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CICLO XXX

Self-sustained periodic behaviors in interacting systems: macroscopic limits and fluctuations

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Riassunto

In questa tesi studiamo comportamenti periodici auto-sostenuti che appaiono nella dinamica macroscopica di certi sistemi interagenti e alcuni fenomeni critici collegati a questo comportamento.

La tesi è organizzata come segue. Nel primo capitolo ci concentriamo sulla comparsa di periodicità in modelli cooperativi a campo medio il cui potenziale di interazione è soggetto a una dissipazione. Definiamo un modello di Curie-Weiss generalizzato con dissipazione ed analizziamo la sua dinamica limite: mostriamo che non solo il comportamento periodico è presente a temperature sufficientemente basse, ma anche che, in certi regimi, diversi cicli limite stabili possono coesistere, purché in numero finito. Nel secondo capitolo ci occupiamo di un modello di Curie-Weiss bipopolato: definiamo due tipi di dinamiche microscopiche, una con ritardo e l'altra senza. Identifichiamo le configurazioni della rete di interazione che possono dare luogo ad oscillazioni macroscopiche nel caso senza ritardo; mostriamo inoltre che il ritardo permette la comparsa di periodicità in configurazioni nelle quali sarebbe altrimenti assente. Nel terzo capitolo consideriamo nuovamente il meccanismo della dissipazione, questa volta lasciando cadere l'ipotesi di interazione a campo medio. Studiamo un sistema di particelle con interazione a corto raggio ottenuto introducendo la dissipazione in un modello di Ising 1-dimensionale. Mostriamo che, in un opportuno limite di temperatura zero e volume infinito, la magnetizzazione totale del sistema presenta oscillazioni regolari tra fasi polarizzate. Infine, il quarto capitolo è dedicato all'analisi delle fluttuazioni critiche di sistemi che esibiscono una biforcazione di Hopf nella dinamica della legge macroscopica. Il comportamento delle fluttuazioni critiche attorno al limite macroscopico riflette il tipo di biforcazione e gli osservabili mostrano fluttuazioni che evolvono su scale temporali differenti. Identifichiamo la variabile lenta e quella veloce ed otteniamo la convergenza della variabile lenta alla sua dinamica limite tramite un *averaging principle*.

Abstract

In this thesis we study the appearance of self-sustained periodic behavior in the macroscopic dynamics of some interacting systems and related critical phenomena.

The thesis is organized as follows. In Chapter 1 we focus on the emergence of periodicity in cooperative mean field models whose interaction potential undergoes a dissipative evolution. We define a generalized Curie-Weiss model with dissipation and we analyse its macroscopic dynamics: we show that not only a periodic behavior is present at sufficiently low temperature, but also that, in certain regimes, any (finite) number of stable limit cycles can coexist. Chapter 2 is concerned with a two-population Curie-Weiss model: we define two types of microscopic dynamics, one with delay and the other without. We identify configurations of the interaction network which can enhance macroscopic oscillations in the case without delay; we also show that delay allows the appearance of a collective periodic behavior in configurations in which periodicity was otherwise absent. In Chapter 3 we consider again the mechanism of dissipation, this time dropping the mean field hypothesis. We study a short-range interacting system obtained introducing dissipation in a 1-dimensional Ising model. We prove that, in a suitable zero-temperature infinite-volume limit, the total magnetization of the system displays regular oscillations between polarized phases. Finally, Chapter 4 is dedicated to the analysis of critical fluctuations for systems exhibiting a Hopf bifurcation in the dynamics of the macroscopic law. The behavior of critical fluctuations around the macroscopic limit reflects the type of bifurcation and the observables display fluctuations evolving at different time scales. We identify the slow and the fast variable and we obtain the convergence of the slow variable to its limiting dynamics via an averaging principle.

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Introduction

This thesis focuses on the appearance of periodic behavior in the macroscopic dynamics of interacting systems and related critical phenomena.

Self-sustained periodic behavior is one of the most commonly observed dynamical patterns in large communities of interacting components: as other emergent behaviors, its appearance is not due to the presence of an external force driving the system, but only to interaction rules at which individuals obey. In fact, even if each agent has no natural tendency to behave periodically, this phenomenon occurs when particles self-organize producing a regular pattern perceived only macroscopically: a collective self-sustained rhythm. Real world examples of systems exhibiting this attitude are widespread in life sciences such as ecology [57] and neuroscience [28], but also in socio-economics [9, 59].

In recent years there has been a growing interest in the mathematical modelling of this peculiar phenomenon in order to understand which interaction mechanisms are capable of enhancing it. A natural approach for this purpose is to define stochastic dynamics for a system comprised by N particles coupled by interaction terms, then to study its dynamics in the limit as $N \uparrow +\infty$: from this point of view, collective periodic behavior is present when the macroscopic law of the system converges to a stable limit cycle in its long-time evolution. As pointed out in [5, 33], this type of behavior is intrinsically in contrast with stochastic reversibility, hence these models belong to the wide branch of non-equilibrium statistical mechanics.

Some universal features have proven to be accountable for the appearance of rhythmic oscillations: first of all, noise plays a central role, since it can both lead to the birth of limit cycles that would otherwise be absent in the macroscopic dynamics (*noise-induced periodicity*, [20, 33, 48, 56]) and facilitate the transition from a disordered phase to a one in which periodic solutions are present (*excitability by noise*, [14, 40]). Existence of self-sustained periodic behavior has also been shown for cooperative systems in which the function describing the interaction potential is subject to its own evolution, in particular when it undergoes a dissipative effect [14, 19, 20]. Oscillations in the dynamics of the macroscopic law have also been observed in multi-populated models with delayed interactions [23, 55].

Mean field models considered in this thesis display deterministic macroscopic dynamics exhibiting (at least) a phase transition, occurring for certain critical values of the models' parameters. It is well-known that interacting systems present peculiar behaviors as they

get close to a phase transition (*critical phenomena*), including long range correlations and large, non normal fluctuations. Critical phenomena are interesting also because there exists interacting systems which are spontaneously attracted to their critical regime, without any external tuning of the parameters (*self-organized criticality* [3]).

The dynamics of fluctuations at critical point in mean field models typically present an anomalous space-time scaling and, possibly, a non-Gaussian limit: this was firstly shown in [16, 22] for reversible mean field dynamics with ferromagnetic interaction. Similar results have been achieved for a class of models in which criticality is achieved by self-organization [35], while in [13] the effects of quenched disorder on critical fluctuations have been investigated in two classical mean field models of statistical mechanics.

In the first part of this thesis, we focus on some mechanisms responsible for the emergence of self-sustained periodic behavior in interacting systems, in particular we deepen the study on the role of dissipation in the interaction energy of cooperative systems. We also examine minimal hypotheses on the interaction network in multi-populated systems leading to the appearance of macroscopic periodicity.

In Chapter 1 we deal with cooperative mean field models in which dissipation is present. As shown in [14, 19, 20], collective periodic behaviors appear in some classes of mean field systems derived as perturbation of classical reversible ferromagnetic models by adding a dissipation term in the interaction energy. The mechanism of dissipation dumps the influence of interaction when no transition occurs for a long time: the result is that the macroscopic observables of the system will oscillate in time between magnetized states rather than polarize in a fixed point (behavior observed in the reversible case). In this chapter, after reviewing the spin model introduced in [19] (a *Curie-Weiss model with dissipation*), we define a diffusive mean field system with dissipation, derived by the so-called *generalized Curie-Weiss model* [24]. We derive the limiting dynamics in a general framework, but then we restrict the analysis to the case in which the nonlinear macroscopic process is Gaussian: this allows for a complete description of the evolution of the macroscopic law in terms of a two-dimensional ODE. We will see that this model not only presents the emergence of periodic behavior at sufficiently low temperature, but also displays a richer behavior: the most interesting aspect is the coexistence of multiple stable limit cycles in the macroscopic dynamics for certain choices of the model's details. These results are due to a joint work with Luisa Andreis [2].

Chapter 2 is dedicated to the emergence of self-sustained periodic behavior in a *two-populated Curie-Weiss model*. This multi-populated extension has naturally arisen in applications: in the 1950s it was introduced to mimic the phase transition undergone by metamagnets [49], while in recent year it has been employed to analyse immigration and cultural coexistence and to explain the role of social groups in influencing an individual's beliefs and preferences [4, 18, 17, 30]. The thermodynamic limit of this model has been rigorously studied both from the statical [31] and the dynamical [12] point of view. In this chapter we continue the analysis of the dynamical features of the model and our main in-

terest is to identify which configurations of the interaction network can enhance oscillations in the macroscopic dynamics. Since collective periodic behavior has already been observed in multi-populated systems with delayed interactions [23, 55], we will define two dynamics for the model: the former is a standard Glauber dynamics, while in the latter we introduce a delay kernel acting on the intra-population interaction. The most interesting result here is that delay is not an essential ingredient for the emergence of macroscopic oscillations in the bi-populated Curie-Weiss model. The key feature seems rather to be the presence of a *frustration* in the interaction network. However, we also show that delay can enhance the appearance of macroscopic rhythm in configurations in which it was otherwise absent. This chapter is based on a joint work with Francesca Collet and Marco Formentin [15].

In Chapter 3 we focus again on the mechanism of dissipation: the aim is to understand whether macroscopic oscillations due to dissipation represent a phenomenon which is strictly related to the mean field setting. To this purpose, we consider a short-range interacting system, obtained by introducing dissipation in a Glauber dynamics for the classical 1-dimensional Ising model. Notice that in Chapter 1 and 2 we can determine the presence of periodic behavior by analysing the macroscopic dynamics described by a finite-dimensional system of ODEs, obtained with a rather general approach (*propagation of chaos*) which only works within the mean field framework. The same machinery can not be applied in case of nearest neighbor interactions and a finite-dimensional description of the macroscopic dynamics is out of reach. Nevertheless, we prove that, choosing proper initial condition and performing the correct time-scale changes, the *magnetization* process, which constitutes an observable of the system, converges to a stochastic process displaying regular oscillations in a suitable zero-temperature infinite-volume limit. Results in this chapter are due to a joint work with Raphaël Cerf, Paolo Dai Pra and Marco Formentin [8].

The second part of the thesis is dedicated to the study of critical fluctuations of mean field spin systems exhibiting a Hopf bifurcation in the dynamical system describing the evolution of the macroscopic law.

In Chapter 4 we consider again the Curie-Weiss model with dissipation and the bi-populated Curie-Weiss model, introduced respectively in Chapter 1 and 2. Depending on the value of their parameters, both of these models display a transition from disorder to periodic behavior through a Hopf bifurcation in the macroscopic dynamics. We aim to understand how the presence of this type of bifurcation effects the dynamics of critical fluctuations. Previous results on dynamical critical fluctuations have been obtained for models which exhibit a *pitchfork bifurcation* at the critical point [16, 22, 13], but the presence of a Hopf bifurcation leads to a different behavior. Indeed, the nature of the bifurcation is relevant, as the dynamics of fluctuations is related to the linearization of the *McKean-Vlasov* equation describing the macroscopic dynamics of the system. In both examples studied in this chapter, critical fluctuations can be reduced to a two-dimensional stochastic process displaying a peculiar behavior: performing a change of variables, we identify a slow and a fast variable, respectively the radial and the angular component in polar coordinates. In

the limit, the fast angular variable averages out, and the convergence of the slow variable to its limiting dynamics is achieved via an averaging principle. This chapter is based on a joint work with Paolo Dai Pra [21].

In more details, the thesis is organized as follows.

Chapter 1 - Mean field models with dissipation

After giving some motivation for considering models displaying dissipative effects in their interaction energy, in Section 1.2 we present the Curie-Weiss model with dissipation introduced in [19] and we fix notations and ideas that will be useful in the rest of the thesis. This section does not contain any new contribution but rather it displays the results of [19]. However, it is convenient to analyse this model since, thanks to its minimality, it allows to fully understand the role of dissipation in enhancing self-sustained periodicity. The Curie-Weiss model with dissipation introduced in [19] is a mean field spin system $(\underline{\sigma}(t))_{t \in [0, T]}$ evolving according to

$$\sigma_i(t) \rightarrow -\sigma_i(t) \text{ at rate } 1 - \tanh(\sigma_i(t)\lambda_N(t)), \quad i = 1, \dots, N$$

where $\lambda_N(t)$ represents the interaction potential for the N -spin system at time t . The process $(\lambda_N(t))_{t \in [0, T]}$ obeys

$$d\lambda_N(t) = -\alpha\lambda_N(t)dt + \beta m_N(t)$$

with $\alpha, \beta > 0$ and $m_N(t)$ being the empirical magnetization of the system, i.e.

$$m_N(t) = \frac{1}{N} \sum_{j=1}^N \sigma_j(t).$$

When $\alpha = 0$, this is a modification of the Glauber dynamics for the classical Curie-Weiss model. Taking $\alpha > 0$, the dynamics of the term driving the interaction does not longer coincide with the jumps of the magnetization process, but a *dissipative* effect is present. Indeed, between two consecutive jumps, the value of λ_N is exponentially attracted toward 0. Notice that if $\lambda_N = 0$ there is no correlation between the particles: hence, the presence of this friction term dumps the influence of interaction when no spin flip occurs for a long time. In other words, it acts as a dissipation in the interaction energy as its effect is to "decorrelate" particles. The choice of bounded rates in the form $1 - \tanh(\cdot)$ has been adopted for technical reasons (see Remark 1.2.1). The two-dimensional process $(m_N(t), \lambda_N(t))_{t \in [0, T]}$ fully characterizes the state of the system and, as $N \uparrow +\infty$, converges to the solution of

$$\begin{cases} \dot{m}(t) &= 2(\tanh(\lambda(t)) - m(t)), \\ \dot{\lambda}(t) &= 2\beta(\tanh(\lambda(t)) - m(t)) - \alpha\lambda(t), \end{cases}$$

which describes the dynamics of the macroscopic law of the model. This dynamical system has the origin as unique equilibrium point, which is globally attractive for $\beta \leq \frac{\alpha}{2} + 1$. At the critical point $\beta = \frac{\alpha}{2} + 1$, the system presents a Hopf bifurcation, which give rise to a stable limit cycle: for $\beta > \frac{\alpha}{2} + 1$, we observe a self-sustained periodic behavior.

In Section 1.3 we present an original contribution, due to a joint work with Luisa Andreis. The aim is to generalise the results of [19] to a continuous model. We start from a reversible diffusion for the *generalized* Curie-Weiss model, so we consider the process $(X^N(t))_{t \in [0, T]}$ with values in \mathbb{R}^N solution of

$$dX_i^N(t) = \frac{\beta}{2} g'(m^N(t)) dt + \frac{\rho'(X_i^N(t))}{2\rho(X_i^N(t))} + dB_i(t), \quad i = 1, \dots, N$$

where $m^N(t)$ again represents the empirical magnetization, i.e.

$$m^N(t) = \frac{1}{N} \sum_{j=1}^N X_j^N(t).$$

Here g is the *interaction* function, which defines the Hamiltonian of the system, while ρ is the density of a probability measure on \mathbb{R} , representing the "natural" distribution of a single particle in absence of interaction. Clearly, both g and ρ have to satisfy some modelling and technical assumptions. We modify this model by introducing dissipative (in sense that it forces particles to decorrelate) and (possibly) noisy dynamics in the interaction energy. In this way, we obtain an N -particle system described by the process $(X^N(t), \lambda^N(t))_{t \in [0, T]}$ taking values in \mathbb{R}^{2N} , solution of

$$\begin{cases} dX_i^N(t) &= \frac{\beta}{2} g'(\lambda_i^N(t)) dt + \frac{\rho'(X_i^N(t))}{2\rho(X_i^N(t))} + dB_i^1(t), \\ d\lambda_i^N(t) &= -\alpha \lambda_i^N(t) dt + dm^N(t) + DdB_i^2(t), \end{cases}$$

for $i = 1, \dots, N$, with $\alpha > 0$ and $D \geq 0$. As before, the vector $X^N(t)$ represents the state of the particles at time t while $\lambda^N(t)$ accounts for the interaction field perceived by each particle. For any $N \geq 1$, we will refer to $(X^N(t), \lambda^N(t))_{t \in [0, T]}$ as a *generalized Curie-Weiss model with dissipation*. We show that this system satisfies a propagation of chaos result, which intuitively means that stochastic independence in the initial condition of any fixed number k of particles in of the N -particle system persists in time as $N \uparrow +\infty$, so they will tend to behave as k independent copies of the same macroscopic process. This macroscopic process is the solution of the *McKean-Vlasov* SDE:

$$\begin{cases} dX(t) &= \frac{\beta}{2} g'(\lambda(t)) dt + \frac{\rho'(X(t))}{2\rho(X(t))} + dB^1(t), \\ d\lambda(t) &= -\alpha \lambda(t) dt + \left\langle \mu_t(dx, dl), \frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right\rangle dt + DdB^2(t), \\ \mu_t &= \text{Law}(X(t), \lambda(t)), \end{cases}$$

where

$$\left\langle \mu_t(dx, dl), \frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right\rangle = \int_{\mathbb{R}^2} \left(\frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right) \mu_t(dx, dl).$$

To analyse the long-time behavior of the system we restrict to the case in which $D = 0$ and $\rho \sim \mathcal{N}(0, \sigma^2)$: with these choices, we obtain a Gaussian macroscopic process and to understand the long-time evolution of the macroscopic law it is sufficient to study the ODEs

$$\begin{cases} \dot{m}(t) = \frac{\beta}{2}g'(\lambda(t)) - \frac{m(t)}{2\sigma_2}, \\ \dot{\lambda}(t) = -\alpha\lambda(t) + \frac{\beta}{2}g'(\lambda(t)) - \frac{m(t)}{2\sigma^2}, \end{cases}$$

which, with a change of variables, can be transformed into a Liénard system. Liénard systems constitute an important class of planar dynamical systems, extensively studied in relation to their limit cycles: in literature we can find sufficient conditions for the existence of *at least* (or *exactly*) $k \geq 0$ limit cycles for systems of this type [10, 43]. Assumptions that have to be taken in account when choosing the interaction function g are rather general and a complete description for the system above is out of reach. Nevertheless, some universal features are studied in Theorem 1.3.6. Finally, we present some peculiar behaviors for the macroscopic dynamics through an explicit example: in particular, the limiting dynamics can present phases in which two (or even more) stable limit cycles coexist. By this, we mean that at least two stable periodic orbit are present in the phase space of the dynamical system describing the evolution of the macroscopic law. Therefore, depending on the initial conditions, the long-time behavior of the system may display oscillations with different amplitudes and frequencies. In general, the dynamics of the macroscopic law of generalized Curie-Weiss model with dissipation present a richer behavior than the model analysed in Section 1.2.

Chapter 2 - A two-populated mean field model

The two-population Curie-Weiss model is a spin system where on the complete graph two populations of spins are present. We divide the whole N -particle system into two disjoint groups, I_1 and I_2 , with $|I_1| = N_1$, $|I_2| = N_2$ and $N_1 + N_2 = N$:

$$\sigma = \left(\begin{array}{c|c} \text{Population } I_1 & \text{Population } I_2 \\ \hline (\sigma_1, \sigma_2, \dots, \sigma_{N_1}) & (\sigma_{N_1+1}, \sigma_{N_1+2}, \dots, \sigma_N) \end{array} \right).$$

Given two spins, their mutual interaction depends on the populations they belong to: J_{11} and J_{22} tune the interaction within sites of the same subsystem, J_{12} and J_{21} control the coupling strength between spins located in different groups. All the interactions can be either positive or negative allowing both ferromagnetic and antiferromagnetic interactions. In Section 2.1 we consider the following dynamics:

$$\begin{aligned} \sigma_i(t) &\rightarrow -\sigma_i(t) \quad \text{at rate } e^{-\sigma_i(t)\mathcal{R}_1(m_{1,N}(t), m_{2,N}(t))}, \quad \text{if } i \in I_1 \\ \sigma_i(t) &\rightarrow -\sigma_i(t) \quad \text{at rate } e^{-\sigma_i(t)\mathcal{R}_2(m_{1,N}(t), m_{2,N}(t))}, \quad \text{if } i \in I_2 \end{aligned}$$

with

$$\begin{aligned} \mathcal{R}_1(x_1, x_2) &= J_{11}x_1 + J_{12}x_2, \\ \mathcal{R}_2(x_1, x_2) &= J_{22}x_2 + J_{21}x_1, \end{aligned}$$

and

$$m_{1,N}(t) = \frac{1}{N} \sum_{j \in I_1} \sigma_j(t), \quad m_{2,N}(t) = \frac{1}{N} \sum_{j \in I_2} \sigma_j(t).$$

The dynamics of the macroscopic law for this system is given by

$$\begin{cases} \dot{m}_1(t) = 2\gamma \sinh(\mathcal{R}_1(m_1(t), m_2(t)) - 2m_1(t) \cosh(\mathcal{R}_1(m_1(t), m_2(t))), \\ \dot{m}_2(t) = 2(1 - \gamma) \sinh(\mathcal{R}_2(m_1(t), m_2(t)) - 2m_2(t) \cosh(\mathcal{R}_2(m_1(t), m_2(t))), \end{cases}$$

where $\gamma := \frac{N_1}{N}$ denotes the fraction of spins belonging to the first population.

As self-sustained periodicity had already been observed in multi-populated systems with delayed interactions [23, 55], in Section 2.2 we introduce a delay in the inter-populations interaction. Our aim is to compare this two types of dynamics, in order to get a better comprehension of the link between periodicity and delay. We assume that at any time t , the influence of each population on the other is given by an average over the magnetization trajectory *up to time* t , weighted through a kernel. The microscopic dynamics is such that

$$\begin{aligned} \sigma_i(t) &\rightarrow -\sigma_i(t) \quad \text{at rate} \quad e^{-\sigma_i(t)\mathcal{R}_1(m_{1,N}(t), \eta_{2,N}^{(n)}(t))}, \quad \text{if } i \in I_1 \\ \sigma_i(t) &\rightarrow -\sigma_i(t) \quad \text{at rate} \quad e^{-\sigma_i(t)\mathcal{R}_2(\eta_{1,N}^{(n)}(t), m_{2,N}(t))}, \quad \text{if } i \in I_2 \end{aligned}$$

where, for $n \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{0\}$, we define

$$\eta_{i,N}^{(n)}(t) = \int_0^t \frac{(t-s)^n}{n!} e^{-k(t-s)} m_{i,N}(s) ds, \quad \text{for } i = 1, 2.$$

The dynamics of the macroscopic law in the case of delayed interaction can be described by a system of $2n + 4$ ODEs:

$$\begin{cases} \dot{m}_1(t) = 2\gamma \sinh(\mathcal{R}_1(m_1(t), \eta_2^{(n)}(t)) - 2m_1(t) \cosh(\mathcal{R}_1(m_1(t), \eta_2^{(n)}(t))), \\ \dot{m}_2(t) = 2(1 - \gamma) \sinh(\mathcal{R}_2(\eta_1^{(n)}(t), m_2(t)) - 2m_2(t) \cosh(\mathcal{R}_2(\eta_1^{(n)}(t), m_2(t))), \\ \dot{\eta}_1^{(0)}(t) = k \left(-\eta_1^{(0)}(t) + m_1(t) \right), \\ \dot{\eta}_1^{(j)}(t) = k \left(-\eta_1^{(j)}(t) + \eta_1^{(j-1)}(t) \right), \quad \text{for } j = 1, \dots, n, \\ \dot{\eta}_2^{(0)}(t) = k \left(-\eta_2^{(0)}(t) + m_2(t) \right), \\ \dot{\eta}_2^{(j)}(t) = k \left(-\eta_2^{(j)}(t) + \eta_2^{(j-1)}(t) \right), \quad \text{for } j = 1, \dots, n. \end{cases}$$

In Section 2.3 we study the presence of Hopf bifurcations for the limiting system with and without delay. We show that in the set $A = \{J_{11}, J_{22} \leq 0\}^c \cap \{J_{12}J_{21} \geq 0\}^c$ we can always choose the model's parameter in order to obtain a Hopf bifurcation for the limiting dynamics of the system without delay. This proves that configurations of interaction network belonging to A can enhance self-sustained periodicity in the case without delay. We also show that in the case with delay we can find periodic solutions within a larger subset of network configurations: a Hopf bifurcation can be present even allowing $J_{11}, J_{22} < 0$, but still remaining in the subspace on which $J_{12}J_{21} < 0$. We conclude the chapter commenting these results.

Chapter 3 - An Ising model with dissipation

The aim of this chapter is to prove that the mechanism of dissipation (deeply studied in Chapter 1 for the mean field case) is capable to produce self-sustained periodicity even when local interactions are considered.

In Section 3.1 we introduce a cooperative spin system with short-range interactions and dissipation. Let

$$\Lambda_N = \{1, 2, \dots, N\} \subseteq \mathbb{Z}$$

represents the set of sites of the spins. Differently from the mean field models studied in Chapter 1 and 2, the geometry of Λ_N is now relevant. For $i \in \Lambda_N$, each spin σ_i only interacts with its first neighbours σ_{i+1} and σ_{i-1} . We assume periodic boundary condition, so that σ_1 and σ_N result to be neighbours and the graph representing the interactions is a circle on which the sites Λ_N are arranged.

We define the process $(\underline{\sigma}(t), \underline{\lambda}(t))_{t \geq 0}$ with values in $\{-1, +1\}^{\Lambda_N} \times \mathbb{R}^N$ evolving according to the following dissipated dynamics:

$$\sigma_i(t) \rightarrow -\sigma_i(t) \quad \text{at rate} \quad e^{-\sigma_i(t)\lambda_i(t)}, \quad i \in \Lambda_N$$

where $\{\lambda_i(t)\}_{i \in \Lambda_N}$ is a family of stochastic processes (representing the *local fields*) evolving according to

$$d\lambda_i(t) = -\alpha\lambda_i(t)dt + \beta dm_i(t), \quad i \in \Lambda_N$$

with $\alpha, \beta > 0$ and

$$m_i(t) = \sum_{j \sim i} \sigma_j(t), \quad i \in \Lambda_N.$$

We assume the following initial conditions:

$$\sigma_i(0) = -1, \quad \lambda_i(0) = -\gamma, \quad i \in \Lambda_N$$

with $\gamma > 0$.

In Section 3.2 we study the distribution of the time in which the first spin-flip occurs, namely

$$T_1 = \inf \left\{ t \geq 0 \mid \exists i \in \Lambda_N \text{ s.t. } \sigma_i(t) = +1 \right\},$$

in the limit as $\gamma, N \uparrow +\infty$. Under some technical conditions, Proposition 3.2.1 establishes a sort of Central Limit Theorem for T_1 :

$$\alpha \log N \left(T_1 - t(\gamma, N) \right) \xrightarrow[\gamma, N \rightarrow +\infty]{d} X,$$

with $t(\gamma, N)$ denoting an implicit deterministic quantity and X being a random variable whose distribution is explicitly known.

In Section 3.3 we show that, in a suitable zero-temperature infinite-volume limit (we require $\beta, N \uparrow +\infty$ in such a way $\frac{\log N}{\beta} \rightarrow c \in [0, 1[$), the positive spin appeared at time T_1 gives rise to a droplet of $+1$ spins which grows very quickly, ending up in covering the whole space

with +1 spins before observing any flip from +1 to -1 or the creation of other droplets. Denote with T_c the covering time, i.e. the time elapsed from T_1 to the moment in which all spins are positive:

$$T_c = \inf \left\{ t > 0 \mid \sigma_i(t) = +1 \ \forall i \in \Lambda_N \right\} - T_1.$$

Ruling out "undesired" behavior on $[T_1, T_1 + T_c]$ (formation of other droplets or flipping of opposite sign), we can obtain a sharp estimate for the time scale of T_c : as shown by Proposition 3.3.1,

$$\frac{T_c}{\frac{N^2}{2\alpha \log N} e^{-2\beta}} \xrightarrow[\gamma, \beta, N \rightarrow +\infty]{d} Z,$$

with Z being a random variable with a strict connection to X (so the fluctuations of time T_1 have an impact on the velocity at which the covering is performed). We remark that this holds under the condition $\frac{\log N}{\beta} \rightarrow c \in [0, 1[$: this ensures that the covering is achieved only by the droplet of +1 spins born at time T_1 (intuitively, if N grows too fast with respect to β , Λ_N becomes too big to be covered by a single droplet).

In Section 3.4 we show that a smart choice for γ allows the iteration of the same phenomena and so the emergence of self-sustained macroscopic oscillations for the system. In fact, by setting

$$\gamma = 4\beta - \log N + \log \log N + \log \alpha,$$

at time $T_1 + T_c$ we recover a situation similar to the initial one: all spins with the same state (clearly +1 at time $T_1 + T_c$) and similar conditions on the flipping rates. Together with results achieved in previous sections, this allows us to formulate Theorem 3.4.1: fix $n \in \mathbb{N}$ and define the following stopping times, for $j = 1, \dots, n$

$$T_{1,j} := \inf \left\{ t > \sum_{k=0}^{j-1} (T_{1,k} + T_{c,k}) \mid \sigma_i(t) = (-1)^{j+1} \text{ for some } i \in \Lambda_N \right\} - \sum_{k=0}^{j-1} (T_{1,k} + T_{c,k}),$$

$$T_{c,j} := \inf \left\{ t > T_{1,j} + \sum_{k=0}^{j-1} (T_{1,k} + T_{c,k}) \mid \sigma_i(t) = (-1)^{j+1} \text{ for all } i \in \Lambda_N \right\} - \sum_{k=0}^{j-1} (T_{1,k} + T_{c,k})$$

with $T_{1,0} = T_{c,0} = 0$. Suppose $\beta, N \uparrow +\infty$ with the condition

$$\lim_{\beta, N \uparrow +\infty} \frac{\log N}{\beta} = c \in [0, 1[.$$

Then, for any $j = 1, \dots, n$,

$$\alpha \log N (T_{1,j} - t(\gamma, N)) \xrightarrow[\gamma, N \uparrow +\infty]{d} X_j$$

$$\frac{T_{c,j}}{\frac{N^2}{2\alpha \log N} e^{-2\beta}} \xrightarrow[\beta, N \uparrow +\infty]{d} Z_j,$$

with a given distribution for the random vectors (X_1, \dots, X_j) and (Z_1, \dots, Z_j) . Our final aim is to prove that the process $(m_N(t))_{t \geq 0}$, representing the total magnetization of the system, i.e.

$$m_N(t) = \frac{1}{N} \sum_{i \in \Lambda_N} \sigma_i(t), \quad t \geq 0$$

converges (in sense of weak convergence of stochastic processes) to an oscillating process. Notice that the two main phenomena, the "first" spin-flip and the covering, occur on very different time scale: letting $\beta, N \uparrow +\infty$ under the condition $\frac{\log N}{\beta} \rightarrow c \in [0, 1[$ implies

$$\lim_{\beta, N \uparrow +\infty} t(\gamma, N) = \begin{cases} +\infty, & \text{if } c = 0, \\ \frac{1}{\alpha} \log \left(\frac{4-c}{c} \right), & \text{otherwise,} \end{cases} \quad \text{and} \quad \lim_{\beta, N \uparrow +\infty} \frac{N^2 e^{-2\beta}}{2\alpha \log N} = 0.$$

Therefore, we consider the time changing process

$$\theta_N(t) = \int_0^t t(\gamma, N) \mathbf{1}_{\{|m_N(s)|=1\}} + \frac{N^2 e^{-2\beta}}{2\alpha \log N} \mathbf{1}_{\{|m_N(s)|<1\}} ds,$$

which "speeds up" time whenever all the spins are equal and we are waiting for the following flip and "slows down" time whenever we are observing the very fast invasion of a droplet of spins opposite sign. Then, we define a time-scaled version of total magnetization process setting $\tilde{m}_N(t) := m_N(\theta_N(t))$. At this point, we expect that the process \tilde{m}_N converges to a stochastic process \tilde{x} with the following behavior: $\tilde{x}(0) = -1$ then it does not move for a unit of time, then it takes a random time Z_1 to linearly grow from -1 to $+1$; after reaching $+1$, it does not move for a unit of time, then it takes a random time Z_2 to linearly decrease from $+1$ to -1 and so on. This is proved in Theorem 3.4.2. Clearly, this limiting process is not periodic, nevertheless it displays some regularity in oscillating between $+1$ and -1 .

