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An algebraic p-adic L-function for ordinary families

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in memory of late Dr. Abir Kumar Adhikari

Abstract, Résumé, Abstract

An algebraic p-adic L-function for ordinary families

Abstract. In this thesis, we construct algebraic p-adic L-functions for families of Galois representations attached to p-adic analytic families of automorphic representations using the formalism of Selmer complexes. This is achieved mainly through making a modification of the Selmer complex to ensure that we deal with perfect complexes and proving a control theorem for the local Euler factors at places not lying above p. The control theorem for local Euler factors is obtained by studying the variation of monodromy under pure specializations of p-adic families of Galois representations restricted to decomposition groups at places of residue characteristic different from p. This allows us to prove a control theorem for the algebraic p-adic L-functions that we construct for Hida families of ordinary cusp forms and ordinary automorphic representations for definite unitary groups. For the Hida family of ordinary cusp forms, we construct a two-variable algebraic p-adic L-function and formulate a conjecture relating it with the analytic p-adic L-function constructed by Emerton, Pollack and Weston. Using results due to Kato, Skinner and Urban, we prove this conjecture in some special cases.

Keywords. p-adic L-functions, Selmer complexes, families of Galois representations, purity, weight-monodromy conjecture.

Une fonction L p-adique algébrique pour les familles ordinaires

Résumé. Dans cette thèse, nous construisons des fonctions L p-adique algébriques pour les familles de représentations galoisiennes attachées aux familles p-adique analytiques de représentations automorphes en utilisant le formalisme des complexes de Selmer. Ce résultat est obtenu principalement en effectuant une modification des complexes de Selmer pour sassurer que nous traitons avec des complexes parfaits et démontrer un théorème de contrôle pour les facteurs d'Euler locaux aux places en dehors de p. Le théoréme de contrôle pour les facteurs d'Euler locaux est obtenu par létude de la variation de la monodromie sous spécialisations purs des familles p-adiques de représentations galoisiennes restreintes aux groupes de décomposition en dehors de p. Cela nous permet de démontrer un théorème de contrôle pour les fonctions algébriques $p$adique que nous construisons pour les familles de Hida de formes paraboliques ordinaires et les représentations automorphes ordinaires pour les groupes unitaires définies. Pour les familles de Hida de formes paraboliques ordinaires, nous construisons un fonction L p-adique algébrique de deux variables et formulons une conjecture la reliant à la fonction L p-adique analytique construite par Emerton, Pollack et Weston. En utilisant des résultats de Kato, Skinner et Urban, nous montrons cette conjecture dans certains cas particuliers.

Mots-clefs. Fonctions L p-adique, complexes de Selmer, familles des représentations galoisienne, pureté, conjecture de monodromie-poids.

Un funzione L p-adiche algebriche per le famiglie ordinario

Abstract. In questa tesi, costruiamo funzioni L p-adiche algebriche per le famiglie di rappresentazioni di Galois associate a famiglie p-adiche analitiche di rappresentazioni automorfe, utilizzando il formalismo dei complessi di Selmer. Questo risultato è ottenuto principalmente attraverso una modifica del complesso di Selmer, attuata in modo tale da garantire che i complessi studiati siano perfetti e attraverso un teorema di controllo per i fattori di Eulero locali nei primi diversi da p. Il teorema di controllo per fattori di Eulero locali si ottiene studiando la monodromia al variare delle specializzazioni pure di famiglie p-adiche di rappresentazioni di Galois ristrette a gruppi di decomposizione a primi di fuori p. Questo ci permette di dimostrare un teorema di controllo per funzioni L p-adiche algebriche, costruite per famiglie di Hida di forme cuspidali ordinarie e rappresentazioni automorfe ordinarie per i gruppi unitari definiti. Per la famiglia di Hida di forme cuspidali ordinarie, costruiamo una funzione L p-adica algebrica di due variabili e formuliamo una congettura che stabilisca il legame con la funzione L p-adica analitica costruita da Emerton, Pollack e Weston. Utilizzando i risultati di Kato, Skinner e Urban, dimostriamo questa congettura in alcuni casi particolari.

Parole chiave. Funzioni L p-adiche, complessi di Selmer, famiglie di rappresentazioni di Galois, purezza, congettura di peso-monodromia.

Contents

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Introduction

Let K_∞/K be a $\mathbb{Z}_p$-extension of a number field K and let $K \subset K_n \subset K_\infty$ denote the sub-extension of degree n. Then K. Iwasawa showed in [**Iwa59**] that the exact power of p dividing the order of the class group X_n of K_n is given for n large enough by the formula

$$\lambda n + \mu p^n + \nu \tag{0.0.1}$$

where $\lambda \geq 0$, $\mu \geq 0$ and ν are integers. Immediately thereafter, J.-P. Serre noticed in [**Ser95**] that this result followed from two general principles: first, the inverse limit X_∞ of the X_n with respect to the norm map is a finite type torsion module over $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$ (a regular ring of dimension 2); second, there exists a specific element ω_n such that X_n is equal to X_∞/ω_n. As the order of the class group is linked via the Dirichlet class number formula to special values of the zeta function, these results suggest that the variation of class groups in $\mathbb{Z}_p$-extensions could be linked with p-adic L-functions and indeed, the Kubota-Leopoldt zeta function was given a new construction in terms of cyclotomic $\mathbb{Z}_p$-extensions in [**Iwa69**]. In [**Maz72**], B. Mazur proved that the formula (0.0.1) admitted an extension to the growth of the Tate-Shafarevich group of abelian varieties in $\mathbb{Z}_p$-extensions and he proposed a bold generalization of these facts to the Galois cohomology of the étale cohomology of varieties over $\mathbb{Q}$. However, already in the context of abelian varieties, a remarkable fact is that the control theorem relating the Selmer group over Λ to the Selmer group over $\mathbb{Z}[\mathrm{Gal}(K_n/K)]$ is true only up to error terms of local origins, the error terms at places above p being sometimes unbounded with n. The analogy mentioned above with p-adic interpolation of special values of L-functions can perhaps account for the strange behavior at p: just as one should not expect to be able to interpolate special values of L-functions without first removing an Euler factor at p, one should presumably not expect p-adic interpolation of Galois cohomology modules to proceed smoothly without modifying the condition at p. The relevance (if any) of error terms at other places, on the other hand, remained mysterious.

In the late 80s and early 90s, several theoretical improvements completely changed our approach to these classical questions. First, R. Greenberg proposed in [**Gre89, Gre91**] that the appropriate context for the study of p-adic variation of special values of L-functions and Selmer modules was the universal deformation of a Galois representation of geometric origin. Second, the conjectures formulated by Bloch, Kato in [**BK90, Kat93**] considerably deepened our understanding of the behavior of special values of L-functions. In particular, they made clear that special values of L-functions should be linked to some integral basis in the determinant of the Galois cohomology complex of motives with coefficients. Seen from this dual perspective, the proper extension of Iwasawa's and Mazur's classical control theorem should be that specialization of some integral basis in the determinant of the Galois cohomology complex of motives with coefficients in universal deformation rings at an arithmetic point x should be equal to the integral basis of the determinant of the Galois cohomology of the motive over $\mathbb{Q}$ corresponding to x coming from the conjectures on special values of L-function. As in [**Kat04**], one can check for instance that this formulation applied to the motive $\mathbb{Q}(1)$ recovers exactly Iwasawa's control theorem and that no prime $\ell \neq p$ can make a contribution to the error term in the setting of $\mathbb{Z}_p$-extensions. However, in the simplest example of p-adic universal families of rank 2 motives, that is p-adic families of ordinary eigenforms parametrized by the Hida-Hecke algebra, even a precise formulation of the conjectural form of the control theorem has been heretofore lacking.

The reasons for this are twofold. To start with, universal deformation rings are typically not known to be regular rings, so complexes of Galois cohomology of p-adic families with coefficients in universal deformation rings are usually not known to be perfect complexes, precluding the possibility of taking unconditionally their determinants. Even in the more classical formulation of Greenberg ([**Gre91**]), one needs to consider the characteristic ideal of some modules and this requires at least the ring to be normal. This for instance is presumably why there is no definition of an algebraic counterpart to the analytic p-adic L-function for Hida families in [**EPW06**] by Emerton, Pollack, Weston. Moreover, even when the complexes are known to be perfect, the error terms ubiquitous in control theorems since [**Maz72**] can be very hard to explicitly control in the universal deformation. This happens for instance in works by Fouquet, Ochiai ([**Och06, FO12**]) and is related to the variation of the inertia invariants in families.

In this manuscript, we prove a perfect control theorem at arithmetic points on a branch of the Hida family for $\mathrm{GL}_2(\mathbb{Q})$ and definite unitary groups with no assumption on the nature of the universal deformation ring, and thus construct unconditionally an algebraic p-adic L-function for the Galois representations attached to these Hida families. The fundamental tool allowing this progress is the recognition of the crucial role played by the weight-monodromy conjecture in the variation of special values of L-function (an idea which we learned from Nekovář [**Nek06**] and Ochiai [**Och06**]). The philosophy behind the conjectures of Bloch, Kato, Fontaine and Perrin-Riou ([**BK90, FPR94**]) is that special values of L-function should encode extension of motives which are not too much ramified. This implies that the local conditions at $\ell \neq p$ conjectured to appear in the definition of algebraic p-adic L-functions will involve ramification. The weight-monodromy conjecture allows to relate inertia invariants of pure modules with eigenvalues of the Frobenius morphisms and this allows at the same time to define unconditionally an algebraic p-adic L-function as well as proving it satisfies a control theorem at arithmetic points.

Statement of results

In this section, we summarize the results obtained in chapter 1, 3, 4.

Purity for big Galois representations. Let p be a rational prime and K denote a finite extension of $\mathbb{Q}_\ell$ with $\ell \neq p$. Let $\mathcal{R}$ be a characteristic zero domain containing $\mathbb{Z}_p$ as a subring. Denote the fraction field of $\mathcal{R}$ by $\mathcal{K}$ and fix an algebraic closure $\overline{\mathcal{K}}$ of $\mathcal{K}$. Denote the integral closure of $\mathcal{R}$ in $\overline{\mathcal{K}}$ by $\mathcal{O}_{\overline{\mathcal{K}}}$. Note that any ring homomorphism ψ from $\mathcal{R}$ to an algebraically closed field Ω of characteristic zero extends to $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$, we fix such an extension and denote it by ψ by abuse of notation. Observe that $\overline{\mathbb{Q}}$ is contained inside $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$. Suppose that $G_K = \mathrm{Gal}(\overline{K}/K)$ acts on a free $\mathcal{R}$-module $\mathcal{T}$ such that its action is monodromic (*i.e.*, a finite index subgroup of I_K acts through its $\mathbb{Z}_p$-quotient via the exponential of a nilpotent matrix, see Definition 1.1.1). Let $\mathcal{M}_\bullet$ denote the associated monodromy filtration on $\mathcal{T}$. Denote the G_K-representation $\mathcal{T} \otimes_{\mathcal{R}} \overline{\mathcal{K}}$ by $\mathcal{V}$. For a $\mathbb{Z}_p$-algebra homomorphism $\lambda : \mathcal{R} \to \overline{\mathbb{Q}}_p$, the G_K-representation $\mathcal{T} \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p$ is denoted by V_λ. The Weil-Deligne parametrization of $\mathcal{V}$ (resp. V_λ) is denoted by $\mathrm{WD}(\mathcal{V})$ (resp. $\mathrm{WD}(V_\lambda)$). For a Weil-Deligne representation V, its Frobenius semisimplification is denoted by $V^{\text{Fr-ss}}$.

Theorem A (Purity for big Galois representations)**.** *Suppose that $\lambda : \mathcal{R} \to \overline{\mathbb{Q}}_p$ is a $\mathbb{Z}_p$-algebra homomorphism such that the G_K-representation V_λ is pure of weight w (see §1.0.1 or definition 1.1.47). Let $\mathfrak{p}_\lambda$ denote the kernel of λ. Then the following hold.*

(1) The terms and gradings of $\mathcal{M}_\bullet$ become free over $\mathcal{R}_{\mathfrak{p}_\lambda}$ after localizing them at $\mathfrak{p}_\lambda$ and for any $i \in \mathbb{Z}$, the map λ induces isomorphisms

$$\mathcal{M}_i \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq M_{\lambda,i}, \quad \mathrm{Gr}_i \mathcal{M}_\bullet \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq \mathrm{Gr}_i M_{\lambda,\bullet}$$

of W_K-modules.

(2) There exist
- *(a) an integer $J \geq 1$,*
- *(b) integers $0 \leq t_1 < \cdots < t_J$,*
- *(c) an integer $I \geq 1$,*
- *(d) (i) unramified characters $\chi_1, \cdots, \chi_I : W_K \to \mathcal{O}_{\overline{\mathcal{K}}}^\times$,*
  (ii) irreducible Frobenius-semisimple representations

$$\rho_1 : W_K \to \mathrm{GL}_{d_1}(\overline{\mathbb{Q}}), \cdots, \rho_I : W_K \to \mathrm{GL}_{d_I}(\overline{\mathbb{Q}})$$

  with finite image and
- *(e) integers $n_{ij} \geq 0$ for $1 \leq i \leq I, 1 \leq j \leq J$ such that the following hold.*

  - *There are isomorphisms of Weil-Deligne representations*

$$\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{I} \bigoplus_{j=1}^{J} \mathrm{Sp}_{t_j}(\chi_i \otimes \rho_i)^{n_{ij}}_{/\overline{\mathcal{K}}},$$

$$\mathrm{WD}(V_\lambda)^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{I} \bigoplus_{j=1}^{J} \mathrm{Sp}_{t_j}(\lambda \circ (\chi_i \otimes \rho_i))^{n_{ij}}_{/\overline{\mathbb{Q}}_p}.$$

  - *The representation $\lambda \circ (\chi_i \otimes \rho_i) : W_K \to \mathrm{GL}_{d_i}(\overline{\mathbb{Q}}_p)$ has image contained in $\mathrm{GL}_{d_i}(\overline{\mathbb{Q}})$ for all $1 \leq i \leq I$.*

Furthermore, the integers I, J, t_i, n_{ij} and the representations χ_i, ρ_i depend on $\mathcal{V}$, but not on λ.

(3) The λ-specialization of the central irreducible summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ (considered over $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$) are strictly pure of weight w.

(4) The polynomial $\mathrm{Eul}(\mathcal{V})^{-1}$ has coefficients in $\mathcal{O}_{\mathcal{K}} \cap \mathcal{R}_{\mathfrak{p}_\lambda}$, its λ-specialization is $\mathrm{Eul}(V_\lambda)^{-1}$.

(5) The $\mathcal{R}_{\mathfrak{p}_\lambda}$-modules $\mathcal{T}_{\mathfrak{p}_\lambda}^{I_K}$, $\mathcal{T}_{\mathfrak{p}_\lambda}/\mathcal{T}_{\mathfrak{p}_\lambda}^{I_K}$ are free and the map λ induces an isomorphism

$$\mathcal{T}^{I_K} \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq \mathcal{T}_{\mathfrak{p}_\lambda}^{I_K} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq V_\lambda^{I_K}.$$

Consequently, the complex $[\mathcal{T}^{I_K} \xrightarrow{\phi-1} \mathcal{T}^{I_K}]$ concentrated in degree 0, 1 descends perfectly to the complex $[V_\lambda^{I_K} \xrightarrow{\phi-1} V_\lambda^{I_K}]$ concentrated in degree 0, 1, i.e.,

$$[\mathcal{T}^{I_K} \xrightarrow{\phi-1} \mathcal{T}^{I_K}] \overset{L}{\otimes}_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq [V_\lambda^{I_K} \xrightarrow{\phi-1} V_\lambda^{I_K}].$$

For a more general version, we refer to theorem 1.2.4 which is the main result of chapter 1. The main upshot of purity for big Galois representations is that using this one can prove control theorems at pure specializations for (the local factors outside p of) the algebraic p-adic L-functions that we construct in chapter 3, 4. Using the same tool and [**Ber13**, Lemma 5.5], we also hope to construct an algebraic p-adic L-function along irreducible components of eigenvarieties. In fact we expect that using purity of big Galois representations, an algebraic p-adic L-function can be constructed for any family of Galois representations and pseudo-representations interpolating Galois representations over $\overline{\mathbb{Q}}_p$ whose restriction to local Galois groups at places not dividing p are pure. We refer to the introduction of chapter 1 for a detailed discussion about an appropriate context of purity for big Galois representations, a sketch of its proof, consequences and explanation of the inevitability of the hypothesis that $\mathcal{R}$ is a domain.

Algebraic p-adic L-functions for the Hida family for $\mathrm{GL}_2(\mathbb{Q})$. The results obtained in chapter 3 are summarized here. In this chapter, we construct algebraic p-adic L-functions $L_{p,\mathrm{Gr}}^{\mathrm{alg}}(-), L_{p',\mathrm{Gr}}^{\mathrm{alg}}(-), L_{p,\mathrm{Kato}}^{\mathrm{alg}}(-)$. Using 1.2.4 and purity of modular Galois representations, we show that they satisfy control theorems at arithmetic specializations (under some hypothesis). We also relate our construction with Greenberg's strict Selmer group (using [**Kat04**, Theorem 17.4], [**Nek06**, Theorem 7.8.6]). Now we state these results referring to chapter 3 for details.

Let $R(\mathfrak{a})$ denote the quotient of the Hida-Hecke algebra h_∞^{ord} by a minimal prime ideal $\mathfrak{a}$. Suppose that the composite map

$$h_\infty^{\mathrm{ord}} \twoheadrightarrow R(\mathfrak{a}) \hookrightarrow \mathrm{Frac}(R(\mathfrak{a}))$$

is *minimal* in the sense of [**Hid88a**, p. 317]. Let $\mathcal{T}(\mathfrak{a})$ denote Hida's big Galois representation of $G_{\mathbb{Q},S}$ over $R(\mathfrak{a})$ where S denotes a finite set of places of $\mathbb{Q}$ containing p and the place at infinity. Assume that the residual representation $\overline{\rho}$ associated with the $G_{\mathbb{Q},S}$-representation $\mathcal{T}(\mathfrak{a})$ is absolutely irreducible (this is assumption 3.2.4). For an arithmetic specialization λ of $R(\mathfrak{a})$, put

$$\mathcal{O}_\lambda = \mathrm{Im}\lambda$$

and let T_λ denote the $G_{\mathbb{Q},S}$-representation $\mathcal{T}(\mathfrak{a}) \otimes_{R(\mathfrak{a}),\lambda} \mathcal{O}_\lambda$.

Let

$$\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}} = \mathcal{T}(\mathfrak{a}) \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]],$$
$$T_{\lambda,\mathrm{Iw}} = T_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$$

denote the cyclotomic deformation of $\mathcal{T}(\mathfrak{a})$ and T_λ respectively where $\mathbb{Q}_\infty$ denotes the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Put

$$\Lambda_{\mathcal{O}_\lambda} = \mathcal{O}_\lambda \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]],$$
$$R(\mathfrak{a})_{\mathrm{Iw}} = R(\mathfrak{a}) \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]].$$

In definition 3.3.4, we define $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$, $L^{\mathrm{alg}}_{p',\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$, $L^{\mathrm{alg}}_{p,\mathrm{Kato}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$, $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}})$, $L^{\mathrm{alg}}_{p',\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}})$, $L^{\mathrm{alg}}_{p,\mathrm{Kato}}(T_{\lambda,\mathrm{Iw}})$ where λ denotes an arithmetic specialization of $R(\mathfrak{a})$.

Theorem B. *Let λ be an arithmetic specialization of $R(\mathfrak{a})$. Then the isomorphisms in propositions 2.1.2, 2.2.1, 2.2.3 induce an isomorphism*

$$L^{\mathrm{alg}}_{p,\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\lambda} \Lambda_{\mathcal{O}_\lambda} \cong L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}})$$

when $\overline{\rho}$ is p-distinguished. They also induce isomorphisms

$$L^{\mathrm{alg}}_{p',\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\lambda} \Lambda_{\mathcal{O}_\lambda} \cong L^{\mathrm{alg}}_{p',\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}}),$$
$$L^{\mathrm{alg}}_{p,\mathrm{Kato}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\lambda} \Lambda_{\mathcal{O}_\lambda} \cong L^{\mathrm{alg}}_{p,\mathrm{Kato}}(T_{\lambda,\mathrm{Iw}}).$$

Theorem C. *Let λ be an arithmetic specialization of $R(\mathfrak{a})$ such that $\mathcal{O}_\lambda$ is a DVR. The Selmer complex $R\Gamma_f(T_{\lambda,\mathrm{Iw}})$ defined with respect to Greenberg's local condition (see definition 2.2.2) is a perfect complex of $\Lambda_{\mathcal{O}_\lambda}$-modules and the map $i_{\Lambda_{\mathcal{O}_\lambda}}(-,-,-)$ (as in equation* (2.1.4)*) induces an isomorphism between $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}})$ and $\left(\det_{\Lambda_{\mathcal{O}_\lambda}} R\Gamma_f(T_{\lambda,\mathrm{Iw}})\right)^{-1}$. For any integer $i<1$ and $i>2$,*

$$\widetilde{H}^i_f(T_{\lambda,\mathrm{Iw}}) = 0.$$

Suppose that p does not divide the level of the ordinary form associated with λ. Then $\widetilde{H}^2_f(T_{\lambda,\mathrm{Iw}})$ is a torsion $\Lambda_{\mathcal{O}_\lambda}$-module and $\widetilde{H}^1_f(T_{\lambda,\mathrm{Iw}})$ is zero. The surjective map

$$\widetilde{H}^1_f(A_{\lambda,\mathrm{Iw}}) \twoheadrightarrow \mathrm{Sel}^{\mathrm{str}}_{A_{\lambda,\mathrm{Iw}}}$$

as in Lemma 3.4.4 induces an injective map

$$D_P\left(\mathrm{Sel}^{\mathrm{str}}_{A_{\lambda,\mathrm{Iw}}}\right) \hookrightarrow \widetilde{H}^2_f(T_{\lambda,\mathrm{Iw}}) \tag{0.0.2}$$

with finite cokernel. Consequently we get a canonical isomorphism

$$L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}}) \cong (\mathrm{char}_{\Lambda_{\mathcal{O}_\lambda}} D_P(\mathrm{Sel}^{\mathrm{str}}_{A_{\lambda,\mathrm{Iw}}}), 0)$$

using equations (2.1.3)*,* (2.1.5) *and* (3.4.2)*.*

The above two theorems correspond to theorem 3.3.7 (resp. 3.4.5). The crucial ingredients of the proof are theorem 1.2.4 and purity of modular Galois representations (resp. [**Kat04**, Theorem 17.4], [**Nek06**, Theorem 7.8.6]).

In §3.5, we show that all the cohomologies of the complex $C^\bullet_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ are zero, except possibly the second cohomology, which is torsion over $R(\mathfrak{a})_{\mathrm{Iw}}$ (proposition 3.5.6). This result allows to construct a *two-variable algebraic p-adic L-function* $\mathscr{L}^{\mathrm{alg}}_p(\mathfrak{a}) \in \mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})$ whose image under mod $\mathfrak{p}$ reduction generates the characteristic ideal of the Pontrjagin dual of the strict Selmer group $\mathrm{Sel}^{\mathrm{str}}_{A_{\lambda_\mathfrak{p},\mathrm{Iw}}}$ for $\mathfrak{p}$ varying in a dense subset of $\mathrm{Spec}^{\mathrm{arith}}(R(\mathfrak{a}))$ ($\lambda_\mathfrak{p}$ denotes an arithmetic specialization of $R(\mathfrak{a})$ whose kernel is $\mathfrak{p}$). On the other hand, these characteristic ideals are generated by the analytic p-adic L-functions of $f_{\lambda_\mathfrak{p}}$ (computed with respect to a canonical period), which are interpolated by an element $L_p^{\mathrm{an}}(\mathfrak{a})$ of $R(\mathfrak{a})_{\mathrm{Iw}}$ (as constructed in [**EPW06**]). This suggests a link between $\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a})$ and $L_p^{\mathrm{an}}(\mathfrak{a})$, which leads to the conjecture below.

Conjecture 1. *The element* $\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a})$ *of* $\mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})$ *is an element of* $R(\mathfrak{a})_{\mathrm{Iw}}^{\mathrm{int}}$ *and*

$$\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a})R(\mathfrak{a})_{\mathrm{Iw}}^{\mathrm{int}} = L_p^{\mathrm{an}}(\mathfrak{a})R(\mathfrak{a})_{\mathrm{Iw}}^{\mathrm{int}}.$$

In the above $R(\mathfrak{a})^{\mathrm{int}}$ denotes the integral closure of $R(\mathfrak{a})$ in its fraction field and $R(\mathfrak{a})_{\mathrm{Iw}}^{\mathrm{int}}$ denotes the completed tensor product $R(\mathfrak{a})^{\mathrm{int}}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$. Assuming Greenberg's conjecture on vanishing of μ-invariants of modular forms (with absolutely irreducible and p-distinguished residual Galois representation), we prove this conjecture in theorem 3.5.22.

Algebraic p-adic L-functions for the Hida family for definite unitary groups. The results obtained in chapter 4 are summarized here. In this chapter, we construct algebraic p-adic L-functions $L_{p',\mathrm{Gr}}^{\mathrm{alg}}(-), L_{p,\mathrm{Kato}}^{\mathrm{alg}}(-)$. Using 1.2.4 and purity of Galois representations associated with automorphic representations (which are of dominant weight and stable) for definite unitary groups, we show that they satisfy control theorems at arithmetic specializations of regular dominant weight whose associated Galois representations are crystalline at each place lying above p and associated automorphic representations are stable. Now we state this result referring to chapter 4 for details.

Let $R(\mathfrak{a})$ denote a partial normalization (as defined in §4.3) of the quotient of the Hida-Hecke algebra $h_{\{\chi_v\}}^{\Sigma,\mathrm{ord}}(U(\mathfrak{p}^\infty),\mathcal{O}_K)$ by a minimal prime ideal $\mathfrak{a}$ (here K denotes a finite extension of $\mathbb{Q}_p$). Let $\mathcal{T}(\mathfrak{a})$ denote Hida's big Galois representation of $G_{F,S}$ over $R(\mathfrak{a})$ where S denotes a finite set of places of a CM field F containing the places above p and the places at infinity. Assume that the residual representation $\overline{\rho}$ associated with the $G_{F,S}$-representation $\mathcal{T}(\mathfrak{a})$ is absolutely irreducible (this is assumption 4.3.1). For an arithmetic specialization ζ of $R(\mathfrak{a})$, put

$$\mathcal{O}_\zeta = \mathrm{Im}\zeta$$

and let T_ζ denote the $G_{F,S}$-representation $\mathcal{T}(\mathfrak{a}) \otimes_{R(\mathfrak{a}),\zeta} \mathcal{O}_\zeta$. Denote the automorphic representation attached to ζ by π_ζ.

Let

$$\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}} = \mathcal{T}(\mathfrak{a}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]],$$
$$T_{\zeta,\mathrm{Iw}} = T_\zeta \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]$$

denote the cyclotomic deformation of $\mathcal{T}(\mathfrak{a})$ and T_ζ respectively where F_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of F. Put

$$\Lambda_{\mathcal{O}_\zeta} = \mathcal{O}_\zeta \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]].$$

In definition 4.3.3, we define $L_{p',\mathrm{Gr}}^{\mathrm{alg}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$, $L_{p,\mathrm{Kato}}^{\mathrm{alg}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$, $L_{p',\mathrm{Gr}}^{\mathrm{alg}}(T_{\zeta,\mathrm{Iw}})$, $L_{p,\mathrm{Kato}}^{\mathrm{alg}}(T_{\zeta,\mathrm{Iw}})$ where ζ denotes an arithmetic specialization of $R(\mathfrak{a})$ of regular dominant weight such that $V_\zeta|_{G_{F_w}}$ is crystalline for any place w of F lying above p. By lemma 4.3.5, the kernels of such specializations form a dense subset of $\mathrm{Spec}(R(\mathfrak{a}))$.

Theorem D. *Let* ζ *be an arithmetic specialization of* $R(\mathfrak{a})$ *of regular dominant weight such that* π_ζ *is stable and* $V_\zeta|_{G_{F_w}}$ *is crystalline for any place* w *of* F *lying above* p*. Then the* *isomorphisms in propositions 2.1.2, 2.2.1, 2.2.3 induce isomorphisms*

$$L^{\mathrm{alg}}_{p',\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\zeta} \Lambda_{\mathcal{O}_\zeta} \cong L^{\mathrm{alg}}_{p',\mathrm{Gr}}(T_{\zeta,\mathrm{Iw}}),$$
$$L^{\mathrm{alg}}_{p,\mathrm{Kato}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\zeta} \Lambda_{\mathcal{O}_\zeta} \cong L^{\mathrm{alg}}_{p,\mathrm{Kato}}(T_{\zeta,\mathrm{Iw}}).$$

The above theorem corresponds to theorem 4.3.6. The crucial ingredient of its proof is theorem 1.2.4 and purity of Galois representations associated with the automorphic forms (which are of dominant weight and stable) for definite unitary groups. Note that though such Galois representations are not known to be motivic, in [**Pin92**, Conjecture 5.4.1], they are conjectured to satisfy properties similar to motivic representations, for example the weight-monodromy conjecture, which is known by [**Car12**].

Organization

This thesis is arranged in four chapters.

The first chapter is the technical heart of this manuscript. Here we develop a tool (theorem 1.2.4) to understand the variation of the inertia invariants (as a Frobenius module) in a family, which we call *purity for big Galois representations*. This describes the Weil-Deligne parametrization of a pure specialization of a big Galois representation in terms of the Weil-Deligne parametrization of the big Galois representation and thus describes the variation of the inertia invariants at pure specializations. This allows to prove control theorems for (the local factors outside p of the) the algebraic p-adic L-function that we construct in chapter 3, 4.

The second chapter recalls the notion of Selmer complexes and the notion of determinant functors as introduced in [**Nek06, KM76**] respectively.

In the third chapter, we construct algebraic p-adic L-functions along irreducible components of the Hida family of ordinary cusp forms and prove that they satisfy perfect control theorems at arithmetic specializations. We also relate our construction with Greenberg's strict Selmer group. In the final section, we conjecture a link between our construction and the analytic p-adic L-function as constructed in [**EPW06**].

In the fourth chapter, we construct algebraic p-adic L-functions along irreducible components of the Hida family for definite unitary groups and prove that they satisfy perfect control theorem at arithmetic specializations which are of regular dominant weight and whose associated Galois representations are crystalline at all the places above p.

Notations

For each field E of characteristic zero, we fix an algebraic closure $\overline{E}$ once and for all and denote the absolute Galois group $\mathrm{Gal}(\overline{E}/E)$ by G_E. We also fix embeddings $\mathbb{C} \overset{i_\infty}{\hookleftarrow} \overline{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \overline{\mathbb{Q}}_p$ once and for all.

Let F be a number field and v denote a finite place of F. Then the decomposition group and inertia group of F at v will be denoted by G_{F_v}, I_{F_v} respectively. When no confusion arise, they will be denoted by G_v, I_v respectively. The geometric Frobenius element of G_v/I_v is denoted by Fr_v.

Throughout this manuscript, the reciprocity isomorphism of local class field theory is normalized by letting uniformizers correspond to geometric Frobenius elements.

CHAPTER 1

Purity for big Galois representations

1.0. Introduction

1.0.1. Weight-Monodromy Conjecture. Let p be a rational prime, K be a finite extension of $\mathbb{Q}_\ell$ with $\ell \neq p$. Denote the residue field of the ring of integers of K by k. Let ϕ denote a lift of the geometric Frobenius to G_K. Suppose that V is a finite dimensional continuous representation of G_K over $\overline{\mathbb{Q}}_p$. Then the Grothendieck monodromy theorem (Theorem 1.1.25) gives a nilpotent endomorphism N of V, called the *monodromy* of V, attached to which there is an increasing filtration $M_\bullet$ on V which is stable under the action of G_K and is called the *monodromy filtration*. The G_K-representation V is said to be *pure of weight* $w \in \mathbb{Z}$ (pure for short) if the characteristic roots of ϕ on $\mathrm{Gr}_i M_\bullet$ are $\#k$-Weil numbers of weight $w+i$. The Weight-Monodromy Conjecture (henceforth WMC) states the following.

Conjecture 1.0.1 ([**Ill94**]). *Let X be a projective smooth variety over K. Then for any integer i, the G_K-representation $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}_\ell}, \mathbb{Q}_p)$ is pure of weight i.*

The Galois representations associated with automorphic representations are expected to come from geometry and hence believed to be pure. The WMC is known for many automorphic Galois representations, see [**Car86**, Théorème A], [**Bla06**, Theorem 2], [**HT01, TY07, Shi11, Car12, Sch12, Clo13**] for example.

1.0.2. Local Euler factors. For a Weil-Deligne representation $V = (r, N)$ of W_K over an algebraically closed field Ω of characteristic zero, its local Euler factor is defined as

$$\mathrm{Eul}((r, N), X) = \det(1 - X\phi|_{V^{I_K, N=0}})^{-1} \in \Omega(X)$$

where $V^{I_K,N=0}$ denotes the subspace of V on which I_K acts trivially and N is zero (*cf.* [**Tay04**, p. 85]).

For a Galois representation $\rho : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}(V)$ of the absolute Galois group of a number field E on a finite dimensional vector space V over an algebraically closed field Ω of characteristic zero, its local Euler factor at a finite place v of E not dividing p is defined by

$$\mathrm{Eul}_v(\rho, X) = \mathrm{Eul}(\mathrm{WD}(V|_{G_v}), X) \in \Omega(X).$$

1.0.3. Families. Following works of Bellaïche, Chenevier, Coleman, Hida, Mazur *et. al.*, it is believed that automorphic Galois representations live in families. In precise terms, we expect to have a tuple

$$\mathcal{F}_p = \{\Pi, E, p, \mathcal{R}, \mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R}), \mathcal{T}\}$$

which we call a *family*, where

(1) Π is a set of automorphic representations of $G(\mathbb{A}_F)$ (where G denotes a reductive group and $\mathbb{A}_F$ denotes the ring of adèles of some number field F) and to each $\pi \in \Pi$,

there is a p-adic Galois representation $\rho_{\pi,p} : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ associated with it (the integer $n \geq 1$ does not depend on λ),

(2) E is a number field, p is a rational prime,

(3) $\mathcal{R}$ is a characteristic zero domain containing $\mathbb{Z}_p$ as a subalgebra, usually of large Krull dimension,

(4) $\mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R})$ is a non-empty subset of $\mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathcal{R}, \overline{\mathbb{Q}}_p)$ and there is a map

$$\mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R}) \to \Pi, \quad \lambda \mapsto \pi_\lambda,$$

(5) $\mathcal{T}$ is a free $\mathcal{R}$-module equipped with an action of the absolute Galois group $G_E = \mathrm{Gal}(\overline{E}/E)$ of E and for any $\lambda \in \mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R})$, the G_E-representations $\mathcal{T} \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p$ and $\rho_{\pi_\lambda,p}$ are isomorphic,

(6) for any finite place v of E not dividing p, the representation $\mathcal{T}|_{G_v}$ is monodromic (see definition 1.1.1).

Let $\mathcal{K}$ denote the fraction field of $\mathcal{R}$. We fix an algebraic closure $\overline{\mathcal{K}}$ of $\mathcal{K}$. The integral closure of $\mathcal{R}$ in $\mathcal{K}$ (resp. $\overline{\mathcal{K}}$) will be denoted by $\mathcal{O}_{\mathcal{K}}$ (resp. $\mathcal{O}_{\overline{\mathcal{K}}}$). By $\mathcal{V}$, we will denote the G_E-representation $\mathcal{T} \otimes_{\mathcal{R}} \overline{\mathcal{K}}$. For an element λ of $\mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathcal{R}, \overline{\mathbb{Q}}_p)$, we set

$$V_\lambda := \mathcal{T} \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p.$$

In the following v will always denote a finite place of E. For such a place not dividing p, we put

$$\mathrm{Spcl}_v^{\mathrm{pure}}(\mathcal{R}) := \{\lambda \in \mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathcal{R}, \overline{\mathbb{Q}}_p) \mid V_\lambda|_{G_v} \text{ is pure}\}.$$

We say *the WMC holds for $\mathcal{F}_p$ at a finite place v of E not dividing p* if for any $\pi \in \Pi$, the G_v-representation $\rho_{\pi,p}|_{G_v}$ is pure. We say *the WMC holds for $\mathcal{F}_p$* if the WMC holds for $\mathcal{F}_p$ at all finite places of E not dividing p. Note that if the WMC holds for $\mathcal{F}_p$, then

$$\mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R}) \subset \mathrm{Spcl}_v^{\mathrm{pure}}(\mathcal{R})$$

for all v not dividing p.

Hida families of ordinary automorphic representations for various reductive groups provide ample examples of families. In chapter 3 and 4, we will consider the Hida families for $\mathrm{GL}_2(\mathbb{Q})$ and definite unitary groups.

For notations used in the example below, we refer to chapter 3.

Example 1.0.2. Hida theory of ordinary forms for $G = \mathrm{GL}_2(\mathbb{Q})$ shows that

$$\mathcal{F}_p = \{\Pi, \mathbb{Q}, p, \mathcal{R}, \mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R}), \mathcal{T}\}$$

is a family where $\mathcal{R} = R(\mathfrak{a}) = h_\infty^{\mathrm{ord}}/\mathfrak{a}$, $\mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R})$ denotes the set of arithmetic specializations of $R(\mathfrak{a})$, Π denotes the set of ordinary automorphic representations of $\mathrm{GL}_2(\mathbb{Q})$ corresponding to the ordinary eigen cusp forms lying on the component $\mathrm{Spec}(h_\infty^{\mathrm{ord}}/\mathfrak{a})$ of $\mathrm{Spec}(h_\infty^{\mathrm{ord}})$, $\mathcal{T}$ denotes $\mathcal{T}(\mathfrak{a})$. The set $\mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R})$ of arithmetic specializations is dense in $\mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathcal{R}, \overline{\mathbb{Q}}_p)$. Moreover the WMC is known for this family (see Prop 3.1.1).

We refer to chapter 4 for the notations and terminologies used in the example below.

Example 1.0.3. Let F be a CM field and F^+ be its maximal totally real subfield. Let G be the definite unitary group defined over F^+ (as in §4.1.1). Let p be a prime, $\mathcal{R} = R(\mathfrak{a})$, $\mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R})$ denote the set of arithmetic specializations of $R(\mathfrak{a})$, Π denote the set of ordinary automorphic representations of $G(\mathbb{A}_{F^+})$ of dominant weights corresponding to the arithmetic specializations of $R(\mathfrak{a})$, $\mathcal{T}$ denote $\mathcal{T}(\mathfrak{a})$ as in chapter 4. Then

$$\mathcal{F}_p = \{\Pi, F, p, \mathcal{R}, \mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R}), \mathcal{T}\}$$

is a family. The set $\mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R})$ of arithmetic specializations is dense in $\mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathcal{R}, \overline{\mathbb{Q}}_p)$. By [**Car12**, Theorem 1.2], WMC is known for the arithmetic specializations of $\mathcal{T}$ which are of dominant weight and whose associated automorphic representation is stable.

1.0.4. Local Euler factors in families. Given a family $\mathcal{F}_p$, we may wonder if the local Euler factors of its specializations are interpolated by the local Euler factors of $\mathcal{V}$, *i.e.*, we may ask if

$$\mathrm{Eul}_v(\mathcal{V}, X)^{-1} \in \mathcal{O}_{\mathcal{K}}[X], \quad \lambda(\mathrm{Eul}_v(\mathcal{V}, X)) = \mathrm{Eul}_v(V_\lambda, X)$$

holds for all $v \nmid p$ and $\lambda \in \mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathcal{R}, \overline{\mathbb{Q}}_p)$. First of all, this need not hold. For example, if G_v acts unipotently on $\mathcal{T}$, then its rank of I_v-invariants, *i.e.*, the degree of $\mathrm{Eul}_v(\mathcal{V}, X)^{-1}$, is equal to the dimension of null space of the monodromy of $\mathcal{T}|_{G_v}$, which might increase under a specialization λ of $\mathcal{R}$, making the degree of $\mathrm{Eul}_v(V_\lambda, X)^{-1}$ larger than that of $\mathrm{Eul}_v(\mathcal{V}, X)^{-1}$.

However the arithmetic specializations of $\mathcal{R}$ are of our interest and we may ask if the local Euler factors of the arithmetic specializations of $\mathcal{F}_p$ are interpolated by the local Euler factors of $\mathcal{V}$, *i.e.*, if

$$\text{(Eul-Interp)} \qquad \mathrm{Eul}_v(\mathcal{V}, X)^{-1} \in \mathcal{O}_{\mathcal{K}}[X], \quad \lambda(\mathrm{Eul}_v(\mathcal{V}, X)) = \mathrm{Eul}_v(V_\lambda, X)$$

holds for all $v \nmid p$ and $\lambda \in \mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R})$. By the theorem below, this is true when the WMC holds for $\mathcal{F}_p$.

1.0.5. Main result. Let $\mathcal{R}$ be a characteristic zero domain containing $\mathbb{Z}_p$ as a subalgebra. Denote the fraction field of $\mathcal{R}$ by $\mathcal{K}$ and fix an algebraic closure $\overline{\mathcal{K}}$ of $\mathcal{K}$. Denote the integral closure of $\mathcal{R}$ in $\overline{\mathcal{K}}$ by $\mathcal{O}_{\overline{\mathcal{K}}}$. Note that any ring homomorphism ψ from $\mathcal{R}$ to an algebraically closed field Ω of characteristic zero extends to $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$, we fix such an extension and denote it by ψ by abuse of notation. Observe that $\overline{\mathbb{Q}}$ is contained inside $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$. Suppose that G_K acts on a free $\mathcal{R}$-module $\mathcal{T}$ such that its action is monodromic (*i.e.*, a finite index subgroup of I_K acts through its $\mathbb{Z}_p$-quotient via the exponential of a nilpotent matrix, see Definition 1.1.1). Let $\mathcal{M}_\bullet$ denote the associated monodromy filtration on $\mathcal{T}$. Denote the G_K-representation $\mathcal{T} \otimes_{\mathcal{R}} \overline{\mathcal{K}}$ by $\mathcal{V}$. For a $\mathbb{Z}_p$-algebra homomorphism $\lambda : \mathcal{R} \to \overline{\mathbb{Q}}_p$, put $V_\lambda = \mathcal{T} \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p$. Since $\mathcal{T}$ is monodromic, V_λ is also monodromic. Denote the associated monodromy filtration on V_λ by $M_{\lambda,\bullet}$.

Theorem 1.0.4 (Purity for big Galois representations)**.** *Suppose that $\lambda : \mathcal{R} \to \overline{\mathbb{Q}}_p$ is a $\mathbb{Z}_p$-algebra homomorphism such that the G_K-representation V_λ is pure of weight w. Let $\mathfrak{p}_\lambda$ denote the kernel of λ. Then the following hold.*

(1) The terms and gradings of $\mathcal{M}_\bullet$ become free over $\mathcal{R}_{\mathfrak{p}_\lambda}$ after localizing them at $\mathfrak{p}_\lambda$ and for any $i \in \mathbb{Z}$, the map λ induces isomorphisms

$$\mathcal{M}_i \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq M_{\lambda,i}, \quad \mathrm{Gr}_i\mathcal{M}_\bullet \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq \mathrm{Gr}_i M_{\lambda,\bullet}$$

of W_K-modules.

(2) There exist
  (a) an integer $J \geq 1$,
  (b) integers $0 \leq t_1 < \cdots < t_J$,
  (c) an integer $I \geq 1$,
  (d) (i) unramified characters $\chi_1, \cdots, \chi_I : W_K \to \mathcal{O}_{\overline{\mathcal{K}}}^\times$,
  (ii) irreducible Frobenius-semisimple representations

$$\rho_1 : W_K \to \mathrm{GL}_{d_1}(\overline{\mathbb{Q}}), \cdots, \rho_I : W_K \to \mathrm{GL}_{d_I}(\overline{\mathbb{Q}})$$

  with finite image and
  (e) integers $n_{ij} \geq 0$ for $1 \leq i \leq I, 1 \leq j \leq J$

such that the following hold.

  (I) There are isomorphisms of Weil-Deligne representations

$$\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{I} \bigoplus_{j=1}^{J} \mathrm{Sp}_{t_j}(\chi_i \otimes \rho_i)^{n_{ij}}_{/\overline{\mathcal{K}}},$$

$$\mathrm{WD}(V_\lambda)^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{I} \bigoplus_{j=1}^{J} \mathrm{Sp}_{t_j}(\lambda \circ (\chi_i \otimes \rho_i))^{n_{ij}}_{/\overline{\mathbb{Q}}_p}.$$

  (II) The representation $\lambda \circ (\chi_i \otimes \rho_i) : W_K \to \mathrm{GL}_{d_i}(\overline{\mathbb{Q}}_p)$ has image contained in $\mathrm{GL}_{d_i}(\overline{\mathbb{Q}})$ for all $1 \leq i \leq I$.

Furthermore, the integers I, J, t_i, n_{ij} and the representations χ_i, ρ_i depend on $\mathcal{V}$, but not on λ.

(3) The λ-specialization of the central irreducible summands (see definition 1.1.24) of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ (considered over $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$) are strictly pure of weight w.

(4) The polynomial $\mathrm{Eul}(\mathcal{V})^{-1}$ has coefficients in $\mathcal{O}_{\mathcal{K}} \cap \mathcal{R}_{\mathfrak{p}_\lambda}$, its λ-specialization is $\mathrm{Eul}(V_\lambda)^{-1}$.

(5) The $\mathcal{R}_{\mathfrak{p}_\lambda}$-modules $\mathcal{T}_{\mathfrak{p}_\lambda}^{I_K}$, $\mathcal{T}_{\mathfrak{p}_\lambda}/\mathcal{T}_{\mathfrak{p}_\lambda}^{I_K}$ are free and the map λ induces an isomorphism

$$\mathcal{T}^{I_K} \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq \mathcal{T}_{\mathfrak{p}_\lambda}^{I_K} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq V_\lambda^{I_K}.$$

Consequently, the complex $[\mathcal{T}^{I_K} \xrightarrow{\phi-1} \mathcal{T}^{I_K}]$ concentrated in degree 0, 1 descends perfectly to the complex $[V_\lambda^{I_K} \xrightarrow{\phi-1} V_\lambda^{I_K}]$ concentrated in degree 0, 1, i.e.,

$$[\mathcal{T}^{I_K} \xrightarrow{\phi-1} \mathcal{T}^{I_K}] \overset{L}{\otimes}_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq [V_\lambda^{I_K} \xrightarrow{\phi-1} V_\lambda^{I_K}].$$

For a more general version, we refer to theorem 1.2.4 which is the main result of this chapter. Its proof is obtained by using theorem 1.2.1, 1.2.2, 1.2.3 (see equation (1.2.1) for the logical order of these results). We establish these four theorems from a sequence of ten main propositions (proposition 1.3.1, 1.3.4, 1.3.5, 1.4.2, 1.4.3, 1.4.5, 1.4.6, 1.5.1, 1.5.3, 1.6.8) among which proposition 1.3.4 is the crux of the proof, which we call *purity for big Galois representations*. Since the full strength of proposition 1.3.4 is realized in theorem 1.2.4, we will also refer to theorem 1.2.4 by *purity for big Galois representations*. The (philosophical) reason behind such a terminology is explained below.

By theorem 1.2.4, the shapes of the indecomposable summands of the Frobenius semisimplification of the Weil-Deligne parametrization of a pure specialization V_λ of the big Galois representation $\mathcal{T}$ determines the shape of the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$. Conversely, the shape of the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ determines the shape of $\mathrm{WD}(V_\xi)^{\text{Fr-ss}}$ for any pure specialization ξ. Moreover for such ξ, the central irreducible summands (see definition 1.1.24) of $\mathrm{WD}(V_\xi)^{\text{Fr-ss}}$ are interpolated by the central irreducible summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ by the same theorem. On the other hand, the purity of a Weil-Deligne representation over $\overline{\mathbb{Q}}_p$ is solely determined by its central irreducible summands.

So by theorem 1.2.4, the central irreducible summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ interpolates the central irreducible summands of $\mathrm{WD}(V_\xi)^{\text{Fr-ss}}$, *i.e.*, the purity determining data of $\mathrm{WD}(V_\xi)^{\text{Fr-ss}}$ for any pure specialization ξ of $\mathcal{R}$. For this reason, we call this theorem *purity for big Galois representations.*

1.0.6. Consequences. We explain some consequences of theorem 1.0.4.

1.0.6.1. *Algebraic p-adic L-functions.* Theorem 1.2.4 is the technical tool that we developed and successfully use in chapter 3, 4 to construct an algebraic p-adic L-function along irreducible components of Hida families of ordinary forms for $\mathrm{GL}_2(\mathbb{Q})$, definite unitary groups. Using similar techniques and [**Ber13**, Lemma 5.5], we also hope to construct an algebraic p-adic L-function along irreducible components of eigenvarieties. In fact we expect that using purity of big Galois representations, an algebraic p-adic L-function can be constructed for any family of Galois representations and pseudo-representations interpolating Galois representations over $\overline{\mathbb{Q}}_p$ whose restriction to local Galois groups at places not dividing p are pure.

1.0.6.2. *Rationality in automorphic families.* Given a family $\mathcal{F}_p$ satisfying the WMC, theorem 1.0.4(2)(I) shows that the indecomposable summands of $\{\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}\}_{\lambda\in\mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R})}$ are interpolated by Weil-Deligne representations defined over $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$ and by theorem 1.0.4(2)(II), the specialization of any of these representations under any λ has image contained in $\mathrm{GL}_d(\overline{\mathbb{Q}})$ for some integer d (depending on the representation). In particular, the structure of the Frobenius semisimplification of the Weil-Deligne parametrizations of arithmetic specializations are rigid in a family satisfying the WMC.

1.0.6.3. *Euler factors.* Given any family $\mathcal{F}_p$ satisfying the WMC, we have

$$\mathrm{Eul}_v(\mathcal{V},X)^{-1}\in\mathcal{O}_{\mathcal{K}}[X],\quad \lambda(\mathrm{Eul}_v(\mathcal{V},X))=\mathrm{Eul}_v(V_\lambda,X)$$

for all $v\nmid p$ and $\lambda\in\mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R})$ by theorem 1.0.4(4), *i.e.*, (Eul-Interp) holds.

Remark 1.0.5. For the Hida family as in example 1.0.2, (Eul-Interp) is proved in [**Nek06**].

Remark 1.0.6. Our proof of this theorem does not assume

$$\mathrm{Spcl}^{\mathrm{pure}}(\mathcal{R}):=\{\lambda\in\mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathcal{R},\overline{\mathbb{Q}}_p)\mid V_\lambda|_{G_K}\text{ is pure}\}$$

to be dense in $\mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathcal{R}, \overline{\mathbb{Q}}_p)$. Note that given a family $\mathcal{F}_p$ for which $\mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R})$ is dense in $\mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathcal{R}, \overline{\mathbb{Q}}_p)$, using Hilbert's nullstellensatz, (Eul-Interp) can be proved for λ in a dense subset of $\mathrm{Spcl}^{\mathrm{arith}}(\mathcal{R})$.

1.0.7. Sketch of the proof. The main idea of the proof of purity for big Galois representations (theorem 1.0.4) lies in the proof of proposition 1.3.4. For simplicity, assume that I_K acts unipotently on $\mathcal{T}$. Then this proposition says that the *central elements* of $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ are the λ-specialization of the *central elements* of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ when V_λ is pure. Its proof is outlined in §1.3.2.1. We explain how this proposition implies theorem 1.0.4(2).

By the conjugation relation of the Frobenius in the tamely ramified Galois group of K, factors of powers of $\#k$ appear in the elements of the multiset $\mathcal{CR}$ of the characteristic roots of ϕ on $\mathcal{V}$ according to the sizes of the Jordan blocks of the monodromy. Under a $\overline{\mathbb{Q}}_p$-specialization λ, the monodromy might degenerate and possibly go to zero making the Jordan blocks of $\lambda(N)$ of size 1×1. However, these factors of powers of $\#k$ present in the elements of the multiset $\mathcal{CR}$ remain intact under such a specialization and the specialization of this multiset gives the multiset CR_λ of the characteristic roots of ϕ on V_λ. When V_λ is pure, its monodromy can be read off from the amount of factors of powers of $\#k$ in the elements of the multiset CR_λ compared to its central elements.

Since the central elements of $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ are the λ-specialization of the central elements of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ (by proposition 1.3.4), the indecomposable summands of $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ are forced to be interpolated by the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ (by lemma 1.1.45).

This gives theorem 1.0.4(2). The proof of proposition 1.3.4 is outlined in §1.3.2.1.

1.0.8. Inevitability of the hypothesis that $\mathcal{R}$ is a domain. In the proof of theorem 1.2.4, we crucially use (through proposition 1.3.1) the hypothesis that the ring $\mathcal{R}$ is a domain. We cannot expect to prove theorem 1.2.4 when the ring $\mathcal{R}$ is replaced by a more general ring, an example being a ring with finitely many minimal primes.

In fact a crucial step in our proof of theorem 1.2.4 is to pin down the factors of powers $\#k$ in the characteristic roots of ϕ on the semistable part of $\mathcal{V}$ and the amount of these factors in them is governed by the size of the Jordan blocks of the monodromy of the semistable part of $\mathcal{V}$. When the coefficient ring $\mathcal{R}$ of $\mathcal{T}$ is not a domain, then the shapes of the Jordan blocks of the images of its monodromy in the stalks of $\mathrm{Spec}(\mathcal{R})$ at the generic points need not be independent of the generic points. Thereby making it impossible to pin down the factors of powers of $\#k$ in the characteristic roots of ϕ on the semistable part of $\mathcal{V}$ in a reasonable manner. In fact one can provide a counterexample even in the very simple case where $\mathcal{R} = \mathbb{Q}_p[[X]] \times \mathbb{Q}_p[[X]] \times \mathbb{Q}_p[[X]]$ by taking

$$N = \begin{pmatrix} (0,0,0) & (X,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,X-1,0) \\ (0,0,0) & (0,0,0) & (0,0,0) \end{pmatrix},$$

letting I_K act unipotently on $\mathcal{T} = \mathcal{R}^3$ (consequently $\mathcal{V}$ is its own semistable part) and ϕ act on $\mathcal{T}$ via a matrix

$$\begin{pmatrix} (\alpha_1,\beta_1,\gamma_1) & (0,0,0) & (0,0,0) \\ (0,0,0) & (\alpha_2,\beta_2,\gamma_2) & (0,0,0) \\ (0,0,0) & (0,0,0) & (\alpha_3,\beta_3,\gamma_3) \end{pmatrix} \in \mathrm{GL}_3(\mathcal{R}).$$

By the Iwasawa relation (as in equation (1.1.1)), we are forced to have

$$\alpha_1 = \alpha_2 q^{-1}, \quad \beta_2 = \beta_3 q^{-1}.$$

Let

$$\begin{aligned} \mathfrak{a}_1 &= \{0\} \times \mathbb{Q}_p[[X]] \times \mathbb{Q}_p[[X]], \\ \mathfrak{a}_2 &= \mathbb{Q}_p[[X]] \times \{0\} \times \mathbb{Q}_p[[X]], \\ \mathfrak{a}_3 &= \mathbb{Q}_p[[X]] \times \mathbb{Q}_p[[X]] \times \{0\} \end{aligned}$$

denote the minimal primes of $\mathcal{R}$. Note that the Jordan decomposition of the image of N in $\mathrm{Frac}(\mathcal{R}/\mathfrak{a}_1)$, $\mathrm{Frac}(\mathcal{R}/\mathfrak{a}_2)$, $\mathrm{Frac}(\mathcal{R}/\mathfrak{a}_3)$ is

$$\left(\begin{array}{cc|c} 0 & X & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & X-1 \\ 0 & 0 & 0 \end{array}\right), \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

respectively. Thus the behaviour of the monodromy N is not uniform along the irreducible components of $\mathrm{Spec}(\mathcal{R})$ and this prohibits us from pinning down the factors of powers of q in the roots of

$$(T - (\alpha_1,\beta_1,\gamma_1))(T - (\alpha_2,\beta_2,\gamma_2))(T - (\alpha_3,\beta_3,\gamma_3))$$

in a uniform manner, *i.e.*, from obtaining an integer e_{ij} for $i \neq j$ such that

$$(\alpha_i,\beta_i,\gamma_i) = q^{e_{ij}}(\alpha_j,\beta_j,\gamma_j).$$

Thus we cannot hope to track the 'right' factors of powers of $\#k$ in the characteristic roots of ϕ on the semistable part of $\mathcal{V}$ unless $\mathcal{R}$ is domain. Thus it seems hard to have a reasonable formulation of the statement of proposition 1.3.1 (together with a proof) that could lead to a proof of theorem 1.2.4 for more general rings $\mathcal{R}$. So we are compelled to assume that $\mathcal{R}$ is a domain.

1.0.9. Organization. In the proof, one needs the notion of Weil-Deligne representations, Weil-Deligne parametrization of Galois representations etc. with coefficients in a domain. This has been given in the first section in a way analogous to [**Del73b**, 8.4–8.6], [**Tay04**, p. 77–78].

The organization of this chapter is as follows. First we recall the structure of the absolute Galois group of ℓ-adic fields. Second, we describe the notion of Weil-Deligne representations, Grothendieck monodromy theorem, Weil-Deligne parametrization, pure modules. In section 1.2, we state the main results of this chapter, which are theorem 1.2.1, 1.2.2, 1.2.3, 1.2.4. In the subsequent sections, we present the proof of these theorems.

1.1. Local Galois representations at $v \nmid p$

1.1.1. Structure of G_K. Let ℓ be a rational prime. Only for this chapter, let K [1] denote a finite extension of $\mathbb{Q}_\ell$ and $\mathcal{O}_K$ denote its ring of integers and k its residue field. Denote the cardinality of k by q. Let ϖ denote a uniformizer of $\mathcal{O}_K$ and $\mathrm{val}_K : K^\times \twoheadrightarrow \mathbb{Z}$ be the ϖ-adic valuation. Let $|\cdot|_K := (\#k)^{-\mathrm{val}_K(\cdot)}$ be the corresponding norm. The action of G_K on K preserves val_K (by [**Neu99**, Theorem 4.8, Chapter II] for instance) and hence induces an action of G_K on k, so that we have a homomorphism $G_K \to G_k$. The *inertia group* I_K is defined as the kernel of this map and is equipped with the subspace topology induced from G_K. Note that we have a short exact sequence

$$0 \to I_K \to G_K \to G_k \to 0.$$

Let $\mathrm{Fr}_k \in G_k$ be the *geometric Frobenius* element. Then the Weil group W_K is defined as the subgroup of G_K consisting of elements which map to an integral power of Fr_k in G_k. Its topology is determined by decreeing that I_K is open, and has its usual topology.

