UNIVERSITÀ DEGLI STUDI DI PADOVA

Sede Amministrativa: Università degli Studi di Padova

Dipartimento di Matematica

SCUOLA DI DOTTORATO DI RICERCA IN SCIENZE MATEMATICHE
INDIRIZZO MATEMATICA
CICLO XXIV

# Singular perturbation and homogenization problems in a periodically perforated domain. A functional analytic approach

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*Alla mia famiglia*

# Riassunto


Questa Tesi è dedicata all'analisi di problemi di perturbazione singolare e omogeneizzazione nello spazio Euclideo periodicamente perforato. Studiamo il comportamento delle soluzioni di problemi al contorno per le equazioni di Laplace, di Poisson e di Helmholtz al tendere a 0 di parametri legati al diametro dei buchi o alla dimensione delle celle di periodicità.

La Tesi è organizzata come segue.

Nel Capitolo 1, presentiamo due costruzioni note di un analogo periodico della soluzione fondamentale dell'equazione di Laplace, e introduciamo potenziali di strato e di volume periodici per l'equazione di Laplace e alcuni risultati basilari di teoria del potenziale periodica. Il Capitolo 2 è dedicato a problemi di perturbazione singolare e omogeneizzazione per le equazioni di Laplace e Poisson con condizioni al bordo di Dirichlet e Neumann. Nel Capitolo 3 consideriamo il caso di problemi al contorno di Robin (lineari e nonlineari) per l'equazione di Laplace, mentre nel Capitolo 4 analizziamo problemi di trasmissione (lineari e nonlineari). Nel Capitolo 5 applichiamo i risultati del Capitolo 4 al fine di provare l'analiticità della conduttività effettiva di un composto periodico. Il Capitolo 6 è dedicato alla costruzione di un analogo periodico della soluzione fondamentale dell'equazione di Helmholtz e dei corrispondenti potenziali di strato. Nel Capitolo 7 raccogliamo alcuni risultati di teoria spettrale per l'operatore di Laplace in domini periodicamente perforati. Nel Capitolo 8 studiamo problemi di perturbazione singolare e di omogeneizzazione per l'equazione di Helmholtz con condizioni al contorno di Neumann. Nel Capitolo 9 consideriamo problemi di perturbazione singolare e di omogeneizzazione con condizioni al contorno di Dirichlet per l'equazione di Helmholtz, mentre nel Capitolo 10 studiamo problemi al contorno di Robin (lineari e nonlineari). Il Capitolo 11 è dedicato allo studio di potenziali di strato periodici per operatori differenziali generali del secondo ordine a coefficienti costanti. Alla fine della Tesi abbiamo incluso delle Appendici con alcuni risultati utilizzati.

# Abstract

This Dissertation is devoted to the singular perturbation and homogenization analysis of boundary value problems in the periodically perforated Euclidean space. We investigate the behaviour of the solutions of boundary value problems for the Laplace, the Poisson, and the Helmholtz equations, as parameters related to diameter of the holes or the size of the periodicity cells tend to 0.

The Dissertation is organized as follows.

In Chapter 1, we present two known constructions of a periodic analogue of the fundamental solution of the Laplace equation and we introduce the periodic layer and volume potentials for the Laplace equation and some basic results of periodic potential theory. Chapter 2 is devoted to singular perturbation and homogenization problems for the Laplace and the Poisson equations with Dirichlet and Neumann boundary conditions. In Chapter 3 we consider the case of (linear and nonlinear) Robin boundary value problems for the Laplace equation, while in Chapter 4 we analyze (linear and nonlinear) transmission problems. In Chapter 5 we apply the results of Chapter 4 in order to prove the real analyticity of the effective conductivity of a periodic dilute composite. Chapter 6 is dedicated to the construction of a periodic analogue of the fundamental solution of the Helmholtz equation and of the corresponding periodic layer potentials. In Chapter 7 we collect some results of spectral theory for the Laplace operator in periodically perforated domains. In Chapter 8 we investigate singular perturbation and homogenization problems for the Helmholtz equation with Neumann boundary conditions. In Chapter 9 we consider singular perturbation and homogenization problems with Dirichlet boundary conditions for the Helmholtz equation, while in Chapter 10 we study (linear and nonlinear) Robin boundary value problems. Chapter 11 is devoted to the study of periodic layer potentials for general second order differential operators with constant coefficients. At the end of the Dissertation we have enclosed some Appendices with some results that we have exploited.

# Contents

# Preface

This Dissertation is devoted to the singular perturbation and homogenization analysis of boundary value problems in the periodically perforated Euclidean space $\mathbb{R}^n$. We consider boundary value problems for the Laplace, the Poisson, and the Helmholtz equations.

Periodical structures and related problems often appear in nature and play an important role in many problems of mechanics and physics. In particular, they have a large variety of applications, especially in connection with composite materials (cf. *e.g.*, Ammari and Kang [3, Chs. 2, 8], Ammari, Kang, and Lee [4, Ch. 3], Milton [97, Ch. 1], Mityushev, Pesetskaya, and Rogosin [101].) More precisely, such problems are relevant in the computation of effective properties, which in turn can be justified by the Homogenization Theory (cf. *e.g.*, Allaire [2, Ch. 1], Bensoussan, Lions, Papanicolaou [10, Ch. 1], Jikov, Kozlov, Oleĭnik [62, Ch. 1].) Furthermore, for composite materials it is interesting to study average or limiting properties corresponding to "small" values of the diameter of the holes or of the size of the periodicity cells.

As is well known, there is a vast literature devoted to the study of singular perturbation and homogenization problems for equations and systems of partial differential equations, especially in the case of linear equations. In this Dissertation we shall consider the case of periodically perforated domains with both linear and nonlinear boundary conditions.

We now briefly describe one of these problems.

Let $n \in \mathbb{N} \setminus \{0, 1\}$. Let $a_{11}, \ldots, a_{nn} \in ]0, +\infty[$. We set $a_i \equiv a_{ii} e_i$ for all $i \in \{1, \ldots, n\}$ and we introduce the fundamental periodicity cell $A \equiv \prod_{i=1}^{n} ]0, a_{ii}[$. Here $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. Then we fix a point $w$ in the fundamental cell $A$ and we take a sufficiently regular bounded connected open subset $\Omega$ of $\mathbb{R}^n$, such that $0 \in \Omega$ and such that $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$ is connected. For each $\epsilon \in ]0, \epsilon'[$, with $\epsilon' > 0$ sufficiently small, we set $\Omega_\epsilon \equiv w + \epsilon\Omega$, $\mathbb{S}_a[\Omega_\epsilon] \equiv \cup_{z \in \mathbb{Z}^n} (\Omega_\epsilon + \sum_{i=1}^{n} z_i a_i)$, and $\mathbb{T}_a[\Omega_\epsilon] \equiv \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]$. Then we denote by $\nu_{\Omega_\epsilon}$ the outward unit normal to $\Omega_\epsilon$. For $\epsilon \in ]0, \epsilon'[$ we consider the following boundary value problem for the Laplace equation in the periodically perforated domain $\mathbb{T}_a[\Omega_\epsilon]$:

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x + a_i) = u(x) & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{ 1, \ldots, n \}, \\ B_\epsilon(x, u(x), \frac{\partial u(x)}{\partial \nu_{\Omega_\epsilon}}) = 0 & \forall x \in \partial\Omega_\epsilon, \end{cases} \tag{0.1}$$

for a suitable function $B_\epsilon$ of $\partial\Omega_\epsilon \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, which represents the boundary condition of problem (0.1). In particular, we consider both linear and nonlinear boundary conditions. Assume that for each $\epsilon \in ]0, \epsilon'[$ boundary value problem (0.1) has a certain solution $u[\epsilon]$ of $\mathbb{T}_a[\Omega_\epsilon]$ to $\mathbb{R}$. Our aim is to investigate the asymptotic behaviour of the solution $u[\epsilon]$ (or of functionals of the solution, such as, for example, the energy integral $\int_{A \setminus \mathrm{cl}\,\Omega_\epsilon} |\nabla u[\epsilon](x)|^2 dx$) as $\epsilon$ tends to 0. Then, it is natural to pose, for example, the following questions:

(i) Let $t \in \mathrm{cl}\,A \setminus \{w\}$ be fixed. What can be said on the map $\epsilon \mapsto u[\epsilon](t)$ for $\epsilon$ small and positive around the degenerate value $\epsilon = 0$?

(ii) Let $t \in \mathbb{R}^n \setminus \Omega$ be fixed. What can be said on the map $\epsilon \mapsto u[\epsilon](w + \epsilon t)$ for $\epsilon$ small and positive around the degenerate value $\epsilon = 0$?

We note that question (i) is related to what we call the "macroscopic behaviour" of the solution, since it is related to the value of the solution at a point which is "far" from the perforations. On the other hand, question (ii) concerns the "microscopic behaviour", since the point $w + \epsilon t$ gets closer to the "singularity" of the domain as the parameter $\epsilon$ tends to 0.

Questions of this type have long been investigated, *e.g.*, for problems on a bounded perforated domain with the methods of Asymptotic Analysis and of Homogenization Theory.

Thus for example, one could resort to Asymptotic Analysis and may succeed to write out an asymptotic expansion of $u[\epsilon](t)$ in terms of the parameter $\epsilon$. In this sense, we mention, *e.g.*, the work of Ammari and Kang [3, Ch. 5], Ammari, Kang, and Lee [4, Ch. 3], Kozlov, Maz'ya, and Movchan [65], Maz'ya and Movchan [89], Maz'ya, Nazarov, and Plamenewskij [91, 92], Maz'ya, Movchan, and Nieves [90], Ozawa [111], Vogelius and Volkov [138], Ward and Keller [141]. For non-linear problems on domains with small holes far less seems to be known; we mention the results which concern the existence of a limiting value of the solutions or of their energy integral as the holes degenerate to points, as those of Ball [9], Sivaloganathan, Spector, and Tilakraj [129]. We also mention the computation of the expansions in the case of quasilinear equations of Titcombe and Ward [134], Ward, Henshaw, and Keller [139], and Ward and Keller [140].

Concerning Homogenization Theory, we mention, *e.g.*, Bakhvalov and Panasenko [8], Cioranescu and Murat [27, 28], Dal Maso and Murat [32], Jikov, Kozlov, and Oleĭnik [62], Marčenko and Khruslov [88], Sánchez-Palencia [122]. Here the interest is focused on the limiting behaviour as the singular perturbation parameters degenerate.

Furthermore, boundary value problems in domains with periodic inclusions have been analyzed, at least for the two dimensional case, with the method of functional and integral equations. Here we mention Castro and Pesetskaya [19], Castro, Pesetskaya, and Rogosin [20], Drygas and Mityushev [48], Mityushev and Adler [100], Rogosin, Dubatovskaya, and Pesetskaya [119].

In connection with doubly periodic problems for composite materials, we mention the monograph of Grigolyuk and Fil'shtinskij [56].

Here instead we wish to characterize the behaviour of $u[\epsilon]$ at $\epsilon = 0$ by a different approach. Thus for example, if we consider a certain functional, say $F(\epsilon)$, relative to the solution (such as, for example, one of those considered in the questions above), we would try to represent it for $\epsilon > 0$ in terms of real analytic functions of the variable $\epsilon$ defined on a whole neighbourhood of 0, and by possibly singular at $\epsilon = 0$ but explictly known functions of $\epsilon$, such as $\log \epsilon$, $\epsilon^{-1}$, etc. ...

We observe that our approach does have certain advantages. Indeed, if we knew, for example, that $u[\epsilon](t)$ equals for positive values of $\epsilon$ a real analytic function of the variable $\epsilon$ defined on a whole neighbourhood of 0, then there would exist $\epsilon'' \in ]0, \epsilon'[$ and a sequence $\{c_j\}_{j=0}^{\infty}$ of real numbers such that

$$u[\epsilon](t) = \sum_{j=0}^{\infty} c_j \epsilon^j \qquad \forall \epsilon \in ]0, \epsilon''[\,,$$

where the series in the right hand side converges absolutely on $]-\epsilon'', \epsilon''[$.

Such a project has been carried out by Lanza de Cristoforis in several papers for problems in a bounded domain with a small hole (cf. *e.g.*, Lanza [68, 69, 71, 72, 73, 74, 75, 77, 76, 78, 79].) In the frame of linearized elastostatics, we also mention, *e.g.*, Dalla Riva [33], Dalla Riva and Lanza [38, 39, 41, 42, 43] and, for the Stokes system, Dalla Riva [35, 36, 37]. Here we note that one of the tools of our analysis is potential theory, and, in particular, the study of the dependence of layer potentials and other related integral operators upon perturbations of the domain (cf. Preciso [115], Lanza and Preciso [83, 84], Lanza and Rossi [85, 86], Dalla Riva [34], Dalla Riva and Lanza [40].)

We note that in this Dissertation we have analyzed problem (0.1), when $B_\epsilon$ represents the Dirichlet, the Neumann, and the (linear and nonlinear) Robin boundary conditions. We have also considered linear and nonlinear transmission problems for the Laplace equation in the pair of domains consisting of $\mathbb{S}_a[\Omega_\epsilon]$ and $\mathbb{T}_a[\Omega_\epsilon]$. We observe that nonlinear transmission problems arise in the study of heat conduction in composite materials with different (non-constant) thermal conductivities (see Mityushev and Rogosin [102, Chapter 5, p. 201].)

We briefly outline the general strategy. We first note that boundary value problem (0.1), which we consider only for positive $\epsilon$, is singular for $\epsilon = 0$. Then by exploiting potential theory, we transform (0.1) into an equivalent integral equation defined on the $\epsilon$-dependent domain $\partial\Omega_\epsilon$. Since the domain $\partial\Omega_\epsilon$ is clearly degenerate for $\epsilon = 0$, we want to get rid of the dependence of the domain on $\epsilon$. By exploiting an appropriate change of variable, we convert the integral equation defined on $\partial\Omega_\epsilon$ into an equivalent integral equation which is defined on the fixed domain $\partial\Omega$. Such an equation makes sense also for $\epsilon = 0$. Then we analyze the solutions of the integral equations around the degenerate case $\epsilon = 0$ by means of the Implicit Function Theorem for real analytic maps. One of the difficulties here is to choose the appropriate functional variables so as to desingularize the problem. By exploiting these results, we can prove our main Theorems on the representation of the solution and of the integral and the energy integral of the solution. Furthermore, we note that in case of nonlinear problems, one of the difficulties concerns the existence and choice of a convenient family of solutions.

Moreover, we have applied the results concerning problem (0.1) to the investigation of homogenization problems in an infinite periodically perforated domain as the parameter related to the "size" of the holes and the one related to the dimension of the periodicity cell tend to 0. If $\epsilon$ is a small positive number and $\delta \in ]0, +\infty[$, we set

$$\Omega(\epsilon, \delta) = \delta\Omega_\epsilon,$$

and

$$\mathbb{T}_a(\epsilon, \delta) = \delta\mathbb{T}_a[\Omega_\epsilon] = \mathbb{R}^n \setminus \cup_{z\in\mathbb{Z}^n}(\delta w + \delta\epsilon \operatorname{cl}\Omega + \sum_{i=1}^{n} \delta z_i a_i).$$

Here, we note that the parameter $\epsilon$ is related, in a sense, to the "size" of the hole with respect to the periodicity cell, whereas the parameter $\delta$ is related to the "size" of the periodicity cell and, as a consequence, also to the distance among the perforations. We note that when $\epsilon$ tends to 0 the holes shrink to points and when $\delta$ tends to 0 the periodicity cell degenerates.

Then, for example, we have considered for each pair $(\epsilon, \delta) \in ]0, \epsilon'[ \times ]0, +\infty[$, with $\epsilon' > 0$ small enough, the problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon, \delta) \\ u(x + \delta a_i) = u(x) & \forall x \in \mathbb{T}_a(\epsilon, \delta) \quad \forall i \in \{1, \dots, n\} \\ B_{(\epsilon,\delta)}(x, u(x), \frac{\partial u(x)}{\partial \nu_{\Omega(\epsilon,\delta)}}) = 0 & \forall x \in \partial\Omega(\epsilon, \delta) \end{cases} \tag{0.2}$$

for a suitable function $B_{(\epsilon,\delta)}$ of $\partial\Omega(\epsilon, \delta) \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ which, as above, represents the boundary condition of problem (0.2). Here $\nu_{\Omega(\epsilon,\delta)}$ denotes the outward unit normal to $\Omega(\epsilon, \delta)$. Under convenient assumptions, we can assume that, for $\epsilon$ and $\delta$ positive and small, problem (0.2) has a solution, which we denote by $u_{(\epsilon,\delta)}$. Then we investigate the asymptotic behaviour of the solution $u_{(\epsilon,\delta)}$ and of functionals of the solution as the pair $(\epsilon, \delta)$ approaches $(0, 0)$, and we try to represent them in terms of real analytic functions and known functions. We observe that our aim is to describe the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon, \delta)$ goes to $(0, 0)$, in terms of real analytic functions (possibly evaluated on "particular" values of $(\epsilon, \delta)$.) Clearly, if $V$ is a non-empty open subset of $\mathbb{R}^n$, then

$$V \not\subseteq \operatorname{cl}\mathbb{T}_a(\epsilon, \delta)$$

for $\epsilon$ and $\delta$ positive and small. Therefore, we cannot hope to describe the behaviour of the restrcition of $u_{(\epsilon,\delta)}$ to the closure of a non-empty open subset in terms of real analytic functions. As a consequence, we need to find a different way to describe the convergence of $u_{(\epsilon,\delta)}$, since the restriction to non-empty open subsets of $\mathbb{R}^n$ is no longer convenient.

We observe that the study of the asymptotic behaviour of the solutions of boundary value problems in periodically perforated bounded domains, as the parameter related to the periodicity of the array of inclusions tends to 0, has been largely investigated in the frame of Homogenization Theory. We mention, for example, the contributions by Ansini and Braides [7], Cioranescu and Murat [27, 28], Marčenko and Khruslov [88]. We also note that in the recent paper by Maz'ya and Movchan [89] the assumption on the periodicity of the array of inclusions is not required.

Problems of the type of (0.1) and (0.2) have been considered not only for the Laplace equation, but also for the Poisson equation and the Helmholtz equation.

One of the aims of this Dissertation is establishing the periodic counterpart of some of the results obtained by Lanza de Cristoforis and his collaborators Dalla Riva, Preciso, and Rossi. The attention here is mainly paid on the mathematical theory of singularly perturbed periodic boundary value problems for the Laplace and the Helmholtz equation, rather than on applications. Therefore we chose to give a detailed description of the main boundary value problems for the Laplace and the Helmholtz equation and we decided to dedicate a chapter to each of them.

This Dissertation is organized as follows.

In Chapter 1, we present two known constructions of a periodic analogue of the fundamental solution of the Laplace equation and we introduce the periodic layer and volume potentials for the Laplace equation and some basic results of periodic potential theory. Chapter 2 is devoted to singular perturbation and homogenization problems for the Laplace and the Poisson equations with Dirichlet and Neumann boundary conditions. In Chapter 3 we consider the case of (linear and nonlinear) Robin boundary value problems for the Laplace equation, while in Chapter 4 we analyze (linear and nonlinear) transmission problems. In Chapter 5 we apply the results of Chapter 4 in order to prove the real analyticity of the effective conductivity of a periodic dilute composite. Chapter 6 is dedicated to the construction of a periodic analogue of the fundamental solution of the Helmholtz equation and of the corresponding periodic layer potentials. In Chapter 7 we collect some results of spectral theory for the Laplace operator in periodically perforated domains. In Chapter 8 we investigate singular perturbation and homogenization problems for the Helmholtz equation with Neumann boundary conditions. In Chapter 9 we consider singular perturbation and homogenization problems with Dirichlet boundary conditions for the Helmholtz equation, while in Chapter 10 we study (linear and nonlinear) Robin boundary value problems. Chapter 11 is devoted to the study of periodic layer potentials for general second order differential operators with constant coefficients. At the end of the Dissertation we have enclosed some Appendices with some results that we have exploited.

**Note:** Some of the results contained in this Dissertation have appeared or will appear on papers by the author (*e.g.*, [103, 104, 105]), and by Lanza de Cristoforis and the author (*e.g.*, [81, 82].) Concerning related topics, see also Lanza de Cristoforis and the author [80], and Dalla Riva and the author [44].

**Acknowledgement:** The author wishes to express his gratitude to Prof. M. Lanza de Cristoforis for his precious and constant help during the Ph.D. program and the preparation of this Dissertation. The author is also indebted to Dr. M. Dalla Riva for many valuable discussions.

# Notation

We denote the norm on a (real) normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the product space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x, y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}} \quad \forall\, (x, y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. We denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the Banach space of linear and continuous maps of $\mathcal{X}$ to $\mathcal{Y}$, equipped with the usual norm of the uniform convergence on the unit sphere of $\mathcal{X}$. We denote by $I$ the identity operator. For standard definitions of Calculus in normed spaces, we refer to Prodi and Ambrosetti [116]. If $\mathcal{X}$ is a vector space, $T$ a linear functional on $\mathcal{X}$ and $x \in \mathcal{X}$, the value of $T$ at $x$ is denoted by $\langle T, x\rangle$. If $\mathcal{X}$ is a topological space and $\mathcal{Y}$ is a subset of $\mathcal{X}$, we denote by $\mathrm{cl}_{\mathcal{X}}\, \mathcal{Y}$, or more simply by $\mathrm{cl}\, \mathcal{Y}$, the closure of $\mathcal{Y}$ in $\mathcal{X}$. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. Throughout the Dissertation,

$$n \in \mathbb{N} \setminus \{0, 1\}.$$

We denote by $\{e_1, \dots, e_n\}$ the canonical basis of $\mathbb{R}^n$. The inverse function of an invertible function $f$ is denoted $f^{(-1)}$, as opposed to the reciprocal of a complex-valued function $g$, or the inverse of a matrix $A$, which are denoted $g^{-1}$ and $A^{-1}$, respectively. If $A$ is a matrix, then we denote by $A^T$ the transpose matrix of $A$ and by $A_{ij}$ the $(i, j)$ entry of $A$. A dot '$\cdot$' denotes the inner product in $\mathbb{R}^n$, or the matrix product between matrices with real entries. If $x \in \mathbb{R}$, then we set

$$\lfloor x \rfloor \equiv \max \{\, l \in \mathbb{Z} \colon l \le x \,\}$$

and

$$\lceil x \rceil \equiv \min \{\, l \in \mathbb{Z} \colon l \ge x \,\}\,.$$

If $x \in \mathbb{R}$, we also set

$$\mathrm{sgn}(x) \equiv \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\mathrm{cl}\, \mathbb{D}$ denotes the closure of $\mathbb{D}$ and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. For all $R > 0$, $x \in \mathbb{R}^n$, $x_j$ denotes the $j$-th coordinate of $x$, $|x|$ denotes the Euclidean modulus of $x$ in $\mathbb{R}^n$ or in $\mathbb{C}$, $\mathbb{B}_n(x, R)$ denotes the open ball $\{\, y \in \mathbb{R}^n \colon |x - y| < R \,\}$ and $\mathbb{B}_n$ denotes the open unit ball $\{\, y \in \mathbb{R}^n \colon |y| < 1 \,\}$. Let $\Omega$ be an open subset of $\mathbb{R}^n$. The space of $m$ times continuously differentiable real-valued (resp. complex-valued) functions on $\Omega$ is denoted by $C^m(\Omega, \mathbb{R})$ (resp. $C^m(\Omega, \mathbb{C})$), or more simply by $C^m(\Omega)$. Let $r \in \mathbb{N} \setminus \{0\}$, $f \in (C^m(\Omega))^r$ or $f \in (C^m(\Omega, \mathbb{C}))^r$. The $s$-th component of $f$ is denoted $f_s$ and the gradient matrix of $f$ is denoted $Df$. Let $\eta \equiv (\eta_1, \dots, \eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \dots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \dots \partial x_n^{\eta_n}}$. If $r = 1$, the symmetric Hessian matrix of the second order partial derivatives of $f$ is denoted $D^2 f$. The subspace of $C^m(\Omega)$ (resp. $C^m(\Omega, \mathbb{C})$) of those functions $f$ such that $f$ and its derivatives $D^\eta f$ of order $|\eta| \le m$ can be extended with continuity to $\mathrm{cl}\, \Omega$ is denoted $C^m(\mathrm{cl}\, \Omega, \mathbb{R})$ (resp. $C^m(\mathrm{cl}\, \Omega, \mathbb{C})$), or more simply $C^m(\mathrm{cl}\, \Omega)$. The subspace of $C^m(\mathrm{cl}\, \Omega)$ (resp. $C^m(\mathrm{cl}\, \Omega, \mathbb{C})$) of those functions which have $m$-th order derivatives that are Hölder continuous with exponent $\alpha \in \,]0, 1]$ is denoted $C^{m,\alpha}(\mathrm{cl}\, \Omega, \mathbb{R})$ (resp. $C^{m,\alpha}(\mathrm{cl}\, \Omega, \mathbb{C})$), or more simply $C^{m,\alpha}(\mathrm{cl}\, \Omega)$ (cf. *e.g.*, Gilbarg and Trudinger [55]). Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $C^{m,\alpha}(\mathrm{cl}\, \Omega, \mathbb{D})$ denotes the set $\{\, f \in (C^{m,\alpha}(\mathrm{cl}\, \Omega))^n \colon f(\mathrm{cl}\, \Omega) \subseteq \mathbb{D} \,\}$. Now let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. Then $C^m(\mathrm{cl}\, \Omega)$ endowed with the norm $\|f\|_{C^m(\mathrm{cl}\, \Omega)} \equiv \sum_{|\eta| \le m} \sup_{\mathrm{cl}\, \Omega} |D^\eta f|$ is a Banach space. The same holds for $C^m(\mathrm{cl}\, \Omega, \mathbb{C})$. If $f \in C^{0,\alpha}(\mathrm{cl}\, \Omega)$ or $f \in C^{0,\alpha}(\mathrm{cl}\, \Omega, \mathbb{C})$, then its Hölder quotient $|f \colon \Omega|_\alpha$ is defined as $\sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} \colon x, y \in \mathrm{cl}\, \Omega, x \ne y \right\}$. The space $C^{m,\alpha}(\mathrm{cl}\, \Omega)$, equipped with its usual norm

$\|f\|_{C^{m,\alpha}(\mathrm{cl}\,\Omega)} = \|f\|_{C^m(\mathrm{cl}\,\Omega)} + \sum_{|\eta|=m} |D^\eta f : \Omega|_\alpha$, is well-known to be a Banach space. The same holds for $C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbb{C})$. We say that a bounded open subset of $\mathbb{R}^n$ is of class $C^m$ or of class $C^{m,\alpha}$, if its closure is a manifold with boundary imbedded in $\mathbb{R}^n$ of class $C^m$ or $C^{m,\alpha}$, respectively (cf. *e.g.*, Gilbarg and Trudinger [55, § 6.2]). For standard properties of the functions of class $C^{m,\alpha}$ both on a domain of $\mathbb{R}^n$ or on a manifold imbedded in $\mathbb{R}^n$ we refer to Gilbarg and Trudinger [55] (see also Lanza [67, §2, Lem. 3.1, 4.26, Thm. 4.28], Lanza and Rossi [85, §2]). We retain the standard notation of $L^p$ spaces and Sobolev spaces $W^{m,p}$ and of corresponding norms (and in particular we set, as usual, $H^m \equiv W^{m,2}$). Let $\Omega$ be a measurable nonempty subset of $\mathbb{R}^n$ and $1 \leq p \leq \infty$. In particular, we write $L^p(\Omega, \mathbb{R})$ (or more simply $L^p(\Omega)$), if we are considering real-valued functions; we write $L^p(\Omega, \mathbb{C})$, if we are considering complex-valued functions. Analogously, we denote by $L^p_{\mathrm{loc}}(\Omega, \mathbb{R})$ (resp. $L^p_{\mathrm{loc}}(\Omega, \mathbb{C})$), or more simply $L^p_{\mathrm{loc}}(\Omega)$, the set of functions $f$ of $\Omega$ to $\mathbb{R}$ (resp. $\mathbb{C}$) such that $f \in L^p(K, \mathbb{R})$ (resp. $f \in L^p(K, \mathbb{C})$) for each compact $K \subseteq \Omega$. We note that throughout the Dissertation 'analytic' means 'real analytic'. For the definition and properties of analytic operators, we refer to Prodi and Ambrosetti [116, p. 89].

We denote by $S_n$ the function of $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(x) \equiv \begin{cases} \frac{1}{s_n} \log|x|, & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n = 2, \\ \frac{1}{(2-n)s_n} |x|^{2-n} & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n > 2, \end{cases}$$

where $s_n$ denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n$. $S_n$ is well known to be the fundamental solution of the Laplace operator.

If $\mathbb{I}$ is an open bounded connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in ]0,1[$, then we denote by $\nu_{\mathbb{I}}$ the outward unit normal to $\partial\mathbb{I}$.

For a multi-index $\alpha \in \mathbb{N}^n$ and $x \in \mathbb{R}^n$, we define

$$|\alpha| \equiv \sum_{i=1}^n \alpha_i, \quad \alpha! \equiv \prod_{i=1}^n \alpha_i!, \quad x^\alpha \equiv \prod_{i=1}^n {x_i}^{\alpha_i}, \quad D^\alpha \equiv \prod_{i=1}^n (\partial_i)^{\alpha_i}.$$

We denote by $d\sigma$ the standard surface measure on a manifold of codimension 1 of $\mathbb{R}^n$. We will sometimes attach to $d\sigma$ a subscript to indicate the integration variable. If $\mathbb{D}$ is a measurable subset of $\mathbb{R}^n$, and $k \in \mathbb{N}$, the $k$-dimensional measure of the set $\mathbb{D}$ is denoted by $|\mathbb{D}|_k$.

For notation and results connected with the Theory of Distributions, we refer to Appendix A. For notation and results from classical potential theory for the Laplace and the Helmholtz equation, we refer to Appendices B and E, respectively.

At the beginning of each Chapter, we refer to the points in the Dissertation where the notation that we adopt has been introduced. In particular, we note that in Chapters 1 and 6 we introduce the notation related to periodic layer potential for the Laplace and Helmholtz equation, respectively. The notation related to periodic domains is introduced in Sections 1.1 and 1.3.

# CHAPTER 1

# Periodic simple and double layer potentials for the Laplace equation

This Chapter is mainly devoted to the definition of periodic analogues of the simple and double layer potentials. Namely, we construct these objects by substituting, in the definition of the classical layer potentials, the fundamental solution of the Laplace operator with a periodic analogue. In the second part of this Chapter, we define a periodic Newtonian potential and we prove some regularity results for the solutions of some integral equations, involved in the resolution of boundary value problems by means of periodic potentials. Some of the results are based on the classical analogous results (cf. *e.g.*, Lanza and Rossi [85].) For a generalization of some results contained in this Chapter, we refer to [81].

For notation, definitions, and properties concerning classical layer potentials for the Laplace equation, we refer to Appendix B.

## 1.1 Notation

First of all, we need to introduce some notation.

We fix

$$a_{11}, \dots, a_{nn} \in \,]0, +\infty[. \tag{1.1}$$

We set

$$a_i \equiv a_{ii} e_i \qquad \forall i \in \{1, \dots, n\}, \tag{1.2}$$

$$a \equiv (a_1 \dots a_n) \in M_{n\times n}(\mathbb{R}). \tag{1.3}$$

In other words, $a$ is the diagonal matrix

$$a \equiv \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

Let $A$, $\tilde{A}$ and $\mathcal{O}$ be the subsets of $\mathbb{R}^n$ defined as follows:

$$A \equiv \prod_{i=1}^{n} ]0, a_{ii}[, \tag{1.4}$$

$$\tilde{A} \equiv \prod_{i=1}^{n} ]-\frac{a_{ii}}{2}, \frac{a_{ii}}{2}[, \tag{1.5}$$

$$\mathcal{O} \equiv \prod_{i=1}^{n} ]-\frac{2a_{ii}}{3}, \frac{2a_{ii}}{3}[. \tag{1.6}$$

We denote by $\nu_A$ the outward unit normal to $\partial A$, where it is defined. We also set

$$a(x) \equiv a \cdot x \qquad \forall x \in \mathbb{R}^n, \tag{1.7}$$

$$a^{-1}(x) \equiv a^{-1} \cdot x \qquad \forall x \in \mathbb{R}^n. \tag{1.8}$$

In other words, $a(\cdot)$, $a^{-1}(\cdot)$ are the linear functions from $\mathbb{R}^n$ to itself associated to the matrices $a$, $a^{-1}$, respectively. Clearly,

$$\det a = |A|_n,$$

where $|A|_n$ is the n-dimensional measure of the set $A$.

Finally, we set

$$Z_n^a \equiv \{\, a(z) \colon z \in \mathbb{Z}^n \,\}. \tag{1.9}$$

Let $z \in \mathbb{Z}^n \setminus \{0\}$. We note that

$$\min\Big\{\, |x - a(z)| \colon x \in \tilde{A} \,\Big\} \geq \frac{|a(z)|}{2}, \tag{1.10}$$

and

$$\min\{\, |x - a(z)| \colon x \in \mathcal{O} \,\} \geq \frac{|a(z)|}{3}. \tag{1.11}$$

Let $\mathbb{D}$ be a subset of $\mathbb{R}^n$ such that

$$x + a(z) \in \mathbb{D} \qquad \forall x \in \mathbb{D}, \quad \forall z \in \mathbb{Z}^n,$$

and $f$ be a function of $\mathbb{D}$ to $\mathbb{R}$. We say that $f$ is *periodic*, if

$$f(x + a_i) = f(x) \qquad \forall x \in \mathbb{D}, \quad \forall i \in \{1, \dots, n\}.$$

## 1.2 Construction of a periodic analogue of the fundamental solution of the Laplace operator

In this Section, we present two ways to construct a periodic analogue of the fundamental solution of the Laplace operator. Even though we shall use only the one constructed in Theorem 1.4, we decided to include also an alternative construction in order to show that it is possible to construct this object by means of different techniques.

### 1.2.1 Construction via Fourier Analysis

In this Subsection we construct a periodic analogue of the fundamental solution of the Laplace operator, by following Ammari and Kang [3, p. 53] (see also Ammari, Kang and Touibi [6].) We briefly outline the strategy. By using the Poisson summation Formula, we deduce the Fourier series of a distributional periodic analogue of the fundamental solution. Then we prove that this distribution is, actually, a function. This and similar constructions can be found, *e.g.*, in Hasimoto [58], Choquard [25], and Poulton, Botten, McPhedran and Movchan [114].

For the notation, definitions and results used in this Subsection, we refer to Appendix A. We start by introducing some other notation.

Let $y \in \mathbb{R}^n$. If $f \in \mathcal{S}(\mathbb{R}^n)$, we denote by $\tau_y f$ the element of $\mathcal{S}(\mathbb{R}^n)$ defined by

$$\tau_y f(x) \equiv f(x - y) \qquad \forall x \in \mathbb{R}^n.$$

If $u \in \mathcal{S}'(\mathbb{R}^n)$, then we denote by $\tau_y u$ the element of $\mathcal{S}'(\mathbb{R}^n)$ defined by

$$\langle \tau_y u, f \rangle \equiv \langle u, \tau_{-y} f \rangle \qquad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Let $y \in \mathbb{R}^n$. We denote by $\delta_y$ the Dirac $\delta$ distribution concentrated at point $y$, *i.e.* the element of $\mathcal{S}'(\mathbb{R}^n)$ defined by

$$\langle \delta_y, f \rangle \equiv f(y) \qquad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Let $y \in \mathbb{R}^n$. We denote by $E_y$ the element of $\mathcal{S}'(\mathbb{R}^n)$ defined by

$$\langle E_y, f\rangle \equiv \int_{\mathbb{R}^n} e^{iy\cdot x} f(x)\,dx \qquad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

In other words, here we denote by $E_y$ the (tempered) distribution associated with the function which takes $x \in \mathbb{R}^n$ to $e^{iy\cdot x}$, in order to emphasize the fact that we think of it as a distribution.

In particular, we note that

$$\langle E_{-2\pi\xi}, f\rangle = \hat{f}(\xi) \qquad \forall f \in \mathcal{S}(\mathbb{R}^n),$$

for all $\xi \in \mathbb{R}^n$.

Let $z \in \mathbb{Z}^n$, $l \in \mathbb{Z}$, $j \in \{1,\dots,n\}$. Clearly,

$$\begin{aligned}\left\langle \tau_{la_j} E_{2\pi a^{-1}(z)}, f\right\rangle &= e^{-2\pi i a^{-1}(z)\cdot l a_j} \left\langle E_{2\pi a^{-1}(z)}, f\right\rangle \\ &= e^{-2\pi i l z_j} \left\langle E_{2\pi a^{-1}(z)}, f\right\rangle \\ &= \left\langle E_{2\pi a^{-1}(z)}, f\right\rangle,\end{aligned} \tag{1.12}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. In other words, $E_{2\pi a^{-1}(z)}$ is a periodic distribution with respect to vectors $a_1,\dots,a_n$.

We have the following variant of a known result (cf. Folland [53, Ex. 22, p. 299], Schmeisser and Triebel [125, p. 143-145].)

**Proposition 1.1.** *Let $g$ be a function of $\mathbb{Z}^n$ to $\mathbb{C}$, such that*

$$|g(z)| \le C(1+|z|)^N \qquad \forall z \in \mathbb{Z}^n,$$

*for some $C, N > 0$. Then the series $\sum_{z\in\mathbb{Z}^n} g(z)E_{2\pi a^{-1}(z)}$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to a tempered distribution $G$, such that*

$$\tau_{la_i} G = G \qquad \forall i \in \{1,\dots,n\}, \quad \forall l \in \mathbb{Z}. \tag{1.13}$$

*Proof.* Let $G$ be the linear functional on $\mathcal{S}(\mathbb{R}^n)$ defined by

$$\langle G, f\rangle \equiv \sum_{z\in\mathbb{Z}^n} g(z) \left\langle E_{2\pi a^{-1}(z)}, f\right\rangle \qquad \forall f \in \mathcal{S}(\mathbb{R}^n). \tag{1.14}$$

First of all, we note that if $f \in \mathcal{S}(\mathbb{R}^n)$, then the generalized series in the right-hand side of (1.14) converges absolutely in $\mathbb{C}$. Indeed,

$$\left\langle E_{2\pi a^{-1}(z)}, f\right\rangle = \int_{\mathbb{R}^n} f(x) e^{2\pi i a^{-1}(z)\cdot x}\,dx = \hat{f}(-a^{-1}(z)).$$

On the other hand, since $f \in \mathcal{S}(\mathbb{R}^n)$, then, by Proposition A.8, we have $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$. In particular, there exists a constant $C' > 0$ such that

$$|\hat{f}(k)| \le C' \frac{1}{(1+|k|)^{n+1+N}} \qquad \forall k \in \mathbb{R}^n.$$

Hence,

$$|g(z)\hat{f}(-a^{-1}(z))| \le C(1+|z|)^N C' \frac{1}{(1+|a^{-1}(z)|)^{n+N+1}} \le \frac{C''}{(1+|z|)^{n+1}} \quad \forall z \in \mathbb{Z}^n,$$

for some $C'' > 0$. Then, by comparison with the convergent integral $\int_{\mathbb{R}^n} 1/(1+|z|)^{n+1}\,dx$, we deduce the convergence of the series in the right-hand side of (1.14). Accordingly, $G$ is a well defined linear map of $\mathcal{S}(\mathbb{R}^n)$ to $\mathbb{C}$, *i.e.* an element of the algebraic dual of $\mathcal{S}(\mathbb{R}^n)$. Then, since $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space, the Banach-Steinhaus Theorem (cf. *e.g.*, Trèves [135, pp. 347, 348]) ensures that $G$ is actually a tempered distribution in $\mathbb{R}^n$. Clearly, $G$ is the limit in $\mathcal{S}'(\mathbb{R}^n)$ of the generalized series $\sum_{z\in\mathbb{Z}^n} g(z)E_{2\pi a^{-1}(z)}$. By (1.12) and by the convergence of the series to $G$, we easily obtain

$$\tau_{la_i} G = G \qquad \forall l \in \mathbb{Z}, \quad \forall i \in \{1,\dots,n\}.$$

Consequently, the proof is now concluded. □

We now prove a slight variant of the Poisson summation Formula (cf. Theorem A.10.)

**Proposition 1.2.** *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\sum_{z\in\mathbb{Z}^n} f(a(z)) = \sum_{z\in\mathbb{Z}^n} \frac{1}{|A|_n}\hat{f}(a^{-1}(z)),$$

*where both series converge absolutely. Moreover,*

$$\sum_{z\in\mathbb{Z}^n} \delta_{a(z)} = \sum_{z\in\mathbb{Z}^n} \frac{1}{|A|_n} E_{2\pi a^{-1}(z)} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

*Proof.* Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then it is clear that the function $f_a$ of $\mathbb{R}^n$ to $\mathbb{C}$, defined by

$$f_a(x) \equiv f(a(x)) \qquad \forall x \in \mathbb{R}^n,$$

satisfies the hypotheses of Theorem A.10. Therefore, by Theorem A.10, we have

$$\sum_{z\in\mathbb{Z}^n} f_a(z) = \sum_{z\in\mathbb{Z}^n} \hat{f}_a(z).$$

On the other hand, $f_a(z) = f(a(z))$ and

$$\begin{aligned}
\hat{f}_a(z) &= \int_{\mathbb{R}^n} f(a(x)) e^{-2\pi i z\cdot x}\, dx \\
&= \frac{1}{|A|_n} \int_{\mathbb{R}^n} f(t) e^{-2\pi i z\cdot a^{-1}(t)}\, dt \\
&= \frac{1}{|A|_n} \int_{\mathbb{R}^n} f(t) e^{-2\pi i a^{-1}(z)\cdot t}\, dt \\
&= \frac{1}{|A|_n} \hat{f}(a^{-1}(z)).
\end{aligned}$$

Accordingly,

$$\sum_{z\in\mathbb{Z}^n} f(a(z)) = \sum_{z\in\mathbb{Z}^n} \frac{1}{|A|_n}\hat{f}(a^{-1}(z)), \tag{1.15}$$

and the series are absolutely convergent. Since the series in (1.15) are absolutely convergent, we have

$$\sum_{z\in\mathbb{Z}^n} f(a(z)) = \sum_{z\in\mathbb{Z}^n} \frac{1}{|A|_n}\hat{f}(-a^{-1}(z)), \qquad \forall f \in \mathcal{S}(\mathbb{R}^n). \tag{1.16}$$

By the definition of $\hat{f}$, we have

$$\hat{f}(-a^{-1}(z)) = \int_{\mathbb{R}^n} f(x) e^{2\pi i a^{-1}(z)\cdot x}\, dx = \left\langle E_{2\pi a^{-1}(z)}, f\right\rangle \qquad \forall z \in \mathbb{Z}^n,$$

and so the equality in (1.16) can be rewritten as

$$\sum_{z\in\mathbb{Z}^n} \left\langle \delta_{a(z)}, f\right\rangle = \sum_{z\in\mathbb{Z}^n} \frac{1}{|A|_n} \left\langle E_{2\pi a^{-1}(z)}, f\right\rangle \qquad \forall f \in \mathcal{S}(\mathbb{R}^n). \tag{1.17}$$

Finally, we observe that the series in (1.17) are absolutely convergent. Furthermore, the equality in (1.17) can be rewritten as

$$\sum_{z\in\mathbb{Z}^n} \delta_{a(z)} = \sum_{z\in\mathbb{Z}^n} \frac{1}{|A|_n} E_{2\pi a^{-1}(z)} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

and thus the proof is complete. □

As a first step, in the following Theorem we prove the existence of a distributional periodic analogue of the fundamental solution of the Laplace operator.

**Theorem 1.3.** *Let $G_n^a$ be the element of $\mathcal{S}'(\mathbb{R}^n)$ defined by*

$$G_n^a \equiv - \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{|A|_n 4\pi^2 |a^{-1}(z)|^2} E_{2\pi a^{-1}(z)}. \tag{1.18}$$

*Then the following statements hold.*

*(i)*

$$\tau_{la_i} G_n^a = G_n^a \qquad \forall l \in \mathbb{Z}, \quad \forall i \in \{1, \ldots, n\}. \tag{1.19}$$

*(ii)*

$$\overline{\langle G_n^a, \overline{f} \rangle} = \langle G_n^a, f \rangle \qquad \forall f \in \mathcal{S}(\mathbb{R}^n), \tag{1.20}$$

*where $\bar{\cdot}$ means complex conjugation.*

*(iii)*

$$\Delta G_n^a = \sum_{z \in \mathbb{Z}^n} \delta_{a(z)} - \frac{1}{|A|_n} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n), \tag{1.21}$$

*in the sense of distributions.*

*Proof.* By virtue of Proposition 1.1, the series in the right-hand side of the equality in (1.18) defines an element of $\mathcal{S}'(\mathbb{R}^n)$ such that $(i)$ holds.

The statement in $(ii)$ is a straightforward consequence of

$$\frac{1}{|A|_n 4\pi^2 |a^{-1}(z)|} = \frac{1}{|A|_n 4\pi^2 |a^{-1}(-z)|} \qquad \forall z \in \mathbb{Z}^n \setminus \{0\},$$

and of

$$\overline{\langle E_{2\pi a^{-1}(z)}, \overline{f} \rangle} = \langle \overline{E_{2\pi a^{-1}(z)}}, f \rangle = \langle E_{2\pi a^{-1}(-z)}, f \rangle \qquad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Now we need to prove (1.21). By continuity of the Laplace operator of $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, we have

$$\Delta G_n^a = \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{|A|_n} E_{2\pi a^{-1}(z)} = \sum_{z \in \mathbb{Z}^n} \frac{1}{|A|_n} E_{2\pi a^{-1}(z)} - \frac{1}{|A|_n} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

On the other hand, by Proposition 1.2 we obtain

$$\sum_{z \in \mathbb{Z}^n} \frac{1}{|A|_n} E_{2\pi a^{-1}(z)} - \frac{1}{|A|_n} = \sum_{z \in \mathbb{Z}^n} \delta_{a(z)} - \frac{1}{|A|_n} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

and so the validity of the statement in $(iii)$ follows. □

The previous Theorem ensures the existence of a distributional periodic analogue of the fundamental solution of the Laplace operator. In the following Theorem we prove that the distribution $G_n^a$ is a function, *i.e.*, can be represented as the distribution associated to a locally integrable function (or, in other words, a regular distribution.) See also Weil [142], Berlyand and Mityushev [13].

**Theorem 1.4.** *Let $G_n^a$ be as in Theorem 1.3. Then the following statements hold.*

*(i) There exists a unique function $S_n^a \in L^1_{\text{loc}}(\mathbb{R}^n, \mathbb{R})$ such that*

$$\int_{\mathbb{R}^n} S_n^a(x) \phi(x)\, dx = \langle G_n^a, \phi \rangle \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}). \tag{1.22}$$

*Therefore, in particular*

$$\Delta S_n^a = \sum_{z \in \mathbb{Z}^n} \delta_{a(z)} - \frac{1}{|A|_n}, \tag{1.23}$$

*in the sense of distributions. Moreover, up to modifications on a set of measure zero, $S_n^a$ is a real analytic function of $\mathbb{R}^n \setminus Z_n^a$ to $\mathbb{R}$, such that*

$$\Delta S_n^a(x) = -\frac{1}{|A|_n} \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a \tag{1.24}$$

*and*

$$S_n^a(x + a_i) = S_n^a(x) \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a, \quad \forall i \in \{1, \ldots, n\}. \tag{1.25}$$

*(ii) There exists a unique real analytic function $R_n^a$ of $(\mathbb{R}^n \setminus Z_n^a) \cup \{0\}$ to $\mathbb{R}$, such that*

$$S_n^a(x) = S_n(x) + R_n^a(x) \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a.$$

*Moreover,*

$$\Delta R_n^a(x) = -\frac{1}{|A|_n} \qquad \forall x \in (\mathbb{R}^n \setminus Z_n^a) \cup \{0\}.$$

*Proof.* We note that, by virtue of Theorem 1.3 *(ii)*, $G_n^a \in \mathcal{D}'(\mathbb{R}^n, \mathbb{R})$. Now let $F \in \mathcal{D}'(\mathbb{R}^n, \mathbb{R})$ be defined by

$$\langle F, \phi \rangle = \langle G_n^a, \phi \rangle - \int_{\mathbb{R}^n} S_n(x)\phi(x)\,dx \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}). \tag{1.26}$$

We have

$$\Delta F = \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \delta_{a(z)} - \frac{1}{|A|_n} \qquad \text{in } \mathcal{D}'(\mathbb{R}^n, \mathbb{R}).$$

Then, by standard elliptic existence and regularity theory (cf. *e.g.*, Friedman [54, Theorem 1.2, p. 205]), there exists a real analytic function $\tilde{R}_n^a$ of $\mathcal{O}$ to $\mathbb{R}$, such that

$$\int_{\mathcal{O}} \tilde{R}_n^a(x)\phi(x)\,dx = \langle F, \phi \rangle \qquad \forall \phi \in \mathcal{D}(\mathcal{O}, \mathbb{R}).$$

Moreover,

$$\Delta \tilde{R}_n^a(x) = -\frac{1}{|A|_n} \qquad \forall x \in \mathcal{O}.$$

Clearly, by (1.26), we have

$$\int_{\mathcal{O}} \big(S_n(x) + \tilde{R}_n^a(x)\big)\phi(x)\,dx = \langle G_n^a, \phi \rangle \qquad \forall \phi \in \mathcal{D}(\mathcal{O}, \mathbb{R}). \tag{1.27}$$

We note that by equality (1.27), we can represent the restriction of $G_n^a$ to the space $\mathcal{D}(\mathcal{O}, \mathbb{R})$ as the distribution associated to a locally integrable function defined on $\mathcal{O}$, namely $S_n + \tilde{R}_n^a$. Our aim is to represent $G_n^a$ as the distribution associated to a locally integrable function defined on $\mathbb{R}^n \setminus Z_n^a$. Since $G_n^a$ is periodic, such a function will be given by extending by periodicity the function $S_n + \tilde{R}_n^a$ to the whole of $\mathbb{R}^n \setminus Z_n^a$.

Set

$$\tilde{\mathcal{O}} \equiv \prod_{i=1}^{n} \left]-\frac{3a_{ii}}{5}, \frac{3a_{ii}}{5}\right[.$$

Next define $S_n^a \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R})$ by imposing

$$S_n^a\big(x + a(z)\big) = S_n(x) + \tilde{R}_n^a(x) \qquad \forall x \in \tilde{\mathcal{O}} \setminus \{0\}, \quad \forall z \in \mathbb{Z}^n. \tag{1.28}$$

In other words, (1.28) means that we define $S_n^a$ by extending by periodicity the restriction to $\tilde{\mathcal{O}}$ of $S_n + \tilde{R}_n^a$. By (1.27) and the periodicity of $G_n^a$, we have that $S_n^a$ is well defined. Indeed, one can easily verify that if $x \in \tilde{\mathcal{O}} \setminus \{0\}$, $z \in \mathbb{Z}^n$, and $x + a(z) \in \tilde{\mathcal{O}}$, then

$$\big(S_n + \tilde{R}_n^a\big)(x) = \big(S_n + \tilde{R}_n^a\big)(x + a(z)).$$

Since $S_n^a$ is well defined, then (1.28) implies the periodicity of $S_n^a$. Furthermore, by (1.27), by the periodicity of $G_n^a$, and by the definition of $S_n^a$, we have

$$\int_{\tilde{\mathcal{O}} + a(z)} S_n^a(x)\phi(x)\,dx = \langle G_n^a, \phi \rangle \qquad \forall \phi \in \mathcal{D}(\tilde{\mathcal{O}} + a(z), \mathbb{R}),$$

for all $z \in \mathbb{Z}^n$. As a consequence,

$$\int_{\mathbb{R}^n} S_n^a(x)\phi(x)\,dx = \langle G_n^a, \phi \rangle \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}).$$

Moreover, since $S_n$ and $R_n^a$ are real analytic in $\tilde{\mathcal{O}} \setminus \{0\}$, $S_n^a$ is a real analytic function of $\mathbb{R}^n \setminus Z_n^a$ to $\mathbb{R}$, such that (1.24) and (1.25) hold.

Finally, if we set

$$R_n^a(x) \equiv S_n^a(x) - S_n(x) \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a,$$

by (1.23) and by ellipticity of the Laplace operator, we have that $R_n^a$ can be extended by continuity to a real analytic function (that we still call $R_n^a$) of $(\mathbb{R}^n \setminus Z_n^a) \cup \{0\}$ to $\mathbb{R}$, such that $(ii)$ holds. □

*Remark* 1.5. We have that

$$S_n^a(x) = S_n^a(-x) \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a.$$

Indeed, let $\phi$ be the function of $\mathbb{R}^n \setminus Z_n^a$ to $\mathbb{R}$, defined by

$$\phi(x) \equiv S_n^a(x) - S_n^a(-x) \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a.$$

By standard elliptic regularity theory, $\phi$ can be extended to a periodic harmonic function $\tilde{\phi}$ of $\mathbb{R}^n$ to $\mathbb{R}$. By Green's Formula and by the periodicity of $\tilde{\phi}$, it is easy to see that

$$\int_A |\nabla \tilde{\phi}(x)|^2 \, dx = \int_{\partial A} \tilde{\phi}(x) \frac{\partial}{\partial \nu_A} \tilde{\phi}(x) \, d\sigma_x = 0.$$

Hence $\tilde{\phi}(x) = c$ for all $x \in \mathbb{R}^n$, for some $c \in \mathbb{R}$. On the other hand, by the periodicity of $S_n^a$, we have

$$S_n^a\Big(-\sum_{i=1}^n \frac{a_i}{2}\Big) = S_n^a\Big(\sum_{i=1}^n a_i - \sum_{i=1}^n \frac{a_i}{2}\Big) = S_n^a\Big(\sum_{i=1}^n \frac{a_i}{2}\Big),$$

and so $\tilde{\phi}(\sum_{i=1}^n \frac{a_i}{2}) = 0$. Hence $\tilde{\phi}(x) = 0$ for all $x \in \mathbb{R}^n$, and thus

$$S_n^a(x) = S_n^a(-x) \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a.$$

We also note that the symmetry of $S_n^a$ could also be deduced by the corresponding (distributional) property of $G_n^a$.

### 1.2.2 An alternative construction

In this Subsection we present an alternative construction of a periodic analogue of the fundamental solution of the Laplace operator. More precisely, we extend to the cases $n = 2$ and $n > 3$ the method of Shcherbina [127]. We briefly outline the strategy. It is natural to start by considering a series made of translations of the fundamental solution of the Laplace operator. However, this series does not converge, but we can manipulate it in such a way to obtain a convergent one. Doing so, we lose periodicity, but we can recover it by adding a suitable function. Similar constructions can be found, *e.g.*, in Berdichevskii [11], Petrina [113] and Shcherbina [128].

In the sequel, we need the following well known result (see, *e.g.*, Schwartz [126, p. 21].)

**Lemma 1.6.** *Let $\beta \in ]n, +\infty[$. Set*

$$y_0 \equiv 0,$$

*and*

$$y_l \equiv \sum_{\substack{z \in \mathbb{Z}^n \setminus \{0\} \\ |z_i| \le l \ \forall i \in \{1,\dots,n\}}} \frac{1}{|z|^\beta} \qquad \forall l \in \mathbb{N} \setminus \{0\}.$$

*Then the sequence $\{y_l\}_{l \in \mathbb{N}}$ is convergent.*

*Proof.* It suffices to observe that there exists a constant $C > 0$, such that

$$y_l \le C \sum_{k=1}^l \frac{k^{n-1}}{k^\beta} = C \sum_{k=1}^l \frac{1}{k^{\beta-n+1}},$$

for all $l \in \mathbb{N} \setminus \{0\}$. □

For each $z \in \mathbb{Z}^n$, we define the function $f_{n,z}$ of $\mathbb{R}^n \setminus Z_n^a$ to $\mathbb{R}$, by setting

$$f_{n,z}(x) \equiv S_n\big(x - a(z)\big) \quad \forall x \in \mathbb{R}^n \setminus Z_n^a.$$

Let $(i,j,k) \in \{1,\dots,n\}^3$, $z \in \mathbb{Z}^n$, and $x \in \mathbb{R}^n \setminus Z_n^a$. We have

$$\partial_k f_{n,z}(x) = \frac{x_k - z_k a_{kk}}{s_n |x - a(z)|^n}, \tag{1.29}$$

$$\partial_j\partial_k f_{n,z}(x) = -n\frac{(x_j - z_j a_{jj})(x_k - z_k a_{kk})}{s_n |x - a(z)|^{n+2}} + \frac{\delta_{jk}}{s_n |x - a(z)|^n}, \tag{1.30}$$

$$\begin{aligned} \partial_i\partial_j\partial_k f_{n,z}(x) = &- n\delta_{jk}\frac{x_i - z_i a_{ii}}{s_n |x - a(z)|^{n+2}} \\ &+ n(n+2)\frac{(x_i - z_i a_{ii})(x_j - z_j a_{jj})(x_k - z_k a_{kk})}{s_n |x - a(z)|^{n+4}} \\ &- n\frac{\delta_{ij}(x_k - z_k a_{kk}) + \delta_{ik}(x_j - z_j a_{jj})}{s_n |x - a(z)|^{n+2}}. \end{aligned} \tag{1.31}$$

Let $z \in \mathbb{Z}^n \setminus \{0\}$, $x \in \mathrm{cl}\,\tilde{A}$. By Taylor's Formula, we have

$$\sum_{|\alpha|=3}\Big(\int_0^1 3(1-t)^2\frac{D^\alpha f_{n,z}(tx)}{\alpha!}\,dt\Big)x^\alpha = f_{n,z}(x) - \sum_{|\alpha|=0}^{2}\frac{D^\alpha f_{n,z}(0)}{\alpha!}x^\alpha. \tag{1.32}$$

Let $l \in \mathbb{N}$. Let $\phi_{n,l}$ be the function of $\mathbb{R}^n \setminus Z_n^a$ to $\mathbb{R}$ defined by

$$\phi_{n,l}(x) \equiv f_{n,0}(x) + \sum_{\substack{z\in\mathbb{Z}^n\setminus\{0\}\\ |z_i|\le l\ \forall i\in\{1,\dots,n\}}}\left(f_{n,z}(x) - \sum_{|\alpha|=0}^{2}\frac{D^\alpha f_{n,z}(0)}{\alpha!}x^\alpha\right), \tag{1.33}$$

for all $x \in \mathbb{R}^n \setminus Z_n^a$. Clearly, $\phi_{n,l}$ is harmonic in $\mathbb{R}^n \setminus Z_n^a$.

By (1.10), (1.31), and (1.32), it is easy to see that there exists a constant $C > 0$, such that

$$|\sum_{|\alpha|=3}\Big(\int_0^1 3(1-t)^2\frac{D^\alpha f_{n,z}(tx)}{\alpha!}\,dt\Big)x^\alpha| \le C\frac{1}{|z|^{n+1}},$$

for all $x \in \mathrm{cl}\,\tilde{A}$ and for all $z \in \mathbb{Z}^n \setminus \{0\}$. Then, by Lemma 1.6, the series in the right-hand side of (1.33) converges uniformly in $\mathrm{cl}\,\tilde{A}$ as $l \to +\infty$. In a similar way, we can prove that, for all $x \in \mathbb{R}^n \setminus Z_n^a$, the limit

$$\lim_{l\to+\infty}\phi_{n,l}(x)$$

exists in $\mathbb{R}$. Accordingly, we can introduce the function $\phi_{n,l}$ of $\mathbb{R}^n \setminus Z_n^a$ to $\mathbb{R}$, by setting

$$\phi_n(x) \equiv \lim_{l\to+\infty}\phi_{n,l}(x) \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a. \tag{1.34}$$

Let $l \in \mathbb{N}$. Let $\psi_{n,l}$ be the function of $\mathbb{R}^n \setminus Z_n^a$ to $\mathbb{R}$ defined by

$$\psi_{n,l}(x) \equiv f_{n,0}(x) + \sum_{\substack{z\in\mathbb{Z}^n\setminus\{0\}\\ |z_i|\le l\ \forall i\in\{1,\dots,n\}}}\big(f_{n,z}(x) - f_{n,z}(0)\big), \tag{1.35}$$

for all $x \in \mathbb{R}^n \setminus Z_n^a$.

By formulas (1.29) and (1.33), we have

$$\phi_{n,l}(x) = f_{n,0}(x) + \sum_{\substack{z\in\mathbb{Z}^n\setminus\{0\}\\ |z_i|\le l\ \forall i\in\{1,\dots,n\}}}\left(f_{n,z}(x) - f_{n,z}(0) - \sum_{|\alpha|=2}\frac{D^\alpha f_{n,z}(0)}{\alpha!}x^\alpha\right), \tag{1.36}$$

for all $x \in \mathbb{R}^n \setminus Z_n^a$ and for all $l \in \mathbb{N}$.

For all $l \in \mathbb{Z}^n$, we denote by $B_{n,l}$ the linear operator of $\mathbb{R}^n$ to $\mathbb{R}^n$, defined by

$$\mathcal{B}_{n,l}(x) \equiv \frac{1}{2} \sum_{\substack{z \in \mathbb{Z}^n \setminus \{0\} \\ |z_i| \leq l \ \forall i \in \{1,\dots,n\}}} \big(D^2 f_{n,z}(0)\big) \cdot x \qquad \forall x \in \mathbb{R}^n. \tag{1.37}$$

It is easy to see that

$$\sum_{i=1}^{n} \mathcal{B}_{n,l}(e_i) \cdot e_i = 0.$$

Moreover,

$$\phi_{n,l}(x) = \psi_{n,l}(x) - \mathcal{B}_{n,l}(x) \cdot x, \tag{1.38}$$

for all $x \in \mathbb{R}^n \setminus Z_n^a$.

Let $l \in \mathbb{N}$, $x \in \mathbb{R}^n \setminus Z_n^a$. We have

$$\begin{aligned}
\psi_{n,l}(x) &= S_n(x) + \sum_{\substack{z \in \mathbb{Z}^n \setminus \{0\} \\ |z_i| \leq l \ \forall i \in \{1,\dots,n\}}} \Big(S_n\big(x - a(z)\big) - S_n\big(a(z)\big)\Big) \\
&= \sum_{\substack{z \in \mathbb{Z}^n \\ |z_i| \leq l \ \forall i \in \{1,\dots,n\}}} S_n\big(x + a(z)\big) - \sum_{\substack{z \in \mathbb{Z}^n \setminus \{0\} \\ |z_i| \leq l \ \forall i \in \{1,\dots,n\}}} S_n\big(a(z)\big) \\
&= \sum_{\substack{(z_2,\dots,z_n) \in \mathbb{Z}^{n-1} \\ |z_i| \leq l \ \forall i \in \{2,\dots,n\}}} \sum_{r_1=-l}^{l} S_n\Big(x + r_1 a_1 + \sum_{i=2}^{n} z_i a_i\Big) - \sum_{\substack{z \in \mathbb{Z}^n \setminus \{0\} \\ |z_i| \leq l \ \forall i \in \{1,\dots,n\}}} S_n\big(a(z)\big),
\end{aligned}$$

and

$$\begin{aligned}
\psi_{n,l}(x + a_1) &= S_n(x + a_1) + \sum_{\substack{z \in \mathbb{Z}^n \setminus \{0\} \\ |z_i| \leq l \ \forall i \in \{1,\dots,n\}}} \Big(S_n\big(x + a_1 - a(z)\big) - S_n\big(a(z)\big)\Big) \\
&= \sum_{\substack{z \in \mathbb{Z}^n \\ |z_i| \leq l \ \forall i \in \{1,\dots,n\}}} S_n\big(x + a_1 + a(z)\big) - \sum_{\substack{z \in \mathbb{Z}^n \setminus \{0\} \\ |z_i| \leq l \ \forall i \in \{1,\dots,n\}}} S_n\big(a(z)\big) \\
&= \sum_{\substack{(z_2,\dots,z_n) \in \mathbb{Z}^{n-1} \\ |z_i| \leq l \ \forall i \in \{2,\dots,n\}}} \sum_{r_1=-l+1}^{l+1} S_n\Big(x + r_1 a_1 + \sum_{i=2}^{n} z_i a_i\Big) - \sum_{\substack{z \in \mathbb{Z}^n \setminus \{0\} \\ |z_i| \leq l \ \forall i \in \{1,\dots,n\}}} S_n\big(a(z)\big).
\end{aligned}$$

Then, if $n = 2$, we have

$$\begin{aligned}
&\psi_{2,l}(x + a_1) - \psi_{2,l}(x) \\
&\qquad = \frac{1}{s_2} \sum_{\substack{z_2 \in \mathbb{Z} \\ |z_2| \leq l}} \Big(\log|x + l a_1 + a_1 + z_2 a_2| - \log|x - l a_1 - z_2 a_2|\Big) \\
&\qquad = \frac{1}{s_2} \sum_{\substack{z_2 \in \mathbb{Z} \\ |z_2| \leq l}} \Big(\log|a_1 + \frac{1}{l} z_2 a_2 + \frac{x}{l} + \frac{a_1}{l}| - \log|a_1 + \frac{1}{l} z_2 a_2 - \frac{x}{l}|\Big) \\
&\qquad = \frac{1}{s_2} \sum_{\substack{z_2 \in \mathbb{Z} \\ |z_2| \leq l}} \frac{1}{l} \frac{(2x + a_1) \cdot (a_1 + \frac{1}{l} z_2 a_2)}{|a_1 + \frac{1}{l} z_2 a_2|^2} + \mathrm{O}(1/l),
\end{aligned} \tag{1.39}$$

for $l$ big enough. Analogously, if $n \geq 3$, we have

$$\begin{aligned}
&\psi_{n,l}(x+a_1)-\psi_{n,l}(x)\\
&=\frac{1}{s_n(2-n)}\sum_{\substack{(z_2,\ldots,z_n)\in\mathbb{Z}^{n-1}\\ |z_i|\leq l\ \forall i\in\{2,\ldots,n\}}}\left(\frac{1}{|x+la_1+a_1+\sum_{i=2}^n z_ia_i|^{n-2}}-\frac{1}{|x-la_1-\sum_{i=2}^n z_ia_i|^{n-2}}\right)\\
&=\frac{1}{s_n}\sum_{\substack{(z_2,\ldots,z_n)\in\mathbb{Z}^{n-1}\\ |z_i|\leq l\ \forall i\in\{2,\ldots,n\}}}\frac{1}{l^{n-1}}\frac{(2x+a_1)\cdot(a_1+\frac{1}{l}\sum_{i=2}^n z_ia_i)}{|a_1+\frac{1}{l}\sum_{i=2}^n z_ia_i|^n}+\mathrm{O}(1/l),
\end{aligned}\tag{1.40}$$

for $l$ big enough.

Indeed, we observe that, if $\bar{x}\in\mathbb{R}^n\setminus Z_n^a$ is fixed and we denote by $[0,\bar{x}]$ the segment in $\mathbb{R}^n$ joining the points 0 and $\bar{x}$, then there exist $\bar{l}\in\mathbb{N}$ and a constant $C>0$ such that

$$\sup\left\{\frac{1}{l^{n-2}|a_1+\frac{1}{l}\sum_{i=2}^n z_ia_i+\frac{\zeta}{l}|^n}\frac{1}{l^2}:(z_2,\ldots,z_n)\in\mathbb{Z}^{n-1},|z_i|\leq l\ \forall i\in\{2,\ldots,n\},\zeta\in[0,\bar{x}]\right\}\leq\frac{C}{l^n},$$

for all $l>\bar{l}$.

In both cases, letting $l\to+\infty$ in (1.39) and in (1.40), we obtain

$$\begin{aligned}
&\lim_{l\to+\infty}[\psi_{n,l}(x+a_1)-\psi_{n,l}(x)]\\
&\qquad=\frac{1}{s_n}\underbrace{\int_{-1}^1\cdots\int_{-1}^1}_{(n-1)\text{ times}}\frac{(2x+a_1)\cdot(a_1+\sum_{i=2}^n y_ia_i)}{|a_1+\sum_{i=2}^n y_ia_i|^n}\,dy_2\ldots dy_n.
\end{aligned}$$

In view of the previous equality, we introduce the linear operator $\mathcal{A}_n$ of $\mathbb{R}^n$ to $\mathbb{R}^n$, by setting

$$\mathcal{A}_n(e_i)\equiv-\frac{1}{a_{ii}}\frac{1}{s_n}\underbrace{\int_{-1}^1\cdots\int_{-1}^1}_{(n-1)\text{ times}}\frac{a_i+\sum_{j\in\{1,\ldots,n\}\setminus\{i\}}y_ja_j}{|a_i+\sum_{j\in\{1,\ldots,n\}\setminus\{i\}}y_ja_j|^n}\,dy_1\ldots\underline{dy_i}\ldots dy_n,\tag{1.41}$$

for all $i\in\{1,\ldots,n\}$, where the symbol $\underline{dy_i}$ means that $dy_i$ is not present. If $x\in\mathbb{R}^n\setminus Z_n^a$, then it is easy to see that

$$\lim_{l\to+\infty}\big[\psi_{n,l}(x+a_i)-\psi_{n,l}(x)\big]=-\big[2(\mathcal{A}_n(a_i)\cdot x)+(A_n(a_i))\cdot a_i\big],\tag{1.42}$$

for all $i\in\{1,\ldots,n\}$. Let $(i,j)\in\{1,\ldots,n\}^2$, with $i\neq j$. If $n=2$, we have

$$\begin{aligned}
\mathcal{A}_2(a_i)\cdot a_j&=-\frac{1}{s_n}\int_{-1}^1\frac{y_ja_{jj}{}^2}{|a_i+y_ja_j|^2}dy_j\\
&=-\frac{1}{s_n}\Big(\log|a_i+a_j|-\log|a_i-a_j|\Big)\\
&=\mathcal{A}_2(a_j)\cdot a_i.
\end{aligned}$$

If $n\geq 3$, we have

$$\begin{aligned}
\mathcal{A}_n(a_i)\cdot a_j=&\frac{1}{s_n(2-n)}\underbrace{\int_{-1}^1\cdots\int_{-1}^1}_{(n-2)\text{ times}}\Big(\frac{1}{|a_i-a_j+\sum_{k\in\{1,\ldots,n\}\setminus\{i,j\}}y_ka_k|^{n-2}}\\
&-\frac{1}{|a_i+a_j+\sum_{k\in\{1,\ldots,n\}\setminus\{i,j\}}y_ka_k|^{n-2}}\Big)\,dy_1\ldots\underline{dy_i}\ldots\underline{dy_j}\ldots dy_n\\
&=\mathcal{A}_n(a_j)\cdot a_i.
\end{aligned}\tag{1.43}$$

Therefore, the linear operator $\mathcal{A}_n$ is symmetric. Hence, if $x\in\mathbb{R}^n\setminus Z_n^a$, we have

$$\lim_{l\to+\infty}\big[\psi_{n,l}(x+a_i)-\psi_{n,l}(x)\big]=\mathcal{A}_n(x)\cdot x-\mathcal{A}_n(x+a_i)\cdot(x+a_i).\tag{1.44}$$

Let $x \in \mathbb{R}^n \setminus Z_n^a$. By (1.38) and (1.44), it follows that

$$\phi_n(x+a_i) - \phi_n(x) + 2\mathcal{A}_n(a_i)\cdot x + \mathcal{A}_n(a_i)\cdot a_i = -\lim_{l\to+\infty}\big[2\mathcal{B}_{n,l}(a_i)\cdot x + \mathcal{B}_{n,l}(a_i)\cdot a_i\big].$$

Consequently, it is easy to see that there exists a linear operator $\mathcal{B}_n$ of $\mathbb{R}^n$ to $\mathbb{R}^n$, such that

$$\lim_{l\to+\infty} \mathcal{B}_{n,l} = \mathcal{B}_n. \tag{1.45}$$

We are now ready to give the following.

**Definition 1.7.** Let $\phi_n$, $\mathcal{A}_n$, and $\mathcal{B}_n$ as in (1.34),(1.41), and (1.45). Then we define the function $\tilde{S}_n^a$ of $\mathbb{R}^n \setminus Z_n^a$ to $\mathbb{R}$, by setting

$$\tilde{S}_n^a(x) \equiv \phi_n(x) + (\mathcal{A}_n + \mathcal{B}_n)(x)\cdot x \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a.$$

*Remark* 1.8. Clearly,

$$\tilde{S}_n^a(x+a_i) = \tilde{S}_n^a(x) \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a, \quad \forall i \in \{1,\dots,n\}.$$

*Remark* 1.9. If $i \in \{1,\dots,n\}$, then

$$\mathcal{A}_n(a_i)\cdot a_i = -\frac{1}{s_n}\underbrace{\int_{-1}^{1}\dots\int_{-1}^{1}}_{(n-1)\text{ times}} \frac{{a_{ii}}^2}{|a_i + \sum_{j\in\{1,\dots,n\}\setminus\{i\}} y_j a_j|^n}\, dy_1 \dots \underline{dy_i} \dots dy_n < 0.$$

Therefore,

$$\sum_{i=1}^{n} \mathcal{A}_n(e_i)\cdot e_i < 0.$$

By known results on the uniform convergence of a sequence of harmonic functions (cf. *e.g.*, Folland [52, Cor. 2.12, p. 71] and Gilbarg and Trudinger [55, Thm. 2.8, p. 21]), we have that there exists a constant $C < 0$, such that

$$\Delta\tilde{S}_n^a(x) = C \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a.$$

In particular, it can be proved that

$$\Delta\tilde{S}_n^a(x) = -\frac{1}{|A|_n} \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a.$$

Furthermore, it can be proved also that the function $S_n^a$ of Theorem 1.4 and $\tilde{S}_n^a$ differ by an additive constant.

## 1.3 Regularity of periodic functions

In this Section we introduce some notation and we collect some elementary results on the regularity of periodic functions.

We shall consider the following assumption for some $\alpha \in ]0,1[$ and $m \in \mathbb{N}\setminus\{0\}$.

*Let $\mathbb{I}$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathrm{cl}\,\mathbb{I} \subseteq A$ and $\mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{I}$ is connected.* (1.46)

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Then we set

$$\mathbb{P}_a[\mathbb{I}] \equiv A \setminus \mathrm{cl}\,\mathbb{I}, \tag{1.47}$$

$$\mathbb{S}_a[\mathbb{I}] \equiv \cup_{z\in\mathbb{Z}^n}(\mathbb{I} + a(z)), \tag{1.48}$$

$$\mathbb{T}_a[\mathbb{I}] \equiv \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{S}[\mathbb{I}]. \tag{1.49}$$

We denote by $\nu_{\mathbb{P}_a[\mathbb{I}]}$ the outward unit normal to $\mathbb{P}_a[\mathbb{I}]$ on $\partial\mathbb{P}_a[\mathbb{I}]$, where it is defined. Clearly,

$$\nu_{\mathbb{P}_a[\mathbb{I}]} = \nu_A \qquad \text{a.e. on } \partial A,$$

and

$$\nu_{\mathbb{P}_a[\mathbb{I}]} = -\nu_{\mathbb{I}} \qquad \text{on } \partial\mathbb{I}.$$

The following elementary Lemmas allow us to deduce the (global) regularity of periodic functions by the regularity of the restrictions to fundamental sets.

**Lemma 1.10.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be as in* (1.46). *Let* $u$ *be a function of* $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$ *to* $\mathbb{R}$ *such that*

$$u(x + a_i) = u(x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \ldots, n\}.$$

*Let* $V$ *be an open bounded subset of* $\mathbb{R}^n$, *such that* $\mathrm{cl}\,A \subseteq V$ *and*

$$\mathrm{cl}\,V \cap \mathrm{cl}(\mathbb{I} + a(z)) = \emptyset \qquad \forall z \in \mathbb{Z}^n \setminus \{0\}.$$

*Set*

$$W \equiv V \setminus \mathrm{cl}\,\mathbb{I}.$$

*Then the following statements hold.*

*(i) Let* $k \in \mathbb{N}$. *Then* $u \in C^k(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$ *if and only if* $u_{|\mathrm{cl}\,W} \in C^k(\mathrm{cl}\,W)$.

*(ii) Let* $k \in \mathbb{N}$, $\beta \in ]0,1]$. *Then* $u \in C^{k,\beta}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$ *if and only if* $u_{|\mathrm{cl}\,W} \in C^{k,\beta}(\mathrm{cl}\,W)$.

*Proof.* Clearly, statement $(i)$ is a straightforward consequence of the periodicity of the function $u$. Consider $(ii)$. For the sake of simplicity, we assume $k = 0$. Obviously, if $u \in C^{0,\beta}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$, then $u_{|\mathrm{cl}\,W} \in C^{0,\beta}(\mathrm{cl}\,W)$. Conversely, assume that $u_{|\mathrm{cl}\,W} \in C^{0,\beta}(\mathrm{cl}\,W)$. Then

$$|u(x) - u(y)| \leq |u : \mathrm{cl}\,W|_\beta |x - y|^\beta \qquad \forall x, y \in \mathrm{cl}\,W.$$

Set

$$\bar{d} \equiv \inf\{|x - y| \colon (x, y) \in \mathrm{cl}\,A \times (\mathbb{R}^n \setminus V)\}.$$

Clearly, $\bar{d} > 0$. Set

$$C \equiv \max\Big\{\frac{2\|u\|_{C^0(\mathrm{cl}\,W)}}{\bar{d}^\beta}, |u : \mathrm{cl}\,W|_\beta\Big\}.$$

Then it is easy to see that

$$|u(x) - u(y)| \leq C|x - y|^\beta \qquad \forall x, y \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],$$

and accordingly $u \in C^{0,\beta}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$. □

**Lemma 1.11.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be as in* (1.46). *Let* $u$ *be a function of* $\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]$ *to* $\mathbb{R}$ *such that*

$$u(x + a_i) = u(x) \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}], \quad \forall i \in \{1, \ldots, n\}.$$

*Then the following statements hold.*

*(i) Let* $k \in \mathbb{N}$. *Then* $u \in C^k(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}])$ *if and only if* $u_{|\mathrm{cl}\,\mathbb{I}} \in C^k(\mathrm{cl}\,\mathbb{I})$.

*(ii) Let* $k \in \mathbb{N}$, $\beta \in ]0,1]$. *Then* $u \in C^{k,\beta}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}])$ *if and only if* $u_{|\mathrm{cl}\,\mathbb{I}} \in C^{k,\beta}(\mathrm{cl}\,\mathbb{I})$.

*Proof.* Clearly, statement $(i)$ is a straightforward consequence of the periodicity of the function $u$. Consider $(ii)$. For the sake of simplicity, we assume $k = 0$. Obviously, if $u \in C^{0,\beta}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}])$, then $u_{|\mathrm{cl}\,\mathbb{I}} \in C^{0,\beta}(\mathrm{cl}\,\mathbb{I})$. Conversely, assume that $u_{|\mathrm{cl}\,\mathbb{I}} \in C^{0,\beta}(\mathrm{cl}\,\mathbb{I})$. Then

$$|u(x) - u(y)| \leq |u : \mathrm{cl}\,\mathbb{I}|_\beta |x - y|^\beta \qquad \forall x, y \in \mathrm{cl}\,\mathbb{I}.$$

Set

$$\bar{d} \equiv \inf\{|x - y| \colon (x, y) \in \mathrm{cl}\,\mathbb{I} \times (\mathbb{R}^n \setminus A)\}.$$

Clearly, $\bar{d} > 0$. Set

$$C \equiv \max\Big\{\frac{2\|u\|_{C^0(\mathrm{cl}\,\mathbb{I})}}{\bar{d}^\beta}, |u : \mathrm{cl}\,\mathbb{I}|_\beta\Big\}.$$

Then it is easy to see that

$$|u(x) - u(y)| \leq C|x - y|^\beta \qquad \forall x, y \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}],$$

and accordingly $u \in C^{0,\beta}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}])$. □

## 1.4 Periodic double layer potential

In this Section we define the periodic double layer potential. The construction is quite natural: we substitute in the definition of the (classical) double layer potential the fundamental solution of the Laplace operator $S_n$ with the function $S_n^a$ introduced in Theorem 1.4. For notation and properties of the (classical) double layer potential for the Laplace equation, we refer to Appendix B.

**Definition 1.12.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\mu \in L^2(\partial\mathbb{I})$. We set

$$w_a[\partial\mathbb{I},\mu](t) \equiv \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(S_n^a(t-s)\big)\mu(s)\,d\sigma_s \qquad \forall t \in \mathbb{R}^n.$$

The function $w_a[\partial\mathbb{I},\mu]$ is called the periodic double layer potential with moment $\mu$.

**Theorem 1.13.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Then the following statements hold.*

*(i) Let $\mu \in C^0(\partial\mathbb{I})$. Then the function $w_a[\partial\mathbb{I},\mu]$ is harmonic in $\mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}]$. Moreover,*

$$w_a[\partial\mathbb{I},\mu](t+a_i) = w_a[\partial\mathbb{I},\mu](t) \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1,\dots,n\}.$$

*The restriction $w_a[\partial\mathbb{I},\mu]_{|\mathbb{S}_a[\mathbb{I}]}$ can be extended uniquely to a continuous periodic function $w_a^+[\partial\mathbb{I},\mu]$ of $\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]$ to $\mathbb{R}$. The restriction $w_a[\partial\mathbb{I},\mu]_{|\mathbb{T}_a[\mathbb{I}]}$ can be extended uniquely to a continuous periodic function $w_a^-[\partial\mathbb{I},\mu]$ of $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$ to $\mathbb{R}$. Moreover, we have the following jump relations*

$$w_a^+[\partial\mathbb{I},\mu](t) = +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(S_n^a(t-s)\big)\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

$$w_a^-[\partial\mathbb{I},\mu](t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(S_n^a(t-s)\big)\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

$$w_a^+[\partial\mathbb{I},\mu](t) - w_a^-[\partial\mathbb{I},\mu](t) = \mu(t) \qquad \forall t \in \partial\mathbb{I}.$$

*(ii) Let $\mu \in C^{m,\alpha}(\partial\mathbb{I})$. Then we have that $w_a^+[\partial\mathbb{I},\mu]$ belongs to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}])$ and $w_a^-[\partial\mathbb{I},\mu]$ belongs to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$. Moreover,*

$$Dw_a^+[\partial\mathbb{I},\mu]\cdot\nu_{\mathbb{I}} - Dw_a^-[\partial\mathbb{I},\mu]\cdot\nu_{\mathbb{I}} = 0 \qquad \text{on } \partial\mathbb{I}.$$

*(iii) The map of $C^{m,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I})$ which takes $\mu$ to $w_a^+[\partial\mathbb{I},\mu]_{|\mathrm{cl}\,\mathbb{I}}$ is linear and continuous. Let $V$ be an open bounded connected subset of $\mathbb{R}^n$, such that $\mathrm{cl}\,A \subseteq V$ and*

$$\mathrm{cl}\,V \cap \mathrm{cl}(\mathbb{I} + a(z)) = \emptyset \qquad \forall z \in \mathbb{Z}^n \setminus \{0\}.$$

*Set*

$$W \equiv V \setminus \mathrm{cl}\,\mathbb{I}.$$

*The map of $C^{m,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,W)$ which takes $\mu$ to $w_a^-[\partial\mathbb{I},\mu]_{|\mathrm{cl}\,W}$ is linear and continuous.*

*(iv) We have*

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(S_n^a(t-s)\big)\,d\sigma_s = \frac{1}{2} - \frac{|\mathbb{I}|_n}{|A|_n} \qquad \forall t \in \partial\mathbb{I},$$

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(S_n^a(t-s)\big)\,d\sigma_s = 1 - \frac{|\mathbb{I}|_n}{|A|_n} \qquad \forall t \in \mathbb{S}_a[\mathbb{I}],$$

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(S_n^a(t-s)\big)\,d\sigma_s = -\frac{|\mathbb{I}|_n}{|A|_n} \qquad \forall t \in \mathbb{T}_a[\mathbb{I}].$$

*Proof.* We start with $(i)$. Let $\mu \in C^0(\partial\mathbb{I})$. Clearly, the periodicity of $w_a[\partial\mathbb{I},\mu]$ follows by the periodicity of $S_n^a$ (see (1.25).) By classical theorems of differentiation under the integral sign, we have that $w_a[\partial\mathbb{I},\mu]$ is harmonic in $\mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}]$. We have

$$w_a[\partial\mathbb{I},\mu](t) = w[\partial\mathbb{I},\mu](t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(R_n^a(t-s)\big)\mu(s)\,d\sigma_s \qquad \forall t \in \mathbb{R}^n.$$

Since $R_n^a$ is real analytic in $(\mathbb{R}^n \setminus Z_n^a) \cup \{0\}$, then the second term in the right-hand side of the previous equality is a function of class $C^\infty$ in a bounded open subset $\tilde{V}$ of $\mathbb{R}^n$, of class $C^\infty$, such that $\mathrm{cl}\, A \subseteq \tilde{V}$ and

$$\mathrm{cl}\,\tilde{V} \cap \mathrm{cl}(\mathbb{I} + a(z)) = \emptyset \qquad \forall z \in \mathbb{Z}^n \setminus \{0\}. \tag{1.50}$$

The existence of such an open set can be proved by a standard argument. Indeed, we take $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \cup_{z\in\mathbb{Z}^n\setminus\{0\}}(\mathrm{cl}\,\mathbb{I} + a(z)))$ such that $0 \leq \varphi \leq 1$ and such that $\varphi = 1$ in a neighborhood of $\mathrm{cl}\, A$. By Sard's Theorem there exists a regular value $c \in ]0,1[$ for $\varphi$. Then we set $\tilde{V} \equiv \{x \in \mathbb{R}^n \setminus \cup_{z\in\mathbb{Z}^n\setminus\{0\}}(\mathrm{cl}\,\mathbb{I} + a(z)) : \varphi(x) > c\}$. Obviously, $\tilde{V}$ is an open subset of $\mathbb{R}^n$ of class $C^\infty$ and $\mathrm{cl}\, A \subseteq \tilde{V} \subseteq \mathrm{cl}\tilde{V} \subseteq \mathbb{R}^n \setminus \cup_{z\in\mathbb{Z}^n\setminus\{0\}}(\mathrm{cl}\,\mathbb{I} + a(z))$. Then we set

$$\tilde{W} \equiv \tilde{V} \setminus \mathrm{cl}\,\mathbb{I}.$$

By Theorem B.1 (*i*),

$$w_a[\partial\mathbb{I},\mu](t) = w^+[\partial\mathbb{I},\mu](t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(R_n^a(t-s)\big)\mu(s)\,d\sigma_s \qquad \forall t \in \mathbb{I},$$

and

$$w_a[\partial\mathbb{I},\mu](t) = w^-[\partial\mathbb{I},\mu](t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(R_n^a(t-s)\big)\mu(s)\,d\sigma_s \qquad \forall t \in \tilde{W}.$$

Furthermore, the terms in the right-hand side of the two previous equalities are continuous functions in $\mathrm{cl}\,\mathbb{I}$ and $\mathrm{cl}\,\tilde{W}$, respectively. Hence, by Lemmas 1.10 and 1.11, we can easily conclude that $w_a[\partial\mathbb{I},\mu]_{|\mathbb{S}_a[\mathbb{I}]}$ can be extended uniquely to a continuous periodic function $w_a^+[\partial\mathbb{I},\mu]$ of $\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]$ to $\mathbb{R}$ and that $w_a[\partial\mathbb{I},\mu]_{|\mathbb{T}_a[\mathbb{I}]}$ can be extended uniquely to a continuous periodic function $w_a^-[\partial\mathbb{I},\mu]$ of $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$ to $\mathbb{R}$. Clearly,

$$\begin{aligned} w_a^+[\partial\mathbb{I},\mu](t) &= w^+[\partial\mathbb{I},\mu](t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(R_n^a(t-s)\big)\mu(s)\,d\sigma_s \\ &= +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(S_n(t-s)\big)\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(R_n^a(t-s)\big)\mu(s)\,d\sigma_s \\ &= +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(S_n^a(t-s)\big)\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I}, \end{aligned}$$

and

$$\begin{aligned} w_a^-[\partial\mathbb{I},\mu](t) &= w^-[\partial\mathbb{I},\mu](t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(R_n^a(t-s)\big)\mu(s)\,d\sigma_s \\ &= -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(S_n(t-s)\big)\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(R_n^a(t-s)\big)\mu(s)\,d\sigma_s \\ &= -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(S_n^a(t-s)\big)\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I}. \end{aligned}$$

Thus, the jump relations hold and the statement in (*i*) is proved. If $\mu \in C^{m,\alpha}(\partial\mathbb{I})$, then, by Theorem B.1 (*ii*), $w^+[\partial\mathbb{I},\mu] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{I})$ and $w^-[\partial\mathbb{I},\mu] \in C^{m,\alpha}(\mathrm{cl}\,\tilde{W})$, and so, by Lemmas 1.10, 1.11, the statement in (*ii*) holds. We now turn to the proof of (*iii*). Let $\tilde{V}$ be a bounded open subset of $\mathbb{R}^n$ of class $C^\infty$ such that $\mathrm{cl}\, V \subseteq \tilde{V}$ and such that (1.50) holds. Set

$$H[\mu](t) \equiv \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(R_n^a(t-s)\big)\mu(s)\,d\sigma_s \qquad \forall t \in \mathrm{cl}\,\tilde{V},$$

for all $\mu \in C^{m,\alpha}(\partial\mathbb{I})$. By Proposition C.1 and by the continuity of the imbedding of $C^{m+1}(\mathrm{cl}\,\tilde{V})$ in $C^{m,\alpha}(\mathrm{cl}\,\tilde{V})$ and of the restriction operator from $C^{m,\alpha}(\mathrm{cl}\,\tilde{V})$ to $C^{m,\alpha}(\mathrm{cl}\,V)$, it is easy to see that $H[\cdot]_{|\mathrm{cl}\,V}$ is a linear and continuous map of $C^{m,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,V)$. We have

$$w_a^+[\partial\mathbb{I},\mu](t) = w^+[\partial\mathbb{I},\mu](t) + H[\mu]_{|\mathrm{cl}\,\mathbb{I}}(t) \qquad \forall t \in \mathrm{cl}\,\mathbb{I},$$

and

$$w_a^-[\partial\mathbb{I},\mu](t) = w^-[\partial\mathbb{I},\mu](t) + H[\mu]_{|\mathrm{cl}\,W}(t) \qquad \forall t \in \mathrm{cl}\,W.$$

Since the linear map of $C^{m,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I})$ which takes $\mu$ to $H[\mu]_{|\mathrm{cl}\,\mathbb{I}}$ is continuous, then, by virtue of Theorem B.1 (*iii*), we conclude that the linear map of $C^{m,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I})$ which takes $\mu$ to $w_a^+[\partial\mathbb{I},\mu]_{|\mathrm{cl}\,\mathbb{I}}$ is continuous. Analogously, since the linear map of $C^{m,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,W)$ which takes $\mu$ to $H[\mu]_{|\mathrm{cl}\,W}$ is continuous, then, by virtue of Theorem B.1 (*iii*), we conclude that the linear map of $C^{m,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,W)$ which takes $\mu$ to $w_a^-[\partial\mathbb{I},\mu]_{|\mathrm{cl}\,W}$ is continuous. We finally consider (*iv*). It suffices to consider the third equality in (*iv*) (the other two can be proved by exploiting the third one and the jump relations.) As a consequence of the periodicity, it suffices to consider $t \in (\mathrm{cl}\,A \setminus \mathrm{cl}\,\mathbb{I})$. By Green's Formula, we have

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(S_n^a(t-s)\big)\,d\sigma_s = \int_{\mathbb{I}} \Delta_s(S_n^a(t-s))\,ds = -\frac{|\mathbb{I}|_n}{|A|_n}.$$

The Theorem is now completely proved. □

## 1.5 Periodic simple layer potential

In this Section we define the periodic simple layer potential. As done for the periodic double layer potential, we substitute in the definition of the (classical) simple layer potential the fundamental solution of the Laplace operator $S_n$ with the function $S_n^a$ introduced in Theorem 1.4. For notation and properties of the (classical) simple layer potential for the Laplace equation, we refer to Appendix B.

**Definition 1.14.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in\, ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\mu \in L^2(\partial\mathbb{I})$. We set

$$v_a[\partial\mathbb{I},\mu](t) \equiv \int_{\partial\mathbb{I}} S_n^a(t-s)\mu(s)\,d\sigma_s \qquad \forall t \in \mathbb{R}^n.$$

The function $v_a[\partial\mathbb{I},\mu]$ is called the periodic simple (or single) layer potential with moment $\mu$.

**Theorem 1.15.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in\, ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Then the following statements hold.*

*(i) Let $\mu \in C^0(\partial\mathbb{I})$. Then the function $v_a[\partial\mathbb{I},\mu]$ is continuous on $\mathbb{R}^n$. Moreover,*

$$v_a[\partial\mathbb{I},\mu](t+a_i) = v_a[\partial\mathbb{I},\mu](t) \qquad \forall t \in \mathbb{R}^n, \quad \forall i \in \{1,\dots,n\},$$

*and*

$$\Delta v_a[\partial\mathbb{I},\mu](t) = -\frac{1}{|A|_n}\int_{\partial\mathbb{I}} \mu(s)\,d\sigma_s \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}].$$

*(ii) Let $v_a^+[\partial\mathbb{I},\mu]$ and $v_a^-[\partial\mathbb{I},\mu]$ denote the restrictions of $v_a[\partial\mathbb{I},\mu]$ to $\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]$ and to $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$, respectively. If $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$, then $v_a^+[\partial\mathbb{I},\mu] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}[\mathbb{I}])$ and $v_a^-[\partial\mathbb{I},\mu] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}[\mathbb{I}])$.*

*(iii) The map of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I})$ which takes $\mu$ to $v_a^+[\partial\mathbb{I},\mu]_{|\mathrm{cl}\,\mathbb{I}}$ is linear and continuous. Let $V$ be an open bounded connected subset of $\mathbb{R}^n$, such that $\mathrm{cl}\,A \subseteq V$ and*

$$\mathrm{cl}\,V \cap \mathrm{cl}(\mathbb{I} + a(z)) = \emptyset \qquad \forall z \in \mathbb{Z}^n \setminus \{0\}.$$

*Set*

$$W \equiv V \setminus \mathrm{cl}\,\mathbb{I}.$$

*The map of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,W)$ which takes $\mu$ to $v_a^-[\partial\mathbb{I},\mu]_{|\mathrm{cl}\,W}$ is linear and continuous.*

*(iv) If $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$, then we have the following jump relations*

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} v_a^+[\partial\mathbb{I},\mu](t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}\big(S_n^a(t-s)\big)\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} v_a^-[\partial\mathbb{I},\mu](t) = +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}\big(S_n^a(t-s)\big)\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} v_a^-[\partial\mathbb{I},\mu](t) - \frac{\partial}{\partial\nu_{\mathbb{I}}} v_a^+[\partial\mathbb{I},\mu](t) = \mu(t) \qquad \forall t \in \partial\mathbb{I}.$$

*Proof.* We start with $(i)$. Let $\mu \in C^0(\partial\mathbb{I})$. Clearly, the periodicity of $v_a[\partial\mathbb{I},\mu]$ follows by the periodicity of $S_n^a$ (see (1.25).) Let $\tilde{V}$ be an open bounded subset of $\mathbb{R}^n$, of class $C^\infty$, such that $\mathrm{cl}\, A \subseteq \tilde{V}$ and

$$\mathrm{cl}\,\tilde{V} \cap \mathrm{cl}(\mathbb{I} + a(z)) = \emptyset \qquad \forall z \in \mathbb{Z}^n \setminus \{0\} \tag{1.51}$$

(cf. the proof of Theorem 1.13.) Set

$$\tilde{W} \equiv \tilde{V} \setminus \mathrm{cl}\,\mathbb{I}.$$

Obviously,

$$v_a[\partial\mathbb{I},\mu](t) = v[\partial\mathbb{I},\mu](t) + \int_{\partial\mathbb{I}} R_n^a(t-s)\mu(s)\,d\sigma_s \qquad \forall t \in \mathrm{cl}\,\tilde{V}.$$

By Theorem B.2 $(i)$, the function $v[\partial\mathbb{I},\mu]$ is continuous on $\mathrm{cl}\,\tilde{V}$. Moreover, the second term in the right-hand side of the previous equality defines a real analytic function on $\mathrm{cl}\,\tilde{V}$. Thus, the restriction of the function $v_a[\partial\mathbb{I},\mu]$ to the set $\mathrm{cl}\,\tilde{V}$ is continuous, and so, by virtue of the periodicity of $v_a[\partial\mathbb{I},\mu]$, we can conclude that $v_a[\partial\mathbb{I},\mu]$ is continuous on $\mathbb{R}^n$. By classical theorems of differentiation under the integral sign, since $\Delta S_n^a = -1/|A|_n$ in $\mathbb{R}^n \setminus Z_n^a$, by arguing in $\tilde{W}$ and in $\mathbb{I}$ and then by exploiting the periodicity of $v_a[\partial\mathbb{I},\mu]$, we have that

$$\Delta v_a[\partial\mathbb{I},\mu](t) = -\frac{1}{|A|_n}\int_{\partial\mathbb{I}} \mu(s)\,d\sigma_s \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}].$$

We now consider $(ii)$. Let $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$. Clearly,

$$v_a^+[\partial\mathbb{I},\mu](t) = v^+[\partial\mathbb{I},\mu](t) + \int_{\partial\mathbb{I}} R_n^a(t-s)\mu(s)\,d\sigma_s \qquad \forall t \in \mathbb{I},$$

and

$$v_a^-[\partial\mathbb{I},\mu](t) = v^-[\partial\mathbb{I},\mu](t) + \int_{\partial\mathbb{I}} R_n^a(t-s)\mu(s)\,d\sigma_s \qquad \forall t \in \mathrm{cl}\,\tilde{W}.$$

Then by Lemma 1.11 and Theorem B.2 $(ii)$, we can conclude that $v_a^+[\partial\mathbb{I},\mu] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}[\mathbb{I}])$. Analogously, by Lemma 1.10 and Theorem B.2 $(iii)$, we can conclude that $v_a^-[\partial\mathbb{I},\mu] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}[\mathbb{I}])$. We now turn to the proof of $(iii)$. Let $\tilde{V}$ be a bounded open subset of $\mathbb{R}^n$ of class $C^\infty$ such that $\mathrm{cl}\,V \subseteq \tilde{V}$ and such that (1.51) holds. Set

$$H[\mu](t) \equiv \int_{\partial\mathbb{I}} R_n^a(t-s)\mu(s)\,d\sigma_s \qquad \forall t \in \mathrm{cl}\,\tilde{V},$$

for all $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$. By Proposition C.1 and by the continuity of the imbedding of $C^{m+1}(\mathrm{cl}\,\tilde{V})$ in $C^{m,\alpha}(\mathrm{cl}\,\tilde{V})$ and of the restriction operator from $C^{m,\alpha}(\mathrm{cl}\,\tilde{V})$ to $C^{m,\alpha}(\mathrm{cl}\,V)$, it is easy to see that $H[\cdot]_{|\mathrm{cl}\,V}$ is a linear and continuous map of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,V)$. We have

$$v_a^+[\partial\mathbb{I},\mu](t) = v^+[\partial\mathbb{I},\mu](t) + H[\mu]_{|\mathrm{cl}\,\mathbb{I}}(t) \qquad \forall t \in \mathrm{cl}\,\mathbb{I},$$

and

$$v_a^-[\partial\mathbb{I},\mu](t) = v^-[\partial\mathbb{I},\mu](t) + H[\mu]_{|\mathrm{cl}\,W}(t) \qquad \forall t \in \mathrm{cl}\,W.$$

Since the linear map of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I})$ which takes $\mu$ to $H[\mu]_{|\mathrm{cl}\,\mathbb{I}}$ is continuous, then, by virtue of Theorem B.2 $(ii)$, we conclude that the map of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I})$ which takes $\mu$ to $v_a^+[\partial\mathbb{I},\mu]_{|\mathrm{cl}\,\mathbb{I}}$ is linear and continuous. Analogously, since the linear map of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,W)$ which takes $\mu$ to $H[\mu]_{|\mathrm{cl}\,W}$ is continuous, then, by virtue of Theorem B.2 $(iii)$, we conclude that the map of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,W)$ which takes $\mu$ to $v_a^-[\partial\mathbb{I},\mu]_{|\mathrm{cl}\,W}$ is linear and continuous. We finally consider $(iv)$. Let $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$. By Theorem B.2 $(v)$, we have

$$\begin{aligned}
\frac{\partial}{\partial\nu_{\mathbb{I}}} v_a^+[\partial\mathbb{I},\mu](t) &= \frac{\partial}{\partial\nu_{\mathbb{I}}} v^+[\partial\mathbb{I},\mu](t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}\big(R_n^a(t-s)\big)\mu(s)\,d\sigma_s \\
&= -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}\big(S_n(t-s)\big)\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}\big(R_n^a(t-s)\big)\mu(s)\,d\sigma_s \\
&= -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}\big(S_n^a(t-s)\big)\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I},
\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial\nu_{\mathbb{I}}}v_a^-[\partial\mathbb{I},\mu](t) &= \frac{\partial}{\partial\nu_{\mathbb{I}}}v^-[\partial\mathbb{I},\mu](t) + \int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(t)}\big(R_n^a(t-s)\big)\mu(s)\,d\sigma_s \\ &= +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(t)}\big(S_n(t-s)\big)\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(t)}\big(R_n^a(t-s)\big)\mu(s)\,d\sigma_s \\ &= +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(t)}\big(S_n^a(t-s)\big)\mu(s)\,d\sigma_s \qquad \forall t\in\partial\mathbb{I}.\end{aligned}$$

Accordingly,

$$\frac{\partial}{\partial\nu_{\mathbb{I}}}v_a^-[\partial\mathbb{I},\mu](t) - \frac{\partial}{\partial\nu_{\mathbb{I}}}v_a^+[\partial\mathbb{I},\mu](t) = \mu(t) \qquad \forall t\in\partial\mathbb{I}.$$

Hence, the proof is now complete. □

## 1.6 Periodic Newtonian potential

In this Section we introduce a periodic analogue of the Newtonian potential, defined, as we did for the layer potentials, by substituting the fundamental solution $S_n$ with its periodic analogue $S_n^a$.

We give the following.

**Definition 1.16.** Let $f\in C^0(\mathbb{R}^n)$ be such that

$$f(t+a_i)=f(t) \qquad \forall t\in\mathbb{R}^n, \quad \forall i\in\{1,\dots,n\}.$$

We set

$$p_a[f](t)\equiv\int_A S_n^a(t-s)f(s)\,ds \qquad \forall t\in\mathbb{R}^n.$$

The function $p_a[f]$ is called the periodic Newtonian potential of $f$.

*Remark* 1.17. Let $f$ be as in Definition 1.16. Let $t\in\mathbb{R}^n$ be fixed. We note that the function $S_n^a(t-\cdot)f(\cdot)$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and so $p_a[f](t)$ is well defined.

In the following Theorem, we prove some elementary properties of the periodic Newtonian potential. Namely, we prove its periodicity and we compute its Laplacian.

**Theorem 1.18.** *Let $m\in\mathbb{N}$, $\alpha\in\,]0,1[$. Let $f\in C^{m,\alpha}(\mathbb{R}^n)$ be such that*

$$f(t+a_i)=f(t) \qquad \forall t\in\mathbb{R}^n, \quad \forall i\in\{1,\dots,n\}.$$

*Then the following statements hold.*

*(i)*

$$p_a[f](t+a_i)=p_a[f](t) \qquad \forall t\in\mathbb{R}^n, \quad \forall i\in\{1,\dots,n\}.$$

*(ii)*

$$p_a[f]\in C^{m+2,\alpha}(\mathbb{R}^n).$$

*(iii)*

$$\Delta p_a[f](t)=f(t)-\frac{1}{|A|_n}\int_A f(s)\,ds \qquad \forall t\in\mathbb{R}^n.$$

*Proof.* Clearly, the statement in $(i)$ is a straightforward consequence of the periodicity of $S_n^a$. We need to prove $(ii)$ and $(iii)$. Obviously,

$$f\in C^{m,\alpha}(\mathrm{cl}\,V),$$

for all bounded open subsets $V$ of $\mathbb{R}^n$. Let $\bar{x}\in\mathbb{R}^n$. By Proposition D.1 $(ii)$ (with $\delta=1$), we have

$$p_a[f](t)=\int_{\tilde{A}+\bar{x}} S_n^a(t-s)f(s)\,ds \qquad \forall t\in\mathbb{R}^n.$$

Now set

$$U\equiv\bar{x}+\mathbb{B}_n\left(0,\min\{a_{11},\dots,a_{nn}\}/3\right).$$

As a first step, we want to prove that $p_a[f]_{|U} \in C^2(U)$ and that $\Delta p_a[f](t) = f(t) - \frac{1}{|A|_n}\int_A f(s)\,ds$ for all $t \in U$. We have

$$p_a[f](t) = \int_{\tilde{A}+\bar{x}} S_n(t-s)f(s)\,ds + \int_{\tilde{A}+\bar{x}} R_n^a(t-s)f(s)\,ds \qquad \forall t \in U.$$

Set

$$u_1(t) \equiv \int_{\tilde{A}+\bar{x}} S_n(t-s)f(s)\,ds \qquad \forall t \in U,$$

and

$$u_2(t) \equiv \int_{\tilde{A}+\bar{x}} R_n^a(t-s)f(s)\,ds \qquad \forall t \in U.$$

By Gilbarg and Trudinger [55, Lemma 4.2, p. 55], we have that $u_1 \in C^2(U)$ and

$$\Delta u_1(t) = f(t) \qquad \forall t \in U.$$

On the other hand, by classical theorems of differentiation under the integral sign, we have that $u_2 \in C^\infty(U)$ and

$$\Delta u_2(t) = -\frac{1}{|A|_n}\int_{\tilde{A}+\bar{x}} f(s)\,ds = -\frac{1}{|A|_n}\int_A f(s)\,ds \qquad \forall t \in U.$$

Hence, $p_a[f]_{|U} \in C^2(U)$ and

$$\Delta p_a[f](t) = f(t) - \frac{1}{|A|_n}\int_A f(s)\,ds \qquad \forall t \in U.$$

Accordingly, $p_a[f] \in C^2(\mathbb{R}^n)$ and

$$\Delta p_a[f](t) = f(t) - \frac{1}{|A|_n}\int_A f(s)\,ds \qquad \forall t \in \mathbb{R}^n,$$

and so the statement in *(iii)* is proved. We need to prove *(ii)*. By *(iii)* we have

$$\Delta p_a[f](\cdot) = f(\cdot) - \frac{1}{|A|_n}\int_A f(s)\,ds \in C^{m,\alpha}(\mathbb{R}^n).$$

Then, by Folland [52, Thm. 2.28, p. 78], we have

$$p_a[f](\cdot) \in C^{m+2,\alpha}(\mathbb{R}^n).$$

The proof is now complete. □

*Remark* 1.19. Let $m$, $\alpha$ and $f$ be as in Theorem 1.18. We observe that the presence of the term

$$-\frac{1}{|A|_n}\int_A f(s)\,ds$$

in the Laplacian of $p_a[f]$ is, somehow, natural. Indeed, by Green's Formula and by the periodicity of $p_a[f]$, it is immediate to see that

$$\int_A \Delta p_a[f](t)\,dt = \int_{\partial A} \frac{\partial}{\partial \nu_A} p_a[f](t)\,d\sigma_t = 0.$$

On the other hand,

$$\int_A \Delta p_a[f](t)\,dt = \int_A \left( f(t) - \frac{1}{|A|_n}\int_A f(s)\,ds \right) dt = 0.$$

In other words, the term

$$-\frac{1}{|A|_n}\int_A f(s)\,ds$$

ensures that

$$\int_A \Delta p_a[f](t)\,dt = 0.$$

*Remark* 1.20. Let $f$ be a real analytic function from $\mathbb{R}^n$ to $\mathbb{R}$ such that

$$f(t + a_i) = f(t) \qquad \forall t \in \mathbb{R}^n, \quad \forall i \in \{1, \dots, n\}.$$

Then, by Theorem 1.18 and by standard elliptic regularity theory, the periodic Newtonian potential $p_a[f]$ is a real analytic function from $\mathbb{R}^n$ to $\mathbb{R}$.

## 1.7 Regularity of the solutions of some integral equations

In this Section, we are interested in proving regularity results for the solutions of some integral equations. Indeed, as in classical potential theory, in order to solve boundary value problems for the Laplace operator by means of periodic simple and double layer potentials, we need to solve particular integral equations. Thus, we prove the following.

**Theorem 1.21.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $b \in C^{m-1,\alpha}(\partial\mathbb{I})$. Then the following statements hold.*

*(i) Let $k \in \{0,1,\dots,m\}$ and $\Gamma \in C^{k,\alpha}(\partial\mathbb{I})$ and $\mu \in L^2(\partial\mathbb{I})$ and*

$$\Gamma(t) = \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(S_n^a(t-s)\big)\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}} S_n^a(t-s)b(s)\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{1.52}$$

*then $\mu \in C^{k,\alpha}(\partial\mathbb{I})$.*

*(ii) Let $k \in \{0,1,\dots,m\}$ and $\Gamma \in C^{k,\alpha}(\partial\mathbb{I})$ and $\mu \in L^2(\partial\mathbb{I})$ and*

$$\Gamma(t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(S_n^a(t-s)\big)\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}} S_n^a(t-s)b(s)\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{1.53}$$

*then $\mu \in C^{k,\alpha}(\partial\mathbb{I})$.*

*(iii) Let $k \in \{1,\dots,m\}$ and $\Gamma \in C^{k-1,\alpha}(\partial\mathbb{I})$ and $\mu \in L^2(\partial\mathbb{I})$ and*

$$\Gamma(t) = \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}\big(S_n^a(t-s)\big)\mu(s)\,d\sigma_s + b(t)\int_{\partial\mathbb{I}} S_n^a(t-s)\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{1.54}$$

*then $\mu \in C^{k-1,\alpha}(\partial\mathbb{I})$.*

*(iv) Let $k \in \{1,\dots,m\}$ and $\Gamma \in C^{k-1,\alpha}(\partial\mathbb{I})$ and $\mu \in L^2(\partial\mathbb{I})$ and*

$$\Gamma(t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}\big(S_n^a(t-s)\big)\mu(s)\,d\sigma_s + b(t)\int_{\partial\mathbb{I}} S_n^a(t-s)\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{1.55}$$

*then $\mu \in C^{k-1,\alpha}(\partial\mathbb{I})$.*

*Proof.* We deduce all the statements by the correspondig results of Theorem B.3. Let $k$, $\Gamma$, and $\mu$ be as in the hypotheses of $(i)$. Set

$$\bar{\Gamma}(t) \equiv \Gamma(t) - \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(R_n^a(t-s)\big)\mu(s)\,d\sigma_s - \int_{\partial\mathbb{I}} R_n^a(t-s)b(s)\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I}.$$

Then, by Theorem C.2, $\bar{\Gamma} \in C^{k,\alpha}(\partial\mathbb{I})$. By (1.52), we have

$$\bar{\Gamma}(t) = \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}\big(S_n(t-s)\big)\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}} S_n(t-s)b(s)\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}.$$

Then, by Theorem B.3 $(i)$, we have $\mu \in C^{k,\alpha}(\partial\mathbb{I})$.

The proofs of statements $(ii)$, $(iii)$, $(iv)$ are very similar, and are accordingly omitted. □

## 1.8 Some technical results for periodic simple and double layer potentials

In this Section we collect some results on periodic simple and double layer potentials that we shall use in the sequel.

Indeed, in order to analyze boundary value problems in the next Chapters, we shall deal with integral equations on 'rescaled' domains, and, as a consequence, we need to study integral operators which arise in these integral equations. Moreover, we have also to undestand how the periodic layer potentials change when we 'rescale' the domains.

### 1.8.1 Notation

We introduce some notation.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. We shall consider the following assumption.

$$\text{Let } \Omega \text{ be a bounded open connected subset of } \mathbb{R}^n \text{ of class } C^{m,\alpha} \text{ such that } 0 \in \Omega \text{ and } \mathbb{R}^n \setminus \mathrm{cl}\,\Omega \text{ is connected.} \tag{1.56}$$

We denote by $\nu_\Omega$ the outward unit normal to $\Omega$ on $\partial\Omega$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$ be as in (1.56). Then there exists $\epsilon_1 > 0$ such that

$$\mathrm{cl}(w + \epsilon\Omega) \subseteq A \qquad \forall \epsilon \in ]-\epsilon_1, \epsilon_1[. \tag{1.57}$$

Let $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. Then there exists $\bar{\epsilon}_1 \in ]0, \epsilon_1[$ such that

$$\bar{x} \in (\mathrm{cl}\,A) \setminus \mathrm{cl}(w + \epsilon\Omega) \qquad \forall \epsilon \in ]-\bar{\epsilon}_1, \bar{\epsilon}_1[. \tag{1.58}$$

We set

$$\Omega_\epsilon \equiv w + \epsilon\Omega \qquad \forall \epsilon \in ]-\epsilon_1, \epsilon_1[ \setminus \{0\}, \tag{1.59}$$

$$\Omega_0 \equiv \{w\}. \tag{1.60}$$

Clearly, if $\epsilon \in ]-\epsilon_1, \epsilon_1[ \setminus \{0\}$, then the subset $\mathbb{I} \equiv \Omega_\epsilon$ satisfies (1.46). If $\epsilon \in ]-\epsilon_1, \epsilon_1[ \setminus \{0\}$, we denote by $\mathbb{P}_a[\Omega_\epsilon]$, $\mathbb{S}_a[\Omega_\epsilon]$, and $\mathbb{T}_a[\Omega_\epsilon]$ the sets $\mathbb{P}_a[\mathbb{I}]$, $\mathbb{S}_a[\mathbb{I}]$, and $\mathbb{T}_a[\mathbb{I}]$, introduced in (1.47), (1.48), and (1.49), with $\mathbb{I} \equiv \Omega_\epsilon$. We set also

$$\mathbb{T}_a[\Omega_0] \equiv \mathbb{R}^n \setminus (w + Z_n^a), \tag{1.61}$$

$$\mathbb{S}_a[\Omega_0] \equiv (w + Z_n^a). \tag{1.62}$$

Moreover, if $\epsilon \in ]-\epsilon_1, \epsilon_1[ \setminus \{0\}$, we denote by $\nu_{\Omega_\epsilon}$ the outward unit normal to $\Omega_\epsilon$ on $\partial\Omega_\epsilon$.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $r \in \{0, \dots, m\}$. We set

$$\mathcal{U}_\epsilon^{r,\alpha} \equiv \left\{ \mu \in C^{r,\alpha}(\partial\Omega_\epsilon) \colon \int_{\partial\Omega_\epsilon} \mu \, d\sigma = 0 \right\} \qquad \forall \epsilon \in ]-\epsilon_1, \epsilon_1[ \setminus \{0\}, \tag{1.63}$$

$$\mathcal{U}_0^{r,\alpha} \equiv \left\{ \theta \in C^{r,\alpha}(\partial\Omega) \colon \int_{\partial\Omega} \theta \, d\sigma = 0 \right\}. \tag{1.64}$$

We observe that if $\epsilon > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$ then we have

$$S_n(\epsilon x) = \begin{cases} \frac{1}{s_2} \log \epsilon + S_2(x), & \text{if } n = 2, \\ \frac{1}{\epsilon^{n-2}} S_n(x), & \text{if } n > 2. \end{cases} \tag{1.65}$$

Let $V$ be a bounded open subset of $\mathbb{R}^n$. We set

$$C_h^0(\mathrm{cl}\,V) \equiv \left\{ u \in C^0(\mathrm{cl}\,V) \cap C^2(V) \colon \Delta u(t) = 0 \quad \forall t \in V \right\}. \tag{1.66}$$

The space $C_h^0(\mathrm{cl}\,V)$ is equipped with the norm of the uniform convergence.

We now find convenient to introduce some notation that we shall use in the next chapters.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. For each $(\epsilon, \delta) \in (]-\epsilon_1, \epsilon_1[ \setminus \{0\}) \times ]0, +\infty[$, we set

$$\Omega(\epsilon, \delta) \equiv \delta w + \delta\epsilon\Omega, \tag{1.67}$$

$$\mathbb{T}_a(\epsilon, \delta) \equiv \mathbb{R}^n \setminus \cup_{z \in \mathbb{Z}^n} \mathrm{cl}(\Omega(\epsilon, \delta) + \delta a(z)), \tag{1.68}$$

$$\mathbb{S}_a(\epsilon, \delta) \equiv \cup_{z \in \mathbb{Z}^n} (\Omega(\epsilon, \delta) + \delta a(z)), \tag{1.69}$$

$$\mathbb{P}_a(\epsilon, \delta) \equiv \delta A \setminus \mathrm{cl}\,\Omega(\epsilon, \delta). \tag{1.70}$$

Clearly, if $\epsilon \in ]0,\epsilon_1[$, then

$$\begin{aligned}\Omega_\epsilon &= \Omega(\epsilon,1),\\ \mathbb{T}_a[\Omega_\epsilon] &= \mathbb{T}_a(\epsilon,1),\\ \mathbb{S}_a[\Omega_\epsilon] &= \mathbb{S}_a(\epsilon,1),\\ \mathbb{P}_a[\Omega_\epsilon] &= \mathbb{P}_a(\epsilon,1).\end{aligned}$$

Moreover, if $(\epsilon,\delta) \in (]-\epsilon_1,\epsilon_1[\setminus\{0\})\times]0,+\infty[$, then we denote by $\nu_{\Omega(\epsilon,\delta)}(\cdot)$ the outward unit normal to $\Omega(\epsilon,\delta)$ on $\partial\Omega(\epsilon,\delta)$.

### 1.8.2 Some technical results for the periodic double layer potential

In the following Proposition, we study some integral operators that are related to the the periodic double layer potential and that appear in integral equations on 'rescaled' domains when we represent the solution of a certain boundary value problem in terms of a periodic double layer potential.

**Proposition 1.22.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Then the following statements hold.*

*(i) There exists $\epsilon_2 \in ]0,\epsilon_1]$ such that the map $N_1$ of $]-\epsilon_2,\epsilon_2[\times C^{m,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$, which takes $(\epsilon,\theta)$ to the function $N_1[\epsilon,\theta]$ of $\partial\Omega$ to $\mathbb{R}$ defined by*

$$N_1[\epsilon,\theta](t) \equiv \int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n(t-s)\theta(s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s \quad \forall t\in\partial\Omega,$$

*is real analytic.*

*(ii) There exists $\epsilon_2' \in ]0,\epsilon_1]$ such that the map $N_2$ of $]-\epsilon_2',\epsilon_2'[\times C^{m,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$, which takes $(\epsilon,\theta)$ to the function $N_2[\epsilon,\theta]$ of $\partial\Omega$ to $\mathbb{R}$ defined by*

$$N_2[\epsilon,\theta](t) \equiv \int_{\partial\Omega}\nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\theta(s)\,d\sigma_s + \epsilon^n\int_{\partial\Omega}\nu_\Omega(t)D^2R_n^a(\epsilon(t-s))\nu_\Omega(s)\theta(s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

*is real analytic.*

*(iii) Let $\bar{x}\in \mathrm{cl}\,A\setminus\{w\}$. Let $\bar{\epsilon}_1$ be as in* (1.58). *There exists $\epsilon_2'' \in ]0,\bar{\epsilon}_1]$ such that the map $N_3$ of $]-\epsilon_2'',\epsilon_2''[\times C^{m,\alpha}(\partial\Omega)$ to $\mathbb{R}$, which takes $(\epsilon,\theta)$ to*

$$N_3[\epsilon,\theta] \equiv \int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n^a(\bar{x}-w-\epsilon s)\theta(s)\,d\sigma_s,$$

*is real analytic.*

*Proof.* We first prove statement $(i)$. Let $j\in\{1,\dots,n\}$. By Theorem C.4, there exists $\epsilon_2\in]0,\epsilon_1]$ small enough, such that the map of $]-\epsilon_2,\epsilon_2[\times C^{m,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$, which takes $(\epsilon,\theta)$ to $\int_{\partial\Omega}\partial_{x_j}R_n^a(\epsilon(\cdot-s))(\nu_\Omega)_j(s)\theta(s)\,d\sigma_s$ is real analytic. Then by standard calculus in Banach space and well known results of classical potential theory, we immediately deduce that there exists $\epsilon_2\in]0,\epsilon_1]$ such that $N_1$ is a real analytic map of $]-\epsilon_2,\epsilon_2[\times C^{m,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$.

By arguing as in the proof of statement $(i)$ and by well known properties of functions in Schauder spaces and well known results of classical potential theory, one can easily prove statement $(ii)$ (cf. also the proof of Lanza [78, Theorem 5.5, p. 287].)

Consider now statement $(iii)$. Let $V$ be a bounded open neighbourhood of $\bar{x}$ such that $\mathrm{cl}\,V\cap(w+Z_n^a)=\emptyset$. By taking $\epsilon_2''\in]0,\bar{\epsilon}_1]$ small enough, we can assume that

$$\mathrm{cl}\,V-(w+\epsilon\partial\Omega)\subseteq\mathbb{R}^n\setminus Z_n^a \qquad \forall\epsilon\in]-\epsilon_2'',\epsilon_2''[.$$

Next we define the map $\tilde{N}$ of $]-\epsilon_2'',\epsilon_2''[\times C^{m,\alpha}(\partial\Omega)$ to $C^0(\mathrm{cl}\,V)$, by setting

$$\tilde{N}[\epsilon,\theta](x) \equiv \int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n^a(x-w-\epsilon s)\theta(s)\,d\sigma_s, \qquad \forall x\in\mathrm{cl}\,V,$$

for all $(\epsilon,\theta) \in ]-\epsilon_2'',\epsilon_2''[ \times C^{m,\alpha}(\partial\Omega)$. Now we observe that if we denote by $\mathrm{id}_{\partial\Omega}$ the identity map in $\partial\Omega$, then the map of $]-\epsilon_2'',\epsilon_2''[$ to $C^0(\partial\Omega,\mathbb{R}^n)$, which takes $\epsilon$ to $w+\epsilon\,\mathrm{id}_{\partial\Omega}$ is real analytic. Hence, by Proposition C.1, $\tilde{N}$ is a real analytic map of $]-\epsilon_2'',\epsilon_2''[$ to $C^0(\mathrm{cl}\,V)$. Then, in order to conclude, it suffices to note that the map of $C^0(\mathrm{cl}\,V)$ to $\mathbb{R}$ which takes $h$ to $h(\bar{x})$ is linear and continuous, and thus real analytic. □

By the previous Proposition, we can deduce the validity of the following.

**Proposition 1.23.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\epsilon_2 \in ]0,\epsilon_1]$. Let $\Theta[\cdot]$ be a real analytic map of $]-\epsilon_2,\epsilon_2[$ to $C^{m,\alpha}(\partial\Omega)$. Then the following statements hold.*

*(i) If $\epsilon \in ]0,\epsilon_2[$, then we have*

$$\begin{aligned} w_a^-\big[\partial\Omega_\epsilon,\Theta[\epsilon](\tfrac{1}{\epsilon}(\cdot-w))\big](w+\epsilon t) = &-\frac{1}{2}\Theta[\epsilon](t) - \int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n(t-s)\Theta[\epsilon](s)\,d\sigma_s \\ &- \epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \quad \forall t\in\partial\Omega. \end{aligned}$$

*Moreover, there exists $\epsilon_3 \in ]0,\epsilon_2]$ such that the map $N_1$ of $]-\epsilon_3,\epsilon_3[$ to $C^{m,\alpha}(\partial\Omega)$ which takes $\epsilon$ to the function of $\partial\Omega$ to $\mathbb{R}$ defined by*

$$\begin{aligned} N_1[\epsilon](t) \equiv &-\frac{1}{2}\Theta[\epsilon](t) - \int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n(t-s)\Theta[\epsilon](s)\,d\sigma_s \\ &- \epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \quad \forall t\in\partial\Omega, \end{aligned}$$

*is real analytic.*

*(ii) If $\epsilon \in ]0,\epsilon_2[$, then we have*

$$\begin{aligned} w_a^+\big[\partial\Omega_\epsilon,\Theta[\epsilon](\tfrac{1}{\epsilon}(\cdot-w))\big](w+\epsilon t) = &\frac{1}{2}\Theta[\epsilon](t) - \int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n(t-s)\Theta[\epsilon](s)\,d\sigma_s \\ &- \epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \quad \forall t\in\partial\Omega. \end{aligned}$$

*Moreover, there exists $\epsilon_3' \in ]0,\epsilon_2]$ such that the map $N_2$ of $]-\epsilon_3',\epsilon_3'[$ to $C^{m,\alpha}(\partial\Omega)$ which takes $\epsilon$ to the function of $\partial\Omega$ to $\mathbb{R}$ defined by*

$$\begin{aligned} N_2[\epsilon](t) \equiv &\frac{1}{2}\Theta[\epsilon](t) - \int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n(t-s)\Theta[\epsilon](s)\,d\sigma_s \\ &- \epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \quad \forall t\in\partial\Omega, \end{aligned}$$

*is real analytic.*

*Proof.* The first part of statements $(i)$, $(ii)$ follows by the Theorem of change of variables in integrals and Theorem 1.13. The second part of statements $(i)$, $(ii)$ is an immediate consequence of Proposition 1.22 $(i)$. □

Then we have the following.

**Proposition 1.24.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\epsilon_2 \in ]0,\epsilon_1]$. Let $\Theta[\cdot]$ be a real analytic map of $]-\epsilon_2,\epsilon_2[$ to $C^{m,\alpha}(\partial\Omega)$. Then the following statements hold.*

*(i) Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exists $\epsilon_3 \in ]0,\epsilon_2]$ such that*

$$\mathrm{cl}\,V \subseteq \mathbb{T}_a[\Omega_\epsilon] \qquad \forall \epsilon \in ]-\epsilon_3,\epsilon_3[. \tag{1.71}$$

*If $\epsilon \in ]0,\epsilon_3[$, then we have*

$$w_a^-\big[\partial\Omega_\epsilon,\Theta[\epsilon](\tfrac{1}{\epsilon}(\cdot-w))\big](x) = -\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n^a(x-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \qquad \forall x\in\mathrm{cl}\,V. \tag{1.72}$$

*Moreover, there exists* $\epsilon_4 \in ]0, \epsilon_3]$ *such that the map* $N_1$ *of* $]-\epsilon_4, \epsilon_4[$ *to* $C^0_h(\mathrm{cl}\, V)$*, which takes* $\epsilon$ *to the function of* $\mathrm{cl}\, V$ *to* $\mathbb{R}$ *defined by*

$$N_1[\epsilon](x) \equiv \int_{\partial\Omega} \nu_\Omega(s) \cdot DS^a_n(x - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V, \tag{1.73}$$

*is real analytic.*

*(ii) Let* $\bar{V}$ *be a bounded open subset of* $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$*. Then there exists* $\bar{\epsilon}_3 \in ]0, \epsilon_2]$ *such that*

$$w + \epsilon\, \mathrm{cl}\, \bar{V} \subseteq \mathrm{cl}\, \mathbb{P}_a[\Omega_\epsilon] \qquad \forall \epsilon \in ]-\bar{\epsilon}_3, \bar{\epsilon}_3[ \setminus \{0\}. \tag{1.74}$$

*If* $\epsilon \in ]0, \bar{\epsilon}_3[$*, then we have*

$$\begin{aligned} w^-_a\big[\partial\Omega_\epsilon, \Theta[\epsilon](\tfrac{1}{\epsilon}(\cdot - w))\big](w + \epsilon t) =& w^-[\partial\Omega, \Theta[\epsilon]](t) \\ & - \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DR^a_n(\epsilon(t - s))\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \mathrm{cl}\, \bar{V}. \end{aligned} \tag{1.75}$$

*Moreover, there exists* $\bar{\epsilon}_4 \in ]0, \bar{\epsilon}_3]$ *such that the map* $N_2$ *of* $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ *to* $C^{m,\alpha}(\mathrm{cl}\, \bar{V})$*, which takes* $\epsilon$ *to the function of* $\mathrm{cl}\, \bar{V}$ *to* $\mathbb{R}$ *defined by*

$$N_2[\epsilon](t) \equiv w^-[\partial\Omega, \Theta[\epsilon]](t) - \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DR^a_n(\epsilon(t - s))\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \mathrm{cl}\, \bar{V}, \tag{1.76}$$

*is real analytic.*

*(iii) If* $\epsilon \in ]0, \epsilon_2[$*, then we have*

$$w^+_a\big[\partial\Omega_\epsilon, \Theta[\epsilon](\tfrac{1}{\epsilon}(\cdot - w))\big](w + \epsilon t) = w^+[\partial\Omega, \Theta[\epsilon]](t) - \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DR^a_n(\epsilon(t - s))\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \mathrm{cl}\, \Omega. \tag{1.77}$$

*Moreover, there exists* $\epsilon'_3 \in ]0, \epsilon_2]$ *such that the map* $N_3$ *of* $]-\epsilon'_3, \epsilon'_3[$ *to* $C^{m,\alpha}(\mathrm{cl}\,\Omega)$*, which takes* $\epsilon$ *to the function of* $\mathrm{cl}\,\Omega$ *to* $\mathbb{R}$ *defined by*

$$N_3[\epsilon](t) \equiv w^+[\partial\Omega, \Theta[\epsilon]](t) - \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DR^a_n(\epsilon(t - s))\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \mathrm{cl}\,\Omega, \tag{1.78}$$

*is real analytic.*

*Proof.* We first prove statement *(i)*. Clearly, by taking $\epsilon_3 \in ]0, \epsilon_2]$ small enough, we can assume that (1.71) holds. Equality (1.72) follows by the Theorem of change of variables in integrals. By arguing as in the proof of Proposition 1.22 *(iii)* and by standard calculus in Banach spaces, we immediately deduce that there exists $\epsilon_4 \in ]0, \epsilon_3[$ such that $N_1$ is a real analytic map of $]-\epsilon_4, \epsilon_4[$ to $C^0_h(\mathrm{cl}\, V)$.

Consider statement *(ii)*. Clearly, by taking $\bar{\epsilon}_3 \in ]0, \epsilon_2]$ small enough, we can assume that (1.74) holds. Equality (1.75) follows by the Theorem of change of variables in integrals. By Theorem B.1 *(iii)*, we easily deduce that the map of $]-\bar{\epsilon}_3, \bar{\epsilon}_3[$ to $C^{m,\alpha}(\mathrm{cl}\, \bar{V})$ which takes $\epsilon$ to $w^-[\partial\Omega, \Theta[\epsilon]]_{|\mathrm{cl}\, \bar{V}}$ is real analytic. Now let $V^\#$ be a bounded connected open subset of $\mathbb{R}^n$ of class $C^1$, such that

$$\mathrm{cl}\, \bar{V} \subseteq V^\# \subseteq \mathrm{cl}\, \bar{V} \subseteq \mathbb{R}^n \setminus \mathrm{cl}\,\Omega.$$

Possibly shrinking $\bar{\epsilon}_3$, we can assume that

$$w + \epsilon\, \mathrm{cl}\, V^\# \subseteq \mathrm{cl}\, \mathbb{P}_a[\Omega_\epsilon] \qquad \forall \epsilon \in ]-\bar{\epsilon}_3, \bar{\epsilon}_3[ \setminus \{0\}.$$

By Proposition C.3 and the continuity of the imbedding of $C^{m+1}(\mathrm{cl}\, V^\#)$ to $C^{m,\alpha}(\mathrm{cl}\, V^\#)$, it is easy to prove that there exists $\bar{\epsilon}_4 \in ]0, \bar{\epsilon}_3]$ such that the map of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ to $C^{m,\alpha}(\mathrm{cl}\, V^\#)$, which takes $\epsilon$ to the function $\epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DR^a_n(\epsilon(t - s))\Theta[\epsilon](s)\, d\sigma_s$ of the variable $t$ is real analytic. Thus, by the continuity of the restriction operator from $C^{m,\alpha}(\mathrm{cl}\,V^{\#})$ to $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$, we can easily conclude that $N_2$ is a real analytic map of $]-\bar{\epsilon}_4,\bar{\epsilon}_4[$ to $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$.

We finally turn to the proof of statement *(iii)*. Equality (1.77) follows by the Theorem of change of variables in integrals. By Theorem B.1 *(iii)*, we easily deduce that the map of $]-\epsilon_2,\epsilon_2[$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ which takes $\epsilon$ to $w^+[\partial\Omega,\Theta[\epsilon]]_{|\mathrm{cl}\,\Omega}$ is real analytic. By Proposition C.3, it is easy to prove that there exists $\epsilon_3'\in]0,\epsilon_2]$ such that the map of $]-\epsilon_3',\epsilon_3'[$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, which takes $\epsilon$ to the function $\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s$ of the variable $t$ is real analytic. Thus we can easily conclude that $N_3$ is a real analytic map of $]-\epsilon_3',\epsilon_3'[$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$. □

### 1.8.3 Some technical results for the periodic simple layer potential

We first prove the following elementary Lemma, concerning the periodic simple layer potential.

**Lemma 1.25.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $\epsilon\in]0,\epsilon_1[$ and $\theta\in\mathcal{U}_0^{m-1,\alpha}$. Then we have*

$$v_a\big[\partial\Omega_\epsilon,\theta(\frac{1}{\epsilon}(\cdot-w))\big](w+\epsilon t)=\epsilon\int_{\partial\Omega}S_n(t-s)\theta(s)\,d\sigma_s+\epsilon^{n-1}\int_{\partial\Omega}R_a^n(\epsilon(t-s))\theta(s)\,d\sigma_s\qquad\forall t\in\partial\Omega.$$

*Proof.* Let $\epsilon\in]0,\epsilon_1[$ and $\theta\in\mathcal{U}_0^{m-1,\alpha}$. By the Theorem of change of variables in integrals, we have

$$v_a\big[\partial\Omega_\epsilon,\theta(\frac{1}{\epsilon}(\cdot-w))\big](w+\epsilon t)=\epsilon^{n-1}\int_{\partial\Omega}S_n(\epsilon(t-s))\theta(s)\,d\sigma_s+\epsilon^{n-1}\int_{\partial\Omega}R_a^n(\epsilon(t-s))\theta(s)\,d\sigma_s\qquad\forall t\in\partial\Omega.$$

If $n=2$, then, by equality (1.65), we have

$$\epsilon\int_{\partial\Omega}S_2(\epsilon(t-s))\theta(s)\,d\sigma_s=\frac{1}{2\pi}\epsilon\log\epsilon\int_{\partial\Omega}\theta(s)\,d\sigma_s+\epsilon\int_{\partial\Omega}S_2(t-s)\theta(s)\,d\sigma_s\qquad\forall t\in\partial\Omega.$$

Accordingly, since $\int_{\partial\Omega}\theta(s)\,d\sigma_s=0$, then

$$\epsilon\int_{\partial\Omega}S_2(\epsilon(t-s))\theta(s)\,d\sigma_s=\epsilon\int_{\partial\Omega}S_2(t-s)\theta(s)\,d\sigma_s\qquad\forall t\in\partial\Omega.$$

If $n\geq 3$, then, by equality (1.65), we have

$$\epsilon^{n-1}\int_{\partial\Omega}S_n(\epsilon(t-s))\theta(s)\,d\sigma_s=\epsilon\int_{\partial\Omega}S_n(t-s)\theta(s)\,d\sigma_s\qquad\forall t\in\partial\Omega.$$

As a consequence, in both cases, we obtain

$$v_a\big[\partial\Omega_\epsilon,\theta(\frac{1}{\epsilon}(\cdot-w))\big](w+\epsilon t)=\epsilon\int_{\partial\Omega}S_n(t-s)\theta(s)\,d\sigma_s+\epsilon^{n-1}\int_{\partial\Omega}R_a^n(\epsilon(t-s))\theta(s)\,d\sigma_s\qquad\forall t\in\partial\Omega,$$

and thus the proof is complete. □

In the following Proposition, we study some integral operators that are related to the periodic simple layer potential and that appear in integral equations on 'rescaled' domains when we represent the solution of a certain boundary value problem in terms of a periodic simple layer potential.

**Proposition 1.26.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Then the following statements hold.*

*(i) There exists $\epsilon_2\in]0,\epsilon_1]$ such that the map $N_1$ of $]-\epsilon_2,\epsilon_2[\times C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$, which takes $(\epsilon,\theta)$ to the function $N_1[\epsilon,\theta]$ of $\partial\Omega$ to $\mathbb{R}$ defined by*

$$N_1[\epsilon,\theta](t)\equiv\int_{\partial\Omega}S_n(t-s)\theta(s)\,d\sigma_s+\epsilon^{n-2}\int_{\partial\Omega}R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\quad\forall t\in\partial\Omega,$$

*is real analytic.*

(ii) *There exists* $\epsilon_2' \in ]0, \epsilon_1]$ *such that the map* $N_2$ *of* $]-\epsilon_2', \epsilon_2'[ \times C^{m-1,\alpha}(\partial\Omega)$ *to* $C^{m-1,\alpha}(\partial\Omega)$*, which takes* $(\epsilon, \theta)$ *to the function* $N_2[\epsilon, \theta]$ *of* $\partial\Omega$ *to* $\mathbb{R}$ *defined by*

$$N_2[\epsilon,\theta](t) \equiv \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s \qquad \forall t \in \partial\Omega,$$

*is real analytic.*

(iii) *Let* $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$*. Let* $\bar{\epsilon}_1$ *be as in* (1.58)*. There exists* $\epsilon_2'' \in ]0, \bar{\epsilon}_1]$ *such that the map* $N_3$ *of* $]-\epsilon_2'', \epsilon_2''[ \times C^{m-1,\alpha}(\partial\Omega)$ *to* $\mathbb{R}$*, which takes* $(\epsilon, \theta)$ *to*

$$N_3[\epsilon, \theta] \equiv \int_{\partial\Omega} S_n^a(\bar{x} - w - \epsilon s)\theta(s)\, d\sigma_s,$$

*is real analytic.*

*Proof.* We first prove statement $(i)$. By Theorem C.4, there exists $\epsilon_2 \in ]0, \epsilon_1]$ small enough, such that the map of $]-\epsilon_2, \epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$, which takes $(\epsilon, \theta)$ to $\int_{\partial\Omega} R_n^a(\epsilon(\cdot - s))\theta(s)\, d\sigma_s$ is real analytic. Then by standard calculus in Banach space and well known results of classical potential theory, we immediately deduce that there exists $\epsilon_2 \in ]0, \epsilon_1]$ such that $N_1$ is a real analytic map of $]-\epsilon_2, \epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$.

Consider statement $(ii)$. Let $j \in \{1, \ldots, n\}$. By continuity and bilinearity of the pointwise product in Schauder spaces and by Theorem C.4, there exists $\epsilon_2 \in ]0, \epsilon_1]$ such that the map of $]-\epsilon_2, \epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$, which takes $(\epsilon, \theta)$ to the function $(\nu_\Omega(\cdot))_j \int_{\partial\Omega} \partial_{x_j} R_n^a(\epsilon(\cdot - s))\theta(s)\, d\sigma_s$ is real analytic. Then by standard calculus in Banach space and well known results of classical potential theory, we immediately deduce that there exists $\epsilon_2' \in ]0, \epsilon_1]$ such that $N_2$ is a real analytic map of $]-\epsilon_2', \epsilon_2'[ \times C^{m-1,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$.

Consider now statement $(iii)$. Let $V$ be a bounded open neighbourhood of $\bar{x}$ such that $\mathrm{cl}\, V \cap (w + Z_n^a)$. Taking $\epsilon_2'' \in ]0, \bar{\epsilon}_1]$ small enough, we can assume that

$$\mathrm{cl}\, V - (w + \epsilon\partial\Omega) \subseteq \mathbb{R}^n \setminus Z_n^a \qquad \forall \epsilon \in ]-\epsilon_2'', \epsilon_2''[.$$

Next we define the map $\tilde{N}$ of $]-\epsilon_2'', \epsilon_2''[ \times C^{m-1,\alpha}(\partial\Omega)$ to $C^0(\mathrm{cl}\, V)$, by setting

$$\tilde{N}[\epsilon, \theta](x) \equiv \int_{\partial\Omega} S_n^a(x - w - \epsilon s)\theta(s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

for all $(\epsilon, \theta) \in ]-\epsilon_2'', \epsilon_2''[ \times C^{m,\alpha}(\partial\Omega)$. Now we observe that if we denote by $\mathrm{id}_{\partial\Omega}$ the identity map in $\partial\Omega$, then the map of $]-\epsilon_2'', \epsilon_2''[$ to $C^0(\partial\Omega, \mathbb{R}^n)$, which takes $\epsilon$ to $w + \epsilon\, \mathrm{id}_{\partial\Omega}$ is real analytic. Hence, by Proposition C.1, $\tilde{N}$ is a real analytic map of $]-\epsilon_2'', \epsilon_2''[$ to $C^0(\mathrm{cl}\, V)$. Then, in order to conclude, it suffices to note that the map of $C^0(\mathrm{cl}\, V)$ to $\mathbb{R}$ which takes $h$ to $h(\bar{x})$ is linear and continuous, and thus real analytic. □

Then we have the following elementary Lemma.

**Lemma 1.27.** *Let* $m \in \mathbb{N} \setminus \{0\}$*,* $\alpha \in ]0, 1[$*. Let* $w \in A$*. Let* $\Omega$*,* $\epsilon_1$ *be as in* (1.56)*,* (1.57)*, respectively. Let* $\epsilon_2 \in ]0, \epsilon_1]$*. Let* $\Theta[\cdot]$ *be a real analytic map of* $]-\epsilon_2, \epsilon_2[$ *to* $C^{m-1,\alpha}(\partial\Omega)$ *such that* $\Theta[\epsilon] \in \mathcal{U}_0^{m-1,\alpha}$ *for all* $\epsilon \in ]0, \epsilon_2[$*. Then* $\Theta[\epsilon] \in \mathcal{U}_0^{m-1,\alpha}$ *for all* $\epsilon \in ]-\epsilon_2, \epsilon_2[$*.*

*Proof.* Let $N$ be the function of $]-\epsilon_2, \epsilon_2[$ to $\mathbb{R}$ defined by

$$N[\epsilon] \equiv \int_{\partial\Omega} \Theta[\epsilon](s)\, d\sigma_s \qquad \forall \epsilon \in ]-\epsilon_2, \epsilon_2[.$$

Clearly, $N$ is a real analytic function of $]-\epsilon_2, \epsilon_2[$ to $\mathbb{R}$. Since $\Theta[\epsilon] \in \mathcal{U}_0^{m-1,\alpha}$ for all $\epsilon \in ]0, \epsilon_2[$, then $N[\epsilon] = 0$ for all $\epsilon \in ]0, \epsilon_2[$. Accordingly, by the identity principle for real analytic functions, we have $N[\epsilon] = 0$ for all $\epsilon \in ]-\epsilon_2, \epsilon_2[$, and, as a consequence, $\Theta[\epsilon] \in \mathcal{U}_0^{m-1,\alpha}$ for all $\epsilon \in ]-\epsilon_2, \epsilon_2[$. □

By the previous results, we deduce the validity of the following.

**Proposition 1.28.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\epsilon_2 \in ]0,\epsilon_1]$. Let $\Theta[\cdot]$ be a real analytic map of $]-\epsilon_2,\epsilon_2[$ to $C^{m-1,\alpha}(\partial\Omega)$ such that $\Theta[\epsilon] \in \mathcal{U}_0^{m-1,\alpha}$ for all $\epsilon \in [0,\epsilon_2[$. Then the following statements hold.*

*(i) If $\epsilon \in ]0,\epsilon_2[$, then we have*

$$v_a\big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](w+\epsilon t) = \epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \quad \forall t \in \partial\Omega.$$

*Moreover, there exists $\epsilon_3 \in ]0,\epsilon_2]$ such that the map $N_1$ of $]-\epsilon_3,\epsilon_3[$ to $C^{m,\alpha}(\partial\Omega)$ which takes $\epsilon$ to the function of $\partial\Omega$ to $\mathbb{R}$ defined by*

$$N_1[\epsilon](t) \equiv \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \quad \forall t \in \partial\Omega,$$

*is real analytic.*

*(ii) If $\epsilon \in ]0,\epsilon_2[$, then we have*

$$\Big(\frac{\partial}{\partial\nu_{\Omega_\epsilon}} v_a^-\big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big]\Big)(w+\epsilon t) = \frac{1}{2}\Theta[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \quad \forall t \in \partial\Omega.$$

*Moreover, there exists $\epsilon_3 \in ]0,\epsilon_2]$ such that the map $N_2$ of $]-\epsilon_3,\epsilon_3[$ to $C^{m-1,\alpha}(\partial\Omega)$ which takes $\epsilon$ to the function of $\partial\Omega$ to $\mathbb{R}$ defined by*

$$N_2[\epsilon](t) \equiv \frac{1}{2}\Theta[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \quad \forall t \in \partial\Omega,$$

*is real analytic.*

*(iii) If $\epsilon \in ]0,\epsilon_2[$, then we have*

$$\Big(\frac{\partial}{\partial\nu_{\Omega_\epsilon}} v_a^+\big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big]\Big)(w+\epsilon t) = -\frac{1}{2}\Theta[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \quad \forall t \in \partial\Omega.$$

*Moreover, there exists $\epsilon_3' \in ]0,\epsilon_2]$ such that the map $N_3$ of $]-\epsilon_3',\epsilon_3'[$ to $C^{m-1,\alpha}(\partial\Omega)$ which takes $\epsilon$ to the function of $\partial\Omega$ to $\mathbb{R}$ defined by*

$$N_3[\epsilon](t) \equiv -\frac{1}{2}\Theta[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \quad \forall t \in \partial\Omega,$$

*is real analytic.*

*Proof.* The first part of statement $(i)$ is an immediate consequence of Lemma 1.25, while the second part follows by Proposition 1.26 $(i)$. The first part of statements $(ii)$, $(iii)$ follows by the Theorem of change of variables in integrals and Theorem 1.15. The second part of statements $(ii)$, $(iii)$ is an immediate consequence of Proposition 1.26 $(ii)$. □

Finally, we have the following.

**Proposition 1.29.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\epsilon_2 \in ]0,\epsilon_1]$. Let $\Theta[\cdot]$ be a real analytic map of $]-\epsilon_2,\epsilon_2[$ to $C^{m-1,\alpha}(\partial\Omega)$ such that $\Theta[\epsilon] \in \mathcal{U}_0^{m-1,\alpha}$ for all $\epsilon \in [0,\epsilon_2[$. Then the following statements hold.*

*(i)* *Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exists $\epsilon_3 \in ]0, \epsilon_2]$ such that*

$$\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon] \qquad \forall \epsilon \in ]-\epsilon_3, \epsilon_3[. \tag{1.79}$$

*If $\epsilon \in ]0, \epsilon_3[$, then we have*

$$v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) = \epsilon^{n-1} \int_{\partial\Omega} S_n^a(x - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V. \tag{1.80}$$

*Moreover, there exists $\epsilon_4 \in ]0, \epsilon_3]$ such that the map $N_1$ of $]-\epsilon_4, \epsilon_4[$ to $C_h^0(\mathrm{cl}\, V)$, which takes $\epsilon$ to the function of $\mathrm{cl}\, V$ to $\mathbb{R}$ defined by*

$$N_1[\epsilon](x) \equiv \int_{\partial\Omega} S_n^a(x - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V, \tag{1.81}$$

*is real analytic. Furthermore, there exist $\epsilon_4' \in ]0, \epsilon_4]$ and a real analytic map $\tilde{N}_1$ of $]-\epsilon_4', \epsilon_4'[$ to $C_h^0(\mathrm{cl}\, V)$ such that*

$$N_1[\epsilon](x) = \epsilon \tilde{N}_1[\epsilon](x) \qquad \forall x \in \mathrm{cl}\, V, \tag{1.82}$$

*for all $\epsilon \in ]-\epsilon_4', \epsilon_4'[$.*

*(ii)* *Let $\bar{V}$ be a bounded open subset of $\mathbb{R}^n \setminus \mathrm{cl}\, \Omega$. Then there exists $\bar{\epsilon}_3 \in ]0, \epsilon_2]$ such that*

$$w + \epsilon\, \mathrm{cl}\, \bar{V} \subseteq \mathrm{cl}\, \mathbb{P}_a[\Omega_\epsilon] \qquad \forall \epsilon \in ]-\bar{\epsilon}_3, \bar{\epsilon}_3[ \setminus \{0\}. \tag{1.83}$$

*If $\epsilon \in ]0, \bar{\epsilon}_3[$, then we have*

$$v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](w + \epsilon t) = \epsilon v^-[\partial\Omega, \Theta[\epsilon]](t) + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t - s))\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \mathrm{cl}\, \bar{V}. \tag{1.84}$$

*Moreover, there exists $\bar{\epsilon}_4 \in ]0, \bar{\epsilon}_3]$ such that the map $N_2$ of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ to $C^{m,\alpha}(\mathrm{cl}\, \bar{V})$, which takes $\epsilon$ to the function of $\mathrm{cl}\, \bar{V}$ to $\mathbb{R}$ defined by*

$$N_2[\epsilon](t) \equiv v^-[\partial\Omega, \Theta[\epsilon]](t) + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t - s))\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \mathrm{cl}\, \bar{V}, \tag{1.85}$$

*is real analytic.*

*(iii)* *If $\epsilon \in ]0, \epsilon_2[$, then we have*

$$v_a^+ \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](w + \epsilon t) = \epsilon v^+[\partial\Omega, \Theta[\epsilon]](t) + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t - s))\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \mathrm{cl}\, \Omega. \tag{1.86}$$

*Moreover, there exists $\epsilon_3' \in ]0, \epsilon_2]$ such that the map $N_3$ of $]-\epsilon_3', \epsilon_3'[$ to $C^{m,\alpha}(\mathrm{cl}\, \Omega)$, which takes $\epsilon$ to the function of $\mathrm{cl}\, \Omega$ to $\mathbb{R}$ defined by*

$$N_3[\epsilon](t) \equiv v^+[\partial\Omega, \Theta[\epsilon]](t) + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t - s))\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \mathrm{cl}\, \Omega, \tag{1.87}$$

*is real analytic.*

*Proof.* We first prove statement $(i)$. Clearly, by taking $\epsilon_3 \in ]0, \epsilon_2]$ small enough, we can assume that (1.79) holds. Equality (1.80) follows by the Theorem of change of variables in integrals. Then we note that, by Lemma 1.27, we have $\Theta[\epsilon] \in \mathcal{U}_0^{m-1,\alpha}$ for all $\epsilon \in ]-\epsilon_2, \epsilon_2[$. As a consequence, one can easily show that $N_1[\epsilon] \in C_h^0(\mathrm{cl}\, V)$ for all $\epsilon \in ]-\epsilon_2, \epsilon_2[$. By arguing as in the proof of Proposition 1.26 $(iii)$ and by standard calculus in Banach spaces, we immediately deduce that there exists $\epsilon_4 \in ]0, \epsilon_3]$ such that $N_1$ is a real analytic map of $]-\epsilon_4, \epsilon_4[$ to $C_h^0(\mathrm{cl}\, V)$. Since $\Theta[0] \in \mathcal{U}_0^{m-1,\alpha}$, then we have

$$N_1[0](x) = S_n^a(x - w) \int_{\partial\Omega} \Theta[0](s)\, d\sigma_s = 0 \qquad \forall x \in \mathrm{cl}\, V$$

Hence, there exist $\epsilon'_4 \in ]0, \epsilon_4]$ and a real analytic map $\tilde{N}_1$ of $]-\epsilon'_4, \epsilon'_4[$ to $C^0_h(\mathrm{cl}\, V)$ such that

$$N_1[\epsilon] = \epsilon \tilde{N}_1[\epsilon] \qquad \text{in } C^0_h(\mathrm{cl}\, V),$$

for all $\epsilon \in ]-\epsilon'_4, \epsilon'_4[$.

Consider statement *(ii)*. Clearly, by taking $\bar{\epsilon}_3 \in ]0, \epsilon_2]$ small enough, we can assume that (1.83) holds. Equality (1.84) follows by the Theorem of change of variables in integrals and by a modification of the proof of Lemma 1.25. By Theorem B.2 *(iii)*, we easily deduce that the map of $]-\bar{\epsilon}_3, \bar{\epsilon}_3[$ to $C^{m,\alpha}(\mathrm{cl}\, \bar{V})$ which takes $\epsilon$ to $v^-[\partial\Omega, \Theta[\epsilon]]_{|\mathrm{cl}\, \bar{V}}$ is real analytic. Now let $V^\#$ be a bounded connected open subset of $\mathbb{R}^n$ of class $C^1$, such that

$$\mathrm{cl}\, \bar{V} \subseteq V^\# \subseteq \mathrm{cl}\, \bar{V} \subseteq \mathbb{R}^n \setminus \mathrm{cl}\, \Omega.$$

Possibly shrinking $\bar{\epsilon}_3$, we can assume that

$$w + \epsilon\, \mathrm{cl}\, V^\# \subseteq \mathrm{cl}\, \mathbb{P}_a[\Omega_\epsilon] \qquad \forall \epsilon \in ]-\bar{\epsilon}_3, \bar{\epsilon}_3[ \setminus \{0\}.$$

By Proposition C.3 and the continuity of the imbedding of $C^{m+1}(\mathrm{cl}\, V^\#)$ to $C^{m,\alpha}(\mathrm{cl}\, V^\#)$, it is easy to prove that there exists $\bar{\epsilon}_4 \in ]0, \bar{\epsilon}_3]$ such that the map of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ to $C^{m,\alpha}(\mathrm{cl}\, V^\#)$, which takes $\epsilon$ to the function $\epsilon^{n-2} \int_{\partial\Omega} R^a_n(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s$ of the variable $t$ is real analytic. Thus, by the continuity of the restriction operator from $C^{m,\alpha}(\mathrm{cl}\, V^\#)$ to $C^{m,\alpha}(\mathrm{cl}\, \bar{V})$, we can easily conclude that $N_2$ is a real analytic map of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ to $C^{m,\alpha}(\mathrm{cl}\, \bar{V})$.

We finally turn to the proof of statement *(iii)*. Equality (1.86) follows by the Theorem of change of variables in integrals and by a modification of the proof of Lemma 1.25. By Theorem B.2 *(ii)*, we easily deduce that the map of $]-\epsilon_2, \epsilon_2[$ to $C^{m,\alpha}(\mathrm{cl}\, \Omega)$ which takes $\epsilon$ to $v^+[\partial\Omega, \Theta[\epsilon]]_{|\mathrm{cl}\, \Omega}$ is real analytic. By Proposition C.3, it is easy to prove that there exists $\epsilon'_3 \in ]0, \epsilon_2]$ such that the map of $]-\epsilon'_3, \epsilon'_3[$ to $C^{m,\alpha}(\mathrm{cl}\, \Omega)$, which takes $\epsilon$ to the function $\epsilon^{n-2} \int_{\partial\Omega} R^a_n(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s$ of the variable $t$ is real analytic. Thus we can easily conclude that $N_3$ is a real analytic map of $]-\epsilon'_3, \epsilon'_3[$ to $C^{m,\alpha}(\mathrm{cl}\, \Omega)$. □

# CHAPTER 2

# Singular perturbation and homogenization problems for the Laplace and Poisson equations with Dirichlet and Neumann boundary conditions

In this Chapter we introduce the periodic Dirichlet and Neumann problems for the Laplace and Poisson equations and we study singular perturbation and homogenization problems for the Laplace and Poisson equations with Dirichlet and Neumann boundary conditions in a periodically perforated domain. First of all, by means of periodic simple and double layer potentials, we show the solvability of these problems (see also Shcherbina [128].) Secondly, we consider singular perturbation problems in a periodically perforated domain with small holes, and we apply the obtained results to homogenization problems. Our strategy follows the functional analytic approach of Lanza [75], where the asymptotic behaviour of the solutions of the Dirichlet problem for the Laplace operator in a domain with a small hole has been studied. On the other hand, as far as the Poisson equation is concerned, we mention in particular Lanza [70]. We also note that Dirichlet (and others) boundary value problems in singularly perturbed domains in the frame of linearized elasticity have been analysed by Dalla Riva in his Ph.D. Dissertation [33]. One of the tools used in our analysis is the study of the dependence of layer potentials upon perturbations (cf. Lanza and Rossi [85] and also Dalla Riva and Lanza [40].) A generalization of the result concerning the singularly perturbed Dirichlet problem for the Laplace equation can be found in [104] (see also [103].) Moreover, for the Dirichlet problem for the Poisson equation, we refer to [105].

We retain the notation of Chapter 1 (see in particular Sections 1.1, 1.3, Theorem 1.4, and Definitions 1.12, 1.14, 1.16.) For notation, definitions, and properties concerning classical layer potentials for the Laplace equation, we refer to Appendix B.

## 2.1 Periodic Dirichlet and Neumann boundary value problems for the Poisson and Laplace equation

In this Section we study periodic Dirichlet and Neumann boundary value problems for the Poisson and Laplace equation.

### 2.1.1 Formulation of the problems

In this Subsection we introduce the periodic Dirichlet and Neumann problems for the Poisson and Laplace equations.

**Definition 2.1.** Let $m \in \mathbb{N} \setminus \{0\}$ Let $\alpha \in ]0, 1[$. Let $\mathbb{I}$ be as in (1.46). Let $f \in C^0(\mathbb{R}^n)$ be such that

$$f(x + a_i) = f(x) \qquad \forall x \in \mathbb{R}^n, \quad \forall i \in \{1, \dots, n\},$$

and

$$\int_A f(y)\, dy = 0.$$

Let $\Gamma \in C^{m,\alpha}(\partial\mathbb{I})$. We say that a function $u \in C^0(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}])$ solves the *periodic Dirichlet problem for the Poisson equation* if

$$\begin{cases} \Delta u(x) = f(x) & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}, \\ u(x) = \Gamma(x) & \forall x \in \partial\mathbb{I}. \end{cases} \tag{2.1}$$

*Remark* 2.2. Boundary value problem (2.1) with $f \equiv 0$ is called the *periodic Dirichlet problem for the Laplace equation.*

**Definition 2.3.** Let $m \in \mathbb{N} \setminus \{0\}$ Let $\alpha \in ]0, 1[$. Let $\mathbb{I}$ be as in (1.46). Let $f \in C^0(\mathbb{R}^n)$ be such that

$$f(x + a_i) = f(x) \qquad \forall x \in \mathbb{R}^n, \quad \forall i \in \{1, \dots, n\},$$

and

$$\int_A f(y)\,dy = 0.$$

Let $\Gamma \in C^{m-1,\alpha}(\partial\mathbb{I})$. We say that a function $u \in C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}])$ solves the *periodic Neumann problem for the Poisson equation* if

$$\begin{cases} \Delta u(x) = f(x) & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) = \Gamma(x) & \forall x \in \partial\mathbb{I}. \end{cases} \tag{2.2}$$

*Remark* 2.4. Boundary value problem (2.2) with $f \equiv 0$ is called the *periodic Neumann problem for the Laplace equation.*

### 2.1.2 Uniqueness results for the solutions of the periodic Dirichlet and Neumann problems

In this Subsection we prove uniqueness results for the solutions of the periodic Dirichlet and Neumann problems for the Laplace equation. Clearly, by these results, we can deduce the analogues for the Poisson equation.

In the following known Theorem, we deduce by the Strong Maximum and Minimum Principles a periodic version for harmonic functions defined on $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$.

**Theorem 2.5** (Strong Maximum and Minimum Principles for periodic harmonic functions)**.** *Let $\mathbb{I}$ be a bounded connected open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,\mathbb{I} \subseteq A$ and $\mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{I}$ is connected. Let $\mathbb{T}_a[\mathbb{I}]$ be as in* (1.49)*. Let $u \in C^0(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}])$ be such that*

$$u(x + a_i) = u(x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\},$$

*and*

$$\Delta u(x) = 0 \qquad \forall x \in \mathbb{T}_a[\mathbb{I}].$$

*Then the following statements hold.*

*(i) If there exists a point $x_0 \in \mathbb{T}_a[\mathbb{I}]$ such that $u(x_0) = \max_{\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} u$, then $u$ is constant within $\mathbb{T}_a[\mathbb{I}]$.*

*(ii) If there exists a point $x_0 \in \mathbb{T}_a[\mathbb{I}]$ such that $u(x_0) = \min_{\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} u$, then $u$ is constant within $\mathbb{T}_a[\mathbb{I}]$.*

*As a consequence,*

*(j)*

$$\max_{\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} u = \max_{\partial\mathbb{I}} u,$$

*(jj)*

$$\min_{\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} u = \min_{\partial\mathbb{I}} u.$$

*Proof.* Clearly, statements $(j)$ and $(jj)$ are straightforward consequences of $(i)$ and $(ii)$. Furthermore, statement $(ii)$ follows by statement $(i)$ by replacing $u$ with $-u$. Therefore, it suffices to prove $(i)$. Let $u$ and $x_0$ be as in the hypotheses. By periodicity of $u$, $\sup_{x \in \mathbb{T}_a[\mathbb{I}]} u(x) < +\infty$. Then by the Maximum Principle, $u$ must be constant in $\mathbb{T}_a[\mathbb{I}]$ (cf. *e.g.*, Folland [52, Theorem 2.13, p. 72].) □

Clearly, by the previous Theorem, we can deduce a uniqueness result for the solutions of a periodic Dirichlet problem for the Laplace equation.

**Corollary 2.6.** *Let* $\mathbb{I}$ *be a bounded connected open subset of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\,\mathbb{I} \subseteq A$ *and* $\mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{I}$ *is connected. Let* $\mathbb{T}_a[\mathbb{I}]$ *be as in* (1.49). *Let* $u \in C^0(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}])$ *be such that*

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}, \\ u(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

*Then* $u = 0$ *in* $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$.

*Proof.* It is a straightforward consequence of Theorem 2.5. □

In the following Proposition, we prove a uniqueness result, up to constant functions, for the periodic Neumann problem for the Laplace equation.

**Proposition 2.7.** *Let* $m \in \mathbb{N}\setminus\{0\}$ *Let* $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be as in* (1.46). *Let* $u \in C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}])$ *be such that*

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

*Then* $u$ *is constant in* $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$.

*Proof.* By Green's Formula and by the harmonicity of $u$, we have

$$\begin{aligned} \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2 \, dx &= - \int_{\mathbb{P}_a[\mathbb{I}]} u(x) \Delta u(x) \, dx + \int_{\partial \mathbb{P}_a[\mathbb{I}]} u(x) \frac{\partial}{\partial \nu_{\mathbb{P}_a[\mathbb{I}]}} u(x) \, d\sigma_x \\ &= \int_{\partial A} u(x) \frac{\partial}{\partial \nu_A} u(x) \, d\sigma_x - \int_{\partial \mathbb{I}} u(x) \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) \, d\sigma_x. \end{aligned}$$

As a consequence, by the periodicity of $u$ and since

$$\frac{\partial}{\partial \nu_{\mathbb{I}}} u = 0 \qquad \text{on } \partial\mathbb{I},$$

we have

$$\int_{\partial A} u(x) \frac{\partial}{\partial \nu_A} u(x) \, d\sigma_x - \int_{\partial \mathbb{I}} u(x) \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) \, d\sigma_x = 0.$$

Accordingly,

$$|\nabla u|^2 = 0 \qquad \text{in } \mathrm{cl}\,\mathbb{P}_a[\mathbb{I}],$$

and so $u$ is constant in $\mathrm{cl}\,\mathbb{P}_a[\mathbb{I}]$ and, consequently, in $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$. □

As for classical Neumann problems for the Poisson and Laplace equations, in the following Proposition, we show a necessary condition on the Neumann datum for the solvability of the periodic Neumann problem for the Poisson equation.

**Proposition 2.8.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be as in* (1.46). *Let* $f$ *and* $\Gamma$ *be as in Definition 2.3. If the periodic Neumann problem for the Poisson equation* (2.2) *has a solution in* $C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}])$, *then*

$$\int_{\partial\mathbb{I}} \Gamma(x) \, d\sigma_x = \int_{\mathbb{I}} f(x) \, dx.$$

*Proof.* By Green's Formula and by the periodicity of $u$, we have

$$\begin{aligned} \int_{\mathbb{P}_a[\mathbb{I}]} \Delta u(x) \, dx &= \int_{\partial \mathbb{P}_a[\mathbb{I}]} \frac{\partial}{\partial \nu_{\mathbb{P}_a[\mathbb{I}]}} u(x) \, d\sigma_x \\ &= \int_{\partial A} \frac{\partial}{\partial \nu_A} u(x) \, d\sigma_x - \int_{\partial \mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) \, d\sigma_x \\ &= - \int_{\partial \mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) \, d\sigma_x \\ &= - \int_{\partial \mathbb{I}} \Gamma(x) \, d\sigma_x. \end{aligned}$$

On the other hand,

$$\int_{\mathbb{P}_a[\mathbb{I}]} \Delta u(x)\,dx = \int_{\mathbb{P}_a[\mathbb{I}]} f(x)\,dx \\ = -\int_{\mathbb{I}} f(x)\,dx.$$

Thus,

$$\int_{\partial\mathbb{I}} \Gamma(x)\,d\sigma_x = \int_{\mathbb{I}} f(x)\,dx,$$

and the proof is complete. □

Clearly, we have the corresponding result for the periodic Neumann problem for the Laplace equation.

**Corollary 2.9.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $\Gamma \in C^{m-1,\alpha}(\partial\mathbb{I})$. If the periodic Neumann problem for the Laplace equation with Neumann datum $\Gamma$ has a solution in $C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}])$, then*

$$\int_{\partial\mathbb{I}} \Gamma(x)\,d\sigma_x = 0.$$

*Proof.* It is an immediate consequence of the previous Theorem with $f \equiv 0$. □

### 2.1.3 Existence results for the solutions of the periodic Dirichlet and Neumann problems

In this Subsection we show the existence of solutions of the periodic Dirichlet and Neumann problems for the Laplace and Poisson equations. We shall solve these problems by means of periodic simple layer, double layer and Newtonian potentials. Clearly, in order to solve boundary value problems by means of periodic layer potentials, we need to study the integral operators describing the behaviour on the boundary of the periodic layer potentials.

We have the following Lemma that we shall need in the sequel.

**Lemma 2.10.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $\mu \in C^{0,\alpha}(\partial\mathbb{I})$. Then the following equalities hold.*

*(i)*
$$\int_{\partial\mathbb{I}}\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y\,d\sigma_x = \left(\frac{1}{2} - \frac{|\mathbb{I}|_n}{|A|_n}\right)\int_{\partial\mathbb{I}}\mu(y)\,d\sigma_y.$$

*(ii)*
$$\int_{\partial\mathbb{I}}\left(-\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y\right)d\sigma_x = -\frac{|\mathbb{I}|_n}{|A|_n}\int_{\partial\mathbb{I}}\mu(y)\,d\sigma_y.$$

*(iii)*
$$\int_{\partial\mathbb{I}}\left(\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y\right)d\sigma_x = \left(1 - \frac{|\mathbb{I}|_n}{|A|_n}\right)\int_{\partial\mathbb{I}}\mu(y)\,d\sigma_y.$$

*Proof.* Consider $(i)$. By Theorem 1.13 $(iv)$, by Fubini's Theorem, and by virtue of the parity of $S_n^a$, we have

$$\int_{\partial\mathbb{I}}\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y\,d\sigma_x = \int_{\partial\mathbb{I}}\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}\big(S_n^a(x-y)\big)\,d\sigma_x\mu(y)\,d\sigma_y \\ = \int_{\partial\mathbb{I}}\left(\frac{1}{2} - \frac{|\mathbb{I}|_n}{|A|_n}\right)\mu(y)\,d\sigma_y \\ = \left(\frac{1}{2} - \frac{|\mathbb{I}|_n}{|A|_n}\right)\int_{\partial\mathbb{I}}\mu(y)\,d\sigma_y.$$

The equalities in $(ii)$ and $(iii)$ are immediate consequences of $(i)$. □

In the following Proposition, we show the compactness of the linear operators that appear in the description of the behaviour on the boundary of the periodic simple and double layer potentials.

**Proposition 2.11.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Then the following statements hold.*

*(i) The map of $L^2(\partial\mathbb{I})$ to $L^2(\partial\mathbb{I})$ which takes $\mu$ to the function of the variable $x \in \partial\mathbb{I}$ defined by*

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\mathbb{I},$$

*is compact. Moreover, its adjoint is the map of $L^2(\partial\mathbb{I})$ to $L^2(\partial\mathbb{I})$, which takes $\mu$ to the function of the variable $x \in \partial\mathbb{I}$ defined by*

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\mathbb{I}.$$

*(ii) The map of $L^2(\partial\mathbb{I})$ to $L^2(\partial\mathbb{I})$ which takes $\mu$ to the function of the variable $x \in \partial\mathbb{I}$ defined by*

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\mathbb{I},$$

*is compact. Moreover, its adjoint is the map of $L^2(\partial\mathbb{I})$ to $L^2(\partial\mathbb{I})$, which takes $\mu$ to the function of the variable $x \in \partial\mathbb{I}$ defined by*

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\mathbb{I}.$$

*Proof.* Clearly, it suffices to consider one of the two statements. Consider $(i)$. Clearly,

$$\begin{aligned}\int_{\partial\mathbb{I}} &\frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y \\ &= \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(S_n(x-y)\big)\mu(y)\,d\sigma_y + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(R_n^a(x-y)\big)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\mathbb{I},\end{aligned}$$

for all $\mu \in L^2(\partial\mathbb{I})$. Since the kernel $\frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(S_n(x-y)\big)$ has a weak singularity and the function $(x,y) \mapsto \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(R_n^a(x-y)\big)$ is continuous in $\partial\mathbb{I} \times \partial\mathbb{I}$, then the linear operator considered in $(i)$ is compact (cf. *e.g.*, Folland [52, Prop. 3.11, p. 121].) A simple computation based on the simmetry of $S_n^a$ shows that its adjoint is the one defined in the second part of $(i)$. Finally, the statement in $(ii)$ is an immediate consequence of $(i)$. □

In the following Propositions, we show the injectivity of some linear operators related to layer potentials.

**Proposition 2.12.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Then the following statements hold.*

*(i) Let $\mu \in L^2(\partial\mathbb{I})$ and*

$$-\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y = 0 \qquad \textit{a.e. on } \partial\mathbb{I}. \tag{2.3}$$

*Then $\mu \equiv 0$.*

*(ii) Let $\mu \in L^2(\partial\mathbb{I})$ and*

$$\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y = 0 \qquad \textit{a.e. on } \partial\mathbb{I}. \tag{2.4}$$

*Then $\mu \equiv 0$.*

*Proof.* We first prove $(i)$. Let $\mu$ be as in $(i)$. By Theorem 1.21 $(iv)$, we have that $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$. By Lemma 2.10 $(ii)$, we have that $\int_{\partial\mathbb{I}} \mu \, d\sigma = 0$. Consequently,

$$\Delta v_a[\partial\mathbb{I}, \mu] = 0 \qquad \text{in } \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}].$$

Then $v_a^+[\partial\mathbb{I}, \mu]_{|\operatorname{cl}\mathbb{I}}$ solves the following interior Neumann problem for the Laplace equation in $\mathbb{I}$

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{I}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} u = 0 & \text{on } \partial\mathbb{I}. \end{cases}$$

Hence, there exists a constant $C \in \mathbb{R}$ such that

$$v_a^+[\partial\mathbb{I}, \mu](x) = C \qquad \forall x \in \mathbb{S}_a[\mathbb{I}].$$

As a consequence, $v_a^-[\partial\mathbb{I}, \mu]$ solves the following periodic Dirichlet problem for the Laplace equation

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_i) = u(x) & \forall x \in \operatorname{cl}\mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}, \\ u(x) = C & \forall x \in \partial\mathbb{I}. \end{cases}$$

Thus, by Theorem 2.5, $v_a^-[\partial\mathbb{I}, \mu] = C$ in $\operatorname{cl}\mathbb{T}_a[\mathbb{I}]$, and so

$$v_a[\partial\mathbb{I}, \mu](x) = C \qquad \forall x \in \mathbb{R}^n.$$

Then, by the third formula in Theorem 1.15 $(iv)$, we have $\mu \equiv 0$.

We now consider $(ii)$. Let $\mu$ be as in $(ii)$. By Theorem 1.21 $(iii)$, we have that $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$. By Lemma 2.10 $(iii)$, we have that $\int_{\partial\mathbb{I}} \mu \, d\sigma = 0$. Consequently,

$$\Delta v_a[\partial\mathbb{I}, \mu] = 0 \qquad \text{in } \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}].$$

Then $v_a^-[\partial\mathbb{I}, \mu]$ solves the following periodic Neumann problem for the Laplace equation

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_i) = u(x) & \forall x \in \operatorname{cl}\mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

Hence, by virtue of Proposition 2.7, there exists a constant $C \in \mathbb{R}$ such that

$$v_a^-[\partial\mathbb{I}, \mu](x) = C \qquad \forall x \in \mathbb{T}_a[\mathbb{I}].$$

Consequently, $v_a^+[\partial\mathbb{I}, \mu]_{|\operatorname{cl}\mathbb{I}}$ solves the following interior Dirichlet problem for the Laplace equation in $\mathbb{I}$

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{I}, \\ u = C & \text{on } \partial\mathbb{I}. \end{cases}$$

Thus, $v_a^+[\partial\mathbb{I}, \mu] = C$ in $\operatorname{cl}\mathbb{S}_a[\mathbb{I}]$, and so

$$v_a[\partial\mathbb{I}, \mu](x) = C \qquad \forall x \in \mathbb{R}^n.$$

Then, by the third formula in Theorem 1.15 $(iv)$, we have $\mu \equiv 0$. □

By the previous Proposition, we deduce the following.

**Proposition 2.13.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $\mathbb{I}$ be as in* (1.46)*. Then the following statements hold.*

*(i) Let $\mu \in L^2(\partial\mathbb{I})$ and*

$$-\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}(y)} \big(S_n^a(x - y)\big)\mu(y) \, d\sigma_y = 0 \qquad a.e. \text{ on } \partial\mathbb{I}. \tag{2.5}$$

*Then $\mu \equiv 0$.*

*(ii) Let $\mu \in L^2(\partial\mathbb{I})$ and*

$$\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y = 0 \qquad \text{a.e. on } \partial\mathbb{I}. \tag{2.6}$$

*Then $\mu \equiv 0$.*

*Proof.* It is an immediate consequence of the Fredholm Theory and of Propositions 2.11 and 2.12. □

In the following Proposition, we show existence and uniqueness results for the solutions of the integral equations that appear in Theorems 1.13 and 1.15.

**Proposition 2.14.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Then the following statements hold.*

*(i) Let $\Gamma \in C^{m,\alpha}(\partial\mathbb{I})$. Then there exists a unique $\mu \in C^{m,\alpha}(\partial\mathbb{I})$, such that*

$$\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y = \Gamma(x) \qquad \text{a.e. on } \partial\mathbb{I}. \tag{2.7}$$

*(ii) Let $\Gamma \in C^{m,\alpha}(\partial\mathbb{I})$. Then there exists a unique $\mu \in C^{m,\alpha}(\partial\mathbb{I})$, such that*

$$-\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y = \Gamma(x) \qquad \text{a.e. on } \partial\mathbb{I}. \tag{2.8}$$

*(iii) Let $\Gamma \in C^{m-1,\alpha}(\partial\mathbb{I})$. Then there exists a unique $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$, such that*

$$\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y = \Gamma(x) \qquad \text{a.e. on } \partial\mathbb{I}. \tag{2.9}$$

*(iv) Let $\Gamma \in C^{m-1,\alpha}(\partial\mathbb{I})$. Then there exists a unique $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$, such that*

$$-\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y = \Gamma(x) \qquad \text{a.e. on } \partial\mathbb{I}. \tag{2.10}$$

*Proof.* Consider $(i)$. By the Fredholm Theory and Proposition 2.13, there exists a unique $\mu \in L^2(\partial\mathbb{I})$ such that (2.7) holds. By Theorem 1.21 $(i)$, we have that $\mu \in C^{m,\alpha}(\partial\mathbb{I})$, and so statement $(i)$ is proved. In the same way, we can easily prove $(ii)$, $(iii)$, and $(iii)$. □

In the following Theorem, we prove that the periodic Dirichlet problem for the Laplace equation has a unique solution.

**Theorem 2.15.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\Gamma \in C^{m,\alpha}(\partial\mathbb{I})$. Then there exists a unique solution $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$ of the following periodic Dirichlet problem for the Laplace equation*

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x+a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1,\dots,n\}, \\ u(x) = \Gamma(x) & \forall x \in \partial\mathbb{I}. \end{cases} \tag{2.11}$$

*In particular, we have*

$$u(x) = w_a^-[\partial\mathbb{I},\mu](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \tag{2.12}$$

*where $\mu$ is the unique solution in $C^{m,\alpha}(\partial\mathbb{I})$ of the following integral equation*

$$\Gamma(x) = -\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\mathbb{I}. \tag{2.13}$$

*Proof.* The uniqueness has already been proved in Corollary 2.6. We need to prove the existence of a solution of (2.11). In particular, we want to prove that the function given by the right-hand side of (2.12) solves (2.11). Clearly, by Proposition 2.14, there exists a unique $\mu \in C^{m,\alpha}(\partial\mathbb{I})$ that solves (2.13). Then, by Theorem 1.13, it is easy to see that $w_a^-[\partial\mathbb{I},\mu]$ is a periodic harmonic function in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$, that solves (2.11). □

By the previous Theorem, we deduce the existence of a unique solution of the periodic Dirichlet problem for the Poisson equation.

**Theorem 2.16.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be as in* (1.46). *Let* $\bar{m} \equiv \max\{0, m-2\}$. *Let* $f \in C^{\bar{m},\alpha}(\mathbb{R}^n)$ *be such that*

$$f(x + a_i) = f(x) \qquad \forall x \in \mathbb{R}^n, \quad \forall i \in \{1, \dots, n\},$$

*and*

$$\int_A f(y)\, dy = 0.$$

*Let* $\Gamma \in C^{m,\alpha}(\partial\mathbb{I})$. *Then there exists a unique solution* $u \in C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}])$ *of the following periodic Dirichlet problem for the Poisson equation*

$$\begin{cases} \Delta u(x) = f(x) & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}, \\ u(x) = \Gamma(x) & \forall x \in \partial\mathbb{I}. \end{cases} \tag{2.14}$$

*In particular, we have*

$$u(x) = w_a^-[\partial\mathbb{I}, \mu](x) + p_a[f](x) \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \tag{2.15}$$

*where* $\mu$ *is the unique solution in* $C^{m,\alpha}(\partial\mathbb{I})$ *of the following integral equation*

$$\Gamma(x) - p_a[f](x) = -\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(S_n^a(x - y)\big)\mu(y)\, d\sigma_y \qquad \forall x \in \partial\mathbb{I}. \tag{2.16}$$

*Proof.* The uniqueness is a straightforward consequence of Corollary 2.6. We need to prove the existence of a solution of (2.14). In particular, we want to prove that the function given by the right-hand side of (2.15) solves (2.14). By Theorem 1.18, $p_a[f] \in C^{m,\alpha}(\mathbb{R}^n)$ and $\Delta p_a[f] = f$ in $\mathbb{R}^n$. Clearly, by Proposition 2.14, there exists a unique $\mu \in C^{m,\alpha}(\partial\mathbb{I})$ that solves (2.16). Then, by Theorem 1.13, it is easy to see that $w_a^-[\partial\mathbb{I}, \mu]$ is a periodic harmonic function in $C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}])$, and that $w_a^-[\partial\mathbb{I}, \mu] + p_a[f]$ solves (2.14). □

As done above, in the following Theorem we prove that the periodic Neumann problem for the Laplace equation has a solution.

**Theorem 2.17.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be as in* (1.46). *Let* $\Gamma \in C^{m-1,\alpha}(\partial\mathbb{I})$ *be such that* $\int_{\partial\mathbb{I}} \Gamma\, d\sigma = 0$. *Then there exists a solution* $u \in C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}])$ *of the following periodic Neumann problem for the Laplace equation*

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}, \\ \frac{\partial}{\partial\nu_{\mathbb{I}}} u(x) = \Gamma(x) & \forall x \in \partial\mathbb{I}. \end{cases} \tag{2.17}$$

*In particular, we have*

$$u(x) = v_a^-[\partial\mathbb{I}, \mu](x) \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \tag{2.18}$$

*where* $\mu$ *is the unique solution in* $C^{m-1,\alpha}(\partial\mathbb{I})$ *of the following integral equation*

$$\Gamma(x) = \frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}\big(S_n^a(x - y)\big)\mu(y)\, d\sigma_y \qquad \forall x \in \partial\mathbb{I}. \tag{2.19}$$

*The set of all the solutions of problem* (2.17) *is given by*

$$\left\{\, v_a^-[\partial\mathbb{I}, \mu] + c\colon c \in \mathbb{R} \,\right\}, \tag{2.20}$$

*where, as above,* $\mu$ *is the unique solution of equation* (2.19)

*Proof.* By virtue of Proposition 2.7, it suffices to prove that the function given by the right-hand side of (2.18) solves boundary value problem (2.17). Clearly, by Proposition 2.14, there exists a unique function $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$ that solves (2.19). Moreover, by Lemma 2.10 (*iii*), since $\int_{\partial\mathbb{I}} \Gamma\, d\sigma = 0$, we have that $\int_{\partial\mathbb{I}} \mu\, d\sigma = 0$. Then, by Theorem 1.15, it is easy to see that $v_a^-[\partial\mathbb{I}, \mu]$ is a periodic harmonic function in $C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}])$, that solves (2.17). □

Obviously, by the previous Theorem, we deduce the existence of a solution of the Neumann problem for the Poisson equation.

**Theorem 2.18.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be as in* (1.46). *Let* $\bar{m} \equiv \max\{0, m-2\}$. *Let* $f \in C^{\bar{m},\alpha}(\mathbb{R}^n)$ *be such that*

$$f(x + a_i) = f(x) \qquad \forall x \in \mathbb{R}^n, \quad \forall i \in \{1, \dots, n\},$$

*and*

$$\int_A f(y)\, dy = 0.$$

*Let* $\Gamma \in C^{m-1,\alpha}(\partial\mathbb{I})$ *be such that* $\int_{\partial\mathbb{I}} \Gamma\, d\sigma = \int_{\mathbb{I}} f(x)\, dx$. *Then there exists a solution* $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$ *of the following periodic Neumann problem for the Poisson equation*

$$\begin{cases} \Delta u(x) = f(x) & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) = \Gamma(x) & \forall x \in \partial\mathbb{I}. \end{cases} \tag{2.21}$$

*In particular, we have*

$$u(x) = v_a^-[\partial\mathbb{I}, \mu](x) + p_a[f] \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \tag{2.22}$$

*where* $\mu$ *is the unique solution in* $C^{m-1,\alpha}(\partial\mathbb{I})$ *of the following integral equation*

$$\Gamma(x) - \frac{\partial}{\partial \nu_{\mathbb{I}}} p_a[f](x) = \frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}(x)} \big(S_n^a(x-y)\big)\mu(y)\, d\sigma_y \qquad \forall x \in \partial\mathbb{I}. \tag{2.23}$$

*The set of all the solutions of problem* (2.21) *is given by*

$$\left\{ v_a^-[\partial\mathbb{I}, \mu] + p_a[f] + c \colon c \in \mathbb{R} \right\}, \tag{2.24}$$

*where, as above,* $\mu$ *is the unique solution of equation* (2.23)

*Proof.* By virtue of Proposition 2.7, it suffices to prove that the function given by the right-hand side of (2.22) solves boundary value problem (2.21). By Theorem 1.18, $p_a[f] \in C^{m,\alpha}(\mathbb{R}^n)$ and $\Delta p_a[f] = f$ in $\mathbb{R}^n$. Clearly, by Proposition 2.14, there exists a function $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$ that solves (2.23). Moreover, since $\int_{\partial\mathbb{I}} \Gamma\, d\sigma = \int_{\mathbb{I}} f(x) dx$, by Green's Formula, we have that

$$\int_{\partial\mathbb{I}} \Gamma\, d\sigma = \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}} p_a[f]\, d\sigma.$$

Accordingly, by Lemma 2.10 (*iii*), $\int_{\partial\mathbb{I}} \mu\, d\sigma = 0$. Then, by Theorem 1.15, it is easy to see that $v_a^-[\partial\mathbb{I}, \mu]$ is a periodic harmonic function in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$, and that $v_a^-[\partial\mathbb{I}, \mu] + p_a[f]$ solves (2.21). □

### 2.1.4 Representation theorems for periodic harmonic functions

In this Subsection we want to prove representation theorems for periodic harmonic functions defined in $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$ and in $\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]$.

In the following four Propositions, we represents periodic harmonic functions by means of periodic double layer potentials and costants.

**Proposition 2.19.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be as in* (1.46). *Let* $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$ *be such that*

$$\Delta u(x) = 0 \qquad \forall x \in \mathbb{T}_a[\mathbb{I}],$$

*and*

$$u(x + a_i) = u(x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}.$$

*Then there exists a unique* $\mu \in C^{m,\alpha}(\partial\mathbb{I})$, *such that*

$$u(x) = w_a^-[\partial\mathbb{I}, \mu](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}].$$

*Proof.* Let $\mu \in C^{m,\alpha}(\partial\mathbb{I})$. By Theorem 1.13 and Theorem 2.5,

$$u(x) = w_a^-[\partial\mathbb{I},\mu](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}].$$

if and only if

$$u(x) = -\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\bigl(S_n^a(x-y)\bigr)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\mathbb{I}.$$

On the other hand, by Proposition 2.14, the previous integral equation has a unique solution $\mu \in C^{m,\alpha}(\partial\mathbb{I})$, and so the conclusion easily follows. □

**Proposition 2.20.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46). *Let $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}])$ be such that*

$$\Delta u(x) = 0 \qquad \forall x \in \mathbb{S}_a[\mathbb{I}],$$

*and*

$$u(x+a_i) = u(x) \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}], \quad \forall i \in \{1,\dots,n\}.$$

*Then there exists a unique $\mu \in C^{m,\alpha}(\partial\mathbb{I})$, such that*

$$u(x) = w_a^+[\partial\mathbb{I},\mu](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}].$$

*Proof.* Let $\mu \in C^{m,\alpha}(\partial\mathbb{I})$. By Theorem 1.13 and the Maximum Principle,

$$u(x) = w_a^+[\partial\mathbb{I},\mu](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]$$

if and only if

$$u(x) = \frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\bigl(S_n^a(x-y)\bigr)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\mathbb{I}.$$

On the other hand, by Proposition 2.14, the previous integral equation has a unique solution $\mu \in C^{m,\alpha}(\partial\mathbb{I})$, and so the conclusion easily follows. □

**Proposition 2.21.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46). *Let $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$ be such that*

$$\Delta u(x) = 0 \qquad \forall x \in \mathbb{T}_a[\mathbb{I}],$$

*and*

$$u(x+a_i) = u(x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1,\dots,n\}.$$

*Then there exists a unique pair $(\mu,c) \in C^{m,\alpha}(\partial\mathbb{I})\times\mathbb{R}$, such that $\int_{\partial\mathbb{I}}\mu\,d\sigma = 0$ and that*

$$u(x) = w_a^-[\partial\mathbb{I},\mu](x) + c \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}].$$

*Proof.* Let $\mathcal{L}$ be the linear and continuous map of $C^{m,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\partial\mathbb{I})$, which takes $\mu$ to the function of the variable $x \in \partial\mathbb{I}$ defined by

$$\mathcal{L}[\mu](x) \equiv -\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\bigl(S_n^a(x-y)\bigr)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\mathbb{I}.$$

By Proposition 2.14, $\mathcal{L}$ is bijective. Moreover, by the Open Mapping Theorem, $\mathcal{L}$ is a linear homeomorphism of $C^{m,\alpha}(\partial\mathbb{I})$ onto itself. By Theorem 1.13 (*iv*), $\mathcal{L}$ takes a constant function to another constant function. In particular,

$$\mathcal{L}[\lambda\chi_{\partial\mathbb{I}}] = -\lambda\frac{|\mathbb{I}|_n}{|A|_n}\chi_{\partial\mathbb{I}} \qquad \forall \lambda \in \mathbb{R}.$$

Then, if we set

$$\mathcal{U}_{\partial\mathbb{I}}^{m,\alpha} \equiv \left\{ \mu \in C^{m,\alpha}(\partial\mathbb{I})\colon \int_{\partial\mathbb{I}} \mu\,d\sigma = 0 \right\},$$

we have

$$C^{m,\alpha}(\partial\mathbb{I}) = \mathcal{U}_{\partial\mathbb{I}}^{m,\alpha} \oplus \langle\chi_{\partial\mathbb{I}}\rangle\,,$$

and so

$$C^{m,\alpha}(\partial\mathbb{I}) = \mathcal{L}[\mathcal{U}_{\partial\mathbb{I}}^{m,\alpha}] \oplus \langle\chi_{\partial\mathbb{I}}\rangle\,.$$

In other words, for each $g \in C^{m,\alpha}(\partial\mathbb{I})$ there exists a unique pair $(\mu, c) \in \mathcal{U}^{m,\alpha}_{\partial\mathbb{I}} \times \mathbb{R}$ such that

$$g(x) = -\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(S_n^a(x-y)\big)\mu(y)\, d\sigma_y + c \qquad \forall x \in \partial\mathbb{I}.$$

Hence, if $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$, then there exists a unique pair $(\mu, c) \in C^{m,\alpha}(\partial\mathbb{I}) \times \mathbb{R}$, such that $\int_{\partial\mathbb{I}} \mu\, d\sigma = 0$ and that

$$u(x) = w_a^-[\partial\mathbb{I}, \mu](x) + c \qquad \forall x \in \partial\mathbb{I}.$$

Consequently, by Theorem 2.5, we have

$$u(x) = w_a^-[\partial\mathbb{I}, \mu](x) + c \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],$$

and the conclusion easily follows. □

**Proposition 2.22.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}])$ be such that*

$$\Delta u(x) = 0 \qquad \forall x \in \mathbb{S}_a[\mathbb{I}],$$

*and*

$$u(x + a_i) = u(x) \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}], \quad \forall i \in \{1, \ldots, n\}.$$

*Then there exists a unique pair $(\mu, c) \in C^{m,\alpha}(\partial\mathbb{I}) \times \mathbb{R}$, such that $\int_{\partial\mathbb{I}} \mu\, d\sigma = 0$ and that*

$$u(x) = w_a^+[\partial\mathbb{I}, \mu](x) + c \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}].$$

*Proof.* Let $\mathcal{L}$ be the linear and continuous map of $C^{m,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\partial\mathbb{I})$, which takes $\mu$ to function of the variable $x \in \partial\mathbb{I}$ defined by

$$\mathcal{L}[\mu](x) \equiv \frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(S_n^a(x-y)\big)\mu(y)\, d\sigma_y \qquad \forall x \in \partial\mathbb{I}.$$

By Proposition 2.14, $\mathcal{L}$ is bijective. Moreover, by the Open Mapping Theorem, $\mathcal{L}$ is a linear homeomorphism of $C^{m,\alpha}(\partial\mathbb{I})$ onto itself. By Theorem 1.13 *(iv)*, $\mathcal{L}$ takes a constant function to another constant function. In particular,

$$\mathcal{L}[\lambda\chi_{\partial\mathbb{I}}] = \lambda\left(1 - \frac{|\mathbb{I}|_n}{|A|_n}\right)\chi_{\partial\mathbb{I}} \qquad \forall \lambda \in \mathbb{R}.$$

Then, if we set

$$\mathcal{U}^{m,\alpha}_{\partial\mathbb{I}} \equiv \left\{ \mu \in C^{m,\alpha}(\partial\mathbb{I}) \colon \int_{\partial\mathbb{I}} \mu\, d\sigma = 0 \right\},$$

we have

$$C^{m,\alpha}(\partial\mathbb{I}) = \mathcal{U}^{m,\alpha}_{\partial\mathbb{I}} \oplus \langle \chi_{\partial\mathbb{I}} \rangle,$$

and so

$$C^{m,\alpha}(\partial\mathbb{I}) = \mathcal{L}[\mathcal{U}^{m,\alpha}_{\partial\mathbb{I}}] \oplus \langle \chi_{\partial\mathbb{I}} \rangle.$$

In other words, for each $g \in C^{m,\alpha}(\partial\mathbb{I})$ there exists a unique pair $(\mu, c) \in \mathcal{U}^{m,\alpha}_{\partial\mathbb{I}} \times \mathbb{R}$ such that

$$g(x) = \frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(y)}\big(S_n^a(x-y)\big)\mu(y)\, d\sigma_y + c \qquad \forall x \in \partial\mathbb{I}.$$

Hence, if $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}])$, then there exists a unique pair $(\mu, c) \in C^{m,\alpha}(\partial\mathbb{I}) \times \mathbb{R}$, such that $\int_{\partial\mathbb{I}} \mu\, d\sigma = 0$ and that

$$u(x) = w_a^+[\partial\mathbb{I}, \mu](x) + c \qquad \forall x \in \partial\mathbb{I}.$$

Consequently, by the Strong Maximum Principle and the periodicity of $u$ and $w_a^+[\partial\mathbb{I}, \mu] + c$, we have

$$u(x) = w_a^+[\partial\mathbb{I}, \mu](x) + c \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}],$$

and the conclusion easily follows. □

In the following Propositions, we show that periodic harmonic functions can be represented also by means of periodic simple layer potentials and constants.

**Proposition 2.23.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$ be such that*

$$\Delta u(x) = 0 \qquad \forall x \in \mathbb{T}_a[\mathbb{I}],$$

*and*

$$u(x + a_i) = u(x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}.$$

*Then there exists a unique pair $(\mu, c) \in C^{m-1,\alpha}(\partial\mathbb{I}) \times \mathbb{R}$, such that*

$$u(x) = v_a^-[\partial\mathbb{I}, \mu](x) + c \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}].$$

*Moreover, $\int_{\partial\mathbb{I}} \mu \, d\sigma = 0$.*

*Proof.* Let $(\mu, c) \in C^{m-1,\alpha}(\partial\mathbb{I}) \times \mathbb{R}$. Let $\bar{x} \in \mathbb{P}_a[\mathbb{I}]$. By Theorem 1.15 and Proposition 2.7,

$$u(x) = v_a^-[\partial\mathbb{I}, \mu](x) + c \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],$$

if and only if

$$\int_{\partial\mathbb{I}} \mu d\,\sigma = 0,$$

and

$$u(\bar{x}) = v_a^-[\partial\mathbb{I}, \mu](\bar{x}) + c,$$

and

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} u(x) = \frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)} \big(S_n^a(x-y)\big)\mu(y)\, d\sigma_y \qquad \forall x \in \partial\mathbb{I}.$$

On the other hand, by Proposition 2.14, the previous integral equation has a unique solution $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$. Moreover, by the periodicity of $u$ and by Green's Formula, we have

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}} u \, d\sigma = 0,$$

and so, by Lemma 2.10, we have

$$\int_{\partial\mathbb{I}} \mu \, d\sigma = 0.$$

Then $c$ must be delivered by

$$c \equiv u(\bar{x}) - v_a^-[\partial\mathbb{I}, \mu](\bar{x}),$$

and so the conclusion easily follows. □

**Proposition 2.24.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}])$ be such that*

$$\Delta u(x) = 0 \qquad \forall x \in \mathbb{S}_a[\mathbb{I}],$$

*and*

$$u(x + a_i) = u(x) \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}.$$

*Then there exists a unique pair $(\mu, c) \in C^{m-1,\alpha}(\partial\mathbb{I}) \times \mathbb{R}$, such that*

$$u(x) = v_a^+[\partial\mathbb{I}, \mu](x) + c \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}].$$

*Moreover, $\int_{\partial\mathbb{I}} \mu \, d\sigma = 0$.*

*Proof.* Let $(\mu, c) \in C^{m-1,\alpha}(\partial\mathbb{I}) \times \mathbb{R}$. Let $\bar{x} \in \mathbb{I}$. By Theorem 1.15,

$$u(x) = v_a^+[\partial\mathbb{I}, \mu](x) + c \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}],$$

if and only if

$$\int_{\partial\mathbb{I}} \mu d\,\sigma = 0,$$

and

$$u(\bar{x}) = v_a^+[\partial\mathbb{I}, \mu](\bar{x}) + c,$$

and

$$\frac{\partial}{\partial\nu_{\mathbb{I}}}u(x) = -\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\mathbb{I}.$$

On the other hand, by Proposition 2.14, the previous integral equation has a unique solution $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$. Moreover, by Green's Formula, we have

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}}u\,d\sigma = 0,$$

and so, by Lemma 2.10, we have

$$\int_{\partial\mathbb{I}} \mu\,d\sigma = 0.$$

Then $c$ must be delivered by

$$c \equiv u(\bar{x}) - v_a^+[\partial\mathbb{I}, \mu](\bar{x}),$$

and so the conclusion easily follows. □

## 2.2 Asymptotic behaviour of the solutions of the Dirichlet problem for the Poisson equation in a periodically perforated domain

In this Section we study the asymptotic behaviour of the solutions of the Dirichlet problem for the Poisson equation in a periodically perforated domain with small holes.

### 2.2.1 Notation

We retain the notation introduced in Subsection 1.8.1.

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). We shall consider also the following assumptions.

$$g \in C^{m,\alpha}(\partial\Omega). \tag{2.25}$$

Let $f$ be real analytic function from $\mathbb{R}^n$ to $\mathbb{R}$ such that $f(x+a_i) = f(x)$ for all $x \in \mathbb{R}^n$ and for all $i \in \{1,\dots,n\}$, and such that $\int_A f(y)\,dy = 0$. (2.26)

### 2.2.2 Preliminaries

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in (1.56), (1.57), (2.25), (2.26), respectively. For each $\epsilon \in ]0,\epsilon_1[$, we consider the following periodic Dirichlet problem for the Poisson equation.

$$\begin{cases} \Delta u(x) = f(x) & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_i) = u(x) & \forall x \in \operatorname{cl}\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1,\dots,n\}, \\ u(x) = g\big(\frac{1}{\epsilon}(x-w)\big) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{2.27}$$

By virtue of Theorems 2.15 and 2.16, we can give the following definitions.

**Definition 2.25.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in (1.56), (1.57), (2.25), (2.26), respectively. For each $\epsilon \in ]0,\epsilon_1[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\operatorname{cl}\mathbb{T}_a[\Omega_\epsilon])$ of boundary value problem (2.27).

**Definition 2.26.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in (1.56), (1.57), (2.25), (2.26), respectively. For each $\epsilon \in ]0,\epsilon_1[$, we denote by $\bar{u}[\epsilon]$ the unique solution in $C^{m,\alpha}(\operatorname{cl}\mathbb{T}_a[\Omega_\epsilon])$ of the following periodic Dirichlet problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_i) = u(x) & \forall x \in \operatorname{cl}\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1,\dots,n\}, \\ u(x) = g\big(\frac{1}{\epsilon}(x-w)\big) - p_a[f](x) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{2.28}$$

*Remark* 2.27. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in (1.56), (1.57), (2.25), (2.26), respectively. For each $\epsilon \in ]0, \epsilon_1[$, we have

$$u[\epsilon](x) \equiv \bar{u}[\epsilon](x) + p_a[f](x) \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon].$$

We now prove the following known Lemma that we shall need in the sequel.

**Lemma 2.28.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in* (1.56)*. Let $\mathcal{U}_0^{m,\alpha}$ be as in* (1.64)*. Then the map $\mathcal{L}$ from $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)$, which takes $(\theta, \xi)$ to*

$$\mathcal{L}[\theta,\xi](t) \equiv -\frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(s)}(S_n(t-s))\theta(s)\, d\sigma_s + \xi \qquad \forall t \in \partial\Omega,$$

*is a linear homeomorphism of $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ onto $C^{m,\alpha}(\partial\Omega)$.*

*Proof.* Clearly, $\mathcal{L}$ is linear and continuous. By the Open Mapping Theorem, it suffices to prove that $\mathcal{L}$ is a bijection. We recall that, by the hypotheses on $\Omega$, we have in particular that $\Omega$ is connected. By well known results of classical potential theory (cf. Folland [52, Chapter 3]), we have

$$C^{m,\alpha}(\partial\Omega) = \left\{ -\frac{1}{2}\theta(\cdot) + \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(s)}(S_n(\cdot - s))\theta(s)\, d\sigma_s \colon \theta \in C^{m,\alpha}(\partial\Omega) \right\} \oplus \langle \chi_{\partial\Omega} \rangle .$$

On the other hand, for each $\psi$ in the set

$$\left\{ -\frac{1}{2}\theta(\cdot) + \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(s)}(S_n(\cdot - s))\theta(s)\, d\sigma_s \colon \theta \in C^{m,\alpha}(\partial\Omega) \right\},$$

there exists a unique $\theta$ in $C^{m,\alpha}(\partial\Omega)$ such that

$$\begin{cases} \psi(t) = -\frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(s)}(S_n(t-s))\theta(s)\, d\sigma_s & \forall t \in \partial\Omega, \\ \int_{\partial\Omega} \theta\, d\sigma = 0. \end{cases}$$

In other words, for each $\phi \in C^{m,\alpha}(\partial\Omega)$, there exists a unique pair $(\theta, \xi)$ in $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$, such that

$$\phi(t) = -\frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(s)}(S_n(t-s))\theta(s)\, d\sigma_s + \xi \qquad \forall t \in \partial\Omega.$$

Hence, $\mathcal{L}$ is bijective. □

As we have seen, by means of the periodic Newtonian potential, we can convert the Dirichlet problem for the Poisson equation, into a Dirichlet problem for the Laplace equation. Since we want to represent the functions $\bar{u}[\epsilon]$ by means of a periodic double layer potential and a constant (cf. Theorem 2.15 and Proposition 2.21), we need to study some integral equations. Indeed, by virtue of Theorem 2.15 and Proposition 2.21, we can transform (2.28) into an integral equation, whose unknowns are the moment of the double layer potential and the additive constant. Moreover, we want to transform these equations defined on the $\epsilon$-dependent domain $\partial\Omega_\epsilon$ into equations defined on the fixed domain $\partial\Omega$. We introduce these integral equations in the following Proposition. The relation between the solution of the integral equations and the solution of boundary value problem (2.28) will be clarified later.

**Proposition 2.29.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in* (1.56), (1.57), (2.25), (2.26)*, respectively. Let $\mathcal{U}_\epsilon^{m,\alpha}$, $\mathcal{U}_0^{m,\alpha}$ be as in* (1.63), (1.64)*, respectively. Let $\Lambda$ be the map of $]-\epsilon_1, \epsilon_1[ \times \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ in $C^{m,\alpha}(\partial\Omega)$ defined by*

$$\begin{aligned} \Lambda[\epsilon,\theta,\xi](t) \equiv &-\frac{1}{2}\theta(t) - \int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n(t-s)\theta(s)\, d\sigma_s - \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s \\ &+ \xi - g(t) + p_a[f](w+\epsilon t) \qquad \forall t \in \partial\Omega, \end{aligned} \tag{2.29}$$

*for all $(\epsilon, \theta, \xi) \in ]-\epsilon_1, \epsilon_1[ \times \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$. Then the following statements hold.*

*(i) If* $\epsilon \in ]0, \epsilon_1[$*, then the pair* $(\theta, \xi) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ *satisfies equation*

$$\Lambda[\epsilon, \theta, \xi] = 0, \tag{2.30}$$

*if and only if the pair* $(\mu, \xi) \in \mathcal{U}_\epsilon^{m,\alpha} \times \mathbb{R}$*, with* $\mu \in \mathcal{U}_\epsilon^{m,\alpha}$ *defined by*

$$\mu(x) \equiv \theta\big(\frac{1}{\epsilon}(x-w)\big) \qquad \forall x \in \partial\Omega_\epsilon, \tag{2.31}$$

*satisfies the equation*

$$\Gamma(x) - p_a[f](x) = -\frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(y)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y + \xi \qquad \forall x \in \partial\Omega_\epsilon, \tag{2.32}$$

*with* $\Gamma \in C^{m,\alpha}(\partial\Omega_\epsilon)$ *defined by*

$$\Gamma(x) \equiv g\big(\frac{1}{\epsilon}(x-w)\big) \qquad \forall x \in \partial\Omega_\epsilon. \tag{2.33}$$

*In particular, equation* (2.30) *has exactly one solution* $(\theta, \xi) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$*, for each* $\epsilon \in ]0, \epsilon_1[$*.*

*(ii) The pair* $(\theta, \xi) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ *satisfies equation*

$$\Lambda[0, \theta, \xi] = 0, \tag{2.34}$$

*if and only if*

$$g(t) - p_a[f](w) = -\frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(s)}(S_n(t-s))\theta(s)\,d\sigma_s + \xi \qquad \forall t \in \partial\Omega. \tag{2.35}$$

*In particular, equation* (2.34) *has exactly one solution* $(\theta, \xi) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$*, which we denote by* $(\tilde{\theta}, \tilde{\xi})$*.*

*Proof.* Consider $(i)$. Let $\theta \in C^{m,\alpha}(\partial\Omega)$. Let $\epsilon \in ]0, \epsilon_1[$. First of all, we note that

$$\int_{\partial\Omega_\epsilon} \theta\big(\frac{1}{\epsilon}(x-w)\big)\,d\sigma_x = \epsilon^{n-1}\int_{\partial\Omega} \theta(t)\,d\sigma_t,$$

and so $\theta \in \mathcal{U}_0^{m,\alpha}$ if and only if $\theta(\frac{1}{\epsilon}(\cdot - w)) \in \mathcal{U}_\epsilon^{m,\alpha}$. The equivalence of equation (2.30) in the unknown $(\theta, \xi) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ and equation (2.32) in the unknown $(\mu, \xi) \in \mathcal{U}_\epsilon^{m,\alpha} \times \mathbb{R}$ follows by a straightforward computation based on the rule of change of variables in integrals and on well known properties of composition of functions in Schauder spaces (cf. *e.g.*, Lanza [67, Sections 3,4].) The existence and uniqueness of a solution of equation (2.32) follows by the proof of Proposition 2.21. Then the existence and uniqueness of a solution of equation (2.30) follows by the equivalence of (2.30) and (2.32). Consider $(ii)$. The equivalence of (2.34) and (2.35) is obvious. The existence of a unique solution of equation (2.34) is an immediate consequence of Lemma 2.28. □

By Proposition 2.29, it makes sense to introduce the following.

**Definition 2.30.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in (1.56), (1.57), (2.25), (2.26), respectively. Let $\mathcal{U}_0^{m,\alpha}$ be as in (1.64). For each $\epsilon \in ]0, \epsilon_1[$, we denote by $(\hat{\theta}[\epsilon], \hat{\xi}[\epsilon])$ the unique pair in $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ that solves (2.30). Analogously, we denote by $(\hat{\theta}[0], \hat{\xi}[0])$ the unique pair in $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ that solves (2.34).

In the following Remark, we show the relation between the solutions of boundary value problem (2.28) and the solutions of equation (2.30).

*Remark* 2.31. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in (1.56), (1.57), (2.25), (2.26), respectively.

Let $\epsilon \in ]0, \epsilon_1[$. We have

$$\bar{u}[\epsilon](x) = -\epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n^a(x - w - \epsilon s)\hat{\theta}[\epsilon](s)\,d\sigma_s + \hat{\xi}[\epsilon] \qquad \forall x \in \mathbb{T}_a[\Omega_\epsilon].$$

Accordingly,

$$u[\epsilon](x) = -\epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n^a(x - w - \epsilon s)\hat{\theta}[\epsilon](s)\,d\sigma_s + \hat{\xi}[\epsilon] + p_a[f](x) \qquad \forall x \in \mathbb{T}_a[\Omega_\epsilon].$$

While the relation between equation (2.30) and boundary value problem (2.28) is now clear, we want to see if (2.34) is related to some (limiting) boundary value problem. We give the following.

**Definition 2.32.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). We denote by $\tau$ the unique solution in $C^{m-1,\alpha}(\partial\Omega)$ of the following system

$$\begin{cases} -\frac{1}{2}\tau(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\tau(s)\,d\sigma_s = 0 \quad \forall t \in \partial\Omega, \\ \int_{\partial\Omega} \tau\,d\sigma = 1. \end{cases} \tag{2.36}$$

*Remark* 2.33. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). The existence and uniqueness of a solution $\tau$ of (2.36) is a well known result of classical potential theory (cf. Folland [52, Chapter 3].) Moreover,

$$\left\{ \theta \in C^{m-1,\alpha}(\partial\Omega) \colon -\frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\,d\sigma_s = 0 \quad \forall t \in \partial\Omega \right\} = \langle\tau\rangle\,.$$

*Remark* 2.34. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in (1.56), (1.57), (2.25), (2.26), respectively. Let $\tilde{\xi}$ be as in Proposition 2.29. By well known results of classical potential theory (cf. Folland [52, Chapter 3]), we have that $\tilde{\xi}$ is the unique $\xi \in \mathbb{R}$, such that

$$\int_{\partial\Omega} \Big(g(x) - p_a[f](w) - \xi\Big)\tau(x)\,d\sigma_x = 0.$$

Hence,

$$\tilde{\xi} = \int_{\partial\Omega} g(x)\tau(x)\,d\sigma_x - p_a[f](w).$$

**Definition 2.35.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $g$ be as in (1.56), (2.25), respectively. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega)$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega, \\ u(x) = g(x) - \int_{\partial\Omega} g(y)\tau(y)\,d\sigma_y & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{2.37}$$

Problem (2.37) will be called the *limiting boundary value problem.*

*Remark* 2.36. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). Let $g^\# \in C^{m,\alpha}(\partial\Omega)$. We note that in general the following exterior Dirichlet problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega, \\ u(x) = g^\#(x) & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0, \end{cases}$$

does not have a solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega)$. However, as can be easily seen by classical potential theory, the particular choice of the Dirichlet datum in (2.37), ensures the existence of a (unique) solution of problem (2.37).

*Remark* 2.37. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in (1.56), (1.57), (2.25), (2.26), respectively. We have

$$\tilde{u}(x) = \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(y)}\big(S_n(x-y)\big)\hat{\theta}[0](y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega.$$

We now prove the following.

**Proposition 2.38.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in* (1.56), (1.57), (2.25), (2.26), *respectively. Let $\mathcal{U}_0^{m,\alpha}$ be as in* (1.64). *Let $\Lambda$ and $(\tilde{\theta}, \tilde{\xi})$ be as in Proposition 2.29. Then there exists $\epsilon_2 \in ]0, \epsilon_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)$. Moreover, if we set $b_0 \equiv (0, \tilde{\theta}, \tilde{\xi})$, then the differential $\partial_{(\theta,\xi)}\Lambda[b_0]$ of $\Lambda$ with respect to the variables $(\theta, \xi)$ at $b_0$ is delivered by the following formula*

$$\partial_{(\theta,\xi)}\Lambda[b_0](\bar{\theta}, \bar{\xi})(t) = -\frac{1}{2}\bar{\theta}(t) - \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n(t-s)\bar{\theta}(s)\,d\sigma_s + \bar{\xi} \qquad \forall t \in \partial\Omega, \tag{2.38}$$

*for all $(\bar{\theta}, \bar{\xi}) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$, and is a linear homeomorphism of $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ onto $C^{m,\alpha}(\partial\Omega)$.*

*Proof.* By Remark 1.20, $p_a[f]$ is a real analytic function. Let $\mathrm{id}_{\mathrm{cl}\,\Omega}$ denote the identity map in $\mathrm{cl}\,\Omega$. Since the map of $]-\epsilon_1,\epsilon_1[$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega, A)$, which takes $\epsilon$ to the function $w+\epsilon\,\mathrm{id}_{\mathrm{cl}\,\Omega}$ is obviously real analytic, then, by a known result on composition operators (cf. Böhme and Tomi [15, p. 10], Henry [60, p. 29], Valent [137, Thm. 5.2, p. 44]), we have that the map of $]-\epsilon_1,\epsilon_1[$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ which takes $\epsilon$ to $p_a[f]\circ(w+\epsilon\,\mathrm{id}_{\mathrm{cl}\,\Omega})$ is a real analytic operator. Since the map of $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ to $C^{m,\alpha}(\partial\Omega)$ which takes a function $h$ to its restriction $h_{|\partial\Omega}$ is linear and continuous, we conclude that the map of $]-\epsilon_1,\epsilon_1[$ to $C^{m,\alpha}(\partial\Omega)$ which takes $\epsilon$ to the function $p_a[f](w+\epsilon t)$ of the variable $t\in\partial\Omega$ is real analytic. Thus, by Proposition 1.22 $(i)$ and standard calculus in Banach spaces, we deduce that there exists $\epsilon_2\in\,]0,\epsilon_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2,\epsilon_2[\times\mathcal{U}_0^{m,\alpha}\times\mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)$. By standard calculus in Banach space, we immediately deduce that (2.38) holds. Finally, by Lemma 2.28, $\partial_{(\theta,\xi)}\Lambda[b_0]$ is a linear homeomorphism. □

We are now ready to prove real analytic continuation properties for $\hat{\theta}[\cdot],\hat{\xi}[\cdot]$.

**Proposition 2.39.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in* (1.56), (1.57), (2.25), (2.26)*, respectively. Let $\mathcal{U}_0^{m,\alpha}$ be as in* (1.64)*. Let $\epsilon_2$ be as in Proposition 2.38. Then there exist $\epsilon_3\in\,]0,\epsilon_2]$ and a real analytic operator $(\Theta,\Xi)$ of $]-\epsilon_3,\epsilon_3[$ to $\mathcal{U}_0^{m,\alpha}\times\mathbb{R}$, such that*

$$(\Theta[\epsilon],\Xi[\epsilon])=(\hat{\theta}[\epsilon],\hat{\xi}[\epsilon]), \tag{2.39}$$

*for all $\epsilon\in[0,\epsilon_3[$.*

*Proof.* It is an immediate consequence of Proposition 2.38 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

### 2.2.3 A functional analytic representation Theorem for the solution of the singularly perturbed Dirichlet problem

By Proposition 2.39 and Remark 2.31, we can deduce the main result of this Section.

**Theorem 2.40.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $g$, $f$ be as in* (1.56), (2.25), (2.26)*, respectively. Let $\epsilon_3$ be as in Proposition 2.39. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V\cap\mathbb{S}_a[\Omega_0]=\emptyset$. Then there exist $\epsilon_4\in\,]0,\epsilon_3]$, a real analytic operator $U_1$ of $]-\epsilon_4,\epsilon_4[$ to the space $C^0_h(\mathrm{cl}\,V)$, and a real analytic operator $U_2$ of $]-\epsilon_4,\epsilon_4[$ to $\mathbb{R}$ such that the following conditions hold.*

*(i)* $\mathrm{cl}\,V\subseteq\mathbb{T}_a[\Omega_\epsilon]$ *for all $\epsilon\in\,]-\epsilon_4,\epsilon_4[$.*

*(ii)*
$$u[\epsilon](x)=\epsilon^{n-1}U_1[\epsilon](x)+U_2[\epsilon]+p_a[f](x)\qquad\forall x\in\mathrm{cl}\,V,$$
*for all $\epsilon\in\,]0,\epsilon_4[$.*

*Proof.* Let $\Theta[\cdot]$, $\Xi[\cdot]$ be as in Proposition 2.39. Choosing $\epsilon_4$ small enough, we can clearly assume that $(i)$ holds. Consider now $(ii)$. Let $\epsilon\in\,]0,\epsilon_4[$. By Remark 2.31 and Proposition 2.39, we have

$$u[\epsilon](x)=-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n^a(x-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s+\Xi[\epsilon]+p_a[f](x)\qquad\forall x\in\mathrm{cl}\,V.$$

Thus, it is natural to set

$$U_1[\epsilon](x)\equiv-\int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n^a(x-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s\qquad\forall x\in\mathrm{cl}\,V,$$
$$U_2[\epsilon]\equiv\Xi[\epsilon],$$

for all $\epsilon\in\,]-\epsilon_4,\epsilon_4[$. By Proposition 2.39, $U_2$ is real analytic. By Proposition 1.24 $(i)$, $U_1[\cdot]$ is a real analytic map of $]-\epsilon_4,\epsilon_4[$ to $C^0_h(\mathrm{cl}\,V)$. Finally, by the definition of $U_1$ and $U_2$, the statement in $(ii)$ holds. □

*Remark* 2.41. We note that the right-hand side of the equality in $(ii)$ of Theorem 2.40 can be continued real analytically in the whole $]-\epsilon_4,\epsilon_4[$. Moreover, if $V$ is a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V\cap\mathbb{S}_a[\Omega_0]=\emptyset$, then

$$\lim_{\epsilon\to0^+}u[\epsilon]=p_a[f]+\int_{\partial\Omega}g\tau\,d\sigma-p_a[f](w)\qquad\text{uniformly in }\mathrm{cl}\,V.$$

### 2.2.4 A real analytic continuation Theorem for the energy integral

We prove the following.

**Lemma 2.42.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, A \subseteq V$. Let $h$ be a real analytic function from $V$ to $\mathbb{R}$. Then there exists a real analytic operator $\tilde{G}_1$ of $]-\epsilon_1, \epsilon_1[$ to $\mathbb{R}$, such that*

$$\int_{\Omega_\epsilon} h(y)\, dy = \epsilon^n \tilde{G}_1[\epsilon],$$

*for all $\epsilon \in ]0, \epsilon_1[$. Moreover,*

$$\tilde{G}_1[0] = |\Omega|_n h(w).$$

*Proof.* Let $\epsilon \in ]0, \epsilon_1[$. We have

$$\int_{\Omega_\epsilon} h(y)\, dy = \epsilon^n \int_\Omega h(w + \epsilon s)\, ds.$$

Let $\tilde{G}$ be the map of $]-\epsilon_1, \epsilon_1[$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ which takes $\epsilon$ to $\tilde{G}[\epsilon]$, where

$$\tilde{G}[\epsilon](s) \equiv h(w + \epsilon s) \qquad \forall s \in \mathrm{cl}\,\Omega.$$

Let $\mathrm{id}_{\mathrm{cl}\,\Omega}$ denote the identity map in $\mathrm{cl}\,\Omega$. Since the map of $]-\epsilon_1, \epsilon_1[$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega, V)$, which takes $\epsilon$ to the function $w + \epsilon\, \mathrm{id}_{\mathrm{cl}\,\Omega}$ is obviously real analytic then, by a known result on composition operators (cf. Böhme and Tomi [15, p. 10], Henry [60, p. 29], Valent [137, Thm. 5.2, p. 44]), we have that $\tilde{G}$ is a real analytic operator. Set

$$\tilde{G}_1[\epsilon] \equiv \int_\Omega \tilde{G}[\epsilon](s)\, ds,$$

for all $\epsilon \in ]-\epsilon_1, \epsilon_1[$. Since the map of $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ to $\mathbb{R}$, which takes $u$ to $\int_\Omega u(s)\, ds$ is linear and continuous (and thus real analytic), we easily conclude that $\tilde{G}_1$ is a real analytic operator of $]-\epsilon_1, \epsilon_1[$ to $\mathbb{R}$. Finally, since

$$\tilde{G}[0](s) = h(w) \qquad \forall s \in \mathrm{cl}\,\Omega,$$

we have

$$\tilde{G}_1[0] = |\Omega|_n h(w).$$

□

As done in Theorem 2.40 for $u[\cdot]$, we can now prove a real analytic continuation Theorem for the energy integral. Namely, we prove the following.

