An alternative approach to generalize the FGM family of copulas is to consider the semi-parametric family of symmetric copulas defined by

$$C^{SP}_{\theta,\phi}(u,v) = uv + \theta\phi(u)\phi(v), \tag{A.5}$$

with $\theta \in [-1,1]$ and ϕ is a function on $I = [0,1]$. It was first introduced in Rodríguez-Lallena (1992), and extensively studied in Amblard and Girard (2002, 2005).

A.3.1 The general case

Many of the dependence properties encountered in earlier sections have natural extensions to the multivariate case. In three or more dimensions, rather than quadrants we have "orthants", and the generalization of quadrant dependence is known as *orthant dependence*.
So we are going to examine the role played by n-copulas in the study of multivariate dependence.

Definition A.3.1 *Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be an n-dimensional random vector.*

1. $\mathbf{X}$ *is positively lower orthant dependent (PLOD) if for all* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *in* $\mathbb{R}^n$*,*

$$P[\mathbf{X} \leq \mathbf{x}] \geq \prod_{i=1}^{n} P[X_i \leq x_i]; \tag{A.6}$$

2. $\mathbf{X}$ *is positively upper orthant dependent (PUOD) if for all* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *in* $\mathbb{R}^n$*,*

$$P[\mathbf{X} > \mathbf{x}] \geq \prod_{i=1}^{n} P[X_i > x_i]; \tag{A.7}$$

3. $\mathbf{X}$ *is positively orthant dependent (POD) if for all* $\mathbf{x}$ *in* $\mathbb{R}^n$*, both (A.6) and (A.7) hold.*

Negative lower orthant dependence (NLOD), negative upper orthant dependence (PUOD) and negative orthant dependence (NOD) are defined analogously, by reversing the sense of the inequalities in (A.6) and (A.7).
For $n = 2$, (A.6) and (A.7) are equivalent to (A.1).
The following definitions are from Brindley and Thompson (1972), Harris (1970), Joe (1997).

Definition A.3.2 *Let $X = (X_1, X_2, \ldots, X_n)$ be an n-dimensional random vector and let the sets A and B partition of {1,2,. . . ,n}.*

1. $LTD(\mathbf{X}_B|\mathbf{X}_A)$ *if* $P[\mathbf{X}_B \leq \mathbf{x}_B|\mathbf{X}_A \leq \mathbf{x}_A]$ *is nonincreasing in* $\mathbf{x}_A$ *for all* $\mathbf{x}_B$*;*
2. $RTI(\mathbf{X}_B|\mathbf{X}_A)$ *if* $P[\mathbf{X}_B > \mathbf{x}_B|\mathbf{X}_A > \mathbf{x}_A]$ *is nondecreasing in* $\mathbf{x}_A$ *for all* $\mathbf{x}_B$*;*
3. $SI(\mathbf{X}_B|\mathbf{X}_A)$ *if* $P[\mathbf{X}_B > \mathbf{x}_B|\mathbf{X}_A = \mathbf{x}_A]$ *is nondecreasing in* $\mathbf{x}_A$ *for all* $\mathbf{x}_B$*;*

4. *LCSD*$(\mathbf{X})$ *if* $P[\mathbf{X} \leq \mathbf{x} | \mathbf{X} \leq \mathbf{x}']$ *is nonincreasing in* $\mathbf{x}'$ *for all* $\mathbf{x}$*;*

5. *RCSI*$(\mathbf{X})$ *if* $P[\mathbf{X} > \mathbf{x} | \mathbf{X} > \mathbf{x}']$ *is nondecreasing in* $\mathbf{x}'$ *for all* $\mathbf{x}$*.*

We recall that for $\mathbf{x} \in \mathbb{R}^n$ a phrase such as "nondecreasing in $\mathbf{x}$" means nondecreasing in each component x_i, $i = 1, 2, \ldots, n$.

In the bivariate case, the corner set monotonicity properties were expressible in terms of total positivity (Corollary 5.2.16 in [81]). The same is true in the multivariate case with the following generalization of total positivity: a function f from $\overline{\mathbb{R}}^n$ to $\mathbb{R}$ is *multivariate totally positive of order two* (MTP_2) if

$$f(\mathbf{x} \vee \mathbf{y}) f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x}) f(\mathbf{y}) \tag{A.8}$$

for all $\mathbf{x}, \mathbf{y} \in \overline{\mathbb{R}}^n$, where

$$\mathbf{x} \vee \mathbf{y} = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots, \max\{x_n, y_n\}),$$

$$\mathbf{x} \wedge \mathbf{y} = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_n, y_n\}).$$

Lastly, $\mathbf{X}$ is positively likelihood ratio dependent if its joint n-dimensional density h is MTP_2. A first one-parameter multivariate extension of the class of copulas given by (A.4) is

$$C^{SP}_{\theta,\phi_i}(\mathbf{u}) = \prod_{i=1}^{n} u_i + \theta \prod_{i=1}^{n} \phi_i(u_i), \qquad \mathbf{u} \in I^n, \tag{A.9}$$

where $\theta \in \mathbb{R}$ and ϕ_i, $1 \leq i \leq n$, are n non-zero absolutely continuous functions such that $\phi_i(0) = \phi_i(1) = 0$. Note that all the $k-$dimensional margins, $2 \leq k < n$, are $\prod^k$. The density function of (A.9) is

$$c^{SP}_{\theta,\phi_i}(\mathbf{u}) = 1 + \theta \prod_{i=1}^{n} \phi_i'(u_i), \tag{A.10}$$