The Artin map

$$\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$$

is normalized so that the uniformizing parameters go to geometric Frobenius elements. Let

$$P_K := \mathrm{Gal}(\overline{K}/K^{\mathrm{tame}})$$

denote the wild inertia subgroup where

$$K^{\mathrm{tame}} = \bigcup_{\ell \nmid n} K^{\mathrm{ur}}(\varpi^{1/n}), \quad K^{\mathrm{ur}} = \overline{K}^{I_K}$$

(see [**Neu99**, Proposition 7.7, Chapter II] for example). Then given a compatible system $\zeta = (\zeta_n)_{\ell \nmid n}$ of primitive roots of unity, we have an isomorphism

$$t_\zeta : I_K/P_K \xrightarrow{\sim} \prod_{p \neq \ell} \mathbb{Z}_p$$

where

$$\frac{\sigma(\varpi^{1/n})}{\varpi^{1/n}} = \zeta_n^{(t_\zeta(\sigma) \bmod n)}.$$

Any other compatible system of roots of unity is of the form ζ^u for some $u \in \prod_{p \neq \ell} \mathbb{Z}_p^\times$, and we have

$$t_{\zeta^u} = u^{-1} t_\zeta.$$

By [**NSW08**, Theorem 7.5.2], for all $\sigma \in W_K$ and $\tau \in I_K$, we have

$$t_\zeta(\sigma\tau\sigma^{-1}) = \varepsilon(\sigma) t_\zeta(\tau) \tag{1.1.1}$$

where

$$\varepsilon := \prod_{p \neq \ell} \varepsilon_p : G_K \to \prod_{p \neq \ell} \mathbb{Z}_p^\times$$

[1] The same notation is introduced in §4.1.2 to denote an extension of $\mathbb{Q}_p$.

is the product of the cyclotomic characters. For a prime $p \neq \ell$, let $t_{\zeta,p}$ denote the composite map

$$t_{\zeta,p} : I_K \to I_K/P_K \xrightarrow{t_\zeta} \prod_{p\neq\ell} \mathbb{Z}_p \to \mathbb{Z}_p \tag{1.1.2}$$

where the first map is the quotient map and the third map is the projection map. Finally define $v_K : W_K \to \mathbb{Z}$ by

$$\sigma|_{K^{\mathrm{ur}}} = \mathrm{Fr}_k^{v_K(\sigma)}$$

for all $\sigma \in W_K$.

We end this section with the following definition.

Definition 1.1.1. *Let A be a commutative $\mathbb{Z}_p$-algebra of characteristic zero. Suppose that M is a free A-module with an A-linear G_K-action*

$$\rho : G_K \to \mathrm{Aut}_A(M)$$

on it. We say M is monodromic *with* monodromy N over K' *if there exists a finite extension K'/K and a nilpotent element $N \in \mathrm{End}_{A[1/p]}(M \otimes_A A[1/p])$ such that for all $\tau \in I_{K'}$*

$$\rho(\tau) = \exp(t_{\zeta,p}(\tau)N)$$

in $\mathrm{End}_{A[1/p]}(M \otimes_A A[1/p])$.

Remark 1.1.2. Note that N is unique when it exists since A is of characteristic zero (*cf.* Theorem 1.1.25).

1.1.2. Weil-Deligne representations.

Definition 1.1.3 ([**Del73b**, 8.4.1], [**Tay04**, p. 77–78])**.** *Let A be a commutative domain of characteristic zero.*

(1) A representation *of W_K over A is a representation of W_K on a free A-module of finite rank which is continuous if the module is endowed with the discrete topology (i.e., a representation with open kernel).*

(2) A Weil-Deligne representation *of W_K on a free A-module M of finite rank is a triple (r, M, N) consisting of a representation $r : W_K \to \mathrm{Aut}_A(M)$ and an endomorphism $N \in \mathrm{End}_A(M)$ such that for all $\sigma \in W_K$,*

$$r(\sigma)Nr(\sigma)^{-1} = (\#k)^{-v_K(\sigma)}N$$

in $\mathrm{End}_{A[1/\ell]}(M \otimes_A A[1/\ell])$.

Note that a Weil representation can be considered as a Weil-Deligne representation with zero N and these two representations will often be identified.

Definition 1.1.4. *Let A be a domain of characteristic zero.*

(1) A representation of I_K on a free A-module of finite rank n is said to semistable *the characteristic polynomial of τ is $(X-1)^n$ for any $\tau \in I_K$.*

(2) A representation of I_K on a free A-module of finite rank is said to totally non-semistable *if there exists an element $\tau \in I_K$ such that the characteristic polynomial of τ does not vanish at 1.*

(3) *A representation of W_K or a Weil-Deligne representation of W_K is said to be* semistable *(resp.* totally non-semistable*) if its restriction to I_K is semistable (resp. totally non-semistable).*

Remark 1.1.5.

(1) Since I_K is compact and open in W_K, if r is a representation of W_K then $r(I_K)$ is finite.
(2) For a Weil-Deligne representation (r, N) of W_K, N is necessarily nilpotent.

Lemma 1.1.6. *Let R be a ring and $r : W_K \to \mathrm{GL}_n(R)$ be a group homomorphism under which I_K has finite image. Then r is trivial on some open subgroup of W_K and hence has open kernel.*

Proof. It suffices to show that $\ker r$ contains an open subgroup H of W_K because then $\ker r$ would be the union of all the translates of H of the form gH with g in $\ker r$. Now H can be taken to be $\ker r|_{I_K}$, which being of finite index in I_K is open in I_K and hence open in W_K. □

Definition 1.1.7. *Given two Weil-Deligne representations (r_1, M_1, N_1) and (r_2, M_2, N_2) of W_K over a domain A, their sum and tensor product is defined by*

$$(r_1, M_1, N_1) \oplus (r_2, M_2, N_2) = (r_1 \oplus r_2, M_1 \oplus M_2, N_1 \oplus N_2),$$
$$(r_1, M_1, N_1) \otimes (r_2, M_2, N_2) = (r_1 \otimes r_2, M_1 \otimes M_2, \mathrm{id}_{M_1} \otimes N_2 + N_1 \otimes \mathrm{id}_{M_2}).$$

Note that the sum and tensor product of Weil-Deligne representations defined over a domain A are Weil-Deligne representations over A (*cf.* [**Del73a**, 3.1.2]).

Definition 1.1.8. *For a finite extension K'/K, the* restriction *of a Weil-Deligne representation (r, M, N) of W_K to $W_{K'}$ is defined by*

$$(r, M, N)|_{W_{K'}} = (r|_{W_{K'}}, M, N).$$

Notice that the above restriction is a Weil-Deligne representation over $W_{K'}$.

1.1.2.1. *Inertia invariants as W_K-summand.* Let $V = (r, N)$ be a Weil-Deligne representation of W_K with coefficient in a field (necessarily of characteristic zero by the definition of Weil-Deligne representation given above) and $\theta \in \mathrm{GL}(V)$ denote the element

$$\theta = \frac{1}{\#\mathrm{Im}(r(I_K))} \sum_{g \in \mathrm{Im}(r(I_K))} g \in \mathrm{End}(V).$$

Lemma 1.1.9. *The element θ is an idempotent and thus V decomposes into an internal direct sum of subspaces*

$$V = \theta V \oplus (1 - \theta)V \tag{1.1.3}$$

with

$$\theta V = V^{I_K}.$$

The above decomposition is an internal direct sum of W_K-stable subspaces and these subspaces are stable under N. Moreover $(r|_{V^{I_K}}, N|_{V^{I_K}})$, $(r|_{V^{I_K,c}}, N|_{V^{I_K,c}})$ are Weil-Deligne representations and

$$(r, N) = (r|_{V^{I_K}}, N|_{V^{I_K}}) \oplus (r|_{V^{I_K,c}}, N|_{V^{I_K,c}})$$

as Weil-Deligne representations where $\oplus$ denotes the internal direct sum and $V^{I_K,c}$ denotes $(1-\theta)V$.

In $V^{I_K,c}$, the letter c stands for complement. We call $V^{I_K,c}$ the *complement of the inertia invariant* of V.

Proof. Since for any $\tau \in I_K$,

$$\begin{aligned} r(\tau)\theta &= \theta, \\ r(\tau)(1-\theta) &= r(\tau) - r(\tau)\theta \\ &= r(\tau) - \theta \\ &= r(\tau) - \theta r(\tau) \\ &= (1-\theta)r(\tau), \end{aligned}$$

the spaces $V^{I_K} = \theta V$ and $V^{I_K,c} = (1-\theta)V$ are stable under the action of I_K.

Since I_K is a normal subgroup of W_K, V^{I_K} is stable under W_K. To prove that $V^{I_K,c}$ is stable under W_K, it suffices to show that it is stable under the action of ϕ. Let $s = \#\mathrm{Im}(r(I_K))$ and $\{\tau_1, \cdots, \tau_s\}$ be a set of lifts of $r(I_K)$ in I_K. Then

$$\begin{aligned} \theta r(\phi)(1-\theta) &= \theta(r(\phi) - r(\phi)\theta) \\ &= \theta(1 - r(\phi)\theta r(\phi)^{-1})r(\phi) \\ &= \theta\left(1 - \frac{1}{s}\sum_{i=1}^{s} r(\phi\tau_i\phi^{-1})\right) r(\phi) \\ &= \left(\theta - \frac{1}{s}\sum_{i=1}^{s} \theta r(\phi\tau_i\phi^{-1})\right) r(\phi) \\ &= \left(\theta - \frac{1}{s}\sum_{i=1}^{s} \theta\right) r(\phi) \qquad\qquad \text{(since } I_K \text{ is normal in } W_K\text{)} \\ &= (\theta - \theta)r(\phi) \\ &= 0. \end{aligned}$$

So θ annihilates $r(\phi)(1-\theta)V$ and hence $r(\phi)(1-\theta)V$ is contained in $(1-\theta)V$.

Since r is a Weil-Deligne representation, θ commutes with N and hence V^{I_K} and $V^{I_K,c}$ are stable under the action of N. Thus the decomposition in equation (1.1.3) is an internal direct sum of W_K-stable subspaces and these subspaces are stable under N.

As a consequence of the above, we have

$$\begin{aligned} r|_{V^{I_K}}(\sigma)N|_{V^{I_K}}r|_{V^{I_K}}(\sigma)^{-1} &= (\#k)^{-v_K(\sigma)}N|_{V^{I_K}}, \\ r|_{V^{I_K,c}}(\sigma)N|_{V^{I_K,c}}r|_{V^{I_K,c}}(\sigma)^{-1} &= (\#k)^{-v_K(\sigma)}N|_{V^{I_K,c}} \end{aligned}$$

for all $\sigma \in W_K$. Since $\ker r$ is open in W_K, the kernels

$$\ker(r|_{V^{I_K}}) = \bigcup_{g \in \ker(r|_{V^{I_K}})} g \ker r, \quad \ker(r|_{V^{I_K,c}}) = \bigcup_{g \in \ker(r|_{V^{I_K,c}})} g \ker r$$

are open subgroups of W_K. So the restriction of (r, N) to V^{I_K} and $V^{I_K,c}$ are Weil-Deligne representations and (r, N) is equal to the internal direct sum of these restrictions as Weil-Deligne representations. □

1.1.2.2. *Frobenius semisimplification.* Let ϕ denote a lift of Fr_k in W_K. Suppose that $(r, N) = (r, V, N)$ is a Weil-Deligne representation with coefficients in a field L of characteristic zero which contains all the characteristic roots of all the elements of $r(W_K)$. Let

$$r(\phi) = r(\phi)^{ss} u = u r(\phi)^{ss}$$

be the Jordan decomposition of $r(\phi)$ as the product of a diagonalizable matrix $r(\phi)^{ss}$ and a unipotent matrix u. Following [**Del73b**, 8.5], [**Tay04**, p. 78], define

$$\tilde{r}(\sigma) = r(\sigma) u^{-v_K(\sigma)}$$

for all $\sigma \in W_K$.

Lemma 1.1.10 (*cf.*[**Del73b**, 8.5])**.** $(\tilde{r}, V, N)$ *is a Weil-Deligne representation.*

Proof. First we show that u and N commute to deduce the appropriate conjugation action of $\tilde{r}$ on N. Let $\mathrm{GL}(V)$ act on $\mathrm{End}_L(V)$ by conjugation and denote this representation by

$$\rho : \mathrm{GL}(V) \to \mathrm{GL}(\mathrm{End}_L(V)).$$

From now on the representation ρ will be considered as an L-algebra homomorphism

$$\rho : L[\mathrm{GL}(V)] \to \mathrm{End}_L(\mathrm{End}_L(V)).$$

The relation

$$r(\phi) N r(\phi)^{-1} = (\#k)^{-1} N$$

shows that

$$\rho(r(\phi)) \cdot N = (\#k)^{-1} N, \tag{1.1.4}$$

i.e., $N \in \mathrm{End}_L(V)$ is an eigenvector for $r(\phi)$ under the representation ρ. Note that

$$\rho(r(\phi)) = \rho(r(\phi)^{ss} u) = \rho(u r(\phi)^{ss}) = \rho(u)\rho(r(\phi)^{ss}) \tag{1.1.5}$$

where $\rho(r(\phi)^{ss})$ is semisimple. Since ρ is a ring homomorphism and $(u-1)^{\dim V} = 0$, the operator $\rho(u)$ is unipotent.

Since N is an eigenvector for $\rho(r(\phi))$, it is also an eigenvector for $\rho(r(\phi)^{ss})$ with the same eigenvalue $(\#k)^{-1}$. So equation (1.1.4) and (1.1.5) give

$$\rho(u) \cdot N = N.$$

In other words u commutes with N. So for all $\sigma \in W_K$ we have

$$\tilde{r}(\sigma) N \tilde{r}(\sigma)^{-1} = (\#k)^{-v_K(\sigma)} N.$$

Since I_K is normal in W_K, $r(I_K)$ is normal in $r(W_K)$ and hence $r(\phi)$ acts on $r(I_K)$ by conjugation. As $r(I_K)$ is a finite group, its automorphism group is finite and hence $r(\phi)^d$ commutes with $r(I_K)$ for some $d \geq 1$. So $r(\phi)^d$ commutes with $r(W_K)$. By the same reasoning as above it follows that u^d commutes with $r(W_K)$.

Recall that $\rho(u)$ is a unipotent operator on $\mathrm{End}_L(V)$. Note that from the Jordan decomposition of a unipotent matrix M, it follows that M fixes a vector v if and only if each positive power of M fixes v. So

$$\ker(\rho(u)^d - 1) = \ker(\rho(u) - 1)$$

and hence u commutes with $r(W_K)$. This shows that for any $\sigma_1, \sigma_2 \in W_K$,

$$\begin{aligned}\tilde{r}(\sigma_1\sigma_2) &= r(\sigma_1\sigma_2)u^{-v_K(\sigma_1\sigma_2)} \\ &= r(\sigma_1)r(\sigma_2)u^{-v_K(\sigma_1)}u^{-v_K(\sigma_2)} \\ &= r(\sigma_1)u^{-v_K(\sigma_1)}r(\sigma_2)u^{-v_K(\sigma_2)} \\ &= \tilde{r}(\sigma_1)\tilde{r}(\sigma_2).\end{aligned}$$

So $\tilde{r}$ is group homomorphism. To establish the lemma it remains to show that $\ker\tilde{r}$ is open which follows from lemma 1.1.6. □

We continue to follow the notations as above and the assumption that L is a field of characteristic zero containing all the characteristic roots of all elements of $r(W_K)$.

Definition 1.1.11 (*cf.* [**Del73b**, 8.6])**.**

(1) *The Weil-Deligne representation* $(\tilde{r}, V, N)$ *is called the* Frobenius semisimplification *of* (r, V, N) *and will be denoted by* $V^{\text{Fr-ss}}$,

(2) (r, N) *is said to be* Frobenius-semisimple *if* $\tilde{r} = r$.

1.1.2.3. *Structure of Frobenius-semisimple Weil-Deligne representations.* Let Ω denote an algebraically closed field of characteristic zero.

Definition 1.1.12.

(1) *A Weil-Deligne representation over* Ω *is said to be* indecomposable *if it is not isomorphic to a direct sum of two nonzero Weil-Deligne representations over* Ω.

(2) *A representation* M *of* W_K *over a commutative domain* A *of characteristic zero is said to be* irreducible *(resp.* Frobenius-semisimple*) if the action of* W_K *(resp. the action of* ϕ*) on* $M \otimes_A \mathrm{Frac}(A)$ *is irreducible (resp. semisimple).*

Lemma 1.1.13. *Let* $\rho : G \to \mathrm{GL}_n(\Omega)$ *be a representation of a finite group* G*. Then there exists a representation* $\rho' : G \to \mathrm{GL}_n(\overline{\mathbb{Q}})$ *such that* ρ *is a conjugate of the composite map*

$$\rho' : G \to \mathrm{GL}_n(\overline{\mathbb{Q}}) \to \mathrm{GL}_n(\Omega).$$

Proof. It follows from [**Tay91**, Theorem 1]. □

Proposition 1.1.14. *Given an irreducible Frobenius-semisimple representation* $r : W_K \to \mathrm{GL}_n(\Omega)$ *of* W_K *over* Ω*, there exists an unramified character*

$$\chi : W_K \to \Omega^\times$$

such that the representation $\chi^{-1} \otimes r : W_K \to \mathrm{GL}_n(\Omega)$ *has finite image. Moreover, there exists an irreducible Frobenius-semisimple representation* $\rho : W_K \to \mathrm{GL}_n(\overline{\mathbb{Q}})$ *with finite image such that*

$$r \simeq \chi \otimes \rho_{/\Omega}$$

where $\rho_{/\Omega}$ *denotes the map* ρ *followed by the map* $\mathrm{GL}_n(\overline{\mathbb{Q}}) \to \mathrm{GL}_n(\Omega)$ *induced by an embedding of* $\overline{\mathbb{Q}}$ *in* Ω.

Proof. The first part follows from the proof of [**BH06**, 28.6 Proposition]. The rest follows from lemma 1.1.13. □

Definition 1.1.15. *For an integer $t \geq 0$, a characteristic zero commutative domain A with $\ell \in A^\times$, a representation (r, M) of W_K over A and a choice of a square root of q in A, let $\mathrm{Sp}_t(r)_{/A}$ denote the Weil-Deligne representation with underlying module M^{t+1} on which W_K acts via*

$$r|\mathrm{Art}_K^{-1}|_K^{t/2} \oplus r|\mathrm{Art}_K^{-1}|_K^{(t-2)/2} \oplus \cdots \oplus r|\mathrm{Art}_K^{-1}|_K^{(-t+2)/2} \oplus r|\mathrm{Art}_K^{-1}|_K^{-t/2}$$

and the monodromy N induces an isomorphism from $r|\mathrm{Art}_K^{-1}|_K^{i-t/2}$ to $r|\mathrm{Art}_K^{-1}|_K^{i+1-t/2}$ for all $0 \leq i \leq t-1$ and is zero on $r|\mathrm{Art}_K^{-1}|_K^{t/2}$.

When A is an algebraically closed field and the W_K-representation r is irreducible, the representation r is called the central irreducible summand *of $\mathrm{Sp}_t(r)_{/A}$.*

When A is understood from the context, we will write $\mathrm{Sp}_t(r)$ to denote $\mathrm{Sp}_t(r)_{/A}$.

Remark 1.1.16. Note that the above definition is independent of the choice of a square root of q when t is even.

Remark 1.1.17. Let r be a Frobenius-semisimple representation of W_K over Ω. Then $\mathrm{Sp}_t(r)_{/\Omega}$ is indecomposable if and only if r is irreducible.

Definition 1.1.18. *Suppose that an indecomposable Weil-Deligne representation V over Ω is isomorphic to $\mathrm{Sp}_t(r)_{/\Omega}$. Then r is called the* central irreducible summand *of V.*

When r is one dimensional, the element $r(\phi)$ is called the central element *of V.*

Remark 1.1.19. In the above, we should have defined the central irreducible summand of V as the W_K-isomorphism class of r. However we will usually fix an isomorphism between V and $\mathrm{Sp}_t(r)_{/\Omega}$ for some r. So calling this r the central irreducible summand of V will not cause much confusion.

Remark 1.1.20. The above definition of $\mathrm{Sp}_t(r)_{/\Omega}$ differs from the definition of $\mathrm{Sp}_t(r)$ given in [**TY07**, p. 471]. In fact we have

$$\mathrm{Sp}_t(r|\mathrm{Art}_K^{-1}|^{t/2})_{/\Omega} = \mathrm{Sp}_{t+1}(r).$$

The reason behind introducing this "twisted" definition is to make the expression of the characteristic roots of ϕ look symmetric.

Theorem 1.1.21. *Any Frobenius-semisimple Weil-Deligne representation over Ω is isomorphic to*

$$\bigoplus_{i \in I} \mathrm{Sp}_{t_i}(r_i)_{/\Omega}$$

for some irreducible Frobenius-semisimple representations $r_i : W_K \to \mathrm{GL}_{n_i}(\Omega)$ and this decomposition is unique up to reordering and replacing factors by isomorphic factors. In this decomposition, the r_i are unramified characters if the original representation is unramified.

Proof. This follows from the proof of [**Del73a**, Proposition 3.1.3 (i)] and remark 1.1.20. □

Remark 1.1.22. We will often drop the subscript $/\Omega$ whenever Ω is understood from the context.

In the above we would like to call the $\mathrm{Sp}_{t_i}(r_i)$ indecomposable summands of V. However they depend on the isomorphism class of r_i, so we make the following definition.

Definition 1.1.23. *An* indecomposable summand *of a Frobenius-semisimple Weil-Deligne representation V over Ω is a Weil-Deligne subrepresentation of V isomorphic to a summand $\mathrm{Sp}_{t_i}(r_i)_{/\Omega}$ via the isomorphism*

$$V \simeq \bigoplus_{i \in I} \mathrm{Sp}_{t_i}(r_i)_{/\Omega}$$

as in theorem 1.1.21.

Notice that V has $\#I$ indecomposable summands.

Definition 1.1.24. *Given a Frobenius-semisimple Weil-Deligne representation V of W_K over Ω, the central irreducible summands of its indecomposable summands are called the* central irreducible summands *of V.*

Given a semistable Frobenius-semisimple Weil-Deligne representation V of W_K over Ω, the central elements of its indecomposable summands are called the central elements *of V.*

1.1.3. Grothendieck monodromy theorem. The following theorem is well-known (see [**ST68**, p. 515] for instance).

Fix $\phi \in G_K$ a lift of Fr_k and a compatible system $(\zeta_n)_{\ell \nmid n}$ of primitive roots of unity. Let $t_{\zeta,p} : I_K \to \mathbb{Z}_p$ denote the map (as in equation (1.1.2)) associated to this compatible system.

Theorem 1.1.25 (Grothendieck monodromy theorem)**.** *Let R be a commutative $\mathbb{Z}_p$-algebra. Suppose that R is a local domain with maximal ideal $\mathfrak{m}$ and finite residue field of characteristic p. Assume that $p \neq 0$ in R and R is complete with respect to the $\mathfrak{m}$-adic topology. Let $\rho : G_K \to \mathrm{GL}_n(R)$ be a continuous representation and $i : \mathrm{GL}_n(R) \to \mathrm{GL}_n(R[1/p])$ denote the inclusion map. Then there is a finite extension K'/K and a unique nilpotent matrix $N \in \mathrm{GL}_n(R[1/p])$ such that for all $\tau \in I_{K'}$, we have*

$$i(\rho(\tau)) = \exp(t_{\zeta,p}(\tau)N)$$

in $\mathrm{GL}_n(R[1/p])$. For all $\sigma \in W_K$, we have

$$\rho(\sigma)N\rho(\sigma)^{-1} = (\#k)^{-v_K(\sigma)}N \tag{1.1.6}$$

in $M_n(R[1/p])$.

Before going through the proof, we recall that for any nilpotent matrix in $M_n(R)$, its matrix exponential is an element of $M_n(R[1/p])$. Also for a unipotent matrix in $M_n(R[1/p])$ (*i.e.*, an element of $M_n(R[1/p])$ which differs from the identity matrix by a nilpotent matrix), its logarithm is an element of $M_n(R[1/p])$. Moreover the composite maps $\exp \circ \log$ and $\log \circ \exp$ are identity maps on the respective domains.

Proof. First we prove the uniqueness of N. Suppose that there is a nilpotent matrix N' and a finite extension K'' of K such that for all $\tau \in I_{K''}$

$$i(\rho(\tau)) = \exp(t_{\zeta,p}(\tau)N')$$

in $\mathrm{GL}_n(R[1/p])$. Then for all $\tau \in I_{K'K''}$, we have

$$\exp(t_{\zeta,p}(\tau)N) = \exp(t_{\zeta,p}(\tau)N').$$

Since $K'K''$ is a finite extension of K, $t_{\zeta,p}(\tau)$ is nonzero for some $\tau \in I_{K'K''}$. Hence by taking logarithm, it follows that

$$N = N'.$$

Now we show the existence of N. Let G_{K_0} denote the kernel of the composite map

$$G_K \to \mathrm{GL}_n(R) \to \mathrm{GL}_n(R/\mathfrak{m})$$

where the last map is mod $\mathfrak{m}$ reduction. Since $R/\mathfrak{m}$ is a finite field, K_0/K is a finite extension. The image of the subgroup G_{K_0} under ρ is contained in

$$1 + \mathfrak{m}M_n(R) = \ker(\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/\mathfrak{m}))$$

and hence is a pro-p-group. Note that the kernel of the map $t_{\zeta,p}$ fits into an exact sequence

$$0 \to P_K \to \ker t_{\zeta,p} \to \prod_{m \neq \ell, p} \mathbb{Z}_m \to 0.$$

So the cardinality of $\ker t_{\zeta,p}$ (as a supernatural number) is not divisible by p as P_K is a pro-ℓ-group. Hence ρ is trivial on $I_{K_0} \cap \ker t_{\zeta,p}$. Thus $\rho|_{I_{K_0}}$ factors through

$$t_{\zeta,p}|_{I_{K_0}} : I_{K_0} \to t_{\zeta,p}(I_{K_0}).$$

Choose $\tau \in I_{K_0}$ such that $t_{\zeta,p}(\tau)$ generates $t_{\zeta,p}(I_{K_0})$. By Iwasawa's relation (1.1.1), the characteristic roots of $\rho(\tau)$ are roots of unity. Since $\rho(I_{K_0}) \subseteq 1 + \mathfrak{m}M_n(R)$ and $R/\mathfrak{m}$ is a finite field of characteristic p, the characteristic roots of $\rho(\tau)$ are p-power roots of unity. So there exists a finite extension K'/K_0 such that all the characteristic roots of the elements of $\rho(I_{K'})$ are 1, *i.e.*, the elements of $\rho(I_{K'})$ are all unipotent.

Let

$$\psi : t_{\zeta,p}(I_{K'}) \to \mathrm{GL}_n(R)$$

be the unique continuous group homomorphism such that the diagram

$$\begin{array}{ccc} I_{K'} & & \\ \downarrow{\scriptstyle t_{\zeta,p}} & \searrow^{\rho|_{I_{K'}}} & \\ t_{\zeta,p}(I_{K'}) & \xrightarrow{\psi} & \mathrm{GL}_n(R) \end{array}$$

commutes. Take $\tau_0 \in I_{K'}$ such that $t_{\zeta,p}(\tau_0)$ generates $t_{\zeta,p}(I_{K'})$. Since $\rho(\tau_0)$ is unipotent, there exists a nilpotent matrix $N_0 \in M_n(R[1/p])$ such that

$$\rho(\tau_0) = \psi(t_{\zeta,p}(\tau_0)) = \exp(N_0).$$

Since K'/K is finite, $t_{\zeta,p}(\tau_0)$ is nonzero. Recall that it is an element of $\mathbb{Z}_p$ by definition of the map $t_{\zeta,p}$ associated with the compatible system $(\zeta_n)_{\ell \nmid n}$ of primitive roots of unity. So the element

$$N = \frac{1}{t_{\zeta,p}(\tau_0)} N_0 \in M_n(R[1/p])$$

is well-defined. Then

$$\rho(\tau_0) = \psi(t_{\zeta,p}(\tau_0)) = \exp(t_{\zeta,p}(\tau_0)N).$$

So for any $m \in \mathbb{Z}$, we have

$$\psi(mt_{\zeta,p}(\tau_0)) = \exp(mt_{\zeta,p}(\tau_0)N).$$

Hence

$$\psi(zt_{\zeta,p}(\tau_0)) = \exp(zt_{\zeta,p}(\tau_0)N)$$

for all $z \in \mathbb{Z}_p$, since ψ is continuous.

Note that for any $\sigma \in I_{K'}$, $t_{\zeta,p}(\sigma)/t_{\zeta,p}(\tau_0) \in \mathbb{Z}_p$ and hence

$$\begin{aligned}\rho(\sigma) &= \psi(t_{\zeta,p}(\sigma)) \\ &= \psi\left(\frac{t_{\zeta,p}(\sigma)}{t_{\zeta,p}(\tau_0)} \cdot t_{\zeta,p}(\tau_0)\right) \\ &= \exp\left(\frac{t_{\zeta,p}(\sigma)}{t_{\zeta,p}(\tau_0)} \cdot t_{\zeta,p}(\tau_0)N\right) \\ &= \exp(t_{\zeta,p}(\sigma)N).\end{aligned}$$

It remains to show the conjugation action of $\rho(\sigma)$ on N for $\sigma \in W_K$. Since K'/K is finite, there exists $\tau_1 \in I_{K'}$ such that $t_{\zeta,p}(\tau_1) \neq 0$. Then for any $\sigma \in W_K$, we have

$$\begin{aligned}\exp(\rho(\sigma)t_{\zeta,p}(\tau_1)N\rho(\sigma)^{-1}) &= \rho(\sigma)\exp(t_{\zeta,p}(\tau_1)N)\rho(\sigma)^{-1} \\ &= \rho(\sigma)\rho(\tau_1)\rho(\sigma)^{-1} \\ &= \rho(\sigma\tau_1\sigma^{-1}) \\ &= \rho(\tau_1^{(\#k)^{-v_K(\sigma)}}) && \text{since } \rho \text{ is trivial on } I_{K_0} \cap \ker t_{\zeta,p} \\ &= \exp(t_{\zeta,p}(\tau_1^{(\#k)^{-v_K(\sigma)}})N) \\ &= \exp((\#k)^{-v_K(\sigma)}t_{\zeta,p}(\tau_1)N).\end{aligned}$$

Since $t_{\zeta,p}(\tau)$ is nonzero, by taking logarithm we obtain the desired result. □

Remark 1.1.26. The endomorphism N above is called the *logarithm of the unipotent part of the local monodromy* (*cf.* [**Ill94**, p. 13].

1.1.4. Weil-Deligne parametrizations.

1.1.4.1. *Weil-Deligne parametrization for $T[1/p]$.* Suppose that R is a commutative $\mathbb{Z}_p$-algebra and is a domain of characteristic zero. Denote its fraction field by $\mathscr{K}$. Let T be a free R-module with an R-linear action of G_K on it via ρ. We assume that T is monodromic with monodromy N over K'. Notice that for all $\sigma \in W_K$

$$\rho(\sigma)N\rho(\sigma)^{-1} = (\#k)^{-v_K(\sigma)}N$$

in $\mathrm{End}_{R[1/p]}(T \otimes_R R[1/p])$. Let $T[1/p]$ denote the G_K-representation $T \otimes_R R[1/p]$.

Definition 1.1.27 ([**Del73b**, 8.4.2])**.** *The* Weil-Deligne parametrization $\mathrm{WD}(T[1/p])$ *of* $T[1/p]$ *is a Weil-Deligne representation given by the pair* (r, N)*, where* $r : W_K \to \mathrm{Aut}_{R[1/p]}(T[1/p])$ *is a group homomorphism defined by*

$$r(\sigma) = \rho(\sigma)\exp(-t_{\zeta,p}(\phi^{-v_K(\sigma)}\sigma)N)$$

for all $\sigma \in W_K$ *and* N *denotes the nilpotent endomorphism in* $\mathrm{End}_{R[1/p]}(T(1/p])$ *mentioned above.*

The lemma below shows that $\mathrm{WD}(T[1/p])$ is well-defined.

Lemma 1.1.28. *The map r is a group homomorphism and the Weil-Deligne parametrization* $\mathrm{WD}(T[1/p])$ *is a Weil-Deligne representation.*

Proof. Let $\sigma_1 = \phi^i\tau, \sigma_2 = \phi^j\nu$ be two elements of W_K with $i, j \in \mathbb{Z}$ and $\tau, \nu \in I_K$. As equations (1.1.1), (1.1.6) give

$$\begin{aligned}
\rho(\phi^j\nu)\exp(-t_{\zeta,p}(\phi^{-j}\tau\phi^j)N)\rho(\phi^j\nu)^{-1} &= \exp\Big(\rho(\phi^j\nu)\cdot t_{\zeta,p}(\phi^{-j}\tau\phi^j)N\cdot\rho(\phi^j\nu)^{-1}\Big)\\
&= \exp\Big(\Big(t_{\zeta,p}(\phi^{-j}\tau\phi^j)\Big)\Big(\rho(\phi^j\nu)N\rho(\phi^j\nu)^{-1}\Big)\Big)\\
&= \exp\Big(q^{-v_K(\phi^{-j})}t_{\zeta,p}(\tau)\Big(q^{-v_K(\phi^j\nu)}N\Big)\Big)\\
&= \exp\Big(q^j t_{\zeta,p}(\tau)\Big(q^{-j}N\Big)\Big)\\
&= \exp(t_{\zeta,p}(\tau)N),
\end{aligned}$$

we have

$$\rho(\phi^j\nu)\exp(-t_{\zeta,p}(\phi^{-j}\tau\phi^j)N) = \exp(-t_{\zeta,p}(\tau)N)\rho(\phi^j\nu). \tag{1.1.7}$$

Then

$$\begin{aligned}
r(\sigma_1\sigma_2) &= r(\phi^i\tau\cdot\phi^j\nu)\\
&= r(\phi^{i+j}\cdot\phi^{-j}\tau\phi^j\nu)\\
&= \rho(\phi^{i+j}\cdot\phi^{-j}\tau\phi^j\nu)\exp(-t_{\zeta,p}(\phi^{-j}\tau\phi^j\nu)N)\\
&= \rho(\phi^i\tau\cdot\phi^j\nu)\exp(-t_{\zeta,p}(\phi^{-j}\tau\phi^j\nu)N)\\
&= \rho(\phi^i\tau)\Big(\rho(\phi^j\nu)\exp(-t_{\zeta,p}(\phi^{-j}\tau\phi^j)N)\Big)\exp(-t_{\zeta,p}(\nu)N)\\
&= \rho(\phi^i\tau)\Big(\exp(-t_{\zeta,p}(\tau)N)\rho(\phi^j\nu)\Big)\exp(-t_{\zeta,p}(\nu)N) \quad \text{(by equation (1.1.7))}\\
&= \Big(\rho(\phi^i\tau)\exp(-t_{\zeta,p}(\tau)N)\Big)\Big(\rho(\phi^j\nu)\exp(-t_{\zeta,p}(\nu)N)\Big)\\
&= r(\sigma_1)r(\sigma_2).
\end{aligned}$$

So r is a group homomorphism. Note that r is trivial on $I_{K'}$ (with K' as in Theorem 1.1.25). So I_K has finite image under r and hence r has open kernel by 1.1.6. Also note that r and N satisfy the appropriate conjugation relation by equation (1.1.6). Thus (r, N) is a Weil-Deligne representation. □

Proposition 1.1.29. *The element*

$$\theta = \frac{1}{\#\mathrm{Im}(r(I_K))}\sum_{g\in\mathrm{Im}(r(I_K))} g$$

in $M_n(R([1/p])$ is an idempotent and we have

$$\mathrm{WD}(T[1/p])^{I_K} = \theta\mathrm{WD}(T[1/p]).$$

The Weil-Deligne parametrization $\mathrm{WD}(T[1/p])$ *of* $T[1/p]$ *decomposes into an internal direct sum of* W_K*-stable* $R[1/p]$*-submodules as*

$$\mathrm{WD}(T[1/p]) = \mathrm{WD}(T[1/p])^{I_K} \oplus_{R[1/p]} \mathrm{WD}(T[1/p])^{I_K,c}$$

where

$$\mathrm{WD}(T[1/p])^{I_K,c} := (1-\theta)\mathrm{WD}(T[1/p]).$$

The above summands are stable under the action of N*. When* $\mathrm{WD}(T[1/p])^{I_K}$ *and* $\mathrm{WD}(T[1/p])^{I_K,c}$ *are free over* $R[1/p]$*,*

$$(r, N) = (r|_{\mathrm{WD}(T[1/p])^{I_K}}, N|_{\mathrm{WD}(T[1/p])^{I_K}}) \oplus (r|_{\mathrm{WD}(T[1/p])^{I_K,c}}, N|_{\mathrm{WD}(T[1/p])^{I_K,c}}) \tag{1.1.8}$$

is a decomposition of Weil-Deligne representations. Moreover for any prime ideal $\mathfrak{p}$ *of* $R[1/p]$*, the* $R[1/p]_{\mathfrak{p}}$*-modules* $\mathrm{WD}(T[1/p])_{\mathfrak{p}}^{I_K}$*,* $\mathrm{WD}(T[1/p])_{\mathfrak{p}}^{I_K,c}$ *are free.*

Proof. The proof of lemma 1.1.9 with V (resp. V^{I_K}, $V^{I_K,c}$) replaced by $\mathrm{WD}(T[1/p])$ (resp. $\mathrm{WD}(T[1/p])^{I_K}$, $\mathrm{WD}(T[1/p])^{I_K,c}$) throughout proves the proposition except the last statement. Since $R[1/p]_{\mathfrak{p}}$ is local, the freeness of $\mathrm{WD}(T[1/p])_{\mathfrak{p}}^{I_K}$, $\mathrm{WD}(T[1/p])_{\mathfrak{p}}^{I_K,c}$ over $R[1/p]_{\mathfrak{p}}$ follows. □

We have an immediate corollary of the above proposition 1.1.29.

Corollary 1.1.30. *Let* T *be as in theorem 1.1.25. Then* $T[1/p]$ *decomposes into an internal direct sum of* G_K*-stable* $R[1/p]$*-submodules*

$$T[1/p] = T[1/p]_{ss} \oplus_{R[1/p]} T[1/p]_{tnss}.$$

The action of I_K *on* $T[1/p]_{ss} \otimes_R \mathscr{K}$ *is semistable and its action on* $T[1/p]_{tnss} \otimes_R \mathscr{K}$ *is totally non-semistable. The* $R[1/p]$*-submodules* $T[1/p]_{ss}$ *and* $T[1/p]_{tnss}$ *of* $T[1/p]$ *are defined by*

$$T[1/p]_{ss} = \mathrm{WD}(T[1/p])^{I_K}, \quad T[1/p]_{tnss} = \mathrm{WD}(T[1/p])^{I_K,c}$$

and the G_K*-action is defined by*

$$\sigma \mapsto r|_{\mathrm{WD}(T[1/p])^{I_K}}(\sigma) \exp(t_{\zeta,p}(\phi^{-v_K(\sigma)}\sigma) N|_{\mathrm{WD}(T[1/p])^{I_K}}),$$
$$\sigma \mapsto r|_{\mathrm{WD}(T[1/p])^{I_K,c}}(\sigma) \exp(t_{\zeta,p}(\phi^{-v_K(\sigma)}\sigma) N|_{\mathrm{WD}(T[1/p])^{I_K,c}})$$

respectively.

Proof. The first part follows from equation (1.1.8). It remains to prove the statement about I_K action on $T[1/p]_{ss}$ and $T[1/p]_{tnss}$. Since I_K acts trivially on $\mathrm{WD}(T[1/p])^{I_K}$, its action on $T[1/p]_{ss} \otimes_R \mathscr{K}$ is semistable. Now suppose that $T[1/p]_{tnss}$ is nonzero and pick a prime ideal $\mathfrak{p}$ of $R[1/p]$. By the above lemma, $\mathrm{WD}(T[1/p])_{\mathfrak{p}}^{I_K,c}$ is free. So for some element $\tau \in I_K$, the characteristic polynomial of τ on $\mathrm{WD}(T[1/p])_{\mathfrak{p}}^{I_K,c}$ is a non-constant polynomial and does not vanish at 1.

Let

$$r' = r|_{(\mathrm{WD}(T[1/p])^{I_K,c})_{\mathfrak{p}}}, \quad N' = N|_{(\mathrm{WD}(T[1/p])^{I_K,c})_{\mathfrak{p}}}.$$

Since N' commutes with $r'(\tau)$ by Proposition 1.1.29, $r'(\tau)$ and N' can be simultaneously upper triangularized over some finite extension of the fraction field $\mathscr{K}$ of R (by [**RR00**, Theorem 1.1.5] for instance). Hence the same holds for $r'(\tau)$ and $\exp(t_{\zeta,p}(\tau)N')$. Since N' is nilpotent, the eigenvalues of $\exp(t_{\zeta,p}(\tau)N')$ are 1. So the characteristic polynomial of

$r'(\tau)\exp(t_{\zeta,p}(\tau)N')$ is equal to the characteristic polynomial of $r'(\tau)$ which does not vanish at 1 by the choice of τ and hence the lemma. □

1.1.4.2. *Weil-Deligne parametrization for V.* We first prove a short lemma.

Let A be a ring, $n \geq 1$ be an integer and

$$A^n = P \oplus Q$$

be a decomposition of A^n into a direct sum of its A-submodules P and Q. For any ring homomorphism $f : A \to B$, we will identify $A^n \otimes_{A,f} B$ with B^n and will denote by $\langle f(P)\rangle$ (resp. $\langle f(Q)\rangle$) the B-submodule of B^n generated by the image of P (resp. Q) in B under f, *i.e.*, under the composite map $P \to A^n \xrightarrow{f^n} B^n$ (resp. $Q \to A^n \xrightarrow{f^n} B^n$).

Lemma 1.1.31. *Let $f : A \to B$ be a ring homomorphism. Then the map*

$$X \otimes_{A,f} B \to A^n \otimes_{A,f} B = B^n$$

induces an isomorphism between $X \otimes_{A,f} B$ and its image $\langle f(X)\rangle$ in B^n for $X = P, Q$.

Proof. It suffices to prove the lemma for $X = P$. Since Q is projective, it is flat. Hence the map

$$P \otimes_{A,f} B \to A^n \otimes_{A,f} B = B^n$$

induces an isomorphism between $P \otimes_{A,f} B$ and its image in B^n, which is $\langle f(P)\rangle$. □

Recall that $\mathscr{K}$ denotes the fraction field of R. Let V denote the G_K-representation $T \otimes_R \mathscr{K} = T[1/p] \otimes_{R[1/p]} \mathscr{K}$. Define its Weil-Deligne parametrization $\mathrm{WD}(V)$ as the pair consisting of the group homomorphism

$$W_K \to \mathrm{Aut}_{\mathscr{K}}(V), \quad \sigma \mapsto i(\rho(\sigma))\exp(-t_{\zeta,p}(\phi^{-v_K(\sigma)}\sigma)N)$$

and the endomorphism N considered as an element of $\mathrm{End}_{\mathscr{K}}(V)$.

From lemma 1.1.9, we have the decomposition

$$\mathrm{WD}(V) = \mathrm{WD}(V)^{I_K} \oplus \mathrm{WD}(V)^{I_K,c}$$

of $\mathrm{WD}(V)$ into an internal direct sum of Weil-Deligne subrepresentations.

Lemma 1.1.32. *We have*

$$\mathrm{WD}(V) = \mathrm{WD}(T[1/p]) \otimes_{R[1/p]} \mathscr{K},$$
$$\mathrm{WD}(V)^{I_K} = \mathrm{WD}(T[1/p])^{I_K} \otimes_{R[1/p]} \mathscr{K},$$
$$\mathrm{WD}(V)^{I_K,c} = \mathrm{WD}(T[1/p])^{I_K,c} \otimes_{R[1/p]} \mathscr{K}.$$

Proof. Follows from lemma 1.1.31. □

We have a corollary in analogy to corollary 1.1.30.

Corollary 1.1.33. *Let V be as above. Then V decomposes into an internal direct sum of $\mathscr{K}[G_K]$-submodules*

$$V = V_{ss} \oplus_{\mathscr{K}} V_{tnss}.$$

The inertia group I_K acts unipotently on V_{ss} and its action on V_{tnss} is totally non-semistable. These $\mathscr{K}[G_K]$-submodules are defined by

$$V_{ss} = \mathrm{WD}(V)^{I_K}, \quad V_{tnss} = \mathrm{WD}(V)^{I_K,c}$$

as $\mathscr{K}$-vector spaces and the G_K-action is defined by

$$\sigma \mapsto r|_{\mathrm{WD}(V)^{I_K}}(\sigma) \exp(t_{\zeta,p}(\phi^{-v_K(\sigma)}\sigma) N|_{\mathrm{WD}(V)^{I_K}}),$$
$$\sigma \mapsto r|_{\mathrm{WD}(V)^{I_K,c}}(\sigma) \exp(t_{\zeta,p}(\phi^{-v_K(\sigma)}\sigma) N|_{\mathrm{WD}(V)^{I_K,c}})$$

respectively.

1.1.5. Semistable part giving inertia invariant.

Proposition 1.1.34. *Let V be as above. Then*

$$V^{I_K} = (V_{ss})^{I_K}$$

and the dimension of V^{I_K} over $\mathscr{K}$ is equal to the number of indecomposable summands of $(\mathrm{WD}(V)^{I_K} \otimes_{\mathscr{K}} \overline{\mathscr{K}})^{\text{Fr-ss}}$. Suppose that $(\mathrm{WD}(V)^{I_K} \otimes_{\mathscr{K}} \overline{\mathscr{K}})^{\text{Fr-ss}}$ is isomorphic to $\oplus_{i\in I}\mathrm{Sp}_{t_i}(r_i)$ as Weil-Deligne representations where the r_i are irreducible Frobenius-semisimple representation of W_K with coefficients in $\overline{\mathscr{K}}$. Then the characteristic polynomial of ϕ on V^{I_K} is

$$\prod_{i\in I}(X - r_i(\phi)q^{-t_j/2}). \tag{1.1.9}$$

Proof. Let v be an element of $(V_{tnss})^{I_K}$. So v is also an element of $(V_{tnss})^{I_{K'}}$ and hence

$$\left(r|_{\mathrm{WD}(V)^{I_K,c}}(\tau') \exp(t_{\zeta,p}(\phi^{-v_K(\tau')}\tau') N|_{\mathrm{WD}(V)^{I_K,c}})\right) v = v$$

for all $\tau' \in I_{K'}$. Since $r|_{I_{K'}}$ is trivial, we get

$$\left(\exp(t_{\zeta,p}(\phi^{-v_K(\tau')}\tau') N|_{\mathrm{WD}(V)^{I_K,c}})\right) v = v$$

for all $\tau' \in I_{K'}$. Since K'/K is finite, there exists $\tau_0 \in I_{K'}$ such that $t_{\zeta,p}(\tau_0) \neq 0$. So we have

$$N|_{\mathrm{WD}(V)^{I_K,c}} v = 0.$$

Since $v \in (V_{tnss})^{I_K}$, for all $\tau \in I_K$, we have

$$\left(r|_{\mathrm{WD}(V)^{I_K,c}}(\tau) \exp(t_{\zeta,p}(\phi^{-v_K(\tau)}\tau) N|_{\mathrm{WD}(V)^{I_K,c}})\right) v = v,$$

i.e.,

$$\left(r|_{\mathrm{WD}(V)^{I_K,c}}(\tau)\right) v = v.$$

Since $v \in V_{tnss} = \mathrm{WD}(V)^{I_K,c}$, we get $v = 0$. So

$$V^{I_K} = (V_{ss})^{I_K}.$$

Recall that the underlying vector spaces of the representations V_{ss}, $\mathrm{WD}(V)^{I_K}$ and $(\mathrm{WD}(V)^{I_K})^{\text{Fr-ss}}$ are the same. Notice that

$$\begin{aligned}
V^{I_K} &= (V_{ss})^{I_K} \\
&= \{v \in V_{ss} |\, \left(r|_{\mathrm{WD}(V)^{I_K}}(\sigma)\exp(t_{\zeta,p}(\phi^{-v_K(\sigma)}\sigma)N|_{\mathrm{WD}(V)^{I_K}})\right) v = v \,\forall \sigma \in I_K\} \\
&= \{v \in V_{ss} |\, \left(r|_{\mathrm{WD}(V)^{I_K}}(\sigma)\exp(t_{\zeta,p}(\sigma)N|_{\mathrm{WD}(V)^{I_K}})\right) v = v \;\forall \sigma \in I_K\} \\
&= \{v \in V_{ss} |\, \left(\exp(t_{\zeta,p}(\sigma)N|_{\mathrm{WD}(V)^{I_K}})\right) v = r|_{\mathrm{WD}(V)^{I_K}}(\sigma^{-1})v \;\forall \sigma \in I_K\} \\
&= \{v \in V_{ss} |\, \left(\exp(t_{\zeta,p}(\sigma)N|_{\mathrm{WD}(V)^{I_K}})\right) v = v \;\forall \sigma \in I_K\} \\
&= \{v \in V_{ss} |\, \left(N|_{\mathrm{WD}(V)^{I_K}}\right) v = 0 \;\forall \sigma \in I_K\} \\
&= \{v \in \mathrm{WD}(V)^{I_K} |\, \left(N|_{\mathrm{WD}(V)^{I_K}}\right) v = 0\} \\
&= \ker\left(N|_{\mathrm{WD}(V)^{I_K}} : \mathrm{WD}(V)^{I_K} \to \mathrm{WD}(V)^{I_K}\right),
\end{aligned}$$

i.e.,

$$V^{I_K} = \ker\left(N|_{\mathrm{WD}(V)^{I_K}} : \mathrm{WD}(V)^{I_K} \to \mathrm{WD}(V)^{I_K}\right). \tag{1.1.10}$$

The above equation gives

$$\begin{aligned}
\dim V^{I_K} &= \dim\ker\left(N|_{\mathrm{WD}(V)^{I_K}} : \mathrm{WD}(V)^{I_K} \to \mathrm{WD}(V)^{I_K}\right) \\
&= \dim\ker\left(N|_{\mathrm{WD}(V)^{I_K}} \otimes_{\mathscr{K}} \overline{\mathscr{K}} : \mathrm{WD}(V)^{I_K} \otimes_{\mathscr{K}} \overline{\mathscr{K}} \to \mathrm{WD}(V)^{I_K} \otimes_{\mathscr{K}} \overline{\mathscr{K}}\right) \\
&= \dim\ker\left(N|_{\mathrm{WD}(V)^{I_K}} \otimes_{\mathscr{K}} \overline{\mathscr{K}} : (\mathrm{WD}(V)^{I_K} \otimes_{\mathscr{K}} \overline{\mathscr{K}})^{\text{Fr-ss}} \to (\mathrm{WD}(V)^{I_K} \otimes_{\mathscr{K}} \overline{\mathscr{K}})^{\text{Fr-ss}}\right)
\end{aligned}$$

Hence the dimension of V^{I_K} over $\mathscr{K}$ is equal to the number of indecomposable summands of $(\mathrm{WD}(V)^{I_K} \otimes_{\mathscr{K}} \overline{\mathscr{K}})^{\text{Fr-ss}}$ by theorem 1.1.21.

Now it remains to find the characteristic polynomial of ϕ on V^{I_K}. Consider the following list of polynomials of $\overline{\mathscr{K}}[X]$.

(1) The characteristic polynomial of ϕ on V^{I_K},
(2) the characteristic polynomial of ϕ on $\ker\left(N|_{\mathrm{WD}(V)^{I_K}} : \mathrm{WD}(V)^{I_K} \to \mathrm{WD}(V)^{I_K}\right)$,
(3) the characteristic polynomial of ϕ on
$$\ker\left(N|_{\mathrm{WD}(V)^{I_K}} \otimes_{\mathscr{K}} \overline{\mathscr{K}} : \mathrm{WD}(V)^{I_K} \otimes_{\mathscr{K}} \overline{\mathscr{K}} \to \mathrm{WD}(V)^{I_K} \otimes_{\mathscr{K}} \overline{\mathscr{K}}\right),$$
(4) the characteristic polynomial of ϕ on
$$\ker\left(N|_{\mathrm{WD}(V)^{I_K}} \otimes_{\mathscr{K}} \overline{\mathscr{K}} : (\mathrm{WD}(V)^{I_K} \otimes_{\mathscr{K}} \overline{\mathscr{K}})^{\text{Fr-ss}} \to (\mathrm{WD}(V)^{I_K} \otimes_{\mathscr{K}} \overline{\mathscr{K}})^{\text{Fr-ss}}\right).$$

We claim that any two consecutive items of the above list are equal. The first equality follows from the fact that the action of ϕ on V_{ss} and on $\mathrm{WD}(V)^{I_K}$ are the same via the maps ρ and r respectively and from the equation (1.1.10). The second equality follows from the flatness of $\overline{\mathscr{K}}$ over $\mathscr{K}$ and the last equality follows since the characteristic polynomial of any operator and its semisimplification are the same. By theorem 1.1.21, the lemma follows. □

From the above proof we have the following corollary.

Corollary 1.1.35. *The characteristic polynomial of ϕ on the spaces*

$$V^{I_K}, \quad \ker\left(N|_{\mathrm{WD}(V)^{I_K}} \otimes_{\mathscr{K}} \overline{\mathscr{K}} : (\mathrm{WD}(V)^{I_K} \otimes_{\mathscr{K}} \overline{\mathscr{K}})^{\text{Fr-ss}} \to (\mathrm{WD}(V)^{I_K} \otimes_{\mathscr{K}} \overline{\mathscr{K}})^{\text{Fr-ss}}\right)$$

are the same.

1.1.6. Indecomposable summands from monodromy filtration. In the following, we recall the definition of monodromy filtration and explain how the structure of a Frobenius-semisimple Weil-Deligne representation is determined by its monodromy filtration.

1.1.6.1. *Generalities on filtrations.* Following [**SZ85**, p. 495–496], we introduce some notions on filtrations.

Definition 1.1.36.

(1) An increasing filtration $M_\bullet$ *on a module* V *is a collection of submodules* $\{M_i\}_{i\in\mathbb{Z}}$*, such that*

$$M_{i-1} \subset M_i$$

for all $i \in \mathbb{Z}$*.*

(2) A increasing filtration $M_\bullet$ *on* V *is said to be* finite *if* $M_i = 0$ *for* i *sufficiently small and* $M_i = V$ *for* i *sufficiently large.*

(3) A decreasing filtration $M^\bullet$ *on a module* V *is a collection of submodules* $\{M_i\}_{i\in\mathbb{Z}}$*, such that*

$$M_{i-1} \supset M_i$$

for all $i \in \mathbb{Z}$*.*

A decreasing filtration $M^\bullet$ on V defines an increasing filtration $M_\bullet$ on V given by

$$M_i = M^{-i}$$

for all $i \in \mathbb{Z}$.

For an increasing filtration $M_\bullet$ on V, we put

$$\mathrm{Gr}_i M_\bullet = M_i/M_{i-1}.$$

Definition 1.1.37. *Given two increasing filtrations* $M_\bullet$ *and* $N_\bullet$ *on a module, their* convolution product $M_\bullet * N_\bullet$ *is defined by*

$$(M_\bullet * N_\bullet)_i = \sum_{j+k=i} M_j \cap N_k.$$

1.1.6.2. *Monodromy filtration.*

Proposition 1.1.38. *Let* N *be a nilpotent endomorphism of a finite dimensional vector space* V*. Then there exists a unique finite increasing filtration* $M_\bullet$ *such that* $NM_i \subset M_{i-2}$ *for all* i *and* N^k *induces an isomorphism* $\mathrm{Gr}_k M_\bullet \xrightarrow{\sim} \mathrm{Gr}_{-k} M_\bullet$ *for all* $k \geq 0$*.*

Proof. See [**Del80**, p. 165]. □

We will call $M_\bullet$ the *monodromy filtration* associated with the nilpotent endomorphism N of V (*cf.* [**Ill94**, p. 13]).

Remark 1.1.39. There is an explicit formula for the above filtration $M_\bullet$ (*cf.* [**SZ85**, p. 499]). Let $K_\bullet$ and $I^\bullet$ denote the kernel filtration and the image filtration defined by

$$K_i = \ker N^{i+1}, \quad I^i = \mathrm{Im} N^i, \qquad i \in \mathbb{Z}.$$

Note that $K_\bullet$ is an increasing filtration and $I^\bullet$ is a decreasing filtration. Consider the increasing filtration $I_\bullet$ associated with $I^\bullet$. Then $M_\bullet$ is equal to the convolution product $K_\bullet * I_\bullet$, *i.e.*, for any $k \in \mathbb{Z}$,

$$M_k = \sum_{i+j=k} \ker N^{i+1} \cap N^{-j}V = \sum_{i+j=k} N^{-j}(\ker N^{i+1-j}) = \sum_i N^{i-k}(\ker N^{2i+1-k}). \tag{1.1.11}$$

Remark 1.1.40. More generally, given a nilpotent endomorphism N on a module V, we will define the associated kernel filtration $K_\bullet$, image filtration $I^\bullet$, monodromy filtration $M_\bullet$ on V by the above formulas, *i.e.*,

$$K_i = \ker N^{i+1}, \quad I^i = \mathrm{Im} N^i, \qquad i \in \mathbb{Z},$$
$$M_\bullet = K_\bullet * I_\bullet$$

where $I_\bullet$ is the increasing filtration associated with $I^\bullet$.

The following example is taken from [**Del80**, I.6.7, p. 166].

Example 1.1.41. Let V denote a vector space of dimension $d+1$ ($d \geq 0$) with a nilpotent operator N on it which is equal to

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with respect to a basis of V of the form $\{e_{-d}, e_{-d+2}, \cdots, e_{d-2}, e_d\}$, *i.e.*, $Ne_{-d} = 0$ and $Ne_{d-2i} = e_{d-2i-2}$ for all $0 \leq i \leq d-1$. From now on, we set e_i to be zero if it is not already defined. The associated filtrations $K_\bullet$ and $I_\bullet$ of N are given by

$$\cdots \subset K_{-1} = \{0\} \subset K_0 = \langle e_{-d} \rangle \subset \cdots \subset K_i = \langle e_{-d}, \cdots, e_{d-2(d-i)} \rangle \subset \cdots \subset K_d = V \subset \cdots,$$
$$\cdots \subset I_{-d-1} = \{0\} \subset I_{-d} = \langle e_{-d} \rangle \subset \cdots \subset I_i = \langle e_{-d}, \cdots, e_{d+2i} \rangle \subset \cdots \subset I_0 = V \subset \cdots.$$

The filtration $M_\bullet$ is given by

$$M_i = \langle e_j \,|\, j \leq i \rangle.$$

Note that

$$\mathrm{Gr}_i M_\bullet = \langle \overline{e}_i \rangle.$$

Also

$$\dim N^a V = \max\{0, d+1-a\} \quad \text{for any integer } a \geq 1, \tag{1.1.12}$$

$$\dim M_i = \max\left\{0, \min\left\{ \left\lfloor \frac{i+d}{2} \right\rfloor + 1, d+1 \right\}\right\} \quad \text{for any } i \in \mathbb{Z}. \tag{1.1.13}$$

Example 1.1.42. Let Ω be a characteristic zero field containing a square root of q. Let $t \geq 0$ denote an integer and $r : W_K \to \Omega$ denote a character. Suppose that $M\big(\mathrm{Sp}_t(r)_{/\Omega}\big)_\bullet$ denote the monodromy filtration on $\mathrm{Sp}_t(r)_{/\Omega}$ associated with its monodromy. Then

$$\mathrm{Gr}_i M\big(\mathrm{Sp}_t(r)_{/\Omega}\big)_\bullet \simeq \begin{cases} r|\mathrm{Art}_K^{-1}|_K^{-i/2} & \text{if } i \equiv t \bmod 2 \text{ and } -t \leq i \leq t, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.1.43. In general, given a nilpotent endomorphism of a vector space V, V is a direct sum of subspaces stable under N and the restriction of N to these subspaces is (a conjugate) of the above form. The filtration $M_\bullet$ on V given by proposition 1.1.38 is the direct sum of the filtrations on the subspaces.