**Theorem 2.43.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in (1.56), (2.25), (2.26), respectively. Let $\epsilon_3$ be as in Proposition 2.39. Then there exist $\epsilon_5 \in ]0, \epsilon_3]$ and two real analytic operators $G_1$, $G_2$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx = \int_A |\nabla p_a[f](x)|^2\, dx - \epsilon^n G_1[\epsilon] + \epsilon^{n-2} G_2[\epsilon], \tag{2.40}$$

*for all $\epsilon \in ]0, \epsilon_5[$. Moreover,*

$$G_1[0] = |\Omega|_n |\nabla p_a[f](w)|^2, \tag{2.41}$$

*and*

$$G_2[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\, dx. \tag{2.42}$$

*Proof.* Let $\Theta[\cdot]$, $\Xi[\cdot]$ be as in Proposition 2.39. Let $\epsilon \in ]0, \epsilon_3[$. Clearly,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx = \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla p_a[f](x)|^2\, dx + \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla \bar{u}[\epsilon](x)|^2\, dx + 2 \int_{\mathbb{P}_a[\Omega_\epsilon]} \nabla \bar{u}[\epsilon](x) \cdot \nabla p_a[f](x)\, dx.$$

Obviously,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla p_a[f](x)|^2 \, dx = \int_A |\nabla p_a[f](x)|^2 \, dx - \int_{\Omega_\epsilon} |\nabla p_a[f](x)|^2 \, dx.$$

By Remark 1.20, it follows that $|\nabla p_a[f](\cdot)|^2$ is real analytic in some bounded open neighbourhood $V'$ of $\mathrm{cl}\, A$. Accordingly, by virtue of Lemma 2.42, there exists a real analytic operator $G_1$ of $]-\epsilon_3, \epsilon_3[$ to $\mathbb{R}$ such that

$$\int_{\Omega_\epsilon} |\nabla p_a[f](x)|^2 \, dx = \epsilon^n G_1[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_3[$, and

$$G_1[0] = |\Omega|_n |\nabla p_a[f](w)|^2.$$

Consequently, (2.41) holds. Now we need to consider

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla \bar{u}[\epsilon](x)|^2 \, dx + 2 \int_{\mathbb{P}_a[\Omega_\epsilon]} \nabla \bar{u}[\epsilon](x) \cdot \nabla p_a[f](x) \, dx.$$

Let $\epsilon \in ]0, \epsilon_3[$. We denote by id the identity map in $\mathbb{R}^n$. By virtue of the periodicity of $\bar{u}[\epsilon]$ and $p_a[f]$ and by the Divergence Theorem, we have

$$\begin{aligned}
\int_{\mathbb{P}_a[\Omega_\epsilon]} &|\nabla \bar{u}[\epsilon](x)|^2 \, dx + 2 \int_{\mathbb{P}_a[\Omega_\epsilon]} \nabla \bar{u}[\epsilon](x) \cdot \nabla p_a[f](x) \, dx \\
&= -\epsilon^{n-2} \int_{\partial\Omega} \epsilon \Big(\frac{\partial}{\partial \nu_{\Omega_\epsilon}} \bar{u}[\epsilon]\Big) \circ (w + \epsilon \,\mathrm{id})(t) (g(t) + p_a[f](w + \epsilon t)) \, d\sigma_t \\
&= -\epsilon^{n-2} \int_{\partial\Omega} D[\bar{u}[\epsilon] \circ (w + \epsilon \,\mathrm{id})](t) \cdot \nu_\Omega(t) (g(t) + p_a[f](w + \epsilon t)) \, d\sigma_t,
\end{aligned}$$

for all $\epsilon \in ]0, \epsilon_3[$.

Now let $\tilde{\Omega}$ be a tubolar open neighbourhood of class $C^{m,\alpha}$ of $\partial\Omega$ as in Lanza and Rossi [86, Lemma 2.4]. Set

$$\tilde{\Omega}^- \equiv \tilde{\Omega} \cap (\mathbb{R}^n \setminus \mathrm{cl}\, \Omega).$$

Choosing $\epsilon_5 \in ]0, \epsilon_3]$ small enough, we can assume that

$$(w + \epsilon \,\mathrm{cl}\, \tilde{\Omega}) \subseteq A,$$

for all $\epsilon \in ]-\epsilon_5, \epsilon_5[$. We have

$$\begin{aligned}
&\bar{u}[\epsilon] \circ (w + \epsilon \,\mathrm{id})(t) \\
&= -\epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n(\epsilon(t - s)) \Theta[\epsilon](s) \, d\sigma_s - \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DR_n^a(\epsilon(t - s)) \Theta[\epsilon](s) \, d\sigma_s + \Xi[\epsilon] \\
&= \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(s)} (S_n(t - s)) \Theta[\epsilon](s) \, d\sigma_s - \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DR_n^a(\epsilon(t - s)) \Theta[\epsilon](s) \, d\sigma_s + \Xi[\epsilon] \ \forall t \in \tilde{\Omega}^-,
\end{aligned}$$

for all $\epsilon \in ]0, \epsilon_5[$. Hence, (cf. Proposition C.3 and Lanza and Rossi [86, Proposition 4.10]) there exists a real analytic operator $\tilde{G}_2$ of $]-\epsilon_5, \epsilon_5[$ to $C^{m,\alpha}(\mathrm{cl}\, \tilde{\Omega}^-)$, such that

$$\bar{u}[\epsilon] \circ (w + \epsilon \,\mathrm{id}) = \tilde{G}_2[\epsilon] \qquad \text{in } \tilde{\Omega}^-,$$

for all $\epsilon \in ]0, \epsilon_5[$. Furthermore, we observe that

$$\tilde{G}_2[0](t) = w^-[\partial\Omega, \Theta[0]](t) + \Xi[0] \qquad \forall t \in \mathrm{cl}\, \tilde{\Omega}^-,$$

and so, by Remark 2.37 and Proposition 2.39,

$$\tilde{G}_2[0](t) = \tilde{u}(t) + \tilde{\xi} \qquad \forall t \in \mathrm{cl}\, \tilde{\Omega}^-.$$

Thus, it is natural to set

$$G_2[\epsilon] \equiv -\int_{\partial\Omega} D[\tilde{G}_2[\epsilon]](t) \cdot \nu_\Omega(t) (g(t) + p_a[f](w + \epsilon t)) \, d\sigma_t,$$

for all $\epsilon \in ]-\epsilon_5, \epsilon_5[$. By the proof of Proposition 2.38, the map of $]-\epsilon_5, \epsilon_5[$ to $C^{m,\alpha}(\partial\Omega)$ which takes $\epsilon$ to $(g(\cdot) + p_a[f](w + \epsilon\cdot))$ is real analytic. By well known properties of the restriction map and pointwise product in Schauder spaces, we can easily conclude that $G_2$ is a real analytic operator of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{R}$ and that (2.40) holds. Finally,

$$\begin{aligned} G_2[0] &= -\int_{\partial\Omega} D[\tilde{G}_2[0]](t) \cdot \nu_\Omega(t)(g(t) + p_a[f](w))\, d\sigma_t \\ &= -\int_{\partial\Omega} D\tilde{u}(t) \cdot \nu_\Omega(t)\tilde{u}(t)\, d\sigma_t - \int_{\partial\Omega} \Big[ Dw^-[\partial\Omega, \Theta[0]](t) \cdot \nu_\Omega(t) \Big( \int_{\partial\Omega} g\tau\, d\sigma + p_a[f](w) \Big) \Big]\, d\sigma_t. \end{aligned}$$

By Theorem B.1 $(ii)$ and Green's Formula, we have

$$\int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega} w^-[\partial\Omega, \Theta[0]](t)\, d\sigma_t = \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega} w^+[\partial\Omega, \Theta[0]](t)\, d\sigma_t = 0.$$

On the other hand, by Folland [52, p. 118], we have

$$\int_{\partial\Omega} D\tilde{u}(t) \cdot \nu_\Omega(t)\tilde{u}(t)\, d\sigma_t = -\int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla\tilde{u}(x)|^2\, dx,$$

and so

$$G_2[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla\tilde{u}(x)|^2\, dx.$$

Accordingly, (2.42) holds. Thus, the Theorem is completely proved. □

*Remark* 2.44. We note that the right-hand side of the equality in (2.40) of Theorem 2.43 can be continued real analytically in the whole $]-\epsilon_5, \epsilon_5[$. Moreover,

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx = \int_A |\nabla p_a[f](x)|^2\, dx + \delta_{2,n} \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla\tilde{u}(x)|^2\, dx,$$

where $\delta_{2,n} = 1$ if $n = 2$, $\delta_{2,n} = 0$ if $n \geq 3$.

### 2.2.5 A real analytic continuation Theorem for the integral of the solution

We have the following.

**Lemma 2.45.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in* (1.56), (2.25), (2.26), *respectively. Let $\epsilon_3$, $\Theta[\cdot]$, $\Xi[\cdot]$ be as in Proposition 2.39. Then there exist $\epsilon_6 \in ]0, \epsilon_3]$ and a real analytic operator $\tilde{J}_1$ of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} w_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\tfrac{1}{\epsilon}(\cdot - w))\big](x)\, dx = \epsilon^n \tilde{J}_1[\epsilon], \tag{2.43}$$

*for all $\epsilon \in ]0, \epsilon_6[$.*

*Proof.* Let $\epsilon \in ]0, \epsilon_3[$. By well known results of classical potential theory, it is easy to see that

$$w_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\tfrac{1}{\epsilon}(\cdot - w))\big](x) = -\sum_{j=1}^n \frac{\partial}{\partial x_j} v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\tfrac{1}{\epsilon}(\cdot - w))(\nu_{\Omega_\epsilon}(\cdot))_j\big](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{P}_a[\Omega_\epsilon].$$

Let $j \in \{1, \ldots, n\}$. By the Divergence Theorem and the periodicity of the periodic simple layer potential, we have

$$\begin{aligned}
\int_{\mathbb{P}_a[\Omega_\epsilon]} \frac{\partial}{\partial x_j} v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))(\nu_{\Omega_\epsilon}(\cdot))_j\big](x)\, dx \\
&= \int_{\partial\mathbb{P}_a[\Omega_\epsilon]} v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))(\nu_{\Omega_\epsilon}(\cdot))_j\big](x)(\nu_{\mathbb{P}_a[\Omega_\epsilon]}(x))_j \, d\sigma_x \\
&= \int_{\partial A} v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))(\nu_{\Omega_\epsilon}(\cdot))_j\big](x)(\nu_A(x))_j \, d\sigma_x \\
&\quad - \int_{\partial\Omega_\epsilon} v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))(\nu_{\Omega_\epsilon}(\cdot))_j\big](x)(\nu_{\Omega_\epsilon}(x))_j \, d\sigma_x \\
&= - \int_{\partial\Omega_\epsilon} v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))(\nu_{\Omega_\epsilon}(\cdot))_j\big](x)(\nu_{\Omega_\epsilon}(x))_j \, d\sigma_x \\
&= - \epsilon^{n-1} \int_{\partial\Omega} v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))(\nu_{\Omega_\epsilon}(\cdot))_j\big](w+\epsilon t)(\nu_\Omega(t))_j \, d\sigma_t .
\end{aligned}$$

Then we note that

$$\begin{aligned}
v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))(\nu_{\Omega_\epsilon}(\cdot))_j\big](w+\epsilon t) =& \epsilon^{n-1} \int_{\partial\Omega} S_n(\epsilon(t-s))\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \\
&+ \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \quad \forall t \in \partial\Omega.
\end{aligned}$$

By equality (1.65), if $n = 2$, we have

$$\begin{aligned}
\int_{\partial\Omega} v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))(\nu_{\Omega_\epsilon}(\cdot))_j\big](w+\epsilon t)(\nu_\Omega(t))_j \, d\sigma_t \\
=& \frac{1}{2\pi} \epsilon \log \epsilon \int_{\partial\Omega} \Big( \int_{\partial\Omega} \Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \Big) (\nu_\Omega(t))_j \, d\sigma_t \\
&+ \epsilon \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_2(t-s)\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \Big) (\nu_\Omega(t))_j \, d\sigma_t \\
&+ \epsilon \int_{\partial\Omega} \Big( \int_{\partial\Omega} R_2^a(\epsilon(t-s))\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \Big) (\nu_\Omega(t))_j \, d\sigma_t .
\end{aligned}$$

On the other hand, by the Divergence Theorem, it is immediate to see that

$$\int_{\partial\Omega} \Big( \int_{\partial\Omega} \Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \Big) (\nu_\Omega(t))_j \, d\sigma_t = \Big( \int_{\partial\Omega} \Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \Big) \Big( \int_{\partial\Omega} (\nu_\Omega(t))_j \, d\sigma_t \Big) = 0.$$

By equality (1.65), if $n \geq 3$, we have

$$\begin{aligned}
\int_{\partial\Omega} v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))(\nu_{\Omega_\epsilon}(\cdot))_j\big](w+\epsilon t)(\nu_\Omega(t))_j \, d\sigma_t \\
=& \epsilon \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \Big) (\nu_\Omega(t))_j \, d\sigma_t \\
&+ \epsilon^{n-1} \int_{\partial\Omega} \Big( \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \Big) (\nu_\Omega(t))_j \, d\sigma_t .
\end{aligned}$$

Hence, if $n \geq 2$ and $\epsilon \in ]0, \epsilon_3[$, we have

$$\begin{aligned}
\int_{\mathbb{P}_a[\Omega_\epsilon]} w_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x)\, dx \\
=& \sum_{j=1}^{n} \epsilon^n \Big[ \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \Big) (\nu_\Omega(t))_j \, d\sigma_t \\
&+ \epsilon^{n-2} \int_{\partial\Omega} \Big( \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \Big) (\nu_\Omega(t))_j \, d\sigma_t \Big].
\end{aligned}$$

Thus it is natural to set

$$\begin{aligned}\tilde{J}_1[\epsilon] \equiv \sum_{j=1}^{n} \Big[ \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \Big) (\nu_\Omega(t))_j \, d\sigma_t \\ + \epsilon^{n-2} \int_{\partial\Omega} \Big( \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \Big) (\nu_\Omega(t))_j \, d\sigma_t \Big],\end{aligned}$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. Clearly,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} w_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) \, dx = \epsilon^n \tilde{J}_1[\epsilon] \qquad \forall \epsilon \in ]0, \epsilon_3[.$$

In order to conclude, it suffices to prove that $\tilde{J}_1$ is real analytic. Indeed, we observe that, if $j \in \mathbb{N}$, then well known properties of functions in Schauder spaces and standard calculus in Banach spaces imply that the map of $]-\epsilon_3, \epsilon_3[$ to $\mathbb{R}$, which takes $\epsilon$ to

$$\int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \Big) (\nu_\Omega(t))_j \, d\sigma_t,$$

is real analytic. Similarly, Theorem C.4, well known properties of functions in Schauder spaces and standard calculus in Banach spaces, imply that there exists $\epsilon_6 \in ]0, \epsilon_3]$, such that the map of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$, which takes $\epsilon$ to

$$\int_{\partial\Omega} \Big( \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \Big) (\nu_\Omega(t))_j \, d\sigma_t,$$

is real analytic, for all $j \in \{1, \dots, n\}$. Hence $\tilde{J}_1$ is a real analytic map of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$, and the proof is complete. □

As done in Theorem 2.43 for the energy integral, we can now prove a real analytic continuation Theorem for the integral of the solution. Namely, we prove the following.

**Theorem 2.46.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in (1.56), (2.25), (2.26), respectively. Let $\epsilon_6$ be as in Lemma 2.45. Then there exists a real analytic operator $J$ of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x) \, dx = J[\epsilon], \tag{2.44}$$

*for all $\epsilon \in ]0, \epsilon_6[$. Moreover,*

$$J[0] = \Big( \int_{\partial\Omega} g\tau \, d\sigma - p_a[f](w) \Big) |A|_n + \int_A p_a[f](x) \, dx. \tag{2.45}$$

*Proof.* Let $\epsilon \in ]0, \epsilon_3[$. We have

$$u[\epsilon](x) = w_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) + \Xi[\epsilon] + p_a[f](x) \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon].$$

As a consequence,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x) \, dx = \int_{\mathbb{P}_a[\Omega_\epsilon]} w_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) \, dx + \int_{\mathbb{P}_a[\Omega_\epsilon]} \Xi[\epsilon] \, dx + \int_{\mathbb{P}_a[\Omega_\epsilon]} p_a[f](x) \, dx.$$

By Lemma 2.45, there exists a real analytic operator $\tilde{J}_1$ of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$ such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} w_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) \, dx = \epsilon^n \tilde{J}_1[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_6[$. On the other hand,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} \Xi[\epsilon] \, dx = \Xi[\epsilon] \Big( |A|_n - \epsilon^n |\Omega|_n \Big),$$

for all $\epsilon \in ]0, \epsilon_3[$. Moreover,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} p_a[f](x)\,dx = \int_A p_a[f](x)\,dx - \int_{\Omega_\epsilon} p_a[f](x)\,dx,$$

for all $\epsilon \in ]0, \epsilon_3[$. By Lemma 2.42, there exists a real analytic operator $\tilde{J}_2$ of $]-\epsilon_3, \epsilon_3[$ to $\mathbb{R}$ such that

$$\int_{\Omega_\epsilon} p_a[f](x)\,dx = \epsilon^n \tilde{J}_2[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_3[$. Thus, if we set

$$J[\epsilon] \equiv \epsilon^n \tilde{J}_1[\epsilon] + \Xi[\epsilon]\big(|A|_n - \epsilon^n |\Omega|_n\big) + \int_A p_a[f](x)\,dx - \epsilon^n \tilde{J}_2[\epsilon],$$

for all $\epsilon \in ]-\epsilon_6, \epsilon_6[$, we have that $J$ is a real analytic map of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$ such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx = J[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_6[$. Finally,

$$\begin{aligned} J[0] &= \Xi[0]|A|_n + \int_A p_a[f](x)\,dx \\ &= \Big(\int_{\partial\Omega} g\tau\,d\sigma - p_a[f](w)\Big)|A|_n + \int_A p_a[f](x)\,dx, \end{aligned}$$

and the proof is complete. □

### 2.2.6 A remark on a Dirichlet problem

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $l \in \mathbb{Z}$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.25), respectively. For each $\epsilon \in ]0, \epsilon_1[$, we consider the following periodic Dirichlet problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1, \dots, n\}, \\ u(x) = \epsilon^{-l} g\big(\frac{1}{\epsilon}(x - w)\big) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{2.46}$$

By virtue of Theorem 2.15, we can give the following definition.

**Definition 2.47.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0, 1[$. Let $l \in \mathbb{Z}$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.25), respectively. For each $\epsilon \in ]0, \epsilon_1[$, we denote by $u_l[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of boundary value problem (2.46).

Then we have the following.

**Theorem 2.48.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $l \in \mathbb{Z}$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.25), respectively. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_2 \in ]0, \epsilon_1]$, a real analytic operator $U_1$ of $]-\epsilon_2, \epsilon_2[$ to the space $C^0_h(\mathrm{cl}\,V)$, and a real analytic operator $U_2$ of $]-\epsilon_2, \epsilon_2[$ to $\mathbb{R}$ such that the following conditions hold.*

*(i)* $\mathrm{cl}\,V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_2, \epsilon_2[$.

*(ii)*
$$u_l[\epsilon](x) = \epsilon^{n-1-l} U_1[\epsilon](x) + \epsilon^{-l} U_2[\epsilon] \qquad \forall x \in \mathrm{cl}\,V,$$
*for all $\epsilon \in ]0, \epsilon_2[$. Moreover,*
$$U_2[0] = \int_{\partial\Omega} g\tau\,d\sigma,$$
*where $\tau$ is as in Definition 2.32.*

*Proof.* It is a straightforward consequence of Theorem 2.40. □

We now show that the energy integral can be continued real analytically when $n \geq 2l+2$. Namely, we prove the following.

**Theorem 2.49.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $l \in \mathbb{Z}$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.25), *respectively. Then there exist $\epsilon_3 \in ]0,\epsilon_1]$ and a real analytic operator $G$ of $]-\epsilon_3,\epsilon_3[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u_l[\epsilon](x)|^2 \, dx = \epsilon^{n-2-2l} G[\epsilon], \tag{2.47}$$

*for all $\epsilon \in ]0,\epsilon_3[$. Moreover,*

$$G[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx, \tag{2.48}$$

*where $\tilde{u}$ is as in Definition 2.35.*

*Proof.* It is a straightforward consequence of Theorem 2.43. □

We have also the following.

**Theorem 2.50.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in* (1.56), (2.25), (2.26), *respectively. Then there exist $\epsilon_4 \in ]0,\epsilon_1]$ and a real analytic operator $J$ of $]-\epsilon_4,\epsilon_4[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u_l[\epsilon](x) \, dx = \epsilon^{-l} J[\epsilon], \tag{2.49}$$

*for all $\epsilon \in ]0,\epsilon_4[$. Moreover,*

$$J[0] = \Big(\int_{\partial\Omega} g\tau \, d\sigma\Big)|A|_n, \tag{2.50}$$

*where $\tau$ is as in Definition 2.32.*

*Proof.* It is a straightforward consequence of Theorem 2.46. □

## 2.3 An homogenization problem for the Laplace equation with Dirichlet boundary conditions in a periodically perforated domain

In this section we consider an homogenization problem for the Laplace equation with Dirichlet boundary conditions in a periodically perforated domain.

### 2.3.1 Notation

In this Section we retain the notation introduced in Subsections 1.8.1, 2.2.1. However, we need to introduce also some other notation.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $(\epsilon,\delta) \in (]-\epsilon_1,\epsilon_1[ \setminus \{0\}) \times ]0,+\infty[$. If $v$ is a function of $\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta)$ to $\mathbb{R}$, then we denote by $\mathbf{E}_{(\epsilon,\delta)}[v]$ the function of $\mathbb{R}^n$ to $\mathbb{R}$, defined by

$$\mathbf{E}_{(\epsilon,\delta)}[v](x) \equiv \begin{cases} v(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta) \\ 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta). \end{cases}$$

### 2.3.2 Preliminaries

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.25), respectively. For each $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we consider the following periodic Dirichlet problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u(x+\delta a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall i \in \{1,\dots,n\}, \\ u(x) = g(\frac{1}{\epsilon\delta}(x-\delta w)) & \forall x \in \partial\Omega(\epsilon,\delta). \end{cases} \tag{2.51}$$

By virtue of Theorem 2.15, we can give the following definition.

**Definition 2.51.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.25), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we denote by $u_{(\epsilon,\delta)}$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta))$ of boundary value problem (2.51).

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. In order to do so we introduce the following.

**Definition 2.52.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.25), respectively. For each $\epsilon \in ]0,\epsilon_1[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of the following periodic Dirichlet problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1,\dots,n\}, \\ u(x) = g\big(\frac{1}{\epsilon}(x-w)\big) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{2.52}$$

*Remark* 2.53. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.25), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we have

$$u_{(\epsilon,\delta)}(x) = u[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).$$

As a first step, we study the behaviour of $u[\epsilon]$ as $\epsilon$ tends to 0. Obviously, we have the following.

**Theorem 2.54.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.25), (2.26)*, respectively. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_2 \in ]0,\epsilon_1]$, a real analytic operator $U_1$ of $]-\epsilon_2,\epsilon_2[$ to the space $C^0_h(\mathrm{cl}\,V)$, and a real analytic operator $U_2$ of $]-\epsilon_2,\epsilon_2[$ to $\mathbb{R}$ such that the following conditions hold.*

*(i)* $\mathrm{cl}\,V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_2,\epsilon_2[$.

*(ii)*
$$u[\epsilon](x) = \epsilon^{n-1} U_1[\epsilon](x) + U_2[\epsilon] \qquad \forall x \in \mathrm{cl}\,V,$$
*for all $\epsilon \in ]0,\epsilon_2[$. Moreover,*
$$U_2[0] = \int_{\partial\Omega} g\tau\, d\sigma,$$
*where $\tau$ is as in Definition 2.32.*

*Proof.* It is Theorem 2.40 in the case $f \equiv 0$. □

**Theorem 2.55.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.25)*, respectively. Then there exist $\epsilon_3 \in ]0,\epsilon_1]$ and a real analytic operator $G$ of $]-\epsilon_3,\epsilon_3[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx = \epsilon^{n-2} G[\epsilon], \tag{2.53}$$

*for all $\epsilon \in ]0,\epsilon_3[$. Moreover,*

$$G[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\, dx, \tag{2.54}$$

*where $\tilde{u}$ is as in Definition 2.35.*

*Proof.* It is Theorem 2.43 in the case $f \equiv 0$. □

**Theorem 2.56.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.25)*, respectively. Then there exist $\epsilon_4 \in ]0,\epsilon_1]$ and a real analytic operator $J$ of $]-\epsilon_4,\epsilon_4[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx = J[\epsilon], \tag{2.55}$$

*for all $\epsilon \in ]0,\epsilon_4[$. Moreover,*

$$J[0] = \Big(\int_{\partial\Omega} g\tau\, d\sigma\Big)|A|_n, \tag{2.56}$$

*where $\tau$ is as in Definition 2.32.*

*Proof.* It is Theorem 2.46 in the case $f \equiv 0$. □

Then we have the following.

**Proposition 2.57.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.25), respectively. Let $1 \leq p < \infty$. Then*

$$\lim_{\epsilon \to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = \int_{\partial\Omega} g\tau \, d\sigma \qquad \text{in } L^p(A),$$

*where $\tau$ is as in Definition 2.32.*

*Proof.* By Theorem 2.5, we have

$$|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]](x)| \leq \sup\{\, |g(t)| \colon t \in \partial\Omega \,\} < +\infty \qquad \forall x \in A, \quad \forall \epsilon \in ]0, \epsilon_1[.$$

By Theorem 2.54, we have

$$\lim_{\epsilon \to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]](x) = \int_{\partial\Omega} g\tau \, d\sigma \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{\epsilon \to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = \int_{\partial\Omega} g\tau \, d\sigma \qquad \text{in } L^p(A).$$

□

### 2.3.3 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorem we deduce by Proposition 2.57 and the results of Appendix D the weak convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to 0.

**Theorem 2.58.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.25), respectively. Let $\tau$ be as in Definition 2.32. Let $1 \leq p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] \rightharpoonup \int_{\partial\Omega} g\tau \, d\sigma \qquad \text{in } L^p(V),$$

*as $(\epsilon,\delta)$ tends to 0 in $]0,\epsilon_1[ \times ]0,+\infty[$.*

*Proof.* It is an immediate consequence of Proposition 2.57 and Theorem D.5. □

However, we can prove something more. Namely, we prove the following.

**Theorem 2.59.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.25), respectively. Let $\tau$ be as in Definition 2.32. Let $1 \leq p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta) \to (0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = \int_{\partial\Omega} g\tau \, d\sigma \qquad \text{in } L^p(V).$$

*Proof.* By virtue of Proposition 2.57, we have

$$\lim_{\epsilon \to 0^+} \|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]] - \int_{\partial\Omega} g\tau \, d\sigma\|_{L^p(A)} = 0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $c > 0$ such that

$$\|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] - \int_{\partial\Omega} g\tau \, d\sigma\|_{L^p(V)} \leq c\|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]] - \int_{\partial\Omega} g\tau \, d\sigma\|_{L^p(A)} \qquad \forall (\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,1[.$$

Thus,

$$\lim_{(\epsilon,\delta) \to (0^+,0^+)} \|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] - \int_{\partial\Omega} g\tau \, d\sigma\|_{L^p(V)} = 0,$$

and the conclusion easily follows. □

Now, our aim is to describe the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ goes to $(0,0)$, in terms of real analytic functions (possibly evaluated on 'particular' values of $(\epsilon,\delta)$.) Clearly, if $V$ is a non-empty open subset of $\mathbb{R}^n$, then

$$V \cap \mathbb{S}_a(\epsilon,\delta) \neq \emptyset$$

if $\epsilon \in ]0,\epsilon_1[$ and $\delta$ is positive and sufficiently small. Therefore, we cannot hope to describe the behaviour of the restrcition of $u_{(\epsilon,\delta)}$ to the closure of an open subset in terms of real analytic functions as we have done for the solution of problems in $\mathbb{T}_a[\Omega_\epsilon]$. As a consequence, we need to find a different way to describe the convergence of $u_{(\epsilon,\delta)}$, since the restriction to non-empty open subsets of $\mathbb{R}^n$ is no longer convenient. So let $1 \leq p < +\infty$. Clearly, if $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, then we can associate to $u_{(\epsilon,\delta)}$ the element of the dual of $L^p(\mathbb{R}^n)$ which takes a function $\phi$ to

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}](x)\phi(x)\,dx.$$

Thus, instead of studying the restriction of $u_{(\epsilon,\delta)}$ to some bounded open subset of $\mathbb{R}^n$, we investigate the behaviour of this element of the dual of $L^p(\mathbb{R}^n)$ associated the function $u_{(\epsilon,\delta)}$. In particular, we want to investigate such a functional evaluated on the functions of a convenient subset, say $\mathcal{S}$, of $L^p(\mathbb{R}^n)$. Then, of course, it will be important to see 'how much large' this subset $\mathcal{S}$ is.

Then we have the following Theorem, where we consider the functional associated to an extension of $u_{(\epsilon,\delta)}$, and we evaluate such a functional on suitable characteristic functions.

**Theorem 2.60.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.25), *respectively. Let $\epsilon_4$, $J$ be as in Theorem 2.56. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J[\epsilon], \tag{2.57}$$

*for all $\epsilon \in ]0,\epsilon_4[$, $l \in \mathbb{N}\setminus\{0\}$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_4[$, $l \in \mathbb{N}\setminus\{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\
&= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\
&= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.
\end{aligned}$$

Then we note that

$$\begin{aligned}
\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\
&= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon]\Big(\frac{l}{r}x\Big)\,dx \\
&= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](t)\,dt \\
&= \frac{r^n}{l^n} J[\epsilon].
\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J[\epsilon],$$

and the conclusion follows. □

In the previous Theorem, we have seen that we can describe the behaviour of the functional associated to $u_{(\epsilon,\delta)}$ evaluated on a convenient characteristic function in terms of real analytic functions.

Therefore, in the following Theorem, we study the vector space of the characteristic functions that appear in Theorem 2.60.

**Theorem 2.61.** *Let $\mathcal{S}_a$ be the vector space defined by*

$$\mathcal{S}_a \equiv \left\{ \sum_{j=1}^{k} \lambda_j \chi_{r_j A + \bar{y}_j} : k \in \mathbb{N} \setminus \{0\}, (\lambda_j, r_j, \bar{y}_j) \in \mathbb{R} \times ]0, +\infty[ \times \mathbb{R}^n \ \forall j \in \{1, \dots, k\} \right\}.$$

*Let $1 \leq p < +\infty$. Let $\phi \in C_c^\infty(\mathbb{R}^n)$. Then there exists a sequence $\{\phi_l\}_{l=1}^\infty \subseteq \mathcal{S}_a$ such that*

$$\lim_{l \to +\infty} \phi_l = \phi \qquad \text{in } L^p(\mathbb{R}^n).$$

*Proof.* Let $\phi \in C_c^\infty(\mathbb{R}^n)$. We first assume that $\operatorname{supp} \phi \subseteq ]0, +\infty[^n$. Then let $\bar{r} > 0$ be such that $\bar{r}A \supseteq \operatorname{supp} \phi$. Then we define the sequence $\{\phi_l\}_{l=1}^\infty \subseteq \mathcal{S}_a$ by setting

$$\phi_l(x) \equiv \sum_{z \in \{0, \dots, l-1\}^n} \phi\big(\frac{\bar{r}}{l} a(z)\big) \chi_{\frac{\bar{r}}{l} A + \frac{\bar{r}}{l} a(z)}(x) \qquad \forall x \in \mathbb{R}^n,$$

for all $l \in \mathbb{N} \setminus \{0\}$. Then it is easy to prove that

$$\lim_{l \to +\infty} \phi_l(x) = \phi(x)$$

for almost every $x \in \mathbb{R}^n$. Moreover,

$$|\phi_l(x)| \leq \|\phi\|_\infty \chi_{\bar{r}A}(x) \qquad \forall x \in \mathbb{R}^n.$$

Thus, by the Dominated Convergence Theorem, we can easily conclude that

$$\lim_{l \to +\infty} \phi_l = \phi \qquad \text{in } L^p(\mathbb{R}^n).$$

Next we observe that if we don't assume that $\operatorname{supp} \phi \subseteq ]0, +\infty[^n$, then there exists $\bar{x} \in \mathbb{R}^n$, such that $\operatorname{supp} \phi(\cdot - \bar{x}) \subseteq ]0, +\infty[^n$. Then, if we set $\tilde{\phi}(\cdot) \equiv \phi(\cdot - \bar{x})$, by the above argument, there exists a sequence $\{\tilde{\phi}_l\}_{l=1}^\infty \subseteq \mathcal{S}_a$, such that

$$\lim_{l \to +\infty} \tilde{\phi}_l = \tilde{\phi} \qquad \text{in } L^p(\mathbb{R}^n).$$

Finally, if we set $\phi_l(\cdot) \equiv \tilde{\phi}_l(\cdot + \bar{x})$, we can easily deduce that

$$\phi_l \in \mathcal{S}_a \qquad \forall l \geq 1,$$

and that

$$\lim_{l \to +\infty} \phi_l = \phi \qquad \text{in } L^p(\mathbb{R}^n).$$

□

In the following Corollary, we prove a density property of the vector space introduced in Theorem 2.61.

**Corollary 2.62.** *Let $1 \leq p < +\infty$. Let $\mathcal{S}_a$ be as in Theorem 2.61. Then the vector space $\mathcal{S}_a$ is dense in $L^p(\mathbb{R}^n)$.*

*Proof.* First of all, we recall the density of $C_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$. Then, in order to conclude, it suffices to apply Theorem 2.61. □

### 2.3.4 Asymptotic behaviour of the energy integral of $u_{(\epsilon,\delta)}$

This Subsection is devoted to the study of the behaviour of the energy integral of $u_{(\epsilon,\delta)}$. We give the following.

**Definition 2.63.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.25), respectively. For each pair $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we set

$$\operatorname{En}(\epsilon, \delta) \equiv \int_{A \cap \mathbb{T}_a(\epsilon,\delta)} \left|\nabla u_{(\epsilon,\delta)}(x)\right|^2 dx.$$

*Remark* 2.64. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.25), respectively. Let $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$. We have

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx = \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2 \, dt$$
$$= \delta^{n-2} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2 \, dt.$$

Then we give the following definition, where we consider $\mathrm{En}(\epsilon, \delta)$, with $\epsilon$ equal to a certain function of $\delta$.

**Definition 2.65.** Let $n \in \mathbb{N} \setminus \{0, 1, 2\}$. For each $\delta \in ]0, +\infty[$, we set

$$\epsilon[\delta] \equiv \delta^{\frac{2}{n-2}}.$$

Let $\epsilon_3$ be as in Theorem 2.55. Let $\delta_1 > 0$ be such that $\epsilon[\delta] \in ]0, \epsilon_3[$, for all $\delta \in ]0, \delta_1[$. Then we set

$$\mathrm{En}[\delta] \equiv \mathrm{En}(\epsilon[\delta], \delta),$$

for all $\delta \in ]0, \delta_1[$.

Here we may note that the 'radius' of the holes is $\delta\epsilon[\delta] = \delta^{\frac{n}{n-2}}$ which is the same which appears in Homogenization Theory (cf. *e.g.*, Ansini and Braides [7] and references therein.)

In the following Proposition we compute the limit of $\mathrm{En}[\delta]$ as $\delta$ tends to 0.

**Proposition 2.66.** *Let $n \in \mathbb{N} \setminus \{0, 1, 2\}$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.25), *respectively. Let $\epsilon_3$ be as in Theorem 2.55. Let $\delta_1 > 0$ be as in Definition 2.65. Then*

$$\lim_{\delta \to 0^+} \mathrm{En}[\delta] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx,$$

*where $\tilde{u}$ is as in Definition 2.35.*

*Proof.* Let $\delta \in ]0, \delta_1[$. By Remark 2.64 and Theorem 2.55, we have

$$\int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx = \delta^{n-2} (\epsilon[\delta])^{n-2} G[\epsilon[\delta]]$$
$$= \delta^n G[\delta^{\frac{2}{n-2}}],$$

where $G$ is as in Theorem 2.55. On the other hand,

$$\lfloor (1/\delta) \rfloor^n \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx \le \mathrm{En}[\delta] \le \lceil (1/\delta) \rceil^n \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx,$$

and so

$$\lfloor (1/\delta) \rfloor^n \delta^n G[\delta^{\frac{2}{n-2}}] \le \mathrm{En}[\delta] \le \lceil (1/\delta) \rceil^n \delta^n G[\delta^{\frac{2}{n-2}}].$$

Thus, since

$$\lim_{\delta \to 0^+} \lfloor (1/\delta) \rfloor^n \delta^n = 1, \qquad \lim_{\delta \to 0^+} \lceil (1/\delta) \rceil^n \delta^n = 1,$$

we have

$$\lim_{\delta \to 0^+} \mathrm{En}[\delta] = G[0].$$

Finally, by equality (2.54), we easily conclude. □

In the following Proposition we represent the function $\mathrm{En}[\cdot]$ by means of a real analytic function.

**Proposition 2.67.** *Let $n \in \mathbb{N} \setminus \{0, 1, 2\}$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.25), *respectively. Let $\epsilon_3$ and $G$ be as in Theorem 2.55. Let $\delta_1 > 0$ be as in Definition 2.65. Then*

$$\mathrm{En}[(1/l)] = G[(1/l)^{\frac{2}{n-2}}],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* It follows by the proof of Proposition 2.66. □

## 2.4 An homogenization problem for the Poisson equation with Dirichlet boundary conditions in a periodically perforated domain

In this section we consider an homogenization problem for the Poisson equation with Dirichlet boundary conditions in a periodically perforated domain.

### 2.4.1 Preliminaries

In this Section we retain the notation introduced in Subsection 2.2.1 and in Subsection 2.3.1 (cf. also Subsection 1.8.1).

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in (1.56), (2.25), (2.26), respectively. For each $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we consider the following periodic Dirichlet problem for the Poisson equation.

$$\begin{cases} \Delta u(x) = f(\frac{x}{\delta}) & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u(x+\delta a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall i \in \{1,\dots,n\}, \\ u(x) = g(\frac{1}{\epsilon\delta}(x-\delta w)) & \forall x \in \partial\Omega(\epsilon,\delta). \end{cases} \tag{2.58}$$

By virtue of Theorems 2.15 and 2.16, we can give the following definition.

**Definition 2.68.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in (1.56), (2.25), (2.26), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we denote by $u_{(\epsilon,\delta)}$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta))$ of boundary value problem (2.58).

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. In order to do so we introduce the following.

**Definition 2.69.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in (1.56), (2.25), (2.26), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we denote by $u^\delta_\epsilon$ the function of $\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon]$ to $\mathbb{R}$ defined by

$$u^\delta_\epsilon(x) \equiv u_{(\epsilon,\delta)}(\delta x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$

**Definition 2.70.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in (1.56), (2.25), (2.26), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we denote by $\bar{u}^\delta_\epsilon$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of the following periodic Dirichlet problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1,\dots,n\}, \\ u(x) = g\big(\frac{1}{\epsilon}(x-w)\big) - \delta^2 p_a[f](x) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{2.59}$$

*Remark* 2.71. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in (1.56), (2.25), (2.26), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we have

$$u_{(\epsilon,\delta)}(x) = \bar{u}^\delta_\epsilon(\frac{x}{\delta}) + \delta^2 p_a[f](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).$$

As a first step, we study the behaviour of $\bar{u}^\delta_\epsilon$ for $(\epsilon,\delta)$ close to $(0,0)$.

As we know, we can convert problem (2.59) into an integral equation. We introduce this equation in the following.

**Proposition 2.72.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $g$, $f$ *be as in* (1.56), (1.57), (2.25), (2.26), *respectively. Let* $\mathcal{U}^{m,\alpha}_\epsilon$, $\mathcal{U}^{m,\alpha}_0$ *be as in* (1.63), (1.64), *respectively. Let* $\Lambda$ *be the map of* $]-\epsilon_1,\epsilon_1[ \times \mathbb{R} \times \mathcal{U}^{m,\alpha}_0 \times \mathbb{R}$ *in* $C^{m,\alpha}(\partial\Omega)$ *defined by*

$$\begin{aligned} \Lambda[\epsilon,\delta,\theta,\xi](t) \equiv & -\frac{1}{2}\theta(t) - \int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n(t-s)\theta(s)\,d\sigma_s - \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DR^a_n(\epsilon(t-s))\theta(s)\,d\sigma_s \\ & + \xi - g(t) + \delta^2 p_a[f](w+\epsilon t) \qquad \forall t \in \partial\Omega, \end{aligned} \tag{2.60}$$

*for all* $(\epsilon,\delta,\theta,\xi) \in ]-\epsilon_1,\epsilon_1[ \times \mathbb{R} \times \mathcal{U}^{m,\alpha}_0 \times \mathbb{R}$. *Then the following statements hold.*

*(i) If* $(\epsilon,\delta)\in ]0,\epsilon_1[\times]0,+\infty[$*, then the pair* $(\theta,\xi)\in\mathcal{U}_0^{m,\alpha}\times\mathbb{R}$ *satisfies equation*

$$\Lambda[\epsilon,\delta,\theta,\xi]=0, \tag{2.61}$$

*if and only if the pair* $(\mu,\xi)\in\mathcal{U}_\epsilon^{m,\alpha}\times\mathbb{R}$*, with* $\mu\in\mathcal{U}_\epsilon^{m,\alpha}$ *defined by*

$$\mu(x)\equiv\theta\big(\frac{1}{\epsilon}(x-w)\big)\qquad\forall x\in\partial\Omega_\epsilon, \tag{2.62}$$

*satisfies the equation*

$$\Gamma(x)=-\frac{1}{2}\mu(x)+\int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}(y)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y+\xi\qquad\forall x\in\partial\Omega_\epsilon, \tag{2.63}$$

*with* $\Gamma\in C^{m,\alpha}(\partial\Omega_\epsilon)$ *defined by*

$$\Gamma(x)\equiv g\big(\frac{1}{\epsilon}(x-w)\big)-\delta^2p_a[f](x)\qquad\forall x\in\partial\Omega_\epsilon. \tag{2.64}$$

*In particular, equation* (2.61) *has exactly one solution* $(\theta,\xi)\in\mathcal{U}_0^{m,\alpha}\times\mathbb{R}$*, for each* $(\epsilon,\delta)\in ]0,\epsilon_1[\times]0,+\infty[$*.*

*(ii) The pair* $(\theta,\xi)\in\mathcal{U}_0^{m,\alpha}\times\mathbb{R}$ *satisfies equation*

$$\Lambda[0,0,\theta,\xi]=0, \tag{2.65}$$

*if and only if*

$$g(t)=-\frac{1}{2}\theta(t)+\int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega(s)}(S_n(t-s))\theta(s)\,d\sigma_s+\xi\qquad\forall t\in\partial\Omega. \tag{2.66}$$

*In particular, equation* (2.65) *has exactly one solution* $(\theta,\xi)\in\mathcal{U}_0^{m,\alpha}\times\mathbb{R}$*, which we denote by* $(\tilde\theta,\tilde\xi)$*.*

*Proof.* Consider $(i)$. Let $\theta\in C^{m,\alpha}(\partial\Omega)$. Let $(\epsilon,\delta)\in ]0,\epsilon_1[\times]0,+\infty[$. First of all, we note that

$$\int_{\partial\Omega_\epsilon}\theta\big(\frac{1}{\epsilon}(x-w)\big)\,d\sigma_x=\epsilon^{n-1}\int_{\partial\Omega}\theta(t)\,d\sigma_t,$$

and so $\theta\in\mathcal{U}_0^{m,\alpha}$ if and only if $\theta(\frac{1}{\epsilon}(\cdot-w))\in\mathcal{U}_\epsilon^{m,\alpha}$. The equivalence of equation (2.61) in the unknown $(\theta,\xi)\in\mathcal{U}_0^{m,\alpha}\times\mathbb{R}$ and equation (2.63) in the unknown $(\mu,\xi)\in\mathcal{U}_\epsilon^{m,\alpha}\times\mathbb{R}$ follows by a straightforward computation based on the rule of change of variables in integrals and on well known properties of composition of functions in Schauder spaces (cf. *e.g.*, Lanza [67, Sections 3,4].) The existence and uniqueness of a solution of equation (2.63) follows by the proof of Proposition 2.21. Then the existence and uniqueness of a solution of equation (2.61) follows by the equivalence of (2.61) and (2.63). Consider $(ii)$. The equivalence of (2.65) and (2.66) is obvious. The existence of a unique solution of equation (2.65) is an immediate consequence of Lemma 2.28. □

By Proposition 2.72, it makes sense to introduce the following.

**Definition 2.73.** Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in (1.56), (1.57), (2.25), (2.26), respectively. Let $\mathcal{U}_0^{m,\alpha}$ be as in (1.64). For each $(\epsilon,\delta)\in]0,\epsilon_1[\times]0,+\infty[$, we denote by $(\hat\theta[\epsilon,\delta],\hat\xi[\epsilon,\delta])$ the unique pair in $\mathcal{U}_0^{m,\alpha}\times\mathbb{R}$ that solves (2.61). Analogously, we denote by $(\hat\theta[0,0],\hat\xi[0,0])$ the unique pair in $\mathcal{U}_0^{m,\alpha}\times\mathbb{R}$ that solves (2.65).

In the following Remark, we show the relation between the solutions of boundary value problem (2.59) and the solutions of equation (2.61).

*Remark* 2.74. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in (1.56), (1.57), (2.25), (2.26), respectively.

Let $(\epsilon,\delta)\in]0,\epsilon_1[\times]0,+\infty[$. We have

$$\bar u_\epsilon^\delta(x)=-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n^a(x-w-\epsilon s)\hat\theta[\epsilon,\delta](s)\,d\sigma_s+\hat\xi[\epsilon,\delta]\qquad\forall x\in\mathbb{T}_a[\Omega_\epsilon].$$

While the relation between equation (2.61) and boundary value problem (2.59) is now clear, we want to see if equation (2.65) is related to some (limiting) boundary value problem. We give the following.

*Remark* 2.75. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in (1.56), (1.57), (2.25), (2.26), respectively. Let $\tilde{\xi}$ be as in Proposition 2.72. Let $\tau$ be as in Definition 2.32. By well known results of classical potential theory (cf. Folland [52, Chapter 3]), we have that $\tilde{\xi}$ is the unique $\xi \in \mathbb{R}$, such that

$$\int_{\partial\Omega} (g(x) - \xi)\tau(x)\, d\sigma_x = 0.$$

Hence,

$$\tilde{\xi} = \int_{\partial\Omega} g(x)\tau(x)\, d\sigma_x.$$

**Definition 2.76.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $g$ be as in (1.56), (2.25), respectively. Let $\tau$ be as in Definition 2.32. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega)$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega, \\ u(x) = g(x) - \int_{\partial\Omega} g(x)\tau(x)\, d\sigma_x & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{2.67}$$

Problem (2.67) will be called the *limiting boundary value problem.*

*Remark* 2.77. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in (1.56), (1.57), (2.25), (2.26), respectively. We have

$$\tilde{u}(x) = \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(y)}\big(S_n(x-y)\big)\hat{\theta}[0,0](y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega.$$

We now prove the following.

**Proposition 2.78.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in* (1.56), (1.57), (2.25), (2.26), *respectively. Let $\mathcal{U}_0^{m,\alpha}$ be as in* (1.64). *Let $\Lambda$ and $(\tilde{\theta}, \tilde{\xi})$ be as in Proposition 2.72. Then there exists $\epsilon_2 \in ]0, \epsilon_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times \mathbb{R} \times \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)$. Moreover, if we set $b_0 \equiv (0,0,\tilde{\theta},\tilde{\xi})$, then the differential $\partial_{(\theta,\xi)}\Lambda[b_0]$ of $\Lambda$ with respect to the variables $(\theta,\xi)$ at $b_0$ is delivered by the following formula*

$$\partial_{(\theta,\xi)}\Lambda[b_0](\bar{\theta},\bar{\xi})(t) = -\frac{1}{2}\bar{\theta}(t) - \int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n(t-s)\bar{\theta}(s)\, d\sigma_s + \bar{\xi} \qquad \forall t \in \partial\Omega, \tag{2.68}$$

*for all $(\bar{\theta},\bar{\xi}) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$, and is a linear homeomorphism of $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ onto $C^{m,\alpha}(\partial\Omega)$.*

*Proof.* By arguing as in the proof of Proposition 2.38, one can show that there exists $\epsilon_2 \in ]0, \epsilon_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times \mathbb{R} \times \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)$. Then by standard calculus in Banach space, we immediately deduce that (2.68) holds. Finally, by Lemma 2.28, $\partial_{(\theta,\xi)}\Lambda[b_0]$ is a linear homeomorphism. □

We are now ready to prove real analytic continuation properties for $\hat{\theta}[\cdot,\cdot], \hat{\xi}[\cdot,\cdot]$.

**Proposition 2.79.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$, $f$ be as in* (1.56), (1.57), (2.25), (2.26), *respectively. Let $\mathcal{U}_0^{m,\alpha}$ be as in* (1.64). *Let $\epsilon_2$ be as in Proposition 2.78. Then there exist $\epsilon_3 \in ]0, \epsilon_2]$, $\delta_1 \in ]0, +\infty[$ and a real analytic operator $(\Theta, \Xi)$ of $]-\epsilon_3, \epsilon_3[ \times ]-\delta_1, \delta_1[$ to $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$, such that*

$$(\Theta[\epsilon,\delta], \Xi[\epsilon,\delta]) = (\hat{\theta}[\epsilon,\delta], \hat{\xi}[\epsilon,\delta]), \tag{2.69}$$

*for all $(\epsilon,\delta) \in (]0,\epsilon_3[ \times ]0,\delta_1[) \cup \{(0,0)\}$.*

*Proof.* It is an immediate consequence of Proposition 2.78 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

By Proposition 2.79, we can deduce the following results.

**Theorem 2.80.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in* (1.56), (2.25), (2.26), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Proposition 2.79. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \epsilon_3]$, a real analytic operator $U_1$ of $]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$ to the space $C^0_h(\mathrm{cl}\, V)$, and a real analytic operator $U_2$ of $]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$ such that the following conditions hold.*

*(i)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(ii)*
$$\bar{u}^\delta_\epsilon(x) = \epsilon^{n-1} U_1[\epsilon,\delta](x) + U_2[\epsilon,\delta] \qquad \forall x \in \mathrm{cl}\, V,$$
*for all* $(\epsilon,\delta) \in ]0,\epsilon_4[ \times ]0,\delta_1[$. *Moreover,*
$$U_2[0,0] = \int_{\partial\Omega} g\tau \, d\sigma,$$
*where $\tau$ is as in Definition 2.32.*

*(iii)*
$$u^\delta_\epsilon(x) = \epsilon^{n-1} U_1[\epsilon,\delta](x) + U_2[\epsilon,\delta] + \delta^2 p_a[f](x) \qquad \forall x \in \mathrm{cl}\, V,$$
*for all* $(\epsilon,\delta) \in ]0,\epsilon_4[ \times ]0,\delta_1[$.

*Proof.* Let $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[$. We have
$$\bar{u}^\delta_\epsilon(x) = -\epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DS^a_n(x - w - \epsilon s)\Theta[\epsilon,\delta](s)\, d\sigma_s + \Xi[\epsilon,\delta] \qquad \forall x \in \mathbb{T}_a[\Omega_\epsilon].$$
Then in order to prove the statements in $(i)$ and $(ii)$ it suffices to follow the proof of Theorem 2.40. Indeed, by choosing $\epsilon_4$ small enough, we can clearly assume that $(i)$ holds. Consider now $(ii)$. As in the proof of Theorem 2.40, it is natural to set
$$U_1[\epsilon,\delta](x) \equiv -\int_{\partial\Omega} \nu_\Omega(s) \cdot DS^a_n(x - w - \epsilon s)\Theta[\epsilon,\delta](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$
$$U_2[\epsilon,\delta] \equiv \Xi[\epsilon,\delta],$$
for all $(\epsilon,\delta) \in ]-\epsilon_4,\epsilon_4[ \times ]-\delta_1,\delta_1[$. By Proposition 2.79, $U_2$ is real analytic. By arguing as in the proof of Proposition 1.24 $(i)$, $U_1[\cdot,\cdot]$ is a real analytic map of $]-\epsilon_4,\epsilon_4[ \times ]-\delta_1,\delta_1[$ to $C^0_h(\mathrm{cl}\, V)$. Finally, by the definition of $U_1$ and $U_2$, the statement in $(ii)$ holds. The statement in $(iii)$ is an immediate consequence of Remark 2.71. □

As far as the energy integral is concerned, we have the following.

**Theorem 2.81.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in* (1.56), (2.25), (2.26), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Proposition 2.79. Then there exist $\epsilon_5 \in ]0,\epsilon_3]$, a real analytic operator $G_1$ of $]-\epsilon_5,\epsilon_5[$ to $\mathbb{R}$ and a real analytic operator $G_2$ of $]-\epsilon_5,\epsilon_5[ \times ]-\delta_1,\delta_1[$ to $\mathbb{R}$, such that*
$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^\delta_\epsilon(x)|^2\, dx = \delta^4 \int_A |\nabla p_a[f](x)|^2\, dx - \delta^4 \epsilon^n G_1[\epsilon] + \epsilon^{n-2} G_2[\epsilon,\delta], \tag{2.70}$$
*for all $\epsilon \in ]0,\epsilon_5[ \times ]0,\delta_1[$. Moreover,*
$$G_1[0] = |\Omega|_n |\nabla p_a[f](w)|^2, \tag{2.71}$$
*and*
$$G_2[0,0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\, dx. \tag{2.72}$$

*Proof.* Let $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[$. Clearly,
$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^\delta_\epsilon(x)|^2\, dx =& \delta^4 \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla p_a[f](x)|^2\, dx + \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla \bar{u}^\delta_\epsilon(x)|^2\, dx \\ &+ 2\delta^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} \nabla \bar{u}^\delta_\epsilon(x) \cdot \nabla p_a[f](x)\, dx.\end{aligned}$$

Obviously,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla p_a[f](x)|^2\,dx = \int_A |\nabla p_a[f](x)|^2\,dx - \int_{\Omega_\epsilon} |\nabla p_a[f](x)|^2\,dx.$$

Then, by arguing as in the proof of Theorem 2.43, we can prove that there exists a real analytic operator $G_1$ of $]-\epsilon_3,\epsilon_3[$ to $\mathbb{R}$ such that

$$\int_{\Omega_\epsilon} |\nabla p_a[f](x)|^2\,dx = \epsilon^n G_1[\epsilon],$$

for all $\epsilon \in ]0,\epsilon_3[$, and

$$G_1[0] = |\Omega|_n |\nabla p_a[f](w)|^2.$$

Consequently, (2.71) holds. Now we need to consider

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla \bar{u}^\delta_\epsilon(x)|^2\,dx + 2\delta^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} \nabla \bar{u}^\delta_\epsilon(x)\cdot \nabla p_a[f](x)\,dx.$$

By arguing as in the proof of Theorem 2.43, we can prove that there exist $\epsilon_5 \in ]0,\epsilon_3]$ and a real analytic operator $G_2$ of $]-\epsilon_5,\epsilon_5[$ to $\mathbb{R}$ such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla \bar{u}^\delta_\epsilon(x)|^2\,dx + 2\delta^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} \nabla \bar{u}^\delta_\epsilon(x)\cdot \nabla p_a[f](x)\,dx = \epsilon^{n-2} G_2[\epsilon,\delta]$$

for all $(\epsilon,\delta) \in ]0,\epsilon_5[\,\times\,]0,\delta_1[$, and

$$G_2[0,0] = \int_{\mathbb{R}^n\setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\,dx.$$

Hence, (2.70), (2.71), and (2.72) follow and the Theorem is completely proved. □

**Theorem 2.82.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in* (1.56), (2.25), (2.26), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Proposition 2.79. Then there exist $\epsilon_6 \in ]0,\epsilon_3]$ and a real analytic operator $J$ of $]-\epsilon_6,\epsilon_6[\,\times\,]-\delta_1,\delta_1[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^\delta_\epsilon(x)\,dx = J[\epsilon,\delta], \tag{2.73}$$

*for all $(\epsilon,\delta) \in ]0,\epsilon_6[\,\times\,]0,\delta_1[$. Moreover,*

$$J[0,0] = \Big(\int_{\partial\Omega} g\tau\,d\sigma\Big)|A|_n, \tag{2.74}$$

*where $\tau$ is as in Definition 2.32.*

*Proof.* Let $(\epsilon,\delta) \in ]0,\epsilon_3[\,\times\,]0,\delta_1[$. We have

$$u^\delta_\epsilon(x) = w_a^-\big[\partial\Omega_\epsilon, \Theta[\epsilon,\delta](\tfrac{1}{\epsilon}(\cdot - w))\big](x) + \Xi[\epsilon,\delta] + \delta^2 p_a[f](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$

As a consequence,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^\delta_\epsilon(x)\,dx = \int_{\mathbb{P}_a[\Omega_\epsilon]} w_a^-\big[\partial\Omega_\epsilon, \Theta[\epsilon,\delta](\tfrac{1}{\epsilon}(\cdot - w))\big](x)\,dx + \int_{\mathbb{P}_a[\Omega_\epsilon]} \Xi[\epsilon,\delta]\,dx + \delta^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} p_a[f](x)\,dx.$$

By arguing as in Lemma 2.45, one can easily show that there exist $\epsilon_6 \in ]0,\epsilon_3]$ and a real analytic operator $\tilde{J}_1$ of $]-\epsilon_6,\epsilon_6[\,\times\,]-\delta_1,\delta_1[$ to $\mathbb{R}$ such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} w_a^-\big[\partial\Omega_\epsilon, \Theta[\epsilon,\delta](\tfrac{1}{\epsilon}(\cdot - w))\big](x)\,dx = \epsilon^n \tilde{J}_1[\epsilon,\delta],$$

for all $(\epsilon,\delta) \in ]0,\epsilon_6[\,\times\,]0,\delta_1[$. On the other hand,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} \Xi[\epsilon,\delta]\,dx = \Xi[\epsilon,\delta]\Big(|A|_n - \epsilon^n|\Omega|_n\Big),$$

for all $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[$. Moreover,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} p_a[f](x)\, dx = \int_A p_a[f](x)\, dx - \int_{\Omega_\epsilon} p_a[f](x)\, dx,$$

for all $\epsilon \in ]0, \epsilon_3[$. By Lemma 2.42, there exists a real analytic operator $\tilde{J}_2$ of $]-\epsilon_3, \epsilon_3[$ to $\mathbb{R}$ such that

$$\int_{\Omega_\epsilon} p_a[f](x)\, dx = \epsilon^n \tilde{J}_2[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_3[$. Thus, if we set

$$J[\epsilon, \delta] \equiv \epsilon^n \tilde{J}_1[\epsilon, \delta] + \Xi[\epsilon, \delta]\big(|A|_n - \epsilon^n |\Omega|_n\big) + \delta^2 \int_A p_a[f](x)\, dx - \delta^2 \epsilon^n \tilde{J}_2[\epsilon],$$

for all $(\epsilon, \delta) \in ]-\epsilon_6, \epsilon_6[ \times ]-\delta_1, \delta_1[$, we have that $J$ is a real analytic map of $]-\epsilon_6, \epsilon_6[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$ such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u_\epsilon^\delta(x)\, dx = J[\epsilon, \delta],$$

for all $(\epsilon, \delta) \in ]0, \epsilon_6[ \times ]0, \delta_1[$. Finally,

$$\begin{aligned} J[0,0] &= \Xi[0,0]|A|_n \\ &= \Big(\int_{\partial\Omega} g\tau\, d\sigma\Big)|A|_n, \end{aligned}$$

and the proof is complete. □

Then we have the following.

**Proposition 2.83.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in* (1.56), (2.25), (2.26), *respectively. Let $1 \le p < \infty$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[\bar{u}_\epsilon^\delta] = \int_{\partial\Omega} g\tau\, d\sigma \qquad \textit{in } L^p(A),$$

*where $\tau$ is as in Definition 2.32.*

*Proof.* By Theorem 2.5, we have

$$|\mathbf{E}_{(\epsilon,1)}[\bar{u}_\epsilon^\delta](x)| \le \sup\{\, |g(t)| \colon t \in \partial\Omega \,\} + \sup\{\, |p_a[f](x)| \colon x \in \mathrm{cl}\, A \,\} < +\infty \qquad \forall x \in A,$$

for all $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, 1[$. By Theorem 2.80, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[\bar{u}_\epsilon^\delta](x) = \int_{\partial\Omega} g\tau\, d\sigma \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[\bar{u}_\epsilon^\delta] = \int_{\partial\Omega} g\tau\, d\sigma \qquad \text{in } L^p(A).$$

□

**Corollary 2.84.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in* (1.56), (2.25), (2.26), *respectively. Let $1 \le p < \infty$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[u_\epsilon^\delta] = \int_{\partial\Omega} g\tau\, d\sigma \qquad \textit{in } L^p(A),$$

*where $\tau$ is as in Definition 2.32.*

*Proof.* Let $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$. We have

$$u_\epsilon^\delta(x) = \bar{u}_\epsilon^\delta(x) + \delta^2 p_a[f](x).$$

Obviously

$$\lim_{\delta\to 0^+} \delta^2 p_a[f] = 0 \qquad \text{in } L^\infty(A).$$

Therefore, by applying Proposition 2.83, we easily conclude. □

### 2.4.2 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorem we deduce by Corollary 2.84 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 2.85.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in* (1.56), (2.25), (2.26), *respectively. Let $\tau$ be as in Definition 2.32. Let $1 \le p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = \int_{\partial\Omega} g\tau\, d\sigma \qquad \text{in } L^p(V).$$

*Proof.* By virtue of Corollary 2.84, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \|\mathbf{E}_{(\epsilon,1)}[u^\delta_\epsilon] - \int_{\partial\Omega} g\tau\, d\sigma\|_{L^p(A)} = 0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $c > 0$ such that

$$\|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] - \int_{\partial\Omega} g\tau\, d\sigma\|_{L^p(V)} \le c\|\mathbf{E}_{(\epsilon,1)}[u^\delta_\epsilon] - \int_{\partial\Omega} g\tau\, d\sigma\|_{L^p(A)} \qquad \forall(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,1[.$$

Thus,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] - \int_{\partial\Omega} g\tau\, d\sigma\|_{L^p(V)} = 0,$$

and the conclusion easily follows. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 2.86.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in* (1.56), (2.25), (2.26), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Proposition 2.79. Let $\epsilon_6$, $J$ be as in Theorem 2.82. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\, dx = r^n J\big[\epsilon, \frac{r}{l}\big], \tag{2.75}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N}\setminus\{0\}$ such that $l > (r/\delta_1)$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_6[$, and let $l \in \mathbb{N}\setminus\{0\}$ be such that $l > (r/\delta_1)$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\, dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\, dx \\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\, dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\, dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\, dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\, dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u^{r/l}_\epsilon\big(\frac{l}{r}x\big)\, dx \\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u^{r/l}_\epsilon(t)\, dt \\ &= \frac{r^n}{l^n} J\big[\epsilon, \frac{r}{l}\big].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\, dx = r^n J\big[\epsilon, \frac{r}{l}\big],$$

and the conclusion follows. □

### 2.4.3 Asymptotic behaviour of the energy integral of $u_{(\epsilon,\delta)}$

This Subsection is devoted to the study of the behaviour of the energy integral of $u_{(\epsilon,\delta)}$. We give the following.

**Definition 2.87.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in (1.56), (2.25), (2.26), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, we set

$$\mathrm{En}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 2.88. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in (1.56), (2.25), (2.26), respectively. Let $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$. We have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2\,dt \\ &= \delta^{n-2}\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^\delta_\epsilon(t)|^2\,dt.\end{aligned}$$

**Definition 2.89.** Let $n \in \mathbb{N}\setminus\{0,1,2\}$. For each $\delta \in ]0,+\infty[$, we set

$$\epsilon[\delta] \equiv \delta^{\frac{2}{n-2}}.$$

Let $\epsilon_5$, $\delta_1$ be as in Theorem 2.81. Let $\delta_2 \in ]0,\delta_1[$ be such that $\epsilon[\delta] \in ]0,\epsilon_5[$, for all $\delta \in ]0,\delta_2[$. Then we set

$$\mathrm{En}[\delta] \equiv \mathrm{En}(\epsilon[\delta],\delta),$$

for all $\delta \in ]0,\delta_2[$.

In the following Proposition we compute the limit of $\mathrm{En}[\delta]$ as $\delta$ tends to 0.

**Proposition 2.90.** *Let $n \in \mathbb{N}\setminus\{0,1,2\}$. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in* (1.56), (2.25), (2.26)*, respectively. Let $\epsilon_5$ be as in Theorem 2.81. Let $\delta_2$ be as in Definition 2.89. Then*

$$\lim_{\delta\to0^+} \mathrm{En}[\delta] = \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\,dx,$$

*where $\tilde{u}$ is as in Definition 2.76.*

*Proof.* Let $\delta \in ]0,\delta_2[$. By Remark 2.88 and Theorem 2.81, we have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2\,dx &= \delta^{n-2}\Big(\delta^4\int_A |\nabla p_a[f](x)|^2\,dx - \delta^4(\epsilon[\delta])^n G_1[\epsilon[\delta]] + (\epsilon[\delta])^{n-2} G_2[\epsilon[\delta],\delta]\Big) \\ &= \delta^n\Big(\delta^2\int_A |\nabla p_a[f](x)|^2\,dx - \delta^2\delta^{\frac{2n}{n-2}} G_1[\delta^{\frac{2}{n-2}}] + G_2[\delta^{\frac{2}{n-2}},\delta]\Big),\end{aligned}$$

where $G_1$, $G_2$ are as in Theorem 2.81. For each $(h_1,h_2) \in ]-\epsilon_5,\epsilon_5[\times]-\delta_1,\delta_1[$, we set

$$G[h_1,h_2] \equiv h_2^2\int_A |\nabla p_a[f](x)|^2\,dx - h_2^2 h_1^n G_1[h_1] + G_2[h_1,h_2].$$

Let $\delta \in ]0,\delta_2[$. We have

$$\lfloor(1/\delta)\rfloor^n \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2\,dx \le \mathrm{En}[\delta] \le \lceil(1/\delta)\rceil^n \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2\,dx,$$

and so

$$\lfloor(1/\delta)\rfloor^n \delta^n G[\delta^{\frac{2}{n-2}},\delta] \le \mathrm{En}[\delta] \le \lceil(1/\delta)\rceil^n \delta^n G[\delta^{\frac{2}{n-2}},\delta].$$

Thus, since

$$\lim_{\delta\to0^+}\lfloor(1/\delta)\rfloor^n\delta^n = 1, \qquad \lim_{\delta\to0^+}\lceil(1/\delta)\rceil^n\delta^n = 1,$$

we have

$$\lim_{\delta\to0^+}\mathrm{En}[\delta] = G_2[0,0].$$

Finally, by equality (2.70), we can easily conclude. □

In the following Proposition we represent the function $\mathrm{En}[\cdot]$ by means of a real analytic function.

**Proposition 2.91.** *Let $n \in \mathbb{N} \setminus \{0,1,2\}$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $g$, $f$ be as in* (1.56), (2.25), (2.26), *respectively. Let $\epsilon_5$ be as in Theorem 2.81. Let $\delta_2 > 0$ be as in Definition 2.89. Then there exists a real analytic operator $G$ of $]-\epsilon_5, \epsilon_5[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$ such that*

$$\mathrm{En}[(1/l)] = G[(1/l)^{\frac{2}{n-2}}, (1/l)],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_2)$.*

*Proof.* It follows by the proof of Proposition 2.90. □

# 2.5 Some remarks about two particular Dirichlet problems for the Laplace equation in a periodically perforated domain

In this Section we study two particular Dirichlet problems for the Laplace equation, that we shall use in the sequel.

## 2.5.1 A particular Dirichlet problem for the Laplace equation in a periodically perforated domain

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. We shall consider also the following assumption.

$$\text{Let } \tilde{\epsilon}_1 \in ]0, \epsilon_1[ \text{ and let } L[\cdot] \text{ be a real analytic map of } ]-\tilde{\epsilon}_1, \tilde{\epsilon}_1[ \text{ to } C^{m,\alpha}(\partial\Omega). \qquad (2.76)$$

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\tilde{\epsilon}_1$, $L$ be as in (2.76). For each $\epsilon \in ]0, \tilde{\epsilon}_1[$, we consider the following periodic Dirichlet problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1, \dots, n\}, \\ u(x) = L[\epsilon]\big(\frac{1}{\epsilon}(x - w)\big) & \forall x \in \partial\Omega_\epsilon. \end{cases} \qquad (2.77)$$

By virtue of Theorem 2.15, we can give the following definition.

**Definition 2.92.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\tilde{\epsilon}_1$, $L$ be as in (2.76). For each $\epsilon \in ]0, \tilde{\epsilon}_1[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of boundary value problem (2.77).

Then we have the following.

**Proposition 2.93.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $\tilde{\epsilon}_1$, $L$ be as in* (2.76). *Let $\mathcal{U}_\epsilon^{m,\alpha}$, $\mathcal{U}_0^{m,\alpha}$ be as in* (1.63), (1.64), *respectively. Let $\Lambda$ be the map of $]-\tilde{\epsilon}_1, \tilde{\epsilon}_1[ \times \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ in $C^{m,\alpha}(\partial\Omega)$ defined by*

$$\begin{aligned} \Lambda[\epsilon, \theta, \xi](t) \equiv & -\frac{1}{2}\theta(t) - \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n(t-s)\theta(s)\, d\sigma_s - \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DR_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s \\ & + \xi - L[\epsilon](t) \qquad \forall t \in \partial\Omega, \end{aligned} \qquad (2.78)$$

*for all $(\epsilon, \theta, \xi) \in ]-\tilde{\epsilon}_1, \tilde{\epsilon}_1[ \times \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$. Then the following statements hold.*

*(i) If $\epsilon \in ]0, \tilde{\epsilon}_1[$, then the pair $(\theta, \xi) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ satisfies equation*

$$\Lambda[\epsilon, \theta, \xi] = 0, \qquad (2.79)$$

*if and only if the pair $(\mu, \xi) \in \mathcal{U}_\epsilon^{m,\alpha} \times \mathbb{R}$, with $\mu \in \mathcal{U}_\epsilon^{m,\alpha}$ defined by*

$$\mu(x) \equiv \theta\big(\frac{1}{\epsilon}(x - w)\big) \qquad \forall x \in \partial\Omega_\epsilon, \qquad (2.80)$$

*satisfies the equation*

$$\Gamma(x) = -\frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(y)}\big(S_n^a(x-y)\big)\mu(y)\,d\sigma_y + \xi \qquad \forall x \in \partial\Omega_\epsilon, \tag{2.81}$$

*with* $\Gamma \in C^{m,\alpha}(\partial\Omega_\epsilon)$ *defined by*

$$\Gamma(x) \equiv L[\epsilon]\big(\frac{1}{\epsilon}(x-w)\big) \qquad \forall x \in \partial\Omega_\epsilon. \tag{2.82}$$

*In particular, equation* (2.79) *has exactly one solution* $(\theta,\xi) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$*, for each* $\epsilon \in ]0,\tilde{\epsilon}_1[$.

*(ii) The pair* $(\theta,\xi) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ *satisfies equation*

$$\Lambda[0,\theta,\xi] = 0, \tag{2.83}$$

*if and only if*

$$L[0](t) = -\frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(s)}(S_n(t-s))\theta(s)\,d\sigma_s + \xi \qquad \forall t \in \partial\Omega. \tag{2.84}$$

*In particular, equation* (2.83) *has exactly one solution* $(\theta,\xi) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$*, which we denote by* $(\tilde{\theta},\tilde{\xi})$.

*Proof.* By arguing exactly so as to prove Proposition 2.29 (i), one can show the validity of the statement in (*i*). Consider (*ii*). As in the proof of Proposition 2.29 (*ii*), the equivalence of (2.83) and (2.84) is obvious. The existence of a unique solution of equation (2.83) is an immediate consequence of Lemma 2.28. □

By Proposition 2.93, it makes sense to introduce the following.

**Definition 2.94.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\mathcal{U}_0^{m,\alpha}$ be as in (1.64). Let $\tilde{\epsilon}_1$, $L$ be as in (2.76). For each $\epsilon \in ]0,\tilde{\epsilon}_1[$, we denote by $(\hat{\theta}[\epsilon],\hat{\xi}[\epsilon])$ the unique pair in $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ that solves (2.79). Analogously, we denote by $(\hat{\theta}[0],\hat{\xi}[0])$ the unique pair in $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ that solves (2.83).

In the following Remark, we show the relation between the solutions of boundary value problem (2.77) and the solutions of equation (2.79).

*Remark* 2.95. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\tilde{\epsilon}_1$, $L$ be as in (2.76).

Let $\epsilon \in ]0,\tilde{\epsilon}_1[$. We have

$$u[\epsilon](x) = -\epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(x-w-\epsilon s)\hat{\theta}[\epsilon](s)\,d\sigma_s + \hat{\xi}[\epsilon] \qquad \forall x \in \mathbb{T}_a[\Omega_\epsilon].$$

While the relation between equation (2.79) and boundary value problem (2.77) is now clear, we want to see if (2.83) is related to some (limiting) boundary value problem. We have the following.

*Remark* 2.96. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\tilde{\epsilon}_1$, $L$ be as in (2.76). Let $\tilde{\xi}$ be as in Proposition 2.93. Let $\tau$ be as in Definition 2.32. By well known results of classical potential theory (cf. Folland [52, Chapter 3]), we have that $\tilde{\xi}$ is the unique $\xi \in \mathbb{R}$, such that

$$\int_{\partial\Omega}(L[0](x)-\xi)\tau(x)\,d\sigma_x = 0.$$

Hence,

$$\tilde{\xi} = \int_{\partial\Omega} L[0](x)\tau(x)\,d\sigma_x.$$

**Definition 2.97.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). Let $\tilde{\epsilon}_1$, $L$ be as in (2.76). Let $\tau$ be as in Definition 2.32. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n\setminus\Omega)$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n\setminus \mathrm{cl}\,\Omega, \\ u(x) = L[0](x) - \int_{\partial\Omega} L[0](x)\tau(x)\,d\sigma_x & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{2.85}$$

Problem (2.85) will be called the *limiting boundary value problem.*

*Remark* 2.98. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\tilde{\epsilon}_1$, $L$ be as in (2.76). We have

$$\tilde{u}(x) = \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(y)} \big(S_n(x-y)\big)\hat{\theta}[0](y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega.$$

We now prove the following.

**Proposition 2.99.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\mathcal{U}_0^{m,\alpha}$ be as in (1.64). Let $\tilde{\epsilon}_1$, $L$ be as in (2.76). Let $\Lambda$ and $(\tilde{\theta},\tilde{\xi})$ be as in Proposition 2.93. Then there exists $\epsilon_2 \in ]0,\tilde{\epsilon}_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2,\epsilon_2[ \times \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)$. Moreover, if we set $b_0 \equiv (0,\tilde{\theta},\tilde{\xi})$, then the differential $\partial_{(\theta,\xi)}\Lambda[b_0]$ of $\Lambda$ with respect to the variables $(\theta,\xi)$ at $b_0$ is delivered by the following formula*

$$\partial_{(\theta,\xi)}\Lambda[b_0](\bar{\theta},\bar{\xi})(t) = -\frac{1}{2}\bar{\theta}(t) - \int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n(t-s)\bar{\theta}(s)\, d\sigma_s + \bar{\xi} \qquad \forall t \in \partial\Omega, \tag{2.86}$$

*for all $(\bar{\theta},\bar{\xi}) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$, and is a linear homeomorphism of $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ onto $C^{m,\alpha}(\partial\Omega)$.*

*Proof.* By the same argument as in the proof of Proposition 2.38, one can show that there exists $\epsilon_2 \in ]0,\tilde{\epsilon}_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2,\epsilon_2[ \times \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)$. Then by standard calculus in Banach space, we immediately deduce that (2.86) holds. Finally, by Lemma 2.28, $\partial_{(\theta,\xi)}\Lambda[b_0]$ is a linear homeomorphism. □

We are now ready to prove real analytic continuation properties for $\hat{\theta}[\cdot], \hat{\xi}[\cdot]$.

**Proposition 2.100.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\mathcal{U}_0^{m,\alpha}$ be as in (1.64). Let $\tilde{\epsilon}_1$, $L$ be as in (2.76). Let $\epsilon_2$ be as in Proposition 2.99. Then there exist $\epsilon_3 \in ]0,\epsilon_2]$ and a real analytic operator $(\Theta,\Xi)$ of $]-\epsilon_3,\epsilon_3[$ to $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$, such that*

$$(\Theta[\epsilon],\Xi[\epsilon]) = (\hat{\theta}[\epsilon],\hat{\xi}[\epsilon]), \tag{2.87}$$

*for all $\epsilon \in [0,\epsilon_3[$.*

*Proof.* It is an immediate consequence of Proposition 2.99 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

By Proposition 2.100 and Remark 2.95, we can deduce the main result of this Section.

**Theorem 2.101.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$ be as in (1.56). Let $\tilde{\epsilon}_1$, $L$ be as in (2.76). Let $\epsilon_3$ be as in Proposition 2.100. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0,\epsilon_3]$, a real analytic operator $U_1$ of $]-\epsilon_4,\epsilon_4[$ to the space $C^0_h(\mathrm{cl}\,V)$, and a real analytic operator $U_2$ of $]-\epsilon_4,\epsilon_4[$ to $\mathbb{R}$ such that the following conditions hold.*

*(i)* $\mathrm{cl}\,V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4,\epsilon_4[$.

*(ii)*

$$u[\epsilon](x) = \epsilon^{n-1}U_1[\epsilon](x) + U_2[\epsilon] \qquad \forall x \in \mathrm{cl}\,V,$$

*for all* $\epsilon \in ]0,\epsilon_4[$.

*Proof.* Let $\epsilon \in ]0,\epsilon_3[$. We have

$$u[\epsilon](x) = -\epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(x-w-\epsilon s)\Theta[\epsilon](s)\, d\sigma_s + \Xi[\epsilon] \qquad \forall x \in \mathbb{T}_a[\Omega_\epsilon].$$

Then in order to prove the Theorem, it suffices to argue as in the proof of Theorem 2.40. Indeed, by choosing $\epsilon_4$ small enough, we can clearly assume that $(i)$ holds. Consider now $(ii)$. As in the proof of Theorem 2.40, it is natural to set

$$U_1[\epsilon](x) \equiv -\int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(x-w-\epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\,V,$$

$$U_2[\epsilon] \equiv \Xi[\epsilon],$$

for all $\epsilon \in ]-\epsilon_4,\epsilon_4[$. By Proposition 2.100, $U_2$ is real analytic. By Proposition 1.24 $(i)$, $U_1[\cdot]$ is a real analytic map of $]-\epsilon_4,\epsilon_4[$ to $C^0_h(\mathrm{cl}\,V)$. Finally, by the definition of $U_1$ and $U_2$, the statement in $(ii)$ holds. □

As done in Theorem 2.101 for $u[\cdot]$, we can now prove a real analytic continuation Theorem for the energy integral. Namely, we prove the following.

**Theorem 2.102.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$ be as in* (1.56). *Let $\tilde{\epsilon}_1$, $L$ be as in* (2.76). *Let $\epsilon_3$ be as in Proposition 2.100. Then there exist $\epsilon_5 \in ]0, \epsilon_3]$ and a real analytic operator $G$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2 \, dx = \epsilon^{n-2} G[\epsilon], \tag{2.88}$$

*for all $\epsilon \in ]0, \epsilon_5[$. Moreover,*

$$G[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx. \tag{2.89}$$

*Proof.* Let $\epsilon \in ]0, \epsilon_3[$. We have

$$u[\epsilon](x) = -\epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n^a(x - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s + \Xi[\epsilon] \qquad \forall x \in \mathbb{T}_a[\Omega_\epsilon].$$

Then in order to prove the Theorem, it suffices to argue as in the part of the proof of Theorem 2.43 concerning $\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla \bar{u}[\epsilon](x)|^2 \, dx$, with $f \equiv 0$ and by replacing $g(\cdot)$ by $L[\epsilon](\cdot)$. □

As done in Theorem 2.102 for the energy integral, we can now prove a real analytic continuation Theorem for the integral of the solution. Namely, we prove the following.

**Theorem 2.103.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$ be as in* (1.56). *Let $\tilde{\epsilon}_1$, $L$ be as in* (2.76). *Let $\epsilon_3$ be as in Proposition 2.100. Then there exist $\epsilon_6 \in ]0, \epsilon_3[$ and a real analytic operator $J$ of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx = J[\epsilon], \tag{2.90}$$

*for all $\epsilon \in ]0, \epsilon_6[$. Moreover,*

$$J[0] = \Big( \int_{\partial\Omega} L[0]\tau \, d\sigma \Big) |A|_n, \tag{2.91}$$

*where $\tau$ is as in Definition 2.32.*

*Proof.* It suffices to follow exactly the same argument of the proof of Theorem 2.46 concerning the integral of $w_a^-\big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) + \Xi[\epsilon]$. □

### 2.5.2 Another particular Dirichlet problem for the Laplace equation in a periodically perforated domain

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. We shall consider also the following assumption.

$$\begin{aligned} &\textit{Let } \tilde{\epsilon}_1 \in ]0, \epsilon_1[,\ \delta_1 \in ]0, +\infty[ \textit{ and let } L[\cdot,\cdot] \textit{ be a real analytic} \\ &\textit{map of } ]-\tilde{\epsilon}_1, \tilde{\epsilon}_1[ \times ]-\delta_1, \delta_1[ \textit{ to } C^{m,\alpha}(\partial\Omega). \end{aligned} \tag{2.92}$$

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\tilde{\epsilon}_1$, $\delta_1$, $L$ be as in (2.92). For each $(\epsilon, \delta) \in ]0, \tilde{\epsilon}_1[ \times ]0, \delta_1[$, we consider the following periodic Dirichlet problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1, \dots, n\}, \\ u(x) = L[\epsilon, \delta]\big(\frac{1}{\epsilon}(x - w)\big) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{2.93}$$

By virtue of Theorem 2.15, we can give the following definition.

**Definition 2.104.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\tilde{\epsilon}_1$, $\delta_1$, $L$ be as in (2.92). For each $(\epsilon, \delta) \in ]0, \tilde{\epsilon}_1[ \times ]0, \delta_1[$, we denote by $u[\epsilon, \delta]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of boundary value problem (2.93).

Then we have the following.

**Proposition 2.105.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $\tilde{\epsilon}_1$, $\delta_1$, $L$ be as in* (2.92). *Let $\mathcal{U}_\epsilon^{m,\alpha}$, $\mathcal{U}_0^{m,\alpha}$ be as in* (1.63), (1.64), *respectively. Let $\Lambda$ be the map of $]-\tilde{\epsilon}_1, \tilde{\epsilon}_1[ \times ]-\delta_1, \delta_1[ \times \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ in $C^{m,\alpha}(\partial\Omega)$ defined by*

$$\Lambda[\epsilon, \delta, \theta, \xi](t) \equiv -\frac{1}{2}\theta(t) - \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n(t-s)\theta(s)\, d\sigma_s - \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DR_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s \\ + \xi - L[\epsilon, \delta](t) \qquad \forall t \in \partial\Omega, \tag{2.94}$$

*for all $(\epsilon, \theta, \xi) \in ]-\tilde{\epsilon}_1, \tilde{\epsilon}_1[ \times ]-\delta_1, \delta_1[ \times \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$. Then the following statements hold.*

*(i) If $(\epsilon, \delta) \in ]0, \tilde{\epsilon}_1[ \times ]0, \delta_1[$, then the pair $(\theta, \xi) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ satisfies equation*

$$\Lambda[\epsilon, \delta, \theta, \xi] = 0, \tag{2.95}$$

*if and only if the pair $(\mu, \xi) \in \mathcal{U}_\epsilon^{m,\alpha} \times \mathbb{R}$, with $\mu \in \mathcal{U}_\epsilon^{m,\alpha}$ defined by*

$$\mu(x) \equiv \theta\big(\frac{1}{\epsilon}(x-w)\big) \qquad \forall x \in \partial\Omega_\epsilon, \tag{2.96}$$

*satisfies the equation*

$$\Gamma(x) = -\frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(y)}\big(S_n^a(x-y)\big)\mu(y)\, d\sigma_y + \xi \qquad \forall x \in \partial\Omega_\epsilon, \tag{2.97}$$

*with $\Gamma \in C^{m,\alpha}(\partial\Omega_\epsilon)$ defined by*

$$\Gamma(x) \equiv L[\epsilon, \delta]\big(\frac{1}{\epsilon}(x-w)\big) \qquad \forall x \in \partial\Omega_\epsilon. \tag{2.98}$$

*In particular, equation* (2.95) *has exactly one solution $(\theta, \xi) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$, for each $(\epsilon, \delta) \in ]0, \tilde{\epsilon}_1[ \times ]0, \delta_1[$.*

*(ii) The pair $(\theta, \xi) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ satisfies equation*

$$\Lambda[0, 0, \theta, \xi] = 0, \tag{2.99}$$

*if and only if*

$$L[0,0](t) = -\frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(s)}(S_n(t-s))\theta(s)\, d\sigma_s + \xi \qquad \forall t \in \partial\Omega. \tag{2.100}$$

*In particular, equation* (2.99) *has exactly one solution $(\theta, \xi) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$, which we denote by $(\tilde{\theta}, \tilde{\xi})$.*

*Proof.* By arguing exactly so as to prove Proposition 2.29 (i), one can show the validity of the statement in (*i*). Consider (*ii*). As in the proof of Proposition 2.29 (*ii*), the equivalence of (2.99) and (2.100) is obvious. The existence of a unique solution of equation (2.99) is an immediate consequence of Lemma 2.28. □

By Proposition 2.105, it makes sense to introduce the following.

**Definition 2.106.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\mathcal{U}_0^{m,\alpha}$ be as in (1.64). Let $\tilde{\epsilon}_1$, $\delta_1$, $L$ be as in (2.92). For each $(\epsilon, \delta) \in ]0, \tilde{\epsilon}_1[ \times ]0, \delta_1[$, we denote by $(\hat{\theta}[\epsilon, \delta], \hat{\xi}[\epsilon, \delta])$ the unique pair in $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ that solves (2.95). Analogously, we denote by $(\hat{\theta}[0,0], \hat{\xi}[0,0])$ the unique pair in $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ that solves (2.99).

In the following Remark, we show the relation between the solutions of boundary value problem (2.93) and the solutions of equation (2.95).

*Remark* 2.107. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\tilde{\epsilon}_1$, $\delta_1$, $L$ be as in (2.92).

Let $(\epsilon, \delta) \in ]0, \tilde{\epsilon}_1[ \times ]0, \delta_1[$. We have

$$u[\epsilon,\delta](x) = -\epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n^a(x - w - \epsilon s)\hat{\theta}[\epsilon,\delta](s)\, d\sigma_s + \hat{\xi}[\epsilon,\delta] \qquad \forall x \in \mathbb{T}_a[\Omega_\epsilon].$$

While the relation between equation (2.95) and boundary value problem (2.93) is now clear, we want to see if (2.99) is related to some (limiting) boundary value problem. We have the following.

*Remark* 2.108. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\tilde{\epsilon}_1$, $\delta_1$, $L$ be as in (2.92). Let $\tilde{\xi}$ be as in Proposition 2.105. Let $\tau$ be as in Definition 2.32. By well known results of classical potential theory (cf. Folland [52, Chapter 3]), we have that $\tilde{\xi}$ is the unique $\xi \in \mathbb{R}$, such that

$$\int_{\partial\Omega} (L[0,0](x) - \xi)\tau(x)\, d\sigma_x = 0.$$

Hence,

$$\tilde{\xi} = \int_{\partial\Omega} L[0,0](x)\tau(x)\, d\sigma_x.$$

**Definition 2.109.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). Let $\tilde{\epsilon}_1$, $\delta_1$, $L$ be as in (2.92). Let $\tau$ be as in Definition 2.32. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega)$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega, \\ u(x) = L[0,0](x) - \int_{\partial\Omega} L[0,0](x)\tau(x)\, d\sigma_x & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{2.101}$$

Problem (2.101) will be called the *limiting boundary value problem.*

*Remark* 2.110. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\tilde{\epsilon}_1$, $\delta_1$, $L$ be as in (2.92). We have

$$\tilde{u}(x) = \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(y)} \big(S_n(x-y)\big)\hat{\theta}[0,0](y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega.$$

We now prove the following.

**Proposition 2.111.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\mathcal{U}_0^{m,\alpha}$ be as in (1.64). Let $\tilde{\epsilon}_1$, $\delta_1$, $L$ be as in (2.92). Let $\Lambda$ and $(\tilde{\theta}, \tilde{\xi})$ be as in Proposition 2.105. Then there exists $\epsilon_2 \in ]0, \tilde{\epsilon}_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times ]-\delta_1, \delta_1[ \times \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)$. Moreover, if we set $b_0 \equiv (0,0,\tilde{\theta},\tilde{\xi})$, then the differential $\partial_{(\theta,\xi)}\Lambda[b_0]$ of $\Lambda$ with respect to the variables $(\theta, \xi)$ at $b_0$ is delivered by the following formula*

$$\partial_{(\theta,\xi)}\Lambda[b_0](\bar{\theta},\bar{\xi})(t) = -\frac{1}{2}\bar{\theta}(t) - \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n(t-s)\bar{\theta}(s)\, d\sigma_s + \bar{\xi} \qquad \forall t \in \partial\Omega, \tag{2.102}$$

*for all $(\bar{\theta}, \bar{\xi}) \in \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$, and is a linear homeomorphism of $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ onto $C^{m,\alpha}(\partial\Omega)$.*

*Proof.* By the same argument as in the proof of Proposition 2.38, one can show that there exists $\epsilon_2 \in ]0, \tilde{\epsilon}_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times ]-\delta_1, \delta_1[ \times \mathcal{U}_0^{m,\alpha} \times \mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)$. Then by standard calculus in Banach space, we immediately deduce that (2.102) holds. Finally, by Lemma 2.28, $\partial_{(\theta,\xi)}\Lambda[b_0]$ is a linear homeomorphism. □

We are now ready to prove that $\hat{\theta}[\cdot,\cdot], \hat{\xi}[\cdot,\cdot]$ can be continued real analytically.

**Proposition 2.112.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\mathcal{U}_0^{m,\alpha}$ be as in (1.64). Let $\tilde{\epsilon}_1$, $\delta_1$, $L$ be as in (2.92). Let $\epsilon_2$ be as in Proposition 2.111. Then there exist $\epsilon_3 \in ]0, \epsilon_2]$, $\delta_2 \in ]0, \delta_1]$ and a real analytic operator $(\Theta, \Xi)$ of $]-\epsilon_3, \epsilon_3[ \times ]-\delta_2, \delta_2[$ to $\mathcal{U}_0^{m,\alpha} \times \mathbb{R}$, such that*

$$(\Theta[\epsilon,\delta], \Xi[\epsilon,\delta]) = (\hat{\theta}[\epsilon,\delta], \hat{\xi}[\epsilon,\delta]), \tag{2.103}$$

*for all $(\epsilon, \delta) \in \big(]0, \epsilon_3[ \times ]0, \delta_2[\big) \cup \{(0,0)\}$.*

*Proof.* It is an immediate consequence of Proposition 2.111 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

By Proposition 2.112 and Remark 2.107, we can deduce the main result of this Section.

**Theorem 2.113.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$ be as in* (1.56)*. Let $\tilde{\epsilon}_1$, $\delta_1$, $L$ be as in* (2.92)*. Let $\epsilon_3$, $\delta_2$ be as in Proposition 2.112. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \epsilon_3]$, a real analytic operator $U_1$ of $]-\epsilon_4, \epsilon_4[ \times ]-\delta_2, \delta_2[$ to the space $C^0_h(\mathrm{cl}\, V)$, and a real analytic operator $U_2$ of $]-\epsilon_4, \epsilon_4[ \times ]-\delta_2, \delta_2[$ to $\mathbb{R}$ such that the following conditions hold.*

*(i)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(ii)*
$$u[\epsilon, \delta](x) = \epsilon^{n-1} U_1[\epsilon, \delta](x) + U_2[\epsilon, \delta] \qquad \forall x \in \mathrm{cl}\, V,$$
*for all* $(\epsilon, \delta) \in ]0, \epsilon_4[ \times ]0, \delta_2[$.

*Proof.* Let $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_2[$. We have

$$u[\epsilon, \delta](x) = -\epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DS^a_n(x - w - \epsilon s) \Theta[\epsilon, \delta](s)\, d\sigma_s + \Xi[\epsilon, \delta] \qquad \forall x \in \mathbb{T}_a[\Omega_\epsilon].$$

Then in order to prove the Theorem, it suffices to argue as in the proof of Theorem 2.40. Indeed, by choosing $\epsilon_4$ small enough, we can clearly assume that $(i)$ holds. Consider now $(ii)$. As in the proof of Theorem 2.40, it is natural to set

$$U_1[\epsilon, \delta](x) \equiv -\int_{\partial\Omega} \nu_\Omega(s) \cdot DS^a_n(x - w - \epsilon s) \Theta[\epsilon, \delta](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

$$U_2[\epsilon, \delta] \equiv \Xi[\epsilon, \delta],$$

for all $(\epsilon, \delta) \in ]-\epsilon_4, \epsilon_4[ \times ]-\delta_2, \delta_2[$. By Proposition 2.112, $U_2$ is real analytic. By arguing as in the proof of Proposition 1.24 $(i)$, $U_1[\cdot, \cdot]$ is a real analytic map of $]-\epsilon_4, \epsilon_4[ \times ]-\delta_2, \delta_2[$ to $C^0_h(\mathrm{cl}\, V)$. Finally, by the definition of $U_1$ and $U_2$, the statement in $(ii)$ holds. □

As done in Theorem 2.113 for $u[\cdot, \cdot]$, we can now prove a real analytic continuation Theorem for the energy integral. Namely, we prove the following.

**Theorem 2.114.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$ be as in* (1.56)*. Let $\tilde{\epsilon}_1$, $\delta_1$, $L$ be as in* (2.92)*. Let $\epsilon_3$, $\delta_2$ be as in Proposition 2.112. Then there exist $\epsilon_5 \in ]0, \epsilon_3]$ and a real analytic operator $G$ of $]-\epsilon_5, \epsilon_5[ \times ]-\delta_2, \delta_2[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon, \delta](x)|^2\, dx = \epsilon^{n-2} G[\epsilon, \delta], \tag{2.104}$$

*for all* $(\epsilon, \delta) \in ]0, \epsilon_5[ \times ]0, \delta_2[$. *Moreover,*

$$G[0,0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\, \Omega} |\nabla \tilde{u}(x)|^2\, dx. \tag{2.105}$$

*Proof.* Let $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_2[$. We have

$$u[\epsilon, \delta](x) = -\epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DS^a_n(x - w - \epsilon s) \Theta[\epsilon, \delta](s)\, d\sigma_s + \Xi[\epsilon, \delta] \qquad \forall x \in \mathbb{T}_a[\Omega_\epsilon].$$

Then in order to prove the Theorem, it suffices to argue as in the part of the proof of Theorem 2.43 concerning $\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla \bar{u}[\epsilon](x)|^2\, dx$, with $f \equiv 0$ and by replacing $g(\cdot)$ by $L[\epsilon, \delta](\cdot)$. □

As done in Theorem 2.114 for the energy integral, we can now prove a real analytic continuation Theorem for the integral of the solution. Namely, we prove the following.

**Theorem 2.115.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\Omega$ *be as in* (1.56). *Let* $\tilde{\epsilon}_1$, $\delta_1$, $L$ *be as in* (2.92). *Let* $\epsilon_3$, $\delta_2$ *be as in Proposition 2.112. Then there exist* $\epsilon_6 \in ]0, \epsilon_3[$ *and a real analytic operator* $J$ *of* $]-\epsilon_6, \epsilon_6[ \times ]-\delta_2, \delta_2[$ *to* $\mathbb{R}$, *such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,\delta](x)\,dx = J[\epsilon,\delta], \tag{2.106}$$

*for all* $(\epsilon,\delta) \in ]0,\epsilon_6[ \times ]0,\delta_2[$. *Moreover,*

$$J[0,0] = \Big(\int_{\partial\Omega} L[0,0]\tau\,d\sigma\Big)|A|_n, \tag{2.107}$$

*where* $\tau$ *is as in Definition 2.32.*

*Proof.* Let $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_2[$. We have

$$u[\epsilon,\delta](x) = -\epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(x-w-\epsilon s)\Theta[\epsilon,\delta](s)\,d\sigma_s + \Xi[\epsilon,\delta] \qquad \forall x \in \mathbb{T}_a[\Omega_\epsilon].$$

Then in order to prove the Theorem, it suffices to follow exactly the same argument of the proof of Theorem 2.46 concerning the integral of $w_a^-\big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) + \Xi[\epsilon]$. □

## 2.6 Asymptotic behaviour of the solutions of the Neumann problem for the Laplace equation in a periodically perforated domain

In this Section we study the asymptotic behaviour of the solutions of the Neumann problem for the Laplace equation in a periodically perforated domain with small holes.

### 2.6.1 Notation

We retain the notation introduced in Subsection 1.8.1.

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). We shall consider also the following assumptions.

$$g \in C^{m-1,\alpha}(\partial\Omega), \qquad \int_{\partial\Omega} g\,d\sigma = 0, \tag{2.108}$$

$$\bar{c} \in \mathbb{R}. \tag{2.109}$$

### 2.6.2 Preliminaries

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.58), (2.108), (2.109), respectively. For each $\epsilon \in ]0, \bar{\epsilon}_1[$, we consider the following periodic Neumann problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1,\dots,n\}, \\ \frac{\partial}{\partial\nu_{\Omega_\epsilon}}u(x) = g\big(\frac{1}{\epsilon}(x-w)\big) & \forall x \in \partial\Omega_\epsilon, \\ u(\bar{x}) = \bar{c}. \end{cases} \tag{2.110}$$

By virtue of Theorem 2.17, we can give the following definition.

**Definition 2.116.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.58), (2.108), (2.109), respectively. For each $\epsilon \in ]0, \bar{\epsilon}_1[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of boundary value problem (2.110).

Our aim is to investigate the behaviour of $u[\epsilon]$ as $\epsilon$ tends to 0.

Since we want to represent the function $u[\epsilon]$ by means of a periodic simple layer potential and a constant (cf. Theorem 2.17), we need to study some integral equations. Indeed, by virtue of Theorem 2.17, we can transform (2.110) into an integral equation, whose unknown is the moment of the simple layer potential. Moreover, we want to transform these equations defined on the $\epsilon$-dependent domain $\partial\Omega_\epsilon$ into equations defined on the fixed domain $\partial\Omega$. We introduce these integral equations in the following Proposition. The relation between the solution of the integral equations and the solution of boundary value problem (2.110) will be clarified later.

**Proposition 2.117.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.108), *respectively. Let $\mathcal{U}_\epsilon^{m-1,\alpha}$, $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.63), (1.64), *respectively. Let $\Lambda$ be the map of $]-\epsilon_1, \epsilon_1[ \times C^{m-1,\alpha}(\partial\Omega)$ in $C^{m-1,\alpha}(\partial\Omega)$ defined by*

$$\begin{aligned}\Lambda[\epsilon,\theta](t) \equiv & \frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\theta(s)\, d\sigma_s \\ & + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s - g(t) \qquad \forall t \in \partial\Omega,\end{aligned} \tag{2.111}$$

*for all $(\epsilon,\theta) \in ]-\epsilon_1,\epsilon_1[ \times C^{m-1,\alpha}(\partial\Omega)$. Then the following statements hold.*

*(i) If $\epsilon \in ]0,\epsilon_1[$, then the function $\theta \in C^{m-1,\alpha}(\partial\Omega)$ satisfies equation*

$$\Lambda[\epsilon,\theta] = 0, \tag{2.112}$$

*if and only if the function $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon)$, defined by*

$$\mu(x) \equiv \theta\big(\frac{1}{\epsilon}(x-w)\big) \qquad \forall x \in \partial\Omega_\epsilon, \tag{2.113}$$

*satisfies the equation*

$$\Gamma(x) = \frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}\big(S_n^a(x-y)\big)\mu(y)\, d\sigma_y \qquad \forall x \in \partial\Omega_\epsilon, \tag{2.114}$$

*with $\Gamma \in C^{m-1,\alpha}(\partial\Omega_\epsilon)$ defined by*

$$\Gamma(x) \equiv g\big(\frac{1}{\epsilon}(x-w)\big) \qquad \forall x \in \partial\Omega_\epsilon. \tag{2.115}$$

*In particular, equation* (2.112) *has exactly one solution $\theta \in C^{m-1,\alpha}(\partial\Omega)$, for each $\epsilon \in ]0,\epsilon_1[$. Moreover, if $\theta$ solves* (2.112), *then $\theta \in \mathcal{U}_0^{m-1,\alpha}$, and so also $\theta(\frac{1}{\epsilon}(\cdot - w)) \in \mathcal{U}_\epsilon^{m-1,\alpha}$.*

*(ii) The function $\theta \in C^{m-1,\alpha}(\partial\Omega)$ satisfies equation*

$$\Lambda[0,\theta] = 0, \tag{2.116}$$

*if and only if*

$$g(t) = \frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\, d\sigma_s \qquad \forall t \in \partial\Omega. \tag{2.117}$$

*In particular, equation* (2.116) *has exactly one solution $\theta \in C^{m-1,\alpha}(\partial\Omega)$, which we denote by $\tilde{\theta}$. Moreover, if $\theta$ solves* (2.116), *then $\theta \in \mathcal{U}_0^{m-1,\alpha}$.*

*Proof.* Consider $(i)$. Let $\theta \in C^{m-1,\alpha}(\partial\Omega)$. Let $\epsilon \in ]0,\epsilon_1[$. First of all, we note that

$$\int_{\partial\Omega_\epsilon} \theta\big(\frac{1}{\epsilon}(x-w)\big)\, d\sigma_x = \epsilon^{n-1} \int_{\partial\Omega} \theta(t)\, d\sigma_t,$$

and so $\theta \in \mathcal{U}_0^{m-1,\alpha}$ if and only if $\theta(\frac{1}{\epsilon}(\cdot - w)) \in \mathcal{U}_\epsilon^{m-1,\alpha}$. The equivalence of equation (2.112) in the unknown $\theta \in C^{m-1,\alpha}(\partial\Omega)$ and equation (2.114) in the unknown $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon)$ follows by a straightforward computation based on the rule of change of variables in integrals and of well known properties of composition of functions in Schauder spaces (cf. *e.g.*, Lanza [67, Sections 3,4].) The existence and uniqueness of a solution of equation (2.114) follows by Proposition 2.14 $(iii)$. Then the existence and uniqueness of a solution of equation (2.112) follows by the equivalence of (2.112) and (2.114). Now, if $\theta \in C^{m-1,\alpha}(\partial\Omega)$ solves equation (2.112), then the function $\mu$, defined as in (2.113), solves equation (2.114). Since $\int_{\partial\Omega} g\, d\sigma = 0$, then $\int_{\partial\Omega_\epsilon} \Gamma\, d\sigma = 0$. By Lemma 2.10, then $\mu \in \mathcal{U}_\epsilon^{m-1,\alpha}$, and, consequently, $\theta \in \mathcal{U}_0^{m-1,\alpha}$. Consider $(ii)$. The equivalence of (2.116) and (2.117) is obvious. The existence of a unique solution $\theta \in C^{m-1,\alpha}(\partial\Omega)$ of equation (2.116) follows by well known results of classical potential theory (cf. Folland [52, Chapter 3].) Moreover, since $\int_{\partial\Omega} g\, d\sigma = 0$, by Folland [52, Lemma 3.30, p. 133], we have $\theta \in \mathcal{U}_0^{m-1,\alpha}$. □

By Proposition 2.117, it makes sense to introduce the following.

**Definition 2.118.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. For each $\epsilon \in ]0,\epsilon_1[$, we denote by $\hat{\theta}[\epsilon]$ the unique function in $C^{m-1,\alpha}(\partial\Omega)$ that solves (2.112). Analogously, we denote by $\hat{\theta}[0]$ the unique function in $C^{m-1,\alpha}(\partial\Omega)$ that solves (2.116).

In the following Remark, we show the relation between the solutions of boundary value problem (2.110) and the solutions of equation (2.112).

*Remark* 2.119. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \operatorname{cl} A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.58), (2.108), (2.109), respectively. Let $\epsilon \in ]0,\bar{\epsilon}_1[$. We have

$$u[\epsilon](x) = \epsilon^{n-1} \int_{\partial\Omega} S_n^a(x - w - \epsilon s)\hat{\theta}[\epsilon](s)\, d\sigma_s + \bar{c} - \epsilon^{n-1} \int_{\partial\Omega} S_n^a(\bar{x} - w - \epsilon s)\hat{\theta}[\epsilon](s)\, d\sigma_s \quad \forall x \in \operatorname{cl} \mathbb{T}_a[\Omega_\epsilon].$$

While the relation between equation (2.112) and boundary value problem (2.110) is now clear, we want to see if (2.116) is related to some (limiting) boundary value problem. We give the following.

**Definition 2.120.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $g$ be as in (1.56), (2.108), respectively. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega)$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \operatorname{cl}\Omega, \\ \frac{\partial}{\partial \nu_\Omega} u(x) = g(x) & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{2.118}$$

Problem (2.118) will be called the *limiting boundary value problem.*

*Remark* 2.121. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. We have

$$\tilde{u}(x) = \int_{\partial\Omega} S_n(x-y)\hat{\theta}[0](y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \Omega.$$

We now prove the following.

**Proposition 2.122.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.108)*, respectively. Let $\Lambda$ and $\tilde{\theta}$ be as in Proposition 2.117. Then there exists $\epsilon_2 \in ]0,\epsilon_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2,\epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$. Moreover, if we set $b_0 \equiv (0,\tilde{\theta})$, then the differential $\partial_\theta\Lambda[b_0]$ of $\Lambda$ with respect to the variable $\theta$ at $b_0$ is delivered by the following formula*

$$\partial_\theta\Lambda[b_0](\bar{\theta})(t) = \frac{1}{2}\bar{\theta}(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\bar{\theta}(s)\, d\sigma_s \qquad \forall t \in \partial\Omega, \tag{2.119}$$

*for all $\bar{\theta} \in C^{m-1,\alpha}(\partial\Omega)$, and is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega)$ onto $C^{m-1,\alpha}(\partial\Omega)$.*

*Proof.* By Proposition 1.26 (*ii*) and standard calculus in Banach spaces, we immediately deduce that there exists $\epsilon_2 \in ]0,\epsilon_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2,\epsilon_2[\times C^{m-1,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$. By standard calculus in Banach space, we immediately deduce that (2.119) holds. Finally, since $\mathbb{R}^n \setminus \operatorname{cl}\Omega$ is connected, by classical potential theory (cf. Folland [52, Chapter 3, Section E]), we have that $\partial_\theta\Lambda[b_0]$ is a linear and continuous bijection of $C^{m-1,\alpha}(\partial\Omega)$ onto itself, and so, by the Open Mapping Theorem, is a linear homeomorphism. □

We are now ready to show that $\hat{\theta}[\cdot]$ can be continued real analytically around 0.

**Proposition 2.123.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.108)*, respectively. Let $\epsilon_2$ be as in Proposition 2.122. Then there exist $\epsilon_3 \in ]0,\epsilon_2]$ and a real analytic operator $\Theta$ of $]-\epsilon_3,\epsilon_3[$ to $C^{m-1,\alpha}(\partial\Omega)$, such that*

$$\Theta[\epsilon] = \hat{\theta}[\epsilon], \tag{2.120}$$

*for all $\epsilon \in [0,\epsilon_3[$.*

*Proof.* It is an immediate consequence of Proposition 2.122 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

### 2.6.3 A functional analytic representation Theorem for the solution of the singularly perturbed Neumann problem

By Proposition 2.123 and Remark 2.119, we can deduce the main result of this Section. Namely, we show that $\{u[\epsilon](\cdot)\}_{\epsilon\in]0,\epsilon_1[}$ can be continued real analytically for negative values of $\epsilon$.

**Theorem 2.124.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let $\epsilon_3$ be as in Proposition 2.123. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \min\{\bar{\epsilon}_1, \epsilon_3\}]$, a real analytic operator $U_1$ of $]-\epsilon_4, \epsilon_4[$ to the space $C^0_h(\mathrm{cl}\, V)$, and a real analytic operator $U_2$ of $]-\epsilon_4,\epsilon_4[$ to $\mathbb{R}$ such that the following conditions hold.*

*(i)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(ii)*
$$u[\epsilon](x) = \epsilon^n U_1[\epsilon](x) + U_2[\epsilon] \qquad \forall x \in \mathrm{cl}\, V,$$
*for all $\epsilon \in ]0, \epsilon_4[$. Moreover,*
$$U_2[0] = \bar{c}.$$

*Proof.* Let $\Theta[\cdot]$ be as in Proposition 2.123. Choosing $\epsilon_4$ small enough, we can clearly assume that (*i*) holds. Consider now (*ii*). Let $\epsilon \in ]0, \epsilon_4[$. By Remark 2.119 and Proposition 2.123, we have

$$u[\epsilon](x) = \epsilon^{n-1} \int_{\partial\Omega} S^a_n(x - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s + \bar{c} - \epsilon^{n-1} \int_{\partial\Omega} S^a_n(\bar{x} - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V.$$

Thus, it is natural to set

$$\tilde{U}_1[\epsilon](x) \equiv \int_{\partial\Omega} S^a_n(x - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$, and

$$U_2[\epsilon] \equiv \bar{c} - \epsilon^{n-1} \int_{\partial\Omega} S^a_n(\bar{x} - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s,$$

for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$. By Proposition 1.29 (*i*), possibly taking a smaller $\epsilon_4$, there exists a real analytic map $U_1$ of $]-\epsilon_4, \epsilon_4[$ to $C^0_h(\mathrm{cl}\, V)$ such that

$$\tilde{U}_1[\epsilon] = \epsilon U_1[\epsilon] \qquad \text{in } C^0_h(\mathrm{cl}\, V),$$

for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$. By Proposition 1.26 (*iii*), possibly choosing a smaller $\epsilon_4$, we have that $U_2$ is a real analytic map of $]-\epsilon_4, \epsilon_4[$ to $\mathbb{R}$. Finally, by the definition of $U_1$ and $U_2$, the statement in (*ii*) holds. □

*Remark* 2.125. We note that the right-hand side of the equality in (*ii*) of Theorem 2.124 can be continued real analytically in the whole $]-\epsilon_4, \epsilon_4[$. Moreover, if $V$ is a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$, then

$$\lim_{\epsilon \to 0^+} u[\epsilon] = \bar{c} \qquad \text{uniformly in } \mathrm{cl}\, V.$$

### 2.6.4 A real analytic continuation Theorem for the energy integral

As done in Theorem 2.124 for $u[\cdot]$, we can now prove a real analytic continuation Theorem for the energy integral. Namely, we prove the following.

**Theorem 2.126.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let $\epsilon_3$ be as in Proposition 2.123. Then there exist $\epsilon_5 \in ]0, \min\{\bar{\epsilon}_1, \epsilon_3\}]$ and a real analytic operator $G$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx = \epsilon^n G[\epsilon], \tag{2.121}$$

*for all $\epsilon \in ]0, \epsilon_5[$. Moreover,*

$$G[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\, \Omega} |\nabla \tilde{u}(x)|^2\, dx. \tag{2.122}$$

*Proof.* Let $\Theta[\cdot]$ be as in Proposition 2.123. Let $\epsilon \in ]0, \min\{\bar{\epsilon}_1, \epsilon_3\}[$. Clearly,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2 \, dx = \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2 \, dx.$$

Let id denote the identity map in $\mathbb{R}^n$. By Green's Formula and by the periodicity of the periodic single layer potential $v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))]$, we have

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2 \, dx = -\epsilon^{n-1} \int_{\partial\Omega} g(t) v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))] \circ (w + \epsilon \, \mathrm{id})(t) \, d\sigma_t.$$

By Proposition 1.28 $(i)$, since $\Theta[\epsilon] \in \mathcal{U}_0^{m-1,\alpha}$, we have

$$\begin{aligned} & v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))] \circ (w + \epsilon \, \mathrm{id})(t) \\ & \quad = \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s) \, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s) \, d\sigma_s \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $\epsilon \in ]0, \epsilon_3[$. Thus, it is natural to set

$$G[\epsilon] = -\int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s) \, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s) \, d\sigma_s \Big) g(t) \, d\sigma_t,$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. By definition of $G[\cdot]$, we have

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2 \, dx = \epsilon^n G[\epsilon]$$

for all $\epsilon \in ]0, \epsilon_3[$ and so (2.121) follows. Moreover,

$$G[0] = -\int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[0](s) \, d\sigma_s + \delta_{n,2} \int_{\partial\Omega} R_n^a(0)\Theta[0](s) \, d\sigma_s \Big) g(t) \, d\sigma_t,$$

where $\delta_{n,2} = 1$ if $n = 2$, and $\delta_{n,2} = 0$ if $n \geq 3$. Since $\Theta[0] \in \mathcal{U}_0^{m-1,\alpha}$, we have

$$G[0] = -\int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[0](s) \, d\sigma_s \Big) g(t) \, d\sigma_t,$$

and so, by Remark 2.121,

$$G[0] = -\int_{\partial\Omega} \tilde{u}(t) g(t) \, d\sigma_t.$$

By Folland [52, p. 118], we have

$$-\int_{\partial\Omega} \tilde{u}(t) g(t) \, d\sigma_t = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx,$$

and accordingly

$$G[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx,$$

and (2.122) holds. Now we need to prove the real analyticity of $G[\cdot]$. By continuity of the pointwise product in Schauder spaces, standard calculus in Banach spaces and Proposition 1.28 $(i)$, we immediately deduce that there exists $\epsilon_5 \in ]0, \epsilon_3]$ such that the map $G$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{R}$ is real analytic. Thus, the Theorem is completely proved. □

*Remark* 2.127. We note that the right-hand side of the equality in (2.121) of Theorem 2.126 can be continued real analytically in the whole $]-\epsilon_5, \epsilon_5[$. Moreover,

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2 \, dx = 0.$$

### 2.6.5 A real analytic continuation Theorem for the integral of the solution

As done in Theorem 2.126 for the energy integral, we can now prove a real analytic continuation Theorem for the integral of the solution. Namely, we prove the following.

**Theorem 2.128.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. *Let* $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ *be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let* $\epsilon_3$ *be as in Proposition 2.123. Then there exist* $\epsilon_6 \in ]0, \min\{\bar{\epsilon}_1, \epsilon_3\}]$ *and a real analytic operator* $J$ *of* $]-\epsilon_6, \epsilon_6[$ *to* $\mathbb{R}$, *such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx = J[\epsilon], \tag{2.123}$$

*for all* $\epsilon \in ]0, \epsilon_6[$. *Moreover,*

$$J[0] = \bar{c}|A|_n. \tag{2.124}$$

*Proof.* Let $\Theta[\cdot]$ be as in Proposition 2.123. Let $\epsilon \in ]0, \min\{\bar{\epsilon}_1, \epsilon_3\}[$. Since

$$u[\epsilon](x) = v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) + \bar{c} - v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](\bar{x}) \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon],$$

then

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx = &\int_{\mathbb{P}_a[\Omega_\epsilon]} \Big\{ v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) - v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](\bar{x}) \Big\}\, dx \\ &+ \bar{c}\big(|A|_n - \epsilon^n |\Omega|_n\big).\end{aligned}$$

On the other hand, by arguing as in the proof of Theorem 2.126, we note that

$$\begin{aligned}&v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](w + \epsilon t) \\ &\qquad = \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \partial\Omega,\end{aligned}$$

and that

$$v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](\bar{x}) = \epsilon^{n-1} \int_{\partial\Omega} S_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s.$$

Then, if we set

$$\begin{aligned}L[\epsilon](t) \equiv &\epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ &- \epsilon^{n-1} \int_{\partial\Omega} S_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \partial\Omega,\end{aligned}$$

for all $\epsilon \in ]-\min\{\epsilon_3, \bar{\epsilon}_1\}, \min\{\epsilon_3, \bar{\epsilon}_1\}[$, we have that there exists $\tilde{\epsilon}_1 \in ]0, \min\{\epsilon_3, \bar{\epsilon}_1\}]$ such that $L[\cdot]$ is a real analytic map of $]-\tilde{\epsilon}_1, \tilde{\epsilon}_1[$ to $C^{m,\alpha}(\partial\Omega)$. In particular, $L[0](t) = 0$ for all $t \in \partial\Omega$. Then, by Theorem 2.103, we easily deduce that there exists $\epsilon_6 \in ]0, \tilde{\epsilon}_1]$ and a real analytic map $J_1$ of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} \Big\{ v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) - v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](\bar{x}) \Big\}\, dx = J_1[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_6[$. Moreover, $J_1[0] = 0$. Then, if we set

$$J[\epsilon] \equiv J_1[\epsilon] + \bar{c}\big(|A|_n - \epsilon^n |\Omega|_n\big),$$

for all $\epsilon \in ]-\epsilon_6, \epsilon_6[$, we can immediately conclude. □

## 2.7 An homogenization problem for the Laplace equation with Neumann boundary conditions in a periodically perforated domain

In this section we consider an homogenization problem for the Laplace equation with Neumann boundary conditions in a periodically perforated domain.

### 2.7.1 Notation

In this Section we retain the notation introduced in Subsections 1.8.1, 2.6.1. However, we need to introduce also some other notation.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $(\epsilon,\delta) \in (]-\epsilon_1,\epsilon_1[ \setminus \{0\}) \times ]0,+\infty[$. If $v$ is a function of $\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta)$ to $\mathbb{R}$, then we denote by $\mathbf{E}_{(\epsilon,\delta)}[v]$ the function of $\mathbb{R}^n$ to $\mathbb{R}$, defined by

$$\mathbf{E}_{(\epsilon,\delta)}[v](x) \equiv \begin{cases} v(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \\ 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta). \end{cases}$$

If $v$ is a function of $\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta)$ to $\mathbb{R}$ and $c \in \mathbb{R}$, then we denote by $\mathbf{E}^{\#}_{(\epsilon,\delta)}[v,c]$ the function of $\mathbb{R}^n$ to $\mathbb{R}$, defined by

$$\mathbf{E}^{\#}_{(\epsilon,\delta)}[v,c](x) \equiv \begin{cases} v(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \\ c & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta). \end{cases}$$

### 2.7.2 Preliminaries

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.57), (1.58), (2.108), (2.109), respectively. For each $(\epsilon,\delta) \in ]0,\bar{\epsilon}_1[ \times ]0,+\infty[$, we consider the following periodic Neumann problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u(x+\delta a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall i \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x) = \frac{1}{\delta} g(\frac{1}{\epsilon\delta}(x-\delta w)) & \forall x \in \partial\Omega(\epsilon,\delta), \\ u(\delta\bar{x}) = \bar{c}. \end{cases} \tag{2.125}$$

By virtue of Theorem 2.17, we can give the following definition.

**Definition 2.129.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.57), (1.58), (2.108), (2.109), respectively. For each pair $(\epsilon,\delta) \in ]0,\bar{\epsilon}_1[ \times ]0,+\infty[$, we denote by $u_{(\epsilon,\delta)}$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta))$ of boundary value problem (2.125).

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. In order to do so we introduce the following.

**Definition 2.130.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.57), (1.58), (2.108), (2.109), respectively. For each $\epsilon \in ]0,\bar{\epsilon}_1[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of the following periodic Neumann problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) = g\big(\frac{1}{\epsilon}(x-w)\big) & \forall x \in \partial\Omega_\epsilon, \\ u(\bar{x}) = \bar{c}. \end{cases} \tag{2.126}$$

*Remark* 2.131. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.57), (1.58), (2.108), (2.109), respectively. For each pair $(\epsilon,\delta) \in ]0,\bar{\epsilon}_1[ \times ]0,+\infty[$, we have

$$u_{(\epsilon,\delta)}(x) = u[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).$$

By the previous remark, we note that the solution of problem (2.125) can be expressed by means of the solution of the auxiliary rescaled problem (2.126), which does not depend on $\delta$. This is due to the presence of the factor $1/\delta$ in front of $g(\frac{1}{\epsilon\delta}(x-\delta w))$ in the third equation of problem (2.125).

As a first step, we study the behaviour of (suitable extensions of) $u[\epsilon]$ as $\epsilon$ tends to 0.

**Proposition 2.132.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let $1 \le p < \infty$. Then*

$$\lim_{\epsilon \to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = \bar{c} \qquad \textit{in } L^p(A).$$

*Proof.* Let $\epsilon_3$, $\Theta$ be as in Proposition 2.123. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $\epsilon \in ]0, \min\{\bar{\epsilon}_1, \epsilon_3\}[$, we have

$$\begin{aligned} u[\epsilon] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) =& \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ &+ \bar{c} - \epsilon^{n-1} \int_{\partial\Omega} S_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \partial\Omega. \end{aligned}$$

We set

$$\begin{aligned} N[\epsilon](t) \equiv& \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ &+ \bar{c} - \epsilon^{n-1} \int_{\partial\Omega} S_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $\epsilon \in ]-\min\{\bar{\epsilon}_1, \epsilon_3\}, \min\{\bar{\epsilon}_1, \epsilon_3\}[$. By taking $\tilde{\epsilon} \in ]0, \min\{\bar{\epsilon}_1, \epsilon_3\}[$ small enough, we can assume (cf. Proposition 1.26 $(i)$, $(iii)$) that $N$ is a real analytic map of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C \equiv \sup_{\epsilon \in ]-\tilde{\epsilon}, \tilde{\epsilon}[} \|N[\epsilon]\|_{C^0(\partial\Omega)} < +\infty.$$

By Theorem 2.5, we have

$$|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]](x)| \leq C \qquad \forall x \in A, \quad \forall \epsilon \in ]0, \tilde{\epsilon}[.$$

By Theorem 2.124, we have

$$\lim_{\epsilon \to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]](x) = \bar{c} \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{\epsilon \to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = \bar{c} \qquad \text{in } L^p(A).$$

□

**Proposition 2.133.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let $\epsilon_3$ be as in Proposition 2.123. Then there exist $\tilde{\epsilon} \in ]0, \min\{\bar{\epsilon}_1, \epsilon_3\}[$ and a real analytic map $N$ of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ such that*

$$\|\mathbf{E}^{\#}_{(\epsilon,1)}[u[\epsilon], \bar{c}] - \bar{c}\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all $\epsilon \in ]0, \tilde{\epsilon}[$. Moreover, as a consequence,*

$$\lim_{\epsilon \to 0^+} \mathbf{E}^{\#}_{(\epsilon,1)}[u[\epsilon], \bar{c}] = \bar{c} \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* Let $\epsilon_3$, $\Theta$ be as in Proposition 2.123. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $\epsilon \in ]0, \min\{\bar{\epsilon}_1, \epsilon_3\}[$, we have

$$\begin{aligned} u[\epsilon] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) - \bar{c} =& \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ &- \epsilon^{n-1} \int_{\partial\Omega} S_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \partial\Omega. \end{aligned}$$

We set

$$\begin{aligned} N[\epsilon](t) \equiv& \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ &- \epsilon^{n-2} \int_{\partial\Omega} S_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $\epsilon \in ]-\min\{\bar{\epsilon}_1, \epsilon_3\}, \min\{\bar{\epsilon}_1, \epsilon_3\}[$. By taking $\tilde{\epsilon} \in ]0, \min\{\bar{\epsilon}_1, \epsilon_3\}[$ small enough, we can assume (cf. Proposition 1.26 $(i)$, $(iii)$) that $N$ is a real analytic map of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$.

By Theorem 2.5, we have

$$\|\mathbf{E}^{\#}_{(\epsilon,1)}[u[\epsilon], \bar{c}] - \bar{c}\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)} \qquad \forall \epsilon \in ]0, \tilde{\epsilon}[,$$

and the conclusion easily follows. □

### 2.7.3 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorem we deduce by Proposition 2.132 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 2.134.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A\setminus\{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let $1 \le p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = \bar{c} \qquad \textit{in } L^p(V).$$

*Proof.* By virtue of Proposition 2.132, we have

$$\lim_{\epsilon\to 0^+} \|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]] - \bar{c}\|_{L^p(A)} = 0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $C > 0$ such that

$$\|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] - \bar{c}\|_{L^p(V)} \le C\|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]] - \bar{c}\|_{L^p(A)} \qquad \forall (\epsilon,\delta) \in ]0,\bar{\epsilon}_1[\times]0,1[.$$

Thus,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] - \bar{c}\|_{L^p(V)} = 0,$$

and we can easily conclude. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 2.135.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A\setminus\{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let $\epsilon_6$, $J$ be as in Theorem 2.128. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J[\epsilon], \tag{2.127}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N}\setminus\{0\}$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N}\setminus\{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\
&= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\
&= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.
\end{aligned}$$

Then we note that

$$\begin{aligned}
\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\
&= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon]\Big(\frac{l}{r}x\Big)\,dx \\
&= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](t)\,dt \\
&= \frac{r^n}{l^n} J[\epsilon].
\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J[\epsilon],$$

and the conclusion follows. □

In the following Theorem we consider the $L^\infty$–distance of a certain extension of $u_{(\epsilon,\delta)}$ and its limit.

**Theorem 2.136.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\bar{x} \in \operatorname{cl} A \setminus \{w\}$. *Let* $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ *be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let* $\tilde{\epsilon}$, $N$ *be as in Proposition 2.133. Then*

$$\|\mathbf{E}^{\#}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}, \bar{c}] - \bar{c}\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all* $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. *Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^{\#}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}, \bar{c}] = \bar{c} \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathbf{E}^{\#}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}, \bar{c}] - \bar{c}\|_{L^\infty(\mathbb{R}^n)} &= \|\mathbf{E}^{\#}_{(\epsilon,1)}[u[\epsilon], \bar{c}] - \bar{c}\|_{L^\infty(\mathbb{R}^n)} \\ &= \epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. □

### 2.7.4 Asymptotic behaviour of the energy integral of $u_{(\epsilon,\delta)}$

This Subsection is devoted to the study of the behaviour of the energy integral of $u_{(\epsilon,\delta)}$. We give the following.

**Definition 2.137.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \operatorname{cl} A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.57), (1.58), (2.108), (2.109), respectively. For each pair $(\epsilon, \delta) \in ]0, \bar{\epsilon}_1[ \times ]0, +\infty[$, we set

$$\mathrm{En}(\epsilon, \delta) \equiv \int_{A \cap \mathbb{T}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx.$$

*Remark* 2.138. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \operatorname{cl} A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.57), (1.58), (2.108), (2.109), respectively. Let $(\epsilon, \delta) \in ]0, \bar{\epsilon}_1[ \times ]0, +\infty[$. We have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2 \, dt \\ &= \delta^{n-2} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2 \, dt.\end{aligned}$$

Then we give the following definition, where we consider $\mathrm{En}(\epsilon, \delta)$, with $\epsilon$ equal to a certain function of $\delta$.

**Definition 2.139.** For each $\delta \in ]0, +\infty[$, we set

$$\epsilon[\delta] \equiv \delta^{\frac{2}{n}}.$$

Let $\epsilon_5$ be as in Theorem 2.126. Let $\delta_1 > 0$ be such that $\epsilon[\delta] \in ]0, \epsilon_5[$, for all $\delta \in ]0, \delta_1[$. Then we set

$$\mathrm{En}[\delta] \equiv \mathrm{En}(\epsilon[\delta], \delta),$$

for all $\delta \in ]0, \delta_1[$.

Here we may note that the 'radius' of the holes is $\delta\epsilon[\delta] = \delta^{\frac{n+2}{n}}$ which is different from the one of Definition 2.65 for the Dirichlet problem.

In the following Proposition we compute the limit of $\mathrm{En}[\delta]$ as $\delta$ tends to 0.

**Proposition 2.140.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\bar{x} \in \operatorname{cl} A \setminus \{w\}$. *Let* $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ *be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let* $\epsilon_5$ *be as in Theorem 2.126. Let* $\delta_1 > 0$ *be as in Definition 2.139. Then*

$$\lim_{\delta\to 0^+} \mathrm{En}[\delta] = \int_{\mathbb{R}^n \setminus \operatorname{cl}\Omega} |\nabla \tilde{u}(x)|^2 \, dx,$$

*where* $\tilde{u}$ *is as in Definition 2.120.*

*Proof.* Let $\delta \in ]0, \delta_1[$. By Remark 2.138 and Theorem 2.126, we have

$$\int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx = \delta^{n-2}(\epsilon[\delta])^n G[\epsilon[\delta]] = \delta^n G[\delta^{\frac{2}{n}}],$$

where $G$ is as in Theorem 2.126. On the other hand,

$$\lfloor (1/\delta) \rfloor^n \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx \leq \mathrm{En}[\delta] \leq \lceil (1/\delta) \rceil^n \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx,$$

and so

$$\lfloor (1/\delta) \rfloor^n \delta^n G[\delta^{\frac{2}{n}}] \leq \mathrm{En}[\delta] \leq \lceil (1/\delta) \rceil^n \delta^n G[\delta^{\frac{2}{n}}].$$

Thus, since

$$\lim_{\delta \to 0^+} \lfloor (1/\delta) \rfloor^n \delta^n = 1, \qquad \lim_{\delta \to 0^+} \lceil (1/\delta) \rceil^n \delta^n = 1,$$

we have

$$\lim_{\delta \to 0^+} \mathrm{En}[\delta] = G[0].$$

Finally, by equality (2.122), we easily conclude. □

In the following Proposition we represent the function $\mathrm{En}[\cdot]$ by means of a real analytic function.

**Proposition 2.141.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let $\epsilon_5$ and $G$ be as in Theorem 2.126. Let $\delta_1 > 0$ be as in Definition 2.139. Then*

$$\mathrm{En}[(1/l)] = G[(1/l)^{\frac{2}{n}}],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* It follows by the proof of Proposition 2.140. □

## 2.8 A variant of an homogenization problem for the Laplace equation with Neumann boundary conditions in a periodically perforated domain

In this section we consider a (slightly) different homogenization problem for the Laplace equation with Neumann boundary conditions in a periodically perforated domain.

### 2.8.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 2.6.1, 2.7.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.57), (1.58), (2.108), (2.109), respectively. For each $(\epsilon, \delta) \in ]0, \bar{\epsilon}_1[ \times ]0, +\infty[$, we consider the following periodic Neumann problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u(x + \delta a_i) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a(\epsilon,\delta), \quad \forall i \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x) = g(\frac{1}{\epsilon\delta}(x - \delta w)) & \forall x \in \partial\Omega(\epsilon,\delta), \\ u(\delta \bar{x}) = \bar{c}. \end{cases} \tag{2.128}$$

In contrast to problem (2.125), we note that in the third equation of problem (2.128) there is not the factor $1/\delta$ in front of $g(\frac{1}{\epsilon\delta}(x - \delta w))$.

By virtue of Theorem 2.17, we can give the following definition.

**Definition 2.142.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.57), (1.58), (2.108), (2.109), respectively. For each pair $(\epsilon, \delta) \in ]0, \bar{\epsilon}_1[ \times ]0, +\infty[$, we denote by $u_{(\epsilon,\delta)}$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a(\epsilon,\delta))$ of boundary value problem (2.128).

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. In order to do so we introduce the following.

**Definition 2.143.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.57), (1.58), (2.108), (2.109), respectively. For each $(\epsilon,\delta) \in ]0,\bar{\epsilon}_1[\times]0,+\infty[$, we denote by $u[\epsilon,\delta]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of the following periodic Neumann problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) = g\big(\frac{1}{\epsilon}(x-w)\big) & \forall x \in \partial\Omega_\epsilon, \\ u(\bar{x}) = \frac{\bar{c}}{\delta}. \end{cases} \tag{2.129}$$

*Remark* 2.144. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.57), (1.58), (2.108), (2.109), respectively. For each pair $(\epsilon,\delta) \in ]0,\bar{\epsilon}_1[\times]0,+\infty[$, we have

$$u_{(\epsilon,\delta)}(x) = \delta u[\epsilon,\delta](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).$$

By the previous remark, in contrast to the solution of problem (2.125), we note that the solution of problem (2.128) can be expressed by means of the solution of the auxiliary rescaled problem (2.129), which does depend on $\delta$.

*Remark* 2.145. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.58), (2.108), (2.109), respectively. Let $\epsilon_3$, $\Theta[\cdot]$ be as in Proposition 2.123. Let $(\epsilon,\delta) \in ]0,\min\{\bar{\epsilon}_1,\epsilon_3\}[\times]0,+\infty[$. We have

$$\begin{aligned} u[\epsilon,\delta](x) = \epsilon^{n-1}\int_{\partial\Omega} S_n^a(x-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \\ + \frac{\bar{c}}{\delta} - \epsilon^{n-1}\int_{\partial\Omega} S_n^a(\bar{x}-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \quad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon]. \end{aligned}$$

As a first step, we study the behaviour of $u[\epsilon,\delta]$ as $(\epsilon,\delta)$ tends to $(0,0)$.

**Proposition 2.146.** *Let* $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. *Let* $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ *be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let* $1 \le p < \infty$. *Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon,\delta]] = \bar{c} \qquad \text{in } L^p(A).$$

*Proof.* Let $\epsilon_3$, $\Theta$ be as in Proposition 2.123. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $\epsilon \in ]0,\min\{\bar{\epsilon}_1,\epsilon_3\}[$, we have

$$\begin{aligned} \delta u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = & \delta\epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \\ & + \bar{c} - \delta\epsilon^{n-1}\int_{\partial\Omega} S_n^a(\bar{x}-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \qquad \forall t \in \partial\Omega. \end{aligned}$$

We set

$$\begin{aligned} N[\epsilon,\delta](t) \equiv & \delta\epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \\ & + \bar{c} - \delta\epsilon^{n-1}\int_{\partial\Omega} S_n^a(\bar{x}-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $(\epsilon,\delta) \in ]-\min\{\bar{\epsilon}_1,\epsilon_3\},\min\{\bar{\epsilon}_1,\epsilon_3\}[\times]-\infty,+\infty[$. By taking $\tilde{\epsilon} \in ]0,\min\{\bar{\epsilon}_1,\epsilon_3\}[$ and $\tilde{\delta} \in ]0,+\infty[$ small enough, we can assume (cf. Proposition 1.26 (*i*), (*iii*)) that $N$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\delta},\tilde{\delta}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C \equiv \sup_{(\epsilon,\delta)\in]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\delta},\tilde{\delta}[} \|N[\epsilon,\delta]\|_{C^0(\partial\Omega)} < +\infty.$$

By Theorem 2.5, we have

$$|\mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon,\delta]](x)| \le C \qquad \forall x \in A, \quad \forall(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\tilde{\delta}[.$$

Clearly (cf. Theorem 2.124), we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon,\delta]](x) = \bar{c} \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon,\delta]] = \bar{c} \qquad \text{in } L^p(A).$$

□

We have also the following.

**Theorem 2.147.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let $\epsilon_3$ be as in Proposition 2.123. Then there exist $\epsilon_6 \in ]0,\min\{\bar{\epsilon}_1,\epsilon_3\}]$ and a real analytic operator $J$ of $]-\epsilon_6,\epsilon_6[\times\mathbb{R}$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} \delta u[\epsilon,\delta](x)\,dx = J[\epsilon,\delta], \tag{2.130}$$

*for all $(\epsilon,\delta) \in ]0,\epsilon_6[\times]0,+\infty[$. Moreover,*

$$J[0,0] = \bar{c}|A|_n. \tag{2.131}$$

*Proof.* Let $\Theta[\cdot]$ be as in Proposition 2.123. Let $(\epsilon,\delta) \in ]0,\min\{\bar{\epsilon}_1,\epsilon_3\}[\times]0,+\infty[$. Since

$$\delta u[\epsilon,\delta](x) = \delta v_a^-\big[\partial\Omega_\epsilon,\Theta[\epsilon](\frac{1}{\epsilon}(\cdot-w))\big](x) + \bar{c} - \delta v_a^-\big[\partial\Omega_\epsilon,\Theta[\epsilon](\frac{1}{\epsilon}(\cdot-w))\big](\bar{x}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],$$

then

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} \delta u[\epsilon,\delta](x)\,dx =& \delta\int_{\mathbb{P}_a[\Omega_\epsilon]} \Big\{v_a^-\big[\partial\Omega_\epsilon,\Theta[\epsilon](\frac{1}{\epsilon}(\cdot-w))\big](x) - v_a^-\big[\partial\Omega_\epsilon,\Theta[\epsilon](\frac{1}{\epsilon}(\cdot-w))\big](\bar{x})\Big\}\,dx \\ &+ \bar{c}\big(|A|_n - \epsilon^n|\Omega|_n\big).\end{aligned}$$

On the other hand, by arguing as in the proof of Theorem 2.128, we easily deduce that there exists $\epsilon_6 \in ]0,\min\{\bar{\epsilon}_1,\epsilon_3\}]$ and a real analytic map $J_1$ of $]-\epsilon_6,\epsilon_6[$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} \Big\{v_a^-\big[\partial\Omega_\epsilon,\Theta[\epsilon](\frac{1}{\epsilon}(\cdot-w))\big](x) - v_a^-\big[\partial\Omega_\epsilon,\Theta[\epsilon](\frac{1}{\epsilon}(\cdot-w))\big](\bar{x})\Big\}\,dx = J_1[\epsilon],$$

for all $\epsilon \in ]0,\epsilon_6[$. Moreover, $J_1[0] = 0$. Then, if we set

$$J[\epsilon,\delta] \equiv \delta J_1[\epsilon] + \bar{c}\big(|A|_n - \epsilon^n|\Omega|_n\big),$$

for all $(\epsilon,\delta) \in ]-\epsilon_6,\epsilon_6[\times\mathbb{R}$, we can immediately conclude. □

**Proposition 2.148.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let $\epsilon_3$ be as in Proposition 2.123. Then there exist $\tilde{\epsilon} \in ]0,\min\{\bar{\epsilon}_1,\epsilon_3\}[$ and a real analytic map $N$ of $]-\tilde{\epsilon},\tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ such that*

$$\|\mathbf{E}^{\#}_{(\epsilon,1)}[\delta u[\epsilon,\delta],\bar{c}] - \bar{c}\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\|N[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,+\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^{\#}_{(\epsilon,1)}[\delta u[\epsilon,\delta],\bar{c}] = \bar{c} \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* Let $\epsilon_3$, $\Theta$ be as in Proposition 2.123. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $(\epsilon,\delta) \in ]0,\min\{\bar{\epsilon}_1,\epsilon_3\}[\times]0,+\infty[$, we have

$$\begin{aligned}\delta u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) - \bar{c} =& \delta\epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \\ &- \delta\epsilon^{n-1}\int_{\partial\Omega} S_n^a(\bar{x}-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \qquad \forall t \in \partial\Omega.\end{aligned}$$

We set

$$N[\epsilon](t) \equiv \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s$$
$$- \epsilon^{n-2}\int_{\partial\Omega} S_n^a(\bar{x}-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $\epsilon \in ]-\min\{\bar{\epsilon}_1,\epsilon_3\},\min\{\bar{\epsilon}_1,\epsilon_3\}[$. By taking $\tilde{\epsilon} \in ]0,\min\{\bar{\epsilon}_1,\epsilon_3\}[$ small enough, we can assume (cf. Proposition 1.26 $(i)$, $(iii)$) that $N$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$.

By Theorem 2.5, we have

$$\|\mathbf{E}^{\#}_{(\epsilon,1)}[\delta u[\epsilon,\delta],\bar{c}] - \bar{c}\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\|N[\epsilon]\|_{C^0(\partial\Omega)} \qquad \forall(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,+\infty[,$$

and the conclusion easily follows. □

### 2.8.2 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorem we deduce by Proposition 2.146 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 2.149.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let $1 \le p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = \bar{c} \qquad \text{in } L^p(V).$$

*Proof.* By virtue of Proposition 2.146, we have

$$\lim_{\epsilon\to 0^+}\|\mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon,\delta]] - \bar{c}\|_{L^p(A)} = 0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $C>0$ such that

$$\|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] - \bar{c}\|_{L^p(V)} \le C\|\mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon,\delta]] - \bar{c}\|_{L^p(A)} \qquad \forall(\epsilon,\delta) \in ]0,\bar{\epsilon}_1[\times]0,1[.$$

Thus,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)}\|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] - \bar{c}\|_{L^p(V)} = 0,$$

and we can easily conclude. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 2.150.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let $\epsilon_6$, $J$ be as in Theorem 2.147. Let $r>0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J\big[\epsilon,\frac{r}{l}\big], \tag{2.132}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N}\setminus\{0\}$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_6[$, and let $l \in \mathbb{N}\setminus\{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= l^n\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} (r/l)u\big[\epsilon,(r/l)\big]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} (r/l)u\big[\epsilon,(r/l)\big](t)\,dt \\ &= \frac{r^n}{l^n} J\big[\epsilon,\frac{r}{l}\big].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J\big[\epsilon,\frac{r}{l}\big],$$

and the conclusion follows. □

In the following Theorem we consider the $L^\infty$–distance of a certain extension of $u_{(\epsilon,\delta)}$ and its limit.

**Theorem 2.151.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let $\tilde{\epsilon}$, $N$ be as in Proposition 2.148. Then*

$$\|\mathbf{E}^\#_{(\epsilon,\delta)}[u_{(\epsilon,\delta)},\bar{c}] - \bar{c}\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\|N[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,+\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^\#_{(\epsilon,\delta)}[u_{(\epsilon,\delta)},\bar{c}] = \bar{c} \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathbf{E}^\#_{(\epsilon,\delta)}[u_{(\epsilon,\delta)},\bar{c}] - \bar{c}\|_{L^\infty(\mathbb{R}^n)} &= \|\mathbf{E}^\#_{(\epsilon,1)}[\delta u[\epsilon,\delta],\bar{c}] - \bar{c}\|_{L^\infty(\mathbb{R}^n)} \\ &= \delta\epsilon\|N[\epsilon]\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,+\infty[$. □

### 2.8.3 Asymptotic behaviour of the energy integral of $u_{(\epsilon,\delta)}$

This Subsection is devoted to the study of the behaviour of the energy integral of $u_{(\epsilon,\delta)}$. We give the following.

**Definition 2.152.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.57), (1.58), (2.108), (2.109), respectively. For each pair $(\epsilon,\delta) \in ]0,\bar{\epsilon}_1[\times]0,+\infty[$, we set

$$\mathrm{En}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 2.153. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.57), (1.58), (2.108), (2.109), respectively. Let $(\epsilon,\delta) \in ]0,\bar{\epsilon}_1[\times]0,+\infty[$. We have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n\int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2\,dt \\ &= \delta^n\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon,\delta](t)|^2\,dt.\end{aligned}$$

*Remark* 2.154. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in (1.56), (1.57), (1.58), (2.108), (2.109), respectively. Let $\epsilon_5$ and $G$ be as in Theorem 2.126. Then we have

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon,\delta](t)|^2\,dt = \epsilon^n G[\epsilon],$$

for all $(\epsilon,\delta) \in ]0,\epsilon_5[\times]0,+\infty[$.

In the following Proposition we represent the function $\mathrm{En}(\cdot,\cdot)$ by means of a real analytic function.

**Proposition 2.155.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\epsilon_1$, $\bar{\epsilon}_1$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (1.58), (2.108), (2.109), *respectively. Let $\epsilon_5$ and $G$ be as in Theorem 2.126. Then*

$$\mathrm{En}\Big(\epsilon, \frac{1}{l}\Big) = \epsilon^n G[\epsilon],$$

*for all $\epsilon \in ]0,\epsilon_5[$ and for all $l \in \mathbb{N}\setminus\{0\}$.*

*Proof.* Let $(\epsilon,\delta) \in ]0,\epsilon_5[\times]0,+\infty[$. By Remark 2.153 and Theorem 2.126, we have

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\, dx = \delta^n \epsilon^n G[\epsilon] \tag{2.133}$$

where $G$ is as in Theorem 2.126. On the other hand, if $\epsilon \in ]0,\epsilon_5[$ and $l \in \mathbb{N}\setminus\{0\}$, then we have

$$\begin{aligned}\mathrm{En}\Big(\epsilon, \frac{1}{l}\Big) &= l^n \frac{1}{l^n}\epsilon^n G[\epsilon] \\ &= \epsilon^n G[\epsilon],\end{aligned}$$

and the conclusion easily follows. □

## 2.9 Asymptotic behaviour of the solutions of an alternative Neumann problem for the Laplace equation in a periodically perforated domain

In this Section we study the asymptotic behaviour of the solutions of an alternative Neumann problem for the Laplace equation in a periodically perforated domain with small holes.

### 2.9.1 Notation and preliminaries

We retain the notation introduced in Subsections 1.8.1, 2.6.1. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. For each $\epsilon \in ]0,\epsilon_1[$, we consider the following periodic Neumann problem for the Laplace equation.

$$\begin{cases}\Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) = g\big(\frac{1}{\epsilon}(x-w)\big) & \forall x \in \partial\Omega_\epsilon, \\ \int_{\partial\Omega_\epsilon} u(x)\, d\sigma_x = 0. \end{cases} \tag{2.134}$$

By virtue of Theorem 2.17, we can give the following definition.

**Definition 2.156.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. For each $\epsilon \in ]0,\epsilon_1[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of boundary value problem (2.134).

*Remark* 2.157. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. For each $\epsilon \in ]0,\epsilon_1[$, let $\hat{\theta}[\epsilon]$ be as in Definition 2.118. Let $\epsilon \in ]0,\epsilon_1[$. We have

$$\begin{aligned}u[\epsilon](x) =& \epsilon^{n-1} \int_{\partial\Omega} S_n^a(x-w-\epsilon s)\hat{\theta}[\epsilon](s)\, d\sigma_s \\ &- \epsilon^{n-1} \frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega}\Big(\int_{\partial\Omega} S_n^a(\epsilon(t-s))\hat{\theta}[\epsilon](s)\, d\sigma_s\Big)\, d\sigma_t \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].\end{aligned}$$

### 2.9.2 A functional analytic representation Theorem for the solution of the alternative singularly perturbed Neumann problem

The following statement shows that $\{u[\epsilon](\cdot)\}_{\epsilon\in]0,\epsilon_1[}$ can be continued real analytically for negative values of $\epsilon$.

**Theorem 2.158.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. Let $\epsilon_3$ be as in Proposition 2.123. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0,\epsilon_3]$, a real analytic operator $U_1$ of $]-\epsilon_4,\epsilon_4[$ to the space $C^0_h(\mathrm{cl}\, V)$, and a real analytic operator $U_2$ of $]-\epsilon_4,\epsilon_4[$ to $\mathbb{R}$ such that the following conditions hold.*

*(i)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4,\epsilon_4[$.

*(ii)*
$$u[\epsilon](x) = \epsilon^n U_1[\epsilon](x) + \epsilon U_2[\epsilon] \qquad \forall x \in \mathrm{cl}\, V,$$
*for all* $\epsilon \in ]0,\epsilon_4[$.

*Proof.* Choosing $\epsilon_4$ small enough, we can clearly assume that $(i)$ holds. Consider now $(ii)$. Let $\epsilon \in ]0,\epsilon_4[$. Let $\Theta$ be as in Proposition 2.123. By Remark 2.157 and Proposition 2.123, we have

$$\begin{aligned} u[\epsilon](x) =& \epsilon^{n-1}\int_{\partial\Omega} S^a_n(x-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \\ &- \epsilon^{n-1}\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(\int_{\partial\Omega} S^a_n(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s\Big)\,d\sigma_t \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon]. \end{aligned}$$

By Proposition 1.28 $(i)$, we have

$$\begin{aligned} &\epsilon^{n-1}\int_{\partial\Omega} S^a_n(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \\ &\qquad = \epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $\epsilon \in ]0,\epsilon_4[$. Thus, it is natural to set

$$\tilde{U}_1[\epsilon](x) \equiv \int_{\partial\Omega} S^a_n(x-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

for all $\epsilon \in ]-\epsilon_4,\epsilon_4[$, and

$$U_2[\epsilon] \equiv -\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s\Big)\,d\sigma_t,$$

for all $\epsilon \in ]-\epsilon_4,\epsilon_4[$. By the proof of Theorem 2.124, we have that, possibly taking a smaller $\epsilon_4$, there exists a real analytic operator $U_1$ of $]-\epsilon_4,\epsilon_4[$ to $C^0_h(\mathrm{cl}\, V)$, such that

$$\tilde{U}_1[\epsilon] = \epsilon U_1[\epsilon] \qquad \text{in } C^0_h(\mathrm{cl}\, V),$$

for all $\epsilon \in ]-\epsilon_4,\epsilon_4[$. Possibly choosing a smaller $\epsilon_4$, by Proposition 1.28 $(i)$ and standard calculus in Banach spaces, we easily deduce that $U_2$ is a real analytic operator of $]-\epsilon_4,\epsilon_4[$ to $\mathbb{R}$. Finally, by the definition of $U_1$ and $U_2$, we immediately deduce that the equality in $(ii)$ holds. □

*Remark* 2.159. We note that the right-hand side of the equality in $(ii)$ of Theorem 2.158 can be continued real analytically in the whole $]-\epsilon_4,\epsilon_4[$. Moreover, if $V$ is a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$, then

$$\lim_{\epsilon\to 0^+} u[\epsilon] = 0 \qquad \text{uniformly in } \mathrm{cl}\, V.$$

### 2.9.3 A real analytic continuation Theorem for the energy integral

As done in Theorem 2.158 for $u[\cdot]$, we can now prove a real analytic continuation Theorem for the energy integral. Namely, we prove the following.

**Theorem 2.160.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.108), *respectively. Let $\epsilon_3$ be as in Proposition 2.123. Then there exist $\epsilon_5 \in ]0,\epsilon_3]$ and a real analytic operator $G$ of $]-\epsilon_5,\epsilon_5[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx = \epsilon^n G[\epsilon], \tag{2.135}$$

*for all $\epsilon \in ]0,\epsilon_5[$. Moreover,*

$$G[0] = \int_{\mathbb{R}^n\setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\, dx, \tag{2.136}$$

*where $\tilde{u}$ is as in Definition 2.120.*

*Proof.* Let $\Theta$ be as in Proposition (2.123). Let $\epsilon \in ]0,\epsilon_3[$. Clearly,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx = \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2\, dx.$$

As a consequence, in order to prove the Theorem, it suffices to follow the proof of Theorem 2.126. □

*Remark* 2.161. We note that the right-hand side of the equality in (2.135) of Theorem 2.160 can be continued real analytically in the whole $]-\epsilon_5,\epsilon_5[$. Moreover,

$$\lim_{\epsilon\to 0^+} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\; dx = 0.$$

### 2.9.4 A real analytic continuation Theorem for the integral of the solution

As done in Theorem 2.160 for the energy integral, we can now prove a real analytic continuation Theorem for the integral of the solution. Namely, we prove the following.

**Theorem 2.162.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.108), *respectively. Let $\epsilon_3$ be as in Proposition 2.123. Then there exist $\epsilon_6 \in ]0,\epsilon_3]$ and a real analytic operator $J$ of $]-\epsilon_6,\epsilon_6[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx = J[\epsilon], \tag{2.137}$$

*for all $\epsilon \in ]0,\epsilon_6[$. Moreover,*

$$J[0] = 0. \tag{2.138}$$

*Proof.* If $\epsilon \in ]0,\epsilon_3]$, we have

$$\begin{aligned} u[\epsilon](x) =& \epsilon^{n-1}\int_{\partial\Omega} S_n^a(x - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \\ &- \epsilon^{n-1}\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(\int_{\partial\Omega} S_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s\Big)\, d\sigma_t \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon]. \end{aligned}$$

Then, if we set

$$\begin{aligned} L[\epsilon](t) \equiv\, & \epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ & - \frac{\epsilon}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s\Big)\, d\sigma_t \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $\epsilon \in ]-\epsilon_3,\epsilon_3[$, we have that there exists $\tilde{\epsilon} \in ]0,\epsilon_3]$ such that $L[\cdot]$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$u[\epsilon](w + \epsilon t) = L[\epsilon](t) \qquad \forall t \in \partial\Omega, \quad \forall \epsilon \in ]0,\tilde{\epsilon}[.$$

In particular, $L[0](t) = 0$ for all $t \in \partial\Omega$. Then, by Theorem 2.103, we easily deduce that there exists $\epsilon_6 \in ]0, \tilde{\epsilon}]$ and a real analytic map $J$ of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx = J[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_6[$, and that $J[0] = 0$. □

## 2.10 Alternative homogenization problem for the Laplace equation with Neumann boundary conditions in a periodically perforated domain

In this section we consider an homogenization problem for the Laplace equation with Neumann boundary conditions in a periodically perforated domain.

### 2.10.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 2.6.1 and 2.7.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. For each $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we consider the following periodic Neumann problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon, \delta), \\ u(x + \delta a_i) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a(\epsilon, \delta), \quad \forall i \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x) = \frac{1}{\delta} g(\frac{1}{\epsilon\delta}(x - \delta w)) & \forall x \in \partial\Omega(\epsilon, \delta), \\ \int_{\partial\Omega(\epsilon,\delta)} u(x)\, d\sigma_x = 0. \end{cases} \tag{2.139}$$

By virtue of Theorem 2.17, we can give the following definition.

**Definition 2.163.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. For each $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we denote by $u_{(\epsilon,\delta)}$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a(\epsilon, \delta))$ of boundary value problem (2.139).

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon, \delta)$ tends to $(0, 0)$. In order to do so we introduce the following.

**Definition 2.164.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. For each $\epsilon \in ]0, \epsilon_1[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon])$ of the following periodic Neumann problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) = g\big(\frac{1}{\epsilon}(x - w)\big) & \forall x \in \partial\Omega_\epsilon, \\ \int_{\partial\Omega_\epsilon} u(x)\, d\sigma_x = 0. \end{cases} \tag{2.140}$$

*Remark* 2.165. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. For each pair $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we have

$$u_{(\epsilon,\delta)}(x) = u[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a(\epsilon, \delta).$$

By the previous remark, we note that the solution of problem (2.139) can be expressed by means of the solution of the auxiliary rescaled problem (2.140), which does not depend on $\delta$. This is due to the presence of the factor $1/\delta$ in front of $g(\frac{1}{\epsilon\delta}(x - \delta w))$ in the third equation of problem (2.139).

As a first step, we study the behaviour of $u[\epsilon]$ as $\epsilon$ tends to 0.

**Proposition 2.166.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.108)*, respectively. Let $\epsilon_3$ be as in Proposition 2.123. Then there exist $\tilde{\epsilon} \in ]0, \epsilon_3[$ and a real analytic map $N$ of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ such that*

$$\|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all* $\epsilon \in ]0, \tilde{\epsilon}[$. *Moreover, as a consequence,*

$$\lim_{\epsilon \to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* Let $\epsilon_3$, $\Theta$ be as in Proposition 2.123. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $\epsilon \in ]0, \epsilon_3[$, we have

$$\begin{aligned} u[\epsilon] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) &= \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ &- \frac{\epsilon}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \Big)\, d\sigma_t \qquad \forall t \in \partial\Omega. \end{aligned}$$

We set

$$\begin{aligned} N[\epsilon](t) \equiv & \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ &- \frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \Big)\, d\sigma_t \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. By taking $\tilde{\epsilon} \in ]0, \epsilon_3[$ small enough, we can assume (cf. Proposition 1.28 $(i)$) that $N$ is a real analytic map of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$.

By Theorem 2.5, we have

$$\|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]](x)\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)}, \qquad \forall \epsilon \in ]0, \tilde{\epsilon}[,$$

and the conclusion easily follows. □

### 2.10.2 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorem we deduce by Proposition 2.166 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon, \delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 2.167.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $g$ *be as in* (1.56), (1.57), (2.108), *respectively. Let* $\tilde{\epsilon}$, $N$ *be as in Proposition 2.166. Then*

$$\|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all* $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. *Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta) \to (0^+, 0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* It suffices to observe that

$$\begin{aligned} \|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\|_{L^\infty(\mathbb{R}^n)} &= \|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\|_{L^\infty(\mathbb{R}^n)} \\ &= \epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)}, \end{aligned}$$

for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 2.168.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $g$ *be as in* (1.56), (1.57), (2.108), *respectively. Let* $\epsilon_6$, $J$ *be as in Theorem 2.162. Let* $r > 0$ *and* $\bar{y} \in \mathbb{R}^n$. *Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\, dx = r^n J[\epsilon], \tag{2.141}$$

*for all* $\epsilon \in ]0, \epsilon_6[$, *and for all* $l \in \mathbb{N} \setminus \{0\}$.

*Proof.* Let $\epsilon \in ]0, \epsilon_6[$, $l \in \mathbb{N} \setminus \{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](t)\,dt \\ &= \frac{r^n}{l^n} J[\epsilon].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J[\epsilon],$$

and the conclusion follows. □

### 2.10.3 Asymptotic behaviour of the energy integral of $u_{(\epsilon,\delta)}$

This Subsection is devoted to the study of the behaviour of the energy integral of $u_{(\epsilon,\delta)}$. We give the following.

**Definition 2.169.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. For each pair $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we set

$$\mathrm{En}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 2.170. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. Let $(\epsilon,\delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$. We have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2\,dt \\ &= \delta^{n-2}\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2\,dt.\end{aligned}$$

Then we give the following definition, where we consider $\mathrm{En}(\epsilon,\delta)$, with $\epsilon$ equal to a certain function of $\delta$.

**Definition 2.171.** For each $\delta \in ]0, +\infty[$, we set

$$\epsilon[\delta] \equiv \delta^{\frac{2}{n}}.$$

Let $\epsilon_5$ be as in Theorem 2.160. Let $\delta_1 > 0$ be such that $\epsilon[\delta] \in ]0, \epsilon_5[$, for all $\delta \in ]0, \delta_1[$. Then we set

$$\mathrm{En}[\delta] \equiv \mathrm{En}(\epsilon[\delta], \delta),$$

for all $\delta \in ]0, \delta_1[$.

In the following Proposition we compute the limit of $\mathrm{En}[\delta]$ as $\delta$ tends to 0.

**Proposition 2.172.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.108)*, respectively. Let $\epsilon_5$ be as in Theorem 2.160. Let $\delta_1 > 0$ be as in Definition 2.171. Then*

$$\lim_{\delta\to 0^+} \mathrm{En}[\delta] = \int_{\mathbb{R}^n\setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\,dx,$$

*where $\tilde{u}$ is as in Definition 2.120.*

*Proof.* Let $\delta \in ]0,\delta_1[$. By Remark 2.170 and Theorem 2.160, we have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2\,dx &= \delta^{n-2}(\epsilon[\delta])^n G[\epsilon[\delta]]\\ &= \delta^n G[\delta^{\frac{2}{n}}],\end{aligned}$$

where $G$ is as in Theorem 2.160. On the other hand,

$$\lfloor (1/\delta)\rfloor^n \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2\,dx \le \mathrm{En}[\delta] \le \lceil (1/\delta)\rceil^n \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2\,dx,$$

and so

$$\lfloor (1/\delta)\rfloor^n \delta^n G[\delta^{\frac{2}{n}}] \le \mathrm{En}[\delta] \le \lceil (1/\delta)\rceil^n \delta^n G[\delta^{\frac{2}{n}}].$$

Thus, since

$$\lim_{\delta\to 0^+} \lfloor (1/\delta)\rfloor^n \delta^n = 1, \qquad \lim_{\delta\to 0^+} \lceil (1/\delta)\rceil^n \delta^n = 1,$$

we have

$$\lim_{\delta\to 0^+} \mathrm{En}[\delta] = G[0].$$

Finally, by equality (2.136), we easily conclude. □

In the following Proposition we represent the function $\mathrm{En}[\cdot]$ by means of a real analytic function.

**Proposition 2.173.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.108)*, respectively. Let $\epsilon_5$ and $G$ be as in Theorem 2.160. Let $\delta_1 > 0$ be as in Definition 2.171. Then*

$$\mathrm{En}[(1/l)] = G[(1/l)^{\frac{2}{n}}],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* It follows by the proof of Proposition 2.172. □

## 2.11 A variant of the alternative homogenization problem for the Laplace equation with Neumann boundary conditions in a periodically perforated domain

In this section we consider a different homogenization problem for the Laplace equation with Neumann boundary conditions in a periodically perforated domain.

### 2.11.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 2.6.1 and 2.7.1.

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. For each $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, we consider the following periodic Neumann problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta),\\ u(x+\delta a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall i \in \{1,\dots,n\},\\ \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x) = g(\frac{1}{\epsilon\delta}(x-\delta w)) & \forall x \in \partial\Omega(\epsilon,\delta),\\ \int_{\partial\Omega(\epsilon,\delta)} u(x)\,d\sigma_x = 0. \end{cases} \tag{2.142}$$

In contrast to problem (2.139), we note that in the third equation of problem (2.142) there is not the factor $1/\delta$ in front of $g(\frac{1}{\epsilon\delta}(x-\delta w))$.

By virtue of Theorem 2.17, we can give the following definition.

**Definition 2.174.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. For each $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we denote by $u_{(\epsilon,\delta)}$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a(\epsilon, \delta))$ of boundary value problem (2.142).

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon, \delta)$ tends to $(0, 0)$. In order to do so we introduce the following.

**Definition 2.175.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. For each $\epsilon \in ]0, \epsilon_1[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon])$ of the following periodic Neumann problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) = g\big(\frac{1}{\epsilon}(x - w)\big) & \forall x \in \partial\Omega_\epsilon, \\ \int_{\partial\Omega_\epsilon} u(x)\, d\sigma_x = 0. \end{cases} \tag{2.143}$$

*Remark* 2.176. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. For each pair $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we have

$$u_{(\epsilon,\delta)}(x) = \delta u[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a(\epsilon, \delta).$$

We have the following.

**Proposition 2.177.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.108)*, respectively. Let $\epsilon_3$ be as in Proposition 2.123. Then there exist $\tilde{\epsilon} \in ]0, \epsilon_3[$ and a real analytic map $N$ of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ such that*

$$\|\mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon]]\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* It is an immediate consequence of Proposition 2.166. □

## 2.11.2 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorem we deduce by Proposition 2.177 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon, \delta)$ tends to $(0, 0)$. Namely, we prove the following.

**Theorem 2.178.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.108)*, respectively. Let $\tilde{\epsilon}$, $N$ be as in Proposition 2.177. Then*

$$\|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* It suffices to observe that

$$\begin{aligned} \|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\|_{L^\infty(\mathbb{R}^n)} &= \|\mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon]]\|_{L^\infty(\mathbb{R}^n)} \\ &= \delta\epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)}, \end{aligned}$$

for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. □

Then we have the following.

**Theorem 2.179.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.108), *respectively. Let $\epsilon_6$, $J$ be as in Theorem 2.162. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l} J[\epsilon], \tag{2.144}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N} \setminus \{0\}$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N} \setminus \{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} (r/l)u[\epsilon]\Big(\frac{l}{r}x\Big)\,dx \\ &= \frac{r^n}{l^n}\frac{r}{l}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](t)\,dt \\ &= \frac{r^{n+1}}{l}\frac{1}{l^n}J[\epsilon].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l} J[\epsilon],$$

and the conclusion follows. □

### 2.11.3 Asymptotic behaviour of the energy integral of $u_{(\epsilon,\delta)}$

This Subsection is devoted to the study of the behaviour of the energy integral of $u_{(\epsilon,\delta)}$. We give the following.

**Definition 2.180.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we set

$$\mathrm{En}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 2.181. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in (1.56), (1.57), (2.108), respectively. Let $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$. We have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2\,dt \\ &= \delta^n \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2\,dt.\end{aligned}$$

In the following Proposition we represent the function $\mathrm{En}(\cdot,\cdot)$ by means of a real analytic function.

**Proposition 2.182.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $g$ be as in* (1.56), (1.57), (2.108), *respectively. Let $\epsilon_5$ and $G$ be as in Theorem 2.160. Then*

$$\mathrm{En}\Big(\epsilon,\frac{1}{l}\Big) = \epsilon^n G[\epsilon],$$

*for all $\epsilon \in ]0,\epsilon_5[$ and for all $l \in \mathbb{N} \setminus \{0\}$.*

*Proof.* Let $(\epsilon, \delta) \in ]0, \epsilon_5[ \times ]0, +\infty[$. By Remark 2.181 and Theorem 2.160, we have

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx = \delta^n \epsilon^n G[\epsilon] \tag{2.145}$$

where $G$ is as in Theorem 2.160. On the other hand, if $\epsilon \in ]0, \epsilon_5[$ and $l \in \mathbb{N} \setminus \{0\}$, then we have

$$\begin{aligned} \mathrm{En}\Big(\epsilon, \frac{1}{l}\Big) &= l^n \frac{1}{l^n} \epsilon^n G[\epsilon] \\ &= \epsilon^n G[\epsilon], \end{aligned}$$

and the conclusion easily follows. □

# CHAPTER 3

## Singular perturbation and homogenization problems for the Laplace equation with Robin boundary condition

In this Chapter we introduce the periodic Robin problem for the Laplace equation and we study singular perturbation and homogenization problems for the Laplace operator with Robin boundary condition in a periodically perforated domain. In particular, we consider both the linear and the nonlinear case. First of all, by means of periodic simple layer potentials, we show the solvability of the linear Robin problem. Secondly, both for the linear and the nonlinear case, we consider singular perturbation problems in a periodically perforated domain with small holes, and we apply the obtained results to homogenization problems. As well as for the Dirichlet and Neumann problems, we follow the approach of Lanza [72], where the asymptotic behaviour of the solutions of a nonlinear Robin problem for the Laplace operator in a domain with a small hole is considered. We also mention that nonlinear traction problems have been analysed by Dalla Riva and Lanza [38, 39, 42, 43] with this approach. One of the tools used in our analysis is the study of the dependence of layer potentials upon perturbations (cf. Lanza and Rossi [85] and also Dalla Riva and Lanza [40].) For a more general result concerning the nonlinear Robin problem, we refer to [82].

We retain the notation of Chapter 1 (see in particular Sections 1.1, 1.3, Theorem 1.4, and Definitions 1.12, 1.14, 1.16.) For notation, definitions, and properties concerning classical layer potentials for the Laplace equation, we refer to Appendix B.

## 3.1 A periodic linear Robin boundary value problem for the Laplace equation

In this Section we introduce a periodic linear Robin problem for the Laplace equation and we show the existence and uniqueness of a solution by means of the periodic simple layer potential.

### 3.1.1 Formulation of the problem

In this Subsection we introduce a periodic linear Robin problem for the Laplace equation.

First of all, we need to introduce some notation. Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0, 1[$. Let $\mathbb{I}$ be as in (1.46). We shall consider the following assumptions.

$$\phi \in C^{m-1,\alpha}(\partial\mathbb{I}),\ \phi \leq 0,\ \int_{\partial\mathbb{I}} \phi \, d\sigma < 0; \tag{3.1}$$

$$\Gamma \in C^{m-1,\alpha}(\partial\mathbb{I}). \tag{3.2}$$

Let $m \in \mathbb{N} \setminus \{0\}$ Let $\alpha \in ]0, 1[$. Let $\mathbb{I}$ be as in (1.46). We also set

$$\mathcal{U}^{m-1,\alpha}_{\partial\mathbb{I}} \equiv \left\{ \mu \in C^{m-1,\alpha}(\partial\mathbb{I}) \colon \int_{\partial\mathbb{I}} \mu \, d\sigma = 0 \right\}. \tag{3.3}$$

We are now ready to give the following.

**Definition 3.1.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\phi$, $\Gamma$ be as in (3.1), (3.2), respectively. We say that a function $u \in C^1(\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}])$ solves the *periodic (linear) Robin problem for the Laplace equation* if

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) + \phi(x) u(x) = \Gamma(x) & \forall x \in \partial \mathbb{I}. \end{cases} \tag{3.4}$$

### 3.1.2 Existence and uniqueness results for the solutions of the periodic Robin problem

In this Subsection we prove uniqueness results for the solutions of the periodic Robin problems for the Laplace equation.

**Proposition 3.2.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $\phi$, $\Gamma$ be as in* (3.1), (3.2), *respectively. Then boundary value problem* (3.4) *has at most one solution in $C^1(\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}])$.*

*Proof.* Let $u_1$, $u_2$ in $C^1(\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}])$ be two solutions of (3.4). We set

$$u(x) \equiv u_1(x) - u_2(x) \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\mathbb{I}].$$

Clearly, the function $u$ solves the following boundary value problem:

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) + \phi(x) u(x) = 0 & \forall x \in \partial \mathbb{I}. \end{cases}$$

By the Divergence Theorem and the periodicity of $u$, we have

$$0 \le \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2 \, dx = - \int_{\partial \mathbb{I}} u(x) \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) \, d\sigma_x = \int_{\partial \mathbb{I}} \phi(x) (u(x))^2 \, d\sigma_x \le 0.$$

Therefore, $u$ is constant in $\mathrm{cl}\, \mathbb{P}_a[\mathbb{I}]$. Now assume that there exists a constant $c \in \mathbb{R} \setminus \{0\}$, such that $u(x) = c$ for all $x \in \mathrm{cl}\, \mathbb{P}_a[\mathbb{I}]$. Then

$$0 \le \int_{\partial \mathbb{I}} \phi(x) c^2 \, d\sigma_x = c^2 \int_{\partial \mathbb{I}} \phi \, d\sigma < 0.$$

Hence, $c$ must be equal to 0, *i.e.*, $u = 0$ in $\mathrm{cl}\, \mathbb{P}_a[\mathbb{I}]$, and, accordingly,

$$u_1(x) = u_2(x) \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\mathbb{I}].$$

□

As we know, in order to solve problem (3.4) by means of periodic simple layer potentials, we need to study some integral equations. Thus, in the following Proposition, we study an operator related to the equations that we shall consider in the sequel.

**Proposition 3.3.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $\phi$ be as in* (3.1)*. Let $\mathcal{U}^{m-1,\alpha}_{\partial \mathbb{I}}$ be as in* (3.3)*. Let $\mathcal{L}_{\mathbb{I},\phi}$ be the map of $\mathcal{U}^{m-1,\alpha}_{\partial \mathbb{I}} \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial \mathbb{I})$, which takes $(\mu, \xi)$ to*

$$\begin{aligned} \mathcal{L}_{\mathbb{I},\phi}[\mu, \xi](t) \equiv & \frac{1}{2} \mu(t) + \int_{\partial \mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}(t)} (S^a_n(t - s)) \mu(s) \, d\sigma_s \\ & + \phi(t) \int_{\partial \mathbb{I}} S^a_n(t - s) \mu(s) \, d\sigma_s + \phi(t) \xi \qquad \forall t \in \partial \mathbb{I}, \end{aligned}$$

*for all $(\mu, \xi) \in \mathcal{U}^{m-1,\alpha}_{\partial \mathbb{I}} \times \mathbb{R}$. Then $\mathcal{L}_{\mathbb{I},\phi}$ is an isomorphism of $\mathcal{U}^{m-1,\alpha}_{\partial \mathbb{I}} \times \mathbb{R}$ onto $C^{m-1,\alpha}(\partial \mathbb{I})$.*

*Proof.* Clearly, if $(\mu,\xi)\in\mathcal{U}^{m-1,\alpha}_{\partial\mathbb{I}}\times\mathbb{R}$, then $\mathcal{L}_{\mathbb{I},\phi}[\mu,\xi]\in C^{m-1,\alpha}(\partial\mathbb{I})$. We need to prove that $\mathcal{L}_{\mathbb{I},\mu}$ is bijective. First of all, we prove its injectivity. Let $(\mu,\xi)\in\mathcal{U}^{m-1,\alpha}_{\partial\mathbb{I}}\times\mathbb{R}$ be such that $\mathcal{L}_{\mathbb{I},\mu}[\mu,\xi]=0$. By the properties of the periodic simple layer potential, we have that the function $u$ of $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$ to $\mathbb{R}$ defined by

$$u(x)\equiv v_a^-[\partial\mathbb{I},\mu](x)+\xi\qquad\forall x\in\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],$$

solves problem (3.4) with $\Gamma\equiv0$. Hence, by virtue of Proposition 3.2,

$$u(x)=0\qquad\forall x\in\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}].$$

On the other hand, the function $\bar u$ of $\mathrm{cl}\,\mathbb{I}$ to $\mathbb{R}$, defined by

$$\bar u(x)\equiv v_a^+[\partial\mathbb{I},\mu](x)+\xi\qquad\forall x\in\mathrm{cl}\,\mathbb{I},$$

solves the problem

$$\begin{cases}\Delta\bar u=0 & \text{in }\mathbb{I},\\ \bar u=0 & \text{on }\partial\mathbb{I},\end{cases}$$

and so

$$\bar u(x)=0\qquad\forall x\in\mathrm{cl}\,\mathbb{I}.$$

By Theorem 1.15 $(iv)$, we have

$$\mu(t)=\frac{\partial}{\partial\nu_{\mathbb{I}}}v_a^-[\partial\mathbb{I},\mu](t)-\frac{\partial}{\partial\nu_{\mathbb{I}}}v_a^+[\partial\mathbb{I},\mu](t)=0\qquad\forall t\in\partial\mathbb{I}.$$

As a consequence, $v_a^-[\partial\mathbb{I},\mu]=0$ on $\partial\mathbb{I}$, and so also $\xi=0$.

Now we need to prove the surjectivity. Let $\mathcal{L}$ be the map of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^{m-1,\alpha}(\partial\mathbb{I})$, defined by

$$\mathcal{L}[\mu](t)\equiv\frac12\mu(t)+\int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n^a(t-s))\mu(s)\,d\sigma_s+\phi(t)\int_{\partial\mathbb{I}}S_n^a(t-s)\mu(s)\,d\sigma_s\qquad\forall t\in\partial\mathbb{I},$$

for all $\mu\in C^{m-1,\alpha}(\partial\mathbb{I})$. Let $\tilde{\mathcal{L}}$ be the map of $L^2(\partial\mathbb{I})$ to $L^2(\partial\mathbb{I})$, defined by

$$\tilde{\mathcal{L}}[\mu](t)\equiv\frac12\mu(t)+\int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n^a(t-s))\mu(s)\,d\sigma_s+\phi(t)\int_{\partial\mathbb{I}}S_n^a(t-s)\mu(s)\,d\sigma_s\qquad\forall t\in\partial\mathbb{I},$$

for all $\mu\in L^2(\partial\mathbb{I})$. Since the singularities of the involved integral operators are weak, $\tilde{\mathcal{L}}$ is linear and continuous in $L^2(\partial\mathbb{I})$. Also, if we denote by $I$ the identity operator, then the operator $-\frac12 I+\tilde{\mathcal{L}}$ is compact in $L^2(\partial\mathbb{I})$.

Assume $\tilde{\mathcal{L}}$ is injective. If $\tilde{\mathcal{L}}$ is injective, then, by the Fredholm Theory, we have that it is an isomorphism of $L^2(\partial\mathbb{I})$ onto $L^2(\partial\mathbb{I})$. Accordingly, also $\mathcal{L}$ is an isomorphism of $C^{m-1,\alpha}(\partial\mathbb{I})$ onto $C^{m-1,\alpha}(\partial\mathbb{I})$ (see Theorem 1.21.) Consequently, the codimension in $C^{m-1,\alpha}(\partial\mathbb{I})$ of the subspace

$$\mathcal{V}\equiv\left\{\,\mathcal{L}_{\mathbb{I},\phi}[\mu,0]\colon\,\mu\in\mathcal{U}^{m-1,\alpha}_{\partial\mathbb{I}}\,\right\}$$

is 1. Since $\mathcal{L}_{\mathbb{I},\phi}$ is injective, we have that

$$\phi\notin\mathcal{V},$$

and so

$$C^{m-1,\alpha}(\partial\mathbb{I})=\mathcal{V}\oplus\langle\phi\rangle\,.$$

Therefore, $\mathcal{L}_{\mathbb{I},\phi}$ is surjective.

Now assume that $\tilde{\mathcal{L}}$ is not injective in $L^2(\partial\mathbb{I})$. By arguing as above, we have

$$\left\{\,\mu\in L^2(\partial\mathbb{I})\colon\,\int_{\partial\mathbb{I}}\mu\,d\sigma=0\,\right\}\cap\ker\tilde{\mathcal{L}}=\{0\}.$$

As a consequence, $\dim\ker\tilde{\mathcal{L}}=1$. By Fredholm Theory and Theorem 1.21, there exists $h\in C^{m,\alpha}(\partial\mathbb{I})$, such that

$$L^2(\partial\mathbb{I})=\left\{\,\tilde{\mathcal{L}}[\mu]\colon\,\mu\in L^2(\partial\mathbb{I}),\int_{\partial\mathbb{I}}\mu\,d\sigma=0\,\right\}\oplus_\perp\langle h\rangle\,.$$

Thus, also by virtue of Theorem 1.21, we have

$$C^{m-1,\alpha}(\partial\mathbb{I})=\left\{\,\mathcal{L}[\mu]\colon\,\mu\in\mathcal{U}^{m-1,\alpha}_{\partial\mathbb{I}}\,\right\}\oplus\langle h\rangle\,,$$

and so the subspace $\mathcal{V}$, defined as above, has codimension 1 in $C^{m-1,\alpha}(\partial\mathbb{I})$, and consequently $\mathcal{L}_{\mathbb{I},\mu}$ is surjective. The Proposition is now completely proved. □

We are now ready to prove the main result of this section.

**Theorem 3.4.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\phi$, $\Gamma$ be as in (3.1), (3.2), respectively. Let $\mathcal{U}^{m-1,\alpha}_{\partial\mathbb{I}}$ be as in (3.3). Then boundary value problem (3.4) has a unique solution $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}])$. Moreover,*

$$u(x) = v_a^-[\partial\mathbb{I}, \mu](x) + \xi \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \tag{3.5}$$

*where $(\mu, \xi)$ is the unique pair in $\mathcal{U}^{m-1,\alpha}_{\partial\mathbb{I}} \times \mathbb{R}$ that solves*

$$\mathcal{L}_{\mathbb{I},\phi}[\mu, \xi](t) = \Gamma(t) \qquad \forall t \in \partial\mathbb{I}, \tag{3.6}$$

*with $\mathcal{L}_{\mathbb{I},\phi}$ as in Proposition 3.3.*

*Proof.* The uniqueness has already been proved in Proposition 3.2. We need to prove the existence. Let $(\mu, \xi) \in \mathcal{U}^{m-1,\alpha}_{\partial\mathbb{I}} \times \mathbb{R}$ be a solution of (3.6). We have

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} v_a^-[\partial\mathbb{I}, \mu](t) + \phi(t)(v_a^-[\partial\mathbb{I}, \mu](t) + \xi) = \Gamma(t) \qquad \forall t \in \partial\mathbb{I}.$$

The existence of such a pair $(\mu, \xi)$ is ensured by Proposition 3.3. Since $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$, then $v_a^-[\partial\mathbb{I}, \mu] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}])$. Since $\int_{\partial\mathbb{I}} \mu \, d\sigma = 0$, then $v_a^-[\partial\mathbb{I}, \mu]$ is harmonic in $\mathbb{T}_a[\mathbb{I}]$. Finally, if $u$ is as in (3.5), then $u$ is a periodic harmonic function, such that

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} u(t) + \phi(t)u(t) = \Gamma(t) \qquad \forall t \in \partial\mathbb{I},$$

and we can immediately conclude. □

## 3.2 Asymptotic behaviour of the solutions of the linear Robin problem for the Laplace equation in a periodically perforated domain

In this Section we study the asymptotic behaviour of the solutions of the Robin problem for the Laplace equation in a periodically perforated domain with small holes.

### 3.2.1 Notation

We retain the notation introduced in Subsection 1.8.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). We shall consider also the following assumptions.

$$f \in C^{m-1,\alpha}(\partial\Omega), \; f \leq 0, \; \int_{\partial\Omega} f \, d\sigma < 0; \tag{3.7}$$

$$g \in C^{m-1,\alpha}(\partial\Omega). \tag{3.8}$$

### 3.2.2 Preliminaries

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. For each $\epsilon \in ]0, \epsilon_1[$, we consider the following periodic linear Robin problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1, \dots, n\}, \\ \frac{\partial}{\partial\nu_{\Omega_\epsilon}} u(x) + f(\frac{1}{\epsilon}(x - w))u(x) = g(\frac{1}{\epsilon}(x - w)) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{3.9}$$

By virtue of Theorem 3.4, we can give the following definition.

**Definition 3.5.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. For each $\epsilon \in ]0, \epsilon_1[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of boundary value problem (3.9).

Since we want to represent the functions $u[\epsilon]$ by means of a periodic simple layer potential and a constant (cf. Theorem 3.4), we need to study some integral equations. Indeed, by virtue of Theorem 3.4, we can transform (3.9) into an integral equation, whose unknowns are the moment of the simple layer potential and the additive constant. Moreover, we want to transform these equations defined on the $\epsilon$-dependent domain $\partial\Omega_\epsilon$ into equations defined on the fixed domain $\partial\Omega$. We introduce these integral equations in the following Proposition. The relation between the solution of the integral equations and the solution of boundary value problem (3.9) will be clarified later.

**Proposition 3.6.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $\mathcal{U}_\epsilon^{m-1,\alpha}$, $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.63), (1.64), *respectively. Let $\Lambda$ be the map of $]-\epsilon_1,\epsilon_1[\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$ in $C^{m-1,\alpha}(\partial\Omega)$ defined by*

$$\begin{aligned}\Lambda[\epsilon,\theta,\xi](t) \equiv \frac{1}{2}\theta(t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot DS_n(t-s)\theta(s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s \\ + f(t)\Big(\epsilon\int_{\partial\Omega}S_n(t-s)\theta(s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega}R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s + \xi\Big) - g(t) \qquad \forall t\in\partial\Omega,\end{aligned} \tag{3.10}$$

*for all $(\epsilon,\theta,\xi)\in]-\epsilon_1,\epsilon_1[\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$. Then the following statements hold.*

*(i) If $\epsilon\in]0,\epsilon_1[$, then the pair $(\theta,\xi)\in\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$ satisfies equation*

$$\Lambda[\epsilon,\theta,\xi]=0, \tag{3.11}$$

*if and only if the pair $(\mu,\xi)\in\mathcal{U}_\epsilon^{m-1,\alpha}\times\mathbb{R}$, with $\mu\in\mathcal{U}_\epsilon^{m-1,\alpha}$ defined by*

$$\mu(x)\equiv\theta(\frac{1}{\epsilon}(x-w)) \qquad \forall x\in\partial\Omega_\epsilon, \tag{3.12}$$

*satisfies the equation*

$$\begin{aligned}\Gamma(x) = \frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}(S_n^a(x-y))\mu(y)\,d\sigma_y \\ + \phi(x)\Big(\int_{\partial\Omega_\epsilon}S_n^a(x-y)\mu(y)\,d\sigma_y+\xi\Big) \qquad \forall x\in\partial\Omega_\epsilon,\end{aligned} \tag{3.13}$$

*with $\Gamma$, $\phi\in C^{m-1,\alpha}(\partial\Omega_\epsilon)$ defined by*

$$\Gamma(x)\equiv g(\frac{1}{\epsilon}(x-w)) \qquad \forall x\in\partial\Omega_\epsilon. \tag{3.14}$$

*and*

$$\phi(x)\equiv f(\frac{1}{\epsilon}(x-w)) \qquad \forall x\in\partial\Omega_\epsilon. \tag{3.15}$$

*In particular, equation* (3.11) *has exactly one solution $(\theta,\xi)\in\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$, for each $\epsilon\in]0,\epsilon_1[$.*

*(ii) The pair $(\theta,\xi)\in\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$ satisfies equation*

$$\Lambda[0,\theta,\xi]=0, \tag{3.16}$$

*if and only if*

$$g(t)=\frac{1}{2}\theta(t)+\int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\,d\sigma_s+f(t)\xi \qquad \forall t\in\partial\Omega. \tag{3.17}$$

*In particular, equation* (3.16) *has exactly one solution $(\theta,\xi)\in\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$, which we denote by $(\tilde\theta,\tilde\xi)$.*

*Proof.* Consider $(i)$. Let $\theta\in C^{m-1,\alpha}(\partial\Omega)$. Let $\epsilon\in]0,\epsilon_1[$. First of all, we note that

$$\int_{\partial\Omega_\epsilon}\theta(\frac{1}{\epsilon}(x-w))\,d\sigma_x=\epsilon^{n-1}\int_{\partial\Omega}\theta(t)\,d\sigma_t,$$

and so $\theta \in \mathcal{U}_0^{m-1,\alpha}$ if and only if $\theta(\frac{1}{\epsilon}(\cdot - w)) \in \mathcal{U}_\epsilon^{m-1,\alpha}$. The equivalence of equation (3.11) in the unknown $(\theta, \xi) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ and equation (3.13) in the unknown $(\mu, \xi) \in \mathcal{U}_\epsilon^{m-1,\alpha} \times \mathbb{R}$ follows by a straightforward computation based on the rule of change of variables in integrals, on well known properties of composition of functions in Schauder spaces (cf. *e.g.*, Lanza [67, Sections 3,4]) and Lemma 1.25. The existence and uniqueness of a solution of equation (3.13) follows by Proposition 3.3. Then the existence and uniqueness of a solution of equation (3.11) follows by the equivalence of (3.11) and (3.13). Consider (*ii*). The equivalence of (3.16) and (3.17) is obvious. The existence of a unique solution of equation (3.17) is an immediate consequence of well known results of classical potential theory. Indeed, there exists a unique $\xi \in \mathbb{R}$, such that

$$\int_{\partial\Omega} g(t)\, d\sigma_t - \xi \int_{\partial\Omega} f(t)\, d\sigma_t = 0.$$

Then there exists a unique $\theta \in \mathcal{U}_0^{m-1,\alpha}$ such that

$$\frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\, d\sigma_s = g(t) - \frac{\int_{\partial\Omega} g(s)\, d\sigma_s}{\int_{\partial\Omega} f(s)\, d\sigma_s} f(t) \qquad \forall t \in \partial\Omega$$

(cf. Folland [52, Chapter 3, Section E, and Lemma 3.30] for the existence of $\theta$ in $C^0(\partial\Omega)$ and Lanza [72, Appendix A] for the $C^{m-1,\alpha}$ regularity.) □

By Proposition 3.6, it makes sense to introduce the following.

**Definition 3.7.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in (1.64). For each $\epsilon \in ]0, \epsilon_1[$, we denote by $(\hat{\theta}[\epsilon], \hat{\xi}[\epsilon])$ the unique pair in $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ that solves (3.11). Analogously, we denote by $(\hat{\theta}[0], \hat{\xi}[0])$ the unique pair in $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ that solves (3.16).

In the following Remark, we show the relation between the solutions of boundary value problem (3.9) and the solutions of equation (3.11).

*Remark* 3.8. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively.

Let $\epsilon \in ]0, \epsilon_1[$. We have

$$u[\epsilon](x) = \epsilon^{n-1} \int_{\partial\Omega} S_n^a(x - w - \epsilon s)\hat{\theta}[\epsilon](s)\, d\sigma_s + \hat{\xi}[\epsilon] \qquad \forall x \in \operatorname{cl} \mathbb{T}_a[\Omega_\epsilon].$$

While the relation between equation (3.11) and boundary value problem (3.9) is now clear, we want to see if (3.16) is related to some (limiting) boundary value problem. We give the following.

**Definition 3.9.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $f$, $g$ be as in (1.56), (3.7), (3.8), respectively. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega)$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \operatorname{cl}\Omega, \\ \frac{\partial}{\partial\nu_\Omega} u(x) = g(x) - \frac{\int_{\partial\Omega} g\, d\sigma}{\int_{\partial\Omega} f\, d\sigma} f(x) & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{3.18}$$

Problem (3.18) will be called the *limiting boundary value problem.*

*Remark* 3.10. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. We have

$$\tilde{u}(x) = \int_{\partial\Omega} S_n(x-y)\hat{\theta}[0](y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \Omega.$$

We now prove the following.

**Proposition 3.11.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8)*, respectively. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.64)*. Let $\Lambda$ and $(\tilde{\theta}, \tilde{\xi})$ be as in Proposition 3.6. Then there exists $\epsilon_2 \in ]0, \epsilon_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$.*

*Moreover, if we set* $b_0 \equiv (0, \tilde{\theta}, \tilde{\xi})$*, then the differential* $\partial_{(\theta,\xi)}\Lambda[b_0]$ *of* $\Lambda$ *with respect to the variables* $(\theta, \xi)$ *at* $b_0$ *is delivered by the following formula*

$$\partial_{(\theta,\xi)}\Lambda[b_0](\bar{\theta}, \bar{\xi})(t) = \frac{1}{2}\bar{\theta}(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\bar{\theta}(s)\, d\sigma_s + f(t)\bar{\xi} \qquad \forall t \in \partial\Omega, \tag{3.19}$$

*for all* $(\bar{\theta}, \bar{\xi}) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$*, and is a linear homeomorphism of* $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ *onto* $C^{m-1,\alpha}(\partial\Omega)$*.*

*Proof.* By Proposition 1.26 (*i*), (*ii*) and by the continuity of the pointwise product in Schauder space, we easily deduce that there exists $\epsilon_2 \in ]0, \epsilon_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$. By standard calculus in Banach space, we immediately deduce that (3.19) holds. Now we need to prove that $\partial_{(\theta,\xi)}\Lambda[b_0]$ is a linear homeomorphism. By the Open Mapping Theorem, it suffices to prove that it is a bijection. Let $\psi \in C^{m-1,\alpha}(\partial\Omega)$. We want to prove that there exists a unique pair $(\bar{\theta}, \bar{\xi}) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$, such that

$$\frac{1}{2}\bar{\theta}(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\bar{\theta}(s)\, d\sigma_s + f(t)\bar{\xi} = \psi(t) \qquad \forall t \in \partial\Omega. \tag{3.20}$$

We first prove uniqueness. Let $(\bar{\theta}, \bar{\xi}) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ solve (3.20). By integrating both sides of (3.20) and by the well known identity

$$\int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}\big(S_n(s-t)\big)\, d\sigma_t = \frac{1}{2} \qquad \forall s \in \partial\Omega,$$

we have that

$$\int_{\partial\Omega} \bar{\theta}(t)\, d\sigma_t + \bar{\xi}\int_{\partial\Omega} f(t)\, d\sigma_t = \int_{\partial\Omega} \psi(t)\, d\sigma_t,$$

and accordingly, since $\int_{\partial\Omega} \bar{\theta}\, d\sigma = 0$,

$$\bar{\xi} = \frac{\int_{\partial\Omega} \psi(t)\, d\sigma_t}{\int_{\partial\Omega} f(t)\, d\sigma_t}. \tag{3.21}$$

Then, by known results of classical potential theory (cf. Folland [52, Chapter 3]), $\bar{\theta}$ is the unique element of $\mathcal{U}_0^{m-1,\alpha}$ such that

$$\frac{1}{2}\bar{\theta}(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\bar{\theta}(s)\, d\sigma_s = \psi(t) - f(t)\frac{\int_{\partial\Omega} \psi(t)\, d\sigma_t}{\int_{\partial\Omega} f(t)\, d\sigma_t} \qquad \forall t \in \partial\Omega. \tag{3.22}$$

Hence uniqueness follows. Conversely, in order to prove existence, it suffices to note that the pair $(\bar{\theta}, \bar{\xi}) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$, with $\bar{\xi}$ delivered by (3.21) and where $\bar{\theta}$ is the unique solution in $\mathcal{U}_0^{m-1,\alpha}$ of (3.22), solves equation (3.20). □

We are now ready to prove that $\hat{\theta}[\cdot], \hat{\xi}[\cdot]$ can be continued real analytically in a whole neighbourhood of 0.

**Proposition 3.12.** *Let* $m \in \mathbb{N} \setminus \{0\}$*,* $\alpha \in ]0, 1[$*. Let* $w \in A$*. Let* $\Omega$*,* $\epsilon_1$*,* $f$*,* $g$ *be as in* (1.56), (1.57), (3.7), (3.8)*, respectively. Let* $\mathcal{U}_0^{m-1,\alpha}$ *be as in* (1.64)*. Let* $\epsilon_2$ *be as in Proposition 3.11. Then there exist* $\epsilon_3 \in ]0, \epsilon_2]$ *and a real analytic operator* $(\Theta, \Xi)$ *of* $]-\epsilon_3, \epsilon_3[$ *to* $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$*, such that*

$$(\Theta[\epsilon], \Xi[\epsilon]) = (\hat{\theta}[\epsilon], \hat{\xi}[\epsilon]), \tag{3.23}$$

*for all* $\epsilon \in [0, \epsilon_3[$*.*

*Proof.* It is an immediate consequence of Proposition 3.11 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

### 3.2.3 A functional analytic representation Theorem for the solution of the singularly perturbed linear Robin problem

By Proposition 3.12 and Remark 3.8, we can deduce the main result of this Subsection. More precisely, we show that $\{u[\epsilon](\cdot)\}_{\epsilon\in]0,\epsilon_1[}$ can be continued real analytically for negative values of $\epsilon$.

We have the following.

**Theorem 3.13.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $\epsilon_3$ be as in Proposition 3.12. Then the following statements hold.*

*(i) Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0,\epsilon_3]$, a real analytic operator $U_1$ of $]-\epsilon_4,\epsilon_4[$ to the space $C^0_h(\mathrm{cl}\, V)$, and a real analytic operator $U_2$ of $]-\epsilon_4,\epsilon_4[$ to $\mathbb{R}$ such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4,\epsilon_4[$.

*(jj)*

$$u[\epsilon](x) = \epsilon^n U_1[\epsilon](x) + U_2[\epsilon] \qquad \forall x \in \mathrm{cl}\, V,$$

*for all $\epsilon \in ]0,\epsilon_4[$. Moreover,*

$$U_2[0] = \frac{\int_{\partial\Omega} g\, d\sigma}{\int_{\partial\Omega} f\, d\sigma}.$$

*(ii) Let $\bar{V}$ be a bounded open subset of $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$. Then there exist $\bar{\epsilon}_4 \in ]0,\epsilon_3]$, a real analytic operator $\bar{U}_1$ of $]-\bar{\epsilon}_4,\bar{\epsilon}_4[$ to the space $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$, and a real analytic operator $\bar{U}_2$ of $]-\bar{\epsilon}_4,\bar{\epsilon}_4[$ to $\mathbb{R}$ such that the following conditions hold.*

*(j')* $w + \epsilon\, \mathrm{cl}\,\bar{V} \subseteq \mathrm{cl}\,\mathbb{P}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\bar{\epsilon}_4,\bar{\epsilon}_4[\setminus\{0\}$.

*(jj')*

$$u[\epsilon](w+\epsilon t) = \epsilon \bar{U}_1[\epsilon](t) + \bar{U}_2[\epsilon] \qquad \forall t \in \mathrm{cl}\,\bar{V},$$

*for all $\epsilon \in ]0,\bar{\epsilon}_4[$. Moreover,*

$$\bar{U}_2[0] = \frac{\int_{\partial\Omega} g\, d\sigma}{\int_{\partial\Omega} f\, d\sigma}.$$

*Proof.* Let $\Theta[\cdot]$, $\Xi[\cdot]$ be as in Proposition 3.12. Consider $(i)$. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $\epsilon \in ]0,\epsilon_4[$. By Remark 3.8 and Proposition 3.12, we have

$$u[\epsilon](x) = \epsilon^{n-1}\int_{\partial\Omega} S^a_n(x-w-\epsilon s)\Theta[\epsilon](s)\, d\sigma_s + \Xi[\epsilon] \qquad \forall x \in \mathrm{cl}\, V.$$

Thus, it is natural to set

$$\tilde{U}_1[\epsilon](x) \equiv \int_{\partial\Omega} S^a_n(x-w-\epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

for all $\epsilon \in ]-\epsilon_4,\epsilon_4[$, and

$$U_2[\epsilon] \equiv \Xi[\epsilon],$$

for all $\epsilon \in ]-\epsilon_4,\epsilon_4[$. By following the proof of Theorem 2.124 and by Proposition 3.12, we have that $U_2$ is a real analytic map of $]-\epsilon_4,\epsilon_4[$ to $\mathbb{R}$ and that, by possibly taking a smaller $\epsilon_4$, there exists a real analytic map $U_1$ of $]-\epsilon_4,\epsilon_4[$ to $C^0_h(\mathrm{cl}\, V)$ such that

$$\tilde{U}_1[\epsilon] = \epsilon U_1[\epsilon] \qquad \text{in } C^0_h(\mathrm{cl}\, V),$$

for all $\epsilon \in ]-\epsilon_4,\epsilon_4[$ and that the equality in $(jj)$ holds. Moreover, by Propositions 3.6 and 3.12, we have that

$$U_2[0] = \frac{\int_{\partial\Omega} g\, d\sigma}{\int_{\partial\Omega} f\, d\sigma}.$$

Consider now $(ii)$. Choosing $\bar{\epsilon}_4$ small enough, we can clearly assume that $(j')$ holds. Consider now $(jj')$. Let $\epsilon \in ]0,\bar{\epsilon}_4[$. By Remark 3.8 and Proposition 3.12, we have

$$u[\epsilon](w+\epsilon t) = \epsilon^{n-1}\int_{\partial\Omega} S^a_n(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s + \Xi[\epsilon] \qquad \forall t \in \mathrm{cl}\,\bar{V}.$$

Since $\int_{\partial\Omega} \Theta[\epsilon](s)\, d\sigma_s = 0$ for all $\epsilon \in [0, \epsilon_3[$, by Proposition 1.29 $(ii)$, it is natural to set

$$\bar{U}_1[\epsilon](t) \equiv \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \mathrm{cl}\, \bar{V},$$

for all $\epsilon \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[$, and

$$\bar{U}_2[\epsilon] \equiv \Xi[\epsilon],$$

for all $\epsilon \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[$. Obviously, the equality in $(jj')$ holds. By Proposition 3.12, we have that $\bar{U}_2$ is a real analytic map of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ to $\mathbb{R}$. Moreover, by Propositions 3.6 and 3.12, we have that

$$\bar{U}_2[0] = \frac{\int_{\partial\Omega} g\, d\sigma}{\int_{\partial\Omega} f\, d\sigma}.$$

Consider now $\bar{U}_1$. By Proposition 1.29 $(ii)$, by possibly taking a smaller $\bar{\epsilon}_4$, we have that $\bar{U}_1$ is a real analytic operator of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ to $C^{m,\alpha}(\mathrm{cl}\, \bar{V})$. Thus the proof is complete. □

*Remark* 3.14. Let the assumptions of Theorem 3.13 $(i)$ hold. Let $\Theta[\cdot]$ be as in Proposition 3.12. If $\epsilon \in ]-\epsilon_4, \epsilon_4[$ and $x \in \mathrm{cl}\, V$, then, by the Taylor Formula, we have

$$S_n^a(x - w - \epsilon s) = S_n^a(x - w) - \epsilon \int_0^1 DS_n^a(x - w - \beta\epsilon s)s\, d\beta \qquad \forall s \in \partial\Omega.$$

As a consequence, since $\int_{\partial\Omega} \Theta[\epsilon]\, d\sigma = 0$ for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$, we have

$$\begin{aligned}
&\int_{\partial\Omega} S_n^a(x - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \\
&= S_n^a(x - w) \int_{\partial\Omega} \Theta[\epsilon](s)\, d\sigma_s - \epsilon \int_{\partial\Omega} \Bigl( \int_0^1 DS_n^a(x - w - \beta\epsilon s)s\, d\beta \Bigr) \Theta[\epsilon](s)\, d\sigma_s \\
&= -\epsilon \int_{\partial\Omega} \Bigl( \int_0^1 DS_n^a(x - w - \beta\epsilon s)s\, d\beta \Bigr) \Theta[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,
\end{aligned}$$

for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$. Thus, if $U_1$ is as in Theorem 3.13 $(i)$, one can easily check that

$$\begin{aligned}
U_1[0](x) &= -\sum_{j=1}^n \partial_{x_j} S_n^a(x - w) \int_{\partial\Omega} s_j \Theta[0](s)\, d\sigma_s \\
&= -\sum_{j=1}^n \partial_{x_j} S_n^a(x - w) \int_{\partial\Omega} s_j \hat{\theta}[0](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V.
\end{aligned}$$

By well known jump formulas for the normal derivative of the classical simple layer potential $v[\partial\Omega, \hat{\theta}[0]]$ (cf. Appendix B and (B.2)), we have

$$\int_{\partial\Omega} s_j \hat{\theta}[0](s)\, d\sigma_s = \int_{\partial\Omega} s_j \frac{\partial}{\partial\nu_\Omega} v^-[\partial\Omega, \hat{\theta}[0]](s)\, d\sigma_s - \int_{\partial\Omega} s_j \frac{\partial}{\partial\nu_\Omega} v^+[\partial\Omega, \hat{\theta}[0]](s)\, d\sigma_s.$$

By the Divergence Theorem,

$$\int_{\partial\Omega} s_j \frac{\partial}{\partial\nu_\Omega} v^+[\partial\Omega, \hat{\theta}[0]](s)\, d\sigma_s = \int_{\partial\Omega} (\nu_\Omega(s))_j v^+[\partial\Omega, \hat{\theta}[0]](s)\, d\sigma_s.$$

As a consequence,

$$\int_{\partial\Omega} s_j \hat{\theta}[0](s)\, d\sigma_s = \int_{\partial\Omega} s_j \frac{\partial}{\partial\nu_\Omega} \tilde{u}(s)\, d\sigma_s - \int_{\partial\Omega} (\nu_\Omega(s))_j \tilde{u}(s)\, d\sigma_s.$$

and accordingly

$$U_1[0](x) = -DS_n^a(x - w) \int_{\partial\Omega} s_j \frac{\partial}{\partial\nu_\Omega} \tilde{u}(s)\, d\sigma_s + DS_n^a(x - w) \int_{\partial\Omega} (\nu_\Omega(s))_j \tilde{u}(s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V.$$

*Remark* 3.15. We note that the right-hand side of the equalities in $(jj)$ and $(jj')$ of Theorem 3.13 can be continued real analytically in the whole $]-\epsilon_4, \epsilon_4[$. Moreover, if $V$ is a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$, then

$$\lim_{\epsilon \to 0^+} u[\epsilon] = \frac{\int_{\partial\Omega} g\, d\sigma}{\int_{\partial\Omega} f\, d\sigma} \qquad \text{uniformly in } \mathrm{cl}\, V.$$

### 3.2.4 A real analytic continuation Theorem for the energy integral

As done in Theorem 3.13 for $u[\cdot]$, we can now prove a real analytic continuation Theorem for the energy integral. Namely, we prove the following.

**Theorem 3.16.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $\epsilon_3$ be as in Proposition 3.12. Then there exist $\epsilon_5 \in ]0, \epsilon_3]$ and a real analytic operator $G$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2 \, dx = \epsilon^n G[\epsilon], \tag{3.24}$$

*for all $\epsilon \in ]0, \epsilon_5[$. Moreover,*

$$G[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx. \tag{3.25}$$

*Proof.* Let $\epsilon \in ]0, \epsilon_3[$. Clearly,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2 \, dx = \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2 \, dx.$$

We have

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} & |\nabla v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2 \, dx \\ &= -\epsilon^{n-1} \int_{\partial\Omega} v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](w + \epsilon t) \Big( \frac{\partial}{\partial \nu_{\Omega_\epsilon}} v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))] \Big)(w + \epsilon t) \, d\sigma_t.\end{aligned}$$

Also

$$\begin{aligned} v_a^-[\partial\Omega_\epsilon, & \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](w + \epsilon t) \\ &= \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s) \, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s) \, d\sigma_s \qquad \forall t \in \partial\Omega,\end{aligned}$$

and

$$\begin{aligned} \Big( \frac{\partial}{\partial \nu_{\Omega_\epsilon}} & v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))] \Big)(w + \epsilon t) \\ &= \frac{1}{2}\Theta[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\Theta[\epsilon](s) \, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s) \, d\sigma_s \\ & \qquad \forall t \in \partial\Omega.\end{aligned}$$

Choosing $\epsilon_5 \in ]0, \epsilon_3]$ small enough, the map of $]-\epsilon_5, \epsilon_5[$ to $C^0(\partial\Omega)$ which takes $\epsilon$ to the function of the variable $t \in \partial\Omega$ defined by

$$\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s) \, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s) \, d\sigma_s \qquad \forall t \in \partial\Omega,$$

and the map of $]-\epsilon_5, \epsilon_5[$ to $C^0(\partial\Omega)$ which takes $\epsilon$ to the function of the variable $t \in \partial\Omega$ defined by

$$\frac{1}{2}\Theta[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\Theta[\epsilon](s) \, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s) \, d\sigma_s \qquad \forall t \in \partial\Omega,$$

are real analytic (cf. Proposition 1.28 (*i*), (*ii*).) Thus the map $G$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{R}$ which takes $\epsilon$ to

$$\begin{aligned} G[\epsilon] \equiv & - \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s) \, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s) \, d\sigma_s \Big) \\ & \times \Big( \frac{1}{2}\Theta[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\Theta[\epsilon](s) \, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s) \, d\sigma_s \Big) \, d\sigma_t \end{aligned}$$

is real analytic. Clearly,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2 \, dx = \epsilon^n G[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_5[$ Moreover, since

$$\tilde{u}(x) = \int_{\partial\Omega} S_n(x-y)\Theta[0](y)\, d\sigma_y \qquad \forall x \in \mathrm{cl}\,\Omega,$$

we have

$$G[0] \equiv \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx$$

(see also Folland [52, p. 118].) □

*Remark* 3.17. We note that the right-hand side of the equality in (3.24) of Theorem 3.16 can be continued real analytically in the whole $]-\epsilon_5, \epsilon_5[$. Moreover,

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2 \, dx = 0$$

### 3.2.5 A real analytic continuation Theorem for the integral of the solution

As done in Theorem 3.16 for the energy integral, we can now prove a real analytic continuation Theorem for the integral of the solution. Namely, we prove the following.

**Theorem 3.18.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $\epsilon_3$ be as in Proposition 3.12. Then there exist $\epsilon_6 \in ]0, \epsilon_3]$ and a real analytic operator $J$ of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx = J[\epsilon], \tag{3.26}$$

*for all $\epsilon \in ]0, \epsilon_6[$. Moreover,*

$$J[0] = \frac{\int_{\partial\Omega} g \, d\sigma}{\int_{\partial\Omega} f \, d\sigma} |A|_n. \tag{3.27}$$

*Proof.* It suffices to modify the proof of Theorem 2.128. Let $\Theta[\cdot]$, $\Xi[\cdot]$ be as in Proposition 3.12. Let $\epsilon \in ]0, \epsilon_3[$. Since

$$u[\epsilon](x) = v_a^-\big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) + \Xi[\epsilon] \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],$$

then

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx = \int_{\mathbb{P}_a[\Omega_\epsilon]} v_a^-\big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x)\, dx + \Xi[\epsilon]\big(|A|_n - \epsilon^n |\Omega|_n\big).$$

On the other hand, by arguing as in the proof of Theorem 2.128, we can show that there exists $\epsilon_6 \in ]0, \epsilon_3]$ and a real analytic map $J_1$ of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} v_a^-\big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x)\, dx = J_1[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_6[$. Moreover, $J_1[0] = 0$. Then, if we set

$$J[\epsilon] \equiv J_1[\epsilon] + \Xi[\epsilon](|A|_n - \epsilon^n |\Omega|_n),$$

for all $\epsilon \in ]-\epsilon_6, \epsilon_6[$, we can immediately conclude. □

# 3.3 An homogenization problem for the Laplace equation with linear Robin boundary condition in a periodically perforated domain

In this section we consider an homogenization problem for the Laplace equation with linear Robin boundary condition in a periodically perforated domain.

## 3.3.1 Notation

In this Section we retain the notation introduced in Subsections 1.8.1, 3.2.1. However, we need to introduce also some other notation.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $(\epsilon,\delta) \in (]-\epsilon_1,\epsilon_1[ \setminus \{0\}) \times ]0,+\infty[$. If $v$ is a function of $\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta)$ to $\mathbb{R}$, then we denote by $\mathbf{E}_{(\epsilon,\delta)}[v]$ the function of $\mathbb{R}^n$ to $\mathbb{R}$, defined by

$$\mathbf{E}_{(\epsilon,\delta)}[v](x) \equiv \begin{cases} v(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta) \\ 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta). \end{cases}$$

If $v$ is a function of $\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta)$ to $\mathbb{R}$ and $c \in \mathbb{R}$, then we denote by $\mathbf{E}^{\#}_{(\epsilon,\delta)}[v,c]$ the function of $\mathbb{R}^n$ to $\mathbb{R}$, defined by

$$\mathbf{E}^{\#}_{(\epsilon,\delta)}[v,c](x) \equiv \begin{cases} v(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \\ c & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta). \end{cases}$$

## 3.3.2 Preliminaries

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. For each $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we consider the following periodic linear Robin problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u(x+\delta a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall i \in \{1,\dots,n\}, \\ \delta \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x) + f(\frac{1}{\epsilon\delta}(x-\delta w))u(x) = g(\frac{1}{\epsilon\delta}(x-\delta w)) & \forall x \in \partial\Omega(\epsilon,\delta). \end{cases} \tag{3.28}$$

By virtue of Theorem 3.4, we can give the following definition.

**Definition 3.19.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we denote by $u_{(\epsilon,\delta)}$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta))$ of boundary value problem (3.28).

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. In order to do so we introduce the following.

**Definition 3.20.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. For each $\epsilon \in ]0,\epsilon_1[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of the following periodic linear Robin problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) + f(\frac{1}{\epsilon}(x-w))u(x) = g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{3.29}$$

*Remark* 3.21. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we have

$$u_{(\epsilon,\delta)}(x) = u[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).$$

By the previous remark, we note that the solution of problem (3.28) can be expressed by means of the solution of the auxiliary rescaled problem (3.29), which does not depend on $\delta$. This is due to the presence of the factor $\delta$ in front of $\frac{\partial}{\partial\nu_{\Omega(\epsilon,\delta)}}u(x)$ in the third equation of problem (3.28).

As a first step, we study the behaviour of $u[\epsilon]$ as $\epsilon$ tends to 0 and we have the following.

**Proposition 3.22.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $1 \leq p < \infty$. Then*

$$\lim_{\epsilon\to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma} \qquad \text{in } L^p(A).$$

*Proof.* It suffices to modify the proof of Proposition 2.132. Let $\epsilon_3$, $\Theta$, $\Xi$ be as in Proposition 3.12. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $\epsilon \in ]0,\epsilon_3[$, we have

$$\begin{aligned} &u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) \\ &\quad = \epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s + \Xi[\epsilon] \qquad \forall t\in\partial\Omega. \end{aligned}$$

We set

$$N[\epsilon](t) \equiv \epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s + \Xi[\epsilon] \qquad \forall t\in\partial\Omega,$$

for all $\epsilon \in ]-\epsilon_3,\epsilon_3[$. By taking $\tilde{\epsilon} \in ]0,\epsilon_3[$ small enough, we can assume (cf. Proposition 1.28 $(i)$) that $N$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C \equiv \sup_{\epsilon\in]-\tilde{\epsilon},\tilde{\epsilon}[} \|N[\epsilon]\|_{C^0(\partial\Omega)} < +\infty.$$

By Theorem 2.5, we have

$$|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]](x)| \leq C \qquad \forall x\in A, \quad \forall \epsilon\in ]0,\tilde{\epsilon}[.$$

By Theorem 3.13, we have

$$\lim_{\epsilon\to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]](x) = \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma} \qquad \forall x\in A\setminus\{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{\epsilon\to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma} \qquad \text{in } L^p(A).$$

□

**Proposition 3.23.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $\epsilon_3$ be as in Proposition 3.12. Then there exist $\tilde{\epsilon} \in ]0,\epsilon_3[$ and a real analytic map $N$ of $]-\tilde{\epsilon},\tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ such that*

$$\Big\|\mathbf{E}^{\#}_{(\epsilon,1)}\Big[u[\epsilon], \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\Big] - \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\Big\|_{L^\infty(\mathbb{R}^n)} = \epsilon\|N[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all $\epsilon \in ]0,\tilde{\epsilon}[$. Moreover, as a consequence,*

$$\lim_{\epsilon\to 0^+} \mathbf{E}^{\#}_{(\epsilon,1)}\Big[u[\epsilon], \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\Big] = \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma} \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* Let $\epsilon_3$, $\Theta$, $\Xi$ be as in Proposition 3.12. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $\epsilon \in ]0,\epsilon_3[$, we have

$$\begin{aligned} &u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) \\ &\quad = \epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s + \Xi[\epsilon] \qquad \forall t\in\partial\Omega. \end{aligned}$$

Since $\Xi[\cdot]$ is a real analytic function and

$$\Xi[0] = \frac{\int_{\partial\Omega} g \, d\sigma}{\int_{\partial\Omega} f \, d\sigma},$$

then there exist $\tilde{\epsilon} \in ]0, \epsilon_3[$ and a real analytic function $R_\Xi$ of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $\mathbb{R}$ such that

$$\Xi[\epsilon] - \frac{\int_{\partial\Omega} g \, d\sigma}{\int_{\partial\Omega} f \, d\sigma} = \epsilon R_\Xi[\epsilon] \qquad \forall \epsilon \in ]-\tilde{\epsilon}, \tilde{\epsilon}[.$$

We set

$$N[\epsilon](t) \equiv \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s + R_\Xi[\epsilon], \qquad \forall t \in \partial\Omega,$$

for all $\epsilon \in ]-\tilde{\epsilon}, \tilde{\epsilon}[$. We have that $N$ is a real analytic map of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$.

By Theorem 2.5, we have

$$\|\mathbf{E}_{(\epsilon,1)}\Big[u[\epsilon], \frac{\int_{\partial\Omega} g \, d\sigma}{\int_{\partial\Omega} f \, d\sigma}\Big] - \frac{\int_{\partial\Omega} g \, d\sigma}{\int_{\partial\Omega} f \, d\sigma}\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)} \qquad \forall \epsilon \in ]0, \tilde{\epsilon}[,$$

and the conclusion easily follows. □

### 3.3.3 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorem we deduce by Proposition 3.22 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon, \delta)$ tends to $(0, 0)$. Namely, we prove the following.

**Theorem 3.24.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $1 \leq p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = \frac{\int_{\partial\Omega} g \, d\sigma}{\int_{\partial\Omega} f \, d\sigma} \qquad \text{in } L^p(V).$$

*Proof.* We modify the proof of Theorem 2.134. By virtue of Proposition 3.22, we have

$$\lim_{\epsilon\to 0^+} \|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]] - \frac{\int_{\partial\Omega} g \, d\sigma}{\int_{\partial\Omega} f \, d\sigma}\|_{L^p(A)} = 0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $C > 0$ such that

$$\|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] - \frac{\int_{\partial\Omega} g \, d\sigma}{\int_{\partial\Omega} f \, d\sigma}\|_{L^p(V)} \leq C \|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]] - \frac{\int_{\partial\Omega} g \, d\sigma}{\int_{\partial\Omega} f \, d\sigma}\|_{L^p(A)} \qquad \forall (\epsilon,\delta) \in ]0, \epsilon_3[ \times ]0, 1[.$$

Thus,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] - \frac{\int_{\partial\Omega} g \, d\sigma}{\int_{\partial\Omega} f \, d\sigma}\|_{L^p(V)} = 0,$$

and we can easily conclude. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 3.25.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $\epsilon_6$, $J$ be as in Theorem 3.18. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\, dx = r^n J[\epsilon], \tag{3.30}$$

*for all $\epsilon \in ]0, \epsilon_6[$, and for all $l \in \mathbb{N} \setminus \{0\}$.*

*Proof.* It is a simple modification of the proof of Theorem 2.60. Indeed, let $\epsilon \in ]0, \epsilon_6[$, $l \in \mathbb{N} \setminus \{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](t)\,dt \\ &= \frac{r^n}{l^n} J[\epsilon].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J[\epsilon],$$

and the conclusion follows. □

In the following Theorem we consider the $L^\infty$–distance of a certain extension of $u_{(\epsilon,\delta)}$ and its limit.

**Theorem 3.26.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8)*, respectively. Let $\tilde{\epsilon}$, $N$ be as in Proposition 3.23. Then*

$$\|\mathbf{E}^{\#}_{(\epsilon,\delta)}\Big[u_{(\epsilon,\delta)}, \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\Big] - \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\|_{L^\infty(\mathbb{R}^n)} = \epsilon\|N[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[ \times ]0,+\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^{\#}_{(\epsilon,\delta)}\Big[u_{(\epsilon,\delta)}, \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\Big] = \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma} \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathbf{E}^{\#}_{(\epsilon,\delta)}\Big[u_{(\epsilon,\delta)}, \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\Big] - \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\|_{L^\infty(\mathbb{R}^n)} &= \|\mathbf{E}^{\#}_{(\epsilon,1)}\Big[u[\epsilon], \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\Big] - \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\|_{L^\infty(\mathbb{R}^n)} \\ &= \epsilon\|N[\epsilon]\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[ \times ]0,+\infty[$. □

### 3.3.4 Asymptotic behaviour of the energy integral of $u_{(\epsilon,\delta)}$

This Subsection is devoted to the study of the behaviour of the energy integral of $u_{(\epsilon,\delta)}$. We give the following.

**Definition 3.27.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we set

$$\mathrm{En}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 3.28. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. Let $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$. We have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} \left|\nabla u_{(\epsilon,\delta)}(x)\right|^2 dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} \left|(\nabla u_{(\epsilon,\delta)})(\delta t)\right|^2 dt \\ &= \delta^{n-2} \int_{\mathbb{P}_a[\Omega_\epsilon]} \left|\nabla u[\epsilon](t)\right|^2 dt.\end{aligned}$$

Then we give the following definition, where we consider $\mathrm{En}(\epsilon, \delta)$, with $\epsilon$ equal to a certain function of $\delta$.

**Definition 3.29.** For each $\delta \in ]0, +\infty[$, we set

$$\epsilon[\delta] \equiv \delta^{\frac{2}{n}}.$$

Let $\epsilon_5$ be as in Theorem 3.16. Let $\delta_1 > 0$ be such that $\epsilon[\delta] \in ]0, \epsilon_5[$, for all $\delta \in ]0, \delta_1[$. Then we set

$$\mathrm{En}[\delta] \equiv \mathrm{En}(\epsilon[\delta], \delta),$$

for all $\delta \in ]0, \delta_1[$.

In the following Proposition we compute the limit of $\mathrm{En}[\delta]$ as $\delta$ tends to 0.

**Proposition 3.30.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $\epsilon_5$ be as in Theorem 3.16. Let $\delta_1 > 0$ be as in Definition 3.29. Then*

$$\lim_{\delta \to 0^+} \mathrm{En}[\delta] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} \left|\nabla \tilde{u}(x)\right|^2 dx,$$

*where $\tilde{u}$ is as in Definition 3.9.*

*Proof.* We follow step by step the proof of Propostion 2.140. Let $G$ be as in Theorem 3.16. Let $\delta \in ]0, \delta_1[$. By Remark 3.28 and Theorem 3.16, we have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon[\delta],\delta)} \left|\nabla u_{(\epsilon[\delta],\delta)}(x)\right|^2 dx &= \delta^{n-2} (\epsilon[\delta])^n G[\epsilon[\delta]] \\ &= \delta^n G[\delta^{\frac{2}{n}}].\end{aligned}$$

On the other hand,

$$\lfloor (1/\delta) \rfloor^n \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} \left|\nabla u_{(\epsilon[\delta],\delta)}(x)\right|^2 dx \leq \mathrm{En}[\delta] \leq \lceil (1/\delta) \rceil^n \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} \left|\nabla u_{(\epsilon[\delta],\delta)}(x)\right|^2 dx,$$

and so

$$\lfloor (1/\delta) \rfloor^n \delta^n G[\delta^{\frac{2}{n}}] \leq \mathrm{En}[\delta] \leq \lceil (1/\delta) \rceil^n \delta^n G[\delta^{\frac{2}{n}}].$$

Thus, since

$$\lim_{\delta \to 0^+} \lfloor (1/\delta) \rfloor^n \delta^n = 1, \qquad \lim_{\delta \to 0^+} \lceil (1/\delta) \rceil^n \delta^n = 1,$$

we have

$$\lim_{\delta \to 0^+} \mathrm{En}[\delta] = G[0].$$

Finally, by equality (3.25), we easily conclude. □

In the following Proposition we represent the function $\mathrm{En}[\cdot]$ by means of a real analytic function.