whose parameter θ has the admissible range

$$-1/\sup_{\mathbf{u} \in D^+} (\prod_{i=1}^{n}) \phi_i'(u_i) \leq \theta \leq -1/\inf_{\mathbf{u} \in D^-} (\prod_{i=1}^{n}) \phi_i'(u_i),$$

where $D^- = \{\mathbf{u} \in I^n : \prod_{i=1}^{n} \phi_i'(u_i) < 0\}$ and $D^+ = \{\mathbf{u} \in I^n : \prod_{i=1}^{n} \phi_i'(u_i) > 0\}$.
The survival function and the survival n-copula associated with the n-copula C^{SP}_{θ,ϕ_i} are given by

$$\overline{C}^{SP}_{\theta,\phi_i}(\mathbf{u}) = \prod_{i=1}^{n} (1 - u_i) + (-1)^n \theta \prod_{i=1}^{n} \phi_i(u_i)$$

and

$$\hat{C}^{SP}_{\theta,\phi_i}(\mathbf{u}) = \prod_{i=1}^{n} u_i + (-1)^n \theta \prod_{i=1}^{n} \phi_i(1 - u_i),$$

respectively, for every $\mathbf{u} \in I^n$. Let C^{SP}_{θ,ϕ_i} be the corresponding family of n-copulas given by (A.9). Then, C^{SP}_{θ,ϕ_i} is positively ordered if and only if $\prod_{i=1}^n \phi_i(u_i) \geq 0$ for all $\mathbf{u}$ in I^n. Let

$$C^{SP}_{\theta_1,\phi_i}(\mathbf{u}) = \prod_{i=1}^n u_i + \theta_1 \prod_{i=1}^n \phi_i(u_i) \text{ and } C^{SP}_{\theta_2,\phi_i}(\mathbf{u}) = \prod_{i=1}^n u_i + \theta_2 \prod_{i=1}^n \gamma_i(u_i)$$

be two $n-$copulas. Then, $C^{SP}_{\theta_1,\phi_i}$ is more PLOD (respectively, PUOD) than $C^{SP}_{\theta_2,\phi_i}$ if and only if

$$\theta_1 \prod_{i=1}^n \phi_i(u_i) \geq \theta_2 \prod_{i=1}^n \gamma_i(u_i)$$

(respectively, $(-1)^n \theta_1 \prod_{i=1}^n \phi_i(1-u_i) \geq (-1)^n \theta_2 \prod_{i=1}^n \gamma_i(1-u_i)$). Much of the theory of bivariate dependence presents considerable difficulty when one attempts to generalize it to more than two dimensions. We want to extend in this paper to more than two random variables, $X_1, \ldots, X_n$ the problem of dependence.
The following theorem is from Dolati and Úbeda-Flores (2006) [33].

Theorem A.3.3 *Let* $\mathbf{X}$ *be an n-dimensional random vector whose associated n-copula* C^{SP}_{θ,ϕ_i} *is defined by* (A.9) *and such that the functions* ϕ_i*,* $i = 1, \ldots, n$ *and* θ *are non-negative. Let* $\mathbf{X}_A$ *and* $\mathbf{X}_B$ *be two subsets of* $\mathbf{X}$ *as in the preceding definition. Then:*

(i) $LTD(\mathbf{X}_B|\mathbf{X}_A)$ *if and only if* $\phi_i(u) \geq u\phi_i'(u)$ *for all* $u \in I$ *and for every* $i \in A$*;*

(ii) $RTI(\mathbf{X}_B|\mathbf{X}_A)$ *if and only if* $\phi_i(u) \geq (u-1)\phi_i'(u)$ *for all* $u \in I$ *and for every* $i \in A$*;*

(iii) $SI(\mathbf{X}_B|\mathbf{X}_A)$ *if and only if* $(-1)^n \phi_i''(u) \prod_{h \in A-\{i\}} \phi_h'(u_h) \geq 0$ *for every* $i \in A$*, and* u*,* $u_h \in I$*.*

A.3.2 Other properties

Now we want to study the previous properties extended to n dimensions, using the copula approach, in particular with regard to the family given by (A.9). So, we prove the following theorem.

Theorem A.3.4 *Let* $\mathbf{X}$ *be an n-dimensional random vector whose associated n-copula* C^{SP}_{θ,ϕ_i} *is defined by* (A.9) *and such that the functions* ϕ_i*,* $i = 1, \ldots, n$ *and* θ *are non-negative. Let* $\mathbf{X}_A$ *and* $\mathbf{X}_B$ *be two subsets of* $\mathbf{X}$ *as in the preceding theorem. Then:*

(i) $\mathbf{X}$ *is PFD if n is even;*

(ii) $\mathbf{X}$ *is PLOD;*

(iii) $\mathbf{X}$ *is* MTP_2 *if* $\mathbf{X}_A$ *and* $\mathbf{X}_B$ *are LTD;*

(iv) $\mathbf{X}$ *is RCSI if* $\mathbf{X}_A$ *and* $\mathbf{X}_B$ *are RTI;*

(v) $\mathbf{X}_A$ *and* $\mathbf{X}_B$ *are SI if and only if* $\mathbf{X}$ *has the* MTP_2 *density property.*

Proof:

(i) Let g be an integrable real-valued function on I. The density distribution c^{SP}_{θ,ϕ_i} of the cumulative distribution C^{SP}_{θ,ϕ_i} is given by (A.10). Routine calculations yield

$$\mathbf{E}_{c^{SP}_{\theta,\phi_i}}[g(X_1)\dots g(X_n)] - \mathbf{E}_{c^{SP}_{\theta,\phi_1}}[g(X_1)]\dots\mathbf{E}_{c^{SP}_{\theta,\phi_n}}[g(X_n)] = \theta\left[\int_0^1 g(t)\phi_i'(t)dt\right]^n \geq 0,$$

since $\theta \geq 0$ and n is even.