Remark 1.1.44. Given a Frobenius-semisimple Weil-Deligne representation of W_K over a field, the terms of the kernel and image filtration on it associated with its monodromy are stable under W_K (by the conjugation relation between the monodromy and the W_K-action). So the monodromy filtration is also stable under W_K.

1.1.6.3. *Indecomposable summands from $M_\bullet$.*

Lemma 1.1.45. *Let V be a Frobenius-semisimple Weil-Deligne representation of W_K over an algebraically closed field Ω of characteristic zero. Let $M_\bullet$ denote its monodromy filtration. Let $\mathfrak{C}$ denote a set of pairwise non-isomorphic irreducible Frobenius-semisimple W_K-representations such that each element in $\mathfrak{C}$ is isomorphic to a central irreducible summand of V and each central irreducible summand of V is isomorphic to an element of $\mathfrak{C}$. Then*

$$V \simeq \bigoplus_{r\in\mathfrak{C}} \bigoplus_{t\geq 0} \mathrm{Sp}_t(r)^{m(r|\mathrm{Art}_K^{-1}|_K^{t/2}, \mathrm{Gr}_{-t}M_\bullet) - m(r|\mathrm{Art}_K^{-1}|_K^{(t+2)/2}, \mathrm{Gr}_{-t-2}M_\bullet)}$$

as Weil-Deligne representations where

$$m(\rho_1, \rho_2) = \dim_\Omega \mathrm{Hom}_{\Omega\text{-linear}}(\rho_1, \rho_2)^{W_K}$$

for finite dimensional W_K-representations ρ_1, ρ_2 over Ω.

Proof. By theorem 1.1.21, there exist a finite set of non-negative integers I and integers $n_{rt} \geq 0$ for $r \in \mathfrak{C}$, $t \in I$ such that there is an isomorphism of Weil-Deligne representations

$$V \simeq \bigoplus_{r\in\mathfrak{C}} \bigoplus_{t\in I} \mathrm{Sp}_t(r)^{n_{rt}}.$$

So

$$M_\bullet \simeq \bigoplus_{r\in\mathfrak{C}} \bigoplus_{t\in I} M\left(\mathrm{Sp}_t(r)^{n_{rt}}\right)_\bullet.$$

As any two elements of $\mathfrak{C}$ are pairwise non-isomorphic, for an integer $t \geq 0$ and any $r \in \mathfrak{C}$, we get

$$m(r|\mathrm{Art}_K^{-1}|_K^{t/2}, \mathrm{Gr}_{-t}M_\bullet) - m(r|\mathrm{Art}_K^{-1}|_K^{(t+2)/2}, \mathrm{Gr}_{-t-2}M_\bullet) = \begin{cases} n_{rt} & \text{if } t \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$V \simeq \bigoplus_{r\in\mathfrak{C}} \bigoplus_{t\geq 0} \mathrm{Sp}_t(r)^{m(r|\mathrm{Art}_K^{-1}|_K^{t/2}, \mathrm{Gr}_{-t}M_\bullet) - m(r|\mathrm{Art}_K^{-1}|_K^{(t+2)/2}, \mathrm{Gr}_{-t-2}M_\bullet)}.$$

□

1.1.7. Pure modules.

Definition 1.1.46. *Let Q be a positive integral power of a rational prime. A Q-*Weil number *of weight $w \in \mathbb{Z}$ is an algebraic number $\alpha \in \overline{\mathbb{Q}}$ such that $Q^i\alpha$ is an algebraic integer for some $i \in \mathbb{Z}$ and $|\sigma(\alpha)| = Q^{w/2}$ for all $\sigma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.*

We will often call them Weil numbers when Q is clear from the context.

Definition 1.1.47.

*(1) (cf. [**Sch11**, p. 1014]) A Frobenius-semisimple Weil-Deligne representation V of W_K over $\overline{\mathbb{Q}}_p$ is said to be* pure of weight w *if the eigenvalues of one (and hence any) lift of the geometric Frobenius element on $\mathrm{Gr}_i M_\bullet$ are $\#k$-Weil numbers of weight $w+i$ where $M_\bullet$ denotes the monodromy filtration on V.*

(2) A p-adic representation of G_K is said to be pure of weight w *if the Frobenius semisimplification of its Weil-Deligne parametrization with respect to one (and hence any) choice of ϕ and ζ is pure of weight w.*

*(3) (cf. [**TY07**, p. 471]) A Weil-Deligne representation V of W_K over $\overline{\mathbb{Q}}_p$ is said to be* strictly pure of weight w *if the eigenvalues of one (and hence any) lift of the geometric Frobenius element on V are $\#k$-Weil numbers of weight w.*

Lemma 1.1.48. *An indecomposable Frobenius-semisimple Weil-Deligne representation V of W_K over $\overline{\mathbb{Q}}_p$ is pure of weight w if and only if for any finite extension K'/K, $V|_{W_{K'}}$ is pure of weight w.*

Proof. See [**Bla06**, p. 42] for instance. ☐

Remark 1.1.49. The weight-monodromy conjecture 1.0.1 predicts that any Galois representation arising from geometry (*i.e.*, from the étale cohomology of projective smooth varieties) is pure of integral weight.

1.2. (Statements of) Purity for big Galois representations with integral models

In this section, we state a generalization of the result about constancy of dimension of inertia invariants under arithmetic specializations (more precisely at the specializations satisfying the Weight-Monodromy conjecture) along irreducible components of Hida families of ordinary cusp forms (as in [**Fou13**, Lemma 3.9] for example). We also prove that the indecomposable summands of the Frobenius semisimplification of the Weil-Deligne parametrization of the pure specializations are of the same shape and interpolated by "big integral Weil-Deligne representations". We call this *rigidity of Galois types* and it is the analogue of rigidity of automorphic types proved in *loc. cit.* for example.

1.2.1. Notations. Let $\mathcal{R} \neq 0$ be a commutative $\mathbb{Z}_p$-algebra. Suppose that $\mathcal{R}$ is a domain of characteristic zero. Denote the fraction field of $\mathcal{R}$ by $\mathcal{K}$ and fix an algebraic closure $\overline{\mathcal{K}}$ of $\mathcal{K}$. The integral closure of $\mathcal{R}$ in $\overline{\mathcal{K}}$ will be denoted by $\mathcal{O}_{\overline{\mathcal{K}}}$. The algebraic closure of $\mathbb{Q}_p$ in $\overline{\mathcal{K}}$ is denoted by $\overline{\mathbb{Q}}_p$ and the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$ is denoted by $\overline{\mathbb{Z}}_p$. Note that $\overline{\mathbb{Z}}_p \subset \mathcal{O}_{\overline{\mathcal{K}}}$. The algebraic closure of $\mathbb{Q}$ inside $\overline{\mathbb{Q}}_p$ will be denoted by $\overline{\mathbb{Q}}$. Notice that $\overline{\mathbb{Q}}$ is contained inside $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$. Recall that K denotes a finite extension of $\mathbb{Q}_\ell$ with $\ell \neq p$ and q denotes the cardinality of the residue field of $\mathcal{O}_K$. By $q^{1/2}$, we will denote a square root of q in

$\overline{\mathbb{Q}}$ and for any $n \in \mathbb{Z}$, $q^{n/2}$ will denote $(q^{1/2})^n$. This determines a choice of a square root of q in $\overline{\mathcal{K}}$ which is required to express $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ as a direct sum of Weil-Deligne representations of the form $\mathrm{Sp}_t(r)_{/\overline{\mathcal{K}}}$ when it has a nonzero indecomposable summand of even dimension. Observe that $q^{1/2}$ is an element of $\mathcal{O}_{\overline{\mathcal{K}}}^{\times}$ as $\ell \neq p$.

For a vector space U with an action of ϕ (which denotes the lift of the geometric Frobenius to G_K as chosen in §1.0.1), the multiset of characteristic roots of ϕ on it is denoted by $CR(U)$. The multisets $CR(\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}})$, $CR(\mathrm{WD}(V_\lambda)^{\text{Fr-ss}})$ are denoted by $\mathcal{CR}, CR_\lambda$.

1.2.2. Statement of theorems. Let $\lambda : \mathcal{R} \to \overline{\mathbb{Q}}_p$ be a $\mathbb{Z}_p$-algebra homomorphism. Then λ extends to a $\mathbb{Z}_p$-algebra homomorphism from $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$ to $\overline{\mathbb{Q}}_p$. We fix one such extension and denote it by λ again. We will use the image of $q^{1/2}$ in $\overline{\mathbb{Q}}_p$ under λ as square root of q in $\overline{\mathbb{Q}}_p$. Denote by $\mathcal{O}_\lambda$ the image of the map $\lambda : \mathcal{R} \to \overline{\mathbb{Q}}_p$.

Let $\mathfrak{p}_\lambda$ denote the kernel of $\lambda : \mathcal{R} \to \overline{\mathbb{Q}}_p$. Note that λ extends to a $\mathbb{Z}_p$-algebra homomorphism $\mathcal{R}[1/p]_{\mathfrak{p}_\lambda} \to \overline{\mathbb{Q}}_p$. By abuse of notation, this map will also be denoted by λ.

Let $i : \mathcal{R} \to \mathcal{K}$ denote the inclusion map. Then by abuse of language, the maps $M_n(i) : M_n(\mathcal{R}) \to M_n(\mathcal{K})$, $\mathrm{GL}_n(i) : \mathrm{GL}_n(\mathcal{R}) \to \mathrm{GL}_n(\mathcal{K})$ will also be denoted by i. Similarly the maps $M_n(\lambda)$, $\mathrm{GL}_n(\lambda)$ will also be denoted by λ.

Let $n \geq 1$ be an integer and $\rho : G_K \to \mathrm{GL}_n(\mathcal{R})$ be a representation which is monodromic with monodromy N over K'. By definition 1.1.1, N is an element of $\mathcal{R}[1/p]$. Define $\mathcal{T} = \mathcal{R}^n$ and let G_K act on it via ρ. Denote by $\mathcal{T}[1/p]$ the G_K-representation $\mathcal{T} \otimes_{\mathcal{R}} \mathcal{R}[1/p]$. Let T_λ denote the G_K-representation $\mathcal{T} \otimes_{\mathcal{R},\lambda} \mathcal{O}_\lambda$ and V_λ denote the representation $T_\lambda \otimes_{\mathcal{O}_\lambda} \overline{\mathbb{Q}}_p$. Define the G_K-representation $\mathcal{V}$ to be $\mathcal{T} \otimes_{\mathcal{R}} \overline{\mathcal{K}}$.

The kernel filtration, the image filtration and the monodromy filtration on $\mathcal{T}$ (resp. V_λ) obtained from the nilpotent operator N on $\mathcal{T}$ (resp. $\lambda(N)$ on V_λ) will be denoted by $\mathcal{K}_\bullet, \mathcal{I}^\bullet, \mathcal{M}_\bullet$ (resp. $K_{\lambda,\bullet}$, $I_\lambda^\bullet$, $M_{\lambda,\bullet}$) respectively (*cf.* Remark 1.1.40). [2]

In the following we say that *the powers of the monodromy N do not degenerate under λ* if the inequality [3]

$$\mathrm{rk} N^a \geq \mathrm{rk} \lambda(N^a)$$

is an equality for all integer $a \geq 1$, *i.e.*, if we have

$$\text{(mono-non-deg)} \qquad \dim_{\overline{\mathcal{K}}} N^a \mathcal{V} = \dim_{\overline{\mathbb{Q}}_p} \lambda(N^a) V_\lambda \quad \forall\, a \in \mathbb{Z}_{\geq 1}.$$

If we have

$$\text{(mono-non-deg-1)} \qquad \dim_{\overline{\mathcal{K}}} N\mathcal{V} = \dim_{\overline{\mathbb{Q}}_p} \lambda(N) V_\lambda,$$

---

[2]Recall that we have used the notations K and $\mathcal{K}$ to denote a finite extension of $\mathbb{Q}_\ell$ and the fraction field of $\mathcal{R}$ respectively and they do not carry any bullets.

[3]If r_a denotes the rank of N^a, then all the minors of N^a of size $r_a + 1$ have determinant zero. So all the minors of $\lambda(N^a)$ of size $r_a + 1$ have determinant zero, *i.e.*, $\mathrm{rk}\lambda(N^a) \leq r_a$.

then we say that *the monodromy N does not degenerate under λ.* When

$$\text{(mono-fil-dim)} \qquad \dim_{\overline{\mathcal{K}}} \mathcal{M}_i \otimes_{\mathcal{R}} \overline{\mathcal{K}} = \dim_{\overline{\mathbb{Q}}_p} M_{\lambda,i} \quad \forall\, i \in \mathbb{Z},$$

we say that *the dimensions of the monodromy filtrations $\mathcal{M}_\bullet$, $M_{\lambda,\bullet}$ match termwise.*

Theorem 1.2.1 (Non-degeneracy of monodromy). *Suppose that V_λ is pure. Then the conditions* (mono-non-deg), (mono-fil-dim) *hold, i.e., the powers of the monodromy N do not degenerate under λ and the dimensions of the monodromy filtrations $\mathcal{M}_\bullet$, $M_{\lambda,\bullet}$ match termwise.*

Theorem 1.2.2 (Compatibility and freeness of filtrations). *If the condition* (mono-non-deg) *holds, then*

1. *the terms of the filtrations $\mathcal{K}_\bullet, \mathcal{I}^\bullet$ on $\mathcal{T}$ become free over $\mathcal{R}_{\mathfrak{p}_\lambda}$ after localizing them at $\mathfrak{p}_\lambda$ and under the map λ, they specialize perfectly to the respective terms of the corresponding filtrations $K_{\lambda,\bullet}, I_\lambda^\bullet$ on V_λ, i.e., for any $i \in \mathbb{Z}$, we have isomorphisms*
$$\mathcal{K}_i \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq K_{\lambda,i}, \quad \mathcal{I}^i \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq I_\lambda^i$$
of W_K-modules.
2. *the gradings of $\mathcal{K}_\bullet, \mathcal{I}^\bullet$ become free over $\mathcal{R}_{\mathfrak{p}_\lambda}$ after localizing them at $\mathfrak{p}_\lambda$ and under the map λ, they specialize perfectly to the corresponding gradings of $K_{\lambda,\bullet}, I_\lambda^\bullet$ respectively, i.e., for any $i \in \mathbb{Z}$, we have isomorphisms*
$$\mathrm{Gr}_i \mathcal{K}_\bullet \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq \mathrm{Gr}_i K_{\lambda,\bullet}, \quad \mathrm{Gr}_i \mathcal{I}^\bullet \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq \mathrm{Gr}_i I_\lambda^\bullet$$
of W_K-modules.

If both the conditions (mono-non-deg), (mono-fil-dim) *hold, then*

3. *the terms and gradings of $\mathcal{M}_\bullet$ become free over $\mathcal{R}_{\mathfrak{p}_\lambda}$ after localizing them at $\mathfrak{p}_\lambda$. Moreover for any $i \in \mathbb{Z}$, the map λ induces isomorphisms*
$$\mathcal{M}_i \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq M_{\lambda,i}, \quad \mathrm{Gr}_i \mathcal{M}_\bullet \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq \mathrm{Gr}_i M_{\lambda,\bullet}$$
of W_K-modules.

Theorem 1.2.3 (Rationality and interpolation of summands). *Suppose that both the conditions* (mono-non-deg), (mono-fil-dim) *hold. Then there are isomorphisms of Weil-Deligne representations*

$$\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{I} \bigoplus_{j=1}^{J} \mathrm{Sp}_{t_j}(\chi_i \otimes \rho_i)_{/\overline{\mathcal{K}}}^{n_{ij}},$$
$$\mathrm{WD}(V_\lambda)^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{I} \bigoplus_{j=1}^{J} \mathrm{Sp}_{t_j}(\lambda \circ (\chi_i \otimes \rho_i))_{/\overline{\mathbb{Q}}_p}^{n_{ij}}$$

for

1. *an integer $J \geq 1$,*
2. *integers $0 \leq t_1 < \cdots < t_J$,*
3. *an integer $I \geq 1$,*
4. *pairwise non-isomorphic W_K-representations $\chi_1 \otimes \rho_1, \cdots, \chi_I \otimes \rho_I$ where*
    - *$\chi_1, \cdots, \chi_I : W_K \to \mathcal{O}_{\overline{\mathcal{K}}}^\times$ are unramified characters,*

- $$\rho_1 : W_K \to \mathrm{GL}_{d_1}(\overline{\mathbb{Q}}), \cdots, \rho_I : W_K \to \mathrm{GL}_{d_I}(\overline{\mathbb{Q}})$$

  are irreducible Frobenius-semisimple representations with finite image

and

(5) integers $n_{ij} \geq 0$ for $1 \leq i \leq I, 1 \leq j \leq J$.

Consequently, the representation $\lambda \circ (\chi_i \otimes \rho_i) : W_K \to \mathrm{GL}_{d_i}(\overline{\mathbb{Q}}_p)$ has image contained in $\mathrm{GL}_{d_i}(\overline{\mathbb{Q}})$ for all $1 \leq i \leq I$. Moreover, the integers I, J, t_i, n_{ij} and the representations χ_i, ρ_i depend on $\mathcal{V}$, but not on λ.

Theorem 1.2.4 (Purity for big Galois representations)**.** *Suppose that V_λ is pure of weight w. Then the following hold.*

(1) The conditions (mono-non-deg), (mono-fil-dim) *are satisfied.*

(2) The terms and gradings of $\mathcal{M}_\bullet$ become free over $\mathcal{R}_{\mathfrak{p}_\lambda}$ after localizing them at $\mathfrak{p}_\lambda$ and for any $i \in \mathbb{Z}$, the map λ induces isomorphisms

$$\mathcal{M}_i \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq M_{\lambda,i}, \quad \mathrm{Gr}_i \mathcal{M}_\bullet \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq \mathrm{Gr}_i M_{\lambda,\bullet}$$

of W_K-modules.

(3) There exist isomorphisms of Weil-Deligne representations

$$\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{I} \bigoplus_{j=1}^{J} \mathrm{Sp}_{t_j}(\chi_i \otimes \rho_i)^{n_{ij}}_{/\overline{\mathcal{K}}},$$

$$\mathrm{WD}(V_\lambda)^{\text{Fr-ss}} \simeq \bigoplus_{i=1}^{I} \bigoplus_{j=1}^{J} \mathrm{Sp}_{t_j}(\lambda \circ (\chi_i \otimes \rho_i))^{n_{ij}}_{/\overline{\mathbb{Q}}_p}$$

for

(a) an integer $J \geq 1$,

(b) integers $0 \leq t_1 < \cdots < t_J$,

(c) an integer $I \geq 1$,

(d) pairwise non-isomorphic W_K-representations $\chi_1 \otimes \rho_1, \cdots, \chi_I \otimes \rho_I$ where

- *$\chi_1, \cdots, \chi_I : W_K \to \mathcal{O}_{\overline{\mathcal{K}}}^\times$ are unramified characters,*
- $$\rho_1 : W_K \to \mathrm{GL}_{d_1}(\overline{\mathbb{Q}}), \cdots, \rho_I : W_K \to \mathrm{GL}_{d_I}(\overline{\mathbb{Q}})$$

  are irreducible Frobenius-semisimple representations with finite image

and

(e) integers $n_{ij} \geq 0$ for $1 \leq i \leq I, 1 \leq j \leq J$.

Consequently, the representation $\lambda \circ (\chi_i \otimes \rho_i) : W_K \to \mathrm{GL}_{d_i}(\overline{\mathbb{Q}}_p)$ has image contained in $\mathrm{GL}_{d_i}(\overline{\mathbb{Q}})$ for all $1 \leq i \leq I$. Moreover, the integers I, J, t_i, n_{ij} and the representations χ_i, ρ_i depend on $\mathcal{V}$, but not on λ.

(4) The λ-specialization of the central irreducible summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ *(considered over $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$) are strictly pure of weight w.*

(5) *The* $\mathcal{R}_{\mathfrak{p}_\lambda}$*-modules* $\mathcal{T}_{\mathfrak{p}_\lambda}^{I_K}$, $\mathcal{T}_{\mathfrak{p}_\lambda}/\mathcal{T}_{\mathfrak{p}_\lambda}^{I_K}$ *are free and the map* λ *induces isomorphisms*

$$\mathcal{T}^{I_K} \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq \mathcal{T}_{\mathfrak{p}_\lambda}^{I_K} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq V_\lambda^{I_K}.$$

Consequently, the complex $[\mathcal{T}^{I_K} \xrightarrow{\phi-1} \mathcal{T}^{I_K}]$ *concentrated in degree 0, 1 descends perfectly to the complex* $[V_\lambda^{I_K} \xrightarrow{\phi-1} V_\lambda^{I_K}]$ *concentrated in degree 0, 1, i.e.,*

$$[\mathcal{T}^{I_K} \xrightarrow{\phi-1} \mathcal{T}^{I_K}] \overset{L}{\otimes}_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq [V_\lambda^{I_K} \xrightarrow{\phi-1} V_\lambda^{I_K}].$$

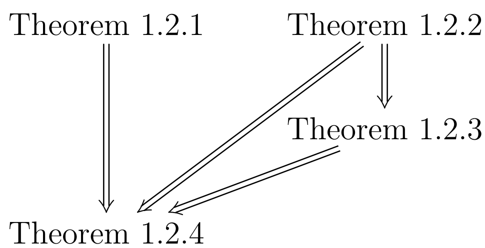
(6) *The polynomial* $\mathrm{Eul}(\mathcal{V})^{-1}$ *has coefficients in* $\mathcal{O}_{\mathcal{K}} \cap \mathcal{R}_{\mathfrak{p}_\lambda}$ *and its* λ*-specialization is* $\mathrm{Eul}(V_\lambda)^{-1}$.

Proposition 1.2.5. *The polynomial* $\mathrm{Eul}(\mathcal{V})^{-1}$ *has coefficients in* $\mathcal{O}_{\mathcal{K}}$ *and we have the inequality*

$$\dim_{\overline{\mathcal{K}}} \mathcal{V}^{I_K} \leq \dim_{\overline{\mathbb{Q}}_p} V_\lambda^{I_K}.$$

The proof of the above theorems rely on few propositions spread over the next sections. In §1.m, we prove theorem 1.2.$(m-2)$ for $m = 3, 4, 5, 6$. Their logical dependence is given below.

(1.2.1)



The above proposition is proved in §1.6.3. This proposition is also proved in [**BC09**, §7.8.1]. Before we go through the proofs, some remarks are in order.

Remark 1.2.6. In theorem 1.2.4, we do not claim that the direct sum

$$\bigoplus_{i=1}^{I} \bigoplus_{j=1}^{J} \mathrm{Sp}_{t_j}(\chi_i \otimes \rho_i)^{n_{ij}}_{/\mathcal{O}_{\overline{\mathcal{K}}}[1/p]}$$

is isomorphic to $(\mathrm{WD}(\mathcal{T}[1/p])^{\text{Fr-ss}} \otimes_{\mathcal{R}[1/p]} \mathcal{O}_{\overline{\mathcal{K}}}[1/p]$. In fact this is not true, otherwise it would imply that monodromy never degenerates under $\overline{\mathbb{Q}}_p$-specializations of $\mathcal{R}$ which is false, for example when N is nonzero and goes to zero under a $\overline{\mathbb{Q}}_p$-specialization of $\mathcal{R}$.

Remark 1.2.7. The proof below does not require $\mathcal{R}$ to be noetherian.

Remark 1.2.8. In the following, we do not require V_λ to be pure unless explicitly mentioned.

1.3. Non-degeneracy of monodromy at pure specializations

1.3.1. Integrality over $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$ and q-power factors in ϕ-characteristic roots. Let (r, N) denote the Weil-Deligne parametrization of $\mathcal{T}[1/p]$.

Proposition 1.3.1 (Rationality over $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$)**.** *Suppose that* $\mathcal{V}$ *is semistable. Then there exist*

(i) an integer $m \geq 1$,

(ii) integers $0 \leq t_1 < \cdots < t_m$,
(iii) an integer $M \geq 1$,
(iv) M *distinct unramified characters* $r_1, \cdots, r_M$ *of* W_K *with* $\mu_i := r_i(\phi) \in \mathcal{O}_{\overline{\mathcal{K}}}^{\times}$ *and*
(v) integers $n_{ij} \geq 0$ *for* $1 \leq i \leq M, 1 \leq j \leq m$ *(with* $\sum_{i=1}^{M} n_{ij} \geq 1$ *for each* j*)*

such that

$$\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}} \simeq \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{M} \mathrm{Sp}_{t_j}(r_i)_{/\overline{\mathcal{K}}}^{n_{ij}} \tag{1.3.1}$$

as Weil-Deligne representations. The map λ *gives an equality*

$$\lambda(\mathcal{CR}) = CR_\lambda$$

of multisets.

Proof. The Weil-Deligne parametrization $\mathrm{WD}(\mathcal{V})$ of $\mathcal{V}$ is a Weil-Deligne representation by lemma 1.1.28. By lemma 1.1.10, $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ is a Weil-Deligne representation. Hence by theorem 1.1.21, there are

(1) integers $m \geq 1$, $0 \leq t_1 < t_2 < \cdots < t_m$,
(2) one-dimensional unramified distinct Weil representations $r_1, \cdots, r_M$ of W_K over $\overline{\mathcal{K}}$ for some integer $M \geq 1$,
(3) integers $n_{ij} \geq 0$ for $1 \leq i \leq M, 1 \leq j \leq m$

such that we have an isomorphism

$$\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}} \simeq \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{M} \mathrm{Sp}_{t_j}(r_i)_{/\overline{\mathcal{K}}}^{n_{ij}},$$

of Weil-Deligne representations. It remains to show that the $r_i(\phi)$ are elements of $\mathcal{O}_{\overline{\mathcal{K}}}^{\times}$.

The characteristic roots of ϕ on $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ are elements of $\mathcal{O}_{\overline{\mathcal{K}}}^{\times}$ since the characteristic polynomial of ϕ on $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ and $\mathrm{WD}(\mathcal{V})$ are the same and $\mathrm{WD}(\mathcal{V})$ is defined over $\mathcal{R}$. So the characteristic roots of ϕ on the $\mathrm{Sp}_{t_j}(r_i)_{/\overline{\mathcal{K}}}$ are elements of $\mathcal{O}_{\overline{\mathcal{K}}}^{\times}$.

If r_i comes from an indecomposable summand of odd dimension (*i.e.*, there exists $1 \leq j \leq m$ with $t_j + 1$ odd and $n_{ij} \neq 0$), then $r_i(\phi) \in \mathcal{O}_{\overline{\mathcal{K}}}^{\times}$. On the other hand, if it comes from an indecomposable summand of even size, then $r_i(\phi) q^{1/2} \in \mathcal{O}_{\overline{\mathcal{K}}}^{\times}$. Since $q^{1/2}$ is a unit in $\mathcal{O}_{\overline{\mathcal{K}}}$, we get $r_i(\phi) \in \mathcal{O}_{\overline{\mathcal{K}}}^{\times}$. □

1.3.2. Determining weights of some Weil numbers. The goal of this subsection is to state and prove proposition 1.3.4. In this subsection, we will assume that $\mathcal{V}$ is semistable and use the notations of proposition 1.3.1.

Denote the number of indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of dimension $t_j + 1$ by c_j. By proposition 1.3.1

$$c_j = \sum_{i=1}^{M} n_{ij}$$

for all $1 \le j \le m$. Denote the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of dimension $t_j + 1$ by $\mathcal{V}_{j1}, \cdots, \mathcal{V}_{jc_j}$.

Definition 1.3.2. *When $\mathcal{V}$ is semistable, let $\mathcal{CE}$ (resp. CE_λ) denote the multiset formed by the central elements (as in definition 1.1.24) of the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ (resp. $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$).*

In the following, the weight of a $\#k$-Weil number α will be called the weight of α and will be denoted by $wt(\alpha)$.

Lemma 1.3.3. *Suppose that $\mathcal{V}$ is semistable and V_λ is pure of weight w. Let $1 \le J \le m$ be an integer such that the λ-specializations of the central elements of the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of dimension at least $t_J + 1$ are Weil numbers of weight w. Then for $J \le j \le m, 1 \le k \le c_j$, there are distinct indecomposable summands V_{jk} of $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ such that*

$$\dim_{\overline{\mathbb{Q}}_p} V_{jk} \ge t_j + 1, \quad \lambda(CR(\mathcal{V}_{jk})) \subset CR(V_{jk}) \tag{1.3.2}$$

for all $J \le j \le m, 1 \le k \le c_j$.

Proof. Since I_K acts trivially on $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$, each indecomposable summand of $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ is a twist of $\mathrm{Sp}_t(1)$ ($t = t_1, \cdots, t_m$) by an unramified character (here 1 denotes the trivial character of W_K). So for any indecomposable summand U of $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$, the elements of $CR(U)$ are Weil numbers of distinct weights. Thus given any number of elements of the multiset CR_λ of the same weight, these elements come from the same number of indecomposable summands of $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$, *i.e.*, each of them is a characteristic root of ϕ on an indecomposable summand of $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ and these summands are distinct. The lemma follows. □

Proposition 1.3.4 (Purity for big Galois representation)**.** *Suppose that $\mathcal{V}$ is semistable and V_λ is pure of weight w. Then the images of the μ_i under λ are $\#k$-Weil numbers of weight w. Consequently the map λ gives an equality of multisets*

$$\lambda(\mathcal{CE}) = CE_\lambda.$$

Before proving this proposition, we first give a sketch of its proof.

1.3.2.1. *Outline of the proof.* The first part of proposition 1.3.4 is proved using induction and then the last part is proved in the last paragraph of the proof. The induction goes in three steps. In step 1, we prove that the λ-specializations of the central elements of the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of largest dimension are Weil numbers of weight w. In step 2, we formulate the induction hypothesis, which says that for an integer $2 \le J \le m$, the λ-specialization of the central elements of the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of dimension $\ge t_J + 1$ are Weil numbers of weight w. In step 3, using the induction hypothesis, we prove that the λ-specializations of the central elements of the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of dimension $t_{J-1} + 1$ are Weil numbers of weight w. These three steps prove the first part of the above proposition.

We give the outline of the proof of step 1 and 3. Step 1 is proved using only the three facts below.

(1) $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ is pure of weight w,
(2) $\lambda(\mathcal{CR}) = CR_\lambda$ as multisets,
(3) $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ is annihilated by the D-th power of its monodromy where D denotes the dimension of an indecomposable summand of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of largest dimension.

Thereafter using the induction hypothesis, we prove that there exists a summand $\mathcal{W}$ (resp. W) of $\mathrm{WD}(\mathcal{W})^{\text{Fr-ss}}$ (resp. $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$) such that

(1) W is pure of weight w,
(2) $\lambda(CR(\mathcal{W})) = CR(W)$ as multisets,
(3) W is annihilated by the D-th power of its monodromy where D denotes the dimension of an indecomposable summand of $\mathcal{W}$ of largest dimension.

So by the proof of step 1, it follows that the λ-specializations of the central elements of the indecomposable summands of $\mathcal{W}$ of largest dimension are Weil numbers of weight w. By the construction of $\mathcal{W}$, its indecomposable summands of largest dimension are precisely the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of dimension $t_{J-1}+1$. This proves step 3. The summand W of $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ with the above-mentioned properties is obtained by applying the above lemma.

Proof. Since

$$V_\lambda = \mathcal{T} \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p,$$

we have

$$N_\lambda = \lambda(N).$$

For any integer $s \geq 0$, all minors of N_λ^s of size $\dim_{\overline{\mathcal{K}}} N^s(\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}) + 1$ has zero determinant (since the same holds for N^s). Hence

$$\dim_{\overline{\mathcal{K}}} N^{t_j+1}(\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}) \geq \dim_{\overline{\mathbb{Q}}_p} N_\lambda^{t_j+1}(\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}) \tag{1.3.3}$$

for $1 \leq j \leq m$.

We first show that the λ-specializations of the μ_i coming from the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of largest dimension (*i.e.*, of dimension t_m+1) are of weight w. Notice that

(1) $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ is pure of weight w,
(2) $\lambda(\mathcal{CR}) = CR_\lambda$ as multisets,
(3) $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ is annihilated by the t_m+1-th power of its monodromy, *i.e.*,

$$\dim_{\overline{\mathbb{Q}}_p} N_\lambda^{t_m+1}(\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}) = 0$$

(by equation (1.3.1) and (1.3.3)).

Since V_λ is pure of weight w, the indecomposable summands of $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ are of dimension at most t_m+1. These summands are of weight w. So the difference of the weights of a highest weight and a lowest element of the multiset CR_λ is at most $2t_m$.

Let μ_i, μ_j denote the central elements of two indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of dimension t_m+1. Then $\lambda(\mu_i q^{t_m/2})$ and $\lambda(\mu_j q^{t_m/2})$ are elements of CR_λ (by theorem 1.3.1) and hence

$$wt(\lambda(\mu_i q^{t_m/2})) - wt(\lambda(\mu_j q^{-t_m/2})) \leq 2t_m,$$

which gives

$$wt(\lambda(\mu_i q^{t_m/2})) - wt(\lambda(\mu_j q^{-t_m/2})) = 2t_m + wt(\lambda(\mu_i)) - wt(\lambda(\mu_j)).$$

So $\lambda(\mu_i)$ and $\lambda(\mu_j)$ are Weil numbers of same weights.

By the same reasoning, $\lambda(\mu_i)\lambda(q^{1/2})^{t_m}$ (resp. $\lambda(\mu_i)\lambda(q^{1/2})^{-t_m}$) is a highest (resp. lowest) weight element of CR_λ. Since V_λ is pure of weight w, the Weil-Deligne representation $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ is also pure of weight w and hence its weight w is equal to the average of the weights of a highest weight and a lowest weight element of CR_λ. Notice that this average weight is the weight of $\lambda(\mu_i)$. So the λ-specialization of the central element of any indecomposable summand of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of dimension $t_m + 1$ is a Weil number of weight w.

Note that if $m = 1$, then the first part of the above proposition follows. So assume that $m \geq 2$. We will use induction to prove the first part of the proposition.

Let $2 \leq J \leq m$ be an integer such that the λ-specializations of the central elements of the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of dimension at least $t_J + 1$ are Weil numbers of weight w. To establish the first part of the proposition, it suffices to show that the λ-specializations of the central elements of the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of dimension $t_{J-1} + 1$ are also Weil numbers of weight w.

For $J \leq j \leq m, 1 \leq k \leq c_j$, let V_{jk} be as prescribed by lemma 1.3.3. So we have

$$\dim_{\overline{\mathbb{Q}}_p} V_{jk} \geq t_j + 1 > t_{J-1} + 1 \tag{1.3.4}$$

for all $J \leq j \leq m, 1 \leq k \leq c_j$. Let

$$\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}} = \mathcal{W} \oplus \bigoplus_{j=J}^{m} \bigoplus_{k=1}^{c_j} \mathcal{V}_{jk}, \quad \mathrm{WD}(V_\lambda)^{\text{Fr-ss}} = W \oplus \bigoplus_{j=J}^{m} \bigoplus_{k=1}^{c_j} V_{jk}$$

be the decomposition of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ and $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ into Weil-Deligne subrepresentations where $\mathcal{W}$ (resp. W) is the internal direct sum of the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ (resp. $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$) apart from the $\mathcal{V}_{jk}$ (resp. V_{jk}).

Then equation (1.3.1) gives

$$\dim_{\overline{\mathcal{K}}} N^{t_{J-1}+1}(\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}) = \sum_{j=J}^{m} c_j(t_j - t_{J-1}).$$

Using equation (1.3.3), we get

$$\sum_{j=J}^{m} c_j(t_j - t_{J-1}) \geq \dim_{\overline{\mathbb{Q}}_p} N_\lambda^{t_{J-1}+1}(\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}). \tag{1.3.5}$$

This decomposition gives

(1.3.6)

$$\dim_{\overline{\mathbb{Q}}_p} N_\lambda^{t_{J-1}+1}(\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}) \geq \dim_{\overline{\mathbb{Q}}_p} N_\lambda^{t_{J-1}+1} W + \sum_{j=J}^{m} c_j(t_j - t_{J-1}) + \sum_{j=J}^{m}\sum_{k=1}^{c_j}\Big(\dim_{\overline{\mathbb{Q}}_p} V_{jk} - (t_j+1)\Big)$$

where

$$\sum_{j=J}^{m}\sum_{k=1}^{c_j}\Big(\dim_{\overline{\mathbb{Q}}_p} V_{jk} - (t_j+1)\Big) \geq 0$$

by equation (1.3.4) and also

$$\dim_{\overline{\mathbb{Q}}_p} N_\lambda^{t_{J-1}+1} W \geq 0.$$

Hence

$$\dim_{\overline{\mathbb{Q}}_p} V_{jk} = t_j + 1 \tag{1.3.7}$$

for all $J \leq j \leq m, 1 \leq k \leq c_j$ and

$$N_\lambda^{t_{J-1}+1}(W) = 0. \tag{1.3.8}$$

Since $\lambda(CR(\mathcal{V}_{jk}))$ is a subset of $CR(V_{jk})$ for all $J \leq j \leq m$, $1 \leq k \leq c_j$, by equation (1.3.7) we get

$$\lambda(CR(\mathcal{W})) = CR(W)$$

as multisets. So we have

(1) W is pure of weight w,
(2) $\lambda(CR(\mathcal{W})) = CR(W)$ as multisets,
(3) W is annihilated by the D-th power of its monodromy where D denotes the dimension of an indecomposable summand of $\mathcal{W}$ of largest dimension.

So the λ-specializations of the central elements of indecomposable summands of $\mathcal{W}$ of largest dimension (*i.e.*, of dimension $t_{J-1}+1$) are Weil numbers of weight w by an argument similar to the proof of the fact that the λ-specializations of the central elements of the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of largest dimension (*i.e.*, of dimension t_m+1) are Weil numbers of weight w.

Now it remains to show that $\lambda(\mathcal{CE}) = CE_\lambda$. Let $\mathcal{CE}_o$ (resp. $\mathcal{CE}_e$) denote the multiset formed by the central elements of the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ of odd (resp. even) dimension. Note that the multisets $\mathcal{CE}_o$, $q^{1/2}\mathcal{CE}_e$ are disjoint sub-multisets of $\mathcal{CR}$ (one of them might be empty but not both). So $\lambda(\mathcal{CE}_o)$ and $\lambda(q^{1/2})\lambda(\mathcal{CE}_e)$ are disjoint sub-multisets of $\lambda(\mathcal{CR}) = CR_\lambda$. Moreover the equality $\lambda(\mathcal{CR}) = CR_\lambda$ also shows that $\lambda(\mathcal{CE}_o)$ (resp. $\lambda(q^{1/2})\lambda(\mathcal{CE}_e)$) is the submultiset of CR_λ of Weil numbers of weight w (resp. $w+1$) by the first part of proposition 1.3.4. Since $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ is pure of weight w, we get

$$CE_\lambda = \lambda(\mathcal{CE}_o) \cup \Big(\lambda(q^{1/2})^{-1}\cdot\big(\lambda(q^{1/2})\lambda(\mathcal{CE}_e)\big)\Big).$$

This gives the desired equality

$$\lambda(\mathcal{CE}) = CE_\lambda.$$

□

1.3.3. Decomposition of $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$**.** The proposition below is a consequence of the above lemma. This lemma allows to determine the gradings of the monodromy filtration of $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ from the set CR_λ using purity of V_λ. Then we get the structure of $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ from lemma 1.1.45.

Proposition 1.3.5. *Suppose that* $\mathcal{V}$ *is semistable and* V_λ *is pure. Then*

$$\mathrm{WD}(V_\lambda)^{\text{Fr-ss}} \simeq \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{M} \mathrm{Sp}_{t_j}(\lambda \circ r_i)^{n_{ij}}_{/\overline{\mathbb{Q}}_p} \tag{1.3.9}$$

as Weil-Deligne representations.

Proof. Since V_λ and $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ have same underlying vector spaces and have same monodromy, the monodromy filtration on $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$ is equal to $M_{\lambda,\bullet}$.

Since I_K acts trivially on $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$, its action is also trivial on the terms of $M_{\lambda,\bullet}$. So $\mathrm{Gr}_k M_{\lambda,\bullet}$ is a Frobenius-semisimple unramified representation of W_K for any $k \in \mathbb{Z}$.

By proposition 1.3.4, the characteristic polynomial of ϕ on $\mathrm{Gr}_k M_{\lambda,\bullet}$ is

$$\prod_{i=1}^{M} \prod_{\substack{1 \le j \le m \\ t_j \equiv k \bmod 2 \\ -t_j \le k \le t_j}} (X - \lambda(\mu_i) q^{k/2})^{n_{ij}}.$$

For $\alpha \in \overline{\mathbb{Q}}_p^\times$, let $\psi_\alpha : W_K \to \overline{\mathbb{Q}}_p^\times$ denote the unramified character which sends ϕ to α. Since $\mathrm{Gr}_k M_{\lambda,\bullet}$ is a Frobenius-semisimple unramified representation of W_K, we get

$$\mathrm{Gr}_k M_{\lambda,\bullet} \simeq \bigoplus_{i=1}^{M} \bigoplus_{\substack{1 \le j \le m \\ t_j \equiv k \bmod 2 \\ -t_j \le k \le t_j}} \left(\psi_{\lambda(\mu_i) q^{k/2}}\right)^{\oplus n_{ij}}.$$

By lemma 1.1.45, the proposition follows. □

1.3.4. Proof of theorem 1.2.1.

Proof of theorem 1.2.1. Since $\mathcal{T}$ is monodromic over K' (as assumed in §1.2.2), the $W_{K'}$-representation $\mathcal{V}|_{W_{K'}}$ is semistable. Since the W_K-representation $\mathcal{V}$ and $W_{K'}$-representation $\mathcal{V}|_{W_{K'}}$ have same underlying vector space and have the same monodromy, their monodromy filtrations are equal. Thus it suffices to prove theorem 1.2.1 when $\mathcal{V}$ is semistable. So assume that the W_K-representation $\mathcal{V}$ is semistable. By equations (1.1.12), (1.1.13) and the equations (1.3.1), (1.3.9) above (the last two equations apply as $\mathcal{V}$ is semistable and V_λ is pure), the powers of the monodromy N do not degenerate under λ and the dimensions of the monodromy filtrations $\mathcal{M}_\bullet$, $M_{\lambda,\bullet}$ match termwise. □

1.4. Compatibility of filtrations

In this subsection, we prove theorem 1.2.2 in the following way. Its part (1) and (2) follow from proposition 1.4.2, 1.4.3, 1.4.5. From proposition 1.4.6 below, its part (3) also follows.

1.4.1. Image filtrations.

Lemma 1.4.1. *Suppose that the condition* (mono-non-deg) *holds. Then for any integer $a \geq 0$, the $\mathcal{R}_{\mathfrak{p}_\lambda}$-module $\mathcal{T}_{\mathfrak{p}_\lambda}/(N^a\mathcal{T})_{\mathfrak{p}_\lambda}$ is free.*

Proof. Consider the exact sequence

$$0 \to (N^a\mathcal{T})_{\mathfrak{p}_\lambda} \to \mathcal{T}_{\mathfrak{p}_\lambda} \to \mathcal{T}_{\mathfrak{p}_\lambda}/(N^a\mathcal{T})_{\mathfrak{p}_\lambda} \to 0 \tag{1.4.1}$$

of $\mathcal{R}_{\mathfrak{p}_\lambda}$-modules where the second map is the inclusion map and the third map is the projection map. This gives

$$\mathrm{rk}_{\mathcal{R}_{\mathfrak{p}_\lambda}} \mathcal{T}_{\mathfrak{p}_\lambda}/(N^a\mathcal{T})_{\mathfrak{p}_\lambda} = \dim_{\overline{\mathcal{K}}} \mathcal{V} - \dim_{\overline{\mathcal{K}}} N^a\mathcal{V}. \tag{1.4.2}$$

Applying $- \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}} L_\lambda$ to the short exact sequence in equation (1.4.1) yields the exact sequence of L_λ-vector spaces below.

$$(N^a\mathcal{T})_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}} L_\lambda \to V'_\lambda \to (\mathcal{T}_{\mathfrak{p}_\lambda}/(N^a\mathcal{T})_{\mathfrak{p}_\lambda}) \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}} L_\lambda \to 0$$

Considering the image of the first term of the exact sequence in its second term, we get the short exact sequence

$$0 \to N_\lambda^a V'_\lambda \to V'_\lambda \to (\mathcal{T}_{\mathfrak{p}_\lambda}/(N^a\mathcal{T})_{\mathfrak{p}_\lambda}) \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}} L_\lambda \to 0.$$

So

$$\dim_{\overline{\mathbb{Q}}_p} \Big((\mathcal{T}_{\mathfrak{p}_\lambda}/(N^a\mathcal{T})_{\mathfrak{p}_\lambda}) \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}} L_\lambda \Big) \otimes_{L_\lambda} \overline{\mathbb{Q}}_p = \dim_{\overline{\mathbb{Q}}_p} V_\lambda - \dim_{\overline{\mathbb{Q}}_p} N_\lambda^a V_\lambda.$$

Thus

$$\dim_{L_\lambda} (\mathcal{T}_{\mathfrak{p}_\lambda}/(N^a\mathcal{T})_{\mathfrak{p}_\lambda}) \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}} L_\lambda = \dim_{\overline{\mathbb{Q}}_p} V_\lambda - \dim_{\overline{\mathbb{Q}}_p} N_\lambda^a V_\lambda. \tag{1.4.3}$$

Since the condition (mono-non-deg) holds, the equations (1.4.2), (1.4.3) show that the rank and the residue dimension of the $\mathcal{R}_{\mathfrak{p}_\lambda}$-module $\mathcal{T}_{\mathfrak{p}_\lambda}/(N^a\mathcal{T})_{\mathfrak{p}_\lambda}$ are the same. So the result follows from Nakayama's lemma. □

Proposition 1.4.2 (Image filtration)**.** *Suppose that the condition* (mono-non-deg) *holds. Then for any integer $a \in \mathbb{Z}$, $(N^a\mathcal{T})_{\mathfrak{p}_\lambda}$ is free over $\mathcal{R}_{\mathfrak{p}_\lambda}$ and the map λ induces an isomorphism*

$$(N^a\mathcal{T})_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}, \lambda} L_\lambda \simeq N_\lambda^a V'_\lambda. \tag{1.4.4}$$

Proof. If $a \leq 0$, then $N^a = \mathrm{id}$ and hence the lemma follows. So assume $a \geq 0$. Then from the exact sequence in equation (1.4.1), it follows that $(N^a\mathcal{T})_{\mathfrak{p}_\lambda}$ is free over $\mathcal{R}_{\mathfrak{p}_\lambda}$ by applying lemma 1.4.1 and Nakayama's lemma. By lemma 1.4.1 above, applying $- \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}, \lambda} L_\lambda$ to the exact sequence in equation (1.4.1) yields the short exact sequence below.

$$0 \to (N^a\mathcal{T})_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}, \lambda} L_\lambda \to V'_\lambda \to V'_\lambda/N_\lambda^a V'_\lambda \to 0 \tag{1.4.5}$$

This proves

$$(N^a\mathcal{T})_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}, \lambda} L_\lambda \simeq \ker (V'_\lambda \to V'_\lambda/N_\lambda^a V'_\lambda),$$

showing

$$(N^a\mathcal{T})_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}, \lambda} L_\lambda \simeq N_\lambda^a V'_\lambda.$$

□

Now using proposition 1.4.2, we generalize lemma 1.4.1.

Proposition 1.4.3 (Gradings of image filtration)**.** *Suppose that the condition* (mono-non-deg) *holds. Then for any $a, b \in \mathbb{Z}$ with $b \geq a$, the $\mathcal{R}_{\mathfrak{p}_\lambda}$-module*

$$(N^a\mathcal{T})_{\mathfrak{p}_\lambda}/(N^b\mathcal{T})_{\mathfrak{p}_\lambda}$$

is free and the map λ induces an isomorphism

$$\big((N^a\mathcal{T})_{\mathfrak{p}_\lambda}/(N^b\mathcal{T})_{\mathfrak{p}_\lambda}\big) \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq N^a_\lambda V_\lambda/N^b_\lambda V_\lambda.$$

Proof. Note that if $r \leq 0$ is an integer, then $N^r = \mathrm{id}$. So if $a \leq 0$ and $b \geq 0$, then lemma 1.4.1 gives the result. If $a \leq 0$ and $b \leq 0$, then the result follows as $N^b = \mathrm{id}$. So from now on, we assume $a \geq 0$.

Consider the exact sequence

$$0 \to (N^b\mathcal{T})_{\mathfrak{p}_\lambda} \to (N^a\mathcal{T})_{\mathfrak{p}_\lambda} \to (N^a\mathcal{T})_{\mathfrak{p}_\lambda}/(N^b\mathcal{T})_{\mathfrak{p}_\lambda} \to 0 \tag{1.4.6}$$

of $\mathcal{R}_{\mathfrak{p}_\lambda}$-modules where the second map is the inclusion map and the third map is the projection map. This gives

$$\mathrm{rk}_{\mathcal{R}_{\mathfrak{p}_\lambda}}(N^a\mathcal{T})_{\mathfrak{p}_\lambda}/(N^b\mathcal{T})_{\mathfrak{p}_\lambda} = \dim_{\overline{\mathcal{K}}} N^a\mathcal{V} - \dim_{\overline{\mathcal{K}}} N^b\mathcal{V}. \tag{1.4.7}$$

Applying $-\otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}} L_\lambda$ to the short exact sequence in equation (1.4.6) gives the exact sequence of L_λ-vector spaces

$$(N^b\mathcal{T})_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}} L_\lambda \to N^a_\lambda V'_\lambda \to \big((N^a\mathcal{T})_{\mathfrak{p}_\lambda}/(N^b\mathcal{T})_{\mathfrak{p}_\lambda}\big) \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}} L_\lambda \to 0$$

by proposition 1.4.2.

Considering the image of the first term of the exact sequence in its second term, we get the short exact sequence

$$0 \to N^b_\lambda V'_\lambda \to N^a_\lambda V'_\lambda \to \big((N^a\mathcal{T})_{\mathfrak{p}_\lambda}/(N^b\mathcal{T})_{\mathfrak{p}_\lambda}\big) \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}} L_\lambda \to 0. \tag{1.4.8}$$

So

$$\dim_{\overline{\mathbb{Q}}_p}\Big(\big((N^a\mathcal{T})_{\mathfrak{p}_\lambda}/(N^b\mathcal{T})_{\mathfrak{p}_\lambda}\big) \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}} L_\lambda\Big) \otimes_{L_\lambda} \overline{\mathbb{Q}}_p = \dim_{\overline{\mathbb{Q}}_p} N^a_\lambda V_\lambda - \dim_{\overline{\mathbb{Q}}_p} N^b_\lambda V_\lambda.$$

Thus

$$\dim_{L_\lambda}\big((N^a\mathcal{T})_{\mathfrak{p}_\lambda}/(N^b\mathcal{T})_{\mathfrak{p}_\lambda}\big) \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda}} L_\lambda = \dim_{\overline{\mathbb{Q}}_p} N^a_\lambda V_\lambda - \dim_{\overline{\mathbb{Q}}_p} N^b_\lambda V_\lambda. \tag{1.4.9}$$

Since the condition (mono-non-deg) holds, the equations (1.4.7), (1.4.9) show that the rank and the residue dimension of the $\mathcal{R}_{\mathfrak{p}_\lambda}$-module $(N^a\mathcal{T})_{\mathfrak{p}_\lambda}/(N^b\mathcal{T})_{\mathfrak{p}_\lambda}$ are the same. So it is free by Nakayama's lemma. Then equation (1.4.8) gives

$$\big((N^a\mathcal{T})_{\mathfrak{p}_\lambda}/(N^b\mathcal{T})_{\mathfrak{p}_\lambda}\big) \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq N^a_\lambda V_\lambda/N^b_\lambda V_\lambda.$$

□

1.4.2. Kernel filtrations. For $i, j \in \mathbb{Z}$, put

$$\mathcal{S}_{ij} = \ker\left(\mathcal{T} \xrightarrow{N^i} \mathcal{T}\right) \cap \mathrm{Im}\left(\mathcal{T} \xrightarrow{N^j} \mathcal{T}\right),$$

$$S'_{\lambda,ij} = \ker\left(V'_\lambda \xrightarrow{N^i_\lambda} V'_\lambda\right) \cap \mathrm{Im}\left(V'_\lambda \xrightarrow{N^j_\lambda} V'_\lambda\right),$$

$$S_{\lambda,ij} = \ker\left(V_\lambda \xrightarrow{N^i_\lambda} V_\lambda\right) \cap \mathrm{Im}\left(V_\lambda \xrightarrow{N^j_\lambda} V_\lambda\right).$$

Note that

$$\mathcal{S}_{ij} = \ker(N^i\mathcal{T} \xrightarrow{N^j} N^{i+j}\mathcal{T}),$$

$$S'_{\lambda,ij} = \ker(N^i_\lambda V'_\lambda \xrightarrow{N^j_\lambda} N^{i+j}_\lambda V'_\lambda),$$

$$S_{\lambda,ij} = \ker(N^i_\lambda V_\lambda \xrightarrow{N^j_\lambda} N^{i+j}_\lambda V_\lambda).$$

Lemma 1.4.4. *Suppose that the condition* (mono-non-deg) *holds. Then for any $i, j \in \mathbb{Z}$, $(\mathcal{S}_{ij})_{\mathfrak{p}_\lambda}$ is free over $\mathcal{R}_{\mathfrak{p}_\lambda}$ and the map λ induces an isomorphism*

$$(\mathcal{S}_{ij})_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} L_\lambda \simeq S'_{\lambda,ij}. \tag{1.4.10}$$

Proof. Note that when $j \le 0$, then $N^j = \mathrm{id}$ and so this lemma follows from proposition 1.4.2. When $i \le 0$, then $\mathcal{S}_{ij} = \{0\}$, $S'_{\lambda,ij} = \{0\}$, so there is nothing to prove. So from now on, we will assume $i \ge 0, j \ge 0$. Then localizing the exact sequence

$$0 \to \mathcal{S}_{ij} \to N^j\mathcal{T} \to N^{i+j}\mathcal{T} \to 0$$

at $\mathfrak{p}_\lambda$ gives the short exact sequence

$$0 \to (\mathcal{S}_{ij})_{\mathfrak{p}_\lambda} \to (N^j\mathcal{T})_{\mathfrak{p}_\lambda} \to (N^{i+j}\mathcal{T})_{\mathfrak{p}_\lambda} \to 0. \tag{1.4.11}$$

The last three terms of this sequence are free over $\mathcal{R}_{\mathfrak{p}_\lambda}$ by proposition 1.4.2. So it follows that $(\mathcal{S}_{ij})_{\mathfrak{p}_\lambda}$ is free over $\mathcal{R}_{\mathfrak{p}_\lambda}$ by Nakayama's lemma.

By proposition 1.4.2, applying $- \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} L_\lambda$ to the exact sequence in equation (1.4.11) yields the short exact sequence below.

$$0 \to (\mathcal{S}_{ij})_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} L_\lambda \to N^j_\lambda V'_\lambda \to N^{i+j}_\lambda V'_\lambda \to 0 \tag{1.4.12}$$

This proves

$$(\mathcal{S}_{ij})_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} L_\lambda \simeq \ker\left(N^j_\lambda V'_\lambda \to N^{i+j}_\lambda V'_\lambda\right) = S'_{\lambda,ij}.$$

□

Proposition 1.4.5 (Kernel filtration and gradings)**.** *Suppose that the condition* (mono-non-deg) *holds. Then for any $a \in \mathbb{Z}$, the $\mathcal{R}_{\mathfrak{p}_\lambda}$-module* $\ker\left(N^a : \mathcal{T} \to \mathcal{T}\right)_{\mathfrak{p}_\lambda}$ *is free and the map λ induces an isomorphism*

$$\ker(N^a : \mathcal{T} \to \mathcal{T})_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq \ker(N^a_\lambda : V_\lambda \to V_\lambda).$$

Moreover for $a, b \in \mathbb{Z}$ with $a \ge b$, the $\mathcal{R}_{\mathfrak{p}_\lambda}$-module $\ker(N^a : \mathcal{T} \to \mathcal{T})_{\mathfrak{p}_\lambda} / \ker(N^b : \mathcal{T} \to \mathcal{T})_{\mathfrak{p}_\lambda}$ *is free and the map λ induces an isomorphism*

$$\Big(\ker(N^a : \mathcal{T} \to \mathcal{T})_{\mathfrak{p}_\lambda} / \ker(N^b : \mathcal{T} \to \mathcal{T})_{\mathfrak{p}_\lambda}\Big) \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq \ker(N^a_\lambda : V_\lambda \to V_\lambda) / \ker(N^b_\lambda : V_\lambda \to V_\lambda).$$

Proof. Note that there is nothing to prove if $a \leq 0$. So we assume that $a \geq 1$. Consider the short exact sequence

$$0 \to \ker(N^a : \mathcal{T} \to \mathcal{T})_{\mathfrak{p}_\lambda} \to \mathcal{T}_{\mathfrak{p}_\lambda} \xrightarrow{N^a} (N^a\mathcal{T})_{\mathfrak{p}_\lambda} \to 0$$

of $\mathcal{R}_{\mathfrak{p}_\lambda}$-modules. From proposition 1.4.2, the first part of the lemma follows.

Note that if $b \leq 0$, then the second part follows from the first part. So we assume $b \geq 1$. Applying snake's lemma on the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\mathcal{T} \xrightarrow{N^b} \mathcal{T}) & \longrightarrow & \mathcal{T} & \xrightarrow{N^b} & N^b\mathcal{T} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow N^{a-b} & & \\
0 & \longrightarrow & \ker(\mathcal{T} \xrightarrow{N^a} \mathcal{T}) & \longrightarrow & \mathcal{T} & \xrightarrow{N^a} & N^a\mathcal{T} & \longrightarrow & 0
\end{array}$$

with exact rows, we get an isomorphism

$$\operatorname{coker}\left(\ker(\mathcal{T} \xrightarrow{N^b} \mathcal{T}) \hookrightarrow \ker(\mathcal{T} \xrightarrow{N^a} \mathcal{T})\right) \simeq \ker\left(N^{a-b} : N^b\mathcal{T} \to N^a\mathcal{T}\right)$$

of $\mathcal{R}$-modules. So we have an exact sequence of $\mathcal{R}$-modules

$$0 \to \ker\left(\mathcal{T} \xrightarrow{N^b} \mathcal{T}\right) \to \ker\left(\mathcal{T} \xrightarrow{N^a} \mathcal{T}\right) \to \mathcal{S}_{a-b,b} \to 0.$$

By lemma 1.4.4, we are done. $\square$

1.4.3. Monodromy filtrations. Note that by equation (1.1.11)

$$\mathcal{M}_k = \sum_{i+j=k} \mathcal{S}_{i+1,-j}, \quad M_{\lambda,k} = \sum_{i+j=k} S_{\lambda,i+1,-j} \tag{1.4.13}$$

for all $k \in \mathbb{Z}$.