**Proposition 3.31.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $\epsilon_5$ and $G$ be as in Theorem 3.16. Let $\delta_1 > 0$ be as in Definition 3.29. Then*

$$\mathrm{En}[(1/l)] = G[(1/l)^{\frac{2}{n}}],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* It follows by the proof of Proposition 3.30. □

## 3.4 A variant of the homogenization problem for the Laplace equation with linear Robin boundary condition in a periodically perforated domain

In this section we consider a slightly different homogenization problem for the Laplace equation with linear Robin boundary condition in a periodically perforated domain.

### 3.4.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 3.2.1, 3.3.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. For each $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, we consider the following periodic linear Robin problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u(x+\delta a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall i \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x) + f(\frac{1}{\epsilon\delta}(x-\delta w))u(x) = g(\frac{1}{\epsilon\delta}(x-\delta w)) & \forall x \in \partial\Omega(\epsilon,\delta). \end{cases} \tag{3.31}$$

In contrast to problem (3.28), we note that in the third equation of problem (3.31) there is not the factor $\delta$ in front of $\frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x)$.

By virtue of Theorem 3.4, we can give the following definition.

**Definition 3.32.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, we denote by $u_{(\epsilon,\delta)}$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta))$ of boundary value problem (3.31).

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. In order to do so we introduce the following.

**Definition 3.33.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. For each $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, we denote by $u[\epsilon,\delta]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of the following auxiliary periodic linear Robin problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) + \delta f(\frac{1}{\epsilon}(x-w))u(x) = \delta g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{3.32}$$

*Remark* 3.34. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, we have

$$u_{(\epsilon,\delta)}(x) = u[\epsilon,\delta](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).$$

By the previous remark, in contrast to the solution of problem (3.28), we note that the solution of problem (3.31) can be expressed by means of the solution of the auxiliary rescaled problem (3.32), which does depend on $\delta$.

As a first step, we study the behaviour of $u[\epsilon,\delta]$ as $(\epsilon,\delta)$ tends to $(0,0)$. As we know, we can convert boundary value problem (3.32) into an integral equation. We introduce this equation in the following.

**Proposition 3.35.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $f$, $g$ *be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let* $\mathcal{U}_\epsilon^{m-1,\alpha}$, $\mathcal{U}_0^{m-1,\alpha}$ *be as in* (1.63), (1.64), *respectively. Let* $\Lambda$ *be the map of* $]-\epsilon_1,\epsilon_1[\times\mathbb{R}\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$ *in* $C^{m-1,\alpha}(\partial\Omega)$ *defined by*

$$\begin{aligned} \Lambda[\epsilon,\delta,\theta,\xi](t) \equiv \frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\theta(s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s \\ + f(t)\Big(\delta\epsilon\int_{\partial\Omega} S_n(t-s)\theta(s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s + \xi\Big) - g(t) \qquad \forall t \in \partial\Omega, \end{aligned} \tag{3.33}$$

*for all* $(\epsilon,\delta,\theta,\xi) \in ]-\epsilon_1,\epsilon_1[\times\mathbb{R}\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$. *Then the following statements hold.*

(i) *If* $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, *then the pair* $(\theta,\xi) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ *satisfies equation*

$$\Lambda[\epsilon,\delta,\theta,\xi] = 0, \tag{3.34}$$

*if and only if the pair* $(\mu,\xi) \in \mathcal{U}_\epsilon^{m-1,\alpha} \times \mathbb{R}$, *with* $\mu \in \mathcal{U}_\epsilon^{m-1,\alpha}$ *defined by*

$$\mu(x) \equiv \delta\theta(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{3.35}$$

*satisfies the equation*

$$\begin{aligned}\delta\Gamma(x) = \frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}(S_n^a(x-y))\mu(y)\,d\sigma_y \\ + \delta\phi(x)\Big(\int_{\partial\Omega_\epsilon} S_n^a(x-y)\mu(y)\,d\sigma_y + \xi\Big) \qquad \forall x \in \partial\Omega_\epsilon,\end{aligned} \tag{3.36}$$

*with* $\Gamma$, $\phi \in C^{m-1,\alpha}(\partial\Omega_\epsilon)$ *defined by*

$$\Gamma(x) \equiv g(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{3.37}$$

*and*

$$\phi(x) \equiv f(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon. \tag{3.38}$$

*In particular, equation* (3.34) *has exactly one solution* $(\theta,\xi) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$, *for each* $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$.

(ii) *The pair* $(\theta,\xi) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ *satisfies equation*

$$\Lambda[0,0,\theta,\xi] = 0, \tag{3.39}$$

*if and only if*

$$g(t) = \frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\,d\sigma_s + f(t)\xi \qquad \forall t \in \partial\Omega. \tag{3.40}$$

*In particular, equation* (3.39) *has exactly one solution* $(\theta,\xi) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$, *which we denote by* $(\tilde{\theta},\tilde{\xi})$.

*Proof.* Consider $(i)$. Let $\theta \in C^{m-1,\alpha}(\partial\Omega)$. Let $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$. First of all, we note that

$$\int_{\partial\Omega_\epsilon} \theta(\frac{1}{\epsilon}(x-w))\,d\sigma_x = \epsilon^{n-1}\int_{\partial\Omega}\theta(t)\,d\sigma_t,$$

and so $\theta \in \mathcal{U}_0^{m-1,\alpha}$ if and only if $\theta(\frac{1}{\epsilon}(\cdot - w)) \in \mathcal{U}_\epsilon^{m-1,\alpha}$. The equivalence of equation (3.34) in the unknown $(\theta,\xi) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ and equation (3.36) in the unknown $(\mu,\xi) \in \mathcal{U}_\epsilon^{m-1,\alpha} \times \mathbb{R}$ follows by a straightforward computation based on the rule of change of variables in integrals, on well known properties of composition of functions in Schauder spaces (cf. *e.g.*, Lanza [67, Sections 3,4]) and Lemma 1.25. The existence and uniqueness of a solution of equation (3.36) follows by Proposition 3.3. Then the existence and uniqueness of a solution of equation (3.34) follows by the equivalence of (3.34) and (3.36). Consider $(ii)$. The equivalence of (3.39) and (3.40) is obvious. The existence of a unique solution of equation (3.40) is an immediate consequence of well known results of classical potential theory and can be proved by exploiting exactly the same argument as in the proof of Proposition 3.6 $(ii)$. □

By Proposition 3.35, it makes sense to introduce the following.

**Definition 3.36.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in (1.64). For each $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, we denote by $(\hat{\theta}[\epsilon,\delta],\hat{\xi}[\epsilon,\delta])$ the unique pair in $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ that solves (3.34). Analogously, we denote by $(\hat{\theta}[0,0],\hat{\xi}[0,0])$ the unique pair in $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ that solves (3.39).

In the following Remark, we show the relation between the solutions of boundary value problem (3.32) and the solutions of equation (3.34).

*Remark* 3.37. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively.

Let $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$. We have

$$u[\epsilon,\delta](x) = \delta\epsilon^{n-1} \int_{\partial\Omega} S_n^a(x - w - \epsilon s)\hat{\theta}[\epsilon,\delta](s)\, d\sigma_s + \hat{\xi}[\epsilon,\delta] \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon].$$

While the relation between equation (3.34) and boundary value problem (3.32) is now clear, we want to see if (3.39) is related to some (limiting) boundary value problem. We give the following.

**Definition 3.38.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $f$, $g$ be as in (1.56), (3.7), (3.8), respectively. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega)$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega, \\ \frac{\partial}{\partial \nu_\Omega} u(x) = g(x) - \frac{\int_{\partial\Omega} g\, d\sigma}{\int_{\partial\Omega} f\, d\sigma} f(x) & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{3.41}$$

Problem (3.41) will be called the *limiting boundary value problem.*

*Remark* 3.39. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. We have

$$\tilde{u}(x) = \int_{\partial\Omega} S_n(x-y)\hat{\theta}[0,0](y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \Omega.$$

Moreover,

$$\hat{\xi}[0,0] = \frac{\int_{\partial\Omega} g\, d\sigma}{\int_{\partial\Omega} f\, d\sigma}.$$

We now prove the following.

**Proposition 3.40.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $f$, $g$ *be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let* $\mathcal{U}_0^{m-1,\alpha}$ *be as in* (1.64). *Let* $\Lambda$ *and* $(\tilde{\theta}, \tilde{\xi})$ *be as in Proposition 3.35. Then there exists* $\epsilon_2 \in ]0, \epsilon_1]$ *such that* $\Lambda$ *is a real analytic operator of* $]-\epsilon_2, \epsilon_2[ \times \mathbb{R} \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ *to* $C^{m-1,\alpha}(\partial\Omega)$. *Moreover, if we set* $b_0 \equiv (0, 0, \tilde{\theta}, \tilde{\xi})$, *then the differential* $\partial_{(\theta,\xi)}\Lambda[b_0]$ *of* $\Lambda$ *with respect to the variables* $(\theta, \xi)$ *at* $b_0$ *is delivered by the following formula*

$$\partial_{(\theta,\xi)}\Lambda[b_0](\bar{\theta}, \bar{\xi})(t) = \frac{1}{2}\bar{\theta}(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\bar{\theta}(s)\, d\sigma_s + f(t)\bar{\xi} \qquad \forall t \in \partial\Omega, \tag{3.42}$$

*for all* $(\bar{\theta}, \bar{\xi}) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$, *and is a linear homeomorphism of* $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ *onto* $C^{m-1,\alpha}(\partial\Omega)$.

*Proof.* By arguing as in the proof of Proposition 3.11, one can show that there exists $\epsilon_2 \in ]0, \epsilon_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times \mathbb{R} \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$. By standard calculus in Banach space, we immediately deduce that (3.42) holds. By the proof of Proposition 3.11, we have that $\partial_{(\theta,\xi)}\Lambda[b_0]$ is a linear homeomorphism. □

We are now ready to prove that $\hat{\theta}[\cdot,\cdot], \hat{\xi}[\cdot,\cdot]$ can be continued real analytically on a whole neighbourhood of $(0, 0)$.

**Proposition 3.41.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $f$, $g$ *be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let* $\mathcal{U}_0^{m-1,\alpha}$ *be as in* (1.64). *Let* $\epsilon_2$ *be as in Proposition 3.40. Then there exist* $\epsilon_3 \in ]0, \epsilon_2]$, $\delta_1 \in ]0, +\infty[$, *and a real analytic operator* $(\Theta, \Xi)$ *of* $]-\epsilon_3, \epsilon_3[ \times ]-\delta_1, \delta_1[$ *to* $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$, *such that*

$$(\Theta[\epsilon,\delta], \Xi[\epsilon,\delta]) = (\hat{\theta}[\epsilon,\delta], \hat{\xi}[\epsilon,\delta]), \tag{3.43}$$

*for all* $(\epsilon, \delta) \in (]0, \epsilon_3[ \times ]0, \delta_1[) \cup \{(0, 0)\}$.

*Proof.* It is an immediate consequence of Proposition 3.40 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

By Proposition 3.41 and Remark 3.37, we can deduce the following results.

**Theorem 3.42.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Proposition 3.41. Then the following statements hold.*

*(i) Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \epsilon_3]$, a real analytic operator $U_1$ of $]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$ to the space $C^0_h(\mathrm{cl}\, V)$, and a real analytic operator $U_2$ of $]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$ such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*
$$u[\epsilon,\delta](x) = \delta \epsilon^{n-1} U_1[\epsilon,\delta](x) + U_2[\epsilon,\delta] \qquad \forall x \in \mathrm{cl}\, V,$$
*for all* $(\epsilon,\delta) \in ]0, \epsilon_4[ \times ]0, \delta_1[$. *Moreover,*
$$U_2[0,0] = \frac{\int_{\partial\Omega} g \, d\sigma}{\int_{\partial\Omega} f \, d\sigma}.$$

*(ii) Let $\bar{V}$ be a bounded open subset of $\mathbb{R}^n \setminus \mathrm{cl}\, \Omega$. Then there exist $\bar{\epsilon}_4 \in ]0, \epsilon_3]$, a real analytic operator $\bar{U}_1$ of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \times ]-\delta_1, \delta_1[$ to the space $C^{m,\alpha}(\mathrm{cl}\, \bar{V})$, and a real analytic operator $\bar{U}_2$ of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$ such that the following conditions hold.*

*(j')* $w + \epsilon\, \mathrm{cl}\, \bar{V} \subseteq \mathrm{cl}\, \mathbb{P}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \setminus \{0\}$.

*(jj')*
$$u[\epsilon,\delta](w + \epsilon t) = \delta \epsilon \bar{U}_1[\epsilon,\delta](t) + \bar{U}_2[\epsilon,\delta] \qquad \forall t \in \mathrm{cl}\, \bar{V},$$
*for all* $(\epsilon,\delta) \in ]0, \bar{\epsilon}_4[ \times ]0, \delta_1[$. *Moreover,*
$$\bar{U}_2[0,0] = \frac{\int_{\partial\Omega} g \, d\sigma}{\int_{\partial\Omega} f \, d\sigma}.$$

*Proof.* Let $(\epsilon,\delta) \in ]0, \epsilon_1[ \times ]0, \delta_1[$. We have
$$u[\epsilon,\delta](x) = \delta \epsilon^{n-1} \int_{\partial\Omega} S^a_n(x - w - \epsilon s) \Theta[\epsilon,\delta](s) \, d\sigma_s + \Xi[\epsilon,\delta] \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon].$$
Then by arguing as in the proof of Theorem 3.13, one can show the validity of the Theorem. Indeed, by choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. By arguing as in the proof of Theorem 3.13, it is natural to set
$$U_1[\epsilon,\delta](x) \equiv \int_{\partial\Omega} S^a_n(x - w - \epsilon s) \Theta[\epsilon,\delta](s) \, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$
for all $(\epsilon,\delta) \in ]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$, and
$$U_2[\epsilon,\delta] \equiv \Xi[\epsilon,\delta],$$
for all $(\epsilon,\delta) \in ]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$. By following the proof of Theorem 2.124 and by Proposition 3.41, we have that $U_1$, $U_2$ are real analytic maps of $]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$, $C^0_h(\mathrm{cl}\, V)$, respectively, such that the equality in $(jj)$ holds. Moreover, by Propositions 3.35 and 3.12, we have that
$$U_2[0,0] = \frac{\int_{\partial\Omega} g \, d\sigma}{\int_{\partial\Omega} f \, d\sigma}.$$
Consider now $(ii)$. Choosing $\bar{\epsilon}_4$ small enough, we can clearly assume that $(j')$ holds. Consider now $(jj')$. Let $(\epsilon,\delta) \in ]0, \bar{\epsilon}_4[ \times ]0, \delta_1[$. We have
$$u[\epsilon,\delta](w + \epsilon t) = \delta \epsilon^{n-1} \int_{\partial\Omega} S^a_n(\epsilon(t - s)) \Theta[\epsilon,\delta](s) \, d\sigma_s + \Xi[\epsilon,\delta] \qquad \forall t \in \mathrm{cl}\, \bar{V}.$$
Since $\int_{\partial\Omega} \Theta[\epsilon,\delta](s) \, d\sigma_s = 0$ for all $(\epsilon,\delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[ \cup \{(0,0)\}$, by arguing as in Proposition 1.29 $(ii)$, it is natural to set
$$\bar{U}_1[\epsilon,\delta](t) \equiv \int_{\partial\Omega} S_n(t - s) \Theta[\epsilon,\delta](s) \, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R^a_n(\epsilon(t - s)) \Theta[\epsilon,\delta](s) \, d\sigma_s \qquad \forall t \in \mathrm{cl}\, \bar{V},$$

for all $(\epsilon, \delta) \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \times ]-\delta_1, \delta_1[$, and

$$\bar{U}_2[\epsilon, \delta] \equiv \Xi[\epsilon, \delta],$$

for all $(\epsilon, \delta) \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \times ]-\delta_1, \delta_1[$. Obviously, the equality in $(jj')$ holds. Then by arguing as in the proof of Theorem 3.13 $(ii)$, we easily conclude. □

**Theorem 3.43.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Proposition 3.41. Then there exist $\epsilon_5 \in ]0, \epsilon_3]$ and a real analytic operator $G$ of $]-\epsilon_5, \epsilon_5[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon, \delta](x)|^2 \, dx = \delta^2 \epsilon^n G[\epsilon, \delta], \tag{3.44}$$

*for all $(\epsilon, \delta) \in ]0, \epsilon_5[ \times ]0, \delta_1[$. Moreover,*

$$G[0, 0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\, \Omega} |\nabla \tilde{u}(x)|^2 \, dx. \tag{3.45}$$

*Proof.* Let $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, \delta_1[$. Clearly,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon, \delta](x)|^2 \, dx = \delta^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon, \delta](\frac{1}{\epsilon}(\cdot - w))](x)|^2 \, dx.$$

Then in order to prove the Theorem, it suffices to exploit the same argument as the proof of Theorem 3.16. □

As done in Theorem 3.43 for the energy integral, we can now prove a real analytic continuation Theorem for the integral of the solution. Namely, we prove the following.

**Theorem 3.44.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Proposition 3.41. Then there exist $\epsilon_6 \in ]0, \epsilon_3]$, $\delta_2 \in ]0, \delta_1]$, and a real analytic operator $J$ of $]-\epsilon_6, \epsilon_6[ \times ]-\delta_2, \delta_2[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon, \delta](x) \, dx = J[\epsilon, \delta], \tag{3.46}$$

*for all $(\epsilon, \delta) \in ]0, \epsilon_6[ \times ]0, \delta_2[$. Moreover,*

$$J[0, 0] = \frac{\int_{\partial\Omega} g \, d\sigma}{\int_{\partial\Omega} f \, d\sigma} |A|_n. \tag{3.47}$$

*Proof.* Let $\Theta[\cdot, \cdot]$, $\Xi[\cdot, \cdot]$ be as in Proposition 3.41. Let $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[$. Since

$$u[\epsilon, \delta](x) = \delta v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon, \delta](\frac{1}{\epsilon}(\cdot - w))\big](x) + \Xi[\epsilon, \delta] \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon],$$

then

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon, \delta](x) \, dx =& \delta \int_{\mathbb{P}_a[\Omega_\epsilon]} v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon, \delta](\frac{1}{\epsilon}(\cdot - w))\big](x) \, dx \\ &+ \Xi[\epsilon, \delta]\big(|A|_n - \epsilon^n |\Omega|_n\big).\end{aligned}$$

On the other hand, by arguing as in the proof of Theorem 3.16, we note that

$$\begin{aligned}&v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon, \delta](\frac{1}{\epsilon}(\cdot - w))\big](w + \epsilon t) \\ &\qquad = \epsilon \int_{\partial\Omega} S_n(t - s)\Theta[\epsilon, \delta](s) \, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t - s))\Theta[\epsilon, \delta](s) \, d\sigma_s \qquad \forall t \in \partial\Omega.\end{aligned}$$

Then, if we set

$$L[\epsilon, \delta](t) \equiv \epsilon \int_{\partial\Omega} S_n(t - s)\Theta[\epsilon, \delta](s) \, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t - s))\Theta[\epsilon, \delta](s) \, d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $(\epsilon,\delta)\in\,]-\epsilon_3,\epsilon_3[\,\times\,]-\delta_1,\delta_1[$, we have that $L[\cdot,\cdot]$ is a real analytic map of $]-\epsilon_3,\epsilon_3[\,\times\,]-\delta_1,\delta_1[$ to $C^{m,\alpha}(\partial\Omega)$. Then, by Theorem 2.115, we easily deduce that there exist $\epsilon_6\in\,]0,\epsilon_3]$, $\delta_2\in\,]0,\delta_1]$, and a real analytic map $J_1$ of $]-\epsilon_6,\epsilon_6[\,\times\,]-\delta_2,\delta_2[$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} v_a^-\big[\partial\Omega_\epsilon,\Theta[\epsilon,\delta](\frac{1}{\epsilon}(\cdot-w))\big](x)\,dx=J_1[\epsilon,\delta],$$

for all $(\epsilon,\delta)\in\,]0,\epsilon_6[\,\times\,]0,\delta_2[$. Then, if we set

$$J[\epsilon,\delta]\equiv\delta J_1[\epsilon,\delta]+\Xi[\epsilon,\delta]\big(|A|_n-\epsilon^n|\Omega|_n\big),$$

for all $(\epsilon,\delta)\in\,]-\epsilon_6,\epsilon_6[\,\times\,]-\delta_2,\delta_2[$, we can immediately conclude. □

We have the following.

**Proposition 3.45.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $1\le p<\infty$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)}\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]=\frac{\int_{\partial\Omega}g\,d\sigma}{\int_{\partial\Omega}f\,d\sigma}\qquad \textit{in } L^p(A).$$

*Proof.* It suffices to modify the proof of Proposition 3.22. Let $\epsilon_3$, $\delta_1$, $\Theta$, $\Xi$ be as in Proposition 3.41. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[$, we have

$$\begin{aligned}&u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\\&\quad=\delta\epsilon\int_{\partial\Omega}S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s+\delta\epsilon^{n-1}\int_{\partial\Omega}R_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s+\Xi[\epsilon,\delta]\quad\forall t\in\partial\Omega.\end{aligned}$$

We set

$$N[\epsilon,\delta](t)\equiv\delta\epsilon\int_{\partial\Omega}S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s+\delta\epsilon^{n-1}\int_{\partial\Omega}R_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s+\Xi[\epsilon,\delta]\qquad\forall t\in\partial\Omega,$$

for all $(\epsilon,\delta)\in\,]-\epsilon_3,\epsilon_3[\,\times\,]-\delta_1,\delta_1[$. By taking $\tilde\epsilon\in\,]0,\epsilon_3[$ and $\tilde\delta\in\,]0,\delta_1[$ small enough, we can assume (cf. Proposition 1.26 $(i)$) that $N$ is a real analytic map of $]-\tilde\epsilon,\tilde\epsilon[\,\times\,]-\tilde\delta,\tilde\delta[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C\equiv\sup_{(\epsilon,\delta)\in]-\tilde\epsilon,\tilde\epsilon[\times]-\tilde\delta,\tilde\delta[}\|N[\epsilon,\delta]\|_{C^0(\partial\Omega)}<+\infty.$$

By Theorem 2.5, we have

$$|\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]](x)|\le C\qquad\forall x\in A,\quad\forall(\epsilon,\delta)\in\,]0,\tilde\epsilon[\,\times\,]0,\tilde\delta[.$$

By Theorem 3.42, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)}\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]](x)=\frac{\int_{\partial\Omega}g\,d\sigma}{\int_{\partial\Omega}f\,d\sigma}\qquad\forall x\in A\setminus\{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)}\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]=\frac{\int_{\partial\Omega}g\,d\sigma}{\int_{\partial\Omega}f\,d\sigma}\qquad\text{in } L^p(A).$$

□

**Proposition 3.46.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Proposition 3.41. Then there exist $\tilde\epsilon\in\,]0,\epsilon_3[$ and a real analytic map $N$ of $]-\tilde\epsilon,\tilde\epsilon[\,\times\,]\delta_1,\delta_1[$ to $C^{m,\alpha}(\partial\Omega)$ such that*

$$\|\mathbf{E}^\#_{(\epsilon,1)}\Big[u[\epsilon,\delta],\frac{\int_{\partial\Omega}g\,d\sigma}{\int_{\partial\Omega}f\,d\sigma}\Big]-\frac{\int_{\partial\Omega}g\,d\sigma}{\int_{\partial\Omega}f\,d\sigma}\|_{L^\infty(\mathbb{R}^n)}=\|N[\epsilon,\delta]\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta)\in\,]0,\tilde\epsilon[\,\times\,]0,\delta_1[$. Moreover,*

$$N[0,0]=0,$$

*and, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)}\mathbf{E}^\#_{(\epsilon,1)}\Big[u[\epsilon,\delta],\frac{\int_{\partial\Omega}g\,d\sigma}{\int_{\partial\Omega}f\,d\sigma}\Big]=\frac{\int_{\partial\Omega}g\,d\sigma}{\int_{\partial\Omega}f\,d\sigma}\qquad\textit{in } L^\infty(\mathbb{R}^n).$$

*Proof.* Let $\epsilon_3$, $\delta_1$, $\Theta$, $\Xi$ be as in Proposition 3.41. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[$, we have

$$\begin{aligned} u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) =&\delta\epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s \\ &+\delta\epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s+\Xi[\epsilon,\delta] \qquad \forall t\in\partial\Omega. \end{aligned}$$

We set

$$\begin{aligned} N[\epsilon,\delta](t) \equiv&\delta\epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s \\ &+\delta\epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s+\Xi[\epsilon,\delta]-\frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}, \qquad \forall t\in\partial\Omega, \end{aligned}$$

for all $(\epsilon,\delta)\in\,]-\epsilon_3,\epsilon_3[\,\times\,]-\delta_1,\delta_1[$. By taking $\tilde\epsilon\in\,]0,\epsilon_3[$ small enough, we can assume (cf. Proposition 1.26 $(i)$) that $N$ is a real analytic map of $]-\tilde\epsilon,\tilde\epsilon[\,\times\,]-\delta_1,\delta_1[$ to $C^{m,\alpha}(\partial\Omega)$. By Theorem 2.5, we have

$$\|\mathbf{E}_{(\epsilon,1)}\Big[u[\epsilon,\delta],\frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\Big]-\frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\|_{L^\infty(\mathbb{R}^n)}=\|N[\epsilon,\delta]\|_{C^0(\partial\Omega)} \qquad \forall(\epsilon,\delta)\in\,]0,\tilde\epsilon[\,\times\,]0,\delta_1[,$$

and the conclusion easily follows. □

### 3.4.2 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorem we deduce by Proposition 3.45 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 3.47.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $1\le p<\infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)}\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]=\frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma} \qquad \text{in } L^p(V).$$

*Proof.* We modify the proof of Theorem 2.134. By virtue of Proposition 3.45, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)}\|\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]-\frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\|_{L^p(A)}=0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $C>0$ such that

$$\begin{aligned} \|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]-\frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\|_{L^p(V)}\le C\|\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]-\frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\|_{L^p(A)} \\ \forall(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\min\{1,\delta_1\}[. \end{aligned}$$

Thus,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)}\|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]-\frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\|_{L^p(V)}=0,$$

and we can easily conclude. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 3.48.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Proposition 3.41. Let $\epsilon_6$, $\delta_2$, $J$ be as in Theorem 3.44. Let $r>0$ and $\bar y\in\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n}\mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar y}(x)\,dx=r^nJ\big[\epsilon,\frac{r}{l}\big], \tag{3.48}$$

*for all $\epsilon\in\,]0,\epsilon_6[$, and for all $l\in\mathbb{N}\setminus\{0\}$ such that $l>(r/\delta_2)$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_6[$, and let $l \in \mathbb{N}\setminus\{0\}$ be such that $l > (r/\delta_2)$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx\\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx\\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx\\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u\big[\epsilon,(r/l)\big]\big(\frac{l}{r}x\big)\,dx\\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u\big[\epsilon,(r/l)\big](t)\,dt\\ &= \frac{r^n}{l^n} J\big[\epsilon,\frac{r}{l}\big].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J\big[\epsilon,\frac{r}{l}\big],$$

and the conclusion follows. □

In the following Theorem we consider the $L^\infty$–distance of a certain extension of $u_{(\epsilon,\delta)}$ and its limit.

**Theorem 3.49.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8)*, respectively. Let $\delta_1$ be as in Proposition 3.41. Let $\tilde{\epsilon}$, $N$ be as in Proposition 3.46. Then*

$$\|\mathbf{E}^{\#}_{(\epsilon,\delta)}\Big[u_{(\epsilon,\delta)},\frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\Big] - \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\|_{L^\infty(\mathbb{R}^n)} = \|N[\epsilon,\delta]\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\delta_1[$. Moreover,*

$$N[0,0] = 0,$$

*and, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^{\#}_{(\epsilon,\delta)}\Big[u_{(\epsilon,\delta)},\frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\Big] = \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma} \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathbf{E}^{\#}_{(\epsilon,\delta)}\Big[u_{(\epsilon,\delta)},\frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\Big] - \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\|_{L^\infty(\mathbb{R}^n)} &= \|\mathbf{E}^{\#}_{(\epsilon,1)}\Big[u[\epsilon,\delta],\frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\Big] - \frac{\int_{\partial\Omega} g\,d\sigma}{\int_{\partial\Omega} f\,d\sigma}\|_{L^\infty(\mathbb{R}^n)}\\ &= \|N[\epsilon,\delta]\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\delta_1[$. □

### 3.4.3 Asymptotic behaviour of the energy integral of $u_{(\epsilon,\delta)}$

This Subsection is devoted to the study of the behaviour of the energy integral of $u_{(\epsilon,\delta)}$. We give the following.

**Definition 3.50.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, we set

$$\mathrm{En}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 3.51. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in (1.56), (1.57), (3.7), (3.8), respectively. Let $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$. We have

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\, dx = \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2\, dt$$
$$= \delta^{n-2} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon,\delta](t)|^2\, dt.$$

In the following Proposition we represent the function $\mathrm{En}(\cdot,\cdot)$ by means of a real analytic function.

**Proposition 3.52.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $f$, $g$ be as in* (1.56), (1.57), (3.7), (3.8), *respectively. Let $\delta_1$ be as in Proposition 3.41. Let $\epsilon_5$ and $G$ be as in Theorem 3.43. Then*

$$\mathrm{En}\Big(\epsilon, \frac{1}{l}\Big) = \epsilon^n G[\epsilon, (1/l)],$$

*for all $\epsilon \in ]0, \epsilon_5[$ and for all $l \in \mathbb{N} \setminus \{0\}$ such that $l > (1/\delta_1)$.*

*Proof.* By Remark 3.51 and Theorem 3.43, we have

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\, dx = \delta^n \epsilon^n G[\epsilon, \delta]. \tag{3.49}$$

On the other hand, if $\epsilon \in ]0, \epsilon_5[$ and $l \in \mathbb{N} \setminus \{0\}$ is such that $l > (1/\delta_1)$, then we have

$$\mathrm{En}\Big(\epsilon, \frac{1}{l}\Big) = l^n \frac{1}{l^n} \epsilon^n G[\epsilon, (1/l)]$$
$$= \epsilon^n G[\epsilon, (1/l)],$$

and the conclusion easily follows. □

## 3.5 Asymptotic behaviour of the solutions of a nonlinear Robin problem for the Laplace equation in a periodically perforated domain

In this Section we study the asymptotic behaviour of the solutions of a nonlinear Robin problem for the Laplace equation in a periodically perforated domain with small holes.

### 3.5.1 Notation and preliminaries

We retain the notation introduced in Subsections 1.8.1, 3.2.1. However, we need to introduce also some other notation. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). If $F \in C^0(\partial\Omega \times \mathbb{R})$, then we denote by $T_F$ the (nonlinear nonautonomous) composition operator of $C^0(\partial\Omega)$ to itself which maps $v \in C^0(\partial\Omega)$ to the function $T_F[v]$ of $\partial\Omega$ to $\mathbb{R}$, defined by

$$T_F[v](t) \equiv F(t, v(t)) \qquad \forall t \in \partial\Omega.$$

If $F \in C^0(\partial\Omega \times \mathbb{R})$ is such that $T_F$ is a real analytic map of $C^{m-1,\alpha}(\partial\Omega)$ to itself, then by Lanza [72, Prop. 6.3, p. 972] the partial derivative $F_u(\cdot,\cdot)$ of $F(\cdot,\cdot)$ with respect to the variable in $\mathbb{R}$ exists, and we have

$$dT_F[v_0](v) = T_{F_u}[v_0]v \qquad \forall v \in C^{m-1,\alpha}(\partial\Omega),$$

for all $v_0 \in C^{m-1,\alpha}(\partial\Omega)$. Moreover, $T_{F_u}$ is a real analytic operator of $C^{m-1,\alpha}(\partial\Omega)$ to itself. Accordingly,

$$F_u(\cdot,\xi) \in C^{m-1,\alpha}(\partial\Omega) \qquad \forall \xi \in \mathbb{R}.$$

Then we shall consider also the following assumption.

$$\begin{aligned} &F \in C^0(\partial\Omega \times \mathbb{R}),\ T_F \text{ is a real analytic map of } C^{m-1,\alpha}(\partial\Omega) \text{ to itself, and} \\ &\text{there exists } \tilde{\xi} \in \mathbb{R} \text{ such that } \int_{\partial\Omega} F(t,\tilde{\xi})\, d\sigma_t = 0 \text{ and } \int_{\partial\Omega} F_u(t,\tilde{\xi})\, d\sigma_t \neq 0. \end{aligned} \tag{3.50}$$

If $F$ is as in (3.50), then we shall consider also the following assumption.

$$\tilde{\xi} \in \mathbb{R} \text{ is such that } \int_{\partial\Omega} F(t, \tilde{\xi})\, d\sigma_t = 0 \text{ and } \int_{\partial\Omega} F_u(t, \tilde{\xi})\, d\sigma_t \neq 0. \tag{3.51}$$

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$ be as in (1.56), (1.57), (3.50), respectively. For each $\epsilon \in ]0, \epsilon_1[$, we consider the following periodic nonlinear Robin problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x + a_i) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) + F\big(\frac{1}{\epsilon}(x - w), u(x)\big) = 0 & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{3.52}$$

We now convert our boundary value problem (3.52) into an integral equation.

**Proposition 3.53.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$ be as in* (1.56), (1.57), (3.50), *respectively. Let $\epsilon \in ]0, \epsilon_1[$. Let $\mathcal{U}_\epsilon^{m-1,\alpha}$ be as in* (1.63). *Then the map of the set of pairs $(\mu, \xi) \in \mathcal{U}_\epsilon^{m-1,\alpha} \times \mathbb{R}$ that solve the equation*

$$\frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \nu_{\Omega_\epsilon}(x) \cdot DS_n^a(x - y)\mu(y)\, d\sigma_y + F\Big(\frac{1}{\epsilon}(x - w), \int_{\partial\Omega_\epsilon} S_n^a(x - y)\mu(y)\, d\sigma_y + \xi\Big) = 0 \qquad \forall x \in \partial\Omega_\epsilon, \tag{3.53}$$

*to the set of $u \in C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon])$ which solve problem* (3.52), *which takes $(\mu, \xi)$ to the function*

$$v_a^-[\partial\Omega_\epsilon, \mu] + \xi \tag{3.54}$$

*is a bijection.*

*Proof.* Assume that the pair $(\mu, \xi) \in \mathcal{U}_\epsilon^{m-1,\alpha} \times \mathbb{R}$ solves equation (3.53). Then, by Theorem 1.15, we immediately deduce that the function $u \equiv v_a^-[\partial\Omega_\epsilon, \mu] + \xi$ is a periodic harmonic function in $C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon])$, that, by equation (3.53), satisfies the third condition of (3.52). Thus, $u$ is a solution of (3.52). Conversely, let $u \in C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon])$ be a solution of problem (3.52). By Proposition 2.23, there exists a unique pair $(\mu, \xi) \in \mathcal{U}_\epsilon^{m-1,\alpha} \times \mathbb{R}$, such that

$$u = v_a^-[\partial\Omega_\epsilon, \mu] + \xi \qquad \text{in } \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon].$$

Then, by Theorem 1.15, since $u$ satisfies in particular the third condition in (3.52), we immediately deduce that the pair $(\mu, \xi)$ solves equation (3.53). □

As we have seen, we can transform (3.52) into an integral equation defined on the $\epsilon$-dependent domain $\partial\Omega_\epsilon$. In order to get rid of such a dependence, we shall introduce the following Theorem.

**Theorem 3.54.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$ be as in* (1.56), (1.57), (3.50), *respectively. Let $\epsilon \in ]0, \epsilon_1[$. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.64). *Then the map $u[\epsilon, \cdot, \cdot]$ of the set of pairs $(\theta, \xi) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ that solve the equation*

$$\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t - s)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^a(\epsilon(t - s))\theta(s)\, d\sigma_s + F\Big(t, \epsilon \int_{\partial\Omega} S_n(t - s)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t - s))\theta(s)\, d\sigma_s + \xi\Big) = 0 \qquad \forall t \in \partial\Omega, \tag{3.55}$$

*to the set of $u \in C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon])$ which solve problem* (3.52), *which takes $(\theta, \xi)$ to the function*

$$u[\epsilon, \theta, \xi] \equiv v_a^-[\partial\Omega_\epsilon, \theta(\frac{1}{\epsilon}(\cdot - w))] + \xi \tag{3.56}$$

*is a bijection.*

*Proof.* It is an immediate consequence of Proposition 3.53, of the Theorem of change of variables in integrals and of Lemma 1.25. □

In the following Proposition we study equation (3.55) for $\epsilon = 0$ and for $\xi = \tilde{\xi}$ (with $\tilde{\xi}$ as in (3.51).)

**Proposition 3.55.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $F$, $\tilde{\xi}$ be as in* (1.56), (3.50), (3.51), *respectively. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.64). *Then the integral equation*

$$\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\theta(s)\, d\sigma_s + F(t,\tilde{\xi}) = 0 \qquad \forall t \in \partial\Omega, \tag{3.57}$$

*which we call the* limiting equation, *has a unique solution $\theta \in \mathcal{U}_0^{m-1,\alpha}$, which we denote by $\tilde{\theta}$.*

*Proof.* By classical potential theory (cf. Folland [52, Chapter 3]), since $\mathbb{R}^n \setminus \operatorname{cl}\Omega$ is connected and by the well known identity

$$\int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}\big(S_n(s-t)\big)\, d\sigma_t = \frac{1}{2} \qquad \forall s \in \partial\Omega,$$

it is immediate to see that equation

$$\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\theta(s)\, d\sigma_s = -F(t,\tilde{\xi}) \qquad \forall t \in \partial\Omega,$$

has a unique solution $\theta \in \mathcal{U}_0^{m-1,\alpha}$. □

Now we want to see if equation (3.57) is related to some (limiting) boundary value problem. We give the following.

**Definition 3.56.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $F$, $\tilde{\xi}$ be as in (1.56), (3.50), (3.51), respectively. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega)$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \operatorname{cl}\Omega, \\ \frac{\partial}{\partial\nu_\Omega} u(x) = -F(x,\tilde{\xi}) & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{3.58}$$

Problem (3.58) will be called the *limiting boundary value problem.*

*Remark* 3.57. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $F$, $\tilde{\xi}$ be as in (1.56), (3.50), (3.51), respectively. Let $\tilde{\theta}$ be as in Proposition 3.55. We have

$$\tilde{u}(x) = \int_{\partial\Omega} S_n(x-y)\tilde{\theta}(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \Omega.$$

We are now ready to analyse equation (3.55) around the degenerate case $\epsilon = 0$.

**Theorem 3.58.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.64). *Let $\Lambda$ be the map of $]-\epsilon_1,\epsilon_1[ \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$, defined by*

$$\begin{aligned}\Lambda[\epsilon,\theta,\xi](t) \equiv \frac{1}{2}\theta(t) &+ \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\theta(s)\, d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s \\ &+ F\Big(t, \epsilon\int_{\partial\Omega} S_n(t-s)\theta(s)\, d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s + \xi\Big) \qquad \forall t \in \partial\Omega, \end{aligned} \tag{3.59}$$

*for all $(\epsilon,\theta,\xi) \in ]-\epsilon_1,\epsilon_1[ \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$. Then the following statements hold.*

*(i) Equation $\Lambda[0,\theta,\tilde{\xi}] = 0$ is equivalent to the limiting equation* (3.57) *and has one and only one solution $\tilde{\theta}$ in $\mathcal{U}_0^{m-1,\alpha}$ (cf. Proposition 3.55.)*

*(ii) If $\epsilon \in ]0,\epsilon_1[$, then equation $\Lambda[\epsilon,\theta,\xi] = 0$ is equivalent to equation* (3.55) *for $(\theta,\xi)$.*

*(iii) There exists $\epsilon_2 \in ]0,\epsilon_1]$, such that the map $\Lambda$ of $]-\epsilon_2,\epsilon_2[ \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$ is real analytic. Moreover, the differential $\partial_{(\theta,\xi)}\Lambda[0,\tilde{\theta},\tilde{\xi}]$ of $\Lambda$ at $(0,\tilde{\theta},\tilde{\xi})$ is a linear homeomorphism of $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ onto $C^{m-1,\alpha}(\partial\Omega)$.*

*(iv) There exist $\epsilon_3 \in ]0,\epsilon_2]$, an open neighbourhood $\tilde{\mathcal{U}}$ of $(\tilde{\theta},\tilde{\xi})$ in $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ and a real analytic map $(\Theta[\cdot],\Xi[\cdot])$ of $]-\epsilon_3,\epsilon_3[$ to $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$, such that the set of zeros of the map $\Lambda$ in $]-\epsilon_3,\epsilon_3[ \times \tilde{\mathcal{U}}$ coincides with the graph of $(\Theta[\cdot],\Xi[\cdot])$. In particular, $(\Theta[0],\Xi[0]) = (\tilde{\theta},\tilde{\xi})$.*

*Proof.* Statements (*i*) and (*ii*) are obvious. We now prove statement (*iii*). By Proposition 1.26 (*i*), (*ii*), by hypothesis (3.50), and standard calculus in Banach spaces, we have that there exists $\epsilon_2 \in ]0, \epsilon_1]$ such that $\Lambda$ is a real analytic map of $]-\epsilon_2, \epsilon_2[ \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$. By standard calculus in Banach spaces, the differential $\partial_{(\theta,\xi)}\Lambda[0, \tilde{\theta}, \tilde{\xi}]$ of $\Lambda$ at $(0, \tilde{\theta}, \tilde{\xi})$ is delivered by the following formula:

$$\partial_{(\theta,\xi)}\Lambda[0, \tilde{\theta}, \tilde{\xi}](\bar{\theta}, \bar{\xi})(t) = \frac{1}{2}\bar{\theta}(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\bar{\theta}(s)\, d\sigma_s + F_u(t, \tilde{\xi})\bar{\xi} \qquad \forall t \in \partial\Omega,$$

for all $(\bar{\theta}, \bar{\xi}) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$. We now show that the above differential is a linear homeomorphism. By the Open Mapping Theorem, it suffices to show that it is a bijection of $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ onto $C^{m-1,\alpha}(\partial\Omega)$. Let $\bar{\psi} \in C^{m-1,\alpha}(\partial\Omega)$. We must show that there exists a unique pair $(\bar{\theta}, \bar{\xi})$ in $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$, such that

$$\partial_{(\theta,\xi)}\Lambda[0, \tilde{\theta}, \tilde{\xi}](\bar{\theta}, \bar{\xi}) = \bar{\psi}.$$

We first prove uniqueness. Let $(\bar{\theta}, \bar{\xi}) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ solve

$$\frac{1}{2}\bar{\theta}(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\bar{\theta}(s)\, d\sigma_s + F_u(t, \tilde{\xi})\bar{\xi} = \bar{\psi}(t) \qquad \forall t \in \partial\Omega. \tag{3.60}$$

Then, by the well known identity

$$\int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}\big(S_n(s-t)\big)\, d\sigma_t = \frac{1}{2} \qquad \forall s \in \partial\Omega,$$

and by integrating both sides of (3.60), it is immediate to see that

$$\bar{\xi} = \frac{\int_{\partial\Omega} \bar{\psi}(t)\, d\sigma_t}{\int_{\partial\Omega} F_u(t, \tilde{\xi})\, d\sigma_t}, \tag{3.61}$$

and that accordingly, by classical potential theory (cf. Folland [52, Chapter 3]), $\bar{\theta}$ is the unique solution in $\mathcal{U}_0^{m-1,\alpha}$ of the following equation:

$$\frac{1}{2}\bar{\theta}(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\bar{\theta}(s)\, d\sigma_s = \bar{\psi}(t) - F_u(t, \tilde{\xi})\frac{\int_{\partial\Omega} \bar{\psi}(t)\, d\sigma_t}{\int_{\partial\Omega} F_u(t, \tilde{\xi})\, d\sigma_t} \qquad \forall t \in \partial\Omega. \tag{3.62}$$

Hence uniqueness follows. Then in order to prove existence it suffices to observe that the pair $(\bar{\theta}, \bar{\xi}) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$, with $\bar{\xi}$ delivered by (3.61) and where $\bar{\theta}$ is the unique solution in $\mathcal{U}_0^{m-1,\alpha}$ of (3.62), solves equation (3.60). Thus the proof of (*iii*) is now concluded. Finally, statement (*iv*) is an immediate consequence of statement (*iii*) and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

We are now in the position to introduce the following.

**Definition 3.59.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $u[\cdot, \cdot, \cdot]$ be as in Theorem 3.54. If $\epsilon \in ]0, \epsilon_3[$, we set

$$u[\epsilon](x) \equiv u[\epsilon, \Theta[\epsilon], \Xi[\epsilon]](x) \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon],$$

where $\epsilon_3$, $\Theta$, $\Xi$ are as in Theorem 3.58 (*iv*).

*Remark* 3.60. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $\epsilon_3$ be as in Theorem 3.58 (*iv*). Let $\epsilon \in ]0, \epsilon_3[$. Then $u[\epsilon]$ is a solution in $C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon])$ of problem (3.52).

### 3.5.2 A functional analytic representation Theorem for the family $\{u[\epsilon]\}_{\epsilon \in ]0, \epsilon_3[}$

The following statement shows that $\{u[\epsilon](\cdot)\}_{\epsilon \in ]0, \epsilon_3[}$ can be continued real analytically for negative values of $\epsilon$. We have the following.

**Theorem 3.61.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ *be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let* $\epsilon_3$ *be as in Theorem 3.58. Then the following statements hold.*

(i) *Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \epsilon_3]$, a real analytic operator $U_1$ of $]-\epsilon_4, \epsilon_4[$ to the space $C^0_h(\mathrm{cl}\,V)$, and a real analytic operator $U_2$ of $]-\epsilon_4, \epsilon_4[$ to $\mathbb{R}$ such that the following conditions hold.*

(j) $\mathrm{cl}\,V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

(jj)
$$u[\epsilon](x) = \epsilon^n U_1[\epsilon](x) + U_2[\epsilon] \qquad \forall x \in \mathrm{cl}\,V,$$
*for all* $\epsilon \in ]0, \epsilon_4[$. *Moreover,*
$$U_2[0] = \tilde{\xi}.$$

(ii) *Let $\bar{V}$ be a bounded open subset of $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$. Then there exist $\bar{\epsilon}_4 \in ]0, \epsilon_3]$, a real analytic operator $\bar{U}_1$ of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ to the space $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$, and a real analytic operator $\bar{U}_2$ of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ to $\mathbb{R}$ such that the following conditions hold.*

(j') $w + \epsilon\,\mathrm{cl}\,\bar{V} \subseteq \mathrm{cl}\,\mathbb{P}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \setminus \{0\}$.

(jj')
$$u[\epsilon](w + \epsilon t) = \epsilon \bar{U}_1[\epsilon](t) + \bar{U}_2[\epsilon] \qquad \forall t \in \mathrm{cl}\,\bar{V},$$
*for all* $\epsilon \in ]0, \bar{\epsilon}_4[$. *Moreover,*
$$\bar{U}_2[0] = \tilde{\xi}.$$

*Proof.* Let $\epsilon \in ]0, \epsilon_3[$. We observe that
$$u[\epsilon](x) = \epsilon^{n-1} \int_{\partial\Omega} S_n^a(x - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s + \Xi[\epsilon] \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$
Thus, in order to prove both statements, it suffices to follow step by step the proof of Theorem 3.13. □

*Remark* 3.62. We note that the right-hand side of the equalities in $(jj)$ and $(jj')$ of Theorem 3.61 can be continued real analytically in a whole neighbourhood of 0. Moreover, if $V$ is a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V \cap \mathbb{S}_a[\Omega_0] = \emptyset$, then
$$\lim_{\epsilon \to 0^+} u[\epsilon] = \tilde{\xi} \qquad \text{uniformly in } \mathrm{cl}\,V.$$

### 3.5.3 A real analytic continuation Theorem for the energy integral

As done in Theorem 3.61 for $u[\cdot]$, we can now prove a real analytic continuation Theorem for the energy integral. Namely, we prove the following.

**Theorem 3.63.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\epsilon_3$ be as in Theorem 3.58. Then there exist $\epsilon_5 \in ]0, \epsilon_3]$ and a real analytic operator $G$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{R}$, such that*
$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx = \epsilon^n G[\epsilon], \tag{3.63}$$
*for all* $\epsilon \in ]0, \epsilon_5[$. *Moreover,*
$$G[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\, dx. \tag{3.64}$$

*Proof.* Let $\epsilon \in ]0, \epsilon_3[$. Clearly,
$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx = \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2\, dx.$$
As a consequence, in order to prove the Theorem, it suffices to follow the proof of Theorem 3.16. □

*Remark* 3.64. We note that the right-hand side of the equality in (3.63) of Theorem 3.63 can be continued real analytically in the whole $]-\epsilon_5, \epsilon_5[$. Moreover,
$$\lim_{\epsilon \to 0^+} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx = 0$$

### 3.5.4 A real analytic continuation Theorem for the integral of the family $\{u[\epsilon]\}_{\epsilon\in]0,\epsilon_3[}$

As done in Theorem 3.63 for the energy integral, we can now prove a real analytic continuation Theorem for the integral of the family $\{u[\epsilon]\}_{\epsilon\in]0,\epsilon_3[}$. Namely, we prove the following.

**Theorem 3.65.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\epsilon_3$ be as in Theorem 3.58. Then there exist $\epsilon_6 \in ]0,\epsilon_3]$ and a real analytic operator $J$ of $]-\epsilon_6,\epsilon_6[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx = J[\epsilon], \tag{3.65}$$

*for all $\epsilon \in ]0,\epsilon_6[$. Moreover,*

$$J[0] = \tilde{\xi}|A|_n. \tag{3.66}$$

*Proof.* It suffices to modify the proof of Theorem 3.18. Let $\Theta[\cdot]$, $\Xi[\cdot]$ be as in Theorem 3.58 $(iv)$. Let $\epsilon \in ]0,\epsilon_3[$. Since

$$u[\epsilon](x) = v_a^-\big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) + \Xi[\epsilon] \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],$$

then

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx = \int_{\mathbb{P}_a[\Omega_\epsilon]} v_a^-\big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x)\,dx + \Xi[\epsilon]\big(|A|_n - \epsilon^n|\Omega|_n\big).$$

On the other hand, by arguing as in the proof of Theorem 2.128, we can show that there exist $\epsilon_6 \in ]0,\epsilon_3]$ and a real analytic map $J_1$ of $]-\epsilon_6,\epsilon_6[$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} v_a^-\big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x)\,dx = J_1[\epsilon],$$

for all $\epsilon \in ]0,\epsilon_6[$. Moreover, $J_1[0] = 0$. Then, if we set

$$J[\epsilon] \equiv J_1[\epsilon] + \Xi[\epsilon]\big(|A|_n - \epsilon^n|\Omega|_n\big),$$

for all $\epsilon \in ]-\epsilon_6,\epsilon_6[$, we can immediately conclude. □

### 3.5.5 A property of local uniqueness of the family $\{u[\epsilon]\}_{\epsilon\in]0,\epsilon_3[}$

In this Subsection, we shall show that the family $\{u[\epsilon]\}_{\epsilon\in]0,\epsilon_3[}$ is essentially unique. Namely, we prove the following.

**Theorem 3.66.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\{\hat{\epsilon}_j\}_{j\in\mathbb{N}}$ be a sequence in $]0,\epsilon_1[$ converging to 0. If $\{u_j\}_{j\in\mathbb{N}}$ is a sequence of functions such that*

$$u_j \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_{\hat{\epsilon}_j}]), \tag{3.67}$$

$$u_j \text{ solves (3.52) with } \epsilon \equiv \hat{\epsilon}_j, \tag{3.68}$$

$$\lim_{j\to\infty} u_j(w + \hat{\epsilon}_j\cdot) = \tilde{\xi} \qquad \text{in } C^{m-1,\alpha}(\partial\Omega), \tag{3.69}$$

*then there exists $j_0 \in \mathbb{N}$ such that*

$$u_j = u[\hat{\epsilon}_j] \qquad \forall j_0 \le j \in \mathbb{N}.$$

*Proof.* By Theorem 3.54, for each $j \in \mathbb{N}$, there exists a unique pair $(\theta_j, \xi_j)$ in $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ such that

$$u_j = u[\hat{\epsilon}_j, \theta_j, \xi_j]. \tag{3.70}$$

We shall now try to show that

$$\lim_{j\to\infty} (\theta_j, \xi_j) = (\tilde{\theta}, \tilde{\xi}) \qquad \text{in } \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}. \tag{3.71}$$

Indeed, if we denote by $\tilde{\mathcal{U}}$ the neighbourhood of Theorem 3.58 (*iv*), the limiting relation of (3.71) implies that there exists $j_0 \in \mathbb{N}$ such that

$$(\hat{\epsilon}_j, \theta_j, \xi_j) \in ]0, \epsilon_3[ \times \tilde{\mathcal{U}},$$

for $j \geq j_0$ and thus Theorem 3.58 (*iv*) would imply that

$$(\theta_j, \xi_j) = (\Theta[\hat{\epsilon}_j], \Xi[\hat{\epsilon}_j]),$$

for $j_0 \leq j \in \mathbb{N}$, and that accordingly the Theorem holds (cf. Definition 3.59.) Thus we now turn to the proof of (3.71). We note that equation $\Lambda[\epsilon, \theta, \xi] = 0$ can be rewritten in the following form

$$\begin{aligned}
&\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR^a_n(\epsilon(t-s))\theta(s)\, d\sigma_s \\
&\quad + F_u(t, \tilde{\xi})\Big(\epsilon \int_{\partial\Omega} S_n(t-s)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R^a_n(\epsilon(t-s))\theta(s)\, d\sigma_s + \xi\Big) \\
&\quad = -F\Big(t, \epsilon \int_{\partial\Omega} S_n(t-s)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R^a_n(\epsilon(t-s))\theta(s)\, d\sigma_s + \xi\Big) \\
&\quad + F_u(t, \tilde{\xi})\Big(\epsilon \int_{\partial\Omega} S_n(t-s)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R^a_n(\epsilon(t-s))\theta(s)\, d\sigma_s + \xi\Big) \qquad \forall t \in \partial\Omega
\end{aligned} \tag{3.72}$$

for all $(\epsilon, \theta, \xi)$ in the domain of $\Lambda$. We define the map $N$ of $]-\epsilon_3, \epsilon_3[ \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$ by setting $N[\epsilon, \theta, \xi]$ equal to the left-hand side of the equality in (3.72), for all $(\epsilon, \theta, \xi) \in ]-\epsilon_3, \epsilon_3[ \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$. By arguing as in the proof of Theorem 3.58, we can prove that $N$ is real analytic. Since $N[\epsilon, \cdot, \cdot]$ is linear for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$, we have

$$N[\epsilon, \theta, \xi] = \partial_{(\theta,\xi)} N[\epsilon, \tilde{\theta}, \tilde{\xi}](\theta, \xi),$$

for all $(\epsilon, \theta, \xi) \in ]-\epsilon_3, \epsilon_3[ \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$, and the map of $]-\epsilon_3, \epsilon_3[$ to $\mathcal{L}(\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}, C^{m-1,\alpha}(\partial\Omega))$ which takes $\epsilon$ to $N[\epsilon, \cdot, \cdot]$ is real analytic. Since

$$N[0, \cdot, \cdot] = \partial_{(\theta,\xi)} \Lambda[0, \tilde{\theta}, \tilde{\xi}](\cdot, \cdot),$$

Theorem 3.58 (*iii*) implies that $N[0, \cdot, \cdot]$ is also a linear homeomorphism. Since the set of linear homeomorphisms of $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$ is open in the space $\mathcal{L}(\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}, C^{m-1,\alpha}(\partial\Omega))$ and since the map which takes a linear invertible operator to its inverse is real analytic (cf. *e.g.*, Hille and Phillips [61, Theorems 4.3.2 and 4.3.4]), there exists $\tilde{\epsilon} \in ]0, \epsilon_3[$ such that the map $\epsilon \mapsto N[\epsilon, \cdot, \cdot]^{(-1)}$ is real analytic from $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega), \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R})$. Next we denote by $S[\epsilon, \theta, \xi]$ the right-hand side of (3.72). Then equation $\Lambda[\epsilon, \theta, \xi] = 0$ (or equivalently equation (3.72)) can be rewritten in the following form:

$$(\theta, \xi) = N[\epsilon, \cdot, \cdot]^{(-1)}[S[\epsilon, \theta, \xi]], \tag{3.73}$$

for all $(\epsilon, \theta, \xi) \in ]-\tilde{\epsilon}, \tilde{\epsilon}[ \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$. Moreover, if $j \in \mathbb{N}$, we observe that by (3.70) we have

$$\begin{aligned}
u_j(w + \hat{\epsilon}_j t) &= u[\hat{\epsilon}_j, \theta_j, \xi_j](w + \hat{\epsilon}_j t) \\
&= \hat{\epsilon}_j \int_{\partial\Omega} S_n(t-s)\theta_j(s)\, d\sigma_s + \hat{\epsilon}_j^{n-1} \int_{\partial\Omega} R^a_n(\hat{\epsilon}_j(t-s))\theta_j(s)\, d\sigma_s + \xi_j \qquad \forall t \in \partial\Omega.
\end{aligned} \tag{3.74}$$

Next we note that condition (3.69), equality (3.74), the proof of Theorem 3.58, the real analyticity of $F$ and standard calculus in Banach space imply that

$$\lim_{j\to\infty} S[\hat{\epsilon}_j, \theta_j, \xi_j] = S[0, \tilde{\theta}, \tilde{\xi}] \qquad \text{in } C^{m-1,\alpha}(\partial\Omega). \tag{3.75}$$

Then by (3.73) and by the real analyticity of $\epsilon \mapsto N[\epsilon, \cdot, \cdot]^{(-1)}$, and by the bilinearity and continuity of the operator of $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega), \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}) \times C^{m-1,\alpha}(\partial\Omega)$ to $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$, which takes a pair $(T_1, T_2)$ to $T_1[T_2]$, by (3.75) we conclude that

$$\begin{aligned}
\lim_{j\to\infty} (\theta_j, \xi_j) &= \lim_{j\to\infty} N[\hat{\epsilon}_j, \cdot, \cdot]^{(-1)}[S[\hat{\epsilon}_j, \theta_j, \xi_j]] \\
&= N[0, \cdot, \cdot]^{(-1)}[S[0, \tilde{\theta}, \tilde{\xi}]] = (\tilde{\theta}, \tilde{\xi}) \qquad \text{in } \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R},
\end{aligned}$$

and, consequently, that (3.71) holds. Thus the proof is complete. □

## 3.6 An homogenization problem for the Laplace equation with nonlinear Robin boundary conditions in a periodically perforated domain

In this section we consider an homogenization problem for the Laplace equation with nonlinear Robin boundary conditions in a periodically perforated domain.

### 3.6.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 3.5.1 and 3.3.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$ be as in (1.56), (1.57), (3.50), respectively. For each $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we consider the following periodic nonlinear Robin problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u(x+\delta a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall i \in \{1,\dots,n\}, \\ \delta \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x) + F\big(\frac{1}{\epsilon\delta}(x-\delta w), u(x)\big) = 0 & \forall x \in \partial\Omega(\epsilon,\delta). \end{cases} \tag{3.76}$$

We give the following definition.

**Definition 3.67.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $\epsilon_3$ be as in Theorem 3.58 (*iv*). Let $u[\cdot]$ be as in Definition 3.59. For each pair $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, +\infty[$, we set

$$u_{(\epsilon,\delta)}(x) \equiv u[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta)$$

*Remark* 3.68. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $\epsilon_3$ be as in Theorem 3.58 (*iv*). For each $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, +\infty[$, $u_{(\epsilon,\delta)}$ is a solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta))$ of problem (3.76).

By the previous remark, we note that a solution of problem (3.76) can be expressed by means of a solution of an auxiliary rescaled problem, which does not depend on $\delta$. This is due to the presence of the factor $\delta$ in front of $\frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x)$ in the third equation of problem (3.76).

By virtue of Theorem (3.66), we have the following.

*Remark* 3.69. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $\epsilon_3$ be as in Theorem 3.58 (*iv*). Let $\bar{\delta} \in ]0, +\infty[$. Let $\{\hat{\epsilon}_j\}_{j\in\mathbb{N}}$ be a sequence in $]0, \epsilon_1[$ converging to 0. If $\{u_j\}_{j\in\mathbb{N}}$ is a sequence of functions such that

$$\begin{aligned} &u_j \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\hat{\epsilon}_j, \bar{\delta})), \\ &u_j \text{ solves (3.76) with } (\epsilon,\delta) \equiv (\hat{\epsilon}_j, \bar{\delta}), \\ &\lim_{j\to\infty} u_j(\bar{\delta} w + \bar{\delta}\hat{\epsilon}_j \cdot) = \tilde{\xi} \qquad \text{in } C^{m-1,\alpha}(\partial\Omega), \end{aligned}$$

then there exists $j_0 \in \mathbb{N}$ such that

$$u_j = u_{(\hat{\epsilon}_j, \bar{\delta})} \qquad \forall j_0 \leq j \in \mathbb{N}.$$

We have the following.

**Proposition 3.70.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $\epsilon_3$ be as in Theorem 3.58 (iv). Let $u[\cdot]$ be as in Definition 3.59. Let $1 \leq p < \infty$. Then*

$$\lim_{\epsilon\to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = \tilde{\xi} \qquad \text{in } L^p(A).$$

*Proof.* It is an easy modification of the proof of Proposition 3.22. Indeed, let $\epsilon_3$, $\Theta$, $\Xi$ be as in Theorem 3.58 (*iv*). Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $\epsilon \in ]0, \epsilon_3[$, we have

$$\begin{aligned} &u[\epsilon] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) \\ &\qquad = \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s + \Xi[\epsilon] \qquad \forall t \in \partial\Omega. \end{aligned}$$

We set

$$N[\epsilon](t) \equiv \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s + \Xi[\epsilon] \qquad \forall t \in \partial\Omega,$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. By taking $\tilde{\epsilon} \in ]0, \epsilon_3[$ small enough, we can assume (cf. Proposition 1.28 $(i)$) that $N$ is a real analytic map of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C \equiv \sup_{\epsilon \in ]-\tilde{\epsilon},\tilde{\epsilon}[} \|N[\epsilon]\|_{C^0(\partial\Omega)} < +\infty.$$

By Theorem 2.5, we have

$$|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]](x)| \le C \qquad \forall x \in A, \quad \forall \epsilon \in ]0, \tilde{\epsilon}[.$$

By Theorem 3.61, we have

$$\lim_{\epsilon \to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]](x) = \tilde{\xi} \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{\epsilon \to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = \tilde{\xi} \qquad \text{in } L^p(A).$$

□

**Proposition 3.71.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\epsilon_3$ be as in Theorem 3.58 $(iv)$. Let $u[\cdot]$ be as in Definition 3.59. Then there exist $\tilde{\epsilon} \in ]0, \epsilon_3[$ and a real analytic map $N$ of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ such that*

$$\|\mathbf{E}^{\#}_{(\epsilon,1)}\left[u[\epsilon], \tilde{\xi}\right] - \tilde{\xi}\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all $\epsilon \in ]0, \tilde{\epsilon}[$. Moreover, as a consequence,*

$$\lim_{\epsilon \to 0^+} \mathbf{E}^{\#}_{(\epsilon,1)}\left[u[\epsilon], \tilde{\xi}\right] = \tilde{\xi} \qquad \textit{in } L^\infty(\mathbb{R}^n).$$

*Proof.* It is an easy modification of the proof of Proposition 3.23. Indeed, Let $\epsilon_3$, $\Theta$, $\Xi$ be as in Theorem 3.58 $(iv)$. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $\epsilon \in ]0, \epsilon_3[$, we have

$$\begin{aligned} &u[\epsilon] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) \\ &\qquad = \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s + \Xi[\epsilon] \qquad \forall t \in \partial\Omega. \end{aligned}$$

Since $\Xi[\cdot]$ is a real analytic function and

$$\Xi[0] = \tilde{\xi},$$

then there exist $\tilde{\epsilon} \in ]0, \epsilon_3[$ and a real analytic function $R_\Xi$ of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $\mathbb{R}$ such that

$$\Xi[\epsilon] - \tilde{\xi} = \epsilon R_\Xi[\epsilon] \qquad \forall \epsilon \in ]-\tilde{\epsilon}, \tilde{\epsilon}[.$$

We set

$$N[\epsilon](t) \equiv \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s + R_\Xi[\epsilon], \qquad \forall t \in \partial\Omega,$$

for all $\epsilon \in ]-\tilde{\epsilon}, \tilde{\epsilon}[$. We have that $N$ is a real analytic map of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$.

By Theorem 2.5, we have

$$\|\mathbf{E}_{(\epsilon,1)}\left[u[\epsilon], \tilde{\xi}\right] - \tilde{\xi}\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)} \qquad \forall \epsilon \in ]0, \tilde{\epsilon}[,$$

and the conclusion easily follows. □

### 3.6.2 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorem we deduce by Proposition 3.70 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 3.72.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\epsilon_3$ be as in Theorem 3.58 (iv). Let $1 \le p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = \tilde{\xi} \qquad \text{in } L^p(V).$$

*Proof.* We modify the proof of Theorem 3.24. By virtue of Proposition 3.70, we have

$$\lim_{\epsilon\to 0^+} \|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]] - \tilde{\xi}\|_{L^p(A)} = 0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $C > 0$ such that

$$\|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] - \tilde{\xi}\|_{L^p(V)} \le C\|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]] - \tilde{\xi}\|_{L^p(A)} \qquad \forall (\epsilon,\delta) \in ]0,\epsilon_3[\times]0,1[.$$

Thus,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] - \tilde{\xi}\|_{L^p(V)} = 0,$$

and we can easily conclude. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 3.73.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\epsilon_6$, $J$ be as in Theorem 3.65. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J[\epsilon], \tag{3.77}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N}\setminus\{0\}$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N}\setminus\{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\
&= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\
&= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.
\end{aligned}$$

Then we note that

$$\begin{aligned}
\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\
&= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon]\big(\frac{l}{r}x\big)\,dx \\
&= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](t)\,dt \\
&= \frac{r^n}{l^n} J[\epsilon].
\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J[\epsilon],$$

and the conclusion follows. □

In the following Theorem we consider the $L^\infty$–distance of a certain extension of $u_{(\epsilon,\delta)}$ and its limit.

**Theorem 3.74.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\epsilon_3$ be as in Theorem 3.58 (iv). Let $\tilde{\epsilon}$, $N$ be as in Proposition 3.71. Then*

$$\|\mathbf{E}^{\#}_{(\epsilon,\delta)}\left[u_{(\epsilon,\delta)}, \tilde{\xi}\right] - \tilde{\xi}\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[ \times ]0,+\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^{\#}_{(\epsilon,\delta)}\left[u_{(\epsilon,\delta)}, \tilde{\xi}\right] = \tilde{\xi} \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* It follows by a simple modification of the proof of Theorem 3.26. Indeed, it suffices to observe that

$$\begin{aligned}\|\mathbf{E}^{\#}_{(\epsilon,\delta)}\left[u_{(\epsilon,\delta)}, \tilde{\xi}\right] - \tilde{\xi}\|_{L^\infty(\mathbb{R}^n)} &= \|\mathbf{E}^{\#}_{(\epsilon,1)}\left[u[\epsilon], \tilde{\xi}\right] - \tilde{\xi}\|_{L^\infty(\mathbb{R}^n)} \\ &= \epsilon \|N[\epsilon]\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[ \times ]0,+\infty[$. □

### 3.6.3 Asymptotic behaviour of the energy integral of $u_{(\epsilon,\delta)}$

This Subsection is devoted to the study of the behaviour of the energy integral of $u_{(\epsilon,\delta)}$. We give the following.

**Definition 3.75.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $\epsilon_3$ be as in Theorem 3.58 (*iv*). For each pair $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,+\infty[$, we set

$$\mathrm{En}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\, dx.$$

*Remark* 3.76. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $\epsilon_3$ be as in Theorem 3.58 (*iv*). Let $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,+\infty[$. We have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\, dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2\, dt \\ &= \delta^{n-2} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2\, dt.\end{aligned}$$

Then we give the following definition, where we consider $\mathrm{En}(\epsilon,\delta)$, with $\epsilon$ equal to a certain function of $\delta$.

**Definition 3.77.** For each $\delta \in ]0,+\infty[$, we set

$$\epsilon[\delta] \equiv \delta^{\frac{2}{n}}.$$

Let $\epsilon_5$ be as in Theorem 3.63. Let $\delta_1 > 0$ be such that $\epsilon[\delta] \in ]0,\epsilon_5[$, for all $\delta \in ]0,\delta_1[$. Then we set

$$\mathrm{En}[\delta] \equiv \mathrm{En}(\epsilon[\delta],\delta),$$

for all $\delta \in ]0,\delta_1[$.

In the following Proposition we compute the limit of $\mathrm{En}[\delta]$ as $\delta$ tends to 0.

**Proposition 3.78.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\epsilon_5$ be as in Theorem 3.63. Let $\delta_1 > 0$ be as in Definition 3.77. Then*

$$\lim_{\delta\to 0^+} \mathrm{En}[\delta] = \int_{\mathbb{R}^n\setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\, dx,$$

*where $\tilde{u}$ is as in Definition 3.56.*

*Proof.* We follow step by step the proof of Propostion 2.140. Let $G$ be as in Theorem 3.63. Let $\delta \in ]0,\delta_1[$. By Remark 3.76 and Theorem 3.63, we have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2\,dx &= \delta^{n-2}(\epsilon[\delta])^n G[\epsilon[\delta]] \\ &= \delta^n G[\delta^{\frac{2}{n}}].\end{aligned}$$

On the other hand,

$$\lfloor (1/\delta)\rfloor^n \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2\,dx \leq \mathrm{En}[\delta] \leq \lceil (1/\delta)\rceil^n \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2\,dx,$$

and so

$$\lfloor (1/\delta)\rfloor^n \delta^n G[\delta^{\frac{2}{n}}] \leq \mathrm{En}[\delta] \leq \lceil (1/\delta)\rceil^n \delta^n G[\delta^{\frac{2}{n}}].$$

Thus, since

$$\lim_{\delta\to 0^+} \lfloor (1/\delta)\rfloor^n \delta^n = 1, \qquad \lim_{\delta\to 0^+} \lceil (1/\delta)\rceil^n \delta^n = 1,$$

we have

$$\lim_{\delta\to 0^+} \mathrm{En}[\delta] = G[0].$$

Finally, by equality (3.64), we easily conclude. □

In the following Proposition we represent the function $\mathrm{En}[\cdot]$ by means of a real analytic function.

**Proposition 3.79.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\epsilon_5$ and $\tilde{\xi}$ be as in Theorem 3.63. Let $\delta_1 > 0$ be as in Definition 3.77. Then*

$$\mathrm{En}[(1/l)] = G[(1/l)^{\frac{2}{n}}],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* It follows by the proof of Proposition 3.78. □

# 3.7 A variant of an homogenization problem for the Laplace equation with nonlinear Robin boundary conditions in a periodically perforated domain

In this section we consider a (slightly) different homogenization problem for the Laplace equation with nonlinear Robin boundary conditions in a periodically perforated domain.

## 3.7.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 3.5.1 and 3.3.1.

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$ be as in (1.56), (1.57), (3.50), respectively. For each $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we consider the following periodic nonlinear Robin problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u(x+\delta a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall i \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x) + F\big(\frac{1}{\epsilon\delta}(x-\delta w), u(x)\big) = 0 & \forall x \in \partial\Omega(\epsilon,\delta). \end{cases} \tag{3.78}$$

In contrast to problem (3.76), we note that in the third equation of problem (3.78) there is not the factor $\delta$ in front of $\frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x)$. As a consequence, we cannot convert problem (3.78) into a rescaled auxiliary problem which does not depend on $\delta$.

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$ be as in (1.56), (1.57), (3.50), respectively. For each $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we introduce the following auxiliary periodic nonlinear Robin problem for the Laplace equation.

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_i) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall i \in \{1,\ldots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) + \delta F\big(\frac{1}{\epsilon}(x-w), u(x)\big) = 0 & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{3.79}$$

We now convert boundary value problem (3.79) into an integral equation.

**Proposition 3.80.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$ be as in* (1.56), (1.57), (3.50), *respectively. Let $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$. Let $\mathcal{U}_\epsilon^{m-1,\alpha}$ be as in* (1.63)*. Then the map of the set of pairs $(\mu,\xi) \in \mathcal{U}_\epsilon^{m-1,\alpha}\times\mathbb{R}$ that solve the equation*

$$\frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \nu_{\Omega_\epsilon}(x)\cdot DS_n^a(x-y)\mu(y)\,d\sigma_y + \delta F\Big(\frac{1}{\epsilon}(x-w), \int_{\partial\Omega_\epsilon} S_n^a(x-y)\mu(y)\,d\sigma_y + \xi\Big) = 0 \qquad \forall x \in \partial\Omega_\epsilon, \tag{3.80}$$

*to the set of $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ which solve problem* (3.79)*, which takes $(\mu,\xi)$ to the function*

$$v_a^-[\partial\Omega_\epsilon,\mu] + \xi \tag{3.81}$$

*is a bijection.*

*Proof.* It follows by Proposition 3.53, by replacing $F$ by $\delta F$. □

As we have seen, we can transform (3.79) into an integral equation defined on the $\epsilon$-dependent domain $\partial\Omega_\epsilon$. In order to get rid of such a dependence, we shall introduce the following Theorem.

**Theorem 3.81.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$ be as in* (1.56), (1.57), (3.50), *respectively. Let $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.64)*. Then the map $u[\epsilon,\delta,\cdot,\cdot]$ of the set of pairs $(\theta,\xi) \in \mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$ that solve the equation*

$$\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\theta(s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s + F\Big(t, \delta\epsilon\int_{\partial\Omega} S_n(t-s)\theta(s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s + \xi\Big) = 0 \qquad \forall t \in \partial\Omega, \tag{3.82}$$

*to the set of $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ which solve problem* (3.79)*, which takes $(\theta,\xi)$ to the function*

$$u[\epsilon,\delta,\theta,\xi] \equiv v_a^-[\partial\Omega_\epsilon, \delta\theta(\frac{1}{\epsilon}(\cdot - w))] + \xi \tag{3.83}$$

*is a bijection.*

*Proof.* It is an immediate consequence of Proposition 3.80, of the Theorem of change of variables in integrals and of Lemma 1.25. □

In the following Proposition we study equation (3.82) for $(\epsilon,\delta) = (0,0)$ and when we set $\xi = \tilde{\xi}$ (where $\tilde{\xi}$ is as in (3.51).)

**Proposition 3.82.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $F$, $\tilde{\xi}$ be as in* (1.56), (3.50), (3.51), *respectively. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.64)*. Then the integral equation*

$$\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\theta(s)\,d\sigma_s + F(t,\tilde{\xi}) = 0 \qquad \forall t \in \partial\Omega, \tag{3.84}$$

*which we call the* limiting equation*, has a unique solution $\theta \in \mathcal{U}_0^{m-1,\alpha}$, which we denote by $\tilde{\theta}$.*

*Proof.* It is Proposition 3.55. □

Now we want to see if equation (3.84) is related to some (limiting) boundary value problem. We give the following.

**Definition 3.83.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $F$, $\tilde{\xi}$ be as in (1.56), (3.50), (3.51), respectively. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega)$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega, \\ \frac{\partial}{\partial \nu_\Omega} u(x) = -F(x, \tilde{\xi}) & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{3.85}$$

Problem (3.85) will be called the *limiting boundary value problem.*

*Remark* 3.84. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $F$, $\tilde{\xi}$ be as in (1.56), (3.50), (3.51), respectively. Let $\tilde{\theta}$ be as in Proposition 3.82. We have

$$\tilde{u}(x) = \int_{\partial\Omega} S_n(x-y)\tilde{\theta}(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \Omega.$$

We are now ready to analyse equation (3.82) around the degenerate case $(\epsilon, \delta) = (0,0)$.

**Theorem 3.85.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.64)*. Let $\Lambda$ be the map of $]-\epsilon_1, \epsilon_1[ \times \mathbb{R} \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$, defined by*

$$\begin{aligned} \Lambda[\epsilon, \delta, \theta, \xi](t) \equiv \frac{1}{2}\theta(t) &+ \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\theta(s)\,d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s \\ &+ F\Big(t, \delta\epsilon \int_{\partial\Omega} S_n(t-s)\theta(s)\,d\sigma_s + \delta\epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s + \xi\Big) \qquad \forall t \in \partial\Omega, \end{aligned} \tag{3.86}$$

*for all $(\epsilon, \delta, \theta, \xi) \in ]-\epsilon_1, \epsilon_1[ \times \mathbb{R} \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$. Then the following statements hold.*

*(i) Equation $\Lambda[0,0,\theta,\tilde{\xi}] = 0$ is equivalent to the limiting equation* (3.84) *and has one and only one solution $\tilde{\theta}$ in $\mathcal{U}_0^{m-1,\alpha}$ (cf. Proposition 3.82.)*

*(ii) If $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, then equation $\Lambda[\epsilon, \delta, \theta, \xi] = 0$ is equivalent to equation* (3.82) *for $(\theta, \xi)$.*

*(iii) There exists $\epsilon_2 \in ]0, \epsilon_1]$, such that the map $\Lambda$ of $]-\epsilon_2, \epsilon_2[ \times \mathbb{R} \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$ is real analytic. Moreover, the differential $\partial_{(\theta,\xi)}\Lambda[0,0,\tilde{\theta},\tilde{\xi}]$ of $\Lambda$ at $(0,0,\tilde{\theta},\tilde{\xi})$ is a linear homeomorphism of $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ onto $C^{m-1,\alpha}(\partial\Omega)$.*

*(iv) There exist $\epsilon_3 \in ]0, \epsilon_2]$, $\delta_1 \in ]0, +\infty[$, an open neighbourhood $\tilde{\mathcal{U}}$ of $(\tilde{\theta}, \tilde{\xi})$ in $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ and a real analytic map $(\Theta[\cdot,\cdot], \Xi[\cdot,\cdot])$ of $]-\epsilon_3, \epsilon_3[ \times ]-\delta_1, \delta_1[$ to $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$, such that the set of zeros of the map $\Lambda$ in $]-\epsilon_3, \epsilon_3[ \times \tilde{\mathcal{U}}$ coincides with the graph of $(\Theta[\cdot,\cdot], \Xi[\cdot,\cdot])$. In particular, $(\Theta[0,0], \Xi[0,0]) = (\tilde{\theta}, \tilde{\xi})$.*

*Proof.* Statements (*i*) and (*ii*) are obvious. We now prove statement (*iii*). By the same argument as the proof of Theorem 3.58, one can show such that there exists $\epsilon_2 \in ]0, \epsilon_1]$, such that the map $\Lambda$ of $]-\epsilon_2, \epsilon_2[ \times \mathbb{R} \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$ is real analytic. By standard calculus in Banach spaces, the differential $\partial_{(\theta,\xi)}\Lambda[0,0,\tilde{\theta},\tilde{\xi}]$ of $\Lambda$ at $(0,0,\tilde{\theta},\tilde{\xi})$ is delivered by the following formula:

$$\partial_{(\theta,\xi)}\Lambda[0,0,\tilde{\theta},\tilde{\xi}](\bar{\theta},\bar{\xi})(t) = \frac{1}{2}\bar{\theta}(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\bar{\theta}(s)\,d\sigma_s + F_u(t,\tilde{\xi})\bar{\xi} \qquad \forall t \in \partial\Omega,$$

for all $(\bar{\theta}, \bar{\xi}) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$. By the proof of Theorem 3.58 (*iii*), we deduce that the differential $\partial_{(\theta,\xi)}\Lambda[0,0,\tilde{\theta},\tilde{\xi}]$ of $\Lambda$ at $(0,0,\tilde{\theta},\tilde{\xi})$ is a linear homeomorphism of $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ onto $C^{m-1,\alpha}(\partial\Omega)$. Finally, statement (*iv*) is an immediate consequence of statement (*iii*) and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

We are now in the position to introduce the following.

**Definition 3.86.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $u[\cdot,\cdot,\cdot,\cdot]$ be as in Theorem 3.81. If $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[$, we set

$$u[\epsilon, \delta](x) \equiv u[\epsilon, \delta, \Theta[\epsilon, \delta], \Xi[\epsilon, \delta]](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],$$

where $\epsilon_3$, $\delta_1$, $\Theta$, $\Xi$ are as in Theorem 3.85 (*iv*).

*Remark* 3.87. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 3.85 (*iv*). Let $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[$. Then $u[\epsilon, \delta]$ is a solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of problem (3.79).

The following statement shows that $\{u[\epsilon, \delta](\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon_3[\times]0,\delta_1[}$ can be continued real analytically for negative values of $\epsilon$ and $\delta$.

**Theorem 3.88.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 3.85. Then the following statements hold.*

*(i) Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \epsilon_3]$, a real analytic operator $U_1$ of $]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$ to the space $C^0_h(\mathrm{cl}\,V)$, and a real analytic operator $U_2$ of $]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$ such that the following conditions hold.*

*(j)* $\mathrm{cl}\,V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*
$$u[\epsilon, \delta](x) = \delta\epsilon^{n-1} U_1[\epsilon, \delta](x) + U_2[\epsilon, \delta] \qquad \forall x \in \mathrm{cl}\,V,$$
*for all* $(\epsilon, \delta) \in ]0, \epsilon_4[ \times ]0, \delta_1[$. *Moreover,*
$$U_2[0, 0] = \tilde{\xi}.$$

*(ii) Let $\bar{V}$ be a bounded open subset of $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$. Then there exist $\bar{\epsilon}_4 \in ]0, \epsilon_3]$, a real analytic operator $\bar{U}_1$ of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \times ]-\delta_1, \delta_1[$ to the space $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$, and a real analytic operator $\bar{U}_2$ of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$ such that the following conditions hold.*

*(j')* $w + \epsilon\,\mathrm{cl}\,\bar{V} \subseteq \mathrm{cl}\,\mathbb{P}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \setminus \{0\}$.

*(jj')*
$$u[\epsilon, \delta](w + \epsilon t) = \delta\epsilon \bar{U}_1[\epsilon, \delta](t) + \bar{U}_2[\epsilon, \delta] \qquad \forall t \in \mathrm{cl}\,\bar{V},$$
*for all* $(\epsilon, \delta) \in ]0, \bar{\epsilon}_4[ \times ]0, \delta_1[$. *Moreover,*
$$\bar{U}_2[0, 0] = \tilde{\xi}.$$

*Proof.* Let $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[$. We observe that
$$u[\epsilon, \delta](x) = \delta\epsilon^{n-1} \int_{\partial\Omega} S^a_n(x - w - \epsilon s)\Theta[\epsilon, \delta](s)\, d\sigma_s + \Xi[\epsilon, \delta] \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$
Thus, in order to prove both statements, it suffices to follow the proof of Theorem 3.42. □

As done in Theorem 3.88 for $u[\cdot, \cdot]$, we can now prove a real analytic continuation Theorem for the energy integral. Namely, we prove the following.

**Theorem 3.89.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 3.85. Then there exist $\epsilon_5 \in ]0, \epsilon_3]$ and a real analytic operator $\tilde{\xi}$ of $]-\epsilon_5, \epsilon_5[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$, such that*
$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon, \delta](x)|^2\, dx = \delta^2 \epsilon^n G[\epsilon, \delta], \tag{3.87}$$
*for all* $(\epsilon, \delta) \in ]0, \epsilon_5[ \times ]0, \delta_1[$. *Moreover,*
$$G[0, 0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\, dx. \tag{3.88}$$

*Proof.* Let $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[$. Clearly,
$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon, \delta](x)|^2\, dx = \delta^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla v^-_a[\partial\Omega_\epsilon, \Theta[\epsilon, \delta](\frac{1}{\epsilon}(\cdot - w))](x)|^2\, dx.$$
As a consequence, in order to prove the Theorem, it suffices to follow the proof of Theorem 3.16. □

As done in Theorem 3.89 for the energy integral, we can now prove a real analytic continuation Theorem for the integral of the solution. Namely, we prove the following.

**Theorem 3.90.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 3.85. Then there exist $\epsilon_6 \in ]0, \epsilon_3]$, $\delta_2 \in ]0, \delta_1]$, and a real analytic operator $J$ of $]-\epsilon_6, \epsilon_6[ \times ]-\delta_2, \delta_2[$ to $\mathbb{R}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon, \delta](x)\, dx = J[\epsilon, \delta], \tag{3.89}$$

*for all $(\epsilon, \delta) \in ]0, \epsilon_6[ \times ]0, \delta_2[$. Moreover,*

$$J[0,0] = \tilde{\xi} |A|_n. \tag{3.90}$$

*Proof.* Let $\Theta[\cdot, \cdot]$, $\Xi[\cdot, \cdot]$ be as in Theorem 3.85 *(iv)*. Let $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[$. Since

$$u[\epsilon, \delta](x) = \delta v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon, \delta](\frac{1}{\epsilon}(\cdot - w))\big](x) + \Xi[\epsilon, \delta] \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon],$$

then

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon, \delta](x)\, dx =& \delta \int_{\mathbb{P}_a[\Omega_\epsilon]} v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon, \delta](\frac{1}{\epsilon}(\cdot - w))\big](x)\, dx \\ &+ \Xi[\epsilon, \delta]\big(|A|_n - \epsilon^n |\Omega|_n\big).\end{aligned}$$

On the other hand, by arguing as in the proof of Theorem 3.16, we note that

$$\begin{aligned}& v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon, \delta](\frac{1}{\epsilon}(\cdot - w))\big](w + \epsilon t) \\ &= \epsilon \int_{\partial\Omega} S_n(t - s)\Theta[\epsilon, \delta](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t - s))\Theta[\epsilon, \delta](s)\, d\sigma_s \qquad \forall t \in \partial\Omega.\end{aligned}$$

Then, if we set

$$L[\epsilon, \delta](t) \equiv \epsilon \int_{\partial\Omega} S_n(t - s)\Theta[\epsilon, \delta](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t - s))\Theta[\epsilon, \delta](s)\, d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $(\epsilon, \delta) \in ]-\epsilon_3, \epsilon_3[ \times ]-\delta_1, \delta_1[$, we have that $L[\cdot, \cdot]$ is a real analytic map of $]-\epsilon_3, \epsilon_3[ \times ]-\delta_1, \delta_1[$ to $C^{m,\alpha}(\partial\Omega)$. Then, by Theorem 2.115, we easily deduce that there exist $\epsilon_6 \in ]0, \epsilon_3]$, $\delta_2 \in ]0, \delta_1]$, and a real analytic map $J_1$ of $]-\epsilon_6, \epsilon_6[ \times ]-\delta_2, \delta_2[$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon, \delta](\frac{1}{\epsilon}(\cdot - w))\big](x)\, dx = J_1[\epsilon, \delta],$$

for all $(\epsilon, \delta) \in ]0, \epsilon_6[ \times ]0, \delta_2[$. Then, if we set

$$J[\epsilon, \delta] \equiv \delta J_1[\epsilon, \delta] + \Xi[\epsilon, \delta]\big(|A|_n - \epsilon^n |\Omega|_n\big),$$

for all $(\epsilon, \delta) \in ]-\epsilon_6, \epsilon_6[ \times ]-\delta_2, \delta_2[$, we can immediately conclude. □

We are now ready to show that the family $\{u[\epsilon, \delta]\}_{(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[}$ is essentially unique. Namely, we prove the following.

**Theorem 3.91.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\{(\hat{\epsilon}_j, \hat{\delta}_j)\}_{j \in \mathbb{N}}$ be a sequence in $]0, \epsilon_1[ \times ]0, +\infty[$ converging to $(0,0)$. If $\{u_j\}_{j \in \mathbb{N}}$ is a sequence of functions such that*

$$u_j \in C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\Omega_{\hat{\epsilon}_j}]), \tag{3.91}$$

$$u_j \text{ solves (3.79) with } (\epsilon, \delta) \equiv (\hat{\epsilon}_j, \hat{\delta}_j), \tag{3.92}$$

$$\lim_{j \to \infty} u_j(w + \hat{\epsilon}_j \cdot) = \tilde{\xi} \qquad \text{in } C^{m-1,\alpha}(\partial\Omega), \tag{3.93}$$

*then there exists $j_0 \in \mathbb{N}$ such that*

$$u_j = u[\hat{\epsilon}_j, \hat{\delta}_j] \qquad \forall j_0 \le j \in \mathbb{N}.$$

*Proof.* By Theorem 3.81, for each $j \in \mathbb{N}$, there exists a unique pair $(\theta_j, \xi_j)$ in $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ such that

$$u_j = u[\hat{\epsilon}_j, \hat{\delta}_j, \theta_j, \xi_j]. \tag{3.94}$$

We shall now try to show that

$$\lim_{j\to\infty} (\theta_j, \xi_j) = (\tilde{\theta}, \tilde{\xi}) \qquad \text{in } \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}. \tag{3.95}$$

Indeed, if we denote by $\tilde{\mathcal{U}}$ the neighbourhood of Theorem 3.85 (*iv*), the limiting relation of (3.95) implies that there exists $j_0 \in \mathbb{N}$ such that

$$(\hat{\epsilon}_j, \hat{\delta}_j, \theta_j, \xi_j) \in ]0, \epsilon_3[\, \times\, ]0, \delta_1[\, \times \tilde{\mathcal{U}},$$

for $j \geq j_0$ and thus Theorem 3.85 (*iv*) would imply that

$$(\theta_j, \xi_j) = (\Theta[\hat{\epsilon}_j, \hat{\delta}_j], \Xi[\hat{\epsilon}_j, \hat{\delta}_j]),$$

for $j_0 \leq j \in \mathbb{N}$, and that accordingly the Theorem holds (cf. Definition 3.86.) Thus we now turn to the proof of (3.95). We note that equation $\Lambda[\epsilon, \delta, \theta, \xi] = 0$ can be rewritten in the following form

$$\begin{aligned}
&\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s \\
&\quad + F_u(t, \tilde{\xi})\Big(\delta\epsilon \int_{\partial\Omega} S_n(t-s)\theta(s)\, d\sigma_s + \delta\epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s + \xi\Big) \\
&\quad = -F\Big(t, \delta\epsilon \int_{\partial\Omega} S_n(t-s)\theta(s)\, d\sigma_s + \delta\epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s + \xi\Big) \\
&\quad + F_u(t, \tilde{\xi})\Big(\delta\epsilon \int_{\partial\Omega} S_n(t-s)\theta(s)\, d\sigma_s + \delta\epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s + \xi\Big) \qquad \forall t \in \partial\Omega
\end{aligned} \tag{3.96}$$

for all $(\epsilon, \delta, \theta, \xi)$ in the domain of $\Lambda$. We define the map $N$ of $]-\epsilon_3, \epsilon_3[\, \times\, ]-\delta_1, \delta_1[\, \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$ by setting $N[\epsilon, \delta, \theta, \xi]$ equal to the left-hand side of the equality in (3.96), for all $(\epsilon, \delta, \theta, \xi) \in\, ]-\epsilon_3, \epsilon_3[\, \times\, ]-\delta_1, \delta_1[\, \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$. By arguing as in the proof of Theorem 3.85, we can prove that $N$ is real analytic. Since $N[\epsilon, \delta, \cdot, \cdot]$ is linear for all $(\epsilon, \delta) \in\, ]-\epsilon_3, \epsilon_3[\, \times\, ]-\delta_1, \delta_1[$, we have

$$N[\epsilon, \delta, \theta, \xi] = \partial_{(\theta,\xi)} N[\epsilon, \delta, \tilde{\theta}, \tilde{\xi}](\theta, \xi),$$

for all $(\epsilon, \delta, \theta, \xi) \in\, ]-\epsilon_3, \epsilon_3[\times]-\delta_1, \delta_1[\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$, and the map of $]-\epsilon_3, \epsilon_3[\times]-\delta_1, \delta_1[$ to $\mathcal{L}(\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}, C^{m-1,\alpha}(\partial\Omega))$ which takes $(\epsilon, \delta)$ to $N[\epsilon, \delta, \cdot, \cdot]$ is real analytic. Since

$$N[0, 0, \cdot, \cdot] = \partial_{(\theta,\xi)} \Lambda[0, 0, \tilde{\theta}, \tilde{\xi}](\cdot, \cdot),$$

Theorem 3.85 (*iii*) implies that $N[0, 0, \cdot, \cdot]$ is also a linear homeomorphism. Since the set of linear homeomorphisms of $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$ is open in the space $\mathcal{L}(\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}, C^{m-1,\alpha}(\partial\Omega))$ and since the map which takes a linear invertible operator to its inverse is real analytic (cf. *e.g.*, Hille and Phillips [61, Theorems 4.3.2 and 4.3.4]), there exists $(\tilde{\epsilon}, \tilde{\delta}) \in\, ]0, \epsilon_3[\, \times\, ]0, \delta_1[$ such that the map $(\epsilon, \delta) \mapsto N[\epsilon, \delta, \cdot, \cdot]^{(-1)}$ is real analytic from $]-\tilde{\epsilon}, \tilde{\epsilon}[\, \times\, ]-\tilde{\delta}, \tilde{\delta}[$ to $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega), \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R})$. Next we denote by $S[\epsilon, \delta, \theta, \xi]$ the right-hand side of (3.96). Then equation $\Lambda[\epsilon, \delta, \theta, \xi] = 0$ (or equivalently equation (3.96)) can be rewritten in the following form:

$$(\theta, \xi) = N[\epsilon, \delta, \cdot, \cdot]^{(-1)}[S[\epsilon, \delta, \theta, \xi]], \tag{3.97}$$

for all $(\epsilon, \delta, \theta, \xi) \in\, ]-\tilde{\epsilon}, \tilde{\epsilon}[\, \times\, ]-\tilde{\delta}, \tilde{\delta}[\, \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$. Moreover, if $j \in \mathbb{N}$, we observe that by (3.94) we have

$$\begin{aligned}
u_j(w + \hat{\epsilon}_j t) &= u[\hat{\epsilon}_j, \hat{\delta}_j, \theta_j, \xi_j](w + \hat{\epsilon}_j t) \\
&= \hat{\delta}_j \hat{\epsilon}_j \int_{\partial\Omega} S_n(t-s)\theta_j(s)\, d\sigma_s + \hat{\delta}_j \hat{\epsilon}_j^{n-1} \int_{\partial\Omega} R_n^a(\hat{\epsilon}_j(t-s))\theta_j(s)\, d\sigma_s + \xi_j \qquad \forall t \in \partial\Omega.
\end{aligned} \tag{3.98}$$

Next we note that condition (3.93), equality (3.98), the proof of Theorem 3.85, the real analyticity of $F$ and standard calculus in Banach space imply that

$$\lim_{j\to\infty} S[\hat{\epsilon}_j,\hat{\delta}_j,\theta_j,\xi_j] = S[0,0,\tilde{\theta},\tilde{\xi}] \qquad \text{in } C^{m-1,\alpha}(\partial\Omega). \tag{3.99}$$

Then by (3.97) and by the real analyticity of $(\epsilon,\delta)\mapsto N[\epsilon,\delta,\cdot,\cdot]^{(-1)}$, and by the bilinearity and continuity of the operator of $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega),\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R})\times C^{m-1,\alpha}(\partial\Omega)$ to $\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$, which takes a pair $(T_1,T_2)$ to $T_1[T_2]$, by (3.99) we conclude that

$$\begin{aligned}\lim_{j\to\infty}(\theta_j,\xi_j) &= \lim_{j\to\infty} N[\hat{\epsilon}_j,\hat{\delta}_j,\cdot,\cdot]^{(-1)}[S[\hat{\epsilon}_j,\hat{\delta}_j,\theta_j,\xi_j]]\\ &= N[0,0,\cdot,\cdot]^{(-1)}[S[0,0,\tilde{\theta},\tilde{\xi}]] = (\tilde{\theta},\tilde{\xi}) \qquad \text{in } \mathcal{U}_0^{m-1,\alpha}\times\mathbb{R},\end{aligned}$$

and, consequently, that (3.95) holds. Thus the proof is complete. □

We give the following definition.

**Definition 3.92.** Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 3.85 (*iv*). Let $u[\cdot,\cdot]$ be as in Definition 3.86. For each pair $(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[$, we set

$$u_{(\epsilon,\delta)}(x)\equiv u[\epsilon,\delta](\frac{x}{\delta}) \qquad \forall x\in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta)$$

*Remark* 3.93. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 3.85 (*iv*). For each $(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[$, $u_{(\epsilon,\delta)}$ is a solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta))$ of problem (3.78).

We have the following.

**Proposition 3.94.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 3.85* (*iv*). *Let $u[\cdot,\cdot]$ be as in Definition 3.86. Let $1\le p<\infty$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]] = \tilde{\xi} \qquad \text{in } L^p(A).$$

*Proof.* t suffices to modify the proof of Proposition 3.22. Let $\epsilon_3$, $\delta_1$, $\Theta$, $\Xi$ be as in Theorem 3.85 (*iv*). Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[$, we have

$$\begin{aligned}&u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\\ &\quad=\delta\epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s+\delta\epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s+\Xi[\epsilon,\delta] \quad \forall t\in\partial\Omega.\end{aligned}$$

We set

$$N[\epsilon,\delta](t)\equiv\delta\epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s+\delta\epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s+\Xi[\epsilon,\delta] \qquad \forall t\in\partial\Omega,$$

for all $(\epsilon,\delta)\in\,]-\epsilon_3,\epsilon_3[\,\times\,]-\delta_1,\delta_1[$. By taking $\tilde{\epsilon}\in\,]0,\epsilon_3[$ and $\tilde{\delta}\in\,]0,\delta_1[$ small enough, we can assume (cf. Proposition 1.26 (*i*)) that $N$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[\,\times\,]-\tilde{\delta},\tilde{\delta}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C\equiv \sup_{(\epsilon,\delta)\in]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\delta},\tilde{\delta}[} \|N[\epsilon,\delta]\|_{C^0(\partial\Omega)}<+\infty.$$

By Theorem 2.5, we have

$$|\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]](x)|\le C \qquad \forall x\in A, \quad \forall(\epsilon,\delta)\in\,]0,\tilde{\epsilon}[\,\times\,]0,\tilde{\delta}[.$$

By Theorem 3.88, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]](x)=\tilde{\xi} \qquad \forall x\in A\setminus\{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]=\tilde{\xi} \qquad \text{in } L^p(A).$$

□

**Proposition 3.95.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 3.85 (iv). Let $u[\cdot,\cdot]$ be as in Definition 3.86. Then there exist $\tilde{\epsilon} \in ]0,\epsilon_3[$ and a real analytic map $N$ of $]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\delta_1,\delta_1[$ to $C^{m,\alpha}(\partial\Omega)$ such that*

$$\|\mathbf{E}^{\#}_{(\epsilon,1)}\Big[u[\epsilon,\delta],\tilde{\xi}\Big] - \tilde{\xi}\|_{L^\infty(\mathbb{R}^n)} = \|N[\epsilon,\delta]\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\delta_1[$. Moreover,*

$$N[0,0] = 0$$

*and, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^{\#}_{(\epsilon,1)}\Big[u[\epsilon,\delta],\tilde{\xi}\Big] = \tilde{\xi} \ \textit{in } L^\infty(\mathbb{R}^n).$$

*Proof.* Let $\epsilon_3$, $\delta_1$, $\Theta$, $\Xi$ be as in Theorem 3.85 (*iv*). Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $(\epsilon,\delta) \in ]0,\epsilon_3[\times]0,\delta_1[$, we have

$$\begin{aligned} u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) =&\delta\epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s \\ &+ \delta\epsilon^{n-1}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s + \Xi[\epsilon,\delta] \qquad \forall t \in \partial\Omega. \end{aligned}$$

We set

$$\begin{aligned} N[\epsilon,\delta](t) \equiv&\delta\epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s \\ &+ \delta\epsilon^{n-1}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s + \Xi[\epsilon,\delta] - \tilde{\xi}, \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $(\epsilon,\delta) \in ]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[$. By taking $\tilde{\epsilon} \in ]0,\epsilon_3[$ small enough, we can assume (cf. Proposition 1.26 (*i*)) that $N$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\delta_1,\delta_1[$ to $C^{m,\alpha}(\partial\Omega)$. By Theorem 2.5, we have

$$\|\mathbf{E}_{(\epsilon,1)}\Big[u[\epsilon,\delta],\tilde{\xi}\Big] - \tilde{\xi}\|_{L^\infty(\mathbb{R}^n)} = \|N[\epsilon,\delta]\|_{C^0(\partial\Omega)} \qquad \forall(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\delta_1[,$$

and the conclusion easily follows. □

### 3.7.2 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorem we deduce by Proposition 3.94 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 3.96.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 3.85 (iv). Let $1 \le p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = \tilde{\xi} \qquad \textit{in } L^p(V).$$

*Proof.* We modify the proof of Theorem 2.134. By virtue of Proposition 3.45, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \|\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]] - \tilde{\xi}\|_{L^p(A)} = 0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $C > 0$ such that

$$\begin{aligned} \|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] - \tilde{\xi}\|_{L^p(V)} \le C\|\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]] - \tilde{\xi}\|_{L^p(A)}& \\ \forall(\epsilon,\delta) \in ]0,\epsilon_3[\times]0,\min\{1,\delta_1\}[.& \end{aligned}$$

Thus,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] - \tilde{\xi}\|_{L^p(V)} = 0,$$

and we can easily conclude. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 3.97.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 3.85. Let $\epsilon_6$, $\delta_2$, $J$ be as in Theorem 3.90. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J\big[\epsilon, \frac{r}{l}\big], \tag{3.100}$$

*for all $\epsilon \in ]0, \epsilon_6[$, and for all $l \in \mathbb{N} \setminus \{0\}$ such that $l > (r/\delta_2)$.*

*Proof.* Let $\epsilon \in ]0, \epsilon_6[$, and let $l \in \mathbb{N} \setminus \{0\}$ be such that $l > (r/\delta_2)$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u\big[\epsilon,(r/l)\big]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u\big[\epsilon,(r/l)\big](t)\,dt \\ &= \frac{r^n}{l^n} J\big[\epsilon,\frac{r}{l}\big].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J\big[\epsilon, \frac{r}{l}\big],$$

and the conclusion follows. □

In the following Theorem we consider the $L^\infty$–distance of a certain extension of $u_{(\epsilon,\delta)}$ and its limit.

**Theorem 3.98.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 3.85 (iv). Let $\tilde{\epsilon}$, $N$ be as in Proposition 3.95. Then*

$$\|\mathbf{E}^{\#}_{(\epsilon,\delta)}\Big[u_{(\epsilon,\delta)},\tilde{\xi}\Big] - \tilde{\xi}\|_{L^\infty(\mathbb{R}^n)} = \|N[\epsilon,\delta]\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[ \times ]0,\delta_1[$. Moreover,*

$$N[0,0] = 0,$$

*and, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^{\#}_{(\epsilon,\delta)}\Big[u_{(\epsilon,\delta)},\tilde{\xi}\Big] = \tilde{\xi} \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathbf{E}^{\#}_{(\epsilon,\delta)}\Big[u_{(\epsilon,\delta)},\tilde{\xi}\Big] - \tilde{\xi}\|_{L^\infty(\mathbb{R}^n)} &= \|\mathbf{E}^{\#}_{(\epsilon,1)}\Big[u[\epsilon,\delta],\tilde{\xi}\Big] - \tilde{\xi}\|_{L^\infty(\mathbb{R}^n)} \\ &= \|N[\epsilon,\delta]\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[ \times ]0,\delta_1[$. □

### 3.7.3 Asymptotic behaviour of the energy integral of $u_{(\epsilon,\delta)}$

This Subsection is devoted to the study of the behaviour of the energy integral of $u_{(\epsilon,\delta)}$. We give the following.

**Definition 3.99.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 3.85 (*iv*). For each pair $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[$, we set

$$\mathrm{En}(\epsilon, \delta) \equiv \int_{A \cap \mathbb{T}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx.$$

*Remark* 3.100. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ be as in (1.56), (1.57), (3.50), (3.51), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 3.85 (*iv*). Let $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[$. We have

$$\begin{aligned} \int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2 \, dt \\ &= \delta^{n-2} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon, \delta](t)|^2 \, dt. \end{aligned}$$

In the following Proposition we represent the function $\mathrm{En}(\cdot, \cdot)$ by means of a real analytic function.

**Proposition 3.101.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $F$, $\tilde{\xi}$ *be as in* (1.56), (1.57), (3.50), (3.51), *respectively. Let* $\epsilon_5$, $\delta_1$, *and* $\tilde{\xi}$ *be as in Theorem 3.89. Then*

$$\mathrm{En}\Big(\epsilon, \frac{1}{l}\Big) = \epsilon^n G[\epsilon, (1/l)],$$

*for all* $\epsilon \in ]0, \epsilon_5[$ *and for all* $l \in \mathbb{N} \setminus \{0\}$ *such that* $l > (1/\delta_1)$.

*Proof.* By Remark 3.100 and Theorem 3.89, we have

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx = \delta^n \epsilon^n G[\epsilon, \delta] \tag{3.101}$$

On the other hand, if $\epsilon \in ]0, \epsilon_5[$ and $l \in \mathbb{N} \setminus \{0\}$ is such that $l > (1/\delta_1)$, then we have

$$\begin{aligned} \mathrm{En}\Big(\epsilon, \frac{1}{l}\Big) &= l^n \frac{1}{l^n} \epsilon^n G[\epsilon, (1/l)] \\ &= \epsilon^n G[\epsilon, (1/l)], \end{aligned}$$

and the conclusion easily follows. □

# CHAPTER 4

# Singular perturbation and homogenization problems for the Laplace equation with transmission boundary condition

In this Chapter we consider periodic transmission problems for the Laplace equation and we study singular perturbation and homogenization problems for the Laplace operator with transmission boundary conditions in a periodically perforated domain. As well as for the Robin problem, we consider both the linear and the nonlinear case. In the first part, by means of periodic simple layer potentials, we show the solvability of a linear transmission problem. Then we consider singular perturbation problems for the Laplace operator, with linear and nonlinear transmission boundary conditions, in a periodically perforated domain with small holes, and we apply the obtained results to homogenization problems. As far as these problems are concerned, the strategy that we follow, in particular for the nonlinear case, is the one of Lanza [78], where the asymptotic behaviour of the solutions of a nonlinear transmission problem for the Laplace operator is investigated. Concerning nonlinear problems, we also mention Dalla Riva and Lanza [38, 39, 42, 43]. One of the tools used in our analysis is the study of the dependence of layer potentials upon perturbations (cf. Lanza and Rossi [85] and also Dalla Riva and Lanza [40].)

We retain the notation of Chapter 1 (see in particular Sections 1.1, 1.3, Theorem 1.4, and Definitions 1.12, 1.14, 1.16.) For notation, definitions, and properties concerning classical layer potentials for the Laplace equation, we refer to Appendix B.

## 4.1 A linear transmission periodic boundary value problem for the Laplace equation

In this Section we introduce a periodic linear transmission problem for the Laplace equation and we show the existence of a solution by means of the periodic simple layer potential.

### 4.1.1 Formulation of the problem

In this Subsection we introduce a periodic linear transmission problem for the Laplace equation.

First of all, we need to introduce some notation. Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). We shall consider the following assumptions.

$$\phi \in ]0,+\infty[, \tag{4.1}$$

$$\gamma \in ]0,+\infty[, \tag{4.2}$$

$$\Gamma \in C^{m-1,\alpha}(\partial\mathbb{I}), \qquad \int_{\partial\mathbb{I}} \Gamma \, d\sigma = 0. \tag{4.3}$$

We are now ready to give the following.

**Definition 4.1.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\phi$, $\gamma$, $\Gamma$ be as in (4.1), (4.2), (4.3), respectively. We say that a pair of functions $(u^i, u^o) \in (C^1(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]) \cap C^2(\mathbb{S}_a[\mathbb{I}])) \times (C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}]))$ solves the *periodic (linear) transmission problem for the Laplace equation* if

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a[\mathbb{I}], \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u^i(x + a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}, \\ u^o(x + a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}, \\ u^o(x) = \phi u^i(x) & \forall x \in \partial\mathbb{I}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} u^o(x) = \gamma \frac{\partial}{\partial \nu_{\mathbb{I}}} u^i(x) + \Gamma(x) & \forall x \in \partial\mathbb{I}. \end{cases} \tag{4.4}$$

### 4.1.2 Existence and uniqueness results for the solutions of the periodic transmission problem

In this Subsection we prove uniqueness and existence results for the solutions of the periodic transmission problems for the Laplace equation.

**Proposition 4.2.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\phi$, $\gamma$ be as in (4.1), (4.2), respectively. Let $\Gamma \in C^{m-1,\alpha}(\partial\mathbb{I})$. If problem (4.4) has a solution in $(C^1(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]) \cap C^2(\mathbb{S}_a[\mathbb{I}])) \times (C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}]))$, then*

$$\int_{\partial\mathbb{I}} \Gamma \, d\sigma = 0.$$

*Proof.* Let $(u^i, u^o) \in (C^1(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]) \cap C^2(\mathbb{S}_a[\mathbb{I}])) \times (C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}]))$ be a solution of (4.4). By Green's Formula,

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}} u^i \, d\sigma = \int_{\mathbb{I}} \Delta u^i(x) \, dx = 0.$$

By the periodicity of $u^o$ and Green's Formula,

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}} u^o \, d\sigma = \int_{\partial A} \frac{\partial}{\partial \nu_A} u^o \, d\sigma - \int_{\mathbb{P}_a[\mathbb{I}]} \Delta u^o(x) \, dx = 0.$$

Hence,

$$\int_{\partial\mathbb{I}} \Gamma \, d\sigma = \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}} u^o \, d\sigma - \gamma \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}} u^i \, d\sigma = 0,$$

and thus the proof is complete. □

**Proposition 4.3.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\phi$, $\gamma$, $\Gamma$ be as in (4.1), (4.2), (4.3), respectively. Let $(u^i_1, u^o_1)$, $(u^i_2, u^o_2)$ be two pairs of functions in $(C^1(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]) \cap C^2(\mathbb{S}_a[\mathbb{I}])) \times (C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}]))$ that solve problem (4.4). Then there exists a constant $c \in \mathbb{R}$ such that*

$$\begin{aligned} u^i_1(x) &= u^i_2(x) + c && \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}], \\ u^o_1(x) &= u^o_2(x) + \phi c && \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \end{aligned}$$

*Proof.* Let $(u^i_1, u^o_1)$, $(u^i_2, u^o_2)$ be two pairs of functions in $(C^1(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]) \cap C^2(\mathbb{S}_a[\mathbb{I}])) \times (C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}]))$ that solve problem (4.4). We set

$$\begin{aligned} v^i(x) &\equiv u^i_1(x) - u^i_2(x) && \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}], \\ v^o(x) &\equiv u^o_1(x) - u^o_2(x) && \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]. \end{aligned}$$

Then

$$\begin{cases} \Delta v^i(x) = 0 & \forall x \in \mathbb{S}_a[\mathbb{I}], \\ \Delta v^o(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ v^i(x + a_j) = v^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}, \\ v^o(x + a_j) = v^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}, \\ v^o(x) = \phi v^i(x) & \forall x \in \partial\mathbb{I}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} v^o(x) = \gamma \frac{\partial}{\partial \nu_{\mathbb{I}}} v^i(x) & \forall x \in \partial\mathbb{I}. \end{cases}$$

By Green's Formula, we have

$$0 \le \int_{\mathbb{I}} |\nabla v^i(x)|^2\,dx = \int_{\partial\mathbb{I}} v^i(x) \frac{\partial}{\partial\nu_{\mathbb{I}}} v^i(x)\,d\sigma_x = \frac{1}{\phi\gamma}\int_{\partial\mathbb{I}} v^o(x)\frac{\partial}{\partial\nu_{\mathbb{I}}} v^o(x)\,d\sigma_x = -\frac{1}{\phi\gamma}\int_{\mathbb{P}_a[\mathbb{I}]} |\nabla v^o(x)|^2\,dx \le 0.$$

Thus

$$\int_{\mathbb{I}} |\nabla v^i(x)|^2\,dx = 0 = \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla v^o(x)|^2\,dx,$$

and so

$$v^i(x) = c^i \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}],$$
$$v^o(x) = c^o \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],$$

for some $c^i$, $c^o$ in $\mathbb{R}$. Finally, since $v^o = \phi v^i$ on $\partial\mathbb{I}$, we must have

$$c^o = \phi c^i,$$

and we can easily conclude. □

As usual, in order to solve problem (4.4) by means of periodic simple layer potentials, we need to study some integral equations. Consequently, we now introduce and study a linear operator that appears in these equations.

**Definition 4.4.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). We denote by $v_{a*}[\partial\mathbb{I},\cdot]$ the linear operator of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^{m-1,\alpha}(\partial\mathbb{I})$ defined by

$$v_{a*}[\partial\mathbb{I},\mu](x) = \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}(S_n^a(x-y))\mu(y)\,d\sigma_y \qquad \forall x \in \partial\mathbb{I},$$

for all $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$.

Then we have the following.

**Proposition 4.5.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $\phi$, $\gamma$ be as in* (4.1)*,* (4.2)*, respectively. Then the following statements hold.*

*(i) The map $v_{a*}[\partial\mathbb{I},\cdot]$ of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^{m-1,\alpha}(\partial\mathbb{I})$ is compact.*

*(ii) Let $\mu \in C^{0,\alpha}(\partial\mathbb{I})$. Then*

$$\int_{\partial\mathbb{I}} \Big(\frac{1}{2}\mu(x) - \frac{\gamma-\phi}{\gamma+\phi} v_{a*}[\partial\mathbb{I},\mu](x)\Big)\,d\sigma_x = \frac{1}{\phi+\gamma}\Big(\phi\frac{|\mathbb{P}_a[\mathbb{I}]|_n}{|A|_n} + \gamma\frac{|\mathbb{I}|_n}{|A|_n}\Big)\int_{\partial\mathbb{I}} \mu\,d\sigma.$$

*(iii) The map*

$$\frac{1}{2}I - \frac{\gamma-\phi}{\gamma+\phi} v_{a*}[\partial\mathbb{I},\cdot]$$

*of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^{m-1,\alpha}(\partial\mathbb{I})$ is a linear homeomorphism of $C^{m-1,\alpha}(\partial\mathbb{I})$ onto itself.*

*Proof.* We first prove statement $(i)$. Let $v_*[\partial\mathbb{I},\cdot]$ denote the map of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^{m-1,\alpha}(\partial\mathbb{I})$, which takes $\mu$ to

$$v_*[\partial\mathbb{I},\mu](x) \equiv \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}(S_n(x-y))\mu(y)\,d\sigma_y \qquad \forall x \in \partial\mathbb{I}.$$

As is well known, the operator $v_*[\partial\mathbb{I},\cdot]$ is compact in $C^{m-1,\alpha}(\partial\mathbb{I})$. Indeed, for $n=3$, case $m=1$ has been proved by Schauder [123, 124] and case $m>1$ by Kirsch [64]. Then as observed in Kirsch [64, p. 789], the case $n \ge 2$ can also be treated. Then we note that

$$v_{a*}[\partial\mathbb{I},\mu](x) = v_*[\partial\mathbb{I},\mu](x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}(R_n^a(x-y))\mu(y)\,d\sigma_y \qquad \forall x \in \partial\mathbb{I},$$

for all $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$. Clearly the second term in the right-hand side of the previous equality defines a compact operator of $C^{m-1,\alpha}(\partial\mathbb{I})$ to itself. Indeed, let $V_1$, $V_2$ be two bounded connected open subsets of $\mathbb{R}^n$ of class $C^\infty$, such that

$$\mathrm{cl}\,\mathbb{I} \subseteq V_1 \subseteq \mathrm{cl}\,V_1 \subseteq V_2 \subseteq \mathrm{cl}\,V_2 \subseteq A.$$

The existence of $V_1$ and $V_2$ can be proved by exploiting a standard argument based on Sard's Theorem. Let the space

$$C^0_h(\mathrm{cl}\,V_2) \equiv \left\{ u \in C^0(\mathrm{cl}\,V_2) \cap C^2(V_2) : \Delta u(t) = 0 \quad \forall t \in V_2 \right\}$$

be endowed with the norm of the uniform convergence. Let $j \in \{1, \ldots, n\}$ The map of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^0_h(\mathrm{cl}\,V_2)$ which takes $\mu$ to $\int_{\partial\mathbb{I}} \partial_{x_j} R^a_n(\cdot - y))\mu(y)\,d\sigma_y$ is linear and continuous. Then, by classical interior estimates for harmonic functions, we have that the map $\mu \mapsto \int_{\partial\mathbb{I}} \partial_{x_j} R^a_n(\cdot - y))\mu(y)\,d\sigma_y$ of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^{m,\alpha}(\mathrm{cl}\,V_1)$ is linear and continuous. By the compactness of the imbedding of $C^{m,\alpha}(\mathrm{cl}\,V_1)$ into $C^{m-1,\alpha}(\mathrm{cl}\,V_1)$ and the continuity of the restriction map of $C^{m-1,\alpha}(\mathrm{cl}\,V_1)$ to $C^{m-1,\alpha}(\partial\mathbb{I})$ and of the pointwise product in Schauder spaces, we easily conclude. Hence, the map $v_{a*}[\partial\mathbb{I}, \cdot]$ of $C^{m-1,\alpha}(\partial\mathbb{I})$ to $C^{m-1,\alpha}(\partial\mathbb{I})$ is compact. We now prove statement $(ii)$. Let $\mu \in C^{0,\alpha}(\partial\mathbb{I})$. By Fubini's Theorem and Theorem 1.13 $(iv)$, we have

$$\begin{aligned}(\phi + \gamma) \int_{\partial\mathbb{I}} \Big(\frac{1}{2}\mu(x) - \frac{\gamma - \phi}{\gamma + \phi} v_{a*}[\partial\mathbb{I}, \mu](x)\Big)\,d\sigma_x &= \Big(\big(\frac{\phi}{2} + \frac{\gamma}{2}\big) - (\gamma - \phi)\big(\frac{1}{2} - \frac{|\mathbb{I}|_n}{|A|_n}\big)\Big) \int_{\partial\mathbb{I}} \mu\,d\sigma \\ &= \Big(\phi\big(1 - \frac{|\mathbb{I}|_n}{|A|_n}\big) + \gamma\frac{|\mathbb{I}|_n}{|A|_n}\Big) \int_{\partial\mathbb{I}} \mu\,d\sigma = \Big(\phi\frac{|\mathbb{P}_a[\mathbb{I}]|_n}{|A|_n} + \gamma\frac{|\mathbb{I}|_n}{|A|_n}\Big) \int_{\partial\mathbb{I}} \mu\,d\sigma.\end{aligned}$$

Therefore, statements $(ii)$ holds. Finally, consider $(iii)$. Case $\phi = \gamma$ is obvious. Thus we can assume that $\phi \neq \gamma$. By Fredholm Theory and the Open Mapping Theorem, it suffices to prove that the map $\frac{1}{2}I - (\frac{\gamma-\phi}{\gamma+\phi})v_{a*}[\partial\mathbb{I}, \cdot]$ is injective. So let $\mu \in C^{0,\alpha}(\partial\mathbb{I})$ be such that

$$(\gamma + \phi)\frac{1}{2}\mu - (\gamma - \phi)v_{a*}[\partial\mathbb{I}, \mu] = 0 \qquad \text{on } \partial\mathbb{I},$$

or equivalently

$$\phi\frac{\partial}{\partial\nu_\mathbb{I}} v^-_a[\partial\mathbb{I}, \mu] - \gamma\frac{\partial}{\partial\nu_\mathbb{I}} v^+_a[\partial\mathbb{I}, \mu] = 0 \qquad \text{on } \partial\mathbb{I}.$$

Now, by virtue of $(ii)$, we have in particular $\int_{\partial\mathbb{I}} \mu\,d\sigma = 0$. Then by Theorem 1.15, it is immediate to see that the pair of functions $(u^i, u^o) \equiv (v^+_a[\partial\mathbb{I}, \mu], \phi v^-_a[\partial\mathbb{I}, \mu])$ solves problem (4.4) with $\Gamma \equiv 0$. Thus, (cf. Proposition 4.3), we have

$$v_a[\partial\mathbb{I}, \mu](x) = c \qquad \forall x \in \mathbb{R}^n,$$

for some $c \in \mathbb{R}$. Then, by Theorem 1.15 $(iv)$, we have $\mu = 0$. Hence $\frac{1}{2}I - (\frac{\gamma-\phi}{\gamma+\phi})v_{a*}[\partial\mathbb{I}, \cdot]$ is injective. □

We are now ready to prove the existence of a solution of problem (4.4).

**Theorem 4.6.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0, 1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $\phi$, $\gamma$, $\Gamma$ be as in* (4.1), (4.2), (4.3)*, respectively. Then boundary value problem* (4.4) *has a solution $(u^i, u^o) \in (C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]) \cap C^2(\mathbb{S}_a[\mathbb{I}])) \times (C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}]))$. More precisely,*

$$u^i(x) \equiv v^+_a[\partial\mathbb{I}, \mu](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}], \tag{4.5}$$

$$u^o(x) \equiv \phi v^-_a[\partial\mathbb{I}, \mu](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \tag{4.6}$$

*where $\mu$ is the unique function in $C^{m-1,\alpha}(\partial\mathbb{I})$ that solves the following equation*

$$\frac{1}{2}\mu(x) - \frac{\gamma - \phi}{\gamma + \phi} \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_\mathbb{I}(x)}(S^a_n(x - y))\mu(y)\,d\sigma_y = \frac{1}{\gamma + \phi}\Gamma(x) \qquad \forall x \in \partial\mathbb{I}. \tag{4.7}$$

*Moreover, the subset of $(C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]) \cap C^2(\mathbb{S}_a[\mathbb{I}])) \times (C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]) \cap C^2(\mathbb{T}_a[\mathbb{I}]))$ of all the solutions of* (4.4)*, is delivered by*

$$\left\{ (v^+_a[\partial\mathbb{I}, \mu] + c, \phi v^-_a[\partial\mathbb{I}, \mu] + \phi c) \colon c \in \mathbb{R} \right\}, \tag{4.8}$$

*with $\mu$ as above.*

*Proof.* By Proposition 4.5 (*iii*), there exists a unique $\mu$ in $C^{m-1,\alpha}(\partial\mathbb{I})$ such that (4.7) holds. Then, by Proposition 4.5 (*ii*), since $\int_{\partial\mathbb{I}}\Gamma\,d\sigma=0$, we have $\int_{\partial\mathbb{I}}\mu\,d\sigma=0$. Then by Theorem 1.15, it is immediate to see that the pair of functions $(u^i,u^o)\equiv(v_a^+[\partial\mathbb{I},\mu],\phi v_a^-[\partial\mathbb{I},\mu])$ solves problem (4.4). Finally, by Proposition 4.3, it is easy to see that the subset of $(C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}])\cap C^2(\mathbb{S}_a[\mathbb{I}]))\times(C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])\cap C^2(\mathbb{T}_a[\mathbb{I}]))$ of all the solutions of (4.4), is delivered by (4.8). □

*Remark* 4.7. Let $m\in\mathbb{N}\setminus\{0\}$. Let $\alpha\in\,]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\phi$, $\gamma$, $\Gamma$ be as in (4.1), (4.2), (4.3), respectively. Let $(u^i,u^o)\in(C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}])\cap C^2(\mathbb{S}_a[\mathbb{I}]))\times(C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])\cap C^2(\mathbb{T}_a[\mathbb{I}]))$ be a solution of problem (4.4). We have

$$\begin{aligned}\int_{\mathbb{I}}|\nabla u^i(x)|^2\,dx&=\int_{\partial\mathbb{I}}u^i(x)\frac{\partial}{\partial\nu_{\mathbb{I}}}u^i(x)\,d\sigma_x\\&=\int_{\partial\mathbb{I}}\frac{1}{\gamma}\Big(\frac{\partial}{\partial\nu_{\mathbb{I}}}u^o(x)-\Gamma(x)\Big)\frac{1}{\phi}u^o(x)\,d\sigma_x\\&=\frac{1}{\phi\gamma}\int_{\partial\mathbb{I}}u^o(x)\frac{\partial}{\partial\nu_{\mathbb{I}}}u^o(x)\,d\sigma_x-\frac{1}{\gamma\phi}\int_{\partial\mathbb{I}}\Gamma(x)u^o(x)\,d\sigma_x,\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{P}_a[\mathbb{I}]}|\nabla u^o(x)|^2\,dx&=-\int_{\partial\mathbb{I}}u^o(x)\frac{\partial}{\partial\nu_{\mathbb{I}}}u^o(x)\,d\sigma_x\\&=-\int_{\partial\mathbb{I}}\Big(\gamma\frac{\partial}{\partial\nu_{\mathbb{I}}}u^i(x)+\Gamma(x)\Big)\phi u^i(x)\,d\sigma_x\\&=-\phi\gamma\int_{\partial\mathbb{I}}u^i(x)\frac{\partial}{\partial\nu_{\mathbb{I}}}u^i(x)\,d\sigma_x-\phi\int_{\partial\mathbb{I}}\Gamma(x)u^i(x)\,d\sigma_x.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\mathbb{I}}|\nabla u^i(x)|^2\,dx+\int_{\mathbb{P}_a[\mathbb{I}]}|\nabla u^o(x)|^2\,dx&=(1-\gamma\phi)\int_{\partial\mathbb{I}}u^i(x)\frac{\partial}{\partial\nu_{\mathbb{I}}}u^i(x)\,d\sigma_x-\phi\int_{\partial\mathbb{I}}\Gamma(x)u^i(x)\,d\sigma_x\\&=\big(-1+\frac{1}{\gamma\phi}\big)\int_{\partial\mathbb{I}}u^o(x)\frac{\partial}{\partial\nu_{\mathbb{I}}}u^o(x)\,d\sigma_x-\frac{1}{\gamma\phi}\int_{\partial\mathbb{I}}\Gamma(x)u^o(x)\,d\sigma_x.\end{aligned}$$

## 4.2 Asymptotic behaviour of the solutions of a linear transmission problem for the Laplace equation in a periodically perforated domain

In this Section we study the asymptotic behaviour of the solutions of a linear transmission problem for the Laplace equation in a periodically perforated domain with small holes.

### 4.2.1 Notation

We retain the notation introduced in Subsection 1.8.1.

Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $\Omega$ be as in (1.56). We shall consider also the following assumptions.

$$\phi\in\,]0,+\infty[,\tag{4.9}$$

$$\gamma\in\,]0,+\infty[,\tag{4.10}$$

$$g\in C^{m-1,\alpha}(\partial\Omega),\qquad\int_{\partial\Omega}g\,d\sigma=0,\tag{4.11}$$

$$\bar{c}\in\mathbb{R}.\tag{4.12}$$

### 4.2.2 Preliminaries

Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. For each $\epsilon\in\,]0,\epsilon_1[$, we consider the following periodic linear transmission problem for the Laplace equation.

$$
\begin{cases}
\Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a[\Omega_\epsilon], \\
\Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\
u^i(x+a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\
u^o(x+a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\
u^o(x) = \phi u^i(x) & \forall x \in \partial\Omega_\epsilon, \\
\frac{\partial}{\partial\nu_{\Omega_\epsilon}} u^o(x) = \gamma \frac{\partial}{\partial\nu_{\Omega_\epsilon}} u^i(x) + g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon, \\
u^i(w) = \bar{c}
\end{cases}
\tag{4.13}
$$

By virtue of Theorem 4.6, we can give the following definition.

**Definition 4.8.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. For each $\epsilon \in ]0, \epsilon_1[$, we denote by $(u^i[\epsilon], u^o[\epsilon])$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of boundary value problem (4.13).

We give the following definition.

**Definition 4.9.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). We denote by $v_*[\partial\Omega, \cdot]$ the linear operator of $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$ defined by

$$
v_*[\partial\mathbb{I}, \theta](t) \equiv \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)} (S_n(t-s))\theta(s)\, d\sigma_s \qquad \forall t \in \partial\Omega,
$$

for all $\theta \in C^{m-1,\alpha}(\partial\Omega)$.

Then we have the following result.

**Proposition 4.10.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $\phi$, $\gamma$ be as in* (1.56), (4.9), (4.10), (4.11), *respectively. Then the following statements hold.*

*(i) The map $v_*[\partial\Omega, \cdot]$ of $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$ is compact.*

*(ii) Let $\theta \in C^{0,\alpha}(\partial\Omega)$. Then*

$$
\int_{\partial\Omega} \Big(\frac{1}{2}\theta(t) - \frac{\gamma-\phi}{\gamma+\phi} v_*[\partial\Omega, \theta](t)\Big)\, d\sigma_t = \frac{1}{2}\Big(1 - (\frac{\gamma-\phi}{\gamma+\phi})\Big) \int_{\partial\Omega} \theta\, d\sigma.
$$

*(iii) The map*

$$
\frac{1}{2}I - \frac{\gamma-\phi}{\gamma+\phi} v_*[\partial\Omega, \cdot]
$$

*of $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$ is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega)$ onto itself.*

*Proof.* The compactness of $v_*[\partial\Omega, \cdot]$ has already been observed (cf. the proof of Proposition 4.5 (*i*).) The statement in (*ii*) is a straightforward consequence of Fubini's Theorem and Theorem B.1 (*iv*). Now we prove the statement in (*iii*). By Fredholm Theory and the Open Mapping Theorem, it suffices to prove that the map $\frac{1}{2}I - (\frac{\gamma-\phi}{\gamma+\phi})v_*[\partial\Omega, \cdot]$ is injective in $C^{0,\alpha}(\partial\Omega)$. So, let $\theta \in C^{0,\alpha}(\partial\Omega)$ be such that

$$
\frac{1}{2}(\phi+\gamma)\theta - (\gamma-\phi)v_*[\partial\Omega, \theta] = 0 \qquad \text{on } \partial\Omega,
$$

or equivalently

$$
\phi\frac{\partial}{\partial\nu_\Omega} v^-[\partial\Omega, \theta] - \gamma\frac{\partial}{\partial\nu_\Omega} v^+[\partial\Omega, \theta] = 0 \qquad \text{on } \partial\Omega.
$$

By (*ii*), we have, in particular, $\int_{\partial\Omega} \theta\, d\sigma = 0$. Thus $v^-[\partial\Omega, \theta]$ is harmonic at infinity and

$$
\lim_{t\to\infty} v^-[\partial\Omega, \theta](t) = 0.
$$

By the Divergence Theorem and Folland [52, p. 118], we have

$$
\begin{aligned}
0 \le \int_\Omega |\nabla v^+[\partial\Omega, \theta](t)|^2\, dt &= \int_{\partial\Omega} v[\partial\Omega, \theta](t) \frac{\partial}{\partial\nu_\Omega} v^+[\partial\Omega, \theta](t)\, d\sigma_t \\
&= \frac{\phi}{\gamma} \int_{\partial\Omega} v[\partial\Omega, \theta](t) \frac{\partial}{\partial\nu_\Omega} v^-[\partial\Omega, \theta](t)\, d\sigma_t = -\frac{\phi}{\gamma} \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla v^-[\partial\Omega, \theta](t)|^2\, dt \le 0.
\end{aligned}
$$

Hence, $v[\partial\Omega,\theta](t) = 0$ for all $t \in \mathbb{R}^n$, and so, by Theorem B.2 $(v)$,

$$\theta(t) = \frac{\partial}{\partial\nu_\Omega} v^-[\partial\Omega,\theta](t) - \frac{\partial}{\partial\nu_\Omega} v^+[\partial\Omega,\theta](t) = 0 \qquad \forall t \in \partial\Omega.$$

□

Since we want to represent the pair of functions $(u^i[\epsilon], u^o[\epsilon])$ by means of periodic simple layer potentials and constants (cf. Theorem 4.6), we need to study some integral equations. Indeed, by virtue of Theorem 4.6, we can transform (4.13) into an integral equation, whose unknown is the moment of the simple layer potential. Moreover, we want to transform these equations defined on the $\epsilon$-dependent domain $\partial\Omega_\epsilon$ into equations defined on the fixed domain $\partial\Omega$. We introduce these integral equations in the following Proposition. The relation between the solution of the integral equation and the solution of boundary value problem (4.13) will be clarified later.

**Proposition 4.11.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\mathcal{U}_\epsilon^{m-1,\alpha}$, $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.63), (1.64), *respectively. Let $\Lambda$ be the map of $]-\epsilon_1,\epsilon_1[ \times C^{m-1,\alpha}(\partial\Omega)$ in $C^{m-1,\alpha}(\partial\Omega)$ defined by*

$$\begin{aligned}\Lambda[\epsilon,\theta](t) \equiv \frac{1}{2}\theta(t) - (\frac{\gamma-\phi}{\gamma+\phi}) \int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\theta(s)\,d\sigma_s \\ - (\frac{\gamma-\phi}{\gamma+\phi})\epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s - \frac{1}{\phi+\gamma} g(t) \qquad \forall t \in \partial\Omega,\end{aligned} \tag{4.14}$$

*for all $(\epsilon,\theta) \in ]-\epsilon_1,\epsilon_1[ \times C^{m-1,\alpha}(\partial\Omega)$. Then the following statements hold.*

*(i) If $\epsilon \in ]0,\epsilon_1[$, then the function $\theta \in C^{m-1,\alpha}(\partial\Omega)$ satisfies equation*

$$\Lambda[\epsilon,\theta] = 0, \tag{4.15}$$

*if and only if the function $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon)$, defined by*

$$\mu(x) \equiv \theta(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{4.16}$$

*satisfies the equation*

$$\frac{1}{\gamma+\phi}\Gamma(x) = \frac{1}{2}\mu(x) - \frac{\gamma-\phi}{\gamma+\phi}\int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}(S_n^a(x-y))\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega_\epsilon, \tag{4.17}$$

*with $\Gamma \in C^{m-1,\alpha}(\partial\Omega_\epsilon)$ defined by*

$$\Gamma(x) \equiv g(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon. \tag{4.18}$$

*In particular, equation (4.15) has exactly one solution $\theta \in C^{m-1,\alpha}(\partial\Omega)$, for each $\epsilon \in ]0,\epsilon_1[$. Moreover, if $\theta$ solves (4.15), then $\theta \in \mathcal{U}_0^{m-1,\alpha}$, and so also $\theta(\frac{1}{\epsilon}(\cdot - w)) \in \mathcal{U}_\epsilon^{m-1,\alpha}$.*

*(ii) The function $\theta \in C^{m-1,\alpha}(\partial\Omega)$ satisfies equation*

$$\Lambda[0,\theta] = 0, \tag{4.19}$$

*if and only if*

$$\frac{1}{\phi+\gamma} g(t) = \frac{1}{2}\theta(t) - \frac{\gamma-\phi}{\gamma+\phi}\int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\,d\sigma_s \qquad \forall t \in \partial\Omega. \tag{4.20}$$

*In particular, if $\epsilon = 0$, equation (4.19) has exactly one solution $\theta \in C^{m-1,\alpha}(\partial\Omega)$, which we denote by $\tilde{\theta}$. Moreover, if $\theta$ solves (4.20), then $\theta \in \mathcal{U}_0^{m-1,\alpha}$.*

*Proof.* Consider $(i)$. Let $\theta \in C^{m-1,\alpha}(\partial\Omega)$. Let $\epsilon \in ]0,\epsilon_1[$. First of all, we note that

$$\int_{\partial\Omega_\epsilon} \theta(\frac{1}{\epsilon}(x-w))\,d\sigma_x = \epsilon^{n-1}\int_{\partial\Omega} \theta(t)\,d\sigma_t,$$

and so $\theta \in \mathcal{U}_0^{m-1,\alpha}$ if and only if $\theta(\frac{1}{\epsilon}(\cdot - w)) \in \mathcal{U}_\epsilon^{m-1,\alpha}$. The equivalence of equation (4.15) in the unknown $\theta \in C^{m-1,\alpha}(\partial\Omega)$ and equation (4.17) in the unknown $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon)$ follows by a straightforward computation based on the rule of change of variables in integrals and on well known properties of composition of functions in Schauder spaces (cf. *e.g.*, Lanza [67, Sections 3,4].) The existence and uniqueness of a solution of equation (4.17) follows by Proposition 4.5 (*iii*). Then the existence and uniqueness of a solution of equation (4.15) follows by the equivalence of (4.15) and (4.17). Moreover, if $\mu \equiv \theta(\frac{1}{\epsilon}(\cdot - w))$ solves (4.17), then, since $\int_{\partial\Omega} g \, d\sigma = 0$, we have $\mu \in \mathcal{U}_\epsilon^{m-1,\alpha}$ (see Proposition 4.5 (*ii*)) and accordingly $\theta \in \mathcal{U}_0^{m-1,\alpha}$. Consider (*ii*). The equivalence of (4.19) and (4.20) is obvious. The existence of a unique solution of equation (4.19) is an immediate consequence of Proposition 4.10 (*iii*). Moreover, if $\theta \in C^{m-1,\alpha}(\partial\Omega)$ solves equation (4.20), then, by Proposition 4.10 (*ii*), we have $\theta \in \mathcal{U}_0^{m-1,\alpha}$. □

By Proposition 3.6, it makes sense to introduce the following.

**Definition 4.12.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. For each $\epsilon \in ]0, \epsilon_1[$, we denote by $\hat{\theta}[\epsilon]$ the unique function in $C^{m-1,\alpha}(\partial\Omega)$ that solves (4.15). Analogously, we denote by $\hat{\theta}[0]$ the unique function in $C^{m-1,\alpha}(\partial\Omega)$ that solves (4.19).

In the following Remark, we show the relation between the solutions of boundary value problem (4.13) and the solutions of equation (4.15).

*Remark* 4.13. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively.

Let $\epsilon \in ]0, \epsilon_1[$. We have

$$u^i[\epsilon](x) = \epsilon^{n-1} \int_{\partial\Omega} S_n^a(x - w - \epsilon s)\hat{\theta}[\epsilon](s) \, d\sigma_s + \Big(\bar{c} - \epsilon^{n-1} \int_{\partial\Omega} S_n^a(-\epsilon s)\hat{\theta}[\epsilon](s) \, d\sigma_s\Big) \qquad \forall x \in \mathrm{cl}\, \mathbb{S}_a[\Omega_\epsilon],$$

and

$$u^o[\epsilon](x) = \phi\epsilon^{n-1} \int_{\partial\Omega} S_n^a(x - w - \epsilon s)\hat{\theta}[\epsilon](s) \, d\sigma_s + \phi\Big(\bar{c} - \epsilon^{n-1} \int_{\partial\Omega} S_n^a(-\epsilon s)\hat{\theta}[\epsilon](s) \, d\sigma_s\Big) \quad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon].$$

While the relation between equation (4.15) and boundary value problem (4.13) is now clear, we want to see if (4.19) is related to some (limiting) boundary value problem. We give the following.

**Definition 4.14.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $\phi$, $\gamma$, $g$ be as in (1.56), (4.9), (4.10), (4.11), respectively. We denote by $(\tilde{u}^i, \tilde{u}^o)$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\Omega) \times C^{m,\alpha}(\mathbb{R}^n \setminus \Omega)$ of the following boundary value problem

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \Omega, \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega, \\ u^o(x) = \phi u^i(x) & \forall x \in \partial\Omega, \\ \frac{\partial}{\partial\nu_\Omega} u^o(x) = \gamma \frac{\partial}{\partial\nu_\Omega} u^i(x) + g(x) & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u^o(x) = 0. \end{cases} \tag{4.21}$$

Problem (4.21) will be called the *limiting boundary value problem.*

*Remark* 4.15. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. We have

$$\tilde{u}^i(x) = \int_{\partial\Omega} S_n(x - y)\hat{\theta}[0](y) \, d\sigma_y \qquad \forall x \in \mathrm{cl}\,\Omega,$$

and

$$\tilde{u}^o(x) = \phi \int_{\partial\Omega} S_n(x - y)\hat{\theta}[0](y) \, d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \Omega.$$

Furthermore,

$$\begin{aligned} \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}^o(x)|^2 \, dx &= -\int_{\partial\Omega} \tilde{u}^o(x) \frac{\partial}{\partial\nu_\Omega} \tilde{u}^o(x) \, d\sigma_x \\ &= -\gamma\phi \int_{\partial\Omega} \tilde{u}^i(x) \frac{\partial}{\partial\nu_\Omega} \tilde{u}^i(x) \, d\sigma_x - \phi \int_{\partial\Omega} g(x)\tilde{u}^i(x) \, d\sigma_x \end{aligned}$$

and

$$\int_{\Omega}|\nabla\tilde{u}^i(x)|^2\,dx = \int_{\partial\Omega}\tilde{u}^i(x)\frac{\partial}{\partial\nu_\Omega}\tilde{u}^i(x)\,d\sigma_x$$
$$= \frac{1}{\gamma\phi}\int_{\partial\Omega}\tilde{u}^o(x)\frac{\partial}{\partial\nu_\Omega}\tilde{u}^o(x)\,d\sigma_x - \frac{1}{\gamma\phi}\int_{\partial\Omega}g(x)\tilde{u}^o(x)\,d\sigma_x.$$

Hence,

$$\int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega}|\nabla\tilde{u}^o(x)|^2\,dx + \int_{\Omega}|\nabla\tilde{u}^i(x)|^2\,dx = (1-\gamma\phi)\int_{\partial\Omega}\tilde{u}^i(x)\frac{\partial}{\partial\nu_\Omega}\tilde{u}^i(x)\,d\sigma_x - \phi\int_{\partial\Omega}g(x)\tilde{u}^i(x)\,d\sigma_x$$
$$= -(1-\frac{1}{\gamma\phi})\int_{\partial\Omega}\tilde{u}^o(x)\frac{\partial}{\partial\nu_\Omega}\tilde{u}^o(x)\,d\sigma_x - \frac{1}{\gamma\phi}\int_{\partial\Omega}g(x)\tilde{u}^o(x)\,d\sigma_x.$$

We now prove the following.

**Proposition 4.16.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\Lambda$ and $\tilde{\theta}$ be as in Proposition 4.11. Then there exists $\epsilon_2 \in ]0,\epsilon_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2,\epsilon_2[\times C^{m-1,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$. Moreover, if we set $b_0 \equiv (0,\tilde{\theta})$, then the differential $\partial_\theta\Lambda[b_0]$ of $\Lambda$ with respect to the variable $\theta$ at $b_0$ is delivered by the following formula*

$$\partial_\theta\Lambda[b_0](\bar{\theta})(t) = \frac{1}{2}\bar{\theta}(t) - (\frac{\gamma-\phi}{\gamma+\phi})\int_{\partial\Omega}\nu_\Omega(t)\cdot DS_n(t-s)\bar{\theta}(s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \tag{4.22}$$

*for all $\bar{\theta} \in C^{m-1,\alpha}(\partial\Omega)$, and is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega)$ onto $C^{m-1,\alpha}(\partial\Omega)$.*

*Proof.* By Proposition 1.26 (*ii*), there exists $\epsilon_2 \in ]0,\epsilon_1]$, such that $\Lambda$ is a real analytic operator of $]-\epsilon_2,\epsilon_2[\times C^{m-1,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$. By standard calculus in Banach space, we immediately deduce that (4.22) holds. By Proposition 4.10 (*iii*), we have that $\partial_\theta\Lambda[b_0]$ is a linear homeomorphism. □

We are now ready to prove that $\hat{\theta}[\cdot]$ can be continued real analytically on a whole neighbourhood of 0.

**Proposition 4.17.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\epsilon_2$ be as in Proposition 4.16. Then there exist $\epsilon_3 \in ]0,\epsilon_2]$ and a real analytic operator $\Theta$ of $]-\epsilon_3,\epsilon_3[$ to $C^{m-1,\alpha}(\partial\Omega)$, such that*

$$\Theta[\epsilon] = \hat{\theta}[\epsilon], \tag{4.23}$$

*for all $\epsilon \in [0,\epsilon_3[$.*

*Proof.* It is an immediate consequence of Proposition 4.16 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

### 4.2.3 A functional analytic representation Theorem for the solution of the singularly perturbed linear transmission problem

By Proposition 4.17 and Remark 4.13, we can deduce the main result of this Subsection. More precisely, we show that $\{(u^i[\epsilon](\cdot),u^o[\epsilon](\cdot))\}_{\epsilon\in]0,\epsilon_1[}$ can be continued real analytically for negative values of $\epsilon$.

We have the following.

**Theorem 4.18.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), *respectively. Let $\epsilon_3$ be as in Proposition 4.17. Then the following statements hold.*

(i) *Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0,\epsilon_3]$, a real analytic operator $U_1^o$ of $]-\epsilon_4,\epsilon_4[$ to the space $C_h^0(\mathrm{cl}\,V)$, and a real analytic operator $U_2^o$ of $]-\epsilon_4,\epsilon_4[$ to $\mathbb{R}$ such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*

$$u^o[\epsilon](x) = \epsilon^n U_1^o[\epsilon](x) + U_2^o[\epsilon] \qquad \forall x \in \mathrm{cl}\, V,$$

*for all* $\epsilon \in ]0, \epsilon_4[$. *Moreover,*

$$U_2^o[0] = \phi \bar{c}.$$

*(ii) Let* $\bar{V}$ *be a bounded open subset of* $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$. *Then there exist* $\bar{\epsilon}_4 \in ]0, \epsilon_3]$, *a real analytic operator* $\bar{U}_1^o$ *of* $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ *to the space* $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$, *and a real analytic operator* $\bar{U}_2^o$ *of* $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ *to* $\mathbb{R}$ *such that the following conditions hold.*

*(j')* $w + \epsilon\, \mathrm{cl}\, \bar{V} \subseteq \mathrm{cl}\, \mathbb{P}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \setminus \{0\}$.

*(jj')*

$$u^o[\epsilon](w + \epsilon t) = \epsilon \bar{U}_1^o[\epsilon](t) + \bar{U}_2^o[\epsilon] \qquad \forall t \in \mathrm{cl}\, \bar{V},$$

*for all* $\epsilon \in ]0, \bar{\epsilon}_4[$. *Moreover,*

$$\bar{U}_2^o[0] = \phi \bar{c}.$$

*(iii) There exist* $\epsilon_4' \in ]0, \epsilon_3]$, *a real analytic operator* $U_1^i$ *of* $]-\epsilon_4', \epsilon_4'[$ *to the space* $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, *and a real analytic operator* $U_2^i$ *of* $]-\epsilon_4', \epsilon_4'[$ *to* $\mathbb{R}$ *such that*

$$u^i[\epsilon](w + \epsilon t) = \epsilon U_1^i[\epsilon](t) + U_2^i[\epsilon] \qquad \forall t \in \mathrm{cl}\,\Omega,$$

*for all* $\epsilon \in ]0, \epsilon_4'[$. *Moreover,*

$$U_2^i[0] = \bar{c}.$$

*Proof.* Let $\Theta[\cdot]$ be as in Proposition 4.17. Consider $(i)$. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $\epsilon \in ]0, \epsilon_4[$. By Remark 4.13 and Proposition 4.17, we have

$$\begin{aligned} u^o[\epsilon](x) =& \phi \epsilon^{n-1} \int_{\partial\Omega} S_n^a(x - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \\ &+ \phi\Big(\bar{c} - \epsilon^{n-1} \int_{\partial\Omega} S_n^a(-\epsilon s)\Theta[\epsilon](s)\, d\sigma_s\Big) \quad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon]. \end{aligned}$$

Thus, it is natural to set

$$\tilde{U}_1^o[\epsilon](x) \equiv \phi \int_{\partial\Omega} S_n^a(x - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$, and

$$U_2^o[\epsilon] \equiv \phi\Big(\bar{c} - \epsilon^{n-1} \int_{\partial\Omega} S_n^a(-\epsilon s)\Theta[\epsilon](s)\, d\sigma_s\Big),$$

for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$. By following the proof of Theorem 2.124 and by possibly taking a smaller $\epsilon_4$, we can show that there exists a real analytic map $U_1^o$ of $]-\epsilon_4, \epsilon_4[$ to $C_h^0(\mathrm{cl}\, V)$ such that

$$\tilde{U}_1^o[\epsilon] = \epsilon U_1^o[\epsilon] \qquad \text{in } C_h^0(\mathrm{cl}\, V),$$

for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$. Consider now $U_2^o$. If $\epsilon \in ]0, \epsilon_4]$, a simple computation shows that

$$U_2^o[\epsilon] = \phi\Big(\bar{c} - \epsilon \int_{\partial\Omega} S_n(-s)\Theta[\epsilon](s)\, d\sigma_s - \epsilon^{n-1} \int_{\partial\Omega} R_n^a(-\epsilon s)\Theta[\epsilon](s)\, d\sigma_s\Big).$$

Then by Proposition 1.29 $(iii)$ and by the continuity of the linear map of $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ to $\mathbb{R}$, which takes a function $h$ to $h(0)$, we immediately deduce that, by possibly taking a smaller $\epsilon_4$, $U_2^o$ is a real analytic map of $]-\epsilon_4, \epsilon_4[$ to $\mathbb{R}$. Consider now $(ii)$. Choosing $\bar{\epsilon}_4$ small enough, we can clearly assume that $(j')$ holds. Consider now $(jj')$. Let $\epsilon \in ]0, \bar{\epsilon}_4[$. By Remark 4.13, we have

$$\begin{aligned} u^o[\epsilon](w + \epsilon t) =& \epsilon^{n-1} \phi \int_{\partial\Omega} S_n^a(\epsilon(t - s))\Theta[\epsilon](s)\, d\sigma_s \\ &+ \phi\Big(\bar{c} - \epsilon \int_{\partial\Omega} S_n(-s)\Theta[\epsilon](s)\, d\sigma_s - \epsilon^{n-1} \int_{\partial\Omega} R_n^a(-\epsilon s)\Theta[\epsilon](s)\, d\sigma_s\Big) \qquad \forall t \in \mathrm{cl}\, \bar{V}. \end{aligned}$$

Thus (cf. Proposition 1.29 (*ii*)), it is natural to set

$$\bar{U}_1^o[\epsilon](t) \equiv \phi \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \qquad \forall t \in \mathrm{cl}\,\bar{V},$$

for all $\epsilon \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[$, and

$$\bar{U}_2^o[\epsilon] \equiv \phi\Big(\bar{c} - \epsilon \int_{\partial\Omega} S_n(-s)\Theta[\epsilon](s)\,d\sigma_s - \epsilon^{n-1}\int_{\partial\Omega} R_n^a(-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s\Big),$$

for all $\epsilon \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[$. By the proof of (*i*), we have that $\bar{U}_2^o$ is a real analytic map of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ to $\mathbb{R}$. Moreover, by Proposition 1.29 (*ii*) we have that $\bar{U}_1^o$ is a real analytic map of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ to $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$. Finally, consider (*iii*). Let $\epsilon \in ]0, \epsilon_3[$. By Remark 4.13, we have

$$\begin{aligned} u^i[\epsilon](w+\epsilon t) =& \epsilon^{n-1}\int_{\partial\Omega} S_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \\ &+ \Big(\bar{c} - \epsilon^{n-1}\int_{\partial\Omega} S_n^a(-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s\Big) \qquad \forall t \in \mathrm{cl}\,\Omega. \end{aligned} \tag{4.24}$$

Thus, by arguing as above, it is natural to set

$$U_1^i[\epsilon](t) \equiv \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \qquad \forall t \in \mathrm{cl}\,\Omega,$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$, and

$$U_2^i[\epsilon] \equiv \Big(\bar{c} - \epsilon \int_{\partial\Omega} S_n(-s)\Theta[\epsilon](s)\,d\sigma_s - \epsilon^{n-1}\int_{\partial\Omega} R_n^a(-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s\Big),$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. Then, by arguing as above (cf. Proposition 1.29 (*iii*)), there exists $\epsilon_4' \in ]0, \epsilon_3]$, such that $U_1^i$ and $U_2^i$ are real analytic maps of $]-\epsilon_4', \epsilon_4'[$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ and $\mathbb{R}$, respectively, such that the equality in (*iii*) holds. □

*Remark* 4.19. We note that the right-hand side of the equalities in (*jj*), (*jj'*) and (*iii*) of Theorem 4.18 can be continued real analytically in a whole neighbourhood of 0. Moreover, if $V$ is a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V \cap \mathbb{S}_a[\Omega_0] = \emptyset$, then

$$\lim_{\epsilon \to 0^+} u^o[\epsilon] = \phi\bar{c} \qquad \text{uniformly in } \mathrm{cl}\,V.$$

### 4.2.4 A real analytic continuation Theorem for the energy integral

As done in Theorem 4.18 for $(u^i[\cdot], u^o[\cdot])$, we can now prove a real analytic continuation Theorem for the energy integral. Namely, we prove the following.

**Theorem 4.20.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), *respectively. Let $\epsilon_3$ be as in Proposition 4.17. Then there exist $\epsilon_5 \in ]0, \epsilon_3]$ and two real analytic operators $G^i$, $G^o$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{R}$, such that*

$$\int_{\Omega_\epsilon} |\nabla u^i[\epsilon](x)|^2\,dx = \epsilon^n G^i[\epsilon], \tag{4.25}$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](x)|^2\,dx = \epsilon^n G^o[\epsilon], \tag{4.26}$$

*for all $\epsilon \in ]0, \epsilon_5[$. Moreover,*

$$G^i[0] = \int_{\Omega} |\nabla \tilde{u}^i(x)|^2\,dx, \tag{4.27}$$

$$G^o[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}^o(x)|^2\,dx. \tag{4.28}$$

*Proof.* Let $\Theta[\cdot]$ be as in Proposition 4.17. Let $\epsilon \in ]0,\epsilon_3[$. Clearly,

$$\int_{\Omega_\epsilon} |\nabla u^i[\epsilon](x)|^2\,dx = \int_{\Omega_\epsilon} |\nabla v_a^+[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2\,dx,$$

and

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](x)|^2\,dx = \phi^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2\,dx.$$

By slightly modifying the proof of Theorem 3.16, we can prove that there exist $\epsilon_5 \in ]0,\epsilon_3]$ and a real analytic operator $G^o$ of $]-\epsilon_5,\epsilon_5[$ to $\mathbb{R}$ such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](x)|^2\,dx = \epsilon^n G^o[\epsilon],$$

for all $\epsilon \in ]0,\epsilon_5[$, and

$$G^o[0] = \int_{\mathbb{R}^n\setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}^o(x)|^2\,dx.$$

Let $\epsilon \in ]0,\epsilon_3[$. We have

$$\begin{aligned}
&\int_{\Omega_\epsilon} |\nabla v_a^+[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2\,dx \\
&\qquad = \epsilon^{n-1}\int_{\partial\Omega} v_a^+[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](w+\epsilon t)\Big(\frac{\partial}{\partial\nu_{\Omega_\epsilon}} v_a^+[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))]\Big)(w+\epsilon t)\,d\sigma_t.
\end{aligned}$$

Also

$$\begin{aligned}
&v_a^+[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](w+\epsilon t) \\
&\qquad = \epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,
\end{aligned}$$

and

$$\begin{aligned}
&\Big(\frac{\partial}{\partial\nu_{\Omega_\epsilon}} v_a^+[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))]\Big)(w+\epsilon t) \\
&\quad = -\frac{1}{2}\Theta[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s.
\end{aligned}$$

By possibly taking a smaller $\epsilon_5$, the map of $]-\epsilon_5,\epsilon_5[$ to $C^0(\partial\Omega)$ which takes $\epsilon$ to the function of the variable $t\in\partial\Omega$ defined by

$$\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

and the map of $]-\epsilon_5,\epsilon_5[$ to $C^0(\partial\Omega)$ which takes $\epsilon$ to the function of the variable $t\in\partial\Omega$ defined by

$$-\frac{1}{2}\Theta[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

are real analytic (cf. Proposition 1.28.) Thus the map $G^i$ of $]-\epsilon_5,\epsilon_5[$ to $\mathbb{R}$ which takes $\epsilon$ to

$$\begin{aligned}
G^i[\epsilon] \equiv &\int_{\partial\Omega}\Big(\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s\Big) \\
\times &\Big(-\frac{1}{2}\Theta[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s\Big)\,d\sigma_t
\end{aligned}$$

is real analytic. Clearly,

$$\int_{\Omega_\epsilon} |\nabla v_a^+[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2\,dx = \epsilon^n G^i[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_5[$. Moreover, since

$$\tilde{u}^i(x) = \int_{\partial\Omega} S_n(x-y)\Theta[0](y)\, d\sigma_y \qquad \forall x \in \mathrm{cl}\,\Omega,$$

we have

$$G^i[0] \equiv \int_{\Omega} |\nabla \tilde{u}^i(x)|^2\, dx.$$

□

*Remark* 4.21. We note that the right-hand side of the equalities in (4.25) and (4.26) of Theorem 4.20 can be continued real analytically in the whole $]-\epsilon_5, \epsilon_5[$. Moreover,

$$\lim_{\epsilon \to 0^+} \Big( \int_{\Omega_\epsilon} |\nabla u^i[\epsilon](x)|^2\, dx + \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](x)|^2\, dx \Big) = 0.$$

### 4.2.5 A real analytic continuation Theorem for the integral of the solution

As done in Theorem 4.20 for the energy integral, we can now prove a real analytic continuation Theorem for the integral of the solution. Namely, we prove the following.

**Theorem 4.22.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), *respectively. Let $\epsilon_3$ be as in Proposition 4.17. Then there exist $\epsilon_6 \in ]0, \epsilon_3]$ and two real analytic operators $J^i$, $J^o$ of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$, such that*

$$\int_{\Omega_\epsilon} u^i[\epsilon](x)\, dx = J^i[\epsilon], \tag{4.29}$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^o[\epsilon](x)\, dx = J^o[\epsilon], \tag{4.30}$$

*for all $\epsilon \in ]0, \epsilon_6[$. Moreover,*

$$J^i[0] = 0, \tag{4.31}$$

$$J^o[0] = \phi \bar{c} |A|_n. \tag{4.32}$$

*Proof.* Let $\Theta[\cdot]$ be as in Proposition 4.17. Let $\epsilon \in ]0, \epsilon_3[$. By Remark 4.13 and Proposition 4.17, we have

$$u^o[\epsilon](x) = \phi v_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\tfrac{1}{\epsilon}(\cdot - w))\big](x) + \phi\Big(\bar{c} - \epsilon^{n-1} \int_{\partial\Omega} S_n^a(-\epsilon s)\Theta[\epsilon](s)\, d\sigma_s\Big) \quad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$

Moreover, by arguing as in Theorem 4.18, we have that

$$\epsilon^{n-1} \int_{\partial\Omega} S_n^a(-\epsilon s)\Theta[\epsilon](s)\, d\sigma_s = \epsilon \int_{\partial\Omega} S_n(-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(-\epsilon s)\Theta[\epsilon](s)\, d\sigma_s.$$

Then, if we set

$$\begin{aligned} L[\epsilon](t) \equiv & \phi\epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \phi\epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ & - \phi\epsilon \int_{\partial\Omega} S_n(-s)\Theta[\epsilon](s)\, d\sigma_s - \phi\epsilon^{n-1} \int_{\partial\Omega} R_n^a(-\epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$, by arguing as in the proof of Theorem 2.128, we can easily show that there exist $\epsilon_6' \in ]0, \epsilon_3]$ and a real analytic map $J_1$ of $]-\epsilon_6', \epsilon_6'[$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^o[\epsilon](x)\, dx = J_1[\epsilon] + \phi\bar{c}\big(|A|_n - \epsilon^n |\Omega|_n\big),$$

for all $\epsilon \in ]0, \epsilon_6'[$, and such that $J_1[0] = 0$. As a consequence, it suffices to set

$$J^o[\epsilon] \equiv J_1[\epsilon] + \phi\bar{c}\big(|A|_n - \epsilon^n |\Omega|_n\big),$$

for all $\epsilon \in ]-\epsilon_6', \epsilon_6'[$.

Let $\epsilon \in ]0, \epsilon_3[$. Clearly,

$$\int_{\Omega_\epsilon} u^i[\epsilon](x)\,dx = \epsilon^n \int_{\Omega} u^i[\epsilon](w+\epsilon t)\,dt.$$

On the other hand, if $\epsilon_4'$, $U_1^i$, $U_2^i$ are as in Theorem 4.18, and we set

$$J^i[\epsilon] \equiv \epsilon^n \int_{\Omega} \left(\epsilon U_1^i[\epsilon](t) + U_2^i[\epsilon]\right) dt$$

for all $\epsilon \in ]-\epsilon_4', \epsilon_4'[$, then we have that $J^i$ is a real analytic map of $]-\epsilon_4', \epsilon_4'[$ to $\mathbb{R}$, such that $J^i[0] = 0$ and that

$$\int_{\Omega_\epsilon} u^i[\epsilon](x)\,dx = J^i[\epsilon]$$

for all $\epsilon \in ]0, \epsilon_4'[$.

Then, by taking $\epsilon_6 \equiv \min\{\epsilon_6', \epsilon_4'\}$, we can easily conclude. □

## 4.3 An homogenization problem for the Laplace equation with a linear transmission boundary condition in a periodically perforated domain

In this section we consider an homogenization problem for the Laplace equation with a linear transmission boundary condition in a periodically perforated domain.

### 4.3.1 Notation

In this Section we retain the notation introduced in Subsections 1.8.1, 4.2.1. However, we need to introduce also some other notation.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $(\epsilon, \delta) \in (]-\epsilon_1, \epsilon_1[ \setminus \{0\}) \times ]0, +\infty[$. If $v$ is a function of $\mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta)$ to $\mathbb{R}$, then we denote by $\mathbf{E}^i_{(\epsilon,\delta)}[v]$ the function of $\mathbb{R}^n$ to $\mathbb{R}$, defined by

$$\mathbf{E}^i_{(\epsilon,\delta)}[v](x) \equiv \begin{cases} v(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta), \\ 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta). \end{cases}$$

Analogously, if $v$ is a function of $\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta)$ to $\mathbb{R}$, then we denote by $\mathbf{E}^o_{(\epsilon,\delta)}[v]$ the function of $\mathbb{R}^n$ to $\mathbb{R}$, defined by

$$\mathbf{E}^o_{(\epsilon,\delta)}[v](x) \equiv \begin{cases} v(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta). \\ 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta). \end{cases}$$

### 4.3.2 Preliminaries

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. For each pair $(\epsilon,\delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we consider the following periodic linear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a(\epsilon,\delta), \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u^i(x+\delta a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\ u^o(x+\delta a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\ u^o(x) = \phi u^i(x) & \forall x \in \partial\Omega(\epsilon,\delta), \\ \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u^o(x) = \gamma \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u^i(x) + \frac{1}{\delta} g(\frac{1}{\epsilon\delta}(x - \delta w)) & \forall x \in \partial\Omega(\epsilon,\delta), \\ u^i(\delta w) = \bar{c}. \end{cases} \tag{4.33}$$

By virtue of Theorem 4.6, we can give the following definition.

**Definition 4.23.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we denote by $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta)) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta))$ of boundary value problem (4.33).

Our aim is to study the asymptotic behaviour of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ as $(\epsilon,\delta)$ tends to $(0,0)$. In order to do so we introduce the following.

**Definition 4.24.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. For each $\epsilon \in ]0,\epsilon_1[$, we denote by $(u^i[\epsilon], u^o[\epsilon])$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of the following periodic linear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a[\Omega_\epsilon], \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u^i(x + a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u^o(x + a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u^o(x) = \phi u^i(x) & \forall x \in \partial\Omega_\epsilon, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u^o(x) = \gamma \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u^i(x) + g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon, \\ u^i(w) = \bar{c}. \end{cases} \tag{4.34}$$

*Remark* 4.25. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we have

$$u^i_{(\epsilon,\delta)}(x) = u^i[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta),$$

and

$$u^o_{(\epsilon,\delta)}(x) = u^o[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).$$

By the previous remark, we note that the solutions of problem (4.33) can be expressed by means of the solutions of the auxiliary rescaled problem (4.34), which does not depend on $\delta$. This is due to the presence of the factor $1/\delta$ in front of $g(\frac{1}{\epsilon\delta}(x-\delta w))$ in the sixth equation of problem (4.33).

As a first step, we study the behaviour of $(u^i[\epsilon], u^o[\epsilon])$ as $\epsilon$ tends to 0.

**Proposition 4.26.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), *respectively. Let $1 \le p < \infty$. Then*

$$\lim_{\epsilon \to 0^+} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]] = 0 \qquad \text{in } L^p(A),$$

*and*

$$\lim_{\epsilon \to 0^+} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]] = \phi\bar{c} \qquad \text{in } L^p(A).$$

*Proof.* It suffices to modify the proof of Proposition 2.132. Let $\epsilon_3$, $\Theta$ be as in Proposition 4.17. If $\epsilon \in ]0,\epsilon_3[$, we have

$$\begin{aligned} u^i[\epsilon](w+\epsilon t) =& \epsilon\Big(\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s\Big) \\ &+ \Big(\bar{c} - \epsilon^{n-1}\int_{\partial\Omega} S^a_n(-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s\Big) \qquad \forall t \in \partial\Omega. \end{aligned} \tag{4.35}$$

We set

$$\begin{aligned} N^i[\epsilon](t) \equiv& \epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \\ &+ \Big(\bar{c} - \epsilon^{n-1}\int_{\partial\Omega} S^a_n(-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s\Big) \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $\epsilon \in ]-\epsilon_3,\epsilon_3[$. By taking $\tilde{\epsilon} \in ]0,\epsilon_3[$ small enough, we can assume (cf. Proposition 1.28 (*i*) and the proof of Theorem 4.18) that $N^i$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C^i \equiv \sup_{\epsilon \in ]-\tilde{\epsilon},\tilde{\epsilon}[} \|N^i[\epsilon]\|_{C^0(\partial\Omega)} < +\infty.$$

By the Maximum Principle for harmonic functions, we have

$$|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]](x)| \le C^i \qquad \forall x \in A, \quad \forall \epsilon \in ]0,\tilde{\epsilon}[.$$

Obviously,

$$\lim_{\epsilon\to 0^+} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]](x) = 0 \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{\epsilon\to 0^+} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]] = 0 \qquad \text{in } L^p(A).$$

If $\epsilon \in ]0,\epsilon_3[$, we have

$$\begin{aligned} u^o[\epsilon](w+\epsilon t) =& \epsilon\phi\Big(\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s\Big) \\ &+ \phi\Big(\bar{c} - \epsilon^{n-1}\int_{\partial\Omega} S^a_n(-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s\Big) \qquad \forall t \in \partial\Omega. \end{aligned} \tag{4.36}$$

We set

$$\begin{aligned} N^o[\epsilon](t) \equiv& \epsilon\phi\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\phi\int_{\partial\Omega} R^a_n(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \\ &+ \phi\Big(\bar{c} - \epsilon^{n-1}\int_{\partial\Omega} S^a_n(-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s\Big) \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $\epsilon \in ]-\epsilon_3,\epsilon_3[$. By taking $\tilde{\epsilon} \in ]0,\epsilon_3[$ small enough, we can assume (cf. Proposition 1.28 $(i)$ and the proof of Theorem 4.18) that $N^o$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C^o \equiv \sup_{\epsilon\in]-\tilde{\epsilon},\tilde{\epsilon}[} \|N^o[\epsilon]\|_{C^0(\partial\Omega)} < +\infty.$$

By Theorem 2.5, we have

$$|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]](x)| \le C^o \qquad \forall x \in A, \quad \forall \epsilon \in ]0,\tilde{\epsilon}[.$$

By Theorem 4.18, we have

$$\lim_{\epsilon\to 0^+} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]](x) = \phi\bar{c} \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{\epsilon\to 0^+} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]] = \phi\bar{c} \qquad \text{in } L^p(A).$$

□

### 4.3.3 Asymptotic behaviour of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

In the following Theorem we deduce by Proposition 4.26 the convergence of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 4.27.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), *respectively. Let $1 \le p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^p(V),$$

*and*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] = \phi\bar{c} \qquad \text{in } L^p(V).$$

*Proof.* We modify the proof of Theorem 2.134. By virtue of Proposition 4.26, we have

$$\lim_{\epsilon\to 0^+} \|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]]\|_{L^p(A)} = 0,$$

and

$$\lim_{\epsilon\to 0^+} \|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]] - \phi\bar{c}\|_{L^p(A)} = 0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $C > 0$ such that

$$\|\mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}]\|_{L^p(V)} \leq C\|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]]\|_{L^p(A)} \qquad \forall(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,1[,$$

and

$$\|\mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] - \phi\bar{c}\|_{L^p(V)} \leq C\|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]] - \phi\bar{c}\|_{L^p(A)} \qquad \forall(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,1[,$$

Thus,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^p(V),$$

and

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] = \phi\bar{c} \qquad \text{in } L^p(V).$$

□

Then we have the following Theorem, where we consider a functional associated to extensions of $u^i_{(\epsilon,\delta)}$ and of $u^o_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 4.28.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), *respectively. Let $\epsilon_3$ be as in Proposition 4.17. Let $\epsilon_6$, $J^i$, $J^o$ be as in Theorem 4.22. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^i[\epsilon], \tag{4.37}$$

*and*

$$\int_{\mathbb{R}^n} \mathbf{E}^o_{(\epsilon,r/l)}[u^o_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^o[\epsilon], \tag{4.38}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N}\setminus\{0\}$.*

*Proof.* We follow the proof of Theorem 2.60. Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N}\setminus\{0\}$. Then, by the periodicity of $u^i_{(\epsilon,r/l)}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx \\
&= \int_{rA} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx \\
&= l^n \int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx.
\end{aligned}$$

Then we note that

$$\begin{aligned}
\int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\Omega_\epsilon} u^i_{(\epsilon,r/l)}(x)\,dx \\
&= \int_{\frac{r}{l}\Omega_\epsilon} u^i[\epsilon]\big(\frac{l}{r}x\big)\,dx \\
&= \frac{r^n}{l^n}\int_{\Omega_\epsilon} u^i[\epsilon](t)\,dt \\
&= \frac{r^n}{l^n} J^i[\epsilon].
\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^i[\epsilon],$$

and the validity of (4.37) follows. The proof of (4.38) is very similar and is accordingly omitted. □

### 4.3.4 Asymptotic behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

This Subsection is devoted to the study of the behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$. We give the following.

**Definition 4.29.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we set

$$\mathrm{En}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{S}_a(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx + \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 4.30. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. Let $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$. We have

$$\begin{aligned}\int_{\Omega(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\Omega(\epsilon,1)} |(\nabla u^i_{(\epsilon,\delta)})(\delta t)|^2\,dt\\ &= \delta^{n-2}\int_{\Omega_\epsilon} |\nabla u^i[\epsilon](t)|^2\,dt\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u^o_{(\epsilon,\delta)})(\delta t)|^2\,dt\\ &= \delta^{n-2}\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](t)|^2\,dt.\end{aligned}$$

Then we give the following definition, where we consider $\mathrm{En}(\epsilon,\delta)$, with $\epsilon$ equal to a certain function of $\delta$.

**Definition 4.31.** For each $\delta \in ]0,+\infty[$, we set

$$\epsilon[\delta] \equiv \delta^{\frac{2}{n}}.$$

Let $\epsilon_5$ be as in Theorem 4.20. Let $\delta_1 > 0$ be such that $\epsilon[\delta] \in ]0,\epsilon_5[$, for all $\delta \in ]0,\delta_1[$. Then we set

$$\mathrm{En}[\delta] \equiv \mathrm{En}(\epsilon[\delta],\delta),$$

for all $\delta \in ]0,\delta_1[$.

In the following Proposition we compute the limit of $\mathrm{En}[\delta]$ as $\delta$ tends to 0.

**Proposition 4.32.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), *respectively. Let $\epsilon_5$ be as in Theorem 4.20. Let $\delta_1 > 0$ be as in Definition 4.31. Then*

$$\lim_{\delta\to 0^+} \mathrm{En}[\delta] = \int_\Omega |\nabla \tilde{u}^i(x)|^2\,dx + \int_{\mathbb{R}^n\setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}^o(x)|^2\,dx,$$

*where $\tilde{u}^i$, $\tilde{u}^o$ are as in Definition 4.14.*

*Proof.* We follow step by step the proof of Propostion 2.140. Let $G^i$, $G^o$ be as in Theorem 4.20. Let $\delta \in ]0,\delta_1[$. By Remark 4.30 and Theorem 4.20, we have

$$\begin{aligned}\int_{\Omega(\epsilon[\delta],\delta)} |\nabla u^i_{(\epsilon[\delta],\delta)}(x)|^2\,dx + \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u^o_{(\epsilon[\delta],\delta)}(x)|^2\,dx &= \delta^{n-2}(\epsilon[\delta])^n (G^i[\epsilon[\delta]] + G^o[\epsilon[\delta]])\\ &= \delta^n (G^i[\delta^{\frac{2}{n}}] + G^o[\delta^{\frac{2}{n}}]).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\lfloor (1/\delta)\rfloor^n \Big(\int_{\Omega(\epsilon[\delta],\delta)} |\nabla u^i_{(\epsilon[\delta],\delta)}(x)|^2\,dx + \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u^o_{(\epsilon[\delta],\delta)}(x)|^2\,dx\Big) \le \mathrm{En}[\delta]\\ &\qquad \le \lceil (1/\delta)\rceil^n \Big(\int_{\Omega(\epsilon[\delta],\delta)} |\nabla u^i_{(\epsilon[\delta],\delta)}(x)|^2\,dx + \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u^o_{(\epsilon[\delta],\delta)}(x)|^2\,dx\Big),\end{aligned}$$

and so

$$\lfloor (1/\delta) \rfloor^n \delta^n (G^i[\delta^{\frac{2}{n}}] + G^o[\delta^{\frac{2}{n}}]) \le \mathrm{En}[\delta] \le \lceil (1/\delta) \rceil^n \delta^n (G^i[\delta^{\frac{2}{n}}] + G^o[\delta^{\frac{2}{n}}]).$$

Thus, since

$$\lim_{\delta \to 0^+} \lfloor (1/\delta) \rfloor^n \delta^n = 1, \qquad \lim_{\delta \to 0^+} \lceil (1/\delta) \rceil^n \delta^n = 1,$$

we have

$$\lim_{\delta \to 0^+} \mathrm{En}[\delta] = (G^i[0] + G^o[0]).$$

Finally, by equalities (4.27) and (4.28), we easily conclude. □

In the following Proposition we represent the function $\mathrm{En}[\cdot]$ by means of real analytic functions.

**Proposition 4.33.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), *respectively. Let $\epsilon_5$, $G^i$, $G^o$ be as in Theorem 4.20. Let $\delta_1 > 0$ be as in Definition 4.31. Then*

$$\mathrm{En}[(1/l)] = G^i[(1/l)^{\frac{2}{n}}] + G^o[(1/l)^{\frac{2}{n}}],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* It follows by the proof of Proposition 4.32. □

## 4.4 A variant of an homogenization problem for the Laplace equation with a linear transmission boundary condition in a periodically perforated domain

In this section we consider a slightly different homogenization problem for the Laplace equation with a linear transmission boundary condition in a periodically perforated domain.

### 4.4.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 4.2.1, 4.3.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. For each pair $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we consider the following periodic linear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a(\epsilon,\delta), \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u^i(x + \delta a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\ u^o(x + \delta a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\ u^o(x) = \phi u^i(x) & \forall x \in \partial\Omega(\epsilon,\delta), \\ \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u^o(x) = \gamma \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u^i(x) + g(\frac{1}{\epsilon\delta}(x - \delta w)) & \forall x \in \partial\Omega(\epsilon,\delta), \\ u^i(\delta w) = \bar{c}. \end{cases} \tag{4.39}$$

In contrast to problem (4.33), we note that in the sixth equation of problem (4.39) there is not the factor $1/\delta$ in front of $g(\frac{1}{\epsilon\delta}(x - \delta w))$.

By virtue of Theorem 4.6, we can give the following definition.

**Definition 4.34.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. For each pair $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we denote by $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta)) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta))$ of boundary value problem (4.39).

Our aim is to study the asymptotic behaviour of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ as $(\epsilon, \delta)$ tends to $(0,0)$. In order to do so we introduce the following.

**Definition 4.35.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. For each $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we denote by $(u^i[\epsilon,\delta], u^o[\epsilon,\delta])$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of the following auxiliary periodic linear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a[\Omega_\epsilon], \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u^i(x+a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u^o(x+a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u^o(x) = \phi u^i(x) & \forall x \in \partial\Omega_\epsilon, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u^o(x) = \gamma \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u^i(x) + g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon, \\ u^i(w) = \frac{\bar{c}}{\delta}. \end{cases} \tag{4.40}$$

*Remark* 4.36. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we have

$$u^i_{(\epsilon,\delta)}(x) = \delta u^i[\epsilon,\delta](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta),$$

and

$$u^o_{(\epsilon,\delta)}(x) = \delta u^o[\epsilon,\delta](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).$$

By the previous remark, in contrast to the solution of problem (4.33), we note that the solution of problem (4.39) can be expressed by means of the solution of the auxiliary rescaled problem (4.40), which does depend on $\delta$.

*Remark* 4.37. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. Let $\epsilon_3$, $\Theta[\cdot]$ be as in Proposition 4.17.

Let $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,+\infty[$. We have

$$\begin{aligned} u^i[\epsilon,\delta](x) = \epsilon^{n-1} &\int_{\partial\Omega} S_n^a(x-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \\ &+ \Big(\frac{\bar{c}}{\delta} - \epsilon^{n-1}\int_{\partial\Omega} S_n^a(-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s\Big) \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon], \end{aligned}$$

and

$$\begin{aligned} u^o[\epsilon,\delta](x) = \phi\epsilon^{n-1} &\int_{\partial\Omega} S_n^a(x-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \\ &+ \phi\Big(\frac{\bar{c}}{\delta} - \epsilon^{n-1}\int_{\partial\Omega} S_n^a(-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s\Big) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon]. \end{aligned}$$

As a first step, we study the behaviour of $(u^i[\epsilon,\delta], u^o[\epsilon,\delta])$ as $(\epsilon,\delta)$ tends to $(0,0)$.

**Proposition 4.38.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), *respectively. Let $1 \le p < \infty$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,1)}[\delta u^i[\epsilon,\delta]] = 0 \qquad \text{in } L^p(A),$$

*and*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,1)}[\delta u^o[\epsilon,\delta]] = \phi\bar{c} \qquad \text{in } L^p(A).$$

*Proof.* Let $\epsilon_3$, $\Theta$ be as in Proposition 4.17. If $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,+\infty[$, we have

$$\begin{aligned} \delta u^i[\epsilon,\delta](w+\epsilon t) =& \delta\epsilon\Big(\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s\Big) \\ &+ \Big(\bar{c} - \delta\epsilon^{n-1}\int_{\partial\Omega} S_n^a(-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s\Big) \qquad \forall t \in \partial\Omega. \end{aligned} \tag{4.41}$$

We set

$$\begin{aligned} N^i[\epsilon,\delta](t) \equiv& \delta\epsilon\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \\ &+ \Big(\bar{c} - \delta\epsilon^{n-1}\int_{\partial\Omega} S_n^a(-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s\Big) \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $(\epsilon, \delta) \in ]-\epsilon_3, \epsilon_3[ \times \mathbb{R}$. By taking $\tilde{\epsilon} \in ]0, \epsilon_3[$, $\tilde{\delta} \in ]0, +\infty[$ small enough, we can assume (cf. Proposition 1.28 (*i*) and the proof of Theorem 4.18) that $N^i$ is a real analytic map of $]-\tilde{\epsilon}, \tilde{\epsilon}[ \times ]-\tilde{\delta}, \tilde{\delta}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C^i \equiv \sup_{(\epsilon,\delta)\in ]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\delta},\tilde{\delta}[} \|N^i[\epsilon, \delta]\|_{C^0(\partial\Omega)} < +\infty.$$

By the Maximum Principle for harmonic functions, we have

$$|\mathbf{E}^i_{(\epsilon,1)}[\delta u^i[\epsilon, \delta]](x)| \le C^i \qquad \forall x \in A, \quad \forall (\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, \tilde{\delta}[.$$

Obviously,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,1)}[\delta u^i[\epsilon, \delta]](x) = 0 \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,1)}[\delta u^i[\epsilon, \delta]] = 0 \qquad \text{in } L^p(A).$$

If $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, +\infty[$, we have

$$\begin{aligned} \delta u^o[\epsilon, \delta](w + \epsilon t) =& \delta\epsilon\phi\Big(\int_{\partial\Omega} S_n(t - s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R^a_n(\epsilon(t - s))\Theta[\epsilon](s)\, d\sigma_s\Big) \\ &+ \phi\Big(\bar{c} - \delta\epsilon^{n-1} \int_{\partial\Omega} S^a_n(-\epsilon s)\Theta[\epsilon](s)\, d\sigma_s\Big) \qquad \forall t \in \partial\Omega. \end{aligned} \tag{4.42}$$

We set

$$\begin{aligned} N^o[\epsilon, \delta](t) \equiv& \delta\epsilon\phi \int_{\partial\Omega} S_n(t - s)\Theta[\epsilon](s)\, d\sigma_s + \delta\epsilon^{n-1}\phi \int_{\partial\Omega} R^a_n(\epsilon(t - s))\Theta[\epsilon](s)\, d\sigma_s \\ &+ \phi\Big(\bar{c} - \delta\epsilon^{n-1} \int_{\partial\Omega} S^a_n(-\epsilon s)\Theta[\epsilon](s)\, d\sigma_s\Big) \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $(\epsilon, \delta) \in ]-\epsilon_3, \epsilon_3[ \times \mathbb{R}$. By taking $\tilde{\epsilon} \in ]0, \epsilon_3[$, $\tilde{\delta} \in ]0, +\infty[$ small enough, we can assume (cf. Proposition 1.28 (*i*) and the proof of Theorem 4.18) that $N^o$ is a real analytic map of $]-\tilde{\epsilon}, \tilde{\epsilon}[ \times ]-\tilde{\delta}, \tilde{\delta}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C^o \equiv \sup_{(\epsilon,\delta)\in ]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\delta},\tilde{\delta}[} \|N^o[\epsilon, \delta]\|_{C^0(\partial\Omega)} < +\infty.$$

By Theorem 2.5, we have

$$|\mathbf{E}^o_{(\epsilon,1)}[\delta u^o[\epsilon, \delta]](x)| \le C^o \qquad \forall x \in A, \quad \forall (\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, \tilde{\delta}[.$$

Clearly (cf. Theorem 4.18), we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,1)}[\delta u^o[\epsilon, \delta]](x) = \phi\bar{c} \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,1)}[\delta u^o[\epsilon, \delta]] = \phi\bar{c} \qquad \text{in } L^p(A).$$

□

Then we have also the following.

**Theorem 4.39.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), *respectively. Let $\epsilon_3$ be as in Proposition 4.17. Then there exist $\epsilon_6 \in ]0, \epsilon_3]$ and two real analytic operators $J^i$, $J^o$ of $]-\epsilon_6, \epsilon_6[ \times \mathbb{R}$ to $\mathbb{R}$, such that*

$$\int_{\Omega_\epsilon} \delta u^i[\epsilon, \delta](x)\, dx = J^i[\epsilon, \delta], \tag{4.43}$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} \delta u^o[\epsilon, \delta](x)\, dx = J^o[\epsilon, \delta], \tag{4.44}$$

*for all $(\epsilon, \delta) \in ]0, \epsilon_6[ \times ]0, +\infty[$. Moreover,*

$$J^i[0, 0] = 0, \tag{4.45}$$

$$J^o[0, 0] = \phi\bar{c}|A|_n. \tag{4.46}$$

*Proof.* Let $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,+\infty[$. We have

$$\begin{aligned}\delta u^o[\epsilon,\delta](x) =& \delta\phi v_a^-\left[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\right](x)\\ &+ \phi\bar{c} - \delta\phi\epsilon^{n-1}\int_{\partial\Omega} S_n^a(-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].\end{aligned}$$

As a consequence

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} \delta u^o[\epsilon,\delta](x)\,dx =& \delta\phi\int_{\mathbb{P}_a[\Omega_\epsilon]} v_a^-\left[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\right](x)\,dx\\ &+ \phi\bar{c}\Big(|A|_n - \epsilon^n|\Omega|_n\Big) - \delta\phi\epsilon^{n-1}\int_{\partial\Omega} S_n^a(-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s\Big(|A|_n - \epsilon^n|\Omega|_n\Big).\end{aligned}$$

Then, by arguing as in the proof of Theorem 4.22, one can easily show that there exist $\epsilon_6' \in ]0,\epsilon_3]$ and a real analytic operator $J^o$ of $]-\epsilon_6',\epsilon_6'[ \times \mathbb{R}$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} \delta u^o[\epsilon,\delta](x)\,dx = J^o[\epsilon,\delta],$$

for all $(\epsilon,\delta) \in ]0,\epsilon_6'[ \times ]0,+\infty[$, and that $J^o[0,0] = \phi\bar{c}|A|_n$.

Let $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,+\infty[$. We have

$$\begin{aligned}\delta u^i[\epsilon,\delta](x) =& \delta v_a^+\left[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\right](x)\\ &+ \bar{c} - \delta\epsilon^{n-1}\int_{\partial\Omega} S_n^a(-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon].\end{aligned}$$

Then, by arguing as in the proof of Theorem 4.22, one can easily show that there exist $\epsilon_6'' \in ]0,\epsilon_3]$ and a real analytic operator $J^i$ of $]-\epsilon_6'',\epsilon_6''[ \times \mathbb{R}$ to $\mathbb{R}$, such that

$$\int_{\Omega_\epsilon} \delta u^i[\epsilon,\delta](x)\,dx = J^i[\epsilon,\delta],$$

for all $(\epsilon,\delta) \in ]0,\epsilon_6''[ \times ]0,+\infty[$, and that $J^i[0,0] = 0$.

Then, by taking $\epsilon_6 \equiv \min\{\epsilon_6',\epsilon_6''\}$, we can easily conclude. □

### 4.4.2 Asymptotic behaviour of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

In the following Theorem we deduce by Proposition 4.38 the convergence of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 4.40.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), *respectively. Let $1 \le p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^p(V),$$

*and*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] = \phi\bar{c} \qquad \text{in } L^p(V).$$

*Proof.* By virtue of Proposition 4.38, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \|\mathbf{E}^i_{(\epsilon,1)}[\delta u^i[\epsilon,\delta]]\|_{L^p(A)} = 0,$$

and

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \|\mathbf{E}^o_{(\epsilon,1)}[\delta u^o[\epsilon,\delta]] - \phi\bar{c}\|_{L^p(A)} = 0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $C > 0$ such that

$$\|\mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}]\|_{L^p(V)} \le C\|\mathbf{E}^i_{(\epsilon,1)}[\delta u^i[\epsilon,\delta]]\|_{L^p(A)} \qquad \forall(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,1[,$$

and

$$\|\mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] - \phi\bar{c}\|_{L^p(V)} \le C\|\mathbf{E}^o_{(\epsilon,1)}[\delta u^o[\epsilon,\delta]] - \phi\bar{c}\|_{L^p(A)} \qquad \forall (\epsilon,\delta) \in ]0,\epsilon_1[\times]0,1[,$$

Thus,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^p(V),$$

and

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] = \phi\bar{c} \qquad \text{in } L^p(V).$$

□

Then we have the following Theorem, where we consider a functional associated to extensions of $u^i_{(\epsilon,\delta)}$ and of $u^o_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 4.41.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), *respectively. Let $\epsilon_3$ be as in Proposition 4.17. Let $\epsilon_6$, $J^i$, $J^o$ be as in Theorem 4.39. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^i\big[\epsilon, \frac{r}{l}\big], \tag{4.47}$$

*and*

$$\int_{\mathbb{R}^n} \mathbf{E}^o_{(\epsilon,r/l)}[u^o_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^o\big[\epsilon, \frac{r}{l}\big], \tag{4.48}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N}\setminus\{0\}$.*

*Proof.* We follow the the proof of Theorem 2.150. Let $\epsilon \in ]0,\epsilon_6[$, and let $l \in \mathbb{N}\setminus\{0\}$. Then, by the periodicity of $u^i_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx\\ &= \int_{rA} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx\\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\Omega_\epsilon} u^i_{(\epsilon,r/l)}(x)\,dx\\ &= \int_{\frac{r}{l}\Omega_\epsilon} (r/l)u^i\big[\epsilon,(r/l)\big]\big(\frac{l}{r}x\big)\,dx\\ &= \frac{r^n}{l^n}\int_{\Omega_\epsilon} (r/l)u^i\big[\epsilon,(r/l)\big](t)\,dt\\ &= \frac{r^n}{l^n} J^i\big[\epsilon,\frac{r}{l}\big].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^i\big[\epsilon, \frac{r}{l}\big],$$

and the validity of (4.47) follows. The proof of (4.48) is very similar and is accordingly omitted. □

### 4.4.3 Asymptotic behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

This Subsection is devoted to the study of the behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$. We give the following.

**Definition 4.42.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. For each pair $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we set

$$\mathrm{En}(\epsilon,\delta) \equiv \int_{A \cap \mathbb{S}_a(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2 \, dx + \int_{A \cap \mathbb{T}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2 \, dx.$$

*Remark* 4.43. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), respectively. Let $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$. We have

$$\begin{aligned}\int_{\Omega(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2 \, dx &= \delta^n \int_{\Omega(\epsilon,1)} |(\nabla u^i_{(\epsilon,\delta)})(\delta t)|^2 \, dt \\ &= \delta^n \int_{\Omega_\epsilon} |\nabla u^i[\epsilon,\delta](t)|^2 \, dt\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2 \, dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u^o_{(\epsilon,\delta)})(\delta t)|^2 \, dt \\ &= \delta^n \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon,\delta](t)|^2 \, dt.\end{aligned}$$

In the following Proposition we represent the function $\mathrm{En}(\cdot,\cdot)$ by means of real analytic functions.

**Proposition 4.44.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$, $\bar{c}$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), (4.12), *respectively. Let $\epsilon_5$, $G^i$, $G^o$ be as in Theorem 4.20. Then*

$$\mathrm{En}\Big(\epsilon, \frac{1}{l}\Big) = \epsilon^n G^i[\epsilon] + \epsilon^n G^o[\epsilon],$$

*for all $\epsilon \in ]0, \epsilon_5[$ and for all $l \in \mathbb{N} \setminus \{0\}$.*

*Proof.* Let $(\epsilon, \delta) \in ]0, \epsilon_5[ \times ]0, +\infty[$. By Remark 4.43 and Theorem 4.20, we have

$$\int_{\Omega(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2 \, dx + \int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2 \, dx = \delta^n \epsilon^n G^i[\epsilon] + \delta^n \epsilon^n G^o[\epsilon] \qquad (4.49)$$

where $G^i$, $G^o$ are as in Theorem 4.20. On the other hand, if $\epsilon \in ]0, \epsilon_5[$ and $l \in \mathbb{N} \setminus \{0\}$, then we have

$$\begin{aligned}\mathrm{En}\Big(\epsilon, \frac{1}{l}\Big) &= l^n \frac{1}{l^n} \Big\{ \epsilon^n G^i[\epsilon] + \epsilon^n G^o[\epsilon] \Big\}, \\ &= \epsilon^n G^i[\epsilon] + \epsilon^n G^o[\epsilon],\end{aligned}$$

and the conclusion easily follows. □

## 4.5 Asymptotic behaviour of the solutions of an alternative linear transmission problem for the Laplace equation in a periodically perforated domain

In this Section we study the asymptotic behaviour of the solutions of an alternative linear transmission problem for the Laplace equation in a periodically perforated domain with small holes.

### 4.5.1 Notation and preliminaries

We retain the notation introduced in Subsections 1.8.1, 4.2.1. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. For each $\epsilon \in ]0, \epsilon_1[$, we consider the following periodic linear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a[\Omega_\epsilon], \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u^i(x+a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u^o(x+a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u^o(x) = \phi u^i(x) & \forall x \in \partial\Omega_\epsilon, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u^o(x) = \gamma \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u^i(x) + g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon, \\ \int_{\partial\Omega_\epsilon} u^i(x)\, d\sigma_x = 0. \end{cases} \tag{4.50}$$

By virtue of Theorem 4.6, we can give the following definition.

**Definition 4.45.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. For each $\epsilon \in ]0,\epsilon_1[$, we denote by $(u^i[\epsilon], u^o[\epsilon])$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of boundary value problem (4.50).

*Remark* 4.46. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. For each $\epsilon \in ]0,\epsilon_1[$, let $\hat{\theta}[\epsilon]$ be as in Definition 4.12. Let $\epsilon \in ]0,\epsilon_1[$. We have

$$\begin{aligned} u^i[\epsilon](x) =& \epsilon^{n-1} \int_{\partial\Omega} S_n^a(x-w-\epsilon s)\hat{\theta}[\epsilon](s)\, d\sigma_s \\ & - \epsilon^{n-1} \frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n^a(\epsilon(t-s))\hat{\theta}[\epsilon](s)\, d\sigma_s \Big) d\sigma_t \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon], \end{aligned}$$

and

$$\begin{aligned} u^o[\epsilon](x) =& \phi\epsilon^{n-1} \int_{\partial\Omega} S_n^a(x-w-\epsilon s)\hat{\theta}[\epsilon](s)\, d\sigma_s \\ & - \phi\Big( \epsilon^{n-1} \frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} \big( \int_{\partial\Omega} S_n^a(\epsilon(t-s))\hat{\theta}[\epsilon](s)\, d\sigma_s \big) d\sigma_t \Big) \quad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon]. \end{aligned}$$

### 4.5.2 A functional analytic representation Theorem for the solution of the alternative singularly perturbed linear transmission problem

In this Subsection, we show that $\{(u^i[\epsilon](\cdot), u^o[\epsilon](\cdot))\}_{\epsilon \in ]0,\epsilon_3[}$ can be continued real analytically for negative values of $\epsilon$.

We have the following.

**Theorem 4.47.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\epsilon_3$ be as in Proposition 4.17. Then the following statements hold.*

*(i) Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0,\epsilon_3]$, a real analytic operator $U_1^o$ of $]-\epsilon_4,\epsilon_4[$ to the space $C_h^0(\mathrm{cl}\,V)$, and a real analytic operator $U_2^o$ of $]-\epsilon_4,\epsilon_4[$ to $\mathbb{R}$ such that the following conditions hold.*

*(j) $\mathrm{cl}\,V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ for all $\epsilon \in ]-\epsilon_4,\epsilon_4[$.*

*(jj)*

$$u^o[\epsilon](x) = \epsilon^n U_1^o[\epsilon](x) + \epsilon U_2^o[\epsilon] \qquad \forall x \in \mathrm{cl}\,V,$$

*for all $\epsilon \in ]0,\epsilon_4[$.*

*(ii) Let $\bar{V}$ be a bounded open subset of $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$. Then there exist $\bar{\epsilon}_4 \in ]0,\epsilon_3]$, a real analytic operator $\bar{U}_1^o$ of $]-\bar{\epsilon}_4,\bar{\epsilon}_4[$ to the space $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$, and a real analytic operator $\bar{U}_2^o$ of $]-\bar{\epsilon}_4,\bar{\epsilon}_4[$ to $\mathbb{R}$ such that the following conditions hold.*

*(j') $w + \epsilon\,\mathrm{cl}\,\bar{V} \subseteq \mathrm{cl}\,\mathbb{P}_a[\Omega_\epsilon]$ for all $\epsilon \in ]-\bar{\epsilon}_4,\bar{\epsilon}_4[ \setminus \{0\}$.*

*(jj')*

$$u^o[\epsilon](w+\epsilon t) = \epsilon \bar{U}_1^o[\epsilon](t) + \epsilon \bar{U}_2^o[\epsilon] \qquad \forall t \in \mathrm{cl}\,\bar{V},$$

*for all $\epsilon \in ]0,\bar{\epsilon}_4[$.*

*(iii)* *There exist* $\epsilon_4' \in ]0,\epsilon_3]$, *a real analytic operator* $U_1^i$ *of* $]-\epsilon_4',\epsilon_4'[$ *to the space* $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, *and a real analytic operator* $U_2^i$ *of* $]-\epsilon_4',\epsilon_4'[$ *to* $\mathbb{R}$ *such that*

$$u^i[\epsilon](w+\epsilon t) = \epsilon U_1^i[\epsilon](t) + \epsilon U_2^i[\epsilon] \qquad \forall t \in \mathrm{cl}\,\Omega,$$

*for all* $\epsilon \in ]0,\epsilon_4'[$.

*Proof.* It is a simple modification of the proof of Theorem 4.18 and Theorem 2.158. Indeed, let $\Theta[\cdot]$ be as in Proposition 4.17. Consider $(i)$. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $\epsilon \in ]0,\epsilon_4[$. By Remark 4.13 and Proposition 4.17, we have

$$\begin{aligned} u^o[\epsilon](x) =&\phi\epsilon^{n-1}\int_{\partial\Omega} S_n^a(x-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \\ &-\phi(\epsilon^{n-1}\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}(\int_{\partial\Omega} S_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s)\,d\sigma_t) \quad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon]. \end{aligned}$$

Thus (cf. the proof of Theorem 2.158), it is natural to set

$$\tilde{U}_1^o[\epsilon](x) \equiv \phi\int_{\partial\Omega} S_n^a(x-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \qquad \forall x \in \mathrm{cl}\,V,$$

for all $\epsilon \in ]-\epsilon_4,\epsilon_4[$, and

$$U_2^o[\epsilon] \equiv \phi(-\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}(\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s)\,d\sigma_t),$$

for all $\epsilon \in ]-\epsilon_4,\epsilon_4[$. Following the proof of Theorem 2.124, by possibly taking a smaller $\epsilon_4$, we have that there exists a real analytic map $U_1^o$ of $]-\epsilon_4,\epsilon_4[$ to $C_h^0(\mathrm{cl}\,V)$ such that

$$\tilde{U}_1^o[\epsilon] = \epsilon U_1^o[\epsilon] \qquad \text{in } C_h^0(\mathrm{cl}\,V),$$

for all $\epsilon \in ]-\epsilon_4,\epsilon_4[$. Furthermore, by arguing as in the proof of Theorem 2.158 we have that $U_2$ is a real analytic operator of $]-\epsilon_4,\epsilon_4[$ to $\mathbb{R}$. Finally, by the definition of $U_1^o$ and $U_2^o$, we immediately deduce that the equality in $(jj)$ holds. Consider now $(ii)$. Choosing $\bar{\epsilon}_4$ small enough, we can clearly assume that $(j')$ holds. Consider now $(jj')$. Let $\epsilon \in ]0,\bar{\epsilon}_4[$. By Remark 4.46, we have

$$\begin{aligned} u^o[\epsilon](w+\epsilon t) = \epsilon^{n-1}\phi\int_{\partial\Omega} S_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s& \\ -\phi(\epsilon^{n-1}\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}(\int_{\partial\Omega} S_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s)\,d\sigma_t) &\qquad \forall t \in \mathrm{cl}\,\bar{V}. \end{aligned}$$

Thus (cf. the proof of Theorem 3.13 $(ii)$), it is natural to set

$$\bar{U}_1^o[\epsilon](t) \equiv \phi\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \qquad \forall t \in \mathrm{cl}\,\bar{V},$$

for all $\epsilon \in ]-\bar{\epsilon}_4,\bar{\epsilon}_4[$, and

$$\bar{U}_2^o[\epsilon] \equiv \phi(-\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}(\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s)\,d\sigma_t),$$

for all $\epsilon \in ]-\bar{\epsilon}_4,\bar{\epsilon}_4[$. By the proof of $(i)$, we have that $\bar{U}_2^o$ is a real analytic map of $]-\bar{\epsilon}_4,\bar{\epsilon}_4[$ to $\mathbb{R}$. Moreover, by arguing as in the proof of Theorem 3.13 $(ii)$ we have that $\bar{U}_1^o$ is a real analytic map of $]-\bar{\epsilon}_4,\bar{\epsilon}_4[$ to $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$. Finally, consider $(iii)$. Let $\epsilon \in ]0,\epsilon_3[$. By Remark 4.46, we have

$$\begin{aligned} u^i[\epsilon](w+\epsilon t) =&\epsilon^{n-1}\int_{\partial\Omega} S_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \\ &-\epsilon^{n-1}\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}(\int_{\partial\Omega} S_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s)\,d\sigma_t \qquad \forall t \in \mathrm{cl}\,\Omega. \end{aligned} \tag{4.51}$$

Thus,by arguing as above, it is natural to set

$$U_1^i[\epsilon](t) \equiv \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s \qquad \forall t \in \mathrm{cl}\,\Omega,$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$, and

$$U_2^i[\epsilon] \equiv -\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s\Big)\,d\sigma_t,$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. Then, by arguing as above (cf. Proposition 1.29 $(iii)$), there exists $\epsilon_4' \in ]0, \epsilon_3]$, such that $U_1^i$ and $U_2^i$ are real analytic maps of $]-\epsilon_4', \epsilon_4'[$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ and $\mathbb{R}$, respectively. □

*Remark* 4.48. We note that the right-hand side of the equalities in $(jj)$, $(jj')$ and $(iii)$ of Theorem 4.18 can be continued real analytically in a whole neighbourhood of 0. Moreover, if $V$ is a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V \cap \mathbb{S}_a[\Omega_0] = \emptyset$, then

$$\lim_{\epsilon\to 0^+} u^o[\epsilon] = 0 \qquad \text{uniformly in } \mathrm{cl}\,V.$$

### 4.5.3 A real analytic continuation Theorem for the energy integral

As done in Theorem 4.47 for $(u^i[\cdot], u^o[\cdot])$, we can now prove a real analytic continuation Theorem for the energy integral. Namely, we prove the following.

**Theorem 4.49.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\epsilon_3$ be as in Proposition 4.17. Then there exist $\epsilon_5 \in ]0, \epsilon_3]$ and two real analytic operators $G^i$, $G^o$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{R}$, such that*

$$\int_{\Omega_\epsilon} |\nabla u^i[\epsilon](x)|^2\,dx = \epsilon^n G^i[\epsilon], \tag{4.52}$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](x)|^2\,dx = \epsilon^n G^o[\epsilon], \tag{4.53}$$

*for all $\epsilon \in ]0, \epsilon_5[$. Moreover,*

$$G^i[0] = \int_{\Omega} |\nabla \tilde{u}^i(x)|^2\,dx, \tag{4.54}$$

$$G^o[0] = \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega} |\nabla \tilde{u}^o(x)|^2\,dx, \tag{4.55}$$

*where $\tilde{u}^i$, $\tilde{u}^o$ are as in Definition 4.14.*

*Proof.* Let $\Theta[\cdot]$ be as in Proposition 4.17. Let $\epsilon \in ]0, \epsilon_3[$. Clearly,

$$\int_{\Omega_\epsilon} |\nabla u^i[\epsilon](x)|^2\,dx = \int_{\Omega_\epsilon} |\nabla v_a^+[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2\,dx,$$

and

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](x)|^2\,dx = \phi^2\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2\,dx.$$

As a consequence, in order to prove the Theorem, it suffices to follow the proof of Theorem 4.20. □

*Remark* 4.50. We note that the right-hand side of the equalities in (4.52) and (4.53) of Theorem 4.49 can be continued real analytically in the whole $]-\epsilon_5, \epsilon_5[$. Moreover,

$$\lim_{\epsilon\to 0^+}\Big(\int_{\Omega_\epsilon} |\nabla u^i[\epsilon](x)|^2\,dx + \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](x)|^2\,dx\Big) = 0.$$

### 4.5.4 A real analytic continuation Theorem for the integral of the solution

As done in Theorem 4.51 for the energy integral, we can now prove a real analytic continuation Theorem for the integral of the solution. Namely, we prove the following.

**Theorem 4.51.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\epsilon_3$ be as in Proposition 4.17. Then there exist $\epsilon_6 \in ]0, \epsilon_3]$ and two real analytic operators $J^i$, $J^o$ of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$, such that*

$$\int_{\Omega_\epsilon} u^i[\epsilon](x)\, dx = J^i[\epsilon], \tag{4.56}$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^o[\epsilon](x)\, dx = J^o[\epsilon], \tag{4.57}$$

*for all $\epsilon \in ]0, \epsilon_6[$. Moreover,*

$$J^i[0] = 0, \tag{4.58}$$

$$J^o[0] = 0. \tag{4.59}$$

*Proof.* Let $\Theta[\cdot]$ be as in Proposition 4.17. Let $\epsilon \in ]0, \epsilon_3[$. We have

$$\begin{aligned} u^o[\epsilon](x) &= \phi v_a^- \left[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\right](x) \\ &-\epsilon\phi(\frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} (\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s)\, d\sigma_t) \quad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon]. \end{aligned}$$

Then, if we set

$$\begin{aligned} L[\epsilon](t) \equiv &\phi\epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \phi\epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ &-\epsilon\phi(\frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} (\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s)\, d\sigma_t) \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$, by arguing as in the proof of Theorem 2.128, we can easily show that there exist $\epsilon_6' \in ]0, \epsilon_3]$ and a real analytic map $J^o$ of $]-\epsilon_6', \epsilon_6'[$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^o[\epsilon](x)\, dx = J^o[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_6'[$, and such that $J^o[0] = 0$.

Let $\epsilon \in ]0, \epsilon_3[$. Clearly,

$$\int_{\Omega_\epsilon} u^i[\epsilon](x)\, dx = \epsilon^n \int_{\Omega} u^i[\epsilon](w + \epsilon t)\, dt.$$

On the other hand, if $\epsilon_4'$, $U_1^i$, $U_2^i$ are as in Theorem 4.47, and we set

$$J^i[\epsilon] \equiv \epsilon^n \int_{\Omega} \left(\epsilon U_1^i[\epsilon](t) + \epsilon U_2^i[\epsilon]\right) dt$$

for all $\epsilon \in ]-\epsilon_4', \epsilon_4'[$, then we have that $J^i$ is a real analytic map of $]-\epsilon_4', \epsilon_4'[$ to $\mathbb{R}$, such that $J^i[0] = 0$ and that

$$\int_{\Omega_\epsilon} u^i[\epsilon](x)\, dx = J^i[\epsilon]$$

for all $\epsilon \in ]0, \epsilon_4'[$.

Then, by taking $\epsilon_6 \equiv \min\{\epsilon_6', \epsilon_4'\}$, we can conclude. □

## 4.6 Alternative homogenization problem for the Laplace equation with a linear transmission condition in a periodically perforated domain

In this section we consider another homogenization problem for the Laplace equation with a linear transmission boundary condition in a periodically perforated domain.

### 4.6.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 4.2.1 and 4.3.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we consider the following periodic linear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a(\epsilon,\delta), \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u^i(x+\delta a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\ u^o(x+\delta a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\ u^o(x) = \phi u^i(x) & \forall x \in \partial\Omega(\epsilon,\delta), \\ \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u^o(x) = \gamma \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u^i(x) + \frac{1}{\delta} g(\frac{1}{\epsilon\delta}(x-\delta w)) & \forall x \in \partial\Omega(\epsilon,\delta), \\ \int_{\partial\Omega(\epsilon,\delta)} u^i(x)\, d\sigma_x = 0. \end{cases} \tag{4.60}$$

By virtue of Theorem 4.6, we can give the following definition.

**Definition 4.52.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we denote by $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta)) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta))$ of boundary value problem (4.60).

Our aim is to study the asymptotic behaviour of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ as $(\epsilon,\delta)$ tends to $(0,0)$. In order to do so we introduce the following.

**Definition 4.53.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. For each $\epsilon \in ]0,\epsilon_1[$, we denote by $(u^i[\epsilon], u^o[\epsilon])$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ of the following periodic linear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a[\Omega_\epsilon], \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u^i(x+a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u^o(x+a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u^o(x) = \phi u^i(x) & \forall x \in \partial\Omega_\epsilon, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u^o(x) = \gamma \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u^i(x) + g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon, \\ \int_{\partial\Omega_\epsilon} u^i(x)\, d\sigma_x = 0. \end{cases} \tag{4.61}$$

*Remark* 4.54. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we have

$$u^i_{(\epsilon,\delta)}(x) = u^i[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta),$$

and

$$u^o_{(\epsilon,\delta)}(x) = u^o[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).$$

By the previous remark, we note that the solutions of problem (4.60) can be expressed by means of the solutions of the auxiliary rescaled problem (4.61), which does not depend on $\delta$. This is due to the presence of the factor $1/\delta$ in front of $g(\frac{1}{\epsilon\delta}(x-\delta w))$ in the sixth equation of problem (4.60).

As a first step, we study the behaviour of $(u^i[\epsilon], u^o[\epsilon])$ as $\epsilon$ tends to 0.

**Proposition 4.55.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\epsilon_3$ be as in Proposition 4.17. Then there exist $\tilde{\epsilon} \in ]0,\epsilon_3[$ and a real analytic map $N^i$ of $]-\tilde{\epsilon},\tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ such that*

$$\|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]]\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|N^i[\epsilon]\|_{C^0(\partial\Omega)},$$
$$\|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]]\|_{L^\infty(\mathbb{R}^n)} = \epsilon\phi \|N^i[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all $\epsilon \in ]0,\tilde{\epsilon}[$. Moreover, as a consequence,*

$$\lim_{\epsilon\to 0^+} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]] = 0 \qquad \textit{in } L^\infty(\mathbb{R}^n),$$
$$\lim_{\epsilon\to 0^+} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]] = 0 \qquad \textit{in } L^\infty(\mathbb{R}^n).$$

*Proof.* Let $\epsilon_3$, $\Theta$ be as in Proposition 4.17. If $\epsilon \in ]0, \epsilon_3[$, we have

$$\begin{aligned} u^i[\epsilon](w+\epsilon t) = \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ - \frac{\epsilon}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \Big) d\sigma_t \quad \forall t \in \partial\Omega. \end{aligned} \tag{4.62}$$

We set

$$\begin{aligned} N^i[\epsilon](t) \equiv \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ - \frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \Big) d\sigma_t \quad \forall t \in \partial\Omega, \end{aligned}$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. By taking $\tilde{\epsilon} \in ]0, \epsilon_3[$ small enough, we can assume (cf. Proposition 1.28 $(i)$) that $N^i$ is a real analytic map of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$.

By the Maximum Principle for harmonic functions, we have

$$\|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]]\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|N^i[\epsilon]\|_{C^0(\partial\Omega)} \qquad \forall \epsilon \in ]0, \tilde{\epsilon}[,$$

and the conclusion, as far as $u^i$ is concerned, easily follows.

If $\epsilon \in ]0, \epsilon_3[$, we note that

$$u^o[\epsilon](w+\epsilon t) = \phi u^i[\epsilon](w+\epsilon t) \qquad \forall t \in \partial\Omega. \tag{4.63}$$

By Theorem 2.5, we have

$$\|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]]\|_{L^\infty(\mathbb{R}^n)} = \epsilon\phi \|N^i[\epsilon]\|_{C^0(\partial\Omega)} \qquad \forall \epsilon \in ]0, \tilde{\epsilon}[,$$

and the conclusion, also for $u^o$, follows. □

### 4.6.2 Asymptotic behaviour of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

In the following Theorem we deduce by Proposition 4.26 the convergence of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ as $(\epsilon, \delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 4.56.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\tilde{\epsilon}$, $N^i$ be as in Proposition 4.55. Then*

$$\begin{aligned} \|\mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}]\|_{L^\infty(\mathbb{R}^n)} &= \epsilon \|N^i[\epsilon]\|_{C^0(\partial\Omega)}, \\ \|\mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}]\|_{L^\infty(\mathbb{R}^n)} &= \epsilon\phi \|N^i[\epsilon]\|_{C^0(\partial\Omega)}, \end{aligned}$$

*for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. Moreover, as a consequence,*

$$\begin{aligned} \lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n), \\ \lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n). \end{aligned}$$

*Proof.* It suffices to observe that

$$\begin{aligned} \|\mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}]\|_{L^\infty(\mathbb{R}^n)} &= \|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]]\|_{L^\infty(\mathbb{R}^n)} \\ &= \epsilon \|N^i[\epsilon]\|_{C^0(\partial\Omega)}, \end{aligned}$$

and that

$$\begin{aligned} \|\mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}]\|_{L^\infty(\mathbb{R}^n)} &= \|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]]\|_{L^\infty(\mathbb{R}^n)} \\ &= \epsilon\phi \|N^i[\epsilon]\|_{C^0(\partial\Omega)}, \end{aligned}$$

for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. □

Then we have the following Theorem, where we consider a functional associated to extensions of $u^i_{(\epsilon,\delta)}$ and of $u^o_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 4.57.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\epsilon_3$ be as in Proposition 4.17. Let $\epsilon_6$, $J^i$, $J^o$ be as in Theorem 4.51. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^i[\epsilon], \tag{4.64}$$

*and*

$$\int_{\mathbb{R}^n} \mathbf{E}^o_{(\epsilon,r/l)}[u^o_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^o[\epsilon], \tag{4.65}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N} \setminus \{0\}$.*

*Proof.* We follow the proof of Theorem 2.60. Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N} \setminus \{0\}$. Then, by the periodicity of $u^i_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\Omega_\epsilon} u^i_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\Omega_\epsilon} u^i[\epsilon]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\Omega_\epsilon} u^i[\epsilon](t)\,dt \\ &= \frac{r^n}{l^n} J^i[\epsilon].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^i[\epsilon],$$

and the validity of (4.64) follows. The proof of (4.65) is very similar and is accordingly omitted. □

### 4.6.3 Asymptotic behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

This Subsection is devoted to the study of the behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$. We give the following.

**Definition 4.58.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we set

$$\mathrm{En}(\epsilon,\delta) \equiv \int_{A\cap \mathbb{S}_a(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx + \int_{A\cap \mathbb{T}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 4.59. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. Let $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$. We have

$$\begin{aligned}\int_{\Omega(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\Omega(\epsilon,1)} |(\nabla u^i_{(\epsilon,\delta)})(\delta t)|^2\,dt \\ &= \delta^{n-2}\int_{\Omega_\epsilon} |\nabla u^i[\epsilon](t)|^2\,dt\end{aligned}$$

and

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\, dx = \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u^o_{(\epsilon,\delta)})(\delta t)|^2\, dt$$
$$= \delta^{n-2} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](t)|^2\, dt.$$

Then we give the following definition, where we consider $\mathrm{En}(\epsilon,\delta)$, with $\epsilon$ equal to a certain function of $\delta$.

**Definition 4.60.** For each $\delta \in ]0,+\infty[$, we set

$$\epsilon[\delta] \equiv \delta^{\frac{2}{n}}.$$

Let $\epsilon_5$ be as in Theorem 4.49. Let $\delta_1 > 0$ be such that $\epsilon[\delta] \in ]0,\epsilon_5[$, for all $\delta \in ]0,\delta_1[$. Then we set

$$\mathrm{En}[\delta] \equiv \mathrm{En}(\epsilon[\delta],\delta),$$

for all $\delta \in ]0,\delta_1[$.

In the following Proposition we compute the limit of $\mathrm{En}[\delta]$ as $\delta$ tends to 0.

**Proposition 4.61.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\epsilon_5$ be as in Theorem 4.49. Let $\delta_1 > 0$ be as in Definition 4.60. Then*

$$\lim_{\delta\to 0^+} \mathrm{En}[\delta] = \int_\Omega |\nabla \tilde{u}^i(x)|^2\, dx + \int_{\mathbb{R}^n\setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}^o(x)|^2\, dx,$$

*where $\tilde{u}^i$, $\tilde{u}^o$ are as in Definition 4.14.*

*Proof.* We follow step by step the proof of Propostion 2.140. Let $G^i$, $G^o$ be as in Theorem 4.49. Let $\delta \in ]0,\delta_1[$. By Remark 4.30 and Theorem 4.49, we have

$$\int_{\Omega(\epsilon[\delta],\delta)} |\nabla u^i_{(\epsilon[\delta],\delta)}(x)|^2\, dx + \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u^o_{(\epsilon[\delta],\delta)}(x)|^2\, dx = \delta^{n-2}(\epsilon[\delta])^n (G^i[\epsilon[\delta]] + G^o[\epsilon[\delta]])$$
$$= \delta^n (G^i[\delta^{\frac{2}{n}}] + G^o[\delta^{\frac{2}{n}}]).$$

On the other hand,

$$\lfloor (1/\delta)\rfloor^n \Big(\int_{\Omega(\epsilon[\delta],\delta)} |\nabla u^i_{(\epsilon[\delta],\delta)}(x)|^2\, dx + \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u^o_{(\epsilon[\delta],\delta)}(x)|^2\, dx\Big) \le \mathrm{En}[\delta]$$
$$\le \lceil (1/\delta)\rceil^n \Big(\int_{\Omega(\epsilon[\delta],\delta)} |\nabla u^i_{(\epsilon[\delta],\delta)}(x)|^2\, dx + \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u^o_{(\epsilon[\delta],\delta)}(x)|^2\, dx\Big),$$

and so

$$\lfloor (1/\delta)\rfloor^n \delta^n (G^i[\delta^{\frac{2}{n}}] + G^o[\delta^{\frac{2}{n}}]) \le \mathrm{En}[\delta] \le \lceil (1/\delta)\rceil^n \delta^n (G^i[\delta^{\frac{2}{n}}] + G^o[\delta^{\frac{2}{n}}]).$$

Thus, since

$$\lim_{\delta\to 0^+} \lfloor (1/\delta)\rfloor^n \delta^n = 1, \qquad \lim_{\delta\to 0^+} \lceil (1/\delta)\rceil^n \delta^n = 1,$$

we have

$$\lim_{\delta\to 0^+} \mathrm{En}[\delta] = (G^i[0] + G^o[0]).$$

Finally, by equalities (4.54) and (4.55), we easily conclude. □

In the following Proposition we represent the function $\mathrm{En}[\cdot]$ by means of real analytic functions.

**Proposition 4.62.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\epsilon_5$, $G^i$, $G^o$ be as in Theorem 4.49. Let $\delta_1 > 0$ be as in Definition 4.60. Then*

$$\mathrm{En}[(1/l)] = G^i[(1/l)^{\frac{2}{n}}] + G^o[(1/l)^{\frac{2}{n}}],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* It follows by the proof of Proposition 4.61. □

## 4.7 A variant of an alternative homogenization problem for the Laplace equation with a linear transmission condition in a periodically perforated domain

In this section we consider an homogenization problem for the Laplace equation with a linear transmission boundary condition in a periodically perforated domain.

### 4.7.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 4.2.1 and 4.3.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. For each pair $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we consider the following periodic linear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a(\epsilon, \delta), \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon, \delta), \\ u^i(x + \delta a_j) = u^i(x) & \forall x \in \operatorname{cl} \mathbb{S}_a(\epsilon, \delta), \quad \forall j \in \{1, \dots, n\}, \\ u^o(x + \delta a_j) = u^o(x) & \forall x \in \operatorname{cl} \mathbb{T}_a(\epsilon, \delta), \quad \forall j \in \{1, \dots, n\}, \\ u^o(x) = \phi u^i(x) & \forall x \in \partial\Omega(\epsilon, \delta), \\ \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u^o(x) = \gamma \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u^i(x) + g(\frac{1}{\epsilon\delta}(x - \delta w)) & \forall x \in \partial\Omega(\epsilon, \delta), \\ \int_{\partial\Omega(\epsilon,\delta)} u^i(x)\, d\sigma_x = 0. \end{cases} \tag{4.66}$$

In contrast to problem (4.60), we note that in the sixth equation of problem (4.66) there is not the factor $1/\delta$ in front of $g(\frac{1}{\epsilon\delta}(x - \delta w))$.

By virtue of Theorem 4.6, we can give the following definition.

**Definition 4.63.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. For each pair $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we denote by $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ the unique solution in $C^{m,\alpha}(\operatorname{cl} \mathbb{S}_a(\epsilon, \delta)) \times C^{m,\alpha}(\operatorname{cl} \mathbb{T}_a(\epsilon, \delta))$ of boundary value problem (4.66).

Our aim is to study the asymptotic behaviour of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ as $(\epsilon, \delta)$ tends to $(0, 0)$. In order to do so we introduce the following.