(ii) The vector $\mathbf{X}$ is *PLOD* if and only if the uniform I-margins vector $\mathbf{U}$ with distribution C^{SP}_{θ,ϕ_i} is *PLOD*. For $\mathbf{U}$, condition (A.6) simply rewrites $C(u_1,\dots,u_n) \geq u_1\dots u_n$, that is $\theta\prod_{i=1}^n \phi_i(u_i) \geq 0$, $\forall u_i \in I$ and the conclusion follows.

(iii) Let the partition of $\{1,2,\dots,n\}$ be in two subsets A and B, such that $\max(u_i,v_i) = u_i$ and $\max(u_j,v_j) = v_j$, $\forall i \in A$ and $\forall j \in B$ respectively. So,

$$C^{SP}_{\theta,\phi_i}(\mathbf{u}\vee\mathbf{v}) = C^{SP}_{\theta,\phi_i}(\dots,u_i,\dots,v_j,\dots) = \prod_{\substack{i\in A\\ j\in B}} u_i v_j + \theta\prod_{\substack{i\in A\\ j\in B}}\phi_i(u_i)\phi_j(v_j) \qquad \mathbf{u},\mathbf{v}\in I^n,$$

and

$$C^{SP}_{\theta,\phi_i}(\mathbf{u}\wedge\mathbf{v}) = C^{SP}_{\theta,\phi_i}(\dots,u_i,\dots,v_j,\dots) = \prod_{\substack{i\in A^C\\ j\in B^C}} u_i v_j + \theta\prod_{\substack{i\in A^C\\ j\in B^C}}\phi_i(u_i)\phi_j(v_j) \qquad \mathbf{u},\mathbf{v}\in I^n.$$

We observe that $A^C = B$ and $A \cup A^C = \{1,\dots,n\}$. Therefore

$$\begin{aligned}
&C^{SP}_{\theta,\phi_i}(\mathbf{u}\vee\mathbf{v})C^{SP}_{\theta,\phi_i}(\mathbf{u}\wedge\mathbf{v}) - C^{SP}_{\theta,\phi_i}(\mathbf{u})C^{SP}_{\theta,\phi_i}(\mathbf{v}) = \\
&= \Big(\prod_{\substack{i\in A\\ j\in B}} u_i v_j + \theta\prod_{\substack{i\in A\\ j\in B}}\phi_i(u_i)\phi_j(v_j)\Big)\Big(\prod_{\substack{i\in A^C\\ j\in B^C}} u_i v_j + \theta\prod_{\substack{i\in A^C\\ j\in B^C}}\phi_i(u_i)\phi_j(v_j)\Big) - \\
&- \Big(\prod_{i=1}^n u_i + \theta\prod_{i=1}^n \phi_i(u_i)\Big)\Big(\prod_{i=1}^n v_i + \theta\prod_{i=1}^n\phi_i(v_i)\Big) = \\
&= \Big(\prod_{i=1}^n u_i v_i + \theta^2\prod_{i=1}^n \phi_i(u_i)\phi_i(v_i) + \theta\prod_{\substack{i\in A\\ j\in B}} u_i v_j \prod_{\substack{i\in A^C\\ j\in B^C}}\phi_i(u_i)\phi_j(v_j) + \\
&+ \theta\prod_{\substack{i\in A^C\\ j\in B^C}} u_i v_j \prod_{\substack{i\in A\\ j\in B}}\phi_i(u_i)\phi_j(v_j)\Big) - \Big(\prod_{i=1}^n u_i v_i + \theta^2\prod_{i=1}^n\phi_i(u_i)\phi_i(v_i) + \\
&+ \theta\prod_{i=1}^n u_i\phi_i(v_i) + \theta\prod_{i=1}^n v_i\phi_i(u_i)\Big).
\end{aligned}$$

So, by rearranging the expression, we have

$$\begin{aligned}
&C^{SP}_{\theta,\phi_i}(\mathbf{u}\vee\mathbf{v})C^{SP}_{\theta,\phi_i}(\mathbf{u}\wedge\mathbf{v})-C^{SP}_{\theta,\phi_i}(\mathbf{u})C^{SP}_{\theta,\phi_i}(\mathbf{v})=\\
&=\theta\prod_{i=1}^{n}u_iv_i\Big(\prod_{\substack{i\in A\\ j\in B}}\frac{\phi_i(u_i)\phi_j(v_j)}{u_iv_j}+\prod_{\substack{i\in A^C\\ j\in B^C}}\frac{\phi_i(u_i)\phi_j(v_j)}{u_iv_j}-\prod_{i=1}^{n}\frac{\phi_i(u_i)}{u_i}-\prod_{i=1}^{n}\frac{\phi_i(v_i)}{v_i}\Big)=\\
&=\theta\prod_{i=1}^{n}u_iv_i\left[\prod_{i\in A}\frac{\phi_i(u_i)}{u_i}-\prod_{i\in A}\frac{\phi_i(v_i)}{v_i}\right]\left[\prod_{j\in B}\frac{\phi_j(v_j)}{v_j}-\prod_{j\in B}\frac{\phi_j(u_j)}{u_j}\right].
\end{aligned}$$

Now $\frac{\phi_i(u)}{u}$ is derivable because the ratio of two derivable functions and we have

$$\frac{d}{du}\Big(\prod_{i\in A}\frac{\phi_i(u)}{u}\Big)=\Big(\frac{\phi_i'(u)u-\phi_i(u)}{u^2}\Big)\prod_{h\in A\setminus\{i\}}\frac{\phi_h(u_h)}{u_h}\leq 0,\qquad \forall u\in I$$

for the hypothesis of LTD. The same happens to the other factor. So we have two monotonically decreasing functions and, as a consequence, MTP_2 property, that is our thesis.