Proposition 1.4.6 (Monodromy filtration and gradings)**.** *Suppose that the conditions* (mono-non-deg), (mono-fil-dim) *hold. Then for any $k \in \mathbb{Z}$, $(\mathcal{M}_k)_{\mathfrak{p}_\lambda}$ is free over $\mathcal{R}_{\mathfrak{p}_\lambda}$ and the map λ induces an isomorphism*

$$(\mathcal{M}_k)_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq M_{\lambda,k}.$$

Moreover for any $i \in \mathbb{Z}$, the $\mathcal{R}_{\mathfrak{p}_\lambda}$-module $(\mathrm{Gr}_i\mathcal{M}_\bullet)_{\mathfrak{p}_\lambda}$ is free and the map λ induces an isomorphism

$$(\mathrm{Gr}_i\mathcal{M}_\bullet)_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq \mathrm{Gr}_i M_{\lambda,\bullet}.$$

Proof. The exact sequence

$$0 \to (\mathcal{M}_k)_{\mathfrak{p}_\lambda} \to \mathcal{T}_{\mathfrak{p}_\lambda} \to (\mathcal{T}/\mathcal{M}_k)_{\mathfrak{p}_\lambda} \to 0 \tag{1.4.14}$$

of $\mathcal{R}_{\mathfrak{p}_\lambda}$-modules show

$$\mathrm{rk}_{\mathcal{R}_{\mathfrak{p}_\lambda}}(\mathcal{T}/\mathcal{M}_k)_{\mathfrak{p}_\lambda} = \dim_{\overline{\mathcal{K}}}\mathcal{V} - \dim_{\overline{\mathcal{K}}}\mathcal{M}_k.$$

Moreover applying $- \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p$ to this exact sequence gives the exact sequence

$$(\mathcal{M}_k)_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \to V_\lambda \to (\mathcal{T}/\mathcal{M}_k)_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \to 0$$

of $\overline{\mathbb{Q}}_p$-vector spaces. Note that lemma 1.4.4 and equation (1.4.13) show that the image of the first term of the above exact sequence in the second term is $M_{\lambda,k}$. So we have an exact sequence

$$0 \to M_{\lambda,k} \to V_\lambda \to (\mathcal{T}/\mathcal{M}_k)_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \to 0$$

and thus

$$\dim_{\overline{\mathbb{Q}}_p} (\mathcal{T}/\mathcal{M}_k)_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p = \dim_{\overline{\mathbb{Q}}_p} V_\lambda - \dim_{\overline{\mathbb{Q}}_p} M_{\lambda,k},$$

i.e.,

$$\dim_{L_\lambda} (\mathcal{T}/\mathcal{M}_k)_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} L_\lambda = \dim_{L_\lambda} V_\lambda - \dim_{L_\lambda} M_{\lambda,k}.$$

Since the condition (mono-fil-dim) holds, we get

$$\mathrm{rk}_{\mathcal{R}_{\mathfrak{p}_\lambda}} (\mathcal{T}/\mathcal{M}_k)_{\mathfrak{p}_\lambda} = \dim_{L_\lambda} (\mathcal{T}/\mathcal{M}_k)_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} L_\lambda.$$

So by Nakayama's lemma, $(\mathcal{T}/\mathcal{M}_k)_{\mathfrak{p}_\lambda}$ is free over $\mathcal{R}_{\mathfrak{p}_\lambda}$ and hence $(\mathcal{M}_k)_{\mathfrak{p}_\lambda}$ is free over $\mathcal{R}_{\mathfrak{p}_\lambda}$.

Thus applying $- \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p$ to the exact sequence in equation (1.4.14) yields

$$(\mathcal{M}_k)_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq \mathrm{Im}\left((\mathcal{M}_k)_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \to V_\lambda\right).$$

Then lemma 1.4.4 and equation (1.4.13) show that

$$(\mathcal{M}_k)_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq M_{\lambda,k}.$$

Now let i be an integer. Then using the above isomorphism for $k = i$ and repeating the proof of freeness of $(\mathcal{T}/\mathcal{M}_k)_{\mathfrak{p}_\lambda}$ over $\mathcal{R}_{\mathfrak{p}_\lambda}$ with $\mathcal{T}, \mathcal{M}_k, V_\lambda, M_{\lambda,k}$ replaced by $\mathcal{M}_i, \mathcal{M}_{i-1}, M_{\lambda,i}, M_{\lambda,i-1}$ respectively, we get $(\mathrm{Gr}_i\mathcal{M}_\bullet)_{\mathfrak{p}_\lambda} = (\mathcal{M}_i/\mathcal{M}_{i-1})_{\mathfrak{p}_\lambda}$ is free over $\mathcal{R}_{\mathfrak{p}_\lambda}$.

Finally the exact sequence

$$0 \to \mathcal{M}_{i-1} \to \mathcal{M}_i \to \mathrm{Gr}_i\mathcal{M}_\bullet \to 0$$

combined with the above equation gives

$$(\mathrm{Gr}_i\mathcal{M}_\bullet)_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq M_{\lambda,i}/M_{\lambda,i-1} = \mathrm{Gr}_i M_{\lambda,\bullet}.$$

□

1.5. Rationality and interpolation of summands

In this section, we prove theorem 1.2.3. It follows from proposition 1.5.1, 1.5.3.

1.5.1. Rationality. Let (r, N) denote the Weil-Deligne parametrization of $\mathcal{T}[1/p]$. By proposition 1.1.29, the representation $\mathrm{WD}(\mathcal{T}[1/p])$ decomposes into an internal direct sum of $\mathcal{R}[1/p]$-submodules as

$$\mathrm{WD}(\mathcal{T}[1/p]) = \mathrm{WD}(\mathcal{T}[1/p])^{I_K} \oplus \mathrm{WD}(\mathcal{T}[1/p])^{I_K,c},$$

both of which are stable under W_K and N.

Proposition 1.5.1 (Rationality over $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$)**.** *There exist*

(i) an integer $m \geq 1$,
(ii) integers $0 \leq t_1 < \cdots < t_m$,
(iii) an integer $M \geq 1$,
(iv) M distinct unramified characters $r_1, \cdots, r_M$ of W_K with $\mu_i := r_i(\phi) \in \mathcal{O}_{\overline{\mathcal{K}}}^\times$ and
(v) integers $n_{ij} \geq 0$ for $1 \leq i \leq M, 1 \leq j \leq m$ (with $\sum_{i=1}^M n_{ij} \geq 1$ for each j)

such that

$$(1.5.1) \qquad (\mathrm{WD}(\mathcal{V})^{I_K})^{\text{Fr-ss}} \simeq \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{M} \mathrm{Sp}_{t_j}(r_i)^{n_{ij}}_{/\overline{\mathcal{K}}}$$

as Weil-Deligne representations. There also exist

(i′) an integer $m' \geq 1$,
(ii′) integers $0 \leq t'_1 < t'_2 < \cdots < t'_{m'}$,
(iii′) an integer $M' \geq 1$,
(iv′)
- *unramified characters* $\chi'_1, \cdots, \chi'_{M'} : W_K \to \mathcal{O}^{\times}_{\overline{\mathcal{K}}}$,
- *irreducible Frobenius-semisimple representations*

$$\rho'_1 : W_K \to \mathrm{GL}_{d_1}(\overline{\mathbb{Q}}), \quad \cdots, \quad \rho'_{M'} : W_K \to \mathrm{GL}_{d_{M'}}(\overline{\mathbb{Q}})$$

with finite image and

(v′) integers $n'_{ij} \geq 0$ *for* $1 \leq i \leq M', 1 \leq j \leq m'$

such that

$$(1.5.2) \qquad (\mathrm{WD}(\mathcal{V})^{I_K,c})^{\text{Fr-ss}} \simeq \bigoplus_{j=1}^{m'} \bigoplus_{i=1}^{M'} \mathrm{Sp}_{t'_j}(\chi'_i \otimes \rho'_i)^{n'_{ij}}_{/\overline{\mathcal{K}}}$$

as Weil-Deligne representations. So

$$(1.5.3) \qquad \mathrm{WD}(\mathcal{V})^{\text{Fr-ss}} \simeq \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{M} \mathrm{Sp}_{t_j}(r_i)^{n_{ij}}_{/\overline{\mathcal{K}}} \oplus \bigoplus_{j=1}^{m'} \bigoplus_{i=1}^{M'} \mathrm{Sp}_{t'_j}(\chi'_i \otimes \rho'_i)^{n'_{ij}}_{/\overline{\mathcal{K}}}$$

and the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ *are defined over* $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$. *The characteristic polynomial of* ϕ *on* $\mathrm{WD}(\mathcal{V})^{I_K}$ *is*

$$\prod_{j=1}^{m} \prod_{i=1}^{M} \prod_{k=0}^{t_j} (X - \mu_i q^{(-t_j+2k)/2})^{n_{ij}}.$$

For any prime ideal $\mathfrak{p}$ *of* $\mathcal{R}[1/p]$, *this polynomial is also the characteristic polynomial of* ϕ *on the free* $\mathcal{R}[1/p]_{\mathfrak{p}}$*-module* $\mathrm{WD}(\mathcal{T}[1/p])^{I_K}_{\mathfrak{p}}$ *and hence it is an element of* $\mathcal{R}[1/p][X] \cap \mathcal{O}_{\overline{\mathcal{K}}}[X]$.

Remark 1.5.2. Henceforth we will consider the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ as defined over $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$.

Proof. The Weil-Deligne parametrization $\mathrm{WD}(\mathcal{V})$ of $\mathcal{V}$ is a Weil-Deligne representation by lemma 1.1.28. Its inertia invariants $\mathrm{WD}(\mathcal{V})^{I_K}$ and its complement $\mathrm{WD}(\mathcal{V})^{I_K,c}$ are also Weil-Deligne representations by lemma 1.1.9. By lemma 1.1.10, $(\mathrm{WD}(\mathcal{V})^{I_K})^{\text{Fr-ss}}$, $(\mathrm{WD}(\mathcal{V})^{I_K,c})^{\text{Fr-ss}}$ are Weil-Deligne representations. Hence by theorem 1.1.21, there are

(1) integers $m \geq 1$, $0 \leq t_1 < t_2 < \cdots < t_m$,
(2) one-dimensional unramified distinct Weil representations $r_1, \cdots, r_M$ of W_K over $\overline{\mathcal{K}}$ for some integer $M \geq 1$,
(3) integers $n_{ij} \geq 0$ for $1 \leq i \leq M, 1 \leq j \leq m$

and

(1′) integers $m' \geq 1$, $0 \leq t'_1 < t'_2 < \cdots < t'_{m'}$, (2′) irreducible Frobenius-semisimple representations $r'_1, \cdots, r'_{M'}$ of W_K over $\overline{\mathcal{K}}$ for some integer $M' \geq 1$,

(3′) integers $n'_{ij} \geq 0$ for $1 \leq i \leq M', 1 \leq j \leq m'$

such that we have isomorphisms

$$(\mathrm{WD}(\mathcal{V})^{I_K})^{\text{Fr-ss}} \simeq \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{M} \mathrm{Sp}_{t_j}(r_i)^{n_{ij}}_{/\overline{\mathcal{K}}}, \quad (\mathrm{WD}(\mathcal{V})^{I_K,c})^{\text{Fr-ss}} \simeq \bigoplus_{j=1}^{m'} \bigoplus_{i=1}^{M'} \mathrm{Sp}_{t'_j}(r'_i)^{n'_{ij}}_{/\overline{\mathcal{K}}}$$

of Weil-Deligne representations. By proposition 1.1.14, for each $1 \leq i \leq M'$, there exists an unramified character $\chi'_i : W_K \to \overline{\mathcal{K}}^\times$ and an irreducible Frobenius-semisimple representation $\rho'_i : W_K \to \mathrm{GL}_{d_i}(\overline{\mathbb{Q}})$ with finite image such that

$$r'_i \simeq \chi'_i \otimes \rho'_{i/\overline{\mathcal{K}}}$$

as W_K-representations over $\overline{\mathcal{K}}$. So to establish equations (1.5.1), (1.5.2), it remains to show that the r_i and χ'_i have image in $\mathcal{O}^\times_{\overline{\mathcal{K}}}$. Since the r_i and χ'_i are unramified, it suffices to show that the $r_i(\phi)$ and $\chi_i(\phi)$ are elements of $\mathcal{O}^\times_{\overline{\mathcal{K}}}$.

The characteristic roots of ϕ on $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ are elements of $\mathcal{O}^\times_{\overline{\mathcal{K}}}$ since the characteristic polynomial of ϕ on $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ and $\mathrm{WD}(\mathcal{V})$ are the same and $\mathrm{WD}(\mathcal{V})$ is defined over $\mathcal{R}$. So the characteristic roots of ϕ on the $\mathrm{Sp}_{t_j}(r_i)_{/\overline{\mathcal{K}}}$ and $\mathrm{Sp}_{t'_j}(r'_i)_{/\overline{\mathcal{K}}}$ are elements of $\mathcal{O}^\times_{\overline{\mathcal{K}}}$.

If r_i comes from an indecomposable summand of odd dimension (*i.e.*, there exists $1 \leq j \leq m$ with $t_j + 1$ odd and $n_{ij} \neq 0$), then $r_i(\phi) \in \mathcal{O}^\times_{\overline{\mathcal{K}}}$. On the other hand, if it comes from a block of even size, then $r_i(\phi)q^{1/2} \in \mathcal{O}^\times_{\overline{\mathcal{K}}}$. Since $q^{1/2}$ is a unit in $\mathcal{O}_{\overline{\mathcal{K}}}$, we get $r_i(\phi) \in \mathcal{O}^\times_{\overline{\mathcal{K}}}$.

Similarly if the χ'_i comes from an indecomposable summand of odd dimension (*i.e.*, there exists $1 \leq j \leq m'$ with $t'_j + 1$ odd and $n'_{ij} \neq 0$), then $\chi'_i(\phi)$ times a root of unity belongs to $\mathcal{O}^\times_{\overline{\mathcal{K}}}$, which shows $\chi'_i(\phi) \in \mathcal{O}^\times_{\overline{\mathcal{K}}}$. On the other hand if it comes from a block of even size, then $\chi'_i(\phi)q^{1/2}$ times a root of unity belongs to $\mathcal{O}^\times_{\overline{\mathcal{K}}}$, which shows $\chi'_i(\phi)q^{1/2} \in \mathcal{O}^\times_{\overline{\mathcal{K}}}$. Since $q^{1/2}$ is a unit in $\mathcal{O}_{\overline{\mathcal{K}}}$, we get $\chi'_i(\phi) \in \mathcal{O}^\times_{\overline{\mathcal{K}}}$. So the equations (1.5.1), (1.5.2) follow.

Recall that there is a decomposition

$$\mathrm{WD}(\mathcal{V}) = \mathrm{WD}(\mathcal{V})^{I_K} \oplus \mathrm{WD}(\mathcal{V})^{I_K,c}$$

as an internal direct sum of Weil-Deligne subrepresentations by lemma 1.1.9. This shows

$$\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}} = (\mathrm{WD}(\mathcal{V})^{I_K})^{\text{Fr-ss}} \oplus (\mathrm{WD}(\mathcal{V})^{I_K,c})^{\text{Fr-ss}}.$$

So equation (1.5.3) holds. Since the ρ'_i are defined over $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}$ has an embedding into $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$, the $\chi'_i \otimes \rho'_i$ can be considered as a representation from W_K to $\mathrm{GL}_{d_i}(\mathcal{O}_{\overline{\mathcal{K}}}[1/p])$. Thus the indecomposable summands of $\mathrm{WD}(\mathcal{V})^{\text{Fr-ss}}$ are defined over $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$.

By equation (1.5.1), the characteristic polynomial of ϕ on $(\mathrm{WD}(\mathcal{V})^{I_K})^{\text{Fr-ss}}$ is

$$\prod_{j=1}^{m} \prod_{i=1}^{M} \prod_{k=0}^{t_j} (X - \mu_i q^{(-t_j+2k)/2})^{n_{ij}}.$$

Since the multiset of characteristic roots of ϕ on $\mathrm{WD}(\mathcal{V})^{I_K}$ and on $(\mathrm{WD}(\mathcal{V})^{I_K})^{\text{Fr-ss}}$ are the same, the above polynomial is the characteristic polynomial of ϕ on $\mathrm{WD}(\mathcal{V})^{I_K}$.

Recall that for any prime ideal $\mathfrak{p}$ of $\mathcal{R}[1/p]$, $\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}}^{I_K}$ is free over $\mathcal{R}[1/p]_{\mathfrak{p}}$ by proposition 1.1.29. So any two consecutive entries of the list

(1) the characteristic polynomial of ϕ on $\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}}^{I_K}$,
(2) the characteristic polynomial of ϕ on $\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}}^{I_K} \otimes_{\mathcal{R}[1/p]_{\mathfrak{p}}} \mathrm{Frac}(\mathcal{R}[1/p]_{\mathfrak{p}})$,
(3) the characteristic polynomial of ϕ on $\mathrm{WD}(\mathcal{T} \otimes_{\mathcal{R}} \mathcal{K})^{I_K}$,
(4) the characteristic polynomial of ϕ on $\mathrm{WD}(\mathcal{T} \otimes_{\mathcal{R}} \overline{\mathcal{K}})^{I_K} = \mathrm{WD}(\mathcal{V})^{I_K}$

are equal where the equality of the last two entries follows from [**Fon04**, proof of Proposition 0.0]. So the characteristic polynomial of ϕ on $\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}}^{I_K}$ is

$$\prod_{j=1}^{m}\prod_{i=1}^{M}\prod_{k=0}^{t_j}(X - \mu_i q^{(-t_j+2k)/2})^{n_{ij}} \in \mathcal{R}[1/p]_{\mathfrak{p}}[X].$$

The last assertion follows since $\mathcal{R}[1/p]$ is equal to the intersection of its localizations at prime ideals (taken inside $\mathcal{K}$). □

1.5.2. Interpolating summands of $\mathrm{WD}(V_\lambda)^{\text{Fr-ss}}$.

Proposition 1.5.3. *Suppose that the conditions* (mono-fil-dim), (mono-non-deg) *hold. Then*

$$\mathrm{WD}(V_\lambda)^{\text{Fr-ss}} \simeq \bigoplus_{j=1}^{m}\bigoplus_{i=1}^{M}\mathrm{Sp}_{t_j}(\lambda \circ r_i)^{n_{ij}}_{/\overline{\mathbb{Q}}_p} \oplus \bigoplus_{j=1}^{m'}\bigoplus_{i=1}^{M'}\mathrm{Sp}_{t'_j}(\lambda \circ (\chi'_i \otimes \rho'_i))^{n'_{ij}}_{/\overline{\mathbb{Q}}_p}$$

as Weil-Deligne representations.

Proof. Let $\mathfrak{q}$ denote a prime of $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]$ lying above the prime $\mathfrak{p}_\lambda$ of $\mathcal{R}$. Denote a lift of $\lambda : \mathcal{O}_{\overline{\mathcal{K}}}[1/p] \to \overline{\mathbb{Q}}_p$ to $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]_{\mathfrak{q}}$ by λ.

Since the conditions (mono-non-deg), (mono-fil-dim) hold, by theorem 1.2.2 the $\mathcal{R}_{\mathfrak{p}_\lambda}$-module $(\mathrm{Gr}_k\mathcal{M}_\bullet)_{\mathfrak{p}_\lambda}$ is free and the map λ induces isomorphism of W_K-representations

$$(\mathrm{Gr}_k\mathcal{M}_\bullet)_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq \mathrm{Gr}_k M_{\lambda,\bullet}$$

for all $k \in \mathbb{Z}$.

So the W_K-module $\mathrm{Gr}_k\mathcal{M}_\bullet \otimes_{\mathcal{R}} \mathcal{O}_{\overline{\mathcal{K}}}[1/p]_{\mathfrak{q}}$ is free over $\mathcal{O}_{\overline{\mathcal{K}}}[1/p]_{\mathfrak{q}}$ and hence it is a W_K-representation. By proposition 1.5.1, the trace of this W_K-representation is same as the trace of the W_K-representation

$$\left(\bigoplus_{i=1}^{M}\bigoplus_{\substack{1\le j\le m\\ t_j\equiv k \bmod 2\\ -t_j\le k\le t_j}}\left(r_i|\mathrm{Art}_K^{-1}|_K^{-k/2}\right)^{n_{ij}}_{/\mathcal{O}_{\overline{\mathcal{K}}}[1/p]_{\mathfrak{q}}}\right) \oplus \left(\bigoplus_{i=1}^{M'}\bigoplus_{\substack{1\le j\le m'\\ t'_j\equiv k \bmod 2\\ -t'_j\le k\le t'_j}}\left((\chi'_i \otimes \rho'_i)|\mathrm{Art}_K^{-1}|_K^{-k/2}\right)^{n'_{ij}}_{/\mathcal{O}_{\overline{\mathcal{K}}}[1/p]_{\mathfrak{q}}}\right).$$

So the trace of the W_K-representation $\mathrm{Gr}_k M_{\lambda,\bullet}$ is same as the trace of the W_K-representation

$$\left(\bigoplus_{i=1}^{M}\bigoplus_{\substack{1\le j\le m\\ t_j\equiv k \bmod 2\\ -t_j\le k\le t_j}} \lambda\left(r_i|\mathrm{Art}_K^{-1}|_K^{-k/2}\right)^{n_{ij}}_{/\overline{\mathbb{Q}}_p}\right)\oplus\left(\bigoplus_{i=1}^{M'}\bigoplus_{\substack{1\le j\le m'\\ t'_j\equiv k \bmod 2\\ -t'_j\le k\le t'_j}} \lambda\left((\chi'_i\otimes\rho'_i)|\mathrm{Art}_K^{-1}|_K^{-k/2}\right)^{n'_{ij}}_{/\overline{\mathbb{Q}}_p}\right).$$

Since $\mathrm{Gr}_k M_{\lambda,\bullet}$ is a semisimple representation of W_K, it is isomorphic to the above representation (by [**Ser98**, Chapter 1, §2] for instance). The proposition follows from lemma 1.1.45. ☐

1.6. (Proof of) Purity for big Galois representations

Before proving theorem 1.2.4, we discuss some properties of $\mathcal{T}^{I_K}$.

1.6.1. Compatibility. Recall that $\mathfrak{p}_\lambda$ denotes the kernel of $\lambda : \mathcal{R}\to\overline{\mathbb{Q}}_p$. Denote the image of this map by L_λ and note that it is a subfield of $\overline{\mathbb{Q}}_p$ as it is isomorphic to $\mathcal{R}_{\mathfrak{p}_\lambda}/\mathfrak{p}_\lambda\mathcal{R}_{\mathfrak{p}_\lambda}$.

Denote the G_K-representation $\mathcal{T}\otimes_{\mathcal{R}} L_\lambda = T_\lambda\otimes_{\mathcal{O}_\lambda} L_\lambda$ by V'_λ. Let (r_λ, N_λ) denote the Weil-Deligne parametrization of V'_λ. Denote by $(r_{\mathfrak{p}_\lambda}, N_{\mathfrak{p}_\lambda})$ the localization $\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}$ of the Weil-Deligne parametrization $\mathrm{WD}(\mathcal{T}[1/p]) = (r, N)$ at $\mathfrak{p}_\lambda$. Denote the image of

$$\theta = \frac{1}{\#\mathrm{Im}(r(I_K))}\sum_{g\in\mathrm{Im}(r(I_K))} g$$

in $M_n(\mathcal{R}[1/p]_{\mathfrak{p}_\lambda})$ by $\theta_{\mathfrak{p}_\lambda}$, which is an idempotent as θ is so. Since

$$V'_\lambda = \mathcal{T}[1/p]_{\mathfrak{p}_\lambda}\otimes_{\mathcal{R}[1/p]_{\mathfrak{p}_\lambda},\lambda} L_\lambda,$$

we have

$$r_\lambda = \lambda\circ r_{\mathfrak{p}_\lambda} \quad \text{and } N_\lambda = \lambda(N_{\mathfrak{p}_\lambda}). \tag{1.6.1}$$

Define the element

$$\theta_\lambda = \frac{1}{\#\mathrm{Im}((\lambda\circ r)(I_K))}\sum_{g\in\mathrm{Im}((\lambda\circ r)(I_K))} g \quad \in M_n(L_\lambda).$$

Then by lemma 1.1.9, θ_λ is an idempotent and $\mathrm{WD}(V'_\lambda)$ decomposes into an internal direct sum of Weil-Deligne subrepresentations as

$$\mathrm{WD}(V'_\lambda) = \mathrm{WD}(V'_\lambda)^{I_K}\oplus\mathrm{WD}(V'_\lambda)^{I_K,c}. \tag{1.6.2}$$

Proposition 1.6.1 (Compatibility)**.** *We have*

$$(r_\lambda, N_\lambda)|_{\mathrm{WD}(V'_\lambda)^{I_K}} = \lambda\left((r_{\mathfrak{p}_\lambda}, N_{\mathfrak{p}_\lambda})|_{\mathrm{WD}(\mathcal{T}[1/p])^{I_K}_{\mathfrak{p}_\lambda}}\right)$$
$$(r_\lambda, N_\lambda)|_{\mathrm{WD}(V'_\lambda)^{I_K,c}} = \lambda\left((r_{\mathfrak{p}_\lambda}, N_{\mathfrak{p}_\lambda})|_{\mathrm{WD}(\mathcal{T}[1/p])^{I_K,c}_{\mathfrak{p}_\lambda}}\right)$$

as Weil-Deligne representations.

Proof. Recall that by proposition 1.1.29, $\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}$ decomposes into an internal direct sum of Weil-Deligne representations as

$$\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda} = \mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}^{I_K} \oplus \mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}^{I_K,c} \tag{1.6.3}$$

with

$$\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}^{I_K} = \theta_{\mathfrak{p}_\lambda}\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}, \quad \mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}^{I_K,c} = (1-\theta_{\mathfrak{p}_\lambda})\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}.$$

By equation (1.6.1), we have

$$\lambda(\theta_{\mathfrak{p}_\lambda}) = \theta_\lambda.$$

Since $\theta_{\mathfrak{p}_\lambda}$, $1-\theta_{\mathfrak{p}_\lambda}$ are idempotents, it follows that

$$\begin{aligned} \theta_\lambda \cdot \lambda(\theta_{\mathfrak{p}_\lambda}) &= \theta_\lambda \\ (1-\theta_\lambda)\cdot\lambda(1-\theta_{\mathfrak{p}_\lambda}) &= 0 \\ (1-\theta_\lambda)\cdot\theta(\theta_{\mathfrak{p}_\lambda}) &= 0 \\ (1-\theta_\lambda)\cdot\lambda(1-\theta_{\mathfrak{p}_\lambda}) &= 1-\theta_\lambda. \end{aligned}$$

So

$$\mathrm{WD}(V'_\lambda)^{I_K} = \lambda(\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}^{I_K}), \quad \mathrm{WD}(V'_\lambda)^{I_K,c} = \lambda(\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}^{I_K,c}). \tag{1.6.4}$$

Now the first part of the proposition follows from equation (1.6.1). □

Corollary 1.6.2. *We have*

$$\dim_{\overline{\mathcal{K}}} \mathcal{V}^{I_K} \leq \dim_{\overline{\mathbb{Q}}_p} V_\lambda^{I_K}.$$

Proof. Put

$$N_1 = N|_{\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}^{I_K}}, \quad N_{\lambda 1} = N_\lambda|_{\mathrm{WD}(V'_\lambda)^{I_K}}. \tag{1.6.5}$$

Then proposition 1.6.1 gives

$$N_{\lambda 1} = \lambda(N_1).$$

By proposition 1.1.34 and [**Fon04**, proof of Proposition 0.0], we obtain the desired inequality. □

Remark 1.6.3. When $\mathcal{V}$ is semistable, this corollary can be deduced from equation (1.3.3) using proposition 1.1.34.

Remark 1.6.4. This corollary is also obtained in [**BC09**, §7.8.1].

1.6.2. Generating inertia invariants. Recall that we have decompositions

$$\begin{aligned} \mathrm{WD}(V'_\lambda) &= \mathrm{WD}(V'_\lambda)^{I_K} \oplus \mathrm{WD}(V'_\lambda)^{I_K,c}, \\ \mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda} &= \mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}^{I_K} \oplus \mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}^{I_K,c}. \end{aligned}$$

as in equation (1.6.2) and (1.6.3). From equation (1.6.4), we have

$$\mathrm{WD}(V'_\lambda)^{I_K} = \lambda(\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}^{I_K}), \quad \mathrm{WD}(V'_\lambda)^{I_K,c} = \lambda(\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}^{I_K,c}). \tag{1.6.6}$$

By definition of semistable and totally non-semistable parts (as given in corollary 1.1.30 and 1.1.33), we get

$$\lambda\left((\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda}\right) = (V'_\lambda)_{ss}, \quad \lambda\left((\mathcal{T}[1/p]_{tnss})_{\mathfrak{p}_\lambda}\right) = (V'_\lambda)_{tnss}$$

From equation (1.6.5), we have

$$N_1 = N|_{\mathrm{WD}(\mathcal{T}[1/p])_{\mathfrak{p}_\lambda}^{I_K}}, \quad N_{\lambda\,1} = N_\lambda|_{\mathrm{WD}(V'_\lambda)^{I_K}}.$$

So this can be rewritten as

$$N_1 = N|_{(\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda}}, \quad N_{\lambda\,1} = N_\lambda|_{(V'_\lambda)_{ss}}$$

and we have

$$N_{\lambda 1} = \lambda(N_2)$$

by proposition 1.6.1. Now put

$$N_2 = N|_{(\mathcal{T}[1/p]_{tnss})_{\mathfrak{p}_\lambda}}, \quad N_{\lambda\,2} = N_\lambda|_{(V'_\lambda)_{tnss}}$$

and note that proposition 1.6.1 gives

$$N_{\lambda 2} = \lambda(N_2).$$

Notice that

$$N = N_1 \oplus N_2, \quad N_\lambda = N_{\lambda 1} \oplus N_{\lambda 2}.$$

In the following we will also use the notation N_1 (resp. N_2) to denote the restriction of N to $\mathcal{T}[1/p]_{ss}$ (resp. $\mathcal{T}[1/p]_{tnss}$).

Recall from corollary 1.1.30 that we have a decomposition

$$\mathcal{T}[1/p] = \mathcal{T}[1/p]_{ss} \oplus \mathcal{T}[1/p]_{tnss}.$$

We will denote the projection maps

$$\mathcal{T}[1/p] \to \mathcal{T}[1/p]_{ss}, \quad \mathcal{T}[1/p] \to \mathcal{T}[1/p]_{tnss}$$

by π_{ss} and π_{tnss} respectively. Similarly the projection maps

$$V_\lambda \to (V_\lambda)_{ss}, V_\lambda \to (V_\lambda)_{tnss}$$

are denoted by $\pi_{\lambda,ss}, \pi_{\lambda,tnss}$ respectively. From lemma 1.1.31 and proposition 1.6.1, we have isomorphisms

$$\mathcal{T}[1/p]_{ss} \otimes_{\mathcal{R}[1/p],\lambda} \overline{\mathbb{Q}}_p = (\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}[1/p]_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq (V_\lambda)_{ss},$$

$$\mathcal{T}[1/p]_{tnss} \otimes_{\mathcal{R}[1/p],\lambda} \overline{\mathbb{Q}}_p = (\mathcal{T}[1/p]_{tnss})_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}[1/p]_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq (V_\lambda)_{tnss}$$

induced by the map λ and consequently

$$\pi_{ss} \otimes_{\mathcal{R}[1/p],\lambda} \overline{\mathbb{Q}}_p = \pi_{\lambda,ss}, \quad \pi_{tnss} \otimes_{\mathcal{R}[1/p],\lambda} \overline{\mathbb{Q}}_p = \pi_{\lambda,tnss}. \tag{1.6.7}$$

Lemma 1.6.5. *We have*

$$\mathcal{T}[1/p]^{I_K} = \ker\left(N_1 : \mathcal{T}[1/p]_{ss} \to \mathcal{T}[1/p]_{ss}\right) \tag{1.6.8}$$

and consequently

$$0 \to \mathcal{T}[1/p]^{I_K} \to \mathcal{T}[1/p]_{ss} \xrightarrow{N_1} N_1\mathcal{T}[1/p]_{ss} \to 0$$

is exact. Moreover when the condition (mono-non-deg-1) *holds, the localizations of all the terms of this exact sequence at* $\mathfrak{p}_\lambda$ *are free over* $\mathcal{R}[1/p]_{\mathfrak{p}_\lambda}$.

Proof. A little modification of the proof of proposition 1.1.34 gives the proof of equation (1.6.8).

First note that equation (1.1.10) gives

$$(\mathcal{T}[1/p] \otimes_{\mathcal{R}[1/p]} \mathcal{K})^{I_K} = \ker\left(N_1 : \mathrm{WD}(\mathcal{T}[1/p] \otimes_{\mathcal{R}[1/p]} \mathcal{K})^{I_K} \to \mathrm{WD}(\mathcal{T}[1/p] \otimes_{\mathcal{R}[1/p]} \mathcal{K})^{I_K}\right).$$

Since

$$\mathrm{WD}(\mathcal{T}[1/p] \otimes_{\mathcal{R}[1/p]} \mathcal{K})^{I_K} = \mathrm{WD}(\mathcal{T}[1/p])^{I_K} \otimes_{\mathcal{R}[1/p]} \mathcal{K}$$

by lemma 1.1.32 and

$$\mathcal{T}[1/p]_{ss} = \mathrm{WD}(\mathcal{T}[1/p])^{I_K}$$

by definition (as given in corollary 1.1.30), we get

$$(\mathcal{T}[1/p] \otimes_{\mathcal{R}[1/p]} \mathcal{K})^{I_K} = \ker\left(N_1 : \mathcal{T}[1/p]_{ss} \otimes_{\mathcal{R}[1/p]} \mathcal{K} \to \mathcal{T}[1/p]_{ss} \otimes_{\mathcal{R}[1/p]} \mathcal{K}\right).$$

Note that $\mathcal{T}[1/p]$ can be considered inside (*i.e.*, can be thought of as an $\mathcal{R}[1/p]$-submodule of) $\mathcal{T}[1/p] \otimes_{\mathcal{R}[1/p]} \mathcal{K}$ as it is torsion free (being free over a domain). So

$$\begin{aligned}
\mathcal{T}[1/p]^{I_K} &= \mathcal{T}[1/p] \cap (\mathcal{T}[1/p] \otimes_{\mathcal{R}[1/p]} \mathcal{K})^{I_K} \\
&= (\mathcal{T}[1/p]_{ss} \oplus \mathcal{T}[1/p]_{tnss}) \cap \ker\left(N_1 : \mathcal{T}[1/p]_{ss} \otimes_{\mathcal{R}[1/p]} \mathcal{K} \to \mathcal{T}[1/p]_{ss} \otimes_{\mathcal{R}[1/p]} \mathcal{K}\right) \\
&= \mathcal{T}[1/p]_{ss} \cap \ker\left(N_1 : \mathcal{T}[1/p]_{ss} \otimes_{\mathcal{R}[1/p]} \mathcal{K} \to \mathcal{T}[1/p]_{ss} \otimes_{\mathcal{R}[1/p]} \mathcal{K}\right) \\
&= \ker(N_1 : \mathcal{T}[1/p]_{ss} \to \mathcal{T}[1/p]_{ss}).
\end{aligned}$$

This proves equation (1.6.8) which in turn shows that the sequence stated in the lemma is exact.

Now it remains to prove the last part of the lemma. Note that by proposition 1.1.29, $(\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda}$ is free over $\mathcal{R}[1/p]_{\mathfrak{p}_\lambda}$. So by Nakayama's lemma, it suffices to prove that $(N_1\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda}$ is free over $\mathcal{R}[1/p]_{\mathfrak{p}_\lambda}$. Again by Nakayama's lemma and the exact sequence

$$0 \to (N_1\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda} \to (\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda} \to (\mathcal{T}[1/p]_{ss}/N_1\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda} \to 0,$$

it is enough to prove that $(\mathcal{T}[1/p]_{ss}/N_1\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda}$ is free over $\mathcal{R}[1/p]_{\mathfrak{p}_\lambda}$. This would follow from Nakayama's lemma, once we prove that

$$\mathrm{rk}_{\mathcal{R}[1/p]_{\mathfrak{p}_\lambda}}(\mathcal{T}[1/p]_{ss}/N_1\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda} = \dim_{\overline{\mathcal{K}}} \mathcal{V}_{ss} - \dim_{\overline{\mathcal{K}}} N_1\mathcal{V}_{ss}$$

is same as

$$\dim_{L_\lambda}(\mathcal{T}[1/p]_{ss}/N_1\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}[1/p]_{\mathfrak{p}_\lambda},\lambda} L_\lambda = \dim_{L_\lambda}(V'_\lambda)_{ss} - \dim_{L_\lambda} N_{\lambda 1}(V'_\lambda)_{ss}.$$

This follows as

$$N = N_1 \oplus N_2, \quad N_\lambda = N_{\lambda 1} \oplus N_{\lambda 2},$$
$$N_\lambda = \lambda(N), \quad N_{\lambda 1} = \lambda(N_1), \quad N_{\lambda 2} = \lambda(N_2)$$

and the condition (mono-non-deg-1) holds. □

We record an immediate corollary of the above proof.

Corollary 1.6.6. *Suppose that the condition* (mono-non-deg-1) *holds. Then the map* λ *induces an isomorphism*

$$N_1\mathcal{T}[1/p]_{ss} \otimes_{\mathcal{R}[1/p],\lambda} \overline{\mathbb{Q}}_p \simeq N_{\lambda 1}(V_\lambda)_{ss}.$$

Proof. In the above proof we have seen that $(\mathcal{T}[1/p]_{ss}/N_1\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda}$ is free over $\mathcal{R}[1/p]_{\mathfrak{p}_\lambda}$. So the exact sequence

$$0 \to (N_1\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda} \to (\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda} \to (\mathcal{T}[1/p]_{ss}/N_1\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda} \to 0$$

gives

$$(N_1\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda}\otimes_{\mathcal{R}[1/p]_{\mathfrak{p}_\lambda},\lambda}\overline{\mathbb{Q}}_p \simeq \mathrm{Im}\Big((N_1\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda}\otimes_{\mathcal{R}[1/p]_{\mathfrak{p}_\lambda},\lambda}\overline{\mathbb{Q}}_p \to (\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda}\otimes_{\mathcal{R}[1/p]_{\mathfrak{p}_\lambda},\lambda}\overline{\mathbb{Q}}_p\Big).$$

Since

$$(\mathcal{T}[1/p]_{ss})_{\mathfrak{p}_\lambda} \otimes_{\mathcal{R}[1/p]_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq (V_\lambda)_{ss}$$

from lemma 1.1.31 and proposition 1.6.1, we get the corollary. □

Lemma 1.6.7. *We have an exact sequence*

$$0 \to \mathcal{T}[1/p]^{I_K} \to \mathcal{T}[1/p] \xrightarrow{(N_1\circ\pi_{ss})\oplus\pi_{tnss}} N_1\mathcal{T}[1/p]_{ss} \oplus \mathcal{T}[1/p]_{tnss} \to 0$$

of representations of W_K *over* $\mathcal{R}[1/p]$*. Moreover when the condition* (mono-non-deg-1) *holds, the localizations of all the terms of this sequence at* $\mathfrak{p}_\lambda$ *are free over* $\mathcal{R}[1/p]_{\mathfrak{p}_\lambda}$*.*

Proof. Exactness of the above sequence follows since

$$\mathcal{T}[1/p] = \mathcal{T}[1/p]_{ss} \oplus \mathcal{T}[1/p]_{tnss}$$

and

$$0 \to \mathcal{T}[1/p]^{I_K} \to \mathcal{T}[1/p]_{ss} \xrightarrow{N_1} N_1\mathcal{T}[1/p]_{ss} \to 0$$

is exact by the above lemma.

By proposition 1.1.29, the localization of $\mathcal{T}[1/p]_{tnss}$ at $\mathfrak{p}_\lambda$ is free over $\mathcal{R}[1/p]_{\mathfrak{p}_\lambda}$. So we are done by the above lemma. □

Proposition 1.6.8. *The* $\mathcal{R}_{\mathfrak{p}_\lambda}$*-modules* $\mathcal{T}_{\mathfrak{p}_\lambda}^{I_K}$*,* $\mathcal{T}_{\mathfrak{p}_\lambda}/\mathcal{T}_{\mathfrak{p}_\lambda}^{I_K}$ *are free and the map* λ *induces an isomorphism*

$$\mathcal{T}_{\mathfrak{p}_\lambda}^{I_K} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq V_\lambda^{I_K}.$$

Proof. From lemma 1.6.7, it follows that

$$(\mathcal{T}[1/p]^{I_K})_{\mathfrak{p}_\lambda}, \quad \mathcal{T}[1/p]_{\mathfrak{p}_\lambda}/(\mathcal{T}[1/p]^{I_K})_{\mathfrak{p}_\lambda}$$

are free over $\mathcal{R}[1/p]_{\mathfrak{p}_\lambda}$. Note that $p \notin \mathfrak{p}_\lambda$ as $\mathfrak{p}_\lambda = \ker(\lambda : \mathcal{R} \to \overline{\mathbb{Q}}_p)$. So $\mathcal{R}[1/p]_{\mathfrak{p}_\lambda} = \mathcal{R}_{\mathfrak{p}_\lambda}$ and the modules

$$\mathcal{T}_{\mathfrak{p}_\lambda}^{I_K} = (\mathcal{T}[1/p]^{I_K})_{\mathfrak{p}_\lambda}, \quad \mathcal{T}_{\mathfrak{p}_\lambda}/\mathcal{T}_{\mathfrak{p}_\lambda}^{I_K} = \mathcal{T}[1/p]_{\mathfrak{p}_\lambda}/(\mathcal{T}[1/p]^{I_K})_{\mathfrak{p}_\lambda}$$

are free over $\mathcal{R}_{\mathfrak{p}_\lambda}$.

Now it remains to prove

$$\mathcal{T}^{I_K} \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \simeq V_\lambda^{I_K}.$$

Note that applying $-\otimes_{\mathcal{R}[1/p],\lambda}\overline{\mathbb{Q}}_p$ to the exact sequence in lemma 1.6.7 yields the short exact sequence

$$0 \to \mathcal{T}[1/p]^{I_K} \otimes_{\mathcal{R}[1/p],\lambda} \overline{\mathbb{Q}}_p \to V_\lambda \to N_1\mathcal{T}[1/p]_{ss} \otimes_{\mathcal{R}[1/p],\lambda} \overline{\mathbb{Q}}_p \bigoplus \mathcal{T}[1/p]_{tnss} \otimes_{\mathcal{R}[1/p],\lambda} \overline{\mathbb{Q}}_p \to 0$$

where the third arrow is

$$((N_1 \circ \pi_{ss}) \oplus \pi_{tnss}) \otimes_{\mathcal{R}[1/p],\lambda} \overline{\mathbb{Q}}_p.$$

In other words

$$0 \to \mathcal{T}^{I_K} \otimes_{\mathcal{R},\lambda} \overline{\mathbb{Q}}_p \to V_\lambda \xrightarrow{(N_{\lambda 1}\circ\pi_\lambda)\oplus\pi_{\lambda,tnss}} N_{\lambda 1}(V_\lambda)_{ss} \bigoplus (V_\lambda)_{tnss} \to 0$$

is exact by corollary 1.6.6, lemma 1.1.31, proposition 1.6.1 and equation (1.6.7). So

$$\begin{aligned}\mathcal{T}_{\mathfrak{p}_\lambda}^{I_K} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p &\simeq \ker(N_{\lambda 1} : (V_\lambda)_{ss} \xrightarrow{N_{\lambda 1}} (V_\lambda)_{ss}) \\ &= \ker(N_{\lambda 1} : \mathrm{WD}(V_\lambda)^{I_K} \xrightarrow{N_{\lambda 1}} \mathrm{WD}(V_\lambda)^{I_K}).\end{aligned}$$

By equation (1.1.10), we get

$$\mathcal{T}_{\mathfrak{p}_\lambda}^{I_K} \otimes_{\mathcal{R}_{\mathfrak{p}_\lambda},\lambda} \overline{\mathbb{Q}}_p \simeq V_\lambda^{I_K}.$$

□

1.6.3. Proof of theorem 1.2.4 and proposition 1.2.5.

Proof of theorem 1.2.4. Suppose that V_λ is pure of weight w. Part (1), (2), (3) of this theorem follow from theorem 1.2.1, 1.2.2, 1.2.3 respectively.

Since V_λ is pure, part (4) of theorem 1.2.4 holds.

The first part of theorem 1.2.4(5) follows from proposition 1.6.8. The rest follows from [**Sta14**, Tag 064K], [**Sta14**, Tag 06Y6].

Note that $\mathrm{Eul}(\mathcal{V})^{-1}$ has coefficients in $\mathcal{R}_{\mathfrak{p}_\lambda}$ by part (5). Since $\mathcal{V}$ is defined over $\mathcal{R}$, the polynomial $\mathrm{Eul}(\mathcal{V})^{-1}$ has coefficients in $\mathcal{O}_{\mathcal{K}} \cap \mathcal{R}_{\mathfrak{p}_\lambda}$. Its λ-specialization is $\mathrm{Eul}(V_\lambda)^{-1}$ by proposition 1.1.34 and theorem 1.2.4(3). So we have part (6) of theorem 1.2.4. □

Proof of proposition 1.2.5. Since $\mathcal{V}$ is defined over $\mathcal{R}$, the polynomial $\mathrm{Eul}(\mathcal{V})^{-1}$ has coefficients in $\mathcal{O}_{\mathcal{K}}$. The inequality of this proposition is from corollary 1.6.2. □

CHAPTER 2

Determinants and Selmer complexes

In this chapter we recall the notion of determinant functor and Selmer complexes referring to [**KM76, Nek06**] for further details. These are used in the next two chapters to construct algebraic p-adic L-functions for Hida families.

2.1. Determinants

2.1.1. Triangulated categories. In this subsection, we review the notion of derived category from [**Sta14**]. Notice that the sign convention of [**Sta14**] agrees with the sign convention of [**BBM82**, §0.3.1, p. 2] (by [**Sta14**, Tag 014L]), which is followed by [**Nek06**, §1.1.3]. So the following is consistent with [**Nek06**].

2.1.1.1. *Cochain complexes.* We first recall some notions and describe some of their properties. Fix an abelian category $\mathcal{A}$ and denote the category of cochain complexes in $\mathcal{A}$ by $C(\mathcal{A})$. There are shift functors $[n]$ on $C(\mathcal{A})$ defined as follows:

(1) for a cochain complex X,

$$X[n] = \begin{cases} X[n]^i = X^{n+i} \\ d^i_{X[n]} = (-1)^n d^{n+i}_X \end{cases},$$

(2) for a morphism of cochain complexes $f : X \to Y$,

$$f[n]^i = f^{n+i}.$$

The cone of a morphism $f : X \to Y$ of cochain complexes in $\mathcal{A}$ is the object of $C(\mathcal{A})$ defined by

$$\mathrm{Cone}(f) = \begin{cases} Y \oplus X[1] \\ d^i_{\mathrm{Cone}(f)} = \begin{pmatrix} d^i_Y & f^{i+1} \\ 0 & d^i_{X[1]} \end{pmatrix} : Y^i \oplus X[1]^i \to Y^{i+1} \oplus X[1]^{i+1}. \end{cases}$$

The cone fits into an exact sequence of complexes

$$0 \to Y \xrightarrow{j} \mathrm{Cone}(f) \xrightarrow{p} X[1] \to 0, \tag{2.1.1}$$

where j and p are the canonical inclusion and projection respectively. The corresponding boundary map

$$\partial : H^i(X[1]) = H^{i+1}(X[1]) \to H^{i+1}(Y)$$

is induced by f^{i+1}. Note that the above exact sequence gives the triangle $(X, Y, \mathrm{Cone}(f), f, j, p)$ in $C(\mathcal{A})$ (*cf.*[**Sta14**, Tag 014E]).

2.1.1.2. *Homotopy category.* The homotopy category $K(\mathcal{A})$ of the abelian category $\mathcal{A}$ has the same objects as $C(\mathcal{A})$ and its morphisms are homotopy classes of maps of complexes (*cf.*[**Sta14**, Tag 013H]). Note that the shift functors $[n]$ on the category of cochain complexes give rise to functors $[n] : K(\mathcal{A}) \to K(\mathcal{A})$ such that $[n] \circ [m] = [n+m]$ and $[0] = \mathrm{id}$ (equality as functors). The category $K(\mathcal{A})$ is a triangulated category with these translation functors and distinguished triangles as the triangles in it isomorphic to the image of the triangle

$$X \xrightarrow{f} Y \xrightarrow{j} \mathrm{Cone}(f) \xrightarrow{-p} X[1], \quad f \in \mathrm{Hom}_{C(\mathcal{A})}(X, Y)$$

in $C(\mathcal{A})$ under the functor $C(\mathcal{A}) \to K(\mathcal{A})$ (*cf.*[**Sta14**, Tag 014S, Tag 014I, Tag 014L]).

2.1.1.3. *Derived category.* Recall that $\mathcal{A}$ denotes an abelian category. The derived category $D(\mathcal{A})$ of $\mathcal{A}$ is the triangulated category defined as the quotient of the triangulated category $K(\mathcal{A})$ by its full triangulated subcategory of acyclic complexes, which is the localization of $K(\mathcal{A})$ at the quasi-isomorphisms (*cf.*[**Sta14**, Tag 05RU, Tag 05RI, Tag 05R6, Tag 05R6]). The additive functors $\{[n]\}_{n\in\mathbb{Z}}$ on $D(\mathcal{A})$ are induced by those of $K(\mathcal{A})$ and the distinguished triangles of $D(\mathcal{A})$ are the triangles in $D(\mathcal{A})$ whice are isomorphic to the image of a distinguished triangle under the localization map (*cf.*[**Sta14**, proof of Tag 05R6]).

2.1.1.4. *Complexes of modules.* For a ring R, let ${}_R\mathrm{Mod}$ denote the category of R-modules, which is an abelian category. Its derived category $D({}_R\mathrm{Mod})$ is a triangulated category. Its full subcategory of cohomologically bounded complexes is denoted by $D^b({}_R\mathrm{Mod})$ and the full subcategory of $D^b({}_R\mathrm{Mod})$ of complexes having cohomology of finite type over R is denoted by $D^b_{ft}({}_R\mathrm{Mod})$. Notice that $D^b({}_R\mathrm{Mod})$, $D^b_{ft}({}_R\mathrm{Mod}))$ are preserved under the translations $[1], [-1]$ and any arrow $f : X \to Y$ in $D^b({}_R\mathrm{Mod})$ (resp. $D^b_{ft}({}_R\mathrm{Mod})$) can be completed to a distinguished triangle (X, Y, Z, f, g, h) in $D({}_R\mathrm{Mod})$ with Z in the objects of $D^b({}_R\mathrm{Mod})$ (resp. $D^b_{ft}({}_R\mathrm{Mod})$). So they are triangulated subcategories of $D({}_R\mathrm{Mod})$ with the restrictions of $\{[n]\}_{n\in\mathbb{Z}}$ as the translations and distinguished triangles as the triangles in it which are distinguished triangles in $D({}_R\mathrm{Mod})$ (*cf.*[**Sta14**, Tag 05QX, Tag 066R, footnote in Tag 05QM]).

2.1.1.5. *Exact sequences.* Recall that $\mathcal{A}$ denotes an abelian category. The functor $C(\mathcal{A})$ to $D(\mathcal{A})$ becomes a δ-functor with the following rule (*cf.*[**Sta14**, Tag 0152]). For every exact sequence of complexes

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

in $C(\mathcal{A})$, define the arrow

$$\delta = \delta_{X \to Y \to Z} : Z \to X[1]$$

in $D(\mathcal{A})$ by

$$Z \xleftarrow{q} \mathrm{Cone}(f) \xrightarrow{-p} X[1]$$

where $q : \mathrm{Cone}(f) \to Z$ denotes the arrow in $C(\mathcal{A})$ which is zero on $X[1]$ and g on Y.

Remark 2.1.1. Note that the map δ associated with the exact sequence of equation (2.1.1) satisfies

$$\delta_{Y \xrightarrow{j} \mathrm{Cone}(f) \xrightarrow{p} X[1]} = f[1] \quad \text{in } \mathrm{Hom}_{D(\mathcal{A})}(Z, X[1])$$

(*cf.*[**Sta14**, Tag 014I]).

2.1.2. Perfect complexes. A complex $M^\bullet$ of R-modules is said to be *perfect* if a bounded complex $P^\bullet$ of projective R-modules of finite type is quasi-isomorphic to it (see [**SGA71**, p. 42–]). An R-module M is said to be *perfect* if it becomes perfect when considered as complex concentrated in degree zero. The derived tensor product of a perfect complex over R with an R-algebra R' is perfect over R' (by [**Sta14**, Tag 066W] for example).

Denote by Parf_R the full subcategory of the derived category of R-modules $D({}_R\mathrm{Mod})$ consisting of perfect complexes. The category Parf_R is equivalent to the category $\mathrm{Parf}_{\mathrm{Spec}(R)}$ (as in [**KM76**, p. 39] for example) by [**KM76**, Proposition 4].

Note that Parf_R is preserved under the translations $[1], [-1]$, it is a full subcategory of the triangulated category $D({}_R\mathrm{Mod})$ and any arrow $f : X \to Y$ in Parf_R can be completed to a distinguished triangle (X, Y, Z, f, g, h) in $D({}_R\mathrm{Mod})$ with Z an object of Parf_R (*cf.*[**Sta14**, Tag 066R]). So it is a triangulated subcategory of $D({}_R\mathrm{Mod})$ with the restrictions of $\{[n]\}_{n\in\mathbb{Z}}$ as the translations and its set of distinguished triangles consists of the distinguished triangles in $D({}_R\mathrm{Mod})$ which are also a triangle in Parf_R (*cf.*[**Sta14**, Tag 05QX, footnote in Tag 05QM]; or alternatively [**Sta14**, Tag 09QH, Tag 07LT]). Similarly, it is also a full triangulated subcategory of the triangulated category $D^b({}_R\mathrm{Mod})$.

A theorem of Auslander-Buchsbaum and Serre (see [**BH93**, Theorem 2.2.7] or [**Sta14**, Tag 066Z]) says that when R is a regular noetherian ring, Parf_R is equal to $D^b_{ft}({}_R\mathrm{Mod})$.

Denote by Parf-is_R the subcategory of Parf_R consisting of all its objects and morphisms as isomorphisms. Evidently, the set of morphisms between two objects in this category is empty if they are not isomorphic in Parf_R.

2.1.3. Graded invertible modules. We recall the notion of graded invertible modules from [**KM76**].

The category of graded invertible R-modules is denoted by $\mathcal{P}_R$. Its objects are pairs (L, α) where L is an invertible R-module and α is a continuous function

$$\alpha : \mathrm{Spec}(R) \to \mathbb{Z},$$

and a morphism $h : (L, \alpha) \to (M, \beta)$ is a homomorphism of R-modules $h : L \to M$ such that for each $\mathfrak{p} \in \mathrm{Spec}(R)$ we have

$$\alpha(\mathfrak{p}) \neq \beta(\mathfrak{p}) \Rightarrow h_{\mathfrak{p}} = 0.$$

The composition of two morphisms is obtained by taking the composition of the maps between the invertible modules. Note that the composition law indeed gives a map in P_R. Thus a morphism $h : (L, \alpha) \to (M, \beta)$ of graded invertible modules is an isomorphism if and only if $h : L \to M$ is an isomorphism and $\alpha = \beta$.

The subcategory of $\mathcal{P}_R$ whose morphisms are isomorphisms is denoted by $\mathcal{P}\mathrm{is}_R$. The tensor product of two objects in P_R is given by

$$(L, \alpha) \otimes (M, \beta) := (L \otimes M, \alpha + \beta).$$

For each pair of objects $(L,\alpha),(M,\beta)$ in P_R we have an isomorphism

$$\psi_{(L,\alpha),(M,\beta)}:(L,\alpha)\otimes(M,\beta)\xrightarrow{\sim}(M,\beta)\otimes(L,\alpha)$$

defined by

$$\psi(l\otimes m)=(-1)^{\alpha(x)\beta(x)}m\otimes l \quad \text{for } l\in L_x, m\in M_x.$$

The object $(R,0)$ of P_R will be denoted by 1. A right inverse of an object (L,α) in P_R will be an object (L',α') together with an isomorphism

$$\delta:(L,\alpha)\otimes(L',\alpha')\xrightarrow{\sim}1.$$

A right inverse will be considered as a left inverse via

$$(L',\alpha')\otimes(L,\alpha)\xrightarrow[\sim]{\psi_{(L',\alpha'),(L,\alpha)}}(L,\alpha)\otimes(L',\alpha')\xrightarrow[\sim]{\delta}1. \tag{2.1.2}$$

2.1.4. Determinant functor.

2.1.4.1. *On* $\mathcal{C}\mathrm{is}_R$. For a commutative ring R, let $\mathcal{C}_R$ denote the category of projective R-modules of finite type. Its full subcategory whose maps are isomorphisms will be denoted by $\mathcal{C}\mathrm{is}_R$.

For a projective R-module M of finite type, we put

$$\det{}^*(M)=(\wedge^{\max}M,\mathrm{rk}F)$$

where

$$(\wedge^{\max}M)_{\mathfrak{p}}=\wedge^{\mathrm{rk}M_{\mathfrak{p}}}M_{\mathfrak{p}}$$

for any prime ideal $\mathfrak{p}$ of R. This defines a functor

$$\det{}^*:\mathcal{C}\mathrm{is}_R\to\mathcal{P}\mathrm{is}_R.$$

Moreover for every short exact sequence

$$0\to F_1\xrightarrow{\alpha}F\xrightarrow{\beta}F_2\to 0$$

in $\mathcal{C}_R$, we have an isomorphism

$$i^*(\alpha,\beta):\det{}^*F_1\otimes\det{}^*F_2\xrightarrow{\sim}\det{}^*F$$

such that locally

$$i^*(\alpha,\beta)\left((e_1\wedge\cdots\wedge e_t)\otimes(\beta f_1\wedge\cdots\wedge\beta f_s)\right)=\alpha e_1\wedge\cdots\wedge\alpha e_t\wedge f_1\wedge\cdots\wedge f_s$$

for e_i (resp. f_j) in the localization of F' (resp. F) at a multiplicative subset of R.

2.1.4.2. *On* $\mathcal{C}^\bullet$is. For a commutative ring R, let $\mathcal{C}^\bullet_R$ denote the category of bounded complexes of objects in $\mathcal{C}_R$, morphisms being all maps of complexes. The full subcategory of $\mathcal{C}^\bullet_R$ whose maps are quasi-isomorphisms will be denoted by $\mathcal{C}^\bullet\mathrm{is}_R$.

A determinant functor from $\mathcal{C}^\bullet$is to $\mathcal{P}$is, denoted (f, i), is a collection of data as defined in [**KM76**, Definition 1]. We describe some of its properties.

For each commutative ring R, this data provides a functor f_R from $\mathcal{C}^\bullet\mathrm{is}_R$ to $\mathcal{P}\mathrm{is}_R$. For each short exact sequence

$$0 \to F_1^\bullet \xrightarrow{\alpha} F^\bullet \xrightarrow{\beta} F_2^\bullet \to 0$$

in $\mathcal{C}^\bullet_R$, this data provides an isomorphism

$$i_R(\alpha, \beta) : f(F_1^\bullet) \otimes f(F_2^\bullet) \xrightarrow{\sim} f(F^\bullet).$$

When $\mathcal{C}\mathrm{is}_R$ is considered as a full subcategory of $\mathcal{C}^\bullet\mathrm{is}_R$ by viewing its objects of $\mathcal{C}\mathrm{is}_R$ as complexes concentrated in degree zero, we have

$$f(F) = \det{}^* F$$

for any object F in $\mathcal{C}\mathrm{is}_R$ and

$$i_R(\alpha, \beta) = i^*(\alpha, \beta)$$

for any short exact sequence

$$0 \to F_1 \xrightarrow{\alpha} F \xrightarrow{\beta} F_2 \to 0$$

in $\mathcal{C}\mathrm{is}_R$.