Chapter 4 - Critical fluctuations at a Hopf bifurcation

In Section 4.1 we study the dynamics of standard and critical fluctuations for the Curie-Weiss model with dissipation (see Section 1.2). In Chapter 1 we presented a Law of Large Number to establish the macroscopic dynamics for this model, achieved showing the convergence of the two-dimensional process $(m_N(t), \lambda_N(t))_{t \in [0, T]}$ to $(m(t), \lambda(t))_{t \in [0, T]}$ as $N \uparrow +\infty$. It is therefore clear that standard dynamical fluctuations are described by the two-dimensional process $(\tilde{m}_N(t), \tilde{\lambda}_N(t))_{t \in [0, T]}$ where

$$\tilde{m}_N(t) = N^{\frac{1}{2}}(m_N(t) - m(t)), \quad \tilde{\lambda}_N(t) = N^{\frac{1}{2}}(\lambda_N(t) - \lambda(t)).$$

As one may expect, this order parameter converges to a Gaussian process, solution of the linear time-inhomogenous stochastic differential equation

$$d \begin{pmatrix} \tilde{m}(t) \\ \tilde{\lambda}(t) \end{pmatrix} = A(t) \begin{pmatrix} \tilde{m}(t) \\ \tilde{\lambda}(t) \end{pmatrix} dt + \sqrt{1 - m(t) \tanh(\lambda(t))} \begin{pmatrix} 2 \\ 2\beta \end{pmatrix} dB(t)$$

with

$$A(t) = \begin{pmatrix} -2 & 2(1 + \tanh(\lambda(t))) \\ -2\beta(1 - \tanh(\lambda(t))) & 2\beta - \alpha \end{pmatrix}.$$

This result is quite standard, and holds for any choice of the parameters $\alpha, \beta > 0$. Our main interest in this chapter is to analyse the *critical fluctuations* at $\beta = \frac{\alpha}{2} + 1$, where the limiting dynamics presents a Hopf bifurcation. Typically, the dynamical critical fluctuations evolve in a time scale of order $N^{\frac{1}{2}}$ in a space scale of $N^{\frac{3}{4}}$, therefore, after setting $\beta = \frac{\alpha}{2} + 1$, we consider the order parameter $(\hat{m}_N(t), \hat{\lambda}_N(t))_{t \in [0, T]}$ such that

$$\hat{m}_N(t) = N^{\frac{1}{4}} m_N(N^{\frac{1}{2}} t), \quad \hat{\lambda}_N(t) = N^{\frac{1}{4}} \lambda_N(N^{\frac{1}{2}} t),$$

which describes the flow of critical fluctuations around the equilibrium. Our main result (Theorem 4.1.2) states that, as $N \uparrow +\infty$ the process $(\kappa_N(t))_{t \in [0, T]}$ defined by

$$\kappa_N(t) = \left(\frac{\beta \hat{m}_N(t) - \hat{\lambda}_N(t)}{\sqrt{\beta - 1}} \right)^2 + (\hat{\lambda}_N(t))^2$$

converges weakly to the solution of

$$d\kappa(t) = \left(4\beta^2 - \frac{\beta}{2} \kappa^2(t) \right) dt + 2\beta \sqrt{2\kappa(t)} dB(t).$$

Section 4.2 is entirely dedicated to the proof of Theorem 4.1.2, which requires several steps. We start with some preliminary computations, among which we give the reason to define $\kappa_N(t)$ as above. Denoting with \mathcal{L}_N the infinitesimal generator of the process $(\hat{m}_N(t), \hat{\lambda}_N(t))$, for a function f regular enough it can be written in the form

$$\mathcal{L}_N f(\hat{m}, \hat{\lambda}) = N^{\frac{1}{2}} \mathcal{L}_1 f(\hat{m}, \hat{\lambda}) + \mathcal{L}_2 f(\hat{m}, \hat{\lambda}) + o(1).$$

Performing the change of variables

$$\begin{cases} z = \hat{\lambda}, \\ u = \frac{\beta \hat{m} - \hat{\lambda}}{\sqrt{\beta - 1}}. \end{cases}$$

and passing to the polar coordinates $\kappa = (z)^2 + (u)^2$, $\theta = \arctan(u/z)$, one obtains that the infinitesimal generator \mathcal{K}_N for the process $(\kappa_N(t), \theta_N(t))$ can be written, for functions regular enough, in the form

$$\mathcal{K}_N f(\kappa, \theta) = N^{\frac{1}{2}} 2\sqrt{\beta - 1} \partial_\theta f(\kappa, \theta) + \mathcal{A}_f(\kappa, \theta) + o(1).$$

This indicates that "fast" component of the generator \mathcal{L}_N , involves only the derivative with respect to the angular variable θ , which therefore plays the role of fast variable compared to the evolution of the "radial" variable κ . This suggests to derive the asymptotic evolution

of κ by an averaging principle, so the candidate limiting process has infinitesimal generator in the form

$$\mathcal{K}f(\kappa) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{A}_f(\kappa, \theta) d\theta.$$

This informal description gives an intuition on how the limiting process has been identified. More formally, the proof of the theorem requires several steps. After having defined the stopping times

$$\tau_{r,R}^N = \inf\{t \in [0, T] \mid \kappa_N(t) \notin]r, R[\}, \quad \text{with } 0 < r < R,$$

we study the convergence of the sequence of stopped processes $(\kappa_N(t \wedge \tau_{r,R}^N))_{t \in [0, T]}$, since it allows us to avoid technical problems due to the singularity of the polar coordinates in the origin. We show that this sequence is tight and, by an averaging principle, we characterize its limit as the solution of a local martingale problem associated with the infinitesimal generator of the limiting process. Finally, we show how this characterization of the limit of stopped processes is sufficient to conclude the proof of Theorem 4.1.2.

Finally, in Section 4.3, we briefly analyse critical fluctuations in presence of a Hopf bifurcation for the bi-populated Curie-Weiss model without delay (see Section 2.1). As pointed out in Section 2.3.1, the parameters $\gamma, J_{11}, J_{12}, J_{21}, J_{22}$ can be adjusted to create an Hopf bifurcation at the origin for the limiting dynamics: it is sufficient to impose that

$$\begin{aligned} \gamma J_{11} - 1 &= -((1 - \gamma)J_{22} - 1), \\ \Gamma &:= (\gamma J_{11} - 1)^2 + \gamma(1 - \gamma)J_{12}J_{21} < 0. \end{aligned}$$

We focus on the critical fluctuations of the process $(m_{1,N}(t), m_{2,N}(t))$ when the above conditions hold, hence in presence of a Hopf bifurcation. The critical fluctuation flow, when starting from the local equilibrium, is described by the two-dimensional process $(x_N(t), y_N(t))_{t \in [0, T]}$ such that

$$x_N(t) = N^{\frac{1}{4}} m_{1,N}(N^{\frac{1}{2}}t), \quad y_N(t) = N^{\frac{1}{4}} m_{2,N}(N^{\frac{1}{2}}t).$$

We consider the change of variables

$$w_N(t) = \frac{y_N(t)}{(1 - \gamma)J_{21}}, \quad v_N(t) = \frac{1}{\sqrt{|\Gamma|}} \left(-x_N(t) + \frac{(\gamma J_{11} - 1)}{(1 - \gamma)J_{21}} y_N(t) \right).$$

and define

$$\kappa_N(t) = w_N(t)^2 + v_N(t)^2.$$

Theorem 4.3.1 establishes the limit process $(\kappa(t))_{t \in [0, T]}$ at which $(\kappa_N(t))_{t \in [0, T]}$ converges as $N \uparrow +\infty$. In both cases (Curie-Weiss model with dissipation and bi-populated Curie-Weiss model), the critical fluctuations belong to the same class of universality, given by limiting dynamics in the form

$$d\kappa(t) = (C_1 - C_2\kappa^2(t))dt + \sqrt{C_3\kappa(t)}dB(t),$$

with $C_1, C_2, C_3 > 0$. This suggests that the behavior of critical fluctuations is strongly related with the presence of the Hopf bifurcation in the macroscopic dynamics, while the microscopic dynamical details of the model are less important. The proof of Theorem 4.3.1 presents the same technical steps of the proof of Theorem 4.1.2, so we will only sketch the computations sufficient to identify the correct change of variables and the limiting equation.

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Part I

Emergence of self-sustained rhythmic behavior

Chapter 1

Mean field models with dissipation

In this chapter we study the role of dissipation in enhancing self-sustained periodic behavior in cooperative mean field systems. We first present a spin flip model studied in [19] which constitutes a minimal example of the phenomenon. The main focus of the chapter will be on a diffusive cooperative model with dissipation, obtained as a modification of the Langevin dynamics for the generalized Curie-Weiss model. We show that the macroscopic behavior for this system is richer than the one of similar models already studied ([14, 19]).

1.1 Why dissipation?

A standard approach in defining an interacting particle system consists in coupling the particles by an *interaction potential* (or *energy*) which is a deterministic function of the particle positions. A prototypical example of stochastic dynamics for a N -particle system $\underline{X} = (X_1, \dots, X_N) \in \mathbb{R}^N$ is given by the SDE

$$dX_i(t) = -\nabla H(\underline{X}(t))dt + \sigma dB_i(t), \quad i = 1, \dots, N$$

with the interaction potential being a function $H : \mathbb{R}^N \rightarrow \mathbb{R}$. In this case, the role of the components of the dynamics is clear: the drift tends to freeze the system in the states of minimal energy while the Brownian motion is a source of disorder.

Even if this general approach is sufficient to describe a huge variety of systems, in recent years there has been a growing interest, motivated by applications in biology and socioeconomics, in models where the interaction energy is subject to its own (possibly stochastic) dynamics. A procedure to define models of this type in a general framework is described in [14]. An interesting example in this context, coming from biology, is given by the following (see [51]): let $\underline{X}(t) = (X_1(t), \dots, X_N(t)) \in \mathbb{R}^N$ describe the positions of a family of cells at time t while $h(x, t)$ represents the concentration of a given chemical in point x at time t . The cells tend to move towards area of higher concentration of the chemical while also releasing chemicals themselves, modifying the concentration h . Moreover, this chemical is subject to dissipation and diffusion in the medium. A system of this type can be described

by the equations

$$\begin{cases} dX_i(t) = \partial_x h(X_i(t), t) dt + \sigma dB_i(t), & i = 1, \dots, N, \\ \partial_t h(x, t) = -\alpha \partial_x h(x, t) + D \partial_{xx}^2 h(x, t) + \beta \sum_{j=1}^N g(X_j(t), x) \end{cases} \quad (1.1.1)$$

with $\alpha, \beta, \sigma, D > 0$. In this case, the interaction energy is represented by $h(t, x)$, whose evolution depends on the behavior of the cells ($\beta > 0$) but also on dissipation ($\alpha > 0$) and diffusion ($D > 0$). Beyond biology, other examples where dissipation has to be accounted come from finance (e.g. [34]).

Long-time behaviors of the macroscopic dynamics of systems such as (1.1.1) present a rich scheme, including patterns like spikes formation and pulsations ([46]), but they are mainly studied on the basis of numerical simulations since rigorous results are often too hard to be obtained due to the infinite-dimensional nature of the problem. Therefore, it is interesting to relax the modelling purposes and to consider the minimal ingredients capable to produce these peculiar patterns. The aim of this chapter is to focus only on the effect of dissipation by introducing it in toy models with cooperative interaction. The minimality of the models allows to describe the evolution of the macroscopic dynamics through a finite-dimensional ODE, therefore to obtain rigorous results on the long-time evolution.

1.2 A spin system with dissipation

In this section we present a Curie-Weiss model with dissipation, introduced in [19]. Thanks to its minimality, it is useful to understand the effect of dissipation in the evolution of the macroscopic law of the system.

1.2.1 A Curie-Weiss model with dissipation

Let $\mathcal{S} = \{-1, +1\}$ and fix $T > 0$. Given a family of N spins $\underline{\sigma} = (\sigma_1, \dots, \sigma_N) \in \mathcal{S}^N$, denote with $\underline{\sigma}^i$ the configuration obtained by flipping the state of the i -th spin, i.e.

$$\sigma_k^i = \begin{cases} -\sigma_k, & i = k, \\ \sigma_k, & i \neq k. \end{cases} \quad (1.2.1)$$

Define the Markov process $(\underline{\sigma}(t), \lambda_N(t))_{t \in [0, T]}$ with values in $\mathcal{S}^N \times \mathbb{R}$ whose infinitesimal generator is given by

$$L_N f(\underline{\sigma}, \lambda) = \sum_{i=1}^N \left[\left(1 - \tanh(\sigma_i \lambda)\right) \left(f\left(\underline{\sigma}^i, \lambda - \frac{2\beta\sigma_i}{N}\right) - f(\underline{\sigma}, \lambda) \right) \right] - \alpha \lambda f_\lambda(\underline{\sigma}, \lambda), \quad (1.2.2)$$

with $\alpha \geq 0$, $\beta > 0$ and f_λ denoting the partial derivative of f with respect to λ .

Let us take a closer look to the dynamics described by (1.2.2): at any time $t \in [0, T]$,

each flip $\sigma_i(t) \rightarrow -\sigma_i(t)$ occurs with rate $1 - \tanh(\sigma_i(t)\lambda_N(t))$, where $(\lambda_N(t))_{t \in [0, T]}$ evolves according to the stochastic differential equation

$$d\lambda_N(t) = -\alpha\lambda_N(t)dt + \beta dm_N(t). \quad (1.2.3)$$

The process $(m_N(t))_{t \in [0, T]}$ appearing in (1.2.3) represents the *magnetization* of the N -spin system: for any $t \in [0, T]$,

$$m_N(t) = \frac{1}{N} \sum_{j=1}^N \sigma_j(t). \quad (1.2.4)$$

To have a better understanding of the behavior of the system, set $\alpha = 0$: in this case, the solution of (1.2.3) reduces to $\lambda_N(t) = h + \beta m_N(t)$ with $h = \lambda_N(0) - \beta m_N(0)$ and $(\underline{\sigma}(t))_{t \in [0, T]}$ becomes a Markov process with infinitesimal generator

$$L_N f(\underline{\sigma}) = \sum_{i=1}^N \left[\left(1 - \tanh(\sigma_i(h + \beta m_N(t))) \right) \left(f(\underline{\sigma}^i) - f(\underline{\sigma}) \right) \right]. \quad (1.2.5)$$

This is a modification of the Glauber dynamics for the classical Curie-Weiss model with external field h at inverse temperature β . Roughly speaking, (1.2.5) describes a system of mean field ferromagnetically coupled spins whose interaction depends only on the state of the magnetization. Taking $\alpha > 0$, the dynamics of the term driving the interaction does not longer coincide with the jumps of the magnetization process, but a *dissipative* effect is present. Indeed, between two consecutive jumps, the value of λ_N is exponentially attracted toward 0. Notice that if $\lambda_N = 0$ there is no correlation between the particles: hence, the presence of this friction term dumps the influence of interaction when no spin flip occurs for a long time, in other words it acts as a dissipation in the interaction energy. With these consideration, we shall refer to the system described by (1.2.2) as a *Curie-Weiss model with dissipation*.

Remark 1.2.1. A model similar to (1.2.2) can be obtained by perturbing the standard Glauber dynamics with a dissipation in the interaction energy. The Glauber dynamics for the classical Curie-Weiss model, which defines a reversible Markov process with respect to the Gibbs measure

$$P_{\beta, N}(\underline{\sigma}) = \frac{1}{Z_{N, \beta}} \exp \left[-\frac{\beta}{2N} \left(\sum_{i=1}^N \sigma_i \right)^2 \right],$$

is given by flipping rates in the form $\exp[-\sigma_i(t)\beta m_N(t)]$. Following this approach, one can consider a Markov process with infinitesimal generator

$$L_N f(\underline{\sigma}, \lambda) = \sum_{i=1}^N \left[\exp[-\sigma_i \lambda] \left(f \left(\underline{\sigma}^i, \lambda - \frac{2\beta \sigma_i}{N} \right) - f(\underline{\sigma}, \lambda) \right) \right] - \alpha \lambda f_\lambda(\underline{\sigma}, \lambda). \quad (1.2.6)$$

Beyond (1.2.2) and (1.2.6), other choices for the form of the flipping rates are possible, all leading to qualitatively similar results for the behavior of the limiting dynamics: the

key ingredient is a cooperative mean field dynamics for $\underline{\sigma}$ modified by a dissipation term as in (1.2.3). Here we chose to analyse the model defined by (1.2.2) to keep uniformity of notations with [19] (where it has been employed for technical reasons) and with the study of critical fluctuation presented in Chapter 4.

Remark 1.2.2. The model studied in [19] is more general than the one defined by (1.2.2), since it also allow the presence of Brownian noise in (1.2.3). However, in that case, the macroscopic evolution cannot be described by a finite set of ODEs and this would prevent us to state rigorous results on the long-time behavior of the limiting dynamics. Since the aim of this section is to give a minimal example of the effect of dissipation at macroscopic level, we will restrict to the study of the system defined by (1.2.2). Nevertheless, an interaction potential with both dissipative and noisy dynamics will be considered for the model analysed in Section 1.3.

1.2.2 Infinite volume dynamics

We now study the dynamics of the process defined by (1.2.2) in the infinite volume limit, i.e. as $N \uparrow +\infty$, in a fixed time interval $[0, T]$.

Let $(\underline{\sigma}(t), \lambda_N(t))_{t \in [0, T]}$ be the Markov process with infinitesimal generator (1.2.2) and initial condition such that

$$\text{Law}(\underline{\sigma}(0), \lambda_N(0)) = \nu_0^{\otimes N} \otimes \delta_{\lambda_0} \quad (1.2.7)$$

where ν_0 is a probability measure on \mathcal{S} and δ_{λ_0} is a Dirac delta centered in $\lambda_0 \in \mathbb{R}$. The trajectories of this process belong to $\mathcal{D}([0, T], \mathcal{S}^N \times \mathbb{R})$, the set of *cadlag* trajectories from $[0, T]$ to $\mathcal{S}^N \times \mathbb{R}$. Let $(\underline{\sigma}[0, T], \lambda_N[0, T])$ denote a path of the system on the time interval $[0, T]$. The limiting dynamics in mean field models is often studied by the *empirical measure* μ_N of the system, defined as

$$\mu_N = \frac{1}{N} \sum_{j=1}^N \delta_{\{\sigma_j[0, T], \lambda_N[0, T]\}}. \quad (1.2.8)$$

Notice that μ_N is a random variable with values in $\mathcal{M}_1(\mathcal{D}([0, T], \mathcal{S} \times \mathbb{R}))$ where $\mathcal{D}([0, T], \mathcal{S} \times \mathbb{R})$ is endowed with the *Skorohod topology* (see [6]) and $\mathcal{M}_1(\mathcal{D}([0, T], \mathcal{S} \times \mathbb{R}))$, the set of probability measure on $\mathcal{D}([0, T], \mathcal{S} \times \mathbb{R})$, is endowed with the weak convergence topology. With this theoretical set up, one can study the limit as $N \uparrow +\infty$ of the empirical measure, proving a *Law of Large Number* of the following type:

$$\mu_N \xrightarrow[N \uparrow +\infty]{w} \mu \quad (1.2.9)$$

with μ being a deterministic element of $\mathcal{M}_1(\mathcal{D}([0, T], \mathcal{S} \times \mathbb{R}))$. If this Law of Large Numbers holds, μ is the law of a stochastic process with trajectories in $\mathcal{D}([0, T], \mathcal{S} \times \mathbb{R})$ which describes the infinite volume dynamics of the system. In fact, the weak convergence of the empirical measure is strictly related to *propagation of chaos*, as shown in the studies of

Sznitman [53] and Tanaka [54]. The concept of propagation of chaos, which traces back to the seminal work of Kac [37], indicates that molecular chaos (i.e. stochastic independence) in the initial condition of any fixed number k of particles in a N -particle system persists in time as $N \uparrow +\infty$. In other words, if we assign independent initial condition and we fix k of its components, in the infinite volume limit they will tend to behave as k copies of the same macroscopic process. This intuitive idea is formalized in Definition 1.2.2.

Definition 1.2.1. Let E be a separable metric space and, for all $N \geq 1$, let ν^N be a symmetric probability measure on E^N . We say that the sequence $\{\nu^N\}_{N \geq 1}$ is ν -chaotic for a measure ν on E if, for any $k \geq 1$ and any $\phi_1, \dots, \phi_k \in C_b(E)$,

$$\lim_{N \rightarrow \infty} \langle \nu^N, \phi_1 \otimes \dots \otimes \phi_k \otimes 1 \otimes \dots \rangle = \prod_{i=1}^k \langle \nu, \phi_i \rangle.$$

Definition 1.2.2. For every $N \geq 1$, let P^N be the law of the solution of a particle system on $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)^N$. We say that *propagation of chaos* holds if, whenever the sequence of initial conditions P_0^N is Q_0 -chaotic for a certain measure Q_0 on \mathbb{R}^d , then for all $T \geq 0$ the sequence of laws P_T^N is Q_T -chaotic, where P_T^N is the law of the N -particle system on $\mathcal{D}([0, T], \mathbb{R}^d)^N$ and Q_T is a law on $\mathbf{D}([0, T], \mathbb{R}^d)$ with initial condition Q_0 .

Theorem 1.2.1 ([53]). *A sequence of symmetric probability measure $\{\nu^N\}_{N \geq 1}$ is ν -chaotic if and only if*

$$\frac{1}{N} \sum_{i=1}^N \delta_{X_i} \xrightarrow[N \uparrow +\infty]{w} \delta_\nu,$$

where $\{X_i\}_{i=1}^N$ indicates the canonical coordinates on E^N .

For the purposes of this section, it is enough to prove a weaker version of (1.2.9): let $(\mu_N(t))_{t \in [0, T]}$ denote the *empirical measure flow*, the measure-valued stochastic process where, for any $t \in [0, T]$, $\mu_N(t)$ corresponds to the marginal distribution of μ_N at time t . Therefore, we are going to prove a Law of Large Number in the form

$$(\mu_N(t))_{t \in [0, T]} \xrightarrow[N \uparrow +\infty]{} (\mu(t))_{t \in [0, T]} \quad (1.2.10)$$

where the convergence is meant in sense of weak convergence of measure-valued stochastic process and $(\mu(t))_{t \in [0, T]}$ is a deterministic flow of measures on $\mathcal{S} \times \mathbb{R}$, representing, for any $t \in [0, T]$, of the law at time t of the limiting system.

Notice that it is possible to find a low-dimensional description of the empirical measure flow: for any $t \in [0, T]$, $\mu_N(t)$ is completely determined by the values of $m_N(t)$ (the magnetization at time t , see (1.2.4)) and $\lambda_N(t)$. In fact, it is easy to check that

$$\mu_N(t)(+1, d\lambda) = \frac{1 + m_N(t)}{2} \delta_{\lambda_N(t)}(d\lambda), \quad \mu_N(t)(-1, d\lambda) = \frac{1 - m_N(t)}{2} \delta_{\lambda_N(t)}(d\lambda).$$

Therefore, the two-dimensional process $(m_N(t), \lambda_N(t))_{t \in [0, T]}$ provides a sufficient statistics for the dynamics of the empirical measure flow: in fact, as proven in Lemma 1.2.2, $(m_N(t), \lambda_N(t))_{t \in [0, T]}$ constitutes a so-called *order parameter* of the system.

Definition 1.2.3. Given a measurable function $f : \mathcal{S} \times \mathbb{R} \rightarrow \mathbb{R}$, we call an **empirical average** the quantity

$$\int_{\mathcal{S} \times \mathbb{R}} f d\mu_N(t) = \frac{1}{N} \sum_{j=1}^N f(\sigma_j(t), \lambda(t)).$$

An **order parameter** is a flow of empirical averages whose dynamics is Markovian.

Lemma 1.2.1. Let $(X_t)_{t \geq 0}$ be a Markov process on a metric space E admitting an infinitesimal generator L . Let $g : E \rightarrow F$ be a function, with F metric space. Assume that, for every $f : F \rightarrow \mathbb{R}$ such that $(f \circ g) \in \text{dom}(L)$, $L(f \circ g)$ is a function of $g(x)$, i.e. $L(f \circ g) = (Kf) \circ g$. Then, $(g(X_t))_{t \geq 0}$ is a Markov process with infinitesimal generator K , defined by $L(f \circ g) = (Kf) \circ g$.

Proof. It is enough to notice that the following equalities hold:

$$\lim_{t \downarrow 0} \frac{E[f(g(X_t^x))] - f(g(x))}{t} = \lim_{t \downarrow 0} \frac{E[(f \circ g)(X_t^x)] - (f \circ g)(x)}{t} = L(f \circ g)(x) = (Kf) \circ g(x).$$

□

Lemma 1.2.2. The stochastic process $(m_N(t), \lambda_N(t))_{t \in [0, T]}$ with values in $[-1, 1] \times \mathbb{R}$ is an order parameter for the model described by (1.2.2).

Proof. Notice that $(m_N(t), \lambda_N(t))$ is an empirical average in the sense defined above since

$$m_N(t) = \int_{\mathcal{S} \times \mathbb{R}} \sigma d\mu_N(t), \quad \lambda_N(t) = \int_{\mathcal{S} \times \mathbb{R}} \lambda d\mu_N(t).$$

So we are left to prove that $(m_N(t), \lambda_N(t))$ is a Markov process: it is enough to compute its infinitesimal generator K_N . Consider the function $g : \mathcal{S}^N \times \mathbb{R} \rightarrow [-1, +1] \times \mathbb{R}$ defined by

$$g(\underline{\sigma}, \lambda) = \left(\frac{1}{N} \sum_{j=1}^N \sigma_j, \lambda \right) =: (m, \lambda).$$

Take $f : [-1, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ such that $(f \circ g) \in \text{dom}(L_N)$, and compute $L_N(f \circ g)$:

$$\begin{aligned} L_N(f \circ g)(\underline{\sigma}, \lambda) &= \\ & \sum_{i=1}^N \left[\left(1 - \tanh(\sigma_i \lambda)\right) \left(f \left(m - \frac{2\sigma_i}{N}, \lambda - \frac{2\beta\sigma_i}{N} \right) - f(m, \lambda) \right) \right] - \alpha \lambda f_\lambda(m, \lambda) = \\ & \sum_{j \in \mathcal{S}} \left[|A_N(j)| \left(1 - \tanh(j\lambda)\right) \left(f \left(m - \frac{2j}{N}, \lambda - \frac{2\beta j}{N} \right) - f(m, \lambda) \right) \right] - \alpha \lambda f_\lambda(m, \lambda) \end{aligned}$$

where $A_N(j)$ is the set of σ_i , $i = 1, \dots, N$, such that $\sigma_i = j$ with $j \in \mathcal{S}$. Then, we have

$$|A_N(j)| = \frac{N \left(1 + j \frac{\sum_{k=1}^N \sigma_k}{N} \right)}{2} = \frac{N(1 + jm)}{2}. \quad (1.2.11)$$

Therefore, thanks to Lemma 1.2.1, $(m_N(t), \lambda_N(t))$ is a Markov process with infinitesimal generator K_N given by:

$$K_N f(m, \lambda) = \sum_{j \in \mathcal{S}} \left[|A_N(j)| (1 - \tanh(j\lambda)) \left(f \left(m - \frac{2j}{N}, \lambda - \frac{2\beta j}{N} \right) - f(m, \lambda) \right) \right] - \alpha \lambda f_\lambda(m, \lambda). \quad (1.2.12)$$

□

The convergence of the order parameter $(m_N(t), \lambda_N(t))_{t \in [0, T]}$ can be proved using an approach based on the uniform convergence of infinitesimal generators: consider the following technical lemma stated in [11] whose proof can be found in [29], Chapter 4, Corollary 8.7.

Lemma 1.2.3. *Let $\{X_n(t)\}_n$ be a sequence of Markov processes with values in \mathcal{X}_n and denote by \mathcal{L}_n the corresponding infinitesimal generators, defined on $\mathcal{D}(L_n)$. Moreover, let \mathcal{L} , defined on $\mathcal{D}(\mathcal{L})$, be the infinitesimal generator of another Markov process $X(t)$ with values on \mathcal{X} , and let \mathcal{C} be a core for \mathcal{L} . Assume that, for every n , $\mathcal{X}_n \subseteq \mathcal{X}$ and each function in \mathcal{C} is an element of $\mathcal{D}(\mathcal{L}_n)$, when restricted to \mathcal{X}_n . If the condition*

$$\lim_{n \rightarrow +\infty} \sup_{x \in \mathcal{X}_n} |\mathcal{L}_n(f(x)) - \mathcal{L}(f(x))| = 0$$

holds for every $f \in \mathcal{C}$ and $X_n(0)$ converges to $X(0)$ in distribution, then the sequence of processes $\{X_n(t)\}_n$ converges to the process $X(t)$ in distribution as $n \rightarrow +\infty$.

Theorem 1.2.2. *The order parameter $(m_N(t), \lambda_N(t))_{t \in [0, T]}$ converges, in sense of weak convergence of stochastic processes, to the solution of the system of ordinary differential equations*

$$\begin{cases} \dot{m}(t) = 2(\tanh(\lambda(t)) - m(t)), \\ \dot{\lambda}(t) = 2\beta(\tanh(\lambda(t)) - m(t)) - \alpha\lambda(t), \end{cases} \quad (1.2.13)$$

with initial conditions $m(0) = E_{\nu_0}[\sigma]$, $\lambda(0) = \lambda_0$, where ν_0 and λ_0 are defined as (1.2.7).