**Definition 4.64.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. For each $\epsilon \in ]0, \epsilon_1[$, we denote by $(u^i[\epsilon], u^o[\epsilon])$ the unique solution in $C^{m,\alpha}(\operatorname{cl} \mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\operatorname{cl} \mathbb{T}_a[\Omega_\epsilon])$ of the following periodic linear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a[\Omega_\epsilon], \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u^i(x + a_j) = u^i(x) & \forall x \in \operatorname{cl} \mathbb{S}_a[\Omega_\epsilon], \quad \forall j \in \{1, \dots, n\}, \\ u^o(x + a_j) = u^o(x) & \forall x \in \operatorname{cl} \mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1, \dots, n\}, \\ u^o(x) = \phi u^i(x) & \forall x \in \partial\Omega_\epsilon, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u^o(x) = \gamma \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u^i(x) + g(\frac{1}{\epsilon}(x - w)) & \forall x \in \partial\Omega_\epsilon, \\ \int_{\partial\Omega_\epsilon} u^i(x)\, d\sigma_x = 0. \end{cases} \tag{4.67}$$

*Remark* 4.65. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. For each pair $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we have

$$u^i_{(\epsilon,\delta)}(x) = \delta u^i[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \operatorname{cl} \mathbb{S}_a(\epsilon, \delta),$$

and

$$u^o_{(\epsilon,\delta)}(x) = \delta u^o[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \operatorname{cl} \mathbb{T}_a(\epsilon, \delta).$$

As a first step, we study the behaviour of $(u^i[\epsilon], u^o[\epsilon])$ as $\epsilon$ tends to 0.

**Proposition 4.66.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\epsilon_3$ be as in Proposition 4.17. Then there exist $\tilde{\epsilon} \in ]0, \epsilon_3[$ and a real analytic map $N^i$ of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ such that*

$$\|\mathbf{E}^i_{(\epsilon,1)}[\delta u^i[\epsilon]]\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon \|N^i[\epsilon]\|_{C^0(\partial\Omega)},$$
$$\|\mathbf{E}^o_{(\epsilon,1)}[\delta u^o[\epsilon]]\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\phi \|N^i[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,1)}[\delta u^i[\epsilon]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$
$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,1)}[\delta u^o[\epsilon]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* It is an immediate consequence of Proposition 4.55. □

### 4.7.2 Asymptotic behaviour of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

In the following Theorem we deduce by Proposition 4.66 the convergence of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ as $(\epsilon, \delta)$ tends to $(0, 0)$. Namely, we prove the following.

**Theorem 4.67.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\tilde{\epsilon}$, $N^i$ be as in Proposition 4.66. Then*

$$\|\mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}]\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon \|N^i[\epsilon]\|_{C^0(\partial\Omega)},$$
$$\|\mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}]\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\phi \|N^i[\epsilon]\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$
$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}]\|_{L^\infty(\mathbb{R}^n)} &= \delta \|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]]\|_{L^\infty(\mathbb{R}^n)} \\ &= \delta\epsilon \|N^i[\epsilon]\|_{C^0(\partial\Omega)},\end{aligned}$$

and that

$$\begin{aligned}\|\mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}]\|_{L^\infty(\mathbb{R}^n)} &= \delta \|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]]\|_{L^\infty(\mathbb{R}^n)} \\ &= \delta\epsilon\phi \|N^i[\epsilon]\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. □

Then we have the following Theorem, where we consider a functional associated to extensions of $u^i_{(\epsilon,\delta)}$ and of $u^o_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 4.68.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\epsilon_3$ be as in Proposition 4.17. Let $\epsilon_6$, $J^i$, $J^o$ be as in Theorem 4.51. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l} J^i[\epsilon], \tag{4.68}$$

*and*

$$\int_{\mathbb{R}^n} \mathbf{E}^o_{(\epsilon,r/l)}[u^o_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l} J^o[\epsilon], \tag{4.69}$$

*for all $\epsilon \in ]0, \epsilon_6[$, and for all $l \in \mathbb{N} \setminus \{0\}$.*

*Proof.* We follow the proof of Theorem 2.179. Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N}\setminus\{0\}$. Then, by the periodicity of $u^i_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\Omega_\epsilon} u^i_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\Omega_\epsilon} (r/l)u^i[\epsilon]\Big(\frac{l}{r}x\Big)\,dx \\ &= \frac{r^n}{l^n}\frac{r}{l}\int_{\Omega_\epsilon} u^i[\epsilon](t)\,dt \\ &= \frac{r^{n+1}}{l}\frac{1}{l^n}J^i[\epsilon].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l}J^i[\epsilon],$$

and the the validity of (4.68) follows. The proof of (4.69) is very similar and is accordingly omitted. □

### 4.7.3 Asymptotic behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

This Subsection is devoted to the study of the behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$. We give the following.

**Definition 4.69.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, we set

$$\mathrm{En}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{S}_a(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx + \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 4.70. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in (1.56), (1.57), (4.9), (4.10), (4.11), respectively. Let $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$. We have

$$\begin{aligned}\int_{\Omega(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\Omega(\epsilon,1)} |(\nabla u^i_{(\epsilon,\delta)})(\delta t)|^2\,dt \\ &= \delta^n \int_{\Omega_\epsilon} |\nabla u^i[\epsilon](t)|^2\,dt\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u^o_{(\epsilon,\delta)})(\delta t)|^2\,dt \\ &= \delta^n \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](t)|^2\,dt.\end{aligned}$$

In the following Proposition we represent the function $\mathrm{En}(\cdot,\cdot)$ by means of real analytic functions.

**Proposition 4.71.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $\phi$, $\gamma$, $g$ be as in* (1.56), (1.57), (4.9), (4.10), (4.11), *respectively. Let $\epsilon_5$, $G^i$, $G^o$ be as in Theorem 4.49. Then*

$$\mathrm{En}\Big(\epsilon,\frac{1}{l}\Big) = \epsilon^n G^i[\epsilon] + \epsilon^n G^o[\epsilon],$$

*for all $\epsilon \in ]0,\epsilon_5[$ and for all $l \in \mathbb{N}\setminus\{0\}$.*

*Proof.* Let $(\epsilon,\delta) \in ]0,\epsilon_5[ \times ]0,+\infty[$. By Remark 4.70 and Theorem 4.49, we have

$$\int_{\Omega(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx + \int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\,dx = \delta^n\epsilon^n G^i[\epsilon] + \delta^n\epsilon^n G^o[\epsilon] \tag{4.70}$$

where $G^i$, $G^o$ are as in Theorem 4.49. On the other hand, if $\epsilon \in ]0,\epsilon_5[$ and $l \in \mathbb{N}\setminus\{0\}$, then we have

$$\begin{aligned}\mathrm{En}\Big(\epsilon,\frac{1}{l}\Big) &= l^n\frac{1}{l^n}\Big\{\epsilon^n G^i[\epsilon] + \epsilon^n G^o[\epsilon]\Big\},\\ &= \epsilon^n G^i[\epsilon] + \epsilon^n G^o[\epsilon],\end{aligned}$$

and the conclusion easily follows. □

## 4.8 Asymptotic behaviour of the solutions of a nonlinear transmission problem for the Laplace equation in a periodically perforated domain

In this Section we study the asymptotic behaviour of the solutions of a nonlinear transmission problem for the Laplace equation in a periodically perforated domain with small holes.

### 4.8.1 Notation and preliminaries

We retain the notation introduced in Subsections 1.8.1, 4.2.1. However, we need to introduce also some other notation. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\bar{x} \in \mathrm{cl}\,A\setminus\{w\}$. Then there exists $\bar{\epsilon}_1 \in ]0,\epsilon_1[$ such that

$$\bar{x} \in \mathrm{cl}\,A \setminus \mathrm{cl}(w+\epsilon\Omega) \qquad \forall \epsilon \in ]-\bar{\epsilon}_1,\bar{\epsilon}_1[. \tag{4.71}$$

We shall consider also the following assumptions.

$$F \text{ is an increasing real analytic diffeomorphism of } \mathbb{R} \text{ onto itself,} \tag{4.72}$$

$$g \in C^{m-1,\alpha}(\partial\Omega), \qquad \int_{\partial\Omega} g\,d\sigma = 0, \tag{4.73}$$

$$\gamma \in ]0,+\infty[, \tag{4.74}$$

$$\bar{c} \in \mathbb{R}. \tag{4.75}$$

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. For each $\epsilon \in ]0,\bar{\epsilon}_1[$, we consider the following periodic nonlinear transmission problem for the Laplace equation.

$$\begin{cases}\Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a[\Omega_\epsilon],\\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon],\\ u^i(x+a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\},\\ u^o(x+a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\},\\ u^o(x) = F(u^i(x)) & \forall x \in \partial\Omega_\epsilon,\\ \frac{\partial}{\partial\nu_{\Omega_\epsilon}}u^o(x) = \gamma\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u^i(x) + g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon,\\ u^o(\bar{x}) = \bar{c}.\end{cases} \tag{4.76}$$

We transform (4.76) into a system of integral equations by means of the following.

**Proposition 4.72.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75)*, respectively. Let $\epsilon \in ]0,\bar{\epsilon}_1[$. Then the map of the set of pairs $(\omega,\mu) \in (C^{m,\alpha}(\partial\Omega_\epsilon))^2$ that solve the following integral equations*

$$F(w_a^+[\partial\Omega_\epsilon,\omega](x) + F^{(-1)}(\bar{c})) = w_a^-[\partial\Omega_\epsilon,\mu](x) + \bar{c} - w_a^-[\partial\Omega_\epsilon,\mu](\bar{x}) \qquad \forall x \in \partial\Omega_\epsilon \tag{4.77}$$

$$\begin{aligned}g(\frac{1}{\epsilon}(x-w)) + \gamma\int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}\frac{\partial}{\partial\nu_{\Omega_\epsilon}(y)}(S_n^a(x-y))\omega(y)\,d\sigma_y&\\ = \int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}\frac{\partial}{\partial\nu_{\Omega_\epsilon}(y)}(S_n^a(x-y))\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega_\epsilon,&\end{aligned} \tag{4.78}$$

$$\int_{\partial\Omega_\epsilon} \mu\, d\sigma = 0, \tag{4.79}$$

*to the set of pairs* $(u^i, u^o)$ *of* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ *which solve problem* (4.76), *which takes* $(\omega,\mu)$ *to the pair of functions*

$$\big(w_a^+[\partial\Omega_\epsilon,\omega] + F^{(-1)}(\bar{c}), w_a^-[\partial\Omega_\epsilon,\mu] + \bar{c} - w_a^-[\partial\Omega_\epsilon,\mu](\bar{x})\big) \tag{4.80}$$

*is a bijection.*

*Proof.* We first assume that the pair $(\omega,\mu)$ satisfies (4.77)-(4.79). Then, by Theorem 1.13, it is easy to verify that the pair of functions

$$\big(w_a^+[\partial\Omega_\epsilon,\omega] + F^{(-1)}(\bar{c}), w_a^-[\partial\Omega_\epsilon,\mu] + \bar{c} - w_a^-[\partial\Omega_\epsilon,\mu](\bar{x})\big)$$

is in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ and solves problem (4.76). Conversely, assume now that $(u^i, u^o) \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ is a solution of problem (4.76). Then, by Proposition 2.20, there exists a unique $\omega \in C^{m,\alpha}(\partial\Omega_\epsilon)$ such that

$$u^i - F^{(-1)}(\bar{c}) = w_a^+[\partial\Omega_\epsilon,\omega] \qquad \text{in } \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon].$$

Analogously, by Proposition 2.21, there exists a unique pair $(\mu,\tau)$ in $C^{m,\alpha}(\partial\Omega_\epsilon) \times \mathbb{R}$, such that $\int_{\partial\Omega_\epsilon} \mu\, d\sigma = 0$ and

$$u^o = w_a^-[\partial\Omega_\epsilon,\mu] + \tau \qquad \text{in } \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$

In particular, by $u^o(\bar{x}) = \bar{c}$, we must have

$$\tau = \bar{c} - w_a^-[\partial\Omega_\epsilon,\mu](\bar{x}).$$

Finally, since $(u^i, u^o)$ solves (4.76), we immediately obtain the validity of (4.77)-(4.79). □

As we have seen, we can convert problem (4.76) into a system of integral equations in the unknown $(\omega,\mu)$. In the following Theorem we introduce a proper change of the functional variables $(\omega,\mu)$.

**Theorem 4.73.** *Let* $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. *Let* $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ *be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let* $\epsilon \in ]0,\bar{\epsilon}_1[$. *Then the map* $(u^i[\epsilon,\cdot,\cdot], u^o[\epsilon,\cdot,\cdot])$ *of the set of pairs* $(\psi,\theta) \in (C^{m,\alpha}(\partial\Omega))^2$ *that solve the following integral equations*

$$\begin{aligned}
&F'(F^{(-1)}(\bar{c}))\Big(w^+[\partial\Omega,\psi](t) - \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\, d\sigma_s\Big)\\
&\quad + \epsilon\Big(w^+[\partial\Omega,\psi](t) - \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\, d\sigma_s\Big)^2\\
&\quad \times \int_0^1 (1-\beta)F''\Big(F^{(-1)}(\bar{c}) + \beta\epsilon\big(w^+[\partial\Omega,\psi](t) - \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\, d\sigma_s\big)\Big)d\beta\\
&\quad - w^-[\partial\Omega,\theta](t) + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s\\
&\quad - \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(\bar{x}-w-\epsilon s)\theta(s)\, d\sigma_s = 0 \qquad \forall t \in \partial\Omega,
\end{aligned} \tag{4.81}$$

$$\begin{aligned}
&g(t) - \gamma\int_{\partial\Omega} \nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\psi(s)\, d\sigma_s - \epsilon^n\gamma\int_{\partial\Omega} \nu_\Omega(t)D^2R_n^a(\epsilon(t-s))\nu_\Omega(s)\psi(s)\, d\sigma_s\\
&\quad + \int_{\partial\Omega} \nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\theta(s)\, d\sigma_s + \epsilon^n\int_{\partial\Omega} \nu_\Omega(t)D^2R_n^a(\epsilon(t-s))\nu_\Omega(s)\theta(s)\, d\sigma_s = 0 \quad \forall t \in \partial\Omega,
\end{aligned} \tag{4.82}$$

$$\int_{\partial\Omega} \theta\, d\sigma = 0, \tag{4.83}$$

*to the set of pairs* $(u^i, u^o)$ *of* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ *which solve problem* (4.76), *which takes* $(\psi,\theta)$ *to the pair of functions*

$$(u^i[\epsilon,\psi,\theta] \equiv w_a^+[\partial\Omega_\epsilon,\omega] + F^{(-1)}(\bar{c}), u^o[\epsilon,\psi,\theta] \equiv w_a^-[\partial\Omega_\epsilon,\mu] + \bar{c} - w_a^-[\partial\Omega_\epsilon,\mu](\bar{x})), \tag{4.84}$$

*where*

$$\mu(x) \equiv \epsilon\theta(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{4.85}$$

$$\omega(x) \equiv \epsilon\psi(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{4.86}$$

*is a bijection.*

*Proof.* Assume that the pair $(u^i, u^o)$ in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ solves problem (4.76). Then, by Proposition 4.72, there exists a unique pair $(\omega, \mu)$ in $(C^{m,\alpha}(\partial\Omega_\epsilon))^2$, which solves (4.77)-(4.79) and such that $(u^i, u^o)$ equals the pair of functions defined in the right-hand side of (4.84). The pair $(\psi, \theta)$ defined by (4.85),(4.86) belongs to $(C^{m,\alpha}(\partial\Omega))^2$. By (4.77), (4.78), (4.79) the pair $(\psi, \theta)$ solves equations (4.82),(4.83) together with the following equation:

$$F\Big(\epsilon\big(w^+[\partial\Omega,\psi](t) - \epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s\big) + F^{(-1)}(\bar{c})\Big)$$
$$= \epsilon\big(w^-[\partial\Omega,\theta](t) - \epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\big) + \bar{c} + \epsilon^n\int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n^a(\bar{x}-w-\epsilon s)\theta(s)\,d\sigma_s$$
$$\forall t \in \partial\Omega. \tag{4.87}$$

We now show that equation (4.87) implies the validity of (4.81). By Taylor Formula, we have

$$F(x + F^{(-1)}(\bar{c})) = \bar{c} + F'(F^{(-1)}(\bar{c}))x + x^2\int_0^1 (1-\beta)F''(F^{(-1)}(\bar{c}) + \beta x)d\beta \qquad \forall x \in \mathbb{R}.$$

Then, by dividing both sides of (4.87) by $\epsilon$, we can rewrite (4.87) as (4.81). Conversely, by reading backward the above argument, one can easily show that if $(\psi, \theta)$ solves (4.81)-(4.83), then the pair $(\omega, \mu)$, with $\omega$, $\mu$ delivered by (4.85),(4.86), satisfies system (4.77)-(4.79). Accordingly, the pair of functions of (4.84) satisfies problem (4.76) by Proposition 4.72. □

Hence we are reduced to analyse system (4.81)-(4.83). As a first step in the analysis of system (4.81)-(4.83), we note that for $\epsilon = 0$ one obtains a system which we address to as the *limiting system* and which has the following form

$$F'(F^{(-1)}(\bar{c}))w^+[\partial\Omega,\psi](t) - w^-[\partial\Omega,\theta](t) = 0 \quad \forall t \in \partial\Omega, \tag{4.88}$$

$$g(t) - \gamma\int_{\partial\Omega}\nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\psi(s)\,d\sigma_s + \int_{\partial\Omega}\nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\theta(s)\,d\sigma_s = 0 \quad \forall t \in \partial\Omega, \tag{4.89}$$

$$\int_{\partial\Omega}\theta\,d\sigma = 0. \tag{4.90}$$

In order to analyse the limiting system, we need the following technical statement from Lanza [78, Theorem 5.2].

**Theorem 4.74.** *Let $m \in \mathbb{N}\setminus\{0\}$ $\alpha \in ]0,1[$. Let $\mathbb{I}$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$, such that $\mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{I}$ is connected and $0 \in \mathbb{I}$. Then the following statements hold.*

*(i) The operator $\frac{1}{2}I - lw[\partial\mathbb{I},\cdot]$ is a linear homeomorphism of $C^{m,\alpha}(\partial\mathbb{I})$ onto itself for all $l \in [-1,1[$.*

*(ii) Let $\phi$, $\gamma \in \mathbb{R}^n\setminus\{0\}$, $\phi\gamma > 0$. Let $\eta \in C^{m-1,\alpha}(\partial\mathbb{I})$, $\int_{\partial\mathbb{I}}\eta\,d\sigma = 1$. If $(f,\Gamma,a) \in C^{m,\alpha}(\partial\mathbb{I}) \times C^{m-1,\alpha}(\partial\mathbb{I}) \times \mathbb{R}$, then the system*

$$\phi w^+[\partial\mathbb{I},\psi] + w^-[\partial\mathbb{I},\theta] = f \qquad \text{on } \partial\mathbb{I},$$
$$-\gamma\int_{\partial\mathbb{I}}\nu_{\mathbb{I}}(t)D^2S_n(t-s)\nu_{\mathbb{I}}(s)\psi(s)\,d\sigma_s$$
$$-\int_{\partial\mathbb{I}}\nu_{\mathbb{I}}(t)D^2S_n(t-s)\nu_{\mathbb{I}}(s)\theta(s)\,d\sigma_s + \int_{\partial\mathbb{I}}\Gamma\,d\sigma\eta(t) = \Gamma(t) \qquad \forall t \in \partial\mathbb{I},$$
$$\int_{\partial\mathbb{I}}\theta\,d\sigma = a,$$

*has one and only one solution* $(\psi,\theta)\in(C^{m,\alpha}(\partial\mathbb{I}))^2$.

*Proof.* See Lanza [78, Theorem 5.2]. □

Then we have the following theorem, which shows the unique solvability of the limiting system, and its link with a boundary value problem which we shall address to as the *limiting boundary value problem.*

**Theorem 4.75.** *Let* $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. *Let* $\Omega$, $F$, $g$, $\gamma$, $\bar{c}$ *be as in* (1.56), (4.72), (4.73), (4.74), (4.75), *respectively. Then the following statements hold.*

*(i) The* limiting system (4.88)-(4.90) *has one and only one solution in* $(C^{m,\alpha}(\partial\Omega))^2$, *which we denote by* $(\tilde{\psi},\tilde{\theta})$.

*(ii) The limiting boundary value problem*

$$\begin{cases}\Delta u^i(x)=0 & \forall x\in\Omega,\\ \Delta u^o(x)=0 & \forall x\in\mathbb{R}^n\setminus\mathrm{cl}\,\Omega,\\ u^o(x)=F'(F^{(-1)}(\bar{c}))u^i(x) & \forall x\in\partial\Omega,\\ \frac{\partial}{\partial\nu_\Omega}u^o(x)=\gamma\frac{\partial}{\partial\nu_\Omega}u^i(x)+g(x) & \forall x\in\partial\Omega,\\ \lim_{x\to\infty}u^o(x)=0,\end{cases}\tag{4.91}$$

*has one and only one solution* $(\tilde{u}^i,\tilde{u}^o)$ *in* $C^{m,\alpha}(\mathrm{cl}\,\Omega)\times C^{m,\alpha}(\mathbb{R}^n\setminus\Omega)$, *and the following formulas hold:*

$$\tilde{u}^i\equiv w^+[\partial\Omega,\tilde{\psi}]\qquad in\ \mathrm{cl}\,\Omega,\tag{4.92}$$

$$\tilde{u}^o\equiv w^-[\partial\Omega,\tilde{\theta}]\qquad in\ \mathbb{R}^n\setminus\Omega.\tag{4.93}$$

*Proof.* The statement in *(i)* is an immediate consequence of Theorem 4.74 (we recall that $\int_{\partial\Omega}g\,d\sigma=0$). We now consider *(ii)*. By Theorem B.1, it is immediate to see that the functions $\tilde{u}^i$, $\tilde{u}^o$ delivered by the right-hand side of (4.92), (4.93), belong to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, $C^{m,\alpha}(\mathbb{R}^n\setminus\Omega)$, respectively and solve problem (4.91). For the uniqueness of the solution of problem (4.91) we refer to the proof of Lanza [78, Theorem 5.3]. □

We are now ready to analyse equations (4.81)-(4.83) around the degenerate case $\epsilon=0$.

**Theorem 4.76.** *Let* $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. *Let* $w\in A$. *Let* $\bar{x}\in\mathrm{cl}\,A\setminus\{w\}$. *Let* $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ *be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let* $\mathcal{U}_0^{m-1,\alpha}$ *be as in* (1.64). *Let* $\Lambda\equiv(\Lambda_j)_{j=1,2,3}$ *be the map of* $]-\bar{\epsilon}_1,\bar{\epsilon}_1[\times(C^{m,\alpha}(\partial\Omega))^2$ *to* $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$, *defined by*

$$\begin{aligned}\Lambda_1[\epsilon,\psi,\theta](t)\equiv{}&F'(F^{(-1)}(\bar{c}))\Big(w^+[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s\Big)\\&+\epsilon\Big(w^+[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s\Big)^2\\&\times\int_0^1(1-\beta)F''\Big(F^{(-1)}(\bar{c})+\beta\epsilon\big(w^+[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s\big)\Big)d\beta\\&-w^-[\partial\Omega,\theta](t)+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\\&-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n^a(\bar{x}-w-\epsilon s)\theta(s)\,d\sigma_s\quad\forall t\in\partial\Omega,\end{aligned}\tag{4.94}$$

$$\begin{aligned}&\Lambda_2[\epsilon,\psi,\theta](t)\\&\equiv g(t)-\gamma\int_{\partial\Omega}\nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\psi(s)\,d\sigma_s-\epsilon^n\gamma\int_{\partial\Omega}\nu_\Omega(t)D^2R_n^a(\epsilon(t-s))\nu_\Omega(s)\psi(s)\,d\sigma_s\\&+\int_{\partial\Omega}\nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\theta(s)\,d\sigma_s+\epsilon^n\int_{\partial\Omega}\nu_\Omega(t)D^2R_n^a(\epsilon(t-s))\nu_\Omega(s)\theta(s)\,d\sigma_s\quad\forall t\in\partial\Omega,\end{aligned}\tag{4.95}$$

$$\Lambda_3[\epsilon,\psi,\theta]\equiv\int_{\partial\Omega}\theta\,d\sigma,\tag{4.96}$$

*for all* $(\epsilon,\psi,\theta)\in]-\bar{\epsilon}_1,\bar{\epsilon}_1[\times(C^{m,\alpha}(\partial\Omega))^2$. *Then the following statements hold.*

*(i)* *Equation* $\Lambda[0,\psi,\theta]=0$ *is equivalent to the limiting system* (4.88)-(4.90) *and has one and only one solution* $(\tilde{\psi},\tilde{\theta})$ *(cf. Theorem 4.75.)*

*(ii)* *If* $\epsilon\in]0,\bar{\epsilon}_1[$, *then equation* $\Lambda[\epsilon,\psi,\theta]=0$ *is equivalent to system* (4.81)-(4.83) *for* $(\psi,\theta)$.

*(iii)* *There exists* $\epsilon_2\in]0,\bar{\epsilon}_1]$ *such that* $\Lambda$ *is a real analytic map of* $]-\epsilon_2,\epsilon_2[\times(C^{m,\alpha}(\partial\Omega))^2$ *to* $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$. *The differential* $\partial_{(\psi,\theta)}\Lambda[0,\tilde{\psi},\tilde{\theta}]$ *of* $\Lambda$ *at* $(0,\tilde{\psi},\tilde{\theta})$ *is a linear homeomorphism of* $(C^{m,\alpha}(\partial\Omega))^2$ *to* $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$.

*(iv)* *There exist* $\epsilon_3\in]0,\epsilon_2]$ *and an open neighbourhood* $\tilde{\mathcal{U}}$ *of* $(\tilde{\psi},\tilde{\theta})$ *in* $(C^{m,\alpha}(\partial\Omega))^2$ *and a real analytic map* $(\Psi[\cdot],\Theta[\cdot])$ *of* $]-\epsilon_3,\epsilon_3[$ *to* $\tilde{\mathcal{U}}$, *such that the set of zeros of the map* $\Lambda$ *in* $]-\epsilon_3,\epsilon_3[\times\tilde{\mathcal{U}}$ *coincides with the graph of* $(\Psi[\cdot],\Theta[\cdot])$. *In particular,* $(\Psi[0],\Theta[0])=(\tilde{\psi},\tilde{\theta})$.

*Proof.* First of all we want to prove that

$$\int_{\partial\Omega}\Lambda_2[\epsilon,\psi,\theta]\,d\sigma=0,\tag{4.97}$$

for all $(\epsilon,\psi,\theta)\in]-\bar{\epsilon}_1,\bar{\epsilon}_1[\times(C^{m,\alpha}(\partial\Omega))^2$. If $\epsilon=0$, by $\int_{\partial\Omega}g\,d\sigma=0$ and

$$\int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega}w^+[\partial\Omega,\psi]\,d\sigma=0,$$

$$\int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega}w^-[\partial\Omega,\theta]\,d\sigma=0,$$

we immediately obtain (4.97). If $\epsilon\neq0$, we need to observe also that the functions

$$t\mapsto\int_{\partial\Omega}DR_n^a(\epsilon(t-s))\cdot\nu_\Omega(s)\psi(s)\,d\sigma_s$$

and

$$t\mapsto\int_{\partial\Omega}DR_n^a(\epsilon(t-s))\cdot\nu_\Omega(s)\theta(s)\,d\sigma_s$$

of $\mathrm{cl}\,\Omega$ to $\mathbb{R}$ are harmonic in $\Omega$. Then, by the Divergence Theorem, we have

$$\int_{\partial\Omega}\int_{\partial\Omega}\nu_\Omega(t)D^2R_n^a(\epsilon(t-s))\nu_\Omega(s)\psi(s)\,d\sigma_s\,d\sigma_t=0$$

and

$$\int_{\partial\Omega}\int_{\partial\Omega}\nu_\Omega(t)D^2R_n^a(\epsilon(t-s))\nu_\Omega(s)\theta(s)\,d\sigma_s\,d\sigma_t=0.$$

Thus, by the above argument for the case $\epsilon=0$, we easily obtain (4.97). The statements in $(i)$ and $(ii)$ are obvious. By the continuity of the pointwise product in Schauder spaces, and by the continuity of the linear maps $w^\pm[\partial\Omega,\cdot]$ in $C^{m,\alpha}(\partial\Omega)$, by the real analyticty of $R_n^a$ and its partial derivatives in a neighbourhood of 0 and by known results on composition operators, we have that the maps $\Lambda_2$, $\Lambda_3$ and the first, third, fourth summands in the right-hand side of the definition of $\Lambda_1$ are real analytic maps in $]-\epsilon_2,\epsilon_2[\times(C^{m,\alpha}(\partial\Omega))^2$, for $\epsilon_2\in]0,\bar{\epsilon_1}]$ small enough (cf. Proposition 1.22 $(i)$, $(ii)$.) Since $S_n^a$ is real analytic in $\mathbb{R}^n\setminus Z_n^a$, by possibly taking a smaller $\epsilon_2$, the map

$$(\epsilon,\theta)\mapsto-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n^a(\bar{x}-w-\epsilon s)\theta(s)\,d\sigma_s$$

of $]-\epsilon_2,\epsilon_2[\times C^{m,\alpha}(\partial\Omega)$ to $\mathbb{R}$ is real analytic (Proposition 1.22 $(iii)$.) By Theorems B.1 and C.4, the map of $]-\epsilon_2,\epsilon_2[\times C^{m,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$ which takes $(\epsilon,\psi)$ to the function of the variable $t\in\partial\Omega$ defined by

$$w^+[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s\qquad\forall t\in\partial\Omega,$$

is real analytic. Hence, the map of $]-\epsilon_2,\epsilon_2[\times C^{m,\alpha}(\partial\Omega)$ to $C^{m,\alpha}([0,1]\times\partial\Omega)$ which takes $(\epsilon,\psi)$ to the function

$$F^{(-1)}(\bar{c})+\beta\epsilon(w^+[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s)$$

of the variables $(\beta, t)$ is real analytic. Since $F''$ is real analytic, known results on composition operators show that the map of $C^{m,\alpha}([0,1]\times\partial\Omega)$ to itself which takes a function $h(\cdot,\cdot)$ to the composite function $F''\circ h(\cdot,\cdot)$ is real analytic (cf. Böhme and Tomi [15, p. 10], Henry [60, p. 29], Valent [137, Thm. 5.2, p. 44].) Finally, the map which takes a function $h$ of $C^{m,\alpha}([0,1]\times\partial\Omega)$ to $\int_0^1(1-\beta)h(\beta,\cdot)d\beta$ in $C^{m,\alpha}(\partial\Omega)$ is linear and continuous, and thus real analytic. Hence, by the continuity of the pointwise product in Schauder spaces, we can easily conclude that the second summand in the definition of $\Lambda_1$ depends real analytically upon $(\epsilon,\psi)$. By standard calculus in Banach space, the differential of $\Lambda$ at $(0,\tilde{\psi},\tilde{\theta})$ with respect to variables $(\psi,\theta)$ is delivered by the following formulas:

$$\partial_{(\psi,\theta)}\Lambda_1[0,\tilde{\psi},\tilde{\theta}](\bar{\psi},\bar{\theta})(t) = F'(F^{(-1)}(\bar{c}))w^+[\partial\Omega,\bar{\psi}](t) - w^-[\partial\Omega,\bar{\theta}](t) \qquad \forall t\in\partial\Omega,$$

$$\begin{aligned}\partial_{(\psi,\theta)}\Lambda_2[0,\tilde{\psi},\tilde{\theta}](\bar{\psi},\bar{\theta})(t) = &-\gamma\int_{\partial\Omega}\nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\bar{\psi}(s)\,d\sigma_s\\ &+\int_{\partial\Omega}\nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\bar{\theta}(s)\,d\sigma_s \qquad \forall t\in\partial\Omega,\end{aligned}$$

$$\partial_{(\psi,\theta)}\Lambda_3[0,\tilde{\psi},\tilde{\theta}](\bar{\psi},\bar{\theta}) = \int_{\partial\Omega}\bar{\theta}(s)\,d\sigma_s,$$

for all $(\bar{\psi},\bar{\theta})\in(C^{m,\alpha}(\partial\Omega))^2$. We now show that the above differential is a linear homeomorphism. By the Open Mapping Theorem, it suffices to show that it is a bijection of $(C^{m,\alpha}(\partial\Omega))^2$ to $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$. Let $(\bar{f},\bar{g},\bar{a})\in C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$. We must show that there exists a unique pair $(\bar{\psi},\bar{\theta})\in (C^{m,\alpha}(\partial\Omega))^2$ such that

$$\partial_{(\psi,\theta)}\Lambda[0,\tilde{\psi},\tilde{\theta}](\bar{\psi},\bar{\theta}) = (\bar{f},\bar{g},\bar{a}). \tag{4.98}$$

By Theorem 4.74, there exists a unique pair $(\bar{\psi},\bar{\theta})\in(C^{m,\alpha}(\partial\Omega))^2$ such that (4.98) holds. Thus the proof of statement (*iii*) is complete. Statement (*iv*) is an immediate consequence of statement (*iii*) and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

We are now in the position to introduce the following.

**Definition 4.77.** Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\bar{x}\in\operatorname{cl}A\setminus\{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. Let $u^i[\cdot,\cdot,\cdot]$ and $u^o[\cdot,\cdot,\cdot]$ be as in Theorem 4.73. If $\epsilon\in\,]0,\epsilon_3[$, we set

$$\begin{aligned}u^i[\epsilon](t) &\equiv u^i[\epsilon,\Psi[\epsilon],\Theta[\epsilon]](t) \qquad \forall t\in\operatorname{cl}\mathbb{S}_a[\Omega_\epsilon],\\ u^o[\epsilon](t) &\equiv u^o[\epsilon,\Psi[\epsilon],\Theta[\epsilon]](t) \qquad \forall t\in\operatorname{cl}\mathbb{T}_a[\Omega_\epsilon],\end{aligned}$$

where $\epsilon_3$, $\Psi$, $\Theta$ are as in Theorem 4.76 (*iv*).

### 4.8.2 A functional analytic representation Theorem for the family of functions $\{(u^i[\epsilon],u^o[\epsilon])\}_{\epsilon\in]0,\epsilon_3[}$

In this Subsection, we show that $\{(u^i[\epsilon](\cdot),u^o[\epsilon](\cdot))\}_{\epsilon\in]0,\epsilon_3[}$ can be continued real analytically for negative values of $\epsilon$.

We have the following.

**Theorem 4.78.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\bar{x}\in\operatorname{cl}A\setminus\{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $\epsilon_3$ be as in Theorem 4.76* (*iv*). *Then the following statements hold.*

(*i*) *Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\operatorname{cl}V\cap\mathbb{S}_a[\Omega_0]=\emptyset$. Then there exist $\epsilon_4\in\,]0,\epsilon_3]$ and a real analytic operator $U_1^o$ of $]-\epsilon_4,\epsilon_4[$ to the space $C_h^0(\operatorname{cl}V)$ such that the following conditions hold.*

(*j*) $\operatorname{cl}V\subseteq\mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon\in\,]-\epsilon_4,\epsilon_4[$.

(*jj*)
$$u^o[\epsilon](x) = \epsilon^n U_1^o[\epsilon](x) + \bar{c} \qquad \forall x\in\operatorname{cl}V,$$
*for all* $\epsilon\in\,]0,\epsilon_4[$.

*(ii)* *Let $\bar{V}$ be a bounded open subset of $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$. Then there exist $\bar{\epsilon}_4 \in ]0, \epsilon_3]$ and a real analytic operator $\bar{U}_1^o$ of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ to the space $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$ such that the following conditions hold.*

*(j')* $w + \epsilon\,\mathrm{cl}\,\bar{V} \subseteq \mathrm{cl}\,\mathbb{P}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \setminus \{0\}$.

*(jj')*

$$u^o[\epsilon](w + \epsilon t) = \epsilon \bar{U}_1^o[\epsilon](t) + \bar{c} \qquad \forall t \in \mathrm{cl}\,\bar{V},$$

*for all* $\epsilon \in ]0, \bar{\epsilon}_4[$. *Moreover,* $\bar{U}_1^o[0]$ *equals the restriction of* $\tilde{u}^o$ *to* $\mathrm{cl}\,\bar{V}$.

*(iii)* *There exist $\epsilon_4' \in ]0, \epsilon_3]$ and a real analytic operator $U_1^i$ of $]-\epsilon_4', \epsilon_4'[$ to the space $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ such that*

$$u^i[\epsilon](w + \epsilon t) = \epsilon U_1^i[\epsilon](t) + F^{(-1)}(\bar{c}) \qquad \forall t \in \mathrm{cl}\,\Omega,$$

*for all* $\epsilon \in ]0, \epsilon_4'[$. *Moreover,* $U_1^i[0]$ *equals* $\tilde{u}^i$ *on* $\mathrm{cl}\,\Omega$.

*Proof.* Let $\Theta[\cdot]$, $\Psi[\cdot]$ be as in Theorem 4.76. We first prove statement $(i)$. Clearly, by taking $\epsilon_4 \in ]0, \epsilon_3]$ small enough, we can assume that $(j)$ holds. Consider $(jj)$. Let $\epsilon \in ]0, \epsilon_4[$. We have

$$\begin{aligned} u^o[\epsilon](x) =& \epsilon^n \Big( -\int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n^a(x - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \\ &+ \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \Big) + \bar{c}, \qquad \forall x \in \mathrm{cl}\,V. \end{aligned}$$

Thus, it is natural to set

$$\begin{aligned} U_1^o[\epsilon](x) \equiv& -\int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n^a(x - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \\ &+ \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\,V, \end{aligned}$$

for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$. Following the proof of Theorem 2.40, one can easily show that, by possibly taking a smaller $\epsilon_4$, the map $U_1^o$ of $]-\epsilon_4, \epsilon_4[$ to $C_h^0(\mathrm{cl}\,V)$ is real analytic and that the equality in $(jj)$ holds (cf. Proposition 1.22 $(iii)$ and Proposition 1.24 $(i)$.) We now prove $(ii)$. Clearly, by taking $\bar{\epsilon}_4 \in ]0, \epsilon_3]$ small enough, we can assume that $(j')$ holds. Consider $(jj')$. Let $\epsilon \in ]0, \bar{\epsilon}_4[$. We have

$$\begin{aligned} u^o[\epsilon](w + \epsilon t) =& \epsilon w^-[\partial\Omega, \Theta[\epsilon]](t) - \epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DR_n^a(\epsilon(t - s))\Theta[\epsilon](s)\, d\sigma_s \\ &+ \epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s + \bar{c}, \qquad \forall t \in \mathrm{cl}\,\bar{V}. \end{aligned}$$

Thus, it is natural to set

$$\begin{aligned} \bar{U}_1^o[\epsilon](t) \equiv& w^-[\partial\Omega, \Theta[\epsilon]](t) - \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DR_n^a(\epsilon(t - s))\Theta[\epsilon](s)\, d\sigma_s \\ &+ \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s, \qquad \forall t \in \mathrm{cl}\,\bar{V}. \end{aligned}$$

for all $\epsilon \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[$. By Proposition 1.22 $(iii)$ and Proposition 1.24 $(ii)$, we can easily conclude that $\bar{U}_1^o$ is a real analytic map of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ to $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$. Moreover, by the definition of $\bar{U}_1^o$, we have

$$\bar{U}_1^o[0](t) = w^-[\partial\Omega, \tilde{\theta}](t) = \tilde{u}^o(t) \qquad \forall t \in \mathrm{cl}\,\bar{V}.$$

Finally, we prove $(iii)$. Let $\epsilon \in ]0, \epsilon_3[$. We have

$$u^i[\epsilon](w + \epsilon t) = \epsilon w^+[\partial\Omega, \Psi[\epsilon]](t) - \epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DR_n^a(\epsilon(t - s))\Psi[\epsilon](s)\, d\sigma_s + F^{(-1)}(\bar{c}), \qquad \forall t \in \mathrm{cl}\,\Omega.$$

Thus, it is natural to set

$$U_1^i[\epsilon](t) \equiv w^+[\partial\Omega, \Psi[\epsilon]](t) - \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s) \cdot DR_n^a(\epsilon(t - s))\Psi[\epsilon](s)\, d\sigma_s, \qquad \forall t \in \mathrm{cl}\,\Omega.$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. By arguing as above, by exploiting Proposition 1.24 (*iii*), one can easily prove that there exists $\epsilon'_4 \in ]0, \epsilon_3]$, such that $U_1^i$ is a real analytic map of $]-\epsilon'_4, \epsilon'_4[$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$. Clearly, the equality in (*iii*) holds. Moreover, by the definition of $U_1^i$, we have

$$U_1^i[0](t) = w^+[\partial\Omega, \tilde{\psi}](t) = \tilde{u}^i(t) \qquad \forall t \in \mathrm{cl}\,\Omega.$$

□

*Remark* 4.79. We note that the right-hand side of the equalities in (*jj*), (*jj'*) and (*iii*) of Theorem 4.78 can be continued real analytically in a whole neighbourhood of 0. Moreover, if $V$ is a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V \cap \mathbb{S}_a[\Omega_0] = \emptyset$, then

$$\lim_{\epsilon \to 0^+} u^o[\epsilon] = \bar{c} \qquad \text{uniformly in } \mathrm{cl}\,V.$$

### 4.8.3 A real analytic continuation Theorem for the energy integral

As done in Theorem 4.78 for $(u^i[\cdot], u^o[\cdot])$, we can now prove a real analytic continuation Theorem for the energy integral. Namely, we prove the following.

**Theorem 4.80.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $\epsilon_3$ be as in Theorem 4.76* (*iv*). *Then there exist $\epsilon_5 \in ]0, \epsilon_3]$ and two real analytic operators $G^i$, $G^o$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{R}$, such that*

$$\int_{\Omega_\epsilon} |\nabla u^i[\epsilon](x)|^2 \, dx = \epsilon^n G^i[\epsilon], \tag{4.99}$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](x)|^2 \, dx = \epsilon^n G^o[\epsilon], \tag{4.100}$$

*for all $\epsilon \in ]0, \epsilon_5[$. Moreover,*

$$G^i[0] = \int_\Omega |\nabla \tilde{u}^i(x)|^2 \, dx, \tag{4.101}$$

$$G^o[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}^o(x)|^2 \, dx. \tag{4.102}$$

*Proof.* Let $\Theta[\cdot]$, $\Psi[\cdot]$ be as in Theorem 4.76. We denote by id the identity map in $\mathbb{R}^n$. We set

$$v^i[\epsilon](x) \equiv u^i[\epsilon](x) - F^{(-1)}(\bar{c}) \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon],$$

and

$$v^o[\epsilon](x) \equiv u^o[\epsilon](x) - \Big(\bar{c} + \epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon](s) \, d\sigma_s\Big) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],$$

for all $\epsilon \in ]0, \epsilon_3[$. Let $\epsilon \in ]0, \epsilon_3[$. We have

$$\begin{aligned}\int_{\Omega_\epsilon} |\nabla u^i[\epsilon](x)|^2 \, dx &= \int_{\Omega_\epsilon} |\nabla v^i[\epsilon](x)|^2 \, dx \\ &= \epsilon^{n-1} \int_{\partial\Omega} \Big(\frac{\partial}{\partial \nu_{\Omega_\epsilon}} v^i[\epsilon]\Big) \circ (w + \epsilon\,\mathrm{id})(t) v^i[\epsilon] \circ (w + \epsilon\,\mathrm{id})(t) \, d\sigma_t,\end{aligned}$$

for all $\epsilon \in ]0, \epsilon_3[$. Let $\epsilon'_4$, $U_1^i[\cdot]$ be as in Theorem 4.78 (*iii*). Let $\epsilon \in ]0, \epsilon'_4[$. We have

$$v^i[\epsilon] \circ (w + \epsilon\,\mathrm{id})(t) = \epsilon U_1^i[\epsilon](t) \qquad \forall t \in \mathrm{cl}\,\Omega,$$

and accordingly

$$D[v^i[\epsilon] \circ (w + \epsilon\,\mathrm{id})](t) = \epsilon D[U_1^i[\epsilon]](t) \qquad \forall t \in \mathrm{cl}\,\Omega.$$

Also,

$$\begin{aligned}\epsilon^{n-1}\int_{\partial\Omega}\Big(\frac{\partial}{\partial\nu_{\Omega_\epsilon}}v^i[\epsilon]\Big)\circ(w+\epsilon\,\mathrm{id})(t)v^i[\epsilon]\circ(w+\epsilon\,\mathrm{id})(t)\,d\sigma_t\\ =\epsilon^{n-2}\int_{\partial\Omega}D[v^i[\epsilon]\circ(w+\epsilon\,\mathrm{id})](t)\cdot\nu_\Omega(t)v^i[\epsilon]\circ(w+\epsilon\,\mathrm{id})(t)\,d\sigma_t\\ =\epsilon^{n}\int_{\partial\Omega}D[U_1^i[\epsilon]](t)\cdot\nu_\Omega(t)U_1^i[\epsilon](t)\,d\sigma_t.\end{aligned}$$

Thus, it is natural to set

$$G^i[\epsilon]\equiv\int_{\partial\Omega}D[U_1^i[\epsilon]](t)\cdot\nu_\Omega(t)U_1^i[\epsilon](t)\,d\sigma_t,$$

for all $\epsilon\in\,]-\epsilon_4',\epsilon_4'[$. Clearly, $G^i$ is a real analytic map of $]-\epsilon_4',\epsilon_4'[$ to $\mathbb{R}$ such that (4.99) holds. Moreover, by Theorem 4.78 (*iii*), we have

$$G^i[0]=\int_\Omega|\nabla\tilde{u}^i(x)|^2\,dx.$$

Let $\epsilon\in\,]0,\epsilon_3[$. We have

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]}|\nabla u^o[\epsilon](x)|^2\,dx=\int_{\mathbb{P}_a[\Omega_\epsilon]}|\nabla v^o[\epsilon](x)|^2\,dx\\ =-\epsilon^{n-1}\int_{\partial\Omega}\Big(\frac{\partial}{\partial\nu_{\Omega_\epsilon}}v^o[\epsilon]\Big)\circ(w+\epsilon\,\mathrm{id})(t)v^o[\epsilon]\circ(w+\epsilon\,\mathrm{id})(t)\,d\sigma_t,\end{aligned}$$

for all $\epsilon\in\,]0,\epsilon_3[$. Now let $\tilde{\Omega}$ be a tubolar open neighbourhood of class $C^{m,\alpha}$ of $\partial\Omega$ as in Lanza and Rossi [86, Lemma 2.4]. Set

$$\tilde{\Omega}^-\equiv\tilde{\Omega}\cap(\mathbb{R}^n\setminus\mathrm{cl}\,\Omega).$$

Choosing $\epsilon_5\in\,]0,\epsilon_3]$ small enough, we can assume that

$$(w+\epsilon\,\mathrm{cl}\,\tilde{\Omega})\subseteq A,$$

for all $\epsilon\in\,]-\epsilon_5,\epsilon_5[$. We have

$$\begin{aligned}&v^o[\epsilon]\circ(w+\epsilon\,\mathrm{id})(t)\\ &=-\epsilon^n\int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s-\epsilon^n\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s\\ &=\epsilon\Big(\int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega(s)}(S_n(t-s))\Theta[\epsilon](s)\,d\sigma_s-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s\Big)\ \forall t\in\tilde{\Omega}^-,\end{aligned}$$

for all $\epsilon\in\,]0,\epsilon_5[$. Hence, (cf. Proposition C.3 and Lanza and Rossi [86, Proposition 4.10]) there exists a real analytic operator $\tilde{G}$ of $]-\epsilon_5,\epsilon_5[$ to $C^{m,\alpha}(\mathrm{cl}\,\tilde{\Omega}^-)$, such that

$$v^o[\epsilon]\circ(w+\epsilon\,\mathrm{id})=\epsilon\tilde{G}[\epsilon]\qquad\text{in }\tilde{\Omega}^-,$$

for all $\epsilon\in\,]0,\epsilon_5[$. Furthermore, we observe that

$$\tilde{G}[0](t)=w^-[\partial\Omega,\Theta[0]](t)\qquad\forall t\in\mathrm{cl}\,\tilde{\Omega}^-,$$

and so

$$\tilde{G}[0](t)=\tilde{u}^o(t)\qquad\forall t\in\mathrm{cl}\,\tilde{\Omega}^-.$$

Thus, it is natural to set

$$G^o[\epsilon]\equiv-\int_{\partial\Omega}D[\tilde{G}[\epsilon]](t)\cdot\nu_\Omega(t)\tilde{G}[\epsilon](t)\,d\sigma_t,$$

for all $\epsilon\in\,]-\epsilon_5,\epsilon_5[$. Accordingly, one can easily show that $G^o$ is a real analytic map of $]-\epsilon_5,\epsilon_5[$ to $\mathbb{R}$ such that (4.100) holds. Moreover, by the above argument and Folland [52, p. 118], we have

$$G^o[0]=\int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega}|\nabla\tilde{u}^o(x)|^2\,dx.$$

Thus the Theorem is completely proved. □

*Remark* 4.81. We note that the right-hand side of the equalities in (4.99) and (4.100) of Theorem 4.80 can be continued real analytically in the whole $]-\epsilon_5, \epsilon_5[$. Moreover,

$$\lim_{\epsilon \to 0^+} \Big( \int_{\Omega_\epsilon} |\nabla u^i[\epsilon](x)|^2 \, dx + \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](x)|^2 \, dx \Big) = 0.$$

### 4.8.4 A real analytic continuation Theorem for the integral of the family $\{(u^i[\epsilon], u^o[\epsilon])\}_{\epsilon \in ]0,\epsilon_3[}$

As done in Theorem 4.80 for the energy integral, we can now prove a real analytic continuation Theorem for the integral of the family $\{(u^i[\epsilon], u^o[\epsilon])\}_{\epsilon \in ]0,\epsilon_3[}$. Namely, we prove the following.

**Theorem 4.82.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $\epsilon_3$ be as in Theorem 4.76 (iv). Then there exist $\epsilon_6 \in ]0, \epsilon_3]$ and two real analytic operators $J^i$, $J^o$ of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$, such that*

$$\int_{\Omega_\epsilon} u^i[\epsilon](x) \, dx = J^i[\epsilon], \tag{4.103}$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^o[\epsilon](x) \, dx = J^o[\epsilon], \tag{4.104}$$

*for all $\epsilon \in ]0, \epsilon_6[$. Moreover,*

$$J^i[0] = 0, \tag{4.105}$$

$$J^o[0] = \bar{c}|A|_n. \tag{4.106}$$

*Proof.* Let $\Theta[\cdot]$, $\Psi[\cdot]$ be as in Theorem 4.76. Let $\epsilon \in ]0, \epsilon_3[$. Clearly,

$$u^o[\epsilon](x) = \epsilon w_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) + \epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon](s) \, d\sigma_s + \bar{c} \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon].$$

Accordingly,

$$\begin{aligned} \int_{\mathbb{P}_a[\Omega_\epsilon]} u^o[\epsilon](x) \, dx =& \epsilon \int_{\mathbb{P}_a[\Omega_\epsilon]} w_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) \, dx \\ &+ \epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon](s) \, d\sigma_s \Big(|A|_n - \epsilon^n |\Omega|_n\Big) + \bar{c}\Big(|A|_n - \epsilon^n |\Omega|_n\Big). \end{aligned}$$

By arguing as in Lemma 2.45, we can easily prove that there exist $\epsilon_6' \in ]0, \epsilon_3]$ and a real analytic operator $\tilde{J}_1$ of $]-\epsilon_6', \epsilon_6'[$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} w_a^- \big[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))\big](x) \, dx = \epsilon^n \tilde{J}_1[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_6'[$. Moreover, by arguing as in Theorem 4.78, we have that, by possibly taking a smaller $\epsilon_6' > 0$, the map $\tilde{J}_2$ of $]-\epsilon_6', \epsilon_6'[$ to $\mathbb{R}$, defined by

$$\tilde{J}_2[\epsilon] \equiv \epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DS_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon](s) \, d\sigma_s \Big(|A|_n - \epsilon^n |\Omega|_n\Big) + \bar{c}\Big(|A|_n - \epsilon^n |\Omega|_n\Big)$$

for all $\epsilon \in ]-\epsilon_6', \epsilon_6'[$, is real analytic. Hence, if we set,

$$J^o[\epsilon] \equiv \epsilon^{n+1} \tilde{J}_1[\epsilon] + \tilde{J}_2[\epsilon]$$

for all $\epsilon \in ]-\epsilon_6', \epsilon_6'[$, we have that $J^o$ is a real analytic map of $]-\epsilon_6', \epsilon_6'[$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^o[\epsilon](x) \, dx = J^o[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_6'[$, and that $J^o[0] = \bar{c}|A|_n$.

Let $\epsilon \in ]0, \epsilon_3[$. Clearly,

$$\int_{\Omega_\epsilon} u^i[\epsilon](x)\,dx = \epsilon^n \int_{\Omega} u^i[\epsilon](w+\epsilon t)\,dt.$$

On the other hand, if $\epsilon_4'$, $U_1^i$ are as in Theorem 4.78, and we set

$$J^i[\epsilon] \equiv \epsilon^n \int_{\Omega} \left(\epsilon U_1^i[\epsilon](t) + F^{(-1)}(\bar{c})\right) dt$$

for all $\epsilon \in ]-\epsilon_4', \epsilon_4'[$, then we have that $J^i$ is a real analytic map of $]-\epsilon_4', \epsilon_4'[$ to $\mathbb{R}$, such that $J^i[0] = 0$ and that

$$\int_{\Omega_\epsilon} u^i[\epsilon](x)\,dx = J^i[\epsilon]$$

for all $\epsilon \in ]0, \epsilon_4'[$.

Then, by taking $\epsilon_6 \equiv \min\{\epsilon_6', \epsilon_4'\}$, we can easily conclude. □

### 4.8.5 A property of local uniqueness of the family $\{(u^i[\epsilon], u^o[\epsilon])\}_{\epsilon\in]0,\epsilon_3[}$

In this Subsection, we shall show that the family $\{(u^i[\epsilon], u^o[\epsilon])\}_{\epsilon\in]0,\epsilon_3[}$ is essentially unique. To do so, we need to introduce a preliminary lemma.

**Lemma 4.83.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $\epsilon \in ]0, \bar{\epsilon}_1[$. Let $(u^i, u^o)$ solve* (4.76). *Let $(\psi, \theta) \in (C^{m,\alpha}(\partial\Omega))^2$ be such that $u^i = u^i[\epsilon, \psi, \theta]$ and $u^o = u^o[\epsilon, \psi, \theta]$. Then*

$$w^+[\partial\Omega, \psi](t) - \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s = \frac{u^i(w+\epsilon t) - F^{(-1)}(\bar{c})}{\epsilon} \qquad \forall t \in \mathrm{cl}\,\Omega.$$

*Proof.* It is an immediate consequence of Theorem 4.73. □

Then we have the following.

**Theorem 4.84.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $\{\hat{\epsilon}_j\}_{j\in\mathbb{N}}$ be a sequence in $]0, \bar{\epsilon}_1[$ converging to 0. If $\{(u^i_j, u^o_j)\}_{j\in\mathbb{N}}$ is a sequence of pairs of functions such that*

$$(u^i_j, u^o_j) \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_{\hat{\epsilon}_j}]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_{\hat{\epsilon}_j}]), \tag{4.107}$$

$$(u^i_j, u^o_j) \text{ solves (4.76) with } \epsilon \equiv \hat{\epsilon}_j, \tag{4.108}$$

$$\lim_{j\to\infty} \frac{u^i_j(w+\hat{\epsilon}_j\cdot) - F^{(-1)}(\bar{c})}{\hat{\epsilon}_j} = \tilde{u}^i(\cdot) \qquad \text{in } C^{m,\alpha}(\partial\Omega), \tag{4.109}$$

*then there exists $j_0 \in \mathbb{N}$ such that*

$$(u^i_j, u^o_j) = (u^i[\hat{\epsilon}_j], u^o[\hat{\epsilon}_j]) \qquad \forall j_0 \le j \in \mathbb{N}.$$

*Proof.* By Theorem 4.73, for each $j \in \mathbb{N}$, there exists a unique pair $(\psi_j, \theta_j)$ in $(C^{m,\alpha}(\partial\Omega))^2$ such that

$$u^i_j = u^i[\hat{\epsilon}_j, \psi_j, \theta_j], \qquad u^o_j = u^o[\hat{\epsilon}_j, \psi_j, \theta_j]. \tag{4.110}$$

We shall now try to show that

$$\lim_{j\to\infty} (\psi_j, \theta_j) = (\tilde{\psi}, \tilde{\theta}) \qquad \text{in } (C^{m,\alpha}(\partial\Omega))^2. \tag{4.111}$$

Indeed, if we denote by $\tilde{\mathcal{U}}$ the neighbourhood of Theorem 4.76 (*iv*), the limiting relation of (4.111) implies that there exists $j_0 \in \mathbb{N}$ such that

$$(\hat{\epsilon}_j, \psi_j, \theta_j) \in ]0, \epsilon_3[ \times \tilde{\mathcal{U}},$$

for $j \ge j_0$ and thus Theorem 4.76 (*iv*) would imply that

$$(\psi_j, \theta_j) = (\Psi[\hat{\epsilon}_j], \Theta[\hat{\epsilon}_j]),$$

for $j_0 \leq j \in \mathbb{N}$, and that accordingly the theorem holds (cf. Definition 4.77.) Thus we now turn to the proof of (4.111). We note that equation $\Lambda[\epsilon, \psi, \theta] = 0$ can be rewritten in the following form

$$
\begin{aligned}
&F'(F^{(-1)}(\bar{c}))\Big(w^+[\partial\Omega, \psi](t) - \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s\Big) \\
&-\Big(w^-[\partial\Omega, \theta](t) - \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(\bar{x}-w-\epsilon s)\theta(s)\,d\sigma_s\Big) \\
&\qquad = -\epsilon\Big(w^+[\partial\Omega, \psi](t) - \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s\Big)^2 \\
&\times \int_0^1 (1-\beta)F''\Big(F^{(-1)}(\bar{c}) + \beta\epsilon\big(w^+[\partial\Omega, \psi](t) - \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s\big)\Big)d\beta \\
&\qquad\qquad \forall t \in \partial\Omega, \quad (4.112)
\end{aligned}
$$

$$
\begin{aligned}
&-\gamma\int_{\partial\Omega} \nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\psi(s)\,d\sigma_s - \epsilon^n\gamma\int_{\partial\Omega} \nu_\Omega(t)D^2R_n^a(\epsilon(t-s))\nu_\Omega(s)\psi(s)\,d\sigma_s \\
&+\int_{\partial\Omega} \nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\theta(s)\,d\sigma_s + \epsilon^n\int_{\partial\Omega} \nu_\Omega(t)D^2R_n^a(\epsilon(t-s))\nu_\Omega(s)\theta(s)\,d\sigma_s = -g(t) \\
&\qquad\qquad \forall t \in \partial\Omega, \quad (4.113)
\end{aligned}
$$

$$\Lambda_3[\epsilon, \psi, \theta] = 0, \qquad (4.114)$$

for all $(\epsilon, \psi, \theta)$ in the domain of $\Lambda$. By arguing so as to prove that the integral of the second component of $\Lambda$ on $\partial\Omega$ equals zero in the beginning of the proof of Theorem 4.76, we can conclude that both hand sides of equation (4.113) have zero integral on $\partial\Omega$. We define the map $N \equiv (N_l)_{l=1,2,3}$ of $]-\epsilon_3, \epsilon_3[ \times (C^{m,\alpha}(\partial\Omega))^2$ to $C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ by setting $N_1[\epsilon, \psi, \theta]$ equal to the left-hand side of the equality in (4.112), $N_2[\epsilon, \psi, \theta]$ equal to the left-hand side of the equality in (4.113) and $N_3[\epsilon, \psi, \theta] = \Lambda_3[\epsilon, \theta, \psi]$ for all $(\epsilon, \psi, \theta) \in ]-\epsilon_3, \epsilon_3[ \times (C^{m,\alpha}(\partial\Omega))^2$. By arguing as in the proof of Theorem 4.76, we can prove that $N$ is real analytic. Since $N[\epsilon, \cdot, \cdot]$ is linear for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$, we have

$$N[\epsilon, \psi, \theta] = \partial_{(\psi,\theta)}N[\epsilon, \tilde{\psi}, \tilde{\theta}](\psi, \theta)$$

for all $(\epsilon, \psi, \theta) \in ]-\epsilon_3, \epsilon_3[ \times (C^{m,\alpha}(\partial\Omega))^2$, and the map of $]-\epsilon_3, \epsilon_3[$ to $\mathcal{L}((C^{m,\alpha}(\partial\Omega))^2, C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R})$ which takes $\epsilon$ to $N[\epsilon, \cdot, \cdot]$ is real analytic. Since

$$N[0, \cdot, \cdot] = \partial_{(\psi,\theta)}\Lambda[0, \tilde{\psi}, \tilde{\theta}](\cdot, \cdot),$$

Theorem 4.76 (*iii*) implies that $N[0, \cdot, \cdot]$ is also a linear homeomorphism. Since the set of linear homeomorphisms of $(C^{m,\alpha}(\partial\Omega))^2$ to $C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ is open in $\mathcal{L}((C^{m,\alpha}(\partial\Omega))^2, C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R})$ and since the map which takes a linear invertible operator to its inverse is real analytic (cf. *e.g.*, Hille and Phillips [61, Theorems 4.3.2 and 4.3.4]), there exists $\tilde{\epsilon} \in ]0, \epsilon_3[$ such that the map $\epsilon \mapsto N[\epsilon, \cdot, \cdot]^{(-1)}$ is real analytic from $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $\mathcal{L}(C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}, (C^{m,\alpha}(\partial\Omega))^2)$. Next we denote by $S[\epsilon, \psi, \theta] \equiv (S_l[\epsilon, \psi, \theta])_{l=1,2,3}$ the triple defined by the right-hand side of (4.112)-(4.114). Then equation $\Lambda[\epsilon, \psi, \theta] = 0$ (or equivalently system (4.112)-(4.114)) can be rewritten in the following form:

$$(\psi, \theta) = N[\epsilon, \cdot, \cdot]^{(-1)}[S[\epsilon, \psi, \theta]], \qquad (4.115)$$

for all $(\epsilon, \psi, \theta) \in ]-\tilde{\epsilon}, \tilde{\epsilon}[ \times (C^{m,\alpha}(\partial\Omega))^2$. Next we note that condition (4.109), the proof of Theorem 4.76, the real analyticity of $F$ and standard calculus in Banach space imply that

$$\lim_{j\to\infty} S[\hat{\epsilon}_j, \psi_j, \theta_j] = S[0, \tilde{\psi}, \tilde{\theta}] \qquad \text{in } C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}. \qquad (4.116)$$

Then by (4.115) and by the real analyticity of $\epsilon \mapsto N[\epsilon, \cdot, \cdot]^{(-1)}$, and by the bilinearity and continuity of the operator of $\mathcal{L}(C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}, (C^{m,\alpha}(\partial\Omega))^2) \times (C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R})$ to $(C^{m,\alpha}(\partial\Omega))^2$, which takes a pair $(T_1, T_2)$ to $T_1[T_2]$, we conclude that (4.111) holds. Thus the proof is complete. □

## 4.9 An homogenization problem for the Laplace equation with a nonlinear transmission boundary condition in a periodically perforated domain

In this section we consider an homogenization problem for the Laplace equation with a nonlinear transmission boundary condition in a periodically perforated domain.

### 4.9.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 4.3.1 and 4.8.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. For each pair $(\epsilon,\delta) \in ]0,\bar{\epsilon}_1[ \times ]0,+\infty[$, we consider the following periodic nonlinear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a(\epsilon,\delta), \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u^i(x+\delta a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\ u^o(x+\delta a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\ u^o(x) = F(u^i(x)) & \forall x \in \partial\Omega(\epsilon,\delta), \\ \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u^o(x) = \gamma \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u^i(x) + \frac{1}{\delta} g(\frac{1}{\epsilon\delta}(x-\delta w)) & \forall x \in \partial\Omega(\epsilon,\delta), \\ u^o(\delta\bar{x}) = \bar{c}. \end{cases} \tag{4.117}$$

We give the following definition.

**Definition 4.85.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. Let $\epsilon_3$ be as in Theorem 4.78 (*iv*). Let $(u^i[\cdot], u^o[\cdot])$ be as in Definition 4.77. For each pair $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,+\infty[$, we set

$$u^i_{(\epsilon,\delta)}(x) \equiv u^i[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta), \qquad u^o_{(\epsilon,\delta)}(x) \equiv u^o[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).$$

*Remark* 4.86. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. Let $\epsilon_3$ be as in Theorem 4.78 (*iv*). For each $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,+\infty[$ the pair $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ is a solution of (4.117).

By the previous remark, we note that a solution of problem (4.117) can be expressed by means of a solution of an auxiliary rescaled problem, which does not depend on $\delta$. This is due to the presence of the factor $1/\delta$ in front of $g(\frac{1}{\epsilon\delta}(x-\delta w))$ in the sixth equation of problem (4.117).

By virtue of Theorem 4.84, we have the following.

*Remark* 4.87. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. Let $\epsilon_3$ be as in Theorem 4.78 (*iv*). Let $\bar{\delta} \in ]0,+\infty[$. Let $\{\hat{\epsilon}_j\}_{j\in\mathbb{N}}$ be a sequence in $]0,\bar{\epsilon}_1[$ converging to 0. If $\{(u^i_j, u^o_j)\}_{j\in\mathbb{N}}$ is a sequence of pairs of functions such that

$$\begin{aligned} &(u^i_j, u^o_j) \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a(\hat{\epsilon}_j,\bar{\delta})) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\hat{\epsilon}_j,\bar{\delta})), \\ &(u^i_j, u^o_j) \text{ solves (4.117) with } (\epsilon,\delta) \equiv (\hat{\epsilon}_j,\bar{\delta}), \\ &\lim_{j\to\infty} \frac{u^i_j(\bar{\delta}w + \bar{\delta}\hat{\epsilon}_j \cdot) - F^{(-1)}(\bar{c})}{\hat{\epsilon}_j} = \tilde{u}^i(\cdot) \qquad \text{in } C^{m,\alpha}(\partial\Omega), \end{aligned}$$

then there exists $j_0 \in \mathbb{N}$ such that

$$(u^i_j, u^o_j) = (u^i_{(\hat{\epsilon}_j,\bar{\delta})}, u^o_{(\hat{\epsilon}_j,\bar{\delta})}) \qquad \forall j_0 \le j \in \mathbb{N}.$$

Our aim is to study the asymptotic behaviour of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ as $(\epsilon,\delta)$ tends to $(0,0)$. As a first step, we study the behaviour of extensions of $u^i[\epsilon]$ and of $u^o[\epsilon]$ as $\epsilon$ tends to 0.

**Proposition 4.88.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $\epsilon_3$ be as in Theorem 4.78 (iv). Let $1 \leq p < \infty$. Then*

$$\lim_{\epsilon \to 0^+} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]] = 0 \qquad \text{in } L^p(A),$$

*and*

$$\lim_{\epsilon \to 0^+} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]] = \bar{c} \qquad \text{in } L^p(A).$$

*Proof.* It suffices to modify the proof of Propositions 2.132, 4.26. Let $\epsilon_3$, $\Psi$, $\Theta$ be as in Theorem 4.76. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $\epsilon \in ]0, \epsilon_3[$, we have

$$u^i[\epsilon] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) = \epsilon w^+[\partial\Omega, \Psi[\epsilon]](t) - \epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DR^a_n(\epsilon(t-s))\Psi[\epsilon](s)\, d\sigma_s + F^{(-1)}(\bar{c}), \qquad \forall t \in \partial\Omega.$$

We set

$$N^i[\epsilon](t) \equiv \epsilon w^+[\partial\Omega, \Psi[\epsilon]](t) - \epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DR^a_n(\epsilon(t-s))\Psi[\epsilon](s)\, d\sigma_s + F^{(-1)}(\bar{c}), \qquad \forall t \in \partial\Omega.$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. By taking $\tilde{\epsilon} \in ]0, \epsilon_3[$ small enough, we can assume (cf. Proposition 1.23 (*ii*)) that $N^i$ is a real analytic map of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C^i \equiv \sup_{\epsilon \in ]-\tilde{\epsilon}, \tilde{\epsilon}[} \|N^i[\epsilon]\|_{C^0(\partial\Omega)} < +\infty.$$

By the Maximum Principle for harmonic functions, we have

$$|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]](x)| \leq C^i \qquad \forall x \in A, \quad \forall \epsilon \in ]0, \tilde{\epsilon}[.$$

Obviously,

$$\lim_{\epsilon \to 0^+} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]](x) = 0 \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{\epsilon \to 0^+} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]] = 0 \qquad \text{in } L^p(A).$$

If $\epsilon \in ]0, \epsilon_3[$, we have

$$\begin{aligned} u^o[\epsilon] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) = &\epsilon w^-[\partial\Omega, \Theta[\epsilon]](t) - \epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DR^a_n(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ &+ \epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DS^a_n(\bar{x} - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s + \bar{c}, \qquad \forall t \in \partial\Omega. \end{aligned}$$

We set

$$\begin{aligned} N^o[\epsilon](t) \equiv &\epsilon w^-[\partial\Omega, \Theta[\epsilon]](t) - \epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DR^a_n(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ &+ \epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DS^a_n(\bar{x} - w - \epsilon s)\Theta[\epsilon](s)\, d\sigma_s + \bar{c}, \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. By taking $\tilde{\epsilon} \in ]0, \epsilon_3[$ small enough, we can assume (cf. Proposition 1.23 (*i*) and the proof of Theorem 4.78) that $N^o$ is a real analytic map of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C^o \equiv \sup_{\epsilon \in ]-\tilde{\epsilon}, \tilde{\epsilon}[} \|N^o[\epsilon]\|_{C^0(\partial\Omega)} < +\infty.$$

By Theorem 2.5, we have

$$|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]](x)| \leq C^o \qquad \forall x \in A, \quad \forall \epsilon \in ]0, \tilde{\epsilon}[.$$

By Theorem 4.78, we have

$$\lim_{\epsilon \to 0^+} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]](x) = \bar{c} \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{\epsilon \to 0^+} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]] = \bar{c} \qquad \text{in } L^p(A).$$

□

### 4.9.2 Asymptotic behaviour of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

In the following Theorem we deduce by Proposition 4.26 the convergence of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 4.89.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $\epsilon_3$ be as in Theorem 4.78 (iv). Let $1 \leq p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] = 0 \qquad \textit{in } L^p(V),$$

*and*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] = \bar{c} \qquad \textit{in } L^p(V).$$

*Proof.* We modify the proof of Theorem 2.134. By virtue of Proposition 4.88, we have

$$\lim_{\epsilon\to 0^+} \|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]]\|_{L^p(A)} = 0,$$

and

$$\lim_{\epsilon\to 0^+} \|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]] - \bar{c}\|_{L^p(A)} = 0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $C > 0$ such that

$$\|\mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}]\|_{L^p(V)} \leq C \|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]]\|_{L^p(A)} \qquad \forall (\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,1[,$$

and

$$\|\mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] - \bar{c}\|_{L^p(V)} \leq C \|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]] - \bar{c}\|_{L^p(A)} \qquad \forall (\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,1[,$$

Thus,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^p(V),$$

and

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] = \bar{c} \qquad \text{in } L^p(V).$$

□

Then we have the following Theorem, where we consider a functional associated to extensions of $u^i_{(\epsilon,\delta)}$ and of $u^o_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 4.90.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $\epsilon_3$ be as in Theorem 4.76 (iv). Let $\epsilon_6$, $J^i$, $J^o$ be as in Theorem 4.82. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^i[\epsilon], \tag{4.118}$$

*and*

$$\int_{\mathbb{R}^n} \mathbf{E}^o_{(\epsilon,r/l)}[u^o_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^o[\epsilon], \tag{4.119}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N}\setminus\{0\}$.*

*Proof.* We follow the proof of Theorem 2.60. Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N}\setminus\{0\}$. Then, by the periodicity of $u^i_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\Omega_\epsilon} u^i_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\Omega_\epsilon} u^i[\epsilon]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\Omega_\epsilon} u^i[\epsilon](t)\,dt \\ &= \frac{r^n}{l^n} J^i[\epsilon].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^i[\epsilon],$$

and the validity of (4.37) follows. The proof of (4.38) is very similar and is accordingly omitted. □

### 4.9.3 Asymptotic behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

This Subsection is devoted to the study of the behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$. We give the following.

**Definition 4.91.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. Let $\epsilon_3$ be as in Theorem 4.78 (*iv*). For each pair $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,+\infty[$, we set

$$\mathrm{En}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{S}_a(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx + \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 4.92. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. Let $\epsilon_3$ be as in Theorem 4.78 (*iv*). Let $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,+\infty[$. We have

$$\begin{aligned}\int_{\Omega(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\Omega(\epsilon,1)} |(\nabla u^i_{(\epsilon,\delta)})(\delta t)|^2\,dt \\ &= \delta^{n-2}\int_{\Omega_\epsilon} |\nabla u^i[\epsilon](t)|^2\,dt\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u^o_{(\epsilon,\delta)})(\delta t)|^2\,dt \\ &= \delta^{n-2}\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](t)|^2\,dt.\end{aligned}$$

Then we give the following definition, where we consider $\mathrm{En}(\epsilon,\delta)$, with $\epsilon$ equal to a certain function of $\delta$.

**Definition 4.93.** For each $\delta \in ]0,+\infty[$, we set

$$\epsilon[\delta] \equiv \delta^{\frac{2}{n}}.$$

Let $\epsilon_5$ be as in Theorem 4.80. Let $\delta_1 > 0$ be such that $\epsilon[\delta] \in ]0,\epsilon_5[$, for all $\delta \in ]0,\delta_1[$. Then we set

$$\mathrm{En}[\delta] \equiv \mathrm{En}(\epsilon[\delta],\delta),$$

for all $\delta \in ]0,\delta_1[$.

In the following Proposition we compute the limit of $\mathrm{En}[\delta]$ as $\delta$ tends to 0.

**Proposition 4.94.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $\epsilon_3$ be as in Theorem 4.78 (iv). Let $\epsilon_5$ be as in Theorem 4.80. Let $\delta_1 > 0$ be as in Definition 4.93. Then*

$$\lim_{\delta \to 0^+} \mathrm{En}[\delta] = \int_{\Omega} |\nabla \tilde{u}^i(x)|^2 \, dx + \int_{\mathbb{R}^n \setminus \mathrm{cl}\, \Omega} |\nabla \tilde{u}^o(x)|^2 \, dx,$$

*where $\tilde{u}^i$, $\tilde{u}^o$ are as in Theorem 4.75.*

*Proof.* Let $G^i$, $G^o$ be as in Theorem 4.80. Let $\delta \in ]0, \delta_1[$. By Remark 4.92 and Theorem 4.80, we have

$$\begin{aligned}\int_{\Omega(\epsilon[\delta],\delta)} |\nabla u^i_{(\epsilon[\delta],\delta)}(x)|^2 \, dx + \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u^o_{(\epsilon[\delta],\delta)}(x)|^2 \, dx &= \delta^{n-2} (\epsilon[\delta])^n (G^i[\epsilon[\delta]] + G^o[\epsilon[\delta]]) \\ &= \delta^n (G^i[\delta^{\frac{2}{n}}] + G^o[\delta^{\frac{2}{n}}]).\end{aligned}$$

On the other hand,

$$\begin{aligned}\lfloor (1/\delta) \rfloor^n \Big( \int_{\Omega(\epsilon[\delta],\delta)} |\nabla u^i_{(\epsilon[\delta],\delta)}(x)|^2 \, dx + \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u^o_{(\epsilon[\delta],\delta)}(x)|^2 \, dx \Big) \leq \mathrm{En}[\delta] \\ \leq \lceil (1/\delta) \rceil^n \Big( \int_{\Omega(\epsilon[\delta],\delta)} |\nabla u^i_{(\epsilon[\delta],\delta)}(x)|^2 \, dx + \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u^o_{(\epsilon[\delta],\delta)}(x)|^2 \, dx \Big),\end{aligned}$$

and so

$$\lfloor (1/\delta) \rfloor^n \delta^n (G^i[\delta^{\frac{2}{n}}] + G^o[\delta^{\frac{2}{n}}]) \leq \mathrm{En}[\delta] \leq \lceil (1/\delta) \rceil^n \delta^n (G^i[\delta^{\frac{2}{n}}] + G^o[\delta^{\frac{2}{n}}]).$$

Thus, since

$$\lim_{\delta \to 0^+} \lfloor (1/\delta) \rfloor^n \delta^n = 1, \qquad \lim_{\delta \to 0^+} \lceil (1/\delta) \rceil^n \delta^n = 1,$$

we have

$$\lim_{\delta \to 0^+} \mathrm{En}[\delta] = (G^i[0] + G^o[0]).$$

Finally, by equalities (4.101) and (4.102), we easily conclude. □

In the following Proposition we represent the function $\mathrm{En}[\cdot]$ by means of real analytic functions.

**Proposition 4.95.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $\epsilon_3$ be as in Theorem 4.78 (iv). Let $\epsilon_5$, $G^i$, $G^o$ be as in Theorem 4.80. Let $\delta_1 > 0$ be as in Definition 4.93. Then*

$$\mathrm{En}[(1/l)] = G^i[(1/l)^{\frac{2}{n}}] + G^o[(1/l)^{\frac{2}{n}}],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* It follows by the proof of Proposition 4.94. □

## 4.10 A variant of an homogenization problem for the Laplace equation with a nonlinear transmission boundary condition in a periodically perforated domain

In this section we consider another homogenization problem for the Laplace equation with a nonlinear transmission boundary condition in a periodically perforated domain.

### 4.10.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 4.3.1 and 4.8.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. For each pair $(\epsilon, \delta) \in ]0, \bar{\epsilon}_1[ \times ]0, +\infty[$, we introduce the following periodic nonlinear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a(\epsilon,\delta), \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u^i(x+\delta a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\ u^o(x+\delta a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\ u^o(x) = F(u^i(x)) & \forall x \in \partial\Omega(\epsilon,\delta), \\ \frac{\partial}{\partial\nu_{\Omega(\epsilon,\delta)}}u^o(x) = \gamma\frac{\partial}{\partial\nu_{\Omega(\epsilon,\delta)}}u^i(x) + g(\frac{1}{\epsilon\delta}(x-\delta w)) & \forall x \in \partial\Omega(\epsilon,\delta), \\ u^o(\delta\bar{x}) = \bar{c}. \end{cases} \tag{4.120}$$

In contrast to problem (4.117), we note that in the sixth equation of problem (4.120) there is not the factor $1/\delta$ in front of $g(\frac{1}{\epsilon\delta}(x-\delta w))$).

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. For each $(\epsilon,\delta) \in ]0,\bar{\epsilon}_1[\times]0,+\infty[$, we consider the following auxiliary periodic nonlinear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a[\Omega_\epsilon], \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u^i(x+a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u^o(x+a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u^o(x) = F(u^i(x)) & \forall x \in \partial\Omega_\epsilon, \\ \frac{\partial}{\partial\nu_{\Omega_\epsilon}}u^o(x) = \gamma\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u^i(x) + \delta g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon, \\ u^o(\bar{x}) = \bar{c}. \end{cases} \tag{4.121}$$

We transform (4.121) into a system of integral equations by means of the following.

**Proposition 4.96.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $(\epsilon,\delta) \in ]0,\bar{\epsilon}_1[\times]0,+\infty[$. Then the map of the set of pairs $(\omega,\mu) \in (C^{m,\alpha}(\partial\Omega_\epsilon))^2$ that solve the following integral equations*

$$F(w_a^+[\partial\Omega_\epsilon,\omega](x) + F^{(-1)}(\bar{c})) = w_a^-[\partial\Omega_\epsilon,\mu](x) + \bar{c} - w_a^-[\partial\Omega_\epsilon,\mu](\bar{x}) \qquad \forall x \in \partial\Omega_\epsilon \tag{4.122}$$

$$\begin{aligned} \delta g(\frac{1}{\epsilon}(x-w)) + \gamma\int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}\frac{\partial}{\partial\nu_{\Omega_\epsilon}(y)}(S_n^a(x-y))\omega(y)\,d\sigma_y \\ = \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}\frac{\partial}{\partial\nu_{\Omega_\epsilon}(y)}(S_n^a(x-y))\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega_\epsilon, \end{aligned} \tag{4.123}$$

$$\int_{\partial\Omega_\epsilon} \mu\,d\sigma = 0, \tag{4.124}$$

*to the set of pairs $(u^i,u^o)$ of $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ which solve problem* (4.121), *which takes $(\omega,\mu)$ to the pair of functions*

$$(w_a^+[\partial\Omega_\epsilon,\omega] + F^{(-1)}(\bar{c}), w_a^-[\partial\Omega_\epsilon,\mu] + \bar{c} - w_a^-[\partial\Omega_\epsilon,\mu](\bar{x})) \tag{4.125}$$

*is a bijection.*

*Proof.* It suffices to modify the proof of Proposition 4.72, by replacing $g$ by $\delta g$. □

As we have seen, we can convert problem (4.121) into a system of integral equations in the unknown $(\omega,\mu)$. In the following Theorem we introduce a proper change of the functional variables $(\omega,\mu)$.

**Theorem 4.97.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $(\epsilon,\delta) \in ]0,\bar{\epsilon}_1[\times]0,+\infty[$. Then the map $(u^i[\epsilon,\delta,\cdot,\cdot], u^o[\epsilon,\delta,\cdot,\cdot])$ of the set of pairs $(\psi,\theta) \in (C^{m,\alpha}(\partial\Omega))^2$ that solve the following integral* *equations*

$$
\begin{aligned}
&F'(F^{(-1)}(\bar{c}))\Big(w^{+}[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_{\Omega}(s)\cdot DR_{n}^{a}(\epsilon(t-s))\psi(s)\,d\sigma_{s}\Big)\\
&+\delta\epsilon\Big(w^{+}[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_{\Omega}(s)\cdot DR_{n}^{a}(\epsilon(t-s))\psi(s)\,d\sigma_{s}\Big)^{2}\\
&\times\int_{0}^{1}(1-\beta)F''\Big(F^{(-1)}(\bar{c})+\beta\delta\epsilon\big(w^{+}[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_{\Omega}(s)\cdot DR_{n}^{a}(\epsilon(t-s))\psi(s)\,d\sigma_{s}\big)\Big)d\beta\\
&-w^{-}[\partial\Omega,\theta](t)+\epsilon^{n-1}\int_{\partial\Omega}\nu_{\Omega}(s)\cdot DR_{n}^{a}(\epsilon(t-s))\theta(s)\,d\sigma_{s}\\
&-\epsilon^{n-1}\int_{\partial\Omega}\nu_{\Omega}(s)\cdot DS_{n}^{a}(\bar{x}-w-\epsilon s)\theta(s)\,d\sigma_{s}=0\qquad\forall t\in\partial\Omega,
\end{aligned}
\tag{4.126}
$$

$$
\begin{aligned}
&g(t)-\gamma\int_{\partial\Omega}\nu_{\Omega}(t)D^{2}S_{n}(t-s)\nu_{\Omega}(s)\psi(s)\,d\sigma_{s}-\epsilon^{n}\gamma\int_{\partial\Omega}\nu_{\Omega}(t)D^{2}R_{n}^{a}(\epsilon(t-s))\nu_{\Omega}(s)\psi(s)\,d\sigma_{s}\\
&+\int_{\partial\Omega}\nu_{\Omega}(t)D^{2}S_{n}(t-s)\nu_{\Omega}(s)\theta(s)\,d\sigma_{s}+\epsilon^{n}\int_{\partial\Omega}\nu_{\Omega}(t)D^{2}R_{n}^{a}(\epsilon(t-s))\nu_{\Omega}(s)\theta(s)\,d\sigma_{s}=0\quad\forall t\in\partial\Omega,
\end{aligned}
\tag{4.127}
$$

$$
\int_{\partial\Omega}\theta\,d\sigma=0,
\tag{4.128}
$$

*to the set of pairs* $(u^i,u^o)$ *of* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon])\times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ *which solve problem* (4.121)*, which takes* $(\psi,\theta)$ *to the pair of functions*

$$
(u^{i}[\epsilon,\delta,\psi,\theta]\equiv w_{a}^{+}[\partial\Omega_{\epsilon},\omega]+F^{(-1)}(\bar{c}),u^{o}[\epsilon,\delta,\psi,\theta]\equiv w_{a}^{-}[\partial\Omega_{\epsilon},\mu]+\bar{c}-w_{a}^{-}[\partial\Omega_{\epsilon},\mu](\bar{x})),
\tag{4.129}
$$

*where*

$$
\mu(x)\equiv\delta\epsilon\theta(\frac{1}{\epsilon}(x-w))\qquad\forall x\in\partial\Omega_{\epsilon},
\tag{4.130}
$$

$$
\omega(x)\equiv\delta\epsilon\psi(\frac{1}{\epsilon}(x-w))\qquad\forall x\in\partial\Omega_{\epsilon},
\tag{4.131}
$$

*is a bijection.*

*Proof.* It suffices to modify the proof of Theorem 4.73. Let the pair $(u^i,u^o)\in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon])\times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ solve problem (4.121). Then, by Proposition 4.96, there exists a unique pair $(\omega,\mu)$ in $(C^{m,\alpha}(\partial\Omega_\epsilon))^2$, which solves (4.122)-(4.124) and such that $(u^i,u^o)$ equals the pair of functions defined in the right-hand side of (4.129). The pair $(\psi,\theta)$ defined by (4.130),(4.131) belongs to $(C^{m,\alpha}(\partial\Omega))^2$. By (4.122), (4.123),(4.124) the pair $(\psi,\theta)$ solves equations (4.127),(4.128) together with the following equation:

$$
\begin{aligned}
&F\Big(\delta\epsilon\big(w^{+}[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_{\Omega}(s)\cdot DR_{n}^{a}(\epsilon(t-s))\psi(s)\,d\sigma_{s}\big)+F^{(-1)}(\bar{c})\Big)\\
&=\delta\epsilon\big(w^{-}[\partial\Omega,\theta](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_{\Omega}(s)\cdot DR_{n}^{a}(\epsilon(t-s))\theta(s)\,d\sigma_{s}\big)+\bar{c}+\delta\epsilon^{n}\int_{\partial\Omega}\nu_{\Omega}(s)\cdot DS_{n}^{a}(\bar{x}-w-\epsilon s)\theta(s)\,d\sigma_{s}\\
&\qquad\forall t\in\partial\Omega.
\end{aligned}
\tag{4.132}
$$

We now show that equation (4.132) implies the validity of (4.126). By Taylor Formula, we have

$$
F(x+F^{(-1)}(\bar{c}))=\bar{c}+F'(F^{(-1)}(\bar{c}))x+x^{2}\int_{0}^{1}(1-\beta)F''(F^{(-1)}(\bar{c})+\beta x)d\beta\qquad\forall x\in\mathbb{R}.
$$

Then, by dividing both sides of (4.132) by $\delta\epsilon$, we can rewrite (4.132) as (4.126). Conversely, by reading backward the above argument, one can easily show that if $(\psi,\theta)$ solves (4.126)-(4.128), then the pair $(\omega,\mu)$, with $\omega$, $\mu$ delivered by (4.130),(4.131), satisfies system (4.122)-(4.124). Accordingly, the pair of functions of (4.129) satisfies problem (4.121) by Proposition 4.96. □

Hence we are reduced to analyse system (4.126)-(4.128). As a first step in the analysis of system (4.126)-(4.128), we note that for $(\epsilon,\delta)=(0,0)$ one obtains a system which we address to as the *limiting system* and which has the following form

$$F'(F^{(-1)}(\bar{c}))w^+[\partial\Omega,\psi](t)-w^-[\partial\Omega,\theta](t)=0 \quad \forall t\in\partial\Omega, \tag{4.133}$$

$$g(t)-\gamma\int_{\partial\Omega}\nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\psi(s)\,d\sigma_s+\int_{\partial\Omega}\nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\theta(s)\,d\sigma_s=0 \quad \forall t\in\partial\Omega, \tag{4.134}$$

$$\int_{\partial\Omega}\theta\,d\sigma=0. \tag{4.135}$$

Then we have the following theorem, which shows the unique solvability of the limiting system, and its link with a boundary value problem which we shall address to as the *limiting boundary value problem.*

**Theorem 4.98.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $\Omega$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.72), (4.73), (4.74), (4.75)*, respectively. Then the following statements hold.*

*(i) The* limiting system (4.133)-(4.135) *has one and only one solution in $(C^{m,\alpha}(\partial\Omega))^2$, which we denote by $(\tilde{\psi},\tilde{\theta})$.*

*(ii) The limiting boundary value problem*

$$\begin{cases} \Delta u^i(x)=0 & \forall x\in\Omega,\\ \Delta u^o(x)=0 & \forall x\in\mathbb{R}^n\setminus\mathrm{cl}\,\Omega,\\ u^o(x)=F'(F^{(-1)}(\bar{c}))u^i(x) & \forall x\in\partial\Omega,\\ \frac{\partial}{\partial\nu_\Omega}u^o(x)=\gamma\frac{\partial}{\partial\nu_\Omega}u^i(x)+g(x) & \forall x\in\partial\Omega,\\ \lim_{x\to\infty}u^o(x)=0, \end{cases} \tag{4.136}$$

*has one and only one solution $(\tilde{u}^i,\tilde{u}^o)$ in $C^{m,\alpha}(\mathrm{cl}\,\Omega)\times C^{m,\alpha}(\mathbb{R}^n\setminus\Omega)$, and the following formulas hold:*

$$\tilde{u}^i\equiv w^+[\partial\Omega,\tilde{\psi}] \qquad in\ \mathrm{cl}\,\Omega, \tag{4.137}$$

$$\tilde{u}^o\equiv w^-[\partial\Omega,\tilde{\theta}] \qquad in\ \mathbb{R}^n\setminus\Omega. \tag{4.138}$$

*Proof.* It is Theorem 4.75. □

We are now ready to analyse equations (4.126)-(4.128) around the degenerate case $(\epsilon,\delta)=(0,0)$.

**Theorem 4.99.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\bar{x}\in\mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75)*, respectively. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.64). *Let $\Lambda\equiv(\Lambda_j)_{j=1,2,3}$ be the map of $]-\bar{\epsilon}_1,\bar{\epsilon}_1[\times\mathbb{R}\times(C^{m,\alpha}(\partial\Omega))^2$ to $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$, defined by*

$$\begin{aligned}\Lambda_1[\epsilon,\delta,\psi,\theta](t)&\equiv F'(F^{(-1)}(\bar{c}))\Big(w^+[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s\Big)\\ &+\delta\epsilon\Big(w^+[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s\Big)^2\\ &\times\int_0^1(1-\beta)F''\Big(F^{(-1)}(\bar{c})+\beta\delta\epsilon\big(w^+[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s\big)\Big)d\beta\\ &-w^-[\partial\Omega,\theta](t)+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\\ &-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n^a(\bar{x}-w-\epsilon s)\theta(s)\,d\sigma_s \quad \forall t\in\partial\Omega,\end{aligned} \tag{4.139}$$

$$\begin{aligned}&\Lambda_2[\epsilon,\delta,\psi,\theta](t)\\ &\equiv g(t)-\gamma\int_{\partial\Omega}\nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\psi(s)\,d\sigma_s-\epsilon^n\gamma\int_{\partial\Omega}\nu_\Omega(t)D^2R_n^a(\epsilon(t-s))\nu_\Omega(s)\psi(s)\,d\sigma_s\\ &+\int_{\partial\Omega}\nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\theta(s)\,d\sigma_s+\epsilon^n\int_{\partial\Omega}\nu_\Omega(t)D^2R_n^a(\epsilon(t-s))\nu_\Omega(s)\theta(s)\,d\sigma_s \quad \forall t\in\partial\Omega,\end{aligned} \tag{4.140}$$

$$\Lambda_3[\epsilon,\delta,\psi,\theta] \equiv \int_{\partial\Omega} \theta \, d\sigma, \tag{4.141}$$

*for all* $(\epsilon,\delta,\psi,\theta) \in ]-\bar{\epsilon}_1,\bar{\epsilon}_1[ \times \mathbb{R} \times (C^{m,\alpha}(\partial\Omega))^2$. *Then the following statements hold.*

*(i) Equation* $\Lambda[0,0,\psi,\theta] = 0$ *is equivalent to the limiting system* (4.133)-(4.135) *and has one and only one solution* $(\tilde{\psi},\tilde{\theta})$ *(cf. Theorem 4.75.)*

*(ii) If* $(\epsilon,\delta) \in ]0,\bar{\epsilon}_1[ \times ]0,+\infty[$, *then equation* $\Lambda[\epsilon,\delta,\psi,\theta] = 0$ *is equivalent to system* (4.126)-(4.128) *for* $(\psi,\theta)$.

*(iii) There exists* $\epsilon_2 \in ]0,\bar{\epsilon}_1]$ *such that* $\Lambda$ *is a real analytic map of* $]-\epsilon_2,\epsilon_2[ \times \mathbb{R} \times (C^{m,\alpha}(\partial\Omega))^2$ *to* $C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$. *The differential* $\partial_{(\psi,\theta)}\Lambda[0,0,\tilde{\psi},\tilde{\theta}]$ *of* $\Lambda$ *at* $(0,0,\tilde{\psi},\tilde{\theta})$ *is a linear homeomorphism of* $(C^{m,\alpha}(\partial\Omega))^2$ *to* $C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$.

*(iv) There exist* $\epsilon_3 \in ]0,\epsilon_2]$, $\delta_1 \in ]0,+\infty[$ *and an open neighbourhood* $\tilde{\mathcal{U}}$ *of* $(\tilde{\psi},\tilde{\theta})$ *in* $(C^{m,\alpha}(\partial\Omega))^2$ *and a real analytic map* $(\Psi[\cdot,\cdot],\Theta[\cdot,\cdot])$ *of* $]-\epsilon_3,\epsilon_3[ \times ]-\delta_1,\delta_1[$ *to* $\tilde{\mathcal{U}}$, *such that the set of zeros of the map* $\Lambda$ *in* $]-\epsilon_3,\epsilon_3[ \times ]-\delta_1,\delta_1[ \times \tilde{\mathcal{U}}$ *coincides with the graph of* $(\Psi[\cdot,\cdot],\Theta[\cdot,\cdot])$. *In particular,* $(\Psi[0,0],\Theta[0,0]) = (\tilde{\psi},\tilde{\theta})$.

*Proof.* It suffices to modify the proof of Theorem 4.76. The statements in (*i*) and (*ii*) are obvious. By arguing as in the proof of statement (*iii*) of Theorem 4.76, we can easily conclude that there exists $\epsilon_2 \in ]0,\bar{\epsilon}_1]$ such that $\Lambda$ is a real analytic map of $]-\epsilon_2,\epsilon_2[\times\mathbb{R}\times(C^{m,\alpha}(\partial\Omega))^2$ to $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$. By standard calculus in Banach space, the differential of $\Lambda$ at $(0,0,\tilde{\psi},\tilde{\theta})$ with respect to variables $(\psi,\theta)$ is delivered by the following formulas:

$$\partial_{(\psi,\theta)}\Lambda_1[0,0,\tilde{\psi},\tilde{\theta}](\bar{\psi},\bar{\theta})(t) = F'(F^{(-1)}(\bar{c}))w^+[\partial\Omega,\bar{\psi}](t) - w^-[\partial\Omega,\bar{\theta}](t) \qquad \forall t \in \partial\Omega,$$

$$\partial_{(\psi,\theta)}\Lambda_2[0,0,\tilde{\psi},\tilde{\theta}](\bar{\psi},\bar{\theta})(t) = -\gamma \int_{\partial\Omega} \nu_\Omega(t) D^2 S_n(t-s)\nu_\Omega(s)\bar{\psi}(s)\,d\sigma_s + \int_{\partial\Omega} \nu_\Omega(t) D^2 S_n(t-s)\nu_\Omega(s)\bar{\theta}(s)\,d\sigma_s \qquad \forall t \in \partial\Omega,$$

$$\partial_{(\psi,\theta)}\Lambda_3[0,0,\tilde{\psi},\tilde{\theta}](\bar{\psi},\bar{\theta}) = \int_{\partial\Omega} \bar{\theta}(s)\,d\sigma_s,$$

for all $(\bar{\psi},\bar{\theta}) \in (C^{m,\alpha}(\partial\Omega))^2$. By the proof of statement (*iii*) of Theorem 4.76, the above differential is a linear homeomorphism. Statement (*iv*) is an immediate consequence of statement (*iii*) and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

We are now in the position to introduce the following.

**Definition 4.100.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. Let $u^i[\cdot,\cdot,\cdot,\cdot]$ and $u^o[\cdot,\cdot,\cdot,\cdot]$ be as in Theorem 4.97. If $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[$, we set

$$u^i[\epsilon,\delta](t) \equiv u^i[\epsilon,\delta,\Psi[\epsilon,\delta],\Theta[\epsilon,\delta]](t) \qquad \forall t \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon],$$
$$u^o[\epsilon,\delta](t) \equiv u^o[\epsilon,\delta,\Psi[\epsilon,\delta],\Theta[\epsilon,\delta]](t) \qquad \forall t \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],$$

where $\epsilon_3$, $\delta_1$, $\Psi$, $\Theta$ are as in Theorem 4.99 (*iv*).

We now show that $\{(u^i[\epsilon,\delta](\cdot),u^o[\epsilon,\delta](\cdot))\}_{(\epsilon,\delta)\in]0,\epsilon_3[\times]0,\delta_1[}$ can be continued real analytically for negative values of $\epsilon$, $\delta$.

Then we have the following.

**Theorem 4.101.** *Let* $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\bar{x} \in \mathrm{cl}\,A\setminus\{w\}$. *Let* $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ *be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let* $\epsilon_3$, $\delta_1$ *be as in Theorem 4.99* (*iv*). *Then the following statements hold.*

*(i) Let* $V$ *be a bounded open subset of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\,V\cap\mathbb{S}_a[\Omega_0] = \emptyset$. *Then there exist* $\epsilon_4 \in ]0,\epsilon_3]$ *and a real analytic operator* $U_1^o$ *of* $]-\epsilon_4,\epsilon_4[ \times ]-\delta_1,\delta_1[$ *to the space* $C_h^0(\mathrm{cl}\,V)$ *such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*

$$u^o[\epsilon,\delta](x) = \delta\epsilon^n U_1^o[\epsilon,\delta](x) + \bar{c} \qquad \forall x \in \mathrm{cl}\, V,$$

*for all* $(\epsilon,\delta) \in ]0,\epsilon_4[ \times ]0,\delta_1[$.

*(ii) Let* $\bar{V}$ *be a bounded open subset of* $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$. *Then there exist* $\bar{\epsilon}_4 \in ]0,\epsilon_3]$ *and a real analytic operator* $\bar{U}_1^o$ *of* $]-\bar{\epsilon}_4,\bar{\epsilon}_4[ \times ]-\delta_1,\delta_1[$ *to the space* $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$ *such that the following conditions hold.*

*(j')* $w + \epsilon\, \mathrm{cl}\, \bar{V} \subseteq \mathrm{cl}\, \mathbb{P}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \setminus \{0\}$.

*(jj')*

$$u^o[\epsilon,\delta](w+\epsilon t) = \delta\epsilon \bar{U}_1^o[\epsilon,\delta](t) + \bar{c} \qquad \forall t \in \mathrm{cl}\, \bar{V},$$

*for all* $(\epsilon,\delta) \in ]0,\bar{\epsilon}_4[ \times ]0,\delta_1[$. *Moreover,* $\bar{U}_1^o[0,0]$ *equals the restriction of* $\tilde{u}^o$ *to* $\mathrm{cl}\,\bar{V}$.

*(iii) There exist* $\epsilon_4' \in ]0,\epsilon_3]$ *and a real analytic operator* $U_1^i$ *of* $]-\epsilon_4',\epsilon_4'[ \times ]-\delta_1,\delta_1[$ *to the space* $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ *such that*

$$u^i[\epsilon,\delta](w+\epsilon t) = \delta\epsilon U_1^i[\epsilon,\delta](t) + F^{(-1)}(\bar{c}) \qquad \forall t \in \mathrm{cl}\,\Omega,$$

*for all* $(\epsilon,\delta) \in ]0,\epsilon_4'[ \times ]0,\delta_1[$. *Moreover,* $U_1^i[0,0]$ *equals* $\tilde{u}^i$ *on* $\mathrm{cl}\,\Omega$.

*Proof.* We modify the proof of Theorem 4.78. Let $\Theta[\cdot,\cdot]$, $\Psi[\cdot,\cdot]$ be as in Theorem 4.99. We first prove statement $(i)$. Clearly, by taking $\epsilon_4 \in ]0,\epsilon_3]$ small enough, we can assume that $(j)$ holds. Consider $(jj)$. Let $(\epsilon,\delta) \in ]0,\epsilon_4[ \times ]0,\delta_1[$. We have

$$\begin{aligned} u^o[\epsilon,\delta](x) =& \delta\epsilon^n\Big(-\int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(x-w-\epsilon s)\Theta[\epsilon,\delta](s)\,d\sigma_s \\ &+ \int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(\bar{x}-w-\epsilon s)\Theta[\epsilon,\delta](s)\,d\sigma_s\Big) + \bar{c}, \qquad \forall x \in \mathrm{cl}\, V. \end{aligned}$$

Thus, it is natural to set

$$\begin{aligned} U_1^o[\epsilon,\delta](x) \equiv& -\int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(x-w-\epsilon s)\Theta[\epsilon,\delta](s)\,d\sigma_s \\ &+ \int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(\bar{x}-w-\epsilon s)\Theta[\epsilon,\delta](s)\,d\sigma_s \qquad \forall x \in \mathrm{cl}\, V, \end{aligned}$$

for all $(\epsilon,\delta) \in ]-\epsilon_4,\epsilon_4[ \times ]-\delta_1,\delta_1[$. Following the proof of Theorem 2.40, one can easily show that, by possibly taking a smaller $\epsilon_4$, the map $U_1^o$ of $]-\epsilon_4,\epsilon_4[ \times ]-\delta_1,\delta_1[$ to $C_h^0(\mathrm{cl}\, V)$ is real analytic and that the equality in $(jj)$ holds (cf. Proposition 1.22 $(iii)$ and Proposition 1.24 $(i)$.) We now prove $(ii)$. Clearly, by taking $\bar{\epsilon}_4 \in ]0,\epsilon_3]$ small enough, we can assume that $(j')$ holds. Consider $(jj')$. Let $(\epsilon,\delta) \in ]0,\bar{\epsilon}_4[ \times ]0,\delta_1[$. We have

$$\begin{aligned} u^o[\epsilon,\delta](w+\epsilon t) =& \delta\epsilon w^-[\partial\Omega,\Theta[\epsilon,\delta]](t) - \delta\epsilon^n \int_{\partial\Omega} \nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s \\ &+ \delta\epsilon^n \int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(\bar{x}-w-\epsilon s)\Theta[\epsilon,\delta](s)\,d\sigma_s + \bar{c}, \qquad \forall t \in \mathrm{cl}\,\bar{V}. \end{aligned}$$

Thus, it is natural to set

$$\begin{aligned} \bar{U}_1^o[\epsilon,\delta](t) \equiv& w^-[\partial\Omega,\Theta[\epsilon,\delta]](t) - \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s \\ &+ \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(\bar{x}-w-\epsilon s)\Theta[\epsilon,\delta](s)\,d\sigma_s, \qquad \forall t \in \mathrm{cl}\,\bar{V}. \end{aligned}$$

for all $(\epsilon,\delta) \in ]-\bar{\epsilon}_4,\bar{\epsilon}_4[ \times ]-\delta_1,\delta_1[$. By Proposition 1.22 $(iii)$ and Proposition 1.24 $(ii)$, we can easily conclude that $\bar{U}_1^o$ is a real analytic map of $]-\bar{\epsilon}_4,\bar{\epsilon}_4[ \times ]-\delta_1,\delta_1[$ to $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$. Moreover, by the definition of $\bar{U}_1^o$, we have

$$\bar{U}_1^o[0,0](t) = w^-[\partial\Omega,\tilde{\theta}](t) = \tilde{u}^o(t) \qquad \forall t \in \mathrm{cl}\,\bar{V}.$$

Finally, we prove $(iii)$. Let $(\epsilon,\delta)\in ]0,\epsilon_3[\times]0,\delta_1[$. We have

$$u^i[\epsilon,\delta](w+\epsilon t)=\delta\epsilon w^+[\partial\Omega,\Psi[\epsilon,\delta]](t)\\ -\delta\epsilon^n\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\Psi[\epsilon,\delta](s)\,d\sigma_s+F^{(-1)}(\bar{c}),\qquad\forall t\in\operatorname{cl}\Omega.$$

Thus, it is natural to set

$$U_1^i[\epsilon,\delta](t)\equiv w^+[\partial\Omega,\Psi[\epsilon,\delta]](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\Psi[\epsilon,\delta](s)\,d\sigma_s,\qquad\forall t\in\operatorname{cl}\Omega.$$

for all $(\epsilon,\delta)\in ]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[$. By arguing as above, by exploiting Proposition 1.24 $(iii)$, one can easily prove that there exists $\epsilon_4'\in ]0,\epsilon_3]$, such that $U_1^i$ is a real analytic map of $]-\epsilon_4',\epsilon_4'[\times]-\delta_1,\delta_1[$ to $C^{m,\alpha}(\operatorname{cl}\Omega)$. Clearly, the equality in $(iii)$ holds. Moreover, by the definition of $U_1^i$, we have

$$U_1^i[0,0](t)=w^+[\partial\Omega,\tilde{\psi}](t)=\tilde{u}^i(t)\qquad\forall t\in\operatorname{cl}\Omega.$$

□

As done in Theorem 4.101 for $(u^i[\cdot,\cdot],u^o[\cdot,\cdot])$, we can now prove a real analytic continuation Theorem for the energy integral. Namely, we prove the following.

**Theorem 4.102.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\bar{x}\in\operatorname{cl}A\setminus\{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.99 $(iv)$. Then there exist $\epsilon_5\in]0,\epsilon_3]$ and two real analytic operators $G^i$, $G^o$ of $]-\epsilon_5,\epsilon_5[\times]-\delta_1,\delta_1[$ to $\mathbb{R}$, such that*

$$\int_{\Omega_\epsilon}|\nabla u^i[\epsilon,\delta](x)|^2\,dx=\delta^2\epsilon^nG^i[\epsilon,\delta],\tag{4.142}$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]}|\nabla u^o[\epsilon,\delta](x)|^2\,dx=\delta^2\epsilon^nG^o[\epsilon,\delta],\tag{4.143}$$

*for all $(\epsilon,\delta)\in]0,\epsilon_5[\times]0,\delta_1[$. Moreover,*

$$G^i[0,0]=\int_\Omega|\nabla\tilde{u}^i(x)|^2\,dx,\tag{4.144}$$

$$G^o[0,0]=\int_{\mathbb{R}^n\setminus\operatorname{cl}\Omega}|\nabla\tilde{u}^o(x)|^2\,dx.\tag{4.145}$$

*Proof.* It suffices to modify the proof of Theorem 4.80. Let $\Theta[\cdot,\cdot]$, $\Psi[\cdot,\cdot]$ be as in Theorem 4.99. We denote by id the identity map in $\mathbb{R}^n$. We set

$$v^i[\epsilon,\delta](x)\equiv u^i[\epsilon,\delta](x)-F^{(-1)}(\bar{c})\qquad\forall x\in\operatorname{cl}\mathbb{S}_a[\Omega_\epsilon],$$

and

$$v^o[\epsilon,\delta](x)\equiv u^o[\epsilon,\delta](x)-\Big(\bar{c}+\delta\epsilon^n\int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n^a(\bar{x}-w-\epsilon s)\Theta[\epsilon,\delta](s)\,d\sigma_s\Big)\qquad\forall x\in\operatorname{cl}\mathbb{T}_a[\Omega_\epsilon],$$

for all $(\epsilon,\delta)\in]0,\epsilon_3[\times]0,\delta_1[$. Let $(\epsilon,\delta)\in]0,\epsilon_3[\times]0,\delta_1[$. We have

$$\int_{\Omega_\epsilon}|\nabla u^i[\epsilon,\delta](x)|^2\,dx=\int_{\Omega_\epsilon}|\nabla v^i[\epsilon,\delta](x)|^2\,dx\\ =\epsilon^{n-1}\int_{\partial\Omega}\Big(\frac{\partial}{\partial\nu_{\Omega_\epsilon}}v^i[\epsilon,\delta]\Big)\circ(w+\epsilon\operatorname{id})(t)v^i[\epsilon,\delta]\circ(w+\epsilon\operatorname{id})(t)\,d\sigma_t,$$

for all $(\epsilon,\delta)\in]0,\epsilon_3[\times]0,\delta_1[$. Let $\epsilon_4'$, $U_1^i[\cdot,\cdot]$ be as in Theorem 4.101 $(iii)$. Let $(\epsilon,\delta)\in]0,\epsilon_4'[\times]0,\delta_1[$. We have

$$v^i[\epsilon,\delta]\circ(w+\epsilon\operatorname{id})(t)=\delta\epsilon U_1^i[\epsilon,\delta](t)\qquad\forall t\in\operatorname{cl}\Omega,$$

and accordingly

$$D[v^i[\epsilon,\delta]\circ(w+\epsilon\operatorname{id})](t)=\delta\epsilon D[U_1^i[\epsilon,\delta]](t)\qquad\forall t\in\operatorname{cl}\Omega.$$

Also,

$$\epsilon^{n-1}\int_{\partial\Omega}\Big(\frac{\partial}{\partial\nu_{\Omega_\epsilon}}v^i[\epsilon,\delta]\Big)\circ(w+\epsilon\,\mathrm{id})(t)v^i[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id})(t)\,d\sigma_t$$
$$=\epsilon^{n-2}\int_{\partial\Omega}D[v^i[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id})](t)\cdot\nu_\Omega(t)v^i[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id})(t)\,d\sigma_t$$
$$=\delta^2\epsilon^n\int_{\partial\Omega}D[U_1^i[\epsilon,\delta]](t)\cdot\nu_\Omega(t)U_1^i[\epsilon,\delta](t)\,d\sigma_t.$$

Thus, it is natural to set

$$G^i[\epsilon,\delta]\equiv\int_{\partial\Omega}D[U_1^i[\epsilon,\delta]](t)\cdot\nu_\Omega(t)U_1^i[\epsilon,\delta](t)\,d\sigma_t,$$

for all $(\epsilon,\delta)\in\,]-\epsilon_4',\epsilon_4'[\,\times\,]-\delta_1,\delta_1[$. Clearly, $G^i$ is a real analytic map of $]-\epsilon_4',\epsilon_4'[\,\times\,]-\delta_1,\delta_1[$ to $\mathbb{R}$ such that (4.142) holds. Moreover,

$$G^i[0,0]=\int_\Omega|\nabla\tilde{u}^i(x)|^2\,dx.$$

Let $(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[$. We have

$$\int_{\mathbb{P}_a[\Omega_\epsilon]}|\nabla u^o[\epsilon,\delta](x)|^2\,dx=\int_{\mathbb{P}_a[\Omega_\epsilon]}|\nabla v^o[\epsilon,\delta](x)|^2\,dx$$
$$=-\epsilon^{n-1}\int_{\partial\Omega}\Big(\frac{\partial}{\partial\nu_{\Omega_\epsilon}}v^o[\epsilon,\delta]\Big)\circ(w+\epsilon\,\mathrm{id})(t)v^o[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id})(t)\,d\sigma_t,$$

for all $(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[$. Now let $\tilde{\Omega}$ be a tubolar open neighbourhood of class $C^{m,\alpha}$ of $\partial\Omega$ as in Lanza and Rossi [86, Lemma 2.4]. Set

$$\tilde{\Omega}^-\equiv\tilde{\Omega}\cap(\mathbb{R}^n\setminus\mathrm{cl}\,\Omega).$$

Choosing $\epsilon_5\in\,]0,\epsilon_3]$ small enough, we can assume that

$$(w+\epsilon\,\mathrm{cl}\,\tilde{\Omega})\subseteq A,$$

for all $\epsilon\in\,]-\epsilon_5,\epsilon_5[$. We have

$$v^o[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id})(t)$$
$$=-\delta\epsilon^n\int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s-\delta\epsilon^n\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s$$
$$=\delta\epsilon\Big(\int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega(s)}(S_n(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s\Big)\ \forall t\in\tilde{\Omega}^-,$$

for all $(\epsilon,\delta)\in\,]0,\epsilon_5[\,\times\,]0,\delta_1[$. Hence, (cf. Proposition C.3 and Lanza and Rossi [86, Proposition 4.10]) there exists a real analytic operator $\tilde{G}$ of $]-\epsilon_5,\epsilon_5[\,\times\,]-\delta_1,\delta_1[$ to $C^{m,\alpha}(\mathrm{cl}\,\tilde{\Omega}^-)$, such that

$$v^o[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id})=\delta\epsilon\tilde{G}[\epsilon,\delta]\qquad\text{in }\tilde{\Omega}^-,$$

for all $(\epsilon,\delta)\in\,]0,\epsilon_5[\,\times\,]0,\delta_1[$. Furthermore, we observe that

$$\tilde{G}[0,0](t)=w^-[\partial\Omega,\Theta[0,0]](t)\qquad\forall t\in\mathrm{cl}\,\tilde{\Omega}^-,$$

and so

$$\tilde{G}[0,0](t)=\tilde{u}^o(t)\qquad\forall t\in\mathrm{cl}\,\tilde{\Omega}^-.$$

Thus, it is natural to set

$$G^o[\epsilon,\delta]\equiv-\int_{\partial\Omega}D[\tilde{G}[\epsilon,\delta]](t)\cdot\nu_\Omega(t)\tilde{G}[\epsilon,\delta](t)\,d\sigma_t,$$

for all $(\epsilon,\delta)\in\,]-\epsilon_5,\epsilon_5[\,\times\,]-\delta_1,\delta_1[$. Accordingly, one can easily show that $G^o$ is a real analytic map of $]-\epsilon_5,\epsilon_5[\,\times\,]-\delta_1,\delta_1[$ to $\mathbb{R}$ such that (4.143) holds. Moreover, by the above argument and Folland [52, p. 118], we have

$$G^o[0]=\int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega}|\nabla\tilde{u}^o(x)|^2\,dx.$$

Thus the Theorem is completely proved. □

We now show that the family $\{(u^i[\epsilon,\delta],u^o[\epsilon,\delta])\}_{(\epsilon,\delta)\in]0,\epsilon_3[\times]0,\delta_1[}$ is essentially unique. To do so, we need to introduce a preliminary lemma.

**Lemma 4.103.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\bar{x}\in\mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $(\epsilon,\delta)\in]0,\bar{\epsilon}_1[\times]0,+\infty[$. Let $(u^i,u^o)$ solve* (4.121). *Let $(\psi,\theta)\in(C^{m,\alpha}(\partial\Omega))^2$ be such that $u^i=u^i[\epsilon,\delta,\psi,\theta]$ and $u^o=u^o[\epsilon,\delta,\psi,\theta]$. Then*

$$w^+[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s=\frac{u^i(w+\epsilon t)-F^{(-1)}(\bar{c})}{\delta\epsilon}\qquad\forall t\in\mathrm{cl}\,\Omega.$$

*Proof.* It is an immediate consequence of Theorem 4.97. □

**Theorem 4.104.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\bar{x}\in\mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $\{(\hat{\epsilon}_j,\hat{\delta}_j)\}_{j\in\mathbb{N}}$ be a sequence in $]0,\bar{\epsilon}_1[\times]0,+\infty[$ converging to $(0,0)$. If $\{(u^i_j,u^o_j)\}_{j\in\mathbb{N}}$ is a sequence of pairs of functions such that*

$$(u^i_j,u^o_j)\in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_{\hat{\epsilon}_j}])\times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_{\hat{\epsilon}_j}]),\tag{4.146}$$

$$(u^i_j,u^o_j)\ \text{solves (4.121) with}\ (\epsilon,\delta)\equiv(\hat{\epsilon}_j,\hat{\delta}_j),\tag{4.147}$$

$$\lim_{j\to\infty}\frac{u^i_j(w+\hat{\epsilon}_j\cdot)-F^{(-1)}(\bar{c})}{\hat{\delta}_j\hat{\epsilon}_j}=\tilde{u}^i(\cdot)\qquad\text{in}\ C^{m,\alpha}(\partial\Omega),\tag{4.148}$$

*then there exists $j_0\in\mathbb{N}$ such that*

$$(u^i_j,u^o_j)=(u^i[\hat{\epsilon}_j,\hat{\delta}_j],u^o[\hat{\epsilon}_j,\hat{\delta}_j])\qquad\forall j_0\le j\in\mathbb{N}.$$

*Proof.* It is a simple modification of the proof of Theorem 4.84. Indeed, by Theorem 4.97, for each $j\in\mathbb{N}$, there exists a unique pair $(\psi_j,\theta_j)$ in $(C^{m,\alpha}(\partial\Omega))^2$ such that

$$u^i_j=u^i[\hat{\epsilon}_j,\hat{\delta}_j,\psi_j,\theta_j],\qquad u^o_j=u^o[\hat{\epsilon}_j,\hat{\delta}_j,\psi_j,\theta_j].\tag{4.149}$$

We shall now try to show that

$$\lim_{j\to\infty}(\psi_j,\theta_j)=(\tilde{\psi},\tilde{\theta})\qquad\text{in}\ (C^{m,\alpha}(\partial\Omega))^2.\tag{4.150}$$

Indeed, if we denote by $\tilde{\mathcal{U}}$ the neighbourhood of Theorem 4.99 *(iv)*, the limiting relation of (4.150) implies that there exists $j_0\in\mathbb{N}$ such that

$$(\hat{\epsilon}_j,\hat{\delta}_j,\psi_j,\theta_j)\in]0,\epsilon_3[\times]0,\delta_1[\times\tilde{\mathcal{U}},$$

for $j\ge j_0$ and thus Theorem 4.99 *(iv)* would imply that

$$(\psi_j,\theta_j)=(\Psi[\hat{\epsilon}_j,\hat{\delta}_j],\Theta[\hat{\epsilon}_j,\hat{\delta}_j]),$$

for $j_0\le j\in\mathbb{N}$, and that accordingly the theorem holds (cf. Definition 4.100.) Thus we now turn to the proof of (4.150). We note that equation $\Lambda[\epsilon,\delta,\psi,\theta]=0$ can be rewritten in the following form

$$\begin{aligned}
&F'(F^{(-1)}(\bar{c}))\Big(w^+[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s\Big)\\
&-\Big(w^-[\partial\Omega,\theta](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DS_n^a(\bar{x}-w-\epsilon s)\theta(s)\,d\sigma_s\Big)\\
&\qquad=-\delta\epsilon\Big(w^+[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s\Big)^2\\
&\times\int_0^1(1-\beta)F''\Big(F^{(-1)}(\bar{c})+\beta\delta\epsilon\big(w^+[\partial\Omega,\psi](t)-\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s\big)\Big)d\beta
\end{aligned}$$
$$\forall t\in\partial\Omega,\tag{4.151}$$

$$
\begin{aligned}
&-\gamma\int_{\partial\Omega}\nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\psi(s)\,d\sigma_s-\epsilon^n\gamma\int_{\partial\Omega}\nu_\Omega(t)D^2R^a_n(\epsilon(t-s))\nu_\Omega(s)\psi(s)\,d\sigma_s\\
&\quad+\int_{\partial\Omega}\nu_\Omega(t)D^2S_n(t-s)\nu_\Omega(s)\theta(s)\,d\sigma_s+\epsilon^n\int_{\partial\Omega}\nu_\Omega(t)D^2R^a_n(\epsilon(t-s))\nu_\Omega(s)\theta(s)\,d\sigma_s=-g(t)\\
&\qquad\qquad\forall t\in\partial\Omega,
\end{aligned}
\tag{4.152}
$$

$$
\Lambda_3[\epsilon,\delta,\psi,\theta]=0, \tag{4.153}
$$

for all $(\epsilon,\delta,\psi,\theta)$ in the domain of $\Lambda$. By arguing so as to prove that the integral of the second component of $\Lambda$ on $\partial\Omega$ equals zero in the beginning of the proof of Theorem 4.76, we can conclude that both hand sides of equation (4.152) have zero integral on $\partial\Omega$. We define the map $N\equiv(N_l)_{l=1,2,3}$ of $]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[\times(C^{m,\alpha}(\partial\Omega))^2$ to $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$ by setting $N_1[\epsilon,\delta,\psi,\theta]$ equal to the left-hand side of the equality in (4.151), $N_2[\epsilon,\delta,\psi,\theta]$ equal to the left-hand side of the equality in (4.152) and $N_3[\epsilon,\delta,\psi,\theta]=\Lambda_3[\epsilon,\delta,\theta,\psi]$ for all $(\epsilon,\delta,\psi,\theta)\in]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[\times(C^{m,\alpha}(\partial\Omega))^2$. By arguing as in the proof of Theorem 4.99, we can prove that $N$ is real analytic. Since $N[\epsilon,\delta,\cdot,\cdot]$ is linear for all $(\epsilon,\delta)\in]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[$, we have

$$
N[\epsilon,\delta,\psi,\theta]=\partial_{(\psi,\theta)}N[\epsilon,\delta,\tilde\psi,\tilde\theta](\psi,\theta)
$$

for all $(\epsilon,\delta,\psi,\theta)\in]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[\times(C^{m,\alpha}(\partial\Omega))^2$, and the map of $]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[$ to the space $\mathcal{L}((C^{m,\alpha}(\partial\Omega))^2,C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R})$ which takes $(\epsilon,\delta)$ to $N[\epsilon,\delta,\cdot,\cdot]$ is real analytic. Since

$$
N[0,0,\cdot,\cdot]=\partial_{(\psi,\theta)}\Lambda[0,0,\tilde\psi,\tilde\theta](\cdot,\cdot),
$$

Theorem 4.99 *(iii)* implies that $N[0,0,\cdot,\cdot]$ is also a linear homeomorphism. Since the set of linear homeomorphisms of $(C^{m,\alpha}(\partial\Omega))^2$ to $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}$ is open in $\mathcal{L}((C^{m,\alpha}(\partial\Omega))^2,C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R})$ and since the map which takes a linear invertible operator to its inverse is real analytic (cf. *e.g.*, Hille and Phillips [61, Theorems 4.3.2 and 4.3.4]), there exists $(\tilde\epsilon,\tilde\delta)\in]0,\epsilon_3[\times]0,\delta_1[$ such that the map $(\epsilon,\delta)\mapsto N[\epsilon,\delta,\cdot,\cdot]^{(-1)}$ is real analytic from $]-\tilde\epsilon,\tilde\epsilon[\times]-\tilde\delta,\tilde\delta[$ to $\mathcal{L}(C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R},(C^{m,\alpha}(\partial\Omega))^2)$. Next we denote by $S[\epsilon,\delta,\psi,\theta]\equiv(S_l[\epsilon,\delta,\psi,\theta])_{l=1,2,3}$ the triple defined by the right-hand side of (4.151)-(4.153). Then equation $\Lambda[\epsilon,\delta,\psi,\theta]=0$ (or equivalently system (4.151)-(4.153)) can be rewritten in the following form:

$$
(\psi,\theta)=N[\epsilon,\delta,\cdot,\cdot]^{(-1)}[S[\epsilon,\delta,\psi,\theta]], \tag{4.154}
$$

for all $(\epsilon,\delta,\psi,\theta)\in]-\tilde\epsilon,\tilde\epsilon[\times]-\tilde\delta,\tilde\delta[\times(C^{m,\alpha}(\partial\Omega))^2$. Next we note that condition (4.148), the proof of Theorem 4.99, the real analyticity of $F$ and standard calculus in Banach space imply that

$$
\lim_{j\to\infty}S[\hat\epsilon_j,\hat\delta_j,\psi_j,\theta_j]=S[0,0,\tilde\psi,\tilde\theta]\qquad\text{in } C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R}. \tag{4.155}
$$

Then by (4.154) and by the real analyticity of $(\epsilon,\delta)\mapsto N[\epsilon,\delta,\cdot,\cdot]^{(-1)}$, and by the bilinearity and continuity of the operator of $\mathcal{L}(C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R},(C^{m,\alpha}(\partial\Omega))^2)\times(C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}\times\mathbb{R})$ to $(C^{m,\alpha}(\partial\Omega))^2$, which takes a pair $(T_1,T_2)$ to $T_1[T_2]$, we conclude that (4.150) holds. Thus the proof is complete. □

We give the following definition.

**Definition 4.105.** Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\bar x\in\mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\bar\epsilon_1$, $F$, $g$, $\gamma$, $\bar c$ be as in (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.101 *(iv)*. Let $(u^i[\cdot,\cdot],u^o[\cdot,\cdot])$ be as in Definition 4.100. For each pair $(\epsilon,\delta)\in]0,\epsilon_3[\times]0,\delta_1[$, we set

$$
u^i_{(\epsilon,\delta)}(x)\equiv u^i[\epsilon,\delta](\frac{x}{\delta})\qquad\forall x\in\mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta),\qquad u^o_{(\epsilon,\delta)}(x)\equiv u^o[\epsilon,\delta](\frac{x}{\delta})\qquad\forall x\in\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).
$$

*Remark* 4.106. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\bar x\in\mathrm{cl}\,A\setminus\{w\}$. Let $\Omega$, $\bar\epsilon_1$, $F$, $g$, $\gamma$, $\bar c$ be as in (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.101 *(iv)*. For each $(\epsilon,\delta)\in]0,\epsilon_3[\times]0,\delta_1[$ the pair $(u^i_{(\epsilon,\delta)},u^o_{(\epsilon,\delta)})$ is a solution of (4.120).

Our aim is to study the asymptotic behaviour of $(u^i_{(\epsilon,\delta)},u^o_{(\epsilon,\delta)})$ as $(\epsilon,\delta)$ tends to $(0,0)$. As a first step, we study the behaviour of $(u^i[\epsilon,\delta],u^o[\epsilon,\delta])$ as $(\epsilon,\delta)$ tends to $(0,0)$.

**Proposition 4.107.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.101 (iv). Let $1 \leq p < \infty$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon,\delta]] = 0 \qquad \text{in } L^p(A),$$

*and*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon,\delta]] = \bar{c} \qquad \text{in } L^p(A).$$

*Proof.* Let $\epsilon_3$, $\delta_1$, $\Psi$, $\Theta$ be as in Theorem 4.99. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[$, we have

$$u^i[\epsilon,\delta] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) = \delta\epsilon w^+[\partial\Omega, \Psi[\epsilon,\delta]](t) - \delta\epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DR^a_n(\epsilon(t-s))\Psi[\epsilon,\delta](s)\, d\sigma_s + F^{(-1)}(\bar{c}), \qquad \forall t \in \partial\Omega.$$

We set

$$N^i[\epsilon,\delta](t) \equiv \delta\epsilon w^+[\partial\Omega, \Psi[\epsilon,\delta]](t) - \delta\epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DR^a_n(\epsilon(t-s))\Psi[\epsilon,\delta](s)\, d\sigma_s + F^{(-1)}(\bar{c}), \quad \forall t \in \partial\Omega,$$

for all $(\epsilon,\delta) \in ]-\epsilon_3,\epsilon_3[ \times ]-\delta_1,\delta_1[$. By taking $\tilde{\epsilon} \in ]0,\epsilon_3[$, $\tilde{\delta} \in ]0,\delta_1[$ small enough, we can assume (cf. Proposition 1.22 (*i*)) that $N^i$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[ \times ]-\tilde{\delta},\tilde{\delta}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C^i \equiv \sup_{(\epsilon,\delta)\in]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\delta},\tilde{\delta}[} \|N^i[\epsilon,\delta]\|_{C^0(\partial\Omega)} < +\infty.$$

By the Maximum Principle for harmonic functions, we have

$$|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon,\delta]](x)| \leq C^i \qquad \forall x \in A, \quad \forall (\epsilon,\delta) \in ]0,\tilde{\epsilon}[ \times ]0,\tilde{\delta}[.$$

Obviously,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon,\delta]](x) = 0 \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon,\delta]] = 0 \qquad \text{in } L^p(A).$$

If $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[$, we have

$$\begin{aligned} u^o[\epsilon,\delta] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) = & \delta\epsilon w^-[\partial\Omega, \Theta[\epsilon,\delta]](t) - \delta\epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DR^a_n(\epsilon(t-s))\Theta[\epsilon,\delta](s)\, d\sigma_s \\ & + \delta\epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DS^a_n(\bar{x} - w - \epsilon s)\Theta[\epsilon,\delta](s)\, d\sigma_s + \bar{c}, \qquad \forall t \in \partial\Omega. \end{aligned}$$

We set

$$\begin{aligned} N^o[\epsilon,\delta](t) \equiv & \delta\epsilon w^-[\partial\Omega, \Theta[\epsilon,\delta]](t) - \delta\epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DR^a_n(\epsilon(t-s))\Theta[\epsilon,\delta](s)\, d\sigma_s \\ & + \delta\epsilon^n \int_{\partial\Omega} \nu_\Omega(s) \cdot DS^a_n(\bar{x} - w - \epsilon s)\Theta[\epsilon,\delta](s)\, d\sigma_s + \bar{c}, \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $(\epsilon,\delta) \in ]-\epsilon_3,\epsilon_3[ \times ]-\delta_1,\delta_1[$. By taking $\tilde{\epsilon} \in ]0,\epsilon_3[$, $\tilde{\delta} \in ]0,\delta_1[$ small enough, we can assume (cf. Proposition 1.22 (*i*), (*iii*) and the proof of Theorem 4.78) that $N^o$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[ \times ]-\tilde{\delta},\tilde{\delta}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C^o \equiv \sup_{(\epsilon,\delta)\in]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\delta},\tilde{\delta}[} \|N^o[\epsilon,\delta]\|_{C^0(\partial\Omega)} < +\infty.$$

By Theorem 2.5, we have

$$|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon,\delta]](x)| \leq C^o \qquad \forall x \in A, \quad \forall (\epsilon,\delta) \in ]0,\tilde{\epsilon}[ \times ]0,\tilde{\delta}[.$$

Clearly (cf. Theorem 4.101), we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon,\delta]](x) = \bar{c} \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon,\delta]] = \bar{c} \qquad \text{in } L^p(A).$$

□

We also have the following.

**Theorem 4.108.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75)*, respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.99 (iv). Then there exist $\epsilon_6 \in ]0,\epsilon_3]$ and two real analytic operators $J^i$, $J^o$ of $]-\epsilon_6,\epsilon_6[\times]-\delta_1,\delta_1[$ to $\mathbb{R}$, such that*

$$\int_{\Omega_\epsilon} u^i[\epsilon,\delta](x)\,dx = J^i[\epsilon,\delta], \tag{4.156}$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^o[\epsilon,\delta](x)\,dx = J^o[\epsilon,\delta], \tag{4.157}$$

*for all $(\epsilon,\delta) \in ]0,\epsilon_6[\times]0,\delta_1[$. Moreover,*

$$J^i[0,0] = 0, \tag{4.158}$$

$$J^o[0,0] = \bar{c}|A|_n. \tag{4.159}$$

*Proof.* Let $\Theta[\cdot,\cdot]$, $\Psi[\cdot,\cdot]$ be as in Theorem 4.99. Let $(\epsilon,\delta) \in ]0,\epsilon_3[\times]0,\delta_1[$. Clearly,

$$u^o[\epsilon,\delta](x) = \delta\epsilon w_a^-\left[\partial\Omega_\epsilon, \Theta[\epsilon,\delta](\frac{1}{\epsilon}(\cdot - w))\right](x) + \delta\epsilon^n \int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon,\delta](s)\,d\sigma_s + \bar{c} \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$

Accordingly,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^o[\epsilon,\delta](x)\,dx = \delta\epsilon \int_{\mathbb{P}_a[\Omega_\epsilon]} w_a^-\left[\partial\Omega_\epsilon, \Theta[\epsilon,\delta](\frac{1}{\epsilon}(\cdot - w))\right](x)\,dx + \delta\epsilon^n \int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon,\delta](s)\,d\sigma_s\Big(|A|_n - \epsilon^n|\Omega|_n\Big) + \bar{c}\Big(|A|_n - \epsilon^n|\Omega|_n\Big).$$

By arguing as in Lemma 2.45, we can easily prove that there exist $\epsilon_6' \in ]0,\epsilon_3]$ and a real analytic operator $\tilde{J}_1$ of $]-\epsilon_6',\epsilon_6'[\times]-\delta_1,\delta_1[$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} w_a^-\left[\partial\Omega_\epsilon, \Theta[\epsilon,\delta](\frac{1}{\epsilon}(\cdot - w))\right](x)\,dx = \epsilon^n \tilde{J}_1[\epsilon,\delta],$$

for all $(\epsilon,\delta) \in ]0,\epsilon_6'[\times]0,\delta_1[$. Moreover, by arguing as in Theorem 4.78, we have that, by possibly taking a smaller $\epsilon_6' > 0$, the map $\tilde{J}_2$ of $]-\epsilon_6',\epsilon_6'[\times]-\delta_1,\delta_1[$ to $\mathbb{R}$, defined by

$$\tilde{J}_2[\epsilon,\delta] \equiv \delta\epsilon^n \int_{\partial\Omega} \nu_\Omega(s)\cdot DS_n^a(\bar{x} - w - \epsilon s)\Theta[\epsilon,\delta](s)\,d\sigma_s\Big(|A|_n - \epsilon^n|\Omega|_n\Big) + \bar{c}\Big(|A|_n - \epsilon^n|\Omega|_n\Big)$$

for all $(\epsilon,\delta) \in ]-\epsilon_6',\epsilon_6'[\times]-\delta_1,\delta_1[$, is real analytic. Hence, if we set,

$$J^o[\epsilon,\delta] \equiv \delta\epsilon^{n+1}\tilde{J}_1[\epsilon,\delta] + \tilde{J}_2[\epsilon,\delta]$$

for all $(\epsilon,\delta) \in ]-\epsilon_6',\epsilon_6'[\times]-\delta_1,\delta_1[$, we have that $J^o$ is a real analytic map of $]-\epsilon_6',\epsilon_6'[\times]-\delta_1,\delta_1[$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^o[\epsilon,\delta](x)\,dx = J^o[\epsilon,\delta],$$

for all $(\epsilon, \delta) \in ]0, \epsilon_6'[ \times ]0, \delta_1[$, and that $J^o[0,0] = \bar{c}|A|_n$.

Let $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[$. Clearly,

$$\int_{\Omega_\epsilon} u^i[\epsilon, \delta](x)\,dx = \epsilon^n \int_\Omega u^i[\epsilon, \delta](w + \epsilon t)\,dt.$$

On the other hand, if $\epsilon_4'$, $U_1^i$ are as in Theorem 4.101, and we set

$$J^i[\epsilon, \delta] \equiv \epsilon^n \int_\Omega \left(\delta\epsilon U_1^i[\epsilon, \delta](t) + F^{(-1)}(\bar{c})\right) dt$$

for all $(\epsilon, \delta) \in ]-\epsilon_4', \epsilon_4'[ \times ]-\delta_1, \delta_1[$, then we have that $J^i$ is a real analytic map of $]-\epsilon_4', \epsilon_4'[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$, such that $J^i[0,0] = 0$ and that

$$\int_{\Omega_\epsilon} u^i[\epsilon, \delta](x)\,dx = J^i[\epsilon, \delta]$$

for all $(\epsilon, \delta) \in ]0, \epsilon_4'[ \times ]0, \delta_1[$.

Then, by taking $\epsilon_6 \equiv \min\{\epsilon_6', \epsilon_4'\}$, we can easily conclude. □

## 4.10.2 Asymptotic behaviour of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

In the following Theorem we deduce by Proposition 4.107 the convergence of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ as $(\epsilon, \delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 4.109.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.101 (iv). Let $1 \le p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^p(V),$$

*and*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] = \bar{c} \qquad \text{in } L^p(V).$$

*Proof.* We modify the proof of Theorem 4.27. By virtue of Proposition 4.107, we have

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon,\delta]]\|_{L^p(A)} = 0,$$

and

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon,\delta]] - \bar{c}\|_{L^p(A)} = 0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $C > 0$ such that

$$\|\mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}]\|_{L^p(V)} \le C\|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon,\delta]]\|_{L^p(A)} \qquad \forall (\epsilon,\delta) \in ]0, \epsilon_1[ \times ]0, \min\{1, \delta_1\}[.$$

and

$$\|\mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] - \bar{c}\|_{L^p(V)} \le C\|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon,\delta]] - \bar{c}\|_{L^p(A)} \qquad \forall (\epsilon,\delta) \in ]0, \epsilon_1[ \times ]0, \min\{1, \delta_1\}[,$$

Thus,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^p(V),$$

and

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] = \bar{c} \qquad \text{in } L^p(V).$$

□

Then we have the following Theorem, where we consider a functional associated to extensions of $u^i_{(\epsilon,\delta)}$ and of $u^o_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 4.110.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56), (4.71), (4.72), (4.73), (4.74), (4.75)*, respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.99 (iv). Let $\epsilon_6$, $J^i$, $J^o$ be as in Theorem 4.108. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^i\big[\epsilon, \frac{r}{l}\big], \tag{4.160}$$

*and*

$$\int_{\mathbb{R}^n} \mathbf{E}^o_{(\epsilon,r/l)}[u^o_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^o\big[\epsilon, \frac{r}{l}\big], \tag{4.161}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N} \setminus \{0\}$ such that $l > (r/\delta_1)$.*

*Proof.* We follow the the proof of Theorem 2.150. Let $\epsilon \in ]0,\epsilon_6[$, and let $l \in \mathbb{N} \setminus \{0\}$, $l > (r/\delta_1)$. Then, by the periodicity of $u^i_{(\epsilon,r/l)}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx \\
&= \int_{rA} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx \\
&= l^n \int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx.
\end{aligned}$$

Then we note that

$$\begin{aligned}
\int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\Omega_\epsilon} u^i_{(\epsilon,r/l)}(x)\,dx \\
&= \int_{\frac{r}{l}\Omega_\epsilon} u^i\big[\epsilon,(r/l)\big]\big(\frac{l}{r}x\big)\,dx \\
&= \frac{r^n}{l^n}\int_{\Omega_\epsilon} u^i\big[\epsilon,(r/l)\big](t)\,dt \\
&= \frac{r^n}{l^n} J^i\big[\epsilon, \frac{r}{l}\big].
\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^i\big[\epsilon, \frac{r}{l}\big],$$

and the validity of (4.160) follows. The proof of (4.161) is very similar and is accordingly omitted. □

### 4.10.3 Asymptotic behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

This Subsection is devoted to the study of the behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$. We give the following.

**Definition 4.111.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.101 (*iv*). For each pair $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[$, we set

$$\mathrm{En}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{S}_a(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx + \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 4.112. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\, A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in (1.56), (1.57), (4.71), (4.72), (4.73), (4.74), (4.75), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.101 (*iv*). Let $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[$. We have

$$\begin{aligned}
\int_{\Omega(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\Omega(\epsilon,1)} |(\nabla u^i_{(\epsilon,\delta)})(\delta t)|^2\,dt \\
&= \delta^{n-2}\int_{\Omega_\epsilon} |\nabla u^i[\epsilon,\delta](t)|^2\,dt
\end{aligned}$$

and

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\,dx = \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u^o_{(\epsilon,\delta)})(\delta t)|^2\,dt$$
$$= \delta^{n-2} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon,\delta](t)|^2\,dt.$$

In the following Proposition we represent the function $\mathrm{En}(\cdot,\cdot)$ by means of real analytic functions.

**Proposition 4.113.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\bar{x} \in \mathrm{cl}\,A \setminus \{w\}$. Let $\Omega$, $\bar{\epsilon}_1$, $F$, $g$, $\gamma$, $\bar{c}$ be as in* (1.56)*,* (1.57)*,* (4.71)*,* (4.72)*,* (4.73)*,* (4.74)*,* (4.75)*, respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.101 (iv). Let $\epsilon_5$, $G^i$, $G^o$ be as in Theorem 4.102. Then*

$$\mathrm{En}\Big(\epsilon, \frac{1}{l}\Big) = \epsilon^n G^i[\epsilon,(1/l)] + \epsilon^n G^o[\epsilon,(1/l)],$$

*for all $\epsilon \in ]0,\epsilon_5[$ and for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* Let $(\epsilon,\delta) \in ]0,\epsilon_5[ \times ]0,\delta_1[$. By Remark 4.112 and Theorem 4.102, we have

$$\int_{\Omega(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx + \int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\,dx = \delta^n\epsilon^n G^i[\epsilon,\delta] + \delta^n\epsilon^n G^o[\epsilon,\delta] \tag{4.162}$$

where $G^i$, $G^o$ are as in Theorem 4.102. On the other hand, if $\epsilon \in ]0,\epsilon_5[$ and $l \in \mathbb{N}\setminus\{0\}$ is such that $l > (1/\delta_1)$, then we have

$$\mathrm{En}\Big(\epsilon,\frac{1}{l}\Big) = l^n \frac{1}{l^n}\Big\{\epsilon^n G^i[\epsilon,(1/l)] + \epsilon^n G^o[\epsilon,(1/l)]\Big\},$$
$$= \epsilon^n G^i[\epsilon,(1/l)] + \epsilon^n G^o[\epsilon,(1/l)],$$

and the conclusion easily follows. □

## 4.11 Asymptotic behaviour of the solutions of an alternative nonlinear transmission problem for the Laplace equation in a periodically perforated domain

In this Section we study the asymptotic behaviour of the solutions of an alternative nonlinear transmission problem for the Laplace equation in a periodically perforated domain with small holes.

### 4.11.1 Notation and preliminaries

We retain the notation introduced in Subsections 1.8.1, 4.2.1. However, we need to introduce also some other notation. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). We shall consider also the following assumptions.

$$F \text{ is an increasing real analytic diffeomorphism of } \mathbb{R} \text{ onto itself,} \tag{4.163}$$

$$g \in C^{m-1,\alpha}(\partial\Omega), \qquad \int_{\partial\Omega} g\,d\sigma = 0, \tag{4.164}$$

$$\gamma \in ]0,+\infty[, \tag{4.165}$$

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. For each $\epsilon \in ]0,\epsilon_1[$, we consider the following periodic nonlinear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a[\Omega_\epsilon], \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u^i(x+a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u^o(x+a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u^o(x) = F(u^i(x)) & \forall x \in \partial\Omega_\epsilon, \\ \frac{\partial}{\partial\nu_{\Omega_\epsilon}} u^o(x) = \gamma \frac{\partial}{\partial\nu_{\Omega_\epsilon}} u^i(x) + g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon, \\ \int_{\partial\Omega_\epsilon} u^o(x)\,d\sigma_x = 0. \end{cases} \tag{4.166}$$

We recall the following.

**Definition 4.114.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). We denote by $v_*[\partial\Omega, \cdot]$ the linear operator of $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$ defined by

$$v_*[\partial\Omega, \theta](t) \equiv \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\,d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $\theta \in C^{m-1,\alpha}(\partial\Omega)$.

We transform (4.166) into a system of integral equations by means of the following.

**Theorem 4.115.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.64). *Let $\epsilon \in ]0, \epsilon_1[$. Then the map $(u^i[\epsilon,\cdot,\cdot,\cdot], u^o[\epsilon,\cdot,\cdot,\cdot])$ of the set of triples $(\psi, \theta, \xi) \in (\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ that solve the following integral equations*

$$\begin{aligned}
&F'(F^{(-1)}(0))\Big(v^+[\partial\Omega,\psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi\Big) \\
&\quad + \epsilon\Big(v^+[\partial\Omega,\psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi\Big)^2 \\
&\quad \times \int_0^1 (1-\beta)F''\Big(F^{(-1)}(0) + \beta\epsilon\big(v^+[\partial\Omega,\psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi\big)\Big)d\beta \qquad (4.167)\\
&\quad - \Big(v^-[\partial\Omega,\theta](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\Big) \\
&\quad + \frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(v^-[\partial\Omega,\theta](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\Big)\,d\sigma_t = 0 \quad \forall t \in \partial\Omega,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\theta(t) + v_*[\partial\Omega,\theta](t) + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s \\
&\quad + \frac{1}{2}\gamma\psi(t) - \gamma v_*[\partial\Omega,\psi](t) - \gamma\epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s - g(t) = 0 \quad \forall t \in \partial\Omega,
\end{aligned} \qquad (4.168)$$

*to the set of pairs $(u^i, u^o)$ of $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ which solve problem* (4.166), *which takes $(\psi, \theta, \xi)$ to the pair of functions*

$$\begin{aligned}
&\Big(u^i[\epsilon,\psi,\theta,\xi] \equiv v_a^+[\partial\Omega_\epsilon, \psi(\tfrac{1}{\epsilon}(\cdot - w))] + \epsilon\xi + F^{(-1)}(0), \\
&\qquad u^o[\epsilon,\psi,\theta,\xi] \equiv v_a^-[\partial\Omega_\epsilon, \theta(\tfrac{1}{\epsilon}(\cdot - w))] - \frac{1}{\int_{\partial\Omega_\epsilon} d\sigma}\int_{\partial\Omega_\epsilon} v_a^-[\partial\Omega_\epsilon, \theta(\tfrac{1}{\epsilon}(\cdot - w))]\,d\sigma\Big), \quad (4.169)
\end{aligned}$$

*is a bijection.*

*Proof.* Let $\epsilon \in ]0, \epsilon_1[$. Assume that the pair $(u^i, u^o)$ in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ solves problem (4.166). Then by Propositions 2.23, 2.24, it is easy to see that there exists a unique triple $(\psi, \theta, \xi)$ in $(\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$, such that

$$u^i = v_a^+[\partial\Omega_\epsilon, \psi(\tfrac{1}{\epsilon}(\cdot - w))] + \epsilon\xi + F^{(-1)}(0) \qquad \text{in } \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon],$$

and

$$u^o = v_a^-[\partial\Omega_\epsilon, \theta(\tfrac{1}{\epsilon}(\cdot - w))] - \frac{1}{\int_{\partial\Omega_\epsilon} d\sigma}\int_{\partial\Omega_\epsilon} v_a^-[\partial\Omega_\epsilon, \theta(\tfrac{1}{\epsilon}(\cdot - w))]\,d\sigma \qquad \text{in } \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$

Then a simple computation based on the Theorem of change of variables in integrals, on identity (1.65), and on the definition of $\mathcal{U}_0^{m-1,\alpha}$, shows that the triple $(\psi, \theta, \xi)$ must solve equation (4.168), together with the following

$$\begin{aligned}
&F\Big(\epsilon\big(v^+[\partial\Omega,\psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi\big) + F^{(-1)}(0)\Big) \\
&\qquad = \epsilon\Big(v^-[\partial\Omega,\theta](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\Big) \qquad (4.170)\\
&\qquad - \frac{\epsilon}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(v^-[\partial\Omega,\theta](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\Big)\,d\sigma_t \qquad \forall t \in \partial\Omega.
\end{aligned}$$

We now show that equation (4.170) implies the validity of (4.167). By Taylor Formula, we have

$$F(x + F^{(-1)}(0)) = 0 + F'(F^{(-1)}(0))x + x^2 \int_0^1 (1-\beta)F''(F^{(-1)}(0) + \beta x)d\beta \qquad \forall x \in \mathbb{R}.$$

Then, by dividing both sides of (4.170) by $\epsilon$, we can rewrite (4.170) as (4.167). Conversely, by reading backward the above argument, one can easily show that if $(\psi, \theta, \xi)$ in $(\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ solves (4.167)-(4.168), then the pair of functions of (4.169) satisfies problem (4.166). □

Hence we are reduced to analyse system (4.167)-(4.168). As a first step in the analysis of system (4.167)-(4.168), we note that for $\epsilon = 0$, since $\psi$, $\theta \in \mathcal{U}_0^{m-1,\alpha}$, one obtains a system which we address to as the *limiting system* and which has the following form

$$F'(F^{(-1)}(0))\Big(v^+[\partial\Omega, \psi](t) + \xi\Big) - \Big(v^-[\partial\Omega, \theta](t) - \frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} v^-[\partial\Omega, \theta]\, d\sigma\Big) = 0 \quad \forall t \in \partial\Omega, \quad (4.171)$$

$$\frac{1}{2}\theta(t) + v_*[\partial\Omega, \theta](t) + \frac{1}{2}\gamma\psi(t) - \gamma v_*[\partial\Omega, \psi](t) - g(t) = 0 \qquad \forall t \in \partial\Omega. \quad (4.172)$$

In order to analyse the limiting system, we need the following technical statement.

**Theorem 4.116.** *Let $m \in \mathbb{N} \setminus \{0\}$ $\alpha \in ]0, 1[$. Let $\Omega$ be as in* (1.56)*. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.64)*. Then the following statements hold.*

*(i) Let $\bar{f} \in C^{m,\alpha}(\partial\Omega)$. Then there exists a unique pair $(\eta, \tau) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$, such that*

$$\bar{f}(t) = v[\partial\Omega, \eta](t) + \tau \qquad \forall t \in \partial\Omega. \quad (4.173)$$

*(ii) Let $\phi,\gamma \in ]0, +\infty[$. If $(\bar{f}, \bar{g}) \in C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha}$, then the system*

$$\begin{cases} \phi(v^+[\partial\Omega, \psi](t) + \xi) - (v^-[\partial\Omega, \theta](t) - \frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} v^-[\partial\Omega, \theta]\, d\sigma) = \bar{f}(t) & \forall t \in \partial\Omega, \\ \frac{1}{2}\theta(t) + v_*[\partial\Omega, \theta](t) + \frac{1}{2}\gamma\psi(t) - \gamma v_*[\partial\Omega, \psi](t) = \bar{g}(t) & \forall t \in \partial\Omega, \end{cases} \quad (4.174)$$

*has one and only one solution $(\psi, \theta, \xi) \in (\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$.*

*Proof.* We first prove statement $(i)$. Let $\bar{f} \in C^{m,\alpha}(\partial\Omega)$. Let $\bar{u} \in C^{m,\alpha}(\mathrm{cl}\,\Omega)$ be the unique solution of the following Dirichlet problem for the Laplace operator

$$\begin{cases} \Delta \bar{u} = 0 & \text{in } \Omega, \\ \bar{u} = \bar{f} & \text{on } \partial\Omega. \end{cases}$$

By classical potential theory (cf. Folland [52, Chapter 3]), there exists a unique $\eta \in C^{m-1,\alpha}(\partial\Omega)$ such that

$$\begin{cases} -\frac{1}{2}\eta + v_*[\partial\Omega, \eta] = \frac{\partial}{\partial\nu_\Omega}\bar{u} & \text{on } \partial\Omega \\ \int_{\partial\Omega} \eta\, d\sigma = 0. \end{cases}$$

Accordingly, $\bar{u} - v^+[\partial\Omega, \eta]$ is constant in $\mathrm{cl}\,\Omega$. Then, if we set

$$\tau \equiv \bar{u}(\bar{x}) - v^+[\partial\Omega, \eta](\bar{x}),$$

for any $\bar{x} \in \mathrm{cl}\,\Omega$, we clearly obtain

$$\tau + v[\partial\Omega, \eta](t) = \bar{f}(t) \qquad \forall t \in \partial\Omega.$$

For the uniqueness of such a pair, it suffices to observe that if $(\eta, \tau) \in \mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$ and

$$\tau + v[\partial\Omega, \eta](t) = 0 \qquad \forall t \in \partial\Omega,$$

then

$$-\frac{1}{2}\eta(t) + v_*[\partial\Omega, \eta](t) = 0 \qquad \forall t \in \partial\Omega,$$

that together with $\int_{\partial\Omega} \eta \, d\sigma = 0$, by classical potential theory, implies $\eta = 0$ and consequently $\tau = 0$. We now prove statement $(ii)$. To do so, we shall assume that system (4.174) has a solution and prove that such a solution must necessarily be delivered by a certain formula. Thus uniqueness will follow. Then we shall exploit such a formula to show existence. By $(i)$, there exists a unique pair $(\eta, \tau)$ in $\mathcal{U}_0^{m-1,\alpha} \times \mathbb{R}$, such that

$$\bar{f} = v[\partial\Omega, \eta] + \tau \qquad \text{on } \partial\Omega.$$

The first equation of (4.174) implies that

$$\phi\psi(t) - \theta(t) = \eta(t) \qquad \forall t \in \partial\Omega. \tag{4.175}$$

Moreover, by integrating both sides of the first equation of (4.174), we obtain

$$\xi = \frac{1}{\int_{\partial\Omega} d\sigma} \Big( \frac{1}{\phi} \int_{\partial\Omega} \bar{f} \, d\sigma - \int_{\partial\Omega} v^+[\partial\Omega, \psi] \, d\sigma \Big). \tag{4.176}$$

By (4.175), we can rewrite the second equation of (4.174) in the following form

$$\frac{1}{2}\psi(t) - \frac{\gamma - \phi}{\gamma + \phi} v_*[\partial\Omega, \psi](t) = \frac{1}{\gamma + \phi} \big(\bar{g}(t) + \frac{1}{2}\eta(t) + v_*[\partial\Omega, \eta](t)\big) \qquad \forall t \in \partial\Omega.$$

Clearly,

$$\int_{\partial\Omega} \big(\bar{g}(t) + \frac{1}{2}\eta(t) + v_*[\partial\Omega, \eta](t)\big) \, d\sigma_t = 0.$$

Hence, by Proposition 4.10 $(ii)$, $(iii)$, we have

$$\psi = \frac{1}{\gamma + \phi} \Big( \frac{1}{2} I - \frac{\gamma - \phi}{\gamma + \phi} v_*[\partial\Omega, \cdot] \Big)^{(-1)} \big(\bar{g} + \frac{1}{2}\eta + v_*[\partial\Omega, \eta]\big). \tag{4.177}$$

Hence $\psi$ in $\mathcal{U}_0^{m-1,\alpha}$ is uniquely determined. Consequently, equalities (4.175),(4.176) uniquely determine $\theta$ in $\mathcal{U}_0^{m-1,\alpha}$ and $\xi$ in $\mathbb{R}$. Conversely, by reading backward the proof above, one can easily check that the triple $(\psi, \theta, \xi)$ delivered by formulas (4.175)-(4.177), belongs to $(\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ and solves system (4.174). □

Then we have the following theorem, which shows the unique solvability of the limiting system, and its link with a boundary value problem which we shall address to as the *limiting boundary value problem.*

**Theorem 4.117.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $F$, $g$, $\gamma$ be as in* (1.56), (4.163), (4.164), (4.165), *respectively. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.64). *Then the following statements hold.*

*(i) The* limiting system (4.171)-(4.172) *has one and only one solution in $(\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$, which we denote by $(\tilde{\psi}, \tilde{\theta}, \tilde{\xi})$.*

*(ii) The limiting boundary value problem*

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \Omega, \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega, \\ u^o(x) - \frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} u^o \, d\sigma = F'(F^{(-1)}(0)) u^i(x) & \forall x \in \partial\Omega, \\ \frac{\partial}{\partial\nu_\Omega} u^o(x) = \gamma \frac{\partial}{\partial\nu_\Omega} u^i(x) + g(x) & \forall x \in \partial\Omega, \\ \lim_{x \to \infty} u^o(x) = 0 \end{cases} \tag{4.178}$$

*has one and only one solution $(\tilde{u}^i, \tilde{u}^o)$ in $C^{m,\alpha}(\mathrm{cl}\,\Omega) \times C^{m,\alpha}(\mathbb{R}^n \setminus \Omega)$, and the following formulas hold:*

$$\tilde{u}^i \equiv v^+[\partial\Omega, \tilde{\psi}] + \tilde{\xi} \qquad \text{in } \mathrm{cl}\,\Omega, \tag{4.179}$$

$$\tilde{u}^o \equiv v^-[\partial\Omega, \tilde{\theta}] \qquad \text{in } \mathbb{R}^n \setminus \Omega. \tag{4.180}$$

*Proof.* The statement in $(i)$ is an immediate consequence of Theorem 4.116. We now consider $(ii)$. By Theorem B.2, it is immediate to see that the functions $\tilde{u}^i$, $\tilde{u}^o$ delivered by the right-hand side of (4.179), (4.180), belong to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega)$, respectively and solve problem (4.178). The uniqueness of the solution of problem (4.178) follows by an easy computation based on the Divergence Theorem and on Folland [52, p. 118]. Indeed, it suffices to observe that if $(v^i, v^o)$ is a pair of functions in $(C^2(\Omega) \cap C^1(\mathrm{cl}\,\Omega)) \times (C^2(\mathbb{R}^n \setminus \mathrm{cl}\,\Omega) \cap C^1(\mathbb{R}^n \setminus \Omega))$, such that

$$\begin{cases} \Delta v^i(x) = 0 & \forall x \in \Omega, \\ \Delta v^o(x) = 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega, \\ v^o(x) - \frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} v^o \, d\sigma = F'(F^{(-1)}(0)) v^i(x) & \forall x \in \partial\Omega, \\ \frac{\partial}{\partial\nu_\Omega} v^o(x) = \gamma \frac{\partial}{\partial\nu_\Omega} v^i(x) & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} v^o(x) = 0, \end{cases}$$

then

$$\begin{aligned} 0 \le \int_\Omega |\nabla v^i(x)|^2 \, dx &= \int_{\partial\Omega} \frac{\partial v^i}{\partial\nu_\Omega} v^i \, d\sigma \\ &= \frac{1}{F'(F^{(-1)}(0))\gamma} \Big( \int_{\partial\Omega} \frac{\partial v^o}{\partial\nu_\Omega} v^o \, d\sigma - \frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} v^o \, d\sigma \int_{\partial\Omega} \frac{\partial v^o}{\partial\nu_\Omega} \, d\sigma \Big) \\ &= \frac{1}{F'(F^{(-1)}(0))\gamma} \int_{\partial\Omega} \frac{\partial v^o}{\partial\nu_\Omega} v^o \, d\sigma \\ &= -\frac{1}{F'(F^{(-1)}(0))\gamma} \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla v^o(x)|^2 \, dx \le 0, \end{aligned}$$

and so

$$v^o(x) = 0 \qquad \forall x \in \mathbb{R}^n \setminus \Omega,$$

and

$$v^i(x) = 0 \qquad \forall x \in \mathrm{cl}\,\Omega.$$

□

We are now ready to analyse equations (4.167)-(4.168) around the degenerate case $\epsilon = 0$.

**Theorem 4.118.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.64). *Let $\Lambda \equiv (\Lambda_j)_{j=1,2}$ be the map of $]-\epsilon_1, \epsilon_1[ \times (\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ to $C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha}$, defined by*

$$\begin{aligned} \Lambda_1[\epsilon,\psi,\theta,\xi](t) \equiv\, & F'(F^{(-1)}(0)) \Big( v^+[\partial\Omega,\psi](t) + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\psi(s)\, d\sigma_s + \xi \Big) \\ & + \epsilon \Big( v^+[\partial\Omega,\psi](t) + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\psi(s)\, d\sigma_s + \xi \Big)^2 \\ & \times \int_0^1 (1-\beta) F'' \Big( F^{(-1)}(0) + \beta\epsilon \big( v^+[\partial\Omega,\psi](t) + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\psi(s)\, d\sigma_s + \xi \big) \Big) d\beta \\ & - \Big( v^-[\partial\Omega,\theta](t) + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s \Big) \\ & + \frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} \Big( v^-[\partial\Omega,\theta](t) + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s \Big) d\sigma_t \quad \forall t \in \partial\Omega, \end{aligned} \tag{4.181}$$

$$\begin{aligned} \Lambda_2[\epsilon,\psi,\theta,\xi](t) \equiv\, & \frac{1}{2}\theta(t) + v_*[\partial\Omega,\theta](t) + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s \\ & + \frac{1}{2}\gamma\psi(t) - \gamma v_*[\partial\Omega,\psi](t) - \gamma\epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^a(\epsilon(t-s))\psi(s)\, d\sigma_s - g(t) \quad \forall t \in \partial\Omega, \end{aligned} \tag{4.182}$$

*for all $(\epsilon,\psi,\theta,\xi) \in ]-\epsilon_1,\epsilon_1[ \times (\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$. Then the following statements hold.*

*(i) Equation $\Lambda[0,\psi,\theta,\xi] = 0$ is equivalent to the limiting system* (4.171)-(4.172) *and has one and only one solution $(\tilde{\psi}, \tilde{\theta}, \tilde{\xi})$ (cf. Theorem 4.117.)*

*(ii) If $\epsilon \in ]0,\epsilon_1[$, then equation $\Lambda[\epsilon,\psi,\theta,\xi] = 0$ is equivalent to system* (4.167)-(4.168) *for $(\psi,\theta,\xi)$.*

*(iii)* *There exists* $\epsilon_2 \in ]0, \epsilon_1]$ *such that* $\Lambda$ *is a real analytic map of* $]-\epsilon_2, \epsilon_2[ \times (\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ *to* $C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha}$. *The differential* $\partial_{(\psi,\theta,\xi)}\Lambda[0, \tilde{\psi}, \tilde{\theta}, \tilde{\xi}]$ *of* $\Lambda$ *at* $(0, \tilde{\psi}, \tilde{\theta}, \tilde{\xi})$ *is a linear homeomorphism of* $(\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ *to* $C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha}$.

*(iv)* *There exist* $\epsilon_3 \in ]0, \epsilon_2]$ *and an open neighbourhood* $\tilde{\mathcal{U}}$ *of* $(\tilde{\psi}, \tilde{\theta}, \tilde{\xi})$ *in* $(\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ *and a real analytic map* $(\Psi[\cdot], \Theta[\cdot], \Xi[\cdot])$ *of* $]-\epsilon_3, \epsilon_3[$ *to* $\tilde{\mathcal{U}}$, *such that the set of zeros of the map* $\Lambda$ *in* $]-\epsilon_3, \epsilon_3[ \times \tilde{\mathcal{U}}$ *coincides with the graph of* $(\Psi[\cdot], \Theta[\cdot], \Xi[\cdot])$. *In particular,* $(\Psi[0], \Theta[0], \Xi[0]) = (\tilde{\psi}, \tilde{\theta}, \tilde{\xi})$.

*Proof.* First of all we want to prove that

$$\int_{\partial\Omega} \Lambda_2[\epsilon, \psi, \theta, \xi]\, d\sigma = 0, \tag{4.183}$$

for all $(\epsilon, \psi, \theta, \xi) \in ]-\epsilon_1, \epsilon_1[ \times (\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$. If $\epsilon = 0$, by Fubini's Theorem and since $\int_{\partial\Omega} \psi\, d\sigma = 0$ and $\int_{\partial\Omega} \theta\, d\sigma = 0$, we have

$$\int_{\partial\Omega} v_*[\partial\Omega, \psi]\, d\sigma = 0,$$

$$\int_{\partial\Omega} v_*[\partial\Omega, \theta]\, d\sigma = 0,$$

and so, since $\int_{\partial\Omega} g\, d\sigma = 0$, we immediately obtain (4.183). If $\epsilon \neq 0$, we need to observe also that the functions

$$t \mapsto \int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s$$

and

$$t \mapsto \int_{\partial\Omega} R_n^a(\epsilon(t-s))\psi(s)\, d\sigma_s$$

of $\mathrm{cl}\,\Omega$ to $\mathbb{R}$ are harmonic in $\Omega$. Then, by the Divergence Theorem, we have

$$\int_{\partial\Omega}\int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^a(\epsilon(t-s))\theta(s)\, d\sigma_s\, d\sigma_t = 0$$

and

$$\int_{\partial\Omega}\int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^a(\epsilon(t-s))\psi(s)\, d\sigma_s\, d\sigma_t = 0.$$

Thus, by the above argument for the case $\epsilon = 0$, we easily obtain (4.183). The statements in *(i)* and *(ii)* are obvious. By an easy modification of the proof of Theorem 4.76 *(iii)*, one can easily show that there exists $\epsilon_2 \in ]0, \epsilon_1]$ such that $\Lambda$ is a real analytic map of $]-\epsilon_2, \epsilon_2[ \times (\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ to $C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha}$. By standard calculus in Banach space, the differential of $\Lambda$ at $(0, \tilde{\psi}, \tilde{\theta}, \tilde{\xi})$ with respect to the variables $(\psi, \theta, \xi)$ is delivered by the following formulas

$$\begin{aligned}\partial_{(\psi,\theta,\xi)}\Lambda_1[0, \tilde{\psi}, \tilde{\theta}, \tilde{\xi}](\bar{\psi}, \bar{\theta}, \bar{\xi})(t) =& F'(F^{(-1)}(0))(v^+[\partial\Omega, \bar{\psi}](t) + \bar{\xi}) \\ & - (v^-[\partial\Omega, \bar{\theta}](t) - \frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} v^-[\partial\Omega, \bar{\theta}]\, d\sigma) \quad \forall t \in \partial\Omega,\end{aligned}$$

$$\partial_{(\psi,\theta,\xi)}\Lambda_2[0, \tilde{\psi}, \tilde{\theta}, \tilde{\xi}](\bar{\psi}, \bar{\theta}, \bar{\xi})(t) = \frac{1}{2}\bar{\theta}(t) + v_*[\partial\Omega, \bar{\theta}](t) + \frac{1}{2}\gamma\psi(t) - \gamma v_*[\partial\Omega, \bar{\psi}](t) \qquad \forall t \in \partial\Omega,$$

for all $(\bar{\psi}, \bar{\theta}, \bar{\xi}) \in (\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$. We now show that the above differential is a linear homeomorphism. By the Open Mapping Theorem, it suffices to show that it is a bijection of $(\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ to $C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha}$. Let $(\bar{f}, \bar{g}) \in C^{m,\alpha}(\partial\Omega) \times \mathcal{U}_0^{m-1,\alpha}$. We must show that there exists a unique triple $(\bar{\psi}, \bar{\theta}, \bar{\xi}) \in (\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ such that

$$\partial_{(\psi,\theta,\xi)}\Lambda[0, \tilde{\psi}, \tilde{\theta}, \tilde{\xi}](\bar{\psi}, \bar{\theta}, \bar{\xi}) = (\bar{f}, \bar{g}). \tag{4.184}$$

By Theorem 4.116, there exists a unique triple $(\bar{\psi}, \bar{\theta}, \bar{\xi}) \in (\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ such that (4.184) holds. Thus the proof of statement *(iii)* is complete. Statement *(iv)* is an immediate consequence of statement *(iii)* and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

We are now in the position to introduce the following.

**Definition 4.119.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $u^i[\cdot,\cdot,\cdot]$ and $u^o[\cdot,\cdot,\cdot]$ be as in Theorem 4.115. If $\epsilon \in ]0,\epsilon_3[$, we set

$$u^i[\epsilon](t) \equiv u^i[\epsilon,\Psi[\epsilon],\Theta[\epsilon],\Xi[\epsilon]](t) \qquad \forall t \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon],$$
$$u^o[\epsilon](t) \equiv u^o[\epsilon,\Psi[\epsilon],\Theta[\epsilon],\Xi[\epsilon]](t) \qquad \forall t \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],$$

where $\epsilon_3$, $\Psi$, $\Theta$, $\Xi$ are as in Theorem 4.118 $(iv)$.

### 4.11.2 A functional analytic representation Theorem for of the family $\{(u^i[\epsilon], u^o[\epsilon])\}_{\epsilon\in]0,\epsilon_3[}$

In this Subsection, we show that $\{(u^i[\epsilon](\cdot), u^o[\epsilon](\cdot))\}_{\epsilon\in]0,\epsilon_3[}$ can be continued real analytically for negative values of $\epsilon$.

We have the following.

**Theorem 4.120.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\epsilon_3$ be as in Theorem 4.118 $(iv)$. Then the following statements hold.*

*(i) Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0,\epsilon_3]$, a real analytic operator $U_1^o$ of $]-\epsilon_4,\epsilon_4[$ to the space $C_h^0(\mathrm{cl}\,V)$, and a real analytic operator $U_2^o$ of $]-\epsilon_4,\epsilon_4[$ to $\mathbb{R}$, such that the following conditions hold.*

*(j) $\mathrm{cl}\,V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ for all $\epsilon \in ]-\epsilon_4,\epsilon_4[$.*

*(jj)*
$$u^o[\epsilon](x) = \epsilon^n U_1^o[\epsilon](x) + \epsilon U_2^o[\epsilon] \qquad \forall x \in \mathrm{cl}\,V,$$
*for all $\epsilon \in ]0,\epsilon_4[$.*

*(ii) Let $\bar{V}$ be a bounded open subset of $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$. Then there exist $\bar{\epsilon}_4 \in ]0,\epsilon_3]$, a real analytic operator $\bar{U}_1^o$ of $]-\bar{\epsilon}_4,\bar{\epsilon}_4[$ to the space $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$, and a real analytic operator $\bar{U}_2^o$ of $]-\bar{\epsilon}_4,\bar{\epsilon}_4[$ to $\mathbb{R}$, such that the following conditions hold.*

*(j') $w + \epsilon\,\mathrm{cl}\,\bar{V} \subseteq \mathrm{cl}\,\mathbb{P}_a[\Omega_\epsilon]$ for all $\epsilon \in ]-\bar{\epsilon}_4,\bar{\epsilon}_4[ \setminus \{0\}$.*

*(jj')*
$$u^o[\epsilon](w+\epsilon t) = \epsilon \bar{U}_1^o[\epsilon](t) + \epsilon \bar{U}_2^o[\epsilon] \qquad \forall t \in \mathrm{cl}\,\bar{V},$$
*for all $\epsilon \in ]0,\bar{\epsilon}_4[$. Moreover, $\bar{U}_1^o[0](\cdot)$ equals the restriction of $\tilde{u}^o(\cdot)$ to $\mathrm{cl}\,\bar{V}$.*

*(iii) There exist $\epsilon_4' \in ]0,\epsilon_3]$, a real analytic operator $U_1^i$ of $]-\epsilon_4',\epsilon_4'[$ to the space $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, and a real analytic operator $U_2^i$ of $]-\epsilon_4',\epsilon_4'[$ to $\mathbb{R}$, such that*
$$u^i[\epsilon](w+\epsilon t) = \epsilon U_1^i[\epsilon](t) + \epsilon U_2^i[\epsilon] + F^{(-1)}(0) \qquad \forall t \in \mathrm{cl}\,\Omega,$$
*for all $\epsilon \in ]0,\epsilon_4'[$. Moreover, $U_1^i[0](\cdot) + U_2^i[0]$ equals $\tilde{u}^i(\cdot)$ on $\mathrm{cl}\,\Omega$.*

*Proof.* Let $\Psi[\cdot]$, $\Theta[\cdot]$, $\Xi[\cdot]$ be as in Theorem 4.118 $(iv)$. Consider $(i)$. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $\epsilon \in ]0,\epsilon_4[$. We have

$$u^o[\epsilon](x) = \epsilon^{n-1}\int_{\partial\Omega} S_n^a(x-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s - \epsilon^{n-1}\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(\int_{\partial\Omega} S_n^a(\epsilon(t-s))\Theta[\epsilon](s)\,d\sigma_s\Big)\,d\sigma_t \qquad \forall x \in \mathrm{cl}\,V.$$

Thus (cf. the proof of Theorem 2.158), it is natural to set

$$\tilde{U}_1^o[\epsilon](x) \equiv \int_{\partial\Omega} S_n^a(x-w-\epsilon s)\Theta[\epsilon](s)\,d\sigma_s \qquad \forall x \in \mathrm{cl}\,V,$$

for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$, and

$$U_2^o[\epsilon] \equiv -\frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \Big)\, d\sigma_t,$$

for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$. Following the proof of Proposition 1.29 $(i)$, by possibly taking a smaller $\epsilon_4$, we have that there exists a real analytic map $U_1^o$ of $]-\epsilon_4, \epsilon_4[$ to $C_h^0(\mathrm{cl}\, V)$ such that

$$\tilde{U}_1^o[\epsilon] = \epsilon U_1^o[\epsilon] \qquad \text{in } C_h^0(\mathrm{cl}\, V),$$

for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$. Furthermore, we have that $U_2^o$ is a real analytic operator of $]-\epsilon_4, \epsilon_4[$ to $\mathbb{R}$. Finally, by the definition of $U_1^o$ and $U_2^o$, we immediately deduce that the equality in $(jj)$ holds. Consider now $(ii)$. Choosing $\bar{\epsilon}_4$ small enough, we can clearly assume that $(j')$ holds. Consider now $(jj')$. Let $\epsilon \in ]0, \bar{\epsilon}_4[$. We have

$$u^o[\epsilon](w+\epsilon t) = \epsilon^{n-1} \int_{\partial\Omega} S_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s - \epsilon^{n-1} \frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \Big)\, d\sigma_t \qquad \forall t \in \mathrm{cl}\, \bar{V}.$$

Thus (cf. Proposition 1.29 $(ii)$), it is natural to set

$$\bar{U}_1^o[\epsilon](t) \equiv \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \qquad \forall t \in \mathrm{cl}\, \bar{V},$$

for all $\epsilon \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[$, and

$$\bar{U}_2^o[\epsilon] \equiv -\frac{1}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \Big)\, d\sigma_t,$$

for all $\epsilon \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[$. By the proof of $(i)$, we have that $\bar{U}_2^o$ is a real analytic map of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ to $\mathbb{R}$. Moreover, (cf. Proposition 1.29 $(ii)$) we have that $\bar{U}_1^o$ is a real analytic map of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[$ to $C^{m,\alpha}(\mathrm{cl}\, \bar{V})$. Finally, consider $(iii)$. Let $\epsilon \in ]0, \epsilon_3[$. We have

$$u^i[\epsilon](w+\epsilon t) = \epsilon^{n-1} \int_{\partial\Omega} S_n^a(\epsilon(t-s))\Psi[\epsilon](s)\, d\sigma_s + \epsilon \Xi[\epsilon] + F^{(-1)}(0) \qquad \forall t \in \mathrm{cl}\, \Omega.$$

Thus, by arguing as above, it is natural to set

$$U_1^i[\epsilon](t) \equiv \int_{\partial\Omega} S_n(t-s)\Psi[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Psi[\epsilon](s)\, d\sigma_s \qquad \forall t \in \mathrm{cl}\, \Omega,$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$, and

$$U_2^i[\epsilon] \equiv \Xi[\epsilon]$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. Then, by arguing as above (cf. Proposition 1.29 $(iii)$), there exists $\epsilon_4' \in ]0, \epsilon_3]$, such that $U_1^i$ and $U_2^i$ are real analytic maps of $]-\epsilon_4', \epsilon_4'[$ to $C^{m,\alpha}(\mathrm{cl}\, \Omega)$ and $\mathbb{R}$, respectively, such that the statement in $(iii)$ holds. □

*Remark* 4.121. We note that the right-hand side of the equalities in $(jj)$, $(jj')$ and $(iii)$ of Theorem 4.120 can be continued real analytically in a whole neighbourhood of 0. Moreover, if $V$ is a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$, then

$$\lim_{\epsilon \to 0^+} u^o[\epsilon] = 0 \qquad \text{uniformly in } \mathrm{cl}\, V.$$

### 4.11.3 A real analytic continuation Theorem for the energy integral

As done in Theorem 4.120 for $(u^i[\cdot], u^o[\cdot])$, we can now prove a real analytic continuation Theorem for the energy integral. Namely, we prove the following.

**Theorem 4.122.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\epsilon_3$ be as in Theorem 4.118* (*iv*). *Then there exist $\epsilon_5 \in ]0, \epsilon_3]$ and two real analytic operators $G^i$, $G^o$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{R}$, such that*

$$\int_{\Omega_\epsilon} |\nabla u^i[\epsilon](x)|^2 \, dx = \epsilon^n G^i[\epsilon], \tag{4.185}$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](x)|^2 \, dx = \epsilon^n G^o[\epsilon], \tag{4.186}$$

*for all $\epsilon \in ]0, \epsilon_5[$. Moreover,*

$$G^i[0] = \int_{\Omega} |\nabla \tilde{u}^i(x)|^2 \, dx, \tag{4.187}$$

$$G^o[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\, \Omega} |\nabla \tilde{u}^o(x)|^2 \, dx. \tag{4.188}$$

*Proof.* Let $\Psi[\cdot]$, $\Theta[\cdot]$, $\Xi[\cdot]$ be as in Theorem 4.118 (*iv*). Let $\epsilon \in ]0, \epsilon_3[$. Clearly,

$$\int_{\Omega_\epsilon} |\nabla u^i[\epsilon](x)|^2 \, dx = \int_{\Omega_\epsilon} |\nabla v_a^+[\partial\Omega_\epsilon, \Psi[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2 \, dx,$$

and

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](x)|^2 \, dx = \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla v_a^-[\partial\Omega_\epsilon, \Theta[\epsilon](\frac{1}{\epsilon}(\cdot - w))](x)|^2 \, dx.$$

As a consequence, by slightly modifying the proof of Theorem 4.20, we can prove that there exist $\epsilon_5 \in ]0, \epsilon_3]$ and two real analytic operators $G^i$ and $G^o$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{R}$ such that

$$\int_{\Omega_\epsilon} |\nabla u^i[\epsilon](x)|^2 \, dx = \epsilon^n G^i[\epsilon],$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](x)|^2 \, dx = \epsilon^n G^o[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_5[$, and

$$G^i[0] = \int_{\Omega} |\nabla \tilde{u}^i(x)|^2 \, dx,$$

$$G^o[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\, \Omega} |\nabla \tilde{u}^o(x)|^2 \, dx.$$

□

*Remark* 4.123. We note that the right-hand side of the equalities in (4.185) and (4.186) of Theorem 4.122 can be continued real analytically in the whole $]-\epsilon_5, \epsilon_5[$. Moreover,

$$\lim_{\epsilon \to 0^+} \Big( \int_{\Omega_\epsilon} |\nabla u^i[\epsilon](x)|^2 \, dx + \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](x)|^2 \, dx \Big) = 0.$$

### 4.11.4 A real analytic continuation Theorem for the integral of the family $\{(u^i[\epsilon], u^o[\epsilon])\}_{\epsilon \in ]0, \epsilon_3[}$

As done in Theorem 4.122 for the energy integral, we can now prove a real analytic continuation Theorem for the integral of the family $\{(u^i[\epsilon], u^o[\epsilon])\}_{\epsilon \in ]0, \epsilon_3[}$. Namely, we prove the following.

**Theorem 4.124.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\epsilon_3$ be as in Theorem 4.118* (*iv*). *Then there exist $\epsilon_6 \in ]0, \epsilon_3]$ and two real analytic operators $J^i$, $J^o$ of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{R}$, such that*

$$\int_{\Omega_\epsilon} u^i[\epsilon](x) \, dx = J^i[\epsilon], \tag{4.189}$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^o[\epsilon](x) \, dx = J^o[\epsilon], \tag{4.190}$$

*for all* $\epsilon \in ]0, \epsilon_6[$. *Moreover,*

$$J^i[0] = 0, \tag{4.191}$$

$$J^o[0] = 0. \tag{4.192}$$

*Proof.* It is a simple modification of the proof of Theorem 4.22. Indeed, let $\Theta[\cdot]$ be as in Theorem 4.118 (*iv*). Let $\epsilon \in ]0, \epsilon_3[$. We have

$$\begin{aligned} u^o[\epsilon](w + \epsilon t) = \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ -\frac{\epsilon}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \Big)\, d\sigma_t \quad \forall t \in \partial\Omega. \end{aligned}$$

Then, if we set

$$\begin{aligned} L[\epsilon](t) \equiv & \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \\ & - \frac{\epsilon}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \Big)\, d\sigma_t \quad \forall t \in \partial\Omega, \end{aligned}$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$, by arguing as in the proof of Theorem 2.128, we can easily show that there exist $\epsilon_6' \in ]0, \epsilon_3]$ and a real analytic map $J^o$ of $]-\epsilon_6', \epsilon_6'[$ to $\mathbb{R}$, such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^o[\epsilon](x)\, dx = J^o[\epsilon],$$

for all $\epsilon \in ]0, \epsilon_6'[$, and such that $J^o[0] = 0$.

Let $\epsilon \in ]0, \epsilon_3[$. Clearly,

$$\int_{\Omega_\epsilon} u^i[\epsilon](x)\, dx = \epsilon^n \int_\Omega u^i[\epsilon](w + \epsilon t)\, dt.$$

On the other hand, if $\epsilon_4'$, $U_1^i$, $U_2^i$ are as in Theorem 4.120, and we set

$$J^i[\epsilon] \equiv \epsilon^n \int_\Omega \big( \epsilon U_1^i[\epsilon](t) + \epsilon U_2^i[\epsilon] + F^{(-1)}(0) \big)\, dt$$

for all $\epsilon \in ]-\epsilon_4', \epsilon_4'[$, then we have that $J^i$ is a real analytic map of $]-\epsilon_4', \epsilon_4'[$ to $\mathbb{R}$, such that $J^i[0] = 0$ and that

$$\int_{\Omega_\epsilon} u^i[\epsilon](x)\, dx = J^i[\epsilon]$$

for all $\epsilon \in ]0, \epsilon_4'[$.

Then, by taking $\epsilon_6 \equiv \min\{\epsilon_6', \epsilon_4'\}$, we can easily conclude. □

### 4.11.5 A property of local uniqueness of the family $\{(u^i[\epsilon], u^o[\epsilon])\}_{\epsilon \in ]0, \epsilon_3[}$

In this Subsection, we shall show that the family $\{(u^i[\epsilon], u^o[\epsilon])\}_{\epsilon \in ]0, \epsilon_3[}$ is essentially unique. To do so, we need to introduce a preliminary lemma.

**Lemma 4.125.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ *be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let* $\epsilon \in ]0, \epsilon_1[$. *Let* $(u^i, u^o)$ *solve* (4.166). *Let* $(\psi, \theta, \xi) \in (\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ *be such that* $u^i = u^i[\epsilon, \psi, \theta, \xi]$ *and* $u^o = u^o[\epsilon, \psi, \theta, \xi]$. *Then*

$$v^+[\partial\Omega, \psi](t) + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\psi(s)\, d\sigma_s + \xi = \frac{u^i(w + \epsilon t) - F^{(-1)}(0)}{\epsilon} \qquad \forall t \in \mathrm{cl}\,\Omega,$$

*Proof.* It is an immediate consequence of Theorem 4.115. □

**Theorem 4.126.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $\{\hat{\epsilon}_j\}_{j\in\mathbb{N}}$ be a sequence in $]0,\epsilon_1[$ converging to 0. If $\{(u^i_j, u^o_j)\}_{j\in\mathbb{N}}$ is a sequence of pairs of functions such that*

$$(u^i_j, u^o_j) \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_{\hat{\epsilon}_j}]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_{\hat{\epsilon}_j}]), \tag{4.193}$$

$$(u^i_j, u^o_j) \text{ solves (4.166) with } \epsilon \equiv \hat{\epsilon}_j, \tag{4.194}$$

$$\lim_{j\to\infty} \frac{u^i_j(w + \hat{\epsilon}_j \cdot) - F^{(-1)}(0)}{\hat{\epsilon}_j} = \tilde{u}^i(\cdot) \qquad \text{in } C^{m,\alpha}(\partial\Omega), \tag{4.195}$$

*then there exists $j_0 \in \mathbb{N}$ such that*

$$(u^i_j, u^o_j) = (u^i[\hat{\epsilon}_j], u^o[\hat{\epsilon}_j]) \qquad \forall j_0 \le j \in \mathbb{N}.$$

*Proof.* By Theorem 4.115, for each $j \in \mathbb{N}$, there exists a unique triple $(\psi_j, \theta_j, \xi_j)$ in $(\mathcal{U}^{m-1,\alpha}_0)^2 \times \mathbb{R}$ such that

$$u^i_j = u^i[\hat{\epsilon}_j, \psi_j, \theta_j, \xi_j], \qquad u^o_j = u^o[\hat{\epsilon}_j, \psi_j, \theta_j, \xi_j]. \tag{4.196}$$

We shall now try to show that

$$\lim_{j\to\infty} (\psi_j, \theta_j, \xi_j) = (\tilde{\psi}, \tilde{\theta}, \tilde{\xi}) \qquad \text{in } (\mathcal{U}^{m-1,\alpha}_0)^2 \times \mathbb{R}. \tag{4.197}$$

Indeed, if we denote by $\tilde{\mathcal{U}}$ the neighbourhood of Theorem 4.118 (*iv*), the limiting relation of (4.197) implies that there exists $j_0 \in \mathbb{N}$ such that

$$(\hat{\epsilon}_j, \psi_j, \theta_j, \xi_j) \in ]0, \epsilon_3[ \times \tilde{\mathcal{U}},$$

for $j \ge j_0$ and thus Theorem 4.118 (*iv*) would imply that

$$(\psi_j, \theta_j, \xi_j) = (\Psi[\hat{\epsilon}_j], \Theta[\hat{\epsilon}_j], \Xi[\hat{\epsilon}_j]),$$

for $j_0 \le j \in \mathbb{N}$, and that accordingly the theorem holds (cf. Definition 4.119.) Thus we now turn to the proof of (4.197). We note that equation $\Lambda[\epsilon, \psi, \theta, \xi] = 0$ can be rewritten in the following form

$$\begin{aligned}
&F'(F^{(-1)}(0))\Big(v^+[\partial\Omega, \psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi\Big) \\
&\quad - \Big(v^-[\partial\Omega, \theta](t) + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\theta(s)\,d\sigma_s\Big) \\
&\quad + \frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(v^-[\partial\Omega, \theta](t) + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\theta(s)\,d\sigma_s\Big)\,d\sigma_t \\
&\quad = -\epsilon\Big(v^+[\partial\Omega, \psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi\Big)^2 \\
&\quad \times \int_0^1 (1-\beta)F''\Big(F^{(-1)}(0) + \beta\epsilon\big(v^+[\partial\Omega, \psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi\big)\Big)d\beta \ \forall t \in \partial\Omega,
\end{aligned} \tag{4.198}$$

$$\begin{aligned}
&\frac{1}{2}\theta(t) + v_*[\partial\Omega, \theta](t) + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR^a_n(\epsilon(t-s))\theta(s)\,d\sigma_s \\
&\quad + \frac{1}{2}\gamma\psi(t) - \gamma v_*[\partial\Omega, \psi](t) - \gamma\epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR^a_n(\epsilon(t-s))\psi(s)\,d\sigma_s = g(t) \qquad \forall t \in \partial\Omega,
\end{aligned} \tag{4.199}$$

for all $(\epsilon, \psi, \theta, \xi)$ in the domain of $\Lambda$. By arguing so as to prove that the integral of the second component of $\Lambda$ on $\partial\Omega$ equals zero in the beginning of the proof of Theorem 4.118, we can conclude that both hand sides of equation (4.199) have zero integral on $\partial\Omega$. We define the map $N \equiv (N_l)_{l=1,2}$ of $]-\epsilon_3, \epsilon_3[ \times (\mathcal{U}^{m-1,\alpha}_0)^2 \times \mathbb{R}$ to $C^{m,\alpha}(\partial\Omega) \times \mathcal{U}^{m-1,\alpha}_0$ by setting $N_1[\epsilon, \psi, \theta, \xi]$ equal to the left-hand side of the equality in (4.198), $N_2[\epsilon, \psi, \theta, \xi]$ equal to the left-hand side of the equality in (4.199) for all $(\epsilon, \psi, \theta, \xi) \in ]-\epsilon_3, \epsilon_3[ \times (\mathcal{U}^{m-1,\alpha}_0)^2 \times \mathbb{R}$. By arguing so as in the proof of Theorem 4.118, we can prove that $N$ is real analytic. Since $N[\epsilon, \cdot, \cdot, \cdot]$ is linear for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$, we have

$$N[\epsilon, \psi, \theta, \xi] = \partial_{(\psi,\theta,\xi)} N[\epsilon, \tilde{\psi}, \tilde{\theta}, \tilde{\xi}](\psi, \theta, \xi)$$

for all $(\epsilon,\psi,\theta,\xi) \in ]-\epsilon_3,\epsilon_3[\times(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$, and the map of $]-\epsilon_3,\epsilon_3[$ to $\mathcal{L}((\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}, C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha})$ which takes $\epsilon$ to $N[\epsilon,\cdot,\cdot,\cdot]$ is real analytic. Since

$$N[0,\cdot,\cdot,\cdot] = \partial_{(\psi,\theta,\xi)}\Lambda[0,\tilde{\psi},\tilde{\theta},\tilde{\xi}](\cdot,\cdot,\cdot),$$

Theorem 4.118 (*iii*) implies that $N[0,\cdot,\cdot,\cdot]$ is also a linear homeomorphism. Since the set of linear homeomorphisms of $(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}$ is open in $\mathcal{L}((\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}, C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha})$ and since the map which takes a linear invertible operator to its inverse is real analytic (cf. *e.g.*, Hille and Phillips [61, Theorems 4.3.2 and 4.3.4]), there exists $\tilde{\epsilon}\in]0,\epsilon_3[$ such that the map $\epsilon\mapsto N[\epsilon,\cdot,\cdot,\cdot]^{(-1)}$ is real analytic from $]-\tilde{\epsilon},\tilde{\epsilon}[$ to $\mathcal{L}(C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha},(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R})$. Next we denote by $S[\epsilon,\psi,\theta,\xi]\equiv(S_l[\epsilon,\psi,\theta,\xi])_{l=1,2}$ the pair defined by the right-hand side of (4.198)-(4.199). Then equation $\Lambda[\epsilon,\psi,\theta,\xi]=0$ (or equivalently system (4.198)-(4.199)) can be rewritten in the following form:

$$(\psi,\theta,\xi) = N[\epsilon,\cdot,\cdot,\cdot]^{(-1)}[S[\epsilon,\psi,\theta,\xi]], \tag{4.200}$$

for all $(\epsilon,\psi,\theta,\xi)\in]-\tilde{\epsilon},\tilde{\epsilon}[\times(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$. Next we note that condition (4.195), the proof of Theorem 4.118, the real analyticity of $F$ and standard calculus in Banach space imply that

$$\lim_{j\to\infty} S[\hat{\epsilon}_j,\psi_j,\theta_j,\xi_j] = S[0,\tilde{\psi},\tilde{\theta},\tilde{\xi}] \qquad \text{in } C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}. \tag{4.201}$$

Then by (4.200) and by the real analyticity of $\epsilon\mapsto N[\epsilon,\cdot,\cdot,\cdot]^{(-1)}$, and by the bilinearity and continuity of the operator of $\mathcal{L}(C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha},(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R})\times(C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha})$ to $(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$, which takes a pair $(T_1,T_2)$ to $T_1[T_2]$, we conclude that (4.197) holds. Thus the proof is complete. □

## 4.12 Alternative homogenization problem for the Laplace equation with a nonlinear transmission boundary condition in a periodically perforated domain

In this section we consider an homogenization problem for the Laplace equation with a nonlinear transmission boundary condition in a periodically perforated domain.

### 4.12.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 4.3.1 and 4.11.1.

Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. For each pair $(\epsilon,\delta)\in]0,\epsilon_1[\times]0,+\infty[$, we consider the following periodic nonlinear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x)=0 & \forall x\in\mathbb{S}_a(\epsilon,\delta), \\ \Delta u^o(x)=0 & \forall x\in\mathbb{T}_a(\epsilon,\delta), \\ u^i(x+\delta a_j)=u^i(x) & \forall x\in\mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta), \quad \forall j\in\{1,\dots,n\}, \\ u^o(x+\delta a_j)=u^o(x) & \forall x\in\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall j\in\{1,\dots,n\}, \\ u^o(x)=F(u^i(x)) & \forall x\in\partial\Omega(\epsilon,\delta), \\ \frac{\partial}{\partial\nu_{\Omega(\epsilon,\delta)}}u^o(x)=\gamma\frac{\partial}{\partial\nu_{\Omega(\epsilon,\delta)}}u^i(x)+\frac{1}{\delta}g(\frac{1}{\epsilon\delta}(x-\delta w)) & \forall x\in\partial\Omega(\epsilon,\delta), \\ \int_{\partial\Omega(\epsilon,\delta)}u^o(x)\,d\sigma_x=0. \end{cases} \tag{4.202}$$

We give the following definition.

**Definition 4.127.** Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $\epsilon_3$ be as in Theorem 4.120 (*iv*). Let $(u^i[\cdot],u^o[\cdot])$ be as in Definition 4.119. For each pair $(\epsilon,\delta)\in]0,\epsilon_3[\times]0,+\infty[$, we set

$$u^i_{(\epsilon,\delta)}(x)\equiv u^i[\epsilon](\frac{x}{\delta}) \quad \forall x\in\mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta), \qquad u^o_{(\epsilon,\delta)}(x)\equiv u^o[\epsilon](\frac{x}{\delta}) \quad \forall x\in\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).$$

*Remark* 4.128. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $\epsilon_3$ be as in Theorem 4.120 (*iv*). For each $(\epsilon,\delta)\in]0,\epsilon_3[\times]0,+\infty[$ the pair $(u^i_{(\epsilon,\delta)},u^o_{(\epsilon,\delta)})$ is a solution of (4.202).

By the previous remark, we note that a solution of problem (4.202) can be expressed by means of a solution of an auxiliary rescaled problem, which does not depend on $\delta$. This is due to the presence of the factor $1/\delta$ in front of $g(\frac{1}{\epsilon\delta}(x-\delta w))$ in the sixth equation of problem (4.202).

By virtue of Theorem 4.126, we have the following.

*Remark* 4.129. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $\epsilon_3$ be as in Theorem 4.120 $(iv)$. Let $\bar{\delta} \in ]0,+\infty[$. Let $\{\hat{\epsilon}_j\}_{j\in\mathbb{N}}$ be a sequence in $]0,\epsilon_1[$ converging to 0. If $\{(u^i_j,u^o_j)\}_{j\in\mathbb{N}}$ is a sequence of pairs of functions such that

$$\begin{aligned}
&(u^i_j,u^o_j) \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a(\hat{\epsilon}_j,\bar{\delta})) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\hat{\epsilon}_j,\bar{\delta})),\\
&(u^i_j,u^o_j) \text{ solves (4.202) with } (\epsilon,\delta) \equiv (\hat{\epsilon}_j,\bar{\delta}),\\
&\lim_{j\to\infty} \frac{u^i_j(\bar{\delta}w+\bar{\delta}\hat{\epsilon}_j\cdot) - F^{(-1)}(0)}{\hat{\epsilon}_j} = \tilde{u}^i(\cdot) \qquad \text{in } C^{m,\alpha}(\partial\Omega),
\end{aligned}$$

then there exists $j_0 \in \mathbb{N}$ such that

$$(u^i_j,u^o_j) = (u^i_{(\hat{\epsilon}_j,\bar{\delta})}, u^o_{(\hat{\epsilon}_j,\bar{\delta})}) \qquad \forall j_0 \le j \in \mathbb{N}.$$

Our aim is to study the asymptotic behaviour of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ as $(\epsilon,\delta)$ tends to $(0,0)$. As a first step, we study the behaviour of $(u^i[\epsilon], u^o[\epsilon])$ as $\epsilon$ tends to 0.

**Proposition 4.130.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\epsilon_3$ be as in Theorem 4.120 $(iv)$. Let $1 \le p < \infty$. Then*

$$\lim_{\epsilon\to 0^+} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]] = 0 \qquad \text{in } L^p(A),$$

*and*

$$\lim_{\epsilon\to 0^+} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]] = 0 \qquad \text{in } L^p(A).$$

*Proof.* It suffices to modify the proof of Propositions 2.132, 4.26. Let $\epsilon_3$, $\Psi$, $\Theta$, $\Xi$ be as in Theorem 4.118. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $\epsilon \in ]0,\epsilon_3[$, we have

$$\begin{aligned}
u^i[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) =& \epsilon\int_{\partial\Omega} S_n(t-s)\Psi[\epsilon](s)\,d\sigma_s\\
&+\epsilon^{n-1}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\Psi[\epsilon](s)\,d\sigma_s + \epsilon\Xi[\epsilon] + F^{(-1)}(0), \qquad \forall t \in \partial\Omega.
\end{aligned}$$

We set

$$\begin{aligned}
N^i[\epsilon](t) \equiv& \epsilon\int_{\partial\Omega} S_n(t-s)\Psi[\epsilon](s)\,d\sigma_s\\
&+\epsilon^{n-1}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\Psi[\epsilon](s)\,d\sigma_s + \epsilon\Xi[\epsilon] + F^{(-1)}(0), \qquad \forall t \in \partial\Omega,
\end{aligned}$$

for all $\epsilon \in ]-\epsilon_3,\epsilon_3[$. By taking $\tilde{\epsilon} \in ]0,\epsilon_3[$ small enough, we can assume (cf. Proposition 1.28 $(i)$) that $N^i$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C^i \equiv \sup_{\epsilon\in]-\tilde{\epsilon},\tilde{\epsilon}[} \|N^i[\epsilon]\|_{C^0(\partial\Omega)} < +\infty.$$

By the Maximum Principle for harmonic functions, we have

$$|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]](x)| \le C^i \qquad \forall x \in A, \quad \forall \epsilon \in ]0,\tilde{\epsilon}[.$$

Obviously,

$$\lim_{\epsilon\to 0^+} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]](x) = 0 \qquad \forall x \in A\setminus\{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{\epsilon\to 0^+} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]] = 0 \qquad \text{in } L^p(A).$$

If $\epsilon \in ]0, \epsilon_3[$, we have

$$u^o[\epsilon] \circ (w + \epsilon \, \mathrm{id}_{\partial\Omega})(t) = \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s$$
$$- \frac{\epsilon}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s\, d\sigma_t + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \Big)\, d\sigma_t, \ \forall t \in \partial\Omega.$$

We set

$$N^o[\epsilon](t) \equiv \epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s$$
$$- \frac{\epsilon}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega} \Big( \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon](s)\, d\sigma_s\, d\sigma_t + \epsilon^{n-2} \int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon](s)\, d\sigma_s \Big)\, d\sigma_t, \ \forall t \in \partial\Omega,$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. By taking $\tilde{\epsilon} \in ]0, \epsilon_3[$ small enough, we can assume (cf. Proposition 1.28 $(i)$) that $N^o$ is a real analytic map of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$C^o \equiv \sup_{\epsilon \in ]-\tilde{\epsilon}, \tilde{\epsilon}[} \| N^o[\epsilon] \|_{C^0(\partial\Omega)} < +\infty.$$

By Theorem 2.5, we have

$$| \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]](x) | \leq C^o \qquad \forall x \in A, \quad \forall \epsilon \in ]0, \tilde{\epsilon}[.$$

By Theorem 4.120, we have

$$\lim_{\epsilon \to 0^+} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]](x) = 0 \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{\epsilon \to 0^+} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]] = 0 \qquad \text{in } L^p(A).$$

□

### 4.12.2 Asymptotic behaviour of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

In the following Theorem we deduce by Proposition 4.130 the convergence of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ as $(\epsilon, \delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 4.131.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\epsilon_3$ be as in Theorem 4.120 (iv). Let $1 \leq p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta) \to (0^+,0^+)} \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^p(V),$$

*and*

$$\lim_{(\epsilon,\delta) \to (0^+,0^+)} \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^p(V).$$

*Proof.* We modify the proof of Theorem 2.134. By virtue of Proposition 4.130, we have

$$\lim_{\epsilon \to 0^+} \| \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]] \|_{L^p(A)} = 0,$$

and

$$\lim_{\epsilon \to 0^+} \| \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]] \|_{L^p(A)} = 0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $C > 0$ such that

$$\| \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] \|_{L^p(V)} \leq C \| \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon]] \|_{L^p(A)} \qquad \forall (\epsilon,\delta) \in ]0, \epsilon_1[ \times ]0,1[,$$

and

$$\| \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] \|_{L^p(V)} \leq C \| \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon]] \|_{L^p(A)} \qquad \forall (\epsilon,\delta) \in ]0, \epsilon_1[ \times ]0,1[,$$

Thus,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^p(V),$$

and

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^p(V).$$

□

Then we have the following Theorem, where we consider a functional associated to extensions of $u^i_{(\epsilon,\delta)}$ and of $u^o_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 4.132.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $\epsilon_3$ be as in Theorem 4.120 (iv). Let $\epsilon_6$, $J^i$, $J^o$ be as in Theorem 4.124. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^i[\epsilon], \tag{4.203}$$

*and*

$$\int_{\mathbb{R}^n} \mathbf{E}^o_{(\epsilon,r/l)}[u^o_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^o[\epsilon], \tag{4.204}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N}\setminus\{0\}$.*

*Proof.* We follow the proof of Theorem 2.60. Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N}\setminus\{0\}$. Then, by the periodicity of $u^i_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\Omega_\epsilon} u^i_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\Omega_\epsilon} u^i[\epsilon]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\Omega_\epsilon} u^i[\epsilon](t)\,dt \\ &= \frac{r^n}{l^n} J^i[\epsilon].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^i[\epsilon],$$

and the validity of (4.203) follows. The proof of (4.204) is very similar and is accordingly omitted. □

*Remark* 4.133. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $\epsilon_3$ be as in Theorem 4.120 (*iv*). We note that it can be easily proved that there exist $\tilde{\epsilon} \in ]0,\epsilon_3[$ and a real analytic map $N^o$ of $]-\tilde{\epsilon},\tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega)$ such that

$$\|\mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}]\|_{L^\infty(\mathbb{R}^n)} = \epsilon\|N^o[\epsilon]\|_{C^0(\partial\Omega)},$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,+\infty[$.

### 4.12.3 Asymptotic behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

This Subsection is devoted to the study of the behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$. We give the following.

**Definition 4.134.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $\epsilon_3$ be as in Theorem 4.120 (*iv*). For each pair $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, +\infty[$, we set

$$\mathrm{En}(\epsilon, \delta) \equiv \int_{A\cap\mathbb{S}_a(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2 \, dx + \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2 \, dx.$$

*Remark* 4.135. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $\epsilon_3$ be as in Theorem 4.120 (*iv*). Let $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, +\infty[$. We have

$$\begin{aligned}\int_{\Omega(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2 \, dx &= \delta^n \int_{\Omega(\epsilon,1)} |(\nabla u^i_{(\epsilon,\delta)})(\delta t)|^2 \, dt \\ &= \delta^{n-2} \int_{\Omega_\epsilon} |\nabla u^i[\epsilon](t)|^2 \, dt\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2 \, dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u^o_{(\epsilon,\delta)})(\delta t)|^2 \, dt \\ &= \delta^{n-2} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon](t)|^2 \, dt.\end{aligned}$$

Then we give the following definition, where we consider $\mathrm{En}(\epsilon, \delta)$, with $\epsilon$ equal to a certain function of $\delta$.

**Definition 4.136.** For each $\delta \in ]0, +\infty[$, we set

$$\epsilon[\delta] \equiv \delta^{\frac{2}{n}}.$$

Let $\epsilon_5$ be as in Theorem 4.122. Let $\delta_1 > 0$ be such that $\epsilon[\delta] \in ]0, \epsilon_5[$, for all $\delta \in ]0, \delta_1[$. Then we set

$$\mathrm{En}[\delta] \equiv \mathrm{En}(\epsilon[\delta], \delta),$$

for all $\delta \in ]0, \delta_1[$.

In the following Proposition we compute the limit of $\mathrm{En}[\delta]$ as $\delta$ tends to 0.

**Proposition 4.137.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\epsilon_3$ be as in Theorem 4.120 (iv). Let $\epsilon_5$ be as in Theorem 4.122. Let $\delta_1 > 0$ be as in Definition 4.136. Then*

$$\lim_{\delta\to 0^+} \mathrm{En}[\delta] = \int_\Omega |\nabla \tilde{u}^i(x)|^2 \, dx + \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega} |\nabla \tilde{u}^o(x)|^2 \, dx,$$

*where $\tilde{u}^i$, $\tilde{u}^o$ are as in Theorem 4.117.*

*Proof.* Let $G^i$, $G^o$ be as in Theorem 4.122. Let $\delta \in ]0, \delta_1[$. By Remark 4.135 and Theorem 4.122, we have

$$\begin{aligned}\int_{\Omega(\epsilon[\delta],\delta)} |\nabla u^i_{(\epsilon[\delta],\delta)}(x)|^2 \, dx + \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u^o_{(\epsilon[\delta],\delta)}(x)|^2 \, dx &= \delta^{n-2}(\epsilon[\delta])^n (G^i[\epsilon[\delta]] + G^o[\epsilon[\delta]]) \\ &= \delta^n (G^i[\delta^{\frac{2}{n}}] + G^o[\delta^{\frac{2}{n}}]).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\lfloor (1/\delta) \rfloor^n \Big( \int_{\Omega(\epsilon[\delta],\delta)} |\nabla u^i_{(\epsilon[\delta],\delta)}(x)|^2 \, dx + \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u^o_{(\epsilon[\delta],\delta)}(x)|^2 \, dx \Big) \le \mathrm{En}[\delta] \\ &\qquad \le \lceil (1/\delta) \rceil^n \Big( \int_{\Omega(\epsilon[\delta],\delta)} |\nabla u^i_{(\epsilon[\delta],\delta)}(x)|^2 \, dx + \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u^o_{(\epsilon[\delta],\delta)}(x)|^2 \, dx \Big),\end{aligned}$$

and so

$$\lfloor(1/\delta)\rfloor^n\delta^n(G^i[\delta^{\frac{2}{n}}]+G^o[\delta^{\frac{2}{n}}]) \leq \mathrm{En}[\delta] \leq \lceil(1/\delta)\rceil^n\delta^n(G^i[\delta^{\frac{2}{n}}]+G^o[\delta^{\frac{2}{n}}]).$$

Thus, since

$$\lim_{\delta\to 0^+}\lfloor(1/\delta)\rfloor^n\delta^n = 1, \qquad \lim_{\delta\to 0^+}\lceil(1/\delta)\rceil^n\delta^n = 1,$$

we have

$$\lim_{\delta\to 0^+}\mathrm{En}[\delta] = (G^i[0]+G^o[0]).$$

Finally, by equalities (4.187) and (4.188), we easily conclude. □

In the following Proposition we represent the function $\mathrm{En}[\cdot]$ by means of real analytic functions.

**Proposition 4.138.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\epsilon_3$ be as in Theorem 4.120 (iv). Let $\epsilon_5$, $G^i$, $G^o$ be as in Theorem 4.122. Let $\delta_1 > 0$ be as in Definition 4.136. Then*

$$\mathrm{En}[(1/l)] = G^i[(1/l)^{\frac{2}{n}}] + G^o[(1/l)^{\frac{2}{n}}],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* It follows by the proof of Proposition 4.137. □

## 4.13 A variant of an alternative homogenization problem for the Laplace equation with a nonlinear transmission boundary condition in a periodically perforated domain

In this section we consider a slightly different homogenization problem for the Laplace equation with linear transmission boundary conditions in a periodically perforated domain.

### 4.13.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 4.3.1 and 4.11.1.

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, we consider the following periodic nonlinear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a(\epsilon,\delta), \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u^i(x+\delta a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\ u^o(x+\delta a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\ u^o(x) = F(u^i(x)) & \forall x \in \partial\Omega(\epsilon,\delta), \\ \frac{\partial}{\partial\nu_{\Omega(\epsilon,\delta)}}u^o(x) = \gamma\frac{\partial}{\partial\nu_{\Omega(\epsilon,\delta)}}u^i(x) + g(\frac{1}{\epsilon\delta}(x-\delta w)) & \forall x \in \partial\Omega(\epsilon,\delta), \\ \int_{\partial\Omega(\epsilon,\delta)} u^o(x)\,d\sigma_x = 0. \end{cases} \tag{4.205}$$

In contrast to problem (4.202), we note that in the sixth equation of problem (4.205) there is not the factor $1/\delta$ in front of $g(\frac{1}{\epsilon\delta}(x-\delta w))$.

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. For each $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, we introduce the following auxiliary periodic nonlinear transmission problem for the Laplace equation.

$$\begin{cases} \Delta u^i(x) = 0 & \forall x \in \mathbb{S}_a[\Omega_\epsilon], \\ \Delta u^o(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u^i(x+a_j) = u^i(x) & \forall x \in \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u^o(x+a_j) = u^o(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u^o(x) = F(u^i(x)) & \forall x \in \partial\Omega_\epsilon, \\ \frac{\partial}{\partial\nu_{\Omega_\epsilon}}u^o(x) = \gamma\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u^i(x) + \delta g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon, \\ \int_{\partial\Omega_\epsilon} u^o(x)\,d\sigma_x = 0. \end{cases} \tag{4.206}$$

We transform (4.206) into a system of integral equations by means of the following.

**Theorem 4.139.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in (1.64). Let $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$. Then the map $(u^i[\epsilon,\delta,\cdot,\cdot,\cdot],u^o[\epsilon,\delta,\cdot,\cdot,\cdot])$ of the set of triples $(\psi,\theta,\xi) \in (\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ that solve the following integral equations*

$$\begin{aligned}
&F'(F^{(-1)}(0))\Big(v^+[\partial\Omega,\psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi\Big)\\
&\quad + \delta\epsilon\Big(v^+[\partial\Omega,\psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi\Big)^2\\
&\quad \times \int_0^1 (1-\beta)F''\Big(F^{(-1)}(0) + \beta\delta\epsilon\big(v^+[\partial\Omega,\psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi\big)\Big)d\beta \qquad (4.207)\\
&\quad - \Big(v^-[\partial\Omega,\theta](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\Big)\\
&\quad + \frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(v^-[\partial\Omega,\theta](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\Big)\,d\sigma_t = 0 \quad \forall t \in \partial\Omega,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\theta(t) + v_*[\partial\Omega,\theta](t) + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\\
&\quad + \frac{1}{2}\gamma\psi(t) - \gamma v_*[\partial\Omega,\psi](t) - \gamma\epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s - g(t) = 0 \quad \forall t \in \partial\Omega,
\end{aligned} \qquad (4.208)$$

*to the set of pairs $(u^i,u^o)$ of $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ which solve problem (4.206), which takes $(\psi,\theta,\xi)$ to the pair of functions*

$$\begin{aligned}
(u^i[\epsilon,\delta,\psi,\theta,\xi] &\equiv v_a^+[\partial\Omega_\epsilon,\delta\psi(\frac{1}{\epsilon}(\cdot-w))] + \delta\epsilon\xi + F^{(-1)}(0),\\
u^o[\epsilon,\delta,\psi,\theta,\xi] &\equiv v_a^-[\partial\Omega_\epsilon,\delta\theta(\frac{1}{\epsilon}(\cdot-w))] - \frac{1}{\int_{\partial\Omega_\epsilon} d\sigma}\int_{\partial\Omega_\epsilon} v_a^-[\partial\Omega_\epsilon,\delta\theta(\frac{1}{\epsilon}(\cdot-w))]\,d\sigma),
\end{aligned} \qquad (4.209)$$

*is a bijection.*

*Proof.* It is a simple modification of the proof of Proposition 4.115. Indeed, let $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$. Assume that the pair $(u^i,u^o)$ in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon])$ solves problem (4.205). Then by Propositions 2.23, 2.24, it is easy to see that there exists a unique triple $(\psi,\theta,\xi)$ in $(\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$, such that

$$u^i = v_a^+[\partial\Omega_\epsilon,\delta\psi(\frac{1}{\epsilon}(\cdot-w))] + \delta\epsilon\xi + F^{(-1)}(0) \qquad \text{in } \mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon],$$

and

$$u^o = v_a^-[\partial\Omega_\epsilon,\delta\theta(\frac{1}{\epsilon}(\cdot-w))] - \frac{1}{\int_{\partial\Omega_\epsilon} d\sigma}\int_{\partial\Omega_\epsilon} v_a^-[\partial\Omega_\epsilon,\delta\theta(\frac{1}{\epsilon}(\cdot-w))]\,d\sigma \qquad \text{in } \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$

Then a simple computation based on the Theorem of change of variables in integrals, on identity (1.65), and on the definition of $\mathcal{U}_0^{m-1,\alpha}$, shows that the triple $(\psi,\theta,\xi)$ must solve equation (4.208), together with the following

$$\begin{aligned}
&F\Big(\delta\epsilon\big(v^+[\partial\Omega,\psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi\big) + F^{(-1)}(0)\Big)\\
&\qquad = \delta\epsilon\Big(v^-[\partial\Omega,\theta](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\Big)\\
&\qquad\quad - \frac{\delta\epsilon}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(v^-[\partial\Omega,\theta](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\Big)\,d\sigma_t \qquad \forall t \in \partial\Omega.
\end{aligned} \qquad (4.210)$$

We now show that equation (4.210) implies the validity of (4.207). By Taylor Formula, we have

$$F(x + F^{(-1)}(0)) = 0 + F'(F^{(-1)}(0))x + x^2\int_0^1 (1-\beta)F''(F^{(-1)}(0) + \beta x)d\beta \qquad \forall x \in \mathbb{R}.$$

Then, by dividing both sides of (4.210) by $\delta\epsilon$, we can rewrite (4.210) as (4.207). Conversely, by reading backward the above argument, one can easily show that if $(\psi,\theta,\xi)$ in $(\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ solves (4.207)-(4.208), then the pair of functions of (4.209) satisfies problem (4.205). □

Hence we are reduced to analyse system (4.207)-(4.208). As a first step in the analysis of system (4.207)-(4.208), we note that for $(\epsilon,\delta)=(0,0)$, since $\psi,\ \theta\in\mathcal{U}_0^{m-1,\alpha}$, one obtains a system which we address to as the *limiting system* and which has the following form

$$F'(F^{(-1)}(0))(v^+[\partial\Omega,\psi](t)+\xi)-(v^-[\partial\Omega,\theta](t)-\frac{1}{\int_{\partial\Omega}d\sigma}\int_{\partial\Omega}v^-[\partial\Omega,\theta]\,d\sigma)=0\quad\forall t\in\partial\Omega,\tag{4.211}$$

$$\frac{1}{2}\theta(t)+v_*[\partial\Omega,\theta](t)+\frac{1}{2}\gamma\psi(t)-\gamma v_*[\partial\Omega,\psi](t)-g(t)=0\qquad\forall t\in\partial\Omega.\tag{4.212}$$

Then we have the following theorem, which shows the unique solvability of the limiting system, and its link with a boundary value problem which we shall address to as the *limiting boundary value problem.*

**Theorem 4.140.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $F$, $g$, $\gamma$ be as in* (1.56), (4.163), (4.164), (4.165), *respectively. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.64). *Then the following statements hold.*

*(i) The* limiting system (4.211)-(4.212) *has one and only one solution in $(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$, which we denote by $(\tilde\psi,\tilde\theta,\tilde\xi)$.*

*(ii) The limiting boundary value problem*

$$\begin{cases}\Delta u^i(x)=0 & \forall x\in\Omega,\\ \Delta u^o(x)=0 & \forall x\in\mathbb{R}^n\setminus\operatorname{cl}\Omega,\\ u^o(x)-\frac{1}{\int_{\partial\Omega}d\sigma}\int_{\partial\Omega}u^o\,d\sigma=F'(F^{(-1)}(0))u^i(x) & \forall x\in\partial\Omega,\\ \frac{\partial}{\partial\nu_\Omega}u^o(x)=\gamma\frac{\partial}{\partial\nu_\Omega}u^i(x)+g(x) & \forall x\in\partial\Omega,\\ \lim_{x\to\infty}u^o(x)=0\end{cases}\tag{4.213}$$

*has one and only one solution $(\tilde u^i,\tilde u^o)$ in $C^{m,\alpha}(\operatorname{cl}\Omega)\times C^{m,\alpha}(\mathbb{R}^n\setminus\Omega)$, and the following formulas hold:*

$$\tilde u^i\equiv v^+[\partial\Omega,\tilde\psi]+\tilde\xi\qquad\text{in }\operatorname{cl}\Omega,\tag{4.214}$$

$$\tilde u^o\equiv v^-[\partial\Omega,\tilde\theta]\qquad\text{in }\mathbb{R}^n\setminus\Omega.\tag{4.215}$$

*Proof.* It is Theorem 4.117. □

We are now ready to analyse equations (4.207)-(4.208) around the degenerate case $(\epsilon,\delta)=(0,0)$.

**Theorem 4.141.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\mathcal{U}_0^{m-1,\alpha}$ be as in* (1.64). *Let $\Lambda\equiv(\Lambda_j)_{j=1,2}$ be the map of $]-\epsilon_1,\epsilon_1[\,\times\mathbb{R}\times(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}$, defined by*

$$\begin{aligned}\Lambda_1[\epsilon,\delta,\psi,\theta,\xi](t)\equiv{}&F'(F^{(-1)}(0))\Big(v^+[\partial\Omega,\psi](t)+\epsilon^{n-2}\int_{\partial\Omega}R_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s+\xi\Big)\\&+\delta\epsilon\Big(v^+[\partial\Omega,\psi](t)+\epsilon^{n-2}\int_{\partial\Omega}R_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s+\xi\Big)^2\\&\times\int_0^1(1-\beta)F''\Big(F^{(-1)}(0)+\beta\delta\epsilon\big(v^+[\partial\Omega,\psi](t)+\epsilon^{n-2}\int_{\partial\Omega}R_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s+\xi\big)\Big)d\beta\\&-\Big(v^-[\partial\Omega,\theta](t)+\epsilon^{n-2}\int_{\partial\Omega}R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\Big)\\&+\frac{1}{\int_{\partial\Omega}d\sigma}\int_{\partial\Omega}\Big(v^-[\partial\Omega,\theta](t)+\epsilon^{n-2}\int_{\partial\Omega}R_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\Big)\,d\sigma_t\quad\forall t\in\partial\Omega,\end{aligned}\tag{4.216}$$

$$\begin{aligned}\Lambda_2[\epsilon,\delta,\psi,\theta,\xi](t)\equiv{}&\frac{1}{2}\theta(t)+v_*[\partial\Omega,\theta](t)+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\theta(s)\,d\sigma_s\\&+\frac{1}{2}\gamma\psi(t)-\gamma v_*[\partial\Omega,\psi](t)-\gamma\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s-g(t)\quad\forall t\in\partial\Omega,\end{aligned}\tag{4.217}$$

*for all $(\epsilon,\delta,\psi,\theta,\xi)\in\,]-\epsilon_1,\epsilon_1[\,\times\mathbb{R}\times(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$. Then the following statements hold.*

(i) *Equation* $\Lambda[0,0,\psi,\theta,\xi]=0$ *is equivalent to the limiting system* (4.211)-(4.212) *and has one and only one solution* $(\tilde{\psi},\tilde{\theta},\tilde{\xi})$ *(cf. Theorem 4.140.)*

(ii) *If* $(\epsilon,\delta)\in\,]0,\epsilon_1[\,\times\,]0,+\infty[$*, then equation* $\Lambda[\epsilon,\delta,\psi,\theta,\xi]=0$ *is equivalent to system* (4.207)-(4.208) *for* $(\psi,\theta,\xi)$.

(iii) *There exists* $\epsilon_2\in\,]0,\epsilon_1]$ *such that* $\Lambda$ *is a real analytic map of* $]-\epsilon_2,\epsilon_2[\,\times\mathbb{R}\times(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$ *to* $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}$*. The differential* $\partial_{(\psi,\theta,\xi)}\Lambda[0,0,\tilde{\psi},\tilde{\theta},\tilde{\xi}]$ *of* $\Lambda$ *at* $(0,0,\tilde{\psi},\tilde{\theta},\tilde{\xi})$ *is a linear homeomorphism of* $(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$ *to* $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}$.

(iv) *There exist* $\epsilon_3\in\,]0,\epsilon_2]$*,* $\delta_1\in\,]0,+\infty[$ *and an open neighbourhood* $\tilde{\mathcal{U}}$ *of* $(\tilde{\psi},\tilde{\theta},\tilde{\xi})$ *in* $(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$ *and a real analytic map* $(\Psi[\cdot,\cdot],\Theta[\cdot,\cdot],\Xi[\cdot,\cdot])$ *of* $]-\epsilon_3,\epsilon_3[\,\times\,]-\delta_1,\delta_1[$ *to* $\tilde{\mathcal{U}}$*, such that the set of zeros of the map* $\Lambda$ *in* $]-\epsilon_3,\epsilon_3[\,\times\,]-\delta_1,\delta_1[\,\times\tilde{\mathcal{U}}$ *coincides with the graph of* $(\Psi[\cdot,\cdot],\Theta[\cdot,\cdot],\Xi[\cdot,\cdot])$*. In particular,* $(\Psi[0,0],\Theta[0,0],\Xi[0,0])=(\tilde{\psi},\tilde{\theta},\tilde{\xi})$.