By [**KM76**, Theorem 1], a determinant functor (as in [**KM76**, Definition 1]) exists and is unique up to canonical isomorphism. We will denote it by $(\det, i)$.

2.1.4.3. *On* Parf-is. The extended determinant functor is a collection of data as defined in [**KM76**, Definition 4] and by [**KM76**, Theorem 2] it exists and is unique up to canonical isomorphism. We describe some of its properties. We have

$$\det_R(0) = 1.$$

For each commutative ring R, this data gives a functor $\det_R$ from Parf-is$_R$ to the category $\mathcal{P}\mathrm{is}_R$. When an object $M^\bullet$ of Parf-is$_R$ is represented by a bounded complex $P^\bullet$ of projective R-modules of finite type, *i.e.*, $P^\bullet$ is quasi-isomorphic to $M^\bullet$, we have a canonical isomorphism

$$\det_R(M^\bullet) \cong \otimes_{n\in\mathbb{Z}} (\det_R(P^n))^{(-1)^n}$$

([**KM76**, Rem a), p. 43]). When the cohomology modules $H^n(M^\bullet)$ are perfect (considered as a complex concentrated in degree zero), there is a canonical isomorphism

$$\det_R(M^\bullet) \cong \otimes_{n\in\mathbb{Z}} (\det_R(H^n(M^\bullet)))^{(-1)^n} \tag{2.1.3}$$

([**KM76**, Rem b), p. 43]). If the ring R is reduced, then for a distinguished triangle

$$M_1^\bullet \xrightarrow{u} M_2^\bullet \xrightarrow{v} M_3^\bullet \xrightarrow{w} M_1^\bullet[1]$$

in Parf$_R$, we have an isomorphism

$$i_R(u, v, w) : \det_R M_1^\bullet \otimes \det_R M_3^\bullet \xrightarrow{\sim} \det_R M_2^\bullet \tag{2.1.4}$$

which is functorial with respect to isomorphism of such triangles ([**KM76**, Proposition 7]). On bounded complexes of projective R-modules of finite type, the extended determinant functor coincides with the determinant functor given in §2.1.4.2. Moreover the extended determinant functor satisfies the following base change property.

Proposition 2.1.2. *Let $\phi : R \to R'$ be a ring homomorphism. Then for an object $M^\bullet \in$ $\mathrm{Parf}\text{-}\mathrm{is}_R$, we have a canonical isomorphism*

$$(\det_R M^\bullet) \otimes_{R,\phi} R' \cong \det_{R'}(M^\bullet \overset{L}{\otimes}_{R,\phi} R').$$

Proof. See [**KM76**, Definition 4 II) iii), p. 42]. □

2.1.4.4. *Choosing an inverse.* Suppose that R is reduced. For an object X of Parf_R, we choose $\det_R(X[1])$ as a right inverse of $\det_R(X)$ via the map

$$i_R(0, 0, -\mathrm{id}_X[1]) : \det_R X \otimes \det_R X[1] \xrightarrow{\sim} \det_R 0 = (R, 0)$$

obtained by applying [**KM76**, Proposition 7] on the exact triangle

$$X \xrightarrow{0} 0 \xrightarrow{0} X[1] \xrightarrow{-\mathrm{id}_X[1]} X[1] \quad \text{in } \mathrm{Parf}_R.$$

This makes $\det_R(X[1])$ into a left inverse of $\det_R(X)$ via the map in equation (2.1.2). From now on, we will consider $\det_R(X[1])$ as both a right and a left inverse of $\det_R X$ and we will denote it by $(\det_R X)^{-1}$.

2.1.4.5. *Determinants of perfect complexes of torsion modules.* Let R be a domain and M be a torsion R-module. Suppose that M is perfect over R. Then

$$\begin{aligned}(\det_R M) \otimes_R \mathrm{Frac}(R) &\cong \det_{\mathrm{Frac}(R)}(M \otimes_R \mathrm{Frac}(R)) && \text{(by proposition 2.1.2)}\\ &= \det_{\mathrm{Frac}(R)}(0)\\ &= (\mathrm{Frac}(R), 0).\end{aligned}$$

Considering the image of $\det_R M$ inside $\mathrm{Frac}(R)$ under the composite map

$$\det_R M \subset (\det_R M) \otimes_R \mathrm{Frac}(R) \simeq (\mathrm{Frac}(R), 0)$$

and forgetting the second factor of the determinant functor, we obtain an R-submodule of $\mathrm{Frac}(R)$, denoted $[\det_R M \hookrightarrow \mathrm{Frac}(R)]$. Suppose that R is a regular ring. Then

$$[\det_R M \hookrightarrow \mathrm{Frac}(R)] = (\mathrm{char}_R M)^{-1},$$

where

$$\mathrm{char}_R M = \prod_{\mathrm{ht}\,\mathfrak{p}=1} \mathfrak{p}^{\mathrm{length}_{R_\mathfrak{p}} M_\mathfrak{p}}$$

This gives an isomorphism

$$\det_R M \cong ((\mathrm{char}_R M)^{-1}, 0).$$

Since $(\mathrm{char} M, 0)$ is an inverse of $((\mathrm{char} M)^{-1}, 0)$, the above isomorphism induces an isomorphism

$$(\det_R M)^{-1} \cong (\mathrm{char}_R M, 0). \tag{2.1.5}$$

2.2. Selmer complexes

2.2.1. Complex of continuous cochains. Throughout this section, T denotes a finitely generated module over a complete local noetherian ring R with residue field k and G denotes a profinite group acting continuously on T. Let $C^\bullet_{\mathrm{cont}}(G,-)$ denote the functor of continuous cochain complex from the category of $R[G]$-modules to the category of bounded below complexes of R-modules. It preserves homotopy, exact sequences and quasi-isomorphisms (see for instance [**Nek06**, Corollary 3.5.6]), and thus defines an exact functor $R\Gamma_{\mathrm{cont}}(G,-)$ from the derived category of $R[G]$-modules to the derived category of bounded below complexes of R-modules.

Proposition 2.2.1. *Assume that* $\mathrm{char}(k) = p > 0$*. Then the functor* $R\Gamma_{\mathrm{cont}}(G,-)$ *takes perfect complexes to perfect complexes. Let* T *be an* $R[G]$*-module which is free as an* R*-module and* $\phi : R \to R'$ *be a ring homomorphism where* R' *is a complete local noetherian ring and both the rings* R *and* R' *have finite residue fields. Then we have an isomorphism between the objects in the derived category of complexes of* R'*-modules:*

$$R\Gamma_{\mathrm{cont}}(G,T)\overset{L}{\otimes}_{R,\phi}R' \xrightarrow{\sim} R\Gamma_{\mathrm{cont}}(G,T\otimes_{R,\phi}R').$$

Proof. See for instance [**Nek06**, proof of proposition 4.2.9] or [**Kat93**, Theorem 3.1.3] for the perfectness of the derived functor $R\Gamma_{\mathrm{cont}}(G,-)$ and for its base change property we refer to [**SGA72**, Exposé XVII Théorème 4.3.1]. □

2.2.2. Local conditions. Fix a rational prime $p \geq 3$ (in chapter 3 (resp. 4), we have $p \geq 3$ by §3.2.2 (resp. §4.1.1, §4.1.2)). Let F be a number field and S denote a finite set of its places containing the places above $p\infty$. Denote by S_f the set of non-archimedean primes in S. Fix an algebraic closure $\overline{F}$ of F. Let F_S be the maximal subextension of $\overline{F}/F$ unramified outside S; denote $G_{F,S} := \mathrm{Gal}(F_S/F)$. Let X denote an admissible (as in [**Nek06**, Definition 3.2.1]) $R[G_{F,S}]$-module (we will consider free R-modules with a continuous action of $G_{F,S}$, which are always admissible). Now for each prime $v \in S$ fix an algebraic closure $\overline{F}_v$ of F_v and an embedding $\overline{F} \hookrightarrow \overline{F}_v$ extending the embedding $F \hookrightarrow F_v$. This defines a continuous homomorphism

$$\rho_v : G_v := \mathrm{Gal}(\overline{F}_v/F_v) \xrightarrow{\alpha_v} G_F = \mathrm{Gal}(\overline{F}/F) \xrightarrow{\pi} G_{F,S},$$

which gives a 'restriction' map

$$\mathrm{res}_v : C^\bullet_{\mathrm{cont}}(G_{F,S},X) \to C^\bullet_{\mathrm{cont}}(G_v,X).$$

For future use, we recall that $\mathrm{cd}_p G_{F,S} = 2$ (as $p \neq 2$), $\mathrm{cd}_p G_v = 2$, $\mathrm{cd}_p G_v/I_v = 1$ for all finite place v of F where $\mathrm{cd}_p G$ denotes the cohomological p-dimension of a topological group G (see for instance [**Ser02**, Corollary to proposition 12, §4.3], [**NSW08**, Theorem 7.1.8, proposition 8.3.18]).

Local conditions for X are given by a collection $\Delta(X) = (\Delta_v(X))_{v\in S_f}$, where each $\Delta_v(X)$ is a local condition at $v \in S_f$, consisting of a morphism of complexes of R-modules

$$i_v^+(X) : U_v^+(X) \to C^\bullet_{\mathrm{cont}}(G_v,X).$$

The *Selmer complex* associated with the local conditions $\Delta(X)$ is denoted by $\widetilde{R\Gamma}_f(G_{F,S},X;\Delta(X))$ (abbreviated as $R\Gamma_f(X)$) and defined to be the object in the derived category of R-modules corresponding to the complex

$$C_f^\bullet(X) := \mathrm{Cone}\left(C_{\mathrm{cont}}^\bullet(G_{F,S},X)\oplus\bigoplus_{v\in S_f}U_v^+(X)\xrightarrow{\mathrm{res}_{S_f}-i_S^+(X)}\bigoplus_{v\in S_f}C_{\mathrm{cont}}^\bullet(G_v,X)\right)[-1],$$

where $\mathrm{res}_{S_f} = (\mathrm{res}_v)_{v\in S_f}$, $i_S^+(X) = (i_v^+(X))_{v\in S_f}$. By equation (2.1.1), we have an exact sequence of complexes

$$0\to\bigoplus_{v\in S_f}C_{\mathrm{cont}}^\bullet(G_v,X)\xrightarrow{j}C_f^\bullet(X)[1]\xrightarrow{p}\left(C_{\mathrm{cont}}^\bullet(G_{F,S},X)\oplus\bigoplus_{v\in S_f}U_v^+(X)\right)[1]\to 0$$

where j and p are the canonical inclusion and projection. The i-th cohomology group of $R\Gamma_f(X)$ is denoted by $\widetilde{H}_f^i(X)$. When X, $U_v^+(X)$ are perfect complexes of R-modules for all $v\in S_f$, then by §2.1.1.5, [**Nek06**, Proposition 4.2.9] and [**Sta14**, Tag 066R], $R\Gamma_f(X)$ is also perfect.

We will also consider the complexes of R-modules

$$C_{c,\mathrm{cont}}^\bullet(X) = \mathrm{Cone}\left(C_{\mathrm{cont}}^\bullet(G_{F,S},X)\xrightarrow{\mathrm{res}_{S_f}}\bigoplus_{v\in S_f}C_{\mathrm{cont}}^\bullet(G_v,X)\right)[-1],$$

$$C_{\mathrm{Gr}}^\bullet(X) = \mathrm{Cone}\left(C_{\mathrm{cont}}^\bullet(G_{F,S},X)\oplus\bigoplus_{\substack{v\in S_f\\ v\mid p}}U_v^+(X)\xrightarrow{\mathrm{res}_{S_f}-(i_v^+(X))_{v\in S_f,v\mid p}}\bigoplus_{v\in S_f}C_{\mathrm{cont}}^\bullet(G_v,X)\right)[-1].$$

The objects in the derived category of R-modules corresponding to them are denoted by $R\Gamma_{c,\mathrm{cont}}(G_{F,S},X)$, $R\Gamma_{\mathrm{Gr}}(G_{F,S},X)$ (or $R\Gamma_{c,\mathrm{cont}}(X)$, $R\Gamma_{\mathrm{Gr}}(X)$, for short) respectively. Their i-th cohomology groups are denoted by $H_{c,\mathrm{cont}}^i(X), H_{\mathrm{Gr}}^i(X)$ respectively.

We are interested in the following local condition as defined in [**Nek06**, §7.1].

Definition 2.2.2 (Greenberg's local condition)**.** *Let X be as above. Then for $v\in S_f$, the* Greenberg's local condition *for X is given by*

$$U_v^+(X) = \begin{cases}C_{\mathrm{cont}}^\bullet(G_v/I_v, X^{I_v}) & \text{if } v\nmid p\\ C_{\mathrm{cont}}^\bullet(G_v, X_v^+) & \text{if } v\mid p,\end{cases}$$

with

$$i_v^+(X) = \begin{cases}U_v^+(X)\xrightarrow{\mathrm{inf}}C_{\mathrm{cont}}^\bullet(G_v,X) & \text{if } v\nmid p\\ U_v^+(X)\to C_{\mathrm{cont}}^\bullet(G_v,X) & \text{if } v\mid p,\end{cases}$$

where X_v^+ denotes a choice of a G_v-stable R-submodule of X for $v\mid p$.

Proposition 2.2.3. *Let X be as above. Then for a finite place v of F not dividing p, the complex $U_v^+(X)$ is quasi-isomorphic to $[X^{I_v}\xrightarrow{\mathrm{Fr}_v-1}X^{I_v}]$ where Fr_v denotes the geometric Frobenius element at v.*

Proof. See [**Nek06**, §7.2.2]. □

CHAPTER 3

Algebraic p-adic L-functions for the Hida family for $\mathrm{GL}_2(\mathbb{Q})$

In this chapter, we construct algebraic p-adic L-functions $L^{\mathrm{alg}}_{p,\mathrm{Kato}}(-), L^{\mathrm{alg}}_{p,\mathrm{Gr}}(-), L^{\mathrm{alg}}_{p',\mathrm{Gr}}(-)$ along branches of the Hida family for $\mathrm{GL}_2(\mathbb{Q})$ and prove that they satisfy a perfect control theorem at arithmetic specializations (theorem 3.3.7). The crucial step of their proof is the recognition of the role of purity in understanding the variation of inertia invariants in families. Since the modular Galois representations are known to be pure, this variation is well-understood by theorem 1.2.4. In this chapter, from §3.3, we assume throughout that the condition 3.2.4 holds.

The local conditions used in $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(-), L^{\mathrm{alg}}_{p',\mathrm{Gr}}(-)$ at places $\ell \neq p$ is a modification $U'_\ell(-)$ of the unramified condition $U^+_\ell(-)$ of Greenberg (as defined in [**Nek06**, §0.8.1] following [**Gre89, Gre91**]). We use the local condition $U'_\ell(-)$ in stead of $U^+_\ell(-)$ as it is pointed out in [**FO12**, Remark 2.17] that the inertia invariants of a big Galois representation ρ may not specialize perfectly to the inertia invariants of a specialization of ρ. The local condition at p used in $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(-)$ is the Greenberg's local condition $U^+_p(-)$ and the control theorem for $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(-)$ is obtained under the p-distinguishedness assumption 3.3.1. This assumption is relaxed while proving the control theorem for $L^{\mathrm{alg}}_{p',\mathrm{Gr}}(-)$, whose construction uses a modification $U'_p(-)$ of the condition $U^+_p(-)$ as its local condition at p. The construction of $L^{\mathrm{alg}}_{p,\mathrm{Kato}}(-)$ uses no condition at p and uses the condition $U'_\ell(-)$ at places $\ell \neq p$.

For any arithmetic specialization λ of $R(\mathfrak{a})$ whose image is a DVR and associated ordinary form is of good ordinary reduction, we show in theorem 3.4.5 that there is a canonical isomorphism (depending only on the isomorphism in equation (3.4.3)) between $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}})$ and the characteristic ideal of the Pontrjagin dual of the Greenberg's Selmer group $\mathrm{Sel}^{\mathrm{str}}_{A_{\lambda,\mathrm{Iw}}}$ (together with a grade). This theorem is a consequence of [**Kat04**, Theorem 17.4].

Using theorem 3.4.5, we prove in proposition 3.5.6 that all the cohomologies of $R\Gamma_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ are zero except possibly the second cohomology, which is torsion over $R(\mathfrak{a})_{\mathrm{Iw}}$. This yields a purely algebraic construction of an element $\mathscr{L}^{\mathrm{alg}}_p(\mathfrak{a})$, called the two-variable algebraic p-adic L-function, using the "factors" of $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ coming from $R\Gamma_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ and the local Euler factors. As a consequence of proposition 3.5.6, we prove in theorem 3.5.11 that the mod $\mathfrak{p}$ reduction of $\mathscr{L}^{\mathrm{alg}}_p(\mathfrak{a})$ generates the characteristic ideal of $\widetilde{H}^2_f(T_{\lambda_\mathfrak{p},\mathrm{Iw}})$ for $\mathfrak{p}$ varying in a dense subset of $\mathrm{Spec}^{\mathrm{arith}}(R(\mathfrak{a}))$. In conjecture 3.5.16, we predict that $\mathscr{L}^{\mathrm{alg}}_p(\mathfrak{a})$ is an integral element and is an associate of the analytic p-adic L-function constructed in [**EPW06**]. When Greenberg's conjecture on vanishing of μ-invariants of modular forms (with absolutely irreducible and p-distinguished residual Galois representation) holds, we prove this conjecture in theorem 3.5.22.

The organization of this chapter is as follows. In the first section, we review cusp forms and associated Galois representations. The second section is about Hida family of ordinary cusp forms. In the third section, we construct algebraic p-adic L-functions and show that they satisfy perfect control theorems. In the fourth section, we relate our construction with the Greenberg's Selmer group. In the final section, we formulate a conjecture relating the two-variable algebraic p-adic L-function $\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a})$ with the analytic p-adic L-function constructed in [**EPW06**]. Under Greenberg's conjecture and assumption 3.5.15, we prove it in theorem 3.5.22.

3.1. Cusp forms and associated representations

In this section, we briefly recall how from a cusp form f, defined as a complex valued function on the upper half plane, one obtains an automorphic representation $\pi(f)$ of GL_2 of the adeles and we describe how the restriction of the Deligne's representation ρ_f to decomposition groups at finite places $\ell \neq p$ can be understood from the local factors of $\pi(f)$. In the end, we describe the action of the Frobenius elements (away from p) under ρ_f.

3.1.1. Automorphic representation attached to a cusp form. Let f be a nonzero cusp form of level N and weight $k \geq 1$ with nebentype character ψ. Suppose that it is an eigenform for every T_p with primes $p \nmid N$. Let χ_ψ denote the grossencharacter defined on $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times$ by restricting ψ to the appropriate factors of the decomposition $\mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times \cdot \mathbb{R}^+ \cdot \prod_{p<\infty} \mathbb{Z}_p^\times$. Using the analogous decomposition $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})\,\mathrm{GL}_2(\mathbb{R}) \prod_{p<\infty} K_p^N$, define the complex valued function φ_f on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ by

$$\varphi_f(g) = f(g_\infty(i)) j(g_\infty, i)^{-k} \chi_\psi(k_0) \qquad \text{for } g = \gamma g_\infty k_0 \text{ with } \gamma \in \mathrm{GL}_2(\mathbb{Q}), g_\infty \in \mathrm{GL}_2(\mathbb{R}), k_0 \in \prod_{p<\infty} K_p^N,$$

where $K_p^N = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_p) : c \equiv 0 \pmod N\}$, χ_ψ on $\prod_{p<\infty} K_p^N$ is defined by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \chi_\psi(a)$ and $j(g_\infty, z) = (cz+d)(\det g_\infty)^{-\frac{1}{2}}$ if $g_\infty = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. This function φ_f is well-defined and belongs to the space of functions $L_0^2(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}), \chi_\psi)$ ([**Gel75**, §3.A]). Its translates under the right regular action of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generates an irreducible unitary representation $\pi(f) = \otimes'_{\ell \leq \infty} \pi(f)_\ell$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ ([**Gel75**, Theorem 5.19]). For each prime number ℓ, the local representation $\pi_\ell = \pi(f)_\ell$ of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ is one of the following types ([**Gel75**, Remark 5.8, Theorem 4.21]) :

(1) *Principal series.* It is the irreducible representation $\pi_\ell = \pi(\mu, \mu')$, in which $\mathrm{GL}_2(\mathbb{Q}_\ell)$ acts by right translation on the space $\mathcal{B}(\mu, \mu')$ of locally constant functions $f : \mathrm{GL}_2(\mathbb{Q}_\ell) \to \mathbb{C}$ satisfying

$$f\left(\begin{pmatrix} a & b \\ 0 & a' \end{pmatrix} g\right) = \mu(a)\mu'(a')|a/a'|^{1/2} f(g),$$

where $\mu, \mu' : \mathbb{Q}_\ell^\times \to \mathbb{C}^\times$ denote characters satisfying $\mu/\mu' \neq |\cdot|^{\pm 1}$ and $|\cdot| = |\cdot|_\ell : \mathbb{Q}_\ell^\times \to \mathbb{R}_+^\times$ is the normalized valuation (*i.e.*, $|\varpi_\ell|_\ell = \ell^{-1}$ for any uniformizer $\varpi_\ell \in \mathbb{Z}_\ell$).

(2) *Twisted Steinberg representation* (= *special representation*).

$$\mathrm{St}(\mu) = \mathrm{St} \otimes \mu \subset \mathcal{B}(\mu|\cdot|^{1/2}, \mu|\cdot|^{-1/2})$$

where $\mu : \mathbb{Q}_\ell^\times \to \mathbb{C}^\times$ is a character.

(3) *Supercuspidal representation.*

We call them the automorphic type of π_ℓ.

3.1.2. Galois representation attached to a cusp form. Let f be as above with weight $k \geq 2$ and K_f denote a finite extension of $\mathbb{Q}_p$ containing the Fourier coefficients a_ℓ of f for primes $\ell \nmid N$ (via i_∞ and i_p). Then by [**Eic54, Shi58**] (for $k = 2$), [**Del69, Car86**], Ohta *et. al.* (for $k > 2$), there exists a continuous two-dimensional p-adic Galois representation (with respect to i_∞ and i_p) $V(f) = V_p(f)$ of $G_\mathbb{Q} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over K_f which is unramified outside Np and satisfies

$$\det(1 - \mathrm{Fr}_\ell X | V(f)) = 1 - a_\ell(f)X + \psi(\ell)\ell^{k-1}X^2 \tag{3.1.1}$$

for each prime $\ell \nmid Np$. Moreover, this representation is absolutely irreducible, by [**Rib77**, Theorem 2.3].

Proposition 3.1.1. *Let f be a cusp form as above. For a rational prime $\ell \neq p$, the restriction $V(f)_\ell$ of $V(f)$ to the decomposition group G_ℓ can be described in terms of the local factor π_ℓ of $\pi(f)$ using the local Langlands correspondence as follows.*

(1) If $\pi_\ell = \pi(\mu, \mu')$, then I_ℓ acts on $V(f)_\ell$ through a finite quotient and the semi-simplification $V(f)_\ell$ is isomorphic to

$$V(f)_\ell^{ss} \xrightarrow{\sim} K_f \otimes \mu|\cdot|^{(1-k)/2} \oplus K_f \otimes \mu'|\cdot|^{(1-k)/2},$$

thus I_ℓ acts on $V(f)_\ell$ by $\mu|_{\mathbb{Z}_\ell^\times} \oplus \mu'|_{\mathbb{Z}_\ell^\times}$.

(2) If $\pi_\ell = \mathrm{St}(\mu)$, then the representation $V(f)_\ell$ is reducible and I_ℓ acts on $V(f)_\ell$ through an infinite quotient. There is an exact sequence of $K_f[G_\ell]$-modules

$$0 \to K_f \otimes \mu|\cdot|^{1-k/2} \to V(f)_\ell \to K_f \otimes \mu|\cdot|^{-k/2} \to 0.$$

In particular, if μ is unramified, then I_ℓ acts on $V(f)_\ell$ through its tame quotient $I_\ell^t = I_\ell/I_\ell^w$, and any topological generator of I_ℓ^t acts on $V(f)_\ell$ by an endomorphism A satisfying $(A-1)^2 = 0 \neq A - 1$. [1]

(3) If π_ℓ is supercuspidal, then G_ℓ acts on $V(f)_\ell$ irreducibly and I_ℓ acts through a finite quotient.

In addition, the eigenvalues of any lift $g \in G_\ell$ of the Frobenius $\mathrm{Fr}_\ell \in G_\ell/I_\ell$ acting of $V(f)$ are Weil numbers of weights

$$\begin{cases} k-1, k-1 & \text{if } \pi(f)_\ell \neq \mathrm{St}(\mu), \\ k-2, k & \text{if } \pi(f)_\ell = \mathrm{St}(\mu). \end{cases}$$

[1] In (1) and (2) above, K_f is assumed to contain the values of μ, μ'. If this is not the case, then the coefficient ring of $V(f)$ can be extended to contain these values and then the above description of $V(f)_\ell$ holds.

Proof. For the proof see [**Car86**]. □

3.2. Hida Theory

In the late 1980's, Hida ([**Hid86a, Hid86b**]) introduced the notion of universal ordinary Hecke algebra to study ordinary cusp forms and their associated Galois representations in p-adic families. In this section we review the necessary results of Hida theory following the presentation of [**Hid87**] and [**Nek06**, §12.7].

3.2.1. Ordinary Hecke algebras. Let p be a rational prime and $\mathcal{O}$ be a discrete valuation ring finite and flat over $\mathbb{Z}_p$. In other words, $\mathcal{O}$ is the p-adic integer ring of a finite extension K of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$.

For positive integers N and k, let $\mathcal{S}_k(\Gamma_1(N))$ denote the space of cusp forms of weight k and level N. An element $f \in \mathcal{S}_k(\Gamma_1(N))$ has the following type of Fourier expansion:

$$f = \sum_{n=1}^{\infty} a_n(f) q^n \quad (q = e^{2\pi i \tau},\ \tau \in \mathfrak{H})$$

which allows to embed $\mathcal{S}_k(\Gamma_1(N))$ into the power series ring $\mathbb{C}[[q]]$. Define $\mathcal{S}_k(\Gamma_1(N);\mathbb{Z})$ as the intersection of $\mathcal{S}_k(\Gamma_1(N))$ with $\mathbb{Z}[[q]]$ inside $\mathbb{C}[[q]]$. For each integer d prime to N, we can let d act on $\mathcal{S}_k(\Gamma_1(N))$ by

$$\langle d \rangle f = d^{k-2} f|[\alpha]_k \quad \text{for any } \alpha = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \in \Gamma_0(N) \text{ with } \delta \equiv d(\bmod N). \tag{3.2.1}$$

The Hecke operators T_n for $n \geq 1$ are endomorphisms of $\mathcal{S}_k(\Gamma_1(N))$ and their effect on the Fourier coefficients can be expressed as

$$a_m(T_n f) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} d a_{mn/d^2}(\langle d \rangle f). \tag{3.2.2}$$

The Hecke algebra $h_k(\Gamma_1(N);\mathbb{Z})$ is the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{S}_k(\Gamma_1(N)))$ generated over $\mathbb{Z}$ by T_n for all n. Define a pairing

$$\langle\, ,\rangle : h_k(\Gamma_1(N);\mathbb{Z}) \times \mathcal{S}_k(\Gamma_1(N);\mathbb{Z}) \to \mathbb{Z} \text{ by } \langle h, f \rangle = a_1(f|h)$$

The following facts are known (*eg.* Section 1, [**Hid86a**])

1. $\mathcal{S}_k(\Gamma_1(N);\mathbb{Z})$ is stable under the action of $h_k(\Gamma_1(N);\mathbb{Z})$,
2. $h_k(\Gamma_1(N);\mathbb{Z})$ is a commutative algebra and T_1 gives the identity,
3. the diamond operator $\langle n \rangle$ belongs to $h_k(\Gamma_1(N);\mathbb{Z})$,
4. the pairing $\langle\, ,\, \rangle$ is perfect over $\mathbb{Z}$,
5. $\mathcal{S}_k(\Gamma_1(N);\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathcal{S}_k(\Gamma_1(N))$ naturally.

We put $h_k(\Gamma_1(N);\mathcal{O}) = h_k(\Gamma_1(N);\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{O}$. By (4), (5) above, the algebra $h_k(\Gamma_1(N);\mathcal{O})$ is free of finite rank over $\mathcal{O}$ and its rank is equal to the dimension of $\mathcal{S}_k(\Gamma_1(N))$. Since $h_k(\Gamma_1(N);\mathcal{O})$ is of finite rank over $\mathcal{O}$, it is a product of finitely many complete local rings R (for instance from [**Eis95**, Corollary 7.6, p. 188]) in a unique way. For such a local ring R, let 1_R denote the idempotent of R and define an idempotent $e_N \in h_k(\Gamma_1(N); \mathcal{O})$ by the sum of 1_R over the local rings R on which the image of T_p is invertible. Then

$$h_k^{\mathrm{ord}}(\Gamma_1(N); \mathcal{O}) := e_N h_k(\Gamma_1(N); \mathcal{O})$$

is the product of all the local rings of $h_k(\Gamma_1(N); \mathcal{O})$ on which the image of T_p is a unit. Thus $h_k^{\mathrm{ord}}(\Gamma_1(N); \mathcal{O})$ is the maximal algebra direct summand of $h_k(\Gamma_1(N); \mathcal{O})$ on which the image of T_p is a unit.

Now the pairing $\langle\ ,\ \rangle$ induces bijections :

(3.2.3) $$\mathrm{Hom}_{\mathcal{O}\text{-alg}}(h_k(\Gamma_1(N); \mathcal{O}), \overline{\mathbb{Q}}_p) \leftrightarrow \{\text{normalized eigenforms in } \mathcal{S}_k(\Gamma_1(N))\},$$

(3.2.4)
$$\mathrm{Hom}_{\mathcal{O}\text{-alg}}(h_k^{\mathrm{ord}}(\Gamma_1(N); \mathcal{O}), \overline{\mathbb{Q}}_p) \leftrightarrow \{\text{normalized eigenforms in } \mathcal{S}_k(\Gamma_1(N)) \text{ with } i_p(i_\infty^{-1}(a_p)) \in \overline{\mathbb{Z}}_p^\times\}.$$

3.2.1.1. *Ordinary forms.* From now on we call a normalized eigenform $f = \sum_{n \geq 1} a_n q^n$ in $\mathcal{S}_k(\Gamma_1(N), \psi)$ to be *p-ordinary* (depending on i_∞ and i_p) if its p-th Fourier coefficient a_p is a p-adic unit (*i.e.*, $i_p(i_\infty^{-1}(a_p)) \in \overline{\mathbb{Z}}_p^\times$). According to [**Wil88**, Theorem 2.2.2, p. 562], for an ordinary form f with $k \geq 2$, there is an exact sequence of $K_f[G_p]$-modules

$$0 \to V(f)^+ \to V(f) \to V(f)^- \to 0$$

where $\dim_{K_f} V(f)^\pm = 1$, $V(f)^+$ is unramified and Fr_ℓ acts on it via the unique p-adic unit root of $X^2 - a_p X + \psi(p)p^{k-1}$, which is a_p if $p \mid N$.

We remark that the notion of ordinariness depends on the embeddings i_∞ and i_p. For example, consider the newform

$$f = q + \alpha q^2 - \alpha q^3 + (\alpha^2 - 2)q^4 + (-\alpha^2 + 1)q^5 - \alpha^2 q^6 + \cdots$$

in $S_2(\Gamma_0(389))$ where α is a root of $X^3 - 4X - 2$ (see [**RS11**, §26.1.1]). The coefficient of q^5, $(-\alpha^2 + 1)$ satisfies $y^3 + 5y^2 + 3y - 5$. By Hensel's lemma, we see that it has a non-unit root in $\mathbb{Z}_5$ and two conjugate roots in a quadratic extension K of $\mathbb{Q}_5$ which are units in $\mathcal{O}_K$.

Note that the notion of ordinariness for a form in $\mathcal{S}_k(\Gamma_1(Np^r))$ with $r \geq 1$ is independent of r by the commutative diagram (3.2.5) below.

3.2.2. The universal ordinary Hecke algebra. From now on we suppose that $p \nmid N, p \neq 2$ and $Np \geq 4$. For integers $r \geq s \geq 1$, we have the following commutative diagram for all $n \geq 1$ (by (3.2.2)):

(3.2.5)
$$\begin{array}{ccc} \mathcal{S}_k(\Gamma_1(Np^s); \mathcal{O}) & \longrightarrow & \mathcal{S}_k(\Gamma_1(Np^r); \mathcal{O}) \\ \downarrow{\scriptstyle T_n} & & \downarrow{\scriptstyle T_n} \\ \mathcal{S}_k(\Gamma_1(Np^s); \mathcal{O}) & \longrightarrow & \mathcal{S}_k(\Gamma_1(Np^r); \mathcal{O}) \end{array}$$

where the horizontal arrows are the natural inclusion and the left (resp., right) vertical arrow is the Hecke operator T_n of level Np^s (resp., Np^r). Then the restriction of each operator in

$h_k(\Gamma_1(Np^r);\mathcal{O})$ to the subspace $\mathcal{S}_k(\Gamma_1(Np^s);\mathcal{O})$ is again contained in $h_k(\Gamma_1(Np^s);\mathcal{O})$; thus, we have a surjective $\mathcal{O}$-algebra homomorphism:

$$h_k(\Gamma_1(Np^r);\mathcal{O}) \to h_k(\Gamma_1(Np^s);\mathcal{O}) \quad \text{for each pair } r \geq s \geq 1. \tag{3.2.6}$$

Since T_p goes to T_p under the above map, the image of e_{Np^r} under this map coincides with e_{Np^s}, and thus the above map induces a map

$$h_k^{\mathrm{ord}}(\Gamma_1(Np^r);\mathcal{O}) \to h_k^{\mathrm{ord}}(\Gamma_1(Np^s);\mathcal{O}).$$

Taking projective limits we obtain the *universal p-ordinary Hecke algebra of tame level N*,

$$h_k^{\mathrm{ord}}(Np^\infty;\mathcal{O}) = \varprojlim_r h_k^{\mathrm{ord}}(\Gamma_1(Np^r);\mathcal{O}).$$

Now the diamond operators are $\mathcal{O}$-algebra homomorphisms

$$\langle\ \rangle_{k,r} : \mathcal{O}[(\mathbb{Z}/Np^r\mathbb{Z})^\times] \to h_k(\Gamma_1(Np^r);\mathcal{O}) \to h_k^{\mathrm{ord}}(\Gamma_1(Np^r);\mathcal{O}) \quad \text{for } r \geq 1,$$

which upon taking limit gives the $\mathcal{O}$-algebra homomorphism

$$\langle\ \rangle_k : \mathcal{O}[[Z_N]] \to h_k^{\mathrm{ord}}(Np^\infty;\mathcal{O})$$

where $Z_N = \varprojlim_r (\mathbb{Z}/Np^r\mathbb{Z})^\times = (1+p\mathbb{Z}_p) \times (\mathbb{Z}/Np\mathbb{Z})^\times$ and $\mathcal{O}[[Z_N]] = \varprojlim_r \mathcal{O}[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$.

Put $\Gamma_r = 1 + p^r\mathbb{Z}_p$ for $r \geq 1$, $\Gamma = \Gamma_1$ and define $\Lambda = \Lambda_{\mathcal{O}} = \mathcal{O}[[\Gamma]] = \varprojlim_r \mathcal{O}[\Gamma/\Gamma_r]$. Let $\chi_\Gamma : \Gamma \hookrightarrow \Lambda^\times$ denote the canonical inclusion. The above implies that $h_k^{\mathrm{ord}}(Np^\infty;\mathcal{O})$ has a canonical Λ-algebra structure.

Fix a topological generator γ of Γ. For an integer $k' \geq 2$ and a finite order character $\varepsilon' : \Gamma \to \overline{\mathbb{Q}}_p^\times$ with values in the ring of integers $\mathcal{O}'$ of a finite extension K' of K, put $P_{k',\varepsilon'} = \chi_\Gamma(\gamma) - \varepsilon'(\gamma)\gamma^{k'-2} \in \Lambda' := \mathcal{O}'[[\Gamma]]$. Note that $P_{k',\varepsilon'}\Lambda'$ is a prime ideal of Λ' and thus induces a prime ideal $P_{k',\varepsilon'}\Lambda' \cap \Lambda$ of Λ.

An *arithmetic prime* of a finite Λ-algebra A is a prime $\wp \in \mathrm{Spec}(A)$ whose contraction to Λ is of the form $P_{k',\varepsilon'}\Lambda' \cap \Lambda$ and an *arithmetic specialization* of A is an $\mathcal{O}$-algebra homomorphism $A \to \overline{\mathbb{Q}}_p$ whose kernel is an arithmetic prime. The set of arithmetic primes of A is denoted by $\mathrm{Spec}^{\mathrm{arith}}(A)$.

Let R be a quotient of $h_k^{\mathrm{ord}}(Np^\infty;\mathcal{O})$ by a minimal prime ideal. Then $\mathrm{Spec}^{\mathrm{arith}}(R)$ is an infinite set since R is of finite type over Λ. Moreover, any infinite subset of $\mathrm{Spec}^{\mathrm{arith}}(R)$ is dense in $\mathrm{Spec}(R)$ since each fibre of $\mathrm{Spec}(R) \to \mathrm{Spec}(\Lambda)$ is finite due to the integrality of R over Λ.

Theorem 3.2.1.

(1) ([**Hid86a**, Theorem 1.1, p. 551]*) For each $k \geq 2$, we have canonical $\mathcal{O}[[Z_N]]$-algebra isomorphism*

$$h_k^{\mathrm{ord}}(Np^\infty;\mathcal{O}) \cong h_2^{\mathrm{ord}}(Np^\infty;\mathcal{O}),$$

which takes T_m of weight k to T_m of weight 2 for all m. We use the above isomorphisms to identify all $h_k^{\mathrm{ord}}(Np^\infty;\mathcal{O})$ $(k \geq 2)$ with $h_\infty^{\mathrm{ord}} := h_2^{\mathrm{ord}}(Np^\infty;\mathcal{O})$.

(2) ([**Hid86b**, Theorem 3.1]*) h_∞^{ord} is free of finite rank over Λ.*

(3) ([**Hid86a**, Theorem 1.2]) *For each $k \geq 2$ and $r \geq 1$, the surjective Λ-algebra homomorphisms $h_\infty^{\mathrm{ord}} \xleftarrow{\sim} h_k^{\mathrm{ord}}(Np^\infty; \mathcal{O}) \to h_k^{\mathrm{ord}}(\Gamma_1(Np^r); \mathcal{O})$ induce Λ-algebra isomorphisms*

$$h_\infty^{\mathrm{ord}}/(\chi_\Gamma(\gamma)^{p^{r-1}} - \gamma^{p^{r-1}(k-2)}) \cong h_k^{\mathrm{ord}}(\Gamma_1(Np^r); \mathcal{O})$$

which sends T_m to T_m for all m.

We have the following corollary of the above theorem.

Corollary 3.2.2 ([**Hid88a**, Corollary 3.5])**.** *Through equation* (3.2.4) *and theorem 3.2.1 (3), the arithmetic primes of h_∞^{ord} of weight $k \geq 2$ are in one-to-one correspondence with the G_K-conjugacy classes of p-ordinary forms (defined over $\overline{\mathbb{Q}}_p$) in $\mathcal{S}_k(\Gamma_1(Np^r))$ for weight $k \geq 2$ and $r \geq 1$ and the arithmetic specializations of h_∞^{ord} of weight $k \geq 2$ are in one-to-one correspondence with the p-ordinary forms in $\mathcal{S}_k(\Gamma_1(Np^r))$ for weight $k \geq 2$ and $r \geq 1$.*

For such an eigenform f, we denote the corresponding arithmetic specialization by λ_f and for such a specialization λ we denote the corresponding ordinary form by f_λ.

3.2.3. Galois representations. Let $f \in \mathcal{S}_k(Np^r, \chi)$ be an ordinary normalized Hecke eigenform of weight $k \geq 2$ such that $K = \mathrm{Frac}(\mathcal{O})$ contains all Hecke eigenvalues of f and all values of χ. Assume, in addition, that f is a *p-stabilized newform* in the sense of [**Wil88**, p. 538]. This means that $r \geq 1$ and that the (necessarily ordinary) normalized newform f' associated with f has level divisible by N. Let $\wp$ denote the arithmetic prime associated with the G_K-conjugacy class of f (which is the set $\{f\}$). Then $\wp$ strictly contains a prime ideal $\mathfrak{a}$ of h_∞^{ord}, necessarily minimal. Put $R(\mathfrak{a}) = h_\infty^{\mathrm{ord}}/\mathfrak{a}$. Then $R(\mathfrak{a})$ is a domain and finite over Λ. Note that $R(\mathfrak{a})$ is local and denote its maximal ideal by $\mathfrak{m}$. Let ψ denote the composite map

$$h_\infty^{\mathrm{ord}} \twoheadrightarrow h_\infty^{\mathrm{ord}}/\mathfrak{a} = R(\mathfrak{a}) \hookrightarrow \mathcal{K}, \quad \mathcal{K} := \mathrm{Frac}(R(\mathfrak{a}))$$

which is *minimal* in the sense of [**Hid88a**, p. 317], since $f = f_\wp$ is a p-stabilized newform. This implies, by [**Hid88a**, Corollary 3.5, theorem 3.6] that the form associated with an arithmetic specialization λ whose kernel contains $P_{k',\varepsilon'}$ is a p-stabilized newform $f_{\wp'} \in \mathcal{S}_{k'}(Np^{r'}, \varepsilon'\psi_0\omega^{-(k'-2)})$ where r' denotes the smallest positive integer for which ε' factors through $\Gamma/\Gamma_{r'}$, ω denotes the Teichmuller character $\mathbb{Z}_p^\times \to (\mathbb{Z}_p^\times)_{\mathrm{tors}} \hookrightarrow \overline{\mathbb{Q}}$ and ψ_0 denotes the restriction of ψ to $(\mathbb{Z}/Np\mathbb{Z})^\times$.

Let S_0 denote the set of all prime of $\mathbb{Q}$ dividing $Np\infty$. Then according to [**Wil88**, Theorem 2.2.1], there is a unique (up to equivalence) continuous Galois representation

$$\rho : G_{\mathbb{Q},S_0} \to \mathrm{GL}_2(\mathcal{K})$$

satisfying

$$\det(1 - \rho(\mathrm{Fr}_\ell)X) = 1 - \psi(T_\ell)X + \psi(\langle \ell \rangle)\ell X^2$$

for all prime $\ell \nmid Np$ where $\langle \ell \rangle$ denotes the image of ℓ under the composite map

$$Z_N \to \mathcal{O}[[Z_N]] \xrightarrow{\langle\,\rangle_2} h_2^{\mathrm{ord}}(Np^\infty; \mathcal{O}) = h_\infty^{\mathrm{ord}}$$

(more precisely, ρ is the dual of the representation constructed in [**Wil88**], as we use the geometric Frobenius instead of the arithmetic Frobenius). This representation is continuous in the sense that its representation space $V(\psi)$ is an admissible $R(\mathfrak{a})[G_{\mathbb{Q},S_0}]$-module (as in

[**Nek06**, Definition 3.2.1]). According to [**Wil88**, Theorem 2.2.2], there is an exact sequence of $\mathcal{K}[G_p]$-modules

$$0 \to V(\psi)^+ \to V(\psi) \to V(\psi)^- \to 0,$$

such that each $V(\psi)^\pm$ is one dimensional over $\mathcal{K}$, I_p acts trivially on $V(\psi)^+$ and Fr_p acts on $V(\psi)^+$ by $\psi(T_p)$.

For an arithmetic specialization λ of $R(\mathfrak{a})$, the λ-specialization of the representation ρ exists and is equivalent to the Deligne's representation attached to the ordinary form f_λ corresponding to λ (see for instance [**Hid87**, p. 440]).

Proposition 3.2.3. *There is a semi-simple representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(R(\mathfrak{a})/\mathfrak{m})$, uniquely determined by the properties:*

(1) $\overline{\rho}$ is unramified away from Np.

(2) If ℓ is a prime not dividing Np then

$$\det(1 - \overline{\rho}(\mathrm{Fr}_\ell)X) = 1 - \psi(T_\ell)X + \psi(\langle \ell \rangle)\ell X^2 \bmod \mathfrak{m} \in (R(\mathfrak{a})/\mathfrak{m})[X].$$

Proof. To construct the representation $\overline{\rho}$, we choose an integral model for ρ over the normalization of $\widehat{R(\mathfrak{a})}$, then reduce modulo its maximal ideal $\widehat{\mathfrak{m}}$ of $\mathfrak{m}$, take semi-simplification and descend (if necessary) from $\widehat{R(\mathfrak{a})}$ to $R(\mathfrak{a})$. It has the required properties since it is obtained from ρ. □

Henceforth we make the following assumption on the above residual representation.

Assumption 3.2.4. *The residual representation $\overline{\rho}$ is absolutely irreducible.*

Then by [**Nys96**], we obtain a uniquely determined representation (denoted by the same symbol ρ)

$$\rho : G_{\mathbb{Q},S_0} \to \mathrm{GL}_2(R(\mathfrak{a})) \tag{3.2.7}$$

characterized by the following property: if ℓ is a prime not dividing Np, then $\rho(\mathrm{Fr}_\ell)$ has trace equal to $T_\ell \in R(\mathfrak{a})$.

3.3. Algebraic p-adic L-function along branches

In this section, we construct algebraic p-adic L-functions $L^{\mathrm{alg}}_{p,\mathrm{Gr}}$, $L^{\mathrm{alg}}_{p',\mathrm{Gr}}$, $L^{\mathrm{alg}}_{p,\mathrm{Kato}}$ along irreducible components of the Hida family and show that it satisfies a control theorem at arithmetic primes.

Recall that under the assumption 3.2.4, we obtained a uniquely determined representation $\rho : G_{\mathbb{Q},S_0} \to \mathrm{GL}_2(R(\mathfrak{a}))$ in equation (3.2.7). From theorem 3.2.1, it follows that $R(\mathfrak{a})$ is a complete local domain and a finite type Λ-module (using [**Eis95**, Corollary 7.6, p. 188] for instance). Let $\mathfrak{m}$ denote its maximal ideal and k denote the residue field. Now we define $\mathcal{T}(\mathfrak{a}) := R(\mathfrak{a})^2$ with a $G_{\mathbb{Q},S_0}$-action on it via ρ. Let $(\mathcal{V}, \rho_{\mathcal{V}}) = \mathcal{T}(\mathfrak{a}) \otimes_{R(\mathfrak{a})} \mathcal{K}$ denote the associated $G_{\mathbb{Q},S_0}$-representation over the fraction field $\mathcal{K}$ of $R(\mathfrak{a})$. Henceforth we make the following assumption on ρ.

Assumption 3.3.1 (p-distinguished)**.** *The representation $\overline{\rho}$ is p-distinguished, i.e., the residual representation of $\rho|_{G_p}$ is non-scalar.*

For a ring homomorphism $\phi : R(\mathfrak{a}) \to R'$, the *$\phi$-specialization* of $\mathcal{T}(\mathfrak{a})$ is denoted by T_ϕ and is defined to be the $G_{\mathbb{Q},S_0}$-representation $\mathcal{T}(\mathfrak{a}) \otimes_{R(\mathfrak{a}),\phi} R'$ with coefficients in R'. From now on we denote the image of an arithmetic specialization $\lambda : R(\mathfrak{a}) \to \overline{\mathbb{Q}}_p$ by $\mathcal{O}_\lambda$ and consider such maps as ring homomorphisms onto their images, *i.e.*, as $\lambda : R(\mathfrak{a}) \twoheadrightarrow \mathcal{O}_\lambda$. Thus for an arithmetic specialization λ of $R(\mathfrak{a})$, the λ-specialization T_λ of $\mathcal{T}(\mathfrak{a})$ will denote the $G_{\mathbb{Q},S_0}$-representation $\mathcal{T}(\mathfrak{a}) \otimes_{R(\mathfrak{a}),\lambda} \mathcal{O}_\lambda$. For such a specialization, we denote by V_λ (resp. V'_λ) the $G_{\mathbb{Q},S_0}$-representation $T_\lambda \otimes_{\mathcal{O}_\lambda} \overline{\mathbb{Q}}_p$ (resp. $T_\lambda \otimes_{\mathcal{O}_\lambda} \mathrm{Frac}(\mathcal{O}_\lambda)$).

3.3.1. Comparing the inertia invariants.

Proposition 3.3.2. *Let $\ell \neq p$ be a rational prime. For any arithmetic specialization λ of $R(\mathfrak{a})$, we have*

$$\mathrm{rk}_{R(\mathfrak{a})} \mathcal{T}(\mathfrak{a})^{I_\ell} = \mathrm{rk}_{\mathcal{O}_\lambda} T_\lambda^{I_\ell}.$$

Suppose that the rank of the $R(\mathfrak{a})$-module $\mathcal{T}(\mathfrak{a})^{I_\ell}$ is one. Then for any two arithmetic specializations λ, λ' of $R(\mathfrak{a})$, the representations $\pi(\lambda)_\ell, \pi(\lambda')_\ell$ are both either singly ramified principal series or unramified Steinberg. Moreover, the ring $R(\mathfrak{a})$ contains the eigenvalue α of Fr_ℓ acting on $\mathcal{V}^{I_\ell}$ and Fr_ℓ acts on $T_\lambda^{I_\ell}$ by the scalar $\lambda(\alpha)$ for any arithmetic specialization λ of $R(\mathfrak{a})$.

Proof. The restriction of the $G_{\mathbb{Q},S_0}$-representation $\mathcal{T}(\mathfrak{a})$ to the decomposition group G_ℓ is continuous and its coefficient ring $R(\mathfrak{a})$ has finite residue field of characteristic $p \neq \ell$. So by theorem 1.1.25, the G_ℓ-representation $\mathcal{T}(\mathfrak{a})$ is monodromic. So theorem 1.2.4 applies to $\mathcal{T}(\mathfrak{a})$. By part (5) of this theorem, we have

$$\mathrm{rk}_{R(\mathfrak{a})} \mathcal{T}(\mathfrak{a})^{I_\ell} = \mathrm{rk}_{\mathcal{O}_\lambda} T_\lambda^{I_\ell}$$

for any arithmetic specialization λ of $R(\mathfrak{a})$.

Now suppose that $\mathrm{rk}_{R(\mathfrak{a})} \mathcal{T}(\mathfrak{a})^{I_\ell} = 1$. So

$$\mathrm{rk}_{\mathcal{O}_\lambda} T_\lambda^{I_\ell} = 1 \tag{3.3.1}$$

for any arithmetic specialization λ of $R(\mathfrak{a})$. Since $V_\lambda^{I_\ell}$ is stable under G_ℓ, the G_ℓ-representation V_λ is reducible. So $\pi(\lambda)_\ell$ is not supercuspidal by proposition 3.1.1 (3). If the monodromy of the G_ℓ-representation $\mathcal{T}$ is zero, then the G_ℓ-representation T_λ has no monodromy and hence $\pi(\lambda)_\ell$ is principal series. By equation (3.3.1) and proposition 3.1.1(1), it is singly ramified principal series. Similarly, $\pi(\lambda')_\ell$ is also singly ramified principal series. On the other hand, if the monodromy of the G_ℓ-representation $\mathcal{T}$ is nonzero, then the G_ℓ-representation T_λ has nonzero monodromy by theorem 1.2.4(1) and hence $\pi(\lambda)_\ell$ is Steinberg. By equation (3.3.1) and proposition 3.1.1(2), it is unramified Steinberg. Similarly, $\pi(\lambda')_\ell$ is also unramified Steinberg.

Note that $\alpha \in \mathcal{K}$ is integral over $R(\mathfrak{a})$. Let $R(\mathfrak{a})[\alpha]$ denote the subring of $\mathcal{K}$ generated by α over $R(\mathfrak{a})$. We extend each arithmetic specialization $\lambda : R(\mathfrak{a}) \to \overline{\mathbb{Q}}_p$ to $R(\mathfrak{a})[\alpha]$ which we denote by λ by abuse of language. Notice that Fr_ℓ acts on $V_\lambda^{I_\ell}$ by $\lambda(\alpha)$ by theorem 1.2.4(6). First suppose that $\pi(f_\lambda)_\ell$ is Steinberg for any arithmetic specialization λ of $R(\mathfrak{a})$. By proposition 3.1.1, this eigenvalue is $\mu_\lambda(\ell)\ell^{k/2-1}$, and since f_λ is new at ℓ (see §3.2.3), this is equal to $a_\ell(f_\lambda)$ by [**Nek06**, 12.3.7, 12.3.8.2]. Since $a_\ell(f_\lambda) = \lambda(T_\ell)$, we get $\lambda(\alpha) = \lambda(T_\ell)$ for any arithmetic specialization λ of $R(\mathfrak{a})$. So $\alpha = T_\ell$ in $R(\mathfrak{a})[\alpha]$, *i.e.*, $\alpha \in R(\mathfrak{a})$. Now suppose that $\pi(\lambda)_\ell$ is principal series for any arithmetic specialization λ of $R(\mathfrak{a})$. Then by a similar argument as above it follows that $\alpha \in R(\mathfrak{a})$. □

3.3.2. Control theorems. Let S denote a finite set of rational primes including the primes dividing Np and the archimedean prime of $\mathbb{Q}$ and S_f denote its subset of finite places.

Recall that $\mathcal{V}$ is reducible as a G_p-representation. Define $\mathcal{T}(\mathfrak{a})^+$ to be the largest $R(\mathfrak{a})$-submodule of $\mathcal{T}(\mathfrak{a})$ on which G_p acts via the unramified character ε which takes Fr_p to T_p and put $\mathcal{T}(\mathfrak{a})^- := \mathcal{T}(\mathfrak{a})/\mathcal{T}(\mathfrak{a})^+$. For an arithmetic specialization λ of $R(\mathfrak{a})$ we define T_λ^+ to be the largest $\mathcal{O}_\lambda$-submodule of T_λ on which G_p acts via the unramified character taking Fr_p to $a_p(f_\lambda)$.

Let $\mathbb{Q}_\infty$ denote the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ which can be regarded as a union of sequence of fields

$$\mathbb{Q} = \mathbb{Q}_0 \subset \mathbb{Q}_1 \subset \cdots \subset \mathbb{Q}_\infty = \cup_n \mathbb{Q}_n \quad \text{with } \Gamma_n := \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \simeq \mathbb{Z}/p^n\mathbb{Z}.$$

We denote the Galois group $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ by Γ and let γ_0 denote a topological generator of Γ. Denote the Iwasawa algebra $\mathcal{O}[[\Gamma]]$ by Λ_{Iw}, which is a $G_{\mathbb{Q},\{p\}}$-module via the map $G_{\mathbb{Q},\{p\}} \twoheadrightarrow \Gamma \hookrightarrow \Lambda_{\mathrm{Iw}}^\times$ since $\mathbb{Q}_\infty$ is unramified at primes $\ell \neq p$. For any finite type $\mathcal{O}$-subalgebra A of $\overline{\mathbb{Z}}_p$, we will write Λ_A to denote $A \otimes_{\mathcal{O}} \Lambda_{\mathrm{Iw}} = A[[\Gamma]]$. We will consider Λ_A as a $G_{\mathbb{Q},\{p\}}$-module via the map $G_{\mathbb{Q},\{p\}} \twoheadrightarrow \Gamma \hookrightarrow \Lambda_A^\times$. The image of an element $g \in G_{\mathbb{Q},\{p\}}$ under this map will be denoted by $[g]$. The completed tensor product $R(\mathfrak{a})\widehat{\otimes}_{\mathcal{O}}\Lambda_{\mathrm{Iw}}$ will be denoted by $R(\mathfrak{a})_{\mathrm{Iw}}$.

Define the *cyclotomic deformation* $\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}$ of $\mathcal{T}(\mathfrak{a})$ as the $G_{\mathbb{Q},S}$-representation $\mathcal{T}(\mathfrak{a})\widehat{\otimes}_{\mathcal{O}}\Lambda_{\mathrm{Iw}}$ over $R(\mathfrak{a})_{\mathrm{Iw}}$ obtained by tensoring the $G_{\mathbb{Q},S}$-representations $\mathcal{T}(\mathfrak{a})$ and Λ_{Iw}. Define the G_p-representation

$$\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}^+ = \mathcal{T}(\mathfrak{a})^+\widehat{\otimes}_{\mathcal{O}}\Lambda_{\mathrm{Iw}}.$$

For an arithmetic specialization λ of $R(\mathfrak{a})$, define the *cyclotomic deformation* $T_{\lambda,\mathrm{Iw}}$ of T_λ as the $G_{\mathbb{Q},S}$-representation $T_\lambda \otimes_{\mathcal{O}} \Lambda_{\mathrm{Iw}}$ over $\mathcal{O}_\lambda \otimes_{\mathcal{O}} \Lambda_{\mathrm{Iw}} = \Lambda_{\mathcal{O}_\lambda}$. Define the G_p-representation

$$T_{\lambda,\mathrm{Iw}}^+ = T_\lambda^+ \otimes_{\mathcal{O}} \Lambda_{\mathrm{Iw}}.$$

Notice that each arithmetic specialization $\lambda : R(\mathfrak{a}) \to \mathcal{O}_\lambda$ of $R(\mathfrak{a})$ extends to a Λ_{Iw}-algebra homomorphism $\lambda\widehat{\otimes}_{\mathcal{O}}\mathrm{id}_{\Lambda_{\mathrm{Iw}}} : R(\mathfrak{a})_{\mathrm{Iw}} \to \mathcal{O}_\lambda \otimes_{\mathcal{O}} \Lambda_{\mathrm{Iw}} = \Lambda_{\mathcal{O}_\lambda}$, which will be denoted by λ by abuse of language.

Definition 3.3.3. *For a complete local noetherian domain R of residue characteristic $p > 0$, let $G_{\mathbb{Q},S}$ act continuously on $T = R^2$ via a representation $G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$. Suppose that G_ℓ/I_ℓ acts on $T^{I_\ell} \otimes_R \mathrm{Frac}(R)$ by an R-valued character χ_ℓ whenever $\mathrm{rk}_R T^{I_\ell} = 1$ for some $\ell \neq p$. For any prime $\ell \neq p$, let $U'_\ell(T)$ denote the object in the derived category of R-modules corresponding to*

$$\begin{cases} C^\bullet_{\mathrm{cont}}(G_\ell/I_\ell, T^{I_\ell}) & \text{if } \mathrm{rk}_R T^{I_\ell} \neq 1, \\ [R \xrightarrow{\mathrm{Fr}_\ell - 1} R] \text{ concentrated in degree 0,1} & \text{if } \mathrm{rk}_R T^{I_\ell} = 1 \end{cases}$$

where Fr_ℓ acts on R via the character χ_ℓ.