Proof. The fact that $(m_N(0), \lambda_N(0)) \rightarrow (m_0, \lambda_0)$ is immediate from the assumption on the initial conditions (1.2.7), since

$$m_N(0) = \frac{1}{N} \sum_{i=1}^N \sigma_i(0).$$

We are left to prove the uniform convergence of infinitesimal generators: let K_N be the infinitesimal generator of $(m_N(t), \lambda_N(t))$ (see (1.2.12)) and let $f \in \mathcal{C}_b^2$, then it is sufficient to perform a first-order Taylor expansion for f around (m, λ) to check that

$$K_N f(m, \lambda) = 2(\tanh(\lambda) - m) f_m(m, \lambda) + (2(\tanh(\lambda) - m) - \alpha\lambda) f_\lambda(m, \lambda) + o(1)$$

where $o(1)$ denotes a quantity converging uniformly to 0. Since the infinitesimal generator \tilde{K}_N of the solution of (1.2.13) reads

$$\tilde{K}(m, \lambda) = 2(\tanh(\lambda) - m) f_m(m, \lambda) + (2(\tanh(\lambda) - m) - \alpha\lambda) f_\lambda(m, \lambda)$$

the proof is concluded by Lemma 1.2.3. \square

By Theorem 1.2.2 and previous considerations, we conclude that the limiting dynamics for our model is represented by a stochastic process $(\Sigma(t), \lambda(t))_{t \in [0, T]}$ with values in $\mathcal{S} \times \mathbb{R}$ whose time-marginal distributions correspond to the measure flow $(\mu(t))_{t \in [0, T]}$ such that, for any $t \in [0, T]$,

$$\mu(t)(+1, d\lambda) = \frac{1 + m(t)}{2} \delta_{\lambda(t)}(d\lambda), \quad \mu(t)(-1, d\lambda) = \frac{1 - m(t)}{2} \delta_{\lambda(t)}(d\lambda),$$

with $(m(t), \lambda(t))_{t \in [0, T]}$ defined by (1.2.13). The process $(\Sigma(t), \lambda(t))$ is a *nonlinear Markov process* ([38]) where $\Sigma(t)$ flips to $-\Sigma(t)$ with rate $1 - \tanh(\Sigma(t)\lambda(t))$ and $\lambda(t)$ evolves according to (1.2.13). The nonlinearity of this process lies in the fact that its dynamics depends on the law of the process itself, since $(m(t))_{t \in [0, T]} \equiv (E[\Sigma(t)])_{t \in [0, T]}$.

1.2.3 Long time behavior and self-sustained rhythm

We now analyse the long time behavior of the macroscopic system by studying in details (1.2.13).

Let us start with the case without dissipation ($\alpha = 0$): assume for simplicity that $\lambda_0 = \beta m_0$, so that $(\lambda(t))_{t \in [0, T]} \equiv (\beta m(t))_{t \in [0, T]}$. In this case, (1.2.13) reduces to

$$\dot{m}(t) = 2(\tanh(\beta m(t)) - m(t)). \quad (1.2.14)$$

Check the following facts:

- if $\beta \leq 1$, 0 is the unique equilibrium solution of (1.2.14) and it is globally asymptotically stable: for any $m(0) \in [-1, 1]$,

$$\lim_{t \rightarrow +\infty} m(t) = 0.$$

- if $\beta > 1$, 0 is an unstable equilibrium while $+m_\beta^*$ and $-m_\beta^*$ are locally stable equilibria with m_β^* being the unique positive solution to the equation

$$m = \tanh(\beta m).$$

In this case, $[-1, 1]$ is bi-partitioned by 0 in two domains of attraction:

$$\lim_{t \rightarrow +\infty} m(t) = \begin{cases} +m_\beta^* & \text{if } m(0) > 0, \\ -m_\beta^* & \text{if } m(0) < 0, \\ 0 & \text{if } m(0) = 0. \end{cases}$$

The picture described above is coherent with the classical results on the macroscopic behavior the Curie-Weiss model: the system presents a *phase transition* for $\beta = 1$ and, for temperature low enough ($\beta > 1$), it polarises to an equilibrium (positive or negative depending on the initial condition). It constitutes a form of self-organization for which, at macroscopic level, the system reaches an equilibrium at which the majority of the spins is aligned with each other (in other words, the system magnetizes).

In the case with dissipation ($\alpha > 0$) this scheme changes dramatically: the system still presents a phase transition but, at temperature low enough, instead of reaching a stable polarized equilibrium as above, it presents a periodic behavior, as stated in the following theorem (see [19] for the proof).

Theorem 1.2.3. *Fix $\alpha > 0$. For any $\beta > 0$, the origin is the unique fixed point of (1.2.13). Moreover:*

- *for $\beta \leq \frac{\alpha}{2} + 1$ the origin is a global attractor for (1.2.13);*
- *for $\beta = \frac{\alpha}{2} + 1$, (1.2.13) presents a Hopf bifurcation;*
- *for $\beta > \frac{\alpha}{2} + 1$, the system (1.2.13) has a unique periodic orbit, which attracts all trajectories except the fixed point.*

Clearly the presence of dissipation has a deep impact on the macroscopic behavior of the system: since the effect of dissipation is to dump the influence of interaction when no transition in the magnetization occurs for an interval of time, the system can no longer relax to a polarized equilibrium. Nevertheless, for the temperature low enough, the particles are still capable to self-organize by producing a regular macroscopic pattern: a stable periodic oscillations of the macroscopic law. Since neither the particles have a natural tendency to behave periodically nor periodic forces are applied to the system, we may refer to this phenomenon as *self-sustained periodic behavior*, or *collective periodic behavior*.

1.3 A diffusive model with dissipation

We consider here a generalization of the model introduced in Section 1.2, starting from a reversible diffusion for the *generalized* Curie-Weiss model and modifying it by introducing dissipation.

1.3.1 The generalized Curie-Weiss model

The classical Curie-Weiss model, in the original formulation (see for example [25]), is a model meant to explain spontaneous magnetization from a statical point of view. In fact, it is defined as a sequence of Gibbs probability measures $P_{N,\beta}$ on the sets of N -spins configurations \mathcal{S}^N by

$$P_{N,\beta}(\underline{\sigma}) = \frac{1}{Z_N(\beta)} \exp[-\beta H_N(\underline{\sigma})], \quad (1.3.1)$$

where $Z_N(\beta)$ is a normalizing constant and H_N is the *Hamiltonian* function

$$H_N(\underline{\sigma}) = -\frac{1}{2N} \left(\sum_{i=1}^N \sigma_i \right)^2. \quad (1.3.2)$$

Notice that when there is no interaction ($\beta = 0$), each single site is distributed as a Bernoulli variable on \mathcal{S} . The quantity $H_N(\underline{\sigma})$ represents the *energy* of a given configuration $\underline{\sigma}$ and the measure $P_{N,\beta}$ gives higher probability to the configurations with minimal energy, but the macroscopic distribution depends on the *entropy-energy balance*. Indeed, under $P_{N,\beta}$, the Law of Large Number presents a phase transition for $\beta = 1$:

$$\text{Law} \left(\frac{1}{N} \sum_{i=1}^N \sigma_i \right) \xrightarrow[N \uparrow +\infty]{w} \begin{cases} \delta_0 & \beta \leq 1, \\ \frac{1}{2} (\delta_{+m(\beta)} + \delta_{-m(\beta)}) & \beta > 1 \end{cases}$$

where $m(\cdot)$ is an increasing positive function $m :]1, +\infty[\rightarrow]0, 1[$ called *spontaneous magnetization*.

A natural extension of this model (introduced in [26] and further analysed in [24],[27]) is obtained by replacing both the quadratic interaction function in (1.3.2) and the Bernoulli single-site distribution by more general terms. However, to define a generalized Curie-Weiss model one has to choose an interaction function g and a probability measure ρ , which have to satisfy the following hypotheses.

Assumptions 1.3.1. (i) The interaction function $g : \mathbb{R} \rightarrow \mathbb{R}_+$ is even, it belongs to $\mathcal{C}^2(\mathbb{R})$ and it is strictly increasing on $[0, +\infty[$, with $g(0) = 0$. It also satisfies the following *two-sided real analyticity* condition: for all $x \in \mathbb{R}$, there exist $\delta > 0$ and two real analytic functions g_1 and g_2 such that

$$g \equiv \begin{cases} g_1 & \text{on }]x - \delta, x], \\ g_2 & \text{on } [x, x + \delta[. \end{cases}$$

(ii) ρ is a symmetric Borel probability measure on \mathbb{R} that is non-degenerate (i.e., $\rho \neq \delta_0$).

(iii) Let $M = \sup\{x \mid x \text{ is in the support of } \rho\}$. Then there exists a symmetric, non-constant, convex function $h : [-M, M] \rightarrow \mathbb{R}_+$ such that $g(x) \leq h(x)$ for any $x \in [-M, M]$ and

$$\int_{-M}^M \exp[\alpha h(x)] \rho(dx) < +\infty, \quad \forall \alpha > 0.$$

Remark 1.3.1. Assumptions 1.3.1 are both of modelling and technical nature. The shape of the function g (evenness, strict increase, $g(0) = 0$) and the requirement of symmetry for ρ are meant to obtain a well-defined generalization of the classical Curie-Weiss model (for which $g(x) = x^2$ and $\rho = \frac{1}{2}(\delta_{-1} + \delta_{+1})$) capable to effectively describe the phenomenon of spontaneous magnetization. The assumptions on the regularity of g are required by the hypothesis of Theorem 1.3.1 and will come on hand also for the dynamical approach in the next subsection. Finally, condition (iii) ensures the integrability of the Gibbs measure (1.3.3).

Under Assumptions 1.3.1, the generalized Curie-Weiss model is defined as a sequence of probability measures $P_{N,\beta}$ on \mathbb{R}^N by

$$P_{N,\beta}(dx_1, \dots, dx_N) = \frac{1}{Z_N(\beta)} \exp \left[\beta N g \left(\frac{1}{N} \sum_{i=1}^N x_i \right) \right] \prod_{i=1}^N \rho(dx_i), \quad (1.3.3)$$

where $Z_N(\beta)$ is again a normalizing constant. In this case, in absence of interactions ($\beta = 0$), the particles constitute a family of i.i.d. random variables on \mathbb{R} distributed according to ρ . Notice that by setting

$$g(x) = \frac{1}{2}x^2, \quad \rho(dx) = \frac{1}{2} (\delta_{+1}(dx) + \delta_{-1}(dx))$$

we recover the classical Curie-Weiss model (1.3.1).

As one would expect, the asymptotic behavior of the generalized Curie-Weiss model, which still depends on the entropy-energy balance, presents a richer picture than the classical one, as stated in the following theorem. We refer to [24] for the proof and other details.

Theorem 1.3.1. *Let Assumptions 1.3.1 hold. Then there exists a non-empty set \mathcal{P} of positive positive points $\{\beta_i\}$, representing the critical values, which are either finite in number ($0 < \beta_1 < \beta_2 < \dots < \beta_n$ for some $n \in \mathbb{N}$) or countably infinite ($0 < \beta_1 < \beta_2 < \dots$) and divergent to $+\infty$. The set \mathcal{P} has the following properties.*

(i) *There exist a function $m :]0, +\infty[\setminus \mathcal{P} \rightarrow [0, M[$, representing the spontaneous magnetization, such that $m(\beta) = 0$ for $0 < \beta < \beta_1$, while, for $\beta \in]\beta_1, +\infty[\setminus \mathcal{P}$, $m(\beta)$ is strictly positive and strictly increasing. The function m is real analytic on each connected sub-interval of $]0, +\infty[\setminus \mathcal{P}$, but cannot be represented as the restriction of one real analytic function in any neighbourhood of a critical value.*

(ii) *Let $i : \mathbb{R} \rightarrow \mathbb{R}_+$ be the entropy function of ρ , i.e.,*

$$i(u) := \sup_{t \in \mathbb{R}} \left\{ tu - \log \int_{\mathbb{R}} \exp[tx] \rho(dx) \right\}.$$

Then, the supremum

$$\sup_{u \in \mathbb{R}} \{ \beta g(u) - i(u) \}$$

is attained at the unique point $u = 0$, while, for $\beta \in]\beta_1, +\infty[\setminus \mathcal{P}$, it is attained at the unique points $u = m(\beta)$ and $u = -m(\beta)$.

(iii) *For any $N \geq 1$, let (X_1, \dots, X_N) be random variables distributed according to $P_{N,\beta}$ given in (1.3.3). Then,*

$$\text{Law} \left(\frac{1}{N} \sum_{i=1}^N X_i \right) \xrightarrow[N \uparrow +\infty]{w} \begin{cases} \delta_0 & \beta \leq \beta_1, \\ \frac{1}{2} (\delta_{+m(\beta)} + \delta_{-m(\beta)}) & \beta \in]\beta_1, +\infty[\setminus \mathcal{P}. \end{cases}$$

1.3.2 A generalized Curie-Weiss model with dissipation

We now consider a Langevin dynamics for the generalized Curie-Weiss model, then we will break its reversibility by introducing dissipation in the same fashion of Section 1.2.

Let ν be a probability measure on \mathbb{R}^d with full support, admitting f as density function. If f is regular enough, one can define the *Langevin diffusion* associated to ν as the stochastic process $x(t)$ on \mathbb{R}^d solution of

$$dx(t) = \frac{1}{2} \nabla \log(f(x(t))) + dB(t) \quad (1.3.4)$$

with $B(t)$ being a d -dimensional Brownian motion. This construction assures that, under suitable conditions, $x(t)$ has μ as unique invariant measure and its time-marginal laws converge to μ in total variation.

Theorem 1.3.2 ([35]). *Let ν be a probability measure with full support on \mathbb{R}^d , $d \geq 1$, admitting a density function f such that $\log(f(x)) \in \mathcal{C}^2(\mathbb{R})$ and*

$$x^\top \nabla \log f(x) \leq C(1 + \|x\|^2) \quad \forall x \in \mathbb{R}^d$$

for a certain $C > 0$. If ξ is a random variable in \mathbb{R}^d with $E[\xi^2] < +\infty$, then there exists a unique strong solution to the stochastic differential equation (1.3.4) with initial condition $x(0) = \xi$. Moreover $(x(t))_{t \geq 0}$ admits ν as unique invariant distribution and, for any $x_0 \in \mathbb{R}^d$,

$$\lim_{t \rightarrow +\infty} \sup_{A \in \mathcal{B}} \left| P(x(t) \in A \mid x(0) = x_0) - \nu(A) \right| = 0,$$

with \mathcal{B} being the Borelian σ -field on \mathbb{R}^d .

Consider now the following set of hypotheses which, together with Assumptions 1.3.1 allows us to define a well-posed Langevin dynamics for the generalized Curie-Weiss model and to prove the results on the limiting dynamics stated in the following.

Assumptions 1.3.2. (i) The derivative of the interaction function g' is uniformly Lipschitz continuous, i.e. there exists a finite constant $L > 0$ such that, for all $x, y \in \mathbb{R}$,

$$|g'(x) - g'(y)| \leq L|x - y|.$$

(ii) The single-site distribution ρ has full support on \mathbb{R} and it admits a density function that we will denote $\rho(x)$ with an abuse of notation. The density function $\rho(x)$ is such that $\log \rho(x) \in \mathcal{C}^2(\mathbb{R})$ and there exists a constant $K > 0$ s.t. for all $x, y \in \mathbb{R}$

$$(x - y) \left(\frac{\rho'(x)}{\rho(x)} - \frac{\rho'(y)}{\rho(y)} \right) \leq K(1 + (x - y)^2). \quad (1.3.5)$$

Remark 1.3.2. Assumptions 1.3.2 are of pure technical nature and, together with Assumptions 1.3.1, are very useful to prove well-posedness of the SDEs describing the N -particle system and the nonlinear macroscopic process. They will also come on hand in proving propagation of chaos (Theorem 1.3.4).

Under Assumptions 1.3.1 and 1.3.2, for any $N \geq 1$, we define the N -particles Langevin diffusion for the generalized Curie-Weiss model with interaction function g and single-site distribution ρ the stochastic process X^N with values in \mathbb{R}^N , unique strong solution of the systems of SDEs

$$dX_i^N(t) = \frac{\beta}{2} g' \left(\frac{\sum_{i=1}^N X_i^N(t)}{N} \right) dt + \frac{\rho'(X_i^N(t))}{2\rho(X_i^N(t))} + dB_i(t), \quad i = 1, \dots, N \quad (1.3.6)$$

with (B^1, \dots, B^N) independent one-dimensional Brownian motion. By Theorem 1.3.2, the measure $P_{N,\beta}$ defined in (1.3.3) is the unique invariant measure for X^N . Equation (1.3.6) describes a mean field system of interacting particles, in which two competing effects are present: the interaction, which tries to freeze the system in the states of minimal energy, and the tendency of each particle to be distributed according to ρ .

We now introduce a dissipation in the interaction energy of (1.3.6): since the interaction depend only on the magnetization

$$m^N(t) = \frac{1}{N} \sum_{i=1}^N X_i^N(t),$$

the straightforward approach would suggest to define a process $\lambda^N(t)$ with values in \mathbb{R} evolving according to

$$d\lambda^N(t) = -\alpha \lambda^N(t) dt + dm^N(t)$$

with $\alpha > 0$ and to consider the system

$$dX_i^N(t) = \frac{\beta}{2} g'(\lambda^N(t)) dt + \frac{\rho'(X_i^N(t))}{2\rho(X_i^N(t))} + dB_i(t), \quad i = 1, \dots, N.$$

We can also generalize this picture by allowing the presence of a Brownian noise in the interaction for each site: in fact, by *generalized Curie-Weiss model with dissipation* we will indicate the process (X^N, λ^N) with values in \mathbb{R}^{2N} solution of the systems of SDEs

$$\begin{cases} dX_i^N(t) &= \frac{\beta}{2} g'(\lambda_i^N(t)) dt + \frac{\rho'(X_i^N(t))}{2\rho(X_i^N(t))} + dB_i^1(t), \\ d\lambda_i^N(t) &= -\alpha \lambda_i^N(t) dt + dm^N(t) + D dB_i^2(t), \end{cases} \quad (1.3.7)$$

for $i = 1, \dots, N$, with $\{B_i^1, B_i^2\}_{i=1}^N$ being a family of two-dimensional independent Brownian motions and $\alpha, D \geq 0$. By definition of magnetization, the term $dm^N(t)$ in (1.3.7) is given by

$$dm^N(t) = \frac{1}{N} \sum_{j=1}^N dX_j^N(t) = \frac{1}{N} \sum_{j=1}^N \left(\frac{\beta}{2} g'(\lambda_j^N(t)) dt + \frac{\rho'(X_j^N(t))}{2\rho(X_j^N(t))} \right) dt + \frac{1}{N} \sum_{j=1}^N dB_j^1(t).$$

Notice that under Assumption 1.3.1 and 1.3.2, by classical results on existence and uniqueness of solution of SDEs (see for example Chapter 5 in [29]), (1.3.7) is well-posed.

Remark 1.3.3. Diffusive mean field models with cooperative interaction subject to dissipation have also been discussed in [14]: here the authors present a general approach to define cooperative models in which the interaction potential is subject to its own dissipative (and possibly diffusive) evolution and they focus on the study of the particle system $(Y_1^N, \dots, Y_N^N, \lambda^N) \in \mathbb{R}^{N+1}$ solution of

$$\begin{cases} dY_i^N(t) &= (-Y_i^N(t))^3 + Y_i^N(t) - \lambda^N(t) dt + \sigma dB_i(t), & i = 1, \dots, N, \\ d\lambda^N(t) &= -\alpha\lambda^N(t) - \theta dm^N(t) \end{cases} \quad (1.3.8)$$

with $m^N(t) = \frac{1}{N} \sum_{j=1}^N Y_j^N(t)$ and $\alpha, \theta, \sigma \geq 0$. The process described by (1.3.8) is obtained introducing dissipation in the cooperative diffusion model studied in [22]. Notice that (1.3.8) can be obtained as particular case of (1.3.7): it is enough to choose $\beta = \theta$, $D = 0$, $g(x) = x^2$,

$$\rho(x) = \frac{1}{Z} \exp \left[\frac{x^2}{2} \left(1 - \frac{x^2\sigma^2}{2} \right) \right] \quad \text{with } Z \text{ normalizing constant}$$

in (1.3.7) and perform the change of variables $Y = \sigma X$ to obtain (1.3.8).

1.3.3 Infinite volume dynamics dynamics

We now study the infinite volume dynamics of the generalized Curie-Weiss model with dissipation on a finite time interval $[0, T]$.

For any $N \geq 1$, fix $T > 0$ and let $(X^N(t), \lambda^N(t))_{t \in [0, T]}$ be the stochastic process strong solution of (1.3.7) with initial condition

$$\text{Law}(X^N(0), \lambda^N(0)) = \mu_0^{\otimes N} \quad (1.3.9)$$

where μ_0 is square-integrable probability measure on \mathbb{R}^2 , i.e.

$$\int_{\mathbb{R}^2} (x^2 + \lambda^2) \mu_0(dx, d\lambda) < +\infty.$$

This process represents a system of mean field interacting particle with trajectories in $\mathcal{C}([0, T], \mathbb{R}^{2N})$, the set of continuous trajectories from $[0, T]$ to \mathbb{R}^{2N} provided of the uniform metric. Let $(X^N[0, T], \lambda^N[0, T])$ denote a path of the system on the time interval $[0, T]$. Consider the empirical measure

$$\mu_N = \frac{1}{N} \sum_{j=1}^N \delta_{\{X_j^N[0, T], \lambda_j^N[0, T]\}}. \quad (1.3.10)$$

which is a random variable with values in $\mathcal{M}_1(\mathcal{C}([0, T], \mathbb{R}^2))$. We aim to prove a Law of Large Number in the form (1.2.9) with the limit representing the law on $\mathcal{C}([0, T], \mathbb{R}^2)$ of the limiting process. Recall that it is equivalent to a propagation of chaos result (Theorem 1.2.1).

First of all, let us identify the macroscopic process. Notice that (1.3.7) can be written in the following way:

$$\begin{cases} dX_i^N(t) &= \frac{\beta}{2} g'(\lambda_i^N(t)) dt + \frac{\rho'(X_i^N(t))}{2\rho(X_i^N(t))} + dB_i^1(t), \\ d\lambda_i^N(t) &= -\alpha \lambda_i^N(t) dt + \left\langle \mu_i^N(dx, dl), \frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right\rangle dt + DdB_i^2(t) + \frac{1}{N} \sum_{j=1}^N dB_j^1(t), \end{cases}$$

with

$$\left\langle \mu_i^N(dx, dl), \frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right\rangle = \int_{\mathbb{R}^2} \left(\frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right) \mu_i^N(dx, dl)$$

and μ_t^N representing the marginal distribution of the empirical measure μ^N at time t . By a simple heuristic argument, one can see that the macroscopic dynamics should be represented by the nonlinear Markov process (X, λ) with values in \mathbb{R}^2 , solution of the following *McKean-Vlasov* equation

$$\begin{cases} dX(t) &= \frac{\beta}{2} g'(\lambda(t)) dt + \frac{\rho'(X(t))}{2\rho(X(t))} + dB^1(t), \\ d\lambda(t) &= -\alpha \lambda(t) dt + \left\langle \mu_t(dx, dl), \frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right\rangle dt + DdB^2(t), \\ \mu_t &= \text{Law}(X(t), \lambda(t)), \end{cases} \quad (1.3.11)$$

with (B^1, B^2) being a two-dimensional Brownian motion. Assumption 1.3.1 and 1.3.2 are not sufficient to guarantee global Lipschitz properties of the coefficients of (1.3.11), which would imply its well-posedness by standard theorems. We will prove both well-posedness of (1.3.11) and propagation of chaos by a *pathwise approach* (see [1] for a complete overview on pathwise propagation of chaos in a general framework).

Definition 1.3.1. Let (M, d) be a metric space and $\mathcal{M}_p(M)$ be the space of probability on M with finite p^{th} moment, for $p \geq 1$:

$$\mathcal{M}_p(M) = \left\{ \mu \in \mathcal{M}(M) : \int d(x, x_0)^p \mu(dx) < +\infty \text{ for some } x_0 \in M \right\}.$$

We define the \mathcal{W}_p Wasserstein metric on $\mathcal{M}_p(M)$ as follows: for all $\mu, \nu \in \mathcal{M}_p(M)$

$$\mathcal{W}_p(\mu, \nu) = \left[\inf_{\gamma \in \Gamma(\mu, \nu)} \left\{ \int_{M \times M} d(x, y)^p \gamma(dx, dy) \right\} \right]^{1/p},$$

with $\Gamma(\mu, \nu)$ is the set of all possible couplings of μ and ν , i.e. the set of all probability measures on $M \times M$ with μ and ν as marginals.

Theorem 1.3.3. *The nonlinear process (1.3.11) is well-defined, i.e. there exists a unique strong solution for all square-integrable initial condition $(X_0, \lambda_0) \in \mathbb{R}^2$.*

Proof. Given any square-integrable law μ_0 on \mathbb{R}^2 , we define a map F that associates to a measure $Q \in \mathcal{M}_2(\mathcal{C}([0, T], \mathbb{R}^2))$ the law of the solution $\{(X(t), \lambda(t))\}_{t \in [0, T]}$ of the SDE

$$\begin{cases} dX(t) = \frac{\beta}{2} g'(\lambda(t)) dt + \frac{\rho'(X(t))}{2\rho(X(t))} dt + dB^1(t) \\ d\lambda(t) = -\alpha \lambda(t) dt + \left\langle Q_t(dx, dl), \frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right\rangle dt + DdB^2(t), \end{cases}$$

that, for μ_0 initial condition, admits a unique strong solution for classical results (see [36]); of course a solution to (1.3.11) is a fixed point of F . We use a coupling argument to prove existence (via a Picard iteration) and uniqueness of the fixed point of F .

Let us start with the proof of uniqueness: let Q^1 and Q^2 be two fixed point of F , i.e. two measures in $\mathcal{M}_2(\mathcal{C}([0, T], \mathbb{R}^2))$ such that $Q^1 = F(Q^1)$ and $Q^2 = F(Q^2)$. We couple them as follows: let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$ be a filtered probability space and $\{B(t)\}_{t \in [0, T]}$ a two-dimensional Brownian motion, then we write

$$\begin{cases} dX^1(t) = \frac{\beta}{2} g'(\lambda^1(t)) dt + \frac{\rho'(X^1(t))}{2\rho(X^1(t))} dt + dB^1(t) \\ d\lambda^1(t) = -\alpha \lambda^1(t) dt + \left\langle Q_t^1(dx, dl), \frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right\rangle dt + DdB^2(t), \end{cases}$$

and

$$\begin{cases} dX^2(t) = \frac{\beta}{2} g'(\lambda^2(t)) dt + \frac{\rho'(X^2(t))}{2\rho(X^2(t))} dt + dB^1(t) \\ d\lambda^2(t) = -\alpha \lambda^2(t) dt + \left\langle Q_t^2(dx, dl), \frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right\rangle dt + DdB^2(t), \end{cases}$$

where the initial conditions are $(X_0^1, \lambda_0^1) = (X_0^2, \lambda_0^2)$ a.s., μ_0 distributed. Notice that on $\mathcal{M}_2(\mathcal{C}([0, T], \mathbb{R}^2))$ the \mathcal{W}_2 distance reads

$$\mathcal{W}_2(\mu, \nu) = \left[\inf_{\gamma \in \Gamma(\mu, \nu)} \left\{ \int_{\mathcal{C} \times \mathcal{C}} \|x - y\|_\infty^2 \gamma(dx, dy) \right\} \right]^{\frac{1}{2}},$$

and the coupling described above is an element of $\Gamma(Q^1, Q^2)$, then we get the following estimate:

$$\mathcal{W}_2(Q^1, Q^2) \leq \sqrt{E \left[\sup_{t \in [0, T]} (X^1(t) - X^2(t))^2 + (\lambda^1(t) - \lambda^2(t))^2 \right]}.$$

The SDE for λ^1 and λ^2 is linear, then we write explicitly

$$\lambda^1(t) - \lambda^2(t) = \int_0^t e^{\alpha(s-t)} \left\langle Q_s^1(dx, dl) - Q_s^2(dx, dl), \frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right\rangle ds.$$

Notice that $\left\langle Q_t^1(dx, dl) - Q_t^2(dx, dl), \frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right\rangle = \frac{d}{dt} E [X^1(t) - X^2(t)]$, that gives

$$\lambda^1(t) - \lambda^2(t) = E [X^1(t) - X^2(t)] - \alpha \int_0^t E [X^1(s) - X^2(s)] e^{-\alpha(t-s)} ds.$$

On the other hand, we use Ito's formula to obtain

$$\begin{aligned} (X^1(t) - X^2(t))^2 &= 2 \int_0^t (X^1(s) - X^2(s)) \left(\frac{\beta}{2} g'(\lambda^1(s)) - \frac{\beta}{2} g'(\lambda^2(s)) \right) ds \\ &\quad + 2 \int_0^t (X^1(s) - X^2(s)) \left(\frac{\rho'(X^1(s))}{2\rho(X^1(s))} - \frac{\rho'(X^2(s))}{2\rho(X^2(s))} \right) ds. \end{aligned}$$

Therefore, using Assumptions 1.3.2, there exists C_T such that

$$\begin{aligned} E \left[\sup_{t \in [0, T]} (X^1(t) - X^2(t))^2 + (\lambda^1(t) - \lambda^2(t))^2 \right] &\leq \\ C_T \int_0^T E \left[\sup_{t \in [0, s]} (X^1(t) - X^2(t))^2 + (\lambda^1(t) - \lambda^2(t))^2 \right] &ds, \end{aligned}$$

and by Gronwall Lemma this gives $\mathcal{W}_2(Q^1, Q^2) = 0$.

Concerning existence, with a Picard iteration of the type $Q^n = F(Q^{n-1})$ and with the above arguments, we obtain that

$$\begin{aligned} E \left[\sup_{t \in [0, T]} (X^n(t) - X^{n-1}(t))^2 + (\lambda^n(t) - \lambda^{n-1}(t))^2 \right] &\leq \\ L \int_0^T E \left[\sup_{t \in [0, s]} (X^n(t) - X^{n-1}(t))^2 + (\lambda^n(t) - \lambda^{n-1}(t))^2 \right] &ds \\ &\quad + \alpha^2 T \int_0^T \mathcal{W}_{2,s}(Q^{n-1}, Q^{n-2})^2 ds, \end{aligned}$$

where $\mathcal{W}_{2,s}(Q^{n-1}, Q^{n-2})$ represents the \mathcal{W}_2 distance of the restrictions of Q^{n-1}, Q^{n-2} to $\mathcal{C}([0, s], \mathbb{R}^2)$. Estimation above gives

$$\mathcal{W}_2(Q^n, Q^{n-1})^2 \leq \frac{(e^{LT} T \alpha^2)^n}{n!} \int_0^T \mathcal{W}_{2,s}(Q^1, Q^0)^2 ds,$$

so $\{Q^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence for \mathcal{W}_2 and therefore for a weaker, but complete, metric on $\mathcal{M}_2(\mathcal{C}([0, T], \mathbb{R}^2))$. \square

Theorem 1.3.4. *Let $(X^N(t), \lambda^N(t))_{t \geq 0}$ be the Markov process solution to (1.3.7) with initial conditions (1.3.9) and denote with P^N its law on $\mathbf{C}([0, T], \mathbb{R}^{2N})$. Let $(X(t), \lambda(t))_{t \geq 0}$ be the solution to (1.3.11) with initial condition μ_0 , and denote with μ its law on $\mathbf{C}([0, T], \mathbb{R}^2)$. Then, the sequence $(P^N)_{N \in \mathbb{N}}$ is μ -chaotic.*

Proof. By Theorem 1.2.1, it is enough to prove $E [\mathcal{W}_2(\mu^N, \mu)] \rightarrow 0$ as $N \rightarrow +\infty$, with μ^N being the empirical measure (1.3.10).