*Proof.* It is a simple modification of the proof of Theorem 4.118. Indeed, the statements in (*i*) and (*ii*) are obvious. By an easy modification of the proof of Theorem 4.118 (*iii*), one can easily show that there exists $\epsilon_2\in\,]0,\epsilon_1]$ such that $\Lambda$ is a real analytic map of $]-\epsilon_2,\epsilon_2[\,\times\mathbb{R}\times(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}$. By standard calculus in Banach space, the differential of $\Lambda$ at $(0,0,\tilde{\psi},\tilde{\theta},\tilde{\xi})$ with respect to the variables $(\psi,\theta,\xi)$ is delivered by the following formulas

$$\begin{aligned}\partial_{(\psi,\theta,\xi)}\Lambda_1[0,0,\tilde{\psi},\tilde{\theta},\tilde{\xi}](\bar{\psi},\bar{\theta},\bar{\xi})(t)=&F'(F^{(-1)}(0))(v^+[\partial\Omega,\bar{\psi}](t)+\bar{\xi})\\&-(v^-[\partial\Omega,\bar{\theta}](t)-\frac{1}{\int_{\partial\Omega}d\sigma}\int_{\partial\Omega}v^-[\partial\Omega,\bar{\theta}]\,d\sigma)\quad\forall t\in\partial\Omega,\end{aligned}$$

$$\partial_{(\psi,\theta,\xi)}\Lambda_2[0,0,\tilde{\psi},\tilde{\theta},\tilde{\xi}](\bar{\psi},\bar{\theta},\bar{\xi})(t)=\frac{1}{2}\bar{\theta}(t)+v_*[\partial\Omega,\bar{\theta}](t)+\frac{1}{2}\gamma\psi(t)-\gamma v_*[\partial\Omega,\bar{\psi}](t)\qquad\forall t\in\partial\Omega,$$

for all $(\bar{\psi},\bar{\theta},\bar{\xi})\in(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$. By the proof of statement (*iii*) of Theorem 4.118, the above differential is a linear homeomorphism. Statement (*iv*) is an immediate consequence of statement (*iii*) and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

We are now in the position to introduce the following.

**Definition 4.142.** Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $u^i[\cdot,\cdot,\cdot,\cdot]$ and $u^o[\cdot,\cdot,\cdot,\cdot]$ be as in Theorem 4.139. If $(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[$, we set

$$\begin{aligned}u^i[\epsilon,\delta](t)&\equiv u^i[\epsilon,\delta,\Psi[\epsilon,\delta],\Theta[\epsilon,\delta],\Xi[\epsilon,\delta]](t)\qquad\forall t\in\mathrm{cl}\,\mathbb{S}_a[\Omega_\epsilon],\\u^o[\epsilon,\delta](t)&\equiv u^o[\epsilon,\delta,\Psi[\epsilon,\delta],\Theta[\epsilon,\delta],\Xi[\epsilon,\delta]](t)\qquad\forall t\in\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],\end{aligned}$$

where $\epsilon_3$, $\delta_1$, $\Psi$, $\Theta$, $\Xi$ are as in Theorem 4.141 (*iv*).

We now show that $\{(u^i[\epsilon,\delta](\cdot),u^o[\epsilon,\delta](\cdot))\}_{(\epsilon,\delta)\in]0,\epsilon_3[\times]0,\delta_1[}$ can be continued real analytically for negative values of $\epsilon$, $\delta$.

We have the following.

**Theorem 4.143.** *Let* $m\in\mathbb{N}\setminus\{0\}$*,* $\alpha\in\,]0,1[$*. Let* $w\in A$*. Let* $\Omega$*,* $\epsilon_1$*,* $F$*,* $g$*,* $\gamma$ *be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let* $\epsilon_3$*,* $\delta_1$ *be as in Theorem 4.141* (*iv*)*. Then the following statements hold.*

(i) *Let* $V$ *be a bounded open subset of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\,V\cap\mathbb{S}_a[\Omega_0]=\emptyset$*. Then there exist* $\epsilon_4\in\,]0,\epsilon_3]$*, a real analytic operator* $U_1^o$ *of* $]-\epsilon_4,\epsilon_4[\,\times\,]-\delta_1,\delta_1[$ *to the space* $C_h^0(\mathrm{cl}\,V)$*, and a real analytic operator* $U_2^o$ *of* $]-\epsilon_4,\epsilon_4[\,\times\,]-\delta_1,\delta_1[$ *to* $\mathbb{R}$*, such that the following conditions hold.*

  (j) $\mathrm{cl}\,V\subseteq\mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon\in\,]-\epsilon_4,\epsilon_4[$.

  (jj)
$$u^o[\epsilon,\delta](x)=\delta\epsilon^{n-1}U_1^o[\epsilon,\delta](x)+\delta\epsilon U_2^o[\epsilon,\delta]\qquad\forall x\in\mathrm{cl}\,V,$$
  *for all* $(\epsilon,\delta)\in\,]0,\epsilon_4[\,\times\,]0,\delta_1[$.

*(ii)* *Let $\bar{V}$ be a bounded open subset of $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$. Then there exist $\bar{\epsilon}_4 \in ]0, \epsilon_3]$, a real analytic operator $\bar{U}_1^o$ of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \times ]-\delta_1, \delta_1[$ to the space $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$, and a real analytic operator $\bar{U}_2^o$ of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$, such that the following conditions hold.*

*(j')* $w + \epsilon\,\mathrm{cl}\,\bar{V} \subseteq \mathrm{cl}\,\mathbb{P}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \setminus \{0\}$.

*(jj')*

$$u^o[\epsilon,\delta](w+\epsilon t) = \delta\epsilon \bar{U}_1^o[\epsilon,\delta](t) + \delta\epsilon\bar{U}_2^o[\epsilon,\delta] \qquad \forall t \in \mathrm{cl}\,\bar{V},$$

*for all* $(\epsilon,\delta) \in ]0, \bar{\epsilon}_4[ \times ]0, \delta_1[$. *Moreover,* $\bar{U}_1^o[0,0](\cdot)$ *equals the restriction of* $\tilde{u}^o(\cdot)$ *to* $\mathrm{cl}\,\bar{V}$.

*(iii)* *There exist $\epsilon_4' \in ]0, \epsilon_3]$, a real analytic operator $U_1^i$ of $]-\epsilon_4', \epsilon_4'[ \times ]-\delta_1, \delta_1[$ to the space $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, and a real analytic operator $U_2^i$ of $]-\epsilon_4', \epsilon_4'[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$, such that*

$$u^i[\epsilon,\delta](w+\epsilon t) = \delta\epsilon U_1^i[\epsilon,\delta](t) + \delta\epsilon U_2^i[\epsilon,\delta] + F^{(-1)}(0) \qquad \forall t \in \mathrm{cl}\,\Omega,$$

*for all* $(\epsilon,\delta) \in ]0, \epsilon_4'[ \times ]0, \delta_1[$. *Moreover,* $U_1^i[0,0](\cdot) + U_2^i[0,0]$ *equals* $\tilde{u}^i(\cdot)$ *on* $\mathrm{cl}\,\Omega$.

*Proof.* We modify the proof of Theorem 4.120. Let $\Psi[\cdot,\cdot]$, $\Theta[\cdot,\cdot]$, $\Xi[\cdot,\cdot]$ be as in Theorem 4.141 $(iv)$. Consider $(i)$. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $(\epsilon,\delta) \in ]0, \epsilon_4[ \times ]0, \delta_1[$. We have

$$u^o[\epsilon,\delta](x) = \delta\epsilon^{n-1}\int_{\partial\Omega} S_n^a(x-w-\epsilon s)\Theta[\epsilon,\delta](s)\,d\sigma_s \\ - \delta\epsilon^{n-1}\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(\int_{\partial\Omega} S_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s\Big)\,d\sigma_t \qquad \forall x \in \mathrm{cl}\,V.$$

Thus (cf. the proof of Theorem 2.158), it is natural to set

$$U_1^o[\epsilon,\delta](x) \equiv \int_{\partial\Omega} S_n^a(x-w-\epsilon s)\Theta[\epsilon,\delta](s)\,d\sigma_s \qquad \forall x \in \mathrm{cl}\,V,$$

for all $(\epsilon,\delta) \in ]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$, and

$$U_2^o[\epsilon,\delta] \equiv -\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s\Big)\,d\sigma_t,$$

for all $(\epsilon,\delta) \in ]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$. Then, by possibly taking a smaller $\epsilon_4$, $U_1^o$ is a real analytic map of $]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$ to $C_h^0(\mathrm{cl}\,V)$. Furthermore, we have that $U_2$ is a real analytic operator of $]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$. Finally, by the definition of $U_1$ and $U_2$, we immediately deduce that the equality in $(jj)$ holds. Consider now $(ii)$. Choosing $\bar{\epsilon}_4$ small enough, we can clearly assume that $(j')$ holds. Consider now $(jj')$. Let $(\epsilon,\delta) \in ]0, \bar{\epsilon}_4[ \times ]0, \delta_1[$. We have

$$u^o[\epsilon,\delta](w+\epsilon t) = \delta\epsilon^{n-1}\int_{\partial\Omega} S_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s \\ - \delta\epsilon^{n-1}\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(\int_{\partial\Omega} S_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s\Big)\,d\sigma_t \qquad \forall t \in \mathrm{cl}\,\bar{V}.$$

Thus (cf. Proposition 1.29 $(ii)$), it is natural to set

$$\bar{U}_1^o[\epsilon,\delta](t) \equiv \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s \qquad \forall t \in \mathrm{cl}\,\bar{V},$$

for all $(\epsilon,\delta) \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \times ]-\delta_1, \delta_1[$, and

$$\bar{U}_2^o[\epsilon,\delta] \equiv -\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s\Big)\,d\sigma_t,$$

for all $(\epsilon,\delta) \in ]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \times ]-\delta_1, \delta_1[$. By the proof of $(i)$, we have that $\bar{U}_2^o$ is a real analytic map of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \times ]-\delta_1, \delta_1[$ to $\mathbb{R}$. Moreover, (cf. Proposition 1.29 $(ii)$) we have that $\bar{U}_1^o$ is a real analytic map of $]-\bar{\epsilon}_4, \bar{\epsilon}_4[ \times ]-\delta_1, \delta_1[$ to $C^{m,\alpha}(\mathrm{cl}\,\bar{V})$. Finally, consider $(iii)$. Let $(\epsilon,\delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[$. We have

$$u^i[\epsilon,\delta](w+\epsilon t) = \delta\epsilon^{n-1}\int_{\partial\Omega} S_n^a(\epsilon(t-s))\Psi[\epsilon,\delta](s)\,d\sigma_s + \delta\epsilon\Xi[\epsilon,\delta] + F^{(-1)}(0) \qquad \forall t \in \mathrm{cl}\,\Omega.$$

Thus, by arguing as above, it is natural to set

$$U_1^i[\epsilon,\delta](t) \equiv \int_{\partial\Omega} S_n(t-s)\Psi[\epsilon,\delta](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\Psi[\epsilon,\delta](s)\,d\sigma_s \qquad \forall t \in \mathrm{cl}\,\Omega,$$

for all $(\epsilon,\delta) \in ]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[$, and

$$U_2^i[\epsilon,\delta] \equiv \Xi[\epsilon,\delta]$$

for all $(\epsilon,\delta) \in ]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[$. Then, by arguing as above (cf. Proposition 1.29 (*iii*)), there exists $\epsilon_4' \in ]0,\epsilon_3]$, such that $U_1^i$ and $U_2^i$ are real analytic maps of $]-\epsilon_4',\epsilon_4'[\times]-\delta_1,\delta_1[$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ and $\mathbb{R}$, respectively, such that the statement in (*iii*) holds. □

As done in Theorem 4.143 for $(u^i[\cdot,\cdot],u^o[\cdot,\cdot])$, we can now prove a real analytic continuation Theorem for the energy integral. Namely, we prove the following.

**Theorem 4.144.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.141 (iv). Then there exist $\epsilon_5 \in ]0,\epsilon_3]$ and two real analytic operators $G^i$, $G^o$ of $]-\epsilon_5,\epsilon_5[\times]-\delta_1,\delta_1[$ to $\mathbb{R}$, such that*

$$\int_{\Omega_\epsilon} |\nabla u^i[\epsilon,\delta](x)|^2\,dx = \delta^2\epsilon^n G^i[\epsilon,\delta], \tag{4.218}$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon,\delta](x)|^2\,dx = \delta^2\epsilon^n G^o[\epsilon,\delta], \tag{4.219}$$

*for all $(\epsilon,\delta) \in ]0,\epsilon_5[\times]0,\delta_1[$. Moreover,*

$$G^i[0,0] = \int_{\Omega} |\nabla \tilde{u}^i(x)|^2\,dx, \tag{4.220}$$

$$G^o[0,0] = \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega} |\nabla \tilde{u}^o(x)|^2\,dx. \tag{4.221}$$

*Proof.* It suffices to modify the proof of Theorem 4.122. Let $\Psi[\cdot,\cdot]$, $\Theta[\cdot,\cdot]$, $\Xi[\cdot,\cdot]$ be as in Theorem 4.141 (*iv*). Let $(\epsilon,\delta) \in ]0,\epsilon_3[\times]0,\delta_1[$. Clearly,

$$\int_{\Omega_\epsilon} |\nabla u^i[\epsilon,\delta](x)|^2\,dx = \delta^2\int_{\Omega_\epsilon} |\nabla v_a^+[\partial\Omega_\epsilon,\Psi[\epsilon,\delta](\frac{1}{\epsilon}(\cdot-w))](x)|^2\,dx,$$

and

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon,\delta](x)|^2\,dx = \delta^2\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla v_a^-[\partial\Omega_\epsilon,\Theta[\epsilon,\delta](\frac{1}{\epsilon}(\cdot-w))](x)|^2\,dx.$$

As a consequence, by slightly modifying the proof of Theorem 4.20, we can prove that there exist $\epsilon_5 \in ]0,\epsilon_3]$ and two real analytic operators $G^i$ and $G^o$ of $]-\epsilon_5,\epsilon_5[\times]-\delta_1,\delta_1[$ to $\mathbb{R}$ such that

$$\int_{\Omega_\epsilon} |\nabla u^i[\epsilon,\delta](x)|^2\,dx = \delta^2\epsilon^n G^i[\epsilon,\delta],$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon,\delta](x)|^2\,dx = \delta^2\epsilon^n G^o[\epsilon,\delta],$$

for all $(\epsilon,\delta) \in ]0,\epsilon_5[\times]0,\delta_1[$, and

$$G^i[0] = \int_{\Omega} |\nabla \tilde{u}^i(x)|^2\,dx,$$

$$G^o[0] = \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega} |\nabla \tilde{u}^o(x)|^2\,dx.$$

□

We also have the following.

**Theorem 4.145.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.141 (iv). Then there exist $\epsilon_6 \in ]0,\epsilon_3]$, $\delta_2 \in ]0,\delta_1]$ and two real analytic operators $J^i$, $J^o$ of $]-\epsilon_6,\epsilon_6[\times]-\delta_2,\delta_2[$ to $\mathbb{R}$, such that*

$$\int_{\Omega_\epsilon} u^i[\epsilon,\delta](x)\,dx = J^i[\epsilon,\delta], \tag{4.222}$$

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^o[\epsilon,\delta](x)\,dx = J^o[\epsilon,\delta], \tag{4.223}$$

*for all $(\epsilon,\delta) \in ]0,\epsilon_6[\times]0,\delta_2[$. Moreover,*

$$J^i[0,0] = 0, \tag{4.224}$$

$$J^o[0,0] = 0. \tag{4.225}$$

*Proof.* Let $\epsilon_3$, $\delta_1$, $\Psi$, $\Theta$, $\Xi$ be as in Theorem 4.141. Let $(\epsilon,\delta) \in ]0,\epsilon_3[\times]0,\delta_1[$. Clearly,

$$\begin{aligned} u^o[\epsilon,\delta](x) =& \delta\epsilon^{n-1}\int_{\partial\Omega} S_n^a(x-w-\epsilon s)\Theta[\epsilon,\delta](s)\,d\sigma_s \\ &- \delta\epsilon^{n-1}\frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(\int_{\partial\Omega} S_n^a(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s\Big)\,d\sigma_t \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon]. \end{aligned}$$

Then, by arguing as in the proof of Theorem 3.44 and by Theorem 2.115, we can easily prove that there exist $\epsilon_6' \in ]0,\epsilon_3[$, $\delta_2 \in ]0,\delta_1]$ and a real analytic map $J^o$ of $]-\epsilon_6',\epsilon_6'[\times]-\delta_2,\delta_2[$ to $\mathbb{R}$ such that

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u^o[\epsilon,\delta](x)\,dx = J^o[\epsilon,\delta],$$

for all $(\epsilon,\delta) \in ]0,\epsilon_6'[\times]0,\delta_2[$, and that $J^o[0,0]=0$.

Let $(\epsilon,\delta) \in ]0,\epsilon_3[\times]0,\delta_1[$. Clearly,

$$\int_{\Omega_\epsilon} u^i[\epsilon,\delta](x)\,dx = \epsilon^n \int_\Omega u^i[\epsilon,\delta](w+\epsilon t)\,dt.$$

On the other hand, if $\epsilon_4'$, $U_1^i$, $U_2^i$ are as in Theorem 4.143, and we set

$$J^i[\epsilon,\delta] \equiv \epsilon^n \int_\Omega \big(\delta\epsilon U_1^i[\epsilon,\delta](t) + \delta\epsilon U_2^i[\epsilon,\delta] + F^{(-1)}(0)\big)\,dt$$

for all $(\epsilon,\delta) \in ]-\epsilon_4',\epsilon_4'[\times]-\delta_1,\delta_1[$, then we have that $J^i$ is a real analytic map of $]-\epsilon_4',\epsilon_4'[\times]-\delta_1,\delta_1[$ to $\mathbb{R}$, such that $J^i[0,0]=0$ and that

$$\int_{\Omega_\epsilon} u^i[\epsilon,\delta](x)\,dx = J^i[\epsilon,\delta]$$

for all $(\epsilon,\delta) \in ]0,\epsilon_4'[\times]0,\delta_1[$.

Then, by taking $\epsilon_6 \equiv \min\{\epsilon_6',\epsilon_4'\}$, we can easily conclude. □

We now show that the family $\{(u^i[\epsilon,\delta],u^o[\epsilon,\delta])\}_{(\epsilon,\delta)\in]0,\epsilon_3[\times]0,\delta_1[}$ is essentially unique. To do so, we need to introduce a preliminary lemma.

**Lemma 4.146.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$. Let $(u^i,u^o)$ solve* (4.206). *Let $(\psi,\theta,\xi) \in (\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ be such that $u^i = u^i[\epsilon,\delta,\psi,\theta,\xi]$ and $u^o = u^o[\epsilon,\delta,\psi,\theta,\xi]$. Then*

$$v^+[\partial\Omega,\psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R_n^a(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi = \frac{u^i(w+\epsilon t) - F^{(-1)}(0)}{\delta\epsilon} \qquad \forall t \in \mathrm{cl}\,\Omega,$$

*Proof.* It is an immediate consequence of Theorem 4.139. □

**Theorem 4.147.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ *be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let* $\{(\hat{\epsilon}_j, \hat{\delta}_j)\}_{j\in\mathbb{N}}$ *be a sequence in* $]0,\epsilon_1[\times]0,+\infty[$ *converging to* $(0,0)$. *If* $\{(u^i_j, u^o_j)\}_{j\in\mathbb{N}}$ *is a sequence of pairs of functions such that*

$$(u^i_j, u^o_j) \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\Omega_{\hat{\epsilon}_j}]) \times C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_{\hat{\epsilon}_j}]), \tag{4.226}$$

$$(u^i_j, u^o_j) \text{ solves (4.206) with } (\epsilon,\delta) \equiv (\hat{\epsilon}_j, \hat{\delta}_j), \tag{4.227}$$

$$\lim_{j\to\infty} \frac{u^i_j(w+\hat{\epsilon}_j\cdot) - F^{(-1)}(0)}{\hat{\delta}_j\hat{\epsilon}_j} = \tilde{u}^i(\cdot) \qquad \text{in } C^{m,\alpha}(\partial\Omega), \tag{4.228}$$

*then there exists* $j_0 \in \mathbb{N}$ *such that*

$$(u^i_j, u^o_j) = (u^i[\hat{\epsilon}_j, \hat{\delta}_j], u^o[\hat{\epsilon}_j, \hat{\delta}_j]) \qquad \forall j_0 \le j \in \mathbb{N}.$$

*Proof.* We modify the proof of Theorem 4.126. By Theorem 4.139, for each $j \in \mathbb{N}$, there exists a unique triple $(\psi_j, \theta_j, \xi_j)$ in $(\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}$ such that

$$u^i_j = u^i[\hat{\epsilon}_j, \hat{\delta}_j, \psi_j, \theta_j, \xi_j], \qquad u^o_j = u^o[\hat{\epsilon}_j, \hat{\delta}_j, \psi_j, \theta_j, \xi_j]. \tag{4.229}$$

We shall now try to show that

$$\lim_{j\to\infty} (\psi_j, \theta_j, \xi_j) = (\tilde{\psi}, \tilde{\theta}, \tilde{\xi}) \qquad \text{in } (\mathcal{U}_0^{m-1,\alpha})^2 \times \mathbb{R}. \tag{4.230}$$

Indeed, if we denote by $\tilde{\mathcal{U}}$ the neighbourhood of Theorem 4.141 (*iv*), the limiting relation of (4.230) implies that there exists $j_0 \in \mathbb{N}$ such that

$$(\hat{\epsilon}_j, \hat{\delta}_j, \psi_j, \theta_j, \xi_j) \in ]0,\epsilon_3[\times]0,\delta_1[\times\tilde{\mathcal{U}},$$

for $j \ge j_0$ and thus Theorem 4.141 (*iv*) would imply that

$$(\psi_j, \theta_j, \xi_j) = (\Psi[\hat{\epsilon}_j, \hat{\delta}_j], \Theta[\hat{\epsilon}_j, \hat{\delta}_j], \Xi[\hat{\epsilon}_j, \hat{\delta}_j]),$$

for $j_0 \le j \in \mathbb{N}$, and that accordingly the theorem holds (cf. Definition 4.142.) Thus we now turn to the proof of (4.230). We note that equation $\Lambda[\epsilon,\delta,\psi,\theta,\xi] = 0$ can be rewritten in the following form

$$\begin{aligned}
&F'(F^{(-1)}(0))\Big(v^+[\partial\Omega,\psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi\Big) \\
&- \Big(v^-[\partial\Omega,\theta](t) + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\theta(s)\,d\sigma_s\Big) \\
&+ \frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\Big(v^-[\partial\Omega,\theta](t) + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\theta(s)\,d\sigma_s\Big)\,d\sigma_t \\
&= -\delta\epsilon\Big(v^+[\partial\Omega,\psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi\Big)^2 \\
&\times \int_0^1 (1-\beta)F''\Big(F^{(-1)}(0) + \beta\delta\epsilon\big(v^+[\partial\Omega,\psi](t) + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\psi(s)\,d\sigma_s + \xi\big)\Big)d\beta \;\; \forall t \in \partial\Omega,
\end{aligned} \tag{4.231}$$

$$\begin{aligned}
&\frac{1}{2}\theta(t) + v_*[\partial\Omega,\theta](t) + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR^a_n(\epsilon(t-s))\theta(s)\,d\sigma_s \\
&+ \frac{1}{2}\gamma\psi(t) - \gamma v_*[\partial\Omega,\psi](t) - \gamma\epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR^a_n(\epsilon(t-s))\psi(s)\,d\sigma_s = g(t) \qquad \forall t \in \partial\Omega,
\end{aligned} \tag{4.232}$$

for all $(\epsilon,\delta,\psi,\theta,\xi)$ in the domain of $\Lambda$. By arguing so as to prove that the integral of the second component of $\Lambda$ on $\partial\Omega$ equals zero in the beginning of the proof of Theorem 4.141, we can conclude that both hand sides of equation (4.199) have zero integral on $\partial\Omega$. We define the map $N \equiv (N_l)_{l=1,2}$ of $]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[\times(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}$ by setting $N_1[\epsilon,\delta,\psi,\theta,\xi]$ equal to the left-hand side of the equality in (4.231), $N_2[\epsilon,\delta,\psi,\theta,\xi]$ equal to the left-hand side of the equality in (4.232) for all $(\epsilon,\delta,\psi,\theta,\xi) \in ]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[\times(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$. By arguing so as in the proof of Theorem

4.141, we can prove that $N$ is real analytic. Since $N[\epsilon,\delta,\cdot,\cdot,\cdot]$ is linear for all $(\epsilon,\delta)\in\,]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[$, we have

$$N[\epsilon,\delta,\psi,\theta,\xi]=\partial_{(\psi,\theta,\xi)}N[\epsilon,\tilde{\psi},\tilde{\theta},\tilde{\xi}](\psi,\theta,\xi)$$

for all $(\epsilon,\delta,\psi,\theta,\xi)\in\,]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[\times(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$, and the map of $]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[$ to $\mathcal{L}((\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R},C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha})$ which takes $(\epsilon,\delta)$ to $N[\epsilon,\delta,\cdot,\cdot,\cdot]$ is real analytic. Since

$$N[0,0,\cdot,\cdot,\cdot]=\partial_{(\psi,\theta,\xi)}\Lambda[0,0,\tilde{\psi},\tilde{\theta},\tilde{\xi}](\cdot,\cdot,\cdot),$$

Theorem 4.141 (*iii*) implies that $N[0,0,\cdot,\cdot,\cdot]$ is also a linear homeomorphism. Since the set of linear homeomorphisms of $(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}$ is open in $\mathcal{L}((\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R},C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha})$ and since the map which takes a linear invertible operator to its inverse is real analytic (cf. *e.g.*, Hille and Phillips [61, Theorems 4.3.2 and 4.3.4]), there exists $(\tilde{\epsilon},\tilde{\delta})\in\,]0,\epsilon_3[\times]0,\delta_1[$ such that the map $(\epsilon,\delta)\mapsto N[\epsilon,\delta,\cdot,\cdot,\cdot]^{(-1)}$ is real analytic from $]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\delta},\tilde{\delta}[$ to $\mathcal{L}(C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha},(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R})$. Next we denote by $S[\epsilon,\delta,\psi,\theta,\xi]\equiv(S_l[\epsilon,\delta,\psi,\theta,\xi])_{l=1,2}$ the pair defined by the right-hand side of (4.231)-(4.232). Then equation $\Lambda[\epsilon,\delta,\psi,\theta,\xi]=0$ (or equivalently system (4.231)-(4.232)) can be rewritten in the following form:

$$(\psi,\theta,\xi)=N[\epsilon,\delta,\cdot,\cdot,\cdot]^{(-1)}[S[\epsilon,\delta,\psi,\theta,\xi]],\tag{4.233}$$

for all $(\epsilon,\delta,\psi,\theta,\xi)\in\,]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\delta},\tilde{\delta}[\times(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$. Next we note that condition (4.228), the proof of Theorem 4.141, the real analyticity of $F$ and standard calculus in Banach space imply that

$$\lim_{j\to\infty}S[\hat{\epsilon}_j,\hat{\delta}_j,\psi_j,\theta_j,\xi_j]=S[0,0,\tilde{\psi},\tilde{\theta},\tilde{\xi}]\qquad\text{in } C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha}.\tag{4.234}$$

Then by (4.233) and by the real analyticity of $(\epsilon,\delta)\mapsto N[\epsilon,\delta,\cdot,\cdot,\cdot]^{(-1)}$, and by the bilinearity and continuity of the operator of $\mathcal{L}(C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha},(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R})\times(C^{m,\alpha}(\partial\Omega)\times\mathcal{U}_0^{m-1,\alpha})$ to $(\mathcal{U}_0^{m-1,\alpha})^2\times\mathbb{R}$, which takes a pair $(T_1,T_2)$ to $T_1[T_2]$, we conclude that (4.230) holds. Thus the proof is complete. □

We give the following definition.

**Definition 4.148.** Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.143 (*iv*). Let $(u^i[\cdot,\cdot],u^o[\cdot,\cdot])$ be as in Definition 4.142. For each pair $(\epsilon,\delta)\in\,]0,\epsilon_3[\times]0,\delta_1[$, we set

$$u^i_{(\epsilon,\delta)}(x)\equiv u^i[\epsilon,\delta](\frac{x}{\delta})\qquad\forall x\in\operatorname{cl}\mathbb{S}_a(\epsilon,\delta),\qquad u^o_{(\epsilon,\delta)}(x)\equiv u^o[\epsilon,\delta](\frac{x}{\delta})\qquad\forall x\in\operatorname{cl}\mathbb{T}_a(\epsilon,\delta).$$

*Remark* 4.149. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.143 (*iv*). For each $(\epsilon,\delta)\in\,]0,\epsilon_3[\times]0,\delta_1[$ the pair $(u^i_{(\epsilon,\delta)},u^o_{(\epsilon,\delta)})$ is a solution of (4.205).

Our aim is to study the asymptotic behaviour of $(u^i_{(\epsilon,\delta)},u^o_{(\epsilon,\delta)})$ as $(\epsilon,\delta)$ tends to $(0,0)$. In order to do so we introduce the following. As a first step, we study the behaviour of $(u^i[\epsilon,\delta],u^o[\epsilon,\delta])$ as $(\epsilon,\delta)$ tends to $(0,0)$.

**Proposition 4.150.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.143 (iv). Let $1\le p<\infty$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)}\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon,\delta]]=0\qquad\text{in } L^p(A),$$

*and*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)}\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon,\delta]]=0\qquad\text{in } L^p(A).$$

*Proof.* Let $\epsilon_3$, $\delta_1$, $\Psi$, $\Theta$, $\Xi$ be as in Theorem 4.141. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $(\epsilon,\delta)\in\,]0,\epsilon_3[\times]0,\delta_1[$, we have

$$u^i[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)=\delta\epsilon\int_{\partial\Omega}S_n(t-s)\Psi[\epsilon,\delta](s)\,d\sigma_s$$
$$+\delta\epsilon^{n-1}\int_{\partial\Omega}R^a_n(\epsilon(t-s))\Psi[\epsilon,\delta](s)\,d\sigma_s+\delta\epsilon\Xi[\epsilon,\delta]+F^{(-1)}(0),\quad\forall t\in\partial\Omega.$$

We set

$$
\begin{aligned}
N^i[\epsilon,\delta](t) \equiv & \delta\epsilon \int_{\partial\Omega} S_n(t-s)\Psi[\epsilon,\delta](s)\,d\sigma_s \\
& + \delta\epsilon^{n-1} \int_{\partial\Omega} R^a_n(\epsilon(t-s))\Psi[\epsilon,\delta](s)\,d\sigma_s + \delta\epsilon\Xi[\epsilon,\delta] + F^{(-1)}(0), \qquad \forall t \in \partial\Omega,
\end{aligned}
$$

for all $(\epsilon,\delta) \in ]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[$. By taking $\tilde{\epsilon} \in ]0,\epsilon_3[$, $\tilde{\delta} \in ]0,\delta_1[$ small enough, we can assume (cf. Proposition 1.26 $(i)$) that $N^i$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\delta},\tilde{\delta}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$
C^i \equiv \sup_{(\epsilon,\delta)\in]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\delta},\tilde{\delta}[} \|N^i[\epsilon,\delta]\|_{C^0(\partial\Omega)} < +\infty.
$$

By the Maximum Principle for harmonic functions, we have

$$
|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon,\delta]](x)| \leq C^i \qquad \forall x \in A, \quad \forall (\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\tilde{\delta}[.
$$

Obviously,

$$
\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon,\delta]](x) = 0 \qquad \forall x \in A \setminus \{w\}.
$$

Therefore, by the Dominated Convergence Theorem, we have

$$
\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon,\delta]] = 0 \qquad \text{in } L^p(A).
$$

If $(\epsilon,\delta) \in ]0,\epsilon_3[\times]0,\delta_1[$, we have

$$
\begin{aligned}
& u^o[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \delta\epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s \\
& - \frac{\delta\epsilon}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega}\Big(\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s\,d\sigma_t + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s\Big)\,d\sigma_t, \ \forall t \in \partial\Omega.
\end{aligned}
$$

We set

$$
\begin{aligned}
& N^o[\epsilon,\delta](t) \equiv \delta\epsilon \int_{\partial\Omega} S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s \\
& - \frac{\delta\epsilon}{\int_{\partial\Omega} d\sigma} \int_{\partial\Omega}\Big(\int_{\partial\Omega} S_n(t-s)\Theta[\epsilon,\delta](s)\,d\sigma_s\,d\sigma_t + \epsilon^{n-2}\int_{\partial\Omega} R^a_n(\epsilon(t-s))\Theta[\epsilon,\delta](s)\,d\sigma_s\Big)\,d\sigma_t, \ \forall t \in \partial\Omega,
\end{aligned}
$$

for all $(\epsilon,\delta) \in ]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[$. By taking $\tilde{\epsilon} \in ]0,\epsilon_3[$, $\tilde{\delta} \in ]0,\delta_1[$ small enough, we can assume that $N^o$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\delta},\tilde{\delta}[$ to $C^{m,\alpha}(\partial\Omega)$ and that

$$
C^o \equiv \sup_{(\epsilon,\delta)\in]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\delta},\tilde{\delta}[} \|N^o[\epsilon,\delta]\|_{C^0(\partial\Omega)} < +\infty.
$$

By Theorem 2.5, we have

$$
|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon,\delta]](x)| \leq C^o \qquad \forall x \in A, \quad \forall (\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\tilde{\delta}[.
$$

By Theorem 4.143, we have

$$
\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon,\delta]](x) = 0 \qquad \forall x \in A \setminus \{w\}.
$$

Therefore, by the Dominated Convergence Theorem, we have

$$
\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon,\delta]] = 0 \qquad \text{in } L^p(A).
$$

□

### 4.13.2 Asymptotic behaviour of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

In the following Theorem we deduce by Proposition 4.150 the convergence of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 4.151.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.143 (iv). Let $1 \le p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] = 0 \qquad \textit{in } L^p(V),$$

*and*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] = 0 \qquad \textit{in } L^p(V).$$

*Proof.* It suffices to modify the proof of Theorem 4.27. By virtue of Proposition 4.150, we have

$$\lim_{\epsilon\to 0^+} \|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon,\delta]]\|_{L^p(A)} = 0,$$

and

$$\lim_{\epsilon\to 0^+} \|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon,\delta]]\|_{L^p(A)} = 0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $C > 0$ such that

$$\|\mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}]\|_{L^p(V)} \le C\|\mathbf{E}^i_{(\epsilon,1)}[u^i[\epsilon,\delta]]\|_{L^p(A)} \qquad \forall(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,\min\{1,\delta_1\}[,$$

and

$$\|\mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}]\|_{L^p(V)} \le C\|\mathbf{E}^o_{(\epsilon,1)}[u^o[\epsilon,\delta]]\|_{L^p(A)} \qquad \forall(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,\min\{1,\delta_1\}[,$$

Thus,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^i_{(\epsilon,\delta)}[u^i_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^p(V),$$

and

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}^o_{(\epsilon,\delta)}[u^o_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^p(V).$$

□

Then we have the following Theorem, where we consider a functional associated to extensions of $u^i_{(\epsilon,\delta)}$ and of $u^o_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 4.152.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.141 (iv). Let $\epsilon_6$, $\delta_2$, $J^i$, $J^o$ be as in Theorem 4.145. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^i\big[\epsilon,\frac{r}{l}\big], \tag{4.235}$$

*and*

$$\int_{\mathbb{R}^n} \mathbf{E}^o_{(\epsilon,r/l)}[u^o_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^o\big[\epsilon,\frac{r}{l}\big], \tag{4.236}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N}\setminus\{0\}$ such that $l > (r/\delta_2)$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_6[$, and let $l \in \mathbb{N}\setminus\{0\}$, $l > (r/\delta_2)$. Then, by the periodicity of $u^i_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx\\ &= \int_{rA} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx\\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\Omega_\epsilon} u^i_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\Omega_\epsilon} u^i\big[\epsilon,(r/l)\big]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\Omega_\epsilon} u^i\big[\epsilon,(r/l)\big](t)\,dt \\ &= \frac{r^n}{l^n} J^i\big[\epsilon,\frac{r}{l}\big].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}^i_{(\epsilon,r/l)}[u^i_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n J^i\big[\epsilon,\frac{r}{l}\big],$$

and the validity of (4.235) follows. The proof of (4.236) is very similar and is accordingly omitted. □

### 4.13.3 Asymptotic behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$

This Subsection is devoted to the study of the behaviour of the energy integral of $(u^i_{(\epsilon,\delta)}, u^o_{(\epsilon,\delta)})$. We give the following.

**Definition 4.153.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in \,]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.143 (*iv*). For each pair $(\epsilon,\delta) \in \,]0,\epsilon_3[\,\times\,]0,\delta_1[$, we set

$$\mathrm{En}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{S}_a(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx + \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 4.154. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in \,]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ be as in (1.56), (1.57), (4.163), (4.164), (4.165), respectively. Let $\epsilon_3$, $\delta_1$ be as in Theorem 4.143 (*iv*). Let $(\epsilon,\delta) \in \,]0,\epsilon_3[\times]0,\delta_1[$. We have

$$\begin{aligned}\int_{\Omega(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\Omega(\epsilon,1)} |(\nabla u^i_{(\epsilon,\delta)})(\delta t)|^2\,dt \\ &= \delta^{n-2}\int_{\Omega_\epsilon} |\nabla u^i[\epsilon,\delta](t)|^2\,dt\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u^o_{(\epsilon,\delta)})(\delta t)|^2\,dt \\ &= \delta^{n-2}\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u^o[\epsilon,\delta](t)|^2\,dt.\end{aligned}$$

In the following Proposition we represent the function $\mathrm{En}(\cdot,\cdot)$ by means of real analytic functions.

**Proposition 4.155.** *Let* $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in \,]0,1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $F$, $g$, $\gamma$ *be as in* (1.56), (1.57), (4.163), (4.164), (4.165), *respectively. Let* $\epsilon_3$, $\delta_1$ *be as in Theorem 4.143* (*iv*). *Let* $\epsilon_5$, $G^i$, $G^o$ *be as in Theorem 4.144. Then*

$$\mathrm{En}\Big(\epsilon,\frac{1}{l}\Big) = \epsilon^n G^i[\epsilon,(1/l)] + \epsilon^n G^o[\epsilon,(1/l)],$$

*for all* $\epsilon \in \,]0,\epsilon_5[$ *and for all* $l \in \mathbb{N}$ *such that* $l > (1/\delta_1)$.

*Proof.* Let $(\epsilon,\delta) \in \,]0,\epsilon_5[\,\times\,]0,\delta_1[$. By Remark 4.154 and Theorem 4.144, we have

$$\int_{\Omega(\epsilon,\delta)} |\nabla u^i_{(\epsilon,\delta)}(x)|^2\,dx + \int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u^o_{(\epsilon,\delta)}(x)|^2\,dx = \delta^n\epsilon^n G^i[\epsilon,\delta] + \delta^n\epsilon^n G^o[\epsilon,\delta] \qquad (4.237)$$

where $G^i$, $G^o$ are as in Theorem 4.144. On the other hand, if $\epsilon \in ]0, \epsilon_5[$ and $l \in \mathbb{N} \setminus \{0\}$ is such that $l > (1/\delta_1)$, then we have

$$\begin{aligned}\mathrm{En}\Big(\epsilon, \frac{1}{l}\Big) &= l^n \frac{1}{l^n}\Big\{\epsilon^n G^i[\epsilon, (1/l)] + \epsilon^n G^o[\epsilon, (1/l)]\Big\}, \\ &= \epsilon^n G^i[\epsilon, (1/l)] + \epsilon^n G^o[\epsilon, (1/l)],\end{aligned}$$

and the conclusion easily follows. □

# CHAPTER 5

# Asymptotic behaviour of the effective electrical conductivity of periodic dilute composites

In this Chapter we study the asymptotic behaviour of the effective electrical conductivity of periodic dilute composites. For a description of this problem and references, we refer, *e.g.*, to Ammari, Kang and Touibi [6] and Ammari and Kang [3]. We briefly outline the problem. Let $V$ be a bounded domain of $\mathbb{R}^n$, with a connected Lipschitz boundary $\partial V$. Let $\sigma$, $\sigma_0$ be two positive constants, with $\sigma \neq \sigma_0$. We consider a periodic dilute composite filling $V$. More precisely, we assume that the material consists of a matrix of conductivity $\sigma_0$, containing a periodic array of small conductivity inhomogeneities. The periodic array has period $\rho > 0$, and each period contains a small inclusion of conductivity $\sigma$ and form $\rho\omega(\rho)\Omega$, where $\Omega$ is a sufficiently regular bounded connected open subset of $\mathbb{R}^n$, such that $0 \in \Omega$ and $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$ is connected, and $\omega$ a suitable real analytic function of a neighbourhood of 0 to $\mathbb{R}$ such that $\lim_{\rho\to 0} \omega(\rho) = 0$. For each $\rho > 0$, small enough, we define the effective conductivity $\tilde{\sigma}^{\omega}[\rho]$ (cf. Definition 5.6, Theorem 5.16, Ammari, Kang and Touibi [6], Milton [97], Jikov, Kozlov and Oleinik [62].) Our aim is to represent $\tilde{\sigma}^{\omega}[\rho]$ by means of real analytic functions of the variable $\rho$ defined in a neighbourhood of 0. In order to do so, we follow the strategy of Ammari, Kang and Touibi [6] and we apply to it our functional analytic approach. For a list of contributions in the computation of the asymptotic expansion of the effective conductivity or other effective properties, we refer to Ammari, Kang and Touibi [6].

We retain the notation of Chapter 1 (see in particular Sections 1.1, 1.3 Theorem 1.4 and Definitions 1.12, 1.14, 1.16.) For notation, definitions, and properties concerning classical layer potentials for the Laplace equation, we refer to Appendix B.

## 5.1 Effective electrical conductivity of periodic composite materials

For the sake of simplicity, throughout this Chapter we shall assume

$$a_{ii} \equiv 1 \qquad \forall i \in \{1, \ldots, n\}.$$

Accordingly,

$$a_i \equiv e_i \qquad \forall i \in \{1, \ldots, n\},$$

and

$$A \equiv ]0, 1[^n.$$

Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0, 1[$. Let $\mathbb{I}$ be as in (1.46). We shall consider also the following assumption.

$$\sigma_0, \sigma \in ]0, +\infty[, \quad \sigma_0 \neq \sigma. \tag{5.1}$$

Let $j \in \{1, \ldots, n\}$. If $x \in \mathbb{R}^n$, we denote by $(x)_j$, or more simply by $x_j$, the j-th coordinate of $x$. Moreover, we denote by $\mathrm{pr}_j$ the function of $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$\mathrm{pr}_j(x) \equiv x_j,$$

for all $x \in \mathbb{R}^n$.

By virtue of Theorem 4.6, it is easy to see that we can state the following.

**Definition 5.1.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\sigma_0$, $\sigma$ be as in (5.1). Let $i \in \{1,\dots,n\}$. We denote by $u_i$ the unique function of $\mathbb{R}^n$ to $\mathbb{R}$ such that $u_{i|\mathbb{S}_a[\mathbb{I}]\cup\mathbb{T}_a[\mathbb{I}]} \in C^2(\mathbb{S}_a[\mathbb{I}]\cup\mathbb{T}_a[\mathbb{I}])$, $u_{i|\,\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]} \in C^1(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}])$, $u_{i|\,\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} \in C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$, and that

$$\begin{cases} \Delta u_i(x) = 0 & \forall x \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}], \\ u_i(x+a_j) - \mathrm{pr}_i(x+a_j) = u_i(x) - \mathrm{pr}_i(x) & \forall x \in \mathbb{R}^n, \quad \forall j \in \{1,\dots,n\}, \\ u_i^-(x) = u_i^+(x) & \forall x \in \partial\mathbb{I}, \\ \sigma_0 \frac{\partial}{\partial\nu_{\mathbb{I}}} u_i^-(x) = \sigma \frac{\partial}{\partial\nu_{\mathbb{I}}} u_i^+(x) & \forall x \in \partial\mathbb{I}, \\ \int_A u_i(x)\,dx = 0, \end{cases} \tag{5.2}$$

where

$$u_i^- \equiv u_{i|\,\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]}, \qquad u_i^+ \equiv u_{i|\,\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]}.$$

We have the following.

**Proposition 5.2.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\sigma_0$, $\sigma$ be as in (5.1). Let $i \in \{1,\dots,n\}$. Let $u_i$, $u_i^-$, $u_i^+$ be as in Definition 5.1. Then $u_i \in C^0(\mathbb{R}^n)$, $u_i^- \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}])$ and $u_i^+ \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}])$. Moreover,*

$$u_i = \mathrm{pr}_i + v_a[\partial\mathbb{I},\mu_i] + C_i \qquad \text{in } \mathbb{R}^n, \tag{5.3}$$

*where $\mu_i$ is the unique function in $C^{m-1,\alpha}(\partial\mathbb{I})$ such that*

$$\frac{1}{2}\mu_i(x) - \frac{\sigma-\sigma_0}{\sigma+\sigma_0}\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}(S_n^a(x-y))\mu_i(y)\,d\sigma_y = \frac{\sigma-\sigma_0}{\sigma+\sigma_0}(\nu_{\mathbb{I}}(x))_i \qquad \forall x \in \partial\mathbb{I}, \tag{5.4}$$

*and $C_i \in \mathbb{R}$ is delivered by the following formula*

$$C_i = -\int_A (\mathrm{pr}_i(x) + v_a[\partial\mathbb{I},\mu_i](x))\,dx. \tag{5.5}$$

*Moreover,*

$$\int_{\partial\mathbb{I}} \mu_i\,d\sigma = 0. \tag{5.6}$$

*Proof.* Clearly, it suffices to prove that the function defined in the right-hand side of equality (5.3) solves problem (5.2). We observe that by Proposition 4.5 (*iii*), there exists a unique $\mu_i$ in $C^{m-1,\alpha}(\partial\mathbb{I})$ such that (5.4) holds. Moreover, by Proposition 4.5 (*ii*), $\int_{\partial\mathbb{I}} \mu_i\,d\sigma = 0$. Then, by Theorem 1.15 (cf. also Theorem 4.6), it is easy to prove that the function defined in the right-hand side of equality (5.3) satisfies (5.2). □

We are now in the position to introduce the following definition (cf. *e.g.*, Ammari, Kang and Touibi [6, p. 121].)

**Definition 5.3.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\sigma_0$, $\sigma$ be as in (5.1). For each $(i,j) \in \{1,\dots,n\}^2$, we set

$$\tilde{\sigma}_{ij}[\mathbb{I},\sigma,\sigma_0] \equiv \Big(\sigma_0 \int_{\mathbb{P}_a[\mathbb{I}]} \nabla u_i(x)\cdot\nabla u_j(x)\,dx + \sigma\int_{\mathbb{I}} \nabla u_i(x)\cdot\nabla u_j(x)\,dx\Big).$$

We also set

$$\tilde{\sigma}[\mathbb{I},\sigma,\sigma_0] \equiv (\tilde{\sigma}_{ij}[\mathbb{I},\sigma,\sigma_0])_{i,j=1,\dots,n}.$$

The matrix $\tilde{\sigma}[\mathbb{I},\sigma,\sigma_0] \in M_{n\times n}(\mathbb{R})$ is called the *effective conductivity matrix*.

Then we have the following Lemma of Ammari, Kang and Touibi [6, Lemma 5.1],

**Lemma 5.4.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\sigma_0$, $\sigma$ be as in (5.1). Let $(i,j) \in \{1,\dots,n\}^2$. Then*

$$\tilde{\sigma}_{ij}[\mathbb{I},\sigma,\sigma_0] = \sigma_0\Big(\delta_{ij} + \int_{\partial\mathbb{I}} x_j\mu_i(x)\,d\sigma_x\Big), \tag{5.7}$$

*where $\mu_i$ is as in Proposition 5.2 and $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.*

*Proof.* We follow Ammari, Kang and Touibi [6, Lemma 5.1]. By Proposition 5.2, the Divergence Theorem and by periodicity, we have

$$\begin{aligned}\tilde{\sigma}_{ij}[\mathbb{I},\sigma,\sigma_0] &= \sigma_0 \int_{\partial A} u_j(x) \frac{\partial}{\partial \nu_A} u_i(x)\, d\sigma_x \\ &= \sigma_0 \int_{\partial A} \big(\mathrm{pr}_j(x) + C_j + v_a[\partial\mathbb{I},\mu_j](x)\big) \frac{\partial}{\partial \nu_A} \big(\mathrm{pr}_i(x) + C_i + v_a[\partial\mathbb{I},\mu_i](x)\big)\, d\sigma_x \\ &= \sigma_0 \Big(\delta_{ij} + \int_{\partial A} x_j \frac{\partial}{\partial \nu_A} v_a[\partial\mathbb{I},\mu_i](x)\, d\sigma_x\Big).\end{aligned}$$

Moreover,

$$\begin{aligned}\int_{\partial A} x_j \frac{\partial}{\partial \nu_A} v_a[\partial\mathbb{I},\mu_i](x)\, d\sigma_x &= \int_{\partial\mathbb{I}} x_j \frac{\partial}{\partial \nu_{\mathbb{I}}} v_a^-[\partial\mathbb{I},\mu_i](x)\, d\sigma_x - \int_{\partial\mathbb{I}} (\nu_{\mathbb{I}}(x))_j v_a[\partial\mathbb{I},\mu_i](x)\, d\sigma_x \\ &= \int_{\partial\mathbb{I}} x_j \frac{\partial}{\partial \nu_{\mathbb{I}}} v_a^-[\partial\mathbb{I},\mu_i](x)\, d\sigma_x - \int_{\partial\mathbb{I}} x_j \frac{\partial}{\partial \nu_{\mathbb{I}}} v_a^+[\partial\mathbb{I},\mu_i](x)\, d\sigma_x \\ &= \int_{\partial\mathbb{I}} x_j \mu_i(x)\, d\sigma_x,\end{aligned}$$

and accordingly (5.7) holds. □

We find convenient to give the following definition (cf. Ammari and Kang [3, Definition 4.1, p. 77].)

**Definition 5.5.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{I}$ is connected. Let $k \in ]0,+\infty[$. For each $(i,j) \in \{1,\dots,n\}^2$, we set

$$m_{ij}[\mathbb{I},k] \equiv \int_{\partial\mathbb{I}} t_j \theta_i(t)\, d\sigma_t,$$

where $\theta_i$ is the unique function in $C^{m-1,\alpha}(\partial\mathbb{I})$ such that

$$\frac{1}{2}\theta_i(t) - \frac{k-1}{k+1} \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}(t)} (S_n(t-s))\theta_i(s)\, d\sigma_s = \frac{k-1}{k+1} (\nu_{\mathbb{I}}(t))_i \qquad \forall t \in \partial\mathbb{I}. \tag{5.8}$$

We also define the matrix $M \in M_{n\times n}(\mathbb{R})$, by setting

$$M[\mathbb{I},k] \equiv (m_{ij}[\mathbb{I},k])_{i,j=1,\dots,n}.$$

## 5.2 Asymptotic behaviour of the effective electrical conductivity

### 5.2.1 Notation and preliminaries

We retain the notation of Section 5.1 and Subsection 1.8.1.

We give the following.

**Definition 5.6.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\sigma_0$, $\sigma$ be as in (5.1). Let $(i,j) \in \{1,\dots,n\}^2$. For each $\epsilon \in ]0,\epsilon_1[$, we set

$$\tilde{\sigma}_{ij}[\epsilon] \equiv \tilde{\sigma}_{ij}[\Omega_\epsilon,\sigma,\sigma_0].$$

We set also

$$\tilde{\sigma}[\epsilon] \equiv \tilde{\sigma}[\Omega_\epsilon,\sigma,\sigma_0].$$

Our aim is to investigate the behaviour of $\tilde{\sigma}_{ij}[\epsilon]$ as $\epsilon$ tends to 0. By Lemma 5.4, we know that $\tilde{\sigma}_{ij}[\epsilon]$ can be expressed by means of the solution of an integral equation defined on the $\epsilon$-dependent domain $\partial\Omega_\epsilon$. In the following Proposition we convert this equation into an integral equation defined on the fixed domain $\partial\Omega$.

**Proposition 5.7.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\sigma_0$, $\sigma$ be as in (5.1). Let $i \in \{1, \dots, n\}$. Let $\Lambda_i$ be the map of $]-\epsilon_1, \epsilon_1[ \times C^{m-1,\alpha}(\partial\Omega)$ in $C^{m-1,\alpha}(\partial\Omega)$ defined by*

$$\begin{aligned}\Lambda_i[\epsilon,\theta](t) \equiv \frac{1}{2}\theta(t) - \Big(\frac{\sigma-\sigma_0}{\sigma+\sigma_0}\Big)\int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\theta(s)\,d\sigma_s \\ - \Big(\frac{\sigma-\sigma_0}{\sigma+\sigma_0}\Big)\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR^a_n(\epsilon(t-s))\theta(s)\,d\sigma_s - \frac{\sigma-\sigma_0}{\sigma+\sigma_0}(\nu_\Omega(t))_i \qquad \forall t\in\partial\Omega,\end{aligned} \tag{5.9}$$

*for all $(\epsilon,\theta) \in ]-\epsilon_1,\epsilon_1[ \times C^{m-1,\alpha}(\partial\Omega)$. Then the following statements hold.*

*(i) If $\epsilon \in ]0,\epsilon_1[$, then the function $\theta \in C^{m-1,\alpha}(\partial\Omega)$ satisfies equation*

$$\Lambda_i[\epsilon,\theta] = 0, \tag{5.10}$$

*if and only if the function $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon)$, defined by*

$$\mu(x) \equiv \theta\Big(\frac{1}{\epsilon}(x-w)\Big) \qquad \forall x \in \partial\Omega_\epsilon, \tag{5.11}$$

*satisfies the equation*

$$\frac{\sigma-\sigma_0}{\sigma+\sigma_0}(\nu_{\Omega_\epsilon}(x))_i = \frac{1}{2}\mu(x) - \frac{\sigma-\sigma_0}{\sigma+\sigma_0}\int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}(S^a_n(x-y))\mu(y)\,d\sigma_y \qquad \forall x\in\partial\Omega_\epsilon. \tag{5.12}$$

*In particular, equation (5.10) has exactly one solution $\theta \in C^{m-1,\alpha}(\partial\Omega)$, for each $\epsilon \in ]0,\epsilon_1[$. Moreover, if $\theta$ solves (5.10), then $\theta \in \mathcal{U}_0^{m-1,\alpha}$, and so also $\theta(\frac{1}{\epsilon}(\cdot - w)) \in \mathcal{U}_\epsilon^{m-1,\alpha}$.*

*(ii) The function $\theta \in C^{m-1,\alpha}(\partial\Omega)$ satisfies equation*

$$\Lambda_i[0,\theta] = 0, \tag{5.13}$$

*if and only if*

$$\frac{\sigma-\sigma_0}{\sigma+\sigma_0}(\nu_\Omega(t))_i = \frac{1}{2}\theta(t) - \frac{\sigma-\sigma_0}{\sigma+\sigma_0}\int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\,d\sigma_s \qquad \forall t\in\partial\Omega. \tag{5.14}$$

*In particular, equation (5.13) has exactly one solution $\theta \in C^{m-1,\alpha}(\partial\Omega)$, which we denote by $\tilde{\theta}_i$. Moreover, if $\theta$ solves (5.14), then $\theta \in \mathcal{U}_0^{m-1,\alpha}$.*

*Proof.* It follows by Proposition 4.11, with

$$\phi \equiv 1, \quad \gamma \equiv \frac{\sigma}{\sigma_0}, \quad g(\cdot) \equiv \frac{\sigma-\sigma_0}{\sigma_0}(\nu_\Omega(\cdot))_i.$$

□

By Proposition 5.7, it makes sense to give the following.

**Definition 5.8.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\sigma_0$, $\sigma$ be as in (5.1). Let $i \in \{1,\dots,n\}$. For each $\epsilon \in ]0,\epsilon_1[$, we denote by $\hat{\theta}_i[\epsilon]$ the unique function in $C^{m-1,\alpha}(\partial\Omega)$ that solves (5.10). Analogously, we denote by $\hat{\theta}_i[0]$ the unique function in $C^{m-1,\alpha}(\partial\Omega)$ that solves (5.13).

The relation between $\hat{\theta}_i[\epsilon]$ and $\tilde{\sigma}_{ij}[\epsilon]$ is explained in the following.

*Remark* 5.9. Let $m \in \mathbb{N}\setminus\{0\}$. Let $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\sigma_0$, $\sigma$ be as in (5.1). Let $(i,j) \in \{1,\dots,n\}^2$. Let $\epsilon \in ]0,\epsilon_1[$. Then we have

$$\begin{aligned}\tilde{\sigma}_{ij}[\epsilon] &= \sigma_0\Big(\delta_{ij} + \epsilon^{n-1}\int_{\partial\Omega}(w+\epsilon t)_j\hat{\theta}_i[\epsilon](t)\,d\sigma_t\Big) \\ &= \sigma_0\Big(\delta_{ij} + \epsilon^{n}\int_{\partial\Omega} t_j\hat{\theta}_i[\epsilon](t)\,d\sigma_t\Big).\end{aligned}$$

While the relation between the solution of equation (5.10) and $\tilde{\sigma}_{ij}[\epsilon]$ is now clear, in the following remark we show that the solution of equation (5.13) is related to the matrix $M\big[\Omega, \frac{\sigma}{\sigma_0}\big]$

*Remark* 5.10. Let $m \in \mathbb{N}\setminus\{0\}$. Let $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\sigma_0$, $\sigma$ be as in (5.1). Let $(i,j) \in \{1,\dots,n\}^2$. Then we have

$$m_{ij}\big[\Omega, \frac{\sigma}{\sigma_0}\big] = \int_{\partial\Omega} t_j \hat{\theta}_i[0](t)\, d\sigma_t,$$

where $m_{ij}$ is as in Definition 5.5.

We now prove the following.

**Proposition 5.11.** *Let $m \in \mathbb{N}\setminus\{0\}$. Let $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $\sigma_0$, $\sigma$ be as in* (5.1). *Let $i \in \{1,\dots,n\}$. Let $\Lambda_i$ and $\tilde{\theta}_i$ be as in Proposition 5.7. Then there exists $\epsilon_2 \in ]0,\epsilon_1]$ such that $\Lambda_i$ is a real analytic operator of $]-\epsilon_2,\epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$. Moreover, if we set $b_0 \equiv (0,\tilde{\theta}_i)$, then the differential $\partial_\theta\Lambda_i[b_0]$ of $\Lambda_i$ with respect to the variable $\theta$ at $b_0$ is delivered by the following formula*

$$\partial_\theta\Lambda_i[b_0](\bar{\theta})(t) = \frac{1}{2}\bar{\theta}(t) - \big(\frac{\sigma-\sigma_0}{\sigma+\sigma_0}\big)\int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\bar{\theta}(s)\, d\sigma_s \qquad \forall t \in \partial\Omega, \tag{5.15}$$

*for all $\bar{\theta} \in C^{m-1,\alpha}(\partial\Omega)$, and is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega)$ onto $C^{m-1,\alpha}(\partial\Omega)$.*

*Proof.* It follows by Proposition 4.16, with

$$\phi \equiv 1, \quad \gamma \equiv \frac{\sigma}{\sigma_0}, \quad g(\cdot) \equiv \frac{\sigma-\sigma_0}{\sigma_0}(\nu_\Omega(\cdot))_i.$$

□

We are now ready to prove that $\hat{\theta}_i[\cdot]$ can be continued real analytically on a whole neighbourhood of 0.

**Proposition 5.12.** *Let $m \in \mathbb{N}\setminus\{0\}$. Let $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $\sigma_0$, $\sigma$ be as in* (5.1). *Let $i \in \{1,\dots,n\}$. Let $\epsilon_2$ be as in Proposition 5.11. Then there exist $\epsilon_3 \in ]0,\epsilon_2]$ and a real analytic operator $\Theta_i$ of $]-\epsilon_3,\epsilon_3[$ to $C^{m-1,\alpha}(\partial\Omega)$, such that*

$$\Theta_i[\epsilon] = \hat{\theta}_i[\epsilon], \tag{5.16}$$

*for all $\epsilon \in [0,\epsilon_3[$.*

*Proof.* It is an immediate consequence of Proposition 5.11 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

### 5.2.2 A representation Theorem for the effective conductivity

By Proposition 5.12 and Remark 5.9, we can deduce the main result of this Subsection.

**Theorem 5.13.** *Let $m \in \mathbb{N}\setminus\{0\}$. Let $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $\sigma_0$, $\sigma$ be as in* (5.1). *Let $(i,j) \in \{1,\dots,n\}^2$. Let $\epsilon_3$ be as in Proposition 5.12. Then there exists a real analytic operator $U_{ij}$ of $]-\epsilon_3,\epsilon_3[$ to $\mathbb{R}$, such that*

$$\tilde{\sigma}_{ij}[\epsilon] = \sigma_0\Big(\delta_{ij} + \epsilon^n U_{ij}[\epsilon]\Big), \tag{5.17}$$

*for all $\epsilon \in ]0,\epsilon_3[$. Moreover,*

$$U_{ij}[0] = m_{ij}\big[\Omega, \frac{\sigma}{\sigma_0}\big]. \tag{5.18}$$

*Proof.* Let $\Theta_i$ be as in Proposition 5.12. Let $\epsilon \in ]0,\epsilon_3[$. By Remark 5.9, we have

$$\tilde{\sigma}_{ij}[\epsilon] = \sigma_0\Big(\delta_{ij} + \epsilon^n \int_{\partial\Omega} t_j \Theta_i[\epsilon](t)\, d\sigma_t\Big).$$

As a consequence, it suffices to set

$$U_{ij}[\epsilon] \equiv \int_{\partial\Omega} t_j \Theta_i[\epsilon](t)\, d\sigma_t,$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. Obviously, $U_{ij}$ is a real analytic map of $]-\epsilon_3, \epsilon_3[$ to $\mathbb{R}$. Moreover, by Remark 5.10, we have

$$m_{ij}\big[\Omega, \frac{\sigma}{\sigma_0}\big] = \int_{\partial\Omega} t_j \Theta_i[0](t)\, d\sigma_t = U_{ij}[0].$$

Hence the proof is complete. □

**Corollary 5.14.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $\sigma_0$, $\sigma$ be as in* (5.1). *Then there exist $\epsilon_4 \in ]0, \epsilon_1[$ and a real analytic operator $U$ of $]-\epsilon_4, \epsilon_4[$ to $M_{n\times n}(\mathbb{R})$, such that*

$$\tilde{\sigma}[\epsilon] = \sigma_0\Big(I_n + \epsilon^n U[\epsilon]\Big), \tag{5.19}$$

*for all $\epsilon \in ]0, \epsilon_4[$, where $I_n$ denotes the identity matrix in $M_{n\times n}(\mathbb{R})$. Moreover,*

$$U[0] = M\big[\Omega, \frac{\sigma}{\sigma_0}\big]. \tag{5.20}$$

*Proof.* It is an immediate consequence of Theorem 5.13. □

*Remark* 5.15. We note that the right-hand side of (5.17) and of (5.19) can be continued real analytically in a whole neighbourhood of 0.

Obviously, we can deduce the following result (see also Ammari and Kang [3, Theorem 8.1, p. 200].)

**Theorem 5.16.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $\sigma_0$, $\sigma$ be as in* (5.1). *Let $\rho_1 \in ]0, +\infty[$. Let $\omega$ be a real analytic function of $]-\rho_1, \rho_1[$ to $]-\epsilon_1, \epsilon_1[$, such that*

$$\omega(\rho) \in ]0, \epsilon_1[ \qquad \forall \rho \in ]0, \rho_1[,$$

*and*

$$\lim_{\rho\to 0} \omega(\rho) = 0.$$

*We set*

$$\tilde{\sigma}^{\omega}[\rho] \equiv \tilde{\sigma}[\omega(\rho)],$$

*for all $\rho \in ]0, \rho_1[$. Then there exist $\rho_2 \in ]0, \rho_1[$ and a real analytic operator $U^{\omega}$ of $]-\rho_2, \rho_2[$ to $M_{n\times n}(\mathbb{R})$, such that*

$$\tilde{\sigma}^{\omega}[\rho] = \sigma_0\Big(I_n + \omega(\rho)^n U^{\omega}[\rho]\Big), \tag{5.21}$$

*for all $\rho \in ]0, \rho_2[$, where $I_n$ denotes the identity matrix in $M_{n\times n}(\mathbb{R})$. Moreover,*

$$U^{\omega}[0] = M\big[\Omega, \frac{\sigma}{\sigma_0}\big]. \tag{5.22}$$

*Proof.* It is an immediate consequence of Theorem 5.13 and Corollary 5.14. □

*Remark* 5.17. Assume, for the sake of simplicity, that $n = 2$. We observe that Theorem 5.2 of Ammari, Kang and Touibi [6, p. 132] (cf. also Ammari and Kang [3, Theorem 8.1, p. 200]) asserts that

$$\tilde{\sigma}[\epsilon] = \sigma_0\Big(I_2 + \epsilon^2 M\big[\Omega, \frac{\sigma}{\sigma_0}\big] + \frac{\epsilon^4}{2} M\big[\Omega, \frac{\sigma}{\sigma_0}\big]^2\Big) + \mathrm{O}(\epsilon^6). \tag{5.23}$$

Then, by combining equation (5.23) with Corollary 5.14, one can prove that there exist $\tilde{\epsilon} > 0$ and a real analytic map $\tilde{U}$ of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $M_{2\times 2}(\mathbb{R})$ such that

$$\tilde{\sigma}[\epsilon] = \sigma_0\Big(I_2 + \epsilon^2 M\big[\Omega, \frac{\sigma}{\sigma_0}\big] + \frac{\epsilon^4}{2} M\big[\Omega, \frac{\sigma}{\sigma_0}\big]^2\Big) + \epsilon^6 \tilde{U}[\epsilon], \tag{5.24}$$

for all $\epsilon \in ]0, \tilde{\epsilon}[$.

# CHAPTER 6

# Periodic simple and double layer potentials for the Helmholtz equation

This Chapter is mainly devoted to the definition of periodic analogues of the simple and double layer potentials for the Helmholtz equation. Namely, we construct these objects by replacing the classical fundamental solution of the Helmholtz operator with a periodic analogue in the definition of the classical layer potentials for the Helmholtz equation. Moreover, we prove some regularity results for the solutions of some integral equations, involved in the resolution of boundary value problems by means of periodic potentials. Some of the results are based on the classical analogous results (cf. *e.g.*, Lanza and Rossi [86].) For a generalization of some results contained in this Chapter, we refer to [81].

We retain the notation introduced in Sections 1.1 and 1.3. For notation, definitions and properties from classical potential theory for the Helmholtz equation we refer to Appendix E.

## 6.1 Construction of a periodic analogue of the fundamental solution for the Helmholtz equation

In this Section, we construct a periodic analogue of the fundamental solution for the Helmholtz equation. In order to do so, we follow the same strategy, based on Fourier Analysis, used for the periodic analogue of the fundamental solution for the Laplace equation. For this and other constructions, we refer, for example, to Ammari, Kang and Lee [4, p. 123], Ammari, Kang, Soussi and Zribi [5], Dienstfrey, Hang and Huang [47], Linton [87], McPhedran, Nicorovici, Botten and Bao [94], Nicorovici, McPhedran and Botten. [106], Poulton, Botten, McPhedran and Movchan [114]. We have the following Theorem.

**Theorem 6.1.** *Let $k \in \mathbb{C}$. We set*

$$Z_a(k) \equiv \left\{ z \in \mathbb{Z}^n \colon k^2 = |2\pi a^{-1}(z)|^2 \right\}. \tag{6.1}$$

*Let $G_n^{a,k}$ be the element of $\mathcal{S}'(\mathbb{R}^n)$ defined by*

$$G_n^{a,k} \equiv \sum_{z \in \mathbb{Z}^n \setminus Z_a(k)} \frac{1}{|A|_n (k^2 - |2\pi a^{-1}(z)|^2)} E_{2\pi a^{-1}(z)}. \tag{6.2}$$

*Then the following statements hold.*

*(i)*

$$\tau_{l a_j} G_n^{a,k} = G_n^{a,k} \qquad \forall l \in \mathbb{Z}, \quad \forall j \in \{1, \dots, n\}. \tag{6.3}$$

*(ii)*

$$(\Delta + k^2) G_n^{a,k} = \sum_{z \in \mathbb{Z}^n} \delta_{a(z)} - \sum_{z \in Z_a(k)} \frac{1}{|A|_n} E_{2\pi a^{-1}(z)} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n), \tag{6.4}$$

*in the sense of distributions.*

*Proof.* By Proposition 1.1, $G_n^{a,k}$ is an element of $\mathcal{S}'(\mathbb{R}^n)$ such that $(i)$ holds. Now we need to prove (6.4). By continuity of the Laplace operator from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, we have

$$\begin{aligned}(\Delta + k^2)G_n^{a,k} &= \sum_{z\in\mathbb{Z}^n\setminus Z_a(k)} \frac{1}{|A|_n} E_{2\pi a^{-1}(z)} \\ &= \sum_{z\in\mathbb{Z}^n} \frac{1}{|A|_n} E_{2\pi a^{-1}(z)} - \sum_{z\in Z_a(k)} \frac{1}{|A|_n} E_{2\pi a^{-1}(z)} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).\end{aligned} \tag{6.5}$$

On the other hand, by Proposition 1.2, we have

$$\sum_{z\in\mathbb{Z}^n} \frac{1}{|A|_n} E_{2\pi a^{-1}(z)} = \sum_{z\in\mathbb{Z}^n} \delta_{a(z)} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

and so the validity of the statement in $(ii)$ follows. □

*Remark* 6.2. We observe that, if $k^2 \neq 4\pi^2|a^{-1}(z)|^2$ for all $z \in \mathbb{Z}^n$, then $Z_a(k) = \emptyset$ and so

$$(\Delta + k^2)G_n^{a,k} = \sum_{z\in\mathbb{Z}^n} \delta_{a(z)} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

**Theorem 6.3.** *Let $k \in \mathbb{C}$. Let $Z_a(k)$ be as in (6.1). Let $G_n^{a,k}$ be as in Theorem 6.1. Let the function $S_n(\cdot,k)$ of $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{C}$ be the fundamental solution of $\Delta + k^2$ defined in Proposition E.3. Then the following statements hold.*

*(i) There exists a unique function $S_n^{a,k}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{C})$ such that*

$$\int_{\mathbb{R}^n} S_n^{a,k}(x)\phi(x)\,dx = \langle G_n^{a,k}, \phi\rangle \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n,\mathbb{C}). \tag{6.6}$$

*In particular,*

$$(\Delta + k^2)S_n^{a,k} = \sum_{z\in\mathbb{Z}^n} \delta_{a(z)} - \sum_{z\in Z_a(k)} \frac{1}{|A|_n} E_{2\pi a^{-1}(z)} \tag{6.7}$$

*in the sense of distributions. Moreover, $S_n^{a,k}$ equals almost everywhere a real analytic function of $\mathbb{R}^n \setminus Z_n^a$ to $\mathbb{C}$ and*

$$(\Delta + k^2)S_n^{a,k}(x) = -\sum_{z\in Z_a(k)} \frac{1}{|A|_n} e^{i2\pi a^{-1}(z)\cdot x} \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a \tag{6.8}$$

*and*

$$S_n^{a,k}(x + a_j) = S_n^{a,k}(x) \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a, \quad \forall j \in \{1,\dots,n\}. \tag{6.9}$$

*(ii) There exists a unique real analytic function $R_n^{a,k}$ of $(\mathbb{R}^n \setminus Z_n^a) \cup \{0\}$ to $\mathbb{C}$, such that*

$$S_n^{a,k}(x) = S_n(x,k) + R_n^{a,k}(x) \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a.$$

*Moreover,*

$$(\Delta + k^2)R_n^{a,k}(x) = -\sum_{z\in Z_a(k)} \frac{1}{|A|_n} e^{i2\pi a^{-1}(z)\cdot x} \qquad \forall x \in (\mathbb{R}^n \setminus Z_n^a) \cup \{0\}.$$

*Proof.* Now let $F \in \mathcal{D}'(\mathbb{R}^n,\mathbb{C})$ be defined by

$$\langle F, \phi\rangle = \langle G_n^{a,k}, \phi\rangle - \int_{\mathbb{R}^n} S_n(x,k)\phi(x)\,dx \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n,\mathbb{C}). \tag{6.10}$$

We have

$$(\Delta + k^2)F = \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \delta_{a(z)} - \sum_{z\in Z_a(k)} \frac{1}{|A|_n} E_{2\pi a^{-1}(z)} \qquad \text{in } \mathcal{D}'(\mathbb{R}^n,\mathbb{C}).$$

By standard elliptic regularity theory (cf. *e.g.*, Friedman [54, Theorem 1.2, p. 205]), there exists a real analytic function $\tilde{R}_n^{a,k}$ of $\mathcal{O}$ to $\mathbb{C}$ (cf. (1.6)), such that

$$\int_{\mathcal{O}} \tilde{R}_n^{a,k}(x)\phi(x)\,dx = \langle F, \phi\rangle \qquad \forall \phi \in \mathcal{D}(\mathcal{O},\mathbb{C}).$$

Moreover,

$$(\Delta + k^2)\tilde{R}_n^{a,k}(x) = -\sum_{z\in Z_a(k)} \frac{1}{|A|_n} e^{i2\pi a^{-1}(z)\cdot x} \qquad \forall x \in \mathcal{O}.$$

Clearly, by (6.10), we have

$$\int_{\mathcal{O}} (S_n(x,k) + \tilde{R}_n^{a,k}(x))\phi(x)\,dx = \left\langle G_n^{a,k}, \phi\right\rangle \qquad \forall \phi \in \mathcal{D}(\mathcal{O},\mathbb{C}). \tag{6.11}$$

Let

$$\tilde{\mathcal{O}} \equiv \prod_{j=1}^{n} ]-\frac{3a_{jj}}{5}, \frac{3a_{jj}}{5}[.$$

Next we define $S_n^{a,k} \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{C})$ by setting

$$S_n^{a,k}\big(x + a(z)\big) = S_n(x,k) + \tilde{R}_n^{a,k}(x) \qquad \forall x \in \tilde{\mathcal{O}} \setminus \{0\}, \quad \forall z \in \mathbb{Z}^n.$$

By (6.11) and the periodicity of $G_n^{a,k}$, the function $S_n^{a,k}$ is well defined. Indeed, one can easily verify that if $x \in \tilde{\mathcal{O}} \setminus \{0\}$, $z \in \mathbb{Z}^n$, and $x + a(z) \in \tilde{\mathcal{O}}$, then

$$S_n(x,k) + \tilde{R}_n^{a,k}(x) = S_n(x + a(z),k) + \tilde{R}_n^{a,k}(x + a(z)).$$

Furthermore, by (6.11), the periodicity of $G_n^{a,k}$, and the definition of $S_n^{a,k}$, we have

$$\int_{\tilde{\mathcal{O}}+a(z)} S_n^{a,k}(x)\phi(x)\,dx = \left\langle G_n^{a,k}, \phi\right\rangle \qquad \forall \phi \in \mathcal{D}(\tilde{\mathcal{O}} + a(z),\mathbb{C}),$$

for all $z \in \mathbb{Z}^n$. As a consequence,

$$\int_{\mathbb{R}^n} S_n^{a,k}(x)\phi(x)\,dx = \left\langle G_n^{a,k}, \phi\right\rangle \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n,\mathbb{C}).$$

Moreover, since $S_n(\cdot,k)$ and $R_n^{a,k}(\cdot)$ are real analytic in $\tilde{\mathcal{O}} \setminus \{0\}$, $S_n^{a,k}$ is a real analytic function of $\mathbb{R}^n \setminus Z_n^a$ to $\mathbb{C}$, such that (6.8) and (6.9) hold.

Finally, if we set

$$R_n^{a,k}(x) \equiv S_n^{a,k}(x) - S_n(x,k) \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a,$$

then, by (6.7) and by standard elliptic regularity theory, we have that $R_n^{a,k}$ can be extended by continuity to a real analytic function (that we still call $R_n^{a,k}$) of $(\mathbb{R}^n \setminus Z_n^a) \cup \{0\}$ to $\mathbb{C}$, such that $(ii)$ holds. □

*Remark* 6.4. By arguing on the definition of $G_n^{a,k}$ and $S_n^{a,k}$, one can easily show that

$$S_n^{a,k}(x) = S_n^{a,k}(-x) \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a.$$

*Remark* 6.5. Let $k \in \mathbb{C}$. Then

$$\sum_{\substack{z\in\mathbb{Z}^n \\ k^2=|2\pi a^{-1}(z)|^2}} e^{i2\pi a^{-1}(z)\cdot x} = \sum_{\substack{z\in\mathbb{N}^n \\ k^2=|2\pi a^{-1}(z)|^2}} \prod_{j=1}^{n} (2 - \delta_{0,z_j}) \cos\frac{2\pi z_j x_j}{a_{jj}},$$

for all $x \in \mathbb{R}^n$.

## 6.2 Periodic double layer potential for the Helmholtz equation

In this Section we define the periodic double layer potential for the Helmholtz equation. The construction is quite natural. We substitute in the definition of the (classical) double layer potential the fundamental solution of the Helmholtz equation $S_n(\cdot, k)$ with the function $S_n^{a,k}$ introduced in Theorem 6.3.

**Definition 6.6.** Let $k \in \mathbb{C}$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\mu \in L^2(\partial\mathbb{I}, \mathbb{C})$. We set

$$w_a[\partial\mathbb{I}, \mu, k](t) \equiv \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n^{a,k}(t-s))\mu(s)\, d\sigma_s \qquad \forall t \in \mathbb{R}^n.$$

The function $w_a[\partial\mathbb{I}, \mu, k]$ is called the periodic double layer potential for the Helmholtz equation with moment $\mu$.

In the following Theorem we collect some properties of the periodic double layer potential for the Helmholtz equation.

**Theorem 6.7.** *Let $k \in \mathbb{C}$. Let $Z_a(k)$ be as in (6.1). Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Then the following statements hold.*

*(i) Let $\mu \in C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$. Then*

$$\begin{aligned}(\Delta + k^2)&w_a[\partial\mathbb{I}, \mu, k](t)\\ =&\frac{1}{|A|_n}\sum_{z \in Z_a(k)} e^{i2\pi a^{-1}(z)\cdot t}\int_{\partial\mathbb{I}} i2\pi\nu_{\mathbb{I}}(s)\cdot a^{-1}(z)e^{-i2\pi a^{-1}(z)\cdot s}\mu(s)\, d\sigma_s \\ &\forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}],\end{aligned} \tag{6.12}$$

*and*

$$w_a[\partial\mathbb{I}, \mu, k](t + a_j) = w_a[\partial\mathbb{I}, \mu, k](t) \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}.$$

*The restriction of $w_a[\partial\mathbb{I}, \mu, k]$ to the set $\mathbb{S}_a[\mathbb{I}]$ can be extended uniquely to a continuous periodic function $w_a^+[\partial\mathbb{I}, \mu, k]$ of $\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]$ to $\mathbb{C}$, and $w_a^+[\partial\mathbb{I}, \mu, k] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}], \mathbb{C})$. The restriction of $w_a[\partial\mathbb{I}, \mu, k]$ to the set $\mathbb{T}_a[\mathbb{I}]$ can be extended uniquely to a continuous periodic function $w_a^-[\partial\mathbb{I}, \mu, k]$ of $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$ to $\mathbb{C}$, and $w_a^-[\partial\mathbb{I}, \mu, k] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \mathbb{C})$. Moreover, we have the following jump relations*

$$w_a^+[\partial\mathbb{I}, \mu, k](t) = +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n^{a,k}(t-s))\mu(s)\, d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

$$w_a^-[\partial\mathbb{I}, \mu, k](t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n^{a,k}(t-s))\mu(s)\, d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

$$Dw_a^+[\partial\mathbb{I}, \mu, k]\cdot\nu_{\mathbb{I}} - Dw_a^-[\partial\mathbb{I}, \mu, k]\cdot\nu_{\mathbb{I}} = 0 \qquad \text{on } \partial\mathbb{I}.$$

*(ii) The map of $C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$ to $C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$ which takes $\mu$ to $w_a^+[\partial\mathbb{I}, \mu, k]_{|\partial\mathbb{I}}$ is linear and continuous (and thus real analytic.) The map of $C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$ to $C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$ which takes $\mu$ to $w_a^-[\partial\mathbb{I}, \mu, k]_{|\partial\mathbb{I}}$ is linear and continuous (and thus real analytic.)*

*Proof.* We start with $(i)$. Let $\mu \in C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$. Clearly, the periodicity of $w_a[\partial\mathbb{I}, \mu, k]$ follows by the periodicity of $S_n^{a,k}$ (see (6.9).) By classical theorems of differentiation under the integral sign and by Theorem 6.3, we have that

$$\begin{aligned}(\Delta + k^2)&w_a[\partial\mathbb{I}, \mu, k](t)\\ =&\frac{1}{|A|_n}\sum_{z \in Z_a(k)} e^{i2\pi a^{-1}(z)\cdot t}\int_{\partial\mathbb{I}} i2\pi\nu_{\mathbb{I}}(s)\cdot a^{-1}(z)e^{-i2\pi a^{-1}(z)\cdot s}\mu(s)\, d\sigma_s \\ &\forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}].\end{aligned}$$

We have

$$w_a[\partial\mathbb{I}, \mu, k](t) = w[\partial\mathbb{I}, \mu, k](t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(R_n^{a,k}(t-s))\mu(s)\, d\sigma_s \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}].$$

Since $R_n^{a,k}$ is real analytic in $(\mathbb{R}^n \setminus Z_n^a) \cup \{0\}$, then the second term in the right-hand side of the previous equality is a function of class $C^\infty$ in an open bounded subset $\tilde{V}$ of $\mathbb{R}^n$, of class $C^\infty$, such that $\mathrm{cl}\, A \subseteq \tilde{V}$ and

$$\mathrm{cl}\, \tilde{V} \cap \mathrm{cl}(\mathbb{I} + a(z)) = \emptyset \qquad \forall z \in \mathbb{Z}^n \setminus \{0\}.$$

(cf. the proof of Theorem 1.13.) We set

$$\tilde{W} \equiv \tilde{V} \setminus \mathrm{cl}\, \mathbb{I}.$$

By Theorem E.4 $(i)$,

$$w_a[\partial\mathbb{I}, \mu, k](t) = w^+[\partial\mathbb{I}, \mu, k](t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(R_n^{a,k}(t-s))\mu(s)\, d\sigma_s \qquad \forall t \in \mathbb{I},$$

and

$$w_a[\partial\mathbb{I}, \mu, k](t) = w^-[\partial\mathbb{I}, \mu, k](t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(R_n^{a,k}(t-s))\mu(s)\, d\sigma_s \qquad \forall t \in \tilde{W}.$$

Furthermore, the terms in the right-hand side of the two previous equalities are continuous functions in $\mathrm{cl}\, \mathbb{I}$ and $\mathrm{cl}\, \tilde{W}$, respectively. Hence, by Lemma 1.11, we can easily conclude that $w_a[\partial\mathbb{I}, \mu, k]_{|\mathbb{S}_a[\mathbb{I}]}$ can be extended uniquely to a continuous periodic function $w_a^+[\partial\mathbb{I}, \mu, k]$ of $\mathrm{cl}\, \mathbb{S}_a[\mathbb{I}]$ to $\mathbb{C}$, and $w_a^+[\partial\mathbb{I}, \mu, k] \in C^{m,\alpha}(\mathrm{cl}\, \mathbb{S}_a[\mathbb{I}], \mathbb{C})$. Analogously, by Lemma 1.10, the restriction of $w_a[\partial\mathbb{I}, \mu, k]$ to the set $\mathbb{T}_a[\mathbb{I}]$ can be extended uniquely to a continuous periodic function $w_a^-[\partial\mathbb{I}, \mu, k]$ of $\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}]$ to $\mathbb{C}$, and $w_a^-[\partial\mathbb{I}, \mu, k] \in C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \mathbb{C})$. Clearly,

$$\begin{aligned} w_a^+[\partial\mathbb{I}, \mu, k](t) &= w^+[\partial\mathbb{I}, \mu, k](t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(R_n^{a,k}(t-s))\mu(s)\, d\sigma_s \\ &= +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n(t-s,k))\mu(s)\, d\sigma_s + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(R_n^{a,k}(t-s))\mu(s)\, d\sigma_s \\ &= +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n^{a,k}(t-s))\mu(s)\, d\sigma_s \qquad \forall t \in \partial\mathbb{I}, \end{aligned}$$

and

$$\begin{aligned} w_a^-[\partial\mathbb{I}, \mu, k](t) &= w^-[\partial\mathbb{I}, \mu, k](t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(R_n^{a,k}(t-s))\mu(s)\, d\sigma_s \\ &= -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n(t-s,k))\mu(s)\, d\sigma_s + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(R_n^{a,k}(t-s))\mu(s)\, d\sigma_s \\ &= -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n^{a,k}(t-s))\mu(s)\, d\sigma_s \qquad \forall t \in \partial\mathbb{I}. \end{aligned}$$

Thus, the jump relations hold and the statement in $(i)$ is proved. We now turn to the proof of $(ii)$. Set

$$H[\mu](t) \equiv \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(R_n^{a,k}(t-s))\mu(s)\, d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

for all $\mu \in C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$. By Theorem C.2, it is easy to see that $H$ is a linear and continuous map of $C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$ to $C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$. We have

$$w_a^+[\partial\mathbb{I}, \mu, k](t) = +\frac{1}{2}\mu(t) + w[\partial\mathbb{I}, \mu, k](t) + H[\mu](t) \qquad \forall t \in \partial\mathbb{I},$$

and

$$w_a^-[\partial\mathbb{I}, \mu, k](t) = -\frac{1}{2}\mu(t) + w[\partial\mathbb{I}, \mu, k](t) + H[\mu](t) \qquad \forall t \in \partial\mathbb{I}.$$

Then, by virtue of Theorem E.6 $(iii)$, we conclude that the map of $C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$ to $C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$ which takes $\mu$ to $w_a^+[\partial\mathbb{I}, \mu, k]_{|\partial\mathbb{I}}$ is linear and continuous. Analogously, by virtue of Theorem E.6 $(iii)$, we conclude that the map of $C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$ to $C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$ which takes $\mu$ to $w_a^-[\partial\mathbb{I}, \mu]_{|\partial\mathbb{I}}$ is linear and continuous. The Theorem is now completely proved. □

**Corollary 6.8.** *Let $k \in \mathbb{C}$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46). *Assume that $k^2 \neq |2\pi a^{-1}(z)|^2$ for all $z \in \mathbb{Z}^n$. Then*

$$(\Delta + k^2) w_a[\partial\mathbb{I}, \mu, k](t) = 0 \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}],$$

*for all $\mu \in C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$.*

*Proof.* It is an immediate consequence of Theorem 6.7. □

Then we have the following.

**Proposition 6.9.** *Let $k \in \mathbb{C}$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46). *Assume that the set $Z_a(k)$ is nonempty (cf.* (6.1).*) Let $\mu \in C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$. Let $\mathcal{V}$ be a nonempty open subset of $\mathbb{R}^n$ such that $\mathcal{V} \subseteq \mathbb{T}_a[\mathbb{I}] \cup \mathbb{S}_a[\mathbb{I}]$. Then*

$$(\Delta + k^2) w_a[\partial\mathbb{I}, \mu, k](t) = 0 \qquad \forall t \in \mathcal{V}, \tag{6.13}$$

*if and only if*

$$\int_{\partial\mathbb{I}} \nu_{\mathbb{I}}(s) \cdot a^{-1}(z) e^{-i2\pi a^{-1}(z)\cdot s} \mu(s)\, d\sigma_s = 0 \qquad \forall z \in Z_a(k). \tag{6.14}$$

*Proof.* Obviously, if (6.14) holds, then, by virtue of Theorem 6.7 ($i$), we have

$$(\Delta + k^2) w_a[\partial\mathbb{I}, \mu, k](t) = 0 \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}],$$

and, as a consequence, (6.13) holds. Conversely, assume that (6.13) holds. Then, by Theorem 6.7 ($i$), we have

$$\frac{1}{|A|_n} \sum_{z \in Z_a(k)} e^{i2\pi a^{-1}(z)\cdot t} \int_{\partial\mathbb{I}} i2\pi \nu_{\mathbb{I}}(s) \cdot a^{-1}(z) e^{-i2\pi a^{-1}(z)\cdot s} \mu(s)\, d\sigma_s = 0 \qquad \forall t \in \mathcal{V}.$$

Consequently, by the identity theorem for real analytic functions, we have

$$\frac{1}{|A|_n} \sum_{z \in Z_a(k)} e^{i2\pi a^{-1}(z)\cdot t} \int_{\partial\mathbb{I}} i2\pi \nu_{\mathbb{I}}(s) \cdot a^{-1}(z) e^{-i2\pi a^{-1}(z)\cdot s} \mu(s)\, d\sigma_s = 0 \qquad \forall t \in \mathbb{R}^n.$$

Now let $z_0 \in Z_a(k)$. By multiplying both sides of the previous equality by $e^{-i2\pi a^{-1}(z_0)\cdot t}$ and then by integrating on $\mathrm{cl}\, A$, we easily obtain

$$\int_{\partial\mathbb{I}} i2\pi \nu_{\mathbb{I}}(s) \cdot a^{-1}(z_0) e^{-i2\pi a^{-1}(z_0)\cdot s} \mu(s)\, d\sigma_s = 0.$$

Accordingly, (6.14) holds and the Proposition is proved. □

## 6.3 Periodic simple layer potential for the Helmholtz equation

In this Section we define the periodic simple layer potential for the Helmholtz equation. As already done for the double layer potential, we substitute in the definition of the (classical) simple layer potential the fundamental solution of the Helmholtz equation $S_n(\cdot, k)$ with the function $S_n^{a,k}$ introduced in Theorem 6.3.

**Definition 6.10.** Let $k \in \mathbb{C}$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\mu \in L^2(\partial\mathbb{I}, \mathbb{C})$. We set

$$v_a[\partial\mathbb{I}, \mu, k](t) \equiv \int_{\partial\mathbb{I}} S_n^{a,k}(t - s)\mu(s)\, d\sigma_s \qquad \forall t \in \mathbb{R}^n.$$

The function $v_a[\partial\mathbb{I}, \mu, k]$ is called the periodic simple layer potential for the Helmholtz equation with moment $\mu$.

In the following Theorem we collect some properties of the periodic simple layer potential for the Helmholtz equation.

**Theorem 6.11.** *Let $k \in \mathbb{C}$. Let $Z_a(k)$ be as in (6.1). Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Then the following statements hold.*

*(i) Let $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$. Then the function $v_a[\partial\mathbb{I}, \mu, k]$ is continuous in $\mathbb{R}^n$, and*

$$(\Delta + k^2) v_a[\partial\mathbb{I}, \mu, k](t) = -\frac{1}{|A|_n} \sum_{z \in Z_a(k)} e^{i2\pi a^{-1}(z)\cdot t} \int_{\partial\mathbb{I}} e^{-i2\pi a^{-1}(z)\cdot s} \mu(s)\, d\sigma_s \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}], \tag{6.15}$$

*and*

$$v_a[\partial\mathbb{I}, \mu, k](t + a_j) = v_a[\partial\mathbb{I}, \mu, k](t) \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}.$$

*Let $v_a^+[\partial\mathbb{I}, \mu, k]$ and $v_a^-[\partial\mathbb{I}, \mu, k]$ denote the restriction of $v_a[\partial\mathbb{I}, \mu, k]$ to the set $\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]$ and to the set $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$, respectively. Then $v_a^+[\partial\mathbb{I}, \mu, k] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}], \mathbb{C})$, and $v_a^-[\partial\mathbb{I}, \mu, k] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \mathbb{C})$. Moreover, we have the following jump relations*

$$\frac{\partial}{\partial \nu_{\mathbb{I}}} v_a^+[\partial\mathbb{I}, \mu, k](t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}(t)} (S_n^{a,k}(t-s)) \mu(s)\, d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

$$\frac{\partial}{\partial \nu_{\mathbb{I}}} v_a^-[\partial\mathbb{I}, \mu, k](t) = +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}(t)} (S_n^{a,k}(t-s)) \mu(s)\, d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

$$\frac{\partial}{\partial \nu_{\mathbb{I}}} v_a^-[\partial\mathbb{I}, \mu, k](t) - \frac{\partial}{\partial \nu_{\mathbb{I}}} v_a^+[\partial\mathbb{I}, \mu, k](t) = \mu(t) \qquad \forall t \in \partial\mathbb{I}.$$

*(ii) The map of $C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$ to $C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$ which takes $\mu$ to $v_a[\partial\mathbb{I}, \mu, k]_{|\partial\mathbb{I}}$ is linear and continuous (and thus real analytic.)*

*(iii) The map of $C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$ which takes $\mu$ to the function $v_{a*}[\partial\mathbb{I}, \mu, k]$ of $\partial\mathbb{I}$ to $\mathbb{C}$, defined by*

$$v_{a*}[\partial\mathbb{I}, \mu, k](t) \equiv \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}(t)} (S_n^{a,k}(t-s)) \mu(s)\, d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

*is linear and continuous (and thus real analytic.)*

*Proof.* We start with $(i)$. Let $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$. Clearly, the periodicity of $v_a[\partial\mathbb{I}, \mu, k]$ follows by the periodicity of $S_n^{a,k}$ (see (6.9).) Let $\tilde{V}$ be an open bounded subset of $\mathbb{R}^n$ of class $C^\infty$, such that $\mathrm{cl}\, A \subseteq \tilde{V}$ and

$$\mathrm{cl}\, \tilde{V} \cap \mathrm{cl}(\mathbb{I} + a(z)) = \emptyset \qquad \forall z \in \mathbb{Z}^n \setminus \{0\}.$$

(cf. the proof of Theorem 1.13.) Set

$$\tilde{W} \equiv \tilde{V} \setminus \mathrm{cl}\, \mathbb{I}.$$

Obviously,

$$v_a[\partial\mathbb{I}, \mu, k](t) = v[\partial\mathbb{I}, \mu, k](t) + \int_{\partial\mathbb{I}} R_n^{a,k}(t-s) \mu(s)\, d\sigma_s \qquad \forall t \in \mathrm{cl}\, \tilde{V}.$$

By Theorem E.5, the function $v[\partial\mathbb{I}, \mu, k]$ is continuous on $\mathrm{cl}\, \tilde{V}$. Moreover, the second term in the right-hand side of the previous equality defines a real analytic function on $\mathrm{cl}\, \tilde{V}$. Thus, the restriction of the function $v_a[\partial\mathbb{I}, \mu, k]$ to the set $\mathrm{cl}\, \tilde{V}$ is continuous, and so, by virtue of the periodicity of $v_a[\partial\mathbb{I}, \mu, k]$, we can conclude that $v_a[\partial\mathbb{I}, \mu, k]$ is continuous on $\mathbb{R}^n$. By classical theorems of differentiation under the integral sign, by Theorem 6.3, by arguing in $\tilde{W}$ and in $\mathbb{I}$ and then exploiting the periodicity of $v_a[\partial\mathbb{I}, \mu, k]$, we have that

$$(\Delta + k^2) v_a[\partial\mathbb{I}, \mu, k](t) = -\frac{1}{|A|_n} \sum_{z \in Z_a(k)} e^{i2\pi a^{-1}(z)\cdot t} \int_{\partial\mathbb{I}} e^{-i2\pi a^{-1}(z)\cdot s} \mu(s)\, d\sigma_s \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}].$$

Clearly,

$$v_a^+[\partial\mathbb{I}, \mu, k](t) = v^+[\partial\mathbb{I}, \mu, k](t) + \int_{\partial\mathbb{I}} R_n^{a,k}(t-s) \mu(s)\, d\sigma_s \qquad \forall t \in \mathbb{I},$$

and

$$v_a^-[\partial\mathbb{I},\mu,k](t) = v^-[\partial\mathbb{I},\mu,k](t) + \int_{\partial\mathbb{I}} R_n^{a,k}(t-s)\mu(s)\,d\sigma_s \qquad \forall t \in \mathrm{cl}\,\tilde{W}.$$

Then by Lemma 1.11 and Theorem E.5, we can conclude that $v_a^+[\partial\mathbb{I},\mu,k] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}[\mathbb{I}],\mathbb{C})$. Analogously, by Lemma 1.10 and Theorem E.5, we conclude that $v_a^-[\partial\mathbb{I},\mu,k] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}[\mathbb{I}],\mathbb{C})$. Moreover, by Theorem E.5, we have

$$\begin{aligned}
\frac{\partial}{\partial\nu_{\mathbb{I}}} v_a^+[\partial\mathbb{I},\mu,k](t) &= \frac{\partial}{\partial\nu_{\mathbb{I}}} v^+[\partial\mathbb{I},\mu,k](t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(R_n^{a,k}(t-s))\mu(s)\,d\sigma_s \\
&= -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n(t-s,k))\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(R_n^{a,k}(t-s))\mu(s)\,d\sigma_s \\
&= -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial\nu_{\mathbb{I}}} v_a^-[\partial\mathbb{I},\mu,k](t) &= \frac{\partial}{\partial\nu_{\mathbb{I}}} v^-[\partial\mathbb{I},\mu,k](t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(R_n^{a,k}(t-s))\mu(s)\,d\sigma_s \\
&= +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n(t-s,k))\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(R_n^{a,k}(t-s))\mu(s)\,d\sigma_s \\
&= +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I}.
\end{aligned}$$

Accordingly,

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} v_a^-[\partial\mathbb{I},\mu,k](t) - \frac{\partial}{\partial\nu_{\mathbb{I}}} v_a^+[\partial\mathbb{I},\mu,k](t) = \mu(t) \qquad \forall t \in \partial\mathbb{I}.$$

We now turn to the proof of $(ii)$. Set

$$H[\mu](t) \equiv \int_{\partial\mathbb{I}} R_n^{a,k}(t-s)\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

for all $\mu \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$. By Theorem C.2, it is easy to see that $H$ is a linear and continuous map of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ to $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$. We have

$$v_a[\partial\mathbb{I},\mu,k](t) = v[\partial\mathbb{I},\mu,k](t) + H[\mu](t) \qquad \forall t \in \partial\mathbb{I}.$$

Since the linear map of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ to $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ which takes $\mu$ to $H[\mu]$ is continuous, then, by virtue of Theorem E.6 $(i)$, we conclude that the map of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ to $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ which takes $\mu$ to $v_a[\partial\mathbb{I},\mu,k]_{|\partial\mathbb{I}}$ is linear and continuous. Consider now $(iii)$. Let $\mu \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$. We have

$$v_{a*}[\partial\mathbb{I},\mu,k](t) = \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n(t-s,k))\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}} \nu_{\mathbb{I}}(t)\cdot DR_n^{a,k}(t-s)\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I}.$$

By Theorem C.2, one can easily show that the map $H'$ of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ to $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ which takes $\mu$ to the function $H'[\mu]$ of $\partial\mathbb{I}$ to $\mathbb{C}$, defined by

$$H'[\mu](t) \equiv \int_{\partial\mathbb{I}} \nu_{\mathbb{I}}(t)\cdot DR_n^{a,k}(t-s)\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

is real continuous. On the other hand, by virtue of Theorem E.6 $(ii)$, the map $v_*[\partial\mathbb{I},\cdot,k]$ of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ to $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ which takes $\mu$ to the function $v_*[\partial\mathbb{I},\mu,k]$ of $\partial\mathbb{I}$ to $\mathbb{C}$, defined by

$$v_*[\partial\mathbb{I},\mu,k](t) \equiv \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n(t-s,k))\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

is continuous. Thus, the map $v_{a*}[\partial\mathbb{I},\cdot,k]$ of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ to $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ is continuous. Hence, the proof is now complete. □

**Corollary 6.12.** *Let* $k \in \mathbb{C}$. *Let* $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be as in* (1.46). *Assume that* $k^2 \neq |2\pi a^{-1}(z)|^2$ *for all* $z \in \mathbb{Z}^n$. *Then*

$$(\Delta + k^2)v_a[\partial\mathbb{I},\mu,k](t) = 0 \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}],$$

*for all* $\mu \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$.

*Proof.* It is an immediate consequence of Theorem 6.11. □

Then we have the following.

**Proposition 6.13.** *Let $k \in \mathbb{C}$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Assume that the set $Z_a(k)$ is nonempty (cf. (6.1).) Let $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$. Let $\mathcal{V}$ be a nonempty open subset of $\mathbb{R}^n$ such that $\mathcal{V} \subseteq \mathbb{T}_a[\mathbb{I}] \cup \mathbb{S}_a[\mathbb{I}]$. Then*

$$(\Delta + k^2) v_a[\partial\mathbb{I}, \mu, k](t) = 0 \qquad \forall t \in \mathcal{V}, \tag{6.16}$$

*if and only if*

$$\int_{\partial\mathbb{I}} e^{-i2\pi a^{-1}(z)\cdot s} \mu(s)\, d\sigma_s = 0 \qquad \forall z \in Z_a(k). \tag{6.17}$$

*Proof.* Obviously, if (6.17) holds, then, by virtue of Theorem 6.11, we have

$$(\Delta + k^2) v_a[\partial\mathbb{I}, \mu, k](t) = 0 \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}],$$

and, as a consequence, (6.16) holds. Conversely, assume that (6.16) holds. Then, by Theorem 6.11, we have

$$-\frac{1}{|A|_n} \sum_{z \in Z_a(k)} e^{i2\pi a^{-1}(z)\cdot t} \int_{\partial\mathbb{I}} e^{-i2\pi a^{-1}(z)\cdot s} \mu(s)\, d\sigma_s = 0 \qquad \forall t \in \mathcal{V}.$$

Consequently, by the identity theorem for real analytic functions, we have

$$-\frac{1}{|A|_n} \sum_{z \in Z_a(k)} e^{i2\pi a^{-1}(z)\cdot t} \int_{\partial\mathbb{I}} e^{-i2\pi a^{-1}(z)\cdot s} \mu(s)\, d\sigma_s = 0 \qquad \forall t \in \mathbb{R}^n.$$

Now let $z_0 \in Z_a(k)$. By multiplying both sides of the previous equality by $e^{-i2\pi a^{-1}(z_0)\cdot t}$ and then by integrating on $\mathrm{cl}\, A$, we easily obtain

$$\int_{\partial\mathbb{I}} e^{-i2\pi a^{-1}(z_0)\cdot s} \mu(s)\, d\sigma_s = 0.$$

Accordingly, (6.17) holds and the Proposition is proved. □

## 6.4 Periodic Helmholtz volume potential

In this Section we introduce an analogue of the periodic Newtonian potential for the Helmholtz equation.

We give the following.

**Definition 6.14.** Let $k \in \mathbb{C}$. Let $f \in C^0(\mathbb{R}^n, \mathbb{C})$ be such that

$$f(t + a_j) = f(t) \qquad \forall t \in \mathbb{R}^n, \quad \forall j \in \{1, \ldots, n\}.$$

We set

$$p_a[f,k](t) \equiv \int_A S_n^{a,k}(t-s) f(s)\, ds \qquad \forall t \in \mathbb{R}^n.$$

The function $p_a[f,k]$ is called the periodic Helmholtz volume potential of $f$.

*Remark* 6.15. Let $k$ and $f$ be as in Definition 6.14. Let $t \in \mathbb{R}^n$ be fixed. We note that the function $S_n^{a,k}(t - \cdot) f(\cdot)$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{C})$, and so $p_a[f,k](t)$ is well defined.

In the following Theorem, we prove some elementary properties of the periodic Helmholtz volume potential.

**Theorem 6.16.** *Let $k \in \mathbb{C}$. Let $Z_a(k)$ be as in (6.1). Let $m \in \mathbb{N}$, $\alpha \in ]0,1[$. Let $f \in C^{m,\alpha}(\mathbb{R}^n, \mathbb{C})$ be such that*

$$f(t + a_j) = f(t) \qquad \forall t \in \mathbb{R}^n, \quad \forall j \in \{1, \ldots, n\}.$$

*Then the following statements hold.*

*(i)*
$$p_a[f,k](t+a_j) = p_a[f,k](t) \qquad \forall t \in \mathbb{R}^n, \quad \forall j \in \{1,\dots,n\}.$$

*(ii)*
$$p_a[f,k] \in C^{m+2,\alpha}(\mathbb{R}^n,\mathbb{C}).$$

*(iii)*
$$(\Delta + k^2)p_a[f,k](t) = f(t) - \sum_{z\in Z_a(k)} \frac{1}{|A|_n} e^{i2\pi a^{-1}(z)\cdot t} \int_A f(s) e^{-i2\pi a^{-1}(z)\cdot s}\, ds \quad \forall t \in \mathbb{R}^n.$$

*Proof.* We modify the proof of Theorem 1.18. Clearly, the statement in $(i)$ is a straightforward consequence of the periodicity of $S_n^{a,k}$. We need to prove $(ii)$ and $(iii)$. We first prove $(iii)$. Obviously,
$$f \in C^{m,\alpha}(\mathrm{cl}\, V, \mathbb{C}),$$
for all bounded open subsets $V$ of $\mathbb{R}^n$. Let $\bar{x} \in \mathbb{R}^n$. By Proposition D.1 $(ii)$ (with $\delta = 1$), we have
$$p_a[f,k](t) = \int_{\tilde{A}+\bar{x}} S_n^{a,k}(t-s) f(s)\, ds \qquad \forall t \in \mathbb{R}^n.$$
Now set
$$U \equiv \bar{x} + \mathbb{B}_n\left(0, \min\{a_{11},\dots,a_{nn}\}/3\right).$$
As a first step, we want to prove that $p_a[f,k]_{|U} \in C^2(U,\mathbb{C})$ and that
$$(\Delta + k^2)p_a[f,k](t) = f(t) - \sum_{z\in Z_a(k)} \frac{1}{|A|_n} e^{i2\pi a^{-1}(z)\cdot t} \int_A f(s) e^{-i2\pi a^{-1}(z)\cdot s}\, ds \qquad \forall t \in U.$$
We have
$$p_a[f,k](t) = \int_{\tilde{A}+\bar{x}} S_n(t-s,k) f(s)\, ds + \int_{\tilde{A}+\bar{x}} R_n^{a,k}(t-s) f(s)\, ds \qquad \forall t \in U.$$
Set
$$u_1(t) \equiv \int_{\tilde{A}+\bar{x}} S_n(t-s,k) f(s)\, ds \qquad \forall t \in U,$$
and
$$u_2(t) \equiv \int_{\tilde{A}+\bar{x}} R_n^{a,k}(t-s) f(s)\, ds \qquad \forall t \in U.$$
Then we have that $u_1 \in C^2(U,\mathbb{C})$ and
$$(\Delta + k^2)u_1(t) = f(t) \qquad \forall t \in U.$$
On the other hand, by classical theorems of differentiation under the integral sign, we have that $u_2 \in C^\infty(U,\mathbb{C})$ and
$$\begin{aligned}(\Delta + k^2)u_2(t) &= -\sum_{z\in Z_a(k)} \frac{1}{|A|_n} \int_{A+\bar{x}} f(s) e^{i2\pi a^{-1}(z)\cdot(t-s)}\, ds \\ &= -\sum_{z\in Z_a(k)} \frac{1}{|A|_n} e^{i2\pi a^{-1}(z)\cdot t} \int_A f(s) e^{-i2\pi a^{-1}(z)\cdot s}\, ds \qquad \forall t \in U.\end{aligned}$$
Hence, $p_a[f,k]_{|U} \in C^2(U,\mathbb{C})$ and
$$(\Delta + k^2)p_a[f,k](t) = f(t) - \sum_{z\in Z_a(k)} \frac{1}{|A|_n} e^{i2\pi a^{-1}(z)\cdot t} \int_A f(s) e^{-i2\pi a^{-1}(z)\cdot s}\, ds \qquad \forall t \in U.$$
Accordingly, $p_a[f,k] \in C^2(\mathbb{R}^n,\mathbb{C})$ and
$$(\Delta + k^2)p_a[f,k](t) = f(t) - \sum_{z\in Z_a(k)} \frac{1}{|A|_n} e^{i2\pi a^{-1}(z)\cdot t} \int_A f(s) e^{-i2\pi a^{-1}(z)\cdot s}\, ds \qquad \forall t \in \mathbb{R}^n,$$
and so the statement in $(iii)$ is proved. We need to prove $(ii)$. We note that if $f \in C^{m,\alpha}(\mathbb{R}^n,\mathbb{C})$, then $(\Delta + k^2)p_a[f,k] \in C^{m,\alpha}(\mathbb{R}^n,\mathbb{C})$. Hence, by standard elliptic regularity theory (cf. Folland [52, p. 82], Stein [130, § VI.5] and Taylor [133, § XI.2]), the statement in $(ii)$ easily follows. □

*Remark* 6.17. Let $k$, $m$, $\alpha$ and $f$ be as in Theorem 6.16. Let $Z_a(k)$ be as in (6.1). As we did for the Laplacian of the periodic Newtonian potential, we observe that the presence of the term

$$-\sum_{z\in Z_a(k)}\frac{1}{|A|_n}e^{i2\pi a^{-1}(z)\cdot t}\int_A f(s)e^{-i2\pi a^{-1}(z)\cdot s}\,ds$$

in $(\Delta+k^2)p_a[f,k](t)$ is, somehow, natural. Indeed, let $u$, $v\in C^2(\mathbb{R}^n,\mathbb{C})$ be such that

$$u(t+a_j)=u(t)\qquad\forall t\in\mathbb{R}^n,\quad\forall j\in\{1,\dots,n\},$$

and

$$v(t+a_j)=v(t)\qquad\forall t\in\mathbb{R}^n,\quad\forall j\in\{1,\dots,n\}.$$

By Green's Formula, we have

$$\int_A\sum_{j=1}^n\frac{\partial u(t)}{\partial t_j}\frac{\partial v(t)}{\partial t_j}\,dt=-\int_A\Delta u(t)v(t)\,dt+\int_{\partial A}\frac{\partial u(t)}{\partial\nu_A}v(t)\,d\sigma_t,$$

and

$$\int_A\sum_{j=1}^n\frac{\partial u(t)}{\partial t_j}\frac{\partial v(t)}{\partial t_j}\,dt=-\int_A\Delta v(t)u(t)\,dt+\int_{\partial A}\frac{\partial v(t)}{\partial\nu_A}u(t)\,d\sigma_t.$$

By the periodicity of $u$ and $v$, we have

$$\int_{\partial A}\frac{\partial u(t)}{\partial\nu_A}v(t)\,d\sigma_t=0,$$

and

$$\int_{\partial A}\frac{\partial v(t)}{\partial\nu_A}u(t)\,d\sigma_t=0.$$

As a consequence,

$$-\int_A\Delta u(t)v(t)\,dt+\int_A u(t)\Delta v(t)\,dt=0.$$

Now, if we also assume that

$$(\Delta+k^2)u(t)=0\qquad\forall t\in\mathbb{R}^n,$$

then we immediately obtain

$$\int_A(\Delta v(t)+k^2v(t))u(t)\,dt=0.$$

Now assume that the set $Z_a(k)$ is nonempty. For each $z\in Z_a(k)$, define the function $u_z$ of $\mathbb{R}^n$ to $\mathbb{C}$ by setting

$$u_z(t)\equiv e^{i2\pi a^{-1}(z)\cdot t}\qquad\forall t\in\mathbb{R}^n.$$

Then clearly

$$\Delta u_z(t)+k^2u_z(t)=0\qquad\forall t\in\mathbb{R}^n,$$

for all $z\in Z_a(k)$. As a consequence

$$\int_A[(\Delta+k^2)v(t)]e^{i2\pi a^{-1}(z)\cdot t}\,dt=0\qquad\forall z\in Z_a(k),$$

for all $v\in C^2(\mathbb{R}^n,\mathbb{C})$ such that

$$v(t+a_j)=v(t)\qquad\forall t\in\mathbb{R}^n,\quad\forall j\in\{1,\dots,n\}.$$

In particular, if $f$ is as in Theorem 6.16, then we must have

$$\int_A[(\Delta+k^2)p_a[f,k](t)]e^{i2\pi a^{-1}(z)\cdot t}\,dt=0\qquad\forall z\in Z_a(k).$$

On the other hand, if $z_0 \in Z_a(k)$, we have

$$
\begin{aligned}
&\int_A [(\Delta + k^2) p_a[f,k](t)] e^{i2\pi a^{-1}(z_0)\cdot t}\,dt \\
&= \int_A \Big( f(t) - \sum_{z\in Z_a(k)} \frac{1}{|A|_n} e^{i2\pi a^{-1}(z)\cdot t} \int_A f(s) e^{-i2\pi a^{-1}(z)\cdot s}\,ds \Big) e^{i2\pi a^{-1}(z_0)\cdot t}\,dt \\
&= \int_A f(t) e^{i2\pi a^{-1}(z_0)\cdot t}\,dt - \sum_{z\in Z_a(k)} \frac{1}{|A|_n} \int_A e^{i2\pi (a^{-1}(z+z_0))\cdot t}\,dt \int_A f(s) e^{-i2\pi a^{-1}(z)\cdot s}\,ds \\
&= \int_A f(t) e^{i2\pi a^{-1}(z_0)\cdot t}\,dt - \int_A f(t) e^{i2\pi a^{-1}(z_0)\cdot t}\,dt = 0,
\end{aligned}
$$

since

$$
\int_A e^{i2\pi (a^{-1}(z+z_0))\cdot t}\,dt = |A|_n \delta_{z,-z_0} \qquad \forall z \in Z_a(k),
$$

where $\delta_{z,-z_0} = 1$ if $z = -z_0$, and $\delta_{z,-z_0} = 0$ if $z \neq -z_0$. In other words, the term

$$
- \sum_{z\in Z_a(k)} \frac{1}{|A|_n} e^{i2\pi a^{-1}(z)\cdot t} \int_A f(s) e^{-i2\pi a^{-1}(z)\cdot s}\,ds
$$

ensures that

$$
\int_A [(\Delta + k^2) p_a[f,k](x)] e^{i2\pi a^{-1}(z)\cdot x}\,dx = 0 \qquad \forall z \in Z_a(k).
$$

## 6.5 Regularity of the solutions of some integral equations

In this Section, we present a variant of Theorem E.7. More precisely, we are interested in proving regularity results for the solutions of the integral equations of Theorem E.7, with $S_n(\cdot,k)$ substituted with $S_n^{a,k}$. Indeed, as in classical potential theory, in order to solve boundary value problems for the Helmholtz equation by means of periodic simple and double layer potentials, we need to solve particular integral equations. Thus, we prove the following.

**Theorem 6.18.** *Let $k \in \mathbb{C}$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46). *Let $b \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$. Then the following statements hold.*

*(i) Let $j \in \{0,1,\dots,m\}$ and $\Gamma \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$ and $\mu \in L^2(\partial\mathbb{I},\mathbb{C})$ and*

$$
\Gamma(t) = \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)} (S_n^{a,k}(t-s))\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}} S_n^{a,k}(t-s) b(s)\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{6.18}
$$

*then $\mu \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$.*

*(ii) Let $j \in \{0,1,\dots,m\}$ and $\Gamma \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$ and $\mu \in L^2(\partial\mathbb{I},\mathbb{C})$ and*

$$
\Gamma(t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)} (S_n^{a,k}(t-s))\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}} S_n^{a,k}(t-s) b(s)\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{6.19}
$$

*then $\mu \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$.*

*(iii) Let $j \in \{1,\dots,m\}$ and $\Gamma \in C^{j-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ and $\mu \in L^2(\partial\mathbb{I},\mathbb{C})$ and*

$$
\Gamma(t) = \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)} (S_n^{a,k}(t-s))\mu(s)\,d\sigma_s + b(t)\int_{\partial\mathbb{I}} S_n^{a,k}(t-s)\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{6.20}
$$

*then $\mu \in C^{j-1,\alpha}(\partial\mathbb{I},\mathbb{C})$.*

*(iv) Let $j \in \{1,\dots,m\}$ and $\Gamma \in C^{j-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ and $\mu \in L^2(\partial\mathbb{I},\mathbb{C})$ and*

$$
\Gamma(t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)} (S_n^{a,k}(t-s))\mu(s)\,d\sigma_s + b(t)\int_{\partial\mathbb{I}} S_n^{a,k}(t-s)\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{6.21}
$$

*then $\mu \in C^{j-1,\alpha}(\partial\mathbb{I},\mathbb{C})$.*

*Proof.* We deduce all the statements by the correspondig results of Theorem E.7. Let $j$, $\Gamma$, and $\mu$ be as in the hypotheses of $(i)$. Set

$$\bar{\Gamma}(t) \equiv \Gamma(t) - \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(R_n^{a,k}(t-s))\mu(s)\,d\sigma_s - \int_{\partial\mathbb{I}} R_n^{a,k}(t-s)b(s)\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I}.$$

Then, by Theorem C.2, $\bar{\Gamma} \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$. By (6.18), we have

$$\bar{\Gamma}(t) = \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n(t-s,k))\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}} S_n(t-s,k)b(s)\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}.$$

Then, by Theorem E.7 $(i)$, we have $\mu \in C^{j,\alpha}(\partial\mathbb{I})$.

The proofs of statements $(ii)$, $(iii)$, $(iv)$ are very similar. □

## 6.6 A remark on the periodic analogue of the fundamental solution for the Helmholtz equation

In this Section we deduce by the previous results an immediate property of the periodic analogue of the fundamental solution for the Helmholtz operator $\Delta + k^2$, when $k \in [0,+\infty[$.

As a first step, we have the following result on $S_n^{a,k}$.

**Proposition 6.19.** *Let $k \in [0,+\infty[$. Let $Z_a(k)$ be as in* (6.1). *Then the following statements hold.*

*(i) The function $S_n(\cdot,k)$ introduced in Proposition E.3 is real-valued.*

*(ii) The function $S_n^{a,k}$ introduced in Theorem 6.3 $(i)$ is real-valued and we have*

$$(\Delta + k^2)S_n^{a,k}(x) = -\sum_{z\in Z_a(k)} \frac{1}{|A|_n}\prod_{j=1}^{n}(2-\delta_{0,z_j})\cos\frac{2\pi z_j x_j}{a_{jj}} \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a. \tag{6.22}$$

*(iii) The function $R_n^{a,k}$ introduced in Theorem 6.3 $(ii)$ is real-valued and we have*

$$(\Delta + k^2)R_n^{a,k}(x) = -\sum_{z\in Z_a(k)} \frac{1}{|A|_n}\prod_{j=1}^{n}(2-\delta_{0,z_j})\cos\frac{2\pi z_j x_j}{a_{jj}} \qquad \forall x \in (\mathbb{R}^n \setminus Z_n^a)\cup\{0\}.$$

*Proof.* Statement $(i)$ is an easy verification based on the definition of $S_n(\cdot,k)$ (cf. Lemma E.1, Definition E.2 and Proposition E.3.) Consider now $(ii)$. Since

$$\overline{\frac{1}{|A|_n(k^2 - |2\pi a^{-1}(z)|^2)}\langle E_{2\pi a^{-1}(z)}, \overline{\phi}\rangle} = \frac{1}{|A|_n(k^2 - |2\pi a^{-1}(-z)|^2)}\langle E_{2\pi a^{-1}(-z)}, \phi\rangle$$
$$\forall \phi \in \mathcal{D}(\mathbb{R}^n,\mathbb{C}), \quad \forall z \in \mathbb{Z}^n \setminus Z_a(k),$$

we can easily conclude that

$$\overline{\langle G_n^{a,k}, \overline{\phi}\rangle} = \langle G_n^{a,k}, \phi\rangle \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n,\mathbb{C}).$$

Accordingly,

$$\int_{\mathbb{R}^n} \overline{S_n^{a,k}}(x)\phi(x)\,dx = \int_{\mathbb{R}^n} S_n^{a,k}(x)\phi(x)\,dx \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n,\mathbb{C}),$$

and, as a consequence, the function $S_n^{a,k}$ is real-valued. Then by Theorem 6.3 $(i)$ and Remark 6.5 we have that (6.22) holds. Statement $(iii)$ is a consequence of statements $(i)$, $(ii)$ and of Theorem 6.3 and Remark 6.5. □

*Remark* 6.20. Let $k$ be a positive real number. Then, by Proposition 6.19, we note that the corresponding periodic layer potentials for the Helmholtz equation are real-valued functions, provided that the densities are real-valued. Clearly, an analogous result holds for the volume potential.

## 6.7 Some technical results for the periodic layer potentials for the Helmholtz equation

In this Section we collect some results that we shall use in the sequel. Indeed, in order to analyze boundary value problems for the Helmholtz equation in the next Chapters, we shall deal with integral equations on 'rescaled' domains, and, as a consequence we need to study integral operators which arise in these integral equations. Moreover, we have also to undestand how the periodic layer potentials change when we 'rescale' the domains.

### 6.7.1 Notation and preliminaries

We retain the notation introduced in Subsection 1.8.1. However, we need also to introduce some other notation.

Let $k \in \mathbb{C}$. Let $S_n(\cdot, k)$ be the function introduced in Proposition E.3. Then, if $x \in \mathbb{R}^n \setminus \{0\}$, we denote by $D_{\mathbb{R}^n} S_n(x,k)$ the gradient of the function $\xi \mapsto S_n(\xi, k)$ computed at point $x$. Let $\mathcal{J}_n$ be the function of $\mathbb{C}$ to $\mathbb{C}$ introduced in Definition E.2. Then we define the function $Q^k_n$ of $\mathbb{R}^n$ to $\mathbb{C}$ by setting

$$Q^k_n(x) \equiv \mathcal{J}_n(k|x|) \qquad \forall x \in \mathbb{R}^n. \tag{6.23}$$

Then $Q^k_n$ is a real analytic function of $\mathbb{R}^n$ to $\mathbb{C}$ (see the proof of Lanza and Rossi [86, Proposition 3.3].) Moreover, if $n$ is odd, then $Q^k_n(x) = 0$ for all $x \in \mathbb{R}^n$. If $\epsilon > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$, then a straightforward computation shows that

$$S_n(\epsilon x, k) = \frac{1}{\epsilon^{n-2}} S_n(x, \epsilon k) + (\log \epsilon) k^{n-2} Q^k_n(\epsilon x), \tag{6.24}$$

and

$$D_{\mathbb{R}^n} S_n(\epsilon x, k) = \frac{1}{\epsilon^{n-1}} D_{\mathbb{R}^n} S_n(x, \epsilon k) + (\log \epsilon) k^{n-2} D Q^k_n(\epsilon x). \tag{6.25}$$

### 6.7.2 Some technical results for the periodic simple layer potential for the Helmholtz equation

In the following Proposition, we study some integral operators that are related to the periodic simple layer potential and that appear in integral equations on 'rescaled' domains when we represent the solution of a certain boundary value problem in terms of a periodic simple layer potential.

**Proposition 6.21.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in \, ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $k \in \mathbb{C}$ be such that*

$$k^2 \neq |2\pi a^{-1}(z)|^2 \quad \forall z \in \mathbb{Z}^n.$$

*Then the following statements hold.*

*(i) There exists $\epsilon_2 \in \, ]0, \epsilon_1]$ such that the maps $N_1$, $N_2$, $N_3$ of $]-\epsilon_2, \epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to the space $C^{m,\alpha}(\partial\Omega, \mathbb{C})$, which take $(\epsilon, \theta)$ to the functions $N_1[\epsilon, \theta]$, $N_2[\epsilon, \theta]$, $N_3[\epsilon, \theta]$, respectively, defined by*

$$N_1[\epsilon, \theta](t) \equiv \int_{\partial\Omega} S_n(t - s, \epsilon k) \theta(s) \, d\sigma_s \quad \forall t \in \partial\Omega, \tag{6.26}$$

$$N_2[\epsilon, \theta](t) \equiv \int_{\partial\Omega} R^{a,k}_n(\epsilon(t - s)) \theta(s) \, d\sigma_s \quad \forall t \in \partial\Omega, \tag{6.27}$$

$$N_3[\epsilon, \theta](t) \equiv \int_{\partial\Omega} Q^k_n(\epsilon(t - s)) \theta(s) \, d\sigma_s \quad \forall t \in \partial\Omega, \tag{6.28}$$

*are real analytic.*

*(ii) There exists $\epsilon'_2 \in \, ]0, \epsilon_1]$ such that the maps $N_4$, $N_5$, $N_6$ of $]-\epsilon'_2, \epsilon'_2[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to the space $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, which take $(\epsilon, \theta)$ to the functions $N_4[\epsilon, \theta]$, $N_5[\epsilon, \theta]$, $N_6[\epsilon, \theta]$, respectively, defined*

*by*

$$N_4[\epsilon,\theta](t) \equiv \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s,\epsilon k)\theta(s)\, d\sigma_s \quad \forall t \in \partial\Omega, \tag{6.29}$$

$$N_5[\epsilon,\theta](t) \equiv \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s \quad \forall t \in \partial\Omega, \tag{6.30}$$

$$N_6[\epsilon,\theta](t) \equiv \int_{\partial\Omega} \nu_\Omega(t) \cdot DQ_n^{k}(\epsilon(t-s))\theta(s)\, d\sigma_s \quad \forall t \in \partial\Omega, \tag{6.31}$$

*are real analytic.*

*Proof.* We first prove statement $(i)$. By Theorem E.6 $(i)$, one can easily show that $N_1$ is a real analytic operator of $]-\epsilon_1,\epsilon_1[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$. We now consider $N_2$. By Theorem C.4, we immediately deduce that there exists $\epsilon_2 \in ]0,\epsilon_1]$ such that the map of $]-\epsilon_2,\epsilon_2[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$, which takes $(\epsilon,\theta)$ to the function $\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s$ of the variable $t\in\partial\Omega$, is real analytic. Since $Q_n^k$ is a real analytic function of $\mathbb{R}^n$ to $\mathbb{C}$, then, by Theorem C.4, one can easily show that $N_3$ is a real analytic operator of $]-\epsilon_1,\epsilon_1[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$. We now turn to the proof of statement $(ii)$. By Theorem E.6 $(ii)$, one can easily show that $N_4$ is a real analytic operator of $]-\epsilon_1,\epsilon_1[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. We now consider $N_5$. By Theorem C.4, we immediately deduce that there exists $\epsilon_2' \in ]0,\epsilon_1]$ such that the map of $]-\epsilon_2',\epsilon_2'[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, which takes $(\epsilon,\theta)$ to the function $\int_{\partial\Omega} \partial_{x_j} R_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s$ of the variable $t\in\partial\Omega$, is real analytic, for all $j\in\{1,\dots,n\}$. By continuity of the pointwise product in Schauder space, we easily deduce that the map of $]-\epsilon_2',\epsilon_2'[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, which takes $(\epsilon,\theta)$ to the function $\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s$ of the variable $t\in\partial\Omega$, is real analytic. Similarly, since $Q_n^k$ is a real analytic function of $\mathbb{R}^n$ to $\mathbb{C}$, then, by Theorem C.4, one can easily show that $N_6$ is a real analytic operator of $]-\epsilon_1,\epsilon_1[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Thus the proof is complete. □

Since the solutions of the boundary value problems that we are going to consider will be represented in terms of periodic simple layer potentials, in the following Proposition we study the real analyticity of an integral operator that is related to the simple layer potential and that we are going to used in the sequel.

**Proposition 6.22.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $k\in\mathbb{C}$ be such that*

$$k^2 \neq |2\pi a^{-1}(z)|^2 \quad \forall z\in\mathbb{Z}^n.$$

*Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V\cap\mathbb{S}_a[\Omega_0]=\emptyset$. Then there exists $\epsilon_2\in]0,\epsilon_1]$ such that*

$$\mathrm{cl}\,V \subseteq \mathbb{T}_a[\Omega_\epsilon] \qquad \forall \epsilon\in]-\epsilon_2,\epsilon_2[. \tag{6.32}$$

*Moreover, the map $N$ of $]-\epsilon_2,\epsilon_2[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^0(\mathrm{cl}\,V,\mathbb{C})$, which takes $(\epsilon,\theta)$ to the function $N[\epsilon,\theta]$ of $\mathrm{cl}\,V$ to $\mathbb{C}$ defined by*

$$N[\epsilon,\theta](x) \equiv \int_{\partial\Omega} S_n^{a,k}(x-w-\epsilon s)\theta(s)\,d\sigma_s \qquad \forall x\in\mathrm{cl}\,V, \tag{6.33}$$

*is real analytic.*

*Proof.* Choosing $\epsilon_2$ small enough, we can clearly assume that (6.32) holds. Then we have

$$\mathrm{cl}\,V - (w+\epsilon\partial\Omega) \subseteq \mathbb{R}^n\setminus Z_n^a \qquad \forall\epsilon\in]-\epsilon_2,\epsilon_2[.$$

Moreover, if we denote by $\mathrm{id}_{\partial\Omega}$ the identity map in $\partial\Omega$, then the map of $]-\epsilon_2,\epsilon_2[$ to $C^0(\partial\Omega,\mathbb{R}^n)$, which takes $\epsilon$ to $w+\epsilon\,\mathrm{id}_{\partial\Omega}$ is real analytic. Hence, by Proposition C.1, $N$ is a real analytic map of $]-\epsilon_2,\epsilon_2[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^0(\mathrm{cl}\,V,\mathbb{C})$. The proof is now complete. □

## 6.8 A remark on the periodic eigenvalues of $-\Delta$ in $\mathbb{R}^n$

Let $\lambda \in \mathbb{C}$. We say that $\lambda$ is a *periodic eigenvalue of* $-\Delta$ *in* $\mathbb{R}^n$ (and we write $\lambda \in \mathrm{Eig}_a(-\Delta)$), if there exists a function $u \in C^2(\mathbb{R}^n)$, $u$ not identically 0, such that

$$\begin{cases} -\Delta u(x) = \lambda u(x) & \forall x \in \mathbb{R}^n, \\ u(x + a_j) = u(x) & \forall x \in \mathbb{R}^n,\ \forall j \in \{1, \dots, n\}. \end{cases}$$

If such a function $u$ exists, then we say that $u$ is a *periodic eigenfunction of* $-\Delta$ *in* $\mathbb{R}^n$.

Clearly, the set $\mathrm{Eig}_a(-\Delta)$ can be seen as the set of the eigenvalues of $-\Delta$ on the (flat) torus $\mathbb{R}^n/Z_n^a$ (cf. *e.g.*, Milnor [96], Chavel [22, p. 29], Berger, Gauduchon, and Mazet [12, pp. 146-148].)

Let $\lambda \in \mathbb{C}$. We set

$$E(\lambda) \equiv \left\{ z \in \mathbb{Z}^n \colon \lambda = |2\pi a^{-1}(z)|^2 \right\}.$$

It is well known (cf. *e.g.*, Milnor [96], Chavel [22, p. 29], Berger, Gauduchon, and Mazet [12, pp. 146-148]) that the set $\mathrm{Eig}_a(-\Delta)$ is delivered by

$$\mathrm{Eig}_a(-\Delta) \equiv \{ \lambda \in \mathbb{C} \colon E(\lambda) \neq \emptyset \}.$$

For each $z \in \mathbb{Z}^n$, define the function $u_z$ of $\mathbb{R}^n$ to $\mathbb{C}$ by setting

$$u_z(x) \equiv e^{i2\pi a^{-1}(z)\cdot x} \qquad \forall x \in \mathbb{R}^n.$$

If $\lambda \in \mathrm{Eig}_a(-\Delta)$, then the corresponding eigenspace is given by the vector space generated by the set of functions $\{u_z\}_{z \in E(\lambda)}$.

In other words, with the notation used in the previous Sections, we have that if $k \in \mathbb{C}$, then $k^2 \in \mathrm{Eig}_a(-\Delta)$ if and only if $Z_a(k)$ is nonempty (cf. (6.1).) Clearly, we observe that the problem of determining the dimension of the eigenspace corresponding to the eigenvalue $\lambda$ can be reformulated as the problem of determining the number of $z \equiv (z_1, \dots, z_n) \in \mathbb{Z}^n$, such that

$$\lambda = 4\pi^2 \Big(\frac{z_1^2}{a_{11}^2} + \dots + \frac{z_n^2}{a_{nn}^2}\Big).$$

Now let

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_j \leq \dots$$

be the sequence of the periodic eigenvalues of $-\Delta$ in $\mathbb{R}^n$ (or, equivalently, the sequence of the eigenvalues of $-\Delta$ in $\mathbb{R}^n/Z_n^a$), where each eigenvalue is repeated as many times as its multiplicity.

For each $\lambda > 0$, we set

$$N(\lambda) \equiv \max \{ j \in \mathbb{N} \colon \lambda_j \leq \lambda \}.$$

Then

$$N(\lambda) = \# \left\{ z \in \mathbb{Z}^n \colon a^{-1}(z) \in \mathrm{cl}\, \mathbb{B}_n(0, \frac{\sqrt{\lambda}}{2\pi}) \right\} \qquad \forall \lambda > 0,$$

where, if $S$ is a set, the symbol $\#S$ denotes the number of elements of $S$.

## 6.9 Maximum principle for the periodic Helmholtz equation

In the following Theorem, we deduce by the classical Maximum Principle for the Helmholtz equation a version for periodic functions defined on $\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}]$.

**Theorem 6.23.** *Let* $\mathbb{I}$ *be a bounded connected open subset of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\, \mathbb{I} \subseteq A$ *and* $\mathbb{R}^n \setminus \mathrm{cl}\, \mathbb{I}$ *is connected. Let* $\mathbb{T}_a[\mathbb{I}]$ *be as in* (1.49). *Let* $k \in \mathbb{C}$ *be such that* $\mathrm{Re}(k) = 0$ *and* $\mathrm{Im}(k) \neq 0$. *Let* $u \in C^0(\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \mathbb{R}) \cap C^2(\mathbb{T}_a[\mathbb{I}], \mathbb{R})$ *be such that*

$$u(x + a_j) = u(x) \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}.$$

*Then the following statements hold.*

(i) *If*

$$\Delta u(x) + k^2 u(x) \geq 0 \qquad \forall x \in \mathbb{T}_a[\mathbb{I}],$$

*and there exists a point* $x_0 \in \mathbb{T}_a[\mathbb{I}]$ *such that* $u(x_0) \geq 0$ *and* $u(x_0) = \max_{\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} u$*, then* $u$ *is constant within* $\mathbb{T}_a[\mathbb{I}]$.

(ii) *If*

$$\Delta u(x) + k^2 u(x) \leq 0 \qquad \forall x \in \mathbb{T}_a[\mathbb{I}],$$

*and there exists a point* $x_0 \in \mathbb{T}_a[\mathbb{I}]$ *such that* $u(x_0) \leq 0$ *and* $u(x_0) = \min_{\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} u$*, then* $u$ *is constant within* $\mathbb{T}_a[\mathbb{I}]$.

*Proof.* Clearly statement $(ii)$ follows by statement $(i)$ by replacing $u$ with $-u$. Therefore, it suffices to prove $(i)$. Let $u$ and $x_0$ be as in the hypotheses. Let $V$ be a bounded connected open neighbourhood of $\mathrm{cl}\,A$, such that

$$\mathrm{cl}\,V \cap \mathrm{cl}(\mathbb{I} + a(z)) = \emptyset \qquad \forall z \in \mathbb{Z}^n \setminus \{0\},$$

and

$$x_0 \in V.$$

Set $W \equiv V \setminus \mathrm{cl}\,\mathbb{I}$. Clearly, $W$ is a bounded connected open subset of $\mathbb{R}^n$. Moreover,

$$x_0 \in W,$$

and

$$0 \leq u(x_0) = \max_{\mathrm{cl}\,W} u.$$

Then, by the Strong Maximum Principle (cf. *e.g.*, Evans [50, Theorem 4, p. 333]), we have that $u$ is constant within $W$ and accordingly $u$ is constant within $\mathbb{P}_a[\mathbb{I}]$. Consequently, by the periodicity of $u$, we have that $u$ is constant in $\mathbb{T}_a[\mathbb{I}]$ and we are done. □

Then we can easily deduce the following result.

**Corollary 6.24.** *Let* $\mathbb{I}$ *be a bounded connected open subset of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\,\mathbb{I} \subseteq A$ *and* $\mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{I}$ *is connected. Let* $\mathbb{T}_a[\mathbb{I}]$ *be as in* (1.49)*. Let* $k \in \mathbb{C}$ *be such that* $\mathrm{Re}(k) = 0$ *and* $\mathrm{Im}(k) \neq 0$*. Let* $u \in C^0(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \mathbb{R}) \cap C^2(\mathbb{T}_a[\mathbb{I}], \mathbb{R})$ *be such that*

$$u(x + a_j) = u(x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\},$$

*and*

$$\Delta u(x) + k^2 u(x) = 0 \qquad \forall x \in \mathbb{T}_a[\mathbb{I}].$$

*Then*

$$\max_{\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} |u| = \max_{\partial\mathbb{I}} |u|.$$

*Proof.* It is a straightforward consequence of Theorem 6.23. □

In particular, as far as complex valued functions are concerned, we have the following.

**Corollary 6.25.** *Let* $\mathbb{I}$ *be a bounded connected open subset of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\,\mathbb{I} \subseteq A$ *and* $\mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{I}$ *is connected. Let* $\mathbb{T}_a[\mathbb{I}]$ *be as in* (1.49)*. Let* $k \in \mathbb{C}$ *be such that* $\mathrm{Re}(k) = 0$ *and* $\mathrm{Im}(k) \neq 0$*. Let* $u \in C^0(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \mathbb{C}) \cap C^2(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ *be such that*

$$u(x + a_j) = u(x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\},$$

*and*

$$\Delta u(x) + k^2 u(x) = 0 \qquad \forall x \in \mathbb{T}_a[\mathbb{I}].$$

*Then*

$$\max_{\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} |u| \leq (\max_{\partial\mathbb{I}} |\mathrm{Re}(u)| + \max_{\partial\mathbb{I}} |\mathrm{Im}(u)|).$$

*As a consequence,*

$$\max_{\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} |u| \leq 2 \max_{\partial\mathbb{I}} |u|.$$

*Proof.* By Corollary 6.24 applied to $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$, we have

$$\max_{\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} |\mathrm{Re}(u)| = \max_{\partial\mathbb{I}} |\mathrm{Re}(u)| \leq \max_{\partial\mathbb{I}} |u|,$$
$$\max_{\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} |\mathrm{Im}(u)| = \max_{\partial\mathbb{I}} |\mathrm{Im}(u)| \leq \max_{\partial\mathbb{I}} |u|.$$

On the other hand,

$$\max_{\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} |u| \leq \big( \max_{\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} |\mathrm{Re}(u)| + \max_{\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]} |\mathrm{Im}(u)| \big)$$
$$= \big( \max_{\partial\mathbb{I}} |\mathrm{Re}(u)| + \max_{\partial\mathbb{I}} |\mathrm{Im}(u)| \big),$$

and the conclusion follows. □

# CHAPTER 7

## Some results of Spectral Theory for the Laplace operator

In this Chapter we collect some well known results of Spectral Theory for the Laplace operator, that we shall use in the sequel. In particular, we present some convergence results for the eigenvalues of the Laplace operator in a periodically perforated domain. These results can be seen as the periodic version of a result of Rauch and Taylor [117], and they can be proved by arguing as in Rauch and Taylor [117].

We retain the notation introduced in Sections 1.1 and 1.3, Chapter 6 and Appendix E.

### 7.1 Some results for the eigenvalues of the Laplace operator in small domains

#### 7.1.1 Notation

We introduce some notation.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. We shall consider the following assumption.

$$\text{Let } \Omega \text{ be a bounded open connected subset of } \mathbb{R}^n \text{ of class } C^{m,\alpha} \text{ such that } 0 \in \Omega \text{ and } \mathbb{R}^n \setminus \mathrm{cl}\,\Omega \text{ is connected.} \tag{7.1}$$

We denote by $\nu_\Omega$ the outward unit normal to $\Omega$ on $\partial\Omega$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$ be as in (7.1). We set

$$\Omega_\epsilon \equiv w + \epsilon\Omega \qquad \forall \epsilon \in \mathbb{R} \setminus \{0\}, \tag{7.2}$$

If $\epsilon \in \mathbb{R} \setminus \{0\}$, we denote by $\nu_{\Omega_\epsilon}$ the outward unit normal to $\Omega_\epsilon$ on $\partial\Omega_\epsilon$.

#### 7.1.2 Asymptotic behaviour of the Dirichlet eigenvalues

We first introduce the following.

**Definition 7.1.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0, 1[$. Let $\mathbb{I}$ be a bounded connected open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $\lambda \in \mathbb{C}$. We say that $\lambda$ is a *Dirichlet eigenvalue of* $-\Delta$ in $\mathbb{I}$ (and we write $\lambda \in \mathrm{Eig}_D[\mathbb{I}]$) if there exists a function $u \in C^0(\mathrm{cl}\,\mathbb{I}, \mathbb{C}) \cap C^2(\mathbb{I}, \mathbb{C})$, $u$ not identically zero, such that

$$\begin{cases} \Delta u(x) + \lambda u(x) = 0 & \forall x \in \mathbb{I}, \\ u(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

If such a function $u$ exists, then $u$ is called a *Dirichlet eigenfunction of* $-\Delta$ in $\mathbb{I}$. Moreover, by elliptic regularity theory, we have $u \in C^1(\mathrm{cl}\,\mathbb{I}, \mathbb{C}) \cap C^2(\mathbb{I}, \mathbb{C})$.

*Remark* 7.2. Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be a bounded connected open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. It is well known that $\mathrm{Eig}_D[\mathbb{I}] \subseteq ]0,+\infty[$. In particular, as a consequence, if $k \in \mathbb{C}$ and $\mathrm{Im}(k) \neq 0$, then $k^2 \notin \mathrm{Eig}_D[\mathbb{I}]$ (cf. also, *e.g.*, Colton and Kress [29, Thm. 3.10, p. 76].) Moreover $\mathrm{Eig}_D[\mathbb{I}] = \cup_{j=1}^{\infty}\{\lambda_j[\mathbb{I}]\}$, where $\{\lambda_j[\mathbb{I}]\}_{j=1}^{+\infty}$ is an increasing sequence of positve (real) numbers accumulating only at infinity.

Then we have the following elementary lemma (cf. e.g., Colton and Kress [29, Lemma 3.26, p. 86], Lanza [79, Proposition 9].)

**Lemma 7.3.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in* (7.1)*. Let $w \in A$. Then*

$$\mathrm{Eig}_D[\Omega_\epsilon] = \frac{1}{\epsilon^2}\,\mathrm{Eig}_D[\Omega] \qquad \forall \epsilon \in ]0,+\infty[.$$

*Proof.* Let $\epsilon \in ]0,+\infty[$. Let $\lambda \in \mathrm{Eig}_D[\Omega]$. Then there exists a function $u \in C^1(\mathrm{cl}\,\Omega,\mathbb{C}) \cap C^2(\Omega,\mathbb{C})$, $u$ not identically zero, such that

$$\begin{cases} \Delta u(x) + \lambda u(x) = 0 & \forall x \in \Omega, \\ u(x) = 0 & \forall x \in \partial\Omega. \end{cases}$$

Set

$$u_\epsilon(x) \equiv u(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \mathrm{cl}\,\Omega_\epsilon.$$

Then $u_\epsilon \in C^1(\mathrm{cl}\,\Omega_\epsilon,\mathbb{C}) \cap C^2(\Omega_\epsilon,\mathbb{C})$, and clearly

$$\Delta u_\epsilon(x) = -\frac{1}{\epsilon^2}\lambda u(\frac{1}{\epsilon}(x-w)) = -\frac{\lambda}{\epsilon^2}u_\epsilon(x) \qquad \forall x \in \Omega_\epsilon.$$

Accordingly,

$$\begin{cases} \Delta u_\epsilon(x) + \frac{\lambda}{\epsilon^2}u_\epsilon(x) = 0 & \forall x \in \Omega_\epsilon, \\ u_\epsilon(x) = 0 & \forall x \in \partial\Omega_\epsilon, \end{cases}$$

and so

$$\frac{\lambda}{\epsilon^2} \in \mathrm{Eig}_D[\Omega_\epsilon].$$

Thus $\mathrm{Eig}_D[\Omega_\epsilon] \supseteq \frac{1}{\epsilon^2}\mathrm{Eig}_D[\Omega]$. Conversely, let $\lambda \in \mathrm{Eig}_D[\Omega_\epsilon]$. Then there exists a function $u \in C^1(\mathrm{cl}\,\Omega_\epsilon,\mathbb{C}) \cap C^2(\Omega_\epsilon,\mathbb{C})$, $u$ not identically zero, such that

$$\begin{cases} \Delta u(x) + \lambda u(x) = 0 & \forall x \in \Omega_\epsilon, \\ u(x) = 0 & \forall x \in \partial\Omega_\epsilon. \end{cases}$$

Set

$$u^\epsilon(x) \equiv u(w+\epsilon x) \qquad \forall x \in \mathrm{cl}\,\Omega.$$

Then $u^\epsilon \in C^1(\mathrm{cl}\,\Omega,\mathbb{C}) \cap C^2(\Omega,\mathbb{C})$, and clearly

$$\Delta u^\epsilon(x) = -\epsilon^2\lambda u(w+\epsilon x) = -\epsilon^2\lambda u^\epsilon(x) \qquad \forall x \in \Omega.$$

Accordingly,

$$\begin{cases} \Delta u^\epsilon(x) + \lambda\epsilon^2 u^\epsilon(x) = 0 & \forall x \in \Omega, \\ u^\epsilon(x) = 0 & \forall x \in \partial\Omega, \end{cases}$$

and so

$$\lambda\epsilon^2 \in \mathrm{Eig}_D[\Omega].$$

Thus $\epsilon^2\,\mathrm{Eig}_D[\Omega_\epsilon] \subseteq \mathrm{Eig}_D[\Omega]$.

Hence,

$$\mathrm{Eig}_D[\Omega_\epsilon] = \frac{1}{\epsilon^2}\,\mathrm{Eig}_D[\Omega],$$

and the proof is complete. □

**Corollary 7.4.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$ be as in* (7.1)*. Let $k \in \mathbb{C}$. Then there exists $\epsilon_D > 0$ such that*

$$k^2 \notin \mathrm{Eig}_D[\Omega_\epsilon] \qquad \forall \epsilon \in ]0,\epsilon_D]. \tag{7.3}$$

*Proof.* It is an immediate consequence of Lemma 7.3. □

### 7.1.3 Asymptotic behaviour of the Neumann eigenvalues

We give the following.

**Definition 7.5.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be a bounded connected open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $\lambda \in \mathbb{C}$. We say that $\lambda$ is a *Neumann eigenvalue of* $-\Delta$ in $\mathbb{I}$ (and we write $\lambda \in \mathrm{Eig}_N[\mathbb{I}]$) if there exists a function $u \in C^1(\mathrm{cl}\,\mathbb{I},\mathbb{C}) \cap C^2(\mathbb{I},\mathbb{C})$, $u$ not identically zero, such that

$$\begin{cases} \Delta u(x) + \lambda u(x) = 0 & \forall x \in \mathbb{I}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) = 0 & \forall x \in \partial \mathbb{I}. \end{cases}$$

If such a function $u$ exists, then $u$ is called a *Neumann eigenfunction of* $-\Delta$ in $\mathbb{I}$.

*Remark* 7.6. Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be a bounded connected open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. It is well known that $0 \in \mathrm{Eig}_N[\mathbb{I}]$ and that $\mathrm{Eig}_N[\mathbb{I}] \subseteq [0,+\infty[$. In particular, as a consequence, if $k \in \mathbb{C}$ and $\mathrm{Im}(k) \neq 0$, then $k^2 \notin \mathrm{Eig}_N[\mathbb{I}]$ (cf. also, *e.g.*, Colton and Kress [29, Thm. 3.10, p. 76].) Moreover $\mathrm{Eig}_N[\mathbb{I}] = \cup_{j=1}^{\infty}\{\lambda_j[\mathbb{I}]\}$, where $\{\lambda_j[\mathbb{I}]\}_{j=1}^{+\infty}$ is an increasing sequence of nonnegative (real) numbers accumulating only at infinity.

Then we have the following elementary lemma (cf. e.g., Colton and Kress [29, Lemma 3.26, p. 86], Lanza [79, Proposition 9].)

**Lemma 7.7.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in* (7.1). *Let $w \in A$. Then*

$$\mathrm{Eig}_N[\Omega_\epsilon] = \frac{1}{\epsilon^2}\,\mathrm{Eig}_N[\Omega] \qquad \forall \epsilon \in ]0,+\infty[.$$

*Proof.* It is a simple modification of the proof of Lemma 7.3. □

**Corollary 7.8.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$ be as in* (7.1). *Let $k \in \mathbb{C} \setminus \{0\}$. Then there exists $\epsilon_N > 0$ such that*

$$k^2 \notin \mathrm{Eig}_N[\Omega_\epsilon] \qquad \forall \epsilon \in ]0,\epsilon_N]. \tag{7.4}$$

*Proof.* It is an immediate consequence of Lemma 7.7. □

## 7.2 Convergence results for the eigenvalues of the Laplace operator in a periodically perforated domain

### 7.2.1 Notation

We retain the notation introduced in Subsection 1.8.1. We note that the results of this Section are the periodic version of some results of Rauch and Taylor [117]. Moreover, we observe that these results are proved, essentially, by replacing some function spaces with their periodic counterparts in the proofs of Rauch and Taylor [117]. However, for the reader's convenience, we shall include the proofs in this Section.

### 7.2.2 A convergence result for the eigenvalues of the Laplace operator in a periodically perforated domain under Dirichlet boundary conditions

First of all, we need to give the following definition.

**Definition 7.9.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\lambda \in \mathbb{C}$. We say that $\lambda$ is a *periodic Dirichlet eigenvalue of* $-\Delta$ in $\mathbb{T}_a[\mathbb{I}]$ (and we write $\lambda \in \mathrm{Eig}_D^a[\mathbb{T}_a[\mathbb{I}]]$) if there exists a function $u \in C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C}) \cap C^2(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$, $u$ not identically zero, such that

$$\begin{cases} \Delta u(x) + \lambda u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1,\dots,n\}, \\ u(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

If such a function $u$ exists, then $u$ is called a *periodic Dirichlet eigenfunction of* $-\Delta$ in $\mathbb{T}_a[\mathbb{I}]$.

Then we have the following, certainly known, Proposition.

**Proposition 7.10.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $k \in \mathbb{C}$. Assume that* $\mathrm{Im}(k) \neq 0$*. Let $u \in C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C}) \cap C^2(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ solve the following boundary value problem*

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x+a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1,\dots,n\}, \\ u(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

*Then $u(x) = 0$ for all $x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$.*

*Proof.* By the Divergence Theorem and the periodicity of $u$, we have

$$\begin{aligned} \int_{\mathbb{P}_a[\mathbb{I}]} \overline{u(x)} \Delta u(x)\, dx &= \int_{\partial\mathbb{P}_a[\mathbb{I}]} \overline{u(x)} \frac{\partial}{\partial\nu_{\mathbb{P}_a[\mathbb{I}]}} u(x)\, d\sigma_x - \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2\, dx \\ &= \int_{\partial A} \overline{u(x)} \frac{\partial}{\partial\nu_A} u(x)\, d\sigma_x - \int_{\partial\mathbb{I}} \overline{u(x)} \frac{\partial}{\partial\nu_{\mathbb{I}}} u(x)\, d\sigma_x - \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2\, dx \\ &= -\int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2\, dx. \end{aligned}$$

On the other hand

$$\int_{\mathbb{P}_a[\mathbb{I}]} \overline{u(x)} \Delta u(x)\, dx = -k^2 \int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\, dx,$$

and accordingly

$$(\mathrm{Re}(k)^2 - \mathrm{Im}(k)^2) \int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\, dx + i2\,\mathrm{Re}(k)\,\mathrm{Im}(k) \int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\, dx = \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2\, dx.$$

Thus,

$$\int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\, dx = 0.$$

Therefore, $u = 0$ in $\mathrm{cl}\,\mathbb{P}_a[\mathbb{I}]$, and, as a consequence, in $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$. □

**Corollary 7.11.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $k \in \mathbb{C}$. Assume that* $\mathrm{Im}(k) \neq 0$*. Then $k^2 \notin \mathrm{Eig}^a_D[\mathbb{T}_a[\mathbb{I}]]$.*

*Proof.* It is an immediate consequence of Proposition 7.10. □

We need to introduce some other notation. We set

$$L^2_a(\mathbb{R}^n,\mathbb{C}) \equiv \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{C}) \colon u(\cdot + a_i) = u(\cdot) \text{ a.e. in } \mathbb{R}^n,\ \forall i \in \{1,\dots,n\} \right\},$$

and we define the norm $\|\cdot\|_{L^2_a(\mathbb{R}^n,\mathbb{C})}$ on $L^2_a(\mathbb{R}^n,\mathbb{C})$ by setting

$$\|u\|_{L^2_a(\mathbb{R}^n,\mathbb{C})} \equiv \Big( \int_A |u(x)|^2\, dx \Big)^{\frac{1}{2}} \qquad \forall u \in L^2_a(\mathbb{R}^n,\mathbb{C}).$$

It is well known that $L^2_a(\mathbb{R}^n,\mathbb{C})$ is a Hilbert space, with the scalar product $(\cdot,\cdot)_{L^2_a(\mathbb{R}^n,\mathbb{C})}$ defined by

$$(u,v)_{L^2_a(\mathbb{R}^n,\mathbb{C})} \equiv \int_A u(x)\overline{v(x)}\, dx \qquad \forall u,v \in L^2_a(\mathbb{R}^n,\mathbb{C}).$$

Let $\{u_l\}_{l=1}^\infty$ be a sequence in $L^2_a(\mathbb{R}^n,\mathbb{C})$ and $u \in L^2_a(\mathbb{R}^n,\mathbb{C})$. We say that $u_l$ *weakly converges to $u$ in* $L^2_a(\mathbb{R}^n,\mathbb{C})$ (and we write $u_l \rightharpoonup u$ in $L^2_a(\mathbb{R}^n,\mathbb{C})$), if

$$\lim_{l\to\infty} \int_A u_l(x)\overline{v(x)}\, dx = \int_A u(x)\overline{v(x)}\, dx \qquad \forall v \in L^2_a(\mathbb{R}^n,\mathbb{C}).$$

We also set

$$H^1_a(\mathbb{R}^n,\mathbb{C}) \equiv \left\{ u \in H^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{C}) \colon u(\cdot + a_i) = u(\cdot) \text{ a.e. in } \mathbb{R}^n,\ \forall i \in \{1,\dots,n\} \right\},$$

and we define the norm $\|\cdot\|_{H^1_a(\mathbb{R}^n,\mathbb{C})}$ on $H^1_a(\mathbb{R}^n,\mathbb{C})$ by setting

$$\|u\|_{H^1_a(\mathbb{R}^n,\mathbb{C})} \equiv \Big(\int_A |u(x)|^2\,dx + \int_A |\nabla u(x)|^2\,dx\Big)^{\frac{1}{2}} \qquad \forall u \in H^1_a(\mathbb{R}^n,\mathbb{C}).$$

It is well known that $H^1_a(\mathbb{R}^n,\mathbb{C})$ is a Hilbert space, with the scalar product $(\cdot,\cdot)_{H^1_a(\mathbb{R}^n,\mathbb{C})}$ defined by

$$(u,v)_{H^1_a(\mathbb{R}^n,\mathbb{C})} \equiv \int_A u(x)\overline{v(x)}\,dx + \int_A \nabla u(x)\cdot\overline{\nabla v(x)}\,dx \qquad \forall u,v \in H^1_a(\mathbb{R}^n,\mathbb{C}).$$

Let $\{u_l\}_{l=1}^{\infty}$ be a sequence in $H^1_a(\mathbb{R}^n,\mathbb{C})$ and $u \in H^1_a(\mathbb{R}^n,\mathbb{C})$. We say that $u_l$ *weakly converges to* $u$ *in* $H^1_a(\mathbb{R}^n,\mathbb{C})$ (and we write $u_l \rightharpoonup u$ in $H^1_a(\mathbb{R}^n,\mathbb{C})$), if

$$\lim_{l\to\infty}\int_A u_l(x)\overline{v(x)}\,dx + \int_A \nabla u_l(x)\cdot\overline{\nabla v(x)}\,dx = \int_A u(x)\overline{v(x)}\,dx + \int_A \nabla u(x)\cdot\overline{\nabla v(x)}\,dx \qquad \forall v \in H^1_a(\mathbb{R}^n,\mathbb{C}).$$

We also set

$$C^\infty_a(\mathbb{R}^n,\mathbb{C}) \equiv \{\,\phi \in C^\infty(\mathbb{R}^n,\mathbb{C})\colon \phi(x+a_j)=\phi(x)\ \forall x\in\mathbb{R}^n,\ \forall j\in\{1,\dots,n\}\,\}.$$

Let $m \in \mathbb{N}\setminus\{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Similarly, we set

$$L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C}) \equiv \big\{\,u \in L^2_{\mathrm{loc}}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})\cap L^2(\mathbb{P}_a[\mathbb{I}],\mathbb{C})\colon u(\cdot+a_i)=u(\cdot) \text{ a.e. in } \mathbb{T}_a[\mathbb{I}],\ \forall i\in\{1,\dots,n\}\,\big\},$$

and we define the norm $\|\cdot\|_{L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})}$ on $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ by setting

$$\|u\|_{L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})} \equiv \Big(\int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\,dx\Big)^{\frac{1}{2}} \qquad \forall u \in L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C}).$$

It is well known that $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ is a Hilbert space, with the scalar product $(\cdot,\cdot)_{L^2_a(\mathbb{R}^n,\mathbb{C})}$ defined by

$$(u,v)_{L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})} \equiv \int_{\mathbb{P}_a[\mathbb{I}]} u(x)\overline{v(x)}\,dx \qquad \forall u,v \in L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C}).$$

We also set

$$H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C}) \equiv \big\{\,u \in H^1_{\mathrm{loc}}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})\cap H^1(\mathbb{P}_a[\mathbb{I}],\mathbb{C})\colon u(\cdot+a_i)=u(\cdot) \ \text{ a.e. in } \mathbb{T}_a[\mathbb{I}],\ \forall i\in\{1,\dots,n\}\,\big\},$$

and we define the norm $\|\cdot\|_{H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})}$ on $H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ by setting

$$\|u\|_{H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})} \equiv \Big(\int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\,dx + \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2\,dx\Big)^{\frac{1}{2}} \qquad \forall u \in H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C}).$$

It is well known that $H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ is a Hilbert space, with the scalar product $(\cdot,\cdot)_{H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})}$ defined by

$$(u,v)_{H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})} \equiv \int_{\mathbb{P}_a[\mathbb{I}]} u(x)\overline{v(x)}\,dx + \int_{\mathbb{P}_a[\mathbb{I}]} \nabla u(x)\cdot\overline{\nabla v(x)}\,dx \qquad \forall u,v \in H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C}).$$

Moreover, we set

$$C^\infty_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C}) \equiv \{\,\phi \in C^\infty(\mathbb{R}^n,\mathbb{C})\colon \operatorname{supp}\phi \subseteq \mathbb{T}_a[\mathbb{I}],\ \phi(x+a_j)=\phi(x)\ \forall x\in\mathbb{R}^n, \forall j\in\{1,\dots,n\}\,\},$$

and then

$$H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C}) \equiv \mathrm{cl}_{H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})}\, C^\infty_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C}).$$

We also set

$$C^\infty_a(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C}) \equiv \{\,\phi \in C^\infty(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C})\colon \phi(x+a_j)=\phi(x)\ \forall x\in\mathbb{R}^n, \forall j\in\{1,\dots,n\}\,\}.$$

Then we have the following results.

**Proposition 7.12.** *Let* $\{u_l\}_{l=1}^{\infty}$ *be a sequence in* $H^1_a(\mathbb{R}^n,\mathbb{C})$ *such that*

$$\|u_l\|_{H^1_a(\mathbb{R}^n,\mathbb{C})} \leq M \qquad \forall l \geq 1,$$

*for some constant* $M > 0$. *Then there exists a subsequence* $\{u_{l_h}\}_{h=1}^{\infty}$ *that weakly converges in* $H^1_a(\mathbb{R}^n,\mathbb{C})$.

*Proof.* It follows by the reflexivity of the Hilbert space $H^1_a(\mathbb{R}^n,\mathbb{C})$. □

**Proposition 7.13.** *Let* $\{u_l\}_{l=1}^{\infty}$ *be a sequence in* $H^1_a(\mathbb{R}^n,\mathbb{C})$ *such that*

$$\|u_l\|_{H^1_a(\mathbb{R}^n,\mathbb{C})} \leq M \qquad \forall l \geq 1,$$

*for some constant* $M > 0$. *Then there exists a subsequence* $\{u_{l_h}\}_{h=1}^{\infty}$ *that converges in* $L^2_a(\mathbb{R}^n,\mathbb{C})$. *As a consequence, the embedding of* $H^1_a(\mathbb{R}^n,\mathbb{C})$ *in* $L^2_a(\mathbb{R}^n,\mathbb{C})$ *is compact.*

*Proof.* Let $V$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^\infty$ such that

$$\operatorname{cl} A \subseteq V.$$

By the periodicity of the elements of $H^1_a(\mathbb{R}^n,\mathbb{C})$, it is easy to see that there exists a constant $M' > 0$ such that

$$\|u_l\|_{H^1(V,\mathbb{C})} \leq M' \qquad \forall l \geq 1.$$

Then, by the Rellich-Kondrachov Compactness Theorem (cf. *e.g.*, Evans [50, Theorem 1, p. 272]), we easily conclude. □

**Corollary 7.14.** *Let* $\{u_l\}_{l=1}^{\infty}$ *be a sequence in* $H^1_a(\mathbb{R}^n,\mathbb{C})$ *such that*

$$\|u_l\|_{H^1_a(\mathbb{R}^n,\mathbb{C})} \leq M \qquad \forall l \geq 1,$$

*for some constant* $M > 0$. *Then there exist a subsequence* $\{u_{l_h}\}_{h=1}^{\infty}$ *and a function* $v \in H^1_a(\mathbb{R}^n,\mathbb{C})$, *such that*

$$\lim_{h\to\infty} u_{l_h} = v \qquad \text{in } L^2_a(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* It is an immediate consequence of Propositions 7.12, 7.13. □

**Proposition 7.15.** *Let* $m \in \mathbb{N}\setminus\{0\}$. *Let* $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be as in* (1.46). *Then the embedding of* $H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ *in* $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ *is compact.*

*Proof.* Let $V_1$, $V_2$ be two bounded open connected subsets of $\mathbb{R}^n$ of class $C^\infty$ such that

$$\operatorname{cl} A \subseteq V_1 \subseteq \operatorname{cl} V_1 \subseteq V_2,$$

and

$$\operatorname{cl} V_2 \cap (\operatorname{cl}\mathbb{I} + a(z)) = \emptyset \qquad \forall z \in \mathbb{Z}^n \setminus \{0\}.$$

We set

$$W \equiv V_2 \setminus \operatorname{cl}\mathbb{I}.$$

We also set

$$A^{\#} \equiv \prod_{j=1}^{n} [0, a_{jj}[.$$

Let $\phi \in C^\infty(\mathbb{R}^n,\mathbb{C})$ be such that $0 \leq \phi \leq 1$, $\phi = 1$ on $\operatorname{cl} V_1$ and $\operatorname{supp}\phi \subseteq V_2$. Let $T_1$ be the map of $H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ to $H^1_0(W,\mathbb{C})$ which takes a function $u$ to $(u\phi)_{|W}$. Clearly, $T_1$ is a linear and continuous map of $H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ to $H^1_0(W,\mathbb{C})$. Since $|W|_n < \infty$, by Tartar [132, Lemma 11.2, p. 56], we have that the embedding $T_2$ of $H^1_0(W,\mathbb{C})$ in $L^2(W,\mathbb{C})$ is compact. Furthermore the map $T_3$ of $L^2(W,\mathbb{C})$ to $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ which takes a function $u$ to the function defined by extending by periodicity $u_{|A^{\#}\setminus\operatorname{cl}\mathbb{I}}$ to the whole $\mathbb{T}_a[\mathbb{I}]$ is linear and continuous. Then, in order to conclude, it suffices to observe that the map $T \equiv T_3 \circ T_2 \circ T_1$ of $H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ to $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ is compact and that

$$T(u) = u \qquad \forall u \in H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C}).$$

□

Then we have the following lemma (cf. Courtois [30, Proposition 2.1, p. 198], Dupuy, Orive and Smaranda [49, Lemma 3.6, p. 234].)

**Lemma 7.16.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Set*

$$\mathcal{V} \equiv \{ \phi \in C^\infty_a(\mathbb{R}^n, \mathbb{C}) \colon \phi \text{ vanishes in an open neighbourhood of } w \} .$$

*Then* $\mathcal{V}$ *is dense in* $H^1_a(\mathbb{R}^n, \mathbb{C})$.

*Proof.* First of all, as a consequence of the density of trigonometric polynomials in $H^1_a(\mathbb{R}^n, \mathbb{C})$ (cf. *e.g.*, Schmeisser and Triebel [125, Theorem 1, p. 163, and p. 168-169]), we observe that $C^\infty_a(\mathbb{R}^n, \mathbb{C})$ is dense in $H^1_a(\mathbb{R}^n, \mathbb{C})$.

We set

$$A^\# \equiv \prod_{j=1}^{n} [0, a_{jj}[.$$

If $\phi \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{C})$, then we denote by $[\phi]_a$ the element of $L^2_a(\mathbb{R}^n, \mathbb{C})$ defined by extending by periodicity $\phi_{|A^\#}$ to the whole of $\mathbb{R}^n$.

In order to prove the lemma, we follow the proof of Tartar [132, Lemmas 17.2, 17.3, p. 85, 86] for the non-periodic case, and we treat separately case $n \geq 3$ and case $n = 2$.

Assume $n \geq 3$. Clearly, it suffices to show that if $u \in C^\infty_a(\mathbb{R}^n, \mathbb{C})$, then there exists a sequence $\{u_l\}_{l=1}^\infty$ in $\mathcal{V}$ such that $u_l \to u$ in $H^1_a(\mathbb{R}^n, \mathbb{C})$. Let $R > 0$ be such that $\mathrm{cl}\,\mathbb{B}_n(w, 2R) \subseteq A$, and let $\theta \in C^\infty(\mathbb{R}^n, \mathbb{R})$ be such that $0 \leq \theta \leq 1$, and

$$\theta(x) = 1 \qquad \forall x \in \mathbb{R}^n \setminus \mathbb{B}_n(0, 2R),$$
$$\theta(x) = 0 \qquad \forall x \in \mathrm{cl}\,\mathbb{B}_n(0, R).$$

Set

$$\theta_l(x) \equiv \theta(l(x - w)) \qquad \forall x \in \mathbb{R}^n,$$

for all $l \geq 1$. Let $u \in C^\infty_a(\mathbb{R}^n, \mathbb{C})$. We set

$$u_l \equiv [u\theta_l]_a = u[\theta_l]_a \qquad \forall l \geq 1.$$

Clearly, $u_l \in \mathcal{V}$, for all $l \geq 1$. Moreover, a simple computation shows that

$$\lim_{l \to \infty} u_l = u \qquad \text{in } H^1_a(\mathbb{R}^n, \mathbb{C}),$$

and the conclusion, if $n \geq 3$, follows.

We now consider case $n = 2$. By the Hahn-Banach Theorem, it suffices to prove that if $T \in (H^1_a(\mathbb{R}^2, \mathbb{C}))'$ is such that

$$\langle T, u \rangle = 0 \qquad \forall u \in \mathcal{V},$$

then $T \equiv 0$. So let $T \in (H^1_a(\mathbb{R}^2, \mathbb{C}))'$ be such that $\langle T, u \rangle = 0$ for all $u \in \mathcal{V}$. Let $R > 0$ be such that

$$\mathrm{cl}\,\mathbb{B}_2(w, 3R) \subseteq A.$$

Let $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R}^2, \mathbb{R})$ be such that $0 \leq \psi \leq 1$, $\psi = 1$ on $\mathrm{cl}\,\mathbb{B}_2(w, 2R)$, and $\mathrm{supp}\,\psi \subseteq \mathrm{cl}\,\mathbb{B}_2(w, 3R)$. Define $\tilde{T} \in (H^1(\mathbb{R}^2, \mathbb{C}))'$ by setting

$$\langle \tilde{T}, \phi \rangle \equiv \langle T, [\psi\phi]_a \rangle \qquad \forall \phi \in H^1(\mathbb{R}^2, \mathbb{C}).$$

Clearly,

$$\langle \tilde{T}, \phi \rangle = 0,$$

for all $\phi \in C^\infty_{\mathrm{c}}(\mathbb{R}^2, \mathbb{C})$ which are 0 in a small ball around $w$. Thus, by a density argument (cf. *e.g.*, Tartar [132, Lemma 17.3, p. 86]), we have

$$\langle \tilde{T}, \phi \rangle = 0 \qquad \forall \phi \in H^1(\mathbb{R}^2, \mathbb{C}).$$

Now set

$$\mathcal{W} \equiv \left\{ u \in C^\infty_a(\mathbb{R}^2, \mathbb{C}) \colon \mathrm{supp}(u\chi_{\mathrm{cl}\,A}) \subseteq \mathrm{cl}\,\mathbb{B}_2(w, 2R) \right\} .$$

Accordingly,

$$\langle T, u\rangle = \langle \tilde{T}, u\chi_{\mathrm{cl}\,A}\rangle = 0 \qquad \forall u \in \mathcal{W}.$$

Let $u \in C^\infty_a(\mathbb{R}^2,\mathbb{C})$. Let $\tilde{\psi} \in C^\infty_c(\mathbb{R}^2,\mathbb{R})$ be such that $0 \le \tilde{\psi} \le 1$, $\tilde{\psi} = 1$ on $\mathrm{cl}\,\mathbb{B}_2(w,R)$, and $\mathrm{supp}\,\tilde{\psi} \subseteq \mathrm{cl}\,\mathbb{B}_2(w,2R)$. Then

$$u = u[\tilde{\psi}]_a + u(1-[\tilde{\psi}]_a), \qquad u[\tilde{\psi}]_a \in \mathcal{W}, \quad u(1-[\tilde{\psi}]_a) \in \mathcal{V}.$$

Hence,

$$\langle T, u\rangle = \langle T, u[\tilde{\psi}]_a\rangle + \langle T, u(1-[\tilde{\psi}]_a)\rangle = 0 + 0 = 0.$$

As a consequence,

$$\langle T, u\rangle = 0 \qquad \forall u \in C^\infty_a(\mathbb{R}^2,\mathbb{C}).$$

Thus, by density,

$$\langle T, u\rangle = 0 \qquad \forall u \in H^1_a(\mathbb{R}^2,\mathbb{C}),$$

and the proof is complete. □

We now introduce some other notation.

Let $m \in \mathbb{N}\setminus\{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Then we denote by $\mathbf{b}_{D,\mathbb{I}}$ the quadratic form on $H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ defined by

$$\mathbf{b}_{D,\mathbb{I}}(u,v) \equiv -\int_{\mathbb{P}_a[\mathbb{I}]} \nabla u(x)\cdot\overline{\nabla v(x)}\,dx \qquad \forall u,v \in H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C}).$$

Then we define the self-adjoint operator $\Delta_{D,\mathbb{I}}$ as follows: $u \in \mathcal{D}(\Delta_{D,\mathbb{I}})$ if and only if $u \in H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ and there is a function $g \in L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ such that

$$\mathbf{b}_{D,\mathbb{I}}(u,f) = \int_{\mathbb{P}_a[\mathbb{I}]} g(x)\overline{f(x)}\,dx \qquad \forall f \in H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$$

(cf. *e.g.*, Rauch and Taylor [117, pp. 29, 37], Reed and Simon [118, Theorem VIII.15, p. 278] and Davies [45, Lemma 4.4.1, p.81, Theorem 4.4.2, p. 82], Kato [63].) In this case we define

$$\Delta_{D,\mathbb{I}}u \equiv g.$$

We shall always think of $\Delta_{D,\mathbb{I}}$ as an operator acting from its domain $\mathcal{D}(\Delta_{D,\mathbb{I}})$ defined as above, a subspace of $L^2_a(\mathbb{R}^n)$, to $L^2_a(\mathbb{R}^n)$.

Similarly, we denote by $\mathbf{b}$ the quadratic form on $H^1_a(\mathbb{R}^n,\mathbb{C})$ defined by

$$\mathbf{b}(u,v) \equiv -\int_A \nabla u(x)\cdot\overline{\nabla v(x)}\,dx \qquad \forall u,v \in H^1_a(\mathbb{R}^n,\mathbb{C}).$$

Then we define the self-adjoint operator $\Delta$ as follows: $u \in \mathcal{D}(\Delta)$ if and only if $u \in H^1_a(\mathbb{R}^n,\mathbb{C})$ and there is a function $g \in L^2_a(\mathbb{R}^n,\mathbb{C})$ such that

$$\mathbf{b}(u,f) = \int_A g(x)\overline{f(x)}\,dx \qquad \forall f \in H^1_a(\mathbb{R}^n,\mathbb{C})$$

(cf. *e.g.*, Rauch and Taylor [117, pp. 29, 37], Reed and Simon [118, Theorem VIII.15, p. 278] and Davies [45, Lemma 4.4.1, p.81, Theorem 4.4.2, p. 82], Kato [63].) In this case we define

$$\Delta u \equiv g.$$

We shall always think of $\Delta$ as an operator acting from its domain $\mathcal{D}(\Delta)$ defined as above, a subspace of $L^2_a(\mathbb{R}^n)$, to $L^2_a(\mathbb{R}^n)$.

Then we have the following well known results.

**Lemma 7.17.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $g \in L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$. Then there exists a unique function $u \in H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ such that*

$$(1-\Delta_{D,\mathbb{I}})u = g.$$

*Hence, $(1-\Delta_{D,\mathbb{I}})$ is invertible, and $(1-\Delta_{D,\mathbb{I}})^{(-1)}$ is a linear and continuous map of $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ to $H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ (and thus a compact operator of $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ to itself.)*

*Proof.* The invertibility of $(1-\Delta_{D,\mathbb{I}})$ and the continuity of $(1-\Delta_{D,\mathbb{I}})^{(-1)}$ as a map of $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ to $H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ is an immediate consequence of the Lax-Milgram Lemma. The compactness of $(1-\Delta_{D,\mathbb{I}})^{(-1)}$ as a map of $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ to $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ is a consequence of Proposition 7.15. □

**Proposition 7.18.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $\mathbb{I}$ be as in (1.46). Then the spectrum $\sigma(\Delta_{D,\mathbb{I}})$ of $\Delta_{D,\mathbb{I}}$ is a subset of $]-\infty,0[$ and consists of a sequence of eigenvalues of finite multiplicity, accumulating only at $-\infty$. Moreover, $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ has a complete orthonormal set of eigenfunctions of $\Delta_{D,\mathbb{I}}$.*

*Proof.* Clearly, it suffices to study $\sigma(-\Delta_{D,\mathbb{I}})$. By Lemma 7.17 and Davies [45, Theorem 4.3.1, p. 78, Corollary 4.2.3, p. 77], we have that the non-negative self-adjoint operator $-\Delta_{D,\mathbb{I}}$ has empty essential spectrum and that there exists a complete orthonormal set of eigenvectors $\{\phi_l\}_{l=1}^{\infty}$ of $-\Delta_{D,\mathbb{I}}$ with corresponding eigenvalues $\lambda_l\geq 0$ which converge to $+\infty$ as $l\to\infty$. Moreover, we observe that 0 is not an eigenvalue. Indeed, it if it were an eigenvalue with eigenvector $\psi\neq 0$, then

$$\int_{\mathbb{P}_a[\mathbb{I}]}|\nabla\psi(x)|^2\,dx=0,$$

and accordingly, since $\psi\in H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$, we would have $\psi=0$, a contradiction. Thus $\sigma(-\Delta_{D,\mathbb{I}})\subseteq\,]0,+\infty[$. Hence the conclusion easily follows. □

*Remark* 7.19. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $\mathbb{I}$ be as in (1.46). It can be proved that, if $-\lambda\in\mathrm{Eig}^a_D[\mathbb{T}_a[\mathbb{I}]]$, then $\lambda\in\sigma(\Delta_{D,\mathbb{I}})$. Indeed, if $-\lambda\in\mathrm{Eig}^a_D[\mathbb{T}_a[\mathbb{I}]]$, then there exists a function $u\in C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C})\cap C^2(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$, $u$ not identically zero, such that

$$\begin{cases}\Delta u(x)-\lambda u(x)=0 & \forall x\in\mathbb{T}_a[\mathbb{I}],\\ u(x+a_j)=u(x) & \forall x\in\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\quad \forall j\in\{1,\dots,n\},\\ u(x)=0 & \forall x\in\partial\mathbb{I}.\end{cases}$$

Then a standard argument based on the Divergence Theorem implies that $\lambda\in\sigma(\Delta_{D,\mathbb{I}})$. In fact, if $\phi\in C^\infty_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$, then, by the Divergence Theorem and the periodicity of $u$ and $\phi$, we have,

$$\begin{aligned}\int_{\mathbb{P}_a[\mathbb{I}]}\nabla u(x)\cdot\overline{\nabla\phi(x)}\,dx&=-\lambda\int_{\mathbb{P}_a[\mathbb{I}]}u(x)\overline{\phi(x)}\,dx-\int_{\partial\mathbb{I}}\frac{\partial u(x)}{\partial\nu_{\mathbb{I}}}\overline{\phi(x)}\,d\sigma_x\\&=-\lambda\int_{\mathbb{P}_a[\mathbb{I}]}u(x)\overline{\phi(x)}\,dx.\end{aligned}$$

Accordingly,

$$-\int_{\mathbb{P}_a[\mathbb{I}]}\nabla u(x)\cdot\overline{\nabla\phi(x)}\,dx=\lambda\int_{\mathbb{P}_a[\mathbb{I}]}u(x)\overline{\phi(x)}\,dx\qquad\forall\phi\in C^\infty_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C}),$$

and thus by density

$$\mathbf{b}_{D,\mathbb{I}}(u,\phi)=\lambda\int_{\mathbb{P}_a[\mathbb{I}]}u(x)\overline{\phi(x)}\,dx\qquad\forall\phi\in H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C}).$$

Hence, since $u\in H^1_{0,a}(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$, $\lambda\in\sigma(\Delta_{D,\mathbb{I}})$.

Moreover, $\sigma(\Delta_{D,\mathbb{I}})$ can be rearranged into the following sequence of negative real numbers

$$0>\lambda_1(\Delta_{D,\mathbb{I}})\geq\lambda_2(\Delta_{D,\mathbb{I}})\geq\lambda_3(\Delta_{D,\mathbb{I}})\geq\cdots\geq\lambda_j(\Delta_{D,\mathbb{I}})\geq\dots,$$

where each eigenvalue is repeated as many times as its multiplicity.

**Lemma 7.20.** *Let $g\in L^2_a(\mathbb{R}^n,\mathbb{C})$. Then there exists a unique function $u\in H^1_a(\mathbb{R}^n,\mathbb{C})$ such that*

$$(1-\Delta)u=g.$$

*Hence, $(1-\Delta)$ is invertible, and $(1-\Delta)^{(-1)}$ is a linear and continuous map of $L^2_a(\mathbb{R}^n,\mathbb{C})$ to $H^1_a(\mathbb{R}^n,\mathbb{C})$ (and thus a compact operator of $L^2_a(\mathbb{R}^n,\mathbb{C})$ to itself.)*

*Proof.* The invertibility of $(1-\Delta)$ and the continuity of $(1-\Delta)^{(-1)}$ as a map of $L^2_a(\mathbb{R}^n,\mathbb{C})$ to $H^1_a(\mathbb{R}^n,\mathbb{C})$ is an immediate consequence of the Lax-Milgram Lemma. The compactness of $(1-\Delta)^{(-1)}$ as a map of $L^2_a(\mathbb{R}^n,\mathbb{C})$ to $L^2_a(\mathbb{R}^n,\mathbb{C})$ is a consequence of Proposition 7.13. □

**Proposition 7.21.** *The spectrum $\sigma(\Delta)$ of $\Delta$ is a subset of $]-\infty,0]$ and consists of a sequence of eigenvalues of finite multiplicity, accumulating only at $-\infty$. In particular $0\in\sigma(\Delta)$. Moreover, $L^2_a(\mathbb{R}^n,\mathbb{C})$ has a complete orthonormal set of eigenfunctions of $\Delta$.*

*Proof.* First of all we observe that this is a well known result (cf. *e.g.*, Milnor [96], Chavel [22, p. 29], Berger, Gauduchon, and Mazet [12, p. 146-148].) However, for the sake of completeness, we prove it here by following the proof of Proposition 7.18. Obviously, it suffices to study $\sigma(-\Delta)$. By Lemma 7.20 and Davies [45, Theorem 4.3.1, p. 78, Corollary 4.2.3, p. 77], we have that the non-negative self-adjoint operator $-\Delta$ has empty essential spectrum and that there exists a complete orthonormal set of eigenvectors $\{\phi_l\}_{l=1}^{\infty}$ of $-\Delta$ with corresponding eigenvalues $\lambda_l\geq 0$ which converge to $+\infty$ as $l\to\infty$. In particular, an easy computation shows that $0\in\sigma(\Delta)$, with corresponding eigenspace given by the set of constant functions. Hence the conclusion easily follows. □

*Remark* 7.22. It can be proved that $\lambda\in\sigma(\Delta)$ if and only if $-\lambda\in\mathrm{Eig}_a(-\Delta)$ (cf. Section 6.8.) Indeed, if $-\lambda\in\mathrm{Eig}_a(-\Delta)$, then there exists a function $u\in C^2(\mathbb{R}^n,\mathbb{C})$ such that

$$\begin{cases}\Delta u(x)-\lambda u(x)=0 & \forall x\in\mathbb{R}^n,\\ u(x+a_j)=u(x) & \forall x\in\mathbb{R}^n,\quad \forall j\in\{1,\dots,n\}.\end{cases}$$

Then a standard argument based on the Divergence Theorem implies that $\lambda\in\sigma(\Delta)$. In fact, if $\phi\in C^\infty_a(\mathbb{R}^n,\mathbb{C})$, then, by the Divergence Theorem and the periodicity of $u$ and $\phi$, we have,

$$\begin{aligned}\int_A \nabla u(x)\cdot\overline{\nabla\phi(x)}\,dx &= -\lambda\int_A u(x)\overline{\phi(x)}\,dx-\int_{\partial A}\frac{\partial u(x)}{\partial\nu_A}\overline{\phi(x)}\,d\sigma_x\\ &= -\lambda\int_A u(x)\overline{\phi(x)}\,dx.\end{aligned}$$

Accordingly,

$$-\int_A \nabla u(x)\cdot\overline{\nabla\phi(x)}\,dx=\lambda\int_A u(x)\overline{\phi(x)}\,dx\qquad \forall\phi\in C^\infty_a(\mathbb{R}^n,\mathbb{C}),$$

and thus by density

$$\mathbf{b}(u,\phi)=\lambda\int_A u(x)\overline{\phi(x)}\,dx\qquad\forall\phi\in H^1_a(\mathbb{R}^n,\mathbb{C}).$$

Hence $\lambda\in\sigma(\Delta)$. Similarly, if $\lambda\in\sigma(\Delta)$, then $-\lambda\in\mathrm{Eig}_a(-\Delta)$. Indeed, let $\lambda\in\sigma(\Delta)$ and $u\in H^1_a(\mathbb{R}^n,\mathbb{C})$ be the corresponding eigenfunction. Then it is easy to prove that for all $y\in\mathbb{R}^n$, there exists $R_y>0$ such that

$$-\int_{\mathbb{B}_n(y,R_y)}\nabla u(x)\cdot\overline{\nabla\phi(x)}\,dx-\lambda\int_{\mathbb{B}_n(y,R_y)}u(x)\overline{\phi(x)}\,dx=0\qquad\forall\phi\in C^\infty_c(\mathbb{B}_n(y,R_y),\mathbb{C}).$$

Then, by standard elliptic regularity theory, we have that $u\in C^2(\mathbb{R}^n,\mathbb{C})$ and that

$$\begin{cases}\Delta u(x)+(-\lambda)u(x)=0 & \forall x\in\mathbb{R}^n,\\ u(x+a_j)=u(x) & \forall x\in\mathbb{R}^n,\quad \forall j\in\{1,\dots,n\},\end{cases}$$

and thus $-\lambda\in\mathrm{Eig}_a(-\Delta)$.

Moreover, $\sigma(\Delta)$ can be rearranged into the following sequence of non-positive real numbers

$$0=\lambda_1(\Delta)\geq\lambda_2(\Delta)\geq\lambda_3(\Delta)\geq\cdots\geq\lambda_j(\Delta)\geq\dots,$$

where each eigenvalue is repeated as many times as its multiplicity.

Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$ be as in (1.56). Let $\epsilon_1$ be as in (1.57). If $\epsilon\in]0,\epsilon_1[$, then, in order to simplify the notation, we set

$$\mathbf{b}_{D,\epsilon}\equiv\mathbf{b}_{D,\Omega_\epsilon},$$

and

$$\Delta_{D,\epsilon}\equiv\Delta_{D,\Omega_\epsilon}.$$

Let $j\in\mathbb{N}\setminus\{0\}$. Then, for all $\epsilon\in]0,\epsilon_1[$, we can consider $\lambda_j(\Delta_{D,\epsilon})$. In particular, we are interested in the limit of $\lambda_j(\Delta_{D,\epsilon})$ as $\epsilon$ tends to 0 in $]0,\epsilon_1[$.

The behaviour of the eigenvalues of the Laplace operator in an open set of $\mathbb{R}^n$ or in a Riemannian manifold, where a small part is removed (the hole), with a Dirichlet condition on the boundary of the hole, has long been investigated by many authors. It is perhaps difficult to provide a complete list of all the contributions. Here we mention, for the case of an open set of $\mathbb{R}^n$, Rauch and Taylor [117], Ozawa [108], Ozawa [109], Maz'ya, Nazarov and Plamenewskii [93], Maz'ya, Nazarov and Plamenewskij [91, Chapter 9], Flucher [51]. As far as Riemannian manifolds are concerned, we refer, *e.g.*, to Chavel and Feldman [23], Chavel and Feldman [24], Chavel [22, Chapter IX], Besson [14], Courtois [30]. Moreover, the periodic case has been considered, for instance, by Dupuy, Orive and Smaranda [49, p. 232], San Martin and Smaranda [121].

In order to study the convergence of $\lambda_j(\Delta_{D,\epsilon})$ as $\epsilon \to 0^+$, we follow Rauch and Taylor [117, Sections 1, 2] (cf. also Dupuy, Orive and Smaranda [49, p. 232].)

We now introduce some other notation.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\epsilon \in ]0,\epsilon_1[$. If $u$ is in $L^2_a(\mathbb{R}^n,\mathbb{C})$, then we denote by $\mathbf{P}_\epsilon u$ the restriction of $u$ to $\mathbb{T}_a[\Omega_\epsilon]$. Similarly, if $u$ is in $L^2_a(\mathbb{T}_a[\Omega_\epsilon],\mathbb{C})$, then we denote by $\mathbf{E}_{0,\epsilon} u$ the element of $L^2_a(\mathbb{R}^n,\mathbb{C})$, defined by $(\mathbf{E}_{0,\epsilon}u)_{|\mathbb{T}_a[\Omega_\epsilon]} = u$ and $(\mathbf{E}_{0,\epsilon}u)_{|\mathbb{R}^n\setminus\mathbb{T}_a[\Omega_\epsilon]} = 0$.

Then we have the following variant of Rauch and Taylor [117, Lemma 1.1, p. 30] (cf. also Dupuy, Orive and Smaranda [49, Theorem 3.1, p. 233].)

**Proposition 7.23.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $\{\epsilon_l\}_{l=1}^\infty$ be a sequence in $]0,\epsilon_1[$, convergent to $0$. Let $u \in L^2_a(\mathbb{R}^n,\mathbb{C})$. Then there exists a subsequence $\{\epsilon_{l_j}\}_{j=1}^\infty$, such that*

$$\lim_{j\to\infty} \|\mathbf{E}_{0,\epsilon_{l_j}}(1-\Delta_{D,\epsilon_{l_j}})^{(-1)}\mathbf{P}_{\epsilon_{l_j}}u - (1-\Delta)^{(-1)}u\|_{L^2_a(\mathbb{R}^n,\mathbb{C})} = 0.$$

*Proof.* It suffices to modify the proof of Rauch and Taylor [117, Lemma 1.1, p. 30]. We set

$$v_l \equiv \mathbf{E}_{0,\epsilon_l}(1-\Delta_{D,\epsilon_l})^{(-1)}\mathbf{P}_{\epsilon_l}u \qquad \forall l \geq 1.$$

Let $l \geq 1$. Then we have

$$\begin{aligned}
\|v_l\|^2_{H^1_a(\mathbb{R}^n,\mathbb{C})} &= \int_{\mathbb{P}_a[\Omega_{\epsilon_l}]} v_l(x)\overline{v_l(x)}\,dx + \int_{\mathbb{P}_a[\Omega_{\epsilon_l}]} \nabla v_l(x)\cdot\overline{\nabla v_l(x)}\,dx \\
&= \int_{\mathbb{P}_a[\Omega_{\epsilon_l}]} \mathbf{P}_{\epsilon_l}u(x)\overline{v_l(x)}\,dx \\
&\leq \|u\|_{L^2_a(\mathbb{R}^n,\mathbb{C})}\|v_l\|_{H^1_a(\mathbb{R}^n,\mathbb{C})},
\end{aligned}$$

and thus

$$\|v_l\|_{H^1_a(\mathbb{R}^n,\mathbb{C})} \leq \|u\|_{L^2_a(\mathbb{R}^n,\mathbb{C})}.$$

Hence, $\{v_l\}_{l=1}^\infty$ is a bounded sequence in $H^1_a(\mathbb{R}^n,\mathbb{C})$.

First of all, we verify that, possibly considering a subsequence, $v_l$ converges to $(1-\Delta)^{(-1)}u$ weakly in $H^1_a(\mathbb{R}^n,\mathbb{C})$. Let $v$ be a weak limit point of $\{v_l\}_{l=1}^\infty$. Possibly relabeling the sequence, we may assume that

$$v_l \rightharpoonup v \qquad \text{in } H^1_a(\mathbb{R}^n,\mathbb{C}).$$

We set

$$\mathcal{V} \equiv \{\, \phi \in C^\infty_a(\mathbb{R}^n,\mathbb{C}) \colon \phi \text{ vanishes in an open neighbourhood of } w \,\}.$$

Let $\phi \in \mathcal{V}$. Then there exists $\bar{l} \in \mathbb{N}$ such that

$$\phi \in H^1_{0,a}(\mathbb{T}_a[\Omega_{\epsilon_l}],\mathbb{C}) \qquad \forall l \geq \bar{l}.$$

Accordingly, if $l \geq \bar{l}$, then

$$\begin{aligned}
\int_A v_l(x)\overline{\phi(x)}\,dx + \int_A \nabla v_l(x)\cdot\overline{\nabla\phi(x)}\,dx &= \int_{\mathbb{P}_a[\Omega_{\epsilon_l}]} \mathbf{P}_{\epsilon_l}u(x)\overline{\phi(x)}\,dx \\
&= \int_A u(x)\overline{\phi(x)}\,dx.
\end{aligned}$$

On the other hand, since $v_l \rightharpoonup v$ in $H^1_a(\mathbb{R}^n,\mathbb{C})$, then

$$\lim_{l\to\infty}\Big[\int_A v_l(x)\overline{\phi(x)}\,dx + \int_A \nabla v_l(x)\cdot\overline{\nabla\phi(x)}\,dx\Big] = \int_A v(x)\overline{\phi(x)}\,dx + \int_A \nabla v(x)\cdot\overline{\nabla\phi(x)}\,dx,$$

and thus

$$\int_A v(x)\overline{\phi(x)}\,dx + \int_A \nabla v(x)\cdot\overline{\nabla\phi(x)}\,dx = \int_A u(x)\overline{\phi(x)}\,dx.$$

As a consequence,

$$\int_A v(x)\overline{\phi(x)}\,dx + \int_A \nabla v(x)\cdot\overline{\nabla\phi(x)}\,dx = \int_A u(x)\overline{\phi(x)}\,dx \qquad \forall\phi\in\mathcal{V}.$$

Hence, by density (cf. Lemma 7.16),

$$\int_A v(x)\overline{\phi(x)}\,dx + \int_A \nabla v(x)\cdot\overline{\nabla\phi(x)}\,dx = \int_A u(x)\overline{\phi(x)}\,dx \qquad \forall\phi\in H^1_a(\mathbb{R}^n,\mathbb{C}),$$

and, consequently, $v = (1-\Delta)^{(-1)}u$ (cf. also Rauch and Taylor [117, Proposition 2.2, p. 36].)

Finally, Proposition 7.13 implies that there exists a subsequence $\{v_{l_j}\}_{j=1}^{\infty}$ such that $v_{l_j}\to v$ in $L^2_a(\mathbb{R}^n,\mathbb{C})$, and then the conclusion easily follows. □

For some basic notions of Borel functional calculus for unbounded self-adjoint operators, we refer, for instance, to Reed and Simon [118, Theorem VIII.5, p. 262] and Davies [45, Chapter 2]. Then we have the following Theorem.

**Theorem 7.24.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57)*, respectively. Let $\{\epsilon_l\}_{l=1}^{\infty}$ be a sequence in $]0,\epsilon_1[$, convergent to 0. Let $F$ be a bounded Borel function on $]-\infty,1/2]$ which is continuous on a neighbourhood of $\sigma(\Delta)$. Let $u\in L^2_a(\mathbb{R}^n,\mathbb{C})$. Then there exists a subsequence $\{\epsilon_{l_j}\}_{j=1}^{\infty}$, such that*

$$\lim_{j\to\infty}\|\mathbf{E}_{0,\epsilon_{l_j}}F(\Delta_{D,\epsilon_{l_j}})\mathbf{P}_{\epsilon_{l_j}}u - F(\Delta)u\|_{L^2_a(\mathbb{R}^n,\mathbb{C})} = 0.$$

*Proof.* We follow the proof of Rauch and Taylor [117, Theorem 1.2, p. 30] verbatim. It suffices to prove the theorem for real-valued functions as $F$. Let $\Gamma$ be the Banach space of continuous real-valued functions on $]-\infty,1/2]$ which vanish at $-\infty$ and let $\mathcal{A}$ be the set of $F\in\Gamma$ for which the theorem is true. First of all, we note that $\mathcal{A}$ is a subalgebra of $\Gamma$, since, if $G$, $F\in\mathcal{A}$, then we have

$$\begin{aligned}\mathbf{E}_{0,\epsilon_l}&F(\Delta_{D,\epsilon_l})G(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}u - F(\Delta)G(\Delta)u\\ &=\Big[\mathbf{E}_{0,\epsilon_l}F(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}G(\Delta)u - F(\Delta)G(\Delta)u\Big]\\ &\quad+\mathbf{E}_{0,\epsilon_l}F(\Delta_{D,\epsilon_l})\Big[G(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}u - \mathbf{P}_{\epsilon_l}G(\Delta)u\Big],\end{aligned}$$

for all $u\in L^2_a(\mathbb{R}^n,\mathbb{C})$. The first term converges to zero (up to subsequences) because $F\in\mathcal{A}$ and the second term converges to zero (up to subsequences) because $G\in\mathcal{A}$, and accordingly $FG\in\mathcal{A}$. Then we observe that $\mathcal{A}$ is clearly closed in $\Gamma$. Moreover, by Proposition 7.23, we have that the function $f(x)\equiv(1-x)^{-1}$ is in $\Gamma$. Then, since $f$ separates points of $]-\infty,1/2]$, by the Stone-Weierstrass Theorem we can conclude that $\mathcal{A}=\Gamma$.

If $F$ is a bounded continuous function on $]-\infty,1/2]$ it suffices to show that, up to subsequences,

$$\mathbf{E}_{0,\epsilon_l}F(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}u\to F(\Delta)u \qquad \text{in } L^2_a(\mathbb{R}^n,\mathbb{C})$$

for all $u$ in a dense subset of $L^2_a(\mathbb{R}^n,\mathbb{C})$, in particular for all $v$ of the form $\exp(\Delta)u$. We observe that

$$\begin{aligned}&\mathbf{E}_{0,\epsilon_l}F(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}\exp(\Delta)u =\\ &\qquad\mathbf{E}_{0,\epsilon_l}F(\Delta_{D,\epsilon_l})\exp(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}u + \mathbf{E}_{0,\epsilon_l}F(\Delta_{D,\epsilon_l})\Big[\mathbf{P}_{\epsilon_l}\exp(\Delta)u - \exp(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}u\Big].\end{aligned}$$

By the above result, up to subsequences, the first term converges to $F(\Delta)\exp(\Delta)u$ and the second to zero since

$$\mathbf{E}_{0,\epsilon_l}\exp(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}u\to\exp(\Delta)u \qquad \text{in } L^2_a(\mathbb{R}^n,\mathbb{C}).$$

Finally suppose that $F$ is a bounded Borel function on $]-\infty,1/2]$, continuous on a neighbourhood $U$ of $\sigma(\Delta)$. Let $\psi_1$, $\psi_2$ be two positive continuous functions on $]-\infty,1/2]$, such that $\psi_1+\psi_2=1$, $\mathrm{supp}\,\psi_1\subseteq U$, and $\psi_1=1$ on a neighbourhood of $\sigma(\Delta)$. Then

$$\mathbf{E}_{0,\epsilon_l}F(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}u=\mathbf{E}_{0,\epsilon_l}(\psi_1F)(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}u+\mathbf{E}_{0,\epsilon_l}(\psi_2F)(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}u.$$

Since $\psi_1F$ is bounded and continuous, then, up to subsequences,

$$\mathbf{E}_{0,\epsilon_l}(\psi_1F)(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon}u\to(\psi_1F)(\Delta)u=F(\Delta)u\qquad\text{in } L^2_a(\mathbb{R}^n,\mathbb{C}).$$

On the other hand,

$$\|(\psi_2F)(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}u\|_{L^2_a(\mathbb{T}_a[\Omega_{\epsilon_l}],\mathbb{C})}\le\sup|F|\|\psi_2(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}u\|_{L^2_a(\mathbb{T}_a[\Omega_{\epsilon_l}],\mathbb{C})},$$

and, up to subsequences,

$$\lim_{l\to\infty}\|\psi_2(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}u\|_{L^2_a(\mathbb{T}_a[\Omega_{\epsilon_l}],\mathbb{C})}=0,$$

since

$$\mathbf{E}_{0,\epsilon_l}\psi_2(\Delta_{D,\epsilon_l})\mathbf{P}_{\epsilon_l}u\to\psi_2(\Delta)u=0\qquad\text{in } L^2_a(\mathbb{R}^n,\mathbb{C}),$$

as $l\to\infty$. Hence we can easily conclude. □

Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $J\subseteq\mathbb{R}$ be a bounded open interval whose endpoints do not belong to $\sigma(\Delta)$. Then we denote by $\Pi^J$ the spectral projection of $\Delta$ on $J$. Then $\mathrm{rank}\,\Pi^J\equiv\dim(\mathrm{range}\,\Pi^J)$ is the number of eigenvalues of $\Delta$ in $J$. Similarly, if $\epsilon\in]0,\epsilon_1[$, then we denote by $\Pi^J_{D,\epsilon}$ the spectral projection of $\Delta_{D,\epsilon}$ on $J$. Then $\mathrm{rank}\,\Pi^J_{D,\epsilon}\equiv\dim(\mathrm{range}\,\Pi^J_{D,\epsilon})$ is the number of eigenvalues of $\Delta_{D,\epsilon}$ in $J$.

We have the following Proposition.

**Proposition 7.25.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $J\subseteq\mathbb{R}$ be a bounded open interval whose endpoints do not belong to $\sigma(\Delta)$. Then there exists $\epsilon_2\in]0,\epsilon_1[$ such that*

$$\mathrm{rank}\,\Pi^J_{D,\epsilon}=\mathrm{rank}\,\Pi^J,$$

*for all $\epsilon\in]0,\epsilon_2[$.*

*Proof.* We proceed as in Rauch and Taylor [117, Theorem 1.5]. First of all, we observe that

$$\mathrm{rank}\,\Pi^J_{D,\epsilon}=\mathrm{rank}\,\mathbf{E}_{0,\epsilon}\Pi^J_{D,\epsilon},\qquad\mathrm{rank}\,\Pi^J_{D,\epsilon}\mathbf{P}_\epsilon=\mathrm{rank}\,\mathbf{E}_{0,\epsilon}\Pi^J_{D,\epsilon}\mathbf{P}_\epsilon,$$

for all $\epsilon\in]0,\epsilon_1[$. Then the proof consists of three steps: for all $\epsilon\in]0,\epsilon_2[$, with $\epsilon_2\in]0,\epsilon_1[$ small enough, we have

(*i*) $\mathrm{rank}(\mathbf{E}_{0,\epsilon}\Pi^J_{D,\epsilon}\mathbf{P}_\epsilon)\ge\mathrm{rank}\,\Pi^J$,

(*ii*) $\mathrm{rank}(\mathbf{E}_{0,\epsilon}\Pi^J_{D,\epsilon})\le\mathrm{rank}\,\Pi^J$,

(*iii*) $\mathrm{range}\,\Pi^J_{D,\epsilon}=\mathrm{range}\,\Pi^J_{D,\epsilon}\mathbf{P}_\epsilon$.

We first prove (*i*). If it were not true, than there would exist a sequence $\{\epsilon_l\}_{l=1}^\infty$ in $]0,\epsilon_1[$ convergent to 0, such that

$$\mathrm{rank}(\mathbf{E}_{0,\epsilon_l}\Pi^J_{D,\epsilon_l}\mathbf{P}_{\epsilon_l})<\mathrm{rank}\,\Pi^J,\qquad\forall l\ge1.$$

Let $\{u_1,\dots,u_k\}$ be an orthonormal basis of the range of $\Pi^J$. Then, by Theorem 7.24, up to subsequences,

$$\lim_{l\to\infty}\|\mathbf{E}_{0,\epsilon_l}\Pi^J_{D,\epsilon_l}\mathbf{P}_{\epsilon_l}u_j-\Pi^Ju_j\|_{L^2_a(\mathbb{R}^n,\mathbb{C})}=0,$$

for all $j\in\{1,\dots,k\}$. It follows that there exists $\bar{l}\in\mathbb{N}$, such that $\{\mathbf{E}_{0,\epsilon_l}\Pi^J_{D,\epsilon_l}\mathbf{P}_{\epsilon_l}u_j\}_{j=1}^k$ is a linear independent set for all $l\ge\bar{l}$, a contradiction.

We now consider (*ii*). If it were not true, than there would exist a sequence $\{\epsilon_l\}_{l=1}^\infty$ in $]0,\epsilon_1[$ convergent to 0, such that

$$\dim\mathrm{range}(\mathbf{E}_{0,\epsilon_l}\Pi^J_{D,\epsilon_l})>\dim\mathrm{range}\,\Pi^J,\qquad\forall l\ge1.$$

For each $l \geq 1$, choose $v_l \in \operatorname{range} \Pi^J_{D,\epsilon_l}$ such that $\|v_l\|_{L^2_a(\mathbb{T}_a[\Omega_{\epsilon_l}],\mathbb{C})} = 1$ and $\mathbf{E}_{0,\epsilon_l} v_l \perp \operatorname{range} \Pi^J$. Then $\operatorname{range} \Pi^J_{D,\epsilon_l} \subseteq H^1_{0,a}(\mathbb{T}_a[\Omega_{\epsilon_l}],\mathbb{C})$, and

$$|\int_{\mathbb{P}_a[\Omega_{\epsilon_l}]} (\Delta_{D,\epsilon_l} v_l(x))\overline{v_l(x)}\, dx| \leq M \|v_l\|^2_{L^2_a(\mathbb{T}_a[\Omega_{\epsilon_l}],\mathbb{C})}, \qquad \forall l \geq 1,$$

with $M \equiv \sup_{x \in J} |x|$. Accordingly $\{\mathbf{E}_{0,\epsilon_l} v_l\}_{l=1}^\infty$ is a bounded sequence in $H^1_a(\mathbb{R}^n,\mathbb{C})$, and then, by Corollary 7.14, there exists a subsequence $\{\mathbf{E}_{0,\epsilon_{l_j}} v_{l_j}\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} \mathbf{E}_{0,\epsilon_{l_j}} v_{l_j} = v \qquad \text{in } L^2_a(\mathbb{R}^n,\mathbb{C}),$$

for some $v \in H^1_a(\mathbb{R}^n,\mathbb{C})$, with $\|v\|_{L^2_a(\mathbb{R}^n,\mathbb{C})} = 1$, and $v \perp \operatorname{range} \Pi^J$. Now we show that $v \in \operatorname{range} \Pi^J$, a contradiction. Indeed, Theorem 7.24 implies that, up to subsequences,

$$\lim_{j\to\infty} \mathbf{E}_{0,\epsilon_{l_j}} \Pi^J_{D,\epsilon_{l_j}} \mathbf{P}_{\epsilon_{l_j}} v = \Pi^J v \qquad \text{in } L^2_a(\mathbb{R}^n,\mathbb{C}).$$

Moreover,

$$\begin{aligned} \|\mathbf{E}_{0,\epsilon_{l_j}} \Pi^J_{D,\epsilon_{l_j}} v_{l_j} - \mathbf{E}_{0,\epsilon_{l_j}} \Pi^J_{D,\epsilon_{l_j}} \mathbf{P}_{\epsilon_{l_j}} v\|_{L^2_a(\mathbb{R}^n,\mathbb{C})} &\leq \|v_{l_j} - \mathbf{P}_{\epsilon_{l_j}} v\|_{L^2_a(\mathbb{T}_a[\Omega_{\epsilon_{l_j}}],\mathbb{C})} \\ &\leq \|\mathbf{E}_{0,\epsilon_{l_j}} v_{l_j} - v\|_{L^2_a(\mathbb{R}^n,\mathbb{C})}, \end{aligned}$$

and thus

$$\lim_{j\to\infty} \mathbf{E}_{0,\epsilon_{l_j}} \Pi^J_{D,\epsilon_{l_j}} v_{l_j} = \Pi^J v \qquad \text{in } L^2_a(\mathbb{R}^n,\mathbb{C}).$$

On the other hand, $\mathbf{E}_{0,\epsilon_{l_j}} \Pi^J_{D,\epsilon_{l_j}} v_{l_j} = \mathbf{E}_{0,\epsilon_{l_j}} v_{l_j}$, and

$$\lim_{j\to\infty} \mathbf{E}_{0,\epsilon_{l_j}} v_{l_j} = v \qquad \text{in } L^2_a(\mathbb{R}^n,\mathbb{C}).$$

Thus $v = \Pi^J v$, and so $v \in \operatorname{range} \Pi^J$.

We finally prove (*iii*). If it were false for some $\epsilon \in ]0,\epsilon_1[$, then there would exist a non-zero $v \in \operatorname{range} \Pi^J_{D,\epsilon}$, with $v \perp \operatorname{range} \Pi^J_{D,\epsilon} \mathbf{P}_\epsilon$. Thus for all $u \in L^2_a(\mathbb{R}^n,\mathbb{C})$, we would have

$$0 = \int_{\mathbb{P}_a[\Omega_\epsilon]} \Pi^J_{D,\epsilon} \mathbf{P}_\epsilon u(x) \overline{v(x)}\, dx = \int_{\mathbb{P}_a[\Omega_\epsilon]} \mathbf{P}_\epsilon u(x) \overline{\Pi^J_{D,\epsilon} v(x)}\, dx = \int_{\mathbb{P}_a[\Omega_\epsilon]} \mathbf{P}_\epsilon u(x) \overline{v(x)}\, dx.$$

As a consequence, $v = 0$, a contradiction.

Hence the proof is complete. □

**Theorem 7.26.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$ be as in* (1.56). *Let $\epsilon_1$ be as in* (1.57). *Let $j \in \mathbb{N} \setminus \{0\}$. Then*

$$\lambda_j(\Delta_{D,\epsilon}) \to \lambda_j(\Delta),$$

*as $\epsilon$ tends to $0$ in $]0,\epsilon_1[$.*

*Proof.* It is a straightforward consequence of Proposition 7.25 □

**Corollary 7.27.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$ be as in* (1.56). *Let $\epsilon_1$ be as in* (1.57). *Let $k \in \mathbb{C}$ be such that $k^2 \neq |2\pi a^{-1}(z)|^2$ for all $z \in \mathbb{Z}^n$. Then there exists $\epsilon^a_D \in ]0,\epsilon_1[$, such that*

$$k^2 \notin \operatorname{Eig}^a_D[\mathbb{T}_a[\Omega_\epsilon]] \qquad \forall \epsilon \in ]0,\epsilon^a_D].$$

*Proof.* It is an immediate consequence of Theorem 7.26, of Remarks 7.19, 7.22, and of the results of Section 6.8. □

### 7.2.3 A convergence result for the eigenvalues of the Laplace operator in a periodically perforated domain under Neumann boundary conditions

As in the previous Subsection, we give the following.

**Definition 7.28.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\lambda \in \mathbb{C}$. We say that $\lambda$ is a *periodic Neumann eigenvalue of* $-\Delta$ in $\mathbb{T}_a[\mathbb{I}]$ (and we write $\lambda \in \mathrm{Eig}^a_N[\mathbb{T}_a[\mathbb{I}]]$) if there exists a function $u \in C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C}) \cap C^2(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$, $u$ not identically zero, such that

$$\begin{cases} \Delta u(x) + \lambda u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x+a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

If such a function $u$ exists, then $u$ is called a *periodic Neumann eigenfunction of* $-\Delta$ in $\mathbb{T}_a[\mathbb{I}]$.

Then we have the following certainly known result.

**Proposition 7.29.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $k \in \mathbb{C}$. Assume that* $\mathrm{Im}(k) \neq 0$*. Let $u \in C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C}) \cap C^2(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ solve the following boundary value problem*

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x+a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

*Then $u(x) = 0$ for all $x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$.*

*Proof.* By the Divergence Theorem and the periodicity of $u$, we have

$$\begin{aligned} \int_{\mathbb{P}_a[\mathbb{I}]} \overline{u(x)} \Delta u(x)\,dx &= \int_{\partial \mathbb{P}_a[\mathbb{I}]} \overline{u(x)} \frac{\partial}{\partial \nu_{\mathbb{P}_a[\mathbb{I}]}} u(x)\,d\sigma_x - \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2\,dx \\ &= \int_{\partial A} \overline{u(x)} \frac{\partial}{\partial \nu_A} u(x)\,d\sigma_x - \int_{\partial \mathbb{I}} \overline{u(x)} \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x)\,d\sigma_x - \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2\,dx \\ &= -\int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2\,dx. \end{aligned}$$

On the other hand

$$\int_{\mathbb{P}_a[\mathbb{I}]} \overline{u(x)} \Delta u(x)\,dx = -k^2 \int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\,dx,$$

and accordingly

$$(\mathrm{Re}(k)^2 - \mathrm{Im}(k)^2) \int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\,dx + i2\,\mathrm{Re}(k)\,\mathrm{Im}(k) \int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\,dx = \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2\,dx.$$

Thus,

$$\int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\,dx = 0.$$

Therefore, $u = 0$ in $\mathrm{cl}\,\mathbb{P}_a[\mathbb{I}]$, and, as a consequence, in $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$. □

**Corollary 7.30.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $k \in \mathbb{C}$. Assume that* $\mathrm{Im}(k) \neq 0$*. Then $k^2 \notin \mathrm{Eig}^a_N[\mathbb{T}_a[\mathbb{I}]]$.*

*Proof.* It is an immediate consequence of Proposition 7.29. □

We have the following results.

**Proposition 7.31.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Then the embedding of $H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ in $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ is compact.*

*Proof.* Let $V$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^\infty$ such that

$$\mathrm{cl}\, A \subseteq V,$$

and

$$\mathrm{cl}\, V \cap (\mathrm{cl}\, \mathbb{I} + a(z)) = \emptyset \qquad \forall z \in \mathbb{Z}^n \setminus \{0\}.$$

We set

$$W \equiv V \setminus \mathrm{cl}\, \mathbb{I}.$$

We also set

$$A^\# \equiv \prod_{j=1}^{n} [0, a_{jj}[.$$

We observe that the restriction map $T_1$ of $H^1_a(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ to $H^1(W, \mathbb{C})$, which takes a function $u$ to the restriction $u_{|W}$ is linear and continuous. By the regularity of the open set $W$ we have that the embedding $T_2$ of $H^1(W, \mathbb{C})$ in $L^2(W, \mathbb{C})$ is compact (cf. *e.g.*, Evans [50, Theorem 1, p. 272].) Furthermore the map $T_3$ of $L^2(W, \mathbb{C})$ to $L^2_a(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ which takes a function $u$ to the function defined by extending by periodicity $u_{|A^\# \setminus \mathrm{cl}\, \mathbb{I}}$ to the whole $\mathbb{T}_a[\mathbb{I}]$ is linear and continuous. Then, in order to conclude, it suffices to observe that the map $T \equiv T_3 \circ T_2 \circ T_1$ of $H^1_a(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ to $L^2_a(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ is compact and that

$$T(u) = u \qquad \forall u \in H^1_a(\mathbb{T}_a[\mathbb{I}], \mathbb{C}).$$

□

**Lemma 7.32.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0, 1[$. Let $\mathbb{I}$ be as in* (1.46)*. Then the set $C^\infty_a(\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \mathbb{C})$ is dense in $H^1_a(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$.*

*Proof.* Let $u \in H^1_a(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$. Then, by arguing for instance as in Lemma 7.36, one can prove that there exists $\tilde{u} \in H^1_a(\mathbb{R}^n, \mathbb{C})$, such that $\tilde{u}_{|\mathbb{T}_a[\mathbb{I}]} = u$. Then, by the density of $C^\infty_a(\mathbb{R}^n, \mathbb{C})$ in $H^1_a(\mathbb{R}^n, \mathbb{C})$, there exists a sequence $\{\tilde{u}_l\}_{l=1}^\infty \subseteq C^\infty_a(\mathbb{R}^n, \mathbb{C})$, such that

$$\lim_{l \to \infty} \tilde{u}_l = \tilde{u} \qquad \text{in } H^1_a(\mathbb{R}^n, \mathbb{C}).$$

Hence, if we set $u_l \equiv \tilde{u}_{l|\mathbb{T}_a[\mathbb{I}]}$ for all $l \geq 1$, then $\{u_l\}_{l=1}^\infty \subseteq C^\infty_a(\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \mathbb{C})$ and

$$\lim_{l \to \infty} u_l = u \qquad \text{in } H^1_a(\mathbb{T}_a[\mathbb{I}], \mathbb{C}).$$

□

We now introduce some other notation.

Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0, 1[$. Let $\mathbb{I}$ be as in (1.46). Then we denote by $\mathbf{b}_{N,\mathbb{I}}$ the quadratic form on $H^1_a(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ defined by

$$\mathbf{b}_{N,\mathbb{I}}(u, v) \equiv -\int_{\mathbb{P}_a[\mathbb{I}]} \nabla u(x) \cdot \overline{\nabla v(x)}\, dx \qquad \forall u, v \in H^1_a(\mathbb{T}_a[\mathbb{I}], \mathbb{C}).$$

Then we define the self-adjoint operator $\Delta_{N,\mathbb{I}}$ as follows: $u \in \mathcal{D}(\Delta_{N,\mathbb{I}})$ if and only if $u \in H^1_a(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ and there is a function $g \in L^2_a(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ such that

$$\mathbf{b}_{N,\mathbb{I}}(u, f) = \int_{\mathbb{P}_a[\mathbb{I}]} g(x) \overline{f(x)}\, dx \qquad \forall f \in H^1_a(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$$

(cf. *e.g.*, Rauch and Taylor [117, pp. 29, 37], Reed and Simon [118, Theorem VIII.15, p. 278] and Davies [45, Lemma 4.4.1, p.81, Theorem 4.4.2, p. 82], Kato [63].) In this case we define

$$\Delta_{N,\mathbb{I}} u \equiv g.$$

We shall always think of $\Delta_{N,\mathbb{I}}$ as an operator acting from its domain $\mathcal{D}(\Delta_{N,\mathbb{I}})$ defined as above, a subspace of $L^2_a(\mathbb{R}^n)$, to $L^2_a(\mathbb{R}^n)$.

As we have already done in the previous Subsection, we denote by $\mathbf{b}$ the quadratic form on $H^1_a(\mathbb{R}^n,\mathbb{C})$ defined by

$$\mathbf{b}(u,v) \equiv -\int_A \nabla u(x)\cdot\overline{\nabla v(x)}\,dx \qquad \forall u,v\in H^1_a(\mathbb{R}^n,\mathbb{C}).$$

Then we define the self-adjoint operator $\Delta$ as follows: $u\in\mathcal{D}(\Delta)$ if and only if $u\in H^1_a(\mathbb{R}^n,\mathbb{C})$ and there is a function $g\in L^2_a(\mathbb{R}^n,\mathbb{C})$ such that

$$\mathbf{b}(u,f)=\int_A g(x)\overline{f(x)}\,dx \qquad \forall f\in H^1_a(\mathbb{R}^n,\mathbb{C}).$$

In this case we define

$$\Delta u\equiv g.$$

We shall always think of $\Delta$ as an operator acting from its domain $\mathcal{D}(\Delta)$ defined as above, a subspace of $L^2_a(\mathbb{R}^n)$, to $L^2_a(\mathbb{R}^n)$.

Then we have the following well known results (cf. also Briane [17, p. 5].)