(iv) It is similar to (iii). In fact **X** is *RCSI* if and only if the survival copula associated to C, $\hat{C}^{SP}_{\theta,\phi_i}(\mathbf{u})=\prod_{i=1}^{n}u_i+(-1)^n\theta\prod_{i=1}^{n}\phi_i(1-u_i)$ is MTP_2. So we have

$$\begin{aligned}
&\hat{C}^{SP}_{\theta,\phi_i}(\mathbf{u}\vee\mathbf{v})\hat{C}^{SP}_{\theta,\phi_i}(\mathbf{u}\wedge\mathbf{v})-\hat{C}^{SP}_{\theta,\phi_i}(\mathbf{u})\hat{C}^{SP}_{\theta,\phi_i}(\mathbf{v})=\\
&=(-1)^n\theta\prod_{i=1}^{n}u_iv_i\left[\prod_{i\in A}\frac{\phi_i(1-u_i)}{u_i}-\prod_{i\in A}\frac{\phi_i(1-v_i)}{v_i}\right]\left[\prod_{j\in B}\frac{\phi_j(1-v_j)}{v_j}-\prod_{j\in B}\frac{\phi_j(1-u_j)}{u_j}\right].
\end{aligned}$$

Now we do the same thought as in the previous case:

$$\Big(\frac{\phi_i(1-u)}{u}\Big)'=\frac{-u\phi_i'(1-u)-\phi_i(1-u)}{u^2}.$$

We use RTI property, by putting $u'=1-u$ and in fact we have

$$-u\phi_i'(1-u)-\phi_i(1-u)=(u'-1)\phi_i'(u')-\phi_i(u')\leq 0,\qquad \forall u'\in I$$

and so we have MTP_2 property again.

(v) X has the MTP_2 density property if and only if the density of the copula verifies

$$c^{SP}_{\theta,\phi_i}(\mathbf{u}\vee\mathbf{v})c^{SP}_{\theta,\phi_i}(\mathbf{u}\wedge\mathbf{v})-c^{SP}_{\theta,\phi_i}(\mathbf{u})c^{SP}_{\theta,\phi_i}(\mathbf{v})\geq 0,\tag{A.11}$$

which rewrites solving the calculations like in the point (iii)

$$c^{SP}_{\theta,\phi_i}(\mathbf{u}\vee\mathbf{v})c^{SP}_{\theta,\phi_i}(\mathbf{u}\wedge\mathbf{v})-c^{SP}_{\theta,\phi_i}(\mathbf{u})c^{SP}_{\theta,\phi_i}(\mathbf{v})=\Big(1+\theta\prod_{\substack{i\in A\\ j\in B}}\phi_i'(u_i)\phi_j'(v_j)\Big)\Big(1+\theta\prod_{\substack{i\in A^C\\ j\in B^C}}\phi_i'(u_i)\phi_j'(v_j)\Big)+$$

$$-\Big(1+\theta\prod_{i=1}^n\phi_i'(u_i)\Big)\Big(1+\theta\prod_{i=1}^n\phi_i'(v_i)\Big)=\Big(1+\theta^2\prod_{i=1}^n\phi_i'(u_i)\phi_i'(v_i)+\theta\prod_{\substack{i\in A^C\\ j\in B^C}}\phi_i'(u_i)\phi_j'(v_j)+$$

$$+\theta\prod_{\substack{i\in A\\ j\in B}}\phi_i'(u_i)\phi_j'(v_j)\Big)-\Big(1+\theta^2\prod_{i=1}^n\phi_i'(u_i)\phi_i'(v_i)+\theta\prod_{i=1}^n\phi_i'(v_i)+\theta\prod_{i=1}^n\phi_i'(u_i)\Big)=$$

$$=\theta\Big(\prod_{\substack{i\in A\\ j\in B}}\phi_i'(u_i)\phi_j'(v_j)+\prod_{\substack{i\in A^C\\ j\in B^C}}\phi_i'(u_i)\phi_j'(v_j)-\prod_{i=1}^n\phi_i'(v_i)-\prod_{i=1}^n\phi_i'(u_i)\Big)=$$

$$=\theta\left[\prod_{i\in A}\phi_i'(u_i)-\prod_{i\in A}\phi_i'(v_i)\right]\left[\prod_{j\in B}\phi_j'(v_j)-\prod_{j\in B}\phi_j'(u_j)\right].$$

Now,

$$\frac{d}{du}\Big(\prod_{i\in A}\phi_i'(u_i)\Big)=\pm\,\phi_i''(u)\prod_{h\in A\setminus\{i\}}\phi_h'(u_h)\geq 0$$

for our hypothesis. The same happens to the other factor and so we have proved our thesis. Conversely, assume that (A.11) holds. So, the function $\prod_{i\in A}\phi_i'$ is either increasing or decreasing and then $\mathbf{X}_A$ and $\mathbf{X}_B$ are *SI*.