The key idea is to couple the particle system with an intermediate process in which the

limiting law μ is substituted to μ^N in the nonlinear term: on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$, for any $N \in \mathbb{N}$, take a $2N$ -dimensional Brownian motion $\{B(t)\}_{t \in [0, T]}$ and consider the coupled processes given by

$$\begin{cases} dX_i^N(t) &= \frac{\beta}{2} g'(\lambda_i^N(t)) dt + \frac{\rho'(X_i^N(t))}{2\rho(X_i^N(t))} + dB_i^1(t), \\ d\lambda_i^N(t) &= -\alpha \lambda_i^N(t) dt + \left\langle \mu_t^N(dx, dl), \frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right\rangle dt + DdB_i^2(t) + \frac{1}{N} \sum_{j=1}^N dB_j^1(t), \end{cases} \quad (1.3.12)$$

$i = 1, \dots, N$, and

$$\begin{cases} d\bar{X}_i^N(t) &= \frac{\beta}{2} g'(\bar{\lambda}_i^N(t)) dt + \frac{\rho'(\bar{X}_i^N(t))}{2\rho(\bar{X}_i^N(t))} dt + dB_t^{1,i} \\ d\bar{\lambda}_i^N(t) &= -\alpha \bar{\lambda}_i^N(t) dt + \left\langle \mu_t(dx, dl), \frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right\rangle dt + DdB_i^2(t), \end{cases}$$

$i = 1, \dots, N$, where the initial conditions are $(X_i^N(0), \lambda_i^N(0)) = (\bar{X}_i^N(0), \bar{\lambda}_i^N(0))$ a.s., $\mu_0^{\otimes N}$ distributed. Similarly to the proof of Theorem 1.3.3, we write explicitly, for all $i = 1, \dots, N$

$$\lambda_i^N(t) - \bar{\lambda}_i^N(t) = \int_0^t e^{\alpha(s-t)} \left\langle \mu_s^N(dx, dl) - \mu_s(dx, dl), \frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right\rangle ds + \frac{1}{N} \sum_{j=1}^N \int_0^t e^{\alpha(s-t)} dB_j^1(s).$$

However, from (1.3.12), we see that, a.s. it holds

$$\int_0^t e^{\alpha(s-t)} \left\langle \mu_s^N(dx, dl), \frac{\beta}{2} g'(l) + \frac{\rho'(x)}{2\rho(x)} \right\rangle ds = \int_0^t e^{\alpha(s-t)} d\langle \mu_s^N(dx, dl), x \rangle - \frac{1}{N} \sum_{j=1}^N \int_0^t e^{\alpha(s-t)} dB_j^1(s),$$

and, consequently, we write

$$\begin{aligned} \lambda_i^N(t) - \bar{\lambda}_i^N(t) &= \frac{1}{N} \sum_{j=1}^N X_j^N(t) - E[\bar{X}_i^N(t)] - e^{-\alpha t} \left(\frac{1}{N} \sum_{j=1}^N X_j^N(0) - E[\bar{X}_i^N(0)] \right) \\ &\quad - \alpha \int_0^t \left(\frac{1}{N} \sum_{j=1}^N X_j^N(s) - E[\bar{X}_i^N(s)] \right) e^{-\alpha(t-s)} ds. \end{aligned}$$

Now consider the empirical measure of the intermediate process: $\bar{\mu}^N = \frac{1}{N} \sum_{j=1}^N \delta_{(\bar{X}^{N,j}, \bar{\lambda}^{N,j})}$; it is easy to see that, as in the proof of Theorem 1.3.3, it holds

$$\begin{aligned} E[\mathcal{W}_2(\mu^N, \bar{\mu}^N)^2] &\leq E \left[\sup_{t \in [0, T]} (X_1^N(t) - \bar{X}_1^N(t))^2 + (\lambda_1^N(t) - \bar{\lambda}_1^N(t))^2 \right] \\ &\leq L \int_0^T E \left[\sup_{t \in [0, s]} (X_1^N(t) - \bar{X}_1^N(t))^2 + (\lambda_1^N(t) - \bar{\lambda}_1^N(t))^2 \right] ds \\ &\quad + \alpha^2 T \int_0^T E[\mathcal{W}_{2,s}(\mu^N, \mu)^2] ds, \end{aligned}$$

which, by an application of Gronwall's Lemma, implies that there exists $C_T > 0$ such that

$$E [\mathcal{W}_2(\mu^N, \bar{\mu}^N)^2] \leq C_T \int_0^T E [\mathcal{W}_{2,s}(\mu^N, \mu)^2] ds. \quad (1.3.13)$$

Moreover, it is easy to see that $E [\mathcal{W}_2(\bar{\mu}^N, \mu)] \leq \beta(N)$ for some sequence $\beta(N)$ such that $\lim_{N \rightarrow \infty} \beta(N) = 0$. Then, using (1.3.13), we have

$$\begin{aligned} E [\mathcal{W}_2(\mu^N, \mu)^2] &\leq E [\mathcal{W}_2(\mu^N, \bar{\mu}^N)^2] + E [\mathcal{W}_2(\bar{\mu}^N, \mu)^2] \\ &\leq C_T \int_0^T E [\mathcal{W}_{2,s}(\mu^N, \mu)^2] ds + \beta(N) \leq K_T \beta(N) \end{aligned}$$

for some $K_T > 0$, which concludes the proof. \square

1.3.4 The Gaussian case without dissipation

The study of the stability and the long-time behavior of (1.3.11) is particularly hard and we aim to focus on some particular cases. From now on, we choose as single site distribution of spins the Normal distribution with mean zero and variance σ^2 . As a consequence, according to Assumptions 1.3.1, we restrict the choice of the interaction function to the class of functions g such that there exists a symmetric, nonconstant, convex function h on \mathbb{R} with $g(x) \leq h(x)$ for all $x \in \mathbb{R}$ and

$$\int_{\mathbb{R}} e^{ah(x)} e^{-x^2} dx < \infty \text{ for all } a > 0. \quad (1.3.14)$$

Moreover, we will take $D = 0$ and let us consider as initial condition measures of the form

$$\mu_0(dx, d\lambda) = \nu_0(dx) \times \delta_{\lambda_0}(d\lambda),$$

where ν_0 is a square-integrable measure on \mathbb{R} and δ_{λ_0} is a Dirac delta centered in $\lambda_0 \in \mathbb{R}$. These hypotheses drastically simplify the treatment, since they allow for a low-dimensional description of the macroscopic law. In fact, the resulting nonlinear process $(X(t), \lambda(t))_{t \geq 0}$ is solution of the following nonlinear SDE:

$$\begin{cases} dX(t) = \frac{\beta}{2} g'(\lambda(t)) dt - \frac{X(t)}{2\sigma^2} dt + dB(t), \\ \dot{\lambda}(t) = -\alpha \lambda(t) + \frac{\beta}{2} g'(\lambda(t)) - \frac{m(t)}{2\sigma^2}, \\ \mu_t = Law(X_t, \lambda_t) \text{ and } m_t = \langle \mu_t(dx, dl), x \rangle, \end{cases} \quad (1.3.15)$$

with $(B(t))_{t \geq 0}$ standard Brownian motion. Since $\lambda(t)$ follows a deterministic dynamics, the law of the process is such that, for any $t \geq 0$,

$$\mu_t(dx, d\lambda) = \nu_t(dx) \times \delta_{\lambda_t}(d\lambda).$$

Moreover, the resulting process is a Gaussian process hence it is completely described by the initial condition μ_0 and the flow $((m(t), V(t), \lambda(t)))_{t \geq 0}$, where $V(t) = \text{Var}[X(t)]$.

In the following we study the behavior of (1.3.15) with and without the dissipation.

We start with the stability study of (1.3.15) without the dissipative term, so let $\alpha = 0$. In this case, the variable $\lambda(t)$ has the same evolution of $m(t)$, then if $\lambda(0) = E[X(0)]$, the nonlinear process (1.3.15) with $\alpha = 0$, coincides with the nonlinear limit of a sequence of particle systems X^N , each of them evolving according to the Langevin dynamics (1.3.6). Therefore, we may consider this process as the dynamical generalized Curie-Weiss model in the Gaussian case, that is explicitly written a.s.

$$X(t) = e^{\frac{t}{2\sigma^2}} \left(X(0) + \int_0^t e^{-\frac{s}{2\sigma^2}} \frac{\beta}{2} g'(m(s)) ds + \int_0^t e^{-\frac{s}{2\sigma^2}} dB(s) \right).$$

We can restrict the study to the following system of ODE:

$$\begin{cases} \dot{m}(t) = \frac{\beta}{2} g'(m(t)) - \frac{m(t)}{2\sigma^2} \\ \dot{V}(t) = 1 - \frac{V(t)}{\sigma^2}. \end{cases} \quad (1.3.16)$$

Since the two variables evolve independently and the long-time behavior of $V(t)$ is trivial, we focus the attention on the one-dimensional ODE for the evolution of $m(t)$. We define the function

$$f_\beta(x) \doteq \beta g'(x) - \frac{x}{\sigma^2} \quad (1.3.17)$$

and notice that, for a fixed $\beta > 0$, the equilibrium points of (1.3.16) belong to the set $\{(x, \sigma^2), x \in \Lambda(\beta)\}$, where

$$\Lambda(\beta) \doteq \{x \in \mathbb{R} \text{ s.t. } f_\beta(x) = 0\}.$$

Remark 1.3.4. Since g' is globally Lipschitz continuous (Assumption 1.3.2) and it does not depend on β , it is clear that for β sufficiently small the origin is the only equilibrium point and it is a global attractor. Indeed, necessarily $g'(0) = 0$ (since by Assumption 1.3.1 g is even) and for β sufficiently small does not exist any $x^* \neq 0$ such that $f_\beta(x^*) = 0$. Moreover, notice that the set of critical points contains all the values of β such that the line $y = \frac{x}{\sigma^2\beta}$ is tangent to the graph $y = g'(x)$. By point (i) of Assumption 1.3.1, we know that g' cannot be identically zero and this ensures that there exists at least one critical point. Depending on the regularity of the function g , we can observe more than one phase transition and the appearance of coexisting stable equilibria. The stability of the equilibrium points of (1.3.16) follows from a standard analysis of the sign of the function f_β .

We plan to study the long-time behavior of the solution to (1.3.15) when $\alpha = 0$. To this aim, we state and prove a lemma concerning the long-time behavior of the following time-inhomogeneous SDE:

$$\begin{cases} dY(t) = a(t)dt - \frac{Y(t)}{2\sigma^2} dt + dB(t) \\ Y_0 \in L^2(\Omega) \end{cases} \quad (1.3.18)$$

where $a(t)$ is a deterministic function such that

$$\lim_{t \rightarrow +\infty} a(t) = a^* \in \mathbb{R}. \quad (1.3.19)$$

The solution of (1.3.18) will be used as an auxiliary process to prove long-behavior of the solution of (1.3.15). Indeed, when $\alpha = 0$, let $(X(t))_{t \geq 0}$ be the first component of a solution to (1.3.15) and $(Y(t))_{t \geq 0}$ be the solution to (1.3.18). Then, if $\text{Law}(X(0)) = \text{Law}(Y(0))$ and

$$\begin{cases} a(t) = \frac{\beta g'(m(t))}{2}, \\ \dot{m}(t) = \frac{\beta}{2} g'(m(t)) - \frac{m(t)}{2\sigma^2}, \end{cases}$$

with $m(0) = E[X_0]$, then, for all $t > 0$

$$\text{Law}(X(t)) = \text{Law}(Y(t)).$$

The same argument holds true when $\alpha > 0$, replacing $m(t)$ with $\lambda(t)$.

Lemma 1.3.1. *Let $(Y(t))_{t \geq 0}$ be the solution of (1.3.18) and $P_t(Y(0), \cdot)$ be its law at time t . Then,*

$$\lim_{t \rightarrow +\infty} \|P_t(Y(0), \cdot) - \nu_{a^*}(\cdot)\|_{TV} = 0$$

where $\nu_{a^*}(dx) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-a^*)^2}{2\sigma^2}} dx$ and $\|\cdot\|_{TV}$ indicates the total variation norm.

Proof. First of all, notice that

$$\lim_{t \rightarrow +\infty} \left| \int_0^t e^{\frac{s-t}{2\sigma^2}} a(s) ds - a^* \right| = 0. \quad (1.3.20)$$

In fact, fix $\varepsilon > 0$, then by (1.3.19), there exists t_ε^* such that $|a(t) - a^*| < \varepsilon$ for any $t > t_\varepsilon^*$. So,

$$\begin{aligned} \left| \int_0^t e^{\frac{s-t}{2\sigma^2}} a(s) ds - a^* \right| &\leq e^{-\frac{t}{2\sigma^2}} \int_0^t e^{\frac{s}{2\sigma^2}} |a(s) - a^*| ds \\ &= e^{-\frac{t}{2\sigma^2}} \int_0^{t_\varepsilon^*} e^{\frac{s}{2\sigma^2}} |a(s) - a^*| ds + e^{-\frac{t}{2\sigma^2}} \int_{t_\varepsilon^*}^t e^{\frac{s}{2\sigma^2}} \varepsilon \sigma^2 ds \\ &\leq e^{-\frac{t}{2\sigma^2}} t_\varepsilon^* e^{\frac{t_\varepsilon^*}{2\sigma^2}} \max_{s \in [0, t_\varepsilon^*]} |a(s) - a^*| + \frac{\varepsilon}{2}, \end{aligned}$$

then, taking t_ε^{**} such that

$$e^{-\frac{t_\varepsilon^{**}}{2\sigma^2}} t_\varepsilon^* e^{\frac{t_\varepsilon^*}{2\sigma^2}} \max_{s \in [0, t_\varepsilon^*]} |a(s) - a^*| < \frac{\varepsilon}{2},$$

for any $t > t_\varepsilon^{**}$ it holds that

$$\left| \int_0^t e^{\frac{s-t}{2\sigma^2}} a(s) ds - a^* \right| < \varepsilon$$

and (1.3.20) is proved. By the theory of linear stochastic differential equations it is well-known that

$$(Y(t)|Y(0) = y) \sim \mathcal{N} \left(ye^{-\frac{t}{2\sigma^2}} + \int_0^t e^{\frac{s-t}{2\sigma^2}} a(s) ds, \sigma^2 \left(1 - e^{-\frac{t}{\sigma^2}}\right) \right),$$

then, if $\nu_0 = \text{Law}(Y(0))$,

$$\begin{aligned} \|P_t(Y(0), \cdot) - \nu_{a^*}(\cdot)\|_{TV} &= \int_{\mathbb{R}} \|P_t(y, \cdot) - \nu_{a^*}(\cdot)\|_{TV} d\nu_0(y) \\ &= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\exp \left(\frac{\left(x - ye^{-\frac{t}{2\sigma^2}} - \int_0^t e^{\frac{s-t}{2\sigma^2}} a(s) ds \right)^2}{2\sigma^2 \left(1 - e^{-\frac{t}{2\sigma^2}}\right)} \right)}{\sqrt{2\pi\sigma^2 \left(1 - e^{-\frac{t}{2\sigma^2}}\right)}} - \frac{e^{-\frac{(x-a^*)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \right| dx d\nu_0(y) \end{aligned}$$

which converges to 0 as $t \rightarrow +\infty$ thanks to (1.3.20) and the Dominated Convergence Theorem. \square

We are now ready to prove the result on long-time behavior of the macroscopic process.

Theorem 1.3.5. *Fix $\beta > 0$, then $(X(t), \lambda(t))_{t \geq 0}$ has exactly $\text{Card}(\Lambda(\beta))$ stationary solutions given by the measures*

$$\mu_{m^*}(dx, dl) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m^*)^2}{2\sigma^2}} dx \times \delta_{m^*}(dl)$$

for all $m^* \in \Lambda(\beta)$. Moreover, for all $\mu_0(dx, d\lambda) = \nu_0(dx) \times \delta_{m_0}(d\lambda)$, square-integrable initial conditions with $m_0 = \langle \mu_0, x \rangle$

$$\lim_{t \rightarrow \infty} \|\mu_t(\cdot) - \mu_{m^*}(\cdot)\|_{TV} = 0, \quad (1.3.21)$$

where m^* is the equilibrium point of

$$\dot{m}(t) = \frac{\beta}{2} g'(m(t)) - \frac{m(t)}{2\sigma^2}$$

such that m_0 belongs to the domain of attraction of $m(t)$.

Proof. It is clear that the evolution given by (1.3.15) when $\alpha = 0$ must have a law $\mu_t(dx, dl) = \nu_t(dx) \times \delta_{m_t}(dl)$ where δ_{m_t} is a Dirac delta centered in $m_t = \int_{\mathbb{R}} y \nu_t(dy)$. Then the stationary Fokker-Planck gives

$$0 = \frac{1}{2} \frac{d^2}{dx^2} \nu^*(x) - \frac{d}{dx} \left[\left(\frac{\beta}{2} g'(m^*) - \frac{x}{2\sigma^2} \right) \nu^*(x) \right]$$

with $m^* = \int_{\mathbb{R}} x\nu^*(x)dx$ and $\mu^*(dx, dl) = \nu^*(dx)\delta_{m^*}(dl)$. So, there exists $K \in \mathbb{R}$ such that

$$\frac{d}{dx}\nu^*(x) = K + \left(\frac{\beta}{2}g'(m^*) - \frac{x}{2\sigma^2}\right)\nu^*(x).$$

Thus $\nu^*(x)$ solves a linear ODE, i.e. there exists $C \in \mathbb{R}$ such that

$$\nu^*(x) = \exp(\beta g'(m^*)x - \frac{x^2}{2\sigma^2}) \left(C + K \int_{\mathbb{R}} \exp(-\beta g'(m^*)y + \frac{y^2}{2\sigma^2}) dy \right).$$

Notice that necessarily we have to choose $K = 0$ and $C = \left(\int_{\mathbb{R}} \exp(\beta g'(m^*)x - \frac{x^2}{2\sigma^2}) dx \right)^{-1}$ in order to have an integrable and normalized density function ν^* . Moreover, the admissible functions ν^* are such that

$$m^* = \int_{\mathbb{R}} x\nu^*(x)dx = \beta\sigma^2 g'(m^*)$$

and this identifies them as the ones corresponding to $m^* \in \Lambda(\beta)$.

Now, let us prove the long-time behavior of μ_t for any square-integrable initial condition of the type $\mu_0 = \nu_0 \times \delta_{m_0}$. As we said, this implies that $\mu_t = \nu_t \times \delta_{\lambda(t)}$ and $\lambda(t) = E[X(t)]$ for all $t \geq 0$. We introduce an auxiliary process $Y(t)$, solution of

$$dY(t) = a(t)dt - \frac{Y(t)}{2\sigma^2}dt + dB_t$$

with initial condition $Y(0) \sim \mu_0$ and $a(t) = \frac{\beta g'(\lambda(t))}{2}$ for all $t \geq 0$. Denoting $P_t(Y(0), \cdot) = \text{Law}(Y(t))$, it is clear that

$$\|P_t(Y(0), \cdot) - \nu_t(\cdot)\|_{TV} = 0,$$

for all $t \geq 0$. Then (1.3.21) follows directly from Lemma 1.3.1. \square

1.3.5 The Gaussian case with dissipation

Let us now focus on the system in presence of the dissipative effect in the interaction. By assuming the same hypotheses as Section 1.3.4, we again reduce the problem to the study of a system of ODEs, that is the following:

$$\begin{cases} \dot{m}(t) = \frac{\beta}{2}g'(\lambda(t)) - \frac{m(t)}{2\sigma^2}, \\ \dot{\lambda}(t) = -\alpha\lambda(t) + \frac{\beta}{2}g'(\lambda(t)) - \frac{m(t)}{2\sigma^2}, \end{cases} \quad (1.3.22)$$

where, as before, the independence of the evolution of $V(t)$ allows us consider a two-dimensional instead of a three-dimensional system. Performing the change of variable $y = \frac{1}{2\sigma^2}(\lambda - m)$, we get the system

$$\begin{cases} \dot{y}(t) = -\frac{\alpha}{2\sigma^2}\lambda(t), \\ \dot{\lambda}(t) = y(t) - \left(\alpha + \frac{1}{2\sigma^2}\right)\lambda(t) + \frac{\beta}{2}g'(\lambda(t)), \end{cases} \quad (1.3.23)$$

which is a *Liénard system*. The link with Liénard systems is important to us since the systems of this class have been extensively studied in relation to their limit cycles [7, 10, 32, 41, 43, 47]. In general, a system of Liénard type has the following form:

$$\begin{cases} \dot{x} = y - A(x), \\ \dot{y} = -b(x), \end{cases}$$

for two suitable functions A, b . The usual hypothesis require that A' and b are C^1 functions, $b(0) = 0$ and $b(x)x > 0$ for x small enough. A detailed and complete study of all Liénard systems, with necessary and sufficient conditions for the existence of exactly $k \geq 0$ limit cycles, is still an open problem. However, in literature we can find sufficient conditions for the existence of *at least* (or *exactly*) $k \geq 0$ limit cycles, [10, 43]. In this setting, by a slight abuse of notation, we define the function

$$f_{\alpha,\beta}(x) := \left(\alpha + \frac{1}{2\sigma^2} \right) x - \frac{\beta}{2} g'(x);$$

of course, this generalizes (1.3.17), indeed $f_{0,\beta} = f_\beta$. For any fixed triplet of parameters $(\alpha, \beta, \sigma^2)$, equation (1.3.23) represents a Liénard system with $A(x) = f_{\alpha,\beta}(x)$ and $b(x) = \frac{\alpha}{2\sigma^2}x$.

In this case, by phase transition we mean any change in the number or in the stability of equilibrium points and limit cycles of the ODE (1.3.23). Assumptions 1.3.1 and 1.3.2 are rather general and allows to choose the interaction function g with enough freedom to make impossible a complete analysis of the phase transitions of (1.3.23). Nevertheless, in the following theorem we are able to describe three interesting behaviors.

Theorem 1.3.6. *Fix $\sigma^2 > 0$ and $\alpha > 0$ and consider the dynamical system (1.3.23) under Assumptions 1.3.1 and 1.3.2.*

- i) There exists $\beta^* > 0$ such that $\forall \beta \in (0, \beta^*)$ the origin is a global attractor for (1.3.23).*
- ii) If $g''(0) > 0$ and g' is not linear around 0, the origin loses stability via a Hopf bifurcation at the critical value $\beta_H = \frac{2\alpha + \frac{1}{\sigma^2}}{g''(0)}$.*
- iii) If $g''(0) > 0$ and there exists $C > 0$ such that for all $x \in]C, +\infty[$ the function $g'(x)$ is concave, then there exists a β_{UC} such that for all $\beta > \beta_{UC}$ there exists a unique limit cycle for (1.3.23).*

Proof. *i)* First of all, notice that for any choice of $\beta > 0$, $(0, 0)$ is the unique equilibrium for (1.3.23). The strategy to prove its stability for $\beta < \beta^*$ consists in finding a Lyapunov function. Let us consider the function

$$W(y, \lambda) = \frac{\alpha}{4\sigma^2} \lambda^2 + \frac{y^2}{2},$$

it is clear that

$$\frac{d}{dt}W(y(t), \lambda(t)) = -\frac{\alpha}{2\sigma^2}\lambda(t) \left(\left(\alpha + \frac{1}{2\sigma^2} \right) \lambda(t) - \frac{\beta}{2} g'(\lambda(t)) \right) = -\frac{\alpha}{2\sigma^2} \lambda(t) f_{\alpha, \beta}(\lambda(t)). \quad (1.3.24)$$

The problem reduces to consider the intersection of the graph of the function

$$y = g'(\lambda)$$

with a line, that in this case is the line

$$y = \frac{2\alpha + \frac{1}{\sigma^2}}{\beta} \lambda.$$

This, indeed, determines the sign of the function $f_{\alpha, \beta}(\lambda)$. We see that there exists a β^* sufficiently small, such that $\forall \beta < \beta^*$ the only intersection is the origin, meaning that (1.3.24) is strictly negative except than at $(0, 0)$, in which it is zero. Therefore W is a global Lyapunov function for the system (1.3.23), proving global attractivity of the origin.

ii) A Hopf bifurcation occurs when a stable periodic orbit arises from an equilibrium point that loses its (local) stability. Such a bifurcation can be detected looking at the linearized system around this stable equilibrium and finding the values of the parameters for which a pair of complex eigenvalues crosses the imaginary axis [44, Theorem 2, Chapter 4.4]. Therefore, consider the system (1.3.23) linearized around the point $(0, 0)$, that gives the linear system:

$$\begin{pmatrix} \dot{y}(t) \\ \dot{\lambda}(t) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\alpha}{2\sigma^2} \\ 1 & -\left(\alpha + \frac{1}{2\sigma^2} \right) + \frac{\beta}{2} g''(0) \end{pmatrix} \begin{pmatrix} y(t) \\ \lambda(t) \end{pmatrix}$$

with eigenvalues

$$x_{\pm} = \frac{1}{2} \left(\frac{\beta}{2} g''(0) - \alpha - \frac{1}{2\sigma^2} \pm \sqrt{\left(\frac{\beta}{2} g''(0) - \alpha - \frac{1}{2\sigma^2} \right)^2 - \frac{2\alpha}{\sigma^2}} \right).$$

It is easy to check that, when $\beta = \beta_H$, the eigenvalues x_{\pm} constitute a pair of conjugate non-zero purely imaginary numbers, crossing the imaginary line with positive velocity. The assumption on g' being not locally linear assure that we are in presence of a local bifurcation, and we can conclude that when $\beta = \beta_H$ we have a Hopf bifurcation.

iii) Recall that by Assumption 1.3.1, g' is an odd function positive on $]0, +\infty[$. Moreover, by Assumption 1.3.2, in the Gaussian case $g'(x)$ has necessarily to be definitively sub-linear (it is clear by (1.3.14)). Suppose that $g''(0) > 0$ and that there exists $C > 0$ such that for all $x \in]C, \infty[$ the function $g'(x)$ is concave; then, there exists a β_{UC} sufficiently large such that $\forall \beta > \beta_{UC}$, the function $f_{\alpha, \beta}$ has exactly three zeros $-x^* < 0 < x^*$ and satisfies the

following: $f_{\alpha,\beta}$ is negative on $]0, x^*[$ and positive and monotonically increasing on $]x^*, \infty[$. In this way, for all $\beta > \beta_{UC}$, the system (1.3.23) satisfies the conditions for the existence and uniqueness of a limit cycle presented in Theorem 1.1 of [7]. The proof follows a usual approach for Liénard systems, sketched also in [19]. \square

As in the case without dissipation, the results on the dynamical system (1.3.22) immediately extend to the Markov process $(X(t), \lambda(t))_{t \geq 0}$ solution to (1.3.15). Whenever the macroscopic law is attracted by a periodic orbit, we will obtain an invariant set of measures, each of them centred in a point of the limit cycle for (1.3.15).

Theorem 1.3.7. *Fix $\alpha, \beta > 0$, then the process $(X(t), \lambda(t))_{t \geq 0}$ described by (1.3.15) has exactly one stationary solution given by the measure*

$$\mu_{(0,0)}^*(dx, dl) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \times \delta_0(dl).$$

Let γ be a limit cycle of (1.3.22), then the set

$$\Gamma = \left\{ \mu_{(m,\lambda)}^*(dx, dl) : = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx \times \delta_\lambda(dl), \text{ for all } (m, \lambda) \in \gamma \right\}$$

is an invariant set for the dynamics (1.3.15). Moreover, for all $\mu_0(dx, d\lambda) = \nu_0(dx) \times \delta_{\lambda_0}(d\lambda)$, square-integrable initial conditions with $\lambda_0 \in \mathbb{R}$,

$$\lim_{t \rightarrow \infty} \inf_{(m,\lambda) \in \gamma} \|\mu_t(\cdot) - \mu_{(m,\lambda)}^*(\cdot)\|_{TV} = 0,$$

where γ is the attractor of the trajectory starting from $(\langle \mu_0, x \rangle, \lambda_0)$ in the dynamical system (1.3.22); here with γ we mean either a limit cycle or simply the origin.

Proof. The proof follows the same approach of the proof of Theorem 1.3.5, using the stationary Fokker-Planck equation and the auxiliary results on long-time behavior given by Lemma 1.3.1. \square

1.3.6 Coexistence of stable limit cycles

Theorem 1.3.6 and 1.3.7 prove the existence of self-sustained periodic behavior for the generalized Curie-Weiss model with dissipation, with the appearance of a unique attractive limit cycle for the macroscopic system under suitable conditions. In view of the long-time behavior of the classical Curie-Weiss model with dissipation presented in Section 1.2, this should not be surprising. Nevertheless, as already mentioned, the picture described by Theorem 1.3.6 is far from being complete: as a consequence, according to the form of the interaction function g , the generalized Curie-Weiss model can display a much richer behavior in the infinite volume dynamics. The most interesting phenomena are the appearance of limit cycles without the origin losing its local stability and the coexistence of any (finite) number of stable limit cycles. These orbits, in general, are created and destroyed through

global bifurcations. The number of limit cycles and their stability mainly depends on the function $f_{\alpha,\beta}(x)$, which plays a key role in the study of a Liénard system such as (1.3.23). In general, some tools to determine the exact number of limit cycles in a Liénard system are available in literature (see [10, 43] and references therein), but their application may be cumbersome in a general setting, since several features of the function $f_{\alpha,\beta}(x)$ should be studied, such as the position of its zeroes, its local minima and maxima, their height and so on. Nevertheless, by playing with the form of the interaction function g , we can always create a Gaussian Curie-Weiss model with dissipation with a customized number of phase transitions and of coexisting limit cycles.

Let us briefly underline the role of the function g in the occurrence of limit cycles in the dynamics of (1.3.22). To this aim, we rewrite the Liénard system (1.3.23):

$$\begin{cases} \dot{y}(t) = -\frac{\alpha}{2\sigma^2}\lambda(t), \\ \dot{\lambda}(t) = y(t) - f_{\alpha,\beta}(\lambda(t)). \end{cases}$$

In the rich literature on Liénard system, we see that the form of the function $f_{\alpha,\beta}$ plays a fundamental role in the number of limit cycles of the system. In particular, from the results in [43], we can state the following.