**Lemma 7.33.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $\epsilon\in\,]0,\epsilon_1[$. Let $g\in L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$. Then there exists a unique function $u\in H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ such that*

$$(1-\Delta_{N,\mathbb{I}})u=g.$$

*Hence, $(1-\Delta_{N,\mathbb{I}})$ is invertible, and $(1-\Delta_{N,\mathbb{I}})^{(-1)}$ is a linear and continuous map of $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ to $H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ (and thus a compact operator of $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ to itself.)*

*Proof.* The invertibility of $(1-\Delta_{N,\mathbb{I}})$ and the continuity of $(1-\Delta_{N,\mathbb{I}})^{(-1)}$ as a map of $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ to $H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ is an immediate consequence of the Lax-Milgram Lemma. The compactness of $(1-\Delta_{N,\mathbb{I}})^{(-1)}$ as a map of $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ to $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ is a consequence of Proposition 7.31. □

**Proposition 7.34.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $\epsilon\in\,]0,\epsilon_1[$. Then the spectrum $\sigma(\Delta_{N,\mathbb{I}})$ of $\Delta_{N,\mathbb{I}}$ is a subset of $]-\infty,0]$ and consists of a sequence of eigenvalues of finite multiplicity, accumulating only at $-\infty$. In particular $0\in\sigma(\Delta_{N,\mathbb{I}})$. Moreover, $L^2_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ has a complete orthonormal set of eigenfunctions of $\Delta_{N,\mathbb{I}}$.*

*Proof.* We slightly modify the proof of Proposition 7.18. Clearly, it suffices to study $\sigma(-\Delta_{N,\mathbb{I}})$. By Lemma 7.33 and Davies [45, Theorem 4.3.1, p. 78, Corollary 4.2.3, p. 77], we have that the non-negative self-adjoint operator $-\Delta_{N,\mathbb{I}}$ has empty essential spectrum and that there exists a complete orthonormal set of eigenvectors $\{\phi_l\}_{l=1}^{\infty}$ of $-\Delta_{N,\mathbb{I}}$ with corresponding eigenvalues $\lambda_l\geq 0$ which converge to $+\infty$ as $l\to\infty$. In particular, an easy computation shows that $0\in\sigma(\Delta_{N,\mathbb{I}})$, with corresponding eigenspace given by the set of constant functions. Hence the conclusion easily follows. □

*Remark* 7.35. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $\mathbb{I}$ be as in (1.46). It can be proved that, if $-\lambda\in\mathrm{Eig}^a_N[\mathbb{T}_a[\mathbb{I}]]$, then $\lambda\in\sigma(\Delta_{N,\mathbb{I}})$. Indeed, if $-\lambda\in\mathrm{Eig}^a_N[\mathbb{T}_a[\mathbb{I}]]$, then there exists a function $u\in C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C})\cap C^2(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$, $u$ not identically zero, such that

$$\begin{cases}\Delta u(x)-\lambda u(x)=0 & \forall x\in\mathbb{T}_a[\mathbb{I}],\\ u(x+a_j)=u(x) & \forall x\in\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\quad \forall j\in\{1,\dots,n\},\\ \frac{\partial}{\partial\nu_{\mathbb{I}}}u(x)=0 & \forall x\in\partial\mathbb{I}.\end{cases}$$

Then a standard argument based on the Divergence Theorem implies that $\lambda\in\sigma(\Delta_{N,\mathbb{I}})$. In fact, if $\phi\in C^\infty_a(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C})$, then, by the Divergence Theorem and the periodicity of $u$ and $\phi$, we have,

$$\begin{aligned}\int_{\mathbb{P}_a[\mathbb{I}]}\nabla u(x)\cdot\overline{\nabla\phi(x)}\,dx &= -\int_{\mathbb{P}_a[\mathbb{I}]}\Delta u(x)\overline{\phi(x)}\,dx-\int_{\partial\mathbb{I}}\frac{\partial u(x)}{\partial\nu_{\mathbb{I}}}\overline{\phi(x)}\,d\sigma_x\\ &= -\lambda\int_{\mathbb{P}_a[\mathbb{I}]}u(x)\overline{\phi(x)}\,dx.\end{aligned}$$

Accordingly,

$$-\int_{\mathbb{P}_a[\mathbb{I}]}\nabla u(x)\cdot\overline{\nabla\phi(x)}\,dx=\lambda\int_{\mathbb{P}_a[\mathbb{I}]}u(x)\overline{\phi(x)}\,dx \qquad \forall\phi\in C^\infty_a(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C}),$$

and thus by density (cf. Lemma 7.32)

$$\mathbf{b}_{N,\mathbb{I}}(u,\phi) = \lambda \int_{\mathbb{P}_a[\mathbb{I}]} u(x)\overline{\phi(x)}\,dx \qquad \forall \phi \in H^1_a(\mathbb{T}_a[\mathbb{I}],\mathbb{C}).$$

Hence $\lambda \in \sigma(\Delta_{N,\mathbb{I}})$.

Moreover, $\sigma(\Delta_{N,\mathbb{I}})$ can be rearranged into the following sequence of non-positive real numbers

$$0 = \lambda_1(\Delta_{N,\mathbb{I}}) \geq \lambda_2(\Delta_{N,\mathbb{I}}) \geq \lambda_3(\Delta_{N,\mathbb{I}}) \geq \cdots \geq \lambda_j(\Delta_{N,\mathbb{I}}) \geq \dots,$$

where each eigenvalue is repeated as many times as its multiplicity.

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$ be as in (1.56). Let $\epsilon_1$ be as in (1.57). If $\epsilon \in ]0,\epsilon_1[$, then, in order to simplify the notation, we set

$$\mathbf{b}_{N,\epsilon} \equiv \mathbf{b}_{N,\Omega_\epsilon},$$

and

$$\Delta_{N,\epsilon} \equiv \Delta_{N,\Omega_\epsilon}.$$

Let $j \in \mathbb{N}\setminus\{0\}$. Then, for all $\epsilon \in ]0,\epsilon_1[$, we can consider $\lambda_j(\Delta_{N,\epsilon})$. In particular, we are interested in the limit of $\lambda_j(\Delta_{N,\epsilon})$ as $\epsilon$ tends to 0 in $]0,\epsilon_1[$.

For the behaviour of the Neumann eigenvalues of the Laplace operator in an open set of $\mathbb{R}^n$ with a small hole, we mention Rauch and Taylor [117], Ozawa [110], Ozawa [112], Maz'ya, Nazarov and Plamenewskij [91, Chapter 9], Hempel [59], Lanza [79].

As for the Dirichlet eigenvalues, in order to study the convergence of $\lambda_j(\Delta_{N,\epsilon})$ as $\epsilon \to 0^+$, we follow Rauch and Taylor [117, Section 3] (cf. also Ortega, San Martin and Smaranda [107, p. 977, 978].) We have the following technical lemma (cf. Rauch and Taylor [117, p. 38, 40].)

**Lemma 7.36.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$ be as in* (1.56). *Let $\epsilon_1$ be as in* (1.57). *Then there exist $\bar\epsilon_1 \in ]0,\epsilon_1[$ and a family of operators $\{\mathbf{E}_{1,\epsilon}\}_{\epsilon\in]0,\bar\epsilon_1[}$ such that, for each $\epsilon \in ]0,\bar\epsilon_1[$, $\mathbf{E}_{1,\epsilon}$ is a continuous extension map of $H^1_a(\mathbb{T}_a[\Omega_\epsilon],\mathbb{C})$ in $H^1_a(\mathbb{R}^n,\mathbb{C})$ (i.e., $(\mathbf{E}_{1,\epsilon}u)_{|\mathbb{T}_a[\Omega_\epsilon]} = u$ for all $u \in H^1_a(\mathbb{T}_a[\Omega_\epsilon],\mathbb{C})$) and such that there exists a constant $M > 0$ such that*

$$\|\mathbf{E}_{1,\epsilon}u\|_{H^1_a(\mathbb{R}^n,\mathbb{C})} \leq M\|u\|_{H^1_a(\mathbb{T}_a[\Omega_\epsilon],\mathbb{C})} \qquad \forall u \in H^1_a(\mathbb{T}_a[\Omega_\epsilon],\mathbb{C}),$$

*for all $\epsilon \in ]0,\bar\epsilon_1[$.*

*Proof.* We follow Rauch and Taylor [117, Example 1, p. 40]. Let $R > 0$ be such that $\mathrm{cl}\,\Omega \subseteq \mathbb{B}_n(0,R)$. Let $\bar\epsilon_1 \in ]0,\epsilon_1[$ be such that $\mathrm{cl}\,\mathbb{B}_n(w,\bar\epsilon_1 R) \subseteq A$. Let $\epsilon \in ]0,\bar\epsilon_1]$. If $u \in H^1_a(\mathbb{T}_a[\Omega_\epsilon],\mathbb{C})$ then we extend $u$ to $\mathbf{E}_{1,\epsilon}u \in H^1_a(\mathbb{R}^n,\mathbb{C})$ by setting

$$\begin{aligned} &\mathbf{E}_{1,\epsilon}u \equiv u && \text{in } \mathbb{T}_a[\Omega_\epsilon],\\ &\mathbf{E}_{1,\epsilon}u \equiv v_z && \text{in } \mathrm{cl}\,\Omega_\epsilon + a(z),\ \forall z \in \mathbb{Z}^n, \end{aligned}$$

where, if $z \in \mathbb{Z}^n$, then $v_z$ is the unique harmonic function inside $\Omega_\epsilon + a(z)$ which agrees with $u$ on $\partial\Omega_\epsilon + a(z)$. By Tartar [132, Lemma 14.4, p. 70], it is easy to see that if $u \in H^1_a(\mathbb{T}_a[\Omega_\epsilon],\mathbb{C})$ then $\mathbf{E}_{1,\epsilon}u \in H^1_a(\mathbb{R}^n,\mathbb{C})$.

If $\epsilon \in ]0,\bar\epsilon_1]$, we set

$$\mathcal{O}_\epsilon \equiv \mathbb{B}_n(w,\epsilon R)\setminus \mathrm{cl}\,\Omega_\epsilon.$$

By arguing as in Rauch and Taylor [117, Examples 1, 2, p. 40, 41], we observe that there exist two positive constants $C$, $C'$, such that

$$\|\mathbf{E}_{1,\bar\epsilon_1}u\|^2_{L^2(\Omega_{\bar\epsilon_1},\mathbb{C})} \leq C\|u\|^2_{L^2(\mathcal{O}_{\bar\epsilon_1},\mathbb{C})} + C\|\nabla u\|^2_{L^2(\mathcal{O}_{\bar\epsilon_1},\mathbb{C}^n)}, \tag{7.5}$$

$$\|\nabla\mathbf{E}_{1,\bar\epsilon_1}u\|^2_{L^2(\Omega_{\bar\epsilon_1},\mathbb{C}^n)} \leq C'\|\nabla u\|^2_{L^2(\mathcal{O}_{\bar\epsilon_1},\mathbb{C}^n)}, \tag{7.6}$$

for all $u \in H^1_a(\mathbb{T}_a[\Omega_{\bar\epsilon_1}],\mathbb{C})$. Indeed, inequality (7.5) follows by the Trace Theorem (cf. *e.g.*, Burenkov [18, Theorem 8, p. 241]) and standard elliptic theory (cf. *e.g.*, Gilbarg and Trudinger [55, Corollary 8.7, p. 183].) Analogously, there exists a positive constant $\tilde C$ such that

$$\|\nabla\mathbf{E}_{1,\bar\epsilon_1}u\|^2_{L^2(\Omega_{\bar\epsilon_1},\mathbb{C}^n)} \leq \tilde C\|u\|^2_{H^1(\mathcal{O}_{\bar\epsilon_1},\mathbb{C})},$$

for all $u \in H^1_a(\mathbb{T}_a[\Omega_{\bar{\epsilon}_1}], \mathbb{C})$. In order to prove inequality (7.6), we observe that if it were false, then there would exists a sequence $\{u_l\}_{l=1}^{\infty} \subseteq H^1_a(\mathbb{T}_a[\Omega_{\bar{\epsilon}_1}], \mathbb{C})$, such that

$$\begin{aligned} &\|\nabla u_l\|_{L^2(\mathcal{O}_{\bar{\epsilon}_1}, \mathbb{C}^n)} \leq \frac{1}{l} && \forall l \geq 1, \\ &\|\nabla \mathbf{E}_{1,\bar{\epsilon}_1} u_l\|_{L^2(\Omega_{\bar{\epsilon}_1}, \mathbb{C}^n)} \geq 1 && \forall l \geq 1. \end{aligned} \tag{7.7}$$

By taking $\mu_l \equiv \frac{1}{|\mathcal{O}_{\bar{\epsilon}_1}|_n} \int_{\mathcal{O}_{\bar{\epsilon}_1}} u_l(x)\, dx$, then, by Poincaré's inequality (cf. *e.g.*, Evans [50, Theorem 1, p. 275]), we have that there exists a constant $c$ (independent of $l$), such that

$$\|u_l - \mu_l\|_{H^1(\mathcal{O}_{\bar{\epsilon}_1}, \mathbb{C})} \leq \frac{c}{l} \qquad \forall l \geq 1.$$

Then, since $\mathbf{E}_{1,\bar{\epsilon}_1}(u_l - \mu_l) = (\mathbf{E}_{1,\bar{\epsilon}_1} u_l) - \mu_l$, we have

$$\begin{aligned} \|\nabla \mathbf{E}_{1,\bar{\epsilon}_1} u_l\|_{L^2(\Omega_{\bar{\epsilon}_1}, \mathbb{C}^n)} &= \|\nabla \mathbf{E}_{1,\bar{\epsilon}_1}(u_l - \mu_l)\|_{L^2(\Omega_{\bar{\epsilon}_1}, \mathbb{C}^n)} \\ &\leq \|\mathbf{E}_{1,\bar{\epsilon}_1}(u_l - \mu_l)\|_{H^1(\Omega_{\bar{\epsilon}_1}, \mathbb{C})} \\ &\leq \sqrt{C + \tilde{C}} \|u_l - \mu_l\|_{H^1(\mathcal{O}_{\bar{\epsilon}_1}, \mathbb{C})} \\ &\leq \frac{c\sqrt{C + \tilde{C}}}{l} \qquad \forall l \geq 1, \end{aligned}$$

a contradiction.

Then, by inequalities (7.5) and (7.6), a simple scaling argument shows that

$$\begin{aligned} &\|\mathbf{E}_{1,\epsilon} u\|^2_{L^2(\Omega_\epsilon, \mathbb{C})} \leq C \|u\|^2_{L^2(\mathcal{O}_\epsilon, \mathbb{C})} + C\Big(\frac{\epsilon}{\bar{\epsilon}_1}\Big)^2 \|\nabla u\|^2_{L^2(\mathcal{O}_\epsilon, \mathbb{C}^n)}, \\ &\|\nabla \mathbf{E}_{1,\epsilon} u\|^2_{L^2(\Omega_\epsilon, \mathbb{C}^n)} \leq C' \|\nabla u\|^2_{L^2(\mathcal{O}_\epsilon, \mathbb{C}^n)}, \end{aligned}$$

for all $u \in H^1_a(\mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$ and for all $\epsilon \in ]0, \bar{\epsilon}_1]$. As a consequence

$$\begin{aligned} &\|\mathbf{E}_{1,\epsilon} u\|^2_{L^2(A, \mathbb{C})} \leq (C+1) \|u\|^2_{L^2(\mathbb{P}[\Omega_\epsilon], \mathbb{C})} + C \|\nabla u\|^2_{L^2(\mathbb{P}[\Omega_\epsilon], \mathbb{C}^n)}, \\ &\|\nabla \mathbf{E}_{1,\epsilon} u\|^2_{L^2(A, \mathbb{C}^n)} \leq (C'+1) \|\nabla u\|^2_{L^2(\mathbb{P}[\Omega_\epsilon], \mathbb{C}^n)}, \end{aligned}$$

for all $u \in H^1_a(\mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$ and for all $\epsilon \in ]0, \bar{\epsilon}_1]$. Hence the conclusion easily follows. □

Then we have the following variant of Rauch and Taylor [117, Theorem 3.1, p. 38].

**Proposition 7.37.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$ *be as in* (1.56), (1.57), *respectively. Let* $\bar{\epsilon}_1$ *be as in Lemma 7.36. Let* $\{\epsilon_l\}_{l=1}^{\infty}$ *be a sequence in* $]0, \bar{\epsilon}_1[$, *convergent to* $0$. *Let* $u \in L^2_a(\mathbb{R}^n, \mathbb{C})$. *Then there exists a subsequence* $\{\epsilon_{l_j}\}_{j=1}^{\infty}$, *such that*

$$\lim_{j \to \infty} \|\mathbf{E}_{0,\epsilon_{l_j}} (1 - \Delta_{N,\epsilon_{l_j}})^{(-1)} \mathbf{P}_{\epsilon_{l_j}} u - (1 - \Delta)^{(-1)} u\|_{L^2_a(\mathbb{R}^n, \mathbb{C})} = 0.$$

*Proof.* It suffices to modify the proof of Rauch and Taylor [117, Theorem 3.1, p. 38]. For each $l \geq 1$ we set

$$v_l \equiv \mathbf{E}_{0,\epsilon_l} (1 - \Delta_{N,\epsilon_l})^{(-1)} \mathbf{P}_{\epsilon_l} u.$$

We also set

$$v_l^\# \equiv (1 - \Delta_{N,\epsilon_l})^{(-1)} \mathbf{P}_{\epsilon_l} u, \qquad \forall l \geq 1.$$

Let $l \geq 1$. We have

$$\begin{aligned} \|v_l^\#\|^2_{H^1_a(\mathbb{T}_a[\Omega_{\epsilon_l}], \mathbb{C})} &= \|v_l^\#\|^2_{L^2_a(\mathbb{T}_a[\Omega_{\epsilon_l}], \mathbb{C})} + \int_{\mathbb{P}_a[\Omega_{\epsilon_l}]} |\nabla v_l^\#(x)|^2\, dx \\ &= \int_{\mathbb{P}_a[\Omega_{\epsilon_l}]} \mathbf{P}_{\epsilon_l} u(x) \overline{v_l^\#(x)}\, dx. \end{aligned}$$

Hence there exists a constant $C > 0$ such that

$$\|v_l^\#\|_{H^1_a(\mathbb{T}_a[\Omega_{\epsilon_l}], \mathbb{C})} \leq C \qquad \forall l \geq 1.$$

Thus if we set $\tilde{v}_l \equiv \mathbf{E}_{1,\epsilon_l}(1-\Delta_{N,\epsilon_l})^{(-1)}\mathbf{P}_{\epsilon_l}u = \mathbf{E}_{1,\epsilon_l}v_l^{\#}$ for all $l \geq 1$, then, by Lemma 7.36, we have that $\{\tilde{v}_l\}_{l=1}^{\infty}$ is a bounded sequence in $H^1_a(\mathbb{R}^n, \mathbb{C})$.

First of all, we verify that, possibly considering a subsequence, $\tilde{v}_l$ converges to $(1-\Delta)^{(-1)}u$ weakly in $H^1_a(\mathbb{R}^n, \mathbb{C})$. Let $\tilde{v}$ be a weak limit point of $\{\tilde{v}_l\}_{l=1}^{\infty}$. Possibly relabeling the sequence, we may assume that

$$\tilde{v}_l \rightharpoonup \tilde{v} \qquad \text{in } H^1_a(\mathbb{R}^n, \mathbb{C}).$$

Let $\phi \in H^1_a(\mathbb{R}^n, \mathbb{C})$. Then

$$\begin{aligned}\int_A \nabla\tilde{v}(x)\cdot\overline{\nabla\phi(x)}\,dx &= \lim_{l\to\infty}\int_A \nabla\tilde{v}_l(x)\cdot\overline{\nabla\phi(x)}\,dx \\ &= \lim_{l\to\infty}\Big[\int_{\mathbb{P}_a[\Omega_{\epsilon_l}]}\nabla v_l^{\#}(x)\cdot\overline{\nabla\phi(x)}\,dx + \int_{\Omega_{\epsilon_l}}\nabla\tilde{v}_l(x)\cdot\overline{\nabla\phi(x)}\,dx\Big].\end{aligned}$$

Now

$$\int_{\mathbb{P}_a[\Omega_{\epsilon_l}]}\nabla v_l^{\#}(x)\cdot\overline{\nabla\phi(x)}\,dx = -\int_{\mathbb{P}_a[\Omega_{\epsilon_l}]} v_l^{\#}(x)\overline{\phi(x)}\,dx + \int_{\mathbb{P}_a[\Omega_{\epsilon_l}]}\mathbf{P}_{\epsilon_l}u(x)\overline{\phi(x)}\,dx,$$

and

$$\begin{aligned}&\lim_{l\to\infty}\Big[-\int_{\mathbb{P}_a[\Omega_{\epsilon_l}]} v_l^{\#}(x)\overline{\phi(x)}\,dx + \int_{\mathbb{P}_a[\Omega_{\epsilon_l}]}\mathbf{P}_{\epsilon_l}u(x)\overline{\phi(x)}\,dx\Big] \\ &\qquad\qquad = -\int_A \tilde{v}(x)\overline{\phi(x)}\,dx + \int_A u(x)\overline{\phi(x)}\,dx.\end{aligned}$$

On the other hand

$$|\int_{\Omega_{\epsilon_l}}\nabla\tilde{v}_l(x)\cdot\overline{\nabla\phi(x)}\,dx| \leq \|\tilde{v}_l\|_{H^1_a(\mathbb{R}^n,\mathbb{C})}\|\phi\|_{H^1(\Omega_{\epsilon_l},\mathbb{C})}.$$

Clearly,

$$\lim_{l\to\infty}\|\tilde{v}_l\|_{H^1_a(\mathbb{R}^n,\mathbb{C})}\|\phi\|_{H^1(\Omega_{\epsilon_l},\mathbb{C})} = 0.$$

Accordingly

$$\int_A \nabla\tilde{v}(x)\cdot\overline{\nabla\phi(x)}\,dx = -\int_A \tilde{v}(x)\overline{\phi(x)}\,dx + \int_A u(x)\overline{\phi(x)}\,dx \qquad \forall\phi\in H^1_a(\mathbb{R}^n,\mathbb{C}),$$

and thus

$$\tilde{v} = (1-\Delta)^{(-1)}u.$$

Then, Proposition 7.13 implies that there exists a subsequence $\{\tilde{v}_{l_j}\}_{j=1}^{\infty}$ such that $\tilde{v}_{l_j} \to \tilde{v}$ in $L^2_a(\mathbb{R}^n, \mathbb{C})$. Finally, in order to prove that $v_{l_j} \to \tilde{v}$, it suffices to observe that

$$\begin{aligned}\|v_{l_j} - \tilde{v}_{l_j}\|^2_{L^2_a(\mathbb{R}^n,\mathbb{C})} &= \int_{\Omega_{\epsilon_{l_j}}} |\tilde{v}_{l_j}(x)|^2\,dx \\ &\leq 2\int_{\Omega_{\epsilon_{l_j}}}|\tilde{v}(x)|^2\,dx + 2\int_{\Omega_{\epsilon_{l_j}}}|\tilde{v}(x)-\tilde{v}_{l_j}(x)|^2\,dx \to 0,\end{aligned}$$

as $j\to\infty$. □

Then we have the following Theorem.

**Theorem 7.38.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $\bar{\epsilon}_1$ be as in Lemma 7.36. Let $\{\epsilon_l\}_{l=1}^{\infty}$ be a sequence in $]0,\bar{\epsilon}_1[$, convergent to 0. Let $F$ be a bounded Borel function on $]-\infty,1/2]$ which is continuous on a neighbourhood of $\sigma(\Delta)$. Let $u \in L^2_a(\mathbb{R}^n,\mathbb{C})$. Then there exists a subsequence $\{\epsilon_{l_j}\}_{j=1}^{\infty}$, such that*

$$\lim_{j\to\infty}\|\mathbf{E}_{0,\epsilon_{l_j}}F(\Delta_{N,\epsilon_{l_j}})\mathbf{P}_{\epsilon_{l_j}}u - F(\Delta)u\|_{L^2_a(\mathbb{R}^n,\mathbb{C})} = 0.$$

*Proof.* It suffices to follow the proof of Theorem 7.24, with Proposition 7.23 replaced by Proposition 7.37 (cf. Rauch and Taylor [117, Theorem 3.1, p. 38].) □

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in (1.56), (1.57), respectively. Let $J \subseteq \mathbb{R}$ be a bounded open interval whose endpoints do not belong to $\sigma(\Delta)$. Then, as in the previous Subsection, we denote by $\Pi^J$ the spectral projection of $\Delta$ on $J$. Then $\operatorname{rank}\Pi^J \equiv \dim(\operatorname{range}\Pi^J)$ is the number of eigenvalues of $\Delta$ in $J$. Similarly, if $\epsilon \in ]0,\epsilon_1[$, then we denote by $\Pi^J_{N,\epsilon}$ the spectral projection of $\Delta_{N,\epsilon}$ on $J$. Then $\operatorname{rank}\Pi^J_{N,\epsilon} \equiv \dim(\operatorname{range}\Pi^J_{N,\epsilon})$ is the number of eigenvalues of $\Delta_{N,\epsilon}$ in $J$.

We have the following Proposition.

**Proposition 7.39.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$ be as in* (1.56), (1.57), *respectively. Let $\bar{\epsilon}_1$ be as in Lemma 7.36. Let $J \subseteq \mathbb{R}$ be a bounded open interval whose endpoints do not belong to $\sigma(\Delta)$. Then there exists $\epsilon_2 \in ]0,\bar{\epsilon}_1[$ such that*

$$\operatorname{rank}\Pi^J_{N,\epsilon} = \operatorname{rank}\Pi^J,$$

*for all $\epsilon \in ]0,\epsilon_2[$.*

*Proof.* We proceed as in Rauch and Taylor [117, Theorem 1.5] and we modify the proof of Proposition 7.25. First of all, we observe that

$$\operatorname{rank}\Pi^J_{N,\epsilon} = \operatorname{rank}\mathbf{E}_{0,\epsilon}\Pi^J_{N,\epsilon}, \qquad \operatorname{rank}\Pi^J_{N,\epsilon}\mathbf{P}_\epsilon = \operatorname{rank}\mathbf{E}_{0,\epsilon}\Pi^J_{N,\epsilon}\mathbf{P}_\epsilon,$$

for all $\epsilon \in ]0,\bar{\epsilon}_1[$. Then the proof consists of three steps: for all $\epsilon \in ]0,\epsilon_2[$, with $\epsilon_2 \in ]0,\bar{\epsilon}_1[$ small enough, we have

(*i*) $\operatorname{rank}(\mathbf{E}_{0,\epsilon}\Pi^J_{N,\epsilon}\mathbf{P}_\epsilon) \geq \operatorname{rank}\Pi^J$,

(*ii*) $\operatorname{rank}(\mathbf{E}_{0,\epsilon}\Pi^J_{N,\epsilon}) \leq \operatorname{rank}\Pi^J$,

(*iii*) $\operatorname{range}\Pi^J_{N,\epsilon} = \operatorname{range}\Pi^J_{N,\epsilon}\mathbf{P}_\epsilon$.

We first prove (*i*). If it were not true, than there would exist a sequence $\{\epsilon_l\}_{l=1}^\infty$ in $]0,\bar{\epsilon}_1[$ convergent to 0, such that

$$\operatorname{rank}(\mathbf{E}_{0,\epsilon_l}\Pi^J_{N,\epsilon_l}\mathbf{P}_{\epsilon_l}) < \operatorname{rank}\Pi^J, \qquad \forall l \geq 1.$$

Let $\{u_1,\dots,u_k\}$ be an orthonormal basis of the range of $\Pi^J$. Then, by Theorem 7.38, up to subsequences,

$$\lim_{l\to\infty}\|\mathbf{E}_{0,\epsilon_l}\Pi^J_{N,\epsilon_l}\mathbf{P}_{\epsilon_l}u_j - \Pi^J u_j\|_{L^2_a(\mathbb{R}^n,\mathbb{C})} = 0,$$

for all $j \in \{1,\dots,k\}$. It follows that there exists $\bar{l} \in \mathbb{N}$, such that $\{\mathbf{E}_{0,\epsilon_l}\Pi^J_{N,\epsilon_l}\mathbf{P}_{\epsilon_l}u_j\}_{j=1}^k$ is a linear independent set for all $l \geq \bar{l}$, a contradiction.

We now consider (*ii*). If it were not true, than there would exist a sequence $\{\epsilon_l\}_{l=1}^\infty$ in $]0,\bar{\epsilon}_1[$ convergent to 0, such that

$$\dim\operatorname{range}(\mathbf{E}_{0,\epsilon_l}\Pi^J_{N,\epsilon_l}) > \dim\operatorname{range}\Pi^J, \qquad \forall l \geq 1.$$

For each $l \geq 1$, choose $v_l \in \operatorname{range}\Pi^J_{N,\epsilon_l}$ such that $\|v_l\|_{L^2_a(\mathbb{T}_a[\Omega_{\epsilon_l}],\mathbb{C})} = 1$ and $\mathbf{E}_{0,\epsilon_l}v_l \perp \operatorname{range}\Pi^J$. Then $\operatorname{range}\Pi^J_{N,\epsilon_l} \subseteq H^1_a(\mathbb{T}_a[\Omega_{\epsilon_l}],\mathbb{C})$, and

$$\left|\int_{\mathbb{P}_a[\Omega_{\epsilon_l}]}(\Delta_{N,\epsilon_l}v_l(x))\overline{v_l(x)}\,dx\right| \leq M\|v_l\|^2_{L^2_a(\mathbb{T}_a[\Omega_{\epsilon_l}],\mathbb{C})}, \qquad \forall l \geq 1,$$

with $M \equiv \sup_{x\in J}|x|$. Accordingly $\{\mathbf{E}_{1,\epsilon_l}v_l\}_{l=1}^\infty$ is a bounded sequence in $H^1_a(\mathbb{R}^n,\mathbb{C})$, and then, by Corollary 7.14, there exists a subsequence $\{\mathbf{E}_{1,\epsilon_{l_j}}v_{l_j}\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty}\mathbf{E}_{1,\epsilon_{l_j}}v_{l_j} = v \qquad \text{in } L^2_a(\mathbb{R}^n,\mathbb{C}),$$

for some $v \in H^1_a(\mathbb{R}^n,\mathbb{C})$. Moreover,

$$\begin{aligned}\|\mathbf{E}_{0,\epsilon_{l_j}}v_{l_j} - \mathbf{E}_{1,\epsilon_{l_j}}v_{l_j}\|^2_{L^2_a(\mathbb{R}^n,\mathbb{C})} &= \int_{\Omega_{\epsilon_{l_j}}}|\mathbf{E}_{1,\epsilon_{l_j}}v_{l_j}(x)|^2\,dx \\ &\leq 2\int_{\Omega_{\epsilon_{l_j}}}|v(x)|^2\,dx + 2\int_{\Omega_{\epsilon_{l_j}}}|v(x) - \mathbf{E}_{1,\epsilon_{l_j}}v_{l_j}(x)|^2\,dx \to 0,\end{aligned}$$

as $j \to \infty$. Thus

$$\lim_{j\to\infty} \mathbf{E}_{0,\epsilon_{l_j}} v_{l_j} = v \qquad \text{in } L^2_a(\mathbb{R}^n, \mathbb{C}),$$

and $\|v\|_{L^2_a(\mathbb{R}^n,\mathbb{C})} = 1$, and $v \perp \operatorname{range} \Pi^J$. Now we show that $v \in \operatorname{range} \Pi^J$, a contradiction. Indeed, Theorem 7.38 implies that, up to subsequences,

$$\lim_{j\to\infty} \mathbf{E}_{0,\epsilon_{l_j}} \Pi^J_{N,\epsilon_{l_j}} \mathbf{P}_{\epsilon_{l_j}} v = \Pi^J v \qquad \text{in } L^2_a(\mathbb{R}^n, \mathbb{C}).$$

Moreover,

$$\begin{aligned}\|\mathbf{E}_{0,\epsilon_{l_j}} \Pi^J_{N,\epsilon_{l_j}} v_{l_j} - \mathbf{E}_{0,\epsilon_{l_j}} \Pi^J_{N,\epsilon_{l_j}} \mathbf{P}_{\epsilon_{l_j}} v\|_{L^2_a(\mathbb{R}^n,\mathbb{C})} &\le \|v_{l_j} - \mathbf{P}_{\epsilon_{l_j}} v\|_{L^2_a(\mathbb{T}_a[\Omega_{\epsilon_{l_j}}],\mathbb{C})} \\ &\le \|\mathbf{E}_{0,\epsilon_{l_j}} v_{l_j} - v\|_{L^2_a(\mathbb{R}^n,\mathbb{C})},\end{aligned}$$

and thus

$$\lim_{j\to\infty} \mathbf{E}_{0,\epsilon_{l_j}} \Pi^J_{N,\epsilon_{l_j}} v_{l_j} = \Pi^J v \qquad \text{in } L^2_a(\mathbb{R}^n, \mathbb{C}).$$

On the other hand, $\mathbf{E}_{0,\epsilon_{l_j}} \Pi^J_{N,\epsilon_{l_j}} v_{l_j} = \mathbf{E}_{0,\epsilon_{l_j}} v_{l_j}$, and

$$\lim_{j\to\infty} \mathbf{E}_{0,\epsilon_{l_j}} v_{l_j} = v \qquad \text{in } L^2_a(\mathbb{R}^n, \mathbb{C}).$$

Thus $v = \Pi^J v$, and so $v \in \operatorname{range} \Pi^J$.

We finally prove (*iii*). If it were false for some $\epsilon \in ]0, \bar{\epsilon}_1[$, then there would exist a non-zero $v \in \operatorname{range} \Pi^J_{N,\epsilon}$, with $v \perp \operatorname{range} \Pi^J_{N,\epsilon} \mathbf{P}_\epsilon$. Thus for all $u \in L^2_a(\mathbb{R}^n, \mathbb{C})$, we would have

$$0 = \int_{\mathbb{P}_a[\Omega_\epsilon]} \Pi^J_{N,\epsilon} \mathbf{P}_\epsilon u(x) \overline{v(x)}\, dx = \int_{\mathbb{P}_a[\Omega_\epsilon]} \mathbf{P}_\epsilon u(x) \overline{\Pi^J_{N,\epsilon} v(x)}\, dx = \int_{\mathbb{P}_a[\Omega_\epsilon]} \mathbf{P}_\epsilon u(x) \overline{v(x)}\, dx.$$

As a consequence, $v = 0$, a contradiction.

Hence the proof is complete. □

Then we have the following result.

**Theorem 7.40.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$ be as in* (1.56). *Let $\epsilon_1$ be as in* (1.57). *Let $j \in \mathbb{N} \setminus \{0\}$. Then*

$$\lambda_j(\Delta_{N,\epsilon}) \to \lambda_j(\Delta),$$

*as $\epsilon$ tends to 0 in $]0, \epsilon_1[$.*

*Proof.* It is a straightforward consequence of Proposition 7.39. □

**Corollary 7.41.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$ be as in* (1.56). *Let $\epsilon_1$ be as in* (1.57). *Let $k \in \mathbb{C}$ be such that $k^2 \neq |2\pi a^{-1}(z)|^2$ for all $z \in \mathbb{Z}^n$. Then there exists $\epsilon^a_N \in ]0, \epsilon_1[$, such that*

$$k^2 \notin \operatorname{Eig}^a_N[\mathbb{T}_a[\Omega_\epsilon]] \qquad \forall \epsilon \in ]0, \epsilon^a_N].$$

*Proof.* It is an immediate consequence of Theorem 7.40, of Remarks 7.35, 7.22, and of the results of Section 6.8. □

## 7.3 A remark on the results of the previous Sections

In this Section we present an immediate consequence of the results of the previous Sections.

**Proposition 7.42.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$ be as in* (1.56). *Let $\epsilon_1$ be as in* (1.57). *Let $k \in \mathbb{C}$ be such that $k^2 \neq |2\pi a^{-1}(z)|^2$ for all $z \in \mathbb{Z}^n$. Then there exists $\epsilon^*_1 \in ]0, \epsilon_1]$ such that*

$$k^2 \notin \Big(\operatorname{Eig}_D[\Omega_\epsilon] \cup \operatorname{Eig}_N[\Omega_\epsilon] \cup \operatorname{Eig}^a_D[\mathbb{T}_a[\Omega_\epsilon]] \cup \operatorname{Eig}^a_N[\mathbb{T}_a[\Omega_\epsilon]]\Big) \qquad \forall \epsilon \in ]0, \epsilon^*_1].$$

*Proof.* It is a straightforward consequence of Corollaries 7.4, 7.8, 7.27, 7.41. □

*Remark* 7.43. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$ be as in (1.56). Let $\epsilon_1$ be as in (1.57). Let $k \in \mathbb{C}$ be such that $\operatorname{Im}(k) \neq 0$. Then we have

$$k^2 \notin \Big(\operatorname{Eig}_D[\Omega_\epsilon] \cup \operatorname{Eig}_N[\Omega_\epsilon] \cup \operatorname{Eig}^a_D[\mathbb{T}_a[\Omega_\epsilon]] \cup \operatorname{Eig}^a_N[\mathbb{T}_a[\Omega_\epsilon]]\Big) \qquad \forall \epsilon \in ]0, \epsilon_1].$$

As a consequence, with the notation of Proposition 7.42, if $\operatorname{Im}(k) \neq 0$, we can take $\epsilon^*_1 \equiv \epsilon_1$.

# CHAPTER 8

# Singular perturbation and homogenization problems for the Helmholtz equation with Neumann boundary conditions

In this Chapter we introduce the periodic Neumann problem for the Helmholtz equation and we study singular perturbation and homogenization problems for the Helmholtz operator with Neumann boundary conditions in a periodically perforated domain. First of all, by means of periodic simple layer potentials, we show the solvability of the Neumann problem. Secondly, we consider singular perturbation problems in a periodically perforated domain with small holes, and we apply the obtained results to homogenization problems. Our strategy follows the functional analytic approach of Lanza [75], where the asymptotic behaviour of the solutions of the Dirichlet problem for the Laplace operator in a domain with a small hole has been studied (see also [70].) We also mention Lanza [79], dealing with a Neumann eigenvalue problem in a perforated domain. We note that linear boundary value problems in singularly perturbed domains in the frame of linearized elasticity have been analysed by Dalla Riva in his Ph.D. Dissertation [33]. One of the tools used in our analysis is the study of the dependence of layer potentials upon perturbations (cf. Lanza and Rossi [86] and also Dalla Riva and Lanza [40].)

We retain the notation introduced in Sections 1.1 and 1.3, Chapter 6 and Appendix E. For the definitions of $\mathrm{Eig}_D[\mathbb{I}]$, $\mathrm{Eig}_N[\mathbb{I}]$, $\mathrm{Eig}^a_D[\mathbb{I}]$, $\mathrm{Eig}^a_N[\mathbb{I}]$, we refer to Chapter 7.

## 8.1 A periodic Neumann boundary value problem for the Helmholtz equation

In this Section we introduce the periodic Neumann problem for the Helmholtz equation and we show the existence and uniqueness of a solution by means of the periodic simple layer potential.

### 8.1.1 Formulation of the problem

In this Subsection we introduce the periodic Neumann problem for the Helmholtz equation.

First of all, we need to introduce some notation. Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). We shall consider the following assumptions.

$$k \in \mathbb{C},\ k^2 \neq |2\pi a^{-1}(z)|^2 \quad \forall z \in \mathbb{Z}^n; \tag{8.1}$$

$$\Gamma \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C}). \tag{8.2}$$

We are now ready to introduce the following.

**Definition 8.1.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $k$, $\Gamma$ be as in (8.1), (8.2), respectively. We say that a function $u \in C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \mathbb{C}) \cap C^2(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ solves the *periodic Neumann* *problem for the Helmholtz equation* if

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) = \Gamma(x) & \forall x \in \partial\mathbb{I}. \end{cases} \tag{8.3}$$

### 8.1.2 Existence and uniqueness results for the solutions of the periodic Neumann problem

In this Subsection we prove existence and uniqueness results for the solutions of the periodic Neumann problem for the Helmholtz equation.

As we know, in order to solve problem (8.3) by means of periodic simple layer potentials, we need to study some integral equations. Thus, in the following Proposition, we study an operator related to the equations that we shall consider in the sequel.

**Proposition 8.2.** *Let* $m \in \mathbb{N} \setminus \{0\}$. *Let* $\alpha \in ]0, 1[$. *Let* $\mathbb{I}$ *be as in* (1.46). *Let* $k$ *be as in* (8.1). *Assume that* $k^2 \notin \mathrm{Eig}_N^a[\mathbb{T}_a[\mathbb{I}]]$ *and* $k^2 \notin \mathrm{Eig}_D[\mathbb{I}]$. *Then the following statements hold.*

*(i)* *Let* $\mu \in L^2(\partial\mathbb{I}, \mathbb{C})$ *and*

$$\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}(t)} (S_n^{a,k}(t - s))\mu(s)\, d\sigma_s = 0 \quad a.e.\ on\ \partial\mathbb{I}, \tag{8.4}$$

*then* $\mu = 0$.

*(ii)* *Let* $\mu \in L^2(\partial\mathbb{I}, \mathbb{C})$ *and*

$$\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}(s)} (S_n^{a,k}(t - s))\mu(s)\, d\sigma_s = 0 \quad a.e.\ on\ \partial\mathbb{I}, \tag{8.5}$$

*then* $\mu = 0$.

*Proof.* We first prove statement $(i)$. By Theorem 6.18 $(iii)$, we have $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$. Then by Theorem 6.11 $(i)$, we have that the function $v^- \equiv v_a^-[\partial\mathbb{I}, \mu, k]$ is in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ and solves the following boundary value problem

$$\begin{cases} \Delta v^-(x) + k^2 v^-(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ v^-(x + a_j) = v^-(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} v^-(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

Accordingly, since $k^2 \notin \mathrm{Eig}_N^a[\mathbb{T}_a[\mathbb{I}]]$, we have $v^- = 0$ in $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$. Then, by Theorem 6.11 $(i)$, the function $v^+ \equiv v_a^+[\partial\mathbb{I}, \mu, k]_{|\mathrm{cl}\,\mathbb{I}}$ is in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I}, \mathbb{C})$ and solves the following boundary value problem

$$\begin{cases} \Delta v^+(x) + k^2 v^+(x) = 0 & \forall x \in \mathbb{I}, \\ v^+(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

Hence, since $k^2 \notin \mathrm{Eig}_D[\mathbb{I}]$, we have $v^+ = 0$ in $\mathrm{cl}\,\mathbb{I}$, and so

$$\frac{\partial}{\partial \nu_{\mathbb{I}}} v^+ = 0 \qquad \text{on } \partial\mathbb{I}.$$

Thus, by Theorem 6.11 $(i)$, we have

$$\mu = \frac{\partial}{\partial \nu_{\mathbb{I}}} v_a^-[\partial\mathbb{I}, \mu, k] - \frac{\partial}{\partial \nu_{\mathbb{I}}} v_a^+[\partial\mathbb{I}, \mu, k] = 0 \qquad \text{on } \partial\mathbb{I},$$

and the proof of $(i)$ is complete. We now turn to the proof of statement $(ii)$. By Theorem 6.18 $(i)$, we have $\mu \in C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$. Then by Theorem 6.7 $(i)$, we have that the function $w^+ \equiv w_a^+[\partial\mathbb{I}, \mu, k]_{|\mathrm{cl}\,\mathbb{I}}$ is in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I}, \mathbb{C})$ and solves the following boundary value problem

$$\begin{cases} \Delta w^+(x) + k^2 w^+(x) = 0 & \forall x \in \mathbb{I}, \\ w^+(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

Hence, since $k^2 \notin \mathrm{Eig}_D[\mathbb{I}]$, we have $w^+ = 0$ in $\mathrm{cl}\,\mathbb{I}$, and so

$$\frac{\partial}{\partial \nu_{\mathbb{I}}} w^+ = 0 \qquad \text{on } \partial\mathbb{I}.$$

Furthermore, by Theorem 6.7 $(i)$, we have

$$\frac{\partial}{\partial \nu_{\mathbb{I}}} w_a^-[\partial\mathbb{I},\mu,k] = \frac{\partial}{\partial \nu_{\mathbb{I}}} w_a^+[\partial\mathbb{I},\mu,k] = 0 \qquad \text{on } \partial\mathbb{I}.$$

Then by Theorem 6.7 $(i)$, we have that the function $w^- \equiv w_a^-[\partial\mathbb{I},\mu,k]$ is in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ and solves the following boundary value problem

$$\begin{cases} \Delta w^-(x) + k^2 w^-(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ w^-(x+a_j) = w^-(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} w^-(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

Accordingly, since $k^2 \notin \mathrm{Eig}_N^a[\mathbb{T}_a[\mathbb{I}]]$, we have $w^- = 0$ in $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$. Thus, by Theorem 6.7 $(i)$, we have

$$\mu = w_a^+[\partial\mathbb{I},\mu,k] - w_a^-[\partial\mathbb{I},\mu,k] = 0 \qquad \text{on } \partial\mathbb{I},$$

and the proof of $(ii)$ is complete. □

*Remark* 8.3. Let $m$, $\alpha$, $\mathbb{I}$, $k$ be as in Proposition 8.2. We observe that statement $(ii)$ of Proposition 8.2 can also be deduced by statement $(i)$. Indeed, set

$$\mathcal{V} \equiv \left\{ \mu \in L^2(\partial\mathbb{I},\mathbb{C}) \colon \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}(t)} (S_n^{a,k}(t-s))\mu(s)\,d\sigma_s = 0 \quad \text{a.e. on } \partial\mathbb{I} \right\},$$

$$\mathcal{W} \equiv \left\{ \mu \in L^2(\partial\mathbb{I},\mathbb{C}) \colon \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \overline{\frac{\partial}{\partial \nu_{\mathbb{I}}(s)} (S_n^{a,k}(t-s))}\mu(s)\,d\sigma_s = 0 \quad \text{a.e. on } \partial\mathbb{I} \right\},$$

and

$$\mathcal{W}' \equiv \left\{ \mu \in L^2(\partial\mathbb{I},\mathbb{C}) \colon \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}(s)} (S_n^{a,k}(t-s))\mu(s)\,d\sigma_s = 0 \quad \text{a.e. on } \partial\mathbb{I} \right\}.$$

By Proposition 8.2 $(i)$, we have $\mathcal{V} = \{0\}$. Consequently, by the Fredholm Theory, we have $\mathcal{W} = \{0\}$. On the other hand, one can easily show that the map of $\mathcal{W}$ to $\mathcal{W}'$ which takes $\phi$ to $\overline{\phi}$ is a bijection. As a consequence, $\mathcal{W}' = \{0\}$, and accordingly statement $(ii)$ holds.

Then we have the following Theorem.

**Theorem 8.4.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $k$ be as in* (8.1)*. Assume that $k^2 \notin \mathrm{Eig}_N^a[\mathbb{T}_a[\mathbb{I}]]$ and $k^2 \notin \mathrm{Eig}_D[\mathbb{I}]$. Then the following statements hold.*

*(i) The map $\mathcal{L}$ of $L^2(\partial\mathbb{I},\mathbb{C})$ to $L^2(\partial\mathbb{I},\mathbb{C})$, which takes $\mu$ to the function $\mathcal{L}[\mu]$ of $\partial\mathbb{I}$ to $\mathbb{C}$, defined by*

$$\mathcal{L}[\mu](t) \equiv \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}(t)} (S_n^{a,k}(t-s))\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{8.6}$$

*is a linear homeomorphism of $L^2(\partial\mathbb{I},\mathbb{C})$ onto itself.*

*(ii) The map $\tilde{\mathcal{L}}$ of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ to $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$, which takes $\mu$ to the function $\tilde{\mathcal{L}}[\mu]$ of $\partial\mathbb{I}$ to $\mathbb{C}$, defined by*

$$\tilde{\mathcal{L}}[\mu](t) \equiv \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}(t)} (S_n^{a,k}(t-s))\mu(s)\,d\sigma_s \quad \forall t \in \partial\mathbb{I}, \tag{8.7}$$

*is a linear homeomorphism of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ onto itself.*

*(iii) The map $\mathcal{L}'$ of $L^2(\partial\mathbb{I},\mathbb{C})$ to $L^2(\partial\mathbb{I},\mathbb{C})$, which takes $\mu$ to the function $\mathcal{L}'[\mu]$ of $\partial\mathbb{I}$ to $\mathbb{C}$, defined by*

$$\mathcal{L}'[\mu](t) \equiv \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial \nu_{\mathbb{I}}(s)} (S_n^{a,k}(t-s))\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{8.8}$$

*is a linear homeomorphism of $L^2(\partial\mathbb{I},\mathbb{C})$ onto itself.*

*(iv) The map* $\tilde{\mathcal{L}}'$ *of* $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ *to* $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$*, which takes* $\mu$ *to the function* $\tilde{\mathcal{L}}'[\mu]$ *of* $\partial\mathbb{I}$ *to* $\mathbb{C}$*, defined by*

$$\tilde{\mathcal{L}}'[\mu](t) \equiv \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s \quad \forall t \in \partial\mathbb{I}, \tag{8.9}$$

*is a linear homeomorphism of* $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ *onto itself.*

*Proof.* We first prove statement $(i)$. By Proposition 8.2 $(i)$, we have that $\mathcal{L}$ is injective. Since the singularity in the involved integral operator is weak, we have that $\mathcal{L}$ is continuous and that $\mathcal{L} - \frac{1}{2}I$ is a compact operator on $L^2(\partial\mathbb{I},\mathbb{C})$ (cf. *e.g.*, Folland [52, Prop. 3.11, p. 121]). Hence, by the Fredholm Theory, we have that $\mathcal{L}$ is surjective and, by the Open Mapping Theorem, we have that it is a linear homeomorphism of $L^2(\partial\mathbb{I},\mathbb{C})$ onto itself. We now consider statement $(ii)$. By Theorem 6.11 $(iii)$, we have that $\tilde{\mathcal{L}}$ is a linear continuous operator of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ to itself. Hence, by the Open Mapping Theorem, in order to prove that it is a linear homeomorphism of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ onto itself, it suffices to prove that it is a bijection. By Proposition 8.2 $(i)$, $\tilde{\mathcal{L}}$ is injective. Now let $\phi \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$. By statement $(i)$, there exists $\mu \in L^2(\partial\mathbb{I},\mathbb{C})$ such that

$$\phi(t) = \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I},$$

and, by Proposition 6.18 $(iii)$, we have $\mu \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$. As a consequence, $\tilde{\mathcal{L}}$ is surjective, and the proof of $(ii)$ is complete. We now turn to the proof of statement $(iii)$. By Proposition 8.2 $(ii)$, we have that $\mathcal{L}'$ is injective. Since the singularity in the involved integral operator is weak, we have that $\mathcal{L}'$ is continuous and that $\mathcal{L}' - \frac{1}{2}I$ is a compact operator on $L^2(\partial\mathbb{I},\mathbb{C})$ (cf. *e.g.*, Folland [52, Prop. 3.11, p. 121].) Hence, by the Fredholm Theory, we have that $\mathcal{L}'$ is surjective and, by the Open Mapping Theorem, we have that it is a linear homeomorphism of $L^2(\partial\mathbb{I},\mathbb{C})$ onto itself. We finally prove statement $(iv)$. By Theorem 6.7 $(ii)$, we have that $\tilde{\mathcal{L}}'$ is a linear continuous operator of $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ to itself. Hence, by the Open Mapping Theorem, in order to prove that it is a linear homeomorphism of $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ onto itself, it suffices to prove that it is a bijection. By Proposition 8.2 $(ii)$, $\tilde{\mathcal{L}}'$ is injective. Now let $\phi \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$. By statement $(iii)$, there exists $\mu \in L^2(\partial\mathbb{I},\mathbb{C})$ such that

$$\phi(t) = \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I},$$

and, by Proposition 6.18 $(i)$, we have $\mu \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$. As a consequence, $\tilde{\mathcal{L}}$ is surjective, and the proof is complete. □

We are now ready to prove the main result of this section.

**Theorem 8.5.** *Let* $m \in \mathbb{N}\setminus\{0\}$*. Let* $\alpha \in ]0,1[$*. Let* $\mathbb{I}$ *be as in* (1.46)*. Let* $k$*,* $\Gamma$ *be as in* (8.1)*,* (8.2)*, respectively. Assume that* $k^2 \notin \mathrm{Eig}_N^a[\mathbb{T}_a[\mathbb{I}]]$ *and* $k^2 \notin \mathrm{Eig}_D[\mathbb{I}]$*. Then boundary value problem* (8.3) *has a unique solution* $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C}) \cap C^2(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$*. Moreover,*

$$u(x) = v_a^-[\mathbb{I},\mu,k](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \tag{8.10}$$

*where* $\mu$ *is the unique function in* $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ *that solves the following equation*

$$\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s = \Gamma(t) \qquad \forall t \in \partial\mathbb{I}. \tag{8.11}$$

*Proof.* Clearly, it suffices to prove the existence. By Theorem 8.4 $(ii)$, there exists a unique $\mu \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ such that (8.11) holds. Then, by Theorem 6.11 $(i)$, we have that $v_a^-[\partial\mathbb{I},\mu,k] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C})$, that

$$\frac{\partial}{\partial\nu_{\mathbb{I}}}v_a^-[\partial\mathbb{I},\mu,k](t) = \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s = \Gamma(t) \qquad \forall t \in \partial\mathbb{I}.$$

and that

$$\Delta v_a^-[\partial\mathbb{I},\mu,k](t) + k^2 v_a^-[\partial\mathbb{I},\mu,k](t) = 0 \qquad \forall t \in \mathbb{T}_a[\mathbb{I}].$$

Finally, by the periodicity of $v_a^-[\partial\mathbb{I},\mu,k]$, we have that $v_a^-[\partial\mathbb{I},\mu,k]$ solves boundary value problem (8.3). □

We now prove the following representation Theorem.

**Theorem 8.6.** *Let* $m \in \mathbb{N} \setminus \{0\}$. *Let* $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be as in* (1.46). *Let* $k$ *be as in* (8.1). *Assume that* $k^2 \notin \mathrm{Eig}_N^a[\mathbb{T}_a[\mathbb{I}]]$ *and* $k^2 \notin \mathrm{Eig}_D[\mathbb{I}]$. *Let* $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ *be such that*

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}. \end{cases}$$

*Then there exists a unique function* $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$ *such that*

$$u(x) = v_a^-[\mathbb{I}, \mu, k](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]. \tag{8.12}$$

*Moreover* $\mu$ *is the unique function in* $C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$ *that solves the following equation*

$$\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n^{a,k}(t - s))\mu(s)\, d\sigma_s = \frac{\partial}{\partial\nu_{\mathbb{I}}} u(t) \qquad \forall t \in \partial\mathbb{I}. \tag{8.13}$$

*Proof.* Let $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$. Clearly, since $k^2 \notin \mathrm{Eig}_N^a[\mathbb{T}_a[\mathbb{I}]]$ and by Theorem 6.11, we have

$$u(x) = v_a^-[\mathbb{I}, \mu, k](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],$$

if and only if

$$\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n^{a,k}(t - s))\mu(s)\, d\sigma_s = \frac{\partial}{\partial\nu_{\mathbb{I}}} u(t) \qquad \forall t \in \partial\mathbb{I}.$$

By Theorem 8.4 (*ii*), there exists a unique $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$ such that (8.13) holds and hence the conclusion easily follows. □

## 8.2 Asymptotic behaviour of the solutions of the Neumann problem for the Helmholtz equation in a periodically perforated domain

In this Section we study the asymptotic behaviour of the solutions of the Neumann problem for the Helmholtz equation in a periodically perforated domain with small holes.

### 8.2.1 Notation

We retain the notation introduced in Subsections 1.8.1, 6.7.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). We shall consider also the following assumptions.

$$k \in \mathbb{C}, \ k^2 \neq |2\pi a^{-1}(z)|^2 \quad \forall z \in \mathbb{Z}^n; \tag{8.14}$$

$$g \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C}). \tag{8.15}$$

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$ be as in (1.56), (1.57), (8.14), respectively. By Proposition 7.42, there exists $\epsilon_1^* \in ]0, \epsilon_1[$ such that

$$k^2 \notin \Big(\mathrm{Eig}_D[\Omega_\epsilon] \cup \mathrm{Eig}_N[\Omega_\epsilon] \cup \mathrm{Eig}_D^a[\mathbb{T}_a[\Omega_\epsilon]] \cup \mathrm{Eig}_N^a[\mathbb{T}_a[\Omega_\epsilon]]\Big) \qquad \forall \epsilon \in ]0, \epsilon_1^*]. \tag{8.16}$$

### 8.2.2 Preliminaries

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). For each $\epsilon \in ]0, \epsilon_1^*[$, we consider the following periodic Neumann problem for the Helmholtz equation.

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x + a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1, \dots, n\}, \\ \frac{\partial}{\partial\nu_{\Omega_\epsilon}} u(x) = g(\frac{1}{\epsilon}(x - w)) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{8.17}$$

By virtue of Theorem 8.5, we can give the following definition.

**Definition 8.7.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). For each $\epsilon \in ]0, \epsilon_1^*[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$ of boundary value problem (8.17).

We have the following Lemmas.

**Lemma 8.8.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $\epsilon \in ]0, \epsilon_1^*[$. Then the function $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ satisfies the following equation*

$$g\Big(\frac{1}{\epsilon}(x-w)\Big) = \frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}(S_n^{a,k}(x-y))\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega_\epsilon, \tag{8.18}$$

*if and only if the function $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, defined by*

$$\theta(t) \equiv \mu(w + \epsilon t) \qquad \forall t \in \partial\Omega, \tag{8.19}$$

*satisfies the following equation*

$$\begin{aligned} g(t) =& \frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k)\theta(s)\,d\sigma_s \\ &+ \epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega} \nu_\Omega(t) \cdot DQ_n^k(\epsilon(t-s))\theta(s)\,d\sigma_s \\ &+ \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s \qquad \forall t \in \partial\Omega. \end{aligned} \tag{8.20}$$

*Proof.* It is a straightforward verification based on the rule of change of variables in integrals, on well known properties of composition of functions in Schauder spaces (cf. *e.g.*, Lanza [67, Sections 3,4]) and on equality (6.25). □

**Lemma 8.9.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $g$ be as in* (1.56), (8.15), *respectively. Then there exists a unique function $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ that solves the following equation*

$$g(t) = \frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\,d\sigma_s \qquad \forall t \in \partial\Omega. \tag{8.21}$$

*We denote the unique solution of equation* (8.21) *by $\tilde{\theta}$. Moreover,*

$$\int_{\partial\Omega} \tilde{\theta}(s)\,d\sigma_s = \int_{\partial\Omega} g(s)\,d\sigma_s. \tag{8.22}$$

*Proof.* The existence and uniqueness of a solution of equation (8.21) is a well known result of classic potential theory (cf. Folland [52, Chapter 3] for the existence and uniqueness of a solution in $L^2(\partial\Omega, \mathbb{C})$ and, *e.g.*, Theorem B.3 for the regularity.) Equality (8.22) follows by Folland [52, Lemma 3.30, p. 133]. □

Since we want to represent the function $u[\epsilon]$ by means of a periodic simple layer potential, we need to study some integral equations. Indeed, by virtue of Theorem 8.5, we can transform (8.17) into an integral equation, whose unknown is the moment of the simple layer potential. Moreover, we want to transform this equation defined on the $\epsilon$-dependent domain $\partial\Omega_\epsilon$ into an equation defined on the fixed domain $\partial\Omega$. We introduce this integral equation in the following Propositions. The relation between the solution of the integral equation and the solution of boundary value problem (8.17) will be clarified later. Anyway, since the function $Q_n^k$ that appears in equation (8.20) (involved in the determination of the moment of the simple layer potential that solves (8.17)) is identically 0 if $n$ is odd, it is preferable to treat separately case $n$ even and case $n$ odd.

**Proposition 8.10.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $\tilde{\theta}$ be as in Lemma 8.9. Let $\Lambda$ be the map of $]-\epsilon_1^*, \epsilon_1^*[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ defined by*

$$\begin{aligned} \Lambda[\epsilon, \theta](t) \equiv& \frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k)\theta(s)\,d\sigma_s \\ &+ \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s - g(t) \qquad \forall t \in \partial\Omega, \end{aligned} \tag{8.23}$$

*for all $(\epsilon, \theta) \in ]-\epsilon_1^*, \epsilon_1^*[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Then the following statements hold.*

(i) *If* $\epsilon \in ]0, \epsilon_1^*[$, *then the function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *satisfies equation*

$$\Lambda[\epsilon, \theta] = 0, \tag{8.24}$$

*if and only if the function* $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$, *defined by*

$$\mu(x) \equiv \theta(\frac{1}{\epsilon}(x - w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{8.25}$$

*satisfies the equation*

$$\Gamma(x) = \frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}(S_n^{a,k}(x - y))\mu(y)\, d\sigma_y \qquad \forall x \in \partial\Omega_\epsilon, \tag{8.26}$$

*with* $\Gamma \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ *defined by*

$$\Gamma(x) \equiv g(\frac{1}{\epsilon}(x - w)) \qquad \forall x \in \partial\Omega_\epsilon. \tag{8.27}$$

*In particular, equation* (8.24) *has exactly one solution* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, *for each* $\epsilon \in ]0, \epsilon_1^*[$.

(ii) *The function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *satisfies equation*

$$\Lambda[0, \theta] = 0, \tag{8.28}$$

*if and only if*

$$g(t) = \frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t - s))\theta(s)\, d\sigma_s \qquad \forall t \in \partial\Omega. \tag{8.29}$$

*In particular, the unique function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *that solves equation* (8.28) *is* $\tilde{\theta}$.

*Proof.* Consider ($i$). The equivalence of equation (8.24) in the unknown $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ and equation (8.26) in the unknown $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ follows by Lemma 8.8 and the definition of $Q_n^k$ for $n$ odd (cf. (6.23) and Definition E.2.) The existence and uniqueness of a solution of equation (8.26) follows by Proposition 8.4 ($ii$). Then the existence and uniqueness of a solution of equation (8.24) follows by the equivalence of (8.24) and (8.26). Consider ($ii$). The equivalence of (8.28) and (8.29) is obvious. The second part of statement ($ii$) is an immediate consequence of Lemma 8.9. □

**Proposition 8.11.** *Let $n$ be even. Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $k$, $g$ *be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let* $\epsilon_1^*$ *be as in* (8.16). *Let* $\tilde{\theta}$ *be as in Lemma 8.9. Let* $\epsilon_1' > 0$ *be such that*

$$\epsilon \log \epsilon \in ]-\epsilon_1', \epsilon_1'[ \qquad \forall \epsilon \in ]0, \epsilon_1^*[. \tag{8.30}$$

*Let* $\Lambda^\#$ *be the map of* $]-\epsilon_1^*, \epsilon_1^*[ \times ]-\epsilon_1', \epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *to* $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *defined by*

$$\begin{aligned} \Lambda^\#[\epsilon, \epsilon', \theta](t) \equiv & \frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t - s, \epsilon k)\theta(s)\, d\sigma_s \\ & + \epsilon^{n-2}\epsilon' k^{n-2} \int_{\partial\Omega} \nu_\Omega(t) \cdot DQ_n^k(\epsilon(t - s))\theta(s)\, d\sigma_s \\ & + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t - s))\theta(s)\, d\sigma_s - g(t) \qquad \forall t \in \partial\Omega, \end{aligned} \tag{8.31}$$

*for all* $(\epsilon, \epsilon', \theta) \in ]-\epsilon_1^*, \epsilon_1^*[ \times ]-\epsilon_1', \epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. *Then the following statements hold.*

(i) *If* $\epsilon \in ]0, \epsilon_1^*[$, *then the function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *satisfies equation*

$$\Lambda^\#[\epsilon, \epsilon \log \epsilon, \theta] = 0, \tag{8.32}$$

*if and only if the function* $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$, *defined by*

$$\mu(x) \equiv \theta(\frac{1}{\epsilon}(x - w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{8.33}$$

*satisfies the equation*

$$\Gamma(x) = \frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}(S_n^{a,k}(x-y))\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega_\epsilon, \tag{8.34}$$

*with* $\Gamma \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ *defined by*

$$\Gamma(x) \equiv g(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon. \tag{8.35}$$

*In particular, equation* (8.32) *has exactly one solution* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$*, for each* $\epsilon \in ]0, \epsilon_1^*[$.

*(ii) The function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *satisfies equation*

$$\Lambda^\#[0, 0, \theta] = 0, \tag{8.36}$$

*if and only if*

$$g(t) = \frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\,d\sigma_s \qquad \forall t \in \partial\Omega. \tag{8.37}$$

*In particular, the unique function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *that solves equation* (8.36) *is* $\tilde{\theta}$.

*Proof.* Consider $(i)$. The equivalence of equation (8.32) in the unknown $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ and equation (8.34) in the unknown $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ follows by Lemma 8.8 and the definition of $Q_n^k$ for $n$ even (cf. (6.23) and Definition E.2.) The existence and uniqueness of a solution of equation (8.34) follows by Proposition 8.4 $(ii)$. Then the existence and uniqueness of a solution of equation (8.32) follows by the equivalence of (8.32) and (8.34). Consider $(ii)$. The equivalence of (8.36) and (8.37) is obvious. The second part of statement $(ii)$ is an immediate consequence of Lemma 8.9. □

By Propositions 8.10, 8.11, it makes sense to introduce the following.

**Definition 8.12.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). For each $\epsilon \in ]0, \epsilon_1^*[$, we denote by $\hat{\theta}_n[\epsilon]$ the unique function in $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ that solves equation (8.24), if $n$ is odd, or equation (8.32), if $n$ is even. Analogously, we denote by $\hat{\theta}_n[0]$ the unique function in $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ that solves equation (8.28), if $n$ is odd, or equation (8.36), if $n$ is even.

In the following Remark, we show the relation between the solutions of boundary value problem (8.17) and the solutions of equations (8.24), (8.32).

*Remark* 8.13. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16).
Let $\epsilon \in ]0, \epsilon_1^*[$. We have

$$u[\epsilon](x) = \epsilon^{n-1} \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s)\hat{\theta}_n[\epsilon](s)\,d\sigma_s \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$

While the relation between equations (8.24), (8.32) and boundary value problem (8.17) is now clear, we want to see if (8.28), (8.36) are related to some (limiting) boundary value problem. We give the following.

**Definition 8.14.** Let $n \geq 3$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $\Omega$, $g$ be as in (1.56), (8.15), respectively. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega, \mathbb{C})$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega, \\ \frac{\partial}{\partial\nu_\Omega} u(x) = g(x) & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{8.38}$$

Problem (8.38) will be called the *limiting boundary value problem*.

*Remark* 8.15. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). If $n \geq 3$, then we have

$$\tilde{u}(x) = \int_{\partial\Omega} S_n(x-y)\hat{\theta}_n[0](y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \Omega.$$

If $n = 2$, in general the (classic) simple layer potential for the Laplace equation with moment $\hat{\theta}_2[0]$ is not harmonic at infinity, and it does not satisfy the third condition of boundary value problem (8.38). Moreover, if $n = 2$, boundary value problem (8.38) does not have in general a solution (unless $\int_{\partial\Omega} g\, d\sigma = 0$.) However, the function $\tilde{v}$ of $\mathbb{R}^2 \setminus \Omega$ to $\mathbb{C}$, defined by

$$\tilde{v}(x) \equiv \int_{\partial\Omega} S_2(x-y)\hat{\theta}_2[0](y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^2 \setminus \Omega,$$

is a solution of the following boundary value problem

$$\begin{cases} \Delta\tilde{v}(x) = 0 & \forall x \in \mathbb{R}^2 \setminus \mathrm{cl}\,\Omega, \\ \frac{\partial}{\partial\nu_\Omega}\tilde{v}(x) = g(x) & \forall x \in \partial\Omega. \end{cases} \tag{8.39}$$

We now prove the following Propositions.

**Proposition 8.16.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $\tilde{\theta}$ be as in Lemma 8.9. Let $\Lambda$ be as in Proposition 8.10. Then there exists $\epsilon_2 \in ]0, \epsilon_1^*]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Moreover, if we set $b_0 \equiv (0, \tilde{\theta})$, then the differential $\partial_\theta\Lambda[b_0]$ of $\Lambda$ with respect to the variable $\theta$ at $b_0$ is delivered by the following formula*

$$\partial_\theta\Lambda[b_0](\tau)(t) = \frac{1}{2}\tau(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\tau(s)\, d\sigma_s \qquad \forall t \in \partial\Omega, \tag{8.40}$$

*for all $\tau \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, and is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ onto itself.*

*Proof.* By Proposition 6.21 (*ii*), we easily deduce that there exists $\epsilon_2 \in ]0, \epsilon_1^*]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. By standard calculus in Banach space, we immediately deduce that (8.40) holds. Now we need to prove that $\partial_\theta\Lambda[b_0]$ is a linear homeomorphism. By the Open Mapping Theorem, it suffices to prove that it is a bijection. Let $\psi \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. By known results of classical potential theory (cf. Folland [52, Chapter 3]), there exists a unique function $\tau \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, such that

$$\frac{1}{2}\tau(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\tau(s)\, d\sigma_s = \psi(t) \qquad \forall t \in \partial\Omega.$$

Hence $\partial_\theta\Lambda[b_0]$ is bijective, and, accordingly, a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ onto itself. □

**Proposition 8.17.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $\tilde{\theta}$ be as in Lemma 8.9. Let $\epsilon_1' > 0$ be as in* (8.30). *Let $\Lambda^\#$ be as in Proposition 8.11. Then there exists $\epsilon_2 \in ]0, \epsilon_1^*]$ such that $\Lambda^\#$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times ]-\epsilon_1', \epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Moreover, if we set $b_0 \equiv (0, 0, \tilde{\theta})$, then the differential $\partial_\theta\Lambda^\#[b_0]$ of $\Lambda^\#$ with respect to the variable $\theta$ at $b_0$ is delivered by the following formula*

$$\partial_\theta\Lambda^\#[b_0](\tau)(t) = \frac{1}{2}\tau(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\tau(s)\, d\sigma_s \qquad \forall t \in \partial\Omega, \tag{8.41}$$

*for all $\tau \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, and is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ onto itself.*

*Proof.* By Proposition 6.21 (*ii*), we easily deduce that there exists $\epsilon_2 \in ]0, \epsilon_1^*]$ such that $\Lambda^\#$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times ]-\epsilon_1', \epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. By standard calculus in Banach space, we immediately deduce that (8.41) holds. Finally, by the proof of Proposition 8.16 and formula (8.41), we have that $\partial_\theta\Lambda^\#[b_0]$ is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ onto itself. □

By the previous Propositions we can now prove the following results.

**Proposition 8.18.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $\epsilon_2$ be as in Proposition 8.16. Then there exist $\epsilon_3 \in ]0, \epsilon_2]$ and a real analytic operator $\Theta_n$ of $]-\epsilon_3, \epsilon_3[$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, such that*

$$\Theta_n[\epsilon] = \hat{\theta}_n[\epsilon], \tag{8.42}$$

*for all $\epsilon \in [0, \epsilon_3[$.*

*Proof.* It is an immediate consequence of Proposition 8.16 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

**Proposition 8.19.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $\epsilon_1' > 0$ be as in* (8.30). *Let $\epsilon_2$ be as in Proposition 8.17. Then there exist $\epsilon_3 \in ]0, \epsilon_2]$, $\epsilon_2' \in ]0, \epsilon_1']$, and a real analytic operator $\Theta_n^\#$ of $]-\epsilon_3, \epsilon_3[ \times ]-\epsilon_2', \epsilon_2'[$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, such that*

$$\epsilon \log \epsilon \in ]-\epsilon_2', \epsilon_2'[ \quad \forall \epsilon \in ]0, \epsilon_3[,$$

$$\Theta_n^\#[\epsilon, \epsilon \log \epsilon] = \hat{\theta}_n[\epsilon] \quad \forall \epsilon \in ]0, \epsilon_3[, \tag{8.43}$$

$$\Theta_n^\#[0,0] = \hat{\theta}_n[0]. \tag{8.44}$$

*Proof.* It is an immediate consequence of Proposition 8.17 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

### 8.2.3 A functional analytic representation Theorem for the solution of the singularly perturbed Neumann problem

By Propositions 8.18, 8.19 and Remark 8.13, we can deduce the main result of this Subsection.

**Theorem 8.20.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $\epsilon_3$ be as in Proposition 8.18. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \epsilon_3]$, and a real analytic operator $U$ of $]-\epsilon_4, \epsilon_4[$ to the space $C^0(\mathrm{cl}\, V, \mathbb{C})$, such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$.*

*(jj)*

$$u[\epsilon](x) = \epsilon^{n-1} U[\epsilon](x) \qquad \forall x \in \mathrm{cl}\, V,$$

*for all $\epsilon \in ]0, \epsilon_4[$. Moreover,*

$$U[0](x) = S_n^{a,k}(x - w) \int_{\partial\Omega} g \, d\sigma \qquad \forall x \in \mathrm{cl}\, V.$$

*Proof.* Let $\Theta_n[\cdot]$ be as in Proposition 8.18. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $\epsilon \in ]0, \epsilon_4[$. By Remark 8.13 and Proposition 8.18, we have

$$u[\epsilon](x) = \epsilon^{n-1} \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s) \Theta_n[\epsilon](s) \, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V.$$

Thus, it is natural to set

$$U[\epsilon](x) \equiv \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s) \Theta_n[\epsilon](s) \, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$. By Proposition 6.22, $U$ is a real analytic map of $]-\epsilon_4, \epsilon_4[$ to $C^0(\mathrm{cl}\, V, \mathbb{C})$. Furthermore, by Lemma 8.9, we have

$$\begin{aligned} U[0](x) &= S_n^{a,k}(x - w) \int_{\partial\Omega} \Theta_n[0](s) \, d\sigma_s \\ &= S_n^{a,k}(x - w) \int_{\partial\Omega} g \, d\sigma \qquad \forall x \in \mathrm{cl}\, V, \end{aligned}$$

since $\Theta_n[0] = \tilde{\theta}$. Hence the proof is now complete. □

*Remark* 8.21. We note that the right-hand side of the equality in $(jj)$ of Theorem 8.20 can be continued real analytically in the whole $]-\epsilon_4, \epsilon_4[$. Moreover, if $V$ is a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$, then

$$\lim_{\epsilon \to 0^+} u[\epsilon] = 0 \qquad \text{uniformly in } \mathrm{cl}\, V.$$

**Theorem 8.22.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $\epsilon_3$, $\epsilon_2'$ be as in Proposition 8.19. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \epsilon_3]$ and a real analytic operator $U^\#$ of $]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[$ to the space $C^0(\mathrm{cl}\, V, \mathbb{C})$, such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*

$$u[\epsilon](x) = \epsilon^{n-1} U^\#[\epsilon, \epsilon \log \epsilon](x) \qquad \forall x \in \mathrm{cl}\, V,$$

*for all $\epsilon \in ]0, \epsilon_4[$. Moreover,*

$$U^\#[0,0](x) = S_n^{a,k}(x-w) \int_{\partial\Omega} g\, d\sigma \qquad \forall x \in \mathrm{cl}\, V.$$

*Proof.* Let $\Theta_n^\#[\cdot,\cdot]$ be as in Proposition 8.19. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $\epsilon \in ]0, \epsilon_4[$. By Remark 8.13 and Proposition 8.19, we have

$$u[\epsilon](x) = \epsilon^{n-1} \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s) \Theta_n^\#[\epsilon, \epsilon \log \epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V.$$

Thus, it is natural to set

$$U^\#[\epsilon, \epsilon'](x) \equiv \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s) \Theta_n^\#[\epsilon, \epsilon'](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

for all $(\epsilon, \epsilon') \in ]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[$. By Proposition 6.22, $U^\#$ is a real analytic map of $]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[$ to $C^0(\mathrm{cl}\, V, \mathbb{C})$. Furthermore, by Lemma 8.9, we have

$$\begin{aligned} U^\#[0,0](x) &= S_n^{a,k}(x-w) \int_{\partial\Omega} \Theta_n^\#[0,0](s)\, d\sigma_s \\ &= S_n^{a,k}(x-w) \int_{\partial\Omega} g\, d\sigma \qquad \forall x \in \mathrm{cl}\, V, \end{aligned}$$

since $\Theta_n^\#[0,0] = \tilde{\theta}$. Accordingly, the Theorem is now completely proved. □

We have also the following Theorems.

**Theorem 8.23.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $\tilde{\theta}$ be as in Lemma 8.9. Let $\epsilon_3$ be as in Proposition 8.18. Then there exist $\epsilon_5 \in ]0, \epsilon_3]$, and a real analytic operator $G$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\, dx = \epsilon^n G[\epsilon], \tag{8.45}$$

*for all $\epsilon \in ]0, \epsilon_5[$. Moreover,*

$$G[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\, \Omega} |\nabla \tilde{u}(x)|^2\, dx, \tag{8.46}$$

*where $\tilde{u}$ is as in Definition 8.14.*

*Proof.* Let $\Theta_n[\cdot]$ be as in Proposition 8.18. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon \in ]0, \epsilon_3[$. Clearly, by the periodicity of $u[\epsilon]$, we have

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\, dx = -\epsilon^{n-1} \int_{\partial\Omega} g(t) \overline{u[\epsilon] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t)}\, d\sigma_t.$$

By equality (6.24) and since $Q^k_n = 0$ for $n$ odd, we have

$$\begin{aligned}u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) &= \epsilon^{n-1}\int_{\partial\Omega} S^{a,k}_n(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s\\ &= \epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R^{a,k}_n(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s \quad \forall t\in\partial\Omega.\end{aligned}$$

By Theorem E.6 $(i)$, one can easily show that the map which takes $\epsilon$ to the function of the variable $t\in\partial\Omega$ defined by

$$\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

is a real analytic operator of $]-\epsilon_3,\epsilon_3[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By Theorem C.4, we immediately deduce that there exists $\epsilon_5\in\,]0,\epsilon_3]$ such that the map of $]-\epsilon_5,\epsilon_5[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, which takes $\epsilon$ to the function $\int_{\partial\Omega} R^{a,k}_n(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s$ of the variable $t\in\partial\Omega$, is real analytic. Hence, if we set

$$G_1[\epsilon] \equiv -\int_{\partial\Omega} g(t)\overline{\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s}\,d\sigma_t,$$

and

$$G_2[\epsilon] \equiv -\int_{\partial\Omega} g(t)\overline{\int_{\partial\Omega} R^{a,k}_n(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s}\,d\sigma_t,$$

for all $\epsilon\in\,]-\epsilon_5,\epsilon_5[$, then by standard properties of functions in Schauder spaces, we have that $G_1$ and $G_2$ are real analytic maps of $]-\epsilon_5,\epsilon_5[$ to $\mathbb{C}$. Thus, if we set

$$G[\epsilon] \equiv G_1[\epsilon] + \epsilon^{n-2}G_2[\epsilon] \qquad \forall \epsilon\in\,]-\epsilon_5,\epsilon_5[,$$

then $G$ is a real analytic map of $]-\epsilon_5,\epsilon_5[$ to $\mathbb{C}$ such that equality (8.45) holds.

Finally, if $\epsilon=0$, by Folland [52, p. 118], we have

$$\begin{aligned}G[0] &= -\int_{\partial\Omega} g(t)\overline{\int_{\partial\Omega} S_n(t-s)\tilde\theta(s)\,d\sigma_s}\,d\sigma_t\\ &= \int_{\mathbb{R}^n\setminus \mathrm{cl}\,\Omega} |\nabla\tilde u(x)|^2\,dx,\end{aligned}$$

and accordingly (8.46) holds. □

**Theorem 8.24.** *Let $n$ be even. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). Let $\tilde\theta$ be as in Lemma 8.9. Let $\epsilon_3$, $\epsilon_2'$ be as in Proposition 8.19. Then there exist $\epsilon_5\in\,]0,\epsilon_3]$, and two real analytic operators $G_1^\#$, $G_2^\#$ of $]-\epsilon_5,\epsilon_5[\,\times\,]-\epsilon_2',\epsilon_2'[$ to $\mathbb{C}$, such that*

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\,dx - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\,dx &=\\ \epsilon^n G_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{2n-2}(\log\epsilon)&G_2^\#[\epsilon,\epsilon\log\epsilon],\end{aligned} \tag{8.47}$$

*for all $\epsilon\in\,]0,\epsilon_5[$. Moreover,*

$$G_1^\#[0,0] = -\int_{\partial\Omega} g(t)\overline{\int_{\partial\Omega} S_n(t-s)\tilde\theta(s)\,d\sigma_s}\,d\sigma_t - \delta_{2,n}\overline{R^{a,k}_n(0)}\left|\int_{\partial\Omega} g\,d\sigma\right|^2, \tag{8.48}$$

$$G_2^\#[0,0] = -\overline{k}^{n-2}\mathcal{J}_n(0)\left|\int_{\partial\Omega} g\,d\sigma\right|^2, \tag{8.49}$$

*where $\mathcal{J}_n(0)$ is as in Proposition E.3 $(i)$. In particular, if $n>2$, then*

$$G_1^\#[0,0] = \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega} |\nabla\tilde u(x)|^2\,dx, \tag{8.50}$$

*where $\tilde u$ is as in Definition 8.14.*

*Proof.* Let $\Theta_n^\#[\cdot,\cdot]$ be as in Proposition 8.19. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon \in ]0,\epsilon_3[$. Clearly, by the periodicity of $u[\epsilon]$, we have

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\,dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\,dx = -\epsilon^{n-1} \int_{\partial\Omega} g(t)\overline{u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)}\,d\sigma_t.$$

By equality (6.24), we have

$$\begin{aligned} u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) =& \epsilon^{n-1}\int_{\partial\Omega} S_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ =& \epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &+ \epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &+ \epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega. \end{aligned}$$

By Theorem E.6 $(i)$, one can easily show that the map which takes $(\epsilon,\epsilon')$ to the function of the variable $t\in\partial\Omega$, defined by

$$\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

is a real analytic operator of $]-\epsilon_3,\epsilon_3[\times]-\epsilon_2',\epsilon_2'[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By Theorem C.4, we immediately deduce that there exists $\epsilon_5\in]0,\epsilon_3]$ such that the map of $]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, which takes $(\epsilon,\epsilon')$ to the function $\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s$ of the variable $t\in\partial\Omega$, is real analytic. By Theorem C.4, we have that the map of $]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, which takes $(\epsilon,\epsilon')$ to the function $\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s$ of the variable $t\in\partial\Omega$, is real analytic. Hence, if we set

$$\begin{aligned} G_1^\#[\epsilon,\epsilon'] \equiv& -\int_{\partial\Omega} g(t)\overline{\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s}\,d\sigma_t \\ &-\epsilon^{n-2}\int_{\partial\Omega} g(t)\overline{\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s}\,d\sigma_t, \end{aligned}$$

and

$$G_2^\#[\epsilon,\epsilon'] \equiv -\int_{\partial\Omega} g(t)\overline{(k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s)}\,d\sigma_t,$$

for all $(\epsilon,\epsilon')\in]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[$, by standard properties of functions in Schauder spaces, we have that $G_1^\#$ and $G_2^\#$ are real analytic maps of $]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[$ to $\mathbb{C}$ such that equality (8.47) holds.

Finally, if $\epsilon=\epsilon'=0$, we have

$$\begin{aligned} G_1^\#[0,0] =& -\int_{\partial\Omega} g(t)\overline{\int_{\partial\Omega} S_n(t-s)\tilde\theta(s)\,d\sigma_s}\,d\sigma_t - \delta_{2,n}\int_{\partial\Omega} g(t)\overline{\int_{\partial\Omega} R_n^{a,k}(0)\tilde\theta(s)\,d\sigma_s}\,d\sigma_t \\ =& -\int_{\partial\Omega} g(t)\overline{\int_{\partial\Omega} S_n(t-s)\tilde\theta(s)\,d\sigma_s}\,d\sigma_t - \delta_{2,n}\overline{R_n^{a,k}(0)}\int_{\partial\Omega} g(t)\overline{\int_{\partial\Omega}\tilde\theta(s)\,d\sigma_s}\,d\sigma_t \\ =& -\int_{\partial\Omega} g(t)\overline{\int_{\partial\Omega} S_n(t-s)\tilde\theta(s)\,d\sigma_s}\,d\sigma_t - \delta_{2,n}\overline{R_n^{a,k}(0)}\int_{\partial\Omega} g(t)\,d\sigma_t\overline{\int_{\partial\Omega} g(s)\,d\sigma_s}, \end{aligned}$$

and

$$\begin{aligned} G_2^\#[0,0] =& -\int_{\partial\Omega} g(t)\overline{k^{n-2}\int_{\partial\Omega} Q_n^k(0)\tilde\theta(s)\,d\sigma_s}\,d\sigma_t \\ =& -\overline{k}^{n-2}\overline{Q_n^k(0)}\int_{\partial\Omega} g(t)\,d\sigma_t\overline{\int_{\partial\Omega} g(s)\,d\sigma_s}, \end{aligned}$$

and accordingly equalities (8.48) and (8.49) hold. Finally, if $n\ge 4$, by Folland [52, p. 118], we have

$$G_1^\#[0,0] = \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega} |\nabla\tilde u(x)|^2\,dx.$$

□

*Remark* 8.25. If $n$ is odd, we note that the right-hand side of the equality in (8.45) of Theorem 8.23 can be continued real analytically in the whole $]-\epsilon_5, \epsilon_5[$.

Moreover,

$$\lim_{\epsilon\to 0^+}\Big[\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\,dx - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\,dx\Big] = 0,$$

for all $n \in \mathbb{N}\setminus\{0,1\}$ ($n$ even or odd.)

### 8.2.4 A real analytic continuation Theorem for the integral of the solution

We now prove a real analytic continuation Theorem for the integral of the solution. Namely, we prove the following.

**Theorem 8.26.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16).

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx = \frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega} g(t)\,d\sigma_t,$$

*for all $\epsilon \in ]0, \epsilon_1^*[$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_1^*[$. By the Divergence Theorem and the periodicity of $u[\epsilon]$, we have

$$\begin{aligned}
\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx &= -\frac{1}{k^2}\int_{\mathbb{P}_a[\Omega_\epsilon]} \Delta u[\epsilon](x)\,dx\\
&= -\frac{1}{k^2}\int_{\partial\mathbb{P}_a[\Omega_\epsilon]} \frac{\partial}{\partial\nu_{\mathbb{P}_a[\Omega_\epsilon]}} u[\epsilon](x)\,d\sigma_x\\
&= -\frac{1}{k^2}\Big[\int_{\partial A}\frac{\partial}{\partial\nu_A}u[\epsilon](x)\,d\sigma_x - \int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon](x)\,d\sigma_x\Big]\\
&= \frac{1}{k^2}\int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon](x)\,d\sigma_x.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx &= \frac{1}{k^2}\int_{\partial\Omega_\epsilon} g\big(\frac{1}{\epsilon}(x-w)\big)\,d\sigma_x\\
&= \frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega} g(t)\,d\sigma_t,
\end{aligned}$$

and the proof is complete. □

## 8.3 An homogenization problem for the Helmholtz equation with Neumann boundary conditions in a periodically perforated domain

In this section we consider an homogenization problem for the Helmhlotz equation with Neumann boundary conditions in a periodically perforated domain. In most of the results we assume that $\mathrm{Im}(k) \neq 0$ and $\mathrm{Re}(k) = 0$.

We note that we shall consider the equation

$$\Delta u(x) + \frac{k^2}{\delta^2}u(x) = 0 \qquad \forall x \in \mathbb{T}_a(\epsilon,\delta),$$

together with the usual periodicity condition and a Neumann boundary condition. We do so, because if $u$ is a solution of the equation above then the function $u_\delta(\cdot) \equiv u(\delta\cdot)$ is a solution of the following equation

$$\Delta u_\delta(x) + k^2 u_\delta(x) = 0 \qquad \forall x \in \mathbb{T}_a[\Omega_\epsilon],$$

which we can analyse by virtue of the results of Section 8.2.

### 8.3.1 Notation

In this Section we retain the notation introduced in Subsections 1.8.1, 6.7.1, 8.2.1. However, we need to introduce also some other notation.

Let $(\epsilon,\delta) \in (]-\epsilon_1,\epsilon_1[\setminus\{0\}) \times ]0,+\infty[$. If $v$ is a function of $\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta)$ to $\mathbb{C}$, then we denote by $\mathbf{E}_{(\epsilon,\delta)}[v]$ the function of $\mathbb{R}^n$ to $\mathbb{C}$, defined by

$$\mathbf{E}_{(\epsilon,\delta)}[v](x) \equiv \begin{cases} v(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \\ 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta). \end{cases}$$

### 8.3.2 Preliminaries

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). For each $(\epsilon,\delta) \in ]0,\epsilon_1^*[ \times ]0,+\infty[$, we consider the following periodic Neumann problem for the Helmholtz equation.

$$\begin{cases} \Delta u(x) + \frac{k^2}{\delta^2} u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u(x+\delta a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x) = \frac{1}{\delta} g(\frac{1}{\epsilon\delta}(x-\delta w)) & \forall x \in \partial\Omega(\epsilon,\delta). \end{cases} \tag{8.51}$$

By virtue of Theorem 8.5, we can give the following definition.

**Definition 8.27.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). For each $(\epsilon,\delta) \in ]0,\epsilon_1^*[ \times ]0,+\infty[$, we denote by $u_{(\epsilon,\delta)}$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta),\mathbb{C})$ of boundary value problem (8.51).

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. In order to do so we introduce the following.

**Definition 8.28.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). For each $\epsilon \in ]0,\epsilon_1^*[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],\mathbb{C})$ of the following periodic Neumann problem for the Helmholtz equation.

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) = g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{8.52}$$

*Remark* 8.29. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). For each pair $(\epsilon,\delta) \in ]0,\epsilon_1^*[ \times ]0,+\infty[$, we have

$$u_{(\epsilon,\delta)}(x) = u[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).$$

By the previous remark, we note that the solution of problem (8.51) can be expressed by means of the solution of the auxiliary rescaled problem (8.52), which does not depend on $\delta$. This is due to the presence of the factor $1/\delta$ in front of $g(\frac{1}{\epsilon\delta}(x-\delta w))$ in the third equation of problem (8.51).

As a first step, we study the behaviour of $u[\epsilon]$ as $\epsilon$ tends to 0.

**Proposition 8.30.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (8.16). *Let $\epsilon_3$ be as in Proposition 8.18. Then there exist $\tilde{\epsilon} \in ]0,\epsilon_3[$ and a real analytic map $N$ of $]-\tilde{\epsilon},\tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$ such that*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},$$

*for all $\epsilon \in ]0,\tilde{\epsilon}[$. Moreover, as a consequence,*

$$\lim_{\epsilon\to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* Let $\epsilon_3$, $\Theta_n$ be as in Proposition 8.18. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $\epsilon \in ]0,\epsilon_3[$, we have

$$u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s \quad \forall t\in\partial\Omega.$$

We set

$$N[\epsilon](t) \equiv \int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $\epsilon\in]-\epsilon_3,\epsilon_3[$. By taking $\tilde\epsilon\in]0,\epsilon_3[$ small enough, we can assume (cf. Theorem C.4 and the proof of Theorem 8.23) that $N$ is a real analytic map of $]-\tilde\epsilon,\tilde\epsilon[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$.

By Corollary 6.24, we have

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon\|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)} \qquad \forall\epsilon\in]0,\tilde\epsilon[,$$

and

$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon\|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)} \qquad \forall\epsilon\in]0,\tilde\epsilon[.$$

Accordingly,

$$\lim_{\epsilon\to0^+}\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and

$$\lim_{\epsilon\to0^+}\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and so the conclusion follows. □

**Proposition 8.31.** *Let $n$ be even. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\mathrm{Im}(k)\neq0$ and $\mathrm{Re}(k)=0$. Let $\epsilon_1^*$ be as in (8.16). Let $\epsilon_3$, $\epsilon_2'$ be as in Proposition 8.19. Then there exist $\tilde\epsilon\in]0,\epsilon_3[$ and two real analytic maps $N_1^\#$, $N_2^\#$ of $]-\tilde\epsilon,\tilde\epsilon[\times]-\epsilon_2',\epsilon_2'[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$ such that*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon]+\epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon]+\epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$

*for all $\epsilon\in]0,\tilde\epsilon[$. Moreover, as a consequence,*

$$\lim_{\epsilon\to0^+}\mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* Let $\epsilon_3$, $\epsilon_2'$, $\Theta_n^\#$ be as in Proposition 8.19. If $\epsilon\in]0,\epsilon_3[$, we have

$$\begin{aligned}u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) =&\epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s\\ &+\epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s\\ &+\epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega.\end{aligned}$$

We set

$$\begin{aligned}N_1^\#[\epsilon,\epsilon'](t) \equiv&\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s\\ &+\epsilon^{n-2}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,\end{aligned}$$

and

$$N_2^\#[\epsilon,\epsilon'](t) \equiv k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $(\epsilon,\epsilon') \in ]-\epsilon_3,\epsilon_3[ \times ]-\epsilon_2',\epsilon_2'[$. By taking $\tilde{\epsilon} \in ]0,\epsilon_3[$ small enough, we can assume (cf. Theorem C.4 and the proof of Theorem 8.24) that $N_1^{\#}$, $N_2^{\#}$ are real analytic maps of $]-\tilde{\epsilon},\tilde{\epsilon}[ \times ]-\epsilon_2',\epsilon_2'[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$. Clearly,

$$u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \epsilon N_1^{\#}[\epsilon,\epsilon\log\epsilon](t) + \epsilon^{n-1}(\log\epsilon)N_2^{\#}[\epsilon,\epsilon\log\epsilon](t) \qquad \forall t\in\partial\Omega,\ \forall\epsilon\in ]0,\tilde{\epsilon}[.$$

By Corollary 6.24, we have

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Re}\big(\epsilon N_1^{\#}[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^{\#}[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$

and

$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Im}\big(\epsilon N_1^{\#}[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^{\#}[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$

for all $\epsilon\in ]0,\tilde{\epsilon}[$.

Accordingly,

$$\lim_{\epsilon\to 0^+}\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and

$$\lim_{\epsilon\to 0^+}\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and so the conclusion follows. □

### 8.3.3 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorems we deduce by Propositions 8.30, 8.31 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 8.32.** *Let $n$ be odd. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in ]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let* $\mathrm{Im}(k)\neq 0$ *and* $\mathrm{Re}(k)=0$. *Let $\epsilon_1^*$ be as in* (8.16). *Let $\tilde{\epsilon}$, $N$ be as in Proposition 8.30. Then*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon\|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon\|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta)\in ]0,\tilde{\epsilon}[ \times ]0,+\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)}\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \epsilon\|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \epsilon\|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta)\in ]0,\tilde{\epsilon}[ \times ]0,+\infty[$. □

**Theorem 8.33.** *Let $n$ be even. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in ]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let* $\mathrm{Im}(k)\neq 0$ *and* $\mathrm{Re}(k)=0$. *Let $\epsilon_1^*$ be as in* (8.16). *Let $\tilde{\epsilon}$, $N_1^{\#}$, $N_2^{\#}$ be as in Proposition 8.31. Then*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Re}\big(\epsilon N_1^{\#}[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^{\#}[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Im}\big(\epsilon N_1^{\#}[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^{\#}[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta)\in ]0,\tilde{\epsilon}[ \times ]0,+\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)}\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \|\mathrm{Re}\big(\epsilon N_1^{\#}[\epsilon,\epsilon\log\epsilon]+\epsilon^{n-1}(\log\epsilon)N_2^{\#}[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \|\mathrm{Im}\big(\epsilon N_1^{\#}[\epsilon,\epsilon\log\epsilon]+\epsilon^{n-1}(\log\epsilon)N_2^{\#}[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta)\in\,]0,\tilde{\epsilon}[\,\times\,]0,+\infty[$. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 8.34.** *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $r>0$ and $\bar{y}\in\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n}\mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n\frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega}g(t)\,d\sigma_t, \tag{8.53}$$

*for all $\epsilon\in\,]0,\epsilon_1^*[$, $l\in\mathbb{N}\setminus\{0\}$.*

*Proof.* Let $\epsilon\in\,]0,\epsilon_1^*[$, $l\in\mathbb{N}\setminus\{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}}\mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx\\ &= \int_{rA}\mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx\\ &= l^n\int_{\frac{r}{l}A}\mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A}\mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]}u_{(\epsilon,r/l)}(x)\,dx\\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]}u[\epsilon]\big(\frac{l}{r}x\big)\,dx\\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]}u[\epsilon](t)\,dt\\ &= \frac{r^n}{l^n}\frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega}g(t)\,d\sigma_t.\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n}\mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n\frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega}g(t)\,d\sigma_t,$$

and the conclusion follows. □

We give the following.

**Definition 8.35.** Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). For each pair $(\epsilon,\delta)\in\,]0,\epsilon_1^*[\,\times\,]0,+\infty[$, we set

$$\mathcal{F}(\epsilon,\delta)\equiv\int_{A\cap\mathbb{T}_a(\epsilon,\delta)}|\nabla u_{(\epsilon,\delta)}(x)|^2\,dx-\frac{k^2}{\delta^2}\int_{A\cap\mathbb{T}_a(\epsilon,\delta)}|u_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 8.36. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). Let $(\epsilon,\delta) \in ]0,\epsilon_1^*[ \times ]0,+\infty[$. We have

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx = \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2\,dt$$
$$= \delta^{n-2} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2\,dt,$$

and

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2\,dx = \delta^n \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](t)|^2\,dt.$$

Accordingly,

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx - \frac{k^2}{\delta^2}\int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2\,dx$$
$$= \delta^{n-2}\Big(\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2\,dt - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](t)|^2\,dt\Big).$$

Then we give the following definition, where we consider $\mathcal{F}(\epsilon,\delta)$, with $\epsilon$ equal to a certain function of $\delta$.

**Definition 8.37.** For each $\delta \in ]0,+\infty[$, we set

$$\epsilon[\delta] \equiv \delta^{\frac{2}{n}}.$$

Let $\epsilon_5$ be as in Theorem 8.23, if $n$ is odd, or as in Theorem 8.24, if $n$ is even. Let $\delta_1 > 0$ be such that $\epsilon[\delta] \in ]0,\epsilon_5[$, for all $\delta \in ]0,\delta_1[$. Then we set

$$\mathcal{F}[\delta] \equiv \mathcal{F}(\epsilon[\delta],\delta),$$

for all $\delta \in ]0,\delta_1[$.

Here we may note that the 'radius' of the holes is $\delta\epsilon[\delta] = \delta^{\frac{n+2}{n}}$ which is different from the one which appears in Homogenization Theory (cf. *e.g.*, Ansini and Braides [7] and references therein.)

In the following Propositions we compute the limit of $\mathcal{F}[\delta]$ as $\delta$ tends to 0.

**Proposition 8.38.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (8.16). *Let $\epsilon_5$ be as in Theorem 8.23. Let $\delta_1 > 0$ be as in Definition 8.37. Then*

$$\lim_{\delta\to 0^+} \mathcal{F}[\delta] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\,dx,$$

*where $\tilde{u}$ is as in Definition 8.14.*

*Proof.* For each $\delta \in ]0,\delta_1[$, we set

$$\mathcal{G}(\delta) \equiv \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2\,dx - \frac{k^2}{\delta^2}\int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |u_{(\epsilon[\delta],\delta)}(x)|^2\,dx.$$

Let $\delta \in ]0,\delta_1[$. By Remark 8.36 and Theorem 8.23, we have

$$\mathcal{G}(\delta) = \delta^{n-2}(\epsilon[\delta])^n G[\epsilon[\delta]]$$
$$= \delta^{n-2}\delta^2 G[\delta^{\frac{2}{n}}],$$

where $G$ is as in Theorem 8.23. On the other hand,

$$\lfloor (1/\delta) \rfloor^n \mathcal{G}(\delta) \le \mathcal{F}[\delta] \le \lceil (1/\delta) \rceil^n \mathcal{G}(\delta).$$

As a consequence, since

$$\lim_{\delta\to 0^+} \lfloor (1/\delta) \rfloor^n \delta^n = 1, \qquad \lim_{\delta\to 0^+} \lceil (1/\delta) \rceil^n \delta^n = 1,$$

then

$$\lim_{\delta\to 0^+} \mathcal{F}[\delta] = G[0].$$

Finally, by equality (8.46), we easily conclude. □

**Proposition 8.39.** *Let $n$ be even and $n > 2$. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (8.16). *Let $\epsilon_5$ be as in Theorem 8.24. Let $\delta_1 > 0$ be as in Definition 8.37. Then*

$$\lim_{\delta\to 0^+} \mathcal{F}[\delta] = \int_{\mathbb{R}^n\setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\, dx,$$

*where $\tilde{u}$ is as in Definition 8.14.*

*Proof.* For each $\delta \in ]0,\delta_1[$, we set

$$\mathcal{G}(\delta) \equiv \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2\, dx - \frac{k^2}{\delta^2}\int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |u_{(\epsilon[\delta],\delta)}(x)|^2\, dx.$$

Let $\delta \in ]0,\delta_1[$. By Remark 8.36 and Theorem 8.24, we have

$$\begin{aligned}\mathcal{G}(\delta) =& \delta^{n-2}(\epsilon[\delta])^n G_1^{\#}[\epsilon[\delta], \epsilon[\delta]\log\epsilon[\delta]] \\ &+ \delta^{n-2}(\epsilon[\delta])^{2n-2}(\log\epsilon[\delta]) G_2^{\#}[\epsilon[\delta], \epsilon[\delta]\log\epsilon[\delta]] \\ =& \delta^{n-2}\delta^2 G_1^{\#}[\delta^{\frac{2}{n}}, \delta^{\frac{2}{n}}\log(\delta^{\frac{2}{n}})] \\ &+ \delta^{n-2}\delta^{4-\frac{4}{n}}(\log(\delta^{\frac{2}{n}}))G_2^{\#}[\delta^{\frac{2}{n}}, \delta^{\frac{2}{n}}\log(\delta^{\frac{2}{n}})],\end{aligned}$$

where $G_1^{\#}$ and $G_2^{\#}$ are as in Theorem 8.24. On the other hand,

$$\lfloor (1/\delta) \rfloor^n \mathcal{G}(\delta) \le \mathcal{F}[\delta] \le \lceil (1/\delta) \rceil^n \mathcal{G}(\delta).$$

As a consequence, since

$$\lim_{\delta\to 0^+} \lfloor (1/\delta) \rfloor^n \delta^n = 1, \qquad \lim_{\delta\to 0^+} \lceil (1/\delta) \rceil^n \delta^n = 1,$$

then

$$\lim_{\delta\to 0^+} \mathcal{F}[\delta] = G_1^{\#}[0,0].$$

Finally, by equality (8.50), we easily conclude. □

In the following Propositions we represent the function $\mathcal{F}[\cdot]$ by means of real analytic functions.

**Proposition 8.40.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $\epsilon_5$, and $G$ be as in Theorem 8.23. Let $\delta_1 > 0$ be as in Definition 8.37. Then*

$$\mathcal{F}[(1/l)] = G[(1/l)^{\frac{2}{n}}],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* By arguing as in the proof of Proposition 8.38, one can easily see that

$$\begin{aligned}\mathcal{F}[(1/l)] &= l^n (1/l)^{n-2}(1/l)^2 G[(1/l)^{\frac{2}{n}}] \\ &= G[(1/l)^{\frac{2}{n}}],\end{aligned}$$

for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$. □

**Proposition 8.41.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $\epsilon_5$, $G_1^\#$, and $G_2^\#$ be as in Theorem 8.24. Let $\delta_1 > 0$ be as in Definition 8.37. Then*

$$\mathcal{F}[(1/l)] = G_1^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})] + (1/l)^{2-\frac{4}{n}} \log((1/l)^{\frac{2}{n}}) G_2^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* By arguing as in the proof of Proposition 8.39, one can easily see that

$$\begin{aligned}\mathcal{F}[(1/l)] &= l^n (1/l)^{n-2} (1/l)^2 \Big\{ G_1^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})] \\ &\quad + (1/l)^{2-\frac{4}{n}} \log((1/l)^{\frac{2}{n}}) G_2^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})] \Big\} \\ &= G_1^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})] + (1/l)^{2-\frac{4}{n}} \log((1/l)^{\frac{2}{n}}) G_2^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})],\end{aligned}$$

for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$. □

## 8.4 A variant of an homogenization problem for the Helmholtz equation with Neumann boundary conditions in a periodically perforated domain

In this section we consider a variant of the previous homogenization problem for the Helmhlotz equation with Neumann boundary conditions in a periodically perforated domain. As above, most of the results are obtained under the assumption that $\mathrm{Im}(k) \neq 0$ and $\mathrm{Re}(k) = 0$.

### 8.4.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 6.7.1, 8.2.1, 8.3.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). For each $(\epsilon, \delta) \in ]0, \epsilon_1^*[ \times ]0, +\infty[$, we consider the following periodic Neumann problem for the Helmholtz equation.

$$\begin{cases} \Delta u(x) + \frac{k^2}{\delta^2} u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\ u(x + \delta a_j) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x) = g(\frac{1}{\epsilon\delta}(x - \delta w)) & \forall x \in \partial\Omega(\epsilon,\delta). \end{cases} \tag{8.54}$$

In contrast to problem (8.51), we note that in the third equation of problem (8.55) there is not the factor $1/\delta$ in front of $g(\frac{1}{\epsilon\delta}(x - \delta w))$.

By virtue of Theorem 8.5, we can give the following definition.

**Definition 8.42.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). For each $(\epsilon, \delta) \in ]0, \epsilon_1^*[ \times ]0, +\infty[$, we denote by $u_{(\epsilon,\delta)}$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a(\epsilon,\delta), \mathbb{C})$ of boundary value problem (8.54).

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. In order to do so we introduce the following.

**Definition 8.43.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). For each $\epsilon \in ]0, \epsilon_1^*[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$ of the following periodic Neumann problem for the Helmholtz equation.

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x + a_j) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) = g(\frac{1}{\epsilon}(x - w)) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{8.55}$$

*Remark* 8.44. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). For each pair $(\epsilon, \delta) \in ]0, \epsilon_1^*[ \times ]0, +\infty[$, we have

$$u_{(\epsilon,\delta)}(x) = \delta u[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a(\epsilon,\delta).$$

We have the following.

**Proposition 8.45.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (8.16). *Let $\epsilon_3$ be as in Proposition 8.18. Then there exist $\tilde{\epsilon} \in ]0, \epsilon_3[$ and a real analytic map $N$ of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$ such that*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon \|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon \|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n, \mathbb{C}).$$

*Proof.* It is an immediate consequence of Proposition 8.30. □

**Proposition 8.46.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (8.16). *Let $\epsilon_3$, $\epsilon_2'$ be as in Proposition 8.19. Then there exist $\tilde{\epsilon} \in ]0, \epsilon_3[$ and two real analytic maps $N_1^\#$, $N_2^\#$ of $]-\tilde{\epsilon}, \tilde{\epsilon}[ \times ]-\epsilon_2', \epsilon_2'[$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$ such that*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta \|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon, \epsilon \log \epsilon] + \epsilon^{n-1}(\log \epsilon) N_2^\#[\epsilon, \epsilon \log \epsilon]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta \|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon, \epsilon \log \epsilon] + \epsilon^{n-1}(\log \epsilon) N_2^\#[\epsilon, \epsilon \log \epsilon]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[\delta u[\epsilon]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n, \mathbb{C}).$$

*Proof.* It is an immediate consequence of Proposition 8.31. □

### 8.4.2 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorems we deduce by Propositions 8.45, 8.46 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon, \delta)$ tends to $(0, 0)$. Namely, we prove the following.

**Theorem 8.47.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (8.16). *Let $\tilde{\epsilon}$, $N$ be as in Proposition 8.45. Then*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon \|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon \|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n, \mathbb{C}).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \delta \|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} \\ &= \delta\epsilon \|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \delta \|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} \\ &= \delta\epsilon \|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, +\infty[$. □

**Theorem 8.48.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (8.16). *Let $\tilde{\epsilon}$, $N_1^\#$, $N_2^\#$ be as in Proposition 8.46. Then*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[ \times ]0,+\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \delta\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \delta\|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \delta\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \delta\|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[ \times ]0,+\infty[$. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 8.49.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l}\frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega} g(t)\,d\sigma_t, \tag{8.56}$$

*for all $\epsilon \in ]0,\epsilon_1^*[$, $l \in \mathbb{N} \setminus \{0\}$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_1^*[$, $l \in \mathbb{N} \setminus \{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx\\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx\\ &= l^n\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx\\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} \frac{r}{l}u[\epsilon]\big(\frac{l}{r}x\big)\,dx\\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} \frac{r}{l}u[\epsilon](t)\,dt\\ &= \frac{1}{l^n}\frac{r^{n+1}}{l}\frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega} g(t)\,d\sigma_t.\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l}\frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega} g(t)\,d\sigma_t,$$

and the conclusion follows. □

We give the following.

**Definition 8.50.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). For each pair $(\epsilon, \delta) \in ]0, \epsilon_1^*[ \times ]0, +\infty[$, we set

$$\mathcal{F}(\epsilon, \delta) \equiv \int_{A \cap \mathbb{T}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{A \cap \mathbb{T}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2 \, dx.$$

*Remark* 8.51. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (8.14), (8.15), respectively. Let $\epsilon_1^*$ be as in (8.16). Let $(\epsilon, \delta) \in ]0, \epsilon_1^*[ \times ]0, +\infty[$. We have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2 \, dt \\ &= \delta^n \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2 \, dt,\end{aligned}$$

and

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2 \, dx = \delta^{n+2} \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](t)|^2 \, dt.$$

Accordingly,

$$\begin{aligned}&\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2 \, dx \\ &\qquad = \delta^n \Big( \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2 \, dt - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](t)|^2 \, dt \Big).\end{aligned}$$

In the following Propositions we represent the function $\mathcal{F}(\cdot, \cdot)$ by means of real analytic functions.

**Proposition 8.52.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $\epsilon_5$, and $G$ be as in Theorem 8.23. Then*

$$\mathcal{F}\Big(\epsilon, \frac{1}{l}\Big) = \epsilon^n G[\epsilon],$$

*for all $\epsilon \in ]0, \epsilon_5[$ and for all $l \in \mathbb{N} \setminus \{0\}$.*

*Proof.* Let $(\epsilon, \delta) \in ]0, \epsilon_5[ \times ]0, +\infty[$. By Remark 8.51 and Theorem 8.23, we have

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2 \, dx = \delta^n \epsilon^n G[\epsilon]$$

where $G$ is as in Theorem 8.23. On the other hand, if $\epsilon \in ]0, \epsilon_5[$ and $l \in \mathbb{N} \setminus \{0\}$, then we have

$$\begin{aligned}\mathcal{F}\Big(\epsilon, \frac{1}{l}\Big) &= l^n \frac{1}{l^n} \epsilon^n G[\epsilon], \\ &= \epsilon^n G[\epsilon],\end{aligned}$$

and the conclusion easily follows. □

**Proposition 8.53.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (8.14), (8.15), *respectively. Let $\epsilon_1^*$ be as in* (8.16). *Let $\epsilon_5$, $G_1^\#$, and $G_2^\#$ be as in Theorem 8.24. Then*

$$\mathcal{F}\Big(\epsilon, \frac{1}{l}\Big) = \epsilon^n G_1^\#[\epsilon, \epsilon \log \epsilon] + \epsilon^{2n-2} (\log \epsilon) G_2^\#[\epsilon, \epsilon \log \epsilon],$$

*for all $\epsilon \in ]0, \epsilon_5[$ and for all $l \in \mathbb{N} \setminus \{0\}$.*

*Proof.* Let $(\epsilon, \delta) \in ]0, \epsilon_5[ \times ]0, +\infty[$. By Remark 8.51 and Theorem 8.24, we have

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2 \, dx \\ = \delta^n \Big\{ \epsilon^n G_1^{\#}[\epsilon, \epsilon \log \epsilon] + \epsilon^{2n-2} (\log \epsilon) G_2^{\#}[\epsilon, \epsilon \log \epsilon] \Big\}$$

where $G_1^{\#}$, $G_2^{\#}$ are as in Theorem 8.24. On the other hand, if $\epsilon \in ]0, \epsilon_5[$ and $l \in \mathbb{N} \setminus \{0\}$, then we have

$$\mathcal{F}\Big(\epsilon, \frac{1}{l}\Big) = l^n \frac{1}{l^n} \Big\{ \epsilon^n G_1^{\#}[\epsilon, \epsilon \log \epsilon] + \epsilon^{2n-2} (\log \epsilon) G_2^{\#}[\epsilon, \epsilon \log \epsilon] \Big\}, \\ = \epsilon^n G_1^{\#}[\epsilon, \epsilon \log \epsilon] + \epsilon^{2n-2} (\log \epsilon) G_2^{\#}[\epsilon, \epsilon \log \epsilon],$$

and the conclusion easily follows. □

# CHAPTER 9

# Singular perturbation and homogenization problems for the Helmholtz equation with Dirichlet boundary conditions

In this Chapter we introduce the periodic Dirichlet problem for the Helmholtz equation and we study singular perturbation and homogenization problems for the Helmholtz operator with Dirichlet boundary conditions in a periodically perforated domain. First of all, by means of periodic double layer potentials, we show the solvability of the Dirichlet problem. Secondly, we consider singular perturbation problems in a periodically perforated domain with small holes, and we apply the obtained results to homogenization problems. Our strategy follows the functional analytic approach of Lanza [75], where the asymptotic behaviour of the solutions of the Dirichlet problem for the Laplace operator in a domain with a small hole has been studied (see also [70].) We also mention Lanza [79], dealing with a Neumann eigenvalue problem in a perforated domain. We note that linear boundary value problems in singularly perturbed domains in the frame of linearized elasticity have been analysed by Dalla Riva in his Ph.D. Dissertation [33]. One of the tools used in our analysis is the study of the dependence of layer potentials upon perturbations (cf. Lanza and Rossi [86] and also Dalla Riva and Lanza [40].)

We retain the notation introduced in Sections 1.1 and 1.3, Chapter 6 and Appendix E. For the definitions of $\mathrm{Eig}_D[\mathbb{I}]$, $\mathrm{Eig}_N[\mathbb{I}]$, $\mathrm{Eig}^a_D[\mathbb{I}]$, $\mathrm{Eig}^a_N[\mathbb{I}]$, we refer to Chapter 7.

## 9.1 A periodic Dirichlet boundary value problem for the Helmholtz equation

In this Section we introduce the periodic Dirichlet problem for the Helmholtz equation and we show the existence and uniqueness of a solution by means of the periodic double layer potential.

### 9.1.1 Formulation of the problem

In this Subsection we introduce the periodic Dirichlet problem for the Helmholtz equation.

First of all, we need to introduce some notation. Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). We shall consider the following assumptions.

$$k \in \mathbb{C},\ k^2 \neq |2\pi a^{-1}(z)|^2 \quad \forall z \in \mathbb{Z}^n; \tag{9.1}$$

$$\Gamma \in C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C}). \tag{9.2}$$

We are now ready to give the following.

**Definition 9.1.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $k$, $\Gamma$ be as in (9.1), (9.2), respectively. We say that a function $u \in C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \mathbb{C}) \cap C^2(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ solves the *periodic Dirichlet* *problem for the Helmholtz equation* if

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}, \\ u(x) = \Gamma(x) & \forall x \in \partial\mathbb{I}. \end{cases} \tag{9.3}$$

### 9.1.2 Existence and uniqueness results for the solutions of the periodic Dirichlet problem

In this Subsection we prove existence and uniqueness results for the solutions of the periodic Dirichlet problem for the Helmholtz equation.

As we know, in order to solve problem (9.3) by means of periodic double layer potentials, we need to study some integral equations. Thus, in the following Proposition, we study an operator related to the equations that we shall consider in the sequel.

**Proposition 9.2.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0, 1[$. Let $\mathbb{I}$ be as in (1.46). Let $k$ be as in (9.1). Assume that $k^2 \notin \mathrm{Eig}^a_D[\mathbb{T}_a[\mathbb{I}]]$ and $k^2 \notin \mathrm{Eig}_N[\mathbb{I}]$. Then the following statements hold.*

*(i) Let $\mu \in L^2(\partial\mathbb{I}, \mathbb{C})$ and*

$$-\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S^{a,k}_n(t - s))\mu(s)\, d\sigma_s = 0 \quad \text{a.e. on } \partial\mathbb{I}, \tag{9.4}$$

*then $\mu = 0$.*

*(ii) Let $\mu \in L^2(\partial\mathbb{I}, \mathbb{C})$ and*

$$-\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S^{a,k}_n(t - s))\mu(s)\, d\sigma_s = 0 \quad \text{a.e. on } \partial\mathbb{I}, \tag{9.5}$$

*then $\mu = 0$.*

*Proof.* We first prove statement $(i)$. By Theorem 6.18 $(iv)$, we have $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$. Then by Theorem 6.11 $(i)$, we have that the function $v^+ \equiv v^+_a[\partial\mathbb{I}, \mu, k]_{|\mathrm{cl}\,\mathbb{I}}$ is in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I}, \mathbb{C})$ and solves the following boundary value problem

$$\begin{cases} \Delta v^+(x) + k^2 v^+(x) = 0 & \forall x \in \mathbb{I}, \\ \frac{\partial}{\partial\nu_{\mathbb{I}}} v^+(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

Hence, since $k^2 \notin \mathrm{Eig}_N[\mathbb{I}]$, we have $v^+ = 0$ in $\mathrm{cl}\,\mathbb{I}$, and so

$$v^+ = 0 \qquad \text{on } \partial\mathbb{I}.$$

Furthermore, by Theorem 6.11 $(i)$, we have

$$v^-_a[\partial\mathbb{I}, \mu, k] = v^+_a[\partial\mathbb{I}, \mu, k] = 0 \qquad \text{on } \partial\mathbb{I}.$$

Then by Theorem 6.11 $(i)$, we have that the function $v^- \equiv v^-_a[\partial\mathbb{I}, \mu, k]$ is in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ and solves the following boundary value problem

$$\begin{cases} \Delta v^-(x) + k^2 v^-(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ v^-(x + a_j) = v^-(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}, \\ v^-(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

Accordingly, since $k^2 \notin \mathrm{Eig}^a_D[\mathbb{T}_a[\mathbb{I}]]$, we have $v^- = 0$ in $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$ and consequently

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} v^- = 0 \qquad \text{on } \partial\mathbb{I}.$$

Thus, by Theorem 6.11 $(i)$, we have

$$\mu = \frac{\partial}{\partial\nu_{\mathbb{I}}} v^-_a[\partial\mathbb{I}, \mu, k] - \frac{\partial}{\partial\nu_{\mathbb{I}}} v^+_a[\partial\mathbb{I}, \mu, k] = 0 \qquad \text{on } \partial\mathbb{I},$$

and the proof of $(i)$ is complete. We now turn to the proof of statement $(ii)$. By Theorem 6.18 $(ii)$, we have $\mu \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$. Then by Theorem 6.7 $(i)$, we have that the function $w^- \equiv w_a^-[\partial\mathbb{I},\mu,k]$ is in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ and solves the following boundary value problem

$$\begin{cases} \Delta w^-(x) + k^2 w^-(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ w^-(x+a_j) = w^-(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1,\dots,n\}, \\ w^-(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

Accordingly, since $k^2 \notin \mathrm{Eig}_D^a[\mathbb{T}_a[\mathbb{I}]]$, we have $w^- = 0$ in $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$. Furthermore, by Theorem 6.7 $(i)$, we have

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} w_a^+[\partial\mathbb{I},\mu,k] = \frac{\partial}{\partial\nu_{\mathbb{I}}} w_a^-[\partial\mathbb{I},\mu,k] = 0 \qquad \text{on } \partial\mathbb{I}.$$

Then, by Theorem 6.7 $(i)$, the function $w^+ \equiv w_a^+[\partial\mathbb{I},\mu,k]_{|\mathrm{cl}\,\mathbb{I}}$ is in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$ and solves the following boundary value problem

$$\begin{cases} \Delta w^+(x) + k^2 w^+(x) = 0 & \forall x \in \mathbb{I}, \\ \frac{\partial}{\partial\nu_{\mathbb{I}}} w^+(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

Hence, since $k^2 \notin \mathrm{Eig}_N[\mathbb{I}]$, we have $w^+ = 0$ in $\mathrm{cl}\,\mathbb{I}$. Thus, by Theorem 6.7 $(i)$, we have

$$\mu = w_a^+[\partial\mathbb{I},\mu,k] - w_a^-[\partial\mathbb{I},\mu,k] = 0 \qquad \text{on } \partial\mathbb{I},$$

and the proof of $(ii)$ is complete. □

Then we have the following Theorem.

**Theorem 9.3.** *Let* $m \in \mathbb{N}\setminus\{0\}$. *Let* $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be as in* (1.46). *Let* $k$ *be as in* (9.1). *Assume that* $k^2 \notin \mathrm{Eig}_D^a[\mathbb{T}_a[\mathbb{I}]]$ *and* $k^2 \notin \mathrm{Eig}_N[\mathbb{I}]$. *Then the following statements hold.*

*(i) The map* $\mathcal{L}$ *of* $L^2(\partial\mathbb{I},\mathbb{C})$ *to* $L^2(\partial\mathbb{I},\mathbb{C})$, *which takes* $\mu$ *to the function* $\mathcal{L}[\mu]$ *of* $\partial\mathbb{I}$ *to* $\mathbb{C}$, *defined by*

$$\mathcal{L}[\mu](t) \equiv -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{9.6}$$

*is a linear homeomorphism of* $L^2(\partial\mathbb{I},\mathbb{C})$ *onto itself.*

*(ii) The map* $\tilde{\mathcal{L}}$ *of* $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ *to* $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$, *which takes* $\mu$ *to the function* $\tilde{\mathcal{L}}[\mu]$ *of* $\partial\mathbb{I}$ *to* $\mathbb{C}$, *defined by*

$$\tilde{\mathcal{L}}[\mu](t) \equiv -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s \quad \forall t \in \partial\mathbb{I}, \tag{9.7}$$

*is a linear homeomorphism of* $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ *onto itself.*

*(iii) The map* $\mathcal{L}'$ *of* $L^2(\partial\mathbb{I},\mathbb{C})$ *to* $L^2(\partial\mathbb{I},\mathbb{C})$, *which takes* $\mu$ *to the function* $\mathcal{L}'[\mu]$ *of* $\partial\mathbb{I}$ *to* $\mathbb{C}$, *defined by*

$$\mathcal{L}'[\mu](t) \equiv -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{9.8}$$

*is a linear homeomorphism of* $L^2(\partial\mathbb{I},\mathbb{C})$ *onto itself.*

*(iv) The map* $\tilde{\mathcal{L}}'$ *of* $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ *to* $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$, *which takes* $\mu$ *to the function* $\tilde{\mathcal{L}}'[\mu]$ *of* $\partial\mathbb{I}$ *to* $\mathbb{C}$, *defined by*

$$\tilde{\mathcal{L}}'[\mu](t) \equiv -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s \quad \forall t \in \partial\mathbb{I}, \tag{9.9}$$

*is a linear homeomorphism of* $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ *onto itself.*

*Proof.* We first prove statement $(i)$. By Proposition 9.2 $(i)$, we have that $\mathcal{L}$ is injective. Since the singularity in the involved integral operator is weak, we have that $\mathcal{L}$ is continuous and that $\mathcal{L} + \frac{1}{2}I$ is a compact operator on $L^2(\partial\mathbb{I},\mathbb{C})$ (cf. *e.g.*, Folland [52, Prop. 3.11, p. 121].) Hence, by the Fredholm Theory, we have that $\mathcal{L}$ is surjective and, by the Open Mapping Theorem, we have that it is a linear homeomorphism of $L^2(\partial\mathbb{I},\mathbb{C})$ onto itself. We now consider statement $(ii)$. By Theorem 6.11 $(iii)$, we have that $\tilde{\mathcal{L}}$ is a linear continuous operator of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ to itself. Hence, by the Open Mapping Theorem, in order to prove that it is a linear homeomorphism of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ onto itself, it suffices to prove that it is a bijection. By Proposition 9.2 $(i)$, $\tilde{\mathcal{L}}$ is injective. Now let $\phi \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$. By statement $(i)$, there exists $\mu \in L^2(\partial\mathbb{I},\mathbb{C})$ such that

$$\phi(t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I},$$

and, by Proposition 6.18 $(iv)$, we have $\mu \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$. As a consequence, $\tilde{\mathcal{L}}$ is surjective, and the proof of $(ii)$ is complete. We now turn to the proof of statement $(iii)$. By Proposition 9.2 $(ii)$, we have that $\mathcal{L}'$ is injective. Since the singularity in the involved integral operator is weak, we have that $\mathcal{L}'$ is continuous and that $\mathcal{L}' + \frac{1}{2}I$ is a compact operator on $L^2(\partial\mathbb{I},\mathbb{C})$ (cf. *e.g.*, Folland [52, Prop. 3.11, p. 121].) Hence, by the Fredholm Theory, we have that $\mathcal{L}'$ is surjective and, by the Open Mapping Theorem, we have that it is a linear homeomorphism of $L^2(\partial\mathbb{I},\mathbb{C})$ onto itself. We finally prove statement $(iv)$. By Theorem 6.7 $(ii)$, we have that $\tilde{\mathcal{L}}'$ is a linear continuous operator of $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ to itself. Hence, by the Open Mapping Theorem, in order to prove that it is a linear homeomorphism of $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ onto itself, it suffices to prove that it is a bijection. By Proposition 9.2 $(ii)$, $\tilde{\mathcal{L}}'$ is injective. Now let $\phi \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$. By statement $(iii)$, there exists $\mu \in L^2(\partial\mathbb{I},\mathbb{C})$ such that

$$\phi(t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I},$$

and, by Proposition 6.18 $(ii)$, we have $\mu \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$. As a consequence, $\tilde{\mathcal{L}}$ is surjective, and the proof is complete. □

We are now ready to prove the main result of this section.

**Theorem 9.4.** *Let $m \in \mathbb{N}\setminus\{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $k$ be as in* (9.1)*. Let $\Gamma \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$. Assume that $k^2 \notin \mathrm{Eig}^a_D[\mathbb{T}_a[\mathbb{I}]]$ and $k^2 \notin \mathrm{Eig}_N[\mathbb{I}]$. Then boundary value problem* (9.3) *has a unique solution $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C}) \cap C^2(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$. Moreover,*

$$u(x) = w_a^-[\mathbb{I},\mu,k](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \tag{9.10}$$

*where $\mu$ is the unique function in $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ that solves the following equation*

$$-\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s = \Gamma(t) \qquad \forall t \in \partial\mathbb{I}. \tag{9.11}$$

*Proof.* Clearly, since $k^2 \notin \mathrm{Eig}^a_D[\mathbb{T}_a[\mathbb{I}]]$, it suffices to prove the existence. By Theorem 9.3 $(iv)$, there exists a unique $\mu \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ such that (9.11) holds. Then, by Theorem 6.7 $(i)$, we have that $w_a^-[\partial\mathbb{I},\mu,k] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C})$, that

$$w_a^-[\partial\mathbb{I},\mu,k](t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s = \Gamma(t) \qquad \forall t \in \partial\mathbb{I}.$$

and that

$$\Delta w_a^-[\partial\mathbb{I},\mu,k](t) + k^2 w_a^-[\partial\mathbb{I},\mu,k](t) = 0 \qquad \forall t \in \mathbb{T}_a[\mathbb{I}].$$

Finally, by the periodicity of $w_a^-[\partial\mathbb{I},\mu,k]$, we have that $w_a^-[\partial\mathbb{I},\mu,k]$ solves boundary value problem (9.3). □

We are now ready to prove the following representation Theorem.

**Theorem 9.5.** *Let $m \in \mathbb{N}\setminus\{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $k$ be as in* (9.1)*. Assume that $k^2 \notin \mathrm{Eig}^a_D[\mathbb{T}_a[\mathbb{I}]]$ and $k^2 \notin \mathrm{Eig}_N[\mathbb{I}]$. Let $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ be such that*

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x+a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1,\dots,n\}. \end{cases}$$

*Then there exists a unique function $\mu \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ such that*

$$u(x) = w_a^-[\mathbb{I},\mu,k](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]. \tag{9.12}$$

*Moreover $\mu$ is the unique function in $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ that solves the following equation*

$$-\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s = u(t) \qquad \forall t \in \partial\mathbb{I}. \tag{9.13}$$

*Proof.* Let $\mu \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$. Clearly, since $k^2 \notin \mathrm{Eig}^a_D[\mathbb{T}_a[\mathbb{I}]]$ and by Theorem 6.11, we have

$$u(x) = w_a^-[\mathbb{I},\mu,k](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],$$

if and only if

$$-\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n^{a,k}(t-s))\mu(s)\,d\sigma_s = u(t) \qquad \forall t \in \partial\mathbb{I}.$$

By Theorem 9.3 (*iv*), there exists a unique $\mu \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ such that (9.13) holds and hence the conclusion easily follows. □

## 9.2 Asymptotic behaviour of the solutions of the Dirichlet problem for the Helmholtz equation in a periodically perforated domain

In this Section we study the asymptotic behaviour of the solutions of the Dirichlet problem for the Helmholtz equation in a periodically perforated domain with small holes.

### 9.2.1 Notation

We retain the notation introduced in Subsections 1.8.1, 6.7.1.

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). We shall consider also the following assumptions.

$$k \in \mathbb{C},\ k^2 \neq |2\pi a^{-1}(z)|^2 \quad \forall z \in \mathbb{Z}^n; \tag{9.14}$$

$$g \in C^{m,\alpha}(\partial\Omega,\mathbb{C}). \tag{9.15}$$

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$ be as in (1.56), (1.57), (9.14), respectively. By Proposition 7.42, there exists $\epsilon_1^* \in ]0,\epsilon_1[$ such that

$$k^2 \notin \Big(\mathrm{Eig}_D[\Omega_\epsilon] \cup \mathrm{Eig}_N[\Omega_\epsilon] \cup \mathrm{Eig}^a_D[\mathbb{T}_a[\Omega_\epsilon]] \cup \mathrm{Eig}^a_N[\mathbb{T}_a[\Omega_\epsilon]]\Big) \qquad \forall \epsilon \in ]0,\epsilon_1^*]. \tag{9.16}$$

### 9.2.2 Preliminaries

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). For each $\epsilon \in ]0,\epsilon_1^*[$, we consider the following periodic Dirichlet problem for the Helmholtz equation.

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ u(x) = g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{9.17}$$

By virtue of Theorem 9.4, we can give the following definition.

**Definition 9.6.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). For each $\epsilon \in ]0,\epsilon_1^*[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],\mathbb{C})$ of boundary value problem (9.17).

Then we have the following Lemmas.

**Lemma 9.7.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15)*, respectively. Let $\epsilon_1^*$ be as in* (9.16)*. Let $\epsilon \in ]0,\epsilon_1^*[$. Let $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon,\mathbb{C})$. Then*

$$u[\epsilon](x) = v_a^-[\partial\Omega_\epsilon,\mu,k](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \tag{9.18}$$

*if and only if the function $\mu$ solves the following integral equation*

$$g(\frac{1}{\epsilon}(x-w)) = \int_{\partial\Omega_\epsilon} S_n^{a,k}(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega_\epsilon. \tag{9.19}$$

*In particular, there exists a unique function $\mu$ in $C^{m-1,\alpha}(\partial\Omega_\epsilon,\mathbb{C})$ such that* (9.19) *holds.*

*Proof.* Assume that equality (9.18) holds. Then, by Theorem 6.11, equality (9.19) holds. Conversely, assume that equality (9.19) holds. Then,

$$u[\epsilon](x) = v_a^-[\partial\Omega_\epsilon, \mu, k](x) \qquad \forall x \in \partial\Omega_\epsilon.$$

Accordingly, since $k^2 \notin \mathrm{Eig}_D^a[\mathbb{T}_a[\Omega_\epsilon]]$, we have

$$u[\epsilon](x) = v_a^-[\partial\Omega_\epsilon, \mu, k](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$

Thus the equivalence of (9.18) and (9.19) is proved. In order to conclude, we need to prove the existence and uniqueness of a solution of equation (9.19). We first prove uniqueness. Let $\mu_1$, $\mu_2 \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ solve equation (9.19). Then, if we set $\tilde{\mu} \equiv \mu_1 - \mu_2$, since $k^2 \notin \mathrm{Eig}_D^a[\mathbb{T}_a[\Omega_\epsilon]]$, we have

$$v_a^-[\partial\Omega_\epsilon, \tilde{\mu}, k] = 0 \qquad \text{in } \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$

Consequently,

$$\frac{\partial}{\partial\nu_{\Omega_\epsilon}} v_a^-[\partial\Omega_\epsilon, \tilde{\mu}, k] = 0 \qquad \text{on } \partial\Omega_\epsilon,$$

and, by Theorem 6.11,

$$\frac{1}{2}\tilde{\mu}(x) + \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}(S_n^{a,k}(x-y))\tilde{\mu}(y)\,d\sigma_y = 0 \qquad \forall x \in \partial\Omega_\epsilon.$$

Hence, by Theorem 8.4 $(i)$, we have $\tilde{\mu} = 0$ on $\partial\Omega_\epsilon$ and therefore

$$\mu_1 = \mu_2 \qquad \text{on } \partial\Omega_\epsilon.$$

We now turn to prove the existence. By virtue of Theorem 8.5, there exists a solution $u[\epsilon]$ in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$ of boundary value problem (9.17). By Theorem 8.6, there exists a unique function $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ such that (9.18) holds, and, accordingly, such that (9.19) holds. Thus the proof is complete. □

**Lemma 9.8.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15)*, respectively. Let $\epsilon_1^*$ be as in* (9.16)*. Let $\epsilon \in ]0, \epsilon_1^*[$. Let $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Then*

$$u[\epsilon](x) = v_a^-[\partial\Omega_\epsilon, \epsilon^{-1}\theta(\frac{1}{\epsilon}(\cdot - w)), k](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \tag{9.20}$$

*if and only if the function $\theta$ solves the following integral equation*

$$\begin{aligned} g(t) = \int_{\partial\Omega} S_n(t-s, \epsilon k)\theta(s)\,d\sigma_s + \epsilon^{n-2}(\log\epsilon)k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\theta(s)\,d\sigma_s \\ + \epsilon^{n-2}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s \qquad \forall t \in \partial\Omega. \end{aligned} \tag{9.21}$$

*In particular, there exists a unique function $\theta$ in $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ such that* (9.21) *holds.*

*Proof.* It is a straightforward consequence of Lemma 9.7, of the rule of change of variables in integrals, of well known properties of functions in Schauder spaces and of equality (6.24). □

Then we have the following well known result of classical potential theory.

**Lemma 9.9.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the following statements hold.*

(*i*) *Let $n = 2$. Then for each $g \in C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$, there exists a unique pair $(\mu, \rho)$ in the space*

$$\{\phi \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C}) \colon \int_{\partial\mathbb{I}} \phi\,d\sigma = 0\} \times \mathbb{C},$$

*such that*

$$\int_{\partial\mathbb{I}} S_2(t-s)\mu(s)\,d\sigma_s + \rho = g(t) \qquad \forall t \in \partial\mathbb{I}.$$

(*ii*) *Let $n \geq 3$. Then for each $g \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$, there exists a unique function $\mu$ in $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$, such that*

$$\int_{\partial\mathbb{I}} S_n(t-s)\mu(s)\,d\sigma_s = g(t) \qquad \forall t \in \partial\mathbb{I}.$$

*Proof.* We first prove statement (*i*). First of all, we note that if the pair $(\mu,\rho)$ is in the space $\{\phi \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})\colon \int_{\partial\mathbb{I}}\phi\,d\sigma = 0\} \times \mathbb{C}$ is such that

$$\int_{\partial\mathbb{I}} S_2(t-s)\mu(s)\,d\sigma_s + \rho = 0 \qquad \forall t \in \partial\mathbb{I}, \tag{9.22}$$

then $(\mu,\rho) = (0,0)$. Indeed, equality (9.22) implies that the function $v^+ \equiv v^+[\partial\mathbb{I},\mu,0] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$ solves the following boundary value problem

$$\begin{cases} \Delta v^+ = 0 & \text{in } \mathbb{I},\\ v^+ = -\rho & \text{on } \partial\mathbb{I}. \end{cases}$$

As a consequence, $v^+ = -\rho$ on $\mathrm{cl}\,\mathbb{I}$, and accordingly

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} v^+[\partial\mathbb{I},\mu,0] = 0 \qquad \text{on } \partial\mathbb{I}.$$

Analogously, the function $v^- \equiv v^-[\partial\mathbb{I},\mu,0] \in C^{m,\alpha}(\mathbb{R}^2 \setminus \mathbb{I},\mathbb{C})$ solves the following boundary value problem

$$\begin{cases} \Delta v^- = 0 & \text{in } \mathbb{R}^2 \setminus \mathrm{cl}\,\mathbb{I},\\ \sup_{x\in\mathbb{R}^2\setminus\mathbb{I}}|v^-(x)| < +\infty, & \\ v^- = -\rho & \text{on } \partial\mathbb{I}, \end{cases}$$

(cf. *e.g.*, Folland [52, Lemma 3.31, p. 133].) Consequently, $v^- = -\rho$ in $\mathbb{R}^2 \setminus \mathbb{I}$, and so

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} v^-[\partial\mathbb{I},\mu,0] = 0 \qquad \text{on } \partial\mathbb{I}.$$

Thus,

$$\mu = \frac{\partial}{\partial\nu_{\mathbb{I}}} v^-[\partial\mathbb{I},\mu,0] - \frac{\partial}{\partial\nu_{\mathbb{I}}} v^+[\partial\mathbb{I},\mu,0] = 0 \qquad \text{on } \partial\mathbb{I},$$

and hence also $\rho = 0$.

Now let $g \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$. Then there exists a unique function $u^+ \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$, such that

$$\begin{cases} \Delta u^+ = 0 & \text{in } \mathbb{I},\\ u^+ = g & \text{on } \partial\mathbb{I}. \end{cases}$$

Analogously, there exists a unique function $u^- \in C^{m,\alpha}(\mathbb{R}^2 \setminus \mathbb{I},\mathbb{C})$, such that

$$\begin{cases} \Delta u^- = 0 & \text{in } \mathbb{R}^2 \setminus \mathrm{cl}\,\mathbb{I},\\ \sup_{x\in\mathbb{R}^2\setminus\mathbb{I}}|u^-(x)| < +\infty, & \\ u^- = g & \text{on } \partial\mathbb{I}. \end{cases}$$

(cf. *e.g.*, Folland [52, Theorem 3.40, p. 138].) Then we set

$$u^-_\infty \equiv \lim_{x\to\infty} u^-(x).$$

Then, by exploiting the Divergence Theorem and the decay properties of $u^-$ and of its radial derivative (cf. *e.g.*, Folland [52, Propositions 2.74, 2.75, p. 114]), it is easy to see that

$$u^+(t) = w^+[\partial\mathbb{I},u^+_{|\partial\mathbb{I}},0](t) - v^+[\partial\mathbb{I},\frac{\partial}{\partial\nu_{\mathbb{I}}}u^+,0](t) \qquad \forall t \in \mathrm{cl}\,\mathbb{I},$$

$$0 = -w^+[\partial\mathbb{I},u^-_{|\partial\mathbb{I}},0](t) + v^+[\partial\mathbb{I},\frac{\partial}{\partial\nu_{\mathbb{I}}}u^-,0](t) + u^-_\infty \qquad \forall t \in \mathrm{cl}\,\mathbb{I}.$$

Then,

$$g(t) = v^+[\partial\mathbb{I},\frac{\partial}{\partial\nu_{\mathbb{I}}}u^- - \frac{\partial}{\partial\nu_{\mathbb{I}}}u^+,0](t) + u^-_\infty \qquad \forall t \in \partial\mathbb{I}.$$

Since $u^+$ is harmonic in $\mathbb{I}$, then

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}} u^+ \, d\sigma = 0,$$

(cf. *e.g.*, Folland [52, Proposition 3.5, p. 119].)

Analogously, since $u^-$ is harmonic in $\mathbb{R}^2 \setminus \mathrm{cl}\,\mathbb{I}$ and harmonic at infinity, then

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}} u^- \, d\sigma = 0,$$

(cf. *e.g.*, Folland [52, Proposition 3.6, p. 119].)

Then, if we set

$$\mu \equiv \frac{\partial}{\partial\nu_{\mathbb{I}}} u^- - \frac{\partial}{\partial\nu_{\mathbb{I}}} u^+ \qquad \text{on } \partial\mathbb{I},$$
$$\rho \equiv u^-_\infty,$$

we have that $(\mu, \rho) \in \{\phi \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})\colon \int_{\partial\mathbb{I}} \phi \, d\sigma = 0\} \times \mathbb{C}$, and that

$$g(t) = \int_{\partial\mathbb{I}} S_2(t-s)\mu(s)\, d\sigma_s + \rho \qquad \forall t \in \partial\mathbb{I}.$$

Then the proof of statement $(i)$ is complete.

We now turn to the proof of statement $(ii)$. First of all, we note that if $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$ is such that

$$\int_{\partial\mathbb{I}} S_n(t-s)\mu(s)\, d\sigma_s = 0 \qquad \forall t \in \partial\mathbb{I}, \tag{9.23}$$

then $\mu = 0$. Indeed, equality (9.23) implies that the function $v^+ \equiv v^+[\partial\mathbb{I}, \mu, 0] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{I}, \mathbb{C})$ solves the following boundary value problem

$$\begin{cases} \Delta v^+ = 0 & \text{in } \mathbb{I}, \\ v^+ = 0 & \text{on } \partial\mathbb{I}. \end{cases}$$

As a consequence, $v^+ = 0$ on $\mathrm{cl}\,\mathbb{I}$, and accordingly

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} v^+[\partial\mathbb{I}, \mu, 0] = 0 \qquad \text{on } \partial\mathbb{I}.$$

Analogously, the function $v^- \equiv v^-[\partial\mathbb{I}, \mu, 0] \in C^{m,\alpha}(\mathbb{R}^n \setminus \mathbb{I}, \mathbb{C})$ solves the following boundary value problem

$$\begin{cases} \Delta v^- = 0 & \text{in } \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{I}, \\ \sup_{x \in \mathbb{R}^n \setminus \mathbb{I}} |v^-(x)||x|^{n-2} < +\infty, & \\ v^- = 0 & \text{on } \partial\mathbb{I}. \end{cases}$$

Consequently, $v^- = 0$ in $\mathbb{R}^n \setminus \mathbb{I}$, and so

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} v^-[\partial\mathbb{I}, \mu, 0] = 0 \qquad \text{on } \partial\mathbb{I}.$$

Thus,

$$\mu = \frac{\partial}{\partial\nu_{\mathbb{I}}} v^-[\partial\mathbb{I}, \mu, 0] - \frac{\partial}{\partial\nu_{\mathbb{I}}} v^+[\partial\mathbb{I}, \mu, 0] = 0 \qquad \text{on } \partial\mathbb{I}.$$

Now let $g \in C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$. Then there exists a unique function $u^+ \in C^{m,\alpha}(\partial\mathbb{I}, \mathbb{C})$, such that

$$\begin{cases} \Delta u^+ = 0 & \text{in } \mathbb{I}, \\ u^+ = g & \text{on } \partial\mathbb{I}. \end{cases}$$

Analogously, there exists a unique function $u^- \in C^{m,\alpha}(\mathbb{R}^n \setminus \mathbb{I}, \mathbb{C})$, such that

$$\begin{cases} \Delta u^- = 0 & \text{in } \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{I}, \\ \sup_{x \in \mathbb{R}^n \setminus \mathbb{I}} |u^-(x)||x|^{n-2} < +\infty, & \\ u^- = g & \text{on } \partial\mathbb{I}, \end{cases}$$

(cf. *e.g.*, Folland [52, Theorem 3.40, p. 138].) Then we note that

$$\lim_{x\to\infty} u^-(x) = 0$$

(cf. Folland [52, Proposition 2.74, p. 114].) Then, by exploiting the Divergence Theorem and the decay properties of $u^-$ and of its radial derivative (cf. *e.g.*, Folland [52, Propositions 2.74, 2.75, p. 114]), it is easy to see that

$$u^+(t) = w^+[\partial\mathbb{I}, u^+_{|\partial\mathbb{I}}, 0](t) - v^+[\partial\mathbb{I}, \frac{\partial}{\partial\nu_{\mathbb{I}}}u^+, 0](t) \qquad \forall t \in \mathrm{cl}\,\mathbb{I},$$
$$0 = -w^+[\partial\mathbb{I}, u^-_{|\partial\mathbb{I}}, 0](t) + v^+[\partial\mathbb{I}, \frac{\partial}{\partial\nu_{\mathbb{I}}}u^-, 0](t) \qquad \forall t \in \mathrm{cl}\,\mathbb{I}.$$

Then,

$$g(t) = v^+[\partial\mathbb{I}, \frac{\partial}{\partial\nu_{\mathbb{I}}}u^- - \frac{\partial}{\partial\nu_{\mathbb{I}}}u^+, 0](t) \qquad \forall t \in \partial\mathbb{I}.$$

Then, if we set

$$\mu \equiv \frac{\partial}{\partial\nu_{\mathbb{I}}}u^- - \frac{\partial}{\partial\nu_{\mathbb{I}}}u^+, \qquad \text{on } \partial\mathbb{I},$$

we have that $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$, and that

$$g(t) = \int_{\partial\mathbb{I}} S_n(t-s)\mu(s)\, d\sigma_s \qquad \forall t \in \partial\mathbb{I}.$$

Then the proof of statement $(ii)$ is complete. □

**Lemma 9.10.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $g$ be as in* (1.56), (9.15)*, respectively. Then the following statements hold.*

*(i) Let $n = 2$. Then there exists a unique pair $(\phi, \xi) \in \{\mu \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C}) \colon \int_{\partial\Omega} \mu\, d\sigma = 0\} \times \mathbb{C}$ that solves the following equation*

$$g(t) = \int_{\partial\Omega} S_2(t-s)\phi(s)\, d\sigma_s + \xi\frac{|\partial\Omega|_1}{2\pi} \qquad \forall t \in \partial\Omega, \tag{9.24}$$

*where, as usual, $|\partial\Omega|_1$ denotes the 1-dimensional measure of the set $\partial\Omega$. We denote the unique solution of equation* (9.24) *by $(\tilde{\phi}, \tilde{\xi})$.*

*(ii) Let $n \geq 3$. Then there exists a unique function $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ that solves the following equation*

$$g(t) = \int_{\partial\Omega} S_n(t-s)\theta(s)\, d\sigma_s \qquad \forall t \in \partial\Omega. \tag{9.25}$$

*We denote the unique solution of equation* (9.25) *by $\tilde{\theta}$.*

*Proof.* It is an immediate consequence of Lemma 9.9. □

We treat separately case $n = 2$ and case $n \geq 3$.

## Case $n \geq 3$

It is preferable to treat separately case $n$ even and case $n$ odd.

Then we have the following Propositions.

**Proposition 9.11.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15)*, respectively. Let $\epsilon_1^*$ be as in* (9.16)*. Let $\tilde{\theta}$ be as in Lemma 9.10 (ii). Let $\Lambda$ be the map of $]-\epsilon_1^*, \epsilon_1^*[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$ defined by*

$$\Lambda[\epsilon, \theta](t) \equiv \int_{\partial\Omega} S_n(t-s, \epsilon k)\theta(s)\, d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s - g(t) \qquad \forall t \in \partial\Omega, \tag{9.26}$$

*for all $(\epsilon, \theta) \in ]-\epsilon_1^*, \epsilon_1^*[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Then the following statements hold.*

(i) *If* $\epsilon \in ]0, \epsilon_1^*[$, *then the function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *satisfies equation*

$$\Lambda[\epsilon, \theta] = 0, \tag{9.27}$$

*if and only if the function* $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$, *defined by*

$$\mu(x) \equiv \epsilon^{-1}\theta(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{9.28}$$

*satisfies the equation*

$$\Gamma(x) = \int_{\partial\Omega_\epsilon} S_n^{a,k}(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega_\epsilon, \tag{9.29}$$

*with* $\Gamma \in C^{m,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ *defined by*

$$\Gamma(x) \equiv g(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon. \tag{9.30}$$

*In particular, equation* (9.27) *has exactly one solution* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, *for each* $\epsilon \in ]0, \epsilon_1^*[$.

(ii) *The function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *satisfies equation*

$$\Lambda[0, \theta] = 0, \tag{9.31}$$

*if and only if*

$$g(t) = \int_{\partial\Omega} S_n(t-s)\theta(s)\,d\sigma_s \qquad \forall t \in \partial\Omega. \tag{9.32}$$

*In particular, the unique function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *that solves equation* (9.31) *is* $\tilde{\theta}$.

*Proof.* Consider $(i)$. The equivalence of equation (9.27) in the unknown $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ and equation (9.29) in the unknown $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ follows by the rule of change of variables in integrals (cf. also Lemmas 9.7, 9.8), well known properties of functions in Schauder spaces, and the definition of $Q_n^k$ for $n$ odd (cf. (6.23) and Definition E.2.) Then the existence and uniqueness of a solution of equation (9.27) follows by Lemma 9.8. Consider $(ii)$. The equivalence of (9.31) and (9.32) is obvious. The second part of statement $(ii)$ is an immediate consequence of Lemma 9.10 $(ii)$. □

**Proposition 9.12.** *Let* $n$ *be even and* $n > 2$. *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $k$, $g$ *be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let* $\epsilon_1^*$ *be as in* (9.16). *Let* $\tilde{\theta}$ *be as in Lemma 9.10* $(ii)$. *Let* $\epsilon_1' > 0$ *be such that*

$$\epsilon \log \epsilon \in ]-\epsilon_1', \epsilon_1'[ \qquad \forall \epsilon \in ]0, \epsilon_1^*[. \tag{9.33}$$

*Let* $\Lambda^\#$ *be the map of* $]-\epsilon_1^*, \epsilon_1^*[ \times ]-\epsilon_1', \epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *to* $C^{m,\alpha}(\partial\Omega, \mathbb{C})$ *defined by*

$$\begin{aligned} \Lambda^\#[\epsilon, \epsilon', \theta](t) \equiv \int_{\partial\Omega} S_n(t-s, \epsilon k)\theta(s)\,d\sigma_s + \epsilon^{n-3}\epsilon' k^{n-2} \int_{\partial\Omega} Q_n^k(\epsilon(t-s))\theta(s)\,d\sigma_s \\ + \epsilon^{n-2} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s) - g(t) \qquad \forall t \in \partial\Omega, \end{aligned} \tag{9.34}$$

*for all* $(\epsilon, \epsilon', \theta) \in ]-\epsilon_1^*, \epsilon_1^*[ \times ]-\epsilon_1', \epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. *Then the following statements hold.*

(i) *If* $\epsilon \in ]0, \epsilon_1^*[$, *then the function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *satisfies equation*

$$\Lambda^\#[\epsilon, \epsilon \log \epsilon, \theta] = 0, \tag{9.35}$$

*if and only if the function* $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$, *defined by*

$$\mu(x) \equiv \epsilon^{-1}\theta(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{9.36}$$

*satisfies the equation*

$$\Gamma(x) = \int_{\partial\Omega_\epsilon} S_n^{a,k}(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega_\epsilon, \tag{9.37}$$

*with* $\Gamma \in C^{m,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ *defined by*

$$\Gamma(x) \equiv g(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon. \tag{9.38}$$

*In particular, equation* (9.35) *has exactly one solution* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, *for each* $\epsilon \in ]0, \epsilon_1^*[$.

*(ii) The function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *satisfies equation*

$$\Lambda^{\#}[0,0,\theta] = 0, \tag{9.39}$$

*if and only if*

$$g(t) = \int_{\partial\Omega} S_n(t-s)\theta(s)\, d\sigma_s \qquad \forall t \in \partial\Omega. \tag{9.40}$$

*In particular, the unique function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *that solves equation* (9.39) *is* $\tilde{\theta}$.

*Proof.* Consider $(i)$. The equivalence of equation (9.35) in the unknown $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ and equation (9.37) in the unknown $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ follows by the rule of change of variables in integrals (cf. also Lemmas 9.7, 9.8), well known properties of functions in Schauder spaces, and the definition of $Q_n^k$ for $n$ even (cf. (6.23) and Definition E.2.) Then the existence and uniqueness of a solution of equation (9.35) follows by Lemma 9.8. Consider $(ii)$. The equivalence of (9.39) and (9.40) is obvious. The second part of statement $(ii)$ is an immediate consequence of Lemma 9.10 $(ii)$. □

By Propositions 9.11, 9.12, it makes sense to introduce the following.

**Definition 9.13.** Let $n \geq 3$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). For each $\epsilon \in ]0, \epsilon_1^*[$, we denote by $\hat{\theta}_n[\epsilon]$ the unique function in $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ that solves equation (9.27), if $n$ is odd, or equation (9.35), if $n$ is even. Analogously, we denote by $\hat{\theta}_n[0]$ the unique function in $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ that solves equation (9.31), if $n$ is odd, or equation (9.39), if $n$ is even.

In the following Remark, we show the relation between the solutions of boundary value problem (9.17) and the solutions of equations (9.27), (9.35).

*Remark* 9.14. Let $n \geq 3$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16).
Let $\epsilon \in ]0, \epsilon_1^*[$. We have

$$u[\epsilon](x) = \epsilon^{n-2} \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s)\hat{\theta}_n[\epsilon](s)\, d\sigma_s \qquad \forall x \in \operatorname{cl} \mathbb{T}_a[\Omega_\epsilon].$$

While the relation between equations (9.27), (9.35) and boundary value problem (9.17) is now clear, we want to see if (9.31), (9.39) are related to some (limiting) boundary value problem. We give the following.

**Definition 9.15.** Let $n \geq 3$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $g$ be as in (1.56), (9.15), respectively. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega, \mathbb{C})$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \operatorname{cl}\Omega, \\ u(x) = g(x) & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{9.41}$$

Problem (9.41) will be called the *limiting boundary value problem.*

*Remark* 9.16. Let $n \geq 3$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). Then we have

$$\tilde{u}(x) = \int_{\partial\Omega} S_n(x-y)\hat{\theta}_n[0](y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \Omega.$$

We now prove the following Propositions.

**Proposition 9.17.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\tilde{\theta}$ be as in Lemma 9.10 (ii). Let $\Lambda$ be as in Proposition 9.11. Then there exists $\epsilon_2 \in ]0, \epsilon_1^*]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$. Moreover, if we set $b_0 \equiv (0, \tilde{\theta})$, then the differential $\partial_\theta \Lambda[b_0]$ of $\Lambda$ with respect to the variable $\theta$ at $b_0$ is delivered by the following formula*

$$\partial_\theta \Lambda[b_0](\tau)(t) = \int_{\partial\Omega} S_n(t-s)\tau(s)\, d\sigma_s \qquad \forall t \in \partial\Omega, \tag{9.42}$$

*for all $\tau \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, and is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ onto $C^{m,\alpha}(\partial\Omega, \mathbb{C})$.*

*Proof.* By Proposition 6.21 ($i$), one can easily prove that there exists $\epsilon_2 \in ]0, \epsilon_1^*]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$. By standard calculus in Banach space, we immediately deduce that (9.42) holds. Now we need to prove that $\partial_\theta \Lambda[b_0]$ is a linear homeomorphism. By the Open Mapping Theorem, it suffices to prove that it is a bijection. Let $\psi \in C^{m,\alpha}(\partial\Omega, \mathbb{C})$. By Lemma 9.9, there exists a unique function $\tau \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, such that

$$\int_{\partial\Omega} S_n(t-s)\tau(s)\, d\sigma_s = \psi(t) \qquad \forall t \in \partial\Omega.$$

Hence $\partial_\theta \Lambda[b_0]$ is bijective, and, accordingly, a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ onto the space $C^{m,\alpha}(\partial\Omega, \mathbb{C})$. □

**Proposition 9.18.** *Let $n$ be even and $n > 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\tilde{\theta}$ be as in Lemma 9.10* ($ii$). *Let $\epsilon_1' > 0$ be as in* (9.33). *Let $\Lambda^\#$ be as in Proposition 9.12. Then there exists $\epsilon_2 \in ]0, \epsilon_1^*]$ such that $\Lambda^\#$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times ]-\epsilon_1', \epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$. Moreover, if we set $b_0 \equiv (0, 0, \tilde{\theta})$, then the differential $\partial_\theta \Lambda^\#[b_0]$ of $\Lambda^\#$ with respect to the variable $\theta$ at $b_0$ is delivered by the following formula*

$$\partial_\theta \Lambda^\#[b_0](\tau)(t) = \int_{\partial\Omega} S_n(t-s)\tau(s)\, d\sigma_s \qquad \forall t \in \partial\Omega, \tag{9.43}$$

*for all $\tau \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, and is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ onto $C^{m,\alpha}(\partial\Omega, \mathbb{C})$.*

*Proof.* By Proposition 6.21 ($i$), one can easily prove that there exists $\epsilon_2 \in ]0, \epsilon_1^*]$ such that $\Lambda^\#$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times ]-\epsilon_1', \epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$. By standard calculus in Banach space, we immediately deduce that (9.43) holds. Finally, by the proof of Proposition 9.17 and formula (9.43), we have that $\partial_\theta \Lambda^\#[b_0]$ is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ onto $C^{m,\alpha}(\partial\Omega, \mathbb{C})$. □

By the previous Propositions we can now prove the following results.

**Proposition 9.19.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_2$ be as in Proposition 9.17. Then there exist $\epsilon_3 \in ]0, \epsilon_2]$ and a real analytic operator $\Theta_n$ of $]-\epsilon_3, \epsilon_3[$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, such that*

$$\Theta_n[\epsilon] = \hat{\theta}_n[\epsilon], \tag{9.44}$$

*for all $\epsilon \in [0, \epsilon_3[$.*

*Proof.* It is an immediate consequence of Proposition 9.17 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

**Proposition 9.20.** *Let $n$ be even and $n > 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_1' > 0$ be as in* (9.33). *Let $\epsilon_2$ be as in Proposition 9.18. Then there exist $\epsilon_3 \in ]0, \epsilon_2]$, $\epsilon_2' \in ]0, \epsilon_1']$, and a real analytic operator $\Theta_n^\#$ of $]-\epsilon_3, \epsilon_3[ \times ]-\epsilon_2', \epsilon_2'[$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, such that*

$$\epsilon \log \epsilon \in ]-\epsilon_2', \epsilon_2'[ \quad \forall \epsilon \in ]0, \epsilon_3[,$$

$$\Theta_n^\#[\epsilon, \epsilon \log \epsilon] = \hat{\theta}_n[\epsilon] \quad \forall \epsilon \in ]0, \epsilon_3[, \tag{9.45}$$

$$\Theta_n^\#[0, 0] = \hat{\theta}_n[0]. \tag{9.46}$$

*Proof.* It is an immediate consequence of Proposition 9.18 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

**Case** $n = 2$

We have the following Lemma.

**Lemma 9.21.** *Let* $n = 2$. *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $k$, $g$ *be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let* $\epsilon_1^*$ *be as in* (9.16). *Let* $\epsilon \in ]0, \min\{\epsilon_1^*, 1\}[$. *Let* $(\phi, \xi) \in \{\mu \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C}) \colon \int_{\partial\Omega} \mu \, d\sigma = 0\} \times \mathbb{C}$. *Then*

$$u[\epsilon](x) = v_a^-[\partial\Omega_\epsilon, \epsilon^{-1}\phi(\frac{1}{\epsilon}(\cdot - w)), k](x) + \frac{1}{\epsilon \log \epsilon} v_a^-[\partial\Omega_\epsilon, \xi, k](x) \qquad \forall x \in \operatorname{cl} \mathbb{T}_a[\Omega_\epsilon], \tag{9.47}$$

*if and only if the pair* $(\phi, \xi)$ *solves the following integral equation*

$$\begin{aligned} g(t) = &\int_{\partial\Omega} S_2(t-s, \epsilon k)\phi(s)\, d\sigma_s + \log\epsilon \int_{\partial\Omega} Q_2^k(\epsilon(t-s))\phi(s)\, d\sigma_s + \int_{\partial\Omega} R_2^{a,k}(\epsilon(t-s))\phi(s)\, d\sigma_s \\ &+ \frac{1}{\log\epsilon}\xi \int_{\partial\Omega} S_2(t-s,\epsilon k)\, d\sigma_s + \xi \int_{\partial\Omega} Q_2^k(\epsilon(t-s))\, d\sigma_s + \frac{1}{\log\epsilon}\xi \int_{\partial\Omega} R_2^{a,k}(\epsilon(t-s))\, d\sigma_s \;\; \forall t \in \partial\Omega. \end{aligned} \tag{9.48}$$

*In particular, there exists a unique pair* $(\phi, \xi)$ *in* $\{\mu \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C}) \colon \int_{\partial\Omega} \mu \, d\sigma = 0\} \times \mathbb{C}$ *such that* (9.48) *holds.*

*Proof.* It is a straightforward consequence of Lemma 9.8 and of the fact that

$$C^{m-1,\alpha}(\partial\Omega, \mathbb{C}) = \left\{ \mu \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C}) \colon \int_{\partial\Omega} \mu \, d\sigma = 0 \right\} \oplus \mathbb{C}.$$

□

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). We find convenient to set

$$\mathcal{U} \equiv \left\{ \mu \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C}) \colon \int_{\partial\Omega} \mu \, d\sigma = 0 \right\}. \tag{9.49}$$

Then we have the following Proposition.

**Proposition 9.22.** *Let* $n = 2$. *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $k$, $g$ *be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let* $\epsilon_1^*$ *be as in* (9.16). *Let* $(\tilde{\phi}, \tilde{\xi})$ *be as in Lemma 9.10 (i). Let* $\epsilon_1' > 0$, $\epsilon_1'' > 0$ *be such that*

$$\epsilon \log \epsilon \in ]-\epsilon_1', \epsilon_1'[ \qquad \forall \epsilon \in ]0, \epsilon_1^*[, \tag{9.50}$$

*and*

$$\frac{1}{\log \epsilon} \in ]-\epsilon_1'', \epsilon_1''[ \qquad \forall \epsilon \in ]0, \min\{\epsilon_1^*, 1\}[. \tag{9.51}$$

*Let* $\tilde{\Lambda}^{\#}$ *be the map of* $]-\min\{\epsilon_1^*, 1\}, \min\{\epsilon_1^*, 1\}[ \times ]-\epsilon_1', \epsilon_1'[ \times ]-\epsilon_1'', \epsilon_1''[ \times \mathcal{U} \times \mathbb{C}$ *to* $C^{m,\alpha}(\partial\Omega, \mathbb{C})$ *defined by*

$$\begin{aligned} \tilde{\Lambda}^{\#}[\epsilon, \epsilon', \epsilon'', \phi, \xi](t) \equiv &\int_{\partial\Omega} S_2(t-s, \epsilon k)\phi(s)\, d\sigma_s + \epsilon' \int_{\partial\Omega} \Big( \int_0^1 DQ_2^k(\beta\epsilon(t-s)) \cdot (t-s) d\beta \Big) \phi(s)\, d\sigma_s \\ &+ \int_{\partial\Omega} R_2^{a,k}(\epsilon(t-s))\phi(s)\, d\sigma_s + \epsilon''\xi \int_{\partial\Omega} S_2(t-s, \epsilon k)\, d\sigma_s \\ &+ \xi \int_{\partial\Omega} Q_2^k(\epsilon(t-s))\, d\sigma_s + \epsilon''\xi \int_{\partial\Omega} R_2^{a,k}(\epsilon(t-s))\, d\sigma_s - g(t) \quad \forall t \in \partial\Omega, \end{aligned} \tag{9.52}$$

*for all* $(\epsilon, \epsilon', \epsilon'', \phi, \xi) \in ]-\min\{\epsilon_1^*, 1\}, \min\{\epsilon_1^*, 1\}[ \times ]-\epsilon_1', \epsilon_1'[ \times ]-\epsilon_1'', \epsilon_1''[ \times \mathcal{U} \times \mathbb{C}$. *Then the following statements hold.*

*(i) If $\epsilon \in ]0, \min\{\epsilon_1^*, 1\}[$, then the pair $(\phi, \xi) \in \mathcal{U} \times \mathbb{C}$ satisfies equation*

$$\tilde{\Lambda}^{\#}[\epsilon, \epsilon \log \epsilon, (\log \epsilon)^{-1}, \phi, \xi] = 0, \tag{9.53}$$

*if and only if the function $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$, defined by*

$$\mu(x) \equiv \epsilon^{-1}\phi(\frac{1}{\epsilon}(x-w)) + \frac{1}{\epsilon \log \epsilon}\xi \qquad \forall x \in \partial\Omega_\epsilon, \tag{9.54}$$

*satisfies the equation*

$$\Gamma(x) = \int_{\partial\Omega_\epsilon} S_2^{a,k}(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega_\epsilon, \tag{9.55}$$

*with $\Gamma \in C^{m,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ defined by*

$$\Gamma(x) \equiv g(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon. \tag{9.56}$$

*In particular, equation (9.53) has exactly one solution $(\phi, \xi) \in \mathcal{U} \times \mathbb{C}$, for each $\epsilon \in ]0, \min\{\epsilon_1^*, 1\}[$.*

*(ii) The pair $(\phi, \xi) \in \mathcal{U} \times \mathbb{C}$ satisfies equation*

$$\tilde{\Lambda}^{\#}[0, 0, 0, \phi, \xi] = 0, \tag{9.57}$$

*if and only if*

$$g(t) = \int_{\partial\Omega} S_2(t-s)\phi(s)\,d\sigma_s + \xi\frac{|\partial\Omega|_1}{2\pi} \qquad \forall t \in \partial\Omega, \tag{9.58}$$

*where $|\partial\Omega|_1$ denotes the 1-dimensional measure of $\partial\Omega$. In particular, if $\epsilon = \epsilon' = \epsilon'' = 0$, then the unique pair $(\phi, \xi) \in \mathcal{U} \times \mathbb{C}$ that solves equation (9.57) is $(\tilde{\phi}, \tilde{\xi})$.*

*Proof.* Consider $(i)$. Let $\epsilon \in ]0, \min\{\epsilon_1^*, 1\}[$. By the Taylor formula and Proposition E.3 and the definition of $Q_2^k$, we have

$$Q_2^k(x) = \frac{1}{2\pi} + \int_0^1 DQ_2^k(\beta x) \cdot x d\beta \qquad \forall x \in \mathbb{R}^2.$$

Thus,

$$Q_2^k(\epsilon(t-s)) = \frac{1}{2\pi} + \epsilon\int_0^1 DQ_2^k(\beta\epsilon(t-s)) \cdot (t-s)d\beta \qquad \forall (t,s) \in \partial\Omega \times \partial\Omega.$$

Hence,

$$\begin{aligned}
(\log\epsilon)&\int_{\partial\Omega} Q_2^k(\epsilon(t-s))\phi(s)\,d\sigma_s \\
&= (\log\epsilon)\frac{1}{2\pi}\int_{\partial\Omega}\phi(s)\,d\sigma_s + \epsilon(\log\epsilon)\int_{\partial\Omega}\Big(\int_0^1 DQ_2^k(\beta\epsilon(t-s)) \cdot (t-s)d\beta\Big)\phi(s)\,d\sigma_s \\
&= \epsilon(\log\epsilon)\int_{\partial\Omega}\Big(\int_0^1 DQ_2^k(\beta\epsilon(t-s)) \cdot (t-s)d\beta\Big)\phi(s)\,d\sigma_s \qquad \forall t \in \partial\Omega,
\end{aligned} \tag{9.59}$$

for all $\phi \in \mathcal{U}$. Then the equivalence of equation (9.53) in the unknown $(\phi, \xi) \in \mathcal{U} \times \mathbb{C}$ and equation (9.55) in the unknown $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ follows by the rule of change of variables in integrals, Lemmas 9.7 and 9.21, equality (9.59), the definition of $\mathcal{U}$, and well known properties of functions in Schauder spaces. Then the existence and uniqueness of a solution of equation (9.53) follows by Lemma 9.21 and equality (9.59). Consider $(ii)$. The equivalence of (9.57) and (9.58) is obvious. The second part of statement $(ii)$ is an immediate consequence of Lemma 9.10 $(i)$. □

By Proposition 9.22 it makes sense to introduce the following.

**Definition 9.23.** Let $n = 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). For each $\epsilon \in ]0, \min\{\epsilon_1^*, 1\}[$, we denote by $(\hat{\phi}_2[\epsilon], \hat{\xi}_2[\epsilon])$ the unique pair in $\mathcal{U} \times \mathbb{C}$ that solves equation (9.53). Analogously, we denote by $(\hat{\phi}_2[0], \hat{\xi}_2[0])$ the unique pair in $\mathcal{U} \times \mathbb{C}$ that solves equation (9.57).

In the following Remark, we show the relation between the solutions of boundary value problem (9.17) and the solutions of equations (9.53).

*Remark* 9.24. Let $n = 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16).
Let $\epsilon \in ]0, \min\{\epsilon_1^*, 1\}[$. We have

$$u[\epsilon](x) = \int_{\partial\Omega} S_2^{a,k}(x - w - \epsilon s)\hat{\phi}_2[\epsilon](s)\, d\sigma_s + \frac{1}{\log \epsilon}\hat{\xi}_2[\epsilon] \int_{\partial\Omega} S_2^{a,k}(x - w - \epsilon s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon].$$

While the relation between equation (9.53) and boundary value problem (9.17) is now clear, we want to see if (9.57) is related to some (limiting) boundary value problem. We give the following.

**Definition 9.25.** Let $n = 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $\Omega$, $g$ be as in (1.56), (9.15), respectively. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^2 \setminus \Omega, \mathbb{C})$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^2 \setminus \mathrm{cl}\, \Omega, \\ u(x) = g(x) & \forall x \in \partial\Omega, \\ \sup_{x \in \mathbb{R}^2 \setminus \Omega} |u(x)| < +\infty. \end{cases} \tag{9.60}$$

Problem (9.60) will be called the *limiting boundary value problem.*

*Remark* 9.26. Let $n = 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). Then we have

$$\tilde{u}(x) = \int_{\partial\Omega} S_2(x - y)\hat{\phi}_2[0](y)\, d\sigma_y + \hat{\xi}_2[0]\frac{|\partial\Omega|_1}{2\pi} \qquad \forall x \in \mathbb{R}^2 \setminus \Omega.$$

Then we have the following.

**Proposition 9.27.** *Let $n = 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $(\tilde{\phi}, \tilde{\xi})$ be as in Lemma 9.10 (i). Let $\epsilon_1' > 0$, $\epsilon_1'' > 0$ be as in* (9.50), (9.51),, *respectively. Let $\tilde{\Lambda}^{\#}$ be as in Proposition 9.22. Then there exists $\epsilon_2 \in ]0, \min\{\epsilon_1^*, 1\}]$ such that $\tilde{\Lambda}^{\#}$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times ]-\epsilon_1', \epsilon_1'[ \times ]-\epsilon_1'', \epsilon_1''[ \times \mathcal{U} \times \mathbb{C}$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$. Moreover, if we set $b_0 \equiv (0, 0, 0, \tilde{\phi}, \tilde{\xi})$, then the differential $\partial_{(\phi,\xi)}\tilde{\Lambda}^{\#}[b_0]$ of $\tilde{\Lambda}^{\#}$ with respect to the variables $(\phi, \xi)$ at $b_0$ is delivered by the following formula*

$$\partial_{(\phi,\xi)}\tilde{\Lambda}^{\#}[b_0](\psi, \zeta)(t) = \int_{\partial\Omega} S_2(t - s)\psi(s)\, d\sigma_s + \zeta\frac{|\partial\Omega|_1}{2\pi} \qquad \forall t \in \partial\Omega, \tag{9.61}$$

*for all $(\psi, \zeta) \in \mathcal{U} \times \mathbb{C}$, and is a linear homeomorphism of $\mathcal{U} \times \mathbb{C}$ onto $C^{m,\alpha}(\partial\Omega, \mathbb{C})$.*

*Proof.* We set

$$\begin{aligned} \tilde{\Lambda}'^{\#}[\epsilon, \epsilon', \epsilon'', \phi, \xi](t) \equiv & \int_{\partial\Omega} S_2(t - s, \epsilon k)\phi(s)\, d\sigma_s + \int_{\partial\Omega} R_2^{a,k}(\epsilon(t - s))\phi(s)\, d\sigma_s \\ & + \epsilon''\xi \int_{\partial\Omega} S_2(t - s, \epsilon k)\, d\sigma_s + \xi \int_{\partial\Omega} Q_2^k(\epsilon(t - s))\, d\sigma_s \\ & + \epsilon''\xi \int_{\partial\Omega} R_2^{a,k}(\epsilon(t - s))\, d\sigma_s - g(t) \qquad \forall t \in \partial\Omega, \end{aligned}$$

and

$$\tilde{\Lambda}''^{\#}[\epsilon, \epsilon', \epsilon'', \phi, \xi](t) \equiv \epsilon' \int_{\partial\Omega} \Big( \int_0^1 DQ_2^k(\beta\epsilon(t - s)) \cdot (t - s) d\beta \Big) \phi(s)\, d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $(\epsilon, \epsilon', \epsilon'', \phi, \xi) \in ]-\min\{\epsilon_1^*, 1\}, \min\{\epsilon_1^*, 1\}[ \times ]-\epsilon_1', \epsilon_1'[ \times ]-\epsilon_1'', \epsilon_1''[ \times \mathcal{U} \times \mathbb{C}$. Clearly,

$$\tilde{\Lambda}^{\#}[\epsilon, \epsilon', \epsilon'', \phi, \xi] = \tilde{\Lambda}'^{\#}[\epsilon, \epsilon', \epsilon'', \phi, \xi] + \tilde{\Lambda}''^{\#}[\epsilon, \epsilon', \epsilon'', \phi, \xi],$$

for all $(\epsilon, \epsilon', \epsilon'', \phi, \xi) \in ]-\min\{\epsilon_1^*, 1\}, \min\{\epsilon_1^*, 1\}[ \times ]-\epsilon_1', \epsilon_1'[ \times ]-\epsilon_1'', \epsilon_1''[ \times \mathcal{U} \times \mathbb{C}$. By arguing as in the proof of Proposition 9.17 and Proposition 9.18, we easily deduce that there exists $\epsilon_2 \in ]0, \min\{\epsilon_1^*, 1\}]$ such that $\tilde{\Lambda}'^{\#}$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times ]-\epsilon_1', \epsilon_1'[ \times ]-\epsilon_1'', \epsilon_1''[ \times \mathcal{U} \times \mathbb{C}$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$. We now consider $\tilde{\Lambda}''^{\#}$. Since $Q_2^k$ is a real analytic function of $\mathbb{R}^2$ to $\mathbb{C}$, then $\partial_{x_j} Q_2^k$ is a real analytic function of $\mathbb{R}^2$ to $\mathbb{C}$, for all $j \in \{1, \dots, n\}$. Let $j \in \{1, \dots, n\}$. Then, by a known result on composition operators (cf. Böhme and Tomi [15, p. 10], Henry [60, p. 29], Valent [137, Thm. 5.2, p. 44]), we have that the map of $]-\epsilon_1^*, \epsilon_1^*[$ to $C^{m,\alpha}([0,1] \times \partial\Omega \times \partial\Omega, \mathbb{C})$, which takes $\epsilon$ to the function $\partial_{x_j} Q_2^k(\beta\epsilon(t-s))(t-s)_j$ of the variable $(\beta, t, s) \in [0,1] \times \partial\Omega \times \partial\Omega$, is real analytic. Moreover, we observe that the map of $C^{m,\alpha}([0,1] \times \partial\Omega \times \partial\Omega, \mathbb{C})$ to $C^{m,\alpha}(\partial\Omega \times \partial\Omega, \mathbb{C})$, which takes $h$ to $\int_0^1 h(\beta, \cdot, \cdot) d\beta$ is linear and continuous, and thus real analytic. Similarly, the bilinear map of $C^{m,\alpha}(\partial\Omega \times \partial\Omega, \mathbb{C}) \times L^1(\partial\Omega, \mathbb{C})$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$ which takes $(h, g)$ to $\int_{\partial\Omega} h(s, \cdot) g(s)\, d\sigma_s$ is continuous, and thus real analytic. Then, well known properties of functions in Schauder spaces and standard calculus in Banach spaces show that $\tilde{\Lambda}''^{\#}$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times ]-\epsilon_1', \epsilon_1'[ \times ]-\epsilon_1'', \epsilon_1''[ \times \mathcal{U} \times \mathbb{C}$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$. Thus, $\tilde{\Lambda}^{\#}$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times ]-\epsilon_1', \epsilon_1'[ \times ]-\epsilon_1'', \epsilon_1''[ \times \mathcal{U} \times \mathbb{C}$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$. By standard calculus in Banach space and since $Q_2^k(0) = \frac{1}{2\pi}$ (cf. Proposition E.3), we immediately deduce that (9.61) holds. Finally, we need to prove that $\partial_{(\phi,\xi)} \tilde{\Lambda}^{\#}$ is a linear homeomorphism of $\mathcal{U} \times \mathbb{C}$ onto $C^{m,\alpha}(\partial\Omega, \mathbb{C})$. Clearly, by the Open Mapping Theorem, it suffices to prove that it is a bijection. By Lemma 9.9 $(i)$, we immediately deduce that $\partial_{(\phi,\xi)} \tilde{\Lambda}^{\#}$ is a bijection of $\mathcal{U} \times \mathbb{C}$ onto $C^{m,\alpha}(\partial\Omega, \mathbb{C})$, and accordingly a linear homeomorphism. □

By the previous Proposition we can now prove the following result.

**Proposition 9.28.** *Let $n = 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_1' > 0$, $\epsilon_1'' > 0$ be as in* (9.50), (9.51),, *respectively. Let $\epsilon_2$ be as in Proposition 9.27. Then there exist $\epsilon_3 \in ]0, \epsilon_2]$, $\epsilon_2' \in ]0, \epsilon_1']$, $\epsilon_2'' \in ]0, \epsilon_1'']$, and a real analytic operator $(\Phi_2^{\#}, \Xi_2^{\#})$ of $]-\epsilon_3, \epsilon_3[ \times ]-\epsilon_2', \epsilon_2'[ \times ]-\epsilon_2'', \epsilon_2''[$ to $\mathcal{U} \times \mathbb{C}$, such that*

$$\epsilon \log \epsilon \in ]-\epsilon_2', \epsilon_2'[ \qquad \forall \epsilon \in ]0, \epsilon_3[,$$

$$(\log \epsilon)^{-1} \in ]-\epsilon_2'', \epsilon_2''[ \qquad \forall \epsilon \in ]0, \epsilon_3[,$$

$$(\Phi_2^{\#}[\epsilon, \epsilon \log \epsilon, (\log \epsilon)^{-1}], \Xi_2^{\#}[\epsilon, \epsilon \log \epsilon, (\log \epsilon)^{-1}]) = (\hat{\phi}_2[\epsilon], \hat{\xi}_2[\epsilon]) \qquad \forall \epsilon \in ]0, \epsilon_3[, \tag{9.62}$$

$$(\Phi_2^{\#}[0,0,0], \Xi_2^{\#}[0,0,0]) = (\hat{\phi}_2[0], \hat{\xi}_2[0]). \tag{9.63}$$

*Proof.* It is an immediate consequence of Proposition 9.27 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

### 9.2.3 A functional analytic representation Theorem for the solution of the singularly perturbed Dirichlet problem

By Propositions 9.19, 9.20, 9.28 and Remarks 9.14, 9.24, we can deduce the main result of this Subsection.

**Theorem 9.29.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_3$ be as in Proposition 9.19. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \epsilon_3]$, and a real analytic operator $U$ of $]-\epsilon_4, \epsilon_4[$ to the space $C^0(\mathrm{cl}\, V, \mathbb{C})$, such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*

$$u[\epsilon](x) = \epsilon^{n-2} U[\epsilon](x) \qquad \forall x \in \mathrm{cl}\, V,$$

*for all* $\epsilon \in ]0, \epsilon_4[$.

*Proof.* Let $\Theta_n[\cdot]$ be as in Proposition 9.19. By choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $\epsilon \in ]0, \epsilon_4[$. By Remark 9.14 and Proposition 9.19, we have

$$u[\epsilon](x) = \epsilon^{n-2} \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s) \Theta_n[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V.$$

Thus, it is natural to set

$$U[\epsilon](x) \equiv \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s) \Theta_n[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$. By Proposition 6.22, $U$ is a real analytic map of $]-\epsilon_4, \epsilon_4[$ to $C^0(\mathrm{cl}\, V, \mathbb{C})$. □

*Remark* 9.30. We note that the right-hand side of the equality in $(jj)$ of Theorem 9.29 can be continued real analytically in the whole $]-\epsilon_4, \epsilon_4[$. Moreover, if $V$ is a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$, then

$$\lim_{\epsilon \to 0^+} u[\epsilon] = 0 \qquad \text{uniformly in } \mathrm{cl}\, V.$$

**Theorem 9.31.** *Let $n$ be even and $n > 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_3$, $\epsilon_2'$ be as in Proposition 9.20. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \epsilon_3]$ and a real analytic operator $U^\#$ of $]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[$ to the space $C^0(\mathrm{cl}\, V, \mathbb{C})$, such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*
$$u[\epsilon](x) = \epsilon^{n-2} U^\#[\epsilon, \epsilon \log \epsilon](x) \qquad \forall x \in \mathrm{cl}\, V,$$
*for all* $\epsilon \in ]0, \epsilon_4[$.

*Proof.* Let $\Theta_n^\#[\cdot,\cdot]$ be as in Proposition 9.20. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $\epsilon \in ]0, \epsilon_4[$. By Remark 9.14 and Proposition 9.20, we have

$$u[\epsilon](x) = \epsilon^{n-2} \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s)\Theta_n^\#[\epsilon, \epsilon \log \epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V.$$

Thus, it is natural to set

$$U^\#[\epsilon, \epsilon'](x) \equiv \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s)\Theta_n^\#[\epsilon, \epsilon'](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

for all $(\epsilon, \epsilon') \in ]-\epsilon_4, \epsilon_4[\times]-\epsilon_2', \epsilon_2'[$. By Proposition 6.22, $U^\#$ is a real analytic map of $]-\epsilon_4, \epsilon_4[\times]-\epsilon_2', \epsilon_2'[$ to $C^0(\mathrm{cl}\, V, \mathbb{C})$. □

**Theorem 9.32.** *Let $n = 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_3$, $\epsilon_2'$, $\epsilon_2''$ be as in Proposition 9.28. Let $V$ be a bounded open subset of $\mathbb{R}^2$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \epsilon_3]$ and two real analytic operators $\tilde{U}_1^\#$, $\tilde{U}_2^\#$ of $]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[ \times ]-\epsilon_2'', \epsilon_2''[$ to the space $C^0(\mathrm{cl}\, V, \mathbb{C})$, such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*
$$u[\epsilon](x) = \tilde{U}_1^\#[\epsilon, \epsilon \log \epsilon, (\log \epsilon)^{-1}](x) + \frac{1}{\log \epsilon} \tilde{U}_2^\#[\epsilon, \epsilon \log \epsilon, (\log \epsilon)^{-1}](x) \qquad \forall x \in \mathrm{cl}\, V,$$
*for all* $\epsilon \in ]0, \epsilon_4[$. *Moreover,* $\tilde{U}_1^\#[0,0,0](x) = 0$ *for all* $x \in \mathrm{cl}\, V$.

*Proof.* Let $\Phi_2^\#[\cdot,\cdot,\cdot]$, $\Xi_2^\#[\cdot,\cdot,\cdot]$ be as in Proposition 9.28. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $\epsilon \in ]0, \epsilon_4[$. By Remark 9.24 and Proposition 9.28, we have

$$\begin{aligned} u[\epsilon](x) = & \int_{\partial\Omega} S_2^{a,k}(x - w - \epsilon s)\Phi_2^\#[\epsilon, \epsilon \log \epsilon, (\log \epsilon)^{-1}](s)\, d\sigma_s \\ & + \frac{1}{\log \epsilon} \int_{\partial\Omega} S_2^{a,k}(x - w - \epsilon s)\Xi_2^\#[\epsilon, \epsilon \log \epsilon, (\log \epsilon)^{-1}]\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V. \end{aligned}$$

Thus, it is natural to set

$$\tilde{U}_1^\#[\epsilon, \epsilon', \epsilon''](x) \equiv \int_{\partial\Omega} S_2^{a,k}(x - w - \epsilon s)\Phi_2^\#[\epsilon, \epsilon', \epsilon''](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

and

$$\tilde{U}_2^\#[\epsilon, \epsilon', \epsilon''](x) \equiv \int_{\partial\Omega} S_2^{a,k}(x - w - \epsilon s)\Xi_2^\#[\epsilon, \epsilon', \epsilon'']\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

for all $(\epsilon,\epsilon',\epsilon'')\in\,]-\epsilon_4,\epsilon_4[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\epsilon_2'',\epsilon_2''[$. By Proposition 6.22, $\tilde{U}_1^{\#}$ and $\tilde{U}_2^{\#}$ are real analytic maps of $]-\epsilon_4,\epsilon_4[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\epsilon_2'',\epsilon_2''[$ to $C^0(\mathrm{cl}\,V,\mathbb{C})$. Finally, since $\Phi_2^{\#}[0,0,0]\in\mathcal{U}$, and accordingly

$$\int_{\partial\Omega}\Phi_2^{\#}[0,0,0]\,d\sigma=0,$$

we have

$$\begin{aligned}\tilde{U}_1^{\#}[0,0,0](x)&=\int_{\partial\Omega}S_2^{a,k}(x-w)\Phi_2^{\#}[0,0,0](s)\,d\sigma_s\\&=S_2^{a,k}(x-w)\int_{\partial\Omega}\Phi_2^{\#}[0,0,0](s)\,d\sigma_s\\&=0\qquad\forall x\in\mathrm{cl}\,V.\end{aligned}$$

Thus the proof is complete. □

We have also the following Theorems.

**Theorem 9.33.** *Let $n$ be odd. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). Let $\epsilon_3$ be as in Proposition 9.19. Then there exist $\epsilon_5\in\,]0,\epsilon_3]$, and a real analytic operator $G$ of $]-\epsilon_5,\epsilon_5[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]}|\nabla u[\epsilon](x)|^2\,dx-k^2\int_{\mathbb{P}_a[\Omega_\epsilon]}|u[\epsilon](x)|^2\,dx=\epsilon^{n-2}G[\epsilon],\tag{9.64}$$

*for all $\epsilon\in\,]0,\epsilon_5[$. Moreover,*

$$G[0]=\int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega}|\nabla\tilde{u}(x)|^2\,dx,\tag{9.65}$$

*where $\tilde{u}$ is as in Definition 9.15.*

*Proof.* Let $\Theta_n[\cdot]$ be as in Proposition 9.19. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon\in\,]0,\epsilon_3[$. Clearly, by the periodicity of $u[\epsilon]$, we have

$$\int_{\mathbb{P}_a[\Omega_\epsilon]}|\nabla u[\epsilon](x)|^2\,dx-k^2\int_{\mathbb{P}_a[\Omega_\epsilon]}|u[\epsilon](x)|^2\,dx=-\epsilon^{n-1}\int_{\partial\Omega}\overline{g(t)}\Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\,d\sigma_t.$$

By equality (6.25) and since $Q_n^k=0$ for $n$ odd, we have

$$\begin{aligned}\Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)=&\frac{1}{2}\epsilon^{-1}\Theta_n[\epsilon](t)+\epsilon^{-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s\\&+\epsilon^{-1}\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s\quad\forall t\in\partial\Omega.\end{aligned}$$

We set

$$\begin{aligned}\tilde{G}[\epsilon](t)\equiv&\frac{1}{2}\Theta_n[\epsilon](t)+\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s\\&+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s\quad\forall t\in\partial\Omega,\end{aligned}$$

for all $\epsilon\in\,]-\epsilon_3,\epsilon_3[$. By Theorem E.6 *(ii)* and Theorem C.4, one can easily show that there exists $\epsilon_5\in\,]0,\epsilon_3]$ such that $\tilde{G}$ is a real analytic map of $]-\epsilon_5,\epsilon_5[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Hence, if we set

$$G[\epsilon]\equiv-\int_{\partial\Omega}\overline{g(t)}\tilde{G}[\epsilon](t)\,d\sigma_t,$$

for all $\epsilon\in\,]-\epsilon_5,\epsilon_5[$, then, by standard properties of functions in Schauder spaces, we have that $G$ is a real analytic map of $]-\epsilon_5,\epsilon_5[$ to $\mathbb{C}$, such that equality (9.64) holds.

Finally, if $\epsilon=0$, by Folland [52, p. 118], we have

$$\begin{aligned}G[0]&=-\int_{\partial\Omega}\overline{g(t)}\frac{\partial}{\partial\nu_\Omega}v^-[\partial\Omega,\tilde{\theta},0](t)\,d\sigma_t\\&=\int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega}|\nabla\tilde{u}(x)|^2\,dx,\end{aligned}$$

and accordingly (9.65) holds. □

**Theorem 9.34.** *Let $n$ be even and $n > 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_3$, $\epsilon_2'$ be as in Proposition 9.20. Then there exist $\epsilon_5 \in ]0, \epsilon_3]$, and two real analytic operators $G_1^\#$, $G_2^\#$ of $]-\epsilon_5, \epsilon_5[ \times ]-\epsilon_2', \epsilon_2'[$ to $\mathbb{C}$, such that*

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\,dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\,dx = \\ \epsilon^{n-2} G_1^\#[\epsilon, \epsilon\log\epsilon] + \epsilon^{2n-3}(\log\epsilon) G_2^\#[\epsilon, \epsilon\log\epsilon],\end{aligned} \tag{9.66}$$

*for all $\epsilon \in ]0, \epsilon_5[$. Moreover,*

$$G_1^\#[0,0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\,dx, \tag{9.67}$$

*where $\tilde{u}$ is as in Definition 9.15.*

*Proof.* Let $\Theta_n^\#[\cdot,\cdot]$ be as in Proposition 9.20. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon \in ]0, \epsilon_3[$. Clearly, by the periodicity of $u[\epsilon]$, we have

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\,dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\,dx = -\epsilon^{n-1} \int_{\partial\Omega} \overline{g(t)} \Big(\frac{\partial u[\epsilon]}{\partial \nu_{\Omega_\epsilon}}\Big) \circ (w + \epsilon\,\mathrm{id}_{\partial\Omega})(t)\,d\sigma_t.$$

By equality (6.25), we have

$$\begin{aligned}&\Big(\frac{\partial u[\epsilon]}{\partial \nu_{\Omega_\epsilon}}\Big) \circ (w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) \\ &= \frac{1}{2}\epsilon^{-1}\Theta_n^\#[\epsilon, \epsilon\log\epsilon](t) + \epsilon^{-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k)\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &\quad + \epsilon^{-1}\epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega} \nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &\quad + \epsilon^{-1}\epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \qquad \forall t \in \partial\Omega.\end{aligned}$$

We set

$$\begin{aligned}\tilde{G}_1^\#[\epsilon,\epsilon'](t) \equiv &\frac{1}{2}\Theta_n^\#[\epsilon,\epsilon'](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^n} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \\ &+ \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,\end{aligned}$$

and

$$\tilde{G}_2^\#[\epsilon,\epsilon'](t) \equiv k^{n-2}\int_{\partial\Omega} \nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $(\epsilon,\epsilon') \in ]-\epsilon_3,\epsilon_3[ \times ]-\epsilon_2',\epsilon_2'[$. By Theorem E.6 (*ii*) and Theorem C.4, one can easily show that there exists $\epsilon_5 \in ]0,\epsilon_3]$ such that $\tilde{G}_1^\#$, $\tilde{G}_2^\#$ are real analytic maps of $]-\epsilon_5,\epsilon_5[ \times ]-\epsilon_2',\epsilon_2'[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Hence, if we set

$$G_1^\#[\epsilon,\epsilon'] \equiv -\int_{\partial\Omega} \overline{g(t)}\tilde{G}_1^\#[\epsilon,\epsilon'](t)\,d\sigma_t,$$

and

$$G_2^\#[\epsilon,\epsilon'] \equiv -\int_{\partial\Omega} \overline{g(t)}\tilde{G}_2^\#[\epsilon,\epsilon'](t)\,d\sigma_t,$$

for all $(\epsilon,\epsilon') \in ]-\epsilon_5,\epsilon_5[ \times ]-\epsilon_2',\epsilon_2'[$, then, by standard properties of functions in Schauder spaces, we have that $G_1^\#$, $G_2^\#$ are real analytic maps of $]-\epsilon_5,\epsilon_5[ \times ]-\epsilon_2',\epsilon_2'[$ to $\mathbb{C}$, such that equality (9.66) holds.

Finally, if $\epsilon = \epsilon' = 0$, by Folland [52, p. 118], we have

$$\begin{aligned}G_1^\#[0,0] &= -\int_{\partial\Omega} \overline{g(t)}\frac{\partial}{\partial\nu_\Omega} v^-[\partial\Omega,\tilde{\theta},0](t)\,d\sigma_t \\ &= \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega} |\nabla\tilde{u}(x)|^2\,dx,\end{aligned}$$

and accordingly (9.67) holds. □

**Theorem 9.35.** *Let $n = 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). Let $\epsilon_3$, $\epsilon_2'$, $\epsilon_2''$ be as in Proposition 9.28. Then there exist $\epsilon_5 \in ]0,\epsilon_3]$, and four real analytic operators $\tilde{G}_1^\#$, $\tilde{G}_2^\#$, $\tilde{G}_3^\#$, $\tilde{G}_4^\#$ of $]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[\times]-\epsilon_2'',\epsilon_2''[$ to $\mathbb{C}$, such that*

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\,dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\,dx \\ = \tilde{G}_1^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] + \epsilon(\log\epsilon)\tilde{G}_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] \\ + \frac{1}{\log\epsilon}\tilde{G}_3^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] + \epsilon\tilde{G}_4^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}],\end{aligned} \tag{9.68}$$

*for all $\epsilon \in ]0,\epsilon_5[$. Moreover,*

$$\tilde{G}_1^\#[0,0,0] = \int_{\mathbb{R}^2\setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\,dx, \tag{9.69}$$

*where $\tilde{u}$ is as in Definition 9.25.*

*Proof.* Let $\Phi_2^\#[\cdot,\cdot,\cdot]$, $\Xi_2^\#[\cdot,\cdot,\cdot]$ be as in Proposition 9.28. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon \in ]0,\epsilon_3[$. Clearly, by the periodicity of $u[\epsilon]$, we have

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\,dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\,dx = -\epsilon \int_{\partial\Omega} \overline{g(t)} \Big(\frac{\partial u[\epsilon]}{\partial \nu_{\Omega_\epsilon}}\Big) \circ (w + \epsilon\,\mathrm{id}_{\partial\Omega})(t)\,d\sigma_t.$$

By equality (6.25), we have

$$\begin{aligned}&\Big(\frac{\partial u[\epsilon]}{\partial \nu_{\Omega_\epsilon}}\Big) \circ (w + \epsilon\,\mathrm{id}_{\partial\Omega})(t) \\ &= \epsilon^{-1}\Big\{\frac{1}{2}\Phi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^2} S_2(t-s,\epsilon k)\Phi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}](s)\,d\sigma_s \\ &\quad + \epsilon(\log\epsilon)\int_{\partial\Omega} \nu_\Omega(t)\cdot DQ_2^k(\epsilon(t-s))\Phi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}](s)\,d\sigma_s \\ &\quad + \epsilon\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_2^{a,k}(\epsilon(t-s))\Phi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}](s)\,d\sigma_s\Big\} \\ &\quad + \epsilon^{-1}\frac{1}{\log\epsilon}\Big\{\frac{1}{2}\Xi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^2} S_2(t-s,\epsilon k)\Xi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}]\,d\sigma_s \\ &\quad + \epsilon(\log\epsilon)\int_{\partial\Omega} \nu_\Omega(t)\cdot DQ_2^k(\epsilon(t-s))\Xi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}]\,d\sigma_s \\ &\quad + \epsilon\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_2^{a,k}(\epsilon(t-s))\Xi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}]\,d\sigma_s\Big\} \qquad \forall t \in \partial\Omega.\end{aligned}$$

We set

$$\begin{aligned}F_1^\#[\epsilon,\epsilon',\epsilon''](t) \equiv &\frac{1}{2}\Phi_2^\#[\epsilon,\epsilon',\epsilon''](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^2} S_2(t-s,\epsilon k)\Phi_2^\#[\epsilon,\epsilon',\epsilon''](s)\,d\sigma_s \\ &+ \epsilon\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_2^{a,k}(\epsilon(t-s))\Phi_2^\#[\epsilon,\epsilon',\epsilon''](s)\,d\sigma_s \qquad \forall t \in \partial\Omega,\end{aligned}$$

$$F_2^\#[\epsilon,\epsilon',\epsilon''](t) \equiv \int_{\partial\Omega} \nu_\Omega(t)\cdot DQ_2^k(\epsilon(t-s))\Phi_2^\#[\epsilon,\epsilon',\epsilon''](s)\,d\sigma_s \qquad \forall t \in \partial\Omega,$$

$$\begin{aligned}F_3^\#[\epsilon,\epsilon',\epsilon''](t) \equiv &\frac{1}{2}\Xi_2^\#[\epsilon,\epsilon',\epsilon''] + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^2} S_2(t-s,\epsilon k)\Xi_2^\#[\epsilon,\epsilon',\epsilon'']\,d\sigma_s \\ &+ \epsilon\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_2^{a,k}(\epsilon(t-s))\Xi_2^\#[\epsilon,\epsilon',\epsilon'']\,d\sigma_s \qquad \forall t \in \partial\Omega,\end{aligned}$$

$$F_4^\#[\epsilon,\epsilon',\epsilon''](t) \equiv \int_{\partial\Omega} \nu_\Omega(t)\cdot DQ_2^k(\epsilon(t-s))\Xi_2^\#[\epsilon,\epsilon',\epsilon'']\,d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $(\epsilon, \epsilon', \epsilon'') \in ]-\epsilon_3, \epsilon_3[ \times ]-\epsilon'_2, \epsilon'_2[ \times ]-\epsilon''_2, \epsilon''_2[$. By Theorem E.6 (*ii*) and Theorem C.4, one can easily show that there exists $\epsilon_5 \in ]0, \epsilon_3]$ such that $F_1^\#$, $F_2^\#$, $F_3^\#$, $F_4^\#$ are real analytic maps of $]-\epsilon_5, \epsilon_5[ \times ]-\epsilon'_2, \epsilon'_2[ \times ]-\epsilon''_2, \epsilon''_2[$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Hence, if we set

$$\tilde{G}_1^\#[\epsilon, \epsilon', \epsilon''] \equiv -\int_{\partial\Omega} \overline{g(t)} F_1^\#[\epsilon, \epsilon', \epsilon''](t)\, d\sigma_t,$$

$$\tilde{G}_2^\#[\epsilon, \epsilon', \epsilon''] \equiv -\int_{\partial\Omega} \overline{g(t)} F_2^\#[\epsilon, \epsilon', \epsilon''](t)\, d\sigma_t,$$

$$\tilde{G}_3^\#[\epsilon, \epsilon', \epsilon''] \equiv -\int_{\partial\Omega} \overline{g(t)} F_3^\#[\epsilon, \epsilon', \epsilon''](t)\, d\sigma_t,$$

$$\tilde{G}_4^\#[\epsilon, \epsilon', \epsilon''] \equiv -\int_{\partial\Omega} \overline{g(t)} F_4^\#[\epsilon, \epsilon', \epsilon''](t)\, d\sigma_t,$$

for all $(\epsilon, \epsilon', \epsilon'') \in ]-\epsilon_5, \epsilon_5[ \times ]-\epsilon'_2, \epsilon'_2[ \times ]-\epsilon''_2, \epsilon''_2[$, then, by standard properties of functions in Schauder spaces, we have that $\tilde{G}_1^\#$, $\tilde{G}_2^\#$, $\tilde{G}_3^\#$, $\tilde{G}_4^\#$ are real analytic maps of $]-\epsilon_5, \epsilon_5[ \times ]-\epsilon'_2, \epsilon'_2[ \times ]-\epsilon''_2, \epsilon''_2[$ to $\mathbb{C}$, such that equality (9.68) holds.

Finally, if $\epsilon = \epsilon' = \epsilon'' = 0$, then by Folland [52, p. 118] we have

$$\begin{aligned} G_1^\#[0,0,0] &= -\int_{\partial\Omega} \overline{g(t)} \frac{\partial}{\partial\nu_\Omega} v^-[\partial\Omega, \tilde{\phi}, 0](t)\, d\sigma_t \\ &= -\int_{\partial\Omega} \overline{\tilde{u}(t)} \frac{\partial \tilde{u}(t)}{\partial\nu_\Omega}\, d\sigma_t \\ &= \int_{\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\, dx, \end{aligned}$$

and accordingly (9.69) holds. □

*Remark* 9.36. If $n$ is odd, we note that the right-hand side of the equality in (9.64) of Theorem 9.33 can be continued real analytically in the whole $]-\epsilon_5, \epsilon_5[$.

Moreover,

$$\lim_{\epsilon \to 0^+} \Big[ \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\, dx \Big] = 0,$$

for all $n \in \mathbb{N} \setminus \{0, 1, 2\}$ ($n$ even or odd.)

### 9.2.4 A real analytic continuation Theorem for the integral of the solution

We now prove real analytic continuation Theorems for the integral of the solution. Namely, we prove the following results.

**Theorem 9.37.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_3$ be as in Proposition 9.19. Then there exist $\epsilon_6 \in ]0, \epsilon_3]$, and a real analytic operator $J$ of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx = \frac{\epsilon^{n-2}}{k^2} J[\epsilon], \tag{9.70}$$

*for all $\epsilon \in ]0, \epsilon_6[$. Moreover,*

$$J[0] = \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega} \tilde{u}(x)\, d\sigma_x, \tag{9.71}$$

*where $\tilde{u}$ is as in Definition 9.15.*

*Proof.* Let $\Theta_n[\cdot]$ be as in Proposition 9.19. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon \in ]0, \epsilon_3[$. Clearly, by the Divergence Theorem and the periodicity of $u[\epsilon]$, we have

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx &= -\frac{1}{k^2}\int_{\mathbb{P}_a[\Omega_\epsilon]} \Delta u[\epsilon](x)\,dx \\ &= -\frac{1}{k^2}\int_{\partial\mathbb{P}_a[\Omega_\epsilon]} \frac{\partial}{\partial\nu_{\mathbb{P}_a[\Omega_\epsilon]}} u[\epsilon](x)\,d\sigma_x \\ &= -\frac{1}{k^2}\Big[\int_{\partial A}\frac{\partial}{\partial\nu_A}u[\epsilon](x)\,d\sigma_x - \int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon](x)\,d\sigma_x\Big] \\ &= \frac{1}{k^2}\int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon](x)\,d\sigma_x.\end{aligned}$$

As a consequence,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx = \frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega}\Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\,d\sigma_t.$$

By equality (6.25) and since $Q_n^k = 0$ for $n$ odd, we have

$$\begin{aligned}\Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) =&\frac{1}{2}\epsilon^{-1}\Theta_n[\epsilon](t) + \epsilon^{-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s \\ &+\epsilon^{-1}\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s \quad \forall t\in\partial\Omega.\end{aligned}$$

We set

$$\begin{aligned}\tilde J[\epsilon](t) \equiv&\frac{1}{2}\Theta_n[\epsilon](t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s \\ &+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s \quad \forall t\in\partial\Omega,\end{aligned}$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. By Theorem E.6 $(ii)$ and Theorem C.4, one can easily show that there exists $\epsilon_6 \in ]0, \epsilon_3]$ such that $\tilde J$ is a real analytic map of $]-\epsilon_6, \epsilon_6[$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Hence, if we set

$$J[\epsilon] \equiv \int_{\partial\Omega}\tilde J[\epsilon](t)\,d\sigma_t,$$

for all $\epsilon \in ]-\epsilon_6, \epsilon_6[$, then, by standard properties of functions in Schauder spaces, we have that $J$ is a real analytic map of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{C}$, such that equality (9.70) holds.

Finally, if $\epsilon = 0$, we have

$$\begin{aligned}J[0] &= \int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega}v^-[\partial\Omega,\tilde\theta,0](t)\,d\sigma_t \\ &= \int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega}\tilde u(x)\,d\sigma_x,\end{aligned}$$

and accordingly (9.71) holds. □

**Theorem 9.38.** *Let $n$ be even and $n > 2$. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_3$, $\epsilon_2'$ be as in Proposition 9.20. Then there exist $\epsilon_6 \in ]0, \epsilon_3]$, and two real analytic operators $J_1^\#$, $J_2^\#$ of $]-\epsilon_6, \epsilon_6[\times]-\epsilon_2', \epsilon_2'[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx = \frac{\epsilon^{n-2}}{k^2}J_1^\#[\epsilon,\epsilon\log\epsilon] + \frac{\epsilon^{2n-3}(\log\epsilon)}{k^2}J_2^\#[\epsilon,\epsilon\log\epsilon], \tag{9.72}$$

*for all $\epsilon \in ]0, \epsilon_6[$. Moreover,*

$$J_1^\#[0,0] = \int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega}\tilde u(x)\,d\sigma_x, \tag{9.73}$$

*where $\tilde u$ is as in Definition 9.15.*

*Proof.* Let $\Theta_n^\#[\cdot,\cdot]$ be as in Proposition 9.20. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon \in ]0,\epsilon_3[$. Clearly, by the Divergence Theorem and the periodicity of $u[\epsilon]$, we have

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx &= -\frac{1}{k^2}\int_{\mathbb{P}_a[\Omega_\epsilon]} \Delta u[\epsilon](x)\,dx\\ &= -\frac{1}{k^2}\int_{\partial\mathbb{P}_a[\Omega_\epsilon]} \frac{\partial}{\partial\nu_{\mathbb{P}_a[\Omega_\epsilon]}} u[\epsilon](x)\,d\sigma_x\\ &= -\frac{1}{k^2}\Big[\int_{\partial A}\frac{\partial}{\partial\nu_A}u[\epsilon](x)\,d\sigma_x - \int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon](x)\,d\sigma_x\Big]\\ &= \frac{1}{k^2}\int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon](x)\,d\sigma_x.\end{aligned}$$

As a consequence,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx = \frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega}\Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\,d\sigma_t.$$

By equality (6.25), we have

$$\begin{aligned}&\Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\\ &\quad=\frac{1}{2}\epsilon^{-1}\Theta_n^\#[\epsilon,\epsilon\log\epsilon](t)+\epsilon^{-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s\\ &\quad\quad+\epsilon^{-1}\epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s\\ &\quad\quad+\epsilon^{-1}\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega.\end{aligned}$$

We set

$$\begin{aligned}\tilde J_1^\#[\epsilon,\epsilon'](t)\equiv&\frac{1}{2}\Theta_n^\#[\epsilon,\epsilon'](t)+\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s\\ &+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,\end{aligned}$$

and

$$\tilde J_2^\#[\epsilon,\epsilon'](t)\equiv k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $(\epsilon,\epsilon')\in]-\epsilon_3,\epsilon_3[\times]-\epsilon_2',\epsilon_2'[$. By Theorem E.6 $(ii)$ and Theorem C.4, one can easily show that there exists $\epsilon_6\in]0,\epsilon_3]$ such that $\tilde J_1^\#$, $\tilde J_2^\#$ are real analytic maps of $]-\epsilon_6,\epsilon_6[\times]-\epsilon_2',\epsilon_2'[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Hence, if we set

$$J_1^\#[\epsilon,\epsilon']\equiv\int_{\partial\Omega}\tilde J_1^\#[\epsilon,\epsilon'](t)\,d\sigma_t,$$

and

$$J_2^\#[\epsilon,\epsilon']\equiv\int_{\partial\Omega}\tilde J_2^\#[\epsilon,\epsilon'](t)\,d\sigma_t,$$

for all $(\epsilon,\epsilon')\in]-\epsilon_6,\epsilon_6[\times]-\epsilon_2',\epsilon_2'[$, then, by standard properties of functions in Schauder spaces, we have that $J_1^\#$, $J_2^\#$ are real analytic maps of $]-\epsilon_6,\epsilon_6[\times]-\epsilon_2',\epsilon_2'[$ to $\mathbb{C}$, such that equality (9.72) holds.

Finally, if $\epsilon=\epsilon'=0$, we have

$$\begin{aligned}J_1^\#[0,0]&=\int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega}v^-[\partial\Omega,\tilde\theta,0](t)\,d\sigma_t\\ &=\int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega}\tilde u(x)\,d\sigma_x,\end{aligned}$$

and accordingly (9.73) holds. □

**Theorem 9.39.** *Let $n=2$. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha\in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). Let $\epsilon_3$, $\epsilon_2'$, $\epsilon_2''$ be as in Proposition 9.28. Then there exist $\epsilon_6 \in ]0,\epsilon_3]$, and four real analytic operators $\tilde{J}_1^\#$, $\tilde{J}_2^\#$, $\tilde{J}_3^\#$, $\tilde{J}_4^\#$ of $]-\epsilon_6,\epsilon_6[\times]-\epsilon_2',\epsilon_2'[\times]-\epsilon_2'',\epsilon_2''[$ to $\mathbb{C}$, such that*

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx =& \frac{1}{k^2}\tilde{J}_1^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] + \frac{\epsilon(\log\epsilon)}{k^2}\tilde{J}_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}]\\ &+\frac{1}{k^2\log\epsilon}\tilde{J}_3^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] + \frac{\epsilon}{k^2}\tilde{J}_4^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}],\end{aligned} \tag{9.74}$$

*for all $\epsilon\in]0,\epsilon_6[$. Moreover,*

$$\tilde{J}_1^\#[0,0,0]=0. \tag{9.75}$$

*Proof.* Let $\Phi_2^\#[\cdot,\cdot,\cdot]$, $\Xi_2^\#[\cdot,\cdot,\cdot]$ be as in Proposition 9.28. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon\in]0,\epsilon_3[$. Clearly, by the Divergence Theorem and the periodicity of $u[\epsilon]$, we have

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx &= -\frac{1}{k^2}\int_{\mathbb{P}_a[\Omega_\epsilon]}\Delta u[\epsilon](x)\,dx\\ &= -\frac{1}{k^2}\int_{\partial\mathbb{P}_a[\Omega_\epsilon]}\frac{\partial}{\partial\nu_{\mathbb{P}_a[\Omega_\epsilon]}}u[\epsilon](x)\,d\sigma_x\\ &= -\frac{1}{k^2}\Big[\int_{\partial A}\frac{\partial}{\partial\nu_A}u[\epsilon](x)\,d\sigma_x - \int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon](x)\,d\sigma_x\Big]\\ &= \frac{1}{k^2}\int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon](x)\,d\sigma_x.\end{aligned}$$

As a consequence,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx = \frac{\epsilon}{k^2}\int_{\partial\Omega}\Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\,d\sigma_t.$$

By equality (6.25), we have

$$\begin{aligned}&\Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\\ &\quad=\epsilon^{-1}\Big\{\frac{1}{2}\Phi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}](t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^2}S_2(t-s,\epsilon k)\Phi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}](s)\,d\sigma_s\\ &\qquad+\epsilon(\log\epsilon)\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_2^k(\epsilon(t-s))\Phi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}](s)\,d\sigma_s\\ &\qquad+\epsilon\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_2^{a,k}(\epsilon(t-s))\Phi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}](s)\,d\sigma_s\Big\}\\ &\qquad+\epsilon^{-1}\frac{1}{\log\epsilon}\Big\{\frac{1}{2}\Xi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] + \int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^2}S_2(t-s,\epsilon k)\Xi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}]\,d\sigma_s\\ &\qquad+\epsilon(\log\epsilon)\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_2^k(\epsilon(t-s))\Xi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}]\,d\sigma_s\\ &\qquad+\epsilon\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_2^{a,k}(\epsilon(t-s))\Xi_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}]\,d\sigma_s\Big\}\qquad\forall t\in\partial\Omega.\end{aligned}$$

We set

$$\begin{aligned}F_1^\#[\epsilon,\epsilon',\epsilon''](t)\equiv&\frac{1}{2}\Phi_2^\#[\epsilon,\epsilon',\epsilon''](t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^2}S_2(t-s,\epsilon k)\Phi_2^\#[\epsilon,\epsilon',\epsilon''](s)\,d\sigma_s\\ &+\epsilon\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_2^{a,k}(\epsilon(t-s))\Phi_2^\#[\epsilon,\epsilon',\epsilon''](s)\,d\sigma_s\qquad\forall t\in\partial\Omega,\end{aligned}$$

$$F_2^\#[\epsilon,\epsilon',\epsilon''](t)\equiv\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_2^k(\epsilon(t-s))\Phi_2^\#[\epsilon,\epsilon',\epsilon''](s)\,d\sigma_s\qquad\forall t\in\partial\Omega,$$

$$F_3^{\#}[\epsilon,\epsilon',\epsilon''](t) \equiv \frac{1}{2}\Xi_2^{\#}[\epsilon,\epsilon',\epsilon''] + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^2}S_2(t-s,\epsilon k)\Xi_2^{\#}[\epsilon,\epsilon',\epsilon'']\,d\sigma_s$$
$$+\epsilon\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_2^{a,k}(\epsilon(t-s))\Xi_2^{\#}[\epsilon,\epsilon',\epsilon'']\,d\sigma_s \qquad \forall t\in\partial\Omega,$$
$$F_4^{\#}[\epsilon,\epsilon',\epsilon''](t) \equiv \int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_2^k(\epsilon(t-s))\Xi_2^{\#}[\epsilon,\epsilon',\epsilon'']\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $(\epsilon,\epsilon',\epsilon'')\in\,]-\epsilon_3,\epsilon_3[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\epsilon_2'',\epsilon_2''[$. By Theorem E.6 $(ii)$ and Theorem C.4, one can easily show that there exists $\epsilon_6\in\,]0,\epsilon_3]$ such that $F_1^{\#}$, $F_2^{\#}$, $F_3^{\#}$, $F_4^{\#}$ are real analytic maps of $]-\epsilon_6,\epsilon_6[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\epsilon_2'',\epsilon_2''[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Hence, if we set

$$\tilde{J}_1^{\#}[\epsilon,\epsilon',\epsilon''] \equiv \int_{\partial\Omega}F_1^{\#}[\epsilon,\epsilon',\epsilon''](t)\,d\sigma_t,$$
$$\tilde{J}_2^{\#}[\epsilon,\epsilon',\epsilon''] \equiv \int_{\partial\Omega}F_2^{\#}[\epsilon,\epsilon',\epsilon''](t)\,d\sigma_t,$$
$$\tilde{J}_3^{\#}[\epsilon,\epsilon',\epsilon''] \equiv \int_{\partial\Omega}F_3^{\#}[\epsilon,\epsilon',\epsilon''](t)\,d\sigma_t,$$
$$\tilde{J}_4^{\#}[\epsilon,\epsilon',\epsilon''] \equiv \int_{\partial\Omega}F_4^{\#}[\epsilon,\epsilon',\epsilon''](t)\,d\sigma_t,$$

for all $(\epsilon,\epsilon',\epsilon'')\in\,]-\epsilon_6,\epsilon_6[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\epsilon_2'',\epsilon_2''[$, then, by standard properties of functions in Schauder spaces, we have that $\tilde{J}_1^{\#}$, $\tilde{J}_2^{\#}$, $\tilde{J}_3^{\#}$, $\tilde{J}_4^{\#}$ are real analytic maps of $]-\epsilon_6,\epsilon_6[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\epsilon_2'',\epsilon_2''[$ to $\mathbb{C}$, such that equality (9.74) holds.

Finally, if $\epsilon=\epsilon'=\epsilon''=0$, by Folland [52, Lemma 3.30, p. 133], we have

$$\begin{aligned} J_1^{\#}[0,0,0] &= \int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega}v^-[\partial\Omega,\tilde{\phi},0](t)\,d\sigma_t \\ &= \int_{\partial\Omega}\tilde{\phi}(t)\,d\sigma_t \\ &= 0, \end{aligned}$$

and accordingly (9.75) holds. □

## 9.2.5 A remark on the Dirichlet problem

In this section we want to observe that, if we multiply the Dirichlet datum by the factor $\epsilon^{-l}$, with $l\in\mathbb{Z}$, then we may still have real analytic continuation properties for the solution and for functionals related to the solution even if $l>0$.

Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $l\in\mathbb{Z}$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). For each $\epsilon\in\,]0,\epsilon_1^*[$, we consider the following periodic Dirichlet problem for the Laplace equation.

$$\begin{cases} \Delta u(x)+k^2u(x)=0 & \forall x\in\mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_j)=u(x) & \forall x\in\operatorname{cl}\mathbb{T}_a[\Omega_\epsilon], \quad \forall j\in\{1,\dots,n\}, \\ u(x)=\epsilon^{-l}g(\frac{1}{\epsilon}(x-w)) & \forall x\in\partial\Omega_\epsilon. \end{cases} \tag{9.76}$$

By virtue of Theorem 9.4, we can give the following definition.

**Definition 9.40.** Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $l\in\mathbb{Z}$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). For each $\epsilon\in\,]0,\epsilon_1^*[$, we denote by $u_l[\epsilon]$ the unique solution in $C^{m,\alpha}(\operatorname{cl}\mathbb{T}_a[\Omega_\epsilon],\mathbb{C})$ of boundary value problem (9.76).

Then we have the following Theorems

**Theorem 9.41.** *Let $n$ be odd. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $l\in\mathbb{Z}$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_3$ be as in Proposition 9.19. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\operatorname{cl}V\cap\mathbb{S}_a[\Omega_0]=\emptyset$. Then there exist $\epsilon_4\in\,]0,\epsilon_3]$, and a real analytic operator $U$ of $]-\epsilon_4,\epsilon_4[$ to the space $C^0(\operatorname{cl}V,\mathbb{C})$, such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*

$$u_l[\epsilon](x) = \epsilon^{n-2-l} U[\epsilon](x) \qquad \forall x \in \mathrm{cl}\, V,$$

*for all* $\epsilon \in ]0, \epsilon_4[$.

*Proof.* It is a straightforward consequence of Theorem 9.29. □

**Theorem 9.42.** *Let $n$ be even and $n > 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $l \in \mathbb{Z}$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_3$, $\epsilon_2'$ be as in Proposition 9.20. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \epsilon_3]$ and a real analytic operator $U^\#$ of $]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[$ to the space $C^0(\mathrm{cl}\, V, \mathbb{C})$, such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*

$$u_l[\epsilon](x) = \epsilon^{n-2-l} U^\#[\epsilon, \epsilon \log \epsilon](x) \qquad \forall x \in \mathrm{cl}\, V,$$

*for all* $\epsilon \in ]0, \epsilon_4[$.

*Proof.* It is a straightforward consequence of Theorem 9.31. □

**Theorem 9.43.** *Let $n = 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $l \in \mathbb{Z}$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_3$, $\epsilon_2'$, $\epsilon_2''$ be as in Proposition 9.28. Let $V$ be a bounded open subset of $\mathbb{R}^2$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \epsilon_3]$ and two real analytic operators $\tilde{U}_1^\#$, $\tilde{U}_2^\#$ of $]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[ \times ]-\epsilon_2'', \epsilon_2''[$ to the space $C^0(\mathrm{cl}\, V, \mathbb{C})$, such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*

$$u_l[\epsilon](x) = \epsilon^{-l} \tilde{U}_1^\#[\epsilon, \epsilon \log \epsilon, (\log \epsilon)^{-1}](x) + \frac{1}{\log \epsilon} \epsilon^{-l} \tilde{U}_2^\#[\epsilon, \epsilon \log \epsilon, (\log \epsilon)^{-1}](x) \qquad \forall x \in \mathrm{cl}\, V,$$

*for all* $\epsilon \in ]0, \epsilon_4[$.