□

Example We can consider the example 2.2 proposed by Dolati and Úbeda-Flores in [33].
Let $f_i(u)=u^b(1-u)^a$, $1\leq i\leq 3$, with $a,b\geq 1$. Then, for all $(u_1,u_2,u_3)\in[0,1]^3$, the function

$$C^{SP}_{\theta,\phi_i}(u_1,u_2,u_3)=u_1u_2u_3+\theta u_1^b(1-u_1)^a u_2^b(1-u_2)^a u_3^b(1-u_3)^a$$

is a 3-copula. In particular, if $a=b=1$, we have a one-parametric trivariate extension of the FGM family with $\theta\in[-1,1]$. Suppose $\theta>0$, then, from theorem 2.1 in [33] we have that C^{SP}_{θ,ϕ_i} is LTD if and only if $b=1$, C^{SP}_{θ,ϕ_i} is RTI if and only if $a=1$, and C^{SP}_{θ,ϕ_i} is *SI* if and only if $a=b=1$.
As a consequence from the theorem 4, we can also conclude that C^{SP}_{θ,ϕ_i} is MTP_2 if $b=1$. If $a=1$ C^{SP}_{θ,ϕ_i} is *RCSI* and it has the MTP_2 density property if and only if $a=b=1$. Moreover C^{SP}_{θ,ϕ_i} is *PLOD*, but it is not PFD.

Appendix B

Open problems

In this overview about the multivariate aggregation functions connected with copulas and fuzzy measures, interesting and important open problems come from our discussion. We divide them into two groups and we wish all the possible solvers great success and satisfaction from their solution.

B.1 Aggregation functions

Problem B.1.1 *An interesting generalization of copulas is the notion of semi-copula, namely a binary operation on* $[0,1]$ *that satisfies the boundary condition* $\forall x \in [0,1]$ $C(x,1) = C(1,x) = x$ *and the property of increasingness in each place, that is* $C(x,y) \leq C(x',y')$ *for all* $x \leq x'$ *and* $y \leq y'$*. But, as it has been shown in [38], the first generalization of copulas has been the concept of quasi-copula. In detail, a quasi-copula* $Q : [0,1]^2 \to [0,1]$ *satisfies the conditions of semi-copula and it is also 1-Lipschitz:* $|C(x,y) - C(x',y')| \leq |x - x'| + |y - y'|$ *for all* $x, x', y, y' \in [0,1]$*. The study of quasi-copula as an aggregation operator is an open problem.*

Problem B.1.2 *There is a close link between supermodularity and Schur-concavity, but this is another open problem.*

Problem B.1.3 *In Chapter 7 we have introduced two properties which are stronger than the monotonicity of aggregation functions, with some representation results and with an application for constructing copulas. Anyway, it is not clear whether there are strongly* 3*-monotone copulas different from the product* Π*. Also it is still open whether/how the conditions of Theorem B.1.6 can be relaxed yielding still the same result — is the strong k-monotonicity of A sufficient?*

Problem B.1.4 *By generalizing the procedures in [35], we denote by* Θ *the class of all increasing functions* $f : [0,1] \to [0,1]$*. Given* $f^i_j, \forall i, j = 1,2,3 \in \Theta$ *and a trivariate operation H* *on $[0,1]$, let F be the mapping defined on $[0,1]^3$ by*

$$F(x_1,x_2,x_3) := H(A(f_1^1(x_1), f_2^1(x_2), f_3^1(x_3)), B(f_1^2(x_1), f_2^2(x_2), f_3^2(x_3)), C(f_1^3(x_1), f_2^3(x_2), f_3^3(x_3))), \tag{B.1}$$

for all A, B and C in the class of trivariate aggregation operators. The function F is called generalized composition of (A,B,C) with respect to the $10-$tuple (f_j^i, H), which is called generating system. The prefix "generalized" is used here to distinguish the function F from the classical composition that is obtained when $f_j^i = id_{[0,1]}$.
Our aim is to establish which conditions on the generating system ensure that, for every choice of A, B and C in a given subset (for instance, in the subset of copulas), F is also $3-$monotone agop.

Problem B.1.5 *Let $f\colon [0,1] \to [0,1]$ be an increasing function, such that $f(0) = 0$ and $f(1) = 1$ and let $B\colon [0,1]^m \to [0,1]$ be a 3-monotone aggregation function. Is the composite function $f(\mathbf{x}) = f(B(\mathbf{x}))$ a 3-monotone aggregation function?*

Problem B.1.6 *An extension of the previous problem for k-monotone aggregation functions is the following one. Let $A\colon [0,1]^n \to [0,1]$ be a totally monotone aggregation function, and let $B_1,\ldots,B_n\colon [0,1]^m \to [0,1]$ be k-monotone aggregation functions. Is the composite function $D\colon [0,1]^m \to [0,1]$ given by $D(\mathbf{x}) = A(B_1(\mathbf{x}),\ldots,B_n(\mathbf{x}))$ a k-monotone aggregation function? This method can be applied to the construction of k-dimensional copulas (i.e., k-monotone aggregation functions $C\colon [0,1]^k \to [0,1]$ satisfying*

$$C(x,1,\ldots,1) = C(1,x,\ldots,1) = C(1,\ldots,1,x) = x$$

for all $x \in [0,1]$) in a way similar to Theorem 4.3.9.

Example B.1.7 *Consider the totally monotone aggregation function $A\colon [0,1]^n \to [0,1]$ given by $A(\mathbf{x}) = x_1^{p_1}\cdots x_n^{p_n}$, where $p_1,\ldots,p_n \in \mathbb{N}\cup\{0\}$ and $p = \sum p_i > 0$. Then for all k-dimensional copulas $C_1,\ldots,C_n\colon [0,1]^k \to [0,1]$, the aggregation function $C\colon [0,1]^k \to [0,1]$ given by $C(\mathbf{x}) = A(C_1(\tau(\mathbf{x})),\ldots,C_n(\tau(\mathbf{x})))$, where $\tau\colon [0,1]^k \to [0,1]^k$ is given by $\tau(\mathbf{x}) = (x_1^{1/p},\ldots,x_k^{1/p})$, is a k-dimensional copula. This result can be derived also from [64]. For example, for $n = 2$ and $A(x,y) = x\cdot y^2$ (i.e., $p = 3$) and for the ternary copulas $C_1 = M$ (i.e., $M(x,y,z) = \min(x,y,z)$) and $C_2 = \Pi$ (i.e., $\Pi(x,y,z) = xyz$), the composite function $C\colon [0,1]^3 \to [0,1]$ given by*

$$C(x,y,z) = \min\left(x(yz)^{\frac{2}{3}}, y(xz)^{\frac{2}{3}}, z(xy)^{\frac{2}{3}}\right)$$

is a ternary copula.