Proposition 1.3.1. *Fix $\sigma^2, \alpha > 0$ and suppose that there exists a β^* such that the following conditions are satisfied:*

i) the function f_{α,β^} has K positive zeros $x_0 = 0 < x_1 < \dots < x_K (< x_{K+1}$ a bound) at which it changes sign;*

ii) for every $k = 1, \dots, K$ there is a C^1 mapping $\phi_k: [x_{k-1}, x_k] \rightarrow [x_k, x_{k+1}]$ such that

$$\phi_k(x)\phi'_k(x) \geq x \text{ and } |f_{\alpha,\beta^*}(\phi_k(x))| \geq |f_{\alpha,\beta^*}(x)|;$$

iii) the function f_{α,β^} on each interval $[x_{k-1}, \phi_{k-1}(x_{k-1})]$ for $2 \leq k \leq K + 1$ has an extremum at a unique point y_k and its derivative is weakly monotone.*

Then the generalized Curie-Weiss model with dissipation has at least one regime in which it has exactly K limit cycles. The outer cycle is stable, then the others alternate as unstable and stable, respectively.

The proof of this result is a simple application of the results in [43]. It is easy to see that the function

$$f_{\alpha,\beta}(x) = \left(\alpha + \frac{1}{2\sigma^2} \right) x - \frac{\beta}{2} g'(x)$$

depends on the choice of the interaction function g . Assumptions 1.3.1 and 1.3.2 are not very restrictive, so g can be chosen with sufficient generality to obtain a system that admits a regime displaying the desired number of limit cycles.

With an explicit example, we now show how we can manipulate the interaction function g in order to observe the peculiar phenomena described above. Let us define the function

$$g(x) = \tanh(ax^2 + bx^4 + cx^6), \quad (1.3.25)$$

with a, b, c suitable constants such that g stays strictly increasing on $[0, \infty)$. Fix $\sigma^2 > 0$, then the pair (g, ρ) , with $\rho \sim \mathcal{N}(0, \sigma^2)$ clearly satisfies Assumptions 1.3.1 and 1.3.2 and it defines a generalized Curie-Weiss model. We consider two triplets of constants $(\frac{1}{2}, -1, 2)$ and $(1, 1, 0)$ in order to observe some particular regimes that do not exist for the classical Curie-Weiss model with dissipation.

Case A: triplet $(\frac{1}{2}, -1, 2)$

We see from Figure 1.1 the changes in the concavity of $g'(x)$. This causes, in the dynamics without dissipation, three critical values of β and the four following regimes:

- for $\beta < \beta_1$ the origin is a global attractor;
- for $\beta \in (\beta_1, \beta_2)$ the origin is locally stable, but there are four other equilibrium points $-x_2 < -x_1 < 0 < x_1 < x_2$, such that $\pm x_2$ are stable and $\pm x_1$ are unstable;
- for $\beta \in (\beta_2, \beta_3)$ the origin becomes unstable and two additional stable equilibrium points appear, $\dots -x_1 < -x_3 < 0 < x_3 < x_1 \dots$;
- for $\beta = \beta_3$ the pairs of equilibrium points $\{x_3, x_1\}$ and $\{-x_3, -x_1\}$ collapse and disappear, such that for $\beta > \beta_3$ there are three equilibrium points $-x_2 < 0 < x_2$, the outer two are stable and the origin is unstable.

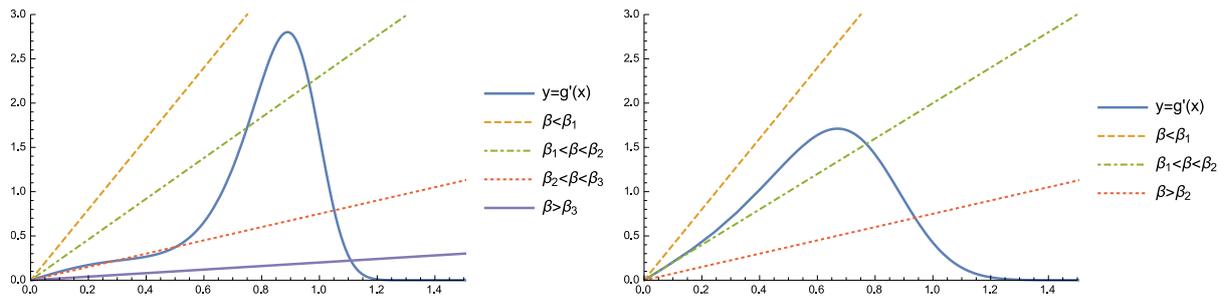


Figure 1.1: The plot of the function g' and of lines $y = \frac{1}{\beta\sigma^2}x$ for different values of β . The number of intersections gives the number of equilibrium points in the positive axes. Left: the case A. Right: the case B.

The exact critical values may be obtained numerically, and the behavior of the dynamical system is clear from a standard analysis on the signs of the function f_β . As we expect, in this case the dissipated dynamics (1.3.22) actually shows four different regimes as well, but the critical values $\hat{\beta}_1(\alpha)$, $\hat{\beta}_2(\alpha)$, $\hat{\beta}_3(\alpha)$ are not straightforwardly obtained with the same

procedure of the elements of CV . To be precise, if β_1 corresponds to the smallest value of β in which the line $y = \frac{x}{\sigma^2\beta}$ is tangent to the graph of $y = g'(x)$, the value $\hat{\beta}_1(\alpha)$ is strictly greater than the smallest value of β such that the line $y = \frac{2\alpha + \frac{1}{\sigma^2}}{\beta}x$ is tangent to the graph of $y = g'(x)$. This means that there exists a β^* such that the line $y = \frac{2\alpha + \frac{1}{\sigma^2}}{\beta^*}x$ intersects the graph of $y = g'(x)$ but any limit cycle occurs. Nevertheless, the system displays a regime of *coexistence of stable limit cycles*. Let us better explain the four regimes that we observe in system (1.3.22) (actually the computations and the plots refer to system (1.3.23), since the link with the function $f_{\alpha,\beta}$ is more clear in this case).

- For $\beta < \hat{\beta}_1(\alpha)$ the origin is a global attractor. Notice that, numerically we can see that $\hat{\beta}_1(\alpha)$ is greater than the β^* obtained in Theorem 1.3.6; indeed it is not necessary that the function $f_{\alpha,\beta}(x)$ is strictly greater than zero for all $x > 0$. It is reasonable to believe, see Figure 1.2(a), that for those β such that the negative part of the function is “small enough” do not necessary give rise to a periodic orbit.
- For $\beta \in (\hat{\beta}_1(\alpha), \hat{\beta}_2(\alpha))$ the origin is locally stable and, through a global bifurcation, two periodic orbits have arisen, the larger one is stable and the smaller one is unstable. Indeed, we can prove numerically that there exists a value $\beta^* \in (\hat{\beta}_1(\alpha), \hat{\beta}_2(\alpha))$ such that the function f_{α,β^*} satisfies the conditions of Theorem A and B in [43] for the existence of exactly two limit cycles. See Figure 1.2(b) in which we can observe the stable outer cycle and the attractivity of the origin.
- Notice that $\beta_H = \hat{\beta}_2(\alpha)$ from Theorem 1.3.6. Therefore, for $\beta \in (\hat{\beta}_2(\alpha), \hat{\beta}_3(\alpha))$ we observe two attractive limit cycles, a smaller one spreading from the origin (that is now unstable) while the bigger one remains from the previous regime: see Figure 1.2(c). The basin of attraction of the two stable orbits are separated by a third unstable periodic orbit. This is the regime in which we see the coexistence of two stable periodic orbits, one inside the other. The existence of this regime is again a consequence of Theorem A and B in [43], because we can numerically find a β^{**} that satisfies the hypothesis for the existence of exactly three limit cycles, two stable and one unstable.
- For $\beta > \hat{\beta}_3(\alpha)$ we see that only the largest periodic orbit has survived. Indeed, for in $\hat{\beta}_3(\alpha)$ the smallest stable orbit and the unstable one collapse and disappear. Of course, we see that this $\hat{\beta}_3(\alpha) = \beta_{UC}$ defined in Theorem 1.3.6; but from numerical evidence we suppose that for this value of β the function $f_{\alpha,\beta}$ has more than one single zero in the positive half-line, but that the other two zeros are not distant enough to admit the existence of the two inner orbits. Of course when β is such that there exists a unique positive zero for $f_{\alpha,\beta}$, we analytically prove the existence and uniqueness of the limit cycle (see Theorem 1.3.6) while for lower values we can only show it numerically, see Figure 1.2(d).

Case B: triplet $(1, 1, 0)$

We see in Figure 1.1 that the shape of g' basically allows three different regimes for the case without dissipation. Indeed, in (1.3.16), the set CV has cardinality 2, i.e. we have $\beta_1 < \beta_2 = \frac{2}{\sigma^2}$. The three regimes are the following:

- for $\beta < \beta_1$ the origin is a global attractor;
- for $\beta \in (\beta_1, \beta_2)$ there are five equilibrium points $-x_2 < -x_1 < 0 < x_1 < x_2$, where $\pm x_1$ are unstable, while the others are stable;
- at $\beta = \beta_1$ the two points $\pm x_1$ collapse in the origin that becomes unstable, such that for $\beta > \beta_1$ the origin is unstable and the points $\pm x_2$ are stable.

We treat this example in the dissipated case (1.3.22) (by means of the Liénard system (1.3.23)). We expect three regimes and, in particular, we will observe an atypical behavior at the Hopf bifurcation, where we will not have a small limit cycle bifurcating from the origin, but the already existing stable limit cycle that becomes a global attractor. In Figure 1.3 we compare the regimes immediately below and above the Hopf bifurcation.

- For $\beta < \beta_1(\alpha)$, the origin is a global attractor. As is Case A the value $\beta_1(\alpha)$ is strictly greater than the value in which the line first touches the graph $y = g'(x)$.
- For $\beta \in (\beta_1(\alpha), \beta_H)$ the origin is stable and we have an unstable periodic orbit contained in a stable periodic one. When β increase the inner orbit shrinks and the outer expands.
- For $\beta = \beta_H$ the Hopf bifurcation is such that the origin loses stability, but this happens simultaneously to the collapse of the unstable periodic orbit on it. Therefore, after the bifurcation, we do not see the usual periodic orbit expanding from the origin because the unique orbit is the stable one (from the previous regime) that becomes globally stable.

This case is interesting because the Hopf bifurcation do not originates a small periodic orbit. However, the phenomenon is still a local one, because it is a small unstable orbit that collapses on the origin changing its stability.

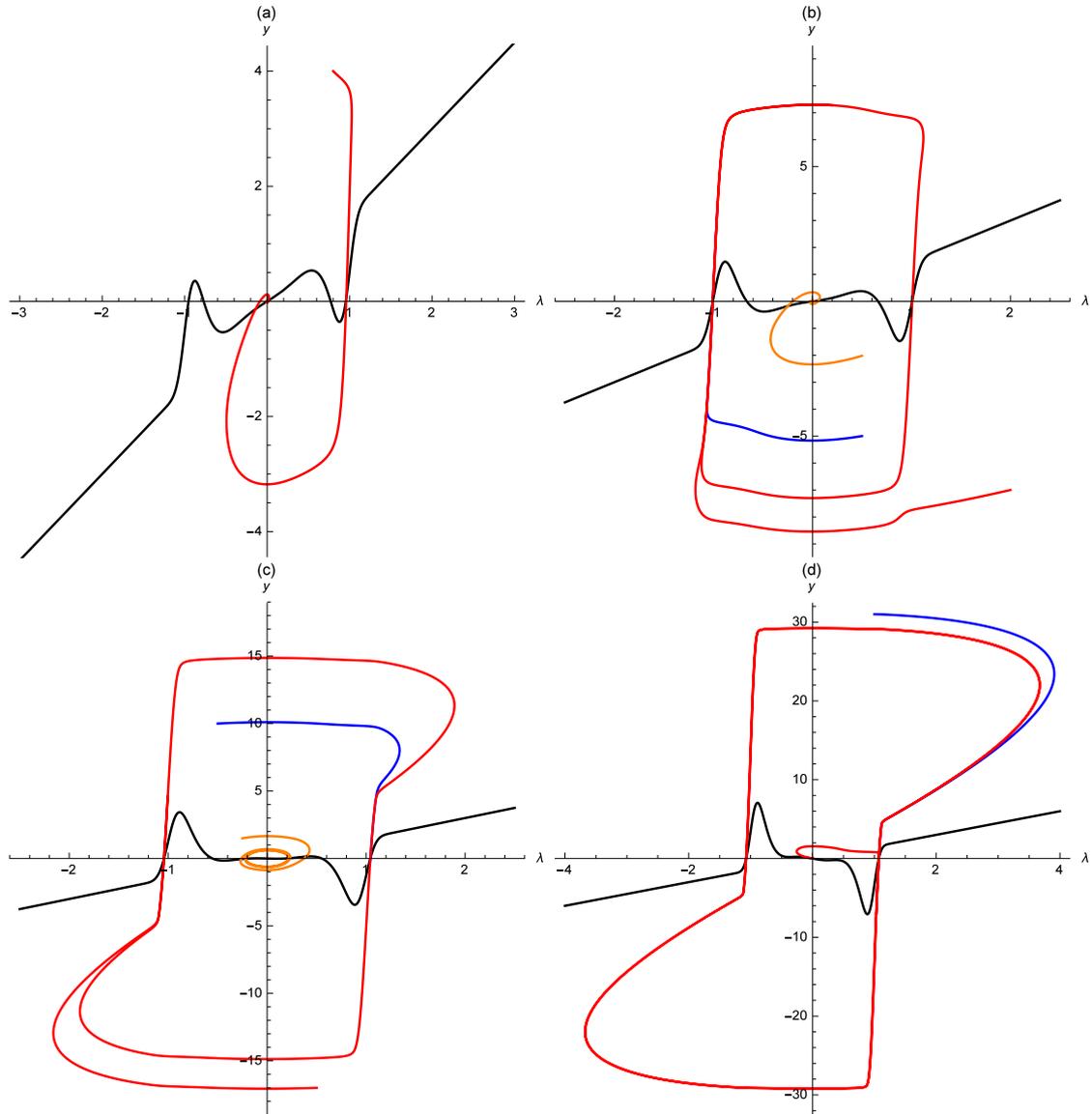


Figure 1.2: Different regimes for case A. In all the pictures, the black line represents the graph of $y = f_{\alpha,\beta}(\lambda)$ and we fixed $\alpha = \sigma^2 = 1$. In (a), the regime $\beta < \hat{\beta}_1(\alpha)$ ($\beta = 1.2$): the red line represents the solution starting from $\lambda(0) = 1$, $y(0) = 4$, which is definitely attracted by the globally stable origin. In (b), the regime $\beta \in (\hat{\beta}_1, \hat{\beta}_2)$ ($\beta = 2$): the red-colored solution, starting from $\lambda(0) = 2$, $y(0) = -7$, and the blue-colored solution, starting from $\lambda(0) = 0.5$, $y(0) = -5$, are attracted by a stable limit cycle. Here, the origin is locally stable (the orange-colored solution with initial condition $\lambda(0) = 0.5$, $y(0) = -2$ is attracted by it) and its basin of attraction is surrounded by an unstable limit cycle. In (c), the regime $\beta \in (\hat{\beta}_2, \hat{\beta}_3)$ ($\beta = 3.4$): the red and blue lines, here representing solution starting from $\lambda(0) = 0.5$, $y(0) = -17$ and $\lambda(0) = -0.5$, $y(0) = 10$ respectively, are again attracted by the outer cycle but now the origin is unstable and another stable cycle is born via the Hopf bifurcation. The orange-colored solution, with initial condition $\lambda(0) = -0.25$, $y(0) = 1.5$, is attracted by the smallest cycle. The basins of attraction of the stable orbits is separated by an unstable cycle, which is not represented in the picture. In (d), the regime $\beta > \hat{\beta}_3$ ($\beta = 6$): only the external orbit is survived and it has become globally attractive, as shown by the red and blue lines, with initial conditions $\lambda(0) = 0$, $y(0) = -0.005$ and $\lambda(0) = 1.5$, $y(0) = 31$ respectively.

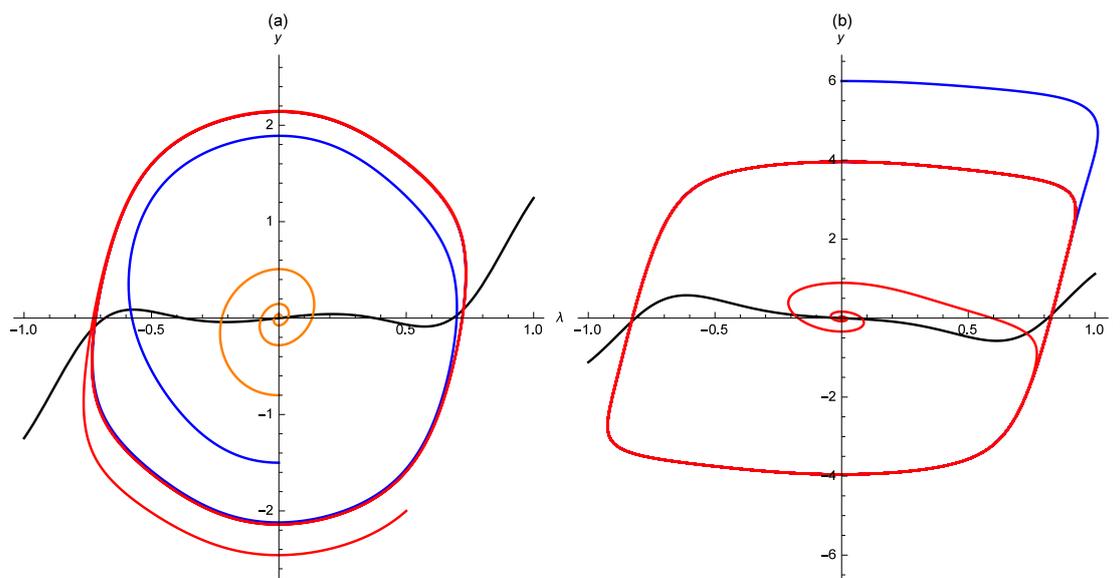


Figure 1.3: Different regimes for case B close to the Hopf bifurcation. In both pictures, the black line represents the graph of $y = f_{\alpha, \beta}(\lambda)$ and we fixed $\alpha = \sigma^2 = 1$. In (a), the regime $\beta \in (\hat{\beta}_1(\alpha), \beta_H)$ ($\beta = 1.2$): the situation is qualitatively the same of Figure 1.2(b). The red, blue and orange lines represent solution starting from $\lambda(0) = 0.5, y(0) = -2$, $\lambda(0) = 0, y(0) = -1.5$ and $\lambda(0) = 0, y(0) = -0.8$ respectively. In (b), the regime $\beta > \beta_H$ ($\beta = 1.8$): the system has undergone through a Hopf bifurcation but the stable limit cycle spreading from the origin is not present here, due to the collapse of the unstable cycle at the origin, leaving the outer orbit to become globally attractive. The red-colored and blue-colored solutions have initial conditions $\lambda(0) = 0, y(0) = -0.005$ and $\lambda(0) = 0, y(0) = 6$ respectively.

Chapter 2

A two-populated mean field model

In this chapter we continue the study of the mechanisms enhancing self-sustained periodic behaviors in mean field models: here the focus is on the role played by the structure of the interaction network in multi-populated systems. The results presented here indicate that having two groups of spins with possibly different size and different inter- and intra-population interactions suffices for the emergence of macroscopic periodicity. Moreover, delay may induce periodic behavior in interaction network configurations where otherwise absent.

2.1 The model and its macroscopic dynamics

The two-population Curie-Weiss model is a spin system where on the complete graph two types of spins are present. Particles are differentiated by their mutual interactions: there are two *intra-group* interactions, tuning how strongly sites in the same group feel each other, and two *inter-group* interactions, giving the magnitude of the influence between particles of distinct populations. This model has been employed to analyse immigration and cultural coexistence and to explain the role of social groups in influencing an individual's beliefs and preferences [4, 18, 17, 30]. In these works, the thermodynamic limit of this model has been studied from the statical point of view. Large volume dynamical behavior has instead been explored in [12], which also contains an analysis on the stationary solutions for the system. In this chapter, we continue this study by looking for interaction networks that can enhance the appearance of self-sustained periodic behavior in the infinite volume dynamics of the two-populated Curie-Weiss model.

We define the model and its microscopical dynamics in the following way. Let $\mathcal{S} = \{-1, +1\}$ let $\underline{\sigma} = (\sigma_j)_{j=1}^N \in \mathcal{S}^N$ be an N -particle configuration. We divide the whole system of size N into two disjoint subsystems of sizes N_1 and N_2 respectively. Let I_1 (resp. I_2) be the set of sites belonging to the first (resp. second) subsystem. We have $|I_1| = N_1$ and $|I_2| = N_2$, with $N_1 + N_2 = N$. To fix notation, let $1, 2, \dots, N_1$ be the indices corresponding to particles

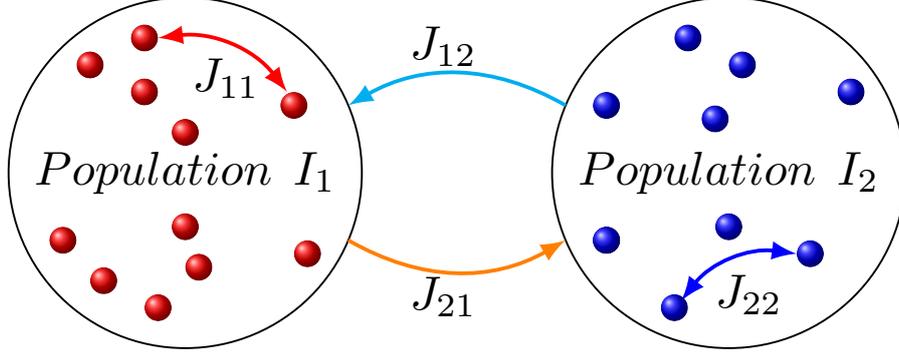


Figure 2.1: A schematic representation of the interaction network for a bipartite Curie-Weiss model. Spins are divided into two populations I_1 and I_2 . Within I_1 (resp. I_2) particles feel a mean field interaction with coupling J_{11} (resp. J_{22}). Besides, population I_1 (resp. I_2) influences the dynamics of the other group through its magnetization with strength J_{21} (resp. J_{12}).

in population I_1 and $N_1 + 1, N_1 + 2, \dots, N$ those of particles in population I_2 , so that

$$\sigma = \left(\begin{array}{c|c} \text{Population } I_1 & \text{Population } I_2 \\ \hline (\sigma_1, \sigma_2, \dots, \sigma_{N_1}) & (\sigma_{N_1+1}, \sigma_{N_1+2}, \dots, \sigma_N) \end{array} \right).$$

Given two spins, their mutual interaction depends on the populations they belong to: J_{11} and J_{22} tune the interaction within sites of the same subsystem; whereas, J_{12} and J_{21} control the coupling strength between spins located in different groups (see Figure 2.1 for a schematic representation). All the interactions can be either positive or negative allowing both ferromagnetic and antiferromagnetic interactions. We consider the Markov process $(\underline{\sigma}(t))_{t \in [0, T]}$ with infinitesimal generator

$$L_N f(\underline{\sigma}) = \sum_{i \in I_1} e^{-\sigma_i \mathcal{R}_1(m_1, m_2)} (f(\underline{\sigma}^i) - f(\underline{\sigma})) + \sum_{i \in I_2} e^{-\sigma_i \mathcal{R}_2(m_1, m_2)} (f(\underline{\sigma}^i) - f(\underline{\sigma})) \quad (2.1.1)$$

where $\underline{\sigma}^i$ is defined by (1.2.1), \mathcal{R}_1 and \mathcal{R}_2 are the functions

$$\mathcal{R}_1(x_1, x_2) = J_{11}x_1 + J_{12}x_2 \quad (2.1.2)$$

$$\mathcal{R}_2(x_1, x_2) = J_{22}x_2 + J_{21}x_1. \quad (2.1.3)$$

while the process $(m_{1,N}(t), m_{2,N}(t))_{t \in [0, T]}$ is given by

$$m_{1,N}(t) = \frac{1}{N} \sum_{i \in I_1} \sigma_i(t), \quad m_{2,N}(t) = \frac{1}{N} \sum_{i \in I_2} \sigma_i(t).$$

Observe that $m_{1,N}(t)$ and $m_{2,N}$ are proportional to the magnetization of the two populations: in fact, if $\gamma := N_1/N$ is the proportion of sites belonging to the first group, the

magnetization of population I_1 at time t is equal to $\gamma^{-1}m_{1,N}(t)$ and the magnetization of population I_2 at time t is equal to $(1-\gamma)^{-1}m_{2,N}(t)$. Notice also that the \mathcal{R}_i 's are comprised of two terms: the first one tells how strong sites in the same population interact, while the second encodes the way one population influences the other.

We now study the dynamics of $(\underline{\sigma}(t))_{t \in [0, T]}$ in the infinite volume limit. We assume that $N \uparrow +\infty$ in such a way the proportions γ and $1-\gamma$ of the two groups remain constant. Similarly to what discussed in Section 1.2.2, since (2.1.1) prescribes a mean field interaction, it is possible to find a low-dimensional description of the empirical measure flow: it is clear that $(m_{1,N}(t), m_{2,N}(t))_{t \in [0, T]}$ is an order parameter of the system (see Definition 1.2.3).

Theorem 2.1.1. *Fix $T > 0$ and let $(\underline{\sigma}(t))_{t \in [0, T]}$ be the Markov process with values in \mathcal{S}^N and infinitesimal generator (2.1.1) and initial conditions*

$$\text{Law}(\sigma_1(0), \dots, \sigma_{N_1}(0), \sigma_{N_1+1}(0), \dots, \sigma_{N_1+N_2}(0)) = \nu_1^{\otimes N_1} \otimes \nu_2^{\otimes N_2}, \quad (2.1.4)$$

with ν_1, ν_2 probability measures on \mathcal{S} . Then, the process $(m_{1,N}(t), m_{2,N}(t))_{t \in [0, T]}$ with values in $[-\gamma, \gamma] \times [-1+\gamma, 1-\gamma]$, where $\gamma = N_1/N$, is an order parameter of the system. Moreover, as $N \uparrow +\infty$ in such a way γ remains constant, $(m_{1,N}(t), m_{2,N}(t))_{t \in [0, T]}$ converges, in sense of weak convergence of stochastic processes, to the solution of the ODEs

$$\begin{cases} \dot{m}_1(t) = 2\gamma \sinh(\mathcal{R}_1(m_1(t), m_2(t))) - 2m_1(t) \cosh(\mathcal{R}_1(m_1(t), m_2(t))), \\ \dot{m}_2(t) = 2(1-\gamma) \sinh(\mathcal{R}_2(m_1(t), m_2(t))) - 2m_2(t) \cosh(\mathcal{R}_2(m_1(t), m_2(t))), \end{cases} \quad (2.1.5)$$

with initial condition $m_1(0) = \gamma(\nu_1(+1) - \nu_1(-1))$ and $m_2(0) = (1-\gamma)(\nu_2(+1) - \nu_2(-1))$.

Proof. Using Lemma 1.2.1 it is simple to check that $(m_{1,N}(t), m_{2,N}(t))_{t \in [0, T]}$ is a Markov process with infinitesimal generator K_N that, for $f : [-\gamma, \gamma] \times [-1+\gamma, 1-\gamma] \rightarrow \mathbb{R}$, reads

$$\begin{aligned} K_N f(m_1, m_2) &= \sum_{j \in \mathcal{S}} |A_{1,N}(j)| e^{-j\mathcal{R}_1(m_1, m_2)} \left(f\left(m_1 - j\frac{2}{N}, m_2\right) - f(m_1, m_2) \right) \\ &\quad + \sum_{j \in \mathcal{S}} |A_{2,N}(j)| e^{-j\mathcal{R}_2(m_1, m_2)} \left(f\left(m_1, m_2 - j\frac{2}{N}\right) - f(m_1, m_2) \right), \end{aligned}$$

where, for $k = 1, 2$ and $j \in \mathcal{S}$, $A_{k,N}(j)$ is the set of sites with spins equal to j in population I_k , for a given configuration. One can see that

$$|A_{1,N}(j)| = \frac{N(\gamma + jm_1)}{2}, \quad |A_{2,N}(j)| = \frac{N((1-\gamma) + jm_2)}{2}.$$

The weak convergence of initial conditions is an application of the Law of Large Numbers: under (2.1.4),

$$\left(\frac{1}{N_1} \sum_{i=1}^{N_1} \sigma_i(0), \frac{1}{N_2} \sum_{i=N_1+1}^N \sigma_i(0) \right) \xrightarrow[N \uparrow +\infty]{w} (\nu_1(+1) - \nu_1(-1), \nu_2(+1) - \nu_2(-1))$$

while $m_1(0) = \gamma \frac{1}{N_1} \sum_{i=1}^{N_1} \sigma_i(0)$ and $m_2(0) = (1 - \gamma) \frac{1}{N_2} \sum_{i=N_1+1}^N \sigma_i(0)$.

We are only left to prove the weak convergence of the process $(m_{1,N}(t), m_{2,N}(t))_{t \in [0, T]}$ to the solution of (2.1.5), but this can be shown by uniform convergence of the infinitesimal generators (see Lemma 1.2.3). This computation is identical to the one performed in the proof of Theorem 1.2.2: given $f : [-\gamma, \gamma] \times [\gamma - 1, 1 - \gamma] \rightarrow \mathbb{R}$, $f \in \mathcal{C}^2$, a first-order Taylor expansion of $K_N f(m_1, m_2)$ provides the infinitesimal generator of (2.1.5), plus a term $o(1)$ uniformly converging to 0. \square

2.2 The model with delay and its macroscopic dynamics

We consider now a modification of the dynamics defined by (2.1.1), obtained by introducing a delay in the inter-populations interactions. We assume that at any time t , the influence of each population on the other is given by an average over the magnetization trajectory *up to time* t , weighted through a kernel. Multi-populated mean field models with interactions weighted through a delay kernel have been considered for example in [23], while in [55] a different type of delay, in which interactions at time t depends only on the state of magnetizations at time $t - \tau$ with $\tau > 0$ fixed, is considered for a bi-populated spin system. The advantage of introducing delay through a suitable kernel consists in the chance to obtain a finite-dimensional macroscopic dynamics.

Informally speaking, we are going to consider a microscopic dynamics in which the transition $\underline{\sigma}(t) \rightarrow \underline{\sigma}^i(t)$ occurs at rate

$$\begin{cases} e^{-\sigma_i(t)} \mathcal{R}_1(m_{1,N}(t), \eta_{2,N}^{(n)}(t)), & \text{if } i \in I_1 \\ e^{-\sigma_i(t)} \mathcal{R}_2(\eta_{1,N}^{(n)}(t), m_{2,N}(t)), & \text{if } i \in I_2, \end{cases} \quad (2.2.1)$$

where, for $n \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{0\}$, we define

$$\eta_{i,N}^{(n)}(t) = \int_0^t \frac{(t-s)^n}{n!} e^{-k(t-s)} m_{i,N}(s) ds, \quad \text{for } i = 1, 2.$$

Notice that the delay kernel is in the form of Erlang distribution: the parameter n is related to the shape of the bump of the function, whereas k tunes how sharp and close to time t the peak is. In particular, for any fixed n , the peak of the kernel becomes closer to t as k grows. This suggests that the addition of delay is relevant for small values of the parameter k .