*Proof.* It is a straightforward consequence of Theorem 9.32. □

**Theorem 9.44.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $l \in \mathbb{Z}$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_3$ be as in Proposition 9.19. Then there exist $\epsilon_5 \in ]0, \epsilon_3]$, and a real analytic operator $G$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u_l[\epsilon](x)|^2 \, dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u_l[\epsilon](x)|^2 \, dx = \epsilon^{n-2-2l} G[\epsilon], \tag{9.77}$$

*for all $\epsilon \in ]0, \epsilon_5[$. Moreover,*

$$G[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\, \Omega} |\nabla \tilde{u}(x)|^2 \, dx, \tag{9.78}$$

*where $\tilde{u}$ is as in Definition 9.15.*

*Proof.* It is a straightforward consequence of Theorem 9.33. □

**Theorem 9.45.** *Let $n$ be even and $n > 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $l \in \mathbb{Z}$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_3$, $\epsilon_2'$ be as in Proposition 9.20. Then there exist $\epsilon_5 \in ]0, \epsilon_3]$, and two real analytic operators $G_1^\#$, $G_2^\#$ of $]-\epsilon_5, \epsilon_5[ \times ]-\epsilon_2', \epsilon_2'[$ to $\mathbb{C}$, such that*

$$\begin{split}\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u_l[\epsilon](x)|^2 \, dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u_l[\epsilon](x)|^2 \, dx = \\ \epsilon^{n-2-2l} G_1^\#[\epsilon, \epsilon \log \epsilon] + \epsilon^{2n-3-2l} (\log \epsilon) G_2^\#[\epsilon, \epsilon \log \epsilon],\end{split} \tag{9.79}$$

*for all* $\epsilon \in ]0, \epsilon_5[$*. Moreover,*

$$G_1^{\#}[0,0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx, \tag{9.80}$$

*where* $\tilde{u}$ *is as in Definition 9.15.*

*Proof.* It is a straightforward consequence of Theorem 9.34. □

**Theorem 9.46.** *Let* $n = 2$*. Let* $m \in \mathbb{N} \setminus \{0\}$*,* $\alpha \in ]0,1[$*. Let* $l \in \mathbb{Z}$*. Let* $w \in A$*. Let* $\Omega$*,* $\epsilon_1$*,* $k$*,* $g$ *be as in* (1.56)*,* (1.57)*,* (9.14)*,* (9.15)*, respectively. Let* $\epsilon_1^*$ *be as in* (9.16)*. Let* $\epsilon_3$*,* $\epsilon_2'$*,* $\epsilon_2''$ *be as in Proposition 9.28. Then there exist* $\epsilon_5 \in ]0, \epsilon_3]$*, and four real analytic operators* $\tilde{G}_1^{\#}$*,* $\tilde{G}_2^{\#}$*,* $\tilde{G}_3^{\#}$*,* $\tilde{G}_4^{\#}$ *of* $]-\epsilon_5, \epsilon_5[ \times ]-\epsilon_2', \epsilon_2'[ \times ]-\epsilon_2'', \epsilon_2''[$ *to* $\mathbb{C}$*, such that*

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} &|\nabla u_l[\epsilon](x)|^2 \, dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u_l[\epsilon](x)|^2 \, dx \\ &= \epsilon^{-2l} \tilde{G}_1^{\#}[\epsilon, \epsilon \log \epsilon, (\log \epsilon)^{-1}] + \epsilon^{1-2l} (\log \epsilon) \tilde{G}_2^{\#}[\epsilon, \epsilon \log \epsilon, (\log \epsilon)^{-1}] \\ &\quad + \frac{1}{\log \epsilon} \epsilon^{-2l} \tilde{G}_3^{\#}[\epsilon, \epsilon \log \epsilon, (\log \epsilon)^{-1}] + \epsilon^{1-2l} \tilde{G}_4^{\#}[\epsilon, \epsilon \log \epsilon, (\log \epsilon)^{-1}],\end{aligned} \tag{9.81}$$

*for all* $\epsilon \in ]0, \epsilon_5[$*. Moreover,*

$$\tilde{G}_1^{\#}[0,0,0] = \int_{\mathbb{R}^2 \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx, \tag{9.82}$$

*where* $\tilde{u}$ *is as in Definition 9.25.*

*Proof.* It is a straightforward consequence of Theorem 9.35. □

## 9.3 An homogenization problem for the Helmholtz equation with Dirichlet boundary conditions in a periodically perforated domain

In this section we consider an homogenization problem for the Helmhlotz equation with Dirichlet boundary conditions in a periodically perforated domain. In most of the results we assume that $\mathrm{Im}(k) \neq 0$ and $\mathrm{Re}(k) = 0$.

We note that we shall consider the equation

$$\Delta u(x) + \frac{k^2}{\delta^2} u(x) = 0 \qquad \forall x \in \mathbb{T}_a(\epsilon, \delta),$$

together with the usual periodicity condition and a Dirichlet boundary condition. We do so, because if $u$ is a solution of the equation above then the function $u_\delta(\cdot) \equiv u(\delta \cdot)$ is a solution of the following equation

$$\Delta u_\delta(x) + k^2 u_\delta(x) = 0 \qquad \forall x \in \mathbb{T}_a[\Omega_\epsilon],$$

which we can analyse by virtue of the results of Section 9.2.

### 9.3.1 Notation

In this Section we retain the notation introduced in Subsections 1.8.1, 6.7.1, 9.2.1. However, we need to introduce also some other notation.

Let $(\epsilon, \delta) \in (]-\epsilon_1, \epsilon_1[ \setminus \{0\}) \times ]0, +\infty[$. If $v$ is a function of $\mathrm{cl}\,\mathbb{T}_a(\epsilon, \delta)$ to $\mathbb{C}$, then we denote by $\mathbf{E}_{(\epsilon,\delta)}[v]$ the function of $\mathbb{R}^n$ to $\mathbb{C}$, defined by

$$\mathbf{E}_{(\epsilon,\delta)}[v](x) \equiv \begin{cases} v(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon, \delta), \\ 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{T}_a(\epsilon, \delta). \end{cases}$$

### 9.3.2 Preliminaries

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). For each $(\epsilon, \delta) \in ]0, \epsilon_1^*[ \times ]0, +\infty[$, we consider the following periodic Dirichlet problem for the Helmholtz equation.

$$\begin{cases} \Delta u(x) + \frac{k^2}{\delta^2} u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon, \delta), \\ u(x + \delta a_j) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a(\epsilon, \delta), \quad \forall j \in \{1, \dots, n\}, \\ u(x) = g(\frac{1}{\epsilon\delta}(x - \delta w)) & \forall x \in \partial\Omega(\epsilon, \delta). \end{cases} \tag{9.83}$$

By virtue of Theorem 9.4, we can give the following definition.

**Definition 9.47.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). For each $(\epsilon, \delta) \in ]0, \epsilon_1^*[ \times ]0, +\infty[$, we denote by $u_{(\epsilon,\delta)}$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a(\epsilon, \delta), \mathbb{C})$ of boundary value problem (9.83).

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon, \delta)$ tends to $(0,0)$. In order to do so we introduce the following.

**Definition 9.48.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). For each $\epsilon \in ]0, \epsilon_1^*[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$ of the following periodic Dirichlet problem for the Helmholtz equation.

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x + a_j) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1, \dots, n\}, \\ u(x) = g(\frac{1}{\epsilon}(x - w)) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{9.84}$$

*Remark* 9.49. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). For each pair $(\epsilon, \delta) \in ]0, \epsilon_1^*[ \times ]0, +\infty[$, we have

$$u_{(\epsilon,\delta)}(x) = u[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a(\epsilon, \delta).$$

As a first step, we study the behaviour of $u[\epsilon]$ as $\epsilon$ tends to 0.

**Proposition 9.50.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\mathrm{Im}(k) \neq 0$ and $\mathrm{Re}(k) = 0$. Let $\epsilon_1^*$ be as in* (9.16). *Let $1 \leq p < \infty$. Then*

$$\lim_{\epsilon \to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = 0 \qquad \text{in } L^p(A, \mathbb{C}).$$

*Proof.* If $\epsilon \in ]0, \epsilon_1[$, we have

$$u[\epsilon] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) = g(t) \quad \forall t \in \partial\Omega.$$

We set

$$c \equiv \sup\{|g(t)| \colon t \in \partial\Omega\}.$$

Then, by Corollary 6.25, we have

$$|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]](x)| \leq 2c < +\infty \qquad \forall x \in A, \quad \forall \epsilon \in ]0, \epsilon_1[.$$

By Theorems 9.29, 9.31, 9.32, we have

$$\lim_{\epsilon \to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]](x) = 0 \qquad \forall x \in A \setminus \{w\}.$$

Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{\epsilon \to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = 0 \qquad \text{in } L^p(A, \mathbb{C}).$$

□

### 9.3.3 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorem we deduce by Proposition 9.50 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 9.51.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (9.16). *Let $1 \le p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^p(V,\mathbb{C}).$$

*Proof.* By virtue of Proposition 9.50, we have

$$\lim_{\epsilon\to 0^+} \|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\|_{L^p(A,\mathbb{C})} = 0.$$

By the same argument as Theorem D.5 (see in particular (D.5)), there exists a constant $C > 0$ such that

$$\|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\|_{L^p(V,\mathbb{C})} \le C\|\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\|_{L^p(A,\mathbb{C})} \qquad \forall(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,1[.$$

Thus,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \|\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\|_{L^p(V,\mathbb{C})} = 0,$$

and we can easily conclude. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 9.52.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_6$, $J$ be as in Theorem 9.37. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n \frac{\epsilon^{n-2}}{k^2} J[\epsilon], \tag{9.85}$$

*for all $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N}\setminus\{0\}$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N}\setminus\{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon]\Big(\frac{l}{r}x\Big)\,dx \\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](t)\,dt \\ &= \frac{r^n}{l^n}\frac{\epsilon^{n-2}}{k^2}J[\epsilon].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n \frac{\epsilon^{n-2}}{k^2} J[\epsilon],$$

and the conclusion follows. □

**Theorem 9.53.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_6$, $J_1^\#$, $J_2^\#$ be as in Theorem 9.38. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n\Big\{\frac{\epsilon^{n-2}}{k^2}J_1^\#[\epsilon,\epsilon\log\epsilon] + \frac{\epsilon^{2n-3}(\log\epsilon)}{k^2}J_2^\#[\epsilon,\epsilon\log\epsilon]\Big\}, \tag{9.86}$$

*for all $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N} \setminus \{0\}$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N} \setminus \{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](t)\,dt \\ &= \frac{r^n}{l^n}\Big\{\frac{\epsilon^{n-2}}{k^2}J_1^\#[\epsilon,\epsilon\log\epsilon] + \frac{\epsilon^{2n-3}(\log\epsilon)}{k^2}J_2^\#[\epsilon,\epsilon\log\epsilon]\Big\}.\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n\Big\{\frac{\epsilon^{n-2}}{k^2}J_1^\#[\epsilon,\epsilon\log\epsilon] + \frac{\epsilon^{2n-3}(\log\epsilon)}{k^2}J_2^\#[\epsilon,\epsilon\log\epsilon]\Big\},$$

and the conclusion follows. □

**Theorem 9.54.** *Let $n = 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_6$, $\tilde{J}_1^\#$, $\tilde{J}_2^\#$, $\tilde{J}_3^\#$, $\tilde{J}_4^\#$ be as in Theorem 9.39. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^2\Big\{&\frac{1}{k^2}\tilde{J}_1^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] + \frac{\epsilon(\log\epsilon)}{k^2}\tilde{J}_2^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] \\ &+ \frac{1}{k^2\log\epsilon}\tilde{J}_3^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] + \frac{\epsilon}{k^2}\tilde{J}_4^\#[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}]\Big\},\end{aligned} \tag{9.87}$$

*for all $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N} \setminus \{0\}$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N} \setminus \{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= l^2 \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^2}{l^2}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](t)\,dt \\ &= \frac{r^2}{l^2}\Big\{\frac{1}{k^2}\tilde{J}_1^{\#}[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] + \frac{\epsilon(\log\epsilon)}{k^2}\tilde{J}_2^{\#}[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] \\ &\quad + \frac{1}{k^2\log\epsilon}\tilde{J}_3^{\#}[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] + \frac{\epsilon}{k^2}\tilde{J}_4^{\#}[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}]\Big\}.\end{aligned}$$

As a consequence,

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^2\Big\{&\frac{1}{k^2}\tilde{J}_1^{\#}[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] + \frac{\epsilon(\log\epsilon)}{k^2}\tilde{J}_2^{\#}[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] \\ &+\frac{1}{k^2\log\epsilon}\tilde{J}_3^{\#}[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}] + \frac{\epsilon}{k^2}\tilde{J}_4^{\#}[\epsilon,\epsilon\log\epsilon,(\log\epsilon)^{-1}]\Big\},\end{aligned}$$

and the conclusion follows. □

We give the following.

**Definition 9.55.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in \,]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). For each pair $(\epsilon,\delta) \in \,]0,\epsilon_1^*[\,\times\,]0,+\infty[$, we set

$$\mathcal{F}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx - \frac{k^2}{\delta^2}\int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 9.56. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in \,]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in (1.56), (1.57), (9.14), (9.15), respectively. Let $\epsilon_1^*$ be as in (9.16). Let $(\epsilon,\delta) \in \,]0,\epsilon_1^*[\,\times\,]0,+\infty[$. We have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2\,dt \\ &= \delta^{n-2}\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2\,dt,\end{aligned}$$

and

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2\,dx = \delta^n \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](t)|^2\,dt.$$

Accordingly,

$$\begin{aligned}&\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx - \frac{k^2}{\delta^2}\int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2\,dx \\ &\qquad = \delta^{n-2}\Big(\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2\,dt - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](t)|^2\,dt\Big).\end{aligned}$$

Then we give the following definition, where we consider $\mathcal{F}(\epsilon,\delta)$, with $\epsilon$ equal to a certain function of $\delta$.

**Definition 9.57.** Let $n \geq 3$. For each $\delta \in \,]0,+\infty[$, we set

$$\epsilon[\delta] \equiv \delta^{\frac{2}{n-2}}.$$

Let $\epsilon_5$ be as in Theorem 9.33, if $n$ is odd, or as in Theorem 9.34, if $n$ is even. Let $\delta_1 > 0$ be such that $\epsilon[\delta] \in \,]0,\epsilon_5[$, for all $\delta \in \,]0,\delta_1[$. Then we set

$$\mathcal{F}[\delta] \equiv \mathcal{F}(\epsilon[\delta],\delta),$$

for all $\delta \in \,]0,\delta_1[$.

Here we may note that the 'radius' of the holes is $\delta\epsilon[\delta] = \delta^{\frac{n}{n-2}}$ which is the same which appears in Homogenization Theory (cf. *e.g.*, Ansini and Braides [7] and references therein.)

In the following Propositions we compute the limit of $\mathcal{F}[\delta]$ as $\delta$ tends to 0.

**Proposition 9.58.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_5$ be as in Theorem 9.33. Let $\delta_1 > 0$ be as in Definition 9.57. Then*

$$\lim_{\delta \to 0^+} \mathcal{F}[\delta] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx,$$

*where $\tilde{u}$ is as in Definition 9.15.*

*Proof.* For each $\delta \in ]0, \delta_1[$, we set

$$\mathcal{G}(\delta) \equiv \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx.$$

Let $\delta \in ]0, \delta_1[$. By Remark 9.56 and Theorem 9.33, we have

$$\begin{aligned} \mathcal{G}(\delta) &= \delta^{n-2} (\epsilon[\delta])^{n-2} G[\epsilon[\delta]] \\ &= \delta^{n-2} \delta^2 G[\delta^{\frac{2}{n-2}}], \end{aligned}$$

where $G$ is as in Theorem 9.33. On the other hand,

$$\lfloor (1/\delta) \rfloor^n \mathcal{G}(\delta) \leq \mathcal{F}[\delta] \leq \lceil (1/\delta) \rceil^n \mathcal{G}(\delta).$$

As a consequence, since

$$\lim_{\delta \to 0^+} \lfloor (1/\delta) \rfloor^n \delta^n = 1, \qquad \lim_{\delta \to 0^+} \lceil (1/\delta) \rceil^n \delta^n = 1,$$

then

$$\lim_{\delta \to 0^+} \mathcal{F}[\delta] = G[0].$$

Finally, by equality (9.65), we easily conclude. □

**Proposition 9.59.** *Let $n$ be even and $n > 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56), (1.57), (9.14), (9.15), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (9.16). *Let $\epsilon_5$ be as in Theorem 9.34. Let $\delta_1 > 0$ be as in Definition 9.57. Then*

$$\lim_{\delta \to 0^+} \mathcal{F}[\delta] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx,$$

*where $\tilde{u}$ is as in Definition 9.15.*

*Proof.* For each $\delta \in ]0, \delta_1[$, we set

$$\mathcal{G}(\delta) \equiv \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx.$$

Let $\delta \in ]0, \delta_1[$. By Remark 9.56 and Theorem 9.34, we have

$$\begin{aligned} \mathcal{G}(\delta) =& \delta^{n-2} (\epsilon[\delta])^{n-2} G_1^{\#}[\epsilon[\delta], \epsilon[\delta] \log \epsilon[\delta]] \\ &+ \delta^{n-2} (\epsilon[\delta])^{2n-3} (\log \epsilon[\delta]) G_2^{\#}[\epsilon[\delta], \epsilon[\delta] \log \epsilon[\delta]] \\ =& \delta^{n-2} \delta^2 G_1^{\#}[\delta^{\frac{2}{n-2}}, \delta^{\frac{2}{n-2}} \log(\delta^{\frac{2}{n-2}})] \\ &+ \delta^{n-2} \delta^{\frac{4n-6}{n-2}} (\log(\delta^{\frac{2}{n-2}})) G_2^{\#}[\delta^{\frac{2}{n-2}}, \delta^{\frac{2}{n-2}} \log(\delta^{\frac{2}{n-2}})], \end{aligned}$$

where $G_1^{\#}$ and $G_2^{\#}$ are as in Theorem 9.34. On the other hand,

$$\lfloor (1/\delta) \rfloor^n \mathcal{G}(\delta) \leq \mathcal{F}[\delta] \leq \lceil (1/\delta) \rceil^n \mathcal{G}(\delta).$$

As a consequence, since

$$\lim_{\delta\to 0^+} \lfloor (1/\delta) \rfloor^n \delta^n = 1, \qquad \lim_{\delta\to 0^+} \lceil (1/\delta) \rceil^n \delta^n = 1,$$

then

$$\lim_{\delta\to 0^+} \mathcal{F}[\delta] = G_1^{\#}[0,0].$$

Finally, by equality (9.67), we easily conclude. □

In the following Propositions we represent the function $\mathcal{F}[\cdot]$ by means of real analytic functions.

**Proposition 9.60.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56)*,* (1.57)*,* (9.14)*,* (9.15)*, respectively. Let $\epsilon_1^*$ be as in* (9.16)*. Let $\epsilon_5$, and $G$ be as in Theorem 9.33. Let $\delta_1 > 0$ be as in Definition 9.57. Then*

$$\mathcal{F}[(1/l)] = G[(1/l)^{\frac{2}{n-2}}],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* By arguing as in the proof of Proposition 9.58, one can easily see that

$$\begin{aligned}\mathcal{F}[(1/l)] &= l^n (1/l)^{n-2} (1/l)^2 G[(1/l)^{\frac{2}{n-2}}] \\ &= G[(1/l)^{\frac{2}{n-2}}],\end{aligned}$$

for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$. □

**Proposition 9.61.** *Let $n$ be even and $n > 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $g$ be as in* (1.56)*,* (1.57)*,* (9.14)*,* (9.15)*, respectively. Let $\epsilon_1^*$ be as in* (9.16)*. Let $\epsilon_5$, $G_1^{\#}$, and $G_2^{\#}$ be as in Theorem 9.34. Let $\delta_1 > 0$ be as in Definition 9.57. Then*

$$\begin{aligned}\mathcal{F}[(1/l)] =& G_1^{\#}[(1/l)^{\frac{2}{n-2}}, (1/l)^{\frac{2}{n-2}} \log((1/l)^{\frac{2}{n-2}})] \\ &+ (1/l)^{\frac{2n-2}{n-2}} \log((1/l)^{\frac{2}{n-2}}) G_2^{\#}[(1/l)^{\frac{2}{n-2}}, (1/l)^{\frac{2}{n-2}} \log((1/l)^{\frac{2}{n-2}})],\end{aligned}$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* By arguing as in the proof of Proposition 9.59, one can easily see that

$$\begin{aligned}\mathcal{F}[(1/l)] =& l^n (1/l)^{n-2} (1/l)^2 \Big\{ G_1^{\#}[(1/l)^{\frac{2}{n-2}}, (1/l)^{\frac{2}{n-2}} \log((1/l)^{\frac{2}{n-2}})] \\ &+ (1/l)^{\frac{4n-6-2n+4}{n-2}} \log((1/l)^{\frac{2}{n-2}}) G_2^{\#}[(1/l)^{\frac{2}{n-2}}, (1/l)^{\frac{2}{n-2}} \log((1/l)^{\frac{2}{n-2}})] \Big\} \\ =& G_1^{\#}[(1/l)^{\frac{2}{n-2}}, (1/l)^{\frac{2}{n-2}} \log((1/l)^{\frac{2}{n-2}})] \\ &+ (1/l)^{\frac{2n-2}{n-2}} \log((1/l)^{\frac{2}{n-2}}) G_2^{\#}[(1/l)^{\frac{2}{n-2}}, (1/l)^{\frac{2}{n-2}} \log((1/l)^{\frac{2}{n-2}})],\end{aligned}$$

for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$. □

# CHAPTER 10

## Singular perturbation and homogenization problems for the Helmholtz equation with Robin boundary conditions

In this Chapter we introduce the periodic Robin problem for the Helmholtz equation and we study singular perturbation and homogenization problems (linear and nonlinear) for the Helmholtz operator with Robin boundary conditions in a periodically perforated domain. First of all, we consider singular perturbation problems in a periodically perforated domain with small holes, and then we apply the obtained results to homogenization problems. As well as for the Dirichlet and Neumann problems, we follow the approach of Lanza [72], where the asymptotic behaviour of the solutions of a nonlinear Robin problem for the Laplace operator in a domain with a small hole is considered. We also mention Lanza [79], dealing with a Neumann eigenvalue problem in a perforated domain. We note that nonlinear traction problems have been analysed by Dalla Riva and Lanza [38, 39, 42, 43] with this approach. One of the tools used in our analysis is the study of the dependence of layer potentials upon perturbations (cf. Lanza and Rossi [86] and also Dalla Riva and Lanza [40].)

We retain the notation introduced in Sections 1.1 and 1.3, Chapter 6 and Appendix E. For the definitions of $\mathrm{Eig}_D[\mathbb{I}]$, $\mathrm{Eig}_N[\mathbb{I}]$, $\mathrm{Eig}^a_D[\mathbb{I}]$, $\mathrm{Eig}^a_N[\mathbb{I}]$, we refer to Chapter 7.

## 10.1 A periodic linear Robin boundary value problem for the Helmholtz equation

In this Section we introduce the periodic linear Robin problem for the Helmholtz equation and we show the existence and uniqueness of a solution by means of the periodic simple layer potential.

### 10.1.1 Formulation of the problem

In this Subsection we introduce the periodic linear Robin problem for the Helmholtz equation.

First of all, we need to introduce some notation. Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). We shall consider the following assumptions.

$$k \in \mathbb{C}, \ k^2 \neq |2\pi a^{-1}(z)|^2 \quad \forall z \in \mathbb{Z}^n; \tag{10.1}$$

$$\phi \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C}), \tag{10.2}$$

$$\Gamma \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C}). \tag{10.3}$$

We are now ready to give the following.

**Definition 10.1.** Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $k$, $\phi$, $\Gamma$ be as in (10.1), (10.2), (10.3), respectively. We say that a function $u \in C^1(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \mathbb{C}) \cap C^2(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ solves the *periodic* *linear Robin problem for the Helmholtz equation* if

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_j) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) + \phi(x) u(x) = \Gamma(x) & \forall x \in \partial \mathbb{I}. \end{cases} \tag{10.4}$$

### 10.1.2 Existence and uniqueness results for the solutions of the periodic linear Robin problem

In this Subsection we prove existence and uniqueness results for the solutions of the periodic linear Robin problem for the Helmholtz equation.

**Proposition 10.2.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0, 1[$. Let $\mathbb{I}$ be as in* (1.46)*. Let $k$, $\phi$, $\Gamma$ be as in* (10.1)*,* (10.2)*,* (10.3)*, respectively. Assume that $\mathrm{Im}(k) \neq 0$ and that $\mathrm{Re}(\phi(x)) \leq 0$ for all $x \in \partial \mathbb{I}$ and that $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(\phi(x)) \geq 0$ for all $x \in \partial \mathbb{I}$. Then boundary value problem* (10.4) *has at most one solution in $C^1(\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \mathbb{C}) \cap C^2(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$.*

*Proof.* Let $u_1$, $u_2 \in C^1(\mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \mathbb{C}) \cap C^2(\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ be two solutions of (10.4). We set

$$u(x) \equiv u_1(x) - u_2(x) \qquad \forall x \in \mathrm{cl}\, \mathbb{T}_a[\mathbb{I}].$$

Clearly, the function $u$ solves the following boundary value problem:

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ u(x + a_j) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x) + \phi(x) u(x) = 0 & \forall x \in \partial \mathbb{I}. \end{cases}$$

By the Divergence Theorem and the periodicity of $u$, we have

$$\begin{aligned} \int_{\mathbb{P}_a[\mathbb{I}]} \overline{u}(x) \Delta u(x)\, dx &= \int_{\partial \mathbb{P}_a[\mathbb{I}]} \overline{u}(x) \frac{\partial}{\partial \nu_{\mathbb{P}_a[\mathbb{I}]}} u(x)\, d\sigma_x - \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2\, dx \\ &= \int_{\partial A} \overline{u}(x) \frac{\partial}{\partial \nu_A} u(x)\, d\sigma_x - \int_{\partial \mathbb{I}} \overline{u}(x) \frac{\partial}{\partial \nu_{\mathbb{I}}} u(x)\, d\sigma_x - \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2\, dx \\ &= \int_{\partial \mathbb{I}} \phi(x) |u(x)|^2\, d\sigma_x - \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2\, dx. \end{aligned}$$

On the other hand

$$\int_{\mathbb{P}_a[\mathbb{I}]} \overline{u}(x) \Delta u(x)\, dx = -k^2 \int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\, dx,$$

and accordingly

$$\begin{aligned} \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2\, dx - (\mathrm{Re}(k)^2 - \mathrm{Im}(k)^2) \int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\, dx - i 2\, \mathrm{Re}(k)\, \mathrm{Im}(k) \int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\, dx \\ = \int_{\partial \mathbb{I}} \phi(x) |u(x)|^2\, d\sigma_x = \int_{\partial \mathbb{I}} \mathrm{Re}(\phi(x)) |u(x)|^2\, d\sigma_x + i \int_{\partial \mathbb{I}} \mathrm{Im}(\phi(x)) |u(x)|^2\, d\sigma_x. \end{aligned}$$

Assume now $\mathrm{Re}(k) = 0$. Then, since $\mathrm{Re}(\phi(x)) \leq 0$ for all $x \in \partial \mathbb{I}$, we have

$$0 \leq \int_{\mathbb{P}_a[\mathbb{I}]} |\nabla u(x)|^2\, dx + \mathrm{Im}(k)^2 \int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\, dx = \int_{\partial \mathbb{I}} \mathrm{Re}(\phi(x)) |u(x)|^2\, d\sigma_x \leq 0.$$

If, on the contrary, we assume $\mathrm{Re}(k) \neq 0$, then we must have

$$-i 2\, \mathrm{Re}(k)\, \mathrm{Im}(k) \int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\, dx = i \int_{\partial \mathbb{I}} \mathrm{Im}(\phi(x)) |u(x)|^2\, d\sigma_x,$$

and consequently, since $\mathrm{Re}(k)\, \mathrm{Im}(k) \neq 0$ and $\mathrm{Re}(k)\, \mathrm{Im}(k)\, \mathrm{Im}(\phi(x)) \geq 0$ for all $x \in \partial \mathbb{I}$, we have

$$0 \leq \int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2\, dx = \int_{\partial \mathbb{I}} -\frac{\mathrm{Im}(\phi(x))}{2\, \mathrm{Re}(k)\, \mathrm{Im}(k)} |u(x)|^2\, d\sigma_x \leq 0.$$

Thus, in both cases ($\mathrm{Re}(k) = 0$ or $\mathrm{Re}(k) \neq 0$), we have

$$\int_{\mathbb{P}_a[\mathbb{I}]} |u(x)|^2 \, dx = 0.$$

Therefore, $u = 0$ in $\mathrm{cl}\,\mathbb{P}_a[\mathbb{I}]$, and, as a consequence, in $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$. Hence,

$$u_1(x) = u_2(x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}].$$

□

As we know, in order to solve problem (10.4) by means of periodic simple layer potentials, we need to study some integral equations. Thus, in the following Proposition, we study an operator related to the equations that we shall consider in the sequel.

**Proposition 10.3.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $k$, $\phi$ be as in (10.1), (10.2), respectively. Assume that $\mathrm{Im}(k) \neq 0$ and that $\mathrm{Re}(\phi(x)) \leq 0$ for all $x \in \partial\mathbb{I}$ and that $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(\phi(x)) \geq 0$ for all $x \in \partial\mathbb{I}$. Let $\mu \in L^2(\partial\mathbb{I}, \mathbb{C})$ and*

$$\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}(S_n^{a,k}(x-y))\mu(y)\,d\sigma_y + \phi(x)\int_{\partial\mathbb{I}} S_n^{a,k}(x-y)\mu(y)\,d\sigma_y = 0 \quad \text{a.e. on } \partial\mathbb{I}. \tag{10.5}$$

*Then $\mu = 0$.*

*Proof.* By Theorem 6.18 (*iii*), we have $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$. Then by Theorem 6.11 (*i*), we have that the function $v^- \equiv v_a^-[\partial\mathbb{I}, \mu, k]$ is in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \mathbb{C})$ and solves the following boundary value problem

$$\begin{cases} \Delta v^-(x) + k^2 v^-(x) = 0 & \forall x \in \mathbb{T}_a[\mathbb{I}], \\ v^-(x + a_j) = v^-(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \quad \forall j \in \{1, \dots, n\}, \\ \frac{\partial}{\partial\nu_{\mathbb{I}}} v^-(x) + \phi(x) v^-(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

Accordingly, by Proposition 10.2, we have $v^- = 0$ in $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$. Then, by Theorem 6.11 (*i*), the function $v^+ \equiv v_a^+[\partial\mathbb{I}, \mu, k]_{|\mathrm{cl}\,\mathbb{I}}$ is in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I}, \mathbb{C})$ and solves the following boundary value problem

$$\begin{cases} \Delta v^+(x) + k^2 v^+(x) = 0 & \forall x \in \mathbb{I}, \\ v^+(x) = 0 & \forall x \in \partial\mathbb{I}. \end{cases}$$

Hence, since $\mathrm{Im}(k) \neq 0$, we have $v^+ = 0$ in $\mathrm{cl}\,\mathbb{I}$, and so

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} v^+ = 0 \qquad \text{on } \partial\mathbb{I}.$$

Thus, by Theorem 6.11 (*i*), we have

$$\mu = \frac{\partial}{\partial\nu_{\mathbb{I}}} v_a^-[\partial\mathbb{I}, \mu, k] - \frac{\partial}{\partial\nu_{\mathbb{I}}} v_a^+[\partial\mathbb{I}, \mu, k] = 0 \qquad \text{on } \partial\mathbb{I},$$

and the proof is complete. □

Then we have the following Theorem.

**Theorem 10.4.** *Let $m \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $k$, $\phi$ be as in (10.1), (10.2), respectively. Assume that $\mathrm{Im}(k) \neq 0$ and that $\mathrm{Re}(\phi(x)) \leq 0$ for all $x \in \partial\mathbb{I}$ and that $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(\phi(x)) \geq 0$ for all $x \in \partial\mathbb{I}$. Then the following statements hold.*

*(i) The map $\mathcal{L}$ of $L^2(\partial\mathbb{I}, \mathbb{C})$ to $L^2(\partial\mathbb{I}, \mathbb{C})$, which takes $\mu$ to the function $\mathcal{L}[\mu]$ of $\partial\mathbb{I}$ to $\mathbb{C}$, defined by*

$$\mathcal{L}[\mu](x) \equiv \frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}(S_n^{a,k}(x-y))\mu(y)\,d\sigma_y + \phi(x)\int_{\partial\mathbb{I}} S_n^{a,k}(x-y)\mu(y)\,d\sigma_y \quad \text{a.e. on } \partial\mathbb{I}, \tag{10.6}$$

*is a linear homeomorphism of $L^2(\partial\mathbb{I}, \mathbb{C})$ onto itself.*

*(ii) The map $\tilde{\mathcal{L}}$ of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ to $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$, which takes $\mu$ to the function $\tilde{\mathcal{L}}[\mu]$ of $\partial\mathbb{I}$ to $\mathbb{C}$, defined by*

$$\tilde{\mathcal{L}}[\mu](x) \equiv \frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}(S_n^{a,k}(x-y))\mu(y)\,d\sigma_y + \phi(x)\int_{\partial\mathbb{I}} S_n^{a,k}(x-y)\mu(y)\,d\sigma_y \quad \forall x \in \partial\mathbb{I}, \tag{10.7}$$

*is a linear homeomorphism of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ onto itself.*

*Proof.* We first prove statement $(i)$. By Proposition 10.3, we have that $\mathcal{L}$ is injective. Since the singularities in the involved integral operators are weak, we have that $\mathcal{L}$ is continuous and that $\mathcal{L} - \frac{1}{2}I$ is a compact operator in $L^2(\partial\mathbb{I},\mathbb{C})$ (cf. *e.g.*, Folland [52, Prop. 3.11, p. 121].) Hence, by the Fredholm Theory, we have that $\mathcal{L}$ is surjective and, by the Open Mapping Theorem, we have that it is a linear homeomorphism of $L^2(\partial\mathbb{I},\mathbb{C})$ onto itself. We now consider statement $(ii)$. By Theorem 6.11 $(ii)$, $(iii)$, we have that $\tilde{\mathcal{L}}$ is a linear continuous operator of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ to itself. Hence, by the Open Mapping Theorem, in order to prove that it is a linear homeomorphism of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ onto itself, it suffices to prove that it is a bijection. By Proposition 10.3, $\tilde{\mathcal{L}}$ is injective. Now let $\psi \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$. By statement $(i)$, there exists $\mu \in L^2(\partial\mathbb{I},\mathbb{C})$ such that

$$\psi(x) = \frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}(S_n^{a,k}(x-y))\mu(y)\,d\sigma_y + \phi(x)\int_{\partial\mathbb{I}} S_n^{a,k}(x-y)\mu(y)\,d\sigma_y \quad \text{a.e. on } \partial\mathbb{I},$$

and, by Proposition 6.18 $(iii)$, we have $\mu \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$. As a consequence, $\tilde{\mathcal{L}}$ is surjective, and the proof is complete. □

We are now ready to prove the main result of this section.

**Theorem 10.5.** *Let $m \in \mathbb{N}\setminus\{0\}$. Let $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in* (1.46). *Let $k$, $\phi$, $\Gamma$ be as in* (10.1), (10.2), (10.3), *respectively. Assume that* $\mathrm{Im}(k) \neq 0$ *and that* $\mathrm{Re}(\phi(x)) \leq 0$ *for all $x \in \partial\mathbb{I}$ and that* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(\phi(x)) \geq 0$ *for all $x \in \partial\mathbb{I}$. Then boundary value problem* (10.4) *has a unique solution $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C}) \cap C^2(\mathbb{T}_a[\mathbb{I}],\mathbb{C})$. Moreover,*

$$u(x) = v_a^-[\mathbb{I},\mu,k](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \tag{10.8}$$

*where $\mu$ is the unique function in $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ that solves the following equation*

$$\frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}(S_n^{a,k}(x-y))\mu(y)\,d\sigma_y + \phi(x)\int_{\partial\mathbb{I}} S_n^{a,k}(x-y)\mu(y)\,d\sigma_y = \Gamma(x) \qquad \forall x \in \partial\mathbb{I}. \tag{10.9}$$

*Proof.* Clearly, it suffices to prove the existence. By Theorem 10.4 $(ii)$, there exists a unique $\mu \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ such that (10.9) holds. Then, by Theorem 6.11 $(i)$, we have that $v_a^-[\partial\mathbb{I},\mu,k] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C})$, that

$$\begin{aligned}
&\frac{\partial}{\partial\nu_{\mathbb{I}}} v_a^-[\partial\mathbb{I},\mu,k](x) + \phi(x)v_a^-[\partial\mathbb{I},\mu,k](x) \\
&\quad = \frac{1}{2}\mu(x) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(x)}(S_n^{a,k}(x-y))\mu(y)\,d\sigma_y + \phi(x)\int_{\partial\mathbb{I}} S_n^{a,k}(x-y)\mu(y)\,d\sigma_y = \Gamma(x) \quad \forall x \in \partial\mathbb{I}.
\end{aligned}$$

and that

$$\Delta v_a^-[\partial\mathbb{I},\mu,k](x) + k^2 v_a^-[\partial\mathbb{I},\mu,k](x) = 0 \qquad \forall x \in \mathbb{T}_a[\mathbb{I}].$$

Finally, by the periodicity of $v_a^-[\partial\mathbb{I},\mu,k]$, we have that $v_a^-[\partial\mathbb{I},\mu,k]$ solves boundary value problem (10.4). □

## 10.2 Asymptotic behaviour of the solutions of a linear Robin problem for the Helmholtz equation in a periodically perforated domain

In this Section we study the asymptotic behaviour of the solutions of the Robin problem for the Helmholtz equation in a periodically perforated domain with small holes.

### 10.2.1 Notation

We retain the notation introduced in Subsections 1.8.1, 6.7.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be as in (1.56). We shall consider also the following assumptions.

$$k \in \mathbb{C}, \ k^2 \neq |2\pi a^{-1}(z)|^2 \quad \forall z \in \mathbb{Z}^n; \tag{10.10}$$

$$f \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), \tag{10.11}$$

$$g \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C}). \tag{10.12}$$

### 10.2.2 Preliminaries

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. For each $\epsilon \in ]0, \epsilon_1[$, we consider the following periodic Robin problem for the Helmholtz equation.

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x + a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) + f(\frac{1}{\epsilon}(x-w))u(x) = g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{10.13}$$

By virtue of Theorem 10.5, we can give the following definition.

**Definition 10.6.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. For each $\epsilon \in ]0, \epsilon_1[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$ of boundary value problem (10.13).

We have the following Lemmas.

**Lemma 10.7.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let $\epsilon \in ]0, \epsilon_1[$. Then the function $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ satisfies the following equation*

$$\begin{aligned} g(\frac{1}{\epsilon}(x-w)) = \frac{1}{2}\mu(x) &+ \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial \nu_{\Omega_\epsilon}(x)} (S_n^{a,k}(x-y))\mu(y)\, d\sigma_y \\ &+ f(\frac{1}{\epsilon}(x-w)) \int_{\partial\Omega_\epsilon} S_n^{a,k}(x-y)\mu(y)\, d\sigma_y \qquad \forall x \in \partial\Omega_\epsilon, \end{aligned} \tag{10.14}$$

*if and only if the function $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, defined by*

$$\theta(t) \equiv \mu(w + \epsilon t) \qquad \forall t \in \partial\Omega, \tag{10.15}$$

*satisfies the following equation*

$$\begin{aligned} g(t) =& \frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k)\theta(s)\, d\sigma_s \\ &+ \epsilon^{n-1}(\log\epsilon)k^{n-2} \int_{\partial\Omega} \nu_\Omega(t) \cdot DQ_n^k(\epsilon(t-s))\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s \\ &+ f(t)\Bigg[\epsilon \int_{\partial\Omega} S_n(t-s, \epsilon k)\theta(s)\, d\sigma_s + \epsilon^{n-1}(\log\epsilon)k^{n-2} \int_{\partial\Omega} Q_n^k(\epsilon(t-s))\theta(s)\, d\sigma_s \\ &+ \epsilon^{n-1} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s\Bigg] \qquad \forall t \in \partial\Omega. \end{aligned} \tag{10.16}$$

*Proof.* It is a straightforward verification based on the rule of change of variables in integrals, on well known properties of composition of functions in Schauder spaces (cf. *e.g.*, Lanza [67, Section 3,4]) and on equalities (6.24), (6.25). □

**Lemma 10.8.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $\Omega$, $g$ *be as in* (1.56), (10.12), *respectively. Then there exists a unique function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *that solves the following equation*

$$g(t) = \frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\, d\sigma_s \qquad \forall t \in \partial\Omega. \tag{10.17}$$

*We denote the unique solution of equation* (10.17) *by* $\tilde{\theta}$. *Moreover,*

$$\int_{\partial\Omega} \tilde{\theta}(s)\, d\sigma_s = \int_{\partial\Omega} g(s)\, d\sigma_s. \tag{10.18}$$

*Proof.* The existence and uniqueness of a solution of equation (10.17) is a well known result of classic potential theory (cf. Folland [52, Chapter 3] for the existence and uniqueness of a solution in $L^2(\partial\Omega, \mathbb{C})$ and, *e.g.*, Theorem B.3 for the regularity.) Equality (10.18) follows by Folland [52, Lemma 3.30, p. 133]. □

Since we want to represent the function $u[\epsilon]$ by means of a periodic simple layer potential, we need to study some integral equations. Indeed, by virtue of Theorem 10.5, we can transform (10.13) into an integral equation, whose unknown is the moment of the simple layer potential. Moreover, we want to transform this equation defined on the $\epsilon$-dependent domain $\partial\Omega_\epsilon$ into an equation defined on the fixed domain $\partial\Omega$. We introduce this integral equation in the following Propositions. The relation between the solution of the integral equation and the solution of boundary value problem (10.13) will be clarified later. Anyway, since the function $Q_n^k$ that appears in equation (10.16) (involved in the determination of the moment of the simple layer potential that solves (10.13)) is identically 0 if $n$ is odd, it is preferable to treat separately case $n$ even and case $n$ odd.

**Proposition 10.9.** *Let* $n$ *be odd. Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $k$, $f$, $g$ *be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all* $t \in \partial\Omega$ *and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all* $t \in \partial\Omega$. *Let* $\tilde{\theta}$ *be as in Lemma 10.8. Let* $\Lambda$ *be the map of* $]-\epsilon_1, \epsilon_1[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *to* $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *defined by*

$$\begin{aligned}
&\Lambda[\epsilon, \theta](t) \\
&\quad\equiv \frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s \\
&\quad + f(t)\left[\epsilon \int_{\partial\Omega} S_n(t-s, \epsilon k)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s\right] - g(t) \quad \forall t \in \partial\Omega,
\end{aligned} \tag{10.19}$$

*for all* $(\epsilon, \theta) \in ]-\epsilon_1, \epsilon_1[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. *Then the following statements hold.*

*(i) If* $\epsilon \in ]0, \epsilon_1[$, *then the function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *satisfies equation*

$$\Lambda[\epsilon, \theta] = 0, \tag{10.20}$$

*if and only if the function* $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$, *defined by*

$$\mu(x) \equiv \theta(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{10.21}$$

*satisfies the equation*

$$\Gamma(x) = \frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}(S_n^{a,k}(x-y))\mu(y)\, d\sigma_y + \phi(x) \int_{\partial\Omega_\epsilon} S_n^{a,k}(x-y)\mu(y)\, d\sigma_y \ \forall x \in \partial\Omega_\epsilon, \tag{10.22}$$

*with* $\Gamma \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$, *and* $\phi \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$, *defined by*

$$\Gamma(x) \equiv g(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{10.23}$$

*and*

$$\phi(x) \equiv f(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon. \tag{10.24}$$

*In particular, equation* (10.20) *has exactly one solution* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, *for each* $\epsilon \in ]0, \epsilon_1[$.

*(ii) The function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *satisfies equation*

$$\Lambda[0, \theta] = 0, \tag{10.25}$$

*if and only if*

$$g(t) = \frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\,d\sigma_s \qquad \forall t \in \partial\Omega. \tag{10.26}$$

*In particular, the unique function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *that solves equation* (10.25) *is* $\tilde{\theta}$.

*Proof.* Consider (*i*). The equivalence of equation (10.20) in the unknown $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ and equation (10.22) in the unknown $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ follows by Lemma 10.7 and the definition of $Q_n^k$ for $n$ odd (cf. (6.23) and Definition E.2.) The existence and uniqueness of a solution of equation (10.22) follows by Proposition 10.4 (*ii*). Then the existence and uniqueness of a solution of equation (10.20) follows by the equivalence of (10.20) and (10.22). Consider (*ii*). The equivalence of (10.25) and (10.26) is obvious. The second part of statement (*ii*) is an immediate consequence of Lemma 10.8. □

**Proposition 10.10.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all* $t \in \partial\Omega$ *and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all* $t \in \partial\Omega$. *Let* $\tilde{\theta}$ *be as in Lemma 10.8. Let* $\epsilon_1' > 0$ *be such that*

$$\epsilon \log \epsilon \in ]-\epsilon_1', \epsilon_1'[ \qquad \forall \epsilon \in ]0, \epsilon_1[. \tag{10.27}$$

*Let* $\Lambda^\#$ *be the map of* $]-\epsilon_1, \epsilon_1[ \times ]-\epsilon_1', \epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *to* $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *defined by*

$$\begin{aligned}
&\Lambda^\#[\epsilon, \epsilon', \theta](t) \\
&\equiv \frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k)\theta(s)\,d\sigma_s + \epsilon^{n-2}\epsilon' k^{n-2} \int_{\partial\Omega} \nu_\Omega(t) \cdot DQ_n^k(\epsilon(t-s))\theta(s)\,d\sigma_s \\
&+ \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s + f(t)\Big[\epsilon \int_{\partial\Omega} S_n(t-s, \epsilon k)\theta(s)\,d\sigma_s \\
&+ \epsilon^{n-2}\epsilon' k^{n-2} \int_{\partial\Omega} Q_n^k(\epsilon(t-s))\theta(s)\,d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s\Big] - g(t) \;\; \forall t \in \partial\Omega,
\end{aligned} \tag{10.28}$$

*for all* $(\epsilon, \epsilon', \theta) \in ]-\epsilon_1, \epsilon_1[ \times ]-\epsilon_1', \epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. *Then the following statements hold.*

*(i) If* $\epsilon \in ]0, \epsilon_1[$, *then the function* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ *satisfies equation*

$$\Lambda^\#[\epsilon, \epsilon \log \epsilon, \theta] = 0, \tag{10.29}$$

*if and only if the function* $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$, *defined by*

$$\mu(x) \equiv \theta(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{10.30}$$

*satisfies the equation*

$$\Gamma(x) = \frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}(S_n^{a,k}(x-y))\mu(y)\,d\sigma_y + \phi(x) \int_{\partial\Omega_\epsilon} S_n^{a,k}(x-y)\mu(y)\,d\sigma_y \;\; \forall x \in \partial\Omega_\epsilon, \tag{10.31}$$

*with* $\Gamma \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$, *and* $\phi \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$, *defined by*

$$\Gamma(x) \equiv g(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{10.32}$$

*and*

$$\phi(x) \equiv f(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon. \tag{10.33}$$

*In particular, equation* (10.29) *has exactly one solution* $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, *for each* $\epsilon \in ]0, \epsilon_1[$.

*(ii) The function $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ satisfies equation*

$$\Lambda^{\#}[0, 0, \theta] = 0, \tag{10.34}$$

*if and only if*

$$g(t) = \frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\, d\sigma_s \qquad \forall t \in \partial\Omega. \tag{10.35}$$

*In particular, the unique function $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ that solves equation* (10.34) *is $\tilde{\theta}$.*

*Proof.* Consider (*i*). The equivalence of equation (10.29) in the unknown $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ and equation (10.31) in the unknown $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ follows by Lemma 10.7 and the definition of $Q_n^k$ for $n$ even (cf. (6.23) and Definition E.2.) The existence and uniqueness of a solution of equation (10.31) follows by Proposition 10.4 (*ii*). Then the existence and uniqueness of a solution of equation (10.29) follows by the equivalence of (10.29) and (10.31). Consider (*ii*). The equivalence of (10.34) and (10.35) is obvious. The second part of statement (*ii*) is an immediate consequence of Lemma 10.8. □

By Propositions 10.9, 10.10, it makes sense to introduce the following.

**Definition 10.11.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. For each $\epsilon \in ]0, \epsilon_1[$, we denote by $\hat{\theta}_n[\epsilon]$ the unique function in $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ that solves equation (10.20), if $n$ is odd, or equation (10.29), if $n$ is even. Analogously, we denote by $\hat{\theta}_n[0]$ the unique function in $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ that solves equation (10.25), if $n$ is odd, or equation (10.34), if $n$ is even.

In the following Remark, we show the relation between the solutions of boundary value problem (10.13) and the solutions of equations (10.20), (10.29).

*Remark* 10.12. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$.

Let $\epsilon \in ]0, \epsilon_1[$. We have

$$u[\epsilon](x) = \epsilon^{n-1} \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s)\hat{\theta}_n[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$

While the relation between equations (10.20), (10.29) and boundary value problem (10.13) is now clear, we want to see if (10.25), (10.34) are related to some (limiting) boundary value problem. We give the following.

**Definition 10.13.** Let $n \geq 3$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $\Omega$, $g$ be as in (1.56), (10.12), respectively. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega, \mathbb{C})$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega, \\ \frac{\partial}{\partial\nu_\Omega} u(x) = g(x) & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{10.36}$$

Problem (10.36) will be called the *limiting boundary value problem.*

*Remark* 10.14. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. If $n \geq 3$, then we have

$$\tilde{u}(x) = \int_{\partial\Omega} S_n(x - y)\hat{\theta}_n[0](y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \Omega.$$

If $n = 2$, in general the (classic) simple layer potential for the Laplace equation with moment $\hat{\theta}_2[0]$ is not harmonic at infinity, and it does not satisfy the third condition of boundary value problem (10.36). Moreover, if $n = 2$, boundary value problem (10.36) does not have in general a solution (unless $\int_{\partial\Omega} g\, d\sigma = 0$.) However, the function $\tilde{v}$ of $\mathbb{R}^2 \setminus \Omega$ to $\mathbb{C}$, defined by

$$\tilde{v}(x) \equiv \int_{\partial\Omega} S_2(x - y)\hat{\theta}_2[0](y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^2 \setminus \Omega,$$

is a solution of the following boundary value problem

$$\begin{cases} \Delta\tilde{v}(x) = 0 & \forall x \in \mathbb{R}^2 \setminus \mathrm{cl}\,\Omega, \\ \frac{\partial}{\partial\nu_\Omega}\tilde{v}(x) = g(x) & \forall x \in \partial\Omega. \end{cases} \tag{10.37}$$

We now prove the following Propositions.

**Proposition 10.15.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all $t \in \partial\Omega$ and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all $t \in \partial\Omega$. Let $\tilde{\theta}$ be as in Lemma 10.8. Let $\Lambda$ be as in Proposition 10.9. Then there exists $\epsilon_2 \in ]0,\epsilon_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2,\epsilon_2[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Moreover, if we set $b_0 \equiv (0,\tilde{\theta})$, then the differential $\partial_\theta\Lambda[b_0]$ of $\Lambda$ with respect to the variable $\theta$ at $b_0$ is delivered by the following formula*

$$\partial_\theta\Lambda[b_0](\tau)(t) = \frac{1}{2}\tau(t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot DS_n(t-s)\tau(s)\,d\sigma_s \qquad \forall t \in \partial\Omega, \tag{10.38}$$

*for all $\tau \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, and is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ onto itself.*

*Proof.* By Proposition 6.21 and by continuity of the pointwise product in Schauder space, we easily deduce that there exists $\epsilon_2 \in ]0,\epsilon_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2,\epsilon_2[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By standard calculus in Banach space, we immediately deduce that (10.38) holds. Now we need to prove that $\partial_\theta\Lambda[b_0]$ is a linear homeomorphism. By the Open Mapping Theorem, it suffices to prove that it is a bijection. Let $\psi \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By known results of classical potential theory (cf. Folland [52, Chapter 3]), there exists a unique function $\tau \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, such that

$$\frac{1}{2}\tau(t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot DS_n(t-s)\tau(s)\,d\sigma_s = \psi(t) \qquad \forall t \in \partial\Omega.$$

Hence $\partial_\theta\Lambda[b_0]$ is bijective, and, accordingly, a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ onto itself. □

**Proposition 10.16.** *Let $n$ be even. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all $t \in \partial\Omega$ and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all $t \in \partial\Omega$. Let $\tilde{\theta}$ be as in Lemma 10.8. Let $\epsilon_1' > 0$ be as in* (10.27). *Let $\Lambda^\#$ be as in Proposition 10.10. Then there exists $\epsilon_2 \in ]0,\epsilon_1]$ such that $\Lambda^\#$ is a real analytic operator of $]-\epsilon_2,\epsilon_2[\times]-\epsilon_1',\epsilon_1'[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Moreover, if we set $b_0 \equiv (0,0,\tilde{\theta})$, then the differential $\partial_\theta\Lambda^\#[b_0]$ of $\Lambda^\#$ with respect to the variable $\theta$ at $b_0$ is delivered by the following formula*

$$\partial_\theta\Lambda^\#[b_0](\tau)(t) = \frac{1}{2}\tau(t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot DS_n(t-s)\tau(s)\,d\sigma_s \qquad \forall t \in \partial\Omega, \tag{10.39}$$

*for all $\tau \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, and is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ onto itself.*

*Proof.* By Proposition 6.21 and by continuity of the pointwise product in Schauder space, we easily deduce that there exists $\epsilon_2 \in ]0,\epsilon_1]$ such that $\Lambda^\#$ is a real analytic operator of $]-\epsilon_2,\epsilon_2[\times]-\epsilon_1',\epsilon_1'[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By standard calculus in Banach space, we immediately deduce that (10.39) holds. Finally, by the proof of Proposition 10.15 and formula (10.39), we have that $\partial_\theta\Lambda^\#[b_0]$ is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ onto itself. □

By the previous Propositions we can now prove the following results.

**Proposition 10.17.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all $t \in \partial\Omega$ and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all $t \in \partial\Omega$. Let $\epsilon_2$ be as in Proposition 10.15. Then there exist $\epsilon_3 \in ]0,\epsilon_2]$ and a real analytic operator $\Theta_n$ of $]-\epsilon_3,\epsilon_3[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, such that*

$$\Theta_n[\epsilon] = \hat{\theta}_n[\epsilon], \tag{10.40}$$

*for all $\epsilon \in [0,\epsilon_3[$.*

*Proof.* It is an immediate consequence of Proposition 10.15 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

**Proposition 10.18.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all* $t \in \partial\Omega$ *and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all* $t \in \partial\Omega$. *Let* $\epsilon_1' > 0$ *be as in* (10.27). *Let* $\epsilon_2$ *be as in Proposition 10.16. Then there exist* $\epsilon_3 \in ]0, \epsilon_2]$, $\epsilon_2' \in ]0, \epsilon_1']$, *and a real analytic operator* $\Theta_n^\#$ *of* $]-\epsilon_3, \epsilon_3[ \times ]-\epsilon_2', \epsilon_2'[$ *to* $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, *such that*

$$\epsilon \log \epsilon \in ]-\epsilon_2', \epsilon_2'[ \quad \forall \epsilon \in ]0, \epsilon_3[,$$

$$\Theta_n^\#[\epsilon, \epsilon \log \epsilon] = \hat{\theta}_n[\epsilon] \quad \forall \epsilon \in ]0, \epsilon_3[, \tag{10.41}$$

$$\Theta_n^\#[0,0] = \hat{\theta}_n[0]. \tag{10.42}$$

*Proof.* It is an immediate consequence of Proposition 10.16 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

### 10.2.3 A functional analytic representation Theorem for the solution of the singularly perturbed Robin problem

By Propositions 10.17, 10.18, and Remark 10.12, we can deduce the main result of this Subsection.

**Theorem 10.19.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all* $t \in \partial\Omega$ *and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all* $t \in \partial\Omega$. *Let* $\epsilon_3$ *be as in Proposition 10.17. Let* $V$ *be a bounded open subset of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. *Then there exist* $\epsilon_4 \in ]0, \epsilon_3]$, *and a real analytic operator* $U$ *of* $]-\epsilon_4, \epsilon_4[$ *to the space* $C^0(\mathrm{cl}\, V, \mathbb{C})$, *such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*

$$u[\epsilon](x) = \epsilon^{n-1} U[\epsilon](x) \qquad \forall x \in \mathrm{cl}\, V,$$

*for all* $\epsilon \in ]0, \epsilon_4[$. *Moreover,*

$$U[0](x) = S_n^{a,k}(x - w) \int_{\partial\Omega} g \, d\sigma \qquad \forall x \in \mathrm{cl}\, V.$$

*Proof.* Let $\Theta_n[\cdot]$ be as in Proposition 10.17. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $\epsilon \in ]0, \epsilon_4[$. By Remark 10.12 and Proposition 10.17, we have

$$u[\epsilon](x) = \epsilon^{n-1} \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s)\Theta_n[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V.$$

Thus, it is natural to set

$$U[\epsilon](x) \equiv \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s)\Theta_n[\epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

for all $\epsilon \in ]-\epsilon_4, \epsilon_4[$. By Proposition 6.22, $U$ is a real analytic map of $]-\epsilon_4, \epsilon_4[$ to $C^0(\mathrm{cl}\, V, \mathbb{C})$. Furthermore, by Lemma 10.8, we have

$$\begin{aligned} U[0](x) &= S_n^{a,k}(x - w) \int_{\partial\Omega} \Theta_n[0](s)\, d\sigma_s \\ &= S_n^{a,k}(x - w) \int_{\partial\Omega} g \, d\sigma \qquad \forall x \in \mathrm{cl}\, V, \end{aligned}$$

since $\Theta_n[0] = \tilde{\theta}$. Hence the proof is now complete. □

*Remark* 10.20. We note that the right-hand side of the equality in $(jj)$ of Theorem 10.19 can be continued real analytically in the whole $]-\epsilon_4, \epsilon_4[$. Moreover, if $V$ is a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$, then

$$\lim_{\epsilon \to 0^+} u[\epsilon] = 0 \qquad \text{uniformly in } \mathrm{cl}\, V.$$

**Theorem 10.21.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all* $t \in \partial\Omega$ *and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all* $t \in \partial\Omega$. *Let* $\epsilon_3$, $\epsilon_2'$ *be as in Proposition 10.18. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that* $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. *Then there exist* $\epsilon_4 \in ]0, \epsilon_3]$ *and a real analytic operator* $U^\#$ *of* $]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[$ *to the space* $C^0(\mathrm{cl}\, V, \mathbb{C})$, *such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*

$$u[\epsilon](x) = \epsilon^{n-1} U^\#[\epsilon, \epsilon \log \epsilon](x) \qquad \forall x \in \mathrm{cl}\, V,$$

*for all* $\epsilon \in ]0, \epsilon_4[$. *Moreover,*

$$U^\#[0,0](x) = S_n^{a,k}(x-w) \int_{\partial\Omega} g \, d\sigma \qquad \forall x \in \mathrm{cl}\, V.$$

*Proof.* Let $\Theta_n^\#[\cdot,\cdot]$ be as in Proposition 10.18. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $\epsilon \in ]0, \epsilon_4[$. By Remark 10.12 and Proposition 10.18, we have

$$u[\epsilon](x) = \epsilon^{n-1} \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s) \Theta_n^\#[\epsilon, \epsilon \log \epsilon](s) \, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V.$$

Thus, it is natural to set

$$U^\#[\epsilon, \epsilon'](x) \equiv \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s) \Theta_n^\#[\epsilon, \epsilon'](s) \, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

for all $(\epsilon, \epsilon') \in ]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[$. By Proposition 6.22, $U^\#$ is a real analytic map of $]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[$ to $C^0(\mathrm{cl}\, V, \mathbb{C})$. Furthermore, by Lemma 10.8, we have

$$\begin{aligned} U^\#[0,0](x) &= S_n^{a,k}(x-w) \int_{\partial\Omega} \Theta_n^\#[0,0](s) \, d\sigma_s \\ &= S_n^{a,k}(x-w) \int_{\partial\Omega} g \, d\sigma \qquad \forall x \in \mathrm{cl}\, V, \end{aligned}$$

since $\Theta_n^\#[0,0] = \tilde{\theta}$. Accordingly, the Theorem is now completely proved. □

We have also the following Theorems.

**Theorem 10.22.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all* $t \in \partial\Omega$ *and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all* $t \in \partial\Omega$. *Let $\tilde{\theta}$ be as in Lemma 10.8. Let $\epsilon_3$ be as in Proposition 10.17. Then there exist* $\epsilon_5 \in ]0, \epsilon_3]$, *and a real analytic operator $G$ of* $]-\epsilon_5, \epsilon_5[$ *to* $\mathbb{C}$, *such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2 \, dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2 \, dx = \epsilon^n G[\epsilon], \tag{10.43}$$

*for all* $\epsilon \in ]0, \epsilon_5[$. *Moreover,*

$$G[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx, \tag{10.44}$$

*where $\tilde{u}$ is as in Definition 10.13.*

*Proof.* Let $\Theta_n[\cdot]$ be as in Proposition 10.17. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon \in ]0, \epsilon_3[$. Clearly, by the periodicity of $u[\epsilon]$, we have

$$\begin{aligned} &\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2 \, dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2 \, dx \\ &\qquad = -\epsilon^{n-1} \int_{\partial\Omega} \Big( \frac{\partial u[\epsilon]}{\partial \nu_{\Omega_\epsilon}} \Big) \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) \overline{u[\epsilon] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t)} \, d\sigma_t. \end{aligned}$$

By equality (6.24) and since $Q_n^k = 0$ for $n$ odd, we have

$$u[\epsilon] \circ (w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \epsilon^{n-1}\int_{\partial\Omega} S_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s$$
$$= \epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s \quad \forall t\in\partial\Omega.$$

By Theorem E.6 $(i)$, one can easily show that the map which takes $\epsilon$ to the function of the variable $t\in\partial\Omega$ defined by

$$\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

is a real analytic operator of $]-\epsilon_3,\epsilon_3[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By Theorem C.4, we immediately deduce that there exists $\epsilon_5\in\,]0,\epsilon_3]$ such that the map of $]-\epsilon_5,\epsilon_5[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, which takes $\epsilon$ to the function $\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s$ of the variable $t\in\partial\Omega$, is real analytic. Analogously, we have

$$\Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \frac{1}{2}\Theta_n[\epsilon](t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s$$
$$+\,\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $\epsilon\in\,]0,\epsilon_3[$. Thus, if we set

$$\tilde{G}[\epsilon](t) \equiv \frac{1}{2}\Theta_n[\epsilon](t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s$$
$$+\,\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,\ \forall\epsilon\in\,]-\epsilon_5,\epsilon_5[$$

then, by arguing as in Proposition 10.15, one can easily show that $\tilde{G}$ is a real analytic map of $]-\epsilon_5,\epsilon_5[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$.

Hence, if we set

$$G[\epsilon] \equiv -\int_{\partial\Omega}\tilde{G}[\epsilon](t)\overline{\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s}\,d\sigma_t$$
$$-\,\epsilon^{n-2}\int_{\partial\Omega}\tilde{G}[\epsilon](t)\overline{\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s}\,d\sigma_t,$$

for all $\epsilon\in\,]-\epsilon_5,\epsilon_5[$, then by standard properties of functions in Schauder spaces, we have that $G$ is a real analytic map of $]-\epsilon_5,\epsilon_5[$ to $\mathbb{C}$ such that equality (10.43) holds.

Finally, if $\epsilon=0$, by Folland [52, p. 118] and since $\tilde{G}[0]=g$, we have

$$G[0] = -\int_{\partial\Omega} g(t)\overline{\int_{\partial\Omega} S_n(t-s)\tilde{\theta}(s)\,d\sigma_s}\,d\sigma_t$$
$$= \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega}|\nabla\tilde{u}(x)|^2\,dx.$$

□

**Theorem 10.23.** *Let $n$ be even. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let $\mathrm{Im}(k)\neq 0$. Let $\mathrm{Re}(f(t))\leq 0$ for all $t\in\partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t))\geq 0$ for all $t\in\partial\Omega$. Let $\tilde{\theta}$ be as in Lemma 10.8. Let $\epsilon_3$, $\epsilon_2'$ be as in Proposition 10.18. Then there exist $\epsilon_5\in\,]0,\epsilon_3]$, and three real analytic operators $G_1^\#$, $G_2^\#$, $G_3^\#$ of $]-\epsilon_5,\epsilon_5[\,\times\,]-\epsilon_2',\epsilon_2'[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]}|\nabla u[\epsilon](x)|^2\,dx - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]}|u[\epsilon](x)|^2\,dx$$
$$= \epsilon^n G_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{2n-2}(\log\epsilon)G_2^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{3n-3}(\log\epsilon)^2 G_3^\#[\epsilon,\epsilon\log\epsilon], \tag{10.45}$$

*for all* $\epsilon \in ]0, \epsilon_5[$. *Moreover,*

$$G_1^\#[0,0] = -\int_{\partial\Omega} g(t)\overline{\int_{\partial\Omega} S_n(t-s)\tilde{\theta}(s)\,d\sigma_s}\,d\sigma_t - \delta_{2,n}\overline{R_n^{a,k}(0)}|\int_{\partial\Omega} g\,d\sigma|^2, \tag{10.46}$$

$$G_2^\#[0,0] = -\overline{k}^{n-2}\mathcal{J}_n(0)|\int_{\partial\Omega} g\,d\sigma|^2, \tag{10.47}$$

$$G_3^\#[0,0] = 0 \tag{10.48}$$

*where* $\mathcal{J}_n(0)$ *is as in Proposition E.3 (i). In particular, if* $n > 2$*, then*

$$G_1^\#[0,0] = \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega} |\nabla\tilde{u}(x)|^2\,dx, \tag{10.49}$$

*where* $\tilde{u}$ *is as in Definition 10.13.*

*Proof.* Let $\Theta_n^\#[\cdot,\cdot]$ be as in Proposition 10.18. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon \in ]0, \epsilon_3[$. Clearly, by the periodicity of $u[\epsilon]$, we have

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\,dx - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\,dx = -\epsilon^{n-1}\int_{\partial\Omega}\Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\overline{u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)}\,d\sigma_t.$$

By equality (6.24), we have

$$\begin{aligned} u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) =& \epsilon^{n-1}\int_{\partial\Omega} S_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ =& \epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &+ \epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &+ \epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega. \end{aligned}$$

Thus it is natural to set

$$\begin{aligned} F_1[\epsilon,\epsilon'](t) &\equiv \int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \\ F_2[\epsilon,\epsilon'](t) &\equiv k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \\ F_3[\epsilon,\epsilon'](t) &\equiv \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \end{aligned}$$

for all $(\epsilon,\epsilon') \in ]-\epsilon_3,\epsilon_3[\times]-\epsilon_2',\epsilon_2'[$.

Then clearly

$$u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \epsilon F_1[\epsilon,\epsilon\log\epsilon](t) + \epsilon^{n-1}(\log\epsilon)F_2[\epsilon,\epsilon\log\epsilon](t) + \epsilon^{n-1}F_3[\epsilon,\epsilon\log\epsilon](t) \quad \forall t\in\partial\Omega,$$

for all $\epsilon \in ]0,\epsilon_3[$. By Theorem E.6 (*i*) and Theorem C.4, we easily deduce that there exists $\epsilon_5 \in ]0,\epsilon_3]$ such that the maps $F_1$, $F_2$, and $F_3$ of $]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ are real analytic. Analogously, we have

$$\begin{aligned} \Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) =& \frac{1}{2}\Theta_n^\#[\epsilon,\epsilon\log\epsilon](t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &+ \epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &+ \epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \end{aligned}$$

for all $\epsilon \in ]0,\epsilon_3[$. Thus, if we set

$$\tilde{G}_1[\epsilon,\epsilon'](t) \equiv \frac{1}{2}\Theta_n^\#[\epsilon,\epsilon'](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^n} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

and

$$\tilde{G}_2[\epsilon,\epsilon'](t) \equiv k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $(\epsilon,\epsilon') \in ]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[$, then, by arguing as in Proposition 10.16, one can easily show that $\tilde{G}_1$ and $\tilde{G}_2$ are real analytic maps of $]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$.

Clearly,

$$\Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \tilde{G}_1[\epsilon,\epsilon\log\epsilon](t) + \epsilon^{n-1}(\log\epsilon)\tilde{G}_2[\epsilon,\epsilon\log\epsilon](t) \qquad \forall t\in\partial\Omega,\ \forall\epsilon\in]0,\epsilon_5[.$$

If $\epsilon\in]0,\epsilon_5[$, then we have

$$\begin{aligned}
&\int_{\mathbb{P}_a[\Omega_\epsilon]}|\nabla u[\epsilon](x)|^2\,dx - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]}|u[\epsilon](x)|^2\,dx\\
&=\epsilon^n\Big(-\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon\log\epsilon]\overline{F_1[\epsilon,\epsilon\log\epsilon]}\,d\sigma - \epsilon^{n-2}\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon\log\epsilon]\overline{F_3[\epsilon,\epsilon\log\epsilon]}\,d\sigma\Big)\\
&\quad+\epsilon^{2n-2}\log\epsilon\Big(-\epsilon\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon\log\epsilon]\overline{F_1[\epsilon,\epsilon\log\epsilon]}\,d\sigma - \int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon\log\epsilon]\overline{F_2[\epsilon,\epsilon\log\epsilon]}\,d\sigma\\
&\qquad-\epsilon^{n-1}\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon\log\epsilon]\overline{F_3[\epsilon,\epsilon\log\epsilon]}\,d\sigma\Big)\\
&\quad+\epsilon^{3n-3}(\log\epsilon)^2\Big(-\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon\log\epsilon]\overline{F_2[\epsilon,\epsilon\log\epsilon]}\,d\sigma\Big).
\end{aligned}$$

If we set

$$\begin{aligned}
G_1^\#[\epsilon,\epsilon'] &\equiv -\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon'](t)\overline{F_1[\epsilon,\epsilon'](t)}\,d\sigma_t - \epsilon^{n-2}\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon'](t)\overline{F_3[\epsilon,\epsilon'](t)}\,d\sigma_t,\\
G_2^\#[\epsilon,\epsilon'] &\equiv -\epsilon\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon'](t)\overline{F_1[\epsilon,\epsilon'](t)}\,d\sigma_t - \int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon'](t)\overline{F_2[\epsilon,\epsilon'](t)}\,d\sigma_t\\
&\quad-\epsilon^{n-1}\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon'](t)\overline{F_3[\epsilon,\epsilon'](t)}\,d\sigma_t,\\
G_3^\#[\epsilon,\epsilon'] &\equiv -\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon'](t)\overline{F_2[\epsilon,\epsilon'](t)}\,d\sigma_t,
\end{aligned}$$

for all $(\epsilon,\epsilon') \in ]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[$, then standard properties of functions in Schauder spaces and a simple computation show that $G_1^\#$, $G_2^\#$, and $G_3^\#$ are real analytic maps of $]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[$ in $\mathbb{C}$ such that equality (10.45) holds for all $\epsilon\in]0,\epsilon_5[$.

Next, we observe that

$$\begin{aligned}
G_1^\#[0,0] &= -\int_{\partial\Omega}g(t)\overline{\int_{\partial\Omega}S_n(t-s)\tilde{\theta}(s)\,d\sigma_s}\,d\sigma_t - \delta_{2,n}\overline{R_n^{a,k}(0)}|\int_{\partial\Omega}g\,d\sigma|^2,\\
G_2^\#[0,0] &= -\overline{k}^{n-2}\overline{Q_n^k(0)}\int_{\partial\Omega}g\,d\sigma\overline{\int_{\partial\Omega}g\,d\sigma},\\
G_3^\#[0,0] &= -k^{n-2}\overline{k^{n-2}Q_n^k(0)}\int_{\partial\Omega}g\,d\sigma\overline{\int_{\partial\Omega}g\,d\sigma}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(0)\,d\sigma_t = 0,
\end{aligned}$$

and accordingly equalities (10.46), (10.47), and (10.48) hold. In particular, if $n\geq 4$, by Folland [52, p. 118], we have

$$G_1^\#[0,0] = \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega}|\nabla\tilde{u}(x)|^2\,dx.$$

□

*Remark* 10.24. If $n$ is odd, we note that the right-hand side of the equality in (10.43) of Theorem 10.22 can be continued real analytically in the whole $]-\epsilon_5, \epsilon_5[$.

Moreover,

$$\lim_{\epsilon \to 0^+} \Big[ \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2 \, dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2 \, dx \Big] = 0,$$

for all $n \in \mathbb{N} \setminus \{0,1\}$ ($n$ even or odd.)

### 10.2.4 A real analytic continuation Theorem for the integral of the solution

We now prove real analytic continuation Theorems for the integral of the solution. Namely, we prove the following results.

**Theorem 10.25.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. Let $\tilde{\theta}$ be as in Lemma 10.8. Let $\epsilon_3$ be as in Proposition 10.17. Then there exist $\epsilon_6 \in ]0, \epsilon_3]$, and a real analytic operator $J$ of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x) \, dx = \frac{\epsilon^{n-1}}{k^2} J[\epsilon], \tag{10.50}$$

*for all $\epsilon \in ]0, \epsilon_6[$. Moreover,*

$$J[0] = \int_{\partial\Omega} g(x) \, d\sigma_x. \tag{10.51}$$

*Proof.* Let $\Theta_n[\cdot]$ be as in Proposition 10.17. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon \in ]0, \epsilon_3[$. Clearly, by the Divergence Theorem and the periodicity of $u[\epsilon]$, we have

$$\begin{aligned} \int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x) \, dx &= -\frac{1}{k^2} \int_{\mathbb{P}_a[\Omega_\epsilon]} \Delta u[\epsilon](x) \, dx \\ &= -\frac{1}{k^2} \int_{\partial\mathbb{P}_a[\Omega_\epsilon]} \frac{\partial}{\partial \nu_{\mathbb{P}_a[\Omega_\epsilon]}} u[\epsilon](x) \, d\sigma_x \\ &= -\frac{1}{k^2} \Big[ \int_{\partial A} \frac{\partial}{\partial \nu_A} u[\epsilon](x) \, d\sigma_x - \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u[\epsilon](x) \, d\sigma_x \Big] \\ &= \frac{1}{k^2} \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u[\epsilon](x) \, d\sigma_x. \end{aligned}$$

As a consequence,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x) \, dx = \frac{\epsilon^{n-1}}{k^2} \int_{\partial\Omega} \Big( \frac{\partial u[\epsilon]}{\partial \nu_{\Omega_\epsilon}} \Big) \circ (w + \epsilon \, \mathrm{id}_{\partial\Omega})(t) \, d\sigma_t.$$

By equality (6.25) and since $Q_n^k = 0$ for $n$ odd, we have

$$\begin{aligned} \Big( \frac{\partial u[\epsilon]}{\partial \nu_{\Omega_\epsilon}} \Big) \circ (w + \epsilon \, \mathrm{id}_{\partial\Omega})(t) &= \frac{1}{2} \Theta_n[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k) \Theta_n[\epsilon](s) \, d\sigma_s \\ &\quad + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot D R_n^{a,k}(\epsilon(t-s)) \Theta_n[\epsilon](s) \, d\sigma_s \qquad \forall t \in \partial\Omega. \end{aligned}$$

We set

$$\begin{aligned} \tilde{J}[\epsilon](t) \equiv & \frac{1}{2} \Theta_n[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k) \Theta_n[\epsilon](s) \, d\sigma_s \\ & + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot D R_n^{a,k}(\epsilon(t-s)) \Theta_n[\epsilon](s) \, d\sigma_s \quad \forall t \in \partial\Omega, \end{aligned}$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. By Theorem E.6 (*ii*) and Theorem C.4, one can easily show that there exists $\epsilon_6 \in ]0, \epsilon_3]$ such that $\tilde{J}$ is a real analytic map of $]-\epsilon_6, \epsilon_6[$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Hence, if we set

$$J[\epsilon] \equiv \int_{\partial\Omega} \tilde{J}[\epsilon](t) \, d\sigma_t,$$

for all $\epsilon \in ]-\epsilon_6, \epsilon_6[$, then, by standard properties of functions in Schauder spaces, we have that $J$ is a real analytic map of $]-\epsilon_6, \epsilon_6[$ to $\mathbb{C}$, such that equality (10.50) holds.

Finally, if $\epsilon = 0$, we have

$$J[0] = \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega} v^-[\partial\Omega, \tilde{\theta}, 0](t)\, d\sigma_t \\ = \int_{\partial\Omega} g(x)\, d\sigma_x,$$

and accordingly (10.51) holds. □

**Theorem 10.26.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all* $t \in \partial\Omega$ *and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all* $t \in \partial\Omega$. *Let $\tilde{\theta}$ be as in Lemma 10.8. Let $\epsilon_3$, $\epsilon_2'$ be as in Proposition 10.18. Then there exist $\epsilon_6 \in ]0, \epsilon_3]$, and two real analytic operators $J_1^\#$, $J_2^\#$ of $]-\epsilon_6, \epsilon_6[ \times ]-\epsilon_2', \epsilon_2'[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx = \frac{\epsilon^{n-1}}{k^2} J_1^\#[\epsilon, \epsilon \log \epsilon] + \frac{\epsilon^{2n-2}(\log \epsilon)}{k^2} J_2^\#[\epsilon, \epsilon \log \epsilon], \tag{10.52}$$

*for all $\epsilon \in ]0, \epsilon_6[$. Moreover,*

$$J_1^\#[0,0] = \int_{\partial\Omega} g(x)\, d\sigma_x. \tag{10.53}$$

*Proof.* Let $\Theta_n^\#[\cdot,\cdot]$ be as in Proposition 10.18. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon \in ]0, \epsilon_3[$. Clearly, by the Divergence Theorem and the periodicity of $u[\epsilon]$, we have

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx = -\frac{1}{k^2} \int_{\mathbb{P}_a[\Omega_\epsilon]} \Delta u[\epsilon](x)\, dx \\ = -\frac{1}{k^2} \int_{\partial\mathbb{P}_a[\Omega_\epsilon]} \frac{\partial}{\partial \nu_{\mathbb{P}_a[\Omega_\epsilon]}} u[\epsilon](x)\, d\sigma_x \\ = -\frac{1}{k^2} \Big[ \int_{\partial A} \frac{\partial}{\partial \nu_A} u[\epsilon](x)\, d\sigma_x - \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u[\epsilon](x)\, d\sigma_x \Big] \\ = \frac{1}{k^2} \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u[\epsilon](x)\, d\sigma_x.$$

As a consequence,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx = \frac{\epsilon^{n-1}}{k^2} \int_{\partial\Omega} \Big( \frac{\partial u[\epsilon]}{\partial \nu_{\Omega_\epsilon}} \Big) \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t)\, d\sigma_t.$$

By equality (6.25), we have

$$\Big( \frac{\partial u[\epsilon]}{\partial \nu_{\Omega_\epsilon}} \Big) \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) = \frac{1}{2} \Theta_n^\#[\epsilon, \epsilon \log \epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k) \Theta_n^\#[\epsilon, \epsilon \log \epsilon](s)\, d\sigma_s \\ + \epsilon^{n-1} (\log \epsilon) k^{n-2} \int_{\partial\Omega} \nu_\Omega(t) \cdot DQ_n^k(\epsilon(t-s)) \Theta_n^\#[\epsilon, \epsilon \log \epsilon](s)\, d\sigma_s \\ + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t-s)) \Theta_n^\#[\epsilon, \epsilon \log \epsilon](s)\, d\sigma_s \qquad \forall t \in \partial\Omega.$$

We set

$$\tilde{J}_1[\epsilon, \epsilon'](t) \equiv \frac{1}{2} \Theta_n^\#[\epsilon, \epsilon'](t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k) \Theta_n^\#[\epsilon, \epsilon'](s)\, d\sigma_s \\ + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t-s)) \Theta_n^\#[\epsilon, \epsilon'](s)\, d\sigma_s \qquad \forall t \in \partial\Omega,$$

and

$$\tilde{J}_2[\epsilon, \epsilon'](t) \equiv k^{n-2} \int_{\partial\Omega} \nu_\Omega(t) \cdot DQ_n^k(\epsilon(t-s)) \Theta_n^\#[\epsilon, \epsilon'](s)\, d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $(\epsilon, \epsilon') \in ]-\epsilon_3, \epsilon_3[ \times ]-\epsilon'_2, \epsilon'_2[$. By Theorem E.6 $(ii)$ and Theorem C.4, one can easily show that there exists $\epsilon_6 \in ]0, \epsilon_3]$ such that $\tilde{J}_1^{\#}$, $\tilde{J}_2^{\#}$ are real analytic maps of $]-\epsilon_6, \epsilon_6[\times]-\epsilon'_2, \epsilon'_2[$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Hence, if we set

$$J_1^{\#}[\epsilon, \epsilon'] \equiv \int_{\partial\Omega} \tilde{J}_1^{\#}[\epsilon, \epsilon'](t)\, d\sigma_t,$$

and

$$J_2^{\#}[\epsilon, \epsilon'] \equiv \int_{\partial\Omega} \tilde{J}_2^{\#}[\epsilon, \epsilon'](t)\, d\sigma_t,$$

for all $(\epsilon, \epsilon') \in ]-\epsilon_6, \epsilon_6[ \times ]-\epsilon'_2, \epsilon'_2[$, then, by standard properties of functions in Schauder spaces, we have that $J_1^{\#}$, $J_2^{\#}$ are real analytic maps of $]-\epsilon_6, \epsilon_6[ \times ]-\epsilon'_2, \epsilon'_2[$ to $\mathbb{C}$, such that equality (10.52) holds.

Finally, if $\epsilon = \epsilon' = 0$, we have

$$\begin{aligned} J_1^{\#}[0,0] &= \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega} v^-[\partial\Omega, \tilde{\theta}, 0](t)\, d\sigma_t \\ &= \int_{\partial\Omega} g(x)\, d\sigma_x, \end{aligned}$$

and accordingly (10.53) holds. □

## 10.3 An homogenization problem for the Helmholtz equation with linear Robin boundary conditions in a periodically perforated domain

In this section we consider an homogenization problem for the Helmhlotz equation with linear Robin boundary conditions in a periodically perforated domain. In most of the results we assume that $\mathrm{Im}(k) \neq 0$ and $\mathrm{Re}(k) = 0$.

We note that we shall consider the equation

$$\Delta u(x) + \frac{k^2}{\delta^2} u(x) = 0 \qquad \forall x \in \mathbb{T}_a(\epsilon, \delta),$$

together with the usual periodicity condition and a Robin boundary condition. We do so, because if $u$ is a solution of the equation above then the function $u_\delta(\cdot) \equiv u(\delta\cdot)$ is a solution of the following equation

$$\Delta u_\delta(x) + k^2 u_\delta(x) = 0 \qquad \forall x \in \mathbb{T}_a[\Omega_\epsilon],$$

which we can analyse by virtue of the results of Section 10.2.

### 10.3.1 Notation

In this Section we retain the notation introduced in Subsections 1.8.1, 6.7.1, 10.2.1. However, we need to introduce also some other notation.

Let $(\epsilon, \delta) \in (]-\epsilon_1, \epsilon_1[ \setminus \{0\}) \times ]0, +\infty[$. If $v$ is a function of $\mathrm{cl}\, \mathbb{T}_a(\epsilon, \delta)$ to $\mathbb{C}$, then we denote by $\mathbf{E}_{(\epsilon,\delta)}[v]$ the function of $\mathbb{R}^n$ to $\mathbb{C}$, defined by

$$\mathbf{E}_{(\epsilon,\delta)}[v](x) \equiv \begin{cases} v(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a(\epsilon, \delta), \\ 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\, \mathbb{T}_a(\epsilon, \delta). \end{cases}$$

### 10.3.2 Preliminaries

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. For each $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we consider the following periodic linear Robin problem for the Helmholtz equation.

$$
\begin{cases}
\Delta u(x) + \frac{k^2}{\delta^2} u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta), \\
u(x+\delta a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\}, \\
\delta \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x) + f(\frac{1}{\epsilon\delta}(x-\delta w))u(x) = g(\frac{1}{\epsilon\delta}(x-\delta w)) & \forall x \in \partial\Omega(\epsilon,\delta).
\end{cases} \tag{10.54}
$$

By virtue of Theorem 10.5, we can give the following definition.

**Definition 10.27.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. For each $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, we denote by $u_{(\epsilon,\delta)}$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta),\mathbb{C})$ of boundary value problem (10.54).

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. In order to do so we introduce the following.

**Definition 10.28.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. For each $\epsilon \in ]0,\epsilon_1[$, we denote by $u[\epsilon]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],\mathbb{C})$ of the following periodic linear Robin problem for the Helmholtz equation.

$$
\begin{cases}
\Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\
u(x+a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\
\frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) + f(\frac{1}{\epsilon}(x-w))u(x) = g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon.
\end{cases} \tag{10.55}
$$

*Remark* 10.29. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, we have

$$
u_{(\epsilon,\delta)}(x) = u[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).
$$

By the previous remark, we note that the solution of problem (10.54) can be expressed by means of the solution of the auxiliary rescaled problem (10.55), which does not depend on $\delta$. This is due to the presence of the factor $\delta$ in front of $\frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x)$ in the third equation of problem (10.54).

As a first step, we study the behaviour of $u[\epsilon]$ as $\epsilon$ tends to 0.

**Proposition 10.30.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all $t \in \partial\Omega$. Let $\epsilon_3$ be as in Proposition 10.17. Then there exist $\tilde{\epsilon} \in ]0,\epsilon_3[$ and a real analytic map $N$ of $]-\tilde{\epsilon},\tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$ such that*

$$
\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},
$$
$$
\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},
$$

*for all $\epsilon \in ]0,\tilde{\epsilon}[$. Moreover, as a consequence,*

$$
\lim_{\epsilon\to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).
$$

*Proof.* Let $\epsilon_3$, $\Theta_n$ be as in Proposition 10.17. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $\epsilon \in ]0,\epsilon_3[$, we have

$$
u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s \quad \forall t \in \partial\Omega.
$$

We set

$$
N[\epsilon](t) \equiv \int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s \qquad \forall t \in \partial\Omega,
$$

for all $\epsilon \in ]-\epsilon_3,\epsilon_3[$. By taking $\tilde{\epsilon} \in ]0,\epsilon_3[$ small enough, we can assume (cf. Theorem C.4 and the proof of Theorem 10.22) that $N$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$. By Corollary 6.24, we have

$$
\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)} \qquad \forall \epsilon \in ]0,\tilde{\epsilon}[,
$$

and

$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon\|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)} \qquad \forall\epsilon \in ]0,\tilde{\epsilon}[.$$

Accordingly,

$$\lim_{\epsilon\to 0^+} \mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and

$$\lim_{\epsilon\to 0^+} \mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and so the conclusion follows. □

**Proposition 10.31.** *Let $n$ be even. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$ and $\mathrm{Re}(k) = 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$. Let $\epsilon_3$, $\epsilon_2'$ be as in Proposition 10.18. Then there exist $\tilde{\epsilon} \in ]0,\epsilon_3[$ and two real analytic maps $N_1^\#$, $N_2^\#$ of $]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\epsilon_2',\epsilon_2'[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$ such that*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$

*for all $\epsilon \in ]0,\tilde{\epsilon}[$. Moreover, as a consequence,*

$$\lim_{\epsilon\to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* Let $\epsilon_3$, $\epsilon_2'$, $\Theta_n^\#$ be as in Proposition 10.18. If $\epsilon \in ]0,\epsilon_3[$, we have

$$\begin{aligned} u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) =&\epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &+ \epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &+ \epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega. \end{aligned}$$

We set

$$\begin{aligned} N_1^\#[\epsilon,\epsilon'](t) \equiv& \int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \\ &+ \epsilon^{n-2}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \end{aligned}$$

and

$$N_2^\#[\epsilon,\epsilon'](t) \equiv k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $(\epsilon,\epsilon') \in ]-\epsilon_3,\epsilon_3[\times]-\epsilon_2',\epsilon_2'[$. By taking $\tilde{\epsilon} \in ]0,\epsilon_3[$ small enough, we can assume (cf. Theorem C.4 and the proof of Theorem 10.23) that $N_1^\#$, $N_2^\#$ are real analytic maps of $]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\epsilon_2',\epsilon_2'[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$. Clearly,

$$u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \epsilon N_1^\#[\epsilon,\epsilon\log\epsilon](t) + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon](t) \qquad \forall t\in\partial\Omega,\ \forall\epsilon\in]0,\tilde{\epsilon}[.$$

By Corollary 6.24, we have

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$

and

$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$

for all $\epsilon \in ]0,\tilde{\epsilon}[$.

Accordingly,

$$\lim_{\epsilon\to 0^+} \mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and

$$\lim_{\epsilon\to 0^+} \mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and so the conclusion follows. □

### 10.3.3 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorems we deduce by Propositions 10.30, 10.31 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 10.32.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let $\mathrm{Im}(k) \neq 0$ and $\mathrm{Re}(k) = 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$. Let $\tilde{\epsilon}$, $N$ be as in Proposition 10.30. Then*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon\|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon\|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,+\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \epsilon\|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \epsilon\|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,+\infty[$. □

**Theorem 10.33.** *Let $n$ be even. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let $\mathrm{Im}(k) \neq 0$ and $\mathrm{Re}(k) = 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$. Let $\tilde{\epsilon}$, $N_1^\#$, $N_2^\#$ be as in Proposition 10.31. Then*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,+\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,+\infty[$. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 10.34.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. Let $\epsilon_6$, $J$ be as in Theorem 10.25. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n\frac{\epsilon^{n-1}}{k^2}J[\epsilon], \tag{10.56}$$

*for all $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N}\setminus\{0\}$.*

*Proof.* Let $\epsilon \in ]0, \epsilon_6[$, $l \in \mathbb{N} \setminus \{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](t)\,dt \\ &= \frac{r^n}{l^n}\frac{\epsilon^{n-1}}{k^2}J[\epsilon].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n\frac{\epsilon^{n-1}}{k^2}J[\epsilon],$$

and the conclusion follows. □

**Theorem 10.35.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all* $t \in \partial\Omega$ *and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all* $t \in \partial\Omega$. *Let* $\epsilon_6$, $J_1^\#$, $J_2^\#$ *be as in Theorem 10.26. Let* $r > 0$ *and* $\bar{y} \in \mathbb{R}^n$. *Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n\Big\{\frac{\epsilon^{n-1}}{k^2}J_1^\#[\epsilon,\epsilon\log\epsilon] + \frac{\epsilon^{2n-2}(\log\epsilon)}{k^2}J_2^\#[\epsilon,\epsilon\log\epsilon]\Big\}, \tag{10.57}$$

*for all* $\epsilon \in ]0, \epsilon_6[$, $l \in \mathbb{N} \setminus \{0\}$.

*Proof.* Let $\epsilon \in ]0, \epsilon_6[$, $l \in \mathbb{N} \setminus \{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](t)\,dt \\ &= \frac{r^n}{l^n}\Big\{\frac{\epsilon^{n-1}}{k^2}J_1^\#[\epsilon,\epsilon\log\epsilon] + \frac{\epsilon^{2n-2}(\log\epsilon)}{k^2}J_2^\#[\epsilon,\epsilon\log\epsilon]\Big\}.\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n\Big\{\frac{\epsilon^{n-1}}{k^2}J_1^\#[\epsilon,\epsilon\log\epsilon] + \frac{\epsilon^{2n-2}(\log\epsilon)}{k^2}J_2^\#[\epsilon,\epsilon\log\epsilon]\Big\},$$

and the conclusion follows. □

We give the following.

**Definition 10.36.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. For each pair $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we set

$$\mathcal{F}(\epsilon,\delta) \equiv \int_{A \cap \mathbb{T}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{A \cap \mathbb{T}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2 \, dx.$$

*Remark* 10.37. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. Let $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$. We have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2 \, dt \\ &= \delta^{n-2} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2 \, dt,\end{aligned}$$

and

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2 \, dx = \delta^n \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](t)|^2 \, dt.$$

Accordingly,

$$\begin{aligned}&\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2 \, dx \\ &\qquad = \delta^{n-2} \Big( \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2 \, dt - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](t)|^2 \, dt \Big).\end{aligned}$$

Then we give the following definition, where we consider $\mathcal{F}(\epsilon, \delta)$, with $\epsilon$ equal to a certain function of $\delta$.

**Definition 10.38.** For each $\delta \in ]0, +\infty[$, we set

$$\epsilon[\delta] \equiv \delta^{\frac{2}{n}}.$$

Let $\epsilon_5$ be as in Theorem 10.22, if $n$ is odd, or as in Theorem 10.23, if $n$ is even. Let $\delta_1 > 0$ be such that $\epsilon[\delta] \in ]0, \epsilon_5[$, for all $\delta \in ]0, \delta_1[$. Then we set

$$\mathcal{F}[\delta] \equiv \mathcal{F}(\epsilon[\delta], \delta),$$

for all $\delta \in ]0, \delta_1[$.

In the following Propositions we compute the limit of $\mathcal{F}[\delta]$ as $\delta$ tends to 0.

**Proposition 10.39.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all $t \in \partial\Omega$. Let $\epsilon_5$ be as in Theorem 10.22. Let $\delta_1 > 0$ be as in Definition 10.38. Then*

$$\lim_{\delta \to 0^+} \mathcal{F}[\delta] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx,$$

*where $\tilde{u}$ is as in Definition 10.13.*

*Proof.* For each $\delta \in ]0, \delta_1[$, we set

$$\mathcal{G}(\delta) \equiv \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx.$$

Let $\delta \in ]0, \delta_1[$. By Remark 10.37 and Theorem 10.22, we have

$$\begin{aligned}\mathcal{G}(\delta) &= \delta^{n-2} (\epsilon[\delta])^n G[\epsilon[\delta]] \\ &= \delta^{n-2} \delta^2 G[\delta^{\frac{2}{n}}],\end{aligned}$$

where $G$ is as in Theorem 10.22. On the other hand,

$$\lfloor (1/\delta) \rfloor^n \mathcal{G}(\delta) \le \mathcal{F}[\delta] \le \lceil (1/\delta) \rceil^n \mathcal{G}(\delta).$$

As a consequence, since

$$\lim_{\delta\to 0^+} \lfloor (1/\delta) \rfloor^n \delta^n = 1, \qquad \lim_{\delta\to 0^+} \lceil (1/\delta) \rceil^n \delta^n = 1,$$

then

$$\lim_{\delta\to 0^+} \mathcal{F}[\delta] = G[0].$$

Finally, by equality (10.44), we easily conclude. □

**Proposition 10.40.** *Let $n$ be even and $n > 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let* $\mathrm{Re}(f(t)) \le 0$ *for all* $t \in \partial\Omega$. *Let $\epsilon_5$ be as in Theorem 10.23. Let $\delta_1 > 0$ be as in Definition 10.38. Then*

$$\lim_{\delta\to 0^+} \mathcal{F}[\delta] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx,$$

*where $\tilde{u}$ is as in Definition 10.13.*

*Proof.* For each $\delta \in ]0, \delta_1[$, we set

$$\mathcal{G}(\delta) \equiv \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx.$$

Let $\delta \in ]0, \delta_1[$. By Remark 10.37 and Theorem 10.23, we have

$$\begin{aligned}
\mathcal{G}(\delta) =& \delta^{n-2} (\epsilon[\delta])^n G_1^{\#}[\epsilon[\delta], \epsilon[\delta] \log \epsilon[\delta]] \\
&+ \delta^{n-2} (\epsilon[\delta])^{2n-2} (\log \epsilon[\delta]) G_2^{\#}[\epsilon[\delta], \epsilon[\delta] \log \epsilon[\delta]] \\
&+ \delta^{n-2} (\epsilon[\delta])^{3n-3} (\log \epsilon[\delta])^2 G_3^{\#}[\epsilon[\delta], \epsilon[\delta] \log \epsilon[\delta]] \\
=& \delta^{n-2} \delta^2 G_1^{\#}[\delta^{\frac{2}{n}}, \delta^{\frac{2}{n}} \log(\delta^{\frac{2}{n}})] \\
&+ \delta^{n-2} \delta^{4-\frac{4}{n}} (\log(\delta^{\frac{2}{n}})) G_2^{\#}[\delta^{\frac{2}{n}}, \delta^{\frac{2}{n}} \log(\delta^{\frac{2}{n}})] \\
&+ \delta^{n-2} \delta^{6-\frac{6}{n}} (\log(\delta^{\frac{2}{n}}))^2 G_3^{\#}[\delta^{\frac{2}{n}}, \delta^{\frac{2}{n}} \log(\delta^{\frac{2}{n}})],
\end{aligned}$$

where $G_1^{\#}$, $G_2^{\#}$, and $G_3^{\#}$ are as in Theorem 10.23. On the other hand,

$$\lfloor (1/\delta) \rfloor^n \mathcal{G}(\delta) \le \mathcal{F}[\delta] \le \lceil (1/\delta) \rceil^n \mathcal{G}(\delta).$$

As a consequence, since

$$\lim_{\delta\to 0^+} \lfloor (1/\delta) \rfloor^n \delta^n = 1, \qquad \lim_{\delta\to 0^+} \lceil (1/\delta) \rceil^n \delta^n = 1,$$

then

$$\lim_{\delta\to 0^+} \mathcal{F}[\delta] = G_1^{\#}[0,0].$$

Finally, by equality (10.49), we easily conclude. □

In the following Propositions we represent the function $\mathcal{F}[\cdot]$ by means of real analytic functions.

**Proposition 10.41.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \le 0$ *for all* $t \in \partial\Omega$ *and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \ge 0$ *for all* $t \in \partial\Omega$. *Let $\epsilon_5$, $G$ be as in Theorem 10.22. Let $\delta_1 > 0$ be as in Definition 10.38. Then*

$$\mathcal{F}[(1/l)] = G[(1/l)^{\frac{2}{n}}],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* By arguing as in the proof of Proposition 10.39, one can easily see that

$$\begin{aligned}\mathcal{F}[(1/l)] &= l^n(1/l)^{n-2}(1/l)^2 G[(1/l)^{\frac{2}{n}}]\\ &= G[(1/l)^{\frac{2}{n}}],\end{aligned}$$

for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$. □

**Proposition 10.42.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all* $t \in \partial\Omega$ *and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all* $t \in \partial\Omega$. *Let* $\epsilon_5$, $G_1^\#$, $G_2^\#$, *and* $G_3^\#$ *be as in Theorem 10.23. Let* $\delta_1 > 0$ *be as in Definition 10.38. Then*

$$\begin{aligned}\mathcal{F}[(1/l)] =& G_1^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}}\log((1/l)^{\frac{2}{n}})]\\ &+ (1/l)^{2-\frac{4}{n}}\log((1/l)^{\frac{2}{n}}) G_2^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}}\log((1/l)^{\frac{2}{n}})]\\ &+ (1/l)^{4-\frac{6}{n}}\Big[\log((1/l)^{\frac{2}{n}})\Big]^2 G_3^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}}\log((1/l)^{\frac{2}{n}})],\end{aligned}$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* By arguing as in the proof of Proposition 10.39, one can easily see that

$$\begin{aligned}\mathcal{F}[(1/l)] &= l^n(1/l)^{n-2}(1/l)^2\Big\{G_1^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}}\log((1/l)^{\frac{2}{n}})]\\ &\quad + (1/l)^{2-\frac{4}{n}}\log((1/l)^{\frac{2}{n}}) G_2^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}}\log((1/l)^{\frac{2}{n}})]\\ &\quad + (1/l)^{4-\frac{6}{n}}\Big[\log((1/l)^{\frac{2}{n}})\Big]^2 G_3^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}}\log((1/l)^{\frac{2}{n}})]\Big\}\\ &= G_1^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}}\log((1/l)^{\frac{2}{n}})]\\ &\quad + (1/l)^{2-\frac{4}{n}}\log((1/l)^{\frac{2}{n}}) G_2^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}}\log((1/l)^{\frac{2}{n}})]\\ &\quad + (1/l)^{4-\frac{6}{n}}\Big[\log((1/l)^{\frac{2}{n}})\Big]^2 G_3^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}}\log((1/l)^{\frac{2}{n}})],\end{aligned}$$

for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$. □

## 10.4 A variant of an homogenization problem for the Helmholtz equation with linear Robin boundary conditions in a periodically perforated domain

In this section we consider a different homogenization problem for the Helmhlotz equation with linear Robin boundary conditions in a periodically perforated domain. As above, most of the results are obtained under the assumption that $\mathrm{Im}(k) \neq 0$ and $\mathrm{Re}(k) = 0$.

### 10.4.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 6.7.1, 10.2.1, 10.3.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. For each $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we consider the following periodic linear Robin problem for the Helmholtz equation.

$$\begin{cases}\Delta u(x) + \frac{k^2}{\delta^2}u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta),\\ u(x+\delta a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\},\\ \frac{\partial}{\partial\nu_{\Omega(\epsilon,\delta)}}u(x) + f(\frac{1}{\epsilon\delta}(x-\delta w))u(x) = g(\frac{1}{\epsilon\delta}(x-\delta w)) & \forall x \in \partial\Omega(\epsilon,\delta).\end{cases} \tag{10.58}$$

In contrast to problem (10.54), we note that in the third equation of problem (10.58) there is not the factor $\delta$ in front of $\frac{\partial}{\partial\nu_{\Omega(\epsilon,\delta)}}u(x)$.

By virtue of Theorem 10.5, we can give the following definition.

**Definition 10.43.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. For each $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we denote by $u_{(\epsilon,\delta)}$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta),\mathbb{C})$ of boundary value problem (10.58).

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. In order to do so we introduce the following.

**Definition 10.44.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. For each $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we denote by $u[\epsilon,\delta]$ the unique solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],\mathbb{C})$ of the following auxiliary periodic linear Robin problem for the Helmholtz equation.

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x+a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) + \delta f(\frac{1}{\epsilon}(x-w))u(x) = \delta g(\frac{1}{\epsilon}(x-w)) & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{10.59}$$

*Remark* 10.45. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$, we have

$$u_{(\epsilon,\delta)}(x) = u[\epsilon,\delta](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).$$

By the previous remark, in contrast to the solution of problem (10.54), we note that the solution of problem (10.58) can be expressed by means of the solution of the auxiliary rescaled problem (10.59), which does depend on $\delta$.

As a first step, we study the behaviour of $u[\epsilon,\delta]$ as $(\epsilon,\delta)$ tends to $(0,0)$.

We have the following Lemmas.

**Lemma 10.46.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56)*,* (1.57)*,* (10.10)*,* (10.11)*,* (10.12)*, respectively. Let $(\epsilon,\delta) \in ]0,\epsilon_1[ \times ]0,+\infty[$. Then the function $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon,\mathbb{C})$ satisfies the following equation*

$$\begin{aligned} \delta g(\frac{1}{\epsilon}(x-w)) = \frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}(S_n^{a,k}(x-y))\mu(y)\,d\sigma_y \\ + \delta f(\frac{1}{\epsilon}(x-w)) \int_{\partial\Omega_\epsilon} S_n^{a,k}(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega_\epsilon, \end{aligned} \tag{10.60}$$

*if and only if the function $\theta \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, defined by*

$$\theta(t) \equiv \frac{1}{\delta}\mu(w+\epsilon t) \qquad \forall t \in \partial\Omega, \tag{10.61}$$

*satisfies the following equation*

$$\begin{aligned} g(t) =& \frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^n} S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s \\ &+ \epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega} \nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\theta(s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s \\ &+ f(t)\Bigg[\delta\epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s + \delta\epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\theta(s)\,d\sigma_s \\ &+ \delta\epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s\Bigg] \qquad \forall t \in \partial\Omega. \end{aligned} \tag{10.62}$$

*Proof.* It is a straightforward verification based on the rule of change of variables in integrals, on well known properties of composition of functions in Schauder spaces (cf. *e.g.*, Lanza [67, Section 3,4]) and on equalities (6.24), (6.25). □

**Lemma 10.47.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $g$ be as in* (1.56), (10.12), *respectively. Then there exists a unique function $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ that solves the following equation*

$$g(t) = \frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\,d\sigma_s \qquad \forall t \in \partial\Omega. \tag{10.63}$$

*We denote the unique solution of equation* (10.63) *by $\tilde{\theta}$. Moreover,*

$$\int_{\partial\Omega} \tilde{\theta}(s)\,d\sigma_s = \int_{\partial\Omega} g(s)\,d\sigma_s. \tag{10.64}$$

*Proof.* It is Lemma 10.8. □

Since we want to represent the function $u[\epsilon,\delta]$ by means of a periodic simple layer potential, we need to study some integral equations. We introduce this integral equation in the following Propositions.

**Proposition 10.48.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. Let $\tilde{\theta}$ be as in Lemma 10.47. Let $\Lambda$ be the map of $]-\epsilon_1,\epsilon_1[\, \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ defined by*

$$\begin{aligned} &\Lambda[\epsilon,\delta,\theta](t) \\ &\equiv \frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s \\ &\quad + f(t)\left[\delta\epsilon \int_{\partial\Omega} S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s\right] - g(t) \quad \forall t \in \partial\Omega, \end{aligned} \tag{10.65}$$

*for all $(\epsilon,\delta,\theta) \in ]-\epsilon_1,\epsilon_1[\, \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Then the following statements hold.*

*(i) If $(\epsilon,\delta) \in ]0,\epsilon_1[\, \times ]0,+\infty[$, then the function $\theta \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ satisfies equation*

$$\Lambda[\epsilon,\delta,\theta] = 0, \tag{10.66}$$

*if and only if the function $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon,\mathbb{C})$, defined by*

$$\mu(x) \equiv \delta\theta(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{10.67}$$

*satisfies the equation*

$$\begin{aligned} \delta\Gamma(x) = \frac{1}{2}\mu(x) &+ \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}(S_n^{a,k}(x-y))\mu(y)\,d\sigma_y \\ &+ \delta\phi(x)\int_{\partial\Omega_\epsilon} S_n^{a,k}(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega_\epsilon, \end{aligned} \tag{10.68}$$

*with $\Gamma \in C^{m-1,\alpha}(\partial\Omega_\epsilon,\mathbb{C})$, and $\phi \in C^{m-1,\alpha}(\partial\Omega_\epsilon,\mathbb{C})$, defined by*

$$\Gamma(x) \equiv g(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{10.69}$$

*and*

$$\phi(x) \equiv f(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon. \tag{10.70}$$

*In particular, equation* (10.66) *has exactly one solution $\theta \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, for each $(\epsilon,\delta) \in ]0,\epsilon_1[\, \times ]0,+\infty[$.*

*(ii) The function $\theta \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ satisfies equation*

$$\Lambda[0,0,\theta] = 0, \tag{10.71}$$

*if and only if*

$$g(t) = \frac{1}{2}\theta(t) + \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\,d\sigma_s \qquad \forall t \in \partial\Omega. \tag{10.72}$$

*In particular, the unique function $\theta \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ that solves equation* (10.71) *is $\tilde{\theta}$.*

*Proof.* Consider $(i)$. The equivalence of equation (10.66) in the unknown $\theta \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ and equation (10.68) in the unknown $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon,\mathbb{C})$ follows by Lemma 10.46 and the definition of $Q_n^k$ for $n$ odd (cf. (6.23) and Definition E.2.) The existence and uniqueness of a solution of equation (10.68) follows by Proposition 10.4 $(ii)$. Then the existence and uniqueness of a solution of equation (10.66) follows by the equivalence of (10.66) and (10.68). Consider $(ii)$. The equivalence of (10.71) and (10.72) is obvious. The second part of statement $(ii)$ is an immediate consequence of Lemma 10.47. □

**Proposition 10.49.** *Let $n$ be even. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. Let $\tilde{\theta}$ be as in Lemma 10.47. Let $\epsilon_1' > 0$ be such that*

$$\epsilon \log \epsilon \in ]-\epsilon_1',\epsilon_1'[ \qquad \forall \epsilon \in ]0,\epsilon_1[. \tag{10.73}$$

*Let $\Lambda^\#$ be the map of $]-\epsilon_1,\epsilon_1[\times]-\epsilon_1',\epsilon_1'[\times\mathbb{R}\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ defined by*

$$\begin{aligned}
&\Lambda^\#[\epsilon,\epsilon',\delta,\theta](t)\\
&\equiv \frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^n} S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s + \epsilon^{n-2}\epsilon' k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\theta(s)\,d\sigma_s\\
&+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s + f(t)\Big[\delta\epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s\\
&+\delta\epsilon^{n-2}\epsilon' k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\theta(s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s\Big] - g(t)\ \forall t\in\partial\Omega,
\end{aligned} \tag{10.74}$$

*for all $(\epsilon,\epsilon',\delta,\theta) \in ]-\epsilon_1,\epsilon_1[\times]-\epsilon_1',\epsilon_1'[\times\mathbb{R}\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Then the following statements hold.*

*(i) If $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, then the function $\theta \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ satisfies equation*

$$\Lambda^\#[\epsilon,\epsilon\log\epsilon,\delta,\theta] = 0, \tag{10.75}$$

*if and only if the function $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon,\mathbb{C})$, defined by*

$$\mu(x) \equiv \delta\theta(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{10.76}$$

*satisfies the equation*

$$\begin{aligned}
\delta\Gamma(x) = \frac{1}{2}\mu(x) &+ \int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}(x)}(S_n^{a,k}(x-y))\mu(y)\,d\sigma_y\\
&+\delta\phi(x)\int_{\partial\Omega_\epsilon} S_n^{a,k}(x-y)\mu(y)\,d\sigma_y \qquad \forall x\in\partial\Omega_\epsilon,
\end{aligned} \tag{10.77}$$

*with $\Gamma \in C^{m-1,\alpha}(\partial\Omega_\epsilon,\mathbb{C})$, and $\phi \in C^{m-1,\alpha}(\partial\Omega_\epsilon,\mathbb{C})$, defined by*

$$\Gamma(x) \equiv g(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon, \tag{10.78}$$

*and*

$$\phi(x) \equiv f(\frac{1}{\epsilon}(x-w)) \qquad \forall x \in \partial\Omega_\epsilon. \tag{10.79}$$

*In particular, equation* (10.75) *has exactly one solution $\theta \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, for each $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$.*

*(ii) The function $\theta \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ satisfies equation*

$$\Lambda^\#[0,0,0,\theta] = 0, \tag{10.80}$$

*if and only if*

$$g(t) = \frac{1}{2}\theta(t) + \int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega(t)}(S_n(t-s))\theta(s)\,d\sigma_s \qquad \forall t \in \partial\Omega. \tag{10.81}$$

*In particular, the unique function $\theta \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ that solves equation* (10.80) *is $\tilde{\theta}$.*

*Proof.* Consider $(i)$. The equivalence of equation (10.75) in the unknown $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ and equation (10.77) in the unknown $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ follows by Lemma 10.46 and the definition of $Q_n^k$ for $n$ even (cf. (6.23) and Definition E.2.) The existence and uniqueness of a solution of equation (10.77) follows by Proposition 10.4 $(ii)$. Then the existence and uniqueness of a solution of equation (10.75) follows by the equivalence of (10.75) and (10.77). Consider $(ii)$. The equivalence of (10.80) and (10.81) is obvious. The second part of statement $(ii)$ is an immediate consequence of Lemma 10.47. □

By Propositions 10.9, 10.49, it makes sense to introduce the following.

**Definition 10.50.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. For each $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$, we denote by $\hat{\theta}_n[\epsilon, \delta]$ the unique function in $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ that solves equation (10.66), if $n$ is odd, or equation (10.75), if $n$ is even. Analogously, we denote by $\hat{\theta}_n[0, 0]$ the unique function in $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ that solves equation (10.71), if $n$ is odd, or equation (10.80), if $n$ is even.

In the following Remark, we show the relation between the solutions of boundary value problem (10.59) and the solutions of equations (10.66), (10.75).

*Remark* 10.51. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$.

Let $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$. We have

$$u[\epsilon, \delta](x) = \delta\epsilon^{n-1} \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s)\hat{\theta}_n[\epsilon, \delta](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$

While the relation between equations (10.66), (10.75) and boundary value problem (10.59) is now clear, we want to see if (10.71), (10.80) are related to some (limiting) boundary value problem. We give the following.

**Definition 10.52.** Let $n \geq 3$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $\Omega$, $g$ be as in (1.56), (10.12), respectively. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega, \mathbb{C})$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega, \\ \frac{\partial}{\partial\nu_\Omega} u(x) = g(x) & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{10.82}$$

Problem (10.82) will be called the *limiting boundary value problem.*

*Remark* 10.53. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. If $n \geq 3$, then we have

$$\tilde{u}(x) = \int_{\partial\Omega} S_n(x - y)\hat{\theta}_n[0, 0](y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \Omega.$$

If $n = 2$, in general the (classic) simple layer potential for the Laplace equation with moment $\hat{\theta}_2[0, 0]$ is not harmonic at infinity, and it does not satisfy the third condition of boundary value problem (10.82). Moreover, if $n = 2$, boundary value problem (10.82) does not have in general a solution (unless $\int_{\partial\Omega} g\, d\sigma = 0$.) However, the function $\tilde{v}$ of $\mathbb{R}^2 \setminus \Omega$ to $\mathbb{C}$, defined by

$$\tilde{v}(x) \equiv \int_{\partial\Omega} S_2(x - y)\hat{\theta}_2[0, 0](y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^2 \setminus \Omega,$$

is a solution of the following boundary value problem

$$\begin{cases} \Delta \tilde{v}(x) = 0 & \forall x \in \mathbb{R}^2 \setminus \mathrm{cl}\,\Omega, \\ \frac{\partial}{\partial\nu_\Omega} \tilde{v}(x) = g(x) & \forall x \in \partial\Omega. \end{cases} \tag{10.83}$$

We now prove the following Propositions.

**Proposition 10.54.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all $t \in \partial\Omega$ and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all $t \in \partial\Omega$. Let $\tilde{\theta}$ be as in Lemma 10.47. Let $\Lambda$ be as in Proposition 10.48. Then there exists $\epsilon_2 \in ]0, \epsilon_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Moreover, if we set $b_0 \equiv (0, 0, \tilde{\theta})$, then the differential $\partial_\theta \Lambda[b_0]$ of $\Lambda$ with respect to the variable $\theta$ at $b_0$ is delivered by the following formula*

$$\partial_\theta \Lambda[b_0](\tau)(t) = \frac{1}{2}\tau(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\tau(s)\, d\sigma_s \qquad \forall t \in \partial\Omega, \tag{10.84}$$

*for all $\tau \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, and is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ onto itself.*

*Proof.* By Proposition 6.21 and by continuity of the pointwise product in Schauder space, we easily deduce that there exists $\epsilon_2 \in ]0, \epsilon_1]$ such that $\Lambda$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. By standard calculus in Banach space, we immediately deduce that (10.84) holds. Now we need to prove that $\partial_\theta \Lambda[b_0]$ is a linear homeomorphism. By the Open Mapping Theorem, it suffices to prove that it is a bijection. Let $\psi \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. By known results of classical potential theory (cf. Folland [52, Chapter 3]), there exists a unique function $\tau \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, such that

$$\frac{1}{2}\tau(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\tau(s)\, d\sigma_s = \psi(t) \qquad \forall t \in \partial\Omega.$$

Hence $\partial_\theta \Lambda[b_0]$ is bijective, and, accordingly, a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ onto itself. □

**Proposition 10.55.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all $t \in \partial\Omega$ and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all $t \in \partial\Omega$. Let $\tilde{\theta}$ be as in Lemma 10.47. Let $\epsilon'_1 > 0$ be as in* (10.73). *Let $\Lambda^\#$ be as in Proposition 10.49. Then there exists $\epsilon_2 \in ]0, \epsilon_1]$ such that $\Lambda^\#$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times ]-\epsilon'_1, \epsilon'_1[ \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Moreover, if we set $b_0 \equiv (0, 0, 0, \tilde{\theta})$, then the differential $\partial_\theta \Lambda^\#[b_0]$ of $\Lambda^\#$ with respect to the variable $\theta$ at $b_0$ is delivered by the following formula*

$$\partial_\theta \Lambda^\#[b_0](\tau)(t) = \frac{1}{2}\tau(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\tau(s)\, d\sigma_s \qquad \forall t \in \partial\Omega, \tag{10.85}$$

*for all $\tau \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, and is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ onto itself.*

*Proof.* By Proposition 6.21 and by continuity of the pointwise product in Schauder space, we easily deduce that there exists $\epsilon_2 \in ]0, \epsilon_1]$ such that $\Lambda^\#$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times ]-\epsilon'_1, \epsilon'_1[ \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. By standard calculus in Banach space, we immediately deduce that (10.85) holds. Finally, by the proof of Proposition 10.54 and formula (10.85), we have that $\partial_\theta \Lambda^\#[b_0]$ is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ onto itself. □

By the previous Propositions we can now prove the following results.

**Proposition 10.56.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all $t \in \partial\Omega$ and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all $t \in \partial\Omega$. Let $\epsilon_2$ be as in Proposition 10.54. Then there exist $\epsilon_3 \in ]0, \epsilon_2]$, $\delta_1 \in ]0, +\infty[$ and a real analytic operator $\Theta_n$ of $]-\epsilon_3, \epsilon_3[ \times ]-\delta_1, \delta_1[$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, such that*

$$\Theta_n[\epsilon, \delta] = \hat{\theta}_n[\epsilon, \delta], \tag{10.86}$$

*for all $(\epsilon, \delta) \in (]0, \epsilon_3[ \times ]0, \delta_1[) \cup \{(0, 0)\}$.*

*Proof.* It is an immediate consequence of Proposition 10.54 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

**Proposition 10.57.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all $t \in \partial\Omega$ and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all $t \in \partial\Omega$. Let $\epsilon'_1 > 0$ be as in* (10.27). *Let $\epsilon_2$ be as in*

*Proposition 10.55. Then there exist* $\epsilon_3 \in ]0, \epsilon_2]$, $\epsilon'_2 \in ]0, \epsilon'_1]$, $\delta_1 \in ]0, +\infty[$ *and a real analytic operator* $\Theta_n^{\#}$ *of* $]-\epsilon_3, \epsilon_3[ \times ]-\epsilon'_2, \epsilon'_2[ \times ]-\delta_1, \delta_1[$ *to* $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$*, such that*

$$\epsilon \log \epsilon \in ]-\epsilon'_2, \epsilon'_2[ \quad \forall \epsilon \in ]0, \epsilon_3[,$$

$$\Theta_n^{\#}[\epsilon, \epsilon \log \epsilon, \delta] = \hat{\theta}_n[\epsilon, \delta] \quad \forall (\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[, \tag{10.87}$$

$$\Theta_n^{\#}[0, 0, 0] = \hat{\theta}_n[0, 0]. \tag{10.88}$$

*Proof.* It is an immediate consequence of Proposition 10.55 and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

By Propositions 10.56, 10.57, and Remark 10.51, we can deduce the following results.

**Theorem 10.58.** *Let* $n$ *be odd. Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $k$, $f$, $g$ *be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all* $t \in \partial\Omega$ *and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all* $t \in \partial\Omega$. *Let* $\epsilon_3$, $\delta_1$ *be as in Proposition 10.56. Let* $V$ *be a bounded open subset of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. *Then there exist* $\epsilon_4 \in ]0, \epsilon_3]$*, and a real analytic operator* $U$ *of* $]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$ *to the space* $C^0(\mathrm{cl}\, V, \mathbb{C})$*, such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*

$$u[\epsilon, \delta](x) = \delta \epsilon^{n-1} U[\epsilon, \delta](x) \qquad \forall x \in \mathrm{cl}\, V,$$

*for all* $(\epsilon, \delta) \in ]0, \epsilon_4[ \times ]0, \delta_1[$. *Moreover,*

$$U[0, 0](x) = S_n^{a,k}(x - w) \int_{\partial\Omega} g \, d\sigma \qquad \forall x \in \mathrm{cl}\, V.$$

*Proof.* Let $\Theta_n[\cdot, \cdot]$ be as in Proposition 10.56. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $(\epsilon, \delta) \in ]0, \epsilon_4[ \times ]0, \delta_1[$. By Remark 10.51 and Proposition 10.56, we have

$$u[\epsilon, \delta](x) = \delta \epsilon^{n-1} \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s) \Theta_n[\epsilon, \delta](s) \, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V.$$

Thus, it is natural to set

$$U[\epsilon](x) \equiv \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s) \Theta_n[\epsilon, \delta](s) \, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

for all $(\epsilon, \delta) \in ]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$. By Proposition 6.22, $U$ is a real analytic map of $]-\epsilon_4, \epsilon_4[ \times ]-\delta_1, \delta_1[$ to $C^0(\mathrm{cl}\, V, \mathbb{C})$. Furthermore, by Lemma 10.47, we have

$$\begin{aligned} U[0, 0](x) &= S_n^{a,k}(x - w) \int_{\partial\Omega} \Theta_n[0, 0](s) \, d\sigma_s \\ &= S_n^{a,k}(x - w) \int_{\partial\Omega} g \, d\sigma \qquad \forall x \in \mathrm{cl}\, V, \end{aligned}$$

since $\Theta_n[0, 0] = \tilde{\theta}$. Hence the proof is now complete. □

**Theorem 10.59.** *Let* $n$ *be even. Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. *Let* $w \in A$. *Let* $\Omega$, $\epsilon_1$, $k$, $f$, $g$ *be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all* $t \in \partial\Omega$ *and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all* $t \in \partial\Omega$. *Let* $\epsilon_3$, $\epsilon'_2$, $\delta_1$ *be as in Proposition 10.57. Let* $V$ *be a bounded open subset of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. *Then there exist* $\epsilon_4 \in ]0, \epsilon_3]$ *and a real analytic operator* $U^{\#}$ *of* $]-\epsilon_4, \epsilon_4[ \times ]-\epsilon'_2, \epsilon'_2[ \times ]-\delta_1, \delta_1[$ *to the space* $C^0(\mathrm{cl}\, V, \mathbb{C})$*, such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*

$$u[\epsilon, \delta](x) = \delta \epsilon^{n-1} U^{\#}[\epsilon, \epsilon \log \epsilon, \delta](x) \qquad \forall x \in \mathrm{cl}\, V,$$

*for all* $(\epsilon, \delta_1) \in ]0, \epsilon_4[ \times ]0, \delta_1[$. *Moreover,*

$$U^{\#}[0, 0, 0](x) = S_n^{a,k}(x - w) \int_{\partial\Omega} g \, d\sigma \qquad \forall x \in \mathrm{cl}\, V.$$

*Proof.* Let $\Theta_n^\#[\cdot,\cdot,\cdot]$ be as in Proposition 10.57. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $(\epsilon,\delta) \in ]0,\epsilon_4[\times]0,\delta_1[$. By Remark 10.51 and Proposition 10.57, we have

$$u[\epsilon,\delta](x) = \delta\epsilon^{n-1}\int_{\partial\Omega} S_n^{a,k}(x-w-\epsilon s)\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s \qquad \forall x \in \mathrm{cl}\,V.$$

Thus, it is natural to set

$$U^\#[\epsilon,\epsilon',\delta](x) \equiv \int_{\partial\Omega} S_n^{a,k}(x-w-\epsilon s)\Theta_n^\#[\epsilon,\epsilon',\delta](s)\,d\sigma_s \qquad \forall x \in \mathrm{cl}\,V,$$

for all $(\epsilon,\epsilon',\delta) \in ]-\epsilon_4,\epsilon_4[\times]-\epsilon_2',\epsilon_2'[\times]-\delta_1,\delta_1[$. By Proposition 6.22, $U^\#$ is a real analytic map of $]-\epsilon_4,\epsilon_4[\times]-\epsilon_2',\epsilon_2'[\times]-\delta_1,\delta_1[$ to $C^0(\mathrm{cl}\,V,\mathbb{C})$. Furthermore, by Lemma 10.47, we have

$$\begin{aligned} U^\#[0,0,0](x) &= S_n^{a,k}(x-w)\int_{\partial\Omega}\Theta_n^\#[0,0,0](s)\,d\sigma_s \\ &= S_n^{a,k}(x-w)\int_{\partial\Omega} g\,d\sigma \qquad \forall x \in \mathrm{cl}\,V, \end{aligned}$$

since $\Theta_n^\#[0,0,0] = \tilde{\theta}$. Accordingly, the Theorem is now completely proved. □

We have also the following Theorems.

**Theorem 10.60.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \le 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \ge 0$ for all $t \in \partial\Omega$. Let $\tilde{\theta}$ be as in Lemma 10.47. Let $\epsilon_3$, $\delta_1$ be as in Proposition 10.56. Then there exist $\epsilon_5 \in ]0,\epsilon_3]$, and a real analytic operator $G$ of $]-\epsilon_5,\epsilon_5[\times]-\delta_1,\delta_1[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon,\delta](x)|^2\,dx - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon,\delta](x)|^2\,dx = \delta^2\epsilon^n G[\epsilon,\delta], \tag{10.89}$$

*for all $(\epsilon,\delta) \in ]0,\epsilon_5[\times]0,\delta_1[$. Moreover,*

$$G[0,0] = \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega} |\nabla\tilde{u}(x)|^2\,dx, \tag{10.90}$$

*where $\tilde{u}$ is as in Definition 10.52.*

*Proof.* Let $\Theta_n[\cdot,\cdot]$ be as in Proposition 10.56. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $(\epsilon,\delta) \in ]0,\epsilon_3[\times]0,\delta_1[$. Clearly, by the periodicity of $u[\epsilon,\delta]$, we have

$$\begin{aligned} &\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon,\delta](x)|^2\,dx - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon,\delta](x)|^2\,dx \\ &\qquad = -\,\epsilon^{n-1}\int_{\partial\Omega}\Big(\frac{\partial u[\epsilon,\delta]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\overline{u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)}\,d\sigma_t. \end{aligned}$$

By equality (6.24) and since $Q_n^k = 0$ for $n$ odd, we have

$$\begin{aligned} u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) &= \delta\epsilon^{n-1}\int_{\partial\Omega} S_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s \\ &= \delta\epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s \quad \forall t \in \partial\Omega. \end{aligned}$$

By Theorem E.6 $(i)$, one can easily show that the map which takes $(\epsilon,\delta)$ to the function of the variable $t \in \partial\Omega$ defined by

$$\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s \qquad \forall t \in \partial\Omega,$$

is a real analytic operator of $]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By Theorem C.4, we immediately deduce that there exists $\epsilon_5 \in ]0,\epsilon_3]$ such that the map of $]-\epsilon_5,\epsilon_5[\times]-\delta_1,\delta_1[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, which takes $(\epsilon,\delta)$ to the function $\int_{\partial\Omega} R^{a,k}_n(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s$ of the variable $t\in\partial\Omega$, is real analytic. Analogously, we have

$$\Big(\frac{\partial u[\epsilon,\delta]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)=\delta\frac{1}{2}\Theta_n[\epsilon,\delta](t)+\delta\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s$$
$$+\delta\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR^{a,k}_n(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s\qquad\forall t\in\partial\Omega,$$

for all $(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[$. Thus, if we set

$$\tilde G[\epsilon,\delta](t)\equiv\frac{1}{2}\Theta_n[\epsilon,\delta](t)+\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s$$
$$+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR^{a,k}_n(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s\qquad\forall t\in\partial\Omega,\ \forall(\epsilon,\delta)\in\,]-\epsilon_5,\epsilon_5[\,\times\,]-\delta_1,\delta_1[$$

then, by arguing as in Proposition 10.54, one can easily show that $\tilde G$ is a real analytic map of $]-\epsilon_5,\epsilon_5[\,\times\,]-\delta_1,\delta_1[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$.

Hence, if we set

$$G[\epsilon,\delta]\equiv-\int_{\partial\Omega}\tilde G[\epsilon,\delta](t)\overline{\int_{\partial\Omega}S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s}\,d\sigma_t$$
$$-\epsilon^{n-2}\int_{\partial\Omega}\tilde G[\epsilon,\delta](t)\overline{\int_{\partial\Omega}R^{a,k}_n(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s}\,d\sigma_t,$$

for all $(\epsilon,\delta)\in\,]-\epsilon_5,\epsilon_5[\,\times\,]-\delta_1,\delta_1[$, then by standard properties of functions in Schauder spaces, we have that $G$ is a real analytic map of $]-\epsilon_5,\epsilon_5[\,\times\,]-\delta_1,\delta_1[$ to $\mathbb{C}$ such that equality (10.89) holds.

Finally, if $(\epsilon,\delta)=(0,0)$, by Folland [52, p. 118] and since $\tilde G[0,0]=g$, we have

$$G[0,0]=-\int_{\partial\Omega}g(t)\overline{\int_{\partial\Omega}S_n(t-s)\tilde\theta(s)\,d\sigma_s}\,d\sigma_t$$
$$=\int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega}|\nabla\tilde u(x)|^2\,dx.$$

□

**Theorem 10.61.** *Let $n$ be even. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k)\neq0$. *Let* $\mathrm{Re}(f(t))\le0$ *for all $t\in\partial\Omega$ and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t))\ge0$ *for all $t\in\partial\Omega$. Let $\tilde\theta$ be as in Lemma 10.8. Let $\epsilon_3$, $\epsilon_2'$, $\delta_1$ be as in Proposition 10.57. Then there exist $\epsilon_5\in\,]0,\epsilon_3]$, and three real analytic operators $G_1^{\#}$, $G_2^{\#}$, $G_3^{\#}$ of $]-\epsilon_5,\epsilon_5[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\delta_1,\delta_1[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]}|\nabla u[\epsilon,\delta](x)|^2\,dx-k^2\int_{\mathbb{P}_a[\Omega_\epsilon]}|u[\epsilon,\delta](x)|^2\,dx$$
$$=\delta^2\epsilon^nG_1^{\#}[\epsilon,\epsilon\log\epsilon,\delta]+\delta^2\epsilon^{2n-2}(\log\epsilon)G_2^{\#}[\epsilon,\epsilon\log\epsilon,\delta]+\delta^2\epsilon^{3n-3}(\log\epsilon)^2G_3^{\#}[\epsilon,\epsilon\log\epsilon,\delta],\tag{10.91}$$

*for all $(\epsilon,\delta)\in\,]0,\epsilon_5[\,\times\,]0,\delta_1[$. Moreover,*

$$G_1^{\#}[0,0,0]=-\int_{\partial\Omega}g(t)\overline{\int_{\partial\Omega}S_n(t-s)\tilde\theta(s)\,d\sigma_s}\,d\sigma_t-\delta_{2,n}\overline{R^{a,k}_n(0)}\Big|\int_{\partial\Omega}g\,d\sigma\Big|^2,\tag{10.92}$$
$$G_2^{\#}[0,0,0]=-\overline{k}^{n-2}\mathcal{J}_n(0)\Big|\int_{\partial\Omega}g\,d\sigma\Big|^2,\tag{10.93}$$
$$G_3^{\#}[0,0,0]=0\tag{10.94}$$

*where $\mathcal{J}_n(0)$ is as in Proposition E.3 (i). In particular, if $n>2$, then*

$$G_1^{\#}[0,0,0]=\int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega}|\nabla\tilde u(x)|^2\,dx,\tag{10.95}$$

*where $\tilde u$ is as in Definition 10.52.*

*Proof.* Let $\Theta_n^\#[\cdot,\cdot,\cdot]$ be as in Proposition 10.57. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $(\epsilon,\delta) \in ]0,\epsilon_3[\times]0,\delta_1[$. Clearly, by the periodicity of $u[\epsilon,\delta]$, we have

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon,\delta](x)|^2\,dx - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon,\delta](x)|^2\,dx = -\epsilon^{n-1}\int_{\partial\Omega}\Big(\frac{\partial u[\epsilon,\delta]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\overline{u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)}\,d\sigma_t.$$

By equality (6.24), we have

$$\begin{aligned} u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) =&\delta\epsilon^{n-1}\int_{\partial\Omega} S_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s\\ =&\delta\epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s\\ &+\delta\epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s\\ &+\delta\epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega. \end{aligned}$$

Thus it is natural to set

$$\begin{aligned} F_1[\epsilon,\epsilon',\delta](t) &\equiv \int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon',\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,\\ F_2[\epsilon,\epsilon',\delta](t) &\equiv k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon',\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,\\ F_3[\epsilon,\epsilon',\delta](t) &\equiv \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon',\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \end{aligned}$$

for all $(\epsilon,\epsilon',\delta) \in ]-\epsilon_3,\epsilon_3[\times]-\epsilon_2',\epsilon_2'[\times]-\delta_1,\delta_1[$.

Then clearly

$$u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \delta\epsilon F_1[\epsilon,\epsilon\log\epsilon,\delta](t) + \delta\epsilon^{n-1}(\log\epsilon)F_2[\epsilon,\epsilon\log\epsilon,\delta](t) + \delta\epsilon^{n-1}F_3[\epsilon,\epsilon\log\epsilon,\delta](t) \qquad \forall t\in\partial\Omega,$$

for all $(\epsilon,\delta) \in ]0,\epsilon_3[\times]0,\delta_1[$. By Theorem E.6 $(i)$ and Theorem C.4, we easily deduce that there exists $\epsilon_5 \in ]0,\epsilon_3]$ such that the maps $F_1$, $F_2$, and $F_3$ of $]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[\times]-\delta_1,\delta_1[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ are real analytic. Analogously, we have

$$\begin{aligned} \Big(\frac{\partial u[\epsilon,\delta]}{\partial\nu_{\Omega_\epsilon}}\Big)&\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\\ &=\delta\frac{1}{2}\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](t)+\delta\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s\\ &+\delta\epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s\\ &+\delta\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\epsilon_3[\times]0,\delta_1[$. Thus, if we set

$$\begin{aligned} \tilde{G}_1[\epsilon,\epsilon',\delta](t) \equiv& \frac{1}{2}\Theta_n^\#[\epsilon,\epsilon',\delta](t)+\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon',\delta](s)\,d\sigma_s\\ &+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon',\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \end{aligned}$$

and

$$\tilde{G}_2[\epsilon,\epsilon',\delta](t) \equiv k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon',\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $(\epsilon,\epsilon',\delta) \in ]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[\times]-\delta_1,\delta_1[$, then, by arguing as in Proposition 10.55, one can easily show that $\tilde{G}_1$ and $\tilde{G}_2$ are real analytic maps of $]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[\times]-\delta_1,\delta_1[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$.

Clearly,

$$\Big(\frac{\partial u[\epsilon,\delta]}{\partial \nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)=\delta\tilde{G}_1[\epsilon,\epsilon\log\epsilon,\delta](t)+\delta\epsilon^{n-1}(\log\epsilon)\tilde{G}_2[\epsilon,\epsilon\log\epsilon,\delta](t)$$
$$\forall t\in\partial\Omega,\ \forall(\epsilon,\delta)\in\,]0,\epsilon_5[\,\times\,]0,\delta_1[.$$

If $(\epsilon,\delta)\in\,]0,\epsilon_5[\,\times\,]0,\delta_1[$, then we have

$$\begin{aligned}
&\int_{\mathbb{P}_a[\Omega_\epsilon]}|\nabla u[\epsilon,\delta](x)|^2\,dx-k^2\int_{\mathbb{P}_a[\Omega_\epsilon]}|u[\epsilon,\delta](x)|^2\,dx\\
&=\delta^2\bigg\{\epsilon^n\Big(-\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon\log\epsilon,\delta]\overline{F_1[\epsilon,\epsilon\log\epsilon,\delta]}\,d\sigma-\epsilon^{n-2}\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon\log\epsilon,\delta]\overline{F_3[\epsilon,\epsilon\log\epsilon,\delta]}\,d\sigma\Big)\\
&\quad+\epsilon^{2n-2}\log\epsilon\Big(-\epsilon\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon\log\epsilon,\delta]\overline{F_1[\epsilon,\epsilon\log\epsilon,\delta]}\,d\sigma-\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon\log\epsilon,\delta]\overline{F_2[\epsilon,\epsilon\log\epsilon,\delta]}\,d\sigma\\
&\qquad-\epsilon^{n-1}\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon\log\epsilon,\delta]\overline{F_3[\epsilon,\epsilon\log\epsilon,\delta]}\,d\sigma\Big)\\
&\quad+\epsilon^{3n-3}(\log\epsilon)^2\Big(-\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon\log\epsilon,\delta]\overline{F_2[\epsilon,\epsilon\log\epsilon,\delta]}\,d\sigma\Big)\bigg\}.
\end{aligned}$$

If we set

$$\begin{aligned}
G_1^{\#}[\epsilon,\epsilon',\delta]\equiv&-\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon',\delta](t)\overline{F_1[\epsilon,\epsilon',\delta](t)}\,d\sigma_t-\epsilon^{n-2}\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon',\delta](t)\overline{F_3[\epsilon,\epsilon',\delta](t)}\,d\sigma_t,\\
G_2^{\#}[\epsilon,\epsilon',\delta]\equiv&-\epsilon\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon',\delta](t)\overline{F_1[\epsilon,\epsilon',\delta](t)}\,d\sigma_t-\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon',\delta](t)\overline{F_2[\epsilon,\epsilon',\delta](t)}\,d\sigma_t\\
&-\epsilon^{n-1}\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon',\delta](t)\overline{F_3[\epsilon,\epsilon',\delta](t)}\,d\sigma_t,\\
G_3^{\#}[\epsilon,\epsilon',\delta]\equiv&-\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon',\delta](t)\overline{F_2[\epsilon,\epsilon',\delta](t)}\,d\sigma_t,
\end{aligned}$$

for all $(\epsilon,\epsilon',\delta)\in\,]-\epsilon_5,\epsilon_5[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\delta_1,\delta_1[$, then standard properties of functions in Schauder spaces and a simple computation show that $G_1^{\#}$, $G_2^{\#}$, and $G_3^{\#}$ are real analytic maps of $]-\epsilon_5,\epsilon_5[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\delta_1,\delta_1[$ in $\mathbb{C}$ such that equality (10.91) holds for all $(\epsilon,\delta)\in\,]0,\epsilon_5[\,\times\,]0,\delta_1[$.

Next, we observe that

$$\begin{aligned}
G_1^{\#}[0,0,0]&=-\int_{\partial\Omega}g(t)\overline{\int_{\partial\Omega}S_n(t-s)\tilde{\theta}(s)\,d\sigma_s}\,d\sigma_t-\delta_{2,n}\overline{R_n^{a,k}(0)}\Big|\int_{\partial\Omega}g\,d\sigma\Big|^2,\\
G_2^{\#}[0,0,0]&=-\overline{k}^{n-2}\overline{Q_n^k(0)}\int_{\partial\Omega}g\,d\sigma\overline{\int_{\partial\Omega}g\,d\sigma},\\
G_3^{\#}[0,0,0]&=-k^{n-2}\overline{k^{n-2}Q_n^k(0)}\int_{\partial\Omega}g\,d\sigma\overline{\int_{\partial\Omega}g\,d\sigma}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(0)\,d\sigma_t=0,
\end{aligned}$$

and accordingly equalities (10.92), (10.93), and (10.94) hold. In particular, if $n\geq 4$, by Folland [52, p. 118], we have

$$G_1^{\#}[0,0,0]=\int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega}|\nabla\tilde{u}(x)|^2\,dx.$$

□

**Theorem 10.62.** *Let $n$ be odd. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k)\neq 0$. *Let* $\mathrm{Re}(f(t))\leq 0$ *for all* $t\in\partial\Omega$ *and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t))\geq 0$ *for all* $t\in\partial\Omega$. *Let $\tilde{\theta}$ be as in Lemma 10.47. Let $\epsilon_3$, $\delta_1$ be as in Proposition* 10.56. *Then there exist $\epsilon_6\in\,]0,\epsilon_3]$, and a real analytic operator $J$ of $]-\epsilon_6,\epsilon_6[\,\times\,]-\delta_1,\delta_1[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]}u[\epsilon,\delta](x)\,dx=\frac{\delta\epsilon^{n-1}}{k^2}J[\epsilon,\delta],\tag{10.96}$$

*for all* $(\epsilon,\delta)\in\,]0,\epsilon_6[\,\times\,]0,\delta_1[$. *Moreover,*

$$J[0,0]=\int_{\partial\Omega} g(x)\,d\sigma_x. \tag{10.97}$$

*Proof.* Let $\Theta_n[\cdot,\cdot]$ be as in Proposition 10.56. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[$. Clearly, by the Divergence Theorem and the periodicity of $u[\epsilon,\delta]$, we have

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,\delta](x)\,dx &= -\frac{1}{k^2}\int_{\mathbb{P}_a[\Omega_\epsilon]}\Delta u[\epsilon,\delta](x)\,dx\\ &= -\frac{1}{k^2}\int_{\partial\mathbb{P}_a[\Omega_\epsilon]}\frac{\partial}{\partial\nu_{\mathbb{P}_a[\Omega_\epsilon]}}u[\epsilon,\delta](x)\,d\sigma_x\\ &= -\frac{1}{k^2}\Big[\int_{\partial A}\frac{\partial}{\partial\nu_A}u[\epsilon,\delta](x)\,d\sigma_x-\int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon,\delta](x)\,d\sigma_x\Big]\\ &= \frac{1}{k^2}\int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon,\delta](x)\,d\sigma_x.\end{aligned}$$

As a consequence,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,\delta](x)\,dx=\frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega}\Big(\frac{\partial u[\epsilon,\delta]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\,d\sigma_t.$$

By equality (6.25) and since $Q_n^k=0$ for $n$ odd, we have

$$\begin{aligned}\Big(\frac{\partial u[\epsilon,\delta]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) &= \delta\frac{1}{2}\Theta_n[\epsilon,\delta](t)+\delta\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s\\ &\quad+\delta\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega.\end{aligned}$$

We set

$$\begin{aligned}\tilde{J}[\epsilon,\delta](t)\equiv&\frac{1}{2}\Theta_n[\epsilon,\delta](t)+\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s\\ &+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s \quad \forall t\in\partial\Omega,\end{aligned}$$

for all $(\epsilon,\delta)\in\,]-\epsilon_3,\epsilon_3[\,\times\,]-\delta_1,\delta_1[$. By Theorem E.6 $(ii)$ and Theorem C.4, one can easily show that there exists $\epsilon_6\in\,]0,\epsilon_3]$ such that $\tilde{J}$ is a real analytic map of $]-\epsilon_6,\epsilon_6[\,\times\,]-\delta_1,\delta_1[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Hence, if we set

$$J[\epsilon,\delta]\equiv\int_{\partial\Omega}\tilde{J}[\epsilon,\delta](t)\,d\sigma_t,$$

for all $(\epsilon,\delta)\in\,]-\epsilon_6,\epsilon_6[\,\times\,]-\delta_1,\delta_1[$, then, by standard properties of functions in Schauder spaces, we have that $J$ is a real analytic map of $]-\epsilon_6,\epsilon_6[\,\times\,]-\delta_1,\delta_1[$ to $\mathbb{C}$, such that equality (10.96) holds.

Finally, if $(\epsilon,\delta)=(0,0)$, we have

$$\begin{aligned}J[0,0]&=\int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega}v^-[\partial\Omega,\tilde{\theta},0](t)\,d\sigma_t\\ &=\int_{\partial\Omega}g(x)\,d\sigma_x,\end{aligned}$$

and accordingly (10.97) holds. □

**Theorem 10.63.** *Let $n$ be even. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k)\neq 0$. *Let* $\mathrm{Re}(f(t))\le 0$ *for all* $t\in\partial\Omega$ *and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t))\ge 0$ *for all* $t\in\partial\Omega$. *Let $\tilde{\theta}$ be as in Lemma 10.47. Let $\epsilon_3$, $\epsilon_2'$, $\delta_1$ be as in Proposition 10.57. Then there exist $\epsilon_6\in\,]0,\epsilon_3]$, and two real analytic operators $J_1^\#$, $J_2^\#$ of $]-\epsilon_6,\epsilon_6[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\delta_1,\delta_1[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,\delta](x)\,dx=\frac{\delta\epsilon^{n-1}}{k^2}J_1^\#[\epsilon,\epsilon\log\epsilon,\delta]+\frac{\delta\epsilon^{2n-2}(\log\epsilon)}{k^2}J_2^\#[\epsilon,\epsilon\log\epsilon,\delta], \tag{10.98}$$

*for all* $(\epsilon,\delta)\in ]0,\epsilon_6[\times]0,\delta_1[$. *Moreover,*

$$J_1^{\#}[0,0,0]=\int_{\partial\Omega} g(x)\,d\sigma_x. \tag{10.99}$$

*Proof.* Let $\Theta_n^{\#}[\cdot,\cdot,\cdot]$ be as in Proposition 10.57. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $(\epsilon,\delta)\in ]0,\epsilon_3[\times]0,\delta_1[$. Clearly, by the Divergence Theorem and the periodicity of $u[\epsilon,\delta]$, we have

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,\delta](x)\,dx &= -\frac{1}{k^2}\int_{\mathbb{P}_a[\Omega_\epsilon]}\Delta u[\epsilon,\delta](x)\,dx\\
&= -\frac{1}{k^2}\int_{\partial\mathbb{P}_a[\Omega_\epsilon]}\frac{\partial}{\partial\nu_{\mathbb{P}_a[\Omega_\epsilon]}}u[\epsilon,\delta](x)\,d\sigma_x\\
&= -\frac{1}{k^2}\Big[\int_{\partial A}\frac{\partial}{\partial\nu_A}u[\epsilon,\delta](x)\,d\sigma_x-\int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon,\delta](x)\,d\sigma_x\Big]\\
&= \frac{1}{k^2}\int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon,\delta](x)\,d\sigma_x.\end{aligned}$$

As a consequence,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,\delta](x)\,dx=\frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega}\Big(\frac{\partial u[\epsilon,\delta]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\,d\sigma_t.$$

By equality (6.25), we have

$$\begin{aligned}&\Big(\frac{\partial u[\epsilon,\delta]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\\
&\quad=\delta\frac{1}{2}\Theta_n^{\#}[\epsilon,\epsilon\log\epsilon,\delta](t)+\delta\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n^{\#}[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s\\
&\quad+\delta\epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s\\
&\quad+\delta\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s\qquad\forall t\in\partial\Omega.\end{aligned}$$

We set

$$\begin{aligned}\tilde J_1[\epsilon,\epsilon',\delta](t)\equiv{}&\frac{1}{2}\Theta_n^{\#}[\epsilon,\epsilon',\delta](t)+\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n^{\#}[\epsilon,\epsilon',\delta](s)\,d\sigma_s\\
&+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon',\delta](s)\,d\sigma_s\qquad\forall t\in\partial\Omega,\end{aligned}$$

and

$$\tilde J_2[\epsilon,\epsilon',\delta](t)\equiv k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon',\delta](s)\,d\sigma_s\qquad\forall t\in\partial\Omega,$$

for all $(\epsilon,\epsilon',\delta)\in ]-\epsilon_3,\epsilon_3[\times]-\epsilon_2',\epsilon_2'[\times]-\delta_1,\delta_1[$. By Theorem E.6 $(ii)$ and Theorem C.4, one can easily show that there exists $\epsilon_6\in]0,\epsilon_3]$ such that $\tilde J_1^{\#}$, $\tilde J_2^{\#}$ are real analytic maps of $]-\epsilon_6,\epsilon_6[\times]-\epsilon_2',\epsilon_2'[\times]-\delta_1,\delta_1[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Hence, if we set

$$J_1^{\#}[\epsilon,\epsilon',\delta]\equiv\int_{\partial\Omega}\tilde J_1^{\#}[\epsilon,\epsilon',\delta](t)\,d\sigma_t,$$

and

$$J_2^{\#}[\epsilon,\epsilon',\delta]\equiv\int_{\partial\Omega}\tilde J_2^{\#}[\epsilon,\epsilon',\delta](t)\,d\sigma_t,$$

for all $(\epsilon,\epsilon',\delta)\in ]-\epsilon_6,\epsilon_6[\times]-\epsilon_2',\epsilon_2'[\times]-\delta_1,\delta_1[$, then, by standard properties of functions in Schauder spaces, we have that $J_1^{\#}$, $J_2^{\#}$ are real analytic maps of $]-\epsilon_6,\epsilon_6[\times]-\epsilon_2',\epsilon_2'[\times]-\delta_1,\delta_1[$ to $\mathbb{C}$, such that equality (10.98) holds.

Finally, if $\epsilon = \epsilon' = \delta = 0$, we have

$$J_1^{\#}[0,0,0] = \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega} v^-[\partial\Omega, \tilde{\theta}, 0](t)\, d\sigma_t = \int_{\partial\Omega} g(x)\, d\sigma_x,$$

and accordingly (10.99) holds. □

We have the following.

**Proposition 10.64.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all $t \in \partial\Omega$. Let $\epsilon_3$, $\delta_1$ be as in Proposition 10.56. Then there exist $\tilde{\epsilon} \in ]0,\epsilon_3[$ and a real analytic map $N$ of $]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\delta_1,\delta_1[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$ such that*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\|\mathrm{Re}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\|\mathrm{Im}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\delta_1[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* Let $\epsilon_3$, $\delta_1$, $\Theta_n$ be as in Proposition 10.56. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $(\epsilon,\delta) \in ]0,\epsilon_3[\times]0,\delta_1[$, we have

$$u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \delta\epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s \qquad \forall t \in \partial\Omega.$$

We set

$$N[\epsilon,\delta](t) \equiv \int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s + \epsilon^{n-2}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $(\epsilon,\delta) \in ]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[$. By taking $\tilde{\epsilon} \in ]0,\epsilon_3[$ small enough, we can assume (cf. Theorem C.4 and the proof of Theorem 10.60) that $N$ is a real analytic map of $]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\delta_1,\delta_1[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$. By Corollary 6.24, we have

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\|\mathrm{Re}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)} \qquad \forall(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\delta_1[,$$

and

$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\|\mathrm{Im}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)} \qquad \forall(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\delta_1[.$$

Accordingly,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and so the conclusion follows. □

**Proposition 10.65.** *Let $n$ be even. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all $t \in \partial\Omega$. Let $\epsilon_3$, $\epsilon_2'$, $\delta_1$ be as in Proposition 10.57. Then there exist $\tilde{\epsilon} \in ]0,\epsilon_3[$ and two real analytic maps $N_1^{\#}$, $N_2^{\#}$ of $]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\epsilon_2',\epsilon_2'[\times]-\delta_1,\delta_1[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$ such that*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\|\mathrm{Re}\big(\epsilon N_1^{\#}[\epsilon,\epsilon\log\epsilon,\delta] + \epsilon^{n-1}(\log\epsilon)N_2^{\#}[\epsilon,\epsilon\log\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\|\mathrm{Im}\big(\epsilon N_1^{\#}[\epsilon,\epsilon\log\epsilon,\delta] + \epsilon^{n-1}(\log\epsilon)N_2^{\#}[\epsilon,\epsilon\log\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\delta_1[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* Let $\epsilon_3$, $\epsilon_2'$, $\delta_1$, $\Theta_n^\#$ be as in Proposition 10.57. If $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[$, we have

$$\begin{aligned} u[\epsilon,\delta] \circ (w+\epsilon\, \mathrm{id}_{\partial\Omega})(t) =& \delta\epsilon \int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](s)\, d\sigma_s \\ &+ \delta\epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](s)\, d\sigma_s \\ &+ \delta\epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](s)\, d\sigma_s \qquad \forall t\in\partial\Omega. \end{aligned}$$

We set

$$\begin{aligned} N_1^\#[\epsilon,\epsilon',\delta](t) \equiv& \int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon',\delta](s)\, d\sigma_s \\ &+ \epsilon^{n-2}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon',\delta](s)\, d\sigma_s \qquad \forall t\in\partial\Omega, \end{aligned}$$

and

$$N_2^\#[\epsilon,\epsilon',\delta](t) \equiv k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon',\delta](s)\, d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $(\epsilon,\epsilon',\delta) \in ]-\epsilon_3,\epsilon_3[ \times ]-\epsilon_2',\epsilon_2'[ \times ]-\delta_1,\delta_1[$. By taking $\tilde\epsilon \in ]0,\epsilon_3[$ small enough, we can assume (cf. Theorem C.4 and the proof of Theorem 10.61) that $N_1^\#$, $N_2^\#$ are real analytic maps of $]-\tilde\epsilon,\tilde\epsilon[ \times ]-\epsilon_2',\epsilon_2'[ \times ]-\delta_1,\delta_1[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$. Clearly,

$$u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \delta\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon,\delta](t) + \delta\epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon,\delta](t) \qquad \forall t\in\partial\Omega, \quad \forall(\epsilon,\delta)\in]0,\tilde\epsilon[\times]0,\delta_1[.$$

By Corollary 6.24, we have

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon,\delta] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},$$

and

$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon,\delta] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},$$

for all $(\epsilon,\delta)\in ]0,\tilde\epsilon[\times]0,\delta_1[$.

Accordingly,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and so the conclusion follows. □

### 10.4.2 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorems we deduce by Propositions 10.64, 10.65 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 10.66.** *Let $n$ be odd. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k)\neq 0$ *and* $\mathrm{Re}(k)=0$. *Let* $\mathrm{Re}(f(t))\le 0$ *for all $t\in\partial\Omega$. Let $\delta_1$ be as in Proposition 10.56. Let $\tilde\epsilon$, $N$ be as in Proposition 10.64. Then*

$$\begin{aligned} \|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \delta\epsilon\|\mathrm{Re}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)}, \\ \|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \delta\epsilon\|\mathrm{Im}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)}, \end{aligned}$$

*for all $(\epsilon,\delta)\in]0,\tilde\epsilon[\times]0,\delta_1[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \textit{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \delta\epsilon\|\mathrm{Re}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \delta\epsilon\|\mathrm{Im}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[ \times ]0,\delta_1[$. □

**Theorem 10.67.** *Let $n$ be even. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all $t \in \partial\Omega$. Let $\delta_1$ be as in Proposition 10.57. Let $\tilde{\epsilon}$, $N_1^\#$, $N_2^\#$ be as in Proposition 10.65. Then*

$$\begin{aligned}\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \delta\|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon,\delta] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},\\ \|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \delta\|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon,\delta] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[ \times ]0,\delta_1[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \textit{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \delta\|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon,\delta] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \delta\|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon,\delta] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[ \times ]0,\delta_1[$. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 10.68.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let* $\mathrm{Im}(k) \neq 0$. *Let* $\mathrm{Re}(f(t)) \leq 0$ *for all $t \in \partial\Omega$ and* $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ *for all $t \in \partial\Omega$. Let $\delta_1$ be as in Proposition 10.56. Let $\epsilon_6$, $J$ be as in Theorem 10.62. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l}\frac{\epsilon^{n-1}}{k^2}J\big[\epsilon,\frac{r}{l}\big], \tag{10.100}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N}\setminus\{0\}$ such that $l > (r/\delta_1)$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N}\setminus\{0\}$ such that $l > (r/\delta_1)$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx\\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx\\ &= l^n\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,(r/l)]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,(r/l)](t)\,dt \\ &= \frac{r^n}{l^n}\frac{r}{l}\frac{\epsilon^{n-1}}{k^2}J\big[\epsilon,\frac{r}{l}\big].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l}\frac{\epsilon^{n-1}}{k^2}J\big[\epsilon,\frac{r}{l}\big],$$

and the conclusion follows. □

**Theorem 10.69.** *Let $n$ be even. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. Let $\delta_1$ be as in Proposition 10.57. Let $\epsilon_6$, $J_1^\#$, $J_2^\#$ be as in Theorem 10.63. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l}\Big\{\frac{\epsilon^{n-1}}{k^2}J_1^\#\big[\epsilon,\epsilon\log\epsilon,\frac{r}{l}\big] + \frac{\epsilon^{2n-2}(\log\epsilon)}{k^2}J_2^\#\big[\epsilon,\epsilon\log\epsilon,\frac{r}{l}\big]\Big\}, \tag{10.101}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N}\setminus\{0\}$ such that $l > (r/\delta_1)$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N}\setminus\{0\}$ such that $l > (r/\delta_1)$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= l^n\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,(r/l)]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,(r/l)](t)\,dt \\ &= \frac{r^n}{l^n}\frac{r}{l}\Big\{\frac{\epsilon^{n-1}}{k^2}J_1^\#\big[\epsilon,\epsilon\log\epsilon,\frac{r}{l}\big] + \frac{\epsilon^{2n-2}(\log\epsilon)}{k^2}J_2^\#\big[\epsilon,\epsilon\log\epsilon,\frac{r}{l}\big]\Big\}.\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l}\Big\{\frac{\epsilon^{n-1}}{k^2}J_1^\#\big[\epsilon,\epsilon\log\epsilon,\frac{r}{l}\big] + \frac{\epsilon^{2n-2}(\log\epsilon)}{k^2}J_2^\#\big[\epsilon,\epsilon\log\epsilon,\frac{r}{l}\big]\Big\},$$

and the conclusion follows. □

We give the following.

**Definition 10.70.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. For each pair $(\epsilon,\delta) \in ]0,\epsilon_1[\times]0,+\infty[$, we set

$$\mathcal{F}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} \left|\nabla u_{(\epsilon,\delta)}(x)\right|^2 dx - \frac{k^2}{\delta^2}\int_{A\cap\mathbb{T}_a(\epsilon,\delta)} \left|u_{(\epsilon,\delta)}(x)\right|^2 dx.$$

*Remark* 10.71. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in (1.56), (1.57), (10.10), (10.11), (10.12), respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. Let $(\epsilon, \delta) \in ]0, \epsilon_1[ \times ]0, +\infty[$. We have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2\,dt \\ &= \delta^{n-2} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon,\delta](t)|^2\,dt,\end{aligned}$$

and

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2\,dx = \delta^n \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon,\delta](t)|^2\,dt.$$

Accordingly,

$$\begin{aligned}&\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx - \frac{k^2}{\delta^2}\int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2\,dx \\ &\qquad = \delta^{n-2}\Big(\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon,\delta](t)|^2\,dt - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon,\delta](t)|^2\,dt\Big).\end{aligned}$$

In the following Propositions we represent the function $\mathcal{F}(\cdot,\cdot)$ by means of real analytic functions.

**Proposition 10.72.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. Let $\delta_1$ be as in Proposition 10.56. Let $\epsilon_5$, $G$ be as in Theorem 10.22. Then*

$$\mathcal{F}\Big(\epsilon, \frac{1}{l}\Big) = \epsilon^n G[\epsilon, (1/l)],$$

*for all $\epsilon \in ]0, \epsilon_5[$ and for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* Let $(\epsilon, \delta) \in ]0, \epsilon_5[ \times ]0, \delta_1[$. By Remark 10.71 and Theorem 10.60, we have

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx - \frac{k^2}{\delta^2}\int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2\,dx = \delta^n \epsilon^n G[\epsilon,\delta]$$

where $G$ is as in Theorem 8.23. On the other hand, if $\epsilon \in ]0, \epsilon_5[$ and $l \in \mathbb{N}$ is such that $l > (1/\delta_1)$, then we have

$$\begin{aligned}\mathcal{F}\Big(\epsilon, \frac{1}{l}\Big) &= l^n \frac{1}{l^n} \epsilon^n G[\epsilon, (1/l)], \\ &= \epsilon^n G[\epsilon, (1/l)],\end{aligned}$$

and the conclusion easily follows. □

**Proposition 10.73.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $f$, $g$ be as in* (1.56), (1.57), (10.10), (10.11), (10.12), *respectively. Let $\mathrm{Im}(k) \neq 0$. Let $\mathrm{Re}(f(t)) \leq 0$ for all $t \in \partial\Omega$ and $\mathrm{Re}(k)\,\mathrm{Im}(k)\,\mathrm{Im}(f(t)) \geq 0$ for all $t \in \partial\Omega$. Let $\delta_1$ be as in Proposition 10.57. Let $\epsilon_5$, $G_1^\#$, $G_2^\#$, and $G_3^\#$ be as in Theorem 10.23. Then*

$$\begin{aligned}\mathcal{F}\Big(\epsilon, \frac{1}{l}\Big) =& \epsilon^n G_1^\#[\epsilon, \epsilon\log\epsilon, (1/l)] \\ &+ \epsilon^{2n-2}(\log\epsilon) G_2^\#[\epsilon, \epsilon\log\epsilon, (1/l)] \\ &+ \epsilon^{3n-3}(\log\epsilon)^2 G_3^\#[\epsilon, \epsilon\log\epsilon, (1/l)],\end{aligned}$$

*for all $\epsilon \in ]0, \epsilon_5[$ and for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* Let $(\epsilon, \delta) \in ]0, \epsilon_5[ \times ]0, \delta_1[$. By Remark 10.71 and Theorem 10.61, we have

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2 \, dx$$
$$= \delta^n \Big\{ \epsilon^n G_1^{\#}[\epsilon, \epsilon \log \epsilon, \delta] + \epsilon^{2n-2} (\log \epsilon) G_2^{\#}[\epsilon, \epsilon \log \epsilon, \delta] + \epsilon^{3n-3} (\log \epsilon)^2 G_3^{\#}[\epsilon, \epsilon \log \epsilon, \delta] \Big\},$$

where $G_1^{\#}$, $G_2^{\#}$, and $G_3^{\#}$ are as in Theorem 8.24. On the other hand, if $\epsilon \in ]0, \epsilon_5[$ and $l \in \mathbb{N}$ is such that $l > (1/\delta_1)$, then we have

$$\begin{aligned} \mathcal{F}\Big(\epsilon, \frac{1}{l}\Big) =& l^n \frac{1}{l^n} \Big\{ \epsilon^n G_1^{\#}[\epsilon, \epsilon \log \epsilon, (1/l)] \\ & + \epsilon^{2n-2} (\log \epsilon) G_2^{\#}[\epsilon, \epsilon \log \epsilon, (1/l)] \\ & + \epsilon^{3n-3} (\log \epsilon)^2 G_3^{\#}[\epsilon, \epsilon \log \epsilon, (1/l)] \Big\}, \\ =& \epsilon^n G_1^{\#}[\epsilon, \epsilon \log \epsilon, (1/l)] \\ & + \epsilon^{2n-2} (\log \epsilon) G_2^{\#}[\epsilon, \epsilon \log \epsilon, (1/l)] \\ & + \epsilon^{3n-3} (\log \epsilon)^2 G_3^{\#}[\epsilon, \epsilon \log \epsilon, (1/l)], \end{aligned}$$

and the conclusion easily follows. □

## 10.5 Asymptotic behaviour of the solutions of a nonlinear Robin problem for the Helmholtz equation in a periodically perforated domain

In this Section we study the asymptotic behaviour of the solutions of a nonlinear Robin problem for the Helmholtz equation in a periodically perforated domain with small holes.

### 10.5.1 Notation and preliminaries

We retain the notation introduced in Subsections 1.8.1, 6.7.1, 10.2.1. However, we need to introduce also some other notation. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $\Omega$ be as in (1.56). If $F \in C^0(\partial\Omega \times \mathbb{C}, \mathbb{C})$, then we denote by $T_F$ the (nonlinear nonautonomous) composition operator of $C^0(\partial\Omega, \mathbb{C})$ to itself which maps $v \in C^0(\partial\Omega, \mathbb{C})$ to the function $T_F[v]$ of $\partial\Omega$ to $\mathbb{C}$, defined by

$$T_F[v](t) \equiv F(t, v(t)) \qquad \forall t \in \partial\Omega.$$

Then we shall consider also the following assumptions.

$$k \in \mathbb{C}, \; k^2 \neq |2\pi a^{-1}(z)|^2 \quad \forall z \in \mathbb{Z}^n; \tag{10.102}$$

$$F \in C^0(\partial\Omega \times \mathbb{C}, \mathbb{C}) \text{ and } T_F \text{ is a real analytic map of } C^{m-1,\alpha}(\partial\Omega, \mathbb{C}) \text{ to itself.} \tag{10.103}$$

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$ be as in (1.56), (1.57), (10.102), respectively. By Proposition 7.42, there exists $\epsilon_1^* \in ]0, \epsilon_1[$ such that

$$k^2 \notin \Big( \mathrm{Eig}_D[\Omega_\epsilon] \cup \mathrm{Eig}_N[\Omega_\epsilon] \cup \mathrm{Eig}_D^a[\mathbb{T}_a[\Omega_\epsilon]] \cup \mathrm{Eig}_N^a[\mathbb{T}_a[\Omega_\epsilon]] \Big) \qquad \forall \epsilon \in ]0, \epsilon_1^*]. \tag{10.104}$$

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). For each $\epsilon \in ]0, \epsilon_1^*[$, we consider the following periodic nonlinear Robin problem for the Helmholtz equation.

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x + a_j) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1, \dots, n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) + F\big(\frac{1}{\epsilon}(x - w), u(x)\big) = 0 & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{10.105}$$

We now convert our boundary value problem (10.105) into an integral equation.

**Proposition 10.74.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon \in ]0, \epsilon_1^*[$. Then the map of the set of functions $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ that solve the equation*

$$\frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \nu_{\Omega_\epsilon}(x) \cdot DS_n^{a,k}(x-y)\mu(y)\, d\sigma_y + F\Big(\frac{1}{\epsilon}(x-w), \int_{\partial\Omega_\epsilon} S_n^{a,k}(x-y)\mu(y)\, d\sigma_y\Big) = 0 \qquad \forall x \in \partial\Omega_\epsilon, \quad (10.106)$$

*to the set of $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$ which solve problem* (10.105), *which takes $\mu$ to the function*

$$v_a^-[\partial\Omega_\epsilon, \mu, k] \quad (10.107)$$

*is a bijection.*

*Proof.* Assume that the function $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ solves equation (10.106). Then, by Theorem 6.11, we immediately deduce that the function $u \equiv v_a^-[\partial\Omega_\epsilon, \mu, k]$ is a periodic function in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$, that satisfies the first condition of (10.105), and, by equation (10.106), also the third condition of (10.105). Thus, $u$ is a solution of (10.105). Conversely, let $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$ be a solution of problem (10.105). By Theorem 8.6, there exists a unique function $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$, such that

$$u = v_a^-[\partial\Omega_\epsilon, \mu, k] \qquad \text{in } \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$

Then, by Theorem 6.11, since $u$ satisfies in particular the third condition in (10.105), we immediately deduce that the function $\mu$ solves equation (10.106). □

As we have seen, we can transform (10.105) into an integral equation defined on the $\epsilon$-dependent domain $\partial\Omega_\epsilon$. In order to get rid of such a dependence, we shall introduce the following Theorem.

**Theorem 10.75.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon \in ]0, \epsilon_1^*[$. Then the map $u[\epsilon, \cdot]$ of the set of functions $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ that solve the equation*

$$\begin{aligned}
&\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k)\theta(s)\, d\sigma_s + \epsilon^{n-1}(\log\epsilon)k^{n-2} \int_{\partial\Omega} \nu_\Omega(t) \cdot DQ_n^k(\epsilon(t-s))\theta(s)\, d\sigma_s \\
&+ \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s + F\Big(t, \epsilon \int_{\partial\Omega} S_n(t-s, \epsilon k)\theta(s)\, d\sigma_s \\
&+ \epsilon^{n-1}(\log\epsilon)k^{n-2} \int_{\partial\Omega} Q_n^k(\epsilon(t-s))\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s\Big) = 0 \ \forall t \in \partial\Omega,
\end{aligned} \quad (10.108)$$

*to the set of $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$ which solve problem* (10.105), *which takes $\theta$ to the function*

$$u[\epsilon, \theta] \equiv v_a^-[\partial\Omega_\epsilon, \theta(\frac{1}{\epsilon}(\cdot - w)), k] \quad (10.109)$$

*is a bijection.*

*Proof.* It is an immediate consequence of Proposition 10.74, of the Theorem of change of variables in integrals, and of equalities (6.24), (6.25). □

In the following Proposition we study equation (10.108) for $\epsilon = 0$.

**Proposition 10.76.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $F$ be as in* (1.56), (10.103), *respectively. Then the integral equation*

$$\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\theta(s)\, d\sigma_s + F(t, 0) = 0 \qquad \forall t \in \partial\Omega, \quad (10.110)$$

*which we call the* limiting equation, *has a unique solution $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, which we denote by $\tilde{\theta}$. Moreover,*

$$\int_{\partial\Omega} \tilde{\theta}(s)\, d\sigma_s = -\int_{\partial\Omega} F(s, 0)\, d\sigma_s. \quad (10.111)$$

*Proof.* The existence and uniqueness of a solution of equation (10.110) is a well known result of classic potential theory (cf. Folland [52, Chapter 3] for the existence and uniqueness of a solution in $L^2(\partial\Omega,\mathbb{C})$ and, *e.g.*, Theorem B.3 for the regularity.) Equality (10.111) follows by Folland [52, Lemma 3.30, p. 133]. □

Now we want to see if equation (10.110) is related to some (limiting) boundary value problem. We give the following.

**Definition 10.77.** Let $n \geq 3$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $F$ be as in (1.56), (10.103), respectively. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega,\mathbb{C})$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega, \\ \frac{\partial}{\partial \nu_\Omega} u(x) = -F(x,0) & \forall x \in \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{10.112}$$

Problem (10.112) will be called the *limiting boundary value problem.*

*Remark* 10.78. Let $n \geq 3$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $F$ be as in (1.56), (10.103), respectively. Let $\tilde{\theta}$ be as in Proposition 10.76. We have

$$\tilde{u}(x) = \int_{\partial\Omega} S_n(x-y)\tilde{\theta}(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \Omega.$$

If $n = 2$, in general the (classic) simple layer potential for the Laplace equation with moment $\tilde{\theta}$ is not harmonic at infinity, and it does not satisfy the third condition of boundary value problem (10.112). Moreover, if $n = 2$, boundary value problem (10.112) does not have in general a solution (unless $\int_{\partial\Omega} F(s,0)\,d\sigma_s = 0$.) However, the function $\tilde{v}$ of $\mathbb{R}^2 \setminus \Omega$ to $\mathbb{C}$, defined by

$$\tilde{v}(x) \equiv \int_{\partial\Omega} S_2(x-y)\tilde{\theta}(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^2 \setminus \Omega,$$

is a solution of the following boundary value problem

$$\begin{cases} \Delta \tilde{v}(x) = 0 & \forall x \in \mathbb{R}^2 \setminus \mathrm{cl}\,\Omega, \\ \frac{\partial}{\partial \nu_\Omega} \tilde{v}(x) = -F(x,0) & \forall x \in \partial\Omega. \end{cases} \tag{10.113}$$

We are now ready to analyse equation (10.108) around the degenerate case $\epsilon = 0$. However, since the function $Q_n^k$ that appears in equation (10.108) (involved in the determination of the moment of the simple layer potential that solves (10.105)) is identically 0 if $n$ is odd, it is preferable to treat separately case $n$ even and case $n$ odd.

**Theorem 10.79.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\tilde{\theta}$ be as in Proposition 10.76. Let $\Lambda$ be the map of $]-\epsilon_1^*,\epsilon_1^*[ \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, defined by*

$$\begin{aligned}\Lambda[\epsilon,\theta](t) \equiv& \frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^n} S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s \\ &+ F\Big(t, \epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s\Big) \ \forall t \in \partial\Omega,\end{aligned} \tag{10.114}$$

*for all $(\epsilon,\theta) \in ]-\epsilon_1^*,\epsilon_1^*[ \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Then the following statements hold.*

*(i) Equation $\Lambda[0,\theta] = 0$ is equivalent to the limiting equation* (10.110) *and has one and only one solution $\tilde{\theta}$ in $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ (cf. Proposition 10.76.)*

*(ii) If $\epsilon \in ]0,\epsilon_1^*[$, then equation $\Lambda[\epsilon,\theta] = 0$ is equivalent to equation* (10.108) *for $\theta$.*

*(iii) There exists $\epsilon_2 \in ]0,\epsilon_1^*]$, such that the map $\Lambda$ of $]-\epsilon_2,\epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ is real analytic. Moreover, the differential $\partial_\theta\Lambda[0,\tilde{\theta}]$ of $\Lambda$ at $(0,\tilde{\theta})$ is a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ onto $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$.*

*(iv) There exist $\epsilon_3 \in ]0, \epsilon_2]$, an open neighbourhood $\tilde{\mathcal{U}}$ of $\tilde{\theta}$ in $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ and a real analytic map $\Theta_n[\cdot]$ of $]-\epsilon_3, \epsilon_3[$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, such that the set of zeros of the map $\Lambda$ in $]-\epsilon_3, \epsilon_3[ \times \tilde{\mathcal{U}}$ coincides with the graph of $\Theta_n[\cdot]$. In particular, $\Theta_n[0] = \tilde{\theta}$.*

*Proof.* Statements $(i)$ and $(ii)$ are obvious. We now prove statement $(iii)$. We set

$$\begin{aligned}\Lambda'[\epsilon, \theta](t) \equiv &\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k)\theta(s)\, d\sigma_s \\ &+ \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s \quad \forall t \in \partial\Omega,\end{aligned}$$

$$\Lambda''[\epsilon, \theta](t) \equiv F\Big(t, \epsilon \int_{\partial\Omega} S_n(t-s, \epsilon k)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s\Big) \quad \forall t \in \partial\Omega,$$

for all $(\epsilon, \theta) \in ]-\epsilon_1^*, \epsilon_1^*[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Clearly,

$$\Lambda[\epsilon, \theta] = \Lambda'[\epsilon, \theta] + \Lambda''[\epsilon, \theta],$$

for all $(\epsilon, \theta) \in ]-\epsilon_1^*, \epsilon_1^*[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. By Proposition 6.21 $(ii)$, we immediately deduce that there exists $\epsilon_2 \in ]0, \epsilon_1^*]$ such that $\Lambda'$ is a real analytic map of $]-\epsilon_2, \epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Analogously, by Proposition 6.21 $(i)$, we easily deduce that the map of $]-\epsilon_2, \epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, which takes $(\epsilon, \theta)$ to the function of $\partial\Omega$ to $\mathbb{C}$ defined by

$$\epsilon \int_{\partial\Omega} S_n(t-s, \epsilon k)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s \quad \forall t \in \partial\Omega,$$

is real analytic. Thus, by hypothesis (10.103) and standard calculus in Banach spaces, $\Lambda''$ is a real analytic operator of $]-\epsilon_2, \epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Hence $\Lambda$ is a real analytic map of $]-\epsilon_2, \epsilon_2[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. By standard calculus in Banach spaces, the differential $\partial_\theta \Lambda[0, \tilde{\theta}]$ of $\Lambda$ at $(0, \tilde{\theta})$ is delivered by the following formula:

$$\partial_\theta \Lambda[0, \tilde{\theta}](\tau)(t) = \frac{1}{2}\tau(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\tau(s)\, d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $\tau \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. We now show that the above differential is a linear homeomorphism. By the Open Mapping Theorem, it suffices to show that it is a bijection of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ onto $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Let $\psi \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. We must show that there exists a unique function $\tau$ in $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, such that

$$\partial_\theta \Lambda[0, \tilde{\theta}](\tau) = \psi.$$

By known results of classical potential theory (cf. Folland [52, Chapter 3]), there exists a unique function $\tau \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, such that

$$\frac{1}{2}\tau(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\tau(s)\, d\sigma_s = \psi(t) \qquad \forall t \in \partial\Omega.$$

Hence $\partial_\theta \Lambda[0, \tilde{\theta}]$ is bijective, and, accordingly, a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ onto itself. Thus the proof of $(iii)$ is now concluded. Finally, statement $(iv)$ is an immediate consequence of statement $(iii)$ and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

**Theorem 10.80.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\tilde{\theta}$ be as in Proposition 10.76. Let $\epsilon_1' > 0$ be such that*

$$\epsilon \log \epsilon \in ]-\epsilon_1', \epsilon_1'[ \qquad \forall \epsilon \in ]0, \epsilon_1^*[. \tag{10.115}$$

*Let $\Lambda^\#$ be the map of $]-\epsilon_1^*, \epsilon_1^*[ \times ]-\epsilon_1', \epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, defined by*

$$\begin{aligned}&\Lambda^\#[\epsilon, \epsilon', \theta](t) \\ &\equiv \frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k)\theta(s)\, d\sigma_s + \epsilon^{n-2}\epsilon' k^{n-2} \int_{\partial\Omega} \nu_\Omega(t) \cdot DQ_n^k(\epsilon(t-s))\theta(s)\, d\sigma_s \\ &+ \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s + F\Big(t, \epsilon \int_{\partial\Omega} S_n(t-s, \epsilon k)\theta(s)\, d\sigma_s \\ &+ \epsilon^{n-2}\epsilon' k^{n-2} \int_{\partial\Omega} Q_n^k(\epsilon(t-s))\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s\Big) \; \forall t \in \partial\Omega,\end{aligned} \tag{10.116}$$

*for all* $(\epsilon,\epsilon',\theta) \in ]-\epsilon_1^*,\epsilon_1^*[ \times ]-\epsilon_1',\epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$*. Then the following statements hold.*

*(i) Equation* $\Lambda^{\#}[0,0,\theta]=0$ *is equivalent to the limiting equation* (10.110) *and has one and only one solution* $\tilde{\theta}$ *in* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *(cf. Proposition 10.76.)*

*(ii) If* $\epsilon \in ]0,\epsilon_1^*[$*, then equation* $\Lambda^{\#}[\epsilon,\epsilon\log\epsilon,\theta]=0$ *is equivalent to equation* (10.108) *for* $\theta$*.*

*(iii) There exists* $\epsilon_2 \in ]0,\epsilon_1^*]$*, such that the map* $\Lambda^{\#}$ *of* $]-\epsilon_2,\epsilon_2[ \times ]-\epsilon_1',\epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *to* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *is real analytic. Moreover, the differential* $\partial_\theta\Lambda^{\#}[0,0,\tilde{\theta}]$ *of* $\Lambda^{\#}$ *at* $(0,0,\tilde{\theta})$ *is a linear homeomorphism of* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *onto* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$*.*

*(iv) There exist* $\epsilon_3 \in ]0,\epsilon_2]$*,* $\epsilon_2' \in ]0,\epsilon_1']$*, an open neighbourhood* $\tilde{\mathcal{U}}$ *of* $\tilde{\theta}$ *in* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *and a real analytic map* $\Theta_n^{\#}[\cdot,\cdot]$ *of* $]-\epsilon_3,\epsilon_3[ \times ]-\epsilon_2',\epsilon_2'[$ *to* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$*, such that* $\epsilon\log\epsilon \in ]-\epsilon_2',\epsilon_2'[$ *for all* $\epsilon \in ]0,\epsilon_3[$ *and such that the set of zeros of the map* $\Lambda^{\#}$ *in* $]-\epsilon_3,\epsilon_3[ \times ]-\epsilon_2',\epsilon_2'[ \times \tilde{\mathcal{U}}$ *coincides with the graph of* $\Theta_n^{\#}[\cdot,\cdot]$*. In particular,* $\Theta_n^{\#}[0,0]=\tilde{\theta}$*.*

*Proof.* Statements *(i)* and *(ii)* are obvious. We now prove statement *(iii)*. We set

$$\begin{aligned}\Lambda'^{\#}[\epsilon,\epsilon',\theta](t) \equiv&\frac{1}{2}\theta(t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s\\ &+\epsilon^{n-2}\epsilon' k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\theta(s)\,d\sigma_s\\ &+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s \quad \forall t\in\partial\Omega,\end{aligned}$$

$$\begin{aligned}\Lambda''^{\#}[\epsilon,\epsilon',\theta](t) \equiv& F\Big(t,\epsilon\int_{\partial\Omega}S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s + \epsilon^{n-2}\epsilon' k^{n-2}\int_{\partial\Omega}Q_n^k(\epsilon(t-s))\theta(s)\,d\sigma_s\\ &+\epsilon^{n-1}\int_{\partial\Omega}R_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s\Big) \quad \forall t\in\partial\Omega,\end{aligned}$$

for all $(\epsilon,\epsilon',\theta) \in ]-\epsilon_1^*,\epsilon_1^*[ \times ]-\epsilon_1',\epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Clearly,

$$\Lambda^{\#}[\epsilon,\epsilon',\theta] = \Lambda'^{\#}[\epsilon,\epsilon',\theta] + \Lambda''^{\#}[\epsilon,\epsilon',\theta],$$

for all $(\epsilon,\epsilon',\theta) \in ]-\epsilon_1^*,\epsilon_1^*[ \times ]-\epsilon_1',\epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By Proposition 6.21 *(ii)*, we immediately deduce that there exists $\epsilon_2 \in ]0,\epsilon_1^*]$ such that $\Lambda'^{\#}$ is a real analytic map of $]-\epsilon_2,\epsilon_2[ \times ]-\epsilon_1',\epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Analogously, by Proposition 6.21 *(i)*, we easily deduce that the map of $]-\epsilon_2,\epsilon_2[ \times ]-\epsilon_1',\epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, which takes $(\epsilon,\epsilon',\theta)$ to the function of $\partial\Omega$ to $\mathbb{C}$ defined by

$$\begin{aligned}\epsilon\int_{\partial\Omega}S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s + \epsilon^{n-2}\epsilon' k^{n-2}\int_{\partial\Omega}Q_n^k(\epsilon(t-s))\theta(s)\,d\sigma_s&\\ +\epsilon^{n-1}\int_{\partial\Omega}R_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s \quad \forall t\in\partial\Omega,&\end{aligned}$$

is real analytic. Thus, by hypothesis (10.103) and standard calculus in Banach spaces, $\Lambda''^{\#}$ is a real analytic operator of $]-\epsilon_2,\epsilon_2[ \times ]-\epsilon_1',\epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Hence $\Lambda^{\#}$ is a real analytic map of $]-\epsilon_2,\epsilon_2[ \times ]-\epsilon_1',\epsilon_1'[ \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By standard calculus in Banach spaces, the differential $\partial_\theta\Lambda^{\#}[0,0,\tilde{\theta}]$ of $\Lambda^{\#}$ at $(0,0,\tilde{\theta})$ is delivered by the following formula:

$$\partial_\theta\Lambda^{\#}[0,0,\tilde{\theta}](\tau)(t) = \frac{1}{2}\tau(t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot DS_n(t-s)\tau(s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $\tau \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. We now show that the above differential is a linear homeomorphism. By the Open Mapping Theorem, it suffices to show that it is a bijection of $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ onto $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Let $\psi \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. We must show that there exists a unique function $\tau$ in $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, such that

$$\partial_\theta\Lambda^{\#}[0,0,\tilde{\theta}](\tau) = \psi.$$

By known results of classical potential theory (cf. Folland [52, Chapter 3]), there exists a unique function $\tau \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, such that

$$\frac{1}{2}\tau(t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot DS_n(t-s)\tau(s)\,d\sigma_s = \psi(t) \qquad \forall t\in\partial\Omega.$$

Hence $\partial_\theta \Lambda^{\#}[0,0,\tilde{\theta}]$ is bijective, and, accordingly, a linear homeomorphism of $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ onto itself. Thus the proof of *(iii)* is now concluded. Finally, statement *(iv)* is an immediate consequence of statement *(iii)* and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

We are now in the position to introduce the following.

**Definition 10.81.** Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $u[\cdot,\cdot]$ be as in Theorem 10.75. If $n$ is odd and $\epsilon \in ]0,\epsilon_3[$, we set

$$u[\epsilon](x) \equiv u[\epsilon,\Theta_n[\epsilon]](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],$$

where $\epsilon_3$, $\Theta_n$ are as in Theorem 10.79 *(iv)*. If $n$ is even and $\epsilon \in ]0,\epsilon_3[$, we set

$$u[\epsilon](x) \equiv u[\epsilon,\Theta_n^{\#}[\epsilon,\epsilon\log\epsilon]](x) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],$$

where $\epsilon_3$, $\Theta_n^{\#}$ are as in Theorem 10.80 *(iv)*.

*Remark* 10.82. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $\epsilon_3$ be as in Theorem 10.79 *(iv)* if $n$ is odd and as in Theorem 10.80 *(iv)* if $n$ is even. Let $\epsilon \in ]0,\epsilon_3[$. Then $u[\epsilon]$ is a solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],\mathbb{C})$ of problem (10.105).

## 10.5.2 A functional analytic representation Theorem for the family of functions $\{u[\epsilon]\}_{\epsilon\in]0,\epsilon_3[}$

By Theorems 10.79, 10.80 and Definition 10.81, we can deduce the main result of this Subsection.

**Theorem 10.83.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_3$ be as in Theorem 10.79 (iv). Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0,\epsilon_3]$, and a real analytic operator $U$ of $]-\epsilon_4,\epsilon_4[$ to the space $C^0(\mathrm{cl}\,V,\mathbb{C})$, such that the following conditions hold.*

*(j)* $\mathrm{cl}\,V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4,\epsilon_4[$.

*(jj)*
$$u[\epsilon](x) = \epsilon^{n-1} U[\epsilon](x) \qquad \forall x \in \mathrm{cl}\,V,$$
*for all* $\epsilon \in ]0,\epsilon_4[$. *Moreover,*
$$U[0](x) = -S_n^{a,k}(x-w)\int_{\partial\Omega} F(s,0)\,d\sigma_s \qquad \forall x \in \mathrm{cl}\,V.$$

*Proof.* Let $\Theta_n[\cdot]$ be as in Theorem 10.79. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $\epsilon \in ]0,\epsilon_4[$. By Definition 10.81, we have

$$u[\epsilon](x) = \epsilon^{n-1}\int_{\partial\Omega} S_n^{a,k}(x-w-\epsilon s)\Theta_n[\epsilon](s)\,d\sigma_s \qquad \forall x \in \mathrm{cl}\,V.$$

Thus, it is natural to set

$$U[\epsilon](x) \equiv \int_{\partial\Omega} S_n^{a,k}(x-w-\epsilon s)\Theta_n[\epsilon](s)\,d\sigma_s \qquad \forall x \in \mathrm{cl}\,V,$$

for all $\epsilon \in ]-\epsilon_4,\epsilon_4[$. By Proposition 6.22, $U$ is a real analytic map of $]-\epsilon_4,\epsilon_4[$ to $C^0(\mathrm{cl}\,V,\mathbb{C})$. Furthermore, by Proposition 10.76 and Theorem 10.79, we have

$$\begin{aligned} U[0](x) &= S_n^{a,k}(x-w)\int_{\partial\Omega} \Theta_n[0](s)\,d\sigma_s \\ &= -S_n^{a,k}(x-w)\int_{\partial\Omega} F(s,0)\,d\sigma_s \qquad \forall x \in \mathrm{cl}\,V, \end{aligned}$$

since $\Theta_n[0] = \tilde{\theta}$. Hence the proof is now complete. □

**Theorem 10.84.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_3$, $\epsilon_2'$ be as in Theorem 10.80 (iv). Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \epsilon_3]$ and a real analytic operator $U^\#$ of $]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[$ to the space $C^0(\mathrm{cl}\, V, \mathbb{C})$, such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*

$$u[\epsilon](x) = \epsilon^{n-1} U^\#[\epsilon, \epsilon \log \epsilon](x) \qquad \forall x \in \mathrm{cl}\, V,$$

*for all $\epsilon \in ]0, \epsilon_4[$. Moreover,*

$$U^\#[0,0](x) = -S_n^{a,k}(x-w) \int_{\partial\Omega} F(s,0)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V.$$

*Proof.* Let $\Theta_n^\#[\cdot,\cdot]$ be as in Theorem 10.80. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $\epsilon \in ]0, \epsilon_4[$. By Definition 10.81, we have

$$u[\epsilon](x) = \epsilon^{n-1} \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s) \Theta_n^\#[\epsilon, \epsilon \log \epsilon](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V.$$

Thus, it is natural to set

$$U^\#[\epsilon, \epsilon'](x) \equiv \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s) \Theta_n^\#[\epsilon, \epsilon'](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$

for all $(\epsilon, \epsilon') \in ]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[$. By Proposition 6.22, $U^\#$ is a real analytic map of $]-\epsilon_4, \epsilon_4[\times]-\epsilon_2', \epsilon_2'[$ to $C^0(\mathrm{cl}\, V, \mathbb{C})$. Furthermore, by Proposition 10.76 and Theorem 10.80, we have

$$\begin{aligned} U^\#[0,0](x) &= S_n^{a,k}(x-w) \int_{\partial\Omega} \Theta_n^\#[0,0](s)\, d\sigma_s \\ &= -S_n^{a,k}(x-w) \int_{\partial\Omega} F(s,0)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V, \end{aligned}$$

since $\Theta_n^\#[0,0] = \tilde{\theta}$. Hence the proof is now complete. □

We have also the following Theorems.

**Theorem 10.85.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_3$ be as in Theorem 10.79 $(iv)$. Then there exist $\epsilon_5 \in ]0, \epsilon_3]$, and a real analytic operator $G$ of $]-\epsilon_5, \epsilon_5[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\, dx = \epsilon^n G[\epsilon], \tag{10.117}$$

*for all $\epsilon \in ]0, \epsilon_5[$. Moreover,*

$$G[0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\, dx, \tag{10.118}$$

*where $\tilde{u}$ is as in Definition 10.77.*

*Proof.* Let $\Theta_n[\cdot]$ be as in Theorem 10.79. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon \in ]0, \epsilon_3[$. Clearly, by the periodicity of $u[\epsilon]$, we have

$$\begin{aligned} &\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\, dx \\ &\qquad = -\epsilon^{n-1} \int_{\partial\Omega} \Big(\frac{\partial u[\epsilon]}{\partial \nu_{\Omega_\epsilon}}\Big) \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) \overline{u[\epsilon] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t)}\, d\sigma_t. \end{aligned}$$

By equality (6.24) and since $Q_n^k = 0$ for $n$ odd, we have

$$u[\epsilon] \circ (w+\epsilon\, \mathrm{id}_{\partial\Omega})(t) = \epsilon^{n-1} \int_{\partial\Omega} S_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\, d\sigma_s$$
$$= \epsilon \int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\, d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\, d\sigma_s \quad \forall t \in \partial\Omega.$$

By Theorem E.6 $(i)$, one can easily show that the map which takes $\epsilon$ to the function of the variable $t \in \partial\Omega$ defined by

$$\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\, d\sigma_s \qquad \forall t \in \partial\Omega,$$

is a real analytic operator of $]-\epsilon_3,\epsilon_3[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By Theorem C.4, we immediately deduce that there exists $\epsilon_5 \in ]0,\epsilon_3]$ such that the map of $]-\epsilon_5,\epsilon_5[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, which takes $\epsilon$ to the function $\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\, d\sigma_s$ of the variable $t\in\partial\Omega$, is real analytic. Analogously, we have

$$\Big(\frac{\partial u[\epsilon]}{\partial \nu_{\Omega_\epsilon}}\Big) \circ (w+\epsilon\, \mathrm{id}_{\partial\Omega})(t) = \frac{1}{2}\Theta_n[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^n} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\, d\sigma_s$$
$$+ \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\, d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $\epsilon \in ]0,\epsilon_3[$. Thus, if we set

$$\tilde{G}[\epsilon](t) \equiv \frac{1}{2}\Theta_n[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^n} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\, d\sigma_s$$
$$+ \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\, d\sigma_s \qquad \forall t\in\partial\Omega,\ \forall \epsilon\in ]-\epsilon_5,\epsilon_5[$$

then, by arguing as in Theorem 10.79, one can easily show that $\tilde{G}$ is a real analytic map of $]-\epsilon_5,\epsilon_5[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$.

Hence, if we set

$$G[\epsilon] \equiv -\int_{\partial\Omega} \tilde{G}[\epsilon](t)\overline{\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\, d\sigma_s}\, d\sigma_t$$
$$-\epsilon^{n-2}\int_{\partial\Omega} \tilde{G}[\epsilon](t)\overline{\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\, d\sigma_s}\, d\sigma_t,$$

for all $\epsilon\in]-\epsilon_5,\epsilon_5[$, then by standard properties of functions in Schauder spaces, we have that $G$ is a real analytic map of $]-\epsilon_5,\epsilon_5[$ to $\mathbb{C}$ such that equality (10.117) holds.

Finally, if $\epsilon = 0$, by Folland [52, p. 118] and since $\tilde{G}[0](\cdot) = -F(\cdot,0)$, we have

$$G[0] = \int_{\partial\Omega} F(t,0)\overline{\int_{\partial\Omega} S_n(t-s)\tilde{\theta}(s)\, d\sigma_s}\, d\sigma_t$$
$$= \int_{\mathbb{R}^n\setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\, dx.$$

□

**Theorem 10.86.** *Let $n$ be even. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $\tilde{\theta}$ be as in Proposition 10.76. Let $\epsilon_3$, $\epsilon_2'$ be as in Theorem 10.80 (iv). Then there exist $\epsilon_5\in]0,\epsilon_3]$, and three real analytic operators $G_1^\#$, $G_2^\#$, $G_3^\#$ of $]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\, dx - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\, dx$$
$$= \epsilon^n G_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{2n-2}(\log\epsilon)G_2^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{3n-3}(\log\epsilon)^2 G_3^\#[\epsilon,\epsilon\log\epsilon], \tag{10.119}$$

*for all $\epsilon\in]0,\epsilon_5[$. Moreover,*

$$G_1^\#[0,0] = \int_{\partial\Omega} F(t,0)\overline{\int_{\partial\Omega} S_n(t-s)\tilde{\theta}(s)\, d\sigma_s}\, d\sigma_t$$
$$-\delta_{2,n}\overline{R_n^{a,k}(0)}\Big|\int_{\partial\Omega} F(s,0)\, d\sigma_s\Big|^2, \tag{10.120}$$

$$G_2^\#[0,0] = -\overline{k}^{n-2}\mathcal{J}_n(0)|\int_{\partial\Omega} F(s,0)\,d\sigma_s|^2, \tag{10.121}$$

$$G_3^\#[0,0] = 0 \tag{10.122}$$

*where $\mathcal{J}_n(0)$ is as in Proposition E.3 (i). In particular, if $n>2$, then*

$$G_1^\#[0,0] = \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega} |\nabla\tilde{u}(x)|^2\,dx, \tag{10.123}$$

*where $\tilde{u}$ is as in Definition 10.77.*

*Proof.* Let $\Theta_n^\#[\cdot,\cdot]$ be as in Theorem 10.80. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon\in]0,\epsilon_3[$. Clearly, by the periodicity of $u[\epsilon]$, we have

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\,dx - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\,dx = -\epsilon^{n-1}\int_{\partial\Omega}\Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\overline{u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)}\,d\sigma_t.$$

By equality (6.24), we have

$$\begin{aligned} u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) =& \epsilon^{n-1}\int_{\partial\Omega} S_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ =& \epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &+\epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &+\epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega. \end{aligned}$$

Thus it is natural to set

$$\begin{aligned} F_1[\epsilon,\epsilon'](t) &\equiv \int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \\ F_2[\epsilon,\epsilon'](t) &\equiv k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \\ F_3[\epsilon,\epsilon'](t) &\equiv \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \end{aligned}$$

for all $(\epsilon,\epsilon')\in]-\epsilon_3,\epsilon_3[\times]-\epsilon_2',\epsilon_2'[$.

Then clearly

$$u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \epsilon F_1[\epsilon,\epsilon\log\epsilon](t) + \epsilon^{n-1}(\log\epsilon)F_2[\epsilon,\epsilon\log\epsilon](t) + \epsilon^{n-1}F_3[\epsilon,\epsilon\log\epsilon](t) \quad \forall t\in\partial\Omega,$$

for all $\epsilon\in]0,\epsilon_3[$. By Theorem E.6 (*i*) and Theorem C.4, we easily deduce that there exists $\epsilon_5\in]0,\epsilon_3]$ such that $F_1$, $F_2$ and $F_3$ are real analytic maps of $]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Analogously, we have

$$\begin{aligned} \Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) =& \frac{1}{2}\Theta_n^\#[\epsilon,\epsilon\log\epsilon](t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &+\epsilon^{n-1}\log\epsilon k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \end{aligned}$$

for all $\epsilon\in]0,\epsilon_3[$. Thus, if we set

$$\begin{aligned} \tilde{G}_1[\epsilon,\epsilon'](t) \equiv& \frac{1}{2}\Theta_n^\#[\epsilon,\epsilon'](t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \\ &+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \end{aligned}$$

and

$$\tilde{G}_2[\epsilon,\epsilon'](t) \equiv k^{n-2}\int_{\partial\Omega} \nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $(\epsilon,\epsilon')\in\,]-\epsilon_5,\epsilon_5[\,\times\,]-\epsilon_2',\epsilon_2'[$, then, by arguing as in Theorem 10.80, one can easily show that $\tilde{G}_1$ and $\tilde{G}_2$ are real analytic maps of $]-\epsilon_5,\epsilon_5[\,\times\,]-\epsilon_2',\epsilon_2'[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$.

Clearly,

$$\Big(\frac{\partial u[\epsilon]}{\partial \nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \tilde{G}_1[\epsilon,\epsilon\log\epsilon](t) + \epsilon^{n-1}\log\epsilon\,\tilde{G}_2[\epsilon,\epsilon\log\epsilon](t) \qquad \forall t\in\partial\Omega,\ \forall\epsilon\in\,]0,\epsilon_5[.$$

If $\epsilon\in\,]0,\epsilon_5[$, then we have

$$\begin{aligned}
&\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](x)|^2\,dx - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](x)|^2\,dx \\
&=\epsilon^n\Big(-\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon\log\epsilon]\overline{F_1[\epsilon,\epsilon\log\epsilon]}\,d\sigma - \epsilon^{n-2}\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon\log\epsilon]\overline{F_3[\epsilon,\epsilon\log\epsilon]}\,d\sigma\Big)\\
&\quad+\epsilon^{2n-2}\log\epsilon\Big(-\epsilon\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon\log\epsilon]\overline{F_1[\epsilon,\epsilon\log\epsilon]}\,d\sigma - \int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon\log\epsilon]\overline{F_2[\epsilon,\epsilon\log\epsilon]}\,d\sigma\\
&\qquad-\epsilon^{n-1}\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon\log\epsilon]\overline{F_3[\epsilon,\epsilon\log\epsilon]}\,d\sigma\Big)\\
&\quad+\epsilon^{3n-3}(\log\epsilon)^2\Big(-\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon\log\epsilon]\overline{F_2[\epsilon,\epsilon\log\epsilon]}\,d\sigma\Big).
\end{aligned}$$

If we set

$$\begin{aligned}
G_1^\#[\epsilon,\epsilon'] &\equiv -\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon'](t)\overline{F_1[\epsilon,\epsilon'](t)}\,d\sigma_t - \epsilon^{n-2}\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon'](t)\overline{F_3[\epsilon,\epsilon'](t)}\,d\sigma_t,\\
G_2^\#[\epsilon,\epsilon'] &\equiv -\epsilon\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon'](t)\overline{F_1[\epsilon,\epsilon'](t)}\,d\sigma_t - \int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon'](t)\overline{F_2[\epsilon,\epsilon'](t)}\,d\sigma_t\\
&\quad-\epsilon^{n-1}\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon'](t)\overline{F_3[\epsilon,\epsilon'](t)}\,d\sigma_t,\\
G_3^\#[\epsilon,\epsilon'] &\equiv -\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon'](t)\overline{F_2[\epsilon,\epsilon'](t)}\,d\sigma_t,
\end{aligned}$$

for all $(\epsilon,\epsilon')\in\,]-\epsilon_5,\epsilon_5[\,\times\,]-\epsilon_2',\epsilon_2'[$, then standard properties of functions in Schauder spaces and a simple computation show that $G_1^\#$, $G_2^\#$, and $G_3^\#$ are real analytic maps of $]-\epsilon_5,\epsilon_5[\,\times\,]-\epsilon_2',\epsilon_2'[$ to $\mathbb{C}$ such that equality (10.119) holds for all $\epsilon\in\,]0,\epsilon_5[$.

Next, we observe that

$$\begin{aligned}
G_1^\#[0,0] &= \int_{\partial\Omega}F(t,0)\overline{\int_{\partial\Omega}S_n(t-s)\tilde{\theta}(s)\,d\sigma_s}\,d\sigma_t - \delta_{2,n}\overline{R_n^{a,k}(0)}\Big|\int_{\partial\Omega}F(t,0)\,d\sigma_t\Big|^2,\\
G_2^\#[0,0] &= -\overline{k}^{n-2}\overline{Q_n^k(0)}\int_{\partial\Omega}\big(-F(t,0)\big)\,d\sigma_t\overline{\int_{\partial\Omega}\big(-F(t,0)\big)\,d\sigma_t},\\
G_3^\#[0,0] &= -|k^{n-2}|^2\overline{Q_n^k(0)}\Big|\int_{\partial\Omega}\big(-F(t,0)\big)\,d\sigma_t\Big|^2\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(0)\,d\sigma_t = 0,
\end{aligned}$$

and accordingly equalities (10.120), (10.121), and (10.122) hold. Finally, if $n\geq 4$, by Folland [52, p. 118], we have

$$G_1^\#[0,0] = \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega}|\nabla\tilde{u}(x)|^2\,dx.$$

□

*Remark* 10.87. If $n$ is odd, we note that the right-hand side of the equality in (10.117) of Theorem 10.85 can be continued real analytically in the whole $]-\epsilon_5,\epsilon_5[$.

Moreover,

$$\lim_{\epsilon\to 0^+}\Big[\int_{\mathbb{P}_a[\Omega_\epsilon]}|\nabla u[\epsilon](x)|^2\,dx - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]}|u[\epsilon](x)|^2\,dx\Big] = 0,$$

for all $n\in\mathbb{N}\setminus\{0,1\}$ ($n$ even or odd.)

### 10.5.3 A real analytic continuation Theorem for the integral of the family $\{u[\epsilon]\}_{\epsilon\in]0,\epsilon_3[}$

We now prove real analytic continuation Theorems for the integral of the solution. Namely, we prove the following results.

**Theorem 10.88.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $\epsilon_3$ be as in Theorem 10.79 (iv). Then there exist $\epsilon_6 \in ]0,\epsilon_3]$, and a real analytic operator $J$ of $]-\epsilon_6,\epsilon_6[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx = \frac{\epsilon^{n-1}}{k^2} J[\epsilon], \tag{10.124}$$

*for all $\epsilon \in ]0,\epsilon_6[$. Moreover,*

$$J[0] = -\int_{\partial\Omega} F(x,0)\,d\sigma_x. \tag{10.125}$$

*Proof.* Let $\Theta_n[\cdot]$ be as in Theorem 10.79. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon \in ]0,\epsilon_3[$. Clearly, by the Divergence Theorem and the periodicity of $u[\epsilon]$, we have

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx &= -\frac{1}{k^2}\int_{\mathbb{P}_a[\Omega_\epsilon]} \Delta u[\epsilon](x)\,dx \\ &= -\frac{1}{k^2}\int_{\partial\mathbb{P}_a[\Omega_\epsilon]} \frac{\partial}{\partial\nu_{\mathbb{P}_a[\Omega_\epsilon]}} u[\epsilon](x)\,d\sigma_x \\ &= -\frac{1}{k^2}\Big[\int_{\partial A} \frac{\partial}{\partial\nu_A} u[\epsilon](x)\,d\sigma_x - \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}} u[\epsilon](x)\,d\sigma_x\Big] \\ &= \frac{1}{k^2}\int_{\partial\Omega_\epsilon} \frac{\partial}{\partial\nu_{\Omega_\epsilon}} u[\epsilon](x)\,d\sigma_x.\end{aligned}$$

As a consequence,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\,dx = \frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega}\Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\,d\sigma_t.$$

By equality (6.25) and since $Q_n^k = 0$ for $n$ odd, we have

$$\begin{aligned}\Big(\frac{\partial u[\epsilon]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) &= \frac{1}{2}\Theta_n[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^n} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s \\ &\quad + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega.\end{aligned}$$

We set

$$\begin{aligned}\tilde{J}[\epsilon](t) \equiv& \frac{1}{2}\Theta_n[\epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^n} S_n(t-s,\epsilon k)\Theta_n[\epsilon](s)\,d\sigma_s \\ &+ \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon](s)\,d\sigma_s \quad \forall t\in\partial\Omega,\end{aligned}$$

for all $\epsilon \in ]-\epsilon_3,\epsilon_3[$. By Theorem E.6 (*ii*) and Theorem C.4, one can easily show that there exists $\epsilon_6 \in ]0,\epsilon_3]$ such that $\tilde{J}$ is a real analytic map of $]-\epsilon_6,\epsilon_6[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Hence, if we set

$$J[\epsilon] \equiv \int_{\partial\Omega} \tilde{J}[\epsilon](t)\,d\sigma_t,$$

for all $\epsilon \in ]-\epsilon_6,\epsilon_6[$, then, by standard properties of functions in Schauder spaces, we have that $J$ is a real analytic map of $]-\epsilon_6,\epsilon_6[$ to $\mathbb{C}$, such that equality (10.124) holds.

Finally, if $\epsilon = 0$, we have

$$\begin{aligned}J[0] &= \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega} v^-[\partial\Omega,\tilde{\theta},0](t)\,d\sigma_t \\ &= -\int_{\partial\Omega} F(x,0)\,d\sigma_x,\end{aligned}$$

and accordingly (10.125) holds. □

**Theorem 10.89.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\tilde{\theta}$ be as in Proposition 10.76. Let $\epsilon_3$, $\epsilon_2'$ be as in Theorem 10.80 (iv). Then there exist $\epsilon_6 \in ]0, \epsilon_3]$, and two real analytic operators $J_1^\#$, $J_2^\#$ of $]-\epsilon_6, \epsilon_6[ \times ]-\epsilon_2', \epsilon_2'[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx = \frac{\epsilon^{n-1}}{k^2} J_1^\#[\epsilon, \epsilon \log \epsilon] + \frac{\epsilon^{2n-2}(\log \epsilon)}{k^2} J_2^\#[\epsilon, \epsilon \log \epsilon], \qquad (10.126)$$

*for all $\epsilon \in ]0, \epsilon_6[$. Moreover,*

$$J_1^\#[0,0] = -\int_{\partial\Omega} F(x,0)\, d\sigma_x. \qquad (10.127)$$

*Proof.* Let $\Theta_n^\#[\cdot,\cdot]$ be as in Theorem 10.80. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $\epsilon \in ]0, \epsilon_3[$. Clearly, by the Divergence Theorem and the periodicity of $u[\epsilon]$, we have

$$\begin{aligned}
\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx &= -\frac{1}{k^2} \int_{\mathbb{P}_a[\Omega_\epsilon]} \Delta u[\epsilon](x)\, dx \\
&= -\frac{1}{k^2} \int_{\partial \mathbb{P}_a[\Omega_\epsilon]} \frac{\partial}{\partial \nu_{\mathbb{P}_a[\Omega_\epsilon]}} u[\epsilon](x)\, d\sigma_x \\
&= -\frac{1}{k^2} \Big[ \int_{\partial A} \frac{\partial}{\partial \nu_A} u[\epsilon](x)\, d\sigma_x - \int_{\partial \Omega_\epsilon} \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u[\epsilon](x)\, d\sigma_x \Big] \\
&= \frac{1}{k^2} \int_{\partial\Omega_\epsilon} \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u[\epsilon](x)\, d\sigma_x.
\end{aligned}$$

As a consequence,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](x)\, dx = \frac{\epsilon^{n-1}}{k^2} \int_{\partial\Omega} \Big( \frac{\partial u[\epsilon]}{\partial \nu_{\Omega_\epsilon}} \Big) \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t)\, d\sigma_t.$$

By equality (6.25), we have

$$\begin{aligned}
\Big( \frac{\partial u[\epsilon]}{\partial \nu_{\Omega_\epsilon}} \Big) \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) &= \frac{1}{2} \Theta_n^\#[\epsilon, \epsilon \log \epsilon](t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k) \Theta_n^\#[\epsilon, \epsilon \log \epsilon](s)\, d\sigma_s \\
&+ \epsilon^{n-1} (\log \epsilon) k^{n-2} \int_{\partial\Omega} \nu_\Omega(t) \cdot D Q_n^k(\epsilon(t-s)) \Theta_n^\#[\epsilon, \epsilon \log \epsilon](s)\, d\sigma_s \\
&+ \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot D R_n^{a,k}(\epsilon(t-s)) \Theta_n^\#[\epsilon, \epsilon \log \epsilon](s)\, d\sigma_s \qquad \forall t \in \partial\Omega.
\end{aligned}$$

We set

$$\begin{aligned}
\tilde{J}_1[\epsilon, \epsilon'](t) \equiv\ & \frac{1}{2} \Theta_n^\#[\epsilon, \epsilon'](t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k) \Theta_n^\#[\epsilon, \epsilon'](s)\, d\sigma_s \\
&+ \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot D R_n^{a,k}(\epsilon(t-s)) \Theta_n^\#[\epsilon, \epsilon'](s)\, d\sigma_s \qquad \forall t \in \partial\Omega,
\end{aligned}$$

and

$$\tilde{J}_2[\epsilon, \epsilon'](t) \equiv k^{n-2} \int_{\partial\Omega} \nu_\Omega(t) \cdot D Q_n^k(\epsilon(t-s)) \Theta_n^\#[\epsilon, \epsilon'](s)\, d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $(\epsilon, \epsilon') \in ]-\epsilon_3, \epsilon_3[ \times ]-\epsilon_2', \epsilon_2'[$. By Theorem E.6 (*ii*) and Theorem C.4, one can easily show that there exists $\epsilon_6 \in ]0, \epsilon_3]$ such that $\tilde{J}_1^\#$, $\tilde{J}_2^\#$ are real analytic maps of $]-\epsilon_6, \epsilon_6[\times]-\epsilon_2', \epsilon_2'[$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Hence, if we set

$$J_1^\#[\epsilon, \epsilon'] \equiv \int_{\partial\Omega} \tilde{J}_1^\#[\epsilon, \epsilon'](t)\, d\sigma_t,$$

and

$$J_2^\#[\epsilon, \epsilon'] \equiv \int_{\partial\Omega} \tilde{J}_2^\#[\epsilon, \epsilon'](t)\, d\sigma_t,$$

for all $(\epsilon, \epsilon') \in ]-\epsilon_6, \epsilon_6[ \times ]-\epsilon_2', \epsilon_2'[$, then, by standard properties of functions in Schauder spaces, we have that $J_1^\#$, $J_2^\#$ are real analytic maps of $]-\epsilon_6, \epsilon_6[ \times ]-\epsilon_2', \epsilon_2'[$ to $\mathbb{C}$, such that equality (10.126) holds.

Finally, if $\epsilon = \epsilon' = 0$, we have

$$\begin{aligned} J_1^{\#}[0,0] &= \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega} v^-[\partial\Omega, \tilde{\theta}, 0](t)\, d\sigma_t \\ &= -\int_{\partial\Omega} F(x,0)\, d\sigma_x, \end{aligned}$$

and accordingly (10.127) holds. □

### 10.5.4 A property of local uniqueness of the family $\{u[\epsilon]\}_{\epsilon\in]0,\epsilon_3[}$

In this Subsection, we shall show that the family $\{u[\epsilon]\}_{\epsilon\in]0,\epsilon_3[}$ is essentially unique. Namely, we prove the following.

**Theorem 10.90.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\{\hat{\epsilon}_j\}_{j\in\mathbb{N}}$ be a sequence in $]0,\epsilon_1^*[$ converging to* 0. *If $\{u_j\}_{j\in\mathbb{N}}$ is a sequence of functions such that*

$$u_j \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_{\hat{\epsilon}_j}], \mathbb{C}), \tag{10.128}$$

$$u_j \text{ solves } (10.105) \text{ with } \epsilon \equiv \hat{\epsilon}_j, \tag{10.129}$$

$$\lim_{j\to\infty} u_j(w + \hat{\epsilon}_j \cdot) = 0 \qquad \text{in } C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), \tag{10.130}$$

*then there exists $j_0 \in \mathbb{N}$ such that*

$$u_j = u[\hat{\epsilon}_j] \qquad \forall j_0 \le j \in \mathbb{N}.$$

*Proof.* By Theorem 10.75, for each $j \in \mathbb{N}$, there exists a unique function $\theta_j$ in $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ such that

$$u_j = u[\hat{\epsilon}_j, \theta_j]. \tag{10.131}$$

We shall now try to show that

$$\lim_{j\to\infty} \theta_j = \tilde{\theta} \qquad \text{in } C^{m-1,\alpha}(\partial\Omega, \mathbb{C}). \tag{10.132}$$

Indeed, if we denote by $\tilde{\mathcal{U}}$ the neighbourhood of Theorems 10.79 (*iv*), 10.80 (*iv*), the limiting relation of (10.132) implies that there exists $j_0 \in \mathbb{N}$ such that

$$\begin{aligned} &(\hat{\epsilon}_j, \theta_j) \in ]0,\epsilon_3[ \times \tilde{\mathcal{U}}, && \text{if } n \text{ is odd,} \\ &(\hat{\epsilon}_j, \hat{\epsilon}_j \log \hat{\epsilon}_j, \theta_j) \in ]0,\epsilon_3[ \times ]-\epsilon_2', \epsilon_2'[ \times \tilde{\mathcal{U}}, && \text{if } n \text{ is even,} \end{aligned}$$

for $j \ge j_0$ and thus Theorems 10.79 (*iv*), 10.80 (*iv*) would imply that

$$\begin{aligned} &\theta_j = \Theta_n[\hat{\epsilon}_j] && \text{if } n \text{ is odd,} \\ &\theta_j = \Theta_n^{\#}[\hat{\epsilon}_j, \hat{\epsilon}_j \log \hat{\epsilon}_j] && \text{if } n \text{ is even,} \end{aligned}$$

for $j_0 \le j \in \mathbb{N}$, and that accordingly the Theorem holds (cf. Definition 10.81.) Thus we now turn to the proof of (10.132). We split our proof into the cases $n$ odd and $n$ even.

We first assume that $n$ is odd. Then we note that equation $\Lambda[\epsilon,\theta] = 0$ can be rewritten in the following form

$$\begin{aligned} &\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s \\ &\qquad = -F\Big(t, \epsilon \int_{\partial\Omega} S_n(t-s,\epsilon k)\theta(s)\, d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s\Big) \quad \forall t \in \partial\Omega, \end{aligned} \tag{10.133}$$

for all $(\epsilon,\theta)$ in the domain of $\Lambda$. Then we define the map $N$ of $]-\epsilon_3,\epsilon_3[ \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ by setting $N[\epsilon,\theta]$ equal to the left-hand side of the equality in (10.133), for all $(\epsilon,\theta) \in ]-\epsilon_3,\epsilon_3[ \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By arguing as in the proof of Theorem 10.79, we can prove that $N$ is real analytic. Since $N[\epsilon,\cdot]$ is linear for all $\epsilon \in ]-\epsilon_3,\epsilon_3[$, we have

$$N[\epsilon,\theta] = \partial_\theta N[\epsilon,\tilde{\theta}](\theta),$$

for all $(\epsilon, \theta) \in ]-\epsilon_3, \epsilon_3[\times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, and the map of $]-\epsilon_3, \epsilon_3[$ to $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), C^{m-1,\alpha}(\partial\Omega, \mathbb{C}))$ which takes $\epsilon$ to $N[\epsilon, \cdot]$ is real analytic. Since

$$N[0, \cdot] = \partial_\theta \Lambda[0, \tilde{\theta}](\cdot),$$

Theorem 10.79 (*iii*) implies that $N[0, \cdot]$ is also a linear homeomorphism. Since the set of linear homeomorphisms of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ is open in $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), C^{m-1,\alpha}(\partial\Omega, \mathbb{C}))$ and since the map which takes a linear invertible operator to its inverse is real analytic (cf. *e.g.*, Hille and Phillips [61, Theorems 4.3.2 and 4.3.4]), there exists $\tilde{\epsilon} \in ]0, \epsilon_3[$ such that the map $\epsilon \mapsto N[\epsilon, \cdot]^{(-1)}$ is real analytic from $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), C^{m-1,\alpha}(\partial\Omega, \mathbb{C}))$. Next we denote by $S[\epsilon, \theta]$ the right-hand side of (10.133). Then equation $\Lambda[\epsilon, \theta] = 0$ (or equivalently equation (10.133)) can be rewritten in the following form:

$$\theta = N[\epsilon, \cdot]^{(-1)}[S[\epsilon, \theta]], \tag{10.134}$$

for all $(\epsilon, \theta) \in ]-\tilde{\epsilon}, \tilde{\epsilon}[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Moreover, if $j \in \mathbb{N}$, we observe that by (10.131) we have

$$\begin{aligned} u_j(w + \hat{\epsilon}_j t) &= u[\hat{\epsilon}_j, \theta_j](w + \hat{\epsilon}_j t) \\ &= \hat{\epsilon}_j \int_{\partial\Omega} S_n(t - s, \hat{\epsilon}_j k)\theta_j(s)\, d\sigma_s + \hat{\epsilon}_j^{n-1} \int_{\partial\Omega} R_n^{a,k}(\hat{\epsilon}_j(t - s))\theta_j(s)\, d\sigma_s \qquad \forall t \in \partial\Omega. \end{aligned} \tag{10.135}$$

Next we note that condition (10.130), equality (10.135), the proof of Theorem 10.79, the real analyticity of $T_F$ and standard calculus in Banach space imply that

$$\lim_{j\to\infty} S[\hat{\epsilon}_j, \theta_j] = S[0, \tilde{\theta}] \qquad \text{in } C^{m-1,\alpha}(\partial\Omega, \mathbb{C}). \tag{10.136}$$

Then by (10.134) and by the real analyticity of $\epsilon \mapsto N[\epsilon, \cdot]^{(-1)}$, and by the bilinearity and continuity of the operator of $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), C^{m-1,\alpha}(\partial\Omega, \mathbb{C})) \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, which takes a pair $(T_1, T_2)$ to $T_1[T_2]$, by (10.136) we conclude that

$$\begin{aligned} \lim_{j\to\infty} \theta_j &= \lim_{j\to\infty} N[\hat{\epsilon}_j, \cdot]^{(-1)}[S[\hat{\epsilon}_j, \theta_j]] \\ &= N[0, \cdot]^{(-1)}[S[0, \tilde{\theta}]] = \tilde{\theta} \qquad \text{in } C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), \end{aligned}$$

and, consequently, that (10.132) holds. Thus the proof of case $n$ odd is complete.

We now consider case $n$ even. We proceed as in case $n$ odd. We note that equation $\Lambda^{\#}[\epsilon, \epsilon', \theta] = 0$ can be rewritten in the following form

$$\begin{aligned} &\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t - s, \epsilon k)\theta(s)\, d\sigma_s + \epsilon^{n-2}\epsilon' k^{n-2} \int_{\partial\Omega} \nu_\Omega(t) \cdot DQ_n^k(\epsilon(t - s))\theta(s)\, d\sigma_s \\ &+ \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t - s))\theta(s)\, d\sigma_s = -F\Big(t, \epsilon \int_{\partial\Omega} S_n(t - s, \epsilon k)\theta(s)\, d\sigma_s \\ &+ \epsilon^{n-2}\epsilon' k^{n-2} \int_{\partial\Omega} Q_n^k(\epsilon(t - s))\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t - s))\theta(s)\, d\sigma_s\Big) \;\forall t \in \partial\Omega, \end{aligned} \tag{10.137}$$

for all $(\epsilon, \epsilon', \theta)$ in the domain of $\Lambda^{\#}$. We define the map $N^{\#}$ of $]-\epsilon_3, \epsilon_3[ \times ]-\epsilon'_2, \epsilon'_2[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ by setting $N^{\#}[\epsilon, \epsilon', \theta]$ equal to the left-hand side of the equality in (10.137), for all $(\epsilon, \epsilon', \theta) \in ]-\epsilon_3, \epsilon_3[ \times ]-\epsilon'_2, \epsilon'_2[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. By arguing as in the proof of Theorem 10.80, we can prove that $N^{\#}$ is real analytic. Since $N^{\#}[\epsilon, \epsilon', \cdot]$ is linear for all $(\epsilon, \epsilon') \in ]-\epsilon_3, \epsilon_3[ \times ]-\epsilon'_2, \epsilon'_2[$, we have

$$N^{\#}[\epsilon, \epsilon', \theta] = \partial_\theta N^{\#}[\epsilon, \epsilon', \tilde{\theta}](\theta),$$

for all $(\epsilon, \epsilon', \theta) \in ]-\epsilon_3, \epsilon_3[ \times ]-\epsilon'_2, \epsilon'_2[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, and the map of $]-\epsilon_3, \epsilon_3[ \times ]-\epsilon'_2, \epsilon'_2[$ to the space $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), C^{m-1,\alpha}(\partial\Omega, \mathbb{C}))$ which takes $(\epsilon, \epsilon')$ to $N^{\#}[\epsilon, \epsilon', \cdot]$ is real analytic. Since

$$N^{\#}[0, 0, \cdot] = \partial_\theta \Lambda^{\#}[0, 0, \tilde{\theta}](\cdot),$$

Theorem 10.80 (*iii*) implies that $N^{\#}[0, 0, \cdot]$ is also a linear homeomorphism. Since the set of linear homeomorphisms of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ is open in $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), C^{m-1,\alpha}(\partial\Omega, \mathbb{C}))$

and since the map which takes a linear invertible operator to its inverse is real analytic (cf. *e.g.*, Hille and Phillips [61, Theorems 4.3.2 and 4.3.4]), there exists $(\tilde{\epsilon}, \tilde{\epsilon}') \in ]0, \epsilon_3[ \times ]0, \epsilon_2'[$ such that the map $(\epsilon, \epsilon') \mapsto N^{\#}[\epsilon, \epsilon', \cdot]^{(-1)}$ is real analytic from $]-\tilde{\epsilon}, \tilde{\epsilon}[ \times ]-\tilde{\epsilon}', \tilde{\epsilon}'[$ to $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), C^{m-1,\alpha}(\partial\Omega, \mathbb{C}))$, and such that

$$\epsilon \log \epsilon \in ]-\tilde{\epsilon}', \tilde{\epsilon}'[ \qquad \forall \epsilon \in ]0, \tilde{\epsilon}[.$$

Next we denote by $S^{\#}[\epsilon, \epsilon', \theta]$ the right-hand side of (10.137). Then equation $\Lambda^{\#}[\epsilon, \epsilon', \theta] = 0$ (or equivalently equation (10.137)) can be rewritten in the following form:

$$\theta = N^{\#}[\epsilon, \epsilon', \cdot]^{(-1)}[S^{\#}[\epsilon, \epsilon', \theta]], \tag{10.138}$$

for all $(\epsilon, \epsilon', \theta) \in ]-\tilde{\epsilon}, \tilde{\epsilon}[ \times ]-\tilde{\epsilon}', \tilde{\epsilon}'[ \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Moreover, if $j \in \mathbb{N}$, we observe that by (10.131) we have

$$\begin{aligned} u_j(w + \hat{\epsilon}_j t) &= u[\hat{\epsilon}_j, \theta_j](w + \hat{\epsilon}_j t) \\ &= \hat{\epsilon}_j \int_{\partial\Omega} S_n(t - s, \hat{\epsilon}_j k)\theta_j(s)\, d\sigma_s + \hat{\epsilon}_j^{n-2}(\hat{\epsilon}_j \log \hat{\epsilon}_j)k^{n-2} \int_{\partial\Omega} Q_n^k(\hat{\epsilon}_j(t-s))\theta_j(s)\, d\sigma_s \\ &\quad + \hat{\epsilon}_j^{n-1} \int_{\partial\Omega} R_n^{a,k}(\hat{\epsilon}_j(t-s))\theta_j(s)\, d\sigma_s \qquad \forall t \in \partial\Omega. \end{aligned} \tag{10.139}$$

Next we note that

$$\lim_{j\to\infty} (\hat{\epsilon}_j, \hat{\epsilon}_j \log \hat{\epsilon}_j) = (0, 0) \tag{10.140}$$

in $]0, \tilde{\epsilon}[ \times ]-\tilde{\epsilon}', \tilde{\epsilon}'[$. Then condition (10.130), equality (10.139), the proof of Theorem 10.80, the real analyticity of $T_F$ and standard calculus in Banach space imply that

$$\lim_{j\to\infty} S^{\#}[\hat{\epsilon}_j, \hat{\epsilon}_j \log \hat{\epsilon}_j, \theta_j] = S^{\#}[0, 0, \tilde{\theta}] \qquad \text{in } C^{m-1,\alpha}(\partial\Omega, \mathbb{C}). \tag{10.141}$$

Then by (10.138) and by the real analyticity of $(\epsilon, \epsilon') \mapsto N^{\#}[\epsilon, \epsilon', \cdot]^{(-1)}$, and by the bilinearity and continuity of the operator of $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), C^{m-1,\alpha}(\partial\Omega, \mathbb{C})) \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, which takes a pair $(T_1, T_2)$ to $T_1[T_2]$ and by (10.140), by (10.141) we conclude that

$$\begin{aligned} \lim_{j\to\infty} \theta_j &= \lim_{j\to\infty} N^{\#}[\hat{\epsilon}_j, \hat{\epsilon}_j \log \hat{\epsilon}_j, \cdot]^{(-1)}[S^{\#}[\hat{\epsilon}_j, \hat{\epsilon}_j \log \hat{\epsilon}_j, \theta_j]] \\ &= N^{\#}[0, 0, \cdot]^{(-1)}[S^{\#}[0, 0, \tilde{\theta}]] = \tilde{\theta} \qquad \text{in } C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), \end{aligned}$$

and, consequently, that (10.132) holds. Thus the proof of case $n$ even is complete. □

## 10.6 An homogenization problem for the Helmholtz equation with nonlinear Robin boundary conditions in a periodically perforated domain

In this section we consider an homogenization problem for the Helmhlotz equation with nonlinear Robin boundary conditions in a periodically perforated domain. In most of the results we assume that $\mathrm{Im}(k) \neq 0$ and $\mathrm{Re}(k) = 0$.