Definition 3.3.4. *Let λ denote an arithmetic specialization of $R(\mathfrak{a})$. Put*

$$U'_p(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) = R\Gamma_{\mathrm{cont}}(G_p, R(\mathfrak{a})_{\mathrm{Iw}})$$
$$U'_p(T_{\lambda,\mathrm{Iw}}) = R\Gamma_{\mathrm{cont}}(G_p, \Lambda_{\mathcal{O}_\lambda})$$

where G_{F_w} acts on $R(\mathfrak{a})_{\mathrm{Iw}}$ (resp. $\Lambda_{\mathcal{O}_\lambda}$) by the character through which it acts on $\mathcal{T}(\mathfrak{a})^+_{\mathrm{Iw}}$ (resp. $T^+_{\lambda,\mathrm{Iw}}$). For $T = \mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}, T_{\lambda,\mathrm{Iw}}$, define the algebraic p-adic L-functions *$L^{\mathrm{alg}}_{p,\mathrm{Kato}}(T)$, $L^{\mathrm{alg}}_{p',\mathrm{Gr}}(T)$, $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T)$ as the object of* $\mathrm{Parf}\text{-}\mathrm{is}_R$ *($R = R(\mathfrak{a})_{\mathrm{Iw}}, \Lambda_{\mathcal{O}_\lambda}$ respectively) given by*

$$L^{\mathrm{alg}}_{p,\mathrm{Kato}}(T) := \det{}_R(R\Gamma_{c,\mathrm{cont}}(G_{\mathbb{Q},S}, T)[1]) \otimes \det{}_R\left(\bigoplus_{\substack{\ell\in S_f\\ \ell\neq p}} U'_\ell(T)[1]\right),$$

$$L^{\mathrm{alg}}_{p',\mathrm{Gr}}(T) := \det{}_R(R\Gamma_{c,\mathrm{cont}}(G_{\mathbb{Q},S}, T)[1]) \otimes \det{}_R\left(\bigoplus_{\ell\in S_f} U'_\ell(T)[1]\right),$$

$$L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T) := \det{}_R(R\Gamma_{\mathrm{Gr}}(G_{\mathbb{Q},S}, T)[1]) \otimes \det{}_R\left(\bigoplus_{\substack{\ell\in S_f\\ \ell\neq p}} U'_\ell(T)[1]\right)$$

respectively. In the definition of $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T)$, we assume that $R\Gamma_{\mathrm{cont}}(G_p, T^+)$ is a perfect complex.

Before showing that the above objects are well-defined, we prove the lemma below.

Lemma 3.3.5. *For an arithmetic specialization λ of $R(\mathfrak{a})$, the inclusion $\mathcal{T}(\mathfrak{a})^+ \hookrightarrow \mathcal{T}(\mathfrak{a})$ induces an isomorphism between*

$$(\mathcal{T}(\mathfrak{a})^+)_\lambda := \mathcal{T}(\mathfrak{a})^+ \otimes_{\mathcal{O}} \mathcal{O}_\lambda$$

and T^+_λ under the assumption 3.3.1.

Proof. Note that G_p acts on $\mathcal{T}(\mathfrak{a})^+$ by ε and on $\mathcal{T}(\mathfrak{a})^-$ by $(\chi_\Gamma \circ \kappa)\psi_0\chi_{cycl}\varepsilon^{-1}$. Since $\overline{\rho}$ is p-distinguished, we have $\dim_k \mathcal{T}(\mathfrak{a})^-/\mathfrak{m} = 1$. Also $\dim_{\mathcal{K}} \mathcal{T}(\mathfrak{a})^- \otimes_R \mathcal{K} = 1$. Hence by Nakayama's lemma, $\mathcal{T}(\mathfrak{a})^-$ is free of rank 1, which implies $\mathcal{T}(\mathfrak{a})^+$ is also free of rank 1. Similarly it follows that T^-_λ is free of rank 1. Now consider the commutative diagram below with exact rows (the exactness of the first row follows from the freeness of $\mathcal{T}(\mathfrak{a})^-$ and the existence of the first vertical arrow follows since $a_p(f_\lambda)$ is equal to the image of T_p under the composite map $h^{\mathrm{ord}}_\infty \to R(\mathfrak{a}) \xrightarrow{\lambda} \overline{\mathbb{Q}}_p$).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{T}(\mathfrak{a})^+)_\lambda & \longrightarrow & \mathcal{T}(\mathfrak{a})_\lambda & \longrightarrow & (\mathcal{T}(\mathfrak{a})^-)_\lambda & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\wr & & \downarrow & & \\ 0 & \longrightarrow & T^+_\lambda & \longrightarrow & T_\lambda & \longrightarrow & T^-_\lambda & \longrightarrow & 0 \end{array}$$

Since the last vertical arrow is a surjection (by snake lemma) between free modules of rank 1 over the domain $\mathcal{O}_\lambda$, it is an isomorphism. So $(\mathcal{T}(\mathfrak{a})^+)_\lambda \xrightarrow{\sim} T^+_\lambda$. ☐

Lemma 3.3.6. *Let λ denote an arithmetic specialization of $R(\mathfrak{a})$. Then for $T = \mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}, T_{\lambda,\mathrm{Iw}}$, the modules $L^{\mathrm{alg}}_{p',\mathrm{Gr}}(T)$ and $L^{\mathrm{alg}}_{p,\mathrm{Kato}}(T)$ are well-defined. Moreover, when $\overline{\rho}$ satisfies assumption 3.3.1, $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T)$ is well-defined for $T = \mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}, T_{\lambda,\mathrm{Iw}}$.*

Proof. Note that the rings $R(\mathfrak{a})$ and $\mathcal{O}_\lambda$ are complete local rings (by [**Eis95**, Corollary 7.6, p. 188] for instance). So $R(\mathfrak{a})_{\mathrm{Iw}}$ and $\Lambda_{\mathcal{O}_\lambda}$ are complete local rings. By proposition 3.3.2, the group G_ℓ/I_ℓ acts on $\mathcal{T}(\mathfrak{a})^{I_\ell}_{\mathrm{Iw}}$ (resp. $T^{I_\ell}_{\lambda,\mathrm{Iw}}$) by an $R(\mathfrak{a})_{\mathrm{Iw}}$-valued (resp. $\Lambda_{\mathcal{O}_\lambda}$-valued) character if $\mathrm{rk}\mathcal{T}(\mathfrak{a})^{I_\ell} = \mathrm{rk}\mathcal{T}(\mathfrak{a})^{I_\ell}_{\mathrm{Iw}} = 1$ (resp. $\mathrm{rk}T^{I_\ell}_\lambda = \mathrm{rk}T^{I_\ell}_{\lambda,\mathrm{Iw}} = 1$). So $U'_\ell(T)$ is well-defined and by proposition 2.2.3, it is a perfect complex for $\ell \in S_f, \ell \neq p$. Then by proposition 2.2.1, $L_{p,\mathrm{Kato}}(T)$ is well-defined.

The action of G_p on $\mathcal{T}(\mathfrak{a})^+$ and T^+_λ are unramified and Fr_p acts on them by $T_p \in R(\mathfrak{a})$ and $a_p(f_\lambda) = \lambda(T_p) \in \mathcal{O}_\lambda$ respectively. So the group G_p acts on $\mathcal{T}(\mathfrak{a})^+_{\mathrm{Iw}}$ (resp. $T^+_{\lambda,\mathrm{Iw}}$) by an $R(\mathfrak{a})_{\mathrm{Iw}}$-valued (resp. $\Lambda_{\mathcal{O}_\lambda}$-valued) character. So $U'_p(T)$ is well-defined and hence $L^{\mathrm{alg}}_{p',\mathrm{Gr}}(T)$ is well-defined.

Under assumption 3.3.1, $R\Gamma_{\mathrm{cont}}(G_p, T^+)$ is perfect by proposition 2.2.1 as T^+ is free (by lemma 3.3.5). So $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T)$ is well-defined under this assumption. □

Now we prove that $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$, $L^{\mathrm{alg}}_{p',\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}), L^{\mathrm{alg}}_{p,\mathrm{Kato}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ satisfy perfect control theorems at arithmetic specializations.

Theorem 3.3.7. *Let λ be an arithmetic specialization of $R(\mathfrak{a})$. Then the isomorphisms in propositions 2.1.2, 2.2.1, 2.2.3 induce an isomorphism*

$$L^{\mathrm{alg}}_{p,\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\,\lambda} \Lambda_{\mathcal{O}_\lambda} \cong L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}}) \tag{3.3.2}$$

under the assumptions 3.2.4 and 3.3.1. They also induce isomorphisms

$$L^{\mathrm{alg}}_{p',\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\,\lambda} \Lambda_{\mathcal{O}_\lambda} \cong L^{\mathrm{alg}}_{p',\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}}), \tag{3.3.3}$$

$$L^{\mathrm{alg}}_{p,\mathrm{Kato}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\,\lambda} \Lambda_{\mathcal{O}_\lambda} \cong L^{\mathrm{alg}}_{p,\mathrm{Kato}}(T_{\lambda,\mathrm{Iw}}) \tag{3.3.4}$$

under the assumption 3.2.4.

Proof. By proposition 2.1.2 and proposition 2.2.1, it remains to prove the control theorem for the factors coming from "local conditions". Notice that lemma 3.3.5 gives the control of $U^+_p(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ and $U'_p(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$.

So it remains to prove the control theorem at $\ell \neq p$, *i.e.*, the λ-specialization of $\det U'_\ell(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is $\det U'_\ell(T_{\lambda,\mathrm{Iw}})$. By proposition 2.1.2, it suffices to prove the control theorem for $U'_\ell(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$. We need to do so only when $\mathrm{rk}_{R(\mathfrak{a})_{\mathrm{Iw}}}\mathcal{T}(\mathfrak{a})^{I_\ell}_{\mathrm{Iw}} = \mathrm{rk}_{R(\mathfrak{a})}\mathcal{T}(\mathfrak{a})^{I_\ell} = 1$ by proposition 2.2.1 and proposition 3.3.2. So assume that $\mathcal{T}(\mathfrak{a})^{I_\ell}$ is of rank one and let Fr_ℓ act on it by $\alpha \in R(\mathfrak{a})$ (by proposition 3.3.2). Since $U'_\ell(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is K-flat by [**Sta14**, Tag 064K], its derived tensor product over $R(\mathfrak{a})_{\mathrm{Iw}}$ with $\Lambda_{\mathcal{O}_\lambda}$ (through λ) is equal to the tensor product by [**Sta14**, Tag 06Y6], *i.e.*, $[\Lambda_{\mathcal{O}_\lambda} \xrightarrow{\lambda(\alpha)\widehat{\otimes}_{\mathcal{O}}[\mathrm{Fr}_\ell]-1} \Lambda_{\mathcal{O}_\lambda}]$ and this is $U'_\ell(T_{\lambda,\mathrm{Iw}})$ by proposition 3.3.2. □

Remark 3.3.8. In the first part of theorem 3.3.7, the assumption 3.3.1 is used only to deduce that $\mathcal{T}(\mathfrak{a})^+$ is free which is not true in general by [**Kil02**]. When $\mathcal{T}(\mathfrak{a})^+$ is not free, the algebraic p-adic L-function $L^{\mathrm{alg}}_{p',\mathrm{Gr}}$ defined using the local condition U'_p at p satisfies a control theorem as proved in theorem 3.3.7.

3.4. Relation with Greenberg's Selmer group

Let λ be an arithmetic specialization of $R(\mathfrak{a})$ such that $\mathcal{O}_\lambda$ is a DVR. Denote its associated ordinary form by f.

Lemma 3.4.1. *The inclusion map* $T_\lambda^{I_\ell} \hookrightarrow T_\lambda$ *tensored with* $\mathrm{id}_{\Lambda_{\mathrm{Iw}}}$ *over* $\mathcal{O}$ *induces an isomorphism*

$$T_\lambda^{I_\ell} \otimes_{\mathcal{O}} \Lambda_{\mathrm{Iw}} \xrightarrow{\sim} T_{\lambda,\mathrm{Iw}}^{I_\ell}.$$

Thus $T_{\lambda,\mathrm{Iw}}^{I_\ell}$ *is free over* $\Lambda_{\mathcal{O}_\lambda}$ *and* $R\Gamma_{\mathrm{cont}}(G_\ell/I_\ell, T_{\lambda,\mathrm{Iw}}^{I_\ell})$ *is a perfect complex over* $\Lambda_{\mathcal{O}_\lambda}$. *The module* $T_{\lambda,\mathrm{Iw}}^{+}$ *is also free over* $\Lambda_{\mathcal{O}_\lambda}$ *and* $R\Gamma_{\mathrm{cont}}(G_p, T_{\lambda,\mathrm{Iw}}^{+})$ *is a perfect complex over* $\Lambda_{\mathcal{O}_\lambda}$.

Proof. Since $\mathcal{O}_\lambda$ is a DVR, T_λ has a free set of generators over $\mathcal{O}_\lambda$ and for any such set $\{e_1, e_2\}$ of free generators, $\{e_1 \otimes 1_{\Lambda_{\mathrm{Iw}}}, e_2 \otimes 1_{\Lambda_{\mathrm{Iw}}}\}$ is a free set of generators for $T_{\lambda,\mathrm{Iw}}$ over Λ_{Iw}. Since Λ_{Iw} is unramified at $\ell \neq p$, the matrices of the I_ℓ action on T_λ and on $T_{\lambda,\mathrm{Iw}}$ are the same. Thus the first isomorphism follows. So $T_{\lambda,\mathrm{Iw}}^{I_\ell}$ is free over Λ_{Iw} and $R\Gamma_{\mathrm{cont}}(G_\ell/I_\ell, T_{\lambda,\mathrm{Iw}}^{I_\ell})$ is a perfect complex by proposition 2.2.3.

Since $\mathcal{O}_\lambda$ is a DVR, T_λ^{+} is a free $\mathcal{O}$-module and hence $T_{\lambda,\mathrm{Iw}}^{+}$ is free over Λ_{Iw}. The perfectness of $R\Gamma_{\mathrm{cont}}(G_p, T_{\lambda,\mathrm{Iw}}^{+})$ follows by proposition 2.2.1. □

Let I denote an injective hull of the residue field $\mathbb{F}$ of $\Lambda_{\mathcal{O}_\lambda}$ and D_M denote the Matlis duality functor $D_M(-) = \mathrm{Hom}_{\Lambda_{\mathcal{O}_\lambda}}(-, I)$. Since $\mathbb{F}$ is finite, by [**Nek06**, §2.9] we have the lemma below.

Lemma 3.4.2. *The Pontrjagin duality functor* $D_P(-) = \mathrm{Hom}_{\mathrm{cont}}(-, \mathbb{Q}_p/\mathbb{Z}_p)$ *and the Matlis duality functor* D_M *coincide on the category of* $\Lambda_{\mathcal{O}_\lambda}$*-modules.*

We put

$$A_{\lambda,\mathrm{Iw}} = D_M(T_{\lambda,\mathrm{Iw}})(1), \quad A_{\lambda,\mathrm{Iw}}^{+} = D_M(T_{\lambda,\mathrm{Iw}}^{-})(1), \quad A_{\lambda,\mathrm{Iw}}^{-} = A_{\lambda,\mathrm{Iw}}/A_{\lambda,\mathrm{Iw}}^{+}.$$

Greenberg [**Gre89, Gre91**] defined the strict Selmer group $\mathrm{Sel}_{A_{\lambda,\mathrm{Iw}}}^{\mathrm{str}}$ by the exact sequence

$$0 \to \mathrm{Sel}_{A_{\lambda,\mathrm{Iw}}}^{\mathrm{str}} \to H^1_{\mathrm{cont}}(G_{\mathbb{Q},S}, A_{\lambda,\mathrm{Iw}}) \to H^1_{\mathrm{cont}}(G_p, A_{\lambda,\mathrm{Iw}}^{-}) \oplus \bigoplus_{\ell \in S_f, \ell \neq p} H^1_{\mathrm{cont}}(I_\ell, A_{\lambda,\mathrm{Iw}})$$

By [**Nek06**, 8.9.6.1], we have the lemma below.

Lemma 3.4.3. *Matlis duality induces an isomorphism of complexes*

$$R\Gamma_f(T_{\lambda,\mathrm{Iw}}) \xrightarrow{\sim} D_M(R\Gamma_f(A_{\lambda,\mathrm{Iw}}))[-3],$$

which induces isomorphisms in cohomology

$$\widetilde{H}^i_f(T_{\lambda,\mathrm{Iw}}) \xrightarrow{\sim} D_M\left(\widetilde{H}^{3-i}_f(A_{\lambda,\mathrm{Iw}})\right). \tag{3.4.1}$$

The next lemma follows from [**Nek06**, Lemma 9.6.3].

Lemma 3.4.4. *The following sequence is exact.*

$$0 \to \widetilde{H}^0_f(A_{\lambda,\mathrm{Iw}}) \to H^0_{\mathrm{cont}}(G_{\mathbb{Q},S}, A_{\lambda,\mathrm{Iw}}) \to H^0_{\mathrm{cont}}(G_p, A_{\lambda,\mathrm{Iw}}^{-}) \to \widetilde{H}^1_f(A_{\lambda,\mathrm{Iw}}) \to \mathrm{Sel}_{A_{\lambda,\mathrm{Iw}}}^{\mathrm{str}} \to 0$$

Note that lemma 3.4.1 combined with proposition 2.2.1 and the fact that $T^{+}_{\lambda,\mathrm{Iw}}$ is free of rank one over $\Lambda_{\mathcal{O}_\lambda}$ shows that the algebraic p-adic L-function $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}})$ for the $G_{\mathbb{Q},S}$-representation $T_{\lambda,\mathrm{Iw}}$ is well-defined. The following theorem describes the determinant of the Selmer complex of $T_{\lambda,\mathrm{Iw}}$ and its relation with the algebraic p-adic L-function $L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}})$.

Theorem 3.4.5. *The Selmer complex $R\Gamma_f(T_{\lambda,\mathrm{Iw}})$ defined with respect to Greenberg's local condition 2.2.2 is a perfect complex of $\Lambda_{\mathcal{O}_\lambda}$-modules and the map $i_{\Lambda_{\mathcal{O}_\lambda}}(-,-,-)$ (as in equation* (2.1.4)*) induces an isomorphism*

$$L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}}) \cong \left(\det{}_{\Lambda_{\mathcal{O}_\lambda}} R\Gamma_f(T_{\lambda,\mathrm{Iw}})\right)^{-1}.$$

Suppose that the assumption 3.2.4 holds. Then $\widetilde{H}^1_f(T_{\lambda,\mathrm{Iw}})$ is a free $\Lambda_{\mathcal{O}_\lambda}$-module and

$$\widetilde{H}^i_f(T_{\lambda,\mathrm{Iw}}) = 0$$

for any integer $i<1$ and $i>2$. Suppose that p does not divide the level of f. Then $\widetilde{H}^2_f(T_{\lambda,\mathrm{Iw}})$ is a torsion $\Lambda_{\mathcal{O}_\lambda}$-module and $\widetilde{H}^1_f(T_{\lambda,\mathrm{Iw}})$ is zero. The surjective map

$$\widetilde{H}^1_f(A_{\lambda,\mathrm{Iw}}) \twoheadrightarrow \mathrm{Sel}^{\mathrm{str}}_{A_{\lambda,\mathrm{Iw}}}$$

as in Lemma 3.4.4 induces an injective map

$$D_P\left(\mathrm{Sel}^{\mathrm{str}}_{A_{\lambda,\mathrm{Iw}}}\right) \hookrightarrow \widetilde{H}^2_f(T_{\lambda,\mathrm{Iw}}) \tag{3.4.2}$$

with finite cokernel. Consequently we get a canonical isomorphism

$$L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}}) \cong (\mathrm{char}_{\Lambda_{\mathcal{O}_\lambda}} D_P(\mathrm{Sel}^{\mathrm{str}}_{A_{\lambda,\mathrm{Iw}}}), 0)$$

using equations (2.1.3)*,* (2.1.5) *and* (3.4.2)*.*

Proof. By lemma 3.4.1, proposition 2.2.1 and [**Sta14**, Tag 066R], it follows that $R\Gamma_f(T_{\lambda,\mathrm{Iw}})$ is a perfect complex of $\Lambda_{\mathcal{O}_\lambda}$-modules.

Since $\Lambda_{\mathcal{O}_\lambda}$ is reduced, by equation (2.1.4) we have an isomorphism

$$i_{\Lambda_{\mathcal{O}_\lambda}}(j,p,(\mathrm{res}_{S_f} - i^+_S(T_{\lambda,\mathrm{Iw}}))[1]) : L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}}) \xrightarrow{\sim} \det{}_{\Lambda_{\mathcal{O}_\lambda}}(R\Gamma_f(T_{\lambda,\mathrm{Iw}})[1]) = \left(\det{}_{\Lambda_{\mathcal{O}_\lambda}} R\Gamma_f(T_{\lambda,\mathrm{Iw}})\right)^{-1},$$

(this isomorphism depends on the choice of an isomorphism

$$T^{I_\ell}_{\lambda,\mathrm{Iw}} \xrightarrow{\sim} \Lambda_{\mathcal{O}_\lambda} \tag{3.4.3}$$

if $\mathrm{rk}_{\Lambda_{\mathcal{O}_\lambda}} T^{I_\ell}_{\lambda,\mathrm{Iw}} = 1$ for some $\ell \neq p$).

As assumption 3.2.4 holds, by [**FO12**, Proposition 2.25],

$$\widetilde{H}^i_f(T_{\lambda,\mathrm{Iw}}) = 0 \tag{3.4.4}$$

for $i<0$ and $i>2$.

Let x denote the element $\gamma_0 - 1 \in \Lambda_{\mathcal{O}_\lambda}$ and y denote an uniformizer of $\mathcal{O}_\lambda$. We now prove that $\widetilde{H}^1_f(T_{\lambda,\mathrm{Iw}})$ is free by first showing that it does not have any x-torsion and then showing that $\widetilde{H}^1_f(T_{\lambda,\mathrm{Iw}})/x$ does not have any y-torsion.

Since $T_\lambda^{I_\ell}, T_\lambda^+$ are free over $\mathcal{O}_\lambda$, they are flat over $\mathcal{O}$. So by [**Nek06**, Proposition 3.4.2], [**Wei94**, Ex 1.2.4], we have an exact sequence

$$0 \to C_f^\bullet(T_{\lambda,\mathrm{Iw}}) \xrightarrow{x} C_f^\bullet(T_{\lambda,\mathrm{Iw}}) \to C_f^\bullet(T_\lambda) \to 0 \tag{3.4.5}$$

obtained from the exact sequence $0 \to \Lambda_{\mathrm{Iw}} \xrightarrow{x} \Lambda_{\mathrm{Iw}} \to \mathcal{O} \to 0$. Hence $\widetilde{H}_f^0(T_\lambda)$ surjects to $\widetilde{H}_f^1(T_{\lambda,\mathrm{Iw}})[x]$. On the other hand, since T_λ is irreducible as a $G_{\mathbb{Q},S}$-representation, we find $\widetilde{H}_f^0(T_\lambda) = 0$. So $\widetilde{H}_f^1(T_{\lambda,\mathrm{Iw}})$ does not have any x-torsion.

Now we will show that $\widetilde{H}_f^1(T_\lambda)$ does not have any y-torsion where y denotes an uniformizer of $\mathcal{O}_\lambda$. By [**Nek06**, 6.1.3.2], we obtain an exact sequence of $\mathcal{O}_\lambda$-modules

$$0 \to H^0_{\mathrm{cont}}(G_p, T_\lambda^-) \to \widetilde{H}_f^1(T_\lambda) \to H^1_{\mathrm{cont}}(G_{\mathbb{Q},S}, T_\lambda),$$

which gives the exact sequence

$$0 \to H^0_{\mathrm{cont}}(G_p, T_\lambda^-)[y] \to \widetilde{H}_f^1(T_\lambda)[y] \to H^1_{\mathrm{cont}}(G_{\mathbb{Q},S}, T_\lambda)[y].$$

As $H^0_{\mathrm{cont}}(G_p, T_\lambda^-)[y]$ is zero, the map

$$\widetilde{H}_f^1(T_\lambda)[y] \to H^1_{\mathrm{cont}}(G_{\mathbb{Q},S}, T_\lambda)[y]$$

is injective. Since the assumption 3.2.4 holds, we have $H^0_{\mathrm{cont}}(G_{\mathbb{Q},S}, T_\lambda/y) = \{0\}$. Then the long exact sequence of cohomologies associated to the exact sequence

$$0 \to T_\lambda \xrightarrow{y} T_\lambda \to T_\lambda/y \to 0$$

gives

$$H^1_{\mathrm{cont}}(G_{\mathbb{Q},S}, T_\lambda)[y] = \{0\}.$$

So $\widetilde{H}_f^1(T_\lambda)[y] = \{0\}$.

From the exact sequence 3.4.5 above, we find that $\widetilde{H}_f^1(T_{\lambda,\mathrm{Iw}})/x$ injects into $\widetilde{H}_f^1(T_\lambda)$. So $\widetilde{H}_f^1(T_{\lambda,\mathrm{Iw}})/x$ is y-torsion free. We have also seen $\widetilde{H}_f^1(T_{\lambda,\mathrm{Iw}})$ does not have any x-torsion. Thus x, y is a regular sequence for the $\Lambda_{\mathcal{O}_\lambda}$-module $\widetilde{H}_f^1(T_{\lambda,\mathrm{Iw}})$. So $\mathrm{depth}_{\Lambda_{\mathcal{O}_\lambda}} \widetilde{H}_f^1(T_{\lambda,\mathrm{Iw}}) = 2$. Thus $\mathrm{pd}_{\Lambda_{\mathcal{O}_\lambda}} \widetilde{H}_f^1(T_{\lambda,\mathrm{Iw}}) = 0$ (by [**Mat89**, Theorem 19.1] and hence $\widetilde{H}_f^1(T_{\lambda,\mathrm{Iw}})$ is projective. So it is free over $\Lambda_{\mathcal{O}_\lambda}$ (by [**Mat80**, Proposition 3.G]).

Since $D_M(-)$ is an exact functor (by [**Nek06**, §2.3.1]), lemma 3.4.4 gives the exact sequence of $\Lambda_{\mathcal{O}_\lambda}$-modules below.

$$0 \to D_M(\mathrm{Sel}^{\mathrm{str}}_{A_{\lambda,\mathrm{Iw}}}) \to D_M(\widetilde{H}_f^1(A_{\lambda,\mathrm{Iw}})) \to D_M(H^0_{\mathrm{cont}}(G_p, A^-_{\lambda,\mathrm{Iw}})).$$

Using lemma 3.4.2 and 3.4.3, we obtain the exact sequence

$$0 \to D_P(\mathrm{Sel}^{\mathrm{str}}_{A_{\lambda,\mathrm{Iw}}}) \to \widetilde{H}_f^2(T_{\lambda,\mathrm{Iw}}) \to D_P(H^0_{\mathrm{cont}}(G_p, A^-_{\lambda,\mathrm{Iw}}))$$

of $\Lambda_{\mathcal{O}_\lambda}$-modules. Now since f is p-ordinary, $a_p(f)$ is a p-adic unit. Also the level of f is not divisible by p. So f is of good ordinary reduction. Hence by [**Kat04**, Theorem 17.4], the Pontrjagin dual of $\mathrm{Sel}^{\mathrm{str}}_{A_{\lambda,\mathrm{Iw}}}$ is a torsion $\Lambda_{\mathcal{O}_\lambda}$-module. Since p does not divide the level of f, $\pi(f)_p$ is principal series by [**Nek06**, Lemma 12.5.4]. So the Pontrjagin dual of

$H^0_{\mathrm{cont}}(G_p, A^-_{\lambda,\mathrm{Iw}})$ is finite. Thus by the above exact sequence, $\widetilde{H}^2_f(T_{\lambda,\mathrm{Iw}})$ is a torsion $\Lambda_{\mathcal{O}_\lambda}$-module and the injective map

$$D_P(\mathrm{Sel}^{\mathrm{str}}_{A_{\lambda,\mathrm{Iw}}}) \hookrightarrow \widetilde{H}^2_f(T_{\lambda,\mathrm{Iw}})$$

has finite cokernel.

Since $\widetilde{H}^2_f(T_{\lambda,\mathrm{Iw}})$ is a torsion $\Lambda_{\mathcal{O}_\lambda}$-module, by [**Nek06**, Theorem 7.8.6, §4.6.5.6], the $\Lambda_{\mathcal{O}_\lambda}$-module $\widetilde{H}^1_f(T_{\lambda,\mathrm{Iw}})$ has rank zero and hence zero (as it is free). So we have

$$\begin{aligned}
L^{\mathrm{alg}}_{p,\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}}) &\cong (\det_{\Lambda_{\mathcal{O}_\lambda}} R\Gamma_f(T_{\lambda,\mathrm{Iw}}))^{-1} && \text{(using theorem 3.4.5)}\\
&\cong \otimes_{n\in\mathbb{Z}}(\det_{\Lambda_{\mathcal{O}_\lambda}}(\widetilde{H}^n_f(T_{\lambda,\mathrm{Iw}})))^{(-1)^{n-1}} && \text{(by equation (2.1.3))}\\
&\cong (\det_{\Lambda_{\mathcal{O}_\lambda}}(\widetilde{H}^2_f(T_{\lambda,\mathrm{Iw}})))^{-1} && \text{(using theorem 3.4.5)}\\
&\cong (\mathrm{char}_{\Lambda_{\mathcal{O}_\lambda}} \widetilde{H}^2_f(T_{\lambda,\mathrm{Iw}}), 0) && \text{(from equation (2.1.5))}\\
&= (\mathrm{char}_{\Lambda_{\mathcal{O}_\lambda}} D_P(\mathrm{Sel}^{\mathrm{str}}_{A_{\lambda,\mathrm{Iw}}}), 0) && \text{(using equation (3.4.2)).}
\end{aligned}$$

In the above, the last equality follows as the map in equation (3.4.2) has finite cokernel and

$$\mathrm{length}_{(\Lambda_{\mathcal{O}_\lambda})_\mathfrak{p}} M_\mathfrak{p} = 0$$

for any height one prime $\mathfrak{p}$ of $\Lambda_{\mathcal{O}_\lambda}$ and a $\Lambda_{\mathcal{O}_\lambda}$-module M of finite cardinality. The first isomorphism above depends only on the choice of the isomorphisms in equation (3.4.3), the rest of the above isomorphisms are canonical. □

3.5. Cohomologies of $R\Gamma_{\mathrm{Gr}}(-)$, $R\Gamma_f(-)$ and $\mathscr{L}^{\mathrm{alg}}_p(\mathfrak{a})$

In this section we assume throughout that the assumptions 3.2.4, 3.3.1 hold. For a domain R, its integral closure in its fraction field is denoted by R^{int}. Until the end of this chapter, the symbol λ (resp. η) will be used to denote arithmetic specializations (resp. $\overline{\mathbb{Z}}_p$-specializations, *i.e.*, $\mathcal{O}$-algebra maps from $R(\mathfrak{a})$ to $\overline{\mathbb{Z}}_p$) of $R(\mathfrak{a})$. We define $\mathcal{O}_\eta, T_\eta$ in the same way $\mathcal{O}_\lambda, T_\lambda$ was defined. Put

$$T^+_\eta := \mathcal{T}(\mathfrak{a})^+ \otimes_{\mathcal{O}} \mathcal{O}_\eta, \quad T^-_\eta := \mathcal{T}(\mathfrak{a})^- \otimes_{\mathcal{O}} \mathcal{O}_\eta,$$

(*cf.* lemma 3.3.5). We define $T_{\eta,\mathrm{Iw}}, T^+_{\eta,\mathrm{Iw}}$ in the same way $T_{\lambda,\mathrm{Iw}}, T^+_{\lambda,\mathrm{Iw}}$ was defined. Put

$$\begin{aligned}
T_{\eta^{\mathrm{int}}} &= T_\eta \otimes_{\mathcal{O}_\eta} \mathcal{O}^{\mathrm{int}}_\eta,\\
T^+_{\eta^{\mathrm{int}}} &= T^+_\eta \otimes_{\mathcal{O}_\eta} \mathcal{O}^{\mathrm{int}}_\eta,\\
T_{\eta^{\mathrm{int}},\mathrm{Iw}} &= T_{\eta^{\mathrm{int}}} \widehat{\otimes}_{\mathcal{O}} \Lambda_{\mathrm{Iw}},\\
T^+_{\eta^{\mathrm{int}},\mathrm{Iw}} &= T^+_{\eta^{\mathrm{int}}} \widehat{\otimes}_{\mathcal{O}} \Lambda_{\mathrm{Iw}}.
\end{aligned}$$

Note that η extends to an $\mathcal{O}_K$-algebra homomorphism $R(\mathfrak{a})^{\mathrm{int}} \to \overline{\mathbb{Z}}_p$, which we denote by η again by abuse of notation. Denote a uniformizer of $\mathcal{O}^{\mathrm{int}}_\eta$ by ϖ_{int} and let k_η denote the residue field $\mathcal{O}^{\mathrm{int}}_\eta/\varpi_\eta$.

3.5.1. Some preliminary results. We begin with a general fact about group representations.

Proposition 3.5.1. *Let A be a ring, $\mathfrak{m}$ be a maximal ideal of A, G be a group and M be an $A[G]$-module such that $M/\mathfrak{m}M$ is a semi-simple $A[G]$-module. Then $M^G \neq 0$ only if M^G is contained in $\mathfrak{m}^n M$ for all $n \geq 0$ or the trivial representation is a sub-object of $M/\mathfrak{m}M$.*

Proof. Denote by k the residue field $A/\mathfrak{m}$. If M^G is contained in $\mathfrak{m}^n M$ for all $n \geq 0$, then there is nothing to prove. Suppose that this not true. So there is an element $x \in M^G$ and an integer $n \geq 0$ such that x belongs to $\mathfrak{m}^n M$, but not to $\mathfrak{m}^{n+1}M$. The k-vector space $\mathfrak{m}^s/\mathfrak{m}^{s+1} \otimes_A M$ is, as $k[G]$-module, a direct sum of copies of the $k[G]$-module $M/\mathfrak{m}$ and thus semi-simple. Hence, $\mathfrak{m}^s M/\mathfrak{m}^{s+1}M$ is the quotient of a semi-simple $k[G]$-module and so semisimple as well. Let $\overline{x}$ be the (nonzero) image of x in $\mathfrak{m}^s M/\mathfrak{m}^{s+1}M$. The $k[G]$-module $\mathfrak{m}^s M/\mathfrak{m}^{s+1}M$ admits the nonzero submodule $k \cdot \overline{x}$ as a sub-$k[G]$-module and so admits the trivial representation as a submodule. The trivial module occurs in a quotient of a semi-simple $k[G]$-module N only if it occurs in N. So the trivial $k[G]$-module occurs in $\mathfrak{m}^s/\mathfrak{m}^{s+1} \otimes_A M$ and thus in $M/\mathfrak{m}$. □

Lemma 3.5.2. *Let $\ell \neq p$ be a rational prime. Then for almost all η,*

$$\mathrm{rk}\mathcal{T}(\mathfrak{a})^{I_\ell} = \mathrm{rk}T_\eta^{I_\ell}.$$

Suppose that $\mathrm{rk}\mathcal{T}(\mathfrak{a})^{I_\ell}$ is one. Then Fr_ℓ acts on $\mathcal{T}(\mathfrak{a})^{I_\ell}$ by an element α_ℓ of $R(\mathfrak{a})$. If the above equality holds for an η, then Fr_ℓ acts on $T_\eta^{I_\ell}$ by $\eta(\alpha_\ell)$.

Proof. By proposition 1.2.5, for any η,

$$\mathrm{rk}\mathcal{T}(\mathfrak{a})^{I_\ell} \leq \mathrm{rk}T_\eta^{I_\ell}.$$

By theorem 1.2.3, this is an equality for almost all η. Now suppose that $\mathrm{rk}\mathcal{T}(\mathfrak{a})^{I_\ell}$ is one. Then α_ℓ is an element of $R(\mathfrak{a})$ by proposition 3.3.2. The rest follows from theorem 1.2.3. □

For each arithmetic prime $\mathfrak{p}$ of $R(\mathfrak{a})$, we fix an arithmetic specialization $\lambda_\mathfrak{p}$ of $R(\mathfrak{a})$ with $\mathfrak{p}$ as its kernel.

Lemma 3.5.3. *Replacing K (as in the beginning of §3.2.1) by a finite extension (if necessary), we may assume that the set of arithmetic primes $\mathfrak{p}$ of $R(\mathfrak{a})$ satisfying the conditions below is dense in $\mathrm{Spec}(R(\mathfrak{a}))$.*

(1) the ordinary form associated with $\lambda_\mathfrak{p}$ has level not divisible by p,

(2) $\mathcal{O}_{\lambda_\mathfrak{p}}$ is a DVR.

Proof. Let $\mathrm{Spec}_0^{\mathrm{arith}}(R(\mathfrak{a}))$ denote the set of arithmetic primes of $R(\mathfrak{a})$ which contain $(\gamma - (1+p)^k)$ for some $k \geq 3$ and $k \equiv 2 \bmod p-1$. Note that $\mathrm{Spec}_0^{\mathrm{arith}}(R(\mathfrak{a}))$ is dense in $\mathrm{Spec}(R(\mathfrak{a}))$ and the ordinary forms associated with the elements of $\mathrm{Spec}_0^{\mathrm{arith}}(R(\mathfrak{a}))$ are of level N by §3.2.3.

Recall that $\mathcal{O}$ denotes the ring of integers of K. Then extending K if necessary, it follows that the elements of $\mathrm{Spec}_0^{\mathrm{arith}}(R(\mathfrak{a}))$ that are kernels of $\mathcal{O}$-valued arithmetic specializations of $R(\mathfrak{a})$ form a dense subset

$$D = \{\ker g \cap R(\mathfrak{a}) \mid g \in \mathrm{Hom}_{\mathcal{O}\text{-alg}}(R(\mathfrak{a})^{\mathrm{int}}, \mathcal{O})\} \cap \mathrm{Spec}_0^{\mathrm{arith}}(R(\mathfrak{a}))$$

of $\mathrm{Spec}(R(\mathfrak{a}))$ (the proof is same as the proof of [**Hid88b**, (3.1b) p. 26]). □

Henceforth we assume that K is so chosen that the arithmetic primes $\mathfrak{p}$ of $R(\mathfrak{a})$ satisfying the conditions of the above lemma form a dense subset of $\mathrm{Spec}(R(\mathfrak{a}))$.

Let $\mathcal{O}'$ denote a finite type $\mathbb{Z}_p$-subalgebra of $\overline{\mathbb{Z}}_p$. Let T be a free $\mathcal{O}'$-module of rank two with a continuous action of $G_{\mathbb{Q},S}$. Put

$$T_{\mathrm{Iw}} = T\widehat{\otimes}_{\mathcal{O}}\Lambda_{\mathrm{Iw}}.$$

Let T^+ be an $\mathcal{O}'$-submodule of T of rank one which is a direct summand of T and is stable under the action of G_p and this action is unramified.

Lemma 3.5.4. *Suppose that $\mathcal{O}'$ is a DVR and the residual representation attached to the $G_{\mathbb{Q},S}$-representation T is irreducible. Then $\widetilde{H}^1_f(T_{\mathrm{Iw}})$ is a free $\Lambda_{\mathcal{O}'}$-module.*

Proof. Since T is residually irreducible, $T^{G_{\mathbb{Q},S}}$ is zero by proposition 3.5.1. So the proof of the freeness of $\widetilde{H}^1_f(T_{\lambda,\mathrm{Iw}})$ over $\Lambda_{\mathcal{O}_\lambda}$ (as in theorem 3.4.5) with $\mathcal{O}_\lambda$, $T_{\lambda,\mathrm{Iw}}$ replaced by $\mathcal{O}'$, T_{Iw} respectively proves this lemma.

□

Lemma 3.5.5. *Suppose that $\mathcal{O}'$ is a DVR and $\widetilde{H}^1_f(T_{\mathrm{Iw}})$ is zero. Then $T_{\mathrm{Iw}}^{G_\ell}$ is zero for any $\ell \neq p$ and*

$$\mathrm{char}_{\Lambda_{\mathcal{O}'}} H^2_{\mathrm{Gr}}(T_{\mathrm{Iw}}) = \Big(\prod_{\substack{\ell\in S_f,\ell\neq p,\\ \mathrm{rk}T^{I_\ell}\geq 1}} \mathrm{Det}\Big((\mathrm{Fr}_\ell - \mathrm{id})|_{T_{\mathrm{Iw}}^{I_\ell}}\Big)\Big)\mathrm{char}_{\Lambda_{\mathcal{O}'}}\widetilde{H}^2_f(T_{\mathrm{Iw}})$$

where $\mathrm{Det}(-)$ *denotes the determinant of a linear operator on a free module.*

Proof. Since for any $\ell \neq p$, the image of ℓ in $1+p\mathbb{Z}_p$ under the projection map $\mathbb{Z}_p^\times \to 1+p\mathbb{Z}_p$ is non-trivial, the group $T_{\mathrm{Iw}}^{G_\ell}$ vanishes for any $\ell \neq p$. The exact sequence

$$0 \to C^\bullet_{\mathrm{Gr}}(T_{\mathrm{Iw}}) \to C^\bullet_f(T_{\mathrm{Iw}}) \to \bigoplus_{\ell\in S_f,\ell\neq p} U^+_\ell(T_{\mathrm{Iw}}) \to 0$$

of complexes of $\Lambda_{\mathcal{O}'}$-modules gives the short exact sequence

$$0 \to \bigoplus_{\ell\in S_f,\,\ell\neq p,\,\mathrm{rk}T_{\mathrm{Iw}}^{I_\ell}\geq 1} T_{\mathrm{Iw}}^{I_\ell}/(\mathrm{Fr}_\ell - \mathrm{id}) \to H^2_{\mathrm{Gr}}(T_{\mathrm{Iw}}) \to \widetilde{H}^2_f(T_{\mathrm{Iw}}) \to 0$$

(by proposition 2.2.3). So the sequence

$$0 \to \bigoplus_{\ell\in S_f,\,\ell\neq p,\,\mathrm{rk}T^{I_\ell}\geq 1} T_{\mathrm{Iw}}^{I_\ell}/(\mathrm{Fr}_\ell - \mathrm{id}) \to H^2_{\mathrm{Gr}}(T_{\mathrm{Iw}}) \to \widetilde{H}^2_f(T_{\mathrm{Iw}}) \to 0$$

is exact. Since $T_{\mathrm{Iw}}^{G_\ell} = 0$ for $\ell \neq p$, the second term in the above sequence is torsion. Since $\widetilde{H}^1_f(T_{\mathrm{Iw}})$ is zero, by [**Nek06**, Theorem 7.8.6, §4.6.5.6], $\widetilde{H}^2_f(T_{\mathrm{Iw}})$ is torsion. So $H^2_{\mathrm{Gr}}(T_{\mathrm{Iw}})$ is torsion. Hence the lemma follows. □

3.5.2. $R\Gamma_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ **and** $\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a})$**.** For each arithmetic specialization λ of $R(\mathfrak{a})$, its kernel will be denoted by $\mathfrak{p}_\lambda$. Pick $x_\lambda \in R(\mathfrak{a})$ such that it generates the maximal ideal of $R(\mathfrak{a})_{\mathfrak{p}_\lambda}$. The kernel of the map $\lambda = \lambda\widehat{\otimes}_{\mathcal{O}}\mathrm{id}_{\Lambda_{\mathrm{Iw}}} : R(\mathfrak{a})_{\mathrm{Iw}} \to \Lambda_{\mathcal{O}_\lambda}$ will be denoted by $\mathfrak{q}_\lambda$. We put

$$V'_\lambda = T_\lambda \otimes_{\mathcal{O}_\lambda} \mathrm{Frac}(\mathcal{O}_\lambda),$$
$$(V'_\lambda)^+ = T_\lambda^+ \otimes_{\mathcal{O}_\lambda} \mathrm{Frac}(\mathcal{O}_\lambda).$$

Proposition 3.5.6. *The $R(\mathfrak{a})_{\mathrm{Iw}}$-modules $\widetilde{H}^1_f(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$, $\widetilde{H}^2_f(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$, $H^2_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ are torsion,*

$$\widetilde{H}^i_f(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) = 0$$

for any integer $i < 1$ and $i > 2$ and

$$H^i_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) = 0$$

for any integer $i \neq 2$.

Proof. By [**FO12**, Proposition 2.25],

$$\widetilde{H}^i_f(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) = 0$$

for any integer $i < 1$ and $i > 2$.

Let λ be such that the conditions of lemma 3.5.3 are satisfied. By theorem 1.2.4(5)

$$0 \to \mathcal{T}(\mathfrak{a})^{I_\ell}_{\mathfrak{p}_\lambda} \xrightarrow{x_\lambda} \mathcal{T}(\mathfrak{a})^{I_\ell}_{\mathfrak{p}_\lambda} \xrightarrow{\lambda} (V'_\lambda)^{I_\ell} \to 0$$

is an exact sequence. The sequence

$$0 \to \mathcal{T}(\mathfrak{a})^+_{\mathfrak{p}_\lambda} \xrightarrow{x_\lambda} \mathcal{T}(\mathfrak{a})^+_{\mathfrak{p}_\lambda} \xrightarrow{\lambda} (V'_\lambda)^+ \to 0$$

is also exact by lemma 3.3.5.

Since

$$0 \to C^\bullet_f((\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})_{\mathfrak{q}_\lambda}) \xrightarrow{x_\lambda\widehat{\otimes}1} C^\bullet_f((\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})_{\mathfrak{q}_\lambda}) \xrightarrow{\lambda} C^\bullet_f((V'_\lambda)_{\mathrm{Iw}}) \to 0$$

is an exact sequence of complexes, we get an injection

$$\widetilde{H}^1_f((\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})_{\mathfrak{q}_\lambda})/x_\lambda\widehat{\otimes}1 \hookrightarrow \widetilde{H}^1_f((V'_\lambda)_{\mathrm{Iw}}).$$

So by theorem 3.4.5

$$\widetilde{H}^1_f(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})_{\mathfrak{q}_\lambda}/x_\lambda\widehat{\otimes}1 = 0.$$

By Nakayama's lemma,

$$\widetilde{H}^1_f(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})_{\mathfrak{q}_\lambda} = 0$$

and hence $\widetilde{H}^1_f(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is a torsion $R(\mathfrak{a})_{\mathrm{Iw}}$-module. By [**Nek06**, Theorem 7.8.6, §4.6.5.6], $\widetilde{H}^2_f(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is also a torsion $R(\mathfrak{a})_{\mathrm{Iw}}$-module. This completes the proof of the statements about the cohomology of $R\Gamma_f(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$.

We have an exact sequence of complexes of $R(\mathfrak{a})_{\mathrm{Iw}}$-modules

$$0 \to C^\bullet_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \to C^\bullet_f(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \to \bigoplus_{\ell \in S_f, \ell \neq p} U^+_\ell(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \to 0$$

(with maps induced by inclusion and projection). This shows

$$H^i_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) = 0$$

for any integer $i \leq 0$ and $i \geq 3$. Also there is an injection

$$H^1_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \hookrightarrow \widetilde{H}^1_f(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$$

and hence $H^1_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is a torsion $R(\mathfrak{a})_{\mathrm{Iw}}$-module. By [**Nek06**, Theorem 7.8.6, §4.6.5.6], $H^2_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is also a torsion $R(\mathfrak{a})_{\mathrm{Iw}}$-module. Now it remains to show that $H^1_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is torsion free. Let x be an element of $R(\mathfrak{a})_{\mathrm{Iw}}$. Define

$$(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}/x)^+ = \mathcal{T}(\mathfrak{a})^+_{\mathrm{Iw}}/x.$$

We have an exact sequence of complexes

$$0 \to C^\bullet_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \xrightarrow{x} C^\bullet_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \to C^\bullet_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}/x) \to 0,$$

which gives a surjective map

$$H^0_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}/x) \twoheadrightarrow H^1_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})[x].$$

Since

$$0 \to H^0_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}/x) \to H^0_{\mathrm{cont}}(G_{\mathbb{Q},S}, \mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}/x) \oplus H^0_{\mathrm{cont}}(G_p, (\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}/x)^+) \to \bigoplus_{\ell \in S_f} H^0_{\mathrm{cont}}(G_\ell, \mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}/x)$$

is an exact sequence and

$$H^0_{\mathrm{cont}}(G_{\mathbb{Q},S}, \mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}/x) = 0$$

(by proposition 3.5.1), we get

$$H^0_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}/x) = 0.$$

So $H^1_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is torsion free. □

Proposition 3.5.7. *There exist non-negative integers $n_m, n_{m+1}, \cdots, n_1, n_2$ and matrices d^i in $M_{n_i \times n_{i-1}}(R(\mathfrak{a})_{\mathrm{Iw}})$, $i = m, m+1, \cdots, 0, 1$ such that there is an isomorphism*

$$R\Gamma_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \simeq [R(\mathfrak{a})^{n_m}_{\mathrm{Iw}} \xrightarrow{d^m} R(\mathfrak{a})^{n_{m+1}}_{\mathrm{Iw}} \xrightarrow{d^{m+1}} \cdots \xrightarrow{d^0} R(\mathfrak{a})^{n_1}_{\mathrm{Iw}} \xrightarrow{d^1} R(\mathfrak{a})^{n_2}_{\mathrm{Iw}}]$$

in the category $\mathrm{Parf}_{R(\mathfrak{a})_{\mathrm{Iw}}}$ (the term $R(\mathfrak{a})^{n_j}_{\mathrm{Iw}}$ is concentrated in degree j). The $R(\mathfrak{a})_{\mathrm{Iw}}$-module $H^2_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is perfect.

Let η be arbitrary. The isomorphism in proposition 2.2.1 together with the above isomorphism induces an isomorphism

$$R\Gamma_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}) \simeq [\Lambda^{n_m}_{\mathcal{O}_\eta} \xrightarrow{\eta(d^m)} \Lambda^{n_{m+1}}_{\mathcal{O}_\eta} \xrightarrow{\eta(d^{m+1})} \cdots \xrightarrow{\eta(d^0)} \Lambda^{n_1}_{\mathcal{O}_\eta} \xrightarrow{\eta(d^1)} \Lambda^{n_2}_{\mathcal{O}_\eta}]$$

in the category $\mathrm{Parf}_{\Lambda_{\mathcal{O}_\eta}}$ (the term $\Lambda^{n_j}_{\mathcal{O}_\eta}$ is concentrated in degree j). The composite map

$$H^2_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\eta} \Lambda_{\mathcal{O}_\eta} \xrightarrow{\sim} \left(R(\mathfrak{a})^{n_2}_{\mathrm{Iw}}/\mathrm{Im}(d^1)\right) \otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\eta} \Lambda_{\mathcal{O}_\eta} \simeq \Lambda^{n_2}_{\mathcal{O}_\eta}/\mathrm{Im}(\eta(d^1)) \xleftarrow{\sim} H^2_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$$

is an isomorphism. Moreover the inclusion map $\mathcal{O}_\eta \to \mathcal{O}^{\mathrm{int}}_\eta$ induces an isomorphism

$$H^2_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}) \otimes_{\Lambda_{\mathcal{O}_\eta}} \Lambda_{\mathcal{O}^{\mathrm{int}}_\eta} \simeq H^2_{\mathrm{Gr}}(T_{\eta^{\mathrm{int}},\mathrm{Iw}}).$$

Proof. Since $\mathcal{T}(\mathfrak{a})^+_{\mathrm{Iw}}$ is a free $R(\mathfrak{a})_{\mathrm{Iw}}$-module, the complex $R\Gamma_{\mathrm{cont}}(G_p, \mathcal{T}(\mathfrak{a})^+_{\mathrm{Iw}})$ is perfect by proposition 2.2.1. So $R\Gamma_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is a perfect complex of $R(\mathfrak{a})_{\mathrm{Iw}}$-modules by [**Sta14**, Tag 066R]. Hence it has perfect amplitude contained in an interval $[m, m']$, *i.e.*, it is isomorphic to a bounded complex $P^\bullet$ of projective $R(\mathfrak{a})_{\mathrm{Iw}}$-modules of finite type (hence free of finite rank by [**Mat80**, Proposition 3.G] as $R(\mathfrak{a})_{\mathrm{Iw}}$ is local) with $P^i = 0$ for every $i < m$ and $i > m'$. If $m' \leq 2$, then automatically $R\Gamma_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ has perfect amplitude contained in

$[m,2]$. When $m'>2$, by [**Nek06**, §4.2.8], $R\Gamma_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ has perfect amplitude contained in $[m,2]$ as

$$H^i_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})=0$$

for all $i\geq 3$. So the first isomorphism follows. Then proposition 2.2.1 gives

$$R\Gamma_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})\simeq R\Gamma_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})\overset{L}{\otimes}_{R(\mathfrak{a})_{\mathrm{Iw}},\eta}\Lambda_{\mathcal{O}_\eta}.$$

So

$$R\Gamma_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})\simeq [R(\mathfrak{a})^{n_m}_{\mathrm{Iw}}\xrightarrow{d^m}R(\mathfrak{a})^{n_{m+1}}_{\mathrm{Iw}}\xrightarrow{d^{m+1}}\cdots\xrightarrow{d^0}R(\mathfrak{a})^{n_1}_{\mathrm{Iw}}\xrightarrow{d^1}R(\mathfrak{a})^{n_2}_{\mathrm{Iw}}]\overset{L}{\otimes}_{R(\mathfrak{a})_{\mathrm{Iw}},\eta}\Lambda_{\mathcal{O}_\eta}.$$

As the complex $[R(\mathfrak{a})^{n_m}_{\mathrm{Iw}}\xrightarrow{d^m}R(\mathfrak{a})^{n_{m+1}}_{\mathrm{Iw}}\xrightarrow{d^{m+1}}\cdots\xrightarrow{d^0}R(\mathfrak{a})^{n_1}_{\mathrm{Iw}}\xrightarrow{d^1}R(\mathfrak{a})^{n_2}_{\mathrm{Iw}}]$ is K-flat (by [**Sta14**, Tag 064K]), its derived tensor product with $\Lambda_{\mathcal{O}_\eta}$ is equal to the tensor product by [**Sta14**, Tag 06Y6]. Thus we get the second isomorphism. The third isomorphism follows from the first two. Since

$$R\Gamma_{\mathrm{Gr}}(T_{\eta^{\mathrm{int}},\mathrm{Iw}})\simeq R\Gamma_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})\overset{L}{\otimes}_{\mathcal{O}_\eta}\mathcal{O}^{\mathrm{int}}_\eta$$

(by proposition 2.2.1), the second isomorphism gives the final isomorphism. □

Remark 3.5.8. From the above proposition, it is not clear if $H^1_{\mathrm{Gr}}(T_{\lambda,\mathrm{Iw}})$ is zero (at least for some λ) because taking cohomology does not commute with taking derived (or usual) tensor product in general. For example, the complex

$$C^\bullet=\mathbb{Z}_p[[X]]\xrightarrow{\begin{pmatrix}p\\-X\end{pmatrix}}\mathbb{Z}_p[[X]]^2\xrightarrow{\begin{pmatrix}-pX & -p^2\\ X^2 & pX\end{pmatrix}}\mathbb{Z}_p[[X]]^2$$

is exact at the middle term (*cf.*[**FO12**, Remark 2.17]). But for each integer $k\geq 2$,

$$C^\bullet\overset{L}{\otimes}_{\mathbb{Z}_p[[X]]}\mathbb{Z}_p[[X]]/(X+1-(1+p)^k)=C^\bullet\otimes_{\mathbb{Z}_p[[X]]}\mathbb{Z}_p[[X]]/(X+1-(1+p)^k)$$

is not exact at the middle term. However applying the Euler-Poincare characteristic formula ([**Nek06**, Theorem 7.8.6, §4.6.5.6]) twice and using the above proposition, we deduce in theorem 3.5.10 that $H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is zero for almost all η. Under Greenberg's conjecture (which is equivalent to conjecture 3.5.21 by [**EPW06**, Theorem 1]), $H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is zero for any η (by lemma 3.5.14 and theorem 3.5.22).

By proposition 3.5.6 and 3.5.7, $H^2_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is torsion and perfect over $R(\mathfrak{a})_{\mathrm{Iw}}$. So $\det_{R(\mathfrak{a})_{\mathrm{Iw}}}H^2_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is well-defined. Its image in $\mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})$ (considered without the grade) under the composite map

$$\begin{aligned}\det_{R(\mathfrak{a})_{\mathrm{Iw}}}H^2_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})&\hookrightarrow\big(\det_{R(\mathfrak{a})_{\mathrm{Iw}}}H^2_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})\big)\otimes_{R(\mathfrak{a})_{\mathrm{Iw}}}\mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})\\&\cong\det_{\mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})}\big(H^2_{\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})\otimes_{R(\mathfrak{a})_{\mathrm{Iw}}}\mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})\big)\quad\text{(by proposition 2.1.2)}\\&=\det_{\mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})}(0)\\&=(\mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}}),0)\end{aligned}$$

is an invertible ideal of $\mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})$. Since $R(\mathfrak{a})_{\mathrm{Iw}}$ is local, this image is free (by [**Mat80**, Proposition 3.G]) and hence equal to $(\beta/\alpha)R(\mathfrak{a})_{\mathrm{Iw}}$ for some nonzero elements α,β in $R(\mathfrak{a})_{\mathrm{Iw}}$.

Note that $\alpha/\beta \in R(\mathfrak{a})^{\mathrm{int}}_{\mathrm{Iw}} = (R(\mathfrak{a})_{\mathrm{Iw}})^{\mathrm{int}}$ (this equality holds as $R(\mathfrak{a})^{\mathrm{int}}$ is finitely generated as an $R(\mathfrak{a})$-module by [**Ser00**, Proposition 11, Chapter III]). Put

$$\beta_{\mathrm{Eul}} = \prod_{\substack{\ell \in S_f, \ell \neq p, \\ \mathrm{rk}\mathcal{T}(\mathfrak{a})^{I_\ell} \geq 1}} \mathrm{Det}\left((\mathrm{Fr}_\ell - \mathrm{id})|_{\mathcal{T}(\mathfrak{a})^{I_\ell}_{\mathrm{Iw}} \otimes \mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}})} \right) \in R(\mathfrak{a})_{\mathrm{Iw}} \setminus \{0\}$$

where $\mathrm{Det}(-)$ denotes the determinant of a linear operator on a free module.

Definition 3.5.9. *The* two-variable algebraic p-adic L-function of $\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}$ *is defined to be*

$$\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a}) = \frac{\alpha}{\beta\beta_{\mathrm{Eul}}} \in \mathrm{Frac}(R(\mathfrak{a})_{\mathrm{Iw}}).$$

3.5.3. $R\Gamma_f(T_{\eta,\mathrm{Iw}}), R\Gamma_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$**.**

Theorem 3.5.10. *For any η,*

$$T_{\eta,\mathrm{Iw}}^{G_\ell} = 0$$

for any $\ell \neq p$,

$$\mathrm{rk}\widetilde{H}^1_f(T_{\eta,\mathrm{Iw}}) = \mathrm{rk}\widetilde{H}^2_f(T_{\eta,\mathrm{Iw}}) = \mathrm{rk}H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}) = \mathrm{rk}H^2_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}), \tag{3.5.1}$$

$$\widetilde{H}^i_f(T_{\eta,\mathrm{Iw}}) = H^i_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}) = 0$$

for any integer $i < 1$ and $i > 2$. The $\Lambda_{\mathcal{O}_\eta}$-module $H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is torsion free, the $\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}}$-module $H^1_{\mathrm{Gr}}(T_{\eta^{\mathrm{int}},\mathrm{Iw}})$ is torsion free and

$$\widetilde{H}^1_f(T_{\eta,\mathrm{Iw}}) = 0 \Longrightarrow H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}) = 0 \Longleftrightarrow H^1_{\mathrm{Gr}}(T_{\eta^{\mathrm{int}},\mathrm{Iw}}) = 0 \Longleftrightarrow \widetilde{H}^1_f(T_{\eta^{\mathrm{int}},\mathrm{Iw}}) = 0. \tag{3.5.2}$$

If the group $H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is zero, then $H^2_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is perfect. For almost all η, the group $H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is zero.