We want to define a Markovian evolution for the two-population Curie-Weiss models in which the flipping rates correspond to (2.2.1). Fix the parameters n and k : clearly, the process $(\underline{\sigma}(t), \eta_{1,N}^{(n)}(t), \eta_{2,N}^{(n)}(t))_{t \in [0, T]}$ is not Markov, since the evolution of the terms $\eta_{i,N}^{(n)}$ at time t depend on the history on $[0, t]$. Nevertheless, we can exploit the properties of the

Erlang kernel : observe that, for fixed k , the families $(\eta_{1,N}^{(j)}(t))_{j=0}^n$ and $(\eta_{2,N}^{(j)}(t))_{j=0}^n$ satisfies

$$\begin{aligned}\dot{\eta}_{1,N}^{(0)}(t) &= k \left(-\eta_{1,N}^{(0)}(t) + m_{1,N}(t) \right), \\ \dot{\eta}_{1,N}^{(j)}(t) &= k \left(-\eta_{1,N}^{(j)}(t) + \eta_{1,N}^{(j-1)}(t) \right), \quad \text{for } j = 1, \dots, n, \\ \dot{\eta}_{2,N}^{(0)}(t) &= k \left(-\eta_{2,N}^{(0)}(t) + m_{2,N}(t) \right), \\ \dot{\eta}_{2,N}^{(j)}(t) &= k \left(-\eta_{2,N}^{(j)}(t) + \eta_{2,N}^{(j-1)}(t) \right), \quad \text{for } j = 1, \dots, n.\end{aligned}$$

Therefore, $(\eta_{1,N}^{(j)}(t))_{j=0}^n$ and $(\eta_{2,N}^{(j)}(t))_{j=0}^n$ constitute a *cascade* of Markovian memory terms: even if the flipping rates at time t depend directly only on $\eta_{1,N}^{(n)}(t)$ and $\eta_{2,N}^{(n)}(t)$, the knowledge of the "auxiliary" processes $(\eta_{1,N}^{(j)}(t))_{j=0}^{n-1}$ and $(\eta_{2,N}^{(j)}(t))_{j=0}^{n-1}$ allows to define a Markovian evolution for the system, as presented below.

After fixing the parameters n, k of the delay kernel, we consider the Markov process

$$\left(\underline{\sigma}(t), \eta_{1,N}^{(0)}(t), \dots, \eta_{1,N}^{(n)}(t), \eta_{2,N}^{(0)}(t), \dots, \eta_{2,N}^{(n)}(t) \right)_{t \in [0, T]} \quad (2.2.2)$$

with values in $\mathcal{S}^N \times \mathbb{R}^{2n+2}$ and with infinitesimal generator

$$\begin{aligned}L_N f(\underline{\sigma}, \bar{\eta}_1, \bar{\eta}_2) &= \sum_{i \in I_1} e^{-\sigma_i \mathcal{R}_1(m_1, \eta_2^{(n)})} \left(f(\underline{\sigma}^i, \bar{\eta}_1, \bar{\eta}_2) - f(\underline{\sigma}, \bar{\eta}_1, \bar{\eta}_2) \right) \\ &+ \sum_{i \in I_2} e^{-\sigma_i \mathcal{R}_2(\eta_1^{(n)}, m_2)} \left(f(\underline{\sigma}^i, \bar{\eta}_1, \bar{\eta}_2) - f(\underline{\sigma}, \bar{\eta}_1, \bar{\eta}_2) \right) \\ &+ \sum_{r=1,2} k \left(-\eta_r^{(0)} + m_r \right) \partial_{\eta_r^{(0)}} f(\underline{\sigma}, \bar{\eta}_1, \bar{\eta}_2) \\ &+ \sum_{r=1,2} \sum_{j=1}^n k \left(-\eta_r^{(j)} + \eta_r^{(j-1)} \right) \partial_{\eta_r^{(j)}} f(\underline{\sigma}, \bar{\eta}_1, \bar{\eta}_2),\end{aligned} \quad (2.2.3)$$

where $\bar{\eta}_r = (\eta_r^{(0)}, \dots, \eta_r^{(n)})$, for $r = 1, 2$. We can study its macroscopic dynamics by means of a $(2n + 4)$ -dimensional order parameter.

Theorem 2.2.1. *Fix $T > 0$ and let $(\underline{\sigma}(t), \bar{\eta}_1(t), \bar{\eta}_2(t))_{t \in [0, T]}$ be the Markov process (2.2.2) with infinitesimal generator (2.2.3) and initial conditions*

$$\text{Law}(\underline{\sigma}(0), \bar{\eta}_1(0), \bar{\eta}_2(0)) = \nu_1^{\otimes N_1} \otimes \nu_2^{\otimes N_2} \otimes \delta_{\xi_1}^{\otimes (n+1)} \otimes \delta_{\xi_2}^{\otimes (n+1)}, \quad (2.2.4)$$

with ν_1, ν_2 probability measures on \mathcal{S} and $\xi_1, \xi_2 \in \mathbb{R}$. Then, the process

$$\left(m_{1,N}(t), m_{2,N}(t), \bar{\eta}_{1,N}(t), \bar{\eta}_{2,N}(t) \right)_{t \in [0, T]}$$

with values in $[-\gamma, \gamma] \times [-1 + \gamma, 1 - \gamma] \times \mathbb{R}^{2n+2}$, where $\gamma = N_1/N$, is an order parameter of the system. Moreover, as $N \uparrow +\infty$ in such a way γ remains constant, the order parameter converges, in sense of weak convergence of stochastic processes, to the solution of the ODEs

$$\begin{cases} \dot{m}_1(t) = 2\gamma \sinh(\mathcal{R}_1(m_1(t), \eta_2^{(n)}(t)) - 2m_1(t) \cosh(\mathcal{R}_1(m_1(t), \eta_2^{(n)}(t))), \\ \dot{m}_2(t) = 2(1 - \gamma) \sinh(\mathcal{R}_2(\eta_1^{(n)}(t), m_2(t)) - 2m_2(t) \cosh(\mathcal{R}_2(\eta_1^{(n)}(t), m_2(t))), \\ \dot{\eta}_1^{(0)}(t) = k \left(-\eta_1^{(0)}(t) + m_1(t) \right), \\ \dot{\eta}_1^{(j)}(t) = k \left(-\eta_1^{(j)}(t) + \eta_1^{(j-1)}(t) \right), \quad \text{for } j = 1, \dots, n, \\ \dot{\eta}_2^{(0)}(t) = k \left(-\eta_2^{(0)}(t) + m_2(t) \right), \\ \dot{\eta}_2^{(j)}(t) = k \left(-\eta_2^{(j)}(t) + \eta_2^{(j-1)}(t) \right), \quad \text{for } j = 1, \dots, n. \end{cases} \quad (2.2.5)$$

with initial conditions $m_1(0) = \gamma(\nu_1(+1) - \nu_1(-1))$, $m_2(0) = (1 - \gamma)(\nu_2(+1) - \nu_2(-1))$ and $\eta_r^{(j)}(0) = \xi_r$ for $j = 0, \dots, n$ and $r = 1, 2$.

Proof. Essentially the same computations of the proof of Theorem 2.1.1. \square

2.3 Emergence of periodic behavior

We now analyse the macroscopic dynamics, both for the model without delay and with delay. We do not seek to provide a complete description of the phase transitions of (2.1.5) and (2.2.5), our goal is rather to understand which configurations of the interaction network can enhance the appearance of periodicity in the macroscopic law.

2.3.1 The case without delay

Consider the pair of ODEs (2.1.5), describing the macroscopic dynamics of the bi-populated Curie-Weiss model (without delay). It is immediate to check that $(0, 0)$ is an equilibrium for any choice of the parameters $\gamma, J_{11}, J_{12}, J_{21}, J_{22}$. By studying the linearization of the dynamical system around the origin, we look for the presence of a Hopf bifurcation, which occurs when a stable periodic orbit arises from an equilibrium point as, at some critical values of the parameters, it loses stability. It is easy to check that the matrix given by the linearization of (2.1.5) around $(0, 0)$ is

$$A = \begin{pmatrix} 2(\gamma J_{11} - 1) & 2\gamma J_{12} \\ 2(1 - \gamma)J_{21} & 2((1 - \gamma)J_{22} - 1) \end{pmatrix}$$

whose characteristic polynomial reads

$$p_A(\lambda) = \lambda^2 - 2\lambda(\gamma J_{11} + (1 - \gamma)J_{22} - 2) + 4(\gamma J_{11} - 1)((1 - \gamma)J_{22} - 1) - 4\gamma(1 - \gamma)J_{12}J_{21}.$$

A Hopf bifurcation can be identified when a pair of distinct eigenvalues of A cross the imaginary axis: so, a Hopf bifurcation occurs at the origin for (2.1.5) if and only if both

the conditions

$$\begin{aligned} \gamma J_{11} - 1 &= -((1 - \gamma)J_{22} - 1), \\ (\gamma J_{11} - 1)^2 + \gamma(1 - \gamma)J_{12}J_{21} &< 0 \end{aligned} \tag{2.3.1}$$

are satisfied. In particular, this means that:

- if $J_{11}, J_{22} \leq 0$, the equality in (2.3.1) is never satisfied, thus the system (2.1.5) never undergoes a Hopf bifurcation.
- If $J_{12}J_{21} \geq 0$, the inequality in (2.3.1) has no solution and again it is not possible to find a Hopf bifurcation.
- The values of parameters to get a Hopf bifurcation can be properly chosen in the set $\{J_{11}, J_{22} \leq 0\}^c \cap \{J_{12}J_{21} \geq 0\}^c$.

2.3.2 The case with delay

We now move to the analysis of (2.2.5), describing the macroscopic evolution of the bi-populated Curie-Weiss model with delay. We expect that whenever a Hopf bifurcation is present for (2.1.5) the same should hold (possibly at a different critical value) for (2.2.5): for example, notice that for k large, (2.2.5) can be reduced to a planar system close to (2.1.5). Therefore, our aim is to answer the following question: can delay enhance the appearance of collective periodic behavior in configurations for which it was not possible for the case without delay? In what follows we show that the answer is yes, in particular we can induce periodicity in a subspace of the phase $\{J_{11}, J_{22} \leq 0\}$, where limit cycles were absent for (2.1.5).

Consider (2.2.5) with $J_{11}, J_{22} < 0$ and $J_{12}J_{21} < 0$. As before, we are going to linearize the dynamics around the origin, which is an equilibrium point for all choice of the parameters. Notice that the dimension of the system is $2n + 4$: since it is non-planar, to detect a super-critical Hopf bifurcation it does not suffice looking for a pair of pure imaginary conjugate eigenvalues, but it is also necessary to check that all the $2n + 2$ others have negative real part [44]. To simplify computations, assume

$$\gamma J_{11} = (1 - \gamma)J_{22} = -\frac{k}{2} + 1 \tag{2.3.2}$$

with $k > 2$ and representing the same constant appearing in the definition of the delay kernel. This choice drastically simplifies computations, but we believe it is not an essential ingredient for the occurrence of a Hopf bifurcation. Under (2.3.2), by re-arranging variables,

the linearization of (2.2.5) reads

$$\begin{pmatrix} \dot{m}_1(t) \\ \dot{m}_2(t) \\ \dot{\eta}_1^{(0)}(t) \\ \dot{\eta}_2^{(0)}(t) \\ \dot{\eta}_1^{(1)}(t) \\ \dot{\eta}_2^{(1)}(t) \\ \vdots \\ \dot{\eta}_1^{(n)}(t) \\ \dot{\eta}_2^{(n)}(t) \end{pmatrix} = \begin{pmatrix} -k & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \tilde{J}_{12} \\ 0 & -k & 0 & 0 & 0 & \cdots & 0 & \tilde{J}_{21} & 0 \\ k & 0 & -k & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & k & 0 & -k & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & k & 0 & -k & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & k & 0 & -k \end{pmatrix} \begin{pmatrix} m_1(t) \\ m_2(t) \\ \eta_1^{(0)}(t) \\ \eta_2^{(0)}(t) \\ \eta_1^{(1)}(t) \\ \eta_2^{(1)}(t) \\ \vdots \\ \eta_1^{(n)}(t) \\ \eta_2^{(n)}(t) \end{pmatrix}$$

with $\tilde{J}_{12} = 2\gamma J_{12}$ and $\tilde{J}_{21} = 2(1 - \gamma)J_{21}$. A tedious computation (but not too long since it rapidly reduces to calculating the determinants of triangular and tridiagonal matrices) yields the characteristic polynomial for the matrix above:

$$p(\lambda) = (k + \lambda)^{2n+4} - k^{2n+2}4\gamma(1 - \gamma)J_{12}J_{21}.$$

Let us denote with x_j , $j = 0, \dots, 2n + 3$, the j -th eigenvalue, so that

$$x_j = -k - |4\gamma(1 - \gamma)J_{12}J_{21}|^{\frac{1}{2n+4}} k^{\frac{n+1}{n+2}} \exp \left[i \frac{(2j + 1)\pi}{2n + 4} \right].$$

At this point, it is easy to check that for

$$k = \sqrt{|4\gamma(1 - \gamma)J_{12}J_{21}|} \left(\cos \left(\frac{\pi}{2n + 4} \right) \right)^{n+2} \quad (2.3.3)$$

x_{n+1} and x_{n+2} is a pair of distinct conjugate eigenvalues crossing the imaginary axis with positive derivative, while all other eigenvalues have strictly negative real parts: therefore, a Hopf bifurcation occur.

2.3.3 Some comments

We were interested in understanding which structures of the interaction network could give rise to a periodic behavior in the macroscopic law of the two-population Curie-Weiss model, without and with delay. Our findings are summarized by Table 2.3.3, which gives information about the possible emergence of macroscopic oscillations for the interaction network configurations depicted in the left column. Notice that by stating that periodic behavior is present, we mean that there exists a choice of the parameters (in details, satisfying (2.3.1) for (2.1.5) and (2.3.2)-(2.3.3) for (2.2.5)) for which a Hopf bifurcation occur at the origin.

As already mentioned, collective periodic behavior in multi-populated systems with delayed interactions are not a novelty in literature [23, 55], hence the most interesting result

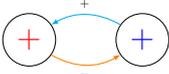
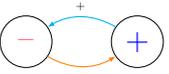
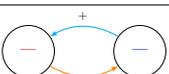
| Dynamics Interactions | <i>without</i> delay | <i>with</i> delay |
|---|------------------------------|-------------------|
|  | Rhythmic behavior | Rhythmic behavior |
|  | Rhythmic behavior | Rhythmic behavior |
|  | Rhythmic behavior | Rhythmic behavior |
|  | Rhythmic behavior | Rhythmic behavior |

Table 2.3.1: Qualitative summary of the results. In the left column a schematic representation of the considered interaction network is displayed. The color convention for couplings is as in Figure 2.1. For each interaction network we highlight the possibility of *observing* or *not observing* periodic behavior when considering the dynamics (2.1.5) (central column) or (2.2.5) (right column). Notice that, in all cases except for one, delay is not necessary to produce rhythmic oscillations.

here is that delay is not an essential ingredient for the emergence of macroscopic oscillations in the bi-populated Curie-Weiss model. The key feature seems rather to be the presence of a *frustration* in the interaction network: this fact, in the case without delay, can be understood in the following terms. If the intra-population interaction strengths J_{11} and J_{22} are large enough, each single population can be seen as a macrospin that, under Glauber dynamics, tends to its own rest state. However, as soon as the two population are linked together with within intra-population interactions such that $J_{12}J_{21} < 0$, they form a *frustrated* pair of macrospins where the rest state of the first is not compatible with the rest position of the second. As a consequence, the dynamics is not driven to a fixed equilibrium and keeps on oscillating. We remark that similar result should hold even in the multi-populated extensions, granted the presence of frustrations in the interaction network. However, the mechanism of frustration enhancing periodic behavior seems to rely strongly on the mean field assumptions (see simulations in [15]).

To sum up, our results suggest that delay is not always needed to create periodicity in multi-populated models. Nevertheless, delay can enhance the appearance of macroscopic rhythm in configurations in which it was absent in the case without delay. In particular, this occurs in the case $J_{11} < 0$, $J_{22} < 0$ and $J_{12}J_{21} < 0$. A possible intuitive explanation of this difference between the dynamics with and without delay is as follows. In the case without delay, both populations have anti-ferromagnetic intra-population interaction ($J_{11} < 0$ and $J_{22} < 0$) so they don't act as macrospins and the rest state for both of them is

zero. Notice that whenever $m_{1,N}$ and $m_{2,N}$ are close to zero, infra-population interactions are very small and frustration in the network can be neglected. Instead, when delay is present, each population receives the information on the other one through a smoothing of the past trajectory (represented by the terms $\eta_{1,N}^{(n)}, \eta_{2,N}^{(n)}$), which regularizes and amplifies the random fluctuations of each population around zero. This mechanism produces a sort of positive feedback thanks to the presence of frustration, and in the end macroscopic oscillations appear.

Chapter 3

An Ising model with dissipation

In Chapter 1 we examined the role of dissipation in enhancing macroscopic periodic behavior in cooperative mean field models. In this chapter, we show that this phenomenon is not strictly related to the mean field setting: we consider a short-range interacting cooperative system (a 1-dimensional Ising model) modified by introducing dissipation and we prove the convergence of a properly time-rescaled version of the magnetization to an oscillating process, in a suitable zero-temperature and infinite-volume limit.

3.1 The model and a description of results

In this section we present an Ising model with dissipation and we describe the results we aim to prove.

Let $\mathcal{S} = \{-1, +1\}$ and consider a configuration of N -spins $\underline{\sigma} \in \mathcal{S}^{\Lambda_N}$, where

$$\Lambda_N = \{1, 2, \dots, N\} \subseteq \mathbb{Z}$$

represents the set of sites of the spins. Now, differently from the mean field models studied so far, the geometry of Λ_N is relevant: for $i \in \Lambda_N$, each spin σ_i only interacts with its first neighbours σ_{i+1} and σ_{i-1} . We assume periodic boundary condition, i.e. $\sigma_{N+1} \equiv \sigma_1$ and $\sigma_0 \equiv \sigma_N$ (one may think of the sites $\sigma_1, \dots, \sigma_N$ sitting on a circle).

We will study the process $(\underline{\sigma}(t), \underline{\lambda}(t))_{t \geq 0}$ with values in $\mathcal{S}^{\Lambda_N} \times \mathbb{R}^N$ evolving according to the following dissipated dynamics: at a given time $t \geq 0$ each transition $\sigma_i(t) \rightarrow -\sigma_i(t)$, $i \in \Lambda_N$, occurs with rate

$$r_i(t) := \exp(-\sigma_i(t)\lambda_i(t)), \quad (3.1.1)$$

where $\{\lambda_i(t)\}_{i \in \Lambda_N}$ is a family of stochastic processes (*local fields*) evolving according to

$$d\lambda_i(t) = -\alpha\lambda_i(t)dt + \beta dm_i(t), \quad i \in \Lambda_N \quad (3.1.2)$$

with $\alpha, \beta > 0$ and

$$m_i(t) = \sum_{j \sim i} \sigma_j(t), \quad i \in \Lambda_N, \quad (3.1.3)$$

where $i \sim j$ denotes the set of sites j which are neighbors of i (namely, $i-1$ and $i+1$). Since we are no longer in a mean field context, there is no need to normalize local magnetization (3.1.3) (one could normalize m_i by a factor 2 and this would bring very little changes in the rest of our study).

Formally speaking, $(\underline{\sigma}(t), \underline{\lambda}(t))_{t \geq 0}$ is a Markov process with infinitesimal generator

$$\mathcal{L}_N f(\underline{\sigma}, \underline{\lambda}) = \sum_{i \in \Lambda_N} \exp[-\sigma_i \lambda_i] (f(\underline{\sigma}^i, \underline{\lambda} - 2\beta \sigma_i \underline{v}^i) - f(\underline{\sigma}, \underline{\lambda})) - \alpha \lambda_i f_{\lambda_i}(\underline{\sigma}, \underline{\lambda}), \quad (3.1.4)$$

where f_{λ_i} represents the partial derivative of f with respect to λ_i , $\underline{\sigma}^i$ is the configuration obtained by flipping the state of the i -th spin (see (1.2.1)) and \underline{v}^i is a N -dimensional vector such that

$$v_k^i = \begin{cases} 1, & k = i + 1 \text{ or } k = i - 1, \\ 0, & \text{otherwise.} \end{cases}$$

In what follows, we will assume the following initial conditions:

$$\sigma_i(0) = -1, \quad \lambda_i(0) = -\gamma, \quad \text{for any } i \in \Lambda_N \text{ with } \gamma > 0. \quad (3.1.5)$$

Remark 3.1.1. By taking $\alpha = 0$ (i.e. ruling out dissipation), we obtain a Glauber dynamics for the classical 1-dimensional Ising model with periodic boundary conditions and inverse temperature β .

Our aim is to show that in a suitable large volume - low temperature limit, the total magnetization of the system has a rhythmic behavior after a proper time scaling: we briefly describe the phenomenon here.

Assuming initial conditions 3.1.5, the analysis of the evolution of $(\underline{\sigma}, \underline{\lambda})_{t \geq 0}$ is divided into two parts. We begin by studying the occurrence time of the first spin flip. Unlike the case with no dissipation ($\alpha = 0$), where this time is exponentially distributed, the dissipation produces a much higher concentration of the distribution of this time: indeed, it will converge to a deterministic value as $\gamma, N \uparrow +\infty$. After the first spin-flip occurs, the change in the local field and the low temperature ($\beta \uparrow +\infty$) favours the growth of a ‘‘droplet’’ (just a segment in the one-dimensionale case) of $+1$ spins, which invades the whole state space in an extremely short time scale. At this point we are back to the situation of all equal spins. We will show that by assigning the initial local fields $\lambda_i(0)$ (i.e. choosing γ) in a suitable way, the local fields at the time the droplet has invaded the space is essentially opposite to the initial one, producing the iteration of the same phenomenon. Since the two parts of the evolution (waiting for the first spin-flip and covering by the droplet) occur on different time scales, we will consider a time-rescaled magnetization process to analyse the macroscopic behavior.

To guarantee the phenomenon described above to occur with overwhelming probability, we will assume $\beta, N \uparrow \infty$ in such a way that $\frac{\log N}{\beta} \rightarrow c \in [0, 1[$. This assumption assures that, after the first spin-flip, the droplet of $+1$ spins covers the whole space *before* the birth of other droplets. As we will see in Section 3.3, this allows a good understanding for the time taken by the droplet to cover Λ_N . Indeed, if $\frac{\log N}{\beta} \rightarrow c \in [1, 2[$, a single droplet cannot

invade the whole space: in this case, the box size N is too big to be covered by a single droplet, and many other droplets of $+1$ spins appear. We believe that this does not rule out periodic behavior, but it makes the analysis considerably harder.

In what follows, we will see that in the regime $\frac{\log N}{\beta} \rightarrow c \in [0, 1[$ the waiting time for the first spin flip is large, but has small fluctuations. These fluctuations, however, have impact on the growth time of the droplet. For this reason, while the waiting time of the first spin flip, rescaled by its mean, has a deterministic limit, the rescaled growth time of the droplet keeps some randomness in the limit. Due to this fact, the macroscopic evolution will not be periodic, but it will present regular oscillations with stochastic rhythm.

In Section 3.2 we analyse the limiting distribution of the stopping time at which the first spin flip is observed, in Section 3.3 we study the time scale at which the covering occurs. Finally, these results will be exploited in Section 3.4, where the limiting behavior of the magnetization is presented.

Remark 3.1.2. Some results in this chapter (such as the analysis of the covering time in Theorem 3.3.1) may resemble statements that can be encountered in metastability studies (for example [42] and references therein). However, there are some fundamental differences between our enquiry and metastability ones. The most important distinction is that the major part of metastability results are achieved for reversible system, for which there exists a Hamiltonian energy which guides the transitions between metastable equilibria. In our case, the mechanism which enhance the phenomenon is dissipation, which makes the dynamics of our system irreversible, therefore a Hamiltonian description of energy is lacking. Moreover, some metastability studies are only concerned with the time at which a droplet of critical size appears. Here instead we have to control the growth of the droplet until it covers all the volume, since, in order to look for periodic orbits, we have to keep track of the evolution of the local fields until all spins have flipped to the opposite sign. In what follows, we obtain sharp estimates on the time scale at which the droplet grows by coupling it with two other droplets, one faster and one slower, a technique present in some metastability studies. However, most of the general metastability techniques have been developed in the reversible context so, as mentioned above, they can not be applied in our case.

3.1.1 Graphical construction of the process

For any $\beta > 0$, the family of processes $\{(\underline{\sigma}(t), \underline{\lambda}(t))_{t \geq 0}\}_{N \geq 1}$ can be defined on a common probability space through the following *graphical construction*. This standard construction is often used to couple the main process with other processes which can be useful in technical proofs (see [50] for a nice example).

Let $\{\mathcal{N}_i\}_{i \in \mathbb{N}}$ be a family of i.i.d. Poisson processes of intensity $e^{4\beta}$ and denote the successive arrival times of the i -th Poisson process with $\{\tau_{i,n}\}_n$. Each arrival time $\tau_{i,n}$ is associated

with a random variable $U_{i,n}$, uniformly distributed on $[0, 1]$. The random variables $\{U_{i,n}\}_{i,n}$ are independent among themselves and independent from the Poisson processes $\{\mathcal{N}_i\}_i$. This concludes the construction of the probability space. For a fixed $N > 1$, the process $(\underline{\sigma}, \underline{\lambda})$ evolves according to what follows: each site $i \in \Lambda_N$ is associated with the process \mathcal{N}_i ; then, each point $\tau_{i,n}$ is *accepted* for a spin flip only if

$$\frac{\exp[-\sigma_i(\tau_{i,n})\lambda_i(\tau_{i,n})]}{e^{4\beta}} > U_{i,n}.$$

Whenever a point $(\tau_{i,n})$ is accepted, the spin at site i is flipped and the values of the local fields updated in the following way:

$$\lambda_k(\tau_{i,n}) = \begin{cases} \lambda_k(\tau_{i,n}^-) - \sigma_i(\tau_{i,n}^-)2\beta, & k = i + 1, i - 1 \\ \lambda_k(\tau_{i,n}^-), & \text{otherwise} \end{cases}$$

At any time in which there is no accepted spin flips, the local fields evolves according to

$$\dot{\lambda}_i(t) = -\alpha\lambda_i(t), \quad i \in \Lambda_N.$$

One can check that this construction provides the rates prescribed by (3.1.4). Other processes will be later coupled with $(\underline{\sigma}, \underline{\lambda})$ using this graphical construction.

3.2 First spin flip

Denote by $-\gamma < 0$, that for the time being we assume independent of the site, the initial value of the local fields (see (3.1.5)). At a later stage, γ will be chose depending on β and with a possible dependence on the site i . The time of the first spin flip T_1 is the time of the first jump in N i.i.d. counting processes $(Z_i)_{i=1}^N$ with intensity

$$\mu(t) = \exp[-\gamma e^{-\alpha t}].$$

In other words

$$T_1 := \inf\{t \geq 0 : \sum_{i=1}^N Z_i(t) = 1\}.$$

Note that

$$P(T_1 > t) = \exp\left[-N \int_0^t e^{-\gamma e^{-\alpha s}} ds\right]. \quad (3.2.1)$$

The asymptotics of the distribution of T_1 , as $N, \gamma \uparrow +\infty$, are computed using (3.2.1), and are described in the following result.

Proposition 3.2.1. *Suppose $N, \gamma \uparrow +\infty$ with the condition $\frac{N}{\gamma}e^{-\gamma} \rightarrow 0$. Then*

$$\alpha \log N \left(T_1 - t(\gamma, N) \right) \xrightarrow[\gamma, N \rightarrow +\infty]{d} X, \quad (3.2.2)$$

where X is a random variable distributed according to

$$P(X > x) = \exp(-e^x), \quad \forall x \in \mathbb{R}$$

and

$$t(\gamma, N) := \frac{1}{\alpha} \log \frac{\gamma}{L^{-1}(L(\gamma) + \frac{\alpha}{N})} \quad (3.2.3)$$

where, for $x > 0$,

$$L(x) = \int_x^{+\infty} \frac{e^{-y}}{y} dy.$$

Moreover, we have the following asymptotics:

$$t(\gamma, N) = \frac{1}{\alpha} \left(\log \gamma - \log \log N + \frac{\log \log N}{\log N} \right) + o\left(\frac{\log \log N}{\log N}\right), \quad (3.2.4)$$

$$\lambda_i(T_1^-) = -\log N + \log \log N + \log \alpha + X_N + o(1), \quad \forall i \in \Lambda_N, \quad (3.2.5)$$

where the term $o(1)$ denotes a remainder that goes to zero in probability as $\gamma, N \uparrow +\infty$ and

$$X_N := \alpha \log N \left(T_1 - t(\gamma, N) \right).$$

Remark 3.2.1. Note that the asymptotics of $\lambda_i(T_1^-)$ do not depend on the initial datum γ : this fact will be useful in the following (see Section 3.4).

The proof of Proposition 3.2.1 is based on the following lemmas.

Lemma 3.2.1. Fix $\gamma > 0$ and $c > 0$ and denote with $\tau(\gamma, c)$ the quantity such that

$$\int_0^{\tau(\gamma, c)} e^{-\gamma e^{-\alpha s}} ds = \frac{1}{c}.$$

Then,

$$\tau(\gamma, c) = \frac{1}{\alpha} \log \frac{\gamma}{L^{-1}(L(\gamma) + \frac{\alpha}{c})}. \quad (3.2.6)$$

Proof. Consider the function,

$$\varphi(t, \gamma) = \int_0^t e^{-\gamma e^{-\alpha s}} ds$$

for $t > 0$ and $\gamma > 0$. Then, fix $c > 0$ and let $\tau(\gamma, c)$ indicate the quantity such that

$$\varphi(\tau(\gamma, c), \gamma) = \frac{1}{c}. \quad (3.2.7)$$

Our aim is to prove that $\tau(\gamma, c)$ satisfies (3.2.6). By (3.2.7) and the implicit function theorem we have that

$$\partial_\gamma \tau(\gamma, c) = -\frac{\partial_\gamma \varphi(\tau(\gamma, c), \gamma)}{\partial_t \varphi(\tau(\gamma, c), \gamma)}.$$

Notice that

$$\partial_t \varphi(t, \gamma) = e^{-\gamma e^{-\alpha t}}$$

while

$$\partial_\gamma \varphi(t, \gamma) = - \int_0^t e^{-\gamma e^{-\alpha s}} e^{-\alpha s} ds = \frac{1}{\alpha} \int_1^{e^{-\alpha t}} e^{-\gamma y} dy = \frac{1}{\alpha \gamma} \left(e^{-\gamma} - e^{-\gamma e^{-\alpha t}} \right).$$

Hence, it holds that

$$\partial_\gamma \tau(\gamma, c) = \frac{1}{\alpha \gamma} \left(1 - e^{-\gamma(1-e^{-\alpha \tau(\gamma, c)})} \right). \quad (3.2.8)$$

Consider now the function

$$w(\gamma) = \frac{\log \gamma}{\alpha} - \tau(\gamma, c) :$$

by (3.2.8) we get

$$w'(\gamma) = \frac{1}{\alpha \gamma} - \partial_\gamma \tau(\gamma, c) = \frac{1}{\alpha \gamma} e^{-\gamma(1-e^{-\alpha \tau(\gamma, c)})} = \frac{1}{\alpha \gamma} e^{e^{\alpha w(\gamma)} - \gamma}.$$

The last equation can be integrated using the separation of variables method, so that

$$\int_{w(1)}^{w(\gamma)} e^{-e^{\alpha y}} dy = \frac{1}{\alpha} \int_1^\gamma \frac{e^{-x}}{x} dx. \quad (3.2.9)$$

It is convenient to write the left-hand side integral of (3.2.9) in the following way:

$$\int_{w(1)}^{w(\gamma)} e^{-e^{\alpha y}} dy = \int_0^{w(\gamma)} e^{-e^{\alpha y}} dy - \int_0^{w(1)} e^{-e^{\alpha y}} dy.$$

Notice that

$$\int_0^{w(1)} e^{-e^{\alpha y}} dy \stackrel{\dagger}{=} - \int_0^{\tau(1, c)} e^{-e^{-\alpha s}} ds = -\varphi(\tau(1, c), 1) = -\frac{1}{c}$$

where in \dagger we used the substitution $s = -y$, while

$$\int_0^{w(\gamma)} e^{-e^{\alpha y}} dy \stackrel{\ddagger}{=} \frac{1}{\alpha} \int_1^{e^{\alpha w(\gamma)}} \frac{e^{-x}}{x} dx$$

where in \ddagger we used the substitution $x = e^{\alpha y}$.