B.2 Fuzzy measures

Problem B.2.1 *Let us consider the following situation which comes from the Theorem 6.3.4, that is a mapping* $\varphi : \mathcal{F}(X) \to [0,1]$ *defined for all* $f \in \mathcal{F}(X)$ *by*

$$\varphi(f) = (C)\int f dP,$$

which is an SM-evaluator on $\mathcal{F}(X)$*, where* $(C)\int f dP$ *stands for the Choquet integral of the fuzzy-valued function* f *with respect to* P*. Is this mapping also a* T_P*-evaluator?*
We know that P *is* T_P*-evaluator, i.e., by posing* $\alpha := \alpha_1 \wedge \alpha_2$

$$P(\{x \in X : (f \wedge g)(x) \geq \alpha\}) \geq P(\{x \in X : f(x) \geq \alpha_1\}) \cdot P(\{x \in X : g(x) \geq \alpha_2\}).$$

We recall that we are working in the discrete case and so we are considering the following partitions of the interval $[0,1]$*:*

1. $s^{(n)} = \{s_1, \cdots, s_n\}$;
2. $t^{(n)} = \{t_1, \cdots, t_n\}$;
3. $y^{(n)} = \{s_1 \wedge t_1, \cdots, s_n \wedge t_n\}$;
4. $z^{(n)} = \{s_1 \vee t_1, \cdots, s_n \vee t_n\}$;

and respectively we have

1. $f = \{s_1 x_1, \cdots, s_n x_n\}$;
2. $g = \{t_1 x_1, \cdots, t_n x_n\}$;
3. $f \wedge g = \{(s_1 \wedge t_1)x_1, \cdots, (s_n \wedge t_n)x_n\}$;
4. $f \vee g = \{(s_1 \vee t_1)x_1, \cdots, (s_n \vee t_n)x_n\}$;

Hence, we see the following situation:

1. $(C)\int f dP = \int_0^1 P(\{x \in X : f(x) \geq \alpha\})d\alpha = \sum_{i=1}^n \int_{s_{\pi(i-1)}}^{s_{\pi(i)}} P(\{x \in X : f(x) \geq s_{\pi(i)}\})d\alpha$;
2. $(C)\int g dP = \int_0^1 P(\{x \in X : g(x) \geq \alpha\})d\alpha = \sum_{i=1}^n \int_{t_{\pi(i-1)}}^{t_{\pi(i)}} P(\{x \in X : g(x) \geq t_{\pi(i)}\})d\alpha$;
3. $(C)\int (f \wedge g) dP = \sum_{i=1}^n \int_{y_{\pi(i-1)}}^{y_{\pi(i)}} P(\{x \in X : (f \wedge g)(x) \geq y_{\pi(i)}\})d\alpha$;
4. $(C)\int (f \vee g) dP = \sum_{i=1}^n \int_{z_{\pi(i-1)}}^{z_{\pi(i)}} P(\{x \in X : (f \vee g)(x) \geq z_{\pi(i)}\})d\alpha$;

where $\pi_s : s^{(n)} \to s^{(n)}$ *is a permutation of* $s^{(n)}$ *with* $s_{\pi(1)} \leq s_{\pi(2)} \leq \cdots \leq s_{\pi(n)}$ *and, by convention,* $s_0 = 0$*. We take a similar permutation also for* $t^{(n)}$*,* $x^{(n)}$ *and* $z^{(n)}$*.*
Now, by posing respectively $p_i = P(\{x \in X : f(x) \geq s_{\pi(i)}\})$*,* $q_i = P(\{x \in X : g(x) \geq t_{\pi(i)}\})$*,* $p'_i = P(\{x \in X : (f \wedge g)(x) \geq y_{\pi(i)}\})$ *and* $q'_i = P(\{x \in X : (f \vee g)(x) \geq z_{\pi(i)}\})$*, we have:*

1. $(C)\int f dP = \sum_{i=1}^{n} p_i a_i$, *where* $a_i = s_{\pi(i)} - s_{\pi(i-1)}$;

2. $(C)\int g dP = \sum_{i=1}^{n} q_i b_i$, *where* $b_i = t_{\pi(i)} - t_{\pi(i-1)}$;

3. $(C)\int (f\wedge g) dP = \sum_{i=1}^{n} p'_i d_i$, *where* $d_i = y_{\pi(i)} - y_{\pi(i-1)}$;

4. $(C)\int (f\vee g) dP = \sum_{i=1}^{n} q'_i e_i$, *where* $e_i = z_{\pi(i)} - z_{\pi(i-1)}$.

and we have

$$T_P\Big((C)\int f dP, (C)\int g dP\Big) = \int_0^1 P(\{x\in X : f(x)\geq \alpha\})d\alpha \cdot \int_0^1 P(\{x\in X : g(x)\geq \alpha\})d\alpha =$$
$$= \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} q_i b_i \leq \sum_{i=1}^{n} p'_i d_i = (C)\int (f\wedge g)dP.$$