### 10.6.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 1.8.1, 6.7.1, 10.5.1 and 10.3.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). For each $(\epsilon, \delta) \in ]0, \epsilon_1^*[ \times ]0, +\infty[$, we consider the following periodic nonlinear Robin problem for the Helmholtz equation.

$$\begin{cases} \Delta u(x) + \frac{k^2}{\delta^2} u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon, \delta), \\ u(x + \delta a_j) = u(x) & \forall x \in \mathrm{cl}\, \mathbb{T}_a(\epsilon, \delta), \quad \forall j \in \{1, \ldots, n\}, \\ \delta \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x) + F(\frac{1}{\epsilon\delta}(x - \delta w), u(x)) = 0 & \forall x \in \partial\Omega(\epsilon, \delta). \end{cases} \tag{10.142}$$

We give the following definition.

**Definition 10.91.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $\epsilon_3$ be as in Theorem 10.79 *(iv)* if $n$ is odd, and as in Theorem 10.80 *(iv)* if $n$ is even. Let $u[\cdot]$ be as in Definition 10.81. For each pair $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, +\infty[$, we set

$$u_{(\epsilon,\delta)}(x) \equiv u[\epsilon](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon, \delta).$$

*Remark* 10.92. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $\epsilon_3$ be as in Theorem 10.79 *(iv)* if $n$ is odd, and as in Theorem 10.80 *(iv)* if $n$ is even. For each $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, +\infty[$, $u_{(\epsilon,\delta)}$ is a solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon, \delta), \mathbb{C})$ of problem (10.142).

By the previous remark, we note that a solution of problem (10.142) can be expressed by means of a solution of an auxiliary rescaled problem, which does not depend on $\delta$. This is due to the presence of the factor $\delta$ in front of $\frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x)$ in the third equation of problem (10.142).

By virtue of Theorem (10.90), we have the following.

*Remark* 10.93. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $\epsilon_3$ be as in Theorem 10.79 *(iv)* if $n$ is odd, and as in Theorem 10.80 *(iv)* if $n$ is even. Let $\bar{\delta} \in ]0, +\infty[$. Let $\{\hat{\epsilon}_j\}_{j \in \mathbb{N}}$ be a sequence in $]0, \epsilon_1^*[$ converging to 0. If $\{u_j\}_{j \in \mathbb{N}}$ is a sequence of functions such that

$$\begin{aligned}
&u_j \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\hat{\epsilon}_j, \bar{\delta}), \mathbb{C}),\\
&u_j \text{ solves (10.142) with } (\epsilon, \delta) \equiv (\hat{\epsilon}_j, \bar{\delta}),\\
&\lim_{j \to \infty} u_j(\bar{\delta} w + \bar{\delta} \hat{\epsilon}_j \cdot) = 0 \qquad \text{in } C^{m-1,\alpha}(\partial\Omega, \mathbb{C}),
\end{aligned}$$

then there exists $j_0 \in \mathbb{N}$ such that

$$u_j = u_{(\hat{\epsilon}_j, \bar{\delta})} \qquad \forall j_0 \leq j \in \mathbb{N}.$$

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon, \delta)$ tends to $(0,0)$. As a first step, we study the behaviour of $u[\epsilon]$ as $\epsilon$ tends to 0.

We have the following.

**Proposition 10.94.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_3$ be as in Theorem 10.79 (iv). Let $u[\cdot]$ be as in Definition 10.81. Then there exist $\tilde{\epsilon} \in ]0, \epsilon_3[$ and a real analytic map $N$ of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$ such that*

$$\begin{aligned}
\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} &= \epsilon \|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},\\
\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} &= \epsilon \|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},
\end{aligned}$$

*for all $\epsilon \in ]0, \tilde{\epsilon}[$. Moreover, as a consequence,*

$$\lim_{\epsilon \to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n, \mathbb{C}).$$

*Proof.* Let $\epsilon_3$, $\Theta_n$ be as in Theorem 10.79 *(iv)*. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $\epsilon \in ]0, \epsilon_3[$, we have

$$u[\epsilon] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) = \epsilon \int_{\partial\Omega} S_n(t - s, \epsilon k) \Theta_n[\epsilon](s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t - s)) \Theta_n[\epsilon](s)\, d\sigma_s \quad \forall t \in \partial\Omega.$$

We set

$$N[\epsilon](t) \equiv \int_{\partial\Omega} S_n(t - s, \epsilon k) \Theta_n[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t - s)) \Theta_n[\epsilon](s)\, d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $\epsilon \in ]-\epsilon_3, \epsilon_3[$. By taking $\tilde{\epsilon} \in ]0, \epsilon_3[$ small enough, we can assume (cf. Theorem C.4 and the proof of Theorem 10.85) that $N$ is a real analytic map of $]-\tilde{\epsilon}, \tilde{\epsilon}[$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$. By Corollary 6.24, we have

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon \|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)} \qquad \forall \epsilon \in ]0, \tilde{\epsilon}[,$$

and

$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon\|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)} \qquad \forall \epsilon \in \,]0,\tilde{\epsilon}[.$$

Accordingly,

$$\lim_{\epsilon\to 0^+} \mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and

$$\lim_{\epsilon\to 0^+} \mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and so the conclusion follows. □

**Proposition 10.95.** *Let $n$ be even. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in \,]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\mathrm{Im}(k) \neq 0$ and $\mathrm{Re}(k) = 0$. Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_3$, $\epsilon_2'$ be as in Theorem 10.80 (iv). Let $u[\cdot]$ be as in Definition 10.81. Then there exist $\tilde{\epsilon} \in \,]0,\epsilon_3[$ and two real analytic maps $N_1^\#$, $N_2^\#$ of $]-\tilde{\epsilon},\tilde{\epsilon}[\,\times\,]-\epsilon_2',\epsilon_2'[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$ such that*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$

*for all $\epsilon \in \,]0,\tilde{\epsilon}[$. Moreover, as a consequence,*

$$\lim_{\epsilon\to 0^+} \mathbf{E}_{(\epsilon,1)}[u[\epsilon]] = 0 \qquad \textit{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* Let $\epsilon_3$, $\epsilon_2'$, $\Theta_n^\#$ be as in Theorem 10.80 (*iv*). If $\epsilon \in \,]0,\epsilon_3[$, we have

$$\begin{aligned} u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) =& \epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &+ \epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \\ &+ \epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon](s)\,d\sigma_s \qquad \forall t\in\partial\Omega. \end{aligned}$$

We set

$$\begin{aligned} N_1^\#[\epsilon,\epsilon'](t) \equiv& \int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \\ &+ \epsilon^{n-2}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \end{aligned}$$

and

$$N_2^\#[\epsilon,\epsilon'](t) \equiv k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon'](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $(\epsilon,\epsilon') \in \,]-\epsilon_3,\epsilon_3[\,\times\,]-\epsilon_2',\epsilon_2'[$. By taking $\tilde{\epsilon} \in \,]0,\epsilon_3[$ small enough, we can assume (cf. Theorem C.4 and the proof of Theorem 10.86) that $N_1^\#$, $N_2^\#$ are real analytic maps of $]-\tilde{\epsilon},\tilde{\epsilon}[\,\times\,]-\epsilon_2',\epsilon_2'[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$. Clearly,

$$u[\epsilon]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \epsilon N_1^\#[\epsilon,\epsilon\log\epsilon](t) + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon](t) \qquad \forall t\in\partial\Omega,\ \forall\epsilon\in\,]0,\tilde{\epsilon}[.$$

By Corollary 6.24, we have

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$

and

$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$

for all $\epsilon \in \,]0,\tilde{\epsilon}[$.

Accordingly,

$$\lim_{\epsilon\to 0^+} \mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and

$$\lim_{\epsilon\to 0^+} \mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and so the conclusion follows. □

### 10.6.2 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorems we deduce by Propositions 10.94, 10.95 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 10.96.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (10.104). *Let $\tilde{\epsilon}$, $N$ be as in Proposition 10.94. Then*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon\|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \epsilon\|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,+\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \epsilon\|\mathrm{Re}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \epsilon\|\mathrm{Im}\big(N[\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,+\infty[$. □

**Theorem 10.97.** *Let $n$ be even. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (10.104). *Let $\tilde{\epsilon}$, $N_1^\#$, $N_2^\#$ be as in Proposition 10.95. Then*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,+\infty[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,+\infty[$. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 10.98.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_6$, $J$ be as in Theorem 10.88. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n\frac{\epsilon^{n-1}}{k^2}J[\epsilon], \tag{10.143}$$

*for all $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N}\setminus\{0\}$.*

*Proof.* Let $\epsilon \in ]0, \epsilon_6[$, $l \in \mathbb{N} \setminus \{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx\\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx\\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx\\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon]\Big(\frac{l}{r}x\Big)\,dx\\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](t)\,dt\\ &= \frac{r^n}{l^n}\frac{\epsilon^{n-1}}{k^2}J[\epsilon].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n\frac{\epsilon^{n-1}}{k^2}J[\epsilon],$$

and the conclusion follows. □

**Theorem 10.99.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_6$, $J_1^\#$, $J_2^\#$ be as in Theorem 10.89. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n\Big\{\frac{\epsilon^{n-1}}{k^2}J_1^\#[\epsilon,\epsilon\log\epsilon] + \frac{\epsilon^{2n-2}(\log\epsilon)}{k^2}J_2^\#[\epsilon,\epsilon\log\epsilon]\Big\}, \quad (10.144)$$

*for all $\epsilon \in ]0, \epsilon_6[$, $l \in \mathbb{N} \setminus \{0\}$.*

*Proof.* Let $\epsilon \in ]0, \epsilon_6[$, $l \in \mathbb{N} \setminus \{0\}$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx\\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx\\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx\\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon]\Big(\frac{l}{r}x\Big)\,dx\\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon](t)\,dt\\ &= \frac{r^n}{l^n}\Big\{\frac{\epsilon^{n-1}}{k^2}J_1^\#[\epsilon,\epsilon\log\epsilon] + \frac{\epsilon^{2n-2}(\log\epsilon)}{k^2}J_2^\#[\epsilon,\epsilon\log\epsilon]\Big\}.\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = r^n\Big\{\frac{\epsilon^{n-1}}{k^2}J_1^\#[\epsilon,\epsilon\log\epsilon] + \frac{\epsilon^{2n-2}(\log\epsilon)}{k^2}J_2^\#[\epsilon,\epsilon\log\epsilon]\Big\},$$

and the conclusion follows. □

We give the following.

**Definition 10.100.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $\epsilon_3$ be as in Theorem 10.79 *(iv)* if $n$ is odd, and as in Theorem 10.80 *(iv)* if $n$ is even. For each pair $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, +\infty[$, we set

$$\mathcal{F}(\epsilon, \delta) \equiv \int_{A \cap \mathbb{T}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{A \cap \mathbb{T}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2 \, dx.$$

*Remark* 10.101. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $\epsilon_3$ be as in Theorem 10.79 *(iv)* if $n$ is odd, and as in Theorem 10.80 *(iv)* if $n$ is even. Let $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, +\infty[$. We have

$$\begin{aligned} \int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2 \, dt \\ &= \delta^{n-2} \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2 \, dt, \end{aligned}$$

and

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2 \, dx = \delta^n \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](t)|^2 \, dt.$$

Accordingly,

$$\begin{aligned} &\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2 \, dx \\ &\qquad = \delta^{n-2} \Big( \int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon](t)|^2 \, dt - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon](t)|^2 \, dt \Big). \end{aligned}$$

Then we give the following definition, where we consider $\mathcal{F}(\epsilon, \delta)$, with $\epsilon$ equal to a certain function of $\delta$.

**Definition 10.102.** For each $\delta \in ]0, +\infty[$, we set

$$\epsilon[\delta] \equiv \delta^{\frac{2}{n}}.$$

Let $\epsilon_5$ be as in Theorem 10.85, if $n$ is odd, or as in Theorem 10.86, if $n$ is even. Let $\delta_1 > 0$ be such that $\epsilon[\delta] \in ]0, \epsilon_5[$, for all $\delta \in ]0, \delta_1[$. Then we set

$$\mathcal{F}[\delta] \equiv \mathcal{F}(\epsilon[\delta], \delta),$$

for all $\delta \in ]0, \delta_1[$.

In the following Propositions we compute the limit of $\mathcal{F}[\delta]$ as $\delta$ tends to 0.

**Proposition 10.103.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_5$ be as in Theorem 10.22. Let $\delta_1 > 0$ be as in Definition 10.102. Then*

$$\lim_{\delta \to 0^+} \mathcal{F}[\delta] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\, \Omega} |\nabla \tilde{u}(x)|^2 \, dx,$$

*where $\tilde{u}$ is as in Definition 10.77.*

*Proof.* For each $\delta \in ]0, \delta_1[$, we set

$$\mathcal{G}(\delta) \equiv \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx.$$

Let $\delta \in ]0, \delta_1[$. By Remark 10.101 and Theorem 10.85, we have

$$\begin{aligned} \mathcal{G}(\delta) &= \delta^{n-2} (\epsilon[\delta])^n G[\epsilon[\delta]] \\ &= \delta^{n-2} \delta^2 G[\delta^{\frac{2}{n}}], \end{aligned}$$

where $G$ is as in Theorem 10.85. On the other hand,

$$\lfloor (1/\delta) \rfloor^n \mathcal{G}(\delta) \leq \mathcal{F}[\delta] \leq \lceil (1/\delta) \rceil^n \mathcal{G}(\delta).$$

As a consequence, since

$$\lim_{\delta\to 0^+} \lfloor (1/\delta) \rfloor^n \delta^n = 1, \qquad \lim_{\delta\to 0^+} \lceil (1/\delta) \rceil^n \delta^n = 1,$$

then

$$\lim_{\delta\to 0^+} \mathcal{F}[\delta] = G[0].$$

Finally, by equality (10.118), we easily conclude. □

**Proposition 10.104.** *Let $n$ be even and $n > 2$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_5$ be as in Theorem 10.23. Let $\delta_1 > 0$ be as in Definition 10.102. Then*

$$\lim_{\delta\to 0^+} \mathcal{F}[\delta] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2 \, dx,$$

*where $\tilde{u}$ is as in Definition 10.77.*

*Proof.* For each $\delta \in ]0, \delta_1[$, we set

$$\mathcal{G}(\delta) \equiv \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |\nabla u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{\mathbb{P}_a(\epsilon[\delta],\delta)} |u_{(\epsilon[\delta],\delta)}(x)|^2 \, dx.$$

Let $\delta \in ]0, \delta_1[$. By Remark 10.101 and Theorem 10.86, we have

$$\begin{aligned}
\mathcal{G}(\delta) =& \delta^{n-2} (\epsilon[\delta])^n G_1^{\#}[\epsilon[\delta], \epsilon[\delta] \log \epsilon[\delta]] \\
&+ \delta^{n-2} (\epsilon[\delta])^{2n-2} (\log \epsilon[\delta]) G_2^{\#}[\epsilon[\delta], \epsilon[\delta] \log \epsilon[\delta]] \\
&+ \delta^{n-2} (\epsilon[\delta])^{3n-3} (\log \epsilon[\delta])^2 G_3^{\#}[\epsilon[\delta], \epsilon[\delta] \log \epsilon[\delta]] \\
=& \delta^{n-2} \delta^2 G_1^{\#}[\delta^{\frac{2}{n}}, \delta^{\frac{2}{n}} \log(\delta^{\frac{2}{n}})] \\
&+ \delta^{n-2} \delta^{4-\frac{4}{n}} (\log(\delta^{\frac{2}{n}})) G_2^{\#}[\delta^{\frac{2}{n}}, \delta^{\frac{2}{n}} \log(\delta^{\frac{2}{n}})] \\
&+ \delta^{n-2} \delta^{6-\frac{6}{n}} (\log(\delta^{\frac{2}{n}}))^2 G_3^{\#}[\delta^{\frac{2}{n}}, \delta^{\frac{2}{n}} \log(\delta^{\frac{2}{n}})],
\end{aligned}$$

where $G_1^{\#}$, $G_2^{\#}$, and $G_3^{\#}$ are as in Theorem 10.86. On the other hand,

$$\lfloor (1/\delta) \rfloor^n \mathcal{G}(\delta) \leq \mathcal{F}[\delta] \leq \lceil (1/\delta) \rceil^n \mathcal{G}(\delta).$$

As a consequence, since

$$\lim_{\delta\to 0^+} \lfloor (1/\delta) \rfloor^n \delta^n = 1, \qquad \lim_{\delta\to 0^+} \lceil (1/\delta) \rceil^n \delta^n = 1,$$

then

$$\lim_{\delta\to 0^+} \mathcal{F}[\delta] = G_1^{\#}[0,0].$$

Finally, by equality (10.123), we easily conclude. □

In the following Propositions we represent the function $\mathcal{F}[\cdot]$ by means of real analytic functions.

**Proposition 10.105.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_5$, $G$ be as in Theorem 10.85. Let $\delta_1 > 0$ be as in Definition 10.102. Then*

$$\mathcal{F}[(1/l)] = G[(1/l)^{\frac{2}{n}}],$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* By arguing as in the proof of Proposition 10.103, one can easily see that

$$\begin{aligned}\mathcal{F}[(1/l)] &= l^n(1/l)^{n-2}(1/l)^2 G[(1/l)^{\frac{2}{n}}]\\ &= G[(1/l)^{\frac{2}{n}}],\end{aligned}$$

for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$. □

**Proposition 10.106.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_5$, $G_1^\#$, $G_2^\#$, and $G_3^\#$ be as in Theorem 10.86. Let $\delta_1 > 0$ be as in Definition 10.102. Then*

$$\begin{aligned}\mathcal{F}[(1/l)] =& G_1^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})]\\ &+ (1/l)^{2-\frac{4}{n}} \log((1/l)^{\frac{2}{n}}) G_2^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})]\\ &+ (1/l)^{4-\frac{6}{n}} \Big[\log((1/l)^{\frac{2}{n}})\Big]^2 G_3^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})],\end{aligned}$$

*for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* By arguing as in the proof of Proposition 10.103, one can easily see that

$$\begin{aligned}\mathcal{F}[(1/l)] &= l^n(1/l)^{n-2}(1/l)^2 \Big\{ G_1^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})]\\ &\quad + (1/l)^{2-\frac{4}{n}} \log((1/l)^{\frac{2}{n}}) G_2^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})]\\ &\quad + (1/l)^{4-\frac{6}{n}} \Big[\log((1/l)^{\frac{2}{n}})\Big]^2 G_3^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})] \Big\}\\ &= G_1^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})]\\ &\quad + (1/l)^{2-\frac{4}{n}} \log((1/l)^{\frac{2}{n}}) G_2^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})]\\ &\quad + (1/l)^{4-\frac{6}{n}} \Big[\log((1/l)^{\frac{2}{n}})\Big]^2 G_3^\#[(1/l)^{\frac{2}{n}}, (1/l)^{\frac{2}{n}} \log((1/l)^{\frac{2}{n}})],\end{aligned}$$

for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$. □

## 10.7 A variant of an homogenization problem for the Helmholtz equation with nonlinear Robin boundary conditions in a periodically perforated domain

In this section we consider another homogenization problem for the Helmhlotz equation with nonlinear Robin boundary conditions in a periodically perforated domain. As above, most of the results are obtained under the assumption that $\mathrm{Im}(k) \neq 0$ and $\mathrm{Re}(k) = 0$.

### 10.7.1 Notation and preliminaries

In this Section we retain the notation introduced in Subsections 10.5.1 and 10.3.1.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). For each $(\epsilon,\delta) \in ]0,\epsilon_1^*[ \times ]0,+\infty[$, we consider the following periodic nonlinear Robin problem for the Helmholtz equation.

$$\begin{cases} \Delta u(x) + \frac{k^2}{\delta^2} u(x) = 0 & \forall x \in \mathbb{T}_a(\epsilon,\delta),\\ u(x+\delta a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta), \quad \forall j \in \{1,\dots,n\},\\ \frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x) + F(\frac{1}{\epsilon\delta}(x-\delta w), u(x)) = 0 & \forall x \in \partial\Omega(\epsilon,\delta). \end{cases} \tag{10.145}$$

In contrast to problem (10.142), we note that in the third equation of problem (10.145) there is not the factor $\delta$ in front of $\frac{\partial}{\partial \nu_{\Omega(\epsilon,\delta)}} u(x)$.

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). For each $(\epsilon,\delta) \in ]0,\epsilon_1^*[ \times ]0,+\infty[$, we introduce the following auxiliary periodic nonlinear Robin problem for the Helmholtz equation.

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & \forall x \in \mathbb{T}_a[\Omega_\epsilon], \\ u(x + a_j) = u(x) & \forall x \in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \quad \forall j \in \{1, \ldots, n\}, \\ \frac{\partial}{\partial \nu_{\Omega_\epsilon}} u(x) + \delta F\big(\frac{1}{\epsilon}(x - w), u(x)\big) = 0 & \forall x \in \partial\Omega_\epsilon. \end{cases} \tag{10.146}$$

We now convert boundary value problem (10.146) into an integral equation.

**Proposition 10.107.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $(\epsilon, \delta) \in ]0, \epsilon_1^*[ \times ]0, +\infty[$. Then the map of the set of functions $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ that solve the equation*

$$\frac{1}{2}\mu(x) + \int_{\partial\Omega_\epsilon} \nu_{\Omega_\epsilon}(x) \cdot DS_n^{a,k}(x - y)\mu(y)\, d\sigma_y + \delta F\Big(\frac{1}{\epsilon}(x - w), \int_{\partial\Omega_\epsilon} S_n^{a,k}(x - y)\mu(y)\, d\sigma_y\Big) = 0 \qquad \forall x \in \partial\Omega_\epsilon, \tag{10.147}$$

*to the set of $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$ which solve problem* (10.146), *which takes $\mu$ to the function*

$$v_a^-[\partial\Omega_\epsilon, \mu, k] \tag{10.148}$$

*is a bijection.*

*Proof.* Assume that the function $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$ solves equation (10.147). Then, by Theorem 6.11, we immediately deduce that the function $u \equiv v_a^-[\partial\Omega_\epsilon, \mu, k]$ is a periodic function in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$, that satisfies the first condition of (10.146), and, by equation (10.147), also the third condition of (10.146). Thus, $u$ is a solution of (10.146). Conversely, let $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$ be a solution of problem (10.146). By Theorem 8.6, there exists a unique function $\mu \in C^{m-1,\alpha}(\partial\Omega_\epsilon, \mathbb{C})$, such that

$$u = v_a^-[\partial\Omega_\epsilon, \mu, k] \qquad \text{in } \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon].$$

Then, by Theorem 6.11, since $u$ satisfies in particular the third condition in (10.146), we immediately deduce that the function $\mu$ solves equation (10.147). □

As we have seen, we can transform (10.146) into an integral equation defined on the $\epsilon$-dependent domain $\partial\Omega_\epsilon$. In order to get rid of such a dependence, we shall introduce the following Theorem.

**Theorem 10.108.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0, 1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $(\epsilon, \delta) \in ]0, \epsilon_1^*[ \times ]0, +\infty[$. Then the map $u[\epsilon, \delta, \cdot]$ of the set of functions $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ that solve the equation*

$$\begin{aligned} &\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t - s, \epsilon k)\theta(s)\, d\sigma_s + \epsilon^{n-1}(\log\epsilon)k^{n-2} \int_{\partial\Omega} \nu_\Omega(t) \cdot DQ_n^k(\epsilon(t - s))\theta(s)\, d\sigma_s \\ &+ \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t - s))\theta(s)\, d\sigma_s + F\Big(t, \delta\epsilon \int_{\partial\Omega} S_n(t - s, \epsilon k)\theta(s)\, d\sigma_s \\ &+ \delta\epsilon^{n-1} \log\epsilon k^{n-2} \int_{\partial\Omega} Q_n^k(\epsilon(t - s))\theta(s)\, d\sigma_s + \delta\epsilon^{n-1} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t - s))\theta(s)\, d\sigma_s\Big) = 0 \ \forall t \in \partial\Omega, \end{aligned} \tag{10.149}$$

*to the set of $u \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon], \mathbb{C})$ which solve problem* (10.146), *which takes $\theta$ to the function*

$$u[\epsilon, \delta, \theta] \equiv v_a^-[\partial\Omega_\epsilon, \delta\theta(\frac{1}{\epsilon}(\cdot - w)), k] \tag{10.150}$$

*is a bijection.*

*Proof.* It is an immediate consequence of Proposition 10.107, of the Theorem of change of variables in integrals, and of equalities (6.24), (6.25). □

In the following Proposition we study equation (10.149) for $(\epsilon, \delta) = (0, 0)$.

**Proposition 10.109.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $F$ be as in* (1.56), (10.103), *respectively. Then the integral equation*

$$\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot DS_n(t-s)\theta(s)\, d\sigma_s + F(t,0) = 0 \qquad \forall t \in \partial\Omega, \tag{10.151}$$

*which we call the* limiting equation, *has a unique solution $\theta \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, which we denote by $\tilde{\theta}$. Moreover,*

$$\int_{\partial\Omega} \tilde{\theta}(s)\, d\sigma_s = -\int_{\partial\Omega} F(s,0)\, d\sigma_s. \tag{10.152}$$

*Proof.* It is Proposition 10.76. □

Now we want to see if equation (10.151) is related to some (limiting) boundary value problem. We give the following.

**Definition 10.110.** Let $n \geq 3$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $F$ be as in (1.56), (10.103), respectively. We denote by $\tilde{u}$ the unique solution in $C^{m,\alpha}(\mathbb{R}^n \setminus \Omega, \mathbb{C})$ of the following boundary value problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{R}^n \setminus \mathrm{cl}\,\Omega, \\ \frac{\partial}{\partial \nu_\Omega} u(x) = -F(x,0) & \forall x \in \partial\Omega, \\ \lim_{x \to \infty} u(x) = 0. \end{cases} \tag{10.153}$$

Problem (10.153) will be called the *limiting boundary value problem.*

*Remark* 10.111. Let $n \geq 3$. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$, $F$ be as in (1.56), (10.103), respectively. Let $\tilde{\theta}$ be as in Proposition 10.109. We have

$$\tilde{u}(x) = \int_{\partial\Omega} S_n(x-y)\tilde{\theta}(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n \setminus \Omega.$$

If $n = 2$, in general the (classic) simple layer potential for the Laplace equation with moment $\tilde{\theta}$ is not harmonic at infinity, and it does not satisfy the third condition of boundary value problem (10.153). Moreover, if $n = 2$, boundary value problem (10.153) does not have in general a solution (unless $\int_{\partial\Omega} F(s,0)\, d\sigma_s = 0$.) However, the function $\tilde{v}$ of $\mathbb{R}^2 \setminus \Omega$ to $\mathbb{C}$, defined by

$$\tilde{v}(x) \equiv \int_{\partial\Omega} S_2(x-y)\tilde{\theta}(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^2 \setminus \Omega,$$

is a solution of the following boundary value problem

$$\begin{cases} \Delta \tilde{v}(x) = 0 & \forall x \in \mathbb{R}^2 \setminus \mathrm{cl}\,\Omega, \\ \frac{\partial}{\partial \nu_\Omega} \tilde{v}(x) = -F(x,0) & \forall x \in \partial\Omega. \end{cases} \tag{10.154}$$

We are now ready to analyse equation (10.149) around the degenerate case $(\epsilon, \delta) = (0,0)$. However, since the function $Q_n^k$ that appears in equation (10.149) (involved in the determination of the moment of the simple layer potential that solves (10.145)) is identically 0 if $n$ is odd, it is preferable to treat separately case $n$ even and case $n$ odd.

**Theorem 10.112.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\tilde{\theta}$ be as in Proposition 10.109. Let $\Lambda$ be the map of $]-\epsilon_1^*, \epsilon_1^*[ \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, defined by*

$$\begin{aligned} &\Lambda[\epsilon,\delta,\theta](t) \\ &\quad \equiv \frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s,\epsilon k)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s \\ &\quad + F\Big(t, \delta\epsilon \int_{\partial\Omega} S_n(t-s,\epsilon k)\theta(s)\, d\sigma_s + \delta\epsilon^{n-1} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\, d\sigma_s\Big) \ \forall t \in \partial\Omega, \end{aligned} \tag{10.155}$$

*for all $(\epsilon, \delta, \theta) \in ]-\epsilon_1^*, \epsilon_1^*[ \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Then the following statements hold.*

*(i) Equation* $\Lambda[0,0,\theta] = 0$ *is equivalent to the limiting equation* (10.151) *and has one and only one solution* $\tilde{\theta}$ *in* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *(cf. Proposition 10.109.)*

*(ii) If* $(\epsilon,\delta) \in ]0,\epsilon_1^*[ \times ]0,+\infty[$*, then equation* $\Lambda[\epsilon,\delta,\theta] = 0$ *is equivalent to equation* (10.149) *for* $\theta$.

*(iii) There exists* $\epsilon_2 \in ]0,\epsilon_1^*]$*, such that the map* $\Lambda$ *of* $]-\epsilon_2,\epsilon_2[ \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *to* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *is real analytic. Moreover, the differential* $\partial_\theta\Lambda[0,0,\tilde{\theta}]$ *of* $\Lambda$ *at* $(0,0,\tilde{\theta})$ *is a linear homeomorphism of* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *onto* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$.

*(iv) There exist* $\epsilon_3 \in ]0,\epsilon_2]$*,* $\delta_1 \in ]0,+\infty[$*, an open neighbourhood* $\tilde{\mathcal{U}}$ *of* $\tilde{\theta}$ *in* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *and a real analytic map* $\Theta_n[\cdot,\cdot]$ *of* $]-\epsilon_3,\epsilon_3[ \times ]-\delta_1,\delta_1[$ *to* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$*, such that the set of zeros of the map* $\Lambda$ *in* $]-\epsilon_3,\epsilon_3[ \times ]-\delta_1,\delta_1[ \times \tilde{\mathcal{U}}$ *coincides with the graph of* $\Theta_n[\cdot,\cdot]$*. In particular,* $\Theta_n[0,0] = \tilde{\theta}$.

*Proof.* Statements (*i*) and (*ii*) are obvious. By arguing as in the proof of statement (*iii*) of Theorem 10.79, we immediately deduce that there exists $\epsilon_2 \in ]0,\epsilon_1^*]$, such that the map $\Lambda$ of $]-\epsilon_2,\epsilon_2[ \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ is real analytic. By standard calculus in Banach spaces, the differential $\partial_\theta\Lambda[0,0,\tilde{\theta}]$ of $\Lambda$ at $(0,0,\tilde{\theta})$ is delivered by the following formula:

$$\partial_\theta\Lambda[0,0,\tilde{\theta}](\tau)(t) = \frac{1}{2}\tau(t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\tau(s)\,d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $\tau \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By the proof of statement (*iii*) of Theorem 10.79, the above differential is a linear homeomorphism. Finally, statement (*iv*) is an immediate consequence of statement (*iii*) and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

**Theorem 10.113.** *Let* $n$ *be even. Let* $m \in \mathbb{N}\setminus\{0\}$*,* $\alpha \in ]0,1[$*. Let* $w \in A$*. Let* $\Omega$*,* $\epsilon_1$*,* $k$*,* $F$ *be as in* (1.56), (1.57), (10.102), (10.103)*, respectively. Let* $\epsilon_1^*$ *be as in* (10.104)*. Let* $\tilde{\theta}$ *be as in Proposition 10.109. Let* $\epsilon_1' > 0$ *be such that*

$$\epsilon\log\epsilon \in ]-\epsilon_1',\epsilon_1'[ \qquad \forall \epsilon \in ]0,\epsilon_1^*[. \tag{10.156}$$

*Let* $\Lambda^\#$ *be the map of* $]-\epsilon_1^*,\epsilon_1^*[ \times ]-\epsilon_1',\epsilon_1'[ \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *to* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$*, defined by*

$$\begin{aligned}
&\Lambda^\#[\epsilon,\epsilon',\delta,\theta](t)\\
&\equiv \frac{1}{2}\theta(t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s + \epsilon^{n-2}\epsilon' k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\theta(s)\,d\sigma_s\\
&+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s + F\Big(t,\delta\epsilon\int_{\partial\Omega}S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s\\
&+\delta\epsilon^{n-2}\epsilon' k^{n-2}\int_{\partial\Omega}Q_n^k(\epsilon(t-s))\theta(s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega}R_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s\Big) \ \forall t\in\partial\Omega,
\end{aligned} \tag{10.157}$$

*for all* $(\epsilon,\epsilon',\delta,\theta) \in ]-\epsilon_1^*,\epsilon_1^*[ \times ]-\epsilon_1',\epsilon_1'[ \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$*. Then the following statements hold.*

*(i) Equation* $\Lambda^\#[0,0,0,\theta] = 0$ *is equivalent to the limiting equation* (10.151) *and has one and only one solution* $\tilde{\theta}$ *in* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *(cf. Proposition 10.109.)*

*(ii) If* $(\epsilon,\delta) \in ]0,\epsilon_1^*[ \times ]0,+\infty[$*, then equation* $\Lambda^\#[\epsilon,\epsilon\log\epsilon,\delta,\theta] = 0$ *is equivalent to equation* (10.149) *for* $\theta$.

*(iii) There exists* $\epsilon_2 \in ]0,\epsilon_1^*]$*, such that the map* $\Lambda^\#$ *of* $]-\epsilon_2,\epsilon_2[ \times ]-\epsilon_1',\epsilon_1'[ \times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *to* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *is real analytic. Moreover, the differential* $\partial_\theta\Lambda^\#[0,0,0,\tilde{\theta}]$ *of* $\Lambda^\#$ *at* $(0,0,0,\tilde{\theta})$ *is a linear homeomorphism of* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *onto* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$.

*(iv) There exist* $\epsilon_3 \in ]0,\epsilon_2]$*,* $\epsilon_2' \in ]0,\epsilon_1']$*,* $\delta_1 \in ]0,+\infty[$*, an open neighbourhood* $\tilde{\mathcal{U}}$ *of* $\tilde{\theta}$ *in* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *and a real analytic map* $\Theta_n^\#[\cdot,\cdot,\cdot]$ *of* $]-\epsilon_3,\epsilon_3[ \times ]-\epsilon_2',\epsilon_2'[ \times ]-\delta_1,\delta_1[$ *to* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$*, such that* $\epsilon\log\epsilon \in ]-\epsilon_2',\epsilon_2'[$ *for all* $\epsilon \in ]0,\epsilon_3[$ *and such that the set of zeros of the map* $\Lambda^\#$ *in* $]-\epsilon_3,\epsilon_3[\times]-\epsilon_2',\epsilon_2'[\times]-\delta_1,\delta_1[\times\tilde{\mathcal{U}}$ *coincides with the graph of* $\Theta_n^\#[\cdot,\cdot,\cdot]$*. In particular,* $\Theta_n^\#[0,0,0] = \tilde{\theta}$.

*Proof.* Statements $(i)$ and $(ii)$ are obvious. By arguing as in the proof of statement $(iii)$ of Theorem 10.80, we deduce that there exists $\epsilon_2 \in \,]0,\epsilon_1^*]$, such that the map $\Lambda^\#$ of $]-\epsilon_2,\epsilon_2[\,\times\,]-\epsilon_1',\epsilon_1'[\,\times\,\mathbb{R}\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ is real analytic. By standard calculus in Banach spaces, the differential $\partial_\theta\Lambda^\#[0,0,0,\tilde{\theta}]$ of $\Lambda^\#$ at $(0,0,0,\tilde{\theta})$ is delivered by the following formula:

$$\partial_\theta\Lambda^\#[0,0,0,\tilde{\theta}](\tau)(t) = \frac{1}{2}\tau(t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot DS_n(t-s)\tau(s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $\tau \in C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By the proof of statement $(iii)$ of Theorem 10.80, the above differential is a linear homeomorphism. Finally, statement $(iv)$ is an immediate consequence of statement $(iii)$ and of the Implicit Function Theorem for real analytic maps in Banach spaces (cf. *e.g.*, Prodi and Ambrosetti [116, Theorem 11.6], Deimling [46, Theorem 15.3].) □

We are now in the position to introduce the following.

**Definition 10.114.** Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $u[\cdot,\cdot,\cdot]$ be as in Theorem 10.108. If $n$ is odd and $(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[$, we set

$$u[\epsilon,\delta](x) \equiv u[\epsilon,\delta,\Theta_n[\epsilon,\delta]](x) \qquad \forall x\in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],$$

where $\epsilon_3$, $\delta_1$, $\Theta_n$ are as in Theorem 10.112 $(iv)$. If $n$ is even and $(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[$, we set

$$u[\epsilon,\delta](x) \equiv u[\epsilon,\delta,\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta]](x) \qquad \forall x\in \mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],$$

where $\epsilon_3$, $\delta_1$, $\Theta_n^\#$ are as in Theorem 10.113 $(iv)$.

*Remark* 10.115. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $\epsilon_3$, $\delta_1$ be as in Theorem 10.112 $(iv)$ if $n$ is odd and as in Theorem 10.113 $(iv)$ if $n$ is even. Let $(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[$. Then $u[\epsilon,\delta]$ is a solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_\epsilon],\mathbb{C})$ of problem (10.146).

By Theorems 10.112, 10.113 and Definition 10.114, we can deduce the following results.

**Theorem 10.116.** *Let $n$ be odd. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_3$, $\delta_1$ be as in Theorem 10.112 $(iv)$. Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\,V\cap\mathbb{S}_a[\Omega_0]=\emptyset$. Then there exist $\epsilon_4\in\,]0,\epsilon_3]$, and a real analytic operator $U$ of $]-\epsilon_4,\epsilon_4[\,\times\,]-\delta_1,\delta_1[$ to the space $C^0(\mathrm{cl}\,V,\mathbb{C})$, such that the following conditions hold.*

*(j)* $\mathrm{cl}\,V\subseteq\mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon\in\,]-\epsilon_4,\epsilon_4[$.

*(jj)*

$$u[\epsilon,\delta](x) = \delta\epsilon^{n-1}U[\epsilon,\delta](x) \qquad \forall x\in\mathrm{cl}\,V,$$

*for all $(\epsilon,\delta)\in\,]0,\epsilon_4[\,\times\,]0,\delta_1[$. Moreover,*

$$U[0,0](x) = -S_n^{a,k}(x-w)\int_{\partial\Omega}F(s,0)\,d\sigma_s \qquad \forall x\in\mathrm{cl}\,V.$$

*Proof.* Let $\Theta_n[\cdot,\cdot]$ be as in Theorem 10.112 $(iv)$. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $(\epsilon,\delta)\in\,]0,\epsilon_4[\,\times\,]0,\delta_1[$. We have

$$u[\epsilon,\delta](x) = \delta\epsilon^{n-1}\int_{\partial\Omega}S_n^{a,k}(x-w-\epsilon s)\Theta_n[\epsilon,\delta](s)\,d\sigma_s \qquad \forall x\in\mathrm{cl}\,V.$$

Thus, it is natural to set

$$U[\epsilon,\delta](x) \equiv \int_{\partial\Omega}S_n^{a,k}(x-w-\epsilon s)\Theta_n[\epsilon,\delta](s)\,d\sigma_s \qquad \forall x\in\mathrm{cl}\,V,$$

for all $(\epsilon,\delta)\in\,]-\epsilon_4,\epsilon_4[\,\times\,]-\delta_1,\delta_1[$. By Proposition 6.22, $U$ is a real analytic map of $]-\epsilon_4,\epsilon_4[\,\times\,]-\delta_1,\delta_1[$ to $C^0(\mathrm{cl}\,V,\mathbb{C})$. Furthermore, by Proposition 10.109, we have

$$\begin{aligned} U[0,0](x) &= S_n^{a,k}(x-w)\int_{\partial\Omega}\Theta_n[0,0](s)\,d\sigma_s \\ &= -S_n^{a,k}(x-w)\int_{\partial\Omega}F(s,0)\,d\sigma_s \qquad \forall x\in\mathrm{cl}\,V, \end{aligned}$$

since $\Theta_n[0,0]=\tilde{\theta}$. Hence the proof is now complete. □

**Theorem 10.117.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_3$, $\epsilon_2'$, $\delta_1$ be as in Theorem 10.113 (iv). Let $V$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\, V \cap \mathbb{S}_a[\Omega_0] = \emptyset$. Then there exist $\epsilon_4 \in ]0, \epsilon_3]$ and a real analytic operator $U^\#$ of $]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[ \times ]-\delta_1, \delta_1[$ to the space $C^0(\mathrm{cl}\, V, \mathbb{C})$, such that the following conditions hold.*

*(j)* $\mathrm{cl}\, V \subseteq \mathbb{T}_a[\Omega_\epsilon]$ *for all* $\epsilon \in ]-\epsilon_4, \epsilon_4[$.

*(jj)*
$$u[\epsilon, \delta](x) = \delta \epsilon^{n-1} U^\#[\epsilon, \epsilon \log \epsilon, \delta](x) \qquad \forall x \in \mathrm{cl}\, V,$$
*for all* $(\epsilon, \delta) \in ]0, \epsilon_4[ \times ]0, \delta_1[$. *Moreover,*
$$U^\#[0,0,0](x) = -S_n^{a,k}(x-w) \int_{\partial\Omega} F(s,0)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V.$$

*Proof.* Let $\Theta_n^\#[\cdot,\cdot,\cdot]$ be as in Theorem 10.113 $(iv)$. Choosing $\epsilon_4$ small enough, we can clearly assume that $(j)$ holds. Consider now $(jj)$. Let $(\epsilon, \delta) \in ]0, \epsilon_4[ \times ]0, \delta_1[$. We have
$$u[\epsilon, \delta](x) = \delta \epsilon^{n-1} \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s) \Theta_n^\#[\epsilon, \epsilon \log \epsilon, \delta](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V.$$
Thus, it is natural to set
$$U^\#[\epsilon, \epsilon', \delta](x) \equiv \int_{\partial\Omega} S_n^{a,k}(x - w - \epsilon s) \Theta_n^\#[\epsilon, \epsilon', \delta](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V,$$
for all $(\epsilon, \epsilon', \delta) \in ]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[ \times ]-\delta_1, \delta_1[$. By Proposition 6.22, $U^\#$ is a real analytic map of $]-\epsilon_4, \epsilon_4[ \times ]-\epsilon_2', \epsilon_2'[ \times ]-\delta_1, \delta_1[$ to $C^0(\mathrm{cl}\, V, \mathbb{C})$. Furthermore, by Proposition 10.109, we have
$$\begin{aligned} U^\#[0,0,0](x) &= S_n^{a,k}(x-w) \int_{\partial\Omega} \Theta_n^\#[0,0,0](s)\, d\sigma_s \\ &= -S_n^{a,k}(x-w) \int_{\partial\Omega} F(s,0)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\, V, \end{aligned}$$
since $\Theta_n^\#[0,0,0] = \tilde{\theta}$. Accordingly, the Theorem is now completely proved. □

We have also the following Theorems.

**Theorem 10.118.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_3$, $\delta_1$ be as in Theorem 10.112 (iv). Then there exist $\epsilon_5 \in ]0, \epsilon_3]$, and a real analytic operator $G$ of $]-\epsilon_5, \epsilon_5[ \times ]-\delta_1, \delta_1[$ to $\mathbb{C}$, such that*
$$\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon, \delta](x)|^2\, dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon, \delta](x)|^2\, dx = \delta^2 \epsilon^n G[\epsilon, \delta], \tag{10.158}$$
*for all* $(\epsilon, \delta) \in ]0, \epsilon_5[ \times ]0, \delta_1[$. *Moreover,*
$$G[0,0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\, \Omega} |\nabla \tilde{u}(x)|^2\, dx, \tag{10.159}$$
*where $\tilde{u}$ is as in Definition 10.110.*

*Proof.* Let $\Theta_n[\cdot,\cdot]$ be as in Theorem 10.112 $(iv)$. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[$. Clearly, by the periodicity of $u[\epsilon, \delta]$, we have
$$\begin{aligned} &\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon, \delta](x)|^2\, dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon, \delta](x)|^2\, dx \\ &\qquad = -\epsilon^{n-1} \int_{\partial\Omega} \Big(\frac{\partial u[\epsilon, \delta]}{\partial \nu_{\Omega_\epsilon}}\Big) \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) \overline{u[\epsilon, \delta] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t)}\, d\sigma_t. \end{aligned}$$

By equality (6.24) and since $Q_n^k = 0$ for $n$ odd, we have

$$\begin{aligned} u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) &= \delta\epsilon^{n-1}\int_{\partial\Omega} S_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s \\ &= \delta\epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s \quad \forall t\in\partial\Omega. \end{aligned}$$

By Theorem E.6 ($i$), one can easily show that the map which takes $(\epsilon,\delta)$ to the function of the variable $t\in\partial\Omega$ defined by

$$\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

is a real analytic operator of $]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By Theorem C.4, we immediately deduce that there exists $\epsilon_5\in]0,\epsilon_3]$ such that the map of $]-\epsilon_5,\epsilon_5[\times]-\delta_1,\delta_1[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, which takes $(\epsilon,\delta)$ to the function $\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s$ of the variable $t\in\partial\Omega$, is real analytic. Analogously, we have

$$\begin{aligned} \Big(\frac{\partial u[\epsilon,\delta]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) &= \delta\frac{1}{2}\Theta_n[\epsilon,\delta](t) + \delta\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s \\ &+ \delta\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega, \end{aligned}$$

for all $(\epsilon,\delta)\in]0,\epsilon_3[\times]0,\delta_1[$. Thus, if we set

$$\begin{aligned} \tilde G[\epsilon,\delta](t) \equiv& \frac{1}{2}\Theta_n[\epsilon,\delta](t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s \\ &+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,\ \forall(\epsilon,\delta)\in]-\epsilon_5,\epsilon_5[\times]-\delta_1,\delta_1[ \end{aligned}$$

then, by arguing as in Theorem 10.112, one can easily show that $\tilde G$ is a real analytic map of $]-\epsilon_5,\epsilon_5[\times]-\delta_1,\delta_1[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$.

Hence, if we set

$$\begin{aligned} G[\epsilon,\delta] \equiv& -\int_{\partial\Omega}\tilde G[\epsilon,\delta](t)\overline{\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s}\,d\sigma_t \\ &-\epsilon^{n-2}\int_{\partial\Omega}\tilde G[\epsilon,\delta](t)\overline{\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s}\,d\sigma_t, \end{aligned}$$

for all $(\epsilon,\delta)\in]-\epsilon_5,\epsilon_5[\times]-\delta_1,\delta_1[$, then by standard properties of functions in Schauder spaces, we have that $G$ is a real analytic map of $]-\epsilon_5,\epsilon_5[\times]-\delta_1,\delta_1[$ to $\mathbb{C}$ such that equality (10.158) holds.

Finally, if $(\epsilon,\delta)=(0,0)$, by Folland [52, p. 118] and since $\tilde G[0,0](\cdot) = -F(\cdot,0)$, we have

$$\begin{aligned} G[0,0] &= \int_{\partial\Omega} F(t,0)\overline{\int_{\partial\Omega} S_n(t-s)\tilde\theta(s)\,d\sigma_s}\,d\sigma_t \\ &= \int_{\mathbb{R}^n\setminus\mathrm{cl}\,\Omega}|\nabla\tilde u(x)|^2\,dx. \end{aligned}$$

□

**Theorem 10.119.** *Let $n$ be even. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\tilde\theta$ be as in Proposition 10.109. Let $\epsilon_3$, $\epsilon_2'$, $\delta_1$ be as in Theorem 10.113 (iv). Then there exist $\epsilon_5\in]0,\epsilon_3]$, and three real analytic operators $G_1^\#$, $G_2^\#$, $G_3^\#$ of $]-\epsilon_5,\epsilon_5[\times]-\epsilon_2',\epsilon_2'[\times]-\delta_1,\delta_1[$ to $\mathbb{C}$, such that*

$$\begin{aligned} &\int_{\mathbb{P}_a[\Omega_\epsilon]}|\nabla u[\epsilon,\delta](x)|^2\,dx - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]}|u[\epsilon,\delta](x)|^2\,dx \\ &\quad= \delta^2\epsilon^n G_1^\#[\epsilon,\epsilon\log\epsilon,\delta] + \delta^2\epsilon^{2n-2}(\log\epsilon)G_2^\#[\epsilon,\epsilon\log\epsilon,\delta] + \delta^2\epsilon^{3n-3}(\log\epsilon)^2G_3^\#[\epsilon,\epsilon\log\epsilon,\delta], \end{aligned} \tag{10.160}$$

*for all* $(\epsilon, \delta) \in ]0, \epsilon_5[ \times ]0, \delta_1[$. *Moreover,*

$$\begin{aligned} G_1^{\#}[0,0,0] = &\int_{\partial\Omega} F(t,0)\overline{\int_{\partial\Omega} S_n(t-s)\tilde{\theta}(s)\,d\sigma_s}\,d\sigma_t \\ &- \delta_{2,n}\overline{R_n^{a,k}(0)}\Big|\int_{\partial\Omega} F(s,0)\,d\sigma_s\Big|^2, \end{aligned} \tag{10.161}$$

$$G_2^{\#}[0,0,0] = -\overline{k}^{n-2}\mathcal{J}_n(0)\Big|\int_{\partial\Omega} F(s,0)\,d\sigma_s\Big|^2, \tag{10.162}$$

$$G_3^{\#}[0,0,0] = 0 \tag{10.163}$$

*where* $\mathcal{J}_n(0)$ *is as in Proposition E.3 (i). In particular, if* $n > 2$*, then*

$$G_1^{\#}[0,0,0] = \int_{\mathbb{R}^n \setminus \mathrm{cl}\,\Omega} |\nabla \tilde{u}(x)|^2\,dx, \tag{10.164}$$

*where* $\tilde{u}$ *is as in Definition 10.110.*

*Proof.* Let $\Theta_n^{\#}[\cdot,\cdot,\cdot]$ be as in Theorem 10.113 (*iv*). Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[$. Clearly, by the periodicity of $u[\epsilon, \delta]$, we have

$$\begin{aligned} \int_{\mathbb{P}_a[\Omega_\epsilon]} &|\nabla u[\epsilon,\delta](x)|^2\,dx - k^2 \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon,\delta](x)|^2\,dx \\ &= -\epsilon^{n-1}\int_{\partial\Omega}\Big(\frac{\partial u[\epsilon,\delta]}{\partial \nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\overline{u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)}\,d\sigma_t. \end{aligned}$$

By equality (6.24), we have

$$\begin{aligned} u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) =& \delta\epsilon^{n-1}\int_{\partial\Omega} S_n^{a,k}(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s \\ =& \delta\epsilon\int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^{\#}[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s \\ &+ \delta\epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s \\ &+ \delta\epsilon^{n-1}\int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s \qquad \forall t \in \partial\Omega. \end{aligned}$$

Thus it is natural to set

$$\begin{aligned} F_1[\epsilon,\epsilon',\delta](t) &\equiv \int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n^{\#}[\epsilon,\epsilon',\delta](s)\,d\sigma_s \qquad \forall t \in \partial\Omega, \\ F_2[\epsilon,\epsilon',\delta](t) &\equiv k^{n-2}\int_{\partial\Omega} Q_n^k(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon',\delta](s)\,d\sigma_s \qquad \forall t \in \partial\Omega, \\ F_3[\epsilon,\epsilon',\delta](t) &\equiv \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon',\delta](s)\,d\sigma_s \qquad \forall t \in \partial\Omega, \end{aligned}$$

for all $(\epsilon,\epsilon',\delta) \in ]-\epsilon_3,\epsilon_3[ \times ]-\epsilon_2',\epsilon_2'[ \times ]-\delta_1,\delta_1[$.

Then clearly

$$u[\epsilon,\delta]\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \delta\epsilon F_1[\epsilon,\epsilon\log\epsilon,\delta](t) + \delta\epsilon^{n-1}(\log\epsilon)F_2[\epsilon,\epsilon\log\epsilon,\delta](t) + \delta\epsilon^{n-1}F_3[\epsilon,\epsilon\log\epsilon,\delta](t) \qquad \forall t \in \partial\Omega,$$

for all $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[$. By Theorem E.6 (*i*) and Theorem C.4, we easily deduce that there exists $\epsilon_5 \in ]0,\epsilon_3]$ such that the maps $F_1$, $F_2$, and $F_3$ of $]-\epsilon_5,\epsilon_5[ \times ]-\epsilon_2',\epsilon_2'[ \times ]-\delta_1,\delta_1[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ are real analytic. Analogously, we have

$$\begin{aligned}
&\Big(\frac{\partial u[\epsilon,\delta]}{\partial \nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\\
&\qquad=\delta\frac{1}{2}\Theta_n^{\#}[\epsilon,\epsilon\log\epsilon,\delta](t)+\delta\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n^{\#}[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s\\
&\qquad+\delta\epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s\\
&\qquad+\delta\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s\qquad\forall t\in\partial\Omega,
\end{aligned}$$

for all $(\epsilon,\delta)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[$. Thus, if we set

$$\begin{aligned}
\tilde{G}_1[\epsilon,\epsilon',\delta](t)\equiv{}&\frac{1}{2}\Theta_n^{\#}[\epsilon,\epsilon',\delta](t)+\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n^{\#}[\epsilon,\epsilon',\delta](s)\,d\sigma_s\\
&+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon',\delta](s)\,d\sigma_s\qquad\forall t\in\partial\Omega,
\end{aligned}$$

and

$$\tilde{G}_2[\epsilon,\epsilon',\delta](t)\equiv k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon',\delta](s)\,d\sigma_s\qquad\forall t\in\partial\Omega,$$

for all $(\epsilon,\epsilon',\delta)\in\,]-\epsilon_5,\epsilon_5[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\delta_1,\delta_1[$, then, by arguing as in Theorem 10.113, one can easily show that $\tilde{G}_1$ and $\tilde{G}_2$ are real analytic maps of $]-\epsilon_5,\epsilon_5[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\delta_1,\delta_1[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$.

Clearly,

$$\Big(\frac{\partial u[\epsilon,\delta]}{\partial \nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)=\delta\tilde{G}_1[\epsilon,\epsilon\log\epsilon,\delta](t)+\delta\epsilon^{n-1}(\log\epsilon)\tilde{G}_2[\epsilon,\epsilon\log\epsilon,\delta](t)$$
$$\forall t\in\partial\Omega,\ \forall(\epsilon,\delta)\in\,]0,\epsilon_5[\,\times\,]0,\delta_1[.$$

If $(\epsilon,\delta)\in\,]0,\epsilon_5[\,\times\,]0,\delta_1[$, then we have

$$\begin{aligned}
&\int_{\mathbb{P}_a[\Omega_\epsilon]}|\nabla u[\epsilon,\delta](x)|^2\,dx-k^2\int_{\mathbb{P}_a[\Omega_\epsilon]}|u[\epsilon,\delta](x)|^2\,dx\\
&=\delta^2\Bigg\{\epsilon^n\Big(-\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon\log\epsilon,\delta]\overline{F_1[\epsilon,\epsilon\log\epsilon,\delta]}\,d\sigma-\epsilon^{n-2}\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon\log\epsilon,\delta]\overline{F_3[\epsilon,\epsilon\log\epsilon,\delta]}\,d\sigma\Big)\\
&\quad+\epsilon^{2n-2}\log\epsilon\Big(-\epsilon\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon\log\epsilon,\delta]\overline{F_1[\epsilon,\epsilon\log\epsilon,\delta]}\,d\sigma-\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon\log\epsilon,\delta]\overline{F_2[\epsilon,\epsilon\log\epsilon,\delta]}\,d\sigma\\
&\qquad-\epsilon^{n-1}\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon\log\epsilon,\delta]\overline{F_3[\epsilon,\epsilon\log\epsilon,\delta]}\,d\sigma\Big)\\
&\quad+\epsilon^{3n-3}(\log\epsilon)^2\Big(-\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon\log\epsilon,\delta]\overline{F_2[\epsilon,\epsilon\log\epsilon,\delta]}\,d\sigma\Big)\Bigg\}.
\end{aligned}$$

If we set

$$\begin{aligned}
G_1^{\#}[\epsilon,\epsilon',\delta]\equiv{}&-\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon',\delta](t)\overline{F_1[\epsilon,\epsilon',\delta](t)}\,d\sigma_t-\epsilon^{n-2}\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon',\delta](t)\overline{F_3[\epsilon,\epsilon',\delta](t)}\,d\sigma_t,\\
G_2^{\#}[\epsilon,\epsilon',\delta]\equiv{}&-\epsilon\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon',\delta](t)\overline{F_1[\epsilon,\epsilon',\delta](t)}\,d\sigma_t-\int_{\partial\Omega}\tilde{G}_1[\epsilon,\epsilon',\delta](t)\overline{F_2[\epsilon,\epsilon',\delta](t)}\,d\sigma_t\\
&-\epsilon^{n-1}\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon',\delta](t)\overline{F_3[\epsilon,\epsilon',\delta](t)}\,d\sigma_t,\\
G_3^{\#}[\epsilon,\epsilon',\delta]\equiv{}&-\int_{\partial\Omega}\tilde{G}_2[\epsilon,\epsilon',\delta](t)\overline{F_2[\epsilon,\epsilon',\delta](t)}\,d\sigma_t,
\end{aligned}$$

for all $(\epsilon,\epsilon',\delta)\in\,]-\epsilon_5,\epsilon_5[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\delta_1,\delta_1[$, then standard properties of functions in Schauder spaces and a simple computation show that $G_1^{\#}$, $G_2^{\#}$, and $G_3^{\#}$ are real analytic maps of $]-\epsilon_5,\epsilon_5[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\delta_1,\delta_1[$ in $\mathbb{C}$ such that equality (10.160) holds for all $(\epsilon,\delta)\in\,]0,\epsilon_5[\,\times\,]0,\delta_1[$.

Next, we observe that

$$G_1^\#[0,0,0] = \int_{\partial\Omega} F(t,0)\overline{\int_{\partial\Omega} S_n(t-s)\tilde{\theta}(s)\,d\sigma_s}\,d\sigma_t - \delta_{2,n}\overline{R_n^{a,k}(0)}|\int_{\partial\Omega} F(s,0)\,d\sigma_s|^2,$$

$$G_2^\#[0,0,0] = -\overline{k}^{n-2}\overline{Q_n^k(0)}\int_{\partial\Omega} F(s,0)\,d\sigma_s\overline{\int_{\partial\Omega} F(s,0)\,d\sigma_s},$$

$$G_3^\#[0,0,0] = -k^{n-2}\overline{k^{n-2}Q_n^k(0)}\int_{\partial\Omega} F(s,0)\,d\sigma_s\overline{\int_{\partial\Omega} F(s,0)\,d\sigma_s}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(0)\,d\sigma_t = 0,$$

and accordingly equalities (10.161), (10.162), and (10.163) hold. In particular, if $n \geq 4$, by Folland [52, p. 118], we have

$$G_1^\#[0,0,0] = \int_{\mathbb{R}^n\setminus \mathrm{cl}\,\Omega} |\nabla\tilde{u}(x)|^2\,dx.$$

□

**Theorem 10.120.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_3$, $\delta_1$ be as in Theorem 10.112 (iv). Then there exist $\epsilon_6 \in ]0,\epsilon_3]$, and a real analytic operator $J$ of $]-\epsilon_6,\epsilon_6[\times]-\delta_1,\delta_1[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,\delta](x)\,dx = \frac{\delta\epsilon^{n-1}}{k^2}J[\epsilon,\delta], \tag{10.165}$$

*for all $(\epsilon,\delta) \in ]0,\epsilon_6[\times]0,\delta_1[$. Moreover,*

$$J[0,0] = -\int_{\partial\Omega} F(x,0)\,d\sigma_x. \tag{10.166}$$

*Proof.* Let $\Theta_n[\cdot,\cdot]$ be as in Theorem 10.112 *(iv)*. Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $(\epsilon,\delta) \in ]0,\epsilon_3[\times]0,\delta_1[$. Clearly, by the Divergence Theorem and the periodicity of $u[\epsilon,\delta]$, we have

$$\begin{aligned}
\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,\delta](x)\,dx &= -\frac{1}{k^2}\int_{\mathbb{P}_a[\Omega_\epsilon]}\Delta u[\epsilon,\delta](x)\,dx \\
&= -\frac{1}{k^2}\int_{\partial\mathbb{P}_a[\Omega_\epsilon]}\frac{\partial}{\partial\nu_{\mathbb{P}_a[\Omega_\epsilon]}}u[\epsilon,\delta](x)\,d\sigma_x \\
&= -\frac{1}{k^2}\Big[\int_{\partial A}\frac{\partial}{\partial\nu_A}u[\epsilon,\delta](x)\,d\sigma_x - \int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon,\delta](x)\,d\sigma_x\Big] \\
&= \frac{1}{k^2}\int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon,\delta](x)\,d\sigma_x.
\end{aligned}$$

As a consequence,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,\delta](x)\,dx = \frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega}\Big(\frac{\partial u[\epsilon,\delta]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\,d\sigma_t.$$

By equality (6.25) and since $Q_n^k = 0$ for $n$ odd, we have

$$\begin{aligned}
\Big(\frac{\partial u[\epsilon,\delta]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t) = \delta\frac{1}{2}\Theta_n[\epsilon,\delta](t) + \delta\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s \\
+\delta\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega.
\end{aligned}$$

We set

$$\begin{aligned}
\tilde{J}[\epsilon,\delta](t) \equiv &\frac{1}{2}\Theta_n[\epsilon,\delta](t) + \int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s \\
&+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s \quad \forall t\in\partial\Omega,
\end{aligned}$$

for all $(\epsilon,\delta)\in ]-\epsilon_3,\epsilon_3[\times]-\delta_1,\delta_1[$. By Theorem E.6 (*ii*) and Theorem C.4, one can easily show that there exists $\epsilon_6\in ]0,\epsilon_3]$ such that $\tilde{J}$ is a real analytic map of $]-\epsilon_6,\epsilon_6[\times]-\delta_1,\delta_1[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Hence, if we set

$$J[\epsilon,\delta]\equiv\int_{\partial\Omega}\tilde{J}[\epsilon,\delta](t)\,d\sigma_t,$$

for all $(\epsilon,\delta)\in ]-\epsilon_6,\epsilon_6[\times]-\delta_1,\delta_1[$, then, by standard properties of functions in Schauder spaces, we have that $J$ is a real analytic map of $]-\epsilon_6,\epsilon_6[\times]-\delta_1,\delta_1[$ to $\mathbb{C}$, such that equality (10.165) holds.

Finally, if $(\epsilon,\delta)=(0,0)$, we have

$$\begin{aligned}J[0,0]&=\int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega}v^-[\partial\Omega,\tilde{\theta},0](t)\,d\sigma_t\\&=-\int_{\partial\Omega}F(x,0)\,d\sigma_x,\end{aligned}$$

and accordingly (10.166) holds. □

**Theorem 10.121.** *Let $n$ be even. Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $w\in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\tilde{\theta}$ be as in Proposition 10.109. Let $\epsilon_3$, $\epsilon_2'$, $\delta_1$ be as in Theorem 10.113 (iv). Then there exist $\epsilon_6\in]0,\epsilon_3]$, and two real analytic operators $J_1^\#$, $J_2^\#$ of $]-\epsilon_6,\epsilon_6[\times]-\epsilon_2',\epsilon_2'[\times]-\delta_1,\delta_1[$ to $\mathbb{C}$, such that*

$$\int_{\mathbb{P}_a[\Omega_\epsilon]}u[\epsilon,\delta](x)\,dx=\frac{\delta\epsilon^{n-1}}{k^2}J_1^\#[\epsilon,\epsilon\log\epsilon,\delta]+\frac{\delta\epsilon^{2n-2}(\log\epsilon)}{k^2}J_2^\#[\epsilon,\epsilon\log\epsilon,\delta],\tag{10.167}$$

*for all $(\epsilon,\delta)\in]0,\epsilon_6[\times]0,\delta_1[$. Moreover,*

$$J_1^\#[0,0,0]=-\int_{\partial\Omega}F(x,0)\,d\sigma_x.\tag{10.168}$$

*Proof.* Let $\Theta_n^\#[\cdot,\cdot,\cdot]$ be as in Theorem 10.113 (*iv*). Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. Let $(\epsilon,\delta)\in]0,\epsilon_3[\times]0,\delta_1[$. Clearly, by the Divergence Theorem and the periodicity of $u[\epsilon,\delta]$, we have

$$\begin{aligned}\int_{\mathbb{P}_a[\Omega_\epsilon]}u[\epsilon,\delta](x)\,dx&=-\frac{1}{k^2}\int_{\mathbb{P}_a[\Omega_\epsilon]}\Delta u[\epsilon,\delta](x)\,dx\\&=-\frac{1}{k^2}\int_{\partial\mathbb{P}_a[\Omega_\epsilon]}\frac{\partial}{\partial\nu_{\mathbb{P}_a[\Omega_\epsilon]}}u[\epsilon,\delta](x)\,d\sigma_x\\&=-\frac{1}{k^2}\Big[\int_{\partial A}\frac{\partial}{\partial\nu_A}u[\epsilon,\delta](x)\,d\sigma_x-\int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon,\delta](x)\,d\sigma_x\Big]\\&=\frac{1}{k^2}\int_{\partial\Omega_\epsilon}\frac{\partial}{\partial\nu_{\Omega_\epsilon}}u[\epsilon,\delta](x)\,d\sigma_x.\end{aligned}$$

As a consequence,

$$\int_{\mathbb{P}_a[\Omega_\epsilon]}u[\epsilon,\delta](x)\,dx=\frac{\epsilon^{n-1}}{k^2}\int_{\partial\Omega}\Big(\frac{\partial u[\epsilon,\delta]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\,d\sigma_t.$$

By equality (6.25), we have

$$\begin{aligned}&\Big(\frac{\partial u[\epsilon,\delta]}{\partial\nu_{\Omega_\epsilon}}\Big)\circ(w+\epsilon\,\mathrm{id}_{\partial\Omega})(t)\\&\quad=\delta\frac{1}{2}\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](t)+\delta\int_{\partial\Omega}\nu_\Omega(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s\\&\quad+\delta\epsilon^{n-1}(\log\epsilon)k^{n-2}\int_{\partial\Omega}\nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s\\&\quad+\delta\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^\#[\epsilon,\epsilon\log\epsilon,\delta](s)\,d\sigma_s\qquad\forall t\in\partial\Omega.\end{aligned}$$

We set

$$\tilde{J}_1[\epsilon,\epsilon',\delta](t) \equiv \frac{1}{2}\Theta_n^{\#}[\epsilon,\epsilon',\delta](t) + \int_{\partial\Omega} \nu_\Omega(t)\cdot D_{\mathbb{R}^n} S_n(t-s,\epsilon k)\Theta_n^{\#}[\epsilon,\epsilon',\delta](s)\,d\sigma_s$$
$$+\epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t)\cdot DR_n^{a,k}(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon',\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

and

$$\tilde{J}_2[\epsilon,\epsilon',\delta](t) \equiv k^{n-2}\int_{\partial\Omega} \nu_\Omega(t)\cdot DQ_n^k(\epsilon(t-s))\Theta_n^{\#}[\epsilon,\epsilon',\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega,$$

for all $(\epsilon,\epsilon',\delta)\in\,]-\epsilon_3,\epsilon_3[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\delta_1,\delta_1[$. By Theorem E.6 (*ii*) and Theorem C.4, one can easily show that there exists $\epsilon_6\in\,]0,\epsilon_3]$ such that $\tilde{J}_1^{\#}$, $\tilde{J}_2^{\#}$ are real analytic maps of $]-\epsilon_6,\epsilon_6[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\delta_1,\delta_1[$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Hence, if we set

$$J_1^{\#}[\epsilon,\epsilon',\delta] \equiv \int_{\partial\Omega} \tilde{J}_1^{\#}[\epsilon,\epsilon',\delta](t)\,d\sigma_t,$$

and

$$J_2^{\#}[\epsilon,\epsilon',\delta] \equiv \int_{\partial\Omega} \tilde{J}_2^{\#}[\epsilon,\epsilon',\delta](t)\,d\sigma_t,$$

for all $(\epsilon,\epsilon',\delta)\in\,]-\epsilon_6,\epsilon_6[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\delta_1,\delta_1[$, then, by standard properties of functions in Schauder spaces, we have that $J_1^{\#}$, $J_2^{\#}$ are real analytic maps of $]-\epsilon_6,\epsilon_6[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]-\delta_1,\delta_1[$ to $\mathbb{C}$, such that equality (10.167) holds.

Finally, if $\epsilon=\epsilon'=\delta=0$, we have

$$J_1^{\#}[0,0,0] = \int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega}v^-[\partial\Omega,\tilde{\theta},0](t)\,d\sigma_t$$
$$= -\int_{\partial\Omega} F(x,0)\,d\sigma_x,$$

and accordingly (10.168) holds. □

We now show that the family $\{u[\epsilon,\delta]\}_{(\epsilon,\delta)\in]0,\epsilon_3[\times]0,\delta_1[}$ is essentially unique. Namely, we prove the following.

**Theorem 10.122.** *Let* $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in\,]0,1[$. *Let* $w\in A$. *Let* $\Omega$, $\epsilon_1$, $k$, $F$ *be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let* $\epsilon_1^*$ *be as in* (10.104). *Let* $\{(\hat{\epsilon}_j,\hat{\delta}_j)\}_{j\in\mathbb{N}}$ *be a sequence in* $]0,\epsilon_1^*[\,\times\,]0,+\infty[$ *converging to* $(0,0)$. *If* $\{u_j\}_{j\in\mathbb{N}}$ *is a sequence of functions such that*

$$u_j\in C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\Omega_{\hat{\epsilon}_j}],\mathbb{C}), \tag{10.169}$$
$$u_j \text{ solves (10.146) with } (\epsilon,\delta)\equiv(\hat{\epsilon}_j,\hat{\delta}_j), \tag{10.170}$$
$$\lim_{j\to\infty} u_j(w+\hat{\epsilon}_j\cdot)=0 \qquad \text{in } C^{m-1,\alpha}(\partial\Omega,\mathbb{C}), \tag{10.171}$$

*then there exists* $j_0\in\mathbb{N}$ *such that*

$$u_j = u[\hat{\epsilon}_j,\hat{\delta}_j] \qquad \forall j_0\le j\in\mathbb{N}.$$

*Proof.* By Theorem 10.108, for each $j\in\mathbb{N}$, there exists a unique function $\theta_j$ in $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ such that

$$u_j = u[\hat{\epsilon}_j,\hat{\delta}_j,\theta_j]. \tag{10.172}$$

We shall now try to show that

$$\lim_{j\to\infty}\theta_j=\tilde{\theta} \qquad \text{in } C^{m-1,\alpha}(\partial\Omega,\mathbb{C}). \tag{10.173}$$

Indeed, if we denote by $\tilde{\mathcal{U}}$ the neighbourhood of Theorems 10.112 (*iv*), 10.113 (*iv*), the limiting relation of (10.173) implies that there exists $j_0\in\mathbb{N}$ such that

$$(\hat{\epsilon}_j,\delta_j,\theta_j)\in\,]0,\epsilon_3[\,\times\,]0,\delta_1[\,\times\tilde{\mathcal{U}}, \qquad \text{if } n \text{ is odd,}$$
$$(\hat{\epsilon}_j,\hat{\epsilon}_j\log\hat{\epsilon}_j,\hat{\delta}_j,\theta_j)\in\,]0,\epsilon_3[\,\times\,]-\epsilon_2',\epsilon_2'[\,\times\,]0,\delta_1[\,\times\tilde{\mathcal{U}}, \qquad \text{if } n \text{ is even,}$$

for $j \geq j_0$ and thus Theorems 10.112 (*iv*), 10.113 (*iv*) would imply that

$$\theta_j = \Theta_n[\hat{\epsilon}_j, \hat{\delta}_j] \qquad \text{if } n \text{ is odd},$$
$$\theta_j = \Theta_n^{\#}[\hat{\epsilon}_j, \hat{\epsilon}_j \log \hat{\epsilon}_j, \hat{\delta}_j] \qquad \text{if } n \text{ is even},$$

for $j_0 \leq j \in \mathbb{N}$, and that accordingly the Theorem holds (cf. Definition 10.114.) Thus we now turn to the proof of (10.173). We split our proof into the cases $n$ odd and $n$ even.

We first assume that $n$ is odd. Then we note that equation $\Lambda[\epsilon, \delta, \theta] = 0$ can be rewritten in the following form

$$\begin{aligned} &\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) \cdot D_{\mathbb{R}^n} S_n(t-s, \epsilon k)\theta(s)\,d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot D R_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s \\ &\quad = -F\Big(t, \delta\epsilon \int_{\partial\Omega} S_n(t-s, \epsilon k)\theta(s)\,d\sigma_s + \delta\epsilon^{n-1} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\theta(s)\,d\sigma_s\Big) \quad \forall t \in \partial\Omega, \end{aligned} \tag{10.174}$$

for all $(\epsilon, \delta, \theta)$ in the domain of $\Lambda$. Then we define the map $N$ of $]-\epsilon_3, \epsilon_3[\, \times\, ]-\delta_1, \delta_1[\, \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ by setting $N[\epsilon, \delta, \theta]$ equal to the left-hand side of the equality in (10.174), for all $(\epsilon, \delta, \theta) \in\, ]-\epsilon_3, \epsilon_3[\, \times\, ]-\delta_1, \delta_1[\, \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. By arguing as in the proof of Theorem 10.112, we can prove that $N$ is real analytic. Since $N[\epsilon, \delta, \cdot]$ is linear for all $(\epsilon, \delta) \in\, ]-\epsilon_3, \epsilon_3[\, \times\, ]-\delta_1, \delta_1[$, we have

$$N[\epsilon, \delta, \theta] = \partial_\theta N[\epsilon, \delta, \tilde{\theta}](\theta),$$

for all $(\epsilon, \delta, \theta) \in\, ]-\epsilon_3, \epsilon_3[\, \times\, ]-\delta_1, \delta_1[\, \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, and the map of $]-\epsilon_3, \epsilon_3[\, \times\, ]-\delta_1, \delta_1[$ to the space $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), C^{m-1,\alpha}(\partial\Omega, \mathbb{C}))$ which takes $(\epsilon, \delta)$ to $N[\epsilon, \delta, \cdot]$ is real analytic. Since

$$N[0, 0, \cdot] = \partial_\theta \Lambda[0, 0, \tilde{\theta}](\cdot),$$

Theorem 10.112 (*iii*) implies that $N[0, 0, \cdot]$ is also a linear homeomorphism. Since the set of linear homeomorphisms of $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ is open in $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), C^{m-1,\alpha}(\partial\Omega, \mathbb{C}))$ and since the map which takes a linear invertible operator to its inverse is real analytic (cf. *e.g.*, Hille and Phillips [61, Theorems 4.3.2 and 4.3.4]), there exist $\tilde{\epsilon} \in\, ]0, \epsilon_3[$, $\tilde{\delta} \in\, ]0, \delta_1[$ such that the map $(\epsilon, \delta) \mapsto N[\epsilon, \delta, \cdot]^{(-1)}$ is real analytic from $]-\tilde{\epsilon}, \tilde{\epsilon}[\, \times\, ]-\tilde{\delta}, \tilde{\delta}[$ to $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), C^{m-1,\alpha}(\partial\Omega, \mathbb{C}))$. Next we denote by $S[\epsilon, \delta, \theta]$ the right-hand side of (10.174). Then equation $\Lambda[\epsilon, \delta, \theta] = 0$ (or equivalently equation (10.174)) can be rewritten in the following form:

$$\theta = N[\epsilon, \delta, \cdot]^{(-1)}[S[\epsilon, \delta, \theta]], \tag{10.175}$$

for all $(\epsilon, \delta, \theta) \in\, ]-\tilde{\epsilon}, \tilde{\epsilon}[\, \times\, ]-\tilde{\delta}, \tilde{\delta}[\, \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Moreover, if $j \in \mathbb{N}$, we observe that by (10.172) we have

$$\begin{aligned} u_j(w + \hat{\epsilon}_j t) &= u[\hat{\epsilon}_j, \hat{\delta}_j, \theta_j](w + \hat{\epsilon}_j t) \\ &= \hat{\delta}_j \hat{\epsilon}_j \int_{\partial\Omega} S_n(t-s, \hat{\epsilon}_j k)\theta_j(s)\,d\sigma_s + \hat{\delta}_j \hat{\epsilon}_j^{n-1} \int_{\partial\Omega} R_n^{a,k}(\hat{\epsilon}_j(t-s))\theta_j(s)\,d\sigma_s \qquad \forall t \in \partial\Omega. \end{aligned} \tag{10.176}$$

Next we note that condition (10.171), equality (10.176), the proof of Theorem 10.112, the real analyticity of $T_F$ and standard calculus in Banach space imply that

$$\lim_{j \to \infty} S[\hat{\epsilon}_j, \hat{\delta}_j, \theta_j] = S[0, 0, \tilde{\theta}] \qquad \text{in } C^{m-1,\alpha}(\partial\Omega, \mathbb{C}). \tag{10.177}$$

Then by (10.175) and by the real analyticity of $(\epsilon, \delta) \mapsto N[\epsilon, \delta, \cdot]^{(-1)}$, and by the bilinearity and continuity of the operator of $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), C^{m-1,\alpha}(\partial\Omega, \mathbb{C})) \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$, which takes a pair $(T_1, T_2)$ to $T_1[T_2]$, by (10.177) we conclude that

$$\begin{aligned} \lim_{j \to \infty} \theta_j &= \lim_{j \to \infty} N[\hat{\epsilon}_j, \hat{\delta}_j, \cdot]^{(-1)}[S[\hat{\epsilon}_j, \hat{\delta}_j, \theta_j]] \\ &= N[0, 0, \cdot]^{(-1)}[S[0, 0, \tilde{\theta}]] = \tilde{\theta} \qquad \text{in } C^{m-1,\alpha}(\partial\Omega, \mathbb{C}), \end{aligned}$$

and, consequently, that (10.173) holds. Thus the proof of case $n$ odd is complete.

We now consider case $n$ even. We proceed as in case $n$ odd. We note that equation $\Lambda^{\#}[\epsilon,\epsilon',\delta,\theta]=0$ can be rewritten in the following form

$$\begin{aligned}&\frac{1}{2}\theta(t)+\int_{\partial\Omega}\nu_{\Omega}(t)\cdot D_{\mathbb{R}^n}S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s+\epsilon^{n-2}\epsilon' k^{n-2}\int_{\partial\Omega}\nu_{\Omega}(t)\cdot DQ^k_n(\epsilon(t-s))\theta(s)\,d\sigma_s\\&+\epsilon^{n-1}\int_{\partial\Omega}\nu_{\Omega}(t)\cdot DR^{a,k}_n(\epsilon(t-s))\theta(s)\,d\sigma_s=-F\Big(t,\delta\epsilon\int_{\partial\Omega}S_n(t-s,\epsilon k)\theta(s)\,d\sigma_s\\&+\delta\epsilon^{n-2}\epsilon' k^{n-2}\int_{\partial\Omega}Q^k_n(\epsilon(t-s))\theta(s)\,d\sigma_s+\delta\epsilon^{n-1}\int_{\partial\Omega}R^{a,k}_n(\epsilon(t-s))\theta(s)\,d\sigma_s\Big)\quad\forall t\in\partial\Omega,\end{aligned}\tag{10.178}$$

for all $(\epsilon,\epsilon',\delta,\theta)$ in the domain of $\Lambda^{\#}$. We define the map $N^{\#}$ of $]-\epsilon_3,\epsilon_3[\times]-\epsilon'_2,\epsilon'_2[\times]-\delta_1,\delta_1[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ by setting $N^{\#}[\epsilon,\epsilon',\delta,\theta]$ equal to the left-hand side of the equality in (10.178), for all $(\epsilon,\epsilon',\delta,\theta)\in]-\epsilon_3,\epsilon_3[\times]-\epsilon'_2,\epsilon'_2[\times]-\delta_1,\delta_1[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. By arguing as in the proof of Theorem 10.113, we can prove that $N^{\#}$ is real analytic. Since $N^{\#}[\epsilon,\epsilon',\delta,\cdot]$ is linear for all $(\epsilon,\epsilon',\delta)\in]-\epsilon_3,\epsilon_3[\times]-\epsilon'_2,\epsilon'_2[\times]-\delta_1,\delta_1[$, we have

$$N^{\#}[\epsilon,\epsilon',\delta,\theta]=\partial_\theta N^{\#}[\epsilon,\epsilon',\delta,\tilde{\theta}](\theta),$$

for all $(\epsilon,\epsilon',\delta,\theta)\in]-\epsilon_3,\epsilon_3[\times]-\epsilon'_2,\epsilon'_2[\times]-\delta_1,\delta_1[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, and the map of $]-\epsilon_3,\epsilon_3[\times]-\epsilon'_2,\epsilon'_2[\times]-\delta_1,\delta_1[$ to the space $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega,\mathbb{C}),C^{m-1,\alpha}(\partial\Omega,\mathbb{C}))$ which takes $(\epsilon,\epsilon',\delta)$ to $N^{\#}[\epsilon,\epsilon',\delta,\cdot]$ is real analytic. Since

$$N^{\#}[0,0,0,\cdot]=\partial_\theta\Lambda^{\#}[0,0,0,\tilde{\theta}](\cdot),$$

Theorem 10.113 (*iii*) implies that $N^{\#}[0,0,0,\cdot]$ is also a linear homeomorphism. Since the set of linear homeomorphisms of $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ is open in $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega,\mathbb{C}),C^{m-1,\alpha}(\partial\Omega,\mathbb{C}))$ and since the map which takes a linear invertible operator to its inverse is real analytic (cf. *e.g.*, Hille and Phillips [61, Theorems 4.3.2 and 4.3.4]), there exists $(\tilde{\epsilon},\tilde{\epsilon}',\tilde{\delta})\in]0,\epsilon_3[\times]0,\epsilon'_2[\times]0,\delta_1[$ such that the map $(\epsilon,\epsilon',\delta)\mapsto N^{\#}[\epsilon,\epsilon',\delta,\cdot]^{(-1)}$ is real analytic from $]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\epsilon}',\tilde{\epsilon}'[\times]-\tilde{\delta},\tilde{\delta}[$ to $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega,\mathbb{C}),C^{m-1,\alpha}(\partial\Omega,\mathbb{C}))$, and such that

$$\epsilon\log\epsilon\in]-\tilde{\epsilon}',\tilde{\epsilon}'[\qquad\forall\epsilon\in]0,\tilde{\epsilon}[.$$

Next we denote by $S^{\#}[\epsilon,\epsilon',\delta,\theta]$ the right-hand side of (10.178). Then equation $\Lambda^{\#}[\epsilon,\epsilon',\delta,\theta]=0$ (or equivalently equation (10.178)) can be rewritten in the following form:

$$\theta=N^{\#}[\epsilon,\epsilon',\delta,\cdot]^{(-1)}[S^{\#}[\epsilon,\epsilon',\delta,\theta]],\tag{10.179}$$

for all $(\epsilon,\epsilon',\delta,\theta)\in]-\tilde{\epsilon},\tilde{\epsilon}[\times]-\tilde{\epsilon}',\tilde{\epsilon}'[\times]-\tilde{\delta},\tilde{\delta}[\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Moreover, if $j\in\mathbb{N}$, we observe that by (10.172) we have

$$\begin{aligned}&u_j(w+\hat{\epsilon}_jt)=u[\hat{\epsilon}_j,\hat{\delta}_j,\theta_j](w+\hat{\epsilon}_jt)\\&=\hat{\delta}_j\hat{\epsilon}_j\int_{\partial\Omega}S_n(t-s,\hat{\epsilon}_jk)\theta_j(s)\,d\sigma_s+\hat{\delta}_j\hat{\epsilon}_j^{n-2}(\hat{\epsilon}_j\log\hat{\epsilon}_j)k^{n-2}\int_{\partial\Omega}Q^k_n(\hat{\epsilon}_j(t-s))\theta_j(s)\,d\sigma_s\\&\quad+\hat{\delta}_j\hat{\epsilon}_j^{n-1}\int_{\partial\Omega}R^{a,k}_n(\hat{\epsilon}_j(t-s))\theta_j(s)\,d\sigma_s\qquad\forall t\in\partial\Omega.\end{aligned}\tag{10.180}$$

Next we note that

$$\lim_{j\to\infty}(\hat{\epsilon}_j,\hat{\epsilon}_j\log\hat{\epsilon}_j,\hat{\delta}_j)=(0,0,0)\tag{10.181}$$

in $]0,\tilde{\epsilon}[\times]-\tilde{\epsilon}',\tilde{\epsilon}'[\times]0,\tilde{\delta}[$. Then condition (10.171), equality (10.180), the proof of Theorem 10.113, the real analyticity of $T_F$ and standard calculus in Banach space imply that

$$\lim_{j\to\infty}S^{\#}[\hat{\epsilon}_j,\hat{\epsilon}_j\log\hat{\epsilon}_j,\hat{\delta}_j,\theta_j]=S^{\#}[0,0,0,\tilde{\theta}]\qquad\text{in }C^{m-1,\alpha}(\partial\Omega,\mathbb{C}).\tag{10.182}$$

Then by (10.179) and by the real analyticity of $(\epsilon,\epsilon',\delta)\mapsto N^{\#}[\epsilon,\epsilon',\delta,\cdot]^{(-1)}$, and by the bilinearity and continuity of the operator of $\mathcal{L}(C^{m-1,\alpha}(\partial\Omega,\mathbb{C}),C^{m-1,\alpha}(\partial\Omega,\mathbb{C}))\times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, which takes a pair $(T_1,T_2)$ to $T_1[T_2]$ and by (10.181), by (10.182) we conclude that

$$\begin{aligned}\lim_{j\to\infty}\theta_j&=\lim_{j\to\infty}N^{\#}[\hat{\epsilon}_j,\hat{\epsilon}_j\log\hat{\epsilon}_j,\hat{\delta}_j,\cdot]^{(-1)}[S^{\#}[\hat{\epsilon}_j,\hat{\epsilon}_j\log\hat{\epsilon}_j,\hat{\delta}_j,\theta_j]]\\&=N^{\#}[0,0,0,\cdot]^{(-1)}[S^{\#}[0,0,0,\tilde{\theta}]]=\tilde{\theta}\qquad\text{in }C^{m-1,\alpha}(\partial\Omega,\mathbb{C}),\end{aligned}$$

and, consequently, that (10.173) holds. Thus the proof of case $n$ even is complete. □

We give the following definition.

**Definition 10.123.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $\epsilon_3$, $\delta_1$ be as in Theorem 10.112 (*iv*) if $n$ is odd, and as in Theorem 10.113 (*iv*) if $n$ is even. Let $u[\cdot,\cdot]$ be as in Definition 10.114. For each pair $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[$, we set

$$u_{(\epsilon,\delta)}(x) \equiv u[\epsilon,\delta](\frac{x}{\delta}) \qquad \forall x \in \mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta).$$

*Remark* 10.124. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $\epsilon_3$, $\delta_1$ be as in Theorem 10.112 (*iv*) if $n$ is odd, and as in Theorem 10.113 (*iv*) if $n$ is even. For each $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[$, $u_{(\epsilon,\delta)}$ is a solution in $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a(\epsilon,\delta),\mathbb{C})$ of problem (10.145).

Our aim is to study the asymptotic behaviour of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. As a first step, we study the behaviour of $u[\epsilon,\delta]$ as $(\epsilon,\delta)$ tends to $(0,0)$.

**Proposition 10.125.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_3$, $\delta_1$ be as in Theorem 10.112 (iv). Let $u[\cdot,\cdot]$ be as in Definition 10.114. Then there exist $\tilde\epsilon \in ]0,\epsilon_3[$ and a real analytic map $N$ of $]-\tilde\epsilon,\tilde\epsilon[ \times ]-\delta_1,\delta_1[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$ such that*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\|\mathrm{Re}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\|\mathrm{Im}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde\epsilon[ \times ]0,\delta_1[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* Let $\epsilon_3$, $\delta_1$, $\Theta_n$ be as in Theorem 10.112 (*iv*). Let $\mathrm{id}_{\partial\Omega}$ denote the identity map in $\partial\Omega$. If $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[$, we have

$$u[\epsilon,\delta] \circ (w + \epsilon\,\mathrm{id}_{\partial\Omega})(t) = \delta\epsilon \int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s + \delta\epsilon^{n-1} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s \qquad \forall t \in \partial\Omega.$$

We set

$$N[\epsilon,\delta](t) \equiv \int_{\partial\Omega} S_n(t-s,\epsilon k)\Theta_n[\epsilon,\delta](s)\,d\sigma_s + \epsilon^{n-2} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s))\Theta_n[\epsilon,\delta](s)\,d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $(\epsilon,\delta) \in ]-\epsilon_3,\epsilon_3[ \times ]-\delta_1,\delta_1[$. By taking $\tilde\epsilon \in ]0,\epsilon_3[$ small enough, we can assume (cf. Theorem C.4 and the proof of Theorem 10.118) that $N$ is a real analytic map of $]-\tilde\epsilon,\tilde\epsilon[ \times ]-\delta_1,\delta_1[$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$. By Corollary 6.24, we have

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\|\mathrm{Re}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)} \qquad \forall (\epsilon,\delta) \in ]0,\tilde\epsilon[ \times ]0,\delta_1[,$$

and

$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\|\mathrm{Im}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)} \qquad \forall (\epsilon,\delta) \in ]0,\tilde\epsilon[ \times ]0,\delta_1[.$$

Accordingly,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and so the conclusion follows. □

**Proposition 10.126.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (10.104). *Let $\epsilon_3$, $\epsilon_2'$, $\delta_1$ be as in Theorem 10.113 (iv). Let $u[\cdot,\cdot]$ be as in Definition 10.114. Then there exist $\tilde{\epsilon} \in ]0, \epsilon_3[$ and two real analytic maps $N_1^\#$, $N_2^\#$ of $]-\tilde{\epsilon}, \tilde{\epsilon}[ \times ]-\epsilon_2', \epsilon_2'[ \times ]-\delta_1, \delta_1[$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$ such that*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta \|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon, \epsilon\log\epsilon, \delta] + \epsilon^{n-1}(\log\epsilon) N_2^\#[\epsilon, \epsilon\log\epsilon, \delta]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta \|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon, \epsilon\log\epsilon, \delta] + \epsilon^{n-1}(\log\epsilon) N_2^\#[\epsilon, \epsilon\log\epsilon, \delta]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, \delta_1[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n, \mathbb{C}).$$

*Proof.* Let $\epsilon_3$, $\epsilon_2'$, $\delta_1$, $\Theta_n^\#$ be as in Theorem 10.113 (*iv*). If $(\epsilon, \delta) \in ]0, \epsilon_3[ \times ]0, \delta_1[$, we have

$$\begin{aligned} u[\epsilon,\delta] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) =& \delta\epsilon \int_{\partial\Omega} S_n(t-s, \epsilon k) \Theta_n^\#[\epsilon, \epsilon\log\epsilon, \delta](s)\, d\sigma_s \\ &+ \delta\epsilon^{n-1}(\log\epsilon) k^{n-2} \int_{\partial\Omega} Q_n^k(\epsilon(t-s)) \Theta_n^\#[\epsilon, \epsilon\log\epsilon, \delta](s)\, d\sigma_s \\ &+ \delta\epsilon^{n-1} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s)) \Theta_n^\#[\epsilon, \epsilon\log\epsilon, \delta](s)\, d\sigma_s \qquad \forall t \in \partial\Omega. \end{aligned}$$

We set

$$\begin{aligned} N_1^\#[\epsilon, \epsilon', \delta](t) \equiv & \int_{\partial\Omega} S_n(t-s, \epsilon k) \Theta_n^\#[\epsilon, \epsilon', \delta](s)\, d\sigma_s \\ &+ \epsilon^{n-2} \int_{\partial\Omega} R_n^{a,k}(\epsilon(t-s)) \Theta_n^\#[\epsilon, \epsilon', \delta](s)\, d\sigma_s \qquad \forall t \in \partial\Omega, \end{aligned}$$

and

$$N_2^\#[\epsilon, \epsilon', \delta](t) \equiv k^{n-2} \int_{\partial\Omega} Q_n^k(\epsilon(t-s)) \Theta_n^\#[\epsilon, \epsilon', \delta](s)\, d\sigma_s \qquad \forall t \in \partial\Omega,$$

for all $(\epsilon, \epsilon', \delta) \in ]-\epsilon_3, \epsilon_3[ \times ]-\epsilon_2', \epsilon_2'[ \times ]-\delta_1, \delta_1[$. By taking $\tilde{\epsilon} \in ]0, \epsilon_3[$ small enough, we can assume (cf. Theorem C.4 and the proof of Theorem 10.119) that $N_1^\#$, $N_2^\#$ are real analytic maps of $]-\tilde{\epsilon}, \tilde{\epsilon}[ \times ]-\epsilon_2', \epsilon_2'[ \times ]-\delta_1, \delta_1[$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$. Clearly,

$$u[\epsilon,\delta] \circ (w + \epsilon\, \mathrm{id}_{\partial\Omega})(t) = \delta\epsilon N_1^\#[\epsilon, \epsilon\log\epsilon, \delta](t) + \delta\epsilon^{n-1}(\log\epsilon) N_2^\#[\epsilon, \epsilon\log\epsilon, \delta](t) \qquad \forall t \in \partial\Omega, \quad \forall (\epsilon,\delta) \in ]0, \tilde{\epsilon}[ \times ]0, \delta_1[.$$

By Corollary 6.24, we have

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta \|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon, \epsilon\log\epsilon, \delta] + \epsilon^{n-1}(\log\epsilon) N_2^\#[\epsilon, \epsilon\log\epsilon, \delta]\big)\|_{C^0(\partial\Omega)},$$

and

$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta \|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon, \epsilon\log\epsilon, \delta] + \epsilon^{n-1}(\log\epsilon) N_2^\#[\epsilon, \epsilon\log\epsilon, \delta]\big)\|_{C^0(\partial\Omega)},$$

for all $(\epsilon, \delta) \in ]0, \tilde{\epsilon}[ \times ]0, \delta_1[$.

Accordingly,

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big) = 0 \qquad \text{in } L^\infty(\mathbb{R}^n),$$

and so the conclusion follows. □

### 10.7.2 Asymptotic behaviour of $u_{(\epsilon,\delta)}$

In the following Theorems we deduce by Propositions 10.125, 10.126 the convergence of $u_{(\epsilon,\delta)}$ as $(\epsilon,\delta)$ tends to $(0,0)$. Namely, we prove the following.

**Theorem 10.127.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (10.104). *Let $\delta_1$ be as in Theorem 10.112 (iv). Let $\tilde{\epsilon}$, $N$ be as in Proposition 10.125. Then*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\|\mathrm{Re}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\epsilon\|\mathrm{Im}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\delta_1[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \delta\epsilon\|\mathrm{Re}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \delta\epsilon\|\mathrm{Im}\big(N[\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\delta_1[$. □

**Theorem 10.128.** *Let $n$ be even. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let* $\mathrm{Im}(k) \neq 0$ *and* $\mathrm{Re}(k) = 0$. *Let $\epsilon_1^*$ be as in* (10.104). *Let $\delta_1$ be as in Theorem 10.113 (iv). Let $\tilde{\epsilon}$, $N_1^\#$, $N_2^\#$ be as in Proposition 10.126. Then*

$$\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon,\delta] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},$$
$$\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} = \delta\|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon,\delta] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},$$

*for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\delta_1[$. Moreover, as a consequence,*

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} \mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}] = 0 \qquad \text{in } L^\infty(\mathbb{R}^n,\mathbb{C}).$$

*Proof.* It suffices to observe that

$$\begin{aligned}\|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Re}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \delta\|\mathrm{Re}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon,\delta] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,\delta)}[u_{(\epsilon,\delta)}]\big)\|_{L^\infty(\mathbb{R}^n)} &= \|\mathrm{Im}\big(\mathbf{E}_{(\epsilon,1)}[u[\epsilon,\delta]]\big)\|_{L^\infty(\mathbb{R}^n)}\\ &= \delta\|\mathrm{Im}\big(\epsilon N_1^\#[\epsilon,\epsilon\log\epsilon,\delta] + \epsilon^{n-1}(\log\epsilon)N_2^\#[\epsilon,\epsilon\log\epsilon,\delta]\big)\|_{C^0(\partial\Omega)},\end{aligned}$$

for all $(\epsilon,\delta) \in ]0,\tilde{\epsilon}[\times]0,\delta_1[$. □

Then we have the following Theorem, where we consider a functional associated to an extension of $u_{(\epsilon,\delta)}$. Moreover, we evaluate such a functional on suitable characteristic functions.

**Theorem 10.129.** *Let $n$ be odd. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\delta_1$ be as in Theorem 10.112 (iv). Let $\epsilon_6$, $J$ be as in Theorem 10.120. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l}\frac{\epsilon^{n-1}}{k^2}J\big[\epsilon,\frac{r}{l}\big], \tag{10.183}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N}\setminus\{0\}$ such that $l > (r/\delta_1)$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N}\setminus\{0\}$ such that $l > (r/\delta_1)$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,(r/l)]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,(r/l)](t)\,dt \\ &= \frac{r^n}{l^n}\frac{r}{l}\frac{\epsilon^{n-1}}{k^2} J\big[\epsilon,\frac{r}{l}\big].\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l}\frac{\epsilon^{n-1}}{k^2} J\big[\epsilon,\frac{r}{l}\big],$$

and the conclusion follows. □

**Theorem 10.130.** *Let $n$ be even. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104)*. Let $\delta_1$ be as in Theorem 10.113 (iv). Let $\epsilon_6$, $J_1^\#$, $J_2^\#$ be as in Theorem 10.121. Let $r > 0$ and $\bar{y} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l}\Big\{\frac{\epsilon^{n-1}}{k^2}J_1^\#\big[\epsilon,\epsilon\log\epsilon,\frac{r}{l}\big] + \frac{\epsilon^{2n-2}(\log\epsilon)}{k^2}J_2^\#\big[\epsilon,\epsilon\log\epsilon,\frac{r}{l}\big]\Big\}, \tag{10.184}$$

*for all $\epsilon \in ]0,\epsilon_6[$, and for all $l \in \mathbb{N}\setminus\{0\}$ such that $l > (r/\delta_1)$.*

*Proof.* Let $\epsilon \in ]0,\epsilon_6[$, $l \in \mathbb{N}\setminus\{0\}$ such that $l > (r/\delta_1)$. Then, by the periodicity of $u_{(\epsilon,r/l)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx &= \int_{rA+\bar{y}} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= \int_{rA} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx \\ &= l^n \int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx.\end{aligned}$$

Then we note that

$$\begin{aligned}\int_{\frac{r}{l}A} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\,dx &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u_{(\epsilon,r/l)}(x)\,dx \\ &= \int_{\frac{r}{l}\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,(r/l)]\big(\frac{l}{r}x\big)\,dx \\ &= \frac{r^n}{l^n}\int_{\mathbb{P}_a[\Omega_\epsilon]} u[\epsilon,(r/l)](t)\,dt \\ &= \frac{r^n}{l^n}\frac{r}{l}\Big\{\frac{\epsilon^{n-1}}{k^2}J_1^\#\big[\epsilon,\epsilon\log\epsilon,\frac{r}{l}\big] + \frac{\epsilon^{2n-2}(\log\epsilon)}{k^2}J_2^\#\big[\epsilon,\epsilon\log\epsilon,\frac{r}{l}\big]\Big\}.\end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,r/l)}[u_{(\epsilon,r/l)}](x)\chi_{rA+\bar{y}}(x)\,dx = \frac{r^{n+1}}{l}\Big\{\frac{\epsilon^{n-1}}{k^2}J_1^\#\big[\epsilon,\epsilon\log\epsilon,\frac{r}{l}\big] + \frac{\epsilon^{2n-2}(\log\epsilon)}{k^2}J_2^\#\big[\epsilon,\epsilon\log\epsilon,\frac{r}{l}\big]\Big\},$$

and the conclusion follows. □

We give the following.

**Definition 10.131.** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $\epsilon_3$, $\delta_1$ be as in Theorem 10.112 *(iv)* if $n$ is odd, and as in Theorem 10.113 *(iv)* if $n$ is even. For each pair $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[$, we set

$$\mathcal{F}(\epsilon,\delta) \equiv \int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx - \frac{k^2}{\delta^2}\int_{A\cap\mathbb{T}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2\,dx.$$

*Remark* 10.132. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in (1.56), (1.57), (10.102), (10.103), respectively. Let $\epsilon_1^*$ be as in (10.104). Let $\epsilon_3$, $\delta_1$ be as in Theorem 10.112 *(iv)* if $n$ is odd, and as in Theorem 10.113 *(iv)* if $n$ is even. Let $(\epsilon,\delta) \in ]0,\epsilon_3[ \times ]0,\delta_1[$. We have

$$\begin{aligned}\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx &= \delta^n \int_{\mathbb{P}_a(\epsilon,1)} |(\nabla u_{(\epsilon,\delta)})(\delta t)|^2\,dt \\ &= \delta^{n-2}\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon,\delta](t)|^2\,dt,\end{aligned}$$

and

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2\,dx = \delta^n \int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon,\delta](t)|^2\,dt.$$

Accordingly,

$$\begin{aligned}&\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx - \frac{k^2}{\delta^2}\int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2\,dx \\ &\qquad = \delta^{n-2}\Big(\int_{\mathbb{P}_a[\Omega_\epsilon]} |\nabla u[\epsilon,\delta](t)|^2\,dt - k^2\int_{\mathbb{P}_a[\Omega_\epsilon]} |u[\epsilon,\delta](t)|^2\,dt\Big).\end{aligned}$$

In the following Propositions we represent the function $\mathcal{F}(\cdot,\cdot)$ by means of real analytic functions.

**Proposition 10.133.** *Let $n$ be odd. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\delta_1$ be as in Theorem 10.112 (iv). Let $\epsilon_5$, $G$ be as in Theorem 10.118. Then*

$$\mathcal{F}\Big(\epsilon,\frac{1}{l}\Big) = \epsilon^n G[\epsilon,(1/l)],$$

*for all $\epsilon \in ]0,\epsilon_5[$ and for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* Let $(\epsilon,\delta) \in ]0,\epsilon_5[ \times ]0,\delta_1[$. By Remark 10.132 and Theorem 10.118, we have

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2\,dx - \frac{k^2}{\delta^2}\int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2\,dx = \delta^n\epsilon^n G[\epsilon,\delta]$$

where $G$ is as in Theorem 10.118. On the other hand, if $\epsilon \in ]0,\epsilon_5[$ and $l \in \mathbb{N}$ is such that $l > (1/\delta_1)$, then we have

$$\begin{aligned}\mathcal{F}\Big(\epsilon,\frac{1}{l}\Big) &= l^n\frac{1}{l^n}\epsilon^n G[\epsilon,(1/l)], \\ &= \epsilon^n G[\epsilon,(1/l)],\end{aligned}$$

and the conclusion easily follows. □

**Proposition 10.134.** *Let $n$ be even. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $w \in A$. Let $\Omega$, $\epsilon_1$, $k$, $F$ be as in* (1.56), (1.57), (10.102), (10.103), *respectively. Let $\epsilon_1^*$ be as in* (10.104). *Let $\delta_1$ be as in Theorem 10.113 (iv). Let $\epsilon_5$, $G_1^\#$, $G_2^\#$, and $G_3^\#$ be as in Theorem 10.119. Then*

$$\begin{aligned}\mathcal{F}\Big(\epsilon,\frac{1}{l}\Big) =& \epsilon^n G_1^\#[\epsilon,\epsilon\log\epsilon,(1/l)] \\ &+ \epsilon^{2n-2}(\log\epsilon)G_2^\#[\epsilon,\epsilon\log\epsilon,(1/l)] \\ &+ \epsilon^{3n-3}(\log\epsilon)^2 G_3^\#[\epsilon,\epsilon\log\epsilon,(1/l)],\end{aligned}$$

*for all $\epsilon \in ]0,\epsilon_5[$ and for all $l \in \mathbb{N}$ such that $l > (1/\delta_1)$.*

*Proof.* Let $(\epsilon, \delta) \in ]0, \epsilon_5[ \times ]0, \delta_1[$. By Remark 10.132 and Theorem 10.119, we have

$$\int_{\mathbb{P}_a(\epsilon,\delta)} |\nabla u_{(\epsilon,\delta)}(x)|^2 \, dx - \frac{k^2}{\delta^2} \int_{\mathbb{P}_a(\epsilon,\delta)} |u_{(\epsilon,\delta)}(x)|^2 \, dx$$
$$= \delta^n \Big\{ \epsilon^n G_1^{\#}[\epsilon, \epsilon \log \epsilon, \delta] + \epsilon^{2n-2} (\log \epsilon) G_2^{\#}[\epsilon, \epsilon \log \epsilon, \delta] + \epsilon^{3n-3} (\log \epsilon)^2 G_3^{\#}[\epsilon, \epsilon \log \epsilon, \delta] \Big\},$$

where $G_1^{\#}$, $G_2^{\#}$, and $G_3^{\#}$ are as in Theorem 10.119. On the other hand, if $\epsilon \in ]0, \epsilon_5[$ and $l \in \mathbb{N}$ is such that $l > (1/\delta_1)$, then we have

$$\begin{aligned} \mathcal{F}\Big(\epsilon, \frac{1}{l}\Big) = & l^n \frac{1}{l^n} \Big\{ \epsilon^n G_1^{\#}[\epsilon, \epsilon \log \epsilon, (1/l)] \\ & + \epsilon^{2n-2} (\log \epsilon) G_2^{\#}[\epsilon, \epsilon \log \epsilon, (1/l)] \\ & + \epsilon^{3n-3} (\log \epsilon)^2 G_3^{\#}[\epsilon, \epsilon \log \epsilon, (1/l)] \Big\}, \\ = & \epsilon^n G_1^{\#}[\epsilon, \epsilon \log \epsilon, (1/l)] \\ & + \epsilon^{2n-2} (\log \epsilon) G_2^{\#}[\epsilon, \epsilon \log \epsilon, (1/l)] \\ & + \epsilon^{3n-3} (\log \epsilon)^2 G_3^{\#}[\epsilon, \epsilon \log \epsilon, (1/l)], \end{aligned}$$

and the conclusion easily follows. □

CHAPTER 11

# Periodic analogue of the fundamental solution and real analyticity of periodic layer potentials of some linear differential operators with constant coefficients

In this Chapter we prove a necessary and sufficient condition on a linear differential operator with constant coefficients for the existence of a periodic analogue of the fundamental solution. Then we deduce by Dalla Riva and Lanza [40] a real analyticity theorem for the periodic layer potentials associated with a strongly elliptic linear differential operator with constant coefficients (see also Lanza and Preciso [83], Lanza and Rossi [85, 86].) The approach adopted is the one of Dalla Riva, Lanza, Preciso, Rossi [115], [83], [84], [85], [86], [40]. For a generalization of some results contained in this Chapter, we refer to [81].

We retain the notation introduced in Sections 1.1 and 1.3.

## 11.1 On the existence of a periodic analogue of the fundamental solution of a linear differential operator with constant coefficients

In this Section we prove a Theorem on the existence of a periodic analogue of the fundamental solution of a linear differential operator with constant coefficients.

We retain the notation of Section 1.1 and of Appendix A (see also the notation introduced in Subsection 1.2.1.)

We recall the following notation (cf. Subsection 1.2.1.)

Let $y \in \mathbb{R}^n$. If $f \in \mathcal{S}(\mathbb{R}^n)$, we denote by $\tau_y f$ the element of $\mathcal{S}(\mathbb{R}^n)$ defined by

$$\tau_y f(x) \equiv f(x-y) \qquad \forall x \in \mathbb{R}^n.$$

If $u \in \mathcal{S}'(\mathbb{R}^n)$ (resp. $u \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$), then we denote by $\tau_y u$ the element of $\mathcal{S}'(\mathbb{R}^n)$ (resp. the element of $\mathcal{D}'(\mathbb{R}^n, \mathbb{C})$) defined by

$$\langle \tau_y u, f \rangle \equiv \langle u, \tau_{-y} f \rangle ,$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ (resp. for all $f \in \mathcal{D}(\mathbb{R}^n, \mathbb{C})$).

Analogosuly, if $u \in \mathcal{S}'(\mathbb{R}^n)$ (resp. $u \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$) and $T$ is an invertible linear map of $\mathbb{R}^n$ to $\mathbb{R}^n$, then we denote by $u \circ T$ the element of $\mathcal{S}'(\mathbb{R}^n)$ (resp. the element of $\mathcal{D}'(\mathbb{R}^n, \mathbb{C})$) defined by

$$\langle u \circ T, f \rangle \equiv |\det T|^{-1} \left\langle u, f \circ T^{-1} \right\rangle ,$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ (resp. for all $f \in \mathcal{D}(\mathbb{R}^n, \mathbb{C})$).

Then we have the following variant of a well known result (cf. Folland [53, pp. 297-299], Schmeisser and Triebel [125, pp. 143-145].)

**Proposition 11.1.** *Let* $G \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$ *be such that*

$$\tau_{la_j} G = G \qquad \forall l \in \mathbb{Z}, \quad \forall j \in \{1, \dots, n\}. \tag{11.1}$$

*Then* $G \in \mathcal{S}'(\mathbb{R}^n)$ *and*

$$G = \sum_{z \in \mathbb{Z}^n} g(z) E_{2\pi a^{-1}(z)} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n), \tag{11.2}$$

*where* $g$ *is a function of* $\mathbb{Z}^n$ *to* $\mathbb{C}$, *such that*

$$|g(z)| \le C(1+|z|)^N \qquad \forall z \in \mathbb{Z}^n,$$

*for some* $C, N > 0$. *Moreover, if* $\tilde{g}$ *is another function of* $\mathbb{Z}^n$ *to* $\mathbb{C}$, *such that*

$$|\tilde{g}(z)| \le \tilde{C}(1+|z|)^{\tilde{N}} \qquad \forall z \in \mathbb{Z}^n,$$

*for some* $\tilde{C}, \tilde{N} > 0$, *and such that*

$$G = \sum_{z \in \mathbb{Z}^n} \tilde{g}(z) E_{2\pi a^{-1}(z)} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

*then we have* $\tilde{g}(z) = g(z)$ *for all* $z \in \mathbb{Z}^n$.

*Proof.* Let $G \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$ be such that (11.1) holds. Set

$$G_a \equiv G \circ a,$$

(cf. (1.7).) Then, clearly,

$$\tau_{le_j} G_a = G_a \qquad \forall l \in \mathbb{Z}, \quad \forall j \in \{1, \dots, n\},$$

where, as usual, $\{e_1, \dots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$. Then, by Folland [53, pp. 297-299], $G_a \in \mathcal{S}'(\mathbb{R}^n)$ and there exists a (unique) function $g_a$ of $\mathbb{Z}^n$ to $\mathbb{C}$, such that

$$|g_a(z)| \le C(1+|z|)^N \qquad \forall z \in \mathbb{Z}^n,$$

for some $C, N > 0$, and such that

$$G_a = \sum_{z \in \mathbb{Z}^n} g_a(z) E_{2\pi z} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Since $G_a \circ a^{-1} \in \mathcal{S}'(\mathbb{R}^n)$ and $G_a \circ a^{-1} = G$ in $\mathcal{D}'(\mathbb{R}^n, \mathbb{C})$, we have $G \in \mathcal{S}'(\mathbb{R}^n)$. Then a simple computation shows that

$$E_{2\pi z} \circ a^{-1} = E_{2\pi a^{-1}(z)} \qquad \forall z \in \mathbb{Z}^n.$$

Then, by continuity of the operator of $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ which takes $u$ to $u \circ a^{-1}$, we have

$$G = G_a \circ a^{-1} = \sum_{z \in \mathbb{Z}^n} g_a(z) E_{2\pi a^{-1}(z)} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Accordingly, the first part of the Proposition is proved. Now let $g$, $\tilde{g}$ be two functions of $\mathbb{Z}^n$ to $\mathbb{C}$ such that

$$|g(z)| \le C(1+|z|)^N, \quad |\tilde{g}(z)| \le \tilde{C}(1+|z|)^{\tilde{N}} \qquad \forall z \in \mathbb{Z}^n,$$

for some $C, N, \tilde{C}, \tilde{N} > 0$, and such that

$$G = \sum_{z \in \mathbb{Z}^n} g(z) E_{2\pi a^{-1}(z)}, \quad G = \sum_{z \in \mathbb{Z}^n} \tilde{g}(z) E_{2\pi a^{-1}(z)} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Then a simple computation shows that

$$E_{2\pi a^{-1}(z)} \circ a = E_{2\pi z} \qquad \forall z \in \mathbb{Z}^n.$$

Then, by continuity of the operator of $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ which takes $u$ in $\mathcal{S}'(\mathbb{R}^n)$ to $u \circ a$, we have

$$G \circ a = \sum_{z \in \mathbb{Z}^n} g(z) E_{2\pi z}, \quad G \circ a = \sum_{z \in \mathbb{Z}^n} \tilde{g}(z) E_{2\pi z} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

As a consequence, by Folland [53, pp. 297-299], we have $g(z) = \tilde{g}(z)$ for all $z \in \mathbb{Z}^n$. Hence the proof is now complete. □

Then we have the following Theorem.

**Theorem 11.2.** *Let $L$ be the linear differential operator with constant coefficients defined by*

$$L \equiv \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|\leq k}} c_\beta D^\beta ,$$

*for some $k \in \mathbb{N}\setminus\{0\}$, $\{c_\beta\}_{|\beta|\leq k} \subseteq \mathbb{C}$. We set*

$$P(x) \equiv \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|\leq k}} c_\beta x^\beta \qquad \forall x \in \mathbb{R}^n .$$

*Then there exists a distribution $G \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$, such that*

*(i)*

$$\tau_{la_j} G = G \qquad \forall l \in \mathbb{Z}, \quad \forall j \in \{1, \dots, n\},$$

*and*

*(ii)*

$$L[G] = \sum_{z\in\mathbb{Z}^n} \delta_{a(z)}$$

*in the sense of distributions,*

*if and only if*

*(j)*

$$P(i2\pi a^{-1}(z)) \neq 0 \qquad \forall z \in \mathbb{Z}^n ,$$

*and*

*(jj)*

$$\frac{1}{|P(i2\pi a^{-1}(z))|} \leq C(1+|z|)^N \qquad \forall z \in \mathbb{Z}^n ,$$

*for some $C$, $N > 0$.*

*In particular, if $(j)$ and $(jj)$ hold, then $G$ is unique, $G \in \mathcal{S}'(\mathbb{R}^n)$, and it is delivered by the following formula*

$$G \equiv \sum_{z\in\mathbb{Z}^n} \frac{1}{|A|_n P(i2\pi a^{-1}(z))} E_{2\pi a^{-1}(z)} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n) \tag{11.3}$$

*(cf. Proposition 1.1.)*

*Proof.* We first assume that there exists a distribution $G \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$ such that $(i)$ and $(ii)$ hold, and we prove that $(j)$ and $(jj)$ must hold, and that $G$ must be delivered by (11.3). Since the distribution $G$ is periodic, then, by Proposition 11.1, we have $G \in \mathcal{S}'(\mathbb{R}^n)$. Moreover, there exists a unique function $g$ of $\mathbb{Z}^n$ to $\mathbb{C}$, such that

$$|g(z)| \leq \tilde{C}(1+|z|)^N \qquad \forall z \in \mathbb{Z}^n ,$$

for some $\tilde{C}$, $N > 0$, and such that

$$G = \sum_{z\in\mathbb{Z}^n} g(z) E_{2\pi a^{-1}(z)} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Consequently,

$$\begin{aligned} L[G] &= \sum_{z\in\mathbb{Z}^n} g(z) \Bigg( \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|\leq k}} c_\beta \big(i2\pi a^{-1}(z)\big)^\beta \Bigg) E_{2\pi a^{-1}(z)} \\ &= \sum_{z\in\mathbb{Z}^n} g(z) P\big(i2\pi a^{-1}(z)\big) E_{2\pi a^{-1}(z)} . \end{aligned}$$

By the Poisson summation Formula (see Theorem A.10 and Proposition 1.2) and Proposition 11.1, we have

$$L[G] = \sum_{z \in \mathbb{Z}^n} \delta_{a(z)} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

if and only if

$$g(z) P\big(i2\pi a^{-1}(z)\big) = \frac{1}{|A|_n} \qquad \forall z \in \mathbb{Z}^n.$$

Thus,

$$P\big(i2\pi a^{-1}(z)\big) \neq 0 \qquad \forall z \in \mathbb{Z}^n,$$

$$g(z) = \frac{1}{|A|_n P\big(i2\pi a^{-1}(z)\big)} \qquad \forall z \in \mathbb{Z}^n,$$

and

$$\frac{1}{|P\big(i2\pi a^{-1}(z)\big)|} \leq |A|_n \tilde{C}(1+|z|)^N \qquad \forall z \in \mathbb{Z}^n.$$

Hence, $(j)$ and $(jj)$ hold, and $G$ must be delivered by (11.3). As a consequence, if such a distribution $G$ exists, it is unique.

Conversely, if $(j)$ and $(jj)$ hold, then, by reading backward the above argument, one can easily show that the distribution $G$ defined in (11.3), satisfies $(i)$ and $(ii)$. □

Now we want to show some conditions on the linear differential operator $L$ that ensure, in particular, that condition $(jj)$ of Theorem 11.2 is satisfied.

**Corollary 11.3.** *Let $L$ be the linear differential operator with constant coefficients defined by*

$$L \equiv \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| \leq 2k}} c_\beta D^\beta,$$

*for some $k \in \mathbb{N} \setminus \{0\}$, $\{c_\beta\}_{|\beta| \leq k} \subseteq \mathbb{C}$. We set*

$$P(x) \equiv \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| \leq 2k}} c_\beta x^\beta \qquad \forall x \in \mathbb{R}^n.$$

*Assume that*

*(i)*

$$P(i2\pi a^{-1}(z)) \neq 0 \qquad \forall z \in \mathbb{Z}^n,$$

*and*

*(ii)*

$$\mathrm{Re}\Big\{ \sum_{\substack{\beta \in \mathbb{N}^n \\ |\beta| = 2k}} c_\beta x^\beta \Big\} \geq C|x|^{2k} \qquad \forall x \in \mathbb{R}^n,$$

*for some $C > 0$.*

*Then there exists a unique distribution $G \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$, delivered by equality (11.3), such that*

*(j)*

$$\tau_{la_j} G = G \qquad \forall l \in \mathbb{Z}, \quad \forall j \in \{1, \dots, n\},$$

*and*

*(jj)*

$$L[G] = \sum_{z \in \mathbb{Z}^n} \delta_{a(z)}$$

*in the sense of distributions.*

*Proof.* Clearly it suffices to prove that by $(ii)$ we have that condition $(jj)$ of Theorem 11.2 holds. In order to do so, we observe that if $(ii)$ holds, then one can easily show that

$$\lim_{\substack{x\in\mathbb{R}^n\\ x\to\infty}} |\mathrm{Re}\{P(i2\pi a^{-1}(x))\}| = +\infty.$$

Accordingly, there exists a constant $C' > 0$ such that

$$\frac{1}{|P(i2\pi a^{-1}(z))|} \leq C' \qquad \forall z \in \mathbb{Z}^n,$$

and so $(jj)$ of Theorem 11.2 holds. □

**Corollary 11.4.** *Let $L$ be the linear differential operator with constant coefficients defined by*

$$L \equiv \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|\leq 2k}} c_\beta D^\beta,$$

*for some $k \in \mathbb{N}\setminus\{0\}$, $\{c_\beta\}_{|\beta|\leq k} \subseteq \mathbb{R}$. We set*

$$P(x) \equiv \sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|\leq 2k}} c_\beta x^\beta \qquad \forall x \in \mathbb{R}^n.$$

*Assume that*

*(i)*

$$P(i2\pi a^{-1}(z)) \neq 0 \qquad \forall z \in \mathbb{Z}^n,$$

*and*

*(ii)*

$$\sum_{\substack{\beta\in\mathbb{N}^n\\ |\beta|=2k}} c_\beta x^\beta \geq C|x|^{2k} \qquad \forall x \in \mathbb{R}^n,$$

*for some $C > 0$.*

*Then there exists a unique distribution $G \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$, delivered by equality* (11.3)*, such that*

*(j)*

$$\tau_{la_j} G = G \qquad \forall l \in \mathbb{Z}, \quad \forall j \in \{1,\ldots,n\},$$

*and*

*(jj)*

$$L[G] = \sum_{z\in\mathbb{Z}^n} \delta_{a(z)}$$

*in the sense of distributions.*

*Moreover,*

$$\langle G, \phi\rangle \in \mathbb{R} \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}). \tag{11.4}$$

*Proof.* Obviously, by virtue of Corollary 11.3, it follows that there exists a unique distribution $G \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$ such that $(j)$ and $(jj)$ hold. We need to prove that (11.4) holds. We note that

$$\overline{P(i2\pi a^{-1}(z))} = P(i2\pi a^{-1}(-z)) \qquad \forall z \in \mathbb{Z}^n.$$

Accordingly,

$$\overline{\frac{1}{P(i2\pi a^{-1}(z))}\langle E_{2\pi a^{-1}(z)}, \phi\rangle} = \frac{1}{P(i2\pi a^{-1}(-z))}\langle E_{2\pi a^{-1}(-z)}, \phi\rangle \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}),$$

for all $z \in \mathbb{Z}^n$. As a consequence,

$$\overline{\langle G, \phi\rangle} = \langle G, \phi\rangle \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}),$$

and the proof is complete. □

## 11.2 Real analyticity of periodic layer potentials of general second order differential operators with constant coefficients

In this Section we deduce by the real analyticity of classic layer potentials of general second order differential operators with constant coefficients, proved by Dalla Riva and Lanza [40], an analogous result for the corresponding periodic layer potentials. We have the following.

**Theorem 11.5.** *Let $L$ be the linear differential operator with constant coefficients defined by*

$$L \equiv \sum_{\substack{\beta\in\mathbb{N}^n \\ |\beta|\le 2}} c_\beta D^\beta,$$

*with $c_\beta \in \mathbb{C}$, for all $\beta \in \mathbb{N}^n$ such that $|\beta| \le 2$. We set*

$$P(x) \equiv \sum_{\substack{\beta\in\mathbb{N}^n \\ |\beta|\le 2}} c_\beta x^\beta \qquad \forall x \in \mathbb{R}^n.$$

*Assume that*

*(i)*

$$P(i2\pi a^{-1}(z)) \neq 0 \qquad \forall z \in \mathbb{Z}^n,$$

*and*

*(ii)*

$$\mathrm{Re}\Big\{ \sum_{\substack{\beta\in\mathbb{N}^n \\ |\beta|= 2}} c_\beta x^\beta \Big\} \ge C|x|^2 \qquad \forall x \in \mathbb{R}^n,$$

*for some $C > 0$.*

*Let $G$ be the element of $\mathcal{D}'(\mathbb{R}^n,\mathbb{C})$ defined by (11.3). Let the function $S_n^L$ of $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{C}$ be a fundamental solution of $L$. Then the following statements hold.*

*(i) There exists a unique function $S_n^{a,L}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{C})$ such that*

$$\int_{\mathbb{R}^n} S_n^{a,L}(x)\phi(x)\,dx = \langle G, \phi\rangle \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n,\mathbb{C}). \tag{11.5}$$

*Therefore, in particular*

$$L[S_n^{a,L}] = \sum_{z\in\mathbb{Z}^n} \delta_{a(z)} \tag{11.6}$$

*in the sense of distributions. Moreover, up to modifications on a set of measure zero, $S_n^{a,L}$ is a real analytic function of $\mathbb{R}^n \setminus Z_n^a$ to $\mathbb{C}$, such that*

$$L[S_n^{a,L}](x) = 0 \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a \tag{11.7}$$

*and*

$$S_n^{a,L}(x + a_i) = S_n^{a,L}(x) \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a, \quad \forall i \in \{1,\dots,n\}. \tag{11.8}$$

*(ii) There exists a unique real analytic function $R_n^{a,L}$ of $(\mathbb{R}^n \setminus Z_n^a) \cup \{0\}$ to $\mathbb{C}$, such that*

$$S_n^{a,L}(x) = S_n^L(x) + R_n^{a,L}(x) \qquad \forall x \in \mathbb{R}^n \setminus Z_n^a.$$

*Moreover,*

$$L[R_n^{a,L}](x) = 0 \qquad \forall x \in (\mathbb{R}^n \setminus Z_n^a) \cup \{0\}.$$

*Proof.* It is a straighforward modification of the proof of Theorem 6.3, where the analogous result has been proved for the Helmholtz operator $\Delta + k^2$. See also the proof of Theorem 1.4 where the Laplace operator is considered. □

From now on, we shall consider only linear differential operators $L$ as in Theorem 11.5. If $L$ is as in Theorem 11.5, then we set

$$c^{(2)}(L) \equiv (c^{(2)}(L)_{lj})_{l,j=1,\dots,n} \qquad c^{(1)}(L) \equiv (c^{(1)}(L)_j)_{j=1,\dots,n}$$

with $c^{(2)}(L)_{lj} \equiv \frac{c_{e_l+e_j}}{2-\delta_{l,j}}$ and $c^{(1)}(L)_j \equiv c_{e_j}$.

We collect in the following statement some facts on the periodic layer potentials associated with $S_n^{a,L}$.

**Theorem 11.6.** *Let $L$ and $S_n^{a,L}$ be as in Theorem 11.5. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be as in (1.46). Let $\mathcal{V}$ be an open bounded connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$, such that $\mathrm{cl}\, A \subseteq \mathcal{V}$ and*

$$\mathrm{cl}\,\mathcal{V} \cap \mathrm{cl}(\mathbb{I} + a(z)) = \emptyset \qquad \forall z \in \mathbb{Z}^n \setminus \{0\}.$$

*Set*

$$\mathcal{W} \equiv \mathcal{V} \setminus \mathrm{cl}\,\mathbb{I}.$$

*Then the following statements hold.*

*(i) If $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$, then the function $v_a[\partial\mathbb{I}, \mu, L]$ of $\mathbb{R}^n$ to $\mathbb{C}$ defined by*

$$v_a[\partial\mathbb{I}, \mu, L](t) \equiv \int_{\partial\mathbb{I}} S_n^{a,L}(t-s)\mu(s)\, d\sigma_s \qquad \forall t \in \mathbb{R}^n,$$

*is continuous. Moreover,*

$$L[v_a[\partial\mathbb{I}, \mu, L]](t) = 0 \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}],$$

*and*

$$v_a[\partial\mathbb{I}, \mu, L](t + a_i) = v_a[\partial\mathbb{I}, \mu, L](t) \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1, \dots, n\}.$$

*(ii) If $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$, then the function $v_a^+[\partial\mathbb{I}, \mu, L] \equiv v_a[\partial\mathbb{I}, \mu, L]_{|\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]}$ belongs to the space $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}], \mathbb{C})$, and the operator which takes $\mu$ to $v_a^+[\partial\mathbb{I}, \mu, L]_{|\mathrm{cl}\,\mathbb{I}}$ is continuous from the space $C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$ to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I}, \mathbb{C})$.*

*(iii) If $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$, then the function $v_a^-[\partial\mathbb{I}, \mu, L] \equiv v_a[\partial\mathbb{I}, \mu, L]_{|\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]}$ belongs to the space $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \mathbb{C})$, and the operator which takes $\mu$ to $v_a^-[\partial\mathbb{I}, \mu, L]_{|\mathrm{cl}\,\mathcal{W}}$ is continuous from the space $C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$ to $C^{m,\alpha}(\mathrm{cl}\,\mathcal{W}, \mathbb{C})$.*

*(iv) If $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$, $l \in \{1, \dots, n\}$, then the integral*

$$v_{a,l}[\partial\mathbb{I}, \mu, L](t) \equiv \int_{\partial\mathbb{I}} \partial_{t_l} S_n^{a,L}(t-s)\mu(s)\, d\sigma_s \qquad \forall t \in \mathbb{R}^n,$$

*converges in the sense of Lebesgue for all $t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}]$ and in the sense of a principal value for all $t \in \mathrm{cl}\,\mathbb{S}_a[\mathbb{I}] \cap \mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$.*

*(v) Let $l \in \{1, \dots, n\}$. If $\mu \in C^{m-1,\alpha}(\partial\mathbb{I}, \mathbb{C})$, then $v_{a,l}[\partial\mathbb{I}, \mu, L]_{|\mathbb{S}_a[\mathbb{I}]}$ admits a continuous extension $v_{a,l}^+[\partial\mathbb{I}, \mu, L]$ to $\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}]$ and $v_{a,l}^+[\partial\mathbb{I}, \mu, L] \in C^{m-1,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}], \mathbb{C})$, and $v_{a,l}[\partial\mathbb{I}, \mu, L]_{|\mathbb{T}_a[\mathbb{I}]}$ admits a continuous extension $v_{a,l}^-[\partial\mathbb{I}, \mu, L]$ to $\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}]$ and $v_{a,l}^-[\partial\mathbb{I}, \mu, L] \in C^{m-1,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}], \mathbb{C})$, and*

$$\begin{aligned} v_{a,l}^{\pm}[\partial\mathbb{I}, \mu, L](t) &= \frac{\partial}{\partial t_l} v_a^{\pm}[\partial\mathbb{I}, \mu, L](t) \\ &= \mp \frac{(\nu_{\mathbb{I}}(t))_l}{2\nu_{\mathbb{I}}(t)^T c^{(2)}(L)\nu_{\mathbb{I}}(t)}\mu(t) + v_{a,l}[\partial\mathbb{I}, \mu, L](t), \end{aligned}$$

$$\begin{aligned} &(Dv_a^{\pm}[\partial\mathbb{I}, \mu, L](t))c(L)^{(2)}\nu_{\mathbb{I}}(t) \\ &\qquad = \mp\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} (DS_n^{a,L}(t-s))c^{(2)}(L)\nu_{\mathbb{I}}(t)\mu(s)\, d\sigma_s \end{aligned}$$

*for all $t \in \partial\mathbb{I}$.*

*(vi)* *Let $l \in \{1,\ldots,n\}$. The operator of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ to $C^{m-1,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$ which takes $\mu$ to the function $v^+_{a,l}[\partial\mathbb{I},\mu,L]_{|\mathrm{cl}\,\mathbb{I}}$ is continuous. The operator of $C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ to $C^{m-1,\alpha}(\mathrm{cl}\,\mathcal{W},\mathbb{C})$ which takes $\mu$ to the function $v^-_{a,l}[\partial\mathbb{I},\mu,L]_{|\mathrm{cl}\,\mathcal{W}}$ is continuous.*

*(vii)* *Let $\mu \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$. Let $w_a[\partial\mathbb{I},\mu,L]$ be the function of $\mathbb{R}^n$ to $\mathbb{C}$ defined by*

$$\begin{aligned} w_a[\partial\mathbb{I},\mu,L](t) \equiv & -\int_{\partial\mathbb{I}} (DS_n^{a,L}(t-s))c^{(2)}(L)\nu_{\mathbb{I}}(s)\mu(s)\,d\sigma_s \\ & -\int_{\partial\mathbb{I}} S_n^{a,L}(t-s)\nu_{\mathbb{I}}^T(s)c^{(1)}(L)\mu(s)\,d\sigma_s \qquad \forall t \in \mathbb{R}^n. \end{aligned}$$

*Then*

$$L[w_a[\partial\mathbb{I},\mu,L]](t) = 0 \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}],$$

*and*

$$w_a[\partial\mathbb{I},\mu,L](t+a_i) = w_a[\partial\mathbb{I},\mu,L](t) \qquad \forall t \in \mathbb{S}_a[\mathbb{I}] \cup \mathbb{T}_a[\mathbb{I}], \quad \forall i \in \{1,\ldots,n\}.$$

*The restriction $w_a[\partial\mathbb{I},\mu,L]_{|\mathbb{S}_a[\mathbb{I}]}$ can be extended uniquely to an element $w_a^+[\partial\mathbb{I},\mu,L]$ of the space $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S}_a[\mathbb{I}],\mathbb{C})$ and the restriction $w_a[\partial\mathbb{I},\mu,L]_{|\mathbb{T}_a[\mathbb{I}]}$ can be extended uniquely to an element $w_a^-[\partial\mathbb{I},\mu,L]$ of the space $C^{m,\alpha}(\mathrm{cl}\,\mathbb{T}_a[\mathbb{I}],\mathbb{C})$ and we have*

$$w_a^+[\partial\mathbb{I},\mu,L] - w_a^-[\partial\mathbb{I},\mu,L] = \mu \qquad \text{on } \partial\mathbb{I},$$

$$(Dw_a^+[\partial\mathbb{I},\mu,L])c^{(2)}(L)\nu_{\mathbb{I}} - (Dw_a^-[\partial\mathbb{I},\mu,L])c^{(2)}(L)\nu_{\mathbb{I}} = 0 \qquad \text{on } \partial\mathbb{I},$$

*(viii)* *The operator of $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$ which takes $\mu$ to $w_a^+[\partial\mathbb{I},\mu,L]_{|\mathrm{cl}\,\mathbb{I}}$ is continuous. The operator of $C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$ to $C^{m,\alpha}(\mathrm{cl}\,\mathcal{W},\mathbb{C})$ which takes $\mu$ to $w_a^-[\partial\mathbb{I},\mu,L]_{|\mathrm{cl}\,\mathcal{W}}$ is continuous.*

*Proof.* It is a straightforward modification of the proof of Theorems 6.7 and 6.11, with Theorems E.4 and E.5 replaced by Dalla Riva and Lanza [40, Theorem 3.1], and Theorem 6.3 replaced by Theorem 11.5. □

Now let $K$ be a compact subset of $\mathbb{R}^n$. Let $C^{0,1}(K,\mathbb{R}^n)$ denote the space of Lipschitz continuous functions of $K$ to $\mathbb{R}^n$. Then we set

$$l_K[f] \equiv \inf\left\{ \frac{|f(x)-f(y)|}{|x-y|} : x,y \in K,\ x \neq y \right\} \qquad \forall f \in C^{0,1}(K,\mathbb{R}^n).$$

We also set

$$\mathcal{A}_K \equiv \left\{ \phi \in C^1(K,\mathbb{R}^n) \colon l_k[\phi] > 0 \right\}.$$

The set $\mathcal{A}_K$ is open in $C^1(K,\mathbb{R}^n)$. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be a bounded open subset of class $C^{m,\alpha}$ of $\mathbb{R}^n$ such that both $\Omega$ and $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$ are connected. Let $\phi \in \mathcal{A}_{\partial\Omega}$. By the Jordan–Leray separation theorem, $\mathbb{R}^n \setminus \phi(\partial\Omega)$ has exactly two open connected components, and we denote by $\mathbb{I}[\phi]$ the bounded connected component. Then we denote by $\nu_\phi$ the outward normal to the set $\mathbb{I}[\phi]$ (cf. Lanza and Rossi [86].)

Then we have the following technical Lemma.

**Lemma 11.7.** *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be a bounded connected open subset of class $C^{m,\alpha}$ of $\mathbb{R}^n$, such that $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$ is connected. Then the following statements hold.*

*(i)* *Let $\phi \in C^{m,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$. Then there exists a positive function $\tilde\sigma[\phi] \in C^{m-1,\alpha}(\partial\Omega)$ such that*

$$\int_{\phi(\partial\Omega)} \omega(s)\,d\sigma_s = \int_{\partial\Omega} \omega\circ\phi(y)\tilde\sigma[\phi](y)\,d\sigma_y \qquad \forall \omega \in L^1(\phi(\partial\Omega),\mathbb{C}).$$

*Moreover, the map $\tilde\sigma[\cdot]$ of $C^{m,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$ to $C^{m-1,\alpha}(\partial\Omega)$ which takes $\phi$ to $\tilde\sigma[\phi]$ is real analytic.*

*(ii)* *The map of $C^{m,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{R}^n)$ which takes $\phi$ to $\nu_\phi\circ\phi$ is real analytic.*

*Proof.* For $(i)$, see the proof of Lanza and Rossi [85, Prop. 3.13] and replace Proposition 2.8 and Lemma 3.3 of Lanza and Rossi [85] with Proposition 2.6 and Lemma 4.2 of Lanza and Rossi [86]. For $(ii)$, see Proposition 2.6 and Lemma 4.2 of Lanza and Rossi [86] (cf. also Lanza e Rossi [85, Prop. 3.13].) □

Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be a bounded connected open subset of class $C^{m,\alpha}$ of $\mathbb{R}^n$, such that $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$ is connected. Then we set

$$\mathcal{E}_a^{m,\alpha}(\partial\Omega) \equiv \{ \phi \in C^{m,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega} \colon \phi(\partial\Omega) \subseteq A \} .$$

The set $\mathcal{E}_a^{m,\alpha}(\partial\Omega)$ is open in $C^{m,\alpha}(\partial\Omega, \mathbb{R}^n)$. If $\phi \in \mathcal{E}_a^{m,\alpha}(\partial\Omega)$, then the set $\mathbb{I}[\phi]$ satisfies (1.46).

We give the following definitions (cf. Dalla Riva and Lanza [40].)

**Definition 11.8.** Let $L$ and $S_n^{a,L}$ be as in Theorem 11.5. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be a bounded connected open subset of class $C^{m,\alpha}$ of $\mathbb{R}^n$, such that $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$ is connected. Let $\phi \in \mathcal{E}_a^{m,\alpha}(\partial\Omega)$, $f \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Then we set

$$V_a[\phi, f, L](x) \equiv v_a[\partial\mathbb{I}[\phi], f \circ \phi^{(-1)}, L] \circ \phi(x) \qquad \forall x \in \partial\Omega.$$

**Definition 11.9.** Let $L$ and $S_n^{a,L}$ be as in Theorem 11.5. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be a bounded connected open subset of class $C^{m,\alpha}$ of $\mathbb{R}^n$, such that $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$ is connected. Let $\phi \in \mathcal{E}_a^{m,\alpha}(\partial\Omega)$, $f \in C^{m,\alpha}(\partial\Omega, \mathbb{C})$. Then we set

$$W_a[\phi, f, L](x) \equiv w_a[\partial\mathbb{I}[\phi], f \circ \phi^{(-1)}, L] \circ \phi(x) \qquad \forall x \in \partial\Omega.$$

**Definition 11.10.** Let $L$ and $S_n^{a,L}$ be as in Theorem 11.5. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be a bounded connected open subset of class $C^{m,\alpha}$ of $\mathbb{R}^n$, such that $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$ is connected. Let $\phi \in \mathcal{E}_a^{m,\alpha}(\partial\Omega)$, $f \in C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Then we set

$$V_{a*}[\phi, f, L](x) \equiv \int_{\phi(\partial\Omega)} DS_n^{a,L}(\phi(x) - s) c^{(2)}(L) \nu_\phi(\phi(x)) f \circ \phi^{(-1)}(s)\, d\sigma_s \qquad \forall x \in \partial\Omega.$$

We are now ready to prove the real analyticity of $W_a[\cdot,\cdot,L]$, $V_a[\cdot,\cdot,L]$, and $V_{a*}[\cdot,\cdot,L]$.

**Theorem 11.11.** *Let $L$ be as in Theorem 11.5. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be a bounded connected open subset of class $C^{m,\alpha}$ of $\mathbb{R}^n$, such that $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$ is connected. Then the following statements hold.*

*(i) The map $V_a[\cdot,\cdot,L]$ of $\mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$ is real analytic.*

*(ii) The map $W_a[\cdot,\cdot,L]$ of $\mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$ is real analytic.*

*(iii) The map $V_{a*}[\cdot,\cdot,L]$ of $\mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ is real analytic.*

*Proof.* Let $S_n^L$ be a fundamental solution of the differential operator $L$. We first prove $(i)$. Let $(\phi, f) \in \mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$. Obviously,

$$\begin{aligned} V_a[\phi, f, L](x) = & \int_{\phi(\partial\Omega)} S_n^L(\phi(x) - s) f \circ \phi^{(-1)}(s)\, d\sigma_s \\ & + \int_{\partial\Omega} R_n^{a,L}(\phi(x) - \phi(y)) f(y) \tilde{\sigma}[\phi](y)\, d\sigma_y \qquad \forall x \in \partial\Omega. \end{aligned} \tag{11.9}$$

By Dalla Riva and Lanza [40, Theorem 5.6], the map $V[\cdot,\cdot,L]$ of $\mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$, which takes $(\phi, f)$ to the function $V[\phi, f, L]$ of $\partial\Omega$ to $\mathbb{C}$, defined by

$$V[\phi, f, L](x) \equiv \int_{\phi(\partial\Omega)} S_n^L(\phi(x) - s) f \circ \phi^{(-1)}(s)\, d\sigma_s \qquad \forall x \in \partial\Omega,$$

is real analytic. By continuity of pointwise product in Schauder spaces, by Theorem C.2 and Lemma 11.7, and by standard calculus in Banach space, we immediately deduce that the second term in the right-hand side of (11.9) defines a real analytic map of $\mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$ in the variable $(\phi, f)$. Thus $V_a[\cdot,\cdot,L]$ is a real analytic map of $\mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega, \mathbb{C})$ to $C^{m,\alpha}(\partial\Omega, \mathbb{C})$. Consider $(ii)$. Let $(\phi, f) \in \mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m,\alpha}(\partial\Omega, \mathbb{C})$. Obviously,

$$\begin{aligned} W_a[\phi, f, L](x) = & - \int_{\phi(\partial\Omega)} DS_n^L(\phi(x) - s) c^{(2)}(L) \nu_\phi(s) f \circ \phi^{(-1)}(s)\, d\sigma_s \\ & - \int_{\phi(\partial\Omega)} S_n^L(\phi(x) - s) \nu_\phi^T(s) c^{(1)}(L) f \circ \phi^{(-1)}(s)\, d\sigma_s \\ & - \int_{\partial\Omega} DR_n^{a,L}(\phi(x) - \phi(y)) c^{(2)}(L) \nu_\phi(\phi(y)) f(y) \tilde{\sigma}[\phi](y)\, d\sigma_y \\ & - \int_{\partial\Omega} R_n^{a,L}(\phi(x) - \phi(y)) \nu_\phi^T(\phi(y)) c^{(1)}(L) f(y) \tilde{\sigma}[\phi](y)\, d\sigma_y \qquad \forall x \in \partial\Omega. \end{aligned} \tag{11.10}$$

By Dalla Riva and Lanza [40, Theorem 5.6], the map $W[\cdot,\cdot,L]$ of $\mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$, which takes $(\phi, f)$ to the function $W[\phi, f, L]$ of $\partial\Omega$ to $\mathbb{C}$, defined by

$$\begin{aligned} W[\phi,f,L](x) \equiv &- \int_{\phi(\partial\Omega)} DS_n^L(\phi(x)-s)c^{(2)}(L)\nu_\phi(s) f\circ\phi^{(-1)}(s)\, d\sigma_s \\ &- \int_{\phi(\partial\Omega)} S_n^L(\phi(x)-s)\nu_\phi^T(s)c^{(1)}(L) f\circ\phi^{(-1)}(s)\, d\sigma_s \qquad \forall x\in\partial\Omega, \end{aligned}$$

is real analytic. By continuity of pointwise product in Schauder spaces, by Theorem C.2 and Lemma 11.7, and by standard calculus in Banach space, we immediately deduce that the third and the fourth term in the right-hand side of (11.10) define real analytic maps of $\mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$ in the variable $(\phi, f)$. Hence $W_a[\cdot,\cdot,L]$ is a real analytic map of $\mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m,\alpha}(\partial\Omega,\mathbb{C})$. We finally prove (*iii*). Let $(\phi, f) \in \mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. Obviously,

$$\begin{aligned} V_{a*}[\phi,f,L](x) = &\int_{\phi(\partial\Omega)} DS_n^L(\phi(x)-s)c^{(2)}(L)\nu_\phi(\phi(x)) f\circ\phi^{(-1)}(s)\, d\sigma_s \\ &+ \int_{\partial\Omega} DR_n^{a,L}(\phi(x)-\phi(y))c^{(2)}(L)\nu_\phi(\phi(x)) f(y)\tilde\sigma[\phi](y)\, d\sigma_y \qquad \forall x\in\partial\Omega. \end{aligned} \tag{11.11}$$

By Dalla Riva and Lanza [40, Theorem 5.6], the map $V_*[\cdot,\cdot,L]$ of $\mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$, which takes $(\phi, f)$ to the function $V_*[\phi, f, L]$ of $\partial\Omega$ to $\mathbb{C}$, defined by

$$V_*[\phi,f,L](x) \equiv \int_{\phi(\partial\Omega)} DS_n^L(\phi(x)-s)c^{(2)}(L)\nu_\phi(\phi(x)) f\circ\phi^{(-1)}(s)\, d\sigma_s \qquad \forall x\in\partial\Omega,$$

is real analytic. By continuity of pointwise product in Schauder spaces, by Theorem C.2 and Lemma 11.7, and by standard calculus in Banach space, we immediately deduce that the second term in the right-hand side of (11.11) defines a real analytic map of $\mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ in the variable $(\phi, f)$. Therefore $V_{a*}[\cdot,\cdot,L]$ is a real analytic map of $\mathcal{E}_a^{m,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ to $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$. □

*Remark* 11.12. We note that, by following the approach of Dalla Riva and Lanza [40] (see also Lanza and Preciso [83], Lanza and Rossi [85, 86]), one can prove a more general result. Namely, one can consider periodic layer potentials corresponding to a family of strongly elliptic differential operators of second order depending on a parameter, and then one can prove a real analyticity theorem for the dependence of the periodic layer potentials also upon variation of the parameter (cf. [81].)

## 11.3 Periodic volume potential

In this Section we introduce an analogue of the periodic Newtonian potential for a general second order elliptic equation with constant coefficients.

We give the following.

**Definition 11.13.** Let $L$ and $S_n^{a,L}$ be as in Theorem 11.5. Let $f \in C^0(\mathbb{R}^n,\mathbb{C})$ be such that

$$f(t+a_i) = f(t) \qquad \forall t\in\mathbb{R}^n, \quad \forall i\in\{1,\dots,n\}.$$

We set

$$p_a[f,L](t) \equiv \int_A S_n^{a,L}(t-s) f(s)\, ds \qquad \forall t\in\mathbb{R}^n.$$

The function $p_a[f, L]$ is called the periodic volume potential of $f$.

**Theorem 11.14.** *Let $L$ and $S_n^{a,L}$ be as in Theorem 11.5. Let $m \in \mathbb{N}$, $\alpha \in ]0, 1[$. Let $f \in C^{m,\alpha}(\mathbb{R}^n,\mathbb{C})$ be such that*

$$f(t+a_i) = f(t) \qquad \forall t\in\mathbb{R}^n, \quad \forall i\in\{1,\dots,n\}.$$

*Then the following statements hold.*

(*i*)

$$p_a[f,L](t+a_i) = p_a[f,L](t) \qquad \forall t\in\mathbb{R}^n, \quad \forall i\in\{1,\dots,n\}.$$

(ii)
$$p_a[f, L] \in C^{m+2,\alpha}(\mathbb{R}^n, \mathbb{C}).$$

(iii)
$$L[p_a[f, L]](t) = f(t) \qquad \forall t \in \mathbb{R}^n.$$

*Proof.* It is a straightforward modification of the proof of Theorem 6.16. □

*Remark* 11.15. Let $L$ and $S_n^{a,L}$ be as in Theorem 11.5. Let $m \in \mathbb{N}$, $\alpha \in ]0, 1[$. We note that by Theorem 11.14 we have that for each function $f \in C^{m,\alpha}(\mathbb{R}^n, \mathbb{C})$ such that

$$f(t + a_i) = f(t) \qquad \forall t \in \mathbb{R}^n, \quad \forall i \in \{1, \ldots, n\},$$

there exists a function $p \in C^{m+2,\alpha}(\mathbb{R}^n, \mathbb{C})$, such that

$$p(t + a_i) = p(t) \qquad \forall t \in \mathbb{R}^n, \quad \forall i \in \{1, \ldots, n\},$$

and

$$L[p](t) = f(t) \qquad \forall t \in \mathbb{R}^n.$$

Finally, we mention a result by Kozlov, Maz'ya and Rossmann [66, Theorem 2.1.1, p. 32], that shows a necessary and sufficient condition for a differential operator $L$ with constant coefficients to be an isomorphism between two suitable Sobolev spaces of periodic functions.

# APPENDIX A

## Results of Fourier Analysis

In this Appendix we collect some definitions and known facts of Fourier Analysis. Throughout this Appendix, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

We recall the following definitions.

**Definition A.1.** Let $\Omega$ be an open subset of $\mathbb{R}^n$. We denote by $\mathcal{D}(\Omega, \mathbb{K})$ the vector space over $\mathbb{K}$ of all $C^\infty$ functions of $\Omega$ to $\mathbb{K}$, whose support is compact and contained in $\Omega$.

We recall that a net $\{\phi_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{D}(\Omega, \mathbb{K})$ converges to $\phi \in \mathcal{D}(\Omega, \mathbb{K})$, if the $\phi_\lambda$'s are all supported in a common compact subset of $\Omega$ and $D^\alpha \phi_\lambda$ converges to $D^\alpha \phi$ uniformly for every multi-index $\alpha$. For a more precise description of $\mathcal{D}(\Omega, \mathbb{K})$, we refer, *e.g.*, to Rudin [120] or Treves [135].

**Definition A.2.** Let $\Omega$ be an open subset of $\mathbb{R}^n$. We denote by $\mathcal{D}'(\Omega, \mathbb{K})$ the vector space over $\mathbb{K}$ of all linear and continuous functionals of $\mathcal{D}(\Omega, \mathbb{K})$ to $\mathbb{K}$, endowed with the weak $*$ topology. The elements of $\mathcal{D}'(\Omega, \mathbb{K})$ are called *distributions*.

**Definition A.3.** A function $f$ of $\mathbb{R}^n$ to $\mathbb{C}$ is said to be *rapidly decreasing* if

$$\sup_{x \in \mathbb{R}^n} |f(x)|(1 + |x|)^m < +\infty \qquad \forall m \in \mathbb{N}.$$

**Definition A.4.** We denote by $\mathcal{S}(\mathbb{R}^n)$ the set of all $C^\infty$ functions $f$ of $\mathbb{R}^n$ to $\mathbb{C}$ such that $D^\alpha f$ is rapidly decreasing for all $\alpha \in \mathbb{N}^n$.

Then we have the following well known result.

**Proposition A.5.** *The vector space $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space for the increasing sequence of norms $\{p_m\}_{m \in \mathbb{N}}$, defined by*

$$p_m(f) \equiv \sup_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} (1 + |x|)^m |D^\alpha f(x)| \qquad \forall f \in \mathcal{S}(\mathbb{R}^n),$$

*for all $m \in \mathbb{N}$.*

*Remark* A.6. It is well known that the family of seminorms $\{p_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}^n}$,where

$$p_{\alpha,\beta}(f) \equiv \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| \qquad \forall f \in \mathcal{S}(\mathbb{R}^n),$$

for all $\alpha, \beta \in \mathbb{N}^n$, is an equivalent set of seminorms on $\mathcal{S}(\mathbb{R}^n)$.

**Definition A.7.** Let $f \in L^1(\mathbb{R}^n, \mathbb{C})$. Let $\hat{f}$ be the function of $\mathbb{R}^n$ to $\mathbb{C}$ defined by

$$\hat{f}(\xi) \equiv \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, dx \qquad \forall \xi \in \mathbb{R}^n.$$

The function $\hat{f}$ is called the *Fourier transform* of $f$.

Then we have the following.

**Proposition A.8.** *The map $\mathcal{F}$ of $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, which takes $f$ to $\mathcal{F}(f) \equiv \hat{f}$ is a homeomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto itself.*

*Proof.* See, *e.g*, Stein and Weiss [131, p. 21]. □

**Definition A.9.** A linear and continuous functional of $\mathcal{S}(\mathbb{R}^n)$ to $\mathbb{C}$ is called a *tempered distribution*. The vector space of all tempered distributions, endowed with the weak $*$ topology, is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

We have the following well known Theorem.

**Theorem A.10** (Poisson summation Formula)**.** *Let $f$ be a continuous function of $\mathbb{R}^n$ to $\mathbb{C}$ such that*

$$|f(x)| \leq C(1+|x|)^{-n-\epsilon} \qquad \forall x \in \mathbb{R}^n$$

*and*

$$|\hat{f}(\xi)| \leq C(1+|\xi|)^{-n-\epsilon} \qquad \forall \xi \in \mathbb{R}^n,$$

*for some $C, \epsilon > 0$. Then, for all $x \in \mathbb{R}^n$,*

$$\sum_{z\in\mathbb{Z}^n} f(x+z) = \sum_{z\in\mathbb{Z}^n} \hat{f}(z) e^{2\pi i z\cdot x}, \tag{A.1}$$

*where both series converge absolutely. In particular,*

$$\sum_{z\in\mathbb{Z}^n} f(z) = \sum_{z\in\mathbb{Z}^n} \hat{f}(z). \tag{A.2}$$

*Proof.* For a proof, we refer, *e.g.*, to Folland [53, 8.32, p.254] and Stein and Weiss [131, Cor. 2.6, p. 252]. □

*Remark* A.11. Clearly, if $f \in \mathcal{S}(\mathbb{R}^n)$, then the hypotheses of Theorem A.10 are satisfied.

# APPENDIX B

## Results of classical potential theory for the Laplace operator

We collect here some notation and results of classical potential theory.

Let $\mathbb{I}$ be an open bounded connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in ]0,1[$. Let $\nu_{\mathbb{I}}$ denote the outward unit normal to $\partial\mathbb{I}$. Set

$$\mathbb{I}^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{I}.$$

We say that a harmonic function $u$ of $\mathbb{I}^-$ to $\mathbb{R}$ is *harmonic at infinity* (cf. *e.g.* Folland [52, Prop. 2.74, p. 114]) if it satisfies the following condition

$$\sup_{|x|\geq R} |x|^{n-2}|u(x)| < \infty, \tag{B.1}$$

for some $R > 0$ such that $\mathrm{cl}\,\mathbb{I} \subseteq \mathbb{B}_n(0, R)$.

We set

$$v[\partial\mathbb{I}, \mu](t) \equiv \int_{\partial\mathbb{I}} S_n(t-s)\mu(s)\, d\sigma_s \qquad \forall t \in \mathbb{R}^n, \tag{B.2}$$

$$w[\partial\mathbb{I}, \mu](t) \equiv \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n(t-s))\mu(s)\, d\sigma_s \qquad \forall t \in \mathbb{R}^n, \tag{B.3}$$

for all $\mu \in L^2(\partial\mathbb{I})$. The function $v[\partial\mathbb{I}, \mu]$ is called the simple (or single) layer potential with moment $\mu$, while $w[\partial\mathbb{I}, \mu]$ is the double layer potential with moment $\mu$.

We have the following well known results.

**Theorem B.1.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be a bounded connected open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $R > 0$ be such that $\mathrm{cl}\,\mathbb{I} \subseteq \mathbb{B}_n(0, R)$. Then the following statements hold.*

*(i) Let $\mu \in C^0(\partial\mathbb{I})$. Then the function $w[\partial\mathbb{I}, \mu]$ is harmonic in $\mathbb{R}^n \setminus \partial\mathbb{I}$. The restriction $w[\partial\mathbb{I}, \mu]_{|\mathbb{I}}$ can be extended uniquely to a continuous function $w^+[\partial\mathbb{I}, \mu]$ of $\mathrm{cl}\,\mathbb{I}$ to $\mathbb{R}$. The restriction $w[\partial\mathbb{I}, \mu]_{|\mathbb{I}^-}$ can be extended uniquely to a continuous function $w^-[\partial\mathbb{I}, \mu]$ of $\mathrm{cl}\,\mathbb{I}^-$ to $\mathbb{R}$ and we have the following jump relations*

$$w^+[\partial\mathbb{I}, \mu](t) = +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n(t-s))\mu(s)\, d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

$$w^-[\partial\mathbb{I}, \mu](t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n(t-s))\mu(s)\, d\sigma_s \qquad \forall t \in \partial\mathbb{I}.$$

*Moreover, the function $w^-[\partial\mathbb{I}, \mu]$ is harmonic at infinity.*

*(ii) Let $\mu \in C^{m,\alpha}(\partial\mathbb{I})$. Then we have that $w^+[\partial\mathbb{I}, \mu]$ belongs to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I})$ and $w^-[\partial\mathbb{I}, \mu]$ belongs to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I}^-)$. Moreover,*

$$Dw^+[\partial\mathbb{I}, \mu] \cdot \nu_{\mathbb{I}} - Dw^-[\partial\mathbb{I}, \mu] \cdot \nu_{\mathbb{I}} = 0 \qquad \text{on } \partial\mathbb{I}.$$

(iii) *The map of* $C^{m,\alpha}(\partial\mathbb{I})$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I})$ *which takes* $\mu$ *to* $w^+[\partial\mathbb{I},\mu]$ *is linear and continuous. The map of* $C^{m,\alpha}(\partial\mathbb{I})$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\mathbb{I})$ *which takes* $\mu$ *to* $w^-[\partial\mathbb{I},\mu]_{|\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\mathbb{I}}$ *is linear and continuous.*

(iv) *We have*

$$\int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n(t-s))\,d\sigma_s = \frac{1}{2} \qquad \forall t \in \partial\mathbb{I}.$$

*Proof.* The above properties of double layer potentials can be found in basically all books on potential theory. For the regularity we refer in particular to Schauder [123], Miranda [98]. For more references we refer to the proof of Lanza and Rossi [85, Thm. 3.1]. □

**Theorem B.2.** *Let* $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be a bounded connected open subset of* $\mathbb{R}^n$ *of class* $C^{m,\alpha}$. *Let* $R > 0$ *be such that* $\mathrm{cl}\,\mathbb{I} \subseteq \mathbb{B}_n(0,R)$. *Then the following statements hold.*

(i) *Let* $\mu \in C^0(\partial\mathbb{I})$. *Then the function* $v[\partial\mathbb{I},\mu]$ *is continuous on* $\mathbb{R}^n$ *and harmonic in* $\mathbb{R}^n\setminus\partial\mathbb{I}$. *Let* $v^+[\partial\mathbb{I},\mu]$ *and* $v^-[\partial\mathbb{I},\mu]$ *denote the restrictions of* $v[\partial\mathbb{I},\mu]$ *to* $\mathrm{cl}\,\mathbb{I}$ *and to* $\mathrm{cl}\,\mathbb{I}^-$, *respectively. If* $n = 2$ *then the function* $v^-[\partial\mathbb{I},\mu]$ *is harmonic at infinity if and only if* $\int_{\partial\mathbb{I}}\mu\,d\sigma = 0$. *If so, then* $\lim_{t\to\infty} v^-[\partial\mathbb{I},\mu](t) = 0$. *If* $n \geq 3$, *then the function* $v^-[\partial\mathbb{I},\mu]$ *is harmonic at infinity.*

(ii) *If* $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$, *then* $v^+[\partial\mathbb{I},\mu] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{I})$, *and the map of* $C^{m-1,\alpha}(\partial\mathbb{I})$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I})$ *which takes* $\mu$ *to* $v^+[\partial\mathbb{I},\mu]$ *is linear and continuous.*

(iii) *If* $\mu \in C^{m-1,\alpha}$, *then* $v^-[\partial\mathbb{I},\mu]_{|\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\mathbb{I}} \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\mathbb{I})$, *and the map of* $C^{m-1,\alpha}(\partial\mathbb{I})$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\mathbb{I})$ *which takes* $\mu$ *to* $v^-[\partial\mathbb{I},\mu]_{|\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\mathbb{I}}$ *is linear and continuous.*

(iv) *Let* $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$. *If* $n = 2$ *and* $\int_{\partial\mathbb{I}}\mu\,d\sigma = 0$, *then the function* $v^-[\partial\mathbb{I},\mu]$ *belongs to* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I}^-)$. *If* $n \geq 3$, *then the function* $v^-[\partial\mathbb{I},\mu]$ *belongs to* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{I}^-)$.

(v) *If* $\mu \in C^{m-1,\alpha}(\partial\mathbb{I})$, *then we have the following jump relations*

$$\frac{\partial}{\partial\nu_{\mathbb{I}}}v^+[\partial\mathbb{I},\mu](t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n(t-s))\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

$$\frac{\partial}{\partial\nu_{\mathbb{I}}}v^-[\partial\mathbb{I},\mu](t) = +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n(t-s))\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I}.$$

*Proof.* The proof of these properties can be found in almost all books on potential theory. For the regularity we refer to Miranda [98]. For more references we refer to the proof of Lanza and Rossi [85, Thm. 3.1]. □

Then we have the following variant of a classical result in potential theory.

**Theorem B.3.** *Let* $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be a bounded connected open subset of* $\mathbb{R}^n$ *of class* $C^{m,\alpha}$. *Let* $b \in C^{m-1,\alpha}(\partial\mathbb{I})$. *Then the following statements hold.*

(i) *Let* $k \in \{0,1,\dots,m\}$ *and* $\bar\Gamma \in C^{k,\alpha}(\partial\mathbb{I})$ *and* $\mu \in L^2(\partial\mathbb{I})$ *and*

$$\bar\Gamma(t) = \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n(t-s))\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}} S_n(t-s)b(s)\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{B.4}$$

*then* $\mu \in C^{k,\alpha}(\partial\mathbb{I})$.

(ii) *Let* $k \in \{0,1,\dots,m\}$ *and* $\bar\Gamma \in C^{k,\alpha}(\partial\mathbb{I})$ *and* $\mu \in L^2(\partial\mathbb{I})$ *and*

$$\bar\Gamma(t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n(t-s))\mu(s)\,d\sigma_s + \int_{\partial\mathbb{I}} S_n(t-s)b(s)\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{B.5}$$

*then* $\mu \in C^{k,\alpha}(\partial\mathbb{I})$.

(iii) *Let* $k \in \{1,\dots,m\}$ *and* $\bar\Gamma \in C^{k-1,\alpha}(\partial\mathbb{I})$ *and* $\mu \in L^2(\partial\mathbb{I})$ *and*

$$\bar\Gamma(t) = \frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n(t-s))\mu(s)\,d\sigma_s + b(t)\int_{\partial\mathbb{I}} S_n(t-s)\mu(s)\,d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{B.6}$$

*then* $\mu \in C^{k-1,\alpha}(\partial\mathbb{I})$.

*(iv) Let* $k \in \{1, \dots, m\}$ *and* $\bar{\Gamma} \in C^{k-1,\alpha}(\partial\mathbb{I})$ *and* $\mu \in L^2(\partial\mathbb{I})$ *and*

$$\bar{\Gamma}(t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n(t-s))\mu(s)\, d\sigma_s + b(t)\int_{\partial\mathbb{I}} S_n(t-s)\mu(s)\, d\sigma_s \quad \text{a.e. on } \partial\mathbb{I}, \tag{B.7}$$

*then* $\mu \in C^{k-1,\alpha}(\partial\mathbb{I})$.

*Proof.* It is based on results by Miranda [99], Agmon, Douglis and Nirenberg [1] and Günter [57]. For a proof see, *e.g.*, Lanza [72, Thm. 5.1]. □

# APPENDIX C

## Technical results on integral and composition operators

In this Appendix, we present some technical facts and some variants of known technical facts, which have been exploited in the Dissertation. For more general results, we refer to [80].

We start by introducing the following elementary Proposition on integral operators.

**Proposition C.1.** *Let $\mathbb{K} \equiv \mathbb{R}$ or $\mathbb{K} \equiv \mathbb{C}$. Let $\alpha \in ]0,1[$. Let $m \in \mathbb{N} \setminus \{0\}$. Let $n$, $n_1$, $r \in \mathbb{N} \setminus \{0\}$, $1 \leq r \leq n_1$. Let $U$ be an open subset of $\mathbb{R}^n$. Let $F$ be a real analytic map of $U$ to $\mathbb{K}$. Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^n$. Let $k \in \mathbb{N}$. Let $\mathbb{M}$ be a compact manifold of class $C^1$ and dimension $r$ imbedded in $\mathbb{R}^{n_1}$. Then the map $H$ of the space $\{ (\phi, f) \in C^0(\mathbb{M}, \mathbb{R}^n) \times L^1(\mathbb{M}, \mathbb{K})\colon \operatorname{cl}\Omega - \phi(\mathbb{M}) \subseteq U \}$ to $C^k(\operatorname{cl}\Omega, \mathbb{K})$ which takes $(\phi, f)$ to the function $H[\phi, f]$ of $\operatorname{cl}\Omega$ to $\mathbb{K}$ defined by*

$$H[\phi, f](t) \equiv \int_{\mathbb{M}} F(t - \phi(y)) f(y)\, d\sigma_y \qquad \forall t \in \operatorname{cl}\Omega,$$

*is real analytic.*

*Proof.* It suffices to modify the proof of Lanza [72, Prop. 6.1]. Clearly, it suffices to show that for each $\gamma \in \mathbb{N}^n$, $|\gamma| \leq k$, the map which takes $(\phi, f)$ to $D_t^\gamma H[\phi, f]$ is real analytic from the domain of $H$ to $C^0(\operatorname{cl}\Omega, \mathbb{K})$. By classical theorems of differentiation under the integral sign, we have

$$D_t^\gamma H[\phi, f](t) = \int_{\mathbb{M}} D^\gamma F(t - \phi(y)) f(y)\, d\sigma_y \qquad \forall t \in \operatorname{cl}\Omega.$$

Now let $\mathrm{id}_{\operatorname{cl}\Omega}$ denote the identity map in $\operatorname{cl}\Omega$. The map of the domain of $H$ to $C^0(\operatorname{cl}\Omega \times \mathbb{M}, U)$, which takes $(\phi, f)$ to the function $\mathrm{id}_{\operatorname{cl}\Omega}(t) - \phi(y)$ of the variable $(t, y) \in \operatorname{cl}\Omega \times \mathbb{M}$ is obviously real analytic. The map of $C^0(\operatorname{cl}\Omega \times \mathbb{M}, U)$ to $C^0(\operatorname{cl}\Omega \times \mathbb{M}, \mathbb{K})$ which takes a function $\Phi$ to its composite function $D^\gamma F \circ \Phi$ is real analytic (cf. Böhme and Tomi [15, p. 10], Henry [60, p. 29], Valent [137, Thm. 5.2, p. 44], who considered the more elaborated case of the Schauder spaces $C^{m,\alpha}$.) Then to complete the proof we just need to observe that the map which takes a pair of functions $(g, f)$ of $C^0(\operatorname{cl}\Omega \times \mathbb{M}, \mathbb{K}) \times L^1(\mathbb{M}, \mathbb{K})$ to $\int_{\mathbb{M}} g(\cdot, y) f(y)\, d\sigma_y$ in $C^0(\operatorname{cl}\Omega, \mathbb{K})$ is real analytic. □

Then we have the following.

**Theorem C.2.** *Let $\mathbb{K} \equiv \mathbb{R}$ or $\mathbb{K} \equiv \mathbb{C}$. Let $\alpha \in ]0,1[$. Let $m \in \mathbb{N} \setminus \{0\}$. Let $n$, $n_1$, $r \in \mathbb{N} \setminus \{0\}$, $1 \leq r \leq n_1$. Let $U$ be an open subset of $\mathbb{R}^n$. Let $F$ be a real analytic map of $U$ to $\mathbb{K}$. Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $\mathbb{M}$ be a compact manifold of class $C^{m,\alpha}$ and dimension $r$ imbedded in $\mathbb{R}^{n_1}$. Then the map $H_1$ of $\{(\psi, \phi, f) \in C^{m,\alpha}(\partial\Omega, \mathbb{R}^n) \times C^{m,\alpha}(\mathbb{M}, \mathbb{R}^n) \times L^1(\mathbb{M}, \mathbb{K})\colon \psi(x) - \phi(y) \subseteq U \quad \forall (x, y) \in \partial\Omega \times \mathbb{M}\}$ to $C^{m,\alpha}(\partial\Omega, \mathbb{K})$, which takes $(\psi, \phi, f)$ to the function $H_1[\psi, \phi, f]$ defined by*

$$H_1[\psi, \phi, f](x) \equiv \int_{\mathbb{M}} F(\psi(x) - \phi(y)) f(y)\, d\sigma_y \qquad \forall x \in \partial\Omega,$$

*is real analytic.*

*Proof.* It suffices to modify the proof of Lanza [72, Thm. 6.2]. The proof follows by a known result on composition operators (cf. Böhme and Tomi [15, p. 10], Henry [60, p. 29], Valent [137, Thm. 5.2, p. 44]), and its proof is a straightforward modification of the corresponding elementary argument of Lanza and Rossi [85, Lem. 3.9]. We just observe that the map which takes $(\psi, \phi, f)$ to $\psi(x) - \phi(y) \in C^{m,\alpha}(\partial\Omega \times \mathbb{M}, U)$, and the map which takes a function of $C^{m,\alpha}(\partial\Omega \times \mathbb{M}, U)$ to its composite function with $F$ in $C^{m,\alpha}(\partial\Omega \times \mathbb{M}, \mathbb{K})$ are real analytic, and that the map which takes a pair of functions $(g, f)$ of $C^{m,\alpha}(\partial\Omega \times \mathbb{M}, \mathbb{K}) \times L^1(\mathbb{M}, \mathbb{K})$ to $\int_{\mathbb{M}} g(\cdot, y) f(y)\, d\sigma_y$ in $C^{m,\alpha}(\partial\Omega, \mathbb{K})$ is bilinear and continuous. □

By modifying the proofs of the previous results, we can prove the following.

**Proposition C.3.** *Let $\mathbb{K} \equiv \mathbb{R}$ or $\mathbb{K} \equiv \mathbb{C}$. Let $\alpha \in ]0, 1[$. Let $m \in \mathbb{N} \setminus \{0\}$. Let $n$, $r \in \mathbb{N} \setminus \{0\}$, $1 \leq r \leq n$. Let $U$ be an open subset of $\mathbb{R}^n$ such that $0 \in U$. Let $F$ be a real analytic map of $U$ to $\mathbb{K}$. Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^n$. Let $k \in \mathbb{N}$. Let $\mathbb{M}$ be a compact manifold of class $C^1$ and dimension $r$ imbedded in $\mathbb{R}^n$. Then there exists $\epsilon' > 0$ such that the map $H_2$ of $]-\epsilon', \epsilon'[ \times L^1(\mathbb{M}, \mathbb{K})$ to $C^k(\mathrm{cl}\,\Omega, \mathbb{K})$ which takes $(\epsilon, f)$ to the function $H_2[\epsilon, f]$ of $\mathrm{cl}\,\Omega$ to $\mathbb{K}$ defined by*

$$H_2[\epsilon, f](t) \equiv \int_{\mathbb{M}} F(\epsilon(t - y)) f(y)\, d\sigma_y \qquad \forall t \in \mathrm{cl}\,\Omega,$$

*is real analytic.*

**Theorem C.4.** *Let $\mathbb{K} \equiv \mathbb{R}$ or $\mathbb{K} \equiv \mathbb{C}$. Let $\alpha \in ]0, 1[$. Let $m \in \mathbb{N} \setminus \{0\}$. Let $n$, $r \in \mathbb{N} \setminus \{0\}$, $1 \leq r \leq n$. Let $U$ be an open subset of $\mathbb{R}^n$ such that $0 \in U$. Let $F$ be a real analytic map of $U$ to $\mathbb{K}$. Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $k \in \mathbb{N}$. Let $\mathbb{M}$ be a compact manifold of class $C^{m,\alpha}$ and dimension $r$ imbedded in $\mathbb{R}^n$. Then there exists $\epsilon'' > 0$ such that the map $H_3$ of $]-\epsilon'', \epsilon''[ \times L^1(\mathbb{M}, \mathbb{K})$ to $C^{m,\alpha}(\partial\Omega, \mathbb{K})$, which takes $(\epsilon, f)$ to the function $H_3[\epsilon, f]$ defined by*

$$H_3[\epsilon, f](x) \equiv \int_{\mathbb{M}} F(\epsilon(x - y)) f(y)\, d\sigma_y \qquad \forall x \in \partial\Omega,$$

*is real analytic.*

# APPENDIX D

## Technical results on periodic functions

In this Appendix we present some technical facts about periodic functions.
Let $\{a_{11}, \dots, a_{nn}\} \subseteq ]0, +\infty[$. We set

$$a_i \equiv a_{ii} e_i \qquad \forall i \in \{1, \dots, n\}.$$

Let $A$ be the open subset of $\mathbb{R}^n$ defined by

$$A \equiv \prod_{i=1}^{n} ]0, a_{ii}[.$$

We recall that we denote by $|A|_n$ the $n$-dimensional measure of $A$. For each $x \in \mathbb{R}^n$ we set

$$a(x) \equiv \sum_{i=1}^{n} x_i a_i.$$

We have the following Proposition.

**Proposition D.1.** *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that*

$$u(x + a_i) = u(x) \qquad a.e \text{ on } \mathbb{R}^n, \quad \forall i \in \{1, \dots, n\}.$$

*Then the following statements hold.*

*(i) Let $\bar{x} \in \mathbb{R}^n$. Then*

$$\int_{\bar{x}+A} u(y)\, dy = \int_A u(y)\, dy.$$

*(ii) Let $\delta > 0$. Then*

$$\int_{\delta(\bar{x}+A)} u\left(\frac{y}{\delta}\right)\, dy = \delta^n \int_A u(y)\, dy.$$

*Proof.* For a proof we refer to Cioranescu and Donato [26, Lemma 2.3, p. 27]. □

We recall that if $p \in [1, \infty]$, we denote by $p'$ the *conjugate exponent* of $p$. In particular, if $1 < p < \infty$, then $p' = p/(p-1)$, if $p = 1$ then $p' = \infty$, and if $p = \infty$ then $p' = 1$. We are now ready to give the following definitions.

**Definition D.2.** Let $1 \leq p < \infty$. Let $V$ be a bounded open subset of $\mathbb{R}^n$. We say that a sequence $\{u_j\}_{j\in\mathbb{N}} \subseteq L^p(V)$ converges weakly to $u \in L^p(V)$ (and we write $u_j \rightharpoonup u$ in $L^p(V)$), if for all $v \in L^{p'}(V)$ we have

$$\lim_{j\to\infty} \int_V v(u_j - u)\, dx = 0. \tag{D.1}$$

In case $p = \infty$ we give the following.

**Definition D.3.** Let $V$ be a bounded open subset of $\mathbb{R}^n$. We say that a sequence $\{u_j\}_{j\in\mathbb{N}} \subseteq L^\infty(V)$ converges weakly $*$ to $u \in L^\infty(V)$ (and we write $u_j \rightharpoonup^* u$ in $L^\infty(V)$), if for all $v \in L^1(V)$ we have

$$\lim_{j\to\infty} \int_V v(u_j - u)\, dx = 0. \tag{D.2}$$

We now state the following generalization of Riemann–Lebesgue Lemma.

**Theorem D.4.** *Let $1 \le p \le \infty$. Let $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ be such that*

$$u(x + a_i) = u(x) \qquad a.e.\ on\ \mathbb{R}^n, \quad \forall i \in \{1, \dots, n\}.$$

*For each $\delta > 0$, define*

$$u_\delta(x) \equiv u\left(\frac{x}{\delta}\right) \qquad a.e.\ on\ \mathbb{R}^n.$$

*Then as $\delta \to 0$*

$$u_\delta \rightharpoonup \frac{1}{|A|_n} \int_A u(y)\, dy \qquad (\rightharpoonup^* \ \ if\ p = \infty) \tag{D.3}$$

*in $L^p(V)$ for every bounded open subset $V$ of $\mathbb{R}^n$.*

*Proof.* For a proof we refer to Braides and Defranceschi [16, Ex. 2.7, p. 20] and to Dacorogna [31, Thm. 1.5, p.21]. □

We now prove a slight variant of Theorem D.4.

**Theorem D.5.** *Let $1 \le p \le \infty$. Let $\epsilon' > 0$ and $\{v_\epsilon\}_{\epsilon\in]0,\epsilon'[} \subseteq L^p_{\mathrm{loc}}(\mathbb{R}^n)$ be such that*

$$v_\epsilon(x + a_i) = v_\epsilon(x) \qquad a.e.\ on\ \mathbb{R}^n, \quad \forall i \in \{1, \dots, n\}, \quad \forall \epsilon \in ]0, \epsilon'[.$$

*Let $v \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ be such that*

$$v(x + a_i) = v(x) \qquad a.e.\ on\ \mathbb{R}^n, \quad \forall i \in \{1, \dots, n\},$$

*and*

$$\lim_{\substack{\epsilon\to 0\\ \epsilon\in]0,\epsilon'[}} v_\epsilon = v \qquad in\ L^p(A).$$

*For each $\delta > 0$, we set*

$$v_{\epsilon,\delta}(x) \equiv v_\epsilon\left(\frac{x}{\delta}\right) \qquad a.e.\ on\ \mathbb{R}^n, \quad \forall \epsilon \in ]0, \epsilon'[.$$

*Then as $(\epsilon, \delta) \to 0$ (with $(\epsilon, \delta) \in ]0, \epsilon'[ \times ]0, +\infty[$), we have*

$$v_{\epsilon,\delta} \rightharpoonup \frac{1}{|A|_n} \int_A v(y)\, dy \qquad (\rightharpoonup^* \ \ if\ p = \infty) \tag{D.4}$$

*in $L^p(V)$ for every bounded open subset $V$ of $\mathbb{R}^n$.*

*Proof.* We slightly modify the proof of Braides and Defranceschi [16, Ex. 2.7, p. 20]. We first treat the case $1 \le p < \infty$. If $\delta < 1$ and $I_\delta \equiv \left\{ k \in \mathbb{Z}^n : (a(k) + A) \cap \frac{1}{\delta} V \neq \emptyset \right\}$, then there exists a constant $c > 0$, independent of $\epsilon$ and $\delta$, such that

$$\begin{aligned}\int_V |v_{\epsilon,\delta}(x) - v\left(\frac{x}{\delta}\right)|^p\, dx &\le \delta^n \int_{\frac{1}{\delta}V} |v_\epsilon(z) - v(z)|^p dz \le \delta^n \sum_{k\in I_\delta} \int_{a(k)+A} |v_\epsilon(z) - v(z)|^p dz \\ &= \delta^n \sum_{k\in I_\delta} \int_A |v_\epsilon(z) - v(z)|^p dz \le c \int_A |v_\epsilon(z) - v(z)|^p dz\end{aligned} \tag{D.5}$$

for all $(\epsilon, \delta) \in ]0, \epsilon'[ \times ]0, 1[$. Now let $\phi \in L^p(V)$. If $(\epsilon, \delta) \in ]0, \epsilon'[ \times ]0, 1[$, we have

$$\int_V v_{\epsilon,\delta}(x)\phi(x)\, dx = \int_V v\left(\frac{x}{\delta}\right) \phi(x)\, dx + \int_V \left(v_{\epsilon,\delta}(x) - v\left(\frac{x}{\delta}\right)\right) \phi(x)\, dx. \tag{D.6}$$

By Theorem D.4, by (D.5) and (D.6), we have

$$v_{\epsilon,\delta} \rightharpoonup \frac{1}{|A|_n} \int_A v(y)\,dy \qquad \text{in } L^p(V),$$

as $(\epsilon, \delta) \to (0, 0)$.

Now let $p = \infty$. We have

$$\|v_{\epsilon,\delta}(\cdot) - v(\cdot/\delta)\|_{L^\infty(V)} \leq \|v_\epsilon - v\|_{L^\infty(A)},$$

for all $(\epsilon, \delta) \in ]0, \epsilon'[ \times ]0, +\infty[$. If $\phi \in L^1(V)$, we have

$$\int_V v_{\epsilon,\delta}(x)\phi(x)\,dx = \int_V v\left(\frac{x}{\delta}\right)\phi(x)\,dx + \int_V \left(v_{\epsilon,\delta}(x) - v\left(\frac{x}{\delta}\right)\right)\phi(x)\,dx.$$

Hence, by Theorem D.4,

$$v_{\epsilon,\delta} \rightharpoonup^* \frac{1}{|A|_n} \int_A v(y)\,dy \qquad \text{in } L^\infty(V),$$

as $(\epsilon, \delta) \to (0, 0)$. □

# APPENDIX E

## Simple and double layer potentials for the Helmholtz equation

We collect here some notation and results of potential theory for the Helmholtz equation from Lanza and Rossi [86] (see also, *e.g.*, Colton and Kress [29], Castro and Speck [21], Meister and Speck [95].)

First of all, we introduce the family $\{S_n(\cdot,k)\}_{k\in\mathbb{C}}$ of fundamental solutions of the family of operators $\{\Delta+k^2\}_{k\in\mathbb{C}}$ defined in Lanza and Rossi [86]. We denote by $\gamma$, $J_\nu$, $N_\nu$ the Euler constant, the Bessel function of order $\nu$ and the Neumann function of order $\nu\in\mathbb{R}$, respectively (as in Schwartz [126, Ch. VIII, IX].) Then we have the following technical Lemma.

**Lemma E.1.** *Let $n\in\mathbb{N}\setminus\{0,1\}$. Then the following statements hold.*

*(i) If $n$ is even, then the map of $]0,+\infty[$ to $\mathbb{R}$ which takes $t$ to*

$$t^{\frac{n-2}{2}}\left\{N_{\frac{n-2}{2}}(t)-\frac{2}{\pi}(\log(t/2)+\gamma)J_{\frac{n-2}{2}}(t)\right\}\qquad\forall t\in\,]0,+\infty[,$$

*admits a unique holomorphic extension $\tilde{N}_{\frac{n-2}{2}}(\cdot)$ of $\mathbb{C}$ to $\mathbb{C}$, and $\tilde{N}_{\frac{n-2}{2}}(0)=-\pi^{-1}2^{\frac{n-2}{2}}(\frac{n-4}{2})!$ for $n>2$, and $\lim_{t\to0}\tilde{N}_{\frac{n-2}{2}}(t)t^{-2}=\frac{1}{2\pi}$ for $n=2$.*

*(ii) If $\nu\in\mathbb{R}$, then the map of $]0,+\infty[$ to $\mathbb{R}$ which takes $t$ to $t^{-\nu}J_\nu(t)$ admits a unique holomorphic extension $\tilde{J}_\nu$ of $\mathbb{C}$ to $\mathbb{C}$.*

*(iii) If $n$ is even, then we have $\tilde{J}_{\frac{n-2}{2}}(0)=2^{-\frac{n-2}{2}}/(\frac{n-2}{2})!$. If $n$ is odd, then we have $\tilde{J}_{-\frac{n-2}{2}}(0)=(-1)^{\frac{n-3}{2}}\frac{2^{\frac{n-2}{2}}}{\pi}(\frac{n-2}{2})^{-1}\Gamma(n/2)$.*

*Proof.* See Lanza and Rossi [86, Lemma 3.1]. □

We are now ready to introduce the fundamental solution of $\Delta+k^2$ (cf. Lanza and Rossi [86, Definition 3.2].)

**Definition E.2.** Let $n\in\mathbb{N}\setminus\{0,1\}$.

(i) If $n$ is even, then we set

$$\begin{aligned}\mathcal{J}_n(z)&\equiv(2\pi)^{-n/2}\tilde{J}_{\frac{n-2}{2}}(z),\\ \mathcal{N}_n(z)&\equiv2^{-(n/2)-1}\pi^{-(n/2)+1}\tilde{N}_{\frac{n-2}{2}}(z),\end{aligned}\tag{E.1}$$

for all $z\in\mathbb{C}$.

(ii) If $n$ is odd, then we set

$$\begin{aligned}\mathcal{J}_n(z)&\equiv0,\\ \mathcal{N}_n(z)&\equiv(-1)^{\frac{n-1}{2}}2^{-(n/2)-1}\pi^{-(n/2)+1}\tilde{J}_{-\frac{n-2}{2}}(z),\end{aligned}\tag{E.2}$$

for all $z\in\mathbb{C}$.

(iii) We set

$$\Upsilon_n(r,k) \equiv k^{n-2}\mathcal{J}_n(rk)\log r + \frac{\mathcal{N}_n(rk)}{r^{n-2}},$$

for all $(r,k) \in ]0,+\infty[ \times \mathbb{C}$.

Then we have the following

**Proposition E.3.** *Let $n \in \mathbb{N} \setminus \{0,1\}$. Let $s_n$ denote the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n(0,1)$. Then the following statements hold.*

*(i)* $\mathcal{J}_2(0) = \frac{1}{2\pi}$, $\mathcal{J}_n(0) = 2^{1-n}\pi^{-n/2}/(\frac{n-2}{2})!$ *if $n > 2$ is even;* $\mathcal{N}_2(0) = 0$, $\mathcal{N}_n(0) = (2-n)^{-1}s_n^{-1}$ *for* $n \geq 3$.

*(ii) $\mathcal{J}_n$ and $\mathcal{N}_n$ are entire holomorphic functions. The function $\Upsilon_n$ is real analytic on $]0,+\infty[ \times \mathbb{C}$.*

*(iii) Let $k \in \mathbb{C}$. The function of $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{C}$ defined by $S_n(x,k) = \Upsilon_n(|x|,k)$ for all $x \in \mathbb{R}^n \setminus \{0\}$ is a fundamental solution of $\Delta + k^2$. In particular, $S_n(\cdot,0)$ is the usual fundamental solution of the Laplace operator.*

*Proof.* See Lanza and Rossi [86, Proposition 3.3]. □

We observe that $S_n(\cdot,k)$ does not coincide with the most commonly used fundamental solution of $\Delta + k^2$ in scattering theory (cf. *e.g.*, Colton and Kress [29].)

Let $\mathbb{I}$ be an open bounded connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in ]0,1[$. Let $\nu_{\mathbb{I}}$ denote the outward unit normal to $\mathbb{I}$ on $\partial\mathbb{I}$. Set

$$\mathbb{I}^- \equiv \mathbb{R}^n \setminus \operatorname{cl}\mathbb{I}.$$

We set

$$v[\partial\mathbb{I},\mu,k](t) \equiv \int_{\partial\mathbb{I}} S_n(t-s,k)\mu(s)\,d\sigma_s \qquad \forall t \in \mathbb{R}^n, \tag{E.3}$$

$$w[\partial\mathbb{I},\mu,k](t) \equiv \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n(t-s,k))\mu(s)\,d\sigma_s \qquad \forall t \in \mathbb{R}^n \setminus \partial\mathbb{I}, \tag{E.4}$$

for all $\mu \in L^2(\partial\mathbb{I},\mathbb{C})$, $k \in \mathbb{C}$. The function $v[\partial\mathbb{I},\mu,k]$ is called the simple (or single) layer potential with moment $\mu$ for the Helmholtz equation, while $w[\partial\mathbb{I},\mu,k]$ is the double layer potential with moment $\mu$. We also set

$$v_*[\partial\mathbb{I},\mu,k](t) \equiv \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n(t-s,k))\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I}, \tag{E.5}$$

for all $\mu \in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C})$, $k \in \mathbb{C}$.

We have the following well known results.

**Theorem E.4.** *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in ]0,1[$. Let $\mathbb{I}$ be a bounded connected open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $R > 0$ be such that $\operatorname{cl}\mathbb{I} \subseteq \mathbb{B}_n(0,R)$. Then the following statements hold.*

*(i) Let $\mu \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$. Then the function $w[\partial\mathbb{I},\mu,k]$ satisfies equation $(\Delta + k^2)w[\partial\mathbb{I},\mu,k] = 0$ in $\mathbb{R}^n \setminus \partial\mathbb{I}$. The restriction $w[\partial\mathbb{I},\mu,k]_{|\mathbb{I}}$ can be extended uniquely to an element $w^+[\partial\mathbb{I},\mu,k]$ of $C^{m,\alpha}(\operatorname{cl}\mathbb{I},\mathbb{C})$. The restriction $w[\partial\mathbb{I},\mu,k]_{|\mathbb{I}^-}$ can be extended uniquely to a continuous function $w^-[\partial\mathbb{I},\mu,k]$ of $\operatorname{cl}\mathbb{I}^-$ to $\mathbb{C}$ and $w^-[\partial\mathbb{I},\mu,k] \in C^{m,\alpha}(\operatorname{cl}\mathbb{B}_n(0,R) \setminus \mathbb{I},\mathbb{C})$. Moreover, we have the following jump relations*

$$w^+[\partial\mathbb{I},\mu,k](t) = +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n(t-s,k))\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

$$w^-[\partial\mathbb{I},\mu,k](t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n(t-s,k))\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

$$Dw^+[\partial\mathbb{I},\mu,k] \cdot \nu_{\mathbb{I}} - Dw^-[\partial\mathbb{I},\mu,k] \cdot \nu_{\mathbb{I}} = 0 \qquad \text{on } \partial\mathbb{I}.$$

*(ii)* *If* $\mu \in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C})$, *then we have*

$$w[\partial\mathbb{I},\mu,k](t) = -\sum_{j=1}^{n} \frac{\partial}{\partial t_j}\Big\{\int_{\partial\mathbb{I}} \mu(s)(\nu_{\mathbb{I}}(s))_j S_n(t-s,k)\,d\sigma_s\Big\},$$

*for all* $t \equiv (t_1,\dots,t_n) \in \mathbb{R}^n \setminus \partial\mathbb{I}$.

*(iii)* *If* $\mu \in C^{m,\alpha}(\partial\mathbb{I},\mathbb{C})$, $U$ *is an open neighbourhood of* $\partial\mathbb{I}$ *in* $\mathbb{R}^n$, $\tilde{\mu} \in C^{m,\alpha}(U,\mathbb{C})$, $\tilde{\mu}_{|\partial\mathbb{I}} = \mu$, *then the following holds*

$$\begin{aligned}\frac{\partial}{\partial t_i} w[\partial\mathbb{I},\mu,k](t) = &\sum_{j=1}^{n} \frac{\partial}{\partial t_j}\Big\{\int_{\partial\mathbb{I}} \big[(\nu_{\mathbb{I}}(s))_i \frac{\partial\tilde{\mu}}{\partial s_j}(s) - (\nu_{\mathbb{I}}(s))_j \frac{\partial\tilde{\mu}}{\partial s_i}(s)\big] S_n(t-s,k)\,d\sigma_s\Big\}\\ &+ k^2 \int_{\partial\mathbb{I}} (\nu_{\mathbb{I}}(s))_i \mu(s) S_n(t-s,k)\,d\sigma_s \qquad \forall t \equiv (t_1,\dots,t_n) \in \mathbb{R}^n \setminus \partial\mathbb{I}.\end{aligned}$$

*Proof.* See Lanza and Rossi [86, Theorem 3.4]. □

**Theorem E.5.** *Let* $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. *Let* $\mathbb{I}$ *be a bounded connected open subset of* $\mathbb{R}^n$ *of class* $C^{m,\alpha}$. *Let* $R > 0$ *be such that* $\mathrm{cl}\,\mathbb{I} \subseteq \mathbb{B}_n(0,R)$. *Let* $\mu \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$. *Then the function* $v[\partial\mathbb{I},\mu,k]$ *is continuous in* $\mathbb{R}^n$ *and satisfies* $(\Delta + k^2)v[\partial\mathbb{I},\mu,k] = 0$ *in* $\mathbb{R}^n \setminus \partial\mathbb{I}$. *Let* $v^+[\partial\mathbb{I},\mu,k]$ *and* $v^-[\partial\mathbb{I},\mu,k]$ *denote the restrictions of* $v[\partial\mathbb{I},\mu,k]$ *to* $\mathrm{cl}\,\mathbb{I}$ *and to* $\mathrm{cl}\,\mathbb{I}^-$, *respectively. Then* $v^+[\partial\mathbb{I},\mu,k] \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$, *and* $v^-[\partial\mathbb{I},\mu,k]_{|\,\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\mathbb{I}} \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R)\setminus\mathbb{I},\mathbb{C})$. *Moreover, we have the following jump relations*

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} v^+[\partial\mathbb{I},\mu,k](t) = -\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n(t-s,k))\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I},$$

$$\frac{\partial}{\partial\nu_{\mathbb{I}}} v^-[\partial\mathbb{I},\mu,k](t) = +\frac{1}{2}\mu(t) + \int_{\partial\mathbb{I}} \frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n(t-s,k))\mu(s)\,d\sigma_s \qquad \forall t \in \partial\mathbb{I}.$$

*Proof.* See Lanza and Rossi [86, Theorem 3.4]. □

Now let $K$ be a compact subset of $\mathbb{R}^n$. Let $C^{0,1}(K,\mathbb{R}^n)$ denote the space of Lipschitz continuous functions of $K$ to $\mathbb{R}^n$. Then we set

$$l_K[f] \equiv \inf\left\{ \frac{|f(x)-f(y)|}{|x-y|} : x,y \in K,\ x \neq y \right\} \qquad \forall f \in C^{0,1}(K,\mathbb{R}^n).$$

We also set

$$\mathcal{A}_K \equiv \left\{ \phi \in C^1(K,\mathbb{R}^n) \colon l_k[\phi] > 0 \right\}.$$

The set $\mathcal{A}_K$ is open in $C^1(K,\mathbb{R}^n)$. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. Let $\Omega$ be a bounded open subset of class $C^{m,\alpha}$ of $\mathbb{R}^n$ such that both $\Omega$ and $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$ are connected. Let $\phi \in \mathcal{A}_{\partial\Omega}$. By the Jordan–Leray separation theorem, $\mathbb{R}^n \setminus \phi(\partial\Omega)$ has exactly two open connected components, and we denote by $\mathbb{I}[\phi]$ the bounded connected component. Then we denote by $\nu_\phi$ the outward normal to the set $\mathbb{I}[\phi]$ (cf. Lanza and Rossi [86].)

We have the following.

**Theorem E.6.** *Let* $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$. *Let* $\Omega$ *be a bounded open subset of class* $C^{m,\alpha}$ *of* $\mathbb{R}^n$ *such that both* $\Omega$ *and* $\mathbb{R}^n \setminus \mathrm{cl}\,\Omega$ *are connected. Then the following statements hold.*

*(i)* *The map* $V[\cdot,\cdot,\cdot]$ *of* $(C^{m,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C}) \times \mathbb{C}$ *to the space* $C^{m,\alpha}(\partial\Omega,\mathbb{C})$ *which takes* $(\phi,f,k)$ *in the domain of* $V[\cdot,\cdot,\cdot]$ *to the function of* $\partial\Omega$ *to* $\mathbb{C}$ *defined by*

$$V[\phi,f,k](t) \equiv \int_{\phi(\partial\Omega)} S_n(\phi(t)-s,k) f\circ\phi^{(-1)}(s)\,d\sigma_s \qquad \forall t \in \partial\Omega,$$

*is real analytic.*

*(ii)* *The map* $V_*[\cdot,\cdot,\cdot]$ *of* $(C^{m,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\alpha}(\partial\Omega,\mathbb{C}) \times \mathbb{C}$ *to the space* $C^{m-1,\alpha}(\partial\Omega,\mathbb{C})$ *which takes* $(\phi,f,k)$ *in the domain of* $V_*[\cdot,\cdot,\cdot]$ *to the function of* $\partial\Omega$ *to* $\mathbb{C}$ *defined by*

$$V_*[\phi,f,k](t) \equiv \Big\{\int_{\phi(\partial\Omega)} \frac{\partial}{\partial\nu_\phi(x)}(S_n(x-s,k)) f\circ\phi^{(-1)}(s)\,d\sigma_s\Big\}_{x=\phi(t)} \qquad \forall t \in \partial\Omega,$$

*is real analytic.*

*(iii)* *The map* $W[\cdot,\cdot,\cdot]$ *of* $(C^{m,\alpha}(\partial\Omega,\mathbb{R}^n)\cap\mathcal{A}_{\partial\Omega})\times C^{m,\alpha}(\partial\Omega,\mathbb{C})\times\mathbb{C}$ *to the space* $C^{m,\alpha}(\partial\Omega,\mathbb{C})$ *which takes* $(\phi,f,k)$ *in the domain of* $W[\cdot,\cdot,\cdot]$ *to the function of* $\partial\Omega$ *to* $\mathbb{C}$ *defined by*

$$W[\phi,f,k](t)\equiv\int_{\phi(\partial\Omega)}\frac{\partial}{\partial\nu_\phi(s)}(S_n(\phi(t)-s,k))f\circ\phi^{(-1)}(s)\,d\sigma_s\qquad\forall t\in\partial\Omega,$$

*is real analytic.*

*Proof.* See Lanza and Rossi [86, Theorem 4.11]. □

Then we have the following variant of a classical result in Potential Theory (cf. *e.g.*, Lanza [72, Theorem 5.1], [79, Theorem B.1].)

**Theorem E.7.** *Let* $k\in\mathbb{C}$. *Let* $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. *Let* $\mathbb{I}$ *be a bounded connected open subset of* $\mathbb{R}^n$ *of class* $C^{m,\alpha}$. *Let* $b\in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})$. *Then the following statements hold.*

*(i)* *Let* $j\in\{0,1,\dots,m\}$ *and* $\bar\Gamma\in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$ *and* $\mu\in L^2(\partial\mathbb{I},\mathbb{C})$ *and*

$$\bar\Gamma(t)=\frac{1}{2}\mu(t)+\int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n(t-s,k))\mu(s)\,d\sigma_s+\int_{\partial\mathbb{I}}S_n(t-s,k)b(s)\mu(s)\,d\sigma_s\quad a.e.\ on\ \partial\mathbb{I},\tag{E.6}$$

*then* $\mu\in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$.

*(ii)* *Let* $j\in\{0,1,\dots,m\}$ *and* $\bar\Gamma\in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$ *and* $\mu\in L^2(\partial\mathbb{I},\mathbb{C})$ *and*

$$\bar\Gamma(t)=-\frac{1}{2}\mu(t)+\int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(s)}(S_n(t-s,k))\mu(s)\,d\sigma_s+\int_{\partial\mathbb{I}}S_n(t-s,k)b(s)\mu(s)\,d\sigma_s\quad a.e.\ on\ \partial\mathbb{I},\tag{E.7}$$

*then* $\mu\in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$.

*(iii)* *Let* $j\in\{1,\dots,m\}$ *and* $\bar\Gamma\in C^{j-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ *and* $\mu\in L^2(\partial\mathbb{I},\mathbb{C})$ *and*

$$\bar\Gamma(t)=\frac{1}{2}\mu(t)+\int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n(t-s,k))\mu(s)\,d\sigma_s+b(t)\int_{\partial\mathbb{I}}S_n(t-s,k)\mu(s)\,d\sigma_s\quad a.e.\ on\ \partial\mathbb{I},\tag{E.8}$$

*then* $\mu\in C^{j-1,\alpha}(\partial\mathbb{I},\mathbb{C})$.

*(iv)* *Let* $j\in\{1,\dots,m\}$ *and* $\bar\Gamma\in C^{j-1,\alpha}(\partial\mathbb{I},\mathbb{C})$ *and* $\mu\in L^2(\partial\mathbb{I},\mathbb{C})$ *and*

$$\bar\Gamma(t)=-\frac{1}{2}\mu(t)+\int_{\partial\mathbb{I}}\frac{\partial}{\partial\nu_{\mathbb{I}}(t)}(S_n(t-s,k))\mu(s)\,d\sigma_s+b(t)\int_{\partial\mathbb{I}}S_n(t-s,k)\mu(s)\,d\sigma_s\quad a.e.\ on\ \partial\mathbb{I},\tag{E.9}$$

*then* $\mu\in C^{j-1,\alpha}(\partial\mathbb{I},\mathbb{C})$.

*Proof.* It suffices to modify the proof of Lanza [72, Theorem 5.1]. We first prove statement (*i*). We proceed by (finite) induction on $j$. Let $j=0$. Since the kernels of the integral operators in the right-hand side of (E.6) display a weak singularity, a standard argument on iterated kernels implies that $\mu\in C^0(\partial\mathbb{I},\mathbb{C})$. By Miranda [99, § 14, III], we have $v[\partial\mathbb{I},b\mu,k]\in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C})$, and consequently $\bar\Gamma-v[\partial\mathbb{I},b\mu,k]\in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C})$. Then a classical property of double layer potentials shows that $\mu\in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C})$ (cf. Miranda [99, § 15, II].) We now assume that the statement holds for $j<m$, and we show it for $j+1$. By inductive assumption we know that $\mu\in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$. Since $b\in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C})\subseteq C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$, we have $b\mu\in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$. Then by known properties of simple layer potentials for the Helmholtz equation (cf. Theorem E.5 and also, *e.g.*, Miranda [98, p. 330]), we have $\bar\Gamma-v[\partial\mathbb{I},b\mu,k]\in C^{j+1,\alpha}(\partial\mathbb{I},\mathbb{C})$. Analogously, by known properties of double layer potentials for the Helmholtz equation (cf. Theorem E.4), we have $w^+[\partial\mathbb{I},\mu,k]\in C^{j,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$. Now we note that equation (E.6) implies

$$\begin{cases}\Delta w^+[\partial\mathbb{I},\mu,k]=-k^2w^+[\partial\mathbb{I},\mu,k]\in C^{j,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C}) & \text{in } \mathbb{I},\\ w^+[\partial\mathbb{I},\mu,k]=\bar\Gamma-v^+[\partial\mathbb{I},b\mu,k]\in C^{j+1,\alpha}(\partial\mathbb{I},\mathbb{C}) & \text{on } \partial\mathbb{I}.\end{cases}$$

By classical results of elliptic regularity theory for the Dirichlet problem (cf. *e.g.*, Gilbarg and Trudinger [55, Thms. 6.19, 8.34]), we have $w^+[\partial\mathbb{I},\mu,k]\in C^{j+1,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$. Hence, we have that $(\partial/\partial\nu_{\mathbb{I}})w^+[\partial\mathbb{I},\mu,k]$ is in $C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$. Now let $R>0$ be such that $\mathrm{cl}\,\mathbb{I}\subseteq\mathbb{B}_n(0,R)$ and $\mathbb{I}_R\equiv\mathbb{B}_n(0,R)\setminus\mathrm{cl}\,\mathbb{I}$.

By known properties of double layer potentials for the Helmholtz equation (cf. Theorem E.4) we have $w^-[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C})$. Then we have

$$\begin{cases} \Delta w^-[\partial\mathbb{I},\mu,k] = -k^2 w^-[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C}) & \text{in } \mathbb{I}_R, \\ \frac{\partial}{\partial\nu_{\mathbb{I}}} w^-[\partial\mathbb{I},\mu,k] = \frac{\partial}{\partial\nu_{\mathbb{I}}} w^+[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C}) & \text{on } \partial\mathbb{I}, \\ \frac{\partial}{\partial\nu_{\mathbb{B}_n(0,R)}} w^-[\partial\mathbb{I},\mu,k]_{|\partial\mathbb{B}_n(0,R)} \in C^\infty(\partial\mathbb{B}_n(0,R),\mathbb{C}) & \text{on } \partial\mathbb{B}_n(0,R). \end{cases}$$

By classical results on elliptic regularity theory for the Neumann problem, we conclude that $w^-[\partial\mathbb{I},\mu,k]$ is in $C^{j+1,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C})$. Hence, $\mu = w^+[\partial\mathbb{I},\mu,k] - w^-[\partial\mathbb{I},\mu,k] \in C^{j+1,\alpha}(\partial\mathbb{I},\mathbb{C})$.

We now prove statement $(ii)$. We proceed by induction on $j$. Let $j = 0$. Since the kernels of the integral operators in the right-hand side of (E.7) display a weak singularity, a standard argument on iterated kernels implies that $\mu \in C^0(\partial\mathbb{I},\mathbb{C})$. As in the proof of statement $(i)$, we have $\bar{\Gamma} - v[\partial\mathbb{I}, b\mu, k] \in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C})$. Then a classical property of double layer potentials shows that $\mu \in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C})$ (cf. Miranda [99, § 15, II].) We now assume that the statement holds for $j < m$, and we show it for $j+1$. By inductive assumption we know that $\mu \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$. Since $b \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C}) \subseteq C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$, we have $b\mu \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$. Then by known properties of simple layer potentials for the Helmholtz equation (cf. Theorem E.5 and also, *e.g.*, Miranda [98, p. 330]), we have $\bar{\Gamma} - v[\partial\mathbb{I}, b\mu, k] \in C^{j+1,\alpha}(\partial\mathbb{I},\mathbb{C})$. Analogously, by known properties of double layer potentials for the Helmholtz equation (cf. Theorem E.4) we have $w^-[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C})$. Now we note that equation (E.7) implies

$$\begin{cases} \Delta w^-[\partial\mathbb{I},\mu,k] = -k^2 w^-[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C}) & \text{in } \mathbb{I}_R, \\ w^-[\partial\mathbb{I},\mu,k]_{|\partial\mathbb{I}} = \bar{\Gamma} - v[\partial\mathbb{I}, b\mu, k]_{|\partial\mathbb{I}} \in C^{j+1,\alpha}(\partial\mathbb{I},\mathbb{C}) & \text{on } \partial\mathbb{I}, \\ w^-[\partial\mathbb{I},\mu,k]_{|\partial\mathbb{B}_n(0,R)} \in C^\infty(\partial\mathbb{B}_n(0,R),\mathbb{C}) & \text{on } \partial\mathbb{B}_n(0,R). \end{cases}$$

By classical results of elliptic regularity theory for the Dirichlet problem (cf. *e.g.*, Gilbarg and Trudinger [55, Thms. 6.19, 8.34]), we have $w^-[\partial\mathbb{I},\mu,k] \in C^{j+1,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C})$. Hence, we have that $(\partial/\partial\nu_{\mathbb{I}})w^-[\partial\mathbb{I},\mu,k]$ is in $C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$. By known properties of double layer potentials for the Helmholtz equation (cf. Theorem E.4) we have $w^+[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$ Then we note that $w^+[\partial\mathbb{I},\mu,k]$ must satisfy

$$\begin{cases} \Delta w^+[\partial\mathbb{I},\mu,k] = -k^2 w^+[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C}) & \text{in } \mathbb{I}, \\ \frac{\partial}{\partial\nu_{\mathbb{I}}} w^+[\partial\mathbb{I},\mu,k] = \frac{\partial}{\partial\nu_{\mathbb{I}}} w^-[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C}) & \text{on } \partial\mathbb{I}. \end{cases}$$

By classical results on elliptic regularity theory for the Neumann problem, we conclude that $w^+[\partial\mathbb{I},\mu,k]$ is in $C^{j+1,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$. Hence, $\mu = w^+[\partial\mathbb{I},\mu,k] - w^-[\partial\mathbb{I},\mu,k] \in C^{j+1,\alpha}(\partial\mathbb{I},\mathbb{C})$.

We now turn to the proof of statement $(iii)$. We proceed by induction on $j$. Let $j = 1$. Since the integral operators which appear in the right-hand side of (E.8) display a weak singularity, then a standard argument on iterated kernels implies that $\mu \in C^0(\partial\mathbb{I},\mathbb{C})$. By Miranda [99, § 14, III], we have $v[\partial\mathbb{I},\mu,k] \in C^{0,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R),\mathbb{C})$. Since $\bar{\Gamma}$, $b \in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C})$, we conclude that $\bar{\Gamma} - bv[\partial\mathbb{I},\mu,k] \in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C})$. Thus equation (E.8) implies that $v^-[\partial\mathbb{I},\mu,k]$ satisfies

$$\begin{cases} \Delta v^-[\partial\mathbb{I},\mu,k] = -k^2 v^-[\partial\mathbb{I},\mu,k] \in C^{0,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C}) & \text{in } \mathbb{I}_R, \\ \frac{\partial}{\partial\nu_{\mathbb{I}}} v^-[\partial\mathbb{I},\mu,k]_{|\partial\mathbb{I}} = \bar{\Gamma} - bv^-[\partial\mathbb{I},\mu,k] \in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C}) & \text{on } \partial\mathbb{I} \\ \frac{\partial}{\partial\nu_{\mathbb{B}_n(0,R)}} v^-[\partial\mathbb{I},\mu,k]_{|\partial\mathbb{B}_n(0,R)} \in C^\infty(\partial\mathbb{B}_n(0,R),\mathbb{C}) & \text{on } \partial\mathbb{B}_n(0,R). \end{cases} \tag{E.10}$$

Thus by classical elliptic regularity theory for the Neumann problem (cf. *e.g.*, Miranda [99, § 16, II], Troianiello [136, Thm. 1.17 (ii), 3.16 (iii)], Agmon, Douglis and Nirenberg [1, Thm. 7.3]), we have $v^-[\partial\mathbb{I},\mu,k] \in C^{1,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C})$. Then we note that

$$\begin{cases} \Delta v^+[\partial\mathbb{I},\mu,k] = -k^2 v^+[\partial\mathbb{I},\mu,k] \in C^{0,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C}) & \text{in } \mathbb{I}, \\ v^+[\partial\mathbb{I},\mu,k] = v^-[\partial\mathbb{I},\mu,k] \in C^{1,\alpha}(\partial\mathbb{I},\mathbb{C}) & \text{on } \partial\mathbb{I}. \end{cases} \tag{E.11}$$

By classical results of elliptic regularity theory for the Dirichlet problem (cf. *e.g.*, Gilbarg and Trudinger [55, Thm. 8.34]), we have $v^+[\partial\mathbb{I},\mu,k] \in C^{1,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$. Hence,

$$\mu = \frac{\partial}{\partial\nu_{\mathbb{I}}} v^-[\partial\mathbb{I},\mu,k] - \frac{\partial}{\partial\nu_{\mathbb{I}}} v^+[\partial\mathbb{I},\mu,k] \in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C}). \tag{E.12}$$

We now assume that the statement is true for $j < m$, and we prove it for $j+1$. By inductive assumption, we know that $\mu \in C^{j-1,\alpha}(\partial\mathbb{I},\mathbb{C})$. By known properties of simple layer potentials for the Helmholtz equation (cf. Theorem E.5 and also, *e.g.*, Miranda [98, p. 330]), we have $v^-[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C})$. Since $b \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C}) \subseteq C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$, $\bar\Gamma \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$, we conclude that $\bar\Gamma - bv^-[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$. Then equation (E.8) implies that $v^-[\partial\mathbb{I},\mu,k]$ satisfies problem (E.10). Then by classical elliptic regularity theory for the Neumann problem (cf. *e.g.*, Miranda [99, § 16, II], Troianiello [136, Thm. 1.17 (ii), 3.16 (iii)], Agmon, Douglis and Nirenberg [1, Thm. 7.3]), we have $v^-[\partial\mathbb{I},\mu,k] \in C^{j+1,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C})$. By known properties of simple layer potentials for the Helmholtz equation (cf. Theorem E.5 and also, *e.g.*, Miranda [98, p. 330]), we have $v^+[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$. Then we have

$$\begin{cases} \Delta v^+[\partial\mathbb{I},\mu,k] = -k^2 v^+[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C}) & \text{in } \mathbb{I}, \\ v^+[\partial\mathbb{I},\mu,k] = v^-[\partial\mathbb{I},\mu,k] \in C^{j+1,\alpha}(\partial\mathbb{I},\mathbb{C}) & \text{on } \partial\mathbb{I}. \end{cases}$$

By classical results of elliptic regularity theory for the Dirichlet problem (cf. *e.g.*, Gilbarg and Trudinger [55, Thms. 6.19, 8.34]), we have $v^+[\partial\mathbb{I},\mu,k] \in C^{j+1,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$. Hence, equality (E.12) implies that $\mu \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$.

We finally prove statement (*iv*). We proceed by induction on $j$. Let $j = 1$. Since the integral operators which appear in the right-hand side of (E.6) display a weak singularity, then a standard argument on iterated kernels implies that $\mu \in C^0(\partial\mathbb{I},\mathbb{C})$. By Miranda [99, § 14, III], we have $v[\partial\mathbb{I},\mu,k] \in C^{0,\alpha}(\mathrm{cl}\,\mathbb{B}_n(0,R),\mathbb{C})$. Since $\bar\Gamma$, $b \in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C})$, we conclude that $\bar\Gamma - bv[\partial\mathbb{I},\mu,k] \in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C})$. Thus equation (E.9) implies that $v^+[\partial\mathbb{I},\mu,k]$ satisfies the Neumann boundary value problem

$$\begin{cases} \Delta v^+[\partial\mathbb{I},\mu,k] = -k^2 v^+[\partial\mathbb{I},\mu,k] \in C^{0,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C}) & \text{in } \mathbb{I}, \\ \frac{\partial}{\partial\nu_{\mathbb{I}}} v^+[\partial\mathbb{I},\mu,k] = \bar\Gamma - bv^+[\partial\mathbb{I},\mu,k] \in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C}) & \text{on } \partial\mathbb{I}. \end{cases} \tag{E.13}$$

Thus by classical elliptic regularity theory for the Neumann problem (cf. *e.g.*, Miranda [99, § 16, II], Troianiello [136, Thm. 1.17 (ii), 3.16 (iii)], Agmon, Douglis and Nirenberg [1, Thm. 7.3]), we have $v^+[\partial\mathbb{I},\mu,k] \in C^{1,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$. Then we have

$$\begin{cases} \Delta v^-[\partial\mathbb{I},\mu,k] = -k^2 v^-[\partial\mathbb{I},\mu,k] \in C^{0,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C}) & \text{in } \mathbb{I}_R, \\ v^-[\partial\mathbb{I},\mu,k]_{|\partial\mathbb{I}} = v^+[\partial\mathbb{I},\mu,k]_{|\partial\mathbb{I}} \in C^{1,\alpha}(\partial\mathbb{I},\mathbb{C}) & \text{on } \partial\mathbb{I}, \\ v^-[\partial\mathbb{I},\mu,k]_{|\partial\mathbb{B}_n(0,R)} \in C^\infty(\partial\mathbb{B}_n(0,R),\mathbb{C}) & \text{on } \partial\mathbb{B}_n(0,R). \end{cases}$$

By classical results of elliptic regularity theory for the Dirichlet problem (cf. *e.g.*, Gilbarg and Trudinger [55, Thm. 8.34]), we have $v^-[\partial\mathbb{I},\mu,k] \in C^{1,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C})$. Hence,

$$\mu = \frac{\partial}{\partial\nu_{\mathbb{I}}} v^-[\partial\mathbb{I},\mu,k] - \frac{\partial}{\partial\nu_{\mathbb{I}}} v^+[\partial\mathbb{I},\mu,k] \in C^{0,\alpha}(\partial\mathbb{I},\mathbb{C}). \tag{E.14}$$

We now assume that the statement is true for $j < m$, and we prove it for $j+1$. By inductive assumption, we know that $\mu \in C^{j-1,\alpha}(\partial\mathbb{I},\mathbb{C})$. By known properties of simple layer potentials for the Helmholtz equation (cf., Theorem E.5 and also, *e.g.*, Miranda [98, p. 330]), we have $v^+[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$. Since $b \in C^{m-1,\alpha}(\partial\mathbb{I},\mathbb{C}) \subseteq C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$, $\bar\Gamma \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$, we conclude that $\bar\Gamma - bv^+[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$. Then equation (E.9) implies that $v^+[\partial\mathbb{I},\mu,k]$ satisfies problem (E.13). Then by classical elliptic regularity theory for the Neumann problem (cf. *e.g.*, Miranda [99, § 16, II], Troianiello [136, Thm. 1.17 (ii), 3.16 (iii)], Agmon, Douglis and Nirenberg [1, Thm. 7.3]), we have $v^+[\partial\mathbb{I},\mu,k] \in C^{j+1,\alpha}(\mathrm{cl}\,\mathbb{I},\mathbb{C})$. By known properties of simple layer potentials for the Helmholtz equation (cf., Theorem E.5 and also, *e.g.*, Miranda [98, p. 330]), we have $v^-[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C})$. Then we have

$$\begin{cases} \Delta v^-[\partial\mathbb{I},\mu,k] = -k^2 v^-[\partial\mathbb{I},\mu,k] \in C^{j,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C}) & \text{in } \mathbb{I}_R, \\ v^-[\partial\mathbb{I},\mu,k]_{|\partial\mathbb{I}} = v^+[\partial\mathbb{I},\mu,k]_{|\partial\mathbb{I}} \in C^{j+1,\alpha}(\partial\mathbb{I},\mathbb{C}) & \text{on } \partial\mathbb{I}, \\ v^-[\partial\mathbb{I},\mu,k]_{|\partial\mathbb{B}_n(0,R)} \in C^\infty(\partial\mathbb{B}_n(0,R),\mathbb{C}) & \text{on } \partial\mathbb{B}_n(0,R). \end{cases}$$

By classical results of elliptic regularity theory for the Dirichlet problem (cf. *e.g.*, Gilbarg and Trudinger [55, Thms. 6.19, 8.34]), we have $v^-[\partial\mathbb{I},\mu,k] \in C^{j+1,\alpha}(\mathrm{cl}\,\mathbb{I}_R,\mathbb{C})$. Hence, equality (E.14) implies that $\mu \in C^{j,\alpha}(\partial\mathbb{I},\mathbb{C})$.

The proof is now complete. □

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