Therefore, from (3.2.9) we get

$$\frac{1}{\alpha} \int_1^{e^{\alpha w(\gamma)}} \frac{e^{-x}}{x} dx + \frac{1}{c} = \frac{1}{\alpha} \int_1^\gamma \frac{e^{-x}}{x} dx,$$

which implies

$$\frac{1}{\alpha} \int_{e^{\alpha w(\gamma)}}^\gamma \frac{e^{-x}}{x} dx = \frac{1}{c}. \quad (3.2.10)$$

Recall that we denoted with $L(x)$ the function

$$L(x) = \int_x^{+\infty} \frac{e^{-y}}{y} dy,$$

so by (3.2.10) we get

$$\begin{aligned} L(e^{\alpha w(\gamma)}) - L(\gamma) &= \frac{\alpha}{c} \\ e^{\alpha w(\gamma)} &= L^{-1}\left(L(\gamma) + \frac{\alpha}{c}\right) \\ w(\gamma) &= \frac{1}{\alpha} \log L^{-1}\left(L(\gamma) + \frac{\alpha}{c}\right) \end{aligned}$$

which, since $w(\gamma) = \frac{\log \gamma}{\alpha} - \tau(\gamma, c)$, implies (3.2.6). \square

It follows from Lemma 3.2.1 and (3.2.3), that

$$t(\gamma, N) = \tau(\gamma, N). \quad (3.2.11)$$

Lemma 3.2.2. *It holds that*

$$\lim_{z \rightarrow 0^+} L^{-1}(z) - \log \frac{1}{z} + \log \log \frac{1}{z} = 0.$$

Proof. Fix $\rho \in \mathbb{R}$: notice that (we are using L'Hôpital's rule)

$$\lim_{x \rightarrow +\infty} \frac{L(x - \log x + \rho)}{e^{-x}} = \lim_{x \rightarrow +\infty} \frac{-\frac{x}{x - \log x + \rho} e^{-x} e^{-\rho} \left(1 - \frac{1}{x}\right)}{-e^{-x}} = e^{-\rho}.$$

Fix now $\rho > 0$ so that

$$\lim_{x \rightarrow +\infty} \frac{L(x - \log x + \rho)}{e^{-x}} = e^{-\rho} < 1;$$

by continuity, there exists \bar{x} such that

$$e^{-x} \geq L(x - \log x + \rho), \quad \forall x > \bar{x}.$$

Let $z = e^{-x}$ and $\bar{z} = e^{-\bar{x}}$ so that

$$z \geq L\left(\log \frac{1}{z} - \log \log \frac{1}{z} + \rho\right), \quad \forall z > \bar{z};$$

since $L^{-1}(\cdot)$ is a decreasing function, we get

$$L^{-1}(z) - \log \frac{1}{z} + \log \log \frac{1}{z} \leq \rho, \quad \forall z > \bar{z}.$$

This is true for any $\rho > 0$, which implies

$$\limsup_{z \rightarrow 0^+} L^{-1}(z) - \log \frac{1}{z} + \log \log \frac{1}{z} \leq 0.$$

By an analogous argument (fixing $\rho < 0$ and following the same steps as above), one also gets

$$\liminf_{z \rightarrow 0^+} L^{-1}(z) - \log \frac{1}{z} + \log \log \frac{1}{z} \geq 0,$$

which concludes the proof. \square

Lemma 3.2.3. *Suppose $\gamma, N \uparrow +\infty$ with the condition $\frac{N}{\gamma}e^{-\gamma} \rightarrow 0$. Then for every $\rho > 0$*

$$\lim_{\gamma, N \uparrow +\infty} \alpha \log N [t(\gamma, N) - t(\gamma, \rho N)] = \log \rho. \quad (3.2.12)$$

Proof. Note first that $L(\gamma) \sim \frac{1}{\gamma}e^{-\gamma}$ as $\gamma \uparrow +\infty$, so that $NL(\gamma) \rightarrow 0$. Using this remark and Lemmas 3.2.1 and 3.2.2, we have the following asymptotic equivalences:

$$\begin{aligned} \alpha \log N [t(\gamma, N) - t(\gamma, \rho N)] &= \log N \left[\log \frac{L^{-1}\left(L(\gamma) + \frac{\alpha}{\rho N}\right)}{L^{-1}\left(L(\gamma) + \frac{\alpha}{N}\right)} \right] \\ &\sim \log N \left[\log \left(\frac{\log\left(L(\gamma) + \frac{\alpha}{\rho N}\right)}{\log\left(L(\gamma) + \frac{\alpha}{N}\right)} \right) \right] \\ &\sim \log N \left[\log \left(\frac{\log\left(\frac{\alpha}{\rho N}\right)}{\log\left(\frac{\alpha}{N}\right)} \right) \right] \\ &\sim \log N \left[\log \left(1 - \frac{\log \rho}{\log \frac{\alpha}{N}} \right) \right] \sim \log \rho. \end{aligned}$$

\square

Proof of Proposition 3.2.1. We begin by observing that, by (3.2.1), (3.2.6) and (3.2.11), for every $\rho > 0$

$$P(T_1 > t(\gamma, \rho N)) = e^{-\frac{1}{\rho}}. \quad (3.2.13)$$

Now, let $x \in \mathbb{R}$. By Lemma 3.2.3,

$$t(\gamma, N) = t(\gamma, e^{-x}N) - \frac{x}{\alpha \log N} + o\left(\frac{1}{\log N}\right).$$

Since $t(\gamma, N)$ is decreasing in the second variable, this implies that for every $a > 1$

$$t(\gamma, ae^{-x}N) \leq t(\gamma, N) + \frac{x}{\alpha \log N} \leq t(\gamma, a^{-1}e^{-x}N)$$

for γ and N sufficiently large. So

$$P(\alpha \log N [T_1 - t(\gamma, N)] > x) = P\left(T_1 > t(\gamma, N) + \frac{x}{\alpha \log N}\right) \leq P(T_1 > t(\gamma, ae^{-x}N)) = e^{-a^{-1}e^x}$$

and, similarly,

$$P(\alpha \log N[T_1 - t(\gamma, N)] > x) \geq P(T_1 > t(\gamma, a^{-1}e^{-x}N)) = e^{-ae^x}.$$

Since a can be chosen arbitrarily close to 1, we obtain

$$\lim P(\alpha \log N[T_1 - t(\gamma, N)] > x) = e^{-e^x}.$$

Finally, notice that (3.2.4) follows readily from (3.2.3) and Lemma 3.2.2 and that, by equation (3.1.2), the local fields $\lambda_i(t)$ decay exponentially with rate α up to time T_1 , i.e.

$$\lambda_i(T_1^-) = \lambda_i(0)e^{-\alpha T_1}.$$

Recalling that $\lambda_i(0) = -\gamma$, letting

$$X_N := \alpha \log N(T_1 - t(\gamma, N)),$$

using the estimate (3.2.4), we have for any $i \in \Lambda_N$,

$$\begin{aligned} \lambda_i(T_1^-) &= -\gamma \exp \left[-\alpha \left(t(\gamma, N) + \frac{X_N}{\alpha \log N} \right) \right] \\ &= -L^{-1} \left(L(\gamma) + \frac{\alpha}{N} \right) \exp \left[-\frac{X_N}{\log N} \right] \\ &= -\log N + \log \log N + \log \alpha + X_N + o(1), \end{aligned}$$

which proves (3.2.5). □

3.2.1 The inhomogeneous case

We now give a formulation of Proposition 3.2.1 taking a more general initial condition for the local fields: we allow

$$\lambda_i(0) = -\gamma + Y_N + \varepsilon_i(\gamma, N), \quad i \in \Lambda_N$$

where $\{Y_N\}_N$ is a sequence of random variables weakly converging to a random variable Y and $\varepsilon_i(\gamma, N)$ are small random perturbations which converge to 0 in probability. The reason for choosing this particular form is that it will be useful to show the convergence of the magnetization to an oscillating process. In Section 3.1 we have already given an intuitive overview of the phenomenon: we wait some time for the first spin-flip, after it we observe the growth of a droplet of $+1$ spins which very quickly covers the whole space. At this point we are back to a situation at which all spins are equal: the oscillatory behavior will originate by iteration of this phenomenon.

In more details, let us call τ the stopping time at which all spins have become positive. At time τ , we recover a uniform state for the spins ($\sigma_i(0) = -1$ and $\sigma_i(\tau) = +1$ for any

$i \in \Lambda_N$): in principle, we would obtain the iteration of the same behavior after time τ (i.e. waiting for a spin flip from $+1$ to -1 and formation of a droplet of -1 spins which covers the space) if the rates at time τ are such that $r_i(\tau) \approx r_i(0)$, for any $i \in \Lambda_N$. We will see that by properly choosing γ , one obtains with very large probability

$$r_i(0) = e^{-\gamma}, \quad r_i(\tau) = e^{-(\gamma + Y_N + \varepsilon_i(\gamma, N))}, \quad i \in \Lambda_N,$$

with Y_N and $\varepsilon_i(\gamma, N)$ as given above. This fact will be extensively explained in Section 3.4. We can anticipate that the randomness Y_N (common to all sites) is due to the fluctuations of time T_1 which have an impact of the growth of the droplet, while $\varepsilon_i(\gamma, N)$ represent some random inhomogeneities due to the action of dissipation during the growth of the positive droplet. Since the covering occurs in a very short time scale, these inhomogeneities are very small and uniformly converge to zero. We refer the reader to Section 3.4 for more details.

We restrict this generalization to the regime $\frac{\log N}{\gamma} \rightarrow k \in [0, 1[$, since will be sufficient for our main purpose.

Proposition 3.2.2. *Assume the following initial conditions for the local fields:*

$$\lambda_i(0) = -(\gamma + Y_N + \varepsilon_i(\gamma, N)), \quad i \in \Lambda_N \quad (3.2.14)$$

where $\{Y_N\}_N$ is a sequence of random variables weakly converging to a random variable Y and $\varepsilon_i(\gamma, N)$ are random variables such that

$$\lim_{\gamma, N \uparrow +\infty} \bar{\varepsilon}(\gamma, N) = \lim_{\gamma, N \uparrow +\infty} \underline{\varepsilon}(\gamma, N) = 0 \quad (3.2.15)$$

in probability, where

$$\bar{\varepsilon}(\gamma, N) := \max_{i=1, \dots, N} \varepsilon_i(\gamma, N) \quad \underline{\varepsilon}(\gamma, N) := \min_{i=1, \dots, N} \varepsilon_i(\gamma, N).$$

Assume that the random variables Y_N and $\varepsilon_i(\gamma, N)$, $i = 1, \dots, N$, are independent from the counting processes driving the evolution of the system for $t > 0$. Then, as $N, \gamma \uparrow +\infty$ with the condition $\frac{\log N}{\gamma} \rightarrow k \in [0, 1[$,

$$\alpha \log N \left(T_1 - t(\gamma, N) \right) \xrightarrow[\gamma, N \rightarrow +\infty]{d} X, \quad (3.2.16)$$

where X is a random variable distributed as $\tilde{X} + kY$, with \tilde{X} independent from Y and such that

$$P(\tilde{X} > x) = \exp(-e^x).$$

Moreover,

$$\lambda_i(T_1^-) = -\log N + \log \log N + \log \alpha + X_N + o_i(1), \quad \forall i \in \Lambda_N, \quad (3.2.17)$$

where the terms $o_i(1)$ denote remainders that uniformly go to zero in probability as $\gamma, N \uparrow +\infty$ and

$$X_N := \alpha \log N \left(T_1 - t(\gamma, N) \right).$$

Before proving Proposition 3.2.2, let us recall the following results.

Lemma 3.2.4 ([45], Section 3.1.5). *Let $\{f_n\}_n$ be a sequence of functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$ converging pointwise to a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$. If this convergence holds uniformly on compacts, then, for every converging sequence $\{x_n\}_n \subset \mathbb{R}$,*

$$\lim_n f_n(x_n) = f(x)$$

where $x = \lim_n x_n$.

Lemma 3.2.5 ([60], Theorem 3.4.4). *Let (S, m) and (S', m') be two metric spaces. Let g and g_n , $n \geq 1$, be measurable functions mapping (S, m) into (S', m') . Let (S', m') be separable. Let E be the set of x in S such that $g_n(x_n) \rightarrow g(x)$ fails for some sequence $\{x_n\}_n$ with $x_n \rightarrow x$ in S . Let X and X_n , $n \geq 1$ be random variables on S , with X_n converging in distribution to X in (S, m) and $P(X \in E) = 0$. Then, $g_n(X_n) \rightarrow g(X)$ in distribution in (S', m') .*

Proof of Proposition 3.2.2. First of all, consider the event

$$E_N = \{|Y_N| \leq \log \gamma\};$$

since the sequence $\{Y_N\}_N$ is tight (see Theorem 1.3. in [6]), $P(E_N) \rightarrow 1$ as $\gamma, N \uparrow +\infty$. Under the event E_N , for γ, N large enough the quantity $t(\gamma + Y_N, N)$ is well-defined. As a first step, we want to prove that

$$\alpha \log N(t(\gamma + Y_N, N)\mathbf{1}_{E_N} - t(\gamma, N)) \xrightarrow[\gamma, N \uparrow +\infty]{d} kY. \quad (3.2.18)$$

Notice that, for any $y \in \mathbb{R}$, by (3.2.3) and Lemma 3.2.1,

$$\begin{aligned} \alpha \log N(t(\gamma + y, N) - t(\gamma, N)) &= \log N\left(\log \frac{\gamma + y}{\gamma} + \log \frac{L^{-1}(L(\gamma + y) + \frac{\alpha}{N})}{L^{-1}(L(\gamma) + \frac{\alpha}{N})}\right) \\ &\approx \frac{\log N}{\gamma} y + o(1) \rightarrow ky. \end{aligned}$$

Since this convergence holds uniformly on compacts, by Lemma 3.2.4, Lemma 3.2.5 and the fact that $P(E_N) \rightarrow 1$, we get (3.2.18).

Now, let us show that taking the initial data without inhomogeneity, i.e.

$$\lambda_i(0) = -(\gamma + Y_N), \quad i \in \Lambda_N,$$

it holds that

$$\alpha \log N[T_1 - t(\gamma + Y_N, N)\mathbf{1}_{E_N}] \xrightarrow[\gamma, N \uparrow +\infty]{d} \tilde{X}. \quad (3.2.19)$$

Similarly to Lemma 3.2.2, we prove that for every $\rho > 0$,

$$\alpha \log N[t(\gamma + Y_N, N) - t(\gamma + Y_N, \rho N)]\mathbf{1}_{E_N} \rightarrow \log \rho \quad (3.2.20)$$

in probability as $\gamma, N \uparrow +\infty$. Conditioned to the realization of Y_N , T_1 is the time of the first jump in N i.i.d. counting processes with intensity

$$\mu(t) = \exp[-(\gamma + Y_N)e^{-\alpha t}].$$

With the same arguments of the proof of Proposition 3.2.1, by (3.2.20) one can see that

$$P(\alpha \log N[T_1 - t(\gamma + Y_N, N)] > x | Y_N) \mathbf{1}_{E_N} \leq e^{-a^{-1}e^x} \mathbf{1}_{E_N}$$

$$P(\alpha \log N[T_1 - t(\gamma + Y_N, N)] > x | Y_N) \mathbf{1}_{E_N} \geq e^{-ae^x} \mathbf{1}_{E_N}$$

with a chosen arbitrarily close to 1, which implies (3.2.19). Now we want to show that (3.2.19) holds as well in the inhomogeneous case, i.e. with initial conditions (3.2.14): consider the event

$$\bar{E}_N = \left\{ |Y_N| \leq \log \gamma, \max(|\bar{\varepsilon}(\gamma, N)|, |\underline{\varepsilon}(\gamma, N)|) \leq \frac{1}{\log \gamma} \right\}.$$

By consideration above and (3.2.15), $P(\bar{E}_N) \rightarrow 1$ as $\gamma, N \uparrow +\infty$. Notice that, in the same way we proved (3.2.18), it holds

$$\alpha \log N |t(\gamma + Y_N, N) - t(\gamma + Y_N + \bar{\varepsilon}(\gamma, N), N)| \mathbf{1}_{\bar{E}_N} \leq \frac{\log N}{\gamma \log \gamma} + o(1) \rightarrow 0,$$

so,

$$\begin{aligned} \log N |t(\gamma + Y_N, N) - t(\gamma + Y_N + \bar{\varepsilon}(\gamma, N), N)| \mathbf{1}_{\bar{E}_N} &\rightarrow 0, \\ \log N |t(\gamma + Y_N, N) - t(\gamma + Y_N + \underline{\varepsilon}(\gamma, N), N)| \mathbf{1}_{\bar{E}_N} &\rightarrow 0, \end{aligned} \quad (3.2.21)$$

both in probability.

Denote by \bar{T}_1 (respectively \underline{T}_1) the time of the first spin flip taking as initial conditions

$$\lambda_i(0) = -(\gamma + Y_N + \bar{\varepsilon}(\gamma, N)) \quad (\text{resp. } \lambda_i(0) = -(\gamma + Y_N + \underline{\varepsilon}(\gamma, N)))$$

for all $i \in \Lambda_N$. Hence, conditioned to Y_N and $\{\varepsilon_i(\gamma, N)\}_i$, \bar{T}_1 (resp. \underline{T}_1) is the first jump time of N i.i.d. counting processes of intensity

$$\exp[-(\gamma + Y_N + \bar{\varepsilon}(\gamma, N))e^{-\alpha t}] \quad (\text{resp. } \exp[-(\gamma + Y_N + \underline{\varepsilon}(\gamma, N))e^{-\alpha t}]).$$

By obvious monotonicity arguments, for every $t > 0$

$$P(\underline{T}_1 > t) \leq P(T_1 > t) \leq P(\bar{T}_1 > t).$$

Thus

$$\begin{aligned} P(\alpha \log N[T_1 - t(\gamma + Y_N, N)] > x | Y_N, \{\varepsilon_i(\gamma, N)\}) \mathbf{1}_{\bar{E}_N} &\leq \\ P(\alpha \log N[\bar{T}_1 - t(\gamma + Y_N, N)] > x | Y_N, \{\varepsilon_i(\gamma, N)\}) \mathbf{1}_{\bar{E}_N} &= \\ P(\alpha \log N[\bar{T}_1 - t(\gamma + Y_N + \bar{\varepsilon}(\gamma, N), N)] > x + \zeta_N | Y_N, \{\varepsilon_i(\gamma, N)\}) \mathbf{1}_{\bar{E}_N}, & \end{aligned}$$

where

$$\zeta_N = \alpha \log N(t(\gamma + Y_N, N) - t(\gamma + Y_N + \bar{\epsilon}(\gamma, N), N)).$$

By (3.2.21), $\zeta_N \rightarrow 0$ under \bar{E}_N so

$$\begin{aligned} \limsup P(\alpha \log N[T_1 - t(\gamma + Y_N, N)\mathbf{1}_{\bar{E}_N}] > x) &\leq \\ \lim P(\alpha \log N[\bar{T}_1 - t(\gamma + Y_N + \bar{\epsilon}(\gamma, N), N)\mathbf{1}_{\bar{E}_N}] > x) &= e^{-e^x}, \end{aligned}$$

where the last limit can be proved in the same way as (3.2.19). The opposite inequality can be readily shown, which let us conclude that under initial conditions (3.2.14),

$$\alpha \log N[T_1 - t(\gamma + Y_N, N)\mathbf{1}_{\bar{E}_N}] \xrightarrow[\gamma, N \uparrow +\infty]{d} \tilde{X}. \quad (3.2.22)$$

Therefore, by (3.2.18) and (3.2.22),

$$\begin{aligned} \alpha \log N[T_1 - t(\gamma, N)] &= \\ \alpha \log N[T_1 - t(\gamma + Y_N, N)\mathbf{1}_{E_N}] + \alpha \log N[t(\gamma + Y_N, N)\mathbf{1}_{\bar{E}_N} - t(\gamma, N)] &\xrightarrow[\gamma, N \uparrow +\infty]{d} \tilde{X} + kY. \end{aligned}$$

Finally, the asymptotics (3.2.17) is obtained as in Proposition 3.2.1. \square

3.3 Covering time

In this section we study the evolution of the spin system after time T_1 , so consider the processes $(\tilde{\sigma}(t), \tilde{\lambda}(t))_{t \geq 0}$ such that

$$\begin{aligned} \tilde{\sigma}_i(t) &= \sigma_i(t + T_1) \\ \tilde{\lambda}_i(t) &= \lambda_i(t + T_1) \end{aligned} \quad t \geq 0, \quad i \in \Lambda_N.$$

By the strong Markov property, the evolution of $(\tilde{\sigma}(t), \tilde{\lambda}(t))_{t \geq 0}$ is still described by (3.1.1) and (3.1.2). Define

$$T_c := \inf\{t > 0 : \tilde{\sigma}_i(t) = 1 \text{ for all } i \in \Lambda_N\}$$

the time needed to reach the homogeneous configuration with all spins equal to +1. The following theorem describes the asymptotic behavior of T_c as $\beta, N \uparrow +\infty$. In what follows we assume (3.2.14) as initial condition for the local fields.

Theorem 3.3.1. *Let $\gamma, \beta, N \uparrow +\infty$ with the condition*

$$\lim_{\beta, N \uparrow +\infty} \frac{\log N}{\beta} = c \quad \lim_{\gamma, N \uparrow +\infty} \frac{\log N}{\gamma} = k, \quad (3.3.1)$$

with $c, k \in [0, 1[$, and assume that conditions (3.2.14) and (3.2.15) hold. Then

$$\frac{T_c}{\frac{N^2}{2\alpha \log N} e^{-2\beta - X_N}} \xrightarrow[\gamma, \beta, N \rightarrow +\infty]{P} 1 \quad (3.3.2)$$

in probability, and

$$\frac{T_c}{\frac{N^2}{2\alpha \log N} e^{-2\beta}} \xrightarrow[\gamma, \beta, N \rightarrow +\infty]{d} Z, \quad (3.3.3)$$

where $X_N := \alpha \log N (T_1 - t(\gamma, N))$ and Z is a random variable distributed as e^{-X} , with X being the random variable introduced in Proposition 3.2.2.

Remark 3.3.1. Note that (3.3.3) follows immediately from (3.3.2) and (3.2.16), since e^{-X_N} converges in distribution to e^{-X} . It is worth noticing that the random fluctuations of the time T_c are only due to the fluctuations of T_1 that, despite of the fact they are small, produce fluctuations for the local fields $\lambda_i(T_1)$. Conditioned to these local fields, the invasion of the space by the +1 spins is essentially deterministic.

Proof of Theorem 3.3.1. Before going into the details of the proof, we give an intuition of what happens during the covering.

Let $\bar{i} \in \{1, 2, \dots, N\}$ be such that

$$\tilde{\sigma}_i(0) = \begin{cases} -1 & \text{for } i \neq \bar{i} \\ 1 & \text{for } i = \bar{i}. \end{cases}$$

The local field profile is given by

$$\tilde{\lambda}_i(0) = \begin{cases} 2\beta + \lambda_i(T_1^-) & \text{for } i = \bar{i} \pm 1 \\ \lambda_i(T_1^-) & \text{otherwise.} \end{cases} \quad (3.3.4)$$

where, by (3.2.17),

$$\lambda_i(T_1^-) = -\log N + \log \log N + \log \alpha + X_N + o_i(1). \quad (3.3.5)$$

Note that the spins at $\bar{i} \pm 1$ are likely to flip first, as, by (3.3.1), $2\beta \gg -\lambda_i(T_1^-)$ with very high probability. Suppose now that the first spin flip occurs at time τ_1 for the spin $\bar{i} + 1$. We have:

$$\tilde{\lambda}_i(\tau_1) = \begin{cases} [2\beta + \lambda_i(T_1^-)] e^{-\alpha\tau_1} & \text{for } i = \bar{i} \pm 1 \\ \lambda_i(T_1^-) e^{-\alpha\tau_1} + 2\beta & \text{for } i = \bar{i}, \bar{i} + 2 \\ \lambda_i(T_1^-) e^{-\alpha\tau_1} & \text{otherwise.} \end{cases} \quad (3.3.6)$$

In terms of the spin-flip rates $\tilde{r}_i(t) = \exp[-\tilde{\sigma}_i(t)\tilde{\lambda}_i(t)]$, note that $\tilde{r}_i(\tau_1) \leq 1$ with high probability for $i \neq \bar{i} - 1, \bar{i} + 2$, while

$$\tilde{r}_{\bar{i}-1}(\tau_1) = \exp[(2\beta + \lambda_{\bar{i}-1}(T_1^-)) e^{-\alpha\tau_1}]$$

and

$$\tilde{r}_{\bar{i}+2}(t) = \exp[\lambda_{\bar{i}+2}(T_1^-) e^{-\alpha\tau_1} + 2\beta].$$

It follows that the spins at $\bar{i} - 1$ and $\bar{i} + 2$ are likely to flip before the others. To have a better understanding, assume the spin at $\bar{i} - 1$ flips first, at time τ_2 . The local field profile at time τ_2 is

$$\tilde{\lambda}_i(\tau_2) = \begin{cases} [\lambda_i(T_1^-)e^{-\alpha\tau_1} + 2\beta] e^{-\alpha(\tau_2-\tau_1)} + 2\beta & \text{for } i = \bar{i} \\ [2\beta + \lambda_i(T_1^-)] e^{-\alpha\tau_2} & \text{for } i = \bar{i} - 1 \\ \lambda_i(T_1^-)e^{-\alpha\tau_2} + 2\beta & \text{for } i = \bar{i} - 2 \\ [2\beta + \lambda_i(T_1^-)] e^{-\alpha\tau_2} & \text{for } i = \bar{i} + 1 \\ [\lambda_i(T_1^-)e^{-\alpha\tau_1} + 2\beta] e^{-\alpha(\tau_2-\tau_1)} & \text{for } i = \bar{i} + 2 \\ \lambda_i(T_1^-)e^{-\alpha\tau_2} & \text{otherwise.} \end{cases} \quad (3.3.7)$$

Again, we see the spins at $\bar{i} \pm 2$ are likely to flip first. Thus with high probability, as we will see in details next, a droplet of consecutive +1 spins forms. Denote by τ_n the time at which a droplet of length $n + 1$ is formed, with $1 \leq n \leq N - 3$. At time τ_n the local field in the *interior* of the droplet is bounded below by $4\beta e^{-\alpha\tau_n} + \lambda_i(T_1^-)$. In the internal boundary of the droplet the local field is bounded below by $2\beta e^{-\alpha\tau_n} + \lambda_i(T_1^-)$. In the external boundary of the droplet the local field satisfies

$$\tilde{\lambda}_i(\tau_n) \in [2\beta + \lambda_i(T_1^-), 2\beta + \lambda_i(T_1^-)e^{-\alpha\tau_n}] \quad (3.3.8)$$

if i is the site neighbor of the last spin flipped, and

$$\tilde{\lambda}_i(\tau_n) \in [2\beta e^{-\alpha\tau_n} + \lambda_i(T_1^-), 2\beta + \lambda_i(T_1^-)e^{-\alpha\tau_n}] \quad (3.3.9)$$

for the other site. Note that the extremes of these interval may be reversed in the unlikely case $\lambda_i(T_1^-) > 0$ (see (3.3.5)). For all other sites the local field equals $\lambda_i(T_1^-)e^{-\alpha\tau_n}$. For $n = N - 2$ the situation is slightly different, since there is only one site in the external boundary of the droplet. This gives, for such site i ,

$$\tilde{\lambda}_i(\tau_{N-2}) \in [4\beta e^{-\alpha\tau_{N-2}} + \lambda_i(T_1^-), 4\beta + \lambda_i(T_1^-)e^{-\alpha\tau_{N-2}}].$$

This gives the intuition on how the local fields change according to the growth of the droplet of +1 spins. Notice that, with very large probability, the covering is performed (excluding the last flip) with subsequent steps each of them occurring with a rate in the form

$$2e^{2\beta - \log N + \log \log N + \log \alpha + X_N + o(1)} = \frac{2\alpha \log N}{N} e^{2\beta + X_N + o(1)}$$

and this is the intuitive reason to choose the time scaling

$$N(2e^{2\beta - \log N + \log \log N + \log \alpha + X_N})^{-1} = \frac{N^2}{2\alpha \log N} e^{-2\beta - X_N} \quad (3.3.10)$$

appearing in (3.3.2).

The strategy of the proof is to show that, in the limit, during the covering process only spins adjacent to the droplet will flip, then to show that (3.3.10) gives the correct time-scaling for the process where all undesired flips are suppressed.

Step 1: Probability of observing an undesired flip

Let $\bar{\tau}$ be the time at which an "undesired" flip occurs, i.e. the time at which we observe a flip of one of the spins that are *not* adjacent to the droplet. Our aim is to show that $P(\bar{\tau} \leq \tau_{N-1})$ converges to zero as $\beta, N \uparrow +\infty$. We estimate this probability conditioned to the event

$$A_N := \{-\log N \leq \lambda_i(T_1^-) \leq -\log N + 2 \log \log N : i = 1, \dots, N\},$$

whose probability tends to one.

Notice that, under A_N , for $t \in [0, \tau_1[$, we have one positive spin with flipping rate at most

$$e^{\log N - 2 \log \log N},$$

$N - 3$ negative spins whose rates are at most

$$e^{[-\log N + 2 \log \log N]e^{-\alpha t}},$$

and two negative spins, adjacent to the droplet of $+1$ spins, whose rates are at least

$$e^{[2\beta - \log N]e^{-\alpha t}}.$$

Then, $P(\bar{\tau} \leq \tau_1 | T_1) \mathbf{1}_{A_N}$ is bounded by the probability that the first point of a Poisson process of time-dependent intensity

$$I_1(t) := e^{\log N - 2 \log \log N} + (N - 3)e^{[-\log N + 2 \log \log N]e^{-\alpha t}}$$

occurs before the first point of a point process with time-dependent intensity

$$J(t) := e^{[2\beta - \log N]e^{-\alpha t}},$$

where the two processes are independent.