Since P is a T_P-evaluator we have $p_i q_i \leq p'_i$ $\forall i = 1,\ldots,n$. In order to prove that the Choquet integral of the fuzzy-valued function f with respect to P is a T_P-evaluator we need to show either that

$$a_i \sum_{i=1}^{n} b_i \leq d_i \qquad \text{or that} \qquad b_i \sum_{i=1}^{n} a_i \leq d_i \qquad \forall i = 1,\ldots,n.$$

Problem B.2.2 *Consider $X = \{x_1,\cdots,x_n\}$. Let P be a fuzzy measure which is both a T_P-(S_P)-evaluator and supermodular on 2^X. Is a mapping $\varphi : \mathcal{F}(X) \to [0,1]$ defined for all $f \in \mathcal{F}(X)$ by*

$$\varphi(f) = (C)\int f dP,$$

both a T_P-(S_P)-evaluator and a T_PS_P-supermodular evaluator on $\mathcal{F}(X)$, where $(C)\int f dP$ stands for the Choquet integral of the fuzzy-valued function f with respect to P? In particular is it possible the following relation?

$$(C)\int T_P(f,g)dP + (C)\int S_P(f,g)dP \geq (C)\int f dP + (C)\int g dP \qquad \text{(B.2)}$$

We can consider the following partitions of the interval $[0,1]$:

1. $s^{(n)} = \{s_1,\cdots,s_n\}$;

2. $t^{(n)} = \{t_1,\cdots,t_n\}$;

3. $y'^{(n)} = \{s_1t_1,\cdots,s_nt_n\}$;

4. $z'^{(n)} = \{s_1+t_1-s_1t_1,\cdots,s_n+t_n-s_nt_n\}$;

and respectively we have

1. $f = \{s_1x_1,\cdots,s_nx_n\}$;

2. $g = \{t_1x_1,\cdots,t_nx_n\}$;

3. $T_P(f,g) = \{(s_1t_1)x_1, \cdots, (s_nt_n)x_n\}$;

4. $S_P(f,g) = \{(s_1+t_1-s_1t_1)x_1, \cdots, (s_n+t_n-s_nt_n)x_n\}$;

Hence, we see the following situation:

1. $(C)\int f dP = \int_0^1 P(\{x \in X : f(x) \geq \alpha\})d\alpha = \sum_{i=1}^n \int_{s_{\pi(i-1)}}^{s_{\pi(i)}} P(\{x \in X : f(x) \geq s_{\pi(i)}\})d\alpha$;

2. $(C)\int g dP = \int_0^1 P(\{x \in X : g(x) \geq \alpha\})d\alpha = \sum_{i=1}^n \int_{t_{\pi(i-1)}}^{t_{\pi(i)}} P(\{x \in X : g(x) \geq t_{\pi(i)}\})d\alpha$;

3. $(C)\int T_P(f,g)dP = \sum_{i=1}^n \int_{y'_{\pi(i-1)}}^{y'_{\pi(i)}} P(\{x \in X : T_P(f,g)(x) \geq y'_{\pi(i)}\})d\alpha$;

4. $(C)\int S_P(f,g)dP = \sum_{i=1}^n \int_{z'_{\pi(i-1)}}^{z'_{\pi(i)}} P(\{x \in X : S_P(f,g)(x) \geq z'_{\pi(i)}\})d\alpha$;

where $\pi_s : s^{(n)} \to s^{(n)}$ *is a permutation of* $s^{(n)}$ *with* $s_{\pi(1)} \leq s_{\pi(2)} \leq \cdots \leq s_{\pi(n)}$ *and, by convention,* $s_0 = 0$. *We take a similar permutation also for* $t^{(n)}$, $y'^{(n)}$ *and* $z'^{(n)}$.
This situation is equivalent to the following one:

1. $(C)\int f dP = \sum_{i=1}^n (C)\int_{A_i} f dP$, *where* $A_i = [s_{\pi(i-1)}, s_{\pi(i)}]$;

2. $(C)\int g dP = \sum_{i=1}^n (C)\int_{B_i} g dP$, *where* $B_i = [t_{\pi(i-1)}, t_{\pi(i)}]$;

3. $(C)\int T_P(f,g)dP = \sum_{i=1}^n (C)\int_{D_i} fg dP$, *where* $D_i = [y'_{\pi(i-1)}, y'_{\pi(i)}]$;

4. $(C)\int S_P(f,g)dP = \sum_{i=1}^n (C)\int_{E_i} f+g-fg dP$, *where* $E_i = [z'_{\pi(i-1)}, z'_{\pi(i)}]$.

Thanks to the following examples we think that our inequalities should work in the general finite case.

Example B.2.3 *Consider* $X = \{x_1, x_2\}$ *and a supermodular measure P, such that* $P(x_1) = \omega_1$ *with* $0 < \omega_1 < 1$ *and* $P(x_2) = 0$.
In the following table we consider the fuzzy sets f and g.

	x_1	x_2
f	0.4	0.7
g	0.8	0.5

So we have

1. $(C)\int f dP = \int_0^1 P(\{x \in X : f(x) \geq t\})dt = \int_0^{0.4} P(\{X\})dt = 0.4$;

2. $(C)\int g dP = \int_0^1 P(\{x \in X : g(x) \geq t\})dt = \int_0^{0.5} P(\{X\})dt + \int_{0.5}^{0.8} P(\{x_1\})dt = 0.5 + 0.3\omega_1$;

Now we consider the fuzzy sets $T_P(f,g)$ *and* $S_P(f,g)$ *and their respective Choquet integrals:*