Proof. The first equality follows from lemma 3.5.5. Note that

$$0 \to C^\bullet_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}) \to C^\bullet_f(T_{\eta,\mathrm{Iw}}) \to \bigoplus_{\ell \in S_f, \ell \neq p} U^+_\ell(T_{\eta,\mathrm{Iw}}) \to 0 \tag{3.5.3}$$

is an exact sequence of complexes of $\Lambda_{\mathcal{O}_\eta}$-modules. By [**FO12**, Proposition 2.25],

$$\widetilde{H}^i_f(T_{\eta,\mathrm{Iw}}) = 0$$

for any integer $i < 1$ and $i > 2$. So for any such integer i, $H^i_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is also zero. Then equation (3.5.3) gives the exact sequence of $\Lambda_{\mathcal{O}_\eta}$-modules below.

(3.5.4)

$$0 \to H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}) \to \widetilde{H}^1_f(T_{\eta,\mathrm{Iw}}) \to \bigoplus_{\substack{\ell \in S_f, \\ \ell \neq p}} H^1_{\mathrm{cont}}(\mathrm{Fr}_\ell, T^{I_\ell}_{\eta,\mathrm{Iw}}) \to H^2_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}) \to \widetilde{H}^2_f(T_{\eta,\mathrm{Iw}}) \to 0$$

Using [**Nek06**, Theorem 7.8.6, §4.6.5.6], we obtain equation (3.5.1).

Now we prove that $H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is torsion free. Let x be an element of $\Lambda_{\mathcal{O}_\eta}$. Define

$$(T_{\eta,\mathrm{Iw}}/x)^+ = T^+_{\eta,\mathrm{Iw}}/x.$$

We have an exact sequence of complexes

$$0 \to C^\bullet_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}) \xrightarrow{x} C^\bullet_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}) \to C^\bullet_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}/x) \to 0,$$

which gives a surjective map

$$H^0_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}/x) \twoheadrightarrow H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})[x].$$

Since

$$0 \to H^0_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}/x) \to H^0_{\mathrm{cont}}(G_{\mathbb{Q},S}, T_{\eta,\mathrm{Iw}}/x) \oplus H^0_{\mathrm{cont}}(G_p, (T_{\eta,\mathrm{Iw}}/x)^+) \to \bigoplus_{\ell\in S_f} H^0_{\mathrm{cont}}(G_\ell, T_{\eta,\mathrm{Iw}}/x)$$

is an exact sequence and

$$H^0_{\mathrm{cont}}(G_{\mathbb{Q},S}, T_{\eta,\mathrm{Iw}}/x) = 0$$

(by proposition 3.5.1), we get

$$H^0_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}/x) = 0.$$

This proves

$$H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})[x] = 0.$$

A similar argument also shows that the $\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}}$-module $H^1_{\mathrm{Gr}}(T_{\eta^{\mathrm{int}},\mathrm{Iw}})$ is torsion free.

Equation (3.5.4) above gives the first implication of equation (3.5.2). The second implication follows from the final isomorphism of proposition 3.5.7 and [**Nek06**, Theorem 7.8.6, §4.6.5.6]. Then lemma 3.5.4 and equation (3.5.4) give the final implication of equation (3.5.2).

If $H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is zero, then $H^2_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is perfect by proposition 3.5.6. For almost all η, $H^2_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is torsion by this proposition. So by [**Nek06**, Theorem 7.8.6, §4.6.5.6], $H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is also torsion. Thus for almost all η, $H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is zero being torsion free. ☐

Theorem 3.5.11. *Let η be such that the following conditions hold.*

(1) $\eta(\alpha/\beta) \neq 0$,
(2) $H^1_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is zero,
(3) for all $\ell \in S_f, \ell \neq p$,

$$\mathrm{rk}\mathcal{T}(\mathfrak{a})^{I_\ell} = \mathrm{rk}T_\eta^{I_\ell}.$$

Then

$$\mathrm{char}_{\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}}} H^2_{\mathrm{Gr}}(T_{\eta^{\mathrm{int}},\mathrm{Iw}}) = \eta(\alpha/\beta)\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}} \tag{3.5.5}$$

and

$$\widetilde{H}^1_f(T_{\eta^{\mathrm{int}},\mathrm{Iw}}) = 0, \quad \widetilde{H}^2_f(T_{\eta^{\mathrm{int}},\mathrm{Iw}}) \otimes_{\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}}} \mathrm{Frac}(\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}}) = 0. \tag{3.5.6}$$

Consequently

$$\mathrm{char}_{\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}}} \widetilde{H}^2_f(T_{\eta^{\mathrm{int}},\mathrm{Iw}}) = \eta(\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a}))\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}} \tag{3.5.7}$$

and $\eta(\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a}))$ belongs to $\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}}$. For almost all η, the first three conditions hold.

Proof. By theorem 3.5.10, $\det_{\Lambda_{\mathcal{O}_\eta}} H^2_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is well-defined. Its image in $\mathrm{Frac}(\Lambda_{\mathcal{O}_\eta})$ (considered without the grade) under the composite map

$$\begin{aligned}\det_{\Lambda_{\mathcal{O}_\eta}} H^2_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}) &\hookrightarrow \left(\det_{\Lambda_{\mathcal{O}_\eta}} H^2_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})\right) \otimes_{\Lambda_{\mathcal{O}_\eta}} \mathrm{Frac}(\Lambda_{\mathcal{O}_\eta}) \\ &\cong \det_{\mathrm{Frac}(\Lambda_{\mathcal{O}_\eta})}\left(H^2_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}) \otimes_{\Lambda_{\mathcal{O}_\eta}} \mathrm{Frac}(\Lambda_{\mathcal{O}_\eta})\right) \quad \text{(by proposition 2.1.2)} \\ &= \det_{\mathrm{Frac}(\Lambda_{\mathcal{O}_\eta})}(0) \\ &= (\mathrm{Frac}(\Lambda_{\mathcal{O}_\eta}), 0)\end{aligned}$$

is equal to $\eta(\beta/\alpha)\Lambda_{\mathcal{O}_\eta}$ (by proposition 3.5.7). Then by proposition A.5.1,

$$\eta(\alpha/\beta)\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}} = \mathrm{char}_{\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}}}\left(H^2_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}}) \otimes_{\Lambda_{\mathcal{O}_\eta}} \Lambda_{\mathcal{O}_\eta^{\mathrm{int}}}\right).$$

Using proposition 3.5.7 again, we get equation (3.5.5).

By theorem 3.5.10, $H^2_{\mathrm{Gr}}(T_{\eta,\mathrm{Iw}})$ is torsion. So by proposition 3.5.7, $H^2_{\mathrm{Gr}}(T_{\eta^{\mathrm{int}},\mathrm{Iw}})$ is torsion and hence by [**Nek06**, Theorem 7.8.6, §4.6.5.6], $H^1_{\mathrm{Gr}}(T_{\eta^{\mathrm{int}},\mathrm{Iw}})$ is also torsion. Since it is torsion free (by theorem 3.5.10), it is zero. Then theorem 3.5.10 shows $\widetilde{H}^1_f(T_{\eta^{\mathrm{int}},\mathrm{Iw}})$ is zero. Then by [**Nek06**, Theorem 7.8.6, §4.6.5.6], $\widetilde{H}^2_f(T_{\eta^{\mathrm{int}},\mathrm{Iw}})$ is torsion over $\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}}$. This proves equation (3.5.6). Equation (3.5.7) follows from equation (3.5.5), lemma 3.5.2 and lemma 3.5.5.

The first condition of the above theorem is immediate for almost all η. The second and the third condition hold for almost all η by theorem 3.5.10 and lemma 3.5.2 respectively. ☐

3.5.4. $R\Gamma_{\mathrm{Gr}}(\overline{\rho}_{\mathrm{Iw}})$. Let S_0 denote the set of places of $\mathbb{Q}$ containing p, ∞ and the places of ramification of $\overline{\rho}$. Put

$$\overline{\rho}^+ = \mathcal{T}(\mathfrak{a})^+ \otimes_{R(\mathfrak{a})} k.$$

Let $\overline{\rho}_{\mathrm{Iw}}$ denote the $G_{\mathbb{Q},S_0}$-representation defined by

$$\overline{\rho}_{\mathrm{Iw}} = \overline{\rho} \otimes_k k[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]].$$

Define

$$\overline{\rho}^+_{\mathrm{Iw}} = \overline{\rho}^+ \otimes_k k[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]].$$

Remark 3.5.12. Let S' denote a finite set of places of $\mathbb{Q}$ containing S_0. The i-th cohomology of $R\Gamma_{\mathrm{Gr}}(G_{\mathbb{Q},S'}, \overline{\rho}_{\mathrm{Iw}})$ is denoted by $H^i_{\mathrm{Gr}}(S', \overline{\rho}_{\mathrm{Iw}})$. When we are interested in the rank or the triviality of $H^1_{\mathrm{Gr}}(S', \overline{\rho}_{\mathrm{Iw}})$, we denote it by $H^1_{\mathrm{Gr}}(\overline{\rho}_{\mathrm{Iw}})$. By lemma 3.5.13, this does not cause any confusion.

Lemma 3.5.13. *Let S' denote a finite set of places of $\mathbb{Q}$ containing S_0. Then $H^1_{\mathrm{Gr}}(S', \overline{\rho}_{\mathrm{Iw}})$ is free over $k[[T]]$ and there exists an exact sequence of complexes*

$$0 \to C^\bullet_{\mathrm{Gr}}(G_{\mathbb{Q},S'}, \overline{\rho}_{\mathrm{Iw}}) \to C^\bullet_{\mathrm{Gr}}(G_{\mathbb{Q},S_0}, \overline{\rho}_{\mathrm{Iw}}) \to \bigoplus_{\ell \in S' \setminus S_0} C^\bullet_{\mathrm{ur}}(G_\ell, \overline{\rho}_{\mathrm{Iw}}) \to 0. \tag{3.5.8}$$

Consequently

$$\mathrm{rk}_{k[[T]]} H^1_{\mathrm{Gr}}(S', \overline{\rho}_{\mathrm{Iw}}) = \mathrm{rk}_{k[[T]]} H^1_{\mathrm{Gr}}(S_0, \overline{\rho}_{\mathrm{Iw}}).$$

Proof. The first exact sequence follows from [**Nek06**, Proposition 7.8.8]. Since no power of Fr_ℓ is one in Λ_{Iw}, equation (3.5.8) gives the exact sequence

$$0 \to H^1_{\mathrm{Gr}}(S', \overline{\rho}_{\mathrm{Iw}}) \to H^1_{\mathrm{Gr}}(S_0, \overline{\rho}_{\mathrm{Iw}}) \to \overline{\rho}_{\mathrm{Iw}}^{I_\ell}/(\mathrm{Fr}_\ell - 1)$$

whose last term is torsion over $k[[T]]$. This proves the lemma. □

Lemma 3.5.14. *For any η,*

$$\mathrm{rk}_{k[[T]]} H^1_{\mathrm{Gr}}(\overline{\rho}_{\mathrm{Iw}}) = \mathrm{rk}_{k_\eta[[T]]} H^1_{\mathrm{Gr}}(T_{\eta^{\mathrm{int}},\mathrm{Iw}})/\varpi_\eta + \mathrm{rk}_{k_\eta[[T]]} H^2_{\mathrm{Gr}}(T_{\eta^{\mathrm{int}},\mathrm{Iw}})[\varpi_\eta].$$

Proof. This follows from the exact sequence

$$0 \to C^\bullet_{\mathrm{Gr}}(T_{\eta^{\mathrm{int}},\mathrm{Iw}}) \xrightarrow{\omega_\eta} C^\bullet_{\mathrm{Gr}}(T_{\eta^{\mathrm{int}},\mathrm{Iw}}) \to C^\bullet_{\mathrm{Gr}}(\overline{\rho}_{\mathrm{Iw}} \otimes_{k[[T]]} k_\eta[[T]]) \to 0.$$

□

3.5.5. A main conjecture. By theorem 3.5.11, for almost all $\mathfrak{p} \in D$ (D as in the proof of lemma 3.5.3), we get

$$\mathrm{char}_{\Lambda_{\mathcal{O}_{\lambda_\mathfrak{p}}}} \widetilde{H}^2_f(T_{\lambda_\mathfrak{p},\mathrm{Iw}}) = \lambda_\mathfrak{p}(\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a}))\Lambda_{\mathcal{O}_{\lambda_\mathfrak{p}}}.$$

On the other hand, by [**EPW06**], there exists an element $L_p^{\mathrm{an}}(\mathfrak{a})$ in $R(\mathfrak{a})_{\mathrm{Iw}}$ which interpolates the analytic p-adic L-function of $f_{\lambda_\mathfrak{p}}$ (computed with respect to certain period) for $\mathfrak{p} \in \mathrm{Spec}^{\mathrm{arith}}(R(\mathfrak{a}))$. Suppose that the conditions below hold.

Assumption 3.5.15.

(1) *The assumptions 3.2.4 and 3.3.1 hold.*
(2) *The character ψ_0 (as in §3.2.3) is trivial.*
(3) *There exists a prime $q||N$ such that $\overline{\rho}$ (as in proposition 3.2.3) is ramified at q.*
(4) *The image of $\overline{\rho}$ contains $\mathrm{SL}_2(R(\mathfrak{a})/\mathfrak{m})$.*

Then this analytic p-adic L-function generates $\mathrm{char}_{\Lambda_{\mathcal{O}_{\lambda_\mathfrak{p}}}} D_P(\mathrm{Sel}^{\mathrm{str}}_{A_{\lambda_\mathfrak{p},\mathrm{Iw}}})$ (by [**SU14**, Theorem 1]), which is equal to $\mathrm{char}_{\Lambda_{\mathcal{O}_{\lambda_\mathfrak{p}}}} \widetilde{H}^2_f(T_{\lambda_\mathfrak{p},\mathrm{Iw}})$ if $\mathfrak{p} \in D$ (by theorem 3.4.5). This shows that the mod $\mathfrak{p}$ reduction of $\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a})$ and $L_p^{\mathrm{an}}(\mathfrak{a})$ are associates for almost all $\mathfrak{p} \in D$.

Conjecture 3.5.16. *The two-variable algebraic p-adic L-function $\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a})$ is an element of $R(\mathfrak{a})^{\mathrm{int}}_{\mathrm{Iw}}$ and*

$$\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a})R(\mathfrak{a})^{\mathrm{int}}_{\mathrm{Iw}} = L_p^{\mathrm{an}}(\mathfrak{a})R(\mathfrak{a})^{\mathrm{int}}_{\mathrm{Iw}}.$$

Remark 3.5.17. This conjecture does not seem to follow from a straightforward argument using density of arithmetic points because there are non-associates in $\mathbb{Z}_p[[X]]$ which become associates modulo every arithmetic prime. As an example, we may consider the elements $p + X^2$ and $p + pX + X^2$. If we have one-side divisibility, then the above conjecture follows since an element of $R(\mathfrak{a})^{\mathrm{int}}_{\mathrm{Iw}}$ ($\simeq R(\mathfrak{a})^{\mathrm{int}}[[T]]$) can become a unit modulo an arithmetic prime only if its constant term is a unit in $R(\mathfrak{a})^{\mathrm{int}}$. Showing one-side divisibility is not immediate either, as there are elements f, g in $\mathbb{Z}_p[[X]]$ (for instance $f = p + X^2$ and $g = p + pX + X^2$) such that $f \nmid g$ and $f \bmod \mathcal{P} \mid g \bmod \mathcal{P}$ for each arithmetic prime $\mathcal{P}$.

Definition 3.5.18. *Let R be a ring. If $f(T) \in R[[T]]$ is a power series, then its* content *is denoted by $I(f(T))$ and is defined as the ideal of R generated by the coefficients of $f(T)$.*

If R is a local ring and $f(T) \in R[[T]]$ has unit content, then the λ-invariant *$\lambda(f(T))$ of $f(T)$ is defined to be the smallest degree in which $f(T)$ has a unit coefficient.*

By choosing a topological generator γ of $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, we identify $R\widehat{\otimes}_{\mathcal{O}}\Lambda_{\mathrm{Iw}}$ with $R[[T]]$ for $R = R(\mathfrak{a})^{\mathrm{int}}, \mathcal{O}_\eta^{\mathrm{int}}$. Recall from §A.4 that $\mathscr{O}[[T]]$ denotes the $\overline{\mathbb{Z}}_p$-subalgebra of $\overline{\mathbb{Z}}_p[[T]]$ spanned by the subsets $\mathcal{O}_L[[T]]$ where L ranges over all finite extensions of $\mathbb{Q}_p$.

Definition 3.5.19. *If $\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a})$ is an element of $R(\mathfrak{a})^{\mathrm{int}}_{\mathrm{Iw}} = R(\mathfrak{a})^{\mathrm{int}}[[T]]$, then $\mu^{\mathrm{alg}}(\mathfrak{a})$ is defined by*

$$\mu^{\mathrm{alg}}(\mathfrak{a}) = I\left(\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a})\right).$$

If $\mu^{\mathrm{alg}}(\mathfrak{a}) = R(\mathfrak{a})^{\mathrm{int}}$, then the algebraic λ-invariant *$\lambda^{\mathrm{alg}}(\mathfrak{a})$ is defined to be $\lambda\left(\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a})\right)$.*

Remark 3.5.20. It would be clear from the context whether λ denotes the λ-invariant or an arithmetic specialization.

By [**EPW06**, Theorem 1], the μ-invariant of (the characteristic ideal of the dual of the Selmer group of) T_{η_0} vanishes for one arithmetic specialization η_0 of $R(\mathfrak{a})$ if and only if the μ-invariant of (the characteristic ideal of the dual of the Selmer group of) T_η vanishes for any arithmetic specialization η of $R(\mathfrak{a})$. If this is the case, following *loc. cit.*, we write $\mu^{\mathrm{alg}}(\overline{\rho}) = 0$. By *loc. cit.*, Greenberg's conjecture on vanishing of μ-invariants of modular forms (with absolutely irreducible and p-distinguished residual Galois representation) is equivalent to the conjecture below.

Conjecture 3.5.21. *If $\overline{\rho}$ satisfies assumption 3.2.4 and 3.3.1, then*

$$\mu^{\mathrm{alg}}(\overline{\rho}) = 0.$$

Theorem 3.5.22. *The two-variable algebraic p-adic L-function $\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a})$ is an element of $R(\mathfrak{a})^{\mathrm{int}}_{\mathrm{Iw}}$. Under assumptions 3.2.4 and 3.3.1, the following conditions are equivalent.*

(1) $\mu^{\mathrm{alg}}(\overline{\rho}) = 0$,

(2) $\mu^{\mathrm{alg}}(\mathfrak{a}) = R(\mathfrak{a})^{\mathrm{int}}$,

(3) $H^1_{\mathrm{Gr}}(\overline{\rho}_{\mathrm{Iw}}) = 0$,

(4) for all η,

$$\widetilde{H}^1_f(T_{\eta^{\mathrm{int}},\mathrm{Iw}}) = 0$$

and the μ-invariant of the $\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}}$-module $\mathrm{char}_{\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}}} \widetilde{H}^2_f(T_{\eta^{\mathrm{int}},\mathrm{Iw}})$ *is zero,*

(5) for some η,

$$\widetilde{H}^1_f(T_{\eta^{\mathrm{int}},\mathrm{Iw}}) = 0$$

and the μ-invariant of the $\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}}$-module $\mathrm{char}_{\Lambda_{\mathcal{O}_\eta^{\mathrm{int}}}} \widetilde{H}^2_f(T_{\eta^{\mathrm{int}},\mathrm{Iw}})$ *is zero.*

Suppose that the assumption 3.5.15 holds. Then the above five conditions are equivalent to

$$\mu^{\mathrm{an}}(\mathfrak{a}) = R(\mathfrak{a})^{\mathrm{int}}.$$

Assume further that $\mu^{\mathrm{alg}}(\overline{\rho}) = 0$. Then

$$\lambda^{\mathrm{alg}}(\mathfrak{a}) = \lambda^{\mathrm{an}}(\mathfrak{a}),$$

$$\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a}) R(\mathfrak{a})^{\mathrm{int}}_{\mathrm{Iw}} = L_p^{\mathrm{an}}(\mathfrak{a}) R(\mathfrak{a})^{\mathrm{int}}_{\mathrm{Iw}}.$$

Proof. By theorem 3.5.11,

$$\eta(\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a})) \in \Lambda_{\mathcal{O}_\eta^{\mathrm{int}}} \subset \mathscr{O}[[T]]$$

for almost all $\eta \in \mathrm{Hom}_{\mathcal{O}_K\text{-alg}}(R(\mathfrak{a}), \overline{\mathbb{Z}}_p)$. So by proposition A.4.1, $\beta\beta_{\mathrm{Eul}}$ divides α in $R(\mathfrak{a})^{\mathrm{int}}_{\mathrm{Iw}}$.

Let D_0 denote the subset of D such that for any $\mathfrak{p}$ in D_0, $\lambda_{\mathfrak{p}}$ satisfies the first three conditions of theorem 3.5.11 (with η replaced by $\lambda_{\mathfrak{p}}$). By theorem 3.5.11, the complement of D_0 in D is finite. Since D is dense in $\mathrm{Spec}(R(\mathfrak{a}))$, D_0 is also dense in it.

Now suppose that the assumptions 3.2.4 and 3.3.1 hold. Then for all $\mathfrak{p} \in D_0$,

$$\mathrm{char}_{\Lambda_{\mathcal{O}_{\lambda_{\mathfrak{p}}}}} \widetilde{H}^2_f(T_{\lambda_{\mathfrak{p}},\mathrm{Iw}}) = \lambda_{\mathfrak{p}}(\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a}))\Lambda_{\mathcal{O}_{\lambda_{\mathfrak{p}}}}$$

$$\mathrm{char}_{\Lambda_{\mathcal{O}_{\lambda_{\mathfrak{p}}}}} \widetilde{H}^2_f(T_{\lambda_{\mathfrak{p}},\mathrm{Iw}}) = \mathrm{char}_{\Lambda_{\mathcal{O}_{\lambda_{\mathfrak{p}}}}} D_P(\mathrm{Sel}^{\mathrm{str}}_{A_{\lambda_{\mathfrak{p}},\mathrm{Iw}}})$$

by theorem 3.5.11 and theorem 3.4.5 respectively. Since D_0 is nonempty, by [**EPW06**, Theorem 1], the first two conditions above are equivalent. Fix an element $\mathfrak{q}$ in D_0. By lemma 3.5.14, $H^1_{\mathrm{Gr}}(\overline{\rho}_{\mathrm{Iw}})$ is zero if and only if $H^2_{\mathrm{Gr}}(T_{\lambda_{\mathfrak{q}},\mathrm{Iw}})[\varpi]$ is zero, which holds if and only if the μ-invariant of a generator of $\mathrm{char}_{\Lambda_{\mathcal{O}_{\lambda_{\mathfrak{q}}}}} \widetilde{H}^2_f(T_{\lambda_{\mathfrak{q}},\mathrm{Iw}})$ is zero (by lemma 3.5.5 and [**EPW06**, Lemma 3.7.4]). Since $\lambda_{\mathfrak{q}}(\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a}))$ generates $\mathrm{char}_{\Lambda_{\mathcal{O}_{\lambda_{\mathfrak{q}}}}} \widetilde{H}^2_f(T_{\lambda_{\mathfrak{q}},\mathrm{Iw}})$, we get

$$H^1_{\mathrm{Gr}}(\overline{\rho}_{\mathrm{Iw}}) = 0 \Longleftrightarrow \mu^{\mathrm{alg}}(\mathfrak{a}) = R(\mathfrak{a})^{\mathrm{int}}.$$

So the first three conditions above are equivalent. By lemma 3.5.14, (3) implies (4) and (5) implies (3). So conditions (3), (4), (5) are equivalent.

First note that for all $\mathfrak{p} \in D_0$,

$$\mathrm{char}_{\Lambda_{\mathcal{O}_{\lambda_{\mathfrak{p}}}}} D_P(\mathrm{Sel}^{\mathrm{str}}_{A_{\lambda_{\mathfrak{p}},\mathrm{Iw}}}) = \lambda_{\mathfrak{p}}(L_p^{\mathrm{an}}(\mathfrak{a}))\Lambda_{\mathcal{O}_{\lambda_{\mathfrak{p}}}}$$

by [**SU14**, Theorem 1] and hence

$$\lambda_{\mathfrak{p}}(\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a}))\Lambda_{\mathcal{O}_{\lambda_{\mathfrak{p}}}} = \lambda_{\mathfrak{p}}(L_p^{\mathrm{an}}(\mathfrak{a}))\Lambda_{\mathcal{O}_{\lambda_{\mathfrak{p}}}}.$$

So the first five conditions are equivalent to

$$\mu^{\mathrm{alg}}(\mathfrak{a}) = R(\mathfrak{a})^{\mathrm{int}}_{\mathrm{Iw}}.$$

Suppose that $\mu^{\mathrm{alg}}(\overline{\rho}) = 0$. Then by [**Och05**, Lemma 3.7],

$$\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a}) = u(T^r + a_{r-1}T^{r-1} + \cdots + a_0), \quad L_p^{\mathrm{an}}(\mathfrak{a}) = v(T^s + b_{s-1}T^{s-1} + \cdots + b_0)$$

with $a_0, \cdots, a_{r-1}, b_0, \cdots, b_{s-1} \in R(\mathfrak{a})^{\mathrm{int}}$ and $u, v \in (R(\mathfrak{a})^{\mathrm{int}}_{\mathrm{Iw}})^\times$. Since for all $\mathfrak{p} \in D_0$, $\lambda_{\mathfrak{p}}(\mathscr{L}_p^{\mathrm{alg}}(\mathfrak{a}))$ and $\lambda_{\mathfrak{p}}(L_p^{\mathrm{an}}(\mathfrak{a}))$ are associates in $\Lambda_{\mathcal{O}_{\lambda_{\mathfrak{p}}}}$, the elements $\lambda_{\mathfrak{p}}(T^r + a_{r-1}T^{r-1} + \cdots + a_0)$, $\lambda_{\mathfrak{p}}(T^s + b_{s-1}T^{s-1} + \cdots + b_0)$ are also associates in $\Lambda_{\mathcal{O}_{\lambda_{\mathfrak{p}}}}$. Hence

$$\lambda_{\mathfrak{p}}(T^r + a_{r-1}T^{r-1} + \cdots + a_0) = \lambda_{\mathfrak{p}}(T^s + b_{s-1}T^{s-1} + \cdots + b_0)$$

for all $\mathfrak{p} \in D_0$. Since D_0 is dense in $\mathrm{Spec}(R(\mathfrak{a}))$, we get

$$T^r + a_{r-1}T^{r-1} + \cdots + a_0 = T^s + b_{s-1}T^{s-1} + \cdots + b_0.$$

This proves the result. □

CHAPTER 4

Algebraic p-adic L-functions for the Hida family for definite unitary groups

In this chapter, we construct algebraic p-adic L-functions $L^{\mathrm{alg}}_{p,\mathrm{Kato}}(-), L^{\mathrm{alg}}_{p',\mathrm{Gr}}(-)$ along branches of the Hida family for definite unitary groups and prove that they satisfy a perfect control theorem at arithmetic specializations of regular dominant weight whose associated automorphic representations are stable and associated Galois representations are crystalline at each place above p (theorem 4.3.6). The crucial step of their proof is the recognition of the role of purity in understanding the variation of inertia invariants in families. Though such Galois representations are not known to be motivic, in [**Pin92**, Conjecture 5.4.1], they are conjectured to satisfy properties similar to motivic representations, for example purity. By [**Car12**], the Galois representations associated with the automorphic forms (which are of dominant weight and stable) for definite unitary groups are pure. So this variation is well-understood by theorem 1.2.4. In this chapter, from §4.3, we assume throughout that the condition 4.3.1 holds.

The local conditions used in $L^{\mathrm{alg}}_{p',\mathrm{Gr}}(-)$ at places $w \nmid p$ is a modification $U'_w(-)$ of the unramified condition $U^+_w(-)$ of Greenberg (as defined in [**Nek06**, §0.8.1] following [**Gre89, Gre91**]). We use the local condition $U'_w(-)$ in stead of $U^+_w(-)$ as it is pointed out in [**FO12**, Remark 2.17] that the inertia invariants of a big Galois representation ρ may not specialize perfectly to the inertia invariants of a specialization of ρ. The construction of $L^{\mathrm{alg}}_{p,\mathrm{Kato}}(-)$ uses no condition at p and uses the condition $U'_w(-)$ at places $w \neq p$.

The organization of this chapter is as follows. In the first section, we review the notion of automorphic representations of a definite unitary group and its associated Galois representation. In the second section, we discuss the set up of Hida theory for unitary groups. For these two sections, we follow [**GG12**, p. 264–268]. However *loc. cit.* often refers to [**Ger10**] for a more detailed exposition and proofs. So we will refer to appropriate results in [**Ger10**] (which uses [**Hid88a, Hid89, Hid95, Hid98, Mau04, TU99**] among others). In the third section, we construct algebraic p-adic L-functions $L^{\mathrm{alg}}_{p,\mathrm{Kato}}(-)$, $L^{\mathrm{alg}}_{p',\mathrm{Gr}}(-)$ along branches of this Hida family and prove that they satisfy perfect control theorems.

4.1. Automorphic representations and Galois representations

4.1.1. Definite Unitary Groups. Let F be a CM field, F^+ be its maximal totally real subfield. Denote the non-trivial element of $\mathrm{Gal}(F/F^+)$ by c. Let $n \geq 2$ be an integer and assume that if n is even, then $n[F^+ : \mathbb{Q}]$ is divisible by 4. Then by the argument of [**HT01**, Lemma I.7.1], there exists an involution $\dagger$ of second kind on $B = M_n(F)$ whose associated reductive algebraic group G over F^+ defined by

$$G(R) = \{g \in B \otimes_{F^+} R \mid g^\dagger g = 1\} \quad \text{for any } F^+\text{-algebra } R$$

has the following properties:

(a) G is an outer form of GL_{n/F^+} with $G_{/F} \simeq \mathrm{GL}_{n/F}$,
(b) for every infinite place v of F^+, $G(F_v^+) \simeq U_n(\mathbb{R})$,
(c) for every finite place v of F^+, G is quasi-split at v.

By [**CHT08**, §3.3], we can choose an order $\mathcal{O}_B$ in B such that $\mathcal{O}_B^\dagger = \mathcal{O}_B$ and $\mathcal{O}_{B,w}$ is a maximal order in B_w for all places w of F which are split over F^+. This choice gives a model of G over $\mathcal{O}_{F^+}$, which we fix from now on.

For every finite place v of F^+ which splits as ww^c in F there is a natural isomorphism

$$\iota_w : G(F_v^+) \xrightarrow{\sim} \mathrm{GL}_n(F_w)$$

which restricts to an isomorphism between $G(\mathcal{O}_{F_v^+})$ and $\mathrm{GL}_n(\mathcal{O}_{F_w})$.

For each embedding $\sigma : F^+ \hookrightarrow \mathbb{R}$ and $\tilde{\sigma} : F \hookrightarrow \mathbb{C}$ an extension of σ, choose an isomorphism

$$\iota_{\tilde{\sigma}} : B \otimes_{F^+,\sigma} \mathbb{R} \xrightarrow{\sim} B \otimes_{F,\tilde{\sigma}} \mathbb{C} = M_n(F_{\tilde{\sigma}})$$

so that $\iota_{\tilde{\sigma}}(x^\dagger) = {}^t(\iota_{\tilde{\sigma}}(x)^c)$. Then $\tilde{\sigma} \circ \iota_{\tilde{\sigma}}$ identifies $G(F_\sigma^+)$ with $U_n(\mathbb{R})$.

4.1.2. Algebraic representations. Let $p > n$ be a rational prime and assume (as in [**HT01**, I.7]) that every prime of F^+ lying above p splits in F. Let K be a finite extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$ which contains the image of every embedding $F \hookrightarrow \overline{\mathbb{Q}}_p$ and a primitive p-th root of unity (as in [**GG12**, p. 266]). Let ϖ denote a uniformizer of the ring of integers $\mathcal{O}_K$ of K and $\mathbb{F}$ denote the residue field.

Let Σ_p denote the set of places of F^+ above p, and I_p the set of embeddings of $F^+ \hookrightarrow K$. For each place $v \in \Sigma_p$, choose once and for all a place $\tilde{v}$ of F lying above v. Let $\tilde{\Sigma}_p$ denote the set of these places $\tilde{v}$ for $v \in \Sigma_p$. Let $\tilde{I}_p$ be the set of embeddings $F \hookrightarrow K$ which give rise to an element of $\tilde{\Sigma}_p$. From now on we will identify I_p and $\tilde{I}_p$. Let $\mathfrak{p}$ denote the product of all places in Σ_p. We write

$$\mathcal{O}_{F^+,p} = \mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_p, \quad F_p^+ = F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

Let $T_n \subset B_n \subset \mathrm{GL}_n$ denote the diagonal torus, the Borel subgroup of upper triangular matrices in GL_n, regarded as algebraic groups over $\mathbb{Z}$. We identify the character group

$$X^*(T_n) \xrightarrow{\sim} \mathbb{Z}^n$$

via the map which sends the character

$$\mathrm{diag}(t_1, \cdots, t_n) \mapsto t_1^{\lambda_1} \cdots t_n^{\lambda_n}$$

to the tuple $(\lambda_1, \cdots, \lambda_n)$. Note that any character of T_n can also be regarded as a character of B_n via the natural homomorphism $B_n \to T_n$. Let ε_i denote the character

$$\mathrm{diag}(t_1, \cdots, t_n) \mapsto t_i.$$

The set of characters

$$\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$$

consists of the roots of GL_n with respect to T_n. Our fixed choice of the Borel subgroup B_n gives us a system Φ^+ of positive roots, *viz.*, the roots $\varepsilon_i - \varepsilon_j$ for $j > i$. The simple roots for this positive system are the roots $\varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \cdots, n-1$. There is a partial order on $X^*(T_n)$ defined by

$$\lambda \geq \mu \Longleftrightarrow \lambda - \mu \in \sum_i \mathbb{N}(\varepsilon_i - \varepsilon_{i+1}).$$

The Weyl group $W_{T_n} := N_{\mathrm{GL}_n}(T_n)/T_n$ acts on T_n by

$$w(t) = wtw^{-1}$$

and on $X^*(T_n)$ via the rule

$$(w\lambda)(t) = \lambda(w^{-1}tw).$$

We identify it with S_n via the rule

$$w(t_1, \cdots, t_n)w^{-1} = (t_{w^{-1}(1)}, \cdots, t_{w^{-1}(n)}).$$

Let w_0 denote the longest element of the Weyl group. It sends the character $(\lambda_1, \cdots, \lambda_n)$ to the character $(\lambda_n, \cdots, \lambda_1)$.

For a character λ of T_n and a ring R, define the induced representation

$$\mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0\lambda)_{/R} := \{f \in R[\mathrm{GL}_n] \mid f(bg) = (w_0\lambda)(b)f(g), \forall R \to A, g \in \mathrm{GL}_n(A), b \in B_n(A)\}$$

on which GL_n acts by right translation. This is a representation of the algebraic group $\mathrm{GL}_{n/R}$. Since K is flat over $\mathcal{O}_K$, we have

$$\mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0\lambda)_{/K} = (\mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0\lambda)_{/\mathcal{O}_K} \otimes_{\mathcal{O}_K} K$$

(see [**Jan03**, Fact 3, §I.3.5]). When $R = \mathcal{O}_K, K$ or $\mathbb{F}$, by the proposition in [**Jan03**, §II.2.6], the induced module $\mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0\lambda)_{/R}$ is nonzero if and only if the character $\lambda = (\lambda_1, \cdots, \lambda_n)$ satisfies

$$\lambda_1 \geq \cdots \geq \lambda_n.$$

Such a character λ is called a *dominant character* for GL_n.

Definition 4.1.1. *For a dominant character λ for GL_n, we define the representation*

$$\xi_\lambda := \mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0\lambda)_{/\mathcal{O}_K}.$$

We let M_λ denote a finite free $\mathcal{O}_K$-module, carrying an action of $\mathrm{GL}_n(\mathcal{O}_K)$, obtained by evaluating ξ_λ on $\mathcal{O}_K$. We let $W_\lambda = M_\lambda \otimes_{\mathcal{O}_K} K$. This space carries an action of $\mathrm{GL}_n(K)$.

We remark that the module M_λ is finite and free over $\mathcal{O}_K$ as it is torsion free by definition and finitely generated by [**Jan03**, Proposition I.5.12(c)].

If W is an algebraic representation of $\mathrm{GL}_{n/R}$ and $\mu \in X^*(T_n)$, we denote by W_μ the subspace of W on which T_n acts via μ. The *weights* of W are those characters μ for which $W_\mu \neq 0$.

Put

$$\mathbb{Z}^n_+ = \{(\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}$$

and let G denote the unitary group as in §4.1.1.

Definition 4.1.2.

(a) *A* dominant weight *for G is a tuple $\lambda = (\lambda_\tau)_\tau \in (\mathbb{Z}^n_+)^{\tilde{I}_p}$. If λ is a dominant weight for G, define*

$$M_\lambda = \otimes_{\tau\in\tilde{I}_p} M_{\lambda_\tau}, \quad W_\lambda = \otimes_{\tau\in\tilde{I}_p} W_{\lambda_\tau} = M_\lambda \otimes_{\mathcal{O}_K} K.$$

Then define representations

$$\xi_\lambda : G(\mathcal{O}_{F^+,p}) \to \mathrm{GL}(M_\lambda) \quad \textit{by} \quad g \mapsto \otimes_{\tau\in\tilde{I}_p} \xi_{\lambda_\tau}(\tau(\iota_{\tilde{v}(\tau)} g)),$$
$$\xi_\lambda : G(F_p^+) \to \mathrm{GL}(W_\lambda) \quad \textit{by} \quad g \mapsto \otimes_{\tau\in\tilde{I}_p} \xi_{\lambda_\tau}(\tau(\iota_{\tilde{v}(\tau)} g))$$

where $\tilde{v}(\tau)$ is the place in $\tilde{\Sigma}_p$ induced by τ.

(b) *If $\lambda = (\lambda_\tau)_\tau \in (\mathbb{Z}^n)^{\tilde{I}_p}$, then we associate to it the character*

$$\lambda : T_n(F_p^+) \simeq \prod_{v\in\tilde{I}_p} T_n(F_{\tilde{v}}) \to K^\times$$

defined by

$$u \mapsto \prod_{\tau\in\tilde{I}_p} \lambda_\tau(\tau(u)).$$

(c) *If $\lambda = (\lambda_\tau)_\tau \in (\mathbb{Z}^n)^{\tilde{I}_p}$ and $w \in W_{T_n}$, we let $w\lambda = (w\lambda_\tau)_\tau \in (\mathbb{Z}^n)^{\tilde{I}_p}$.*

(d) *A dominant weight λ for G is* regular *if for each $v \in \Sigma_p$ and each $j = 1, \cdots, n-1$, there exists $\tau \in \tilde{I}_p$ giving rise to $\tilde{v}$ with $\lambda_{\tau,j} > \lambda_{\tau,j+1}$.*

4.1.3. Automorphic forms on G. Let Σ' denote a finite set of finite places of F^+ disjoint from Σ_p and consisting of places which split in F. Choose once and for all a place $\tilde{v}$ of F over each place $v \in \Sigma'$. For each $v \in \Sigma' \cup \Sigma_p$, we will identify the groups $G(F_v^+)$ and $\mathrm{GL}_n(F_{\tilde{v}})$ via $\iota_{\tilde{v}}$ (as defined in §4.1.1). If v is a place of F^+ split over F and $\tilde{v}$ is a place of F dividing v, then we let

(a) $\mathrm{Iw}(\tilde{v})$ denote the subgroup of $\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$ consisting of matrices which reduce to an upper triangular matrix modulo $\tilde{v}$,

(b) $\mathrm{Iw}(\tilde{v}^{b,c})$, for $0 \le b \le c$, denote the subgroup of $\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$ consisting of matrices which reduce to an upper triangular matrix modulo $\tilde{v}^c$ and to a unipotent matrix modulo $\tilde{v}^b$.

Note that if $k(\tilde{v})$ denotes the residue field of $\tilde{v}$, then we have a natural isomorphism

$$\mathrm{Iw}(\tilde{v})/\mathrm{Iw}(\tilde{v}^{1,1}) \simeq (k(\tilde{v})^\times)^n$$

given by $g = (g_{ij}) \mapsto (\bar{g}_{11}, \cdots, \bar{g}_{nn})$ where the bars denote mod $\tilde{v}$ reduction. For each $v \in \Sigma'$, we have a character

$$\chi_v = \chi_{v,1} \times \cdots \times \chi_{v,n} : \mathrm{Iw}(\tilde{v})/\mathrm{Iw}(\tilde{v}^{1,1}) \to \mathcal{O}_K^\times.$$

Define

$$M_{\{\chi_v\}} := \otimes_{v\in\Sigma'} \mathcal{O}_K(\chi_v).$$

It has an action of $\prod_{v\in\Sigma'} \mathrm{Iw}(\tilde{v})$. If λ is a dominant weight for G, define

$$M_{\lambda,\{\chi_v\}} := M_\lambda \otimes_{\mathcal{O}_K} M_{\{\chi_v\}}.$$

This also carries an action of $G(\mathcal{O}_{F^+,p})$.

Definition 4.1.3. *For an $\mathcal{O}_K$-module A and a dominant weight λ for G, we define $S_{\lambda,\{\chi_v\}}(A)$ to be the space of functions $f : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty) \to M_{\lambda,\{\chi_v\}} \otimes_{\mathcal{O}_K} A$ such that there exists a compact open subgroup*

$$U \subset G(\mathbb{A}_{F^+}^{\infty,\Sigma'\cup\Sigma_p}) \times G(\mathcal{O}_{F^+,p}) \times \prod_{v\in\Sigma'} \mathrm{Iw}(\tilde{v})$$

with

$$(u_{\Sigma'\cup\Sigma_p})f(gu) = f(g)$$

for all $u \in U, g \in G(\mathbb{A}_{F^+}^\infty)$ where $u_{\Sigma\cup\Sigma'}$ is the projection of u to $\prod_{v\in\Sigma'\cup\Sigma_p} G(F_v^+)$. The group $G(\mathbb{A}_{F^+}^{\infty,\Sigma'\cup\Sigma_p}) \times G(\mathcal{O}_{F^+,p}) \times \prod_{v\in\Sigma'} \mathrm{Iw}(\tilde{v})$ acts on $S_{\lambda,\{\chi_v\}}(A)$ via

$$(g \cdot f)(h) = (g_{\Sigma'\cup\Sigma_p})f(hg).$$

If A is a K-module, then the group $G(\mathbb{A}_{F^+}^{\infty,\Sigma'}) \times \prod_{v\in\Sigma'} \mathrm{Iw}(\tilde{v})$ acts on $S_{\lambda,\{\chi_v\}}(A)$ via the same formula.

If U is a subgroup of $G(\mathbb{A}_{F^+}^{\infty,\Sigma'\cup\Sigma_p}) \times G(\mathcal{O}_{F^+,p}) \times \prod_{v\in\Sigma'} \mathrm{Iw}(\tilde{v})$, or if U is a subgroup of $G(\mathbb{A}_{F^+}^{\infty,\Sigma'}) \times \prod_{v\in\Sigma'} \mathrm{Iw}(\tilde{v})$ and A is a K-module, then we define $S_{\lambda,\{\chi_v\}}(U, A)$ by

$$S_{\lambda,\{\chi_v\}}(U, A) = S_{\lambda,\{\chi_v\}}(A)^U.$$

Now we recall the relation between these spaces and the space of automorphic forms on G as defined for example in [**BJ79**]. Let $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ be a field isomorphism. Via this isomorphism, $\mathbb{C}$ becomes a K-algebra. For each embedding $\sigma : F^+ \hookrightarrow \mathbb{R}$, there is a unique embedding $\tilde{\sigma} : F \hookrightarrow \mathbb{C}$ extending σ such that $\iota^{-1}\tilde{\sigma} \in \tilde{I}_p$. There is an induced action of $G(F_\infty^+)$ on $W_\lambda \otimes_{K,\iota} \mathbb{C}$ via

$$g \mapsto \otimes_\sigma \xi_{\lambda_{\iota^{-1}\tilde{\sigma}}}(\tilde{\sigma}(\iota_{\tilde{\sigma}}(g))).$$

Denote this representation by $\xi_{\lambda,\iota}$.

Proposition 4.1.4. *There is an isomorphism of $G(\mathbb{A}_{F^+}^{\infty,\Sigma'}) \times \prod_{v\in\Sigma'} \mathrm{Iw}(\tilde{v})$-modules*

$$S_{\lambda,\{\chi_v\}}(\overline{\mathbb{Q}}_p) \xrightarrow{\sim} \mathrm{Hom}_{G(F_\infty^+)}((\otimes_{v\in\Sigma'}\mathbb{C}(\iota\chi_v^{-1})) \otimes \xi_{\lambda,\iota}^\vee, \mathcal{A})$$

where $\mathcal{A}$ denotes the space of automorphic forms on $G(F^+)\backslash G(\mathbb{A}_{F^+})$.

Proof. Follows from the proof of [**CHT08**, Proposition 3.3.2]. □

4.1.4. Galois representations. We normalize the local Langlands correspondence as in [**CHT08**, §3.1]. If w is a finite place of F and π is an irreducible, admissible, representation of $\mathrm{GL}_n(F_w)$ defined over $\overline{\mathbb{Q}}_p$, we let $r_p(\pi)$ denote the p-adic representation of G_{F_w} associated (as in [**Tat79**]) with the Weil-Deligne representation $\mathrm{rec}_p(\pi^\vee \otimes |\ \ |^{(1-n)/2})$ when it exists (*i.e.*, when the eigenvalues of $\mathrm{rec}_p(\pi^\vee \otimes |\ \ |^{(1-n)/2})(\phi_w)$ are p-adic units for some lift ϕ_w of Fr_w). Here rec_p is as in [**HT01**]. We will denote the p-adic cyclotomic character by ε.

Proposition 4.1.5. *Let λ be a dominant weight for G and π be an irreducible constituent of the $G(\mathbb{A}_{F^+}^{\infty,\Sigma'}) \times \prod_{v\in\Sigma'} \mathrm{Iw}(\tilde{v})$-representation $S_{\lambda,\{\chi_v\}}(\overline{\mathbb{Q}}_p)$. Then there exists a continuous semi-simple representation*

$$\rho_\pi : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p),$$

which is uniquely determined by the following two properties.

(1) $\rho_\pi^c \simeq \rho_\pi^\vee \varepsilon^{1-n}$,

(2) if $v \notin \Sigma' \cup \Sigma_p$ is a finite place of F^+ which splits as ww^c in F, then

$$\rho_\pi|_{G_{F_w}}^{\mathrm{ss}} \simeq (r_p(\pi_v \circ \iota_w^{-1})^\vee(1-n))^{\mathrm{ss}}.$$

If the weak base change of π to $\mathrm{GL}_n(\mathbb{A}_F)$ is cuspidal, then for any finite place w of F not dividing p, the restriction of ρ_π to G_{F_w} is pure.

Proof. From [**Lab11**, Corollaire 5.3], we get a weak base change $\mathrm{WBC}(\pi)$ of π to $\mathrm{GL}_n(\mathbb{A}_F)$. Then [**CH**, Theorem 3.2.5] associates a Galois representation ρ to $\mathrm{WBC}(\pi)$. We define ρ_π to be ρ, which satisfies the stated properties by *loc. cit.* The last part follows from [**Car12**, Theorem 1.1, 1.2] and proofs of theorem 5.8 and corollary 5.9 of *loc. cit.* □

Definition 4.1.6. *Let π be as in the statement of the above proposition. It is said to be* stable *if its weak base change* $\mathrm{WBC}(\pi)$ *to* $\mathrm{GL}_n(\mathbb{A}_F)$ *is cuspidal.*

In the main theorem of this chapter (theorem 4.3.6), we will consider stable automorphic representations.

4.2. Hida Theory

4.2.1. Hecke algebras. Let Σ denote a finite set of finite places of F^+ containing $\Sigma' \cup \Sigma_p$ and such that every place in Σ splits in F. Recall that for every $v \in \Sigma' \cup \Sigma_p$, we have fixed a place $\tilde{v}$ of F lying above v. Now for every place $v \in \Sigma \setminus (\Sigma' \cup \Sigma_p)$, fix a place $\tilde{v}$ of F above v. For $v \in \Sigma$, we will henceforth identify $G(F_v^+)$ with $\mathrm{GL}_n(F_{\tilde{v}})$ via $\iota_{\tilde{v}}$.

Let $U = \prod_v U_v$ be a compact open subgroup of $G(\mathbb{A}_{F^+}^\infty)$ where $U_v \subset G(F_v^+)$ for each finite place v of F^+ and

(a) if $v \notin \Sigma$ splits in F, then $U_v = G(\mathcal{O}_{F_v^+})$,
(b) if $v \in \Sigma'$, then $U_v = \mathrm{Iw}(\tilde{v})$,
(c) if $v \in \Sigma_p$, then $U_v = G(\mathcal{O}_{F_v^+})$.

We do not specify U_v for $v \in \Sigma \setminus (\Sigma' \cup \Sigma_p)$ or for $v \notin \Sigma$ not split in F. For $0 \le b \le c$, define

$$U(\mathfrak{p}^{b,c}) = U^p \times \prod_{v \in \Sigma_p} \mathrm{Iw}(\tilde{v}^{b,c}).$$

4.2.1.1. *Hecke operators.* Let $V, V' \subset G(\mathbb{A}_{F^+}^{\infty,\Sigma'}) \times \prod_{v\in\Sigma'} \mathrm{Iw}(\tilde{v})$ be compact open subgroups of $G(\mathbb{A}_{F^+}^\infty)$. Let λ be a dominant weight for G.

Let A be a K-module. Then for every $g \in G(\mathbb{A}_{F^+}^{\infty,\Sigma'}) \times \prod_{v\in\Sigma'} \mathrm{Iw}(\tilde{v})$ there is an operator

$$[V'gV] : S_{\lambda,\{\chi_v\}}(V, A) \to S_{\lambda,\{\chi_v\}}(V', A)$$

defined by

$$[V'gV]f = \sum_i x_i \cdot f, \quad f \in S_{\lambda,\{\chi_v\}}(V, A)$$

using a decomposition $V'gV = \coprod_i x_i V$. This definition is independent of the choice of x_i.

If A is an $\mathcal{O}_K$-module, but not a K-module, then assume that $v_p, v'_p \in G(\mathcal{O}_{F^+,p})$ for all $v \in V, v' \in V'$. In this case, for every $g \in G(\mathbb{A}_{F^+}^{\infty,\Sigma'\cup\Sigma_p}) \times G(\mathcal{O}_{F^+,p}) \times \prod_{v\in\Sigma'} \mathrm{Iw}(\tilde{v})$, there is an operator

$$[V'gV] : S_{\lambda,\{\chi_v\}}(V, A) \to S_{\lambda,\{\chi_v\}}(V', A)$$

defined as above.

Hecke operators at unramified places. Let w be a place of F, split over over F^+ and lying over a place of F^+ outside Σ. Let λ be a dominant weight for G and A be an $\mathcal{O}_K$-module. Let ϖ_w be a uniformizer in $\mathcal{O}_{F_w}$. For each $j = 1, \cdots, n$, we let $T_w^{(j)}$ denote the endomorphism

$$\left[\iota_w^{-1}\left(\mathrm{GL}_n(\mathcal{O}_{F_w})\begin{pmatrix}\varpi_w 1_j & 0\\ 0 & 1_{n-j}\end{pmatrix}\mathrm{GL}_n(\mathcal{O}_{F_w})\right) \times U(\mathfrak{p}^{b,c})^v\right]$$

of $S_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A))$. It is independent of the choice of the uniformizer. The operators $T_w^{(j)}$, for varying w and j, all commute with each other. Also note that

$$T_{w^c}^{(j)} = (T_w^{(n)})^{-1} T_w^{(n-j)}.$$

Hecke operators at places dividing p. For each $0 \le b \le c$ with $c \ge 1$, and each $v \in \Sigma_p$, the algebra

$$\mathcal{O}_K[\mathrm{Iw}(\tilde{v}^{b,c})\backslash \mathrm{GL}_n(F_{\tilde{v}})/\mathrm{Iw}(\tilde{v}^{b,c})]$$

is non-commutative and acts on $S_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A)$ only when A is a K-module. Following Hida, we consider a commutative subalgebra of this algebra and modify the usual action of the Hecke operators to define an action of this commutative subalgebra on $S_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A)$ for any $\mathcal{O}_K$-module A. This modified action depends on the weight λ.

Let A be an $\mathcal{O}_K$-module and λ be a dominant weight for G. Suppose that $0 \le b \le c$ with $c \ge 1$. For each $v \in \Sigma_p$ and $j = 1, \cdots, n$, put

$$\alpha_{\varpi_{\tilde{v}}}^{(j)} = \begin{pmatrix}\varpi_{\tilde{v}} 1_j & 0\\ 0 & 1_{n-j}\end{pmatrix} \in \mathrm{GL}_n(F_{\tilde{v}}).$$

We will also regard $\alpha_{\varpi_{\tilde{v}}}^{(j)}$ as an element of $G(F_v^+)$ and $G(\mathbb{A}_{F^+}^\infty)$ via $\iota_{\tilde{v}}$. If $v \in \Sigma_p$ then we let $U_{\lambda,\varpi_{\tilde{v}}}^{(j)}$ be the operator which acts on $S_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A)$ via

$$(w_0\lambda)(\alpha_{\varpi_{\tilde{v}}}^{(j)})^{-1}[U(\mathfrak{p}^{b,c})\alpha_{\varpi_{\tilde{v}}}^{(j)}U(\mathfrak{p}^{b,c})].$$

Explicitly, if we write $U(\mathfrak{p}^{b,c})\alpha_{\varpi_{\tilde{v}}}^{(j)}U(\mathfrak{p}^{b,c})$ as a disjoint union $\coprod_i x_i\alpha_{\varpi_{\tilde{v}}}^{(j)}U(\mathfrak{p}^{b,c})$, then for any $f \in S_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A)$ we define

$$U_{\lambda,\varpi_{\tilde{v}}}^{(j)} f = (w_0\lambda)(\alpha_{\varpi_{\tilde{v}}}^{(j)})^{-1}\sum_i (x_i\alpha_{\varpi_{\tilde{v}}}^{(j)}) \cdot f$$

where $w_0\lambda$ is considered as a character $T_n(F_v^+) \to K^\times$ as in Definition 4.1.2. This is an element of $S_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A)$ and is independent of the choice of x_i.

Now for $v \in \Sigma_p$ and $u \in T_n(\mathcal{O}_{F_{\tilde{v}}})$, let $\langle u\rangle$ denote the operator

$$[U(\mathfrak{p}^{b,c})uU(\mathfrak{p}^{b,c})]$$

acting on $S_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)$. For

$$u \in T_n(\mathcal{O}_{F^+,p}) = \prod_{v\in\Sigma_p} T_n(\mathcal{O}_{F_v^+}) \cong \prod_{v\in\Sigma_p} T_n(\mathcal{O}_{F_{\tilde{v}}}),$$

we define

$$\langle u\rangle := \prod_{v\in\Sigma_p} \langle u_v\rangle.$$

4.2.1.2. *Unitary Group Hecke algebras.*

Lemma 4.2.1. *For $0 \le b \le c$ with $c \ge 1$, a dominant weight λ for G and an $\mathcal{O}_K$-module A, the operators $T_w^{(j)}$, $U_{\lambda,\varpi_{\tilde{v}}}^{(j)}$ and $\langle u\rangle$ on $S_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)$ defined above commute with each other. Moreover, if $b \le b'$ and $c \le c'$, then the inclusion*

$$S_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A) \hookrightarrow S_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b',c'}),A)$$

is equivariant for all of the operators $T_w^{(j)}$, $U_{\lambda,\varpi_{\tilde{v}}}^{(j)}$ and $\langle u\rangle$.

Proof. Follows from the proof of [**Hid95**, Proposition 2.2] (*cf.* [**Ger10**, Lemma 2.3.3]). □

Definition 4.2.2. *For $0 \le b \le c$ with $c \ge 1$, a dominant weight λ for G and an $\mathcal{O}_K$-algebra A, let*

$$h^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A) \subset \tilde{h}^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A) \subset \mathrm{End}(S_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A))$$

be the A-subalgebras generated by the operators $T_w^{(j)}, (T_w^{(n)})^{-1}$ and $\langle u\rangle$ in the first case and the operators $T_w^{(j)}$, $(T_w^{(n)})^{-1}$, $U_{\lambda,\varpi_{\tilde{v}}}^{(j)}$ and $\langle u\rangle$ in the second case.

Note that the map $u \mapsto \langle u\rangle$ defines a homomorphism

$$T_n(\mathcal{O}_{F^+,p}/\mathfrak{p}^b) \to h^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)^{\times}. \tag{4.2.1}$$

4.2.2. Ordinary Hecke algebras. Let A be an $\mathcal{O}_K$-algebra of finite type. Since $\tilde{h}^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)$ is a finite type $\mathcal{O}_K$-algebra, it decomposes as a direct product

$$\tilde{h}^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A) = \prod_{\mathfrak{m}} \tilde{h}^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)_{\mathfrak{m}}$$

where $\mathfrak{m}$ runs over the set of maximal ideals of $\tilde{h}^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)$ (by [**Eis95**, Corollary 7.6, p. 188] for instance).

Definition 4.2.3. *A maximal ideal $\mathfrak{m}$ of $\tilde{h}^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)$ is called* ordinary *if for each $v \in \Sigma_p$ and for each $j = 1,\cdots,n$, the image of $U_{\lambda,\varpi_{\tilde{v}}}^{(j)}$ is nonzero in $\tilde{h}^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)/\mathfrak{m}$.*

We define the *ordinary Hecke algebra*

$$\tilde{h}^{\Sigma,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A) = \prod_{\mathfrak{m}} \tilde{h}^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)_{\mathfrak{m}}$$

where $\mathfrak{m}$ runs over the ordinary maximal ideals. We let $h^{\Sigma,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)$ denote the image of $h^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)$ in $\tilde{h}^{\Sigma,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)$. Since $\tilde{h}^{\Sigma,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)$ is a direct factor of $\tilde{h}^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)$, it corresponds to an idempotent $e \in \tilde{h}^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A)$ with the property that

$$\tilde{h}^{\Sigma,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A) = e\tilde{h}^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}),A).$$

If we let $U(\mathfrak{p})$ denote the product

$$U(\mathfrak{p}) := \prod_{v\in\Sigma_p}\prod_{j=1}^{n} U^{(j)}_{\lambda,\varpi_{\tilde{v}}} \in \tilde{h}^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A),$$

then one can check that

$$e = \lim_{r\to\infty} U(\mathfrak{p})^{r!} \in \tilde{h}^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A).$$

Now define the *ordinary parts* of $S_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A)$ by

$$S^{\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A) = eS_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A) = \bigoplus_{\mathfrak{m}\,\mathrm{ord}} S_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A)_{\mathfrak{m}},$$

where $\mathfrak{m}$ runs over the ordinary maximal ideals of $\tilde{h}^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A)$. The algebras $h^{\Sigma,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A)$ and $\tilde{h}^{\Sigma,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A)$ act faithfully on $S^{\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), A)$. The lemma below guarantees that ordinary forms exist.

Recall that an open compact subgroup of $G(\mathbb{A}^{\infty}_{F^+})$ is said to be *sufficiently small* if for some place v of F^+, its projection to $G(F^+_v)$ contains no element of finite order other than the identity.

Lemma 4.2.4. *Suppose that U is sufficiently small and $c \geq n-1$. Then $S^{\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), \mathcal{O}_K) \neq 0$.*

Proof. This lemma can be deduced from [**Hid95**, Proposition 2.2] (see [**Ger10**, Lemma 2.4.3] for details). □

Remark 4.2.5. Each $\mathcal{O}_K$-algebra homomorphism from $h^{\Sigma}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), \mathcal{O}_K)$ to $\overline{\mathbb{Q}}_p$ determines an irreducible constituent π of the $G(\mathbb{A}^{\infty,\Sigma'}_{F^+}) \times \prod_{v\in\Sigma'} \mathrm{Iw}(\tilde{v})$-representation $S_{\lambda,\{\chi_v\}}(\overline{\mathbb{Q}}_p)$ such that

$$\pi^{U(\mathfrak{p}^{b,c})} \cap S_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), \overline{\mathbb{Q}}_p) \neq 0.$$

Such representations π are called *ordinary automorphic representations* (of weight λ).

4.2.3. Universal ordinary Hecke algebras.

4.2.3.1. *Vertical control theorem.*

Lemma 4.2.6. *For $1 \leq b \leq c$, the natural inclusion*

$$S^{\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,b}), \mathcal{O}_K) \to S^{\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{b,c}), \mathcal{O}_K)$$

is an isomorphism.

Proof. This follows from [**Ger10**, Lemma 2.5.2]. □

For each $b \geq 1$, we let $T_n(\mathfrak{p}^b)$ denote the subgroup of $T_n(\mathcal{O}_{F^+,p})$ defined by the exact sequence

$$0 \to T_n(\mathfrak{p}^b) \to T_n(\mathcal{O}_{F^+,p}) \to T_n(\mathcal{O}_{F^+}/\mathfrak{p}^b) \to 0.$$

We let $T_n(\mathfrak{p}) = T_n(\mathfrak{p}^1)$ and we define the completed group algebras

$$\Lambda_b = \mathcal{O}_K[[T_n(\mathfrak{p}^b)]] = \varprojlim_{b' \geq b} \mathcal{O}_K[T_n(\mathfrak{p}^b)/T_n(\mathfrak{p}^{b'})] \quad \text{for } b \geq 1, \quad \Lambda = \Lambda_1,$$
$$\Lambda^+ = \mathcal{O}_K[[T_n(\mathcal{O}_{F^+,p})]] = \varprojlim_{b \geq 1} \mathcal{O}_K[T_n(\mathcal{O}_{F^+,p})/T_n(\mathfrak{p}^b)] \simeq \Lambda[T_n(\mathcal{O}_{F^+}/\mathfrak{p})].$$

Note that Λ^+ is automatically a Λ_b algebra for $b \geq 1$. Let

$$h^{\Sigma,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K) := \varprojlim_{c \geq 1} h^{\Sigma,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{c,c}), \mathcal{O}_K)$$

and note that it naturally has a Λ^+-algebra structure by equation (4.2.1).

Lemma 4.2.7. *The Hecke algebra* $h^{\Sigma,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)$ *is a finite faithful* Λ_{b_0}*-algebra where* $b_0 \geq 1$ *is large enough so that* $U(\mathfrak{p}^{b_0,b_0})$ *is sufficiently small.*

Proof. It follows from [**Ger10**, Corollary 2.5.4]. □

4.2.3.2. *Weight independence.*

Theorem 4.2.8. *There is an* $\mathcal{O}_K$*-algebra isomorphism*

$$\varphi_\lambda : h^{\Sigma,\mathrm{ord}}_{0,\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K) \xrightarrow{\sim} h^{\Sigma,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)$$

which satisfies

(a) $\varphi_\lambda(T^{(j)}_w) = T^{(j)}_w$ *and* $\varphi_\lambda(U^{(j)}_{0,\varpi_{\tilde{v}}}) = U^{(j)}_{\lambda,\varpi_{\tilde{v}}}$,
(b) $\varphi_\lambda(\langle u \rangle) = (w_0\lambda)(u^{-1})\langle u \rangle$ *for all* $u \in T_n(\mathcal{O}_{F^+,p})$.

Proof. Follows from [**Ger10**, Proposition 2.6.1, Corollary 2.5.4]. □

Now we renormalize the Λ-algebra structure on $h^{\Sigma,\mathrm{ord}}_{0,\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)$.

Definition 4.2.9. *Let* $\nu = (\nu_\tau)_\tau \in (\mathbb{Z}^n_+)^{\tilde{I}_p}$ *be the element with* $\nu_\tau = (n-1, n-2, \cdots, 0)$ *for all* τ*. Define a homomorphism*

$$T_n(\mathfrak{p}) \to h^{\Sigma,\mathrm{ord}}_{0,\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)^\times$$

by

$$u \mapsto (w_0\nu)^{-1}(u)\langle u \rangle.$$

This gives rise to an $\mathcal{O}_K$*-algebra homomorphism* $\Lambda \to h^{\Sigma,\mathrm{ord}}_{0,\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)$*. We define the* universal ordinary Hecke algebra $h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)$ *to be* $h^{\Sigma,\mathrm{ord}}_{0,\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)$ *equipped with this new* Λ*-algebra structure.*

We give it the structure of a $\Lambda^+ = \Lambda \otimes_{\mathcal{O}_K} \mathcal{O}_K[T_n(\mathcal{O}_{F^+}/\mathfrak{p})]$*-algebra using the new* Λ*-algebra structure and the original* $\mathcal{O}_K[T_n(\mathcal{O}_{F^+}/\mathfrak{p})]$*-structure.*

4.2.3.3. *Control theorem.* Let A be a finite type $\mathcal{O}_K$-subalgebra of $\overline{\mathbb{Z}}_p$,

$$\alpha : T_n(\mathfrak{p}) \to A^\times$$

be a finite order character. Suppose that $r \geq 1$ is large enough so that

$$T_n(\mathfrak{p}^r) \subset \ker(\alpha).$$

Denote by $S^{\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{r,r}), \alpha, A)$ the maximal subspace of $S^{\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{r,r}), A)$ on which $\langle u \rangle = \alpha(u)$ for all $u \in T_n(\mathfrak{p})$. Let $h^{\Sigma,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{r,r}), \alpha, A)$ denote the quotient of $h^{\Sigma,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{r,r}), A)$ obtained by restricting operators to $S^{\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{r,r}), \alpha, A)$. These algebras are independent of the choice of r.

For a finite order character $\alpha : T_n(\mathfrak{p}) \to \overline{\mathbb{Q}}_p^\times$ and a dominant weight λ for G, define $\wp_{\lambda,\alpha}$ to be the kernel of the $\mathcal{O}_K$-algebra homomorphism $\Lambda \to \overline{\mathbb{Q}}_p$ induced by the character

$$\alpha(w_0\nu)^{-1}(w_0\lambda)^{-1} : T_n(\mathfrak{p}) \to \overline{\mathbb{Q}}_p^\times.$$

Theorem 4.2.10. *Let λ be a dominant weight for G and $\alpha : T_n(\mathfrak{p}) \to \overline{\mathbb{Q}}_p^\times$ be a finite order character with $T_n(\mathfrak{p}^r) \subset \ker(\alpha)$ for some integer $r \geq 1$. Let K' denote the fraction field of $\Lambda/\wp_{\lambda,\alpha}$. Then the map φ_λ induces surjection of finite K'-algebras*

$$h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K) \otimes_\Lambda \Lambda_{\wp_{\lambda,\alpha}}/\wp_{\lambda,\alpha} \twoheadrightarrow h^{\Sigma,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{p}^{r,r}), \alpha, K')$$

whose kernel is nilpotent.

4.2.3.4. *Arithmetic primes.* An *arithmetic prime* of a finite Λ-algebra R is a prime $\wp \in \mathrm{Spec}(R)$ whose contraction to Λ is of the form $\wp_{\lambda,\alpha}$. In this case, λ is said to be the *weight* of $\wp$. An *arithmetic specialization* of R is an $\mathcal{O}_K$-algebra homomorphism $R \to \overline{\mathbb{Q}}_p$ whose kernel is an arithmetic prime. The *weight* of an arithmetic specialization is the weight of its kernel. The set of arithmetic primes of R is denoted by $\mathrm{Spec}^{\mathrm{arith}}(R)$.

Since $h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)$ is a finite type Λ-algebra and $\mathrm{Spec}^{\mathrm{arith}}(\Lambda)$ is dense in $\mathrm{Spec}(\Lambda)$ by [**Hid88a**, Lemma 10.2, p. 371], it follows that $\mathrm{Spec}^{\mathrm{arith}}(h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K))$ is dense in $\mathrm{Spec}(h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K))$.

By the above theorem and remark 4.2.5, an arithmetic specialization of $h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)$ of weight λ determines an ordinary automorphic representation π_ζ of weight λ.

4.2.4. Galois representations.

Proposition 4.2.11. *Let $\mathfrak{m}$ be a maximal ideal of $h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)$. Then there is a unique semisimple representation*

$$\overline{r}_\mathfrak{m} : G_F \to \mathrm{GL}_n(h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)/\mathfrak{m})$$

characterized by the following properties:

(a) if $v \notin \Sigma$ is a finite place of F^+ which splits as ww^c in F, then ρ is unramified at w and w^c,

(b) if $v \notin \Sigma$ is a place of F^+ which splits as ww^c in F and Fr_w is the geometric Frobenius element of G_{F_w}/I_{F_w}, then $r_{\mathfrak{m}}(\mathrm{Fr}_w)$ has characteristic polynomial

$$X^n - T_w^{(1)} X^{n-1} + \cdots + (-1)^j (\mathbf{N}w)^{j(j-1)/2} T_w^{(j)} X^{n-j} + \cdots + (-1)^n (\mathbf{N}w)^{n(n-1)/2} T_w^{(n)}.$$

Proof. Follows from [**Ger10**, Proposition 2.7.3]. □

A maximal ideal $\mathfrak{m}$ of $h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)$ is said to be *non-Eisenstein* if $\overline{r}_{\mathfrak{m}}$ is absolutely irreducible.

Proposition 4.2.12. *Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)$. Then there is a continuous lifting*

$$r_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)_{\mathfrak{m}})$$

of $\overline{r}_{\mathfrak{m}}$ satisfying the following properties. The first two properties determine the lifting $r_{\mathfrak{m}}$ uniquely up to conjugation by elements of $\mathrm{GL}_n(h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)_{\mathfrak{m}})$ which are trivial modulo $\mathfrak{m}$.

(a) If $v \notin \Sigma$ is a finite place of F^+ which splits as ww^c in F, then ρ is unramified at w and w^c.

(b) If $v \notin \Sigma$ is a place of F^+ which splits as ww^c in F and Fr_w is the geometric Frobenius element of G_{F_w}/I_{F_w}, then $r_{\mathfrak{m}}(\mathrm{Fr}_w)$ has characteristic polynomial

$$X^n - T_w^{(1)} X^{n-1} + \cdots + (-1)^j (\mathbf{N}w)^{j(j-1)/2} T_w^{(j)} X^{n-j} + \cdots + (-1)^n (\mathbf{N}w)^{n(n-1)/2} T_w^{(n)}.$$

(c) For each place w of F lying above p, there exists an n-tuple of characters $(\chi_{w1}, \cdots, \chi_{wn})$ such that $r_{\mathfrak{m}}|_{G_{F_w}}$ is conjugate to an upper triangular representation with the ordered tuple $(\chi_{w1}, \cdots, \chi_{wn})$ along the diagonal. In particular, for any $\mathcal{O}_K$-algebra homomorphism $\zeta : h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)_{\mathfrak{m}} \to \overline{\mathbb{Z}}_p$, the representation $(\zeta \circ r_{\mathfrak{m}})|_{G_{F_w}}$ is conjugate to an upper triangular representation with the ordered tuple $(\zeta \circ \chi_{w1}, \cdots, \zeta \circ \chi_{wn})$ along the diagonal.

Proof. It follows from [**GG12**, p. 267–268] (which relies on [**Ger10**, Proposition 2.7.4] for part (a), (b), and on [**Ger10**, Corollary 3.1.4, Prop 2.7.2(2)] for part (c)). □

If $\mathfrak{m}$ is a non-Eisenstein ideal of $h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)$, then the representation $r_{\mathfrak{m}}$ interpolates the Galois representations attached to the ordinary automorphic representations corresponding to the arithmetic primes of $h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)_{\mathfrak{m}}$.

4.3. Algebraic p-adic L-function along branches

In this section, we construct algebraic p-adic L-functions $L^{\mathrm{alg}}_{p',\mathrm{Gr}}$, $L^{\mathrm{alg}}_{p,\mathrm{Kato}}$ along irreducible components of the Hida family and show that it satisfies a control theorem at arithmetic primes.

Let $\mathfrak{m}$ be a maximal ideal of $h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)$ satisfying the following.

Assumption 4.3.1. *The maximal ideal $\mathfrak{m}$ is non-Eisenstein.*

Suppose that $\mathfrak{a}$ is a minimal prime of $h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)$ contained in $\mathfrak{m}$. Then from §4.2.4, we obtain a uniquely determined representation $\rho : G_F \to \mathrm{GL}_n(R(\mathfrak{a})')$ where $R(\mathfrak{a})' = h^{\Sigma,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{p}^\infty), \mathcal{O}_K)_{\mathfrak{m}}/\mathfrak{a}$. Let $R(\mathfrak{a})$ denote the subalgebra of $\mathcal{K} := \mathrm{Frac}(R(\mathfrak{a})')$ obtained by adjoining to $R(\mathfrak{a})'$ the coefficients of the characteristic polynomial of Fr_w on the I_{F_w}-invariants of ρ for the places w of F at which ρ is ramified and has nonzero I_{F_w}-invariants. The ring $R(\mathfrak{a})$ is a complete local domain and a finite type Λ-module ([**Eis95**, Corollary 7.6, p. 188]). Now define $\mathcal{T}(\mathfrak{a}) := R(\mathfrak{a})^n$ with a G_F-action on it via ρ.

Let S denote a finite set of places of F containing the places of ramification of $\mathcal{T}(\mathfrak{a})$, the archimedean places of F and the places of F above p. Denote by S_f the set of finite places in S. We will consider $\mathcal{T}(\mathfrak{a})$ as a representation of $G_{F,S}$.

For a ring homomorphism $\phi : R(\mathfrak{a}) \to R'$, the *$\phi$-specialization* of $\mathcal{T}(\mathfrak{a})$ is denoted by T_ϕ and is defined to be the $G_{F,S}$-representation $\mathcal{T}(\mathfrak{a}) \otimes_{R(\mathfrak{a}),\phi} R'$ with coefficients in R'. From now on we denote the image of an arithmetic specialization $\zeta : R(\mathfrak{a}) \to \overline{\mathbb{Q}}_p$ by $\mathcal{O}_\zeta$ and consider such maps as ring homomorphisms onto their images, *i.e.*, as $\zeta : R(\mathfrak{a}) \twoheadrightarrow \mathcal{O}_\zeta$. Thus for an arithmetic specialization ζ of $R(\mathfrak{a})$, the ζ-specialization T_ζ of $\mathcal{T}(\mathfrak{a})$ will denote the $G_{F,S}$-representation $\mathcal{T}(\mathfrak{a}) \otimes_{R(\mathfrak{a}),\zeta} \mathcal{O}_\zeta$. For such a specialization, we denote by V_ζ the $G_{F,S}$-representation $T_\zeta \otimes_{\mathcal{O}_\zeta} \overline{\mathbb{Q}}_p$.

In the following, w will denote a finite place of F.

For $w \mid p$, let $\mathcal{T}(\mathfrak{a})^+$ (resp. T_ζ^+) denote the largest R-submodule of $\mathcal{T}(\mathfrak{a})$ (resp. T_ζ where ζ denotes an arithmetic specialization of $R(\mathfrak{a})$ of regular dominant weight such that $V_\zeta|_{G_{F_w}}$ is crystalline) on which G_{F_w}-acts by the character χ_{w1} (resp. $\zeta \circ \chi_{w1}$).

Let F_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of F. We denote the Galois group $\mathrm{Gal}(F_\infty/F)$ by Γ. Denote the Iwasawa algebra $\mathcal{O}_K[[\Gamma]]$ by Λ_{Iw}, which is a $G_{F,\{w|p\}}$-module via the map $G_{F,\{w|p\}} \twoheadrightarrow \Gamma \hookrightarrow \Lambda_{\mathrm{Iw}}^\times$ since F_∞ is unramified at places $w \nmid p$. For any finite type $\mathcal{O}_K$-subalgebra A of $\overline{\mathbb{Z}}_p$, we will write Λ_A to denote $A \otimes_{\mathcal{O}_K} \Lambda_{\mathrm{Iw}} = A[[\Gamma]]$. We will consider Λ_A as a $G_{F,\{w|p\}}$-module via the map $G_{F,\{w|p\}} \twoheadrightarrow \Gamma \hookrightarrow \Lambda_A^\times$. The image of an element $g \in G_{F,\{w|p\}}$ under this map will be denoted by $[g]$. The completed tensor product $R(\mathfrak{a})\widehat{\otimes}_{\mathcal{O}_K}\Lambda_{\mathrm{Iw}}$ will be denoted by $R(\mathfrak{a})_{\mathrm{Iw}}$.

Define the *cyclotomic deformation* $\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}$ of $\mathcal{T}(\mathfrak{a})$ as the $G_{F,S}$-representation $\mathcal{T}(\mathfrak{a})\widehat{\otimes}_{\mathcal{O}_K}\Lambda_{\mathrm{Iw}}$ over $R(\mathfrak{a})_{\mathrm{Iw}}$ obtained by tensoring the $G_{F,S}$-representations $\mathcal{T}(\mathfrak{a})$ and Λ_{Iw}. Define the G_p-representation

$$\mathcal{T}(\mathfrak{a})^+_{\mathrm{Iw}} = \mathcal{T}(\mathfrak{a})^+\widehat{\otimes}_{\mathcal{O}_K}\Lambda_{\mathrm{Iw}}.$$

For an arithmetic specialization ζ of $R(\mathfrak{a})$, define the *cyclotomic deformation* $T_{\zeta,\mathrm{Iw}}$ of T_ζ as the $G_{F,S}$-representation $T_\zeta \otimes_{\mathcal{O}_K} \Lambda_{\mathrm{Iw}}$ over $\mathcal{O}_\zeta \otimes_{\mathcal{O}_K} \Lambda_{\mathrm{Iw}} = \Lambda_{\mathcal{O}_\zeta}$. Define the G_p-representation

$$T^+_{\zeta,\mathrm{Iw}} = T^+_\zeta \otimes_{\mathcal{O}_K} \Lambda_{\mathrm{Iw}}.$$

Note that each arithmetic specialization $\zeta : R(\mathfrak{a}) \to \mathcal{O}_\zeta$ of $R(\mathfrak{a})$ extends to a Λ_{Iw}-algebra homomorphism $\zeta\widehat{\otimes}_{\mathcal{O}_K}\mathrm{id}_{\Lambda_{\mathrm{Iw}}} : R(\mathfrak{a})_{\mathrm{Iw}} \to \mathcal{O}_\zeta \otimes_{\mathcal{O}_K} \Lambda_{\mathrm{Iw}} = \Lambda_{\mathcal{O}_\zeta}$, which will be denoted by ζ by abuse of language.

Definition 4.3.2. *Let T be a free module of rank $n \in \mathbb{Z}_{\geq 1}$ over a complete local noetherian domain R. Let $G_{F,S}$ act continuously on T via a representation $G_{F,S} \to \mathrm{Aut}_R(T)$. Suppose that the characteristic polynomial of Fr_w on $T^{I_{F_w}} \otimes_R \mathrm{Frac}(R)$, denoted $CP_w(X,T)$, has coefficients in R whenever $0 < \mathrm{rk}_R T^{I_{F_w}} < n$ for $w \nmid p$.*

For any w not dividing p, let $U'_w(T)$ denote the object in the derived category of R-modules corresponding to

$$\begin{cases} [R \xrightarrow{CP_w(1,T)} R] \text{ concentrated in degree 0,1} & \text{if } 0 < \mathrm{rk}_R T^{I_{F_w}} < n, \\ C^\bullet_{\mathrm{cont}}(G_{F_w}/I_{F_w}, T^{I_{F_w}}) & \text{otherwise.} \end{cases}$$

Definition 4.3.3. *Let ζ denote an arithmetic specialization of $R(\mathfrak{a})$ such that $V_\zeta|_{G_{F_w}}$ is crystalline for any $w \mid p$. For $w \mid p$, put*

$$U'_w(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) = R\Gamma_{\mathrm{cont}}(G_{F_w}, R(\mathfrak{a})_{\mathrm{Iw}})$$
$$U'_w(T_{\zeta,\mathrm{Iw}}) = R\Gamma_{\mathrm{cont}}(G_{F_w}, \Lambda_{\mathcal{O}_\zeta})$$

where G_{F_w} acts on $R(\mathfrak{a})_{\mathrm{Iw}}$ (resp. $\Lambda_{\mathcal{O}_\zeta}$) by the character through which it acts on $\mathcal{T}(\mathfrak{a})^+_{\mathrm{Iw}}$ (resp. $T^+_{\zeta,\mathrm{Iw}}$). For $T = \mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}, T_{\zeta,\mathrm{Iw}}$, define the algebraic p-adic L-functions $L^{\mathrm{alg}}_{p,\mathrm{Kato}}(T)$, $L^{\mathrm{alg}}_{p',\mathrm{Gr}}(T)$ *as the objects of* $\mathrm{Parf\text{-}is}_R$ *($R = R(\mathfrak{a})_{\mathrm{Iw}}, \Lambda_{\mathcal{O}_\zeta}$ respectively) given by*

$$L^{\mathrm{alg}}_{p,\mathrm{Kato}}(T) := \det\nolimits_R(R\Gamma_{c,\mathrm{cont}}(G_{F,S},T)[1]) \otimes \det\nolimits_R\left(\bigoplus_{\substack{w\in S_f\\ w\nmid p}} U'_w(T)[1]\right), \tag{4.3.1}$$

$$L^{\mathrm{alg}}_{p',\mathrm{Gr}}(T) := \det\nolimits_R(R\Gamma_{c,\mathrm{cont}}(G_{F,S},T)[1]) \otimes \det\nolimits_R\left(\bigoplus_{w\mid p} U'_w(T)[1] \oplus \bigoplus_{\substack{w\in S_f\\ w\nmid p}} U'_w(T)[1]\right) \tag{4.3.2}$$

respectively.

Lemma 4.3.4. *The above objects $L^{\mathrm{alg}}_{p,\mathrm{Kato}}(T)$ and $L^{\mathrm{alg}}_{p',\mathrm{Gr}}(T)$ are well-defined for $T = \mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}, T_{\zeta,\mathrm{Iw}}$, where ζ is as in the above definition.*

Proof. The rings $R(\mathfrak{a})$ and $\mathcal{O}_\zeta$ are complete local rings (by [**Eis95**, Corollary 7.6, p. 188] for instance). So $R(\mathfrak{a})_{\mathrm{Iw}}$ and $\Lambda_{\mathcal{O}_\zeta}$ are complete local rings.

By definition of $R(\mathfrak{a})$ and $\mathcal{O}_\zeta$, the polynomials $CP_w(X, \mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ and $CP_w(X, T_{\zeta,\mathrm{Iw}})$ have coefficients in $R(\mathfrak{a})_{\mathrm{Iw}}$ and $\Lambda_{\mathcal{O}_\zeta}$ respectively for any $w \nmid p$ (by theorem 1.2.4(6) and proposition 4.1.5). So $U'_w(T)$ is well-defined and by proposition 2.2.1, it is a perfect complex for $w \in S_f, w \nmid p$. So $L^{\mathrm{alg}}_{p,\mathrm{Kato}}(T)$ is well-defined (using the same proposition again).

By proposition 4.2.12, for $w \mid p$, the group G_{F_w} acts on $\mathcal{T}(\mathfrak{a})^+_{\mathrm{Iw}}$ (resp. $T^+_{\zeta,\mathrm{Iw}}$) by an $R(\mathfrak{a})_{\mathrm{Iw}}$-valued (resp. $\Lambda_{\mathcal{O}_\zeta}$-valued) character. So $U'_w(T)$ is well-defined for $w \mid p$ and they are perfect complexes by proposition 2.2.1. Using this proposition again, it follows that $L^{\mathrm{alg}}_{p',\mathrm{Gr}}(T)$ is well-defined. □

Lemma 4.3.5. *The arithmetic primes of $R(\mathfrak{a})$ which are kernels of the arithmetic specializations $\zeta : R(\mathfrak{a}) \to \overline{\mathbb{Z}}_p$ satisfying*

(1) ζ is of regular dominant weight,

(2) $V_\zeta|_{G_{F_w}}$ is crystalline for any place w of F lying above p,

form a dense subset of $\mathrm{Spec}(R(\mathfrak{a}))$.

Proof. By the comment after the proof of [**Ger10**, Lemma 2.6.4], [**Ger10**, Lemma 2.7.5(2), Proposition 2.7.2(2), (4)] and the last paragraph of the proof of [**Ger10**, Corollary 3.1.4], the lemma follows. $\square$

Theorem 4.3.6. *Let ζ be an arithmetic specialization of $R(\mathfrak{a})$ of regular dominant weight such that π_ζ is stable and $V_\zeta|_{G_{F_w}}$ is crystalline for all $w \mid p$. Then the isomorphisms in propositions 2.1.2, 2.2.1, 2.2.3 induce isomorphisms*

$$L^{\mathrm{alg}}_{p',\mathrm{Gr}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\zeta} \Lambda_{\mathcal{O}_\zeta} \cong L^{\mathrm{alg}}_{p',\mathrm{Gr}}(T_{\zeta,\mathrm{Iw}}), \tag{4.3.3}$$

$$L^{\mathrm{alg}}_{p,\mathrm{Kato}}(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}}) \otimes_{R(\mathfrak{a})_{\mathrm{Iw}},\zeta} \Lambda_{\mathcal{O}_\zeta} \cong L^{\mathrm{alg}}_{p,\mathrm{Kato}}(T_{\zeta,\mathrm{Iw}}) \tag{4.3.4}$$

under the assumption 4.3.1.

Proof. By proposition 2.1.2 and proposition 2.2.1, it remains to prove the control theorem for the factors coming from "local conditions". For $w \mid p$, the complex $U'_w(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is K-flat by [**Sta14**, Tag 064K] and hence the control of $U'_w(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ follows from [**Sta14**, Tag 06Y6]. So it remains to prove the control theorem at $w \nmid p$, *i.e.*, the ζ-specialization of $\det U'_w(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is $\det U'_w(T_{\zeta,\mathrm{Iw}})$. Let $w \nmid p$ denote a finite place of F. By proposition 2.1.2, it suffices to prove the control theorem for $U'_w(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$.

The restriction of the $G_{F,S}$-representation $\mathcal{T}(\mathfrak{a})$ to the decomposition group G_{F_w} is continuous and its coefficient ring $R(\mathfrak{a})$ has finite residue field of characteristic $p \neq \ell$. So by theorem 1.1.25, the G_{F_w}-representation $\mathcal{T}(\mathfrak{a})$ is monodromic. Moreover $V_\zeta|_{G_{F_w}}$ is pure for any arithmetic specialization ζ of $R(\mathfrak{a})$ and $w \nmid p$ (by proposition 4.1.5). So theorem 1.2.4 applies to $\mathcal{T}(\mathfrak{a})$ and its arithmetic specializations. By theorem 1.2.4(5) and proposition 2.2.1, we need to prove the control theorem for $U'_w(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ only when $0 < \mathrm{rk}_{R(\mathfrak{a})}\mathcal{T}(\mathfrak{a})^{I_{F_w}} < n$. Assume that this inequality holds. Then $U'_w(\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})$ is K-flat by [**Sta14**, Tag 064K]. So its derived tensor product over $R(\mathfrak{a})_{\mathrm{Iw}}$ with $\Lambda_{\mathcal{O}_\zeta}$ (through ζ) is equal to the tensor product by [**Sta14**, Tag 06Y6], *i.e.*, $[\Lambda_{\mathcal{O}_\zeta} \xrightarrow{\zeta(CP_w(1,\mathcal{T}(\mathfrak{a})_{\mathrm{Iw}})} \Lambda_{\mathcal{O}_\zeta}]$ and this is $U'_w(T_{\zeta,\mathrm{Iw}})$ by theorem 1.2.4(6). $\square$

APPENDIX A

Divisibility

A.1. Valuations

Let

$$v_p : \overline{\mathbb{Q}}_p \to \mathbb{Q} \cup \{\infty\}$$

denote the valuation normalized so that $v_p(p) = 1$. If ζ_{p^r} denotes a primitive p^r-th root of unity in $\overline{\mathbb{Q}}_p$ ($r \geq 1$), then

$$v_p(\zeta_{p^r} - 1) = \frac{1}{\varphi(p^r)} = \frac{1}{p^{r-1}(p-1)} \tag{A.1.1}$$

by [**Neu99**, Proposition 7.13, Chapter II]. For any integer $k \geq 2$,

$$v_p((1+p)^k - 1) \geq 1. \tag{A.1.2}$$

For any integer $k \geq 2$ and $1 \neq \zeta \in \mu_{p^\infty}(\overline{\mathbb{Z}}_p)$,

$$\zeta(1+p)^k - 1 = \big((\zeta - 1)(1+p)^k\big) + \big((1+p)^k - 1\big)$$

gives

$$v_p(\zeta(1+p)^k - 1) = v_p(\zeta - 1) \tag{A.1.3}$$

by equations (A.1.1), (A.1.2).

Let $K/\mathbb{Q}_p$ denote a finite extension contained inside $\overline{\mathbb{Q}}_p$. Let ϖ denote a uniformizer of $\mathcal{O}_K$.

Lemma A.1.1. *Let $f(X) \in \mathcal{O}_K[X]$ be a distinguished polynomial of degree $d \geq 1$. Let $k \geq 2$ denote an integer and ζ_{p^r} denote a primitive p^r-th root of unity. Then*

$$v_p(f(\zeta_{p^r}(1+p)^k - 1)) = \frac{d}{p^{r-1}(p-1)}$$

for $r \gg 0$.

Proof. Write

$$f(X) = c_0 + c_1X + \cdots + c_{d-1}X^{d-1} + X^d$$

with $c_0, \cdots, c_{d-1} \in \varpi\mathcal{O}_K$. Let t denote the least nonnegative integer such that $c_t \neq 0$. Put $c_d = 1$. So

$$f(X) = c_tX^t + \cdots + c_dX^d.$$

If $t = m$, then

$$\begin{aligned} v_p(f(\zeta_{p^r}(1+p)^k - 1)) &= v_p\Big(c_d(\zeta_{p^r}(1+p)^k - 1)^d\Big) \\ &= dv_p(\zeta_{p^r}(1+p)^k - 1) \\ &= dv_p(\zeta_{p^r} - 1) && \text{(by equation (A.1.3))} \\ &= \frac{d}{p^{r-1}(p-1)} && \text{(by equation (A.1.1)).} \end{aligned}$$

Now let $t < d$. Note that

$$v_p(c_t(\zeta_{p^r}(1+p)^k - 1)^t) = v_p(c_t) + \frac{t}{\varphi(p^r)}, \cdots, v_p(c_d(\zeta_{p^r}(1+p)^k - 1)^d) = v_p(c_d) + \frac{d}{\varphi(p^r)}.$$

So for any $t \leq s < d$,

$$v_p(c_s(\zeta_{p^r}(1+p)^k - 1)^s) > v_p(c_d(\zeta_{p^r}(1+p)^k - 1)^d)$$

as $r \gg 0$. Hence the lemma. □

A.2. Divisibility in $\mathcal{O}_K[[X]]$

Let $\mathfrak{m}_{\overline{\mathbb{Z}}_p}$ denote the maximal ideal of $\overline{\mathbb{Z}}_p$. The symbol η will be used to denote elements of $\mathfrak{m}_{\overline{\mathbb{Z}}_p}$. For $\eta \in \mathfrak{m}_{\overline{\mathbb{Z}}_p}$ and any finite extension $L/\mathbb{Q}_p$, the map

$$\mathcal{O}_L[[X]] \to \overline{\mathbb{Z}}_p, \quad X \mapsto \eta$$

is denoted by η by abuse of notation.

Lemma A.2.1. *Let α, β be two elements of $\mathcal{O}_K[[X]]$ with $\beta \neq 0$. Suppose that $\eta(\beta)$ divides $\eta(\alpha)$ for almost all $\eta \in \mathfrak{m}_{\overline{\mathbb{Z}}_p}$. Then β divides α in $\mathcal{O}_K[[X]]$.*

Proof. Suppose that α is zero. By Weierstrass preparation theorem,

$$\alpha(X) = \varpi^a P(X) U(X), \quad \beta(X) = \varpi^b Q(X) V(X)$$

where a, b are nonnegative integers, $U(X), V(X)$ are units in $\mathcal{O}_K[[X]]$ and $P(X), Q(X) \in \mathcal{O}_K[X]$ are distinguished polynomials. Without loss of generality, we assume that $U(X), V(X)$ are equal to 1. Put

$$\begin{aligned} P(X) &= a_0 + a_1 X + \cdots + a_{m-1} X^{m-1} + X^m, \\ Q(X) &= b_0 + b_1 X + \cdots + b_{n-1} X^{n-1} + X^n \end{aligned}$$

with $a_i, b_j \in \varpi\mathcal{O}_K$, $m, n \in \mathbb{Z}_{\geq 0}$. When m, n are zero, we interpret $P(X), Q(X)$ as 1.

We have

$$v_p(\alpha(\zeta_{p^r}(1+p)^k - 1)) \geq v_p(\beta(\zeta_{p^r}(1+p)^k - 1))$$

whenever $k \gg 0, r \gg 0$. Note that lemma A.1.1 remains valid even when $d = 0$. So lemma A.1.1 gives

$$av_p(\varpi) + \frac{m}{p^{r-1}(p-1)} \geq bv_p(\varpi) + \frac{n}{p^{r-1}(p-1)}$$

for $r \gg 0$. Thus $a \geq b$. So we may assume that $a \geq 0, b = 0$.

Write

$$\beta(X) = Q(X) = \prod_{i=1}^{I}(X - \alpha_i)^{n_i}$$

with $\alpha_i \in \overline{\mathbb{Z}}_p$. Note that $\alpha \in \mathfrak{m}_{\overline{\mathbb{Z}}_p}$. Let L/K denote a finite extension containing $\alpha_1, \cdots, \alpha_I$. So it suffices to prove that if $\eta(X - \alpha)$ divides $\eta(\varpi^a P(X))$ in $\overline{\mathbb{Z}}_p$ for almost all $\eta \in \mathfrak{m}_{\overline{\mathbb{Z}}_p}$ (with $\alpha \in \mathfrak{m}_{\overline{\mathbb{Z}}_p} \cap \mathcal{O}_L$), then $X - \alpha$ divides $\varpi^a P(X)$ in $\mathcal{O}_L[X]$, which is immediate. □

A.3. Divisibility in $\mathcal{R}$

Let $\mathcal{K}$ denote the fraction field of $\mathcal{O}_K[[X]]$. For an extension $\mathcal{L}/\mathcal{K}$ contained in $\overline{\mathcal{K}}$, the integral closure of $\mathcal{O}_K[[X]]$ in $\mathcal{L}$ is denoted by $\mathcal{O}_{\mathcal{L}}$. Let $\mathcal{R}$ denote a finite type $\mathcal{O}_K[[X]]$-subalgebra of $\mathcal{O}_{\overline{\mathcal{K}}}$. Its integral closure in its fraction field is denoted by $\mathcal{R}^{\text{int}}$.

Lemma A.3.1. *Let α, β be two elements of $\mathcal{R}$ with $\beta \neq 0$. Suppose that for almost all $\xi \in \text{Hom}_{\mathcal{O}_K\text{-alg}}(\mathcal{R}, \overline{\mathbb{Z}}_p)$, $\xi(\beta)$ divides $\xi(\alpha)$ in $\overline{\mathbb{Z}}_p$. Then β divides α in $\mathcal{R}^{\text{int}}$.*

Proof. Let $\mathcal{L}/\mathcal{K}$ denote a finite Galois extension containing α, β. Since $\mathcal{O}_{\mathcal{L}}$ is a finite type $\mathcal{R}$-algebra, $\xi(\beta)$ divides $\xi(\alpha)$ in $\overline{\mathbb{Z}}_p$ for almost all $\xi \in \text{Hom}_{\mathcal{O}_K\text{-alg}}(\mathcal{O}_{\mathcal{L}}, \overline{\mathbb{Z}}_p)$.

For each $\xi \in \text{Hom}_{\mathcal{O}_K\text{-alg}}(\mathcal{O}_K[[X]], \overline{\mathbb{Z}}_p)$, we fix a lift $\widetilde{\xi} \in \text{Hom}_{\mathcal{O}_K\text{-alg}}(\mathcal{O}_{\mathcal{L}}, \overline{\mathbb{Z}}_p)$. Note that for any $\sigma \in \text{Gal}(\mathcal{L}/\mathcal{K})$, $\widetilde{\xi} \circ \sigma$ is also an element of $\text{Hom}_{\mathcal{O}_K\text{-alg}}(\mathcal{O}_{\mathcal{L}}, \overline{\mathbb{Z}}_p)$. For almost all $\xi \in \text{Hom}_{\mathcal{O}_K\text{-alg}}(\mathcal{O}_K[[X]], \overline{\mathbb{Z}}_p)$, the images of the coefficients of

$$P(Y) = \prod_{\sigma \in \text{Gal}(\mathcal{L}/\mathcal{K})} (Y - \sigma(\alpha/\beta))$$

under $\widetilde{\xi}$ are elements of $\overline{\mathbb{Z}}_p$. Since $P(Y)$ has coefficients in $K((X))$, the images of its coefficients under ξ are elements of $\overline{\mathbb{Z}}_p$ for almost all $\xi \in \text{Hom}_{\mathcal{O}_K\text{-alg}}(\mathcal{O}_K[[X]], \overline{\mathbb{Z}}_p)$. In particular, the images of the coefficients of $P(Y)$ under η are elements of $\overline{\mathbb{Z}}_p$ for almost all $\eta \in \mathfrak{m}_{\overline{\mathbb{Z}}_p}$. By lemma A.2.1, $P(Y)$ has coefficients in $\mathcal{O}_K[[X]]$. So the element α/β of $\text{Frac}(\mathcal{R})$ is integral over $\mathcal{O}_K[[X]]$ and hence is an element of $\mathcal{R}^{\text{int}}$.

□

A.4. Divisibility in $\mathcal{R}[[T]]$

Let $\mathscr{O}[[T]]$ denote the $\overline{\mathbb{Z}}_p$-subalgebra of $\overline{\mathbb{Z}}_p[[T]]$ spanned by the subsets $\mathcal{O}_L[[T]]$ where L ranges over all finite extensions of $\mathbb{Q}_p$. Note that $\mathscr{O}[[T]]$ is smaller than $\overline{\mathbb{Z}}_p[[T]]$ and each element of $\mathscr{O}[[T]]$ lie in $\mathcal{O}_L[[T]]$ for some finite extension $L/\mathbb{Q}_p$ (depending on the element).

Proposition A.4.1. *Let $f(T), g(T)$ be two elements of $\mathcal{R}[[T]]$ where $g(T) \neq 0$. Suppose that $\xi(g(T))$ divides $\xi(f(T))$ in $\mathscr{O}[[T]]$ for almost all ξ in $\text{Hom}_{\mathcal{O}_K\text{-alg}}(\mathcal{R}, \overline{\mathbb{Z}}_p)$. Then $g(T)$ divides $f(T)$ in $\mathcal{R}^{\text{int}}[[T]]$.*

Proof. Write

$$f(T) = a_0 + a_1 T + \cdots, \quad g(T) = b_0 + b_1 T + \cdots.$$

Note that for an integer $r \geq 1$, if T^r divides $g(T)$, then it also divides $f(T)$. So without loss of generality, we may assume that $b_0 \neq 0$. Let

$$h(T) = c_0 + c_1 T + \cdots \in \text{Frac}(\mathcal{R})[[T]]$$

be such that

$$f(T) = h(T)g(T),$$

i.e., $c_0, c_1, \cdots \in \mathrm{Frac}(\mathcal{R})$ are defined by

$$\sum_{i+j=n} c_i b_j = a_n.$$

Since $\xi(c_0)$ is an element of $\overline{\mathbb{Z}}_p$ for almost all $\xi \in \mathrm{Hom}_{\mathcal{O}_K\text{-alg}}(\mathcal{R}, \overline{\mathbb{Z}}_p)$, by lemma A.3.1, c_0 belongs to $\mathcal{R}^{\mathrm{int}}$. Suppose that $c_0, \cdots, c_n$ are elements of $\mathcal{R}^{\mathrm{int}}$. Then the image of

$$c_{n+1} = \frac{a_{n+1} - \sum_{i=0}^{n} c_i b_{n+1-i}}{b_0}$$

under ξ is an element of $\overline{\mathbb{Z}}_p$ for almost all $\xi \in \mathrm{Hom}_{\mathcal{O}_K\text{-alg}}(\mathcal{R}, \overline{\mathbb{Z}}_p)$. By lemma A.3.1, $c_{n+1} \in \mathcal{R}^{\mathrm{int}}$. By induction, $c_i \in \mathcal{R}^{\mathrm{int}}$ for all $i \in \mathbb{Z}_{\geq 0}$.

□

A.5. Integrality of determinants

Let $\mathcal{O}$ be a finite type $\mathcal{O}_K$-subalgebra of $\overline{\mathbb{Z}}_p$. Let $\mathcal{O}^{\mathrm{int}}$ denote the integral closure of $\mathcal{O}$ in its fraction field and M denote a finitely generated torsion $\mathcal{O}[[T]]$-module. Suppose that M is a perfect $\mathcal{O}[[T]]$-module. The image of $\det_{\mathcal{O}[[T]]} M$ in $\mathrm{Frac}(\mathcal{O}[[T]])$ (considered without the grade) under the composite map

$$\begin{aligned}
\det\nolimits_{\mathcal{O}[[T]]} M &\hookrightarrow \big(\det\nolimits_{\mathcal{O}[[T]]} M\big) \otimes_{\mathcal{O}[[T]]} \mathrm{Frac}(\mathcal{O}[[T]]) \\
&\cong \det\nolimits_{\mathrm{Frac}(\mathcal{O}[[T]])} \big(M \otimes_{\mathcal{O}[[T]]} \mathrm{Frac}(\mathcal{O}[[T]])\big) \qquad \text{(by proposition 2.1.2)} \\
&= \det\nolimits_{\mathrm{Frac}(\mathcal{O}[[T]])}(0) \\
&= (\mathrm{Frac}(\mathcal{O}[[T]]), 0)
\end{aligned}$$

is free and hence equal to $(\beta/\alpha)\mathcal{O}[[T]]$ for some nonzero elements α, β of $\mathcal{O}[[T]]$.

Proposition A.5.1. *We have*

$$\mathrm{char}_{\mathcal{O}^{\mathrm{int}}[[T]]}(M \otimes_{\mathcal{O}[[T]]} \mathcal{O}^{\mathrm{int}}[[T]]) = \frac{\alpha}{\beta}\mathcal{O}^{\mathrm{int}}[[T]]. \tag{A.5.1}$$

Consequently, the element β divides α in $\mathcal{O}^{\mathrm{int}}[[T]]$.

Proof. The image of $\det_{\mathcal{O}[[T]]} M$ in $\mathrm{Frac}(\mathcal{O}[[T]])$ (considered without the grade) under the composite map

$$\begin{aligned}
\det\nolimits_{\mathcal{O}^{\mathrm{int}}[[T]]} \big(M \otimes_{\mathcal{O}[[T]]} \mathcal{O}^{\mathrm{int}}[[T]]\big) &\hookrightarrow \big(\det\nolimits_{\mathcal{O}^{\mathrm{int}}[[T]]}(M \otimes_{\mathcal{O}[[T]]} \mathcal{O}^{\mathrm{int}}[[T]])\big) \otimes_{\mathcal{O}^{\mathrm{int}}[[T]]} \mathrm{Frac}(\mathcal{O}[[T]]) \\
&\cong \det\nolimits_{\mathrm{Frac}(\mathcal{O}[[T]])} \big((M \otimes_{\mathcal{O}[[T]]} \mathcal{O}^{\mathrm{int}}[[T]]) \otimes_{\mathcal{O}^{\mathrm{int}}[[T]]} \mathrm{Frac}(\mathcal{O}[[T]])\big) \\
&= \det\nolimits_{\mathrm{Frac}(\mathcal{O}[[T]])}(0) \\
&= (\mathrm{Frac}(\mathcal{O}[[T]]), 0)
\end{aligned}$$

is $\big(\mathrm{char}_{\mathcal{O}^{\mathrm{int}}[[T]]}(M \otimes_{\mathcal{O}[[T]]} \mathcal{O}^{\mathrm{int}}[[T]])\big)^{-1}$. So equation (A.5.1) holds and hence β divides α in $\mathcal{O}^{\mathrm{int}}[[T]]$. □

Bibliography

[BBM82] Pierre Berthelot, Lawrence Breen, and William Messing, *Théorie de Dieudonné cristalline. II*, Lecture Notes in Mathematics, vol. 930, Springer-Verlag, Berlin, 1982. MR 667344 (85k:14023)

[BC09] Joël Bellaïche and Gaëtan Chenevier, *Families of Galois representations and Selmer groups*, Astérisque (2009), no. 324, xii+314. MR 2656025 (2011m:11105)

[Ber13] John Bergdall, *On the variation of (φ, Γ)-modules over p-adic families of automorphic forms*, ProQuest LLC, Ann Arbor, MI, 2013, Thesis (Ph.D.)–Brandeis University. MR 3192943

[BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR 1251956 (95h:13020)

[BH06] Colin J. Bushnell and Guy Henniart, *The local Langlands conjecture for* GL(2), Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006. MR 2234120 (2007m:22013)

[BJ79] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, With a supplement "On the notion of an automorphic representation" by R. P. Langlands, pp. 189–207. MR 546598 (81m:10055)

[BK90] Spencer Bloch and Kazuya Kato, *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 333–400. MR 1086888 (92g:11063)

[Bla06] Don Blasius, *Hilbert modular forms and the Ramanujan conjecture*, Noncommutative geometry and number theory, Aspects Math., E37, Vieweg, Wiesbaden, 2006, pp. 35–56. MR 2327298 (2008g:11077)

[Car86] Henri Carayol, *Sur les représentations l-adiques associées aux formes modulaires de Hilbert*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 3, 409–468. MR 870690 (89c:11083)

[Car12] Ana Caraiani, *Local-global compatibility and the action of monodromy on nearby cycles*, Duke Math. J. **161** (2012), no. 12, 2311–2413. MR 2972460

[CH] Gaëtan Chenevier and Michael Harris, *Construction of automorphic Galois representations*, preprint.

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 108, 1–181, With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras. MR 2470687 (2010j:11082)

[Clo13] Laurent Clozel, *Purity reigns supreme*, Int. Math. Res. Not. IMRN (2013), no. 2, 328–346. MR 3010691

[Del69] Pierre Deligne, *Formes modulaires et représentations l-adiques*, Seminaire Bourbaki **179** (Février 1969), 139–172 (1971).

[Del73a] P. Deligne, *Formes modulaires et représentations de* GL(2), Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1973, pp. 55–105. Lecture Notes in Math., Vol. 349. MR 0347738 (50 #240)

[Del73b] ———, *Les constantes des équations fonctionnelles des fonctions L*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1973, pp. 501–597. Lecture Notes in Math., Vol. 349. MR 0349635 (50 #2128)

[Del80] Pierre Deligne, *La conjecture de Weil. II*, Inst. Hautes Études Sci. Publ. Math. (1980), no. 52, 137–252. MR 601520 (83c:14017)

[Eic54] Martin Eichler, *Quaternäre quadratische Formen und die Riemannsche Vermutung für die Kongruenzzetafunktion*, Arch. Math. **5** (1954), 355–366. MR 0063406 (16,116d)

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 1322960 (97a:13001)

[EPW06] Matthew Emerton, Robert Pollack, and Tom Weston, *Variation of Iwasawa invariants in Hida families*, Invent. Math. **163** (2006), no. 3, 523–580. MR 2207234 (2007a:11059)

[FO12] Olivier Fouquet and Tadashi Ochiai, *Control theorems for Selmer groups of nearly ordinary deformations*, J. Reine Angew. Math. **666** (2012), 163–187. MR 2920885

[Fon04] Jean-Marc Fontaine, *Arithmétique des représentations galoisiennes p-adiques*, Astérisque (2004), no. 295, xi, 1–115, Cohomologies p-adiques et applications arithmétiques. III. MR 2104360 (2005i:11074)

[Fou13] Olivier Fouquet, *Dihedral Iwasawa theory of nearly ordinary quaternionic automorphic forms*, Compos. Math. **149** (2013), no. 3, 356–416. MR 3040744

[FPR94] Jean-Marc Fontaine and Bernadette Perrin-Riou, *Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 599–706. MR 1265546 (95j:11046)

[Gel75] Stephen S. Gelbart, *Automorphic forms on adèle groups*, Princeton University Press, Princeton, N.J., 1975, Annals of Mathematics Studies, No. 83. MR 0379375 (52 #280)

[Ger10] David James Geraghty, *Modularity lifting theorems for ordinary Galois representations*, ProQuest LLC, Ann Arbor, MI, 2010, Thesis (Ph.D.)–Harvard University. MR 2941425

[GG12] Toby Gee and David Geraghty, *Companion forms for unitary and symplectic groups*, Duke Math. J. **161** (2012), no. 2, 247–303. MR 2876931

[Gre89] Ralph Greenberg, *Iwasawa theory for p-adic representations*, Algebraic number theory, Adv. Stud. Pure Math., vol. 17, Academic Press, Boston, MA, 1989, pp. 97–137. MR 1097613 (92c:11116)

[Gre91] ———, *Iwasawa theory for motives*, L-functions and arithmetic (Durham, 1989), London Math. Soc. Lecture Note Ser., vol. 153, Cambridge Univ. Press, Cambridge, 1991, pp. 211–233. MR 1110394 (93a:11092)

[Hid86a] Haruzo Hida, *Galois representations into* $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ *attached to ordinary cusp forms*, Invent. Math. **85** (1986), no. 3, 545–613. MR 848685 (87k:11049)

[Hid86b] ———, *Iwasawa modules attached to congruences of cusp forms*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 2, 231–273. MR 868300 (88i:11023)

[Hid87] ———, *On p-adic Hecke algebras for* GL_2, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986) (Providence, RI), Amer. Math. Soc., 1987, pp. 434–443. MR 934243 (89d:11039)

[Hid88a] ———, *On p-adic Hecke algebras for* GL_2 *over totally real fields*, Ann. of Math. (2) **128** (1988), no. 2, 295–384. MR 960949 (89m:11046)

[Hid88b] ———, *A p-adic measure attached to the zeta functions associated with two elliptic modular forms. II*, Ann. Inst. Fourier (Grenoble) **38** (1988), no. 3, 1–83. MR 976685 (89k:11120)

[Hid89] ———, *On nearly ordinary Hecke algebras for* $\mathrm{GL}(2)$ *over totally real fields*, Algebraic number theory, Adv. Stud. Pure Math., vol. 17, Academic Press, Boston, MA, 1989, pp. 139–169. MR 1097614 (92f:11064)

[Hid95] ———, *Control theorems of p-nearly ordinary cohomology groups for* $\mathrm{SL}(n)$, Bull. Soc. Math. France **123** (1995), no. 3, 425–475. MR 1373742 (97c:11053)

[Hid98] ———, *Automorphic induction and Leopoldt type conjectures for* $\mathrm{GL}(n)$, Asian J. Math. **2** (1998), no. 4, 667–710, Mikio Sato: a great Japanese mathematician of the twentieth century. MR 1734126 (2000k:11064)

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR 1876802 (2002m:11050)

[Ill94] Luc Illusie, *Autour du théorème de monodromie locale*, Astérisque (1994), no. 223, 9–57, Périodes p-adiques (Bures-sur-Yvette, 1988). MR 1293970 (95k:14032)

[Iwa59] Kenkichi Iwasawa, *On Γ-extensions of algebraic number fields*, Bull. Amer. Math. Soc. **65** (1959), 183–226. MR 0124316 (23 #A1630)

[Iwa69] ———, *On p-adic L-functions*, Ann. of Math. (2) **89** (1969), 198–205. MR 0269627 (42 #4522)

[Jan03] Jens Carsten Jantzen, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003. MR 2015057 (2004h:20061)

[Kat93] Kazuya Kato, *Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR}. I*, Arithmetic algebraic geometry (Trento, 1991), Lecture Notes in Math., vol. 1553, Springer, Berlin, 1993, pp. 50–163. MR 1338860 (96f:11067)

[Kat04] ———, *p-adic Hodge theory and values of zeta functions of modular forms*, Astérisque (2004), no. 295, ix, 117–290, Cohomologies p-adiques et applications arithmétiques. III. MR 2104361 (2006b:11051)

[Kil02] L. J. P. Kilford, *Some non-Gorenstein Hecke algebras attached to spaces of modular forms*, J. Number Theory **97** (2002), no. 1, 157–164. MR 1939142 (2003j:11046)

[KM76] Finn Faye Knudsen and David Mumford, *The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"*, Math. Scand. **39** (1976), no. 1, 19–55. MR 0437541 (55 #10465)

[Lab11] J.-P. Labesse, *Changement de base CM et séries discrètes*, On the stabilization of the trace formula, Stab. Trace Formula Shimura Var. Arith. Appl., vol. 1, Int. Press, Somerville, MA, 2011, pp. 429–470. MR 2856380

[Mat80] Hideyuki Matsumura, *Commutative algebra*, second ed., Mathematics Lecture Note Series, vol. 56, Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980. MR 575344 (82i:13003)

[Mat89] ———, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR 1011461 (90i:13001)

[Mau04] David Mauger, *Algèbres de Hecke quasi-ordinaires universelles*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 2, 171–222. MR 2061780 (2005i:11072)

[Maz72] Barry Mazur, *Rational points of abelian varieties with values in towers of number fields*, Invent. Math. **18** (1972), 183–266. MR 0444670 (56 #3020)

[Nek06] Jan Nekovář, *Selmer complexes*, Astérisque (2006), no. 310, viii+559. MR 2333680 (2009c:11176)

[Neu99] Jürgen Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999, Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder. MR 1697859 (2000m:11104)

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, *Cohomology of number fields*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008. MR 2392026 (2008m:11223)

[Nys96] Louise Nyssen, *Pseudo-représentations*, Math. Ann. **306** (1996), no. 2, 257–283. MR 1411348 (98a:20013)

[Och05] Tadashi Ochiai, *Euler system for Galois deformations*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 113–146. MR 2141691 (2006a:11068)

[Och06] ———, *On the two-variable Iwasawa main conjecture*, Compos. Math. **142** (2006), no. 5, 1157–1200. MR 2264660 (2007i:11146)

[Pin92] Richard Pink, *On l-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification*, Math. Ann. **292** (1992), no. 2, 197–240. MR 1149032 (93c:11042)

[Rib77] Kenneth A. Ribet, *Galois representations attached to eigenforms with Nebentypus*, Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), Springer, Berlin, 1977, pp. 17–51. Lecture Notes in Math., Vol. 601. MR 0453647 (56 #11907)

[RR00] Heydar Radjavi and Peter Rosenthal, *Simultaneous triangularization*, Universitext, Springer-Verlag, New York, 2000. MR 1736065 (2001e:47001)

[RS11] Kenneth A. Ribet and William A. Stein, *Lectures on Modular Forms and Hecke Operators*, `http://wstein.org/books/ribet-stein/main.pdf` (2011).

[Sch11] Anthony J. Scholl, *Hypersurfaces and the Weil conjectures*, Int. Math. Res. Not. IMRN (2011), no. 5, 1010–1022. MR 2775874 (2012g:14039)

[Sch12] Peter Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313. MR 3090258

[Ser95] Jean-Pierre Serre, *Classes des corps cyclotomiques (d'après K. Iwasawa)*, Séminaire Bourbaki, Vol. 5, Soc. Math. France, Paris, 1995, pp. Exp. No. 174, 83–93. MR 1603459

[Ser98] ———, *Abelian l-adic representations and elliptic curves*, Research Notes in Mathematics, vol. 7, A K Peters Ltd., Wellesley, MA, 1998, With the collaboration of Willem Kuyk and John Labute, Revised reprint of the 1968 original. MR 1484415 (98g:11066)

[Ser00] ———, *Local algebra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000, Translated from the French by CheeWhye Chin and revised by the author. MR 1771925 (2001b:13001)

[Ser02] ———, *Galois cohomology*, english ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002, Translated from the French by Patrick Ion and revised by the author. MR 1867431 (2002i:12004)

[SGA71] *Théorie des intersections et théorème de Riemann-Roch*, Lecture Notes in Mathematics, Vol. 225, Springer-Verlag, Berlin, 1971, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre. MR 0354655 (50 #7133)

[SGA72] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*, Lecture Notes in Mathematics, Vol. 269, Springer-Verlag, Berlin, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. MR 0354652 (50 #7130)

[Shi58] Goro Shimura, *Correspondances modulaires et les fonctions ζ de courbes algébriques*, J. Math. Soc. Japan **10** (1958), 1–28. MR 0095173 (20 #1679)

[Shi11] Sug Woo Shin, *Galois representations arising from some compact Shimura varieties*, Ann. of Math. (2) **173** (2011), no. 3, 1645–1741. MR 2800722

[ST68] Jean-Pierre Serre and John Tate, *Good reduction of abelian varieties*, Ann. of Math. (2) **88** (1968), 492–517. MR 0236190 (38 #4488)

[Sta14] The Stacks Project Authors, *stacks project*, `http://stacks.math.columbia.edu`, 2014.

[SU14] Christopher Skinner and Eric Urban, *The Iwasawa Main Conjectures for GL_2*, Invent. Math. **195** (2014), no. 1, 1–277. MR 3148103

[SZ85] Joseph Steenbrink and Steven Zucker, *Variation of mixed Hodge structure. I*, Invent. Math. **80** (1985), no. 3, 489–542. MR 791673 (87h:32050a)

[Tat79] J. Tate, *Number theoretic background*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 3–26. MR 546607 (80m:12009)

[Tay91] Richard Taylor, *Galois representations associated to Siegel modular forms of low weight*, Duke Math. J. **63** (1991), no. 2, 281–332. MR 1115109 (92j:11044)

[Tay04] ———, *Galois representations*, Ann. Fac. Sci. Toulouse Math. (6) **13** (2004), no. 1, 73–119. MR 2060030 (2005a:11071)

[TU99] J. Tilouine and E. Urban, *Several-variable p-adic families of Siegel-Hilbert cusp eigensystems and their Galois representations*, Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 4, 499–574. MR 1693583 (2000j:11064)

[TY07] Richard Taylor and Teruyoshi Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. **20** (2007), no. 2, 467–493. MR 2276777 (2007k:11193)

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR 1269324 (95f:18001)

[Wil88] A. Wiles, *On ordinary λ-adic representations associated to modular forms*, Invent. Math. **94** (1988), no. 3, 529–573. MR 969243 (89j:11051)