Then, under $A_N \cap (\bar{\tau} > \tau_1)$, for $t \in [\tau_1, \tau_2[$, we have a droplet comprised by two positive spins whose rates are at most

$$e^{[-2\beta + \log N]e^{-\alpha t}},$$

$N - 4$ negative spins whose rates are at most

$$e^{[-\log N + 2 \log \log N]e^{-\alpha t}},$$

and two negative spins, adjacent to the droplet of $+1$ spins, whose rates are at least

$$e^{[2\beta - \log N]e^{-\alpha t}}.$$

So, $P(\bar{\tau} \in]\tau_1, \tau_2][T_1, (\bar{\tau} > \tau_1))\mathbf{1}_{A_N}$ is bounded by the probability that the first point of a Poisson process of time-dependent intensity

$$I_2(t) := 2e^{[-2\beta + \log N]e^{-\alpha t}} + (N - 4)e^{[-\log N + 2 \log \log N]e^{-\alpha t}}$$

occurs before the first point of a point process with time-dependent intensity $J(t)$, where the two processes are independent.

Moreover, for any $k = 3, \dots, N - 2$, under $A_N \cap (\bar{\tau} > \tau_{k-1})$, for $t \in [\tau_{k-1}, \tau_k[$, we have a droplet comprised by k positive spins: two of them with rates at most

$$e^{[-2\beta + \log N]e^{-\alpha t}},$$

and $k - 2$ of them with rates at most

$$e^{[-4\beta + \log N]e^{-\alpha t}}.$$

The system will also present $N - k - 2$ negative spins whose rates are at most

$$e^{[-\log N + 2 \log \log N]e^{-\alpha t}},$$

and two negative spins, adjacent to the droplet of $+1$ spins, whose rates are at least

$$e^{[2\beta - \log N]e^{-\alpha t}}.$$

Then, for any $k = 3, \dots, N - 2$, $P(\bar{\tau} \in]\tau_{k-1}, \tau_k][T_1, (\bar{\tau} > \tau_{k-1}))\mathbf{1}_{A_N}$ is bounded by the probability that the first point of a Poisson process of time-dependent intensity

$$I_k(t) := 2e^{[-2\beta + \log N]e^{-\alpha t}} + (k - 2)e^{[-4\beta + \log N]e^{-\alpha t}} + (N - k - 2)e^{[-\log N + 2 \log \log N]e^{-\alpha t}}$$

occurs before the first point of a point process with time-dependent intensity $J(t)$, where the two processes are independent.

Finally, under $A_N \cap (\bar{\tau} > \tau_{N-2})$, for $t \in [\tau_{N-2}, \tau_{N-1}[$, we have a droplet comprised by $N - 1$ positive spins: two of them with rates at most

$$e^{[-2\beta + \log N]e^{-\alpha t}},$$

and $N - 3$ of them with rates at most

$$e^{[-4\beta + \log N]e^{-\alpha t}}.$$

The system also presents one negative spin, with rate at least

$$e^{[4\beta - \log N]e^{-\alpha t}}.$$

Then, $P(\bar{\tau} \in]\tau_{N-2}, \tau_{N-1}[T_1, (\bar{\tau} > \tau_{N-2}))\mathbf{1}_{A_N}$ is bounded by the probability that the first point of a Poisson process of time-dependent intensity

$$I_{N-1}(t) := 2e^{[-2\beta + \log N]e^{-\alpha t}} + (N - 3)e^{[-4\beta + \log N]e^{-\alpha t}}$$

occurs before the first point of a point process with time-dependent intensity $J(t)$, where the two processes are independent.

By the analysis above, we can consider a family of Poisson processes $\{\zeta_k\}_{k=1}^{N-1}$ with time-dependent intensities $\{I_k(t)\}_{k=1}^{N-1}$ and a Poisson process η with time dependent intensity $J(t)$. We also assume that, for any $k = 1, \dots, N-1$, the process ζ_k and the process η are independent. Let us denote with X_k the first point of the process ζ_k for any $k = 1, \dots, N-1$, and with Y the first point of the process η . In this way, we deduce that

$$\begin{aligned} P(\bar{\tau} \leq \tau_{N-1} | T_1) \mathbf{1}_{A_N} &= P(\bar{\tau} \leq \tau_1 | T_1) \mathbf{1}_{A_N} + \sum_{k=2}^{N-1} P(\bar{\tau} \in]\tau_{k-1}, \tau_k] | T_1) \mathbf{1}_{A_N} \\ &\leq P(\bar{\tau} \leq \tau_1 | T_1) \mathbf{1}_{A_N} + \sum_{k=2}^{N-1} P(\bar{\tau} \leq \tau_k | T_1, (\bar{\tau} > \tau_{k-1})) \mathbf{1}_{A_N} \\ &\leq \sum_{k=1}^{N-1} P(X_k < Y). \end{aligned}$$

Let us fix $T_m = e^{-d\beta}$ with d positive constant such that $2c + d < 2$ and observe that, for any $k = 1, \dots, N-1$,

$$\begin{aligned} P(X_k < Y) &\leq P(X_k < Y | Y \leq T_m) + P(Y > T_m) \\ &= \frac{P(X_k < Y, Y \leq T_m)}{P(Y \leq T_m)} + P(Y > T_m) \\ &= \frac{P(X_k < Y, Y \leq T_m, X_k \leq T_m)}{P(Y \leq T_m)} + P(Y > T_m) \\ &= \frac{P(X_k < Y, Y \leq T_m, X_k \leq T_m) P(X_k \leq T_m)}{P(Y \leq T_m) P(X_k \leq T_m)} + P(Y > T_m) \\ &= \frac{P(X_k < Y, Y \leq T_m, X_k \leq T_m) P(X_k \leq T_m)}{P(Y \leq T_m, X_k \leq T_m)} + P(Y > T_m) \\ &= P(X_k < Y | Y \leq T_m, X_k \leq T_m) P(X_k \leq T_m) + P(Y > T_m) \\ &\leq P(X_k < Y | Y \leq T_m, X_k \leq T_m) + P(Y > T_m) \end{aligned}$$

where we used the independence of X_k and Y , hence

$$P(\bar{\tau} \leq \tau_{N-1} | T_1) \mathbf{1}_{A_N} \leq \sum_{k=1}^{N-1} P(X_k < Y | Y \leq T_m, X_k \leq T_m) + (N-1)P(Y > T_m).$$

Notice that, conditioned to $(X_k \leq T_m, Y \leq T_m)$ the distribution of X_k is stochastically bigger than the one of an exponential random variable of parameter $I_k(T_m)$ and the distribution of Y is stochastically smaller than the one of an exponential r.v. of parameter $J(T_m)$. This means that, for any $k = 1, \dots, N-1$,

$$P(X_k < Y | Y \leq T_m, X_k \leq T_m) \leq \frac{I_k(T_m)}{I_k(T_m) + J(T_m)},$$

so we get

$$P(\bar{\tau} \leq \tau_{N-1}|T_1)\mathbf{1}_{A_N} \leq \sum_{k=1}^{N-1} \frac{I_k(T_m)}{I_k(T_m) + J(T_m)} + (N-1)P(Y > T_m). \quad (3.3.11)$$

Let's consider the second term on the right-hand side of (3.3.11):

$$\begin{aligned} (N-1)P(Y > e^{-d\beta}) &\leq (N-1) \exp \left[- \int_0^{e^{-d\beta}} e^{[2\beta - \log N]e^{-\alpha t}} dt \right] \\ &\leq \exp \left[\log N - e^{-d\beta} e^{[2\beta - \log N]e^{-\alpha e^{-d\beta}}} \right] \\ &\approx \exp \left[\log N - e^{(2-d)\beta - \log N + \alpha e^{-d\beta} [2\beta - \log N] + o(\beta e^{-d\beta})} \right] \xrightarrow{\beta, N \uparrow +\infty} 0, \end{aligned}$$

thanks to (3.3.1) and the fact that $c + d < 2$.

Consider now the first term in the right-hand side of (3.3.11). The following limits hold:

$$\begin{aligned} \frac{I_1(T_m)}{I_1(T_m) + J(T_m)} &= \\ &= \frac{e^{\log N - 2 \log \log N} + (N-3)e^{[-\log N + 2 \log \log N]e^{-\alpha T_m}}}{e^{\log N - 2 \log \log N} + (N-3)e^{[-\log N + 2 \log \log N]e^{-\alpha T_m}} + e^{[2\beta - \log N]e^{-\alpha T_m}}} \xrightarrow{\beta, N \uparrow +\infty} 0, \end{aligned}$$

$$\begin{aligned} \frac{I_2(T_m)}{I_2(T_m) + J(T_m)} &= \\ &= \frac{2e^{[-2\beta + \log N]e^{-\alpha T_m}} + (N-4)e^{[-\log N + 2 \log \log N]e^{-\alpha T_m}}}{2e^{[-2\beta + \log N]e^{-\alpha T_m}} + (N-4)e^{[-\log N + 2 \log \log N]e^{-\alpha T_m}} + e^{[2\beta - \log N]e^{-\alpha T_m}}} \xrightarrow{\beta, N \uparrow +\infty} 0, \end{aligned}$$

$$\begin{aligned} \frac{I_{N-1}(T_m)}{I_{N-1}(T_m) + J(T_m)} &= \\ &= \frac{2e^{[-2\beta + \log N]e^{-\alpha T_m}} + (N-3)e^{[-4\beta + \log N]e^{-\alpha T_m}}}{2e^{[-2\beta + \log N]e^{-\alpha T_m}} + (N-3)e^{[-4\beta + \log N]e^{-\alpha T_m}} + e^{[2\beta - \log N]e^{-\alpha T_m}}} \xrightarrow{\beta, N \uparrow +\infty} 0. \end{aligned}$$

Moreover, it holds that

$$\begin{aligned} \sum_{k=3}^{N-2} \frac{I_k(T_m)}{I_k(T_m) + J(T_m)} &= \\ &= \sum_{k=3}^{N-2} \frac{2e^{[-2\beta + \log N]e^{-\alpha T_m}} + (k-2)e^{[-4\beta + \log N]e^{-\alpha T_m}} + (N-k-2)e^{[-\log N + 2 \log \log N]e^{-\alpha T_m}}}{I_k(T_m) + J(T_m)} \leq \\ &= \frac{N2e^{[-2\beta + \log N]e^{-\alpha T_m}}}{J(T_m)} + \frac{N^2e^{[-4\beta + \log N]e^{-\alpha T_m}}}{J(T_m)} + \sum_{k=3}^{N-2} \frac{(N-k-2)e^{[-\log N + 2 \log \log N]e^{-\alpha T_m}}}{I_k(T_m) + J(T_m)}, \end{aligned}$$

where

$$\frac{N2e^{[-2\beta+\log N]e^{-\alpha T_m}}}{J(T_m)} = \frac{2e^{[-2\beta+\log N]e^{-\alpha T_m} + \log N}}{e^{[2\beta-\log N]e^{-\alpha T_m}}} \xrightarrow{\beta, N \uparrow +\infty} 0,$$

$$\frac{N^2e^{[-4\beta+\log N]e^{-\alpha T_m}}}{J(T_m)} = \frac{e^{[-4\beta+\log N]e^{-\alpha T_m} + 2\log N}}{e^{[2\beta-\log N]e^{-\alpha T_m}}} \xrightarrow{\beta, N \uparrow +\infty} 0,$$

while

$$\begin{aligned} & \sum_{k=3}^{N-2} \frac{(N-k-2)e^{[-\log N + 2\log \log N]e^{-\alpha T_m}}}{I_k(T_m) + J(T_m)} \leq \\ & \sum_{k=1}^{N-5} \frac{ke^{[-\log N + 2\log \log N]e^{-\alpha T_m}}}{ke^{[-\log N + 2\log \log N]e^{-\alpha T_m}} + J(T_m)} = \\ & \sum_{k=1}^{N-5} \frac{k}{k + e^{[2\beta - 2\log \log N]e^{-\alpha T_m}}} \leq \\ & \int_0^N \frac{x}{x + e^{[2\beta - 2\log \log N]e^{-\alpha T_m}}} dx = \\ & N + e^{[2\beta - 2\log \log N]e^{-\alpha T_m}} \log \left[\frac{e^{[2\beta - 2\log \log N]e^{-\alpha T_m}}}{N + e^{[2\beta - 2\log \log N]e^{-\alpha T_m}}} \right]. \end{aligned}$$

Notice that

$$\begin{aligned} & \lim_{\beta, N \uparrow +\infty} \left(N + e^{[2\beta - 2\log \log N]e^{-\alpha T_m}} \log \left[\frac{e^{[2\beta - 2\log \log N]e^{-\alpha T_m}}}{N + e^{[2\beta - 2\log \log N]e^{-\alpha T_m}}} \right] \right) = \\ & \lim_{\beta, N \uparrow +\infty} \left(N + e^{[2\beta - 2\log \log N]e^{-\alpha T_m}} \log \left[1 - \frac{N}{N + e^{[2\beta - 2\log \log N]e^{-\alpha T_m}}} \right] \right) = \\ & \lim_{\beta, N \uparrow +\infty} \left(N + e^{[2\beta - 2\log \log N]e^{-\alpha T_m}} \left(\frac{1}{2} \left(\frac{N}{N + e^{[2\beta - 2\log \log N]e^{-\alpha T_m}}} \right)^2 - \frac{N}{N + e^{[2\beta - 2\log \log N]e^{-\alpha T_m}}} \right) \right) = \\ & \lim_{\beta, N \uparrow +\infty} \left(\frac{N^2}{e^{[2\beta - 2\log \log N]e^{-\alpha T_m}}} + \frac{N^2 e^{[2\beta - 2\log \log N]e^{-\alpha T_m}}}{2e^{[4\beta - 4\log \log N]e^{-\alpha T_m}}} \right) = \\ & \lim_{\beta, N \uparrow +\infty} \frac{3}{2} \exp \left[2\log N - [2\beta - 2\log \log N]e^{-\alpha T_m} \right] = \\ & \lim_{\beta, N \uparrow +\infty} \frac{3}{2} \exp \left[2\log N - 2\beta + 2\log \log N + \alpha e^{-d\beta} (2\beta - 2\log \log N) \right] = 0, \end{aligned}$$

thanks to (3.3.1).

All these considerations imply that the probability that an undesired spin flip occurs before

the droplet of +1 spins invades the whole space converges to zero:

$$\begin{aligned}
P(\bar{\tau} \leq \tau_{N-1}) &= E(P(\bar{\tau} \leq \tau_{N-1}|T_1)\mathbf{1}_{A_N}) + P(A_N^c) \\
&\leq E\left(P(\bar{\tau} \leq \tau_1|T_1)\mathbf{1}_{A_N} + \sum_{k=2}^{N-1} P(\bar{\tau} \leq \tau_k|T_1, (\bar{\tau} > \tau_{k-1}))\mathbf{1}_{A_N}\right) + P(A_N^c) \\
&\leq \sum_{k=1}^{N-1} P(X_k < Y) + P(A_N^c) \\
&\leq \sum_{k=1}^{N-1} \frac{I_k(T_m)}{I_k(T_m) + J(T_m)} + (N-1)P(Y > T_m) + P(A_N^c) \xrightarrow{\beta, N \uparrow +\infty} 0.
\end{aligned}$$

Step 2: Time-scaling for the covering process

On the probability space described in Section 3.1.1, we can couple the process $(\tilde{\sigma}, \tilde{\lambda})$ with the process $(\hat{\sigma}, \hat{\lambda})$ obtained by suppressing all undesired spin flip; in other words $\tilde{\sigma}(0) = \hat{\sigma}(0)$, and

$$\hat{\lambda}_i(t) = \begin{cases} \tilde{\lambda}_i(t) & \text{if } \hat{\sigma}_i(t) = -1, \text{ and } \hat{\sigma}_j(t) = 1 \text{ for at least one } j \in \{i-1, i+1\} \\ 0 & \text{otherwise} \end{cases}$$

By the estimates above, we have

$$\lim_{\beta, N \uparrow +\infty} P(\tilde{\sigma}(t) = \hat{\sigma}(t) \text{ for } t \in [0, T_c]) = 1. \quad (3.3.12)$$

Thus, to compute the distribution of T_c we can use the process $\hat{\sigma}$ in place of $\tilde{\sigma}$. Note that the times τ_n introduced above are well defined for the process $(\hat{\sigma}, \hat{\lambda})$. Moreover, $T_c = \tau_{N-1}$ on the event $\{\tilde{\sigma}(t) = \hat{\sigma}(t) \text{ for } t \in [0, T_c]\}$. Using the same estimate as in Step 1, for $n = 1, \dots, N-1$, in A_N we have

$$P\left(\tau_n - \tau_{n-1} > \frac{e^{-d\beta}}{N} \mid T_1\right) \leq \exp\left[-\frac{e^{-d\beta}}{N} e^{[2\beta - \log N]e^{-\alpha} \frac{e^{-d\beta}}{N}}\right]$$

which implies, defining

$$B_N := \left\{ \tau_n - \tau_{n-1} \leq \frac{e^{-d\beta}}{N} : n = 1, \dots, N-1 \right\}, \quad (3.3.13)$$

the estimate

$$\begin{aligned}
P(B_N^c | T_1)\mathbf{1}_{A_N} &\leq N \exp\left[-\frac{e^{-d\beta}}{N} e^{[2\beta - \log N]e^{-\alpha} \frac{e^{-d\beta}}{N}}\right] \\
&= \exp\left[\log N - e^{-d\beta - \log N + [2\beta - \log N]e^{-\alpha} \frac{e^{-d\beta}}{N}}\right] \\
&= \exp\left[\log N - e^{(2-d)\beta - 2\log N + \alpha \frac{e^{-d\beta}}{N} [2\beta - \log N] + o\left(\beta \frac{e^{-d\beta}}{N}\right)}\right] \rightarrow 0 \quad (3.3.14)
\end{aligned}$$

as $\beta, N \uparrow +\infty$ since $2c + d < 2$. This estimate, together with (3.3.12), gives also

$$\lim_{\beta, N \uparrow +\infty} P(T_c > e^{-d\beta}|T_1)\mathbf{1}_{A_N} = \lim_{\beta, N \uparrow +\infty} P(\tau_{N-1} > e^{-d\beta}|T_1)\mathbf{1}_{A_N} \rightarrow 0 \quad (3.3.15)$$

as $\beta, N \uparrow +\infty$.

Having all these preliminary estimates, we now aim at giving sharp estimates on the distribution of τ_{N-1} . The key idea is to write

$$\tau_{N-1} = \tau_1 + \sum_{k=2}^{N-1} (\tau_k - \tau_{k-1}),$$

and show that the random variables in the sum above are nearly independent and identically distributed. We define

$$L_N := -\log N + \log \log N + \log \alpha + X_N,$$

so that $\lambda_i(T_1^-) = L_N + o_i(1)$, where for each $\epsilon > 0$

$$P\left(\max_{i=1, \dots, N} |o_i(1)| > \epsilon\right) \rightarrow 0$$

as $\beta, N \uparrow +\infty$, see (3.2.17). By (3.3.4),

$$P(\tau_1 > t|T_1) = P(\min(X_+, X_-) > t|T_1),$$

where X_+, X_- are random variables which are independent conditionally to \mathcal{F}_{T_1} , and

$$P(X_{\pm} > t|T_1) = \exp\left[-\int_0^t \left(e^{(2\beta + \lambda_{i\pm 1}(T_1^-))e^{-\alpha s}}\right) ds\right].$$

Therefore

$$\begin{aligned} P(\tau_1 > t|T_1) &= \exp\left[-\int_0^t \left(e^{(2\beta + \lambda_{i+1}(T_1^-))e^{-\alpha s}}\right) ds - \int_0^t \left(e^{(2\beta + \lambda_{i-1}(T_1^-))e^{-\alpha s}}\right) ds\right] \\ &= \exp\left[-2\int_0^t \left(e^{(2\beta + L_N + o(1))e^{-\alpha s}}\right) ds\right] \end{aligned} \quad (3.3.16)$$

where, as usual, $o(1)$ denotes a T_1 -measurable random variable which goes to zero in probability. More generally, using (3.3.8) and (3.3.9), for $2 \leq n \leq N-3$

$$P(\tau_{n+1} - \tau_n > t|\tau_n, \tau_{n-1}, \dots, \tau_1, T_1) \geq \exp\left[-2\int_0^t e^{(2\beta + (L_N + o(1))e^{-\alpha \tau_n})e^{-\alpha s}} ds\right] \quad (3.3.17)$$

and

$$P(\tau_{n+1} - \tau_n > t|\tau_n, \tau_{n-1}, \dots, \tau_1, T_1) \leq \exp\left[-2\int_0^t e^{(2\beta e^{-\alpha \tau_n} + L_N + o(1))e^{-\alpha s}} ds\right]. \quad (3.3.18)$$

The case $n = N - 2$ is similar, the 2 multiplying the \int_0^t must be removed and 2β replaced by 4β . By (3.3.17), on $B_N \cap A_N$ we have, for $n = 1, \dots, N - 3$ and the above correction for $n = N - 2$,

$$P(\tau_{n+1} - \tau_n > t | \tau_n, \tau_{n-1}, \dots, \tau_1, T_1) \geq \exp \left[-2te^{2\beta e^{-\alpha e^{-d\beta}} + L_N + o(1)} \right] = P(Y_n > t | T_1) \quad (3.3.19)$$

where (Y_1, \dots, Y_{N-2}) , conditioned to T_1 are independent and have exponential distribution with mean

$$E(Y_n | T_1) = \frac{1}{2} e^{-[2\beta e^{-\alpha e^{-d\beta}} + L_N + o(1)]}$$

for $n \leq N - 3$, while

$$E(Y_{N-2} | T_1) = e^{-[4\beta e^{-\alpha e^{-d\beta}} + L_N + o(1)]}.$$

Thus, by Lemma 3.3.1, the following inequality holds on A_N , for every $t > 0$:

$$P(\tau_{N-1} > t | T_1) \geq P(Y_0 + \dots + Y_{N-2} > t | T_1) - NP(B_N^c | T_1). \quad (3.3.20)$$

Since, for $j \leq N - 3$

$$E(Y_j | T_1) = \frac{N}{2\alpha \log N} e^{-2\beta - X_N} (1 + o(1)). \quad (3.3.21)$$

the Law of Large Numbers for (Y_n) gives

$$\lim_{\beta, N \uparrow +\infty} P \left(\frac{Y_0 + \dots + Y_{N-2}}{\frac{N^2}{\alpha \log N} e^{-2\beta - X_N}} > 1 - \epsilon \right) = 1$$

for every $\epsilon > 0$. Inserting this in (3.3.20), using the fact that by (3.3.14)

$$NP(B_N^c | T_1) \mathbf{1}_{A_N} \rightarrow 0,$$

$P(A_N^c) \rightarrow 0$ and $T_c = \tau_{N-1}$ on A_N , we obtain

$$\lim_{\beta, N \uparrow +\infty} P \left(\frac{T_c}{\frac{N^2}{\alpha \log N} e^{-2\beta - X_N}} > 1 - \epsilon \right) = 1. \quad (3.3.22)$$

To obtain a corresponding upper bound define $\xi_n := [\tau_{n+1} - \tau_n] \mathbf{1} \left(\tau_{n+1} - \tau_n \leq \frac{e^{-d\beta}}{N} \right)$. By (3.3.18),

$$P(\xi_n > t | \tau_n, \tau_{n-1}, \dots, \tau_1, T_1) \leq \exp \left[-2te^{(2\beta + L_N + o(1))e^{-\alpha \frac{e^{-\beta}}{N}}} \right] = P(Z_n > t | T_1),$$

where Z_1, \dots, Z_{N-2} are, conditionally to T_1 , independent, exponentially distributed with mean

$$\frac{1}{2} e^{-(2\beta + L_N + o(1))e^{-\alpha \frac{e^{-d\beta}}{N}}} = \frac{N}{2\alpha \log N} e^{-2\beta - X_N} (1 + o(1)).$$

Using Lemma 3.3.1 as above and observing that $\xi_1 + \dots + \xi_{N-1} = \tau_{N-1} = T_c$ on $B_N \cap A_N$, we obtain

$$\lim_{\beta, N \uparrow +\infty} P \left(\frac{T_c}{\frac{N^2}{\alpha \log N} e^{-2\beta - X_N}} < 1 + \epsilon \right) = 1 \quad (3.3.23)$$

for all $\epsilon > 0$ that, together with (3.3.22), completes the proof. \square

In the proof of Theorem 3.3.1 we obtained that the sequence of times taken to perform each step in the covering can be stochastically dominated, both by above and by below, by two different families of i.i.d. random variables obeying the same Law of Large Number. This was sufficient to conclude the proof thanks to the following technical lemma.

Lemma 3.3.1. *Let $X = (X_n)_{n=1}^N$ and $Y = (Y_n)_{n=1}^N$ be two random vectors, such that X is adapted to a filtration $(\mathcal{F}_n)_{n=1}^N$, and Y has independent components. Define $S_n^X := X_1 + \dots + X_n$, and similarly S_n^Y . Assume there is an event B such that for every $t \in \mathbb{R}$ and $n \in \{1, \dots, N\}$ and $\omega \in B$*

$$P(X_n > t | \mathcal{F}_{n-1})(\omega) \leq P(Y_n > t). \quad (3.3.24)$$

Then for every $t \in \mathbb{R}$ and $n \in \{1, \dots, N\}$

$$P(S_n^X > t) \leq P(S_n^Y > t) + nP(B^c) \quad (3.3.25)$$

Similarly, if (3.3.24) is replaced by

$$P(X_n > t | \mathcal{F}_{n-1})(\omega) \geq P(Y_n > t), \quad (3.3.26)$$

we have

$$P(S_n^X > t) \geq P(S_n^Y > t) - nP(B^c) \quad (3.3.27)$$

Proof. We prove the desired statement by induction on n . For $n = 1$ there is nothing to prove. Note that, without loss of generality, we can assume Y to be independent of X . We have, in B ,

$$\begin{aligned} P(S_{n+1}^X > t | \mathcal{F}_n) &= P(X_{n+1} > t - s | \mathcal{F}_n) \Big|_{s=S_n^X} \\ &\leq P(Y_{n+1} + S_n^X > t | \mathcal{F}_n), \end{aligned}$$

so that

$$P(S_{n+1}^X > t) \leq P(Y_{n+1} + S_n^X > t) + P(B^c).$$

On the other hand, denoting by $\mathcal{L}_{Y_{n+1}}$ the law of Y_{n+1} , using the inductive assumption we obtain

$$\begin{aligned} P(Y_{n+1} + S_n^X > t) &= \int P(S_n^X > t - y) \mathcal{L}_{Y_{n+1}}(dy) \\ &\leq \int P(S_n^Y > t - y) \mathcal{L}_{Y_{n+1}}(dy) + nP(B^c) = P(S_{n+1}^Y > t) + nP(B^c). \end{aligned}$$

The proof with the reversed inequalities is identical. \square

3.4 Convergence to an oscillating process

For sake of clearness, assume (3.1.5) for initial condition, so

$$\sigma_i(0) = -1, \quad \lambda_i(0) = -\gamma, \quad i \in \Lambda_N.$$

By the analysis performed in Section 3.2 and 3.3, it should be clear that at time $T_1 + T_c$, at which all spins are equal to $+1$, with very large probability we will observe the following profile for the local fields:

$$\lambda_i(T_1 + T_c) = 4\beta - \log N + \log \log N + \log \alpha + X_N + o_i(1), \quad i \in \Lambda_N,$$

where X_N is a \mathcal{F}_{T_1} -measurable random variable, converging in distribution to X , as stated in Proposition 3.2.1, and the terms $o_i(1)$ represents $\mathcal{F}_{(T_1+T_c)}$ -measurable random terms uniformly converging to zero in probability. This is due to the asymptotics for $\lambda_i(T_1^-)$ given by (3.2.5), plus the term $+4\beta$ which appears since any spin has seen both of its neighbours flipping from -1 to $+1$ during the interval $[T_1, T_1 + T_c]$. The (random) inhomogeneity appearing in the remainders $o_i(1)$ is due to the dissipation acting on the spins during the growth of the droplet of $+1$ spins. Since with very large probability $T_c \leq e^{-d\beta}$ for a suitable positive constant d , the remainders $o_i(1)$ can be uniformly dominated by a term of order $O(\beta e^{-d\beta})$.

As already pointed out in Remark 3.2.1, the dominant part of the local fields at time T_1 does not depend on γ . This holds true also at time $T_1 + T_c$ and it allows us to choose γ in a suitable way to obtain the iteration of the same phenomenon after $T_1 + T_c$. In fact, set

$$\gamma = 4\beta - \log N + \log \log N + \log \alpha.$$

By this choice we have, at $t = 0$, for any $i \in \Lambda_N$

$$\sigma_i(0) = -1 \quad \text{with rate} \quad e^{-\sigma_i(0)\lambda_i(0)} = e^{-\gamma},$$

while, at time $T_1 + T_c$, with very large probability, for any $i \in \Lambda_N$

$$\sigma_i(T_1 + T_c) = -1 \quad \text{with rate} \quad e^{-\sigma_i(T_1+T_c)\lambda_i(T_1+T_c)} = e^{-(\gamma + X_N + \varepsilon_i(\gamma, N))}.$$

Notice that we are exactly in the situation analysed in Proposition 3.2.2: if we define

$$T_2 := \inf\{t > T_1 + T_c \mid \sigma_i(t) = -1 \text{ for some } i \in \Lambda_N\} - (T_1 + T_c)$$

then, by Proposition 3.2.2 it is immediate to check that

$$\alpha \log N [T_2 - t(\gamma, N)] \xrightarrow[\gamma, N \uparrow +\infty]{d} \tilde{X} + \frac{c}{4-c} X,$$

where \tilde{X}, X are i.i.d. random variables distributed according to

$$P(X > x) = \exp(-e^x), \quad \forall x \in \mathbb{R}$$

and c such that $\frac{\log N}{\beta} \rightarrow c \in [0, 1[$, in fact

$$\frac{\log N}{\gamma} = \frac{\log N}{4\beta - \log N + \log \log N + \log \alpha} \rightarrow \frac{c}{4 - c}.$$

So, the distribution of fluctuations of time T_2 depends also on the fluctuations of time T_1 but, beyond that, the same phenomenon will iterate for any finite number of time.

Formally speaking, by Proposition 3.2.2 and Theorem 3.3.1 we have also proved the following result.

Theorem 3.4.1. *Let $\gamma = 4\beta - \log N + \log \log N + \log \alpha$ and take the initial conditions*

$$\sigma_i(0) = -1, \quad \lambda_i(0) = -\gamma, \quad i \in \Lambda_N.$$

Fix $n \in \mathbb{N}$ and define the following stopping times, for $j = 1, \dots, n