1. $(C)\int T_P(f,g)dP = \int_0^1 P(\{x \in X : (f \cdot g)(x) \geq t\})dt = \int_0^{0.32} P(\{X\})dt = 0.32$;

	x_1	x_2
$T_P(f,g)$	0.32	0.35
$S_P(f,g)$	0.88	0.85

2. *(C)* $\int S_P(f,g)dP = \int_0^1 P(\{x \in X : (f+g-f\cdot g)(x) \geq t\})dt = \int_0^{0.85} P(\{X\})dt + + \int_{0.85}^{0.88} P(\{x_1\})dt = 0.85 + 0.03\omega_1$*;*

and we see $1.17 + 0.03\omega_1 \geq 0.9 + 0.3\omega_1$, *i.e.* $0.27 - 0.27\omega_1 \geq 0$*. So the Choquet integral is a* $T_P S_P$*-supermodular evaluator on* $\mathcal{F}(X)$*.*
Now we check if the Choquet integral is also a T_P*-evaluator on* $\mathcal{F}(X)$*. We need to prove that* $\forall f,g \in \mathcal{F}(X)$*, we have that*

$$T_P(\varphi(f),\varphi(g))) \leq \varphi(f \wedge g),$$

In our case we have the following values for $f \wedge g$*:*

	x_1	x_2
$f \wedge g$	0.4	0.5

and

$$\varphi(f \wedge g) = \int_0^1 P(\{x \in X : (f \wedge g)(x) \geq t\})dt = \int_0^{0.4} P(\{X\})dt = 0.4.$$

On the other side

$$T_P(\varphi(f),\varphi(g))) = 0.4(0.5 + 0.3\omega_1) = 0.2 + 0.12\omega_1 \leq 0.4 = \varphi(f \wedge g) = \varphi(f),$$

We have proved that the Choquet integral is also a T_P*-evaluator on* $\mathcal{F}(X)$*. Moreover*

$$(C)\int f\cdot gdP + (C)\int f+g-f\cdot gdP = 2\cdot(C)\int fdP + (C)\int gdP =$$
$$= (C)\int fdP + (C)\int gdP + (C)\int f\wedge gdP$$

i.e. the inequality (B.2) *is satisfied.*

Example B.2.4 *Let* $X = \{x_1,x_2,x_3\}$ *and a supermodular measure like in the example 5.4.8, i.e.* $P : 2^X \to [0,1]$ *be given for all* $A \in 2^X$ *by*

$$P(A) = \begin{cases} \frac{1}{4-|A|} & \text{if} \quad x_1 \in A, \\ 0 & \text{otherwise}. \end{cases}$$

In the following table we consider the fuzzy sets f *and* g*.*

	x_1	x_2	x_3
f	0.1	0.2	0.5
g	0.8	0.3	0.4

First of all we verify the following inequality: $\varphi(f)+\varphi(g) \leq \varphi(T_P(f,g))+\varphi(S_P(f,g))$, *i.e.*

$$(C)\int f dP+(C)\int g dP \leq (C)\int f\cdot g dP+(C)\int f+g-f\cdot g dP$$

In fact we have

1. *(C)* $\int f dP=\int_0^1 P(\{x\in X: f(x)\geq t\})dt=\int_0^{0.1} P(\{X\})dt=0.1$;
2. *(C)* $\int g dP=\int_0^1 P(\{x\in X: g(x)\geq t\})dt=\int_0^{0.3} P(\{X\})dt+\int_{0.3}^{0.4} P(\{x_1,x_3\})dt+$ $+\int_{0.4}^{0.8} P(\{x_1\})dt=0.3+0.05+0.13=0.48$;

Now we consider the fuzzy sets $T_P(f,g)$ *and* $S_P(f,g)$ *and their respective Choquet integrals:*

	x_1	x_2	x_3
$T_P(f,g)$	0.08	0.06	0.2
$S_P(f,g)$	0.82	0.44	0.7

1. *(C)* $\int T_P(f,g)dP=\int_0^1 P(\{x\in X:(f\cdot g)(x)\geq t\})dt=\int_0^{0.06} P(\{X\})dt+\int_{0.06}^{0.08} P(\{x_1,x_3\})dt=$ $0.06+0.01=0.07$;
2. *(C)* $\int S_P(f,g)dP=\int_0^1 P(\{x\in X:(f+g-f\cdot g)(x)\geq t\})dt=\int_0^{0.44} P(\{X\})dt+$ $+\int_{0.44}^{0.7} P(\{x_1,x_3\})dt+\int_{0.7}^{0.82} P(\{x_1\})dt=0.44+0.13+0.04=0.61$;

Finally we can see $0.61+0.07=0.68\geq 0.58$ *and so the Choquet integral is a* T_PS_P*-supermodular evaluator on* $\mathcal{F}(X)$.
Now we check that the Choquet integral is also a T_P*-evaluator on* $\mathcal{F}(X)$. *We need to prove that* $\forall f,g\in\mathcal{F}(X)$, *we have that*

$$T_P(\varphi(f),\varphi(g)))\leq\varphi(f\wedge g),$$

In our case we have the following values for $f\wedge g$:

	x_1	x_2	x_3
$f\wedge g$	0.1	0.2	0.4

and

$$\varphi(f\wedge g)=\int_0^1 P(\{x\in X:(f\wedge g)(x)\geq t\})dt=\int_0^{0.1} P(\{X\})dt=0.1,$$

So we can see

$$T_P(\varphi(f),\varphi(g)))=0.048\leq 0.1=\varphi(f\wedge g)=\varphi(f),$$

So, also in this case we have proved that the Choquet integral is a T_P*-evaluator on* $\mathcal{F}(X)$. *Moreover*

$$(C)\int f\cdot g dP+(C)\int f+g-f\cdot g dP=2\cdot(C)\int f dP+(C)\int g dP=$$
$$=(C)\int f dP+(C)\int g dP+(C)\int f\wedge g dP$$

i.e. the inequality (B.2) *is satisfied.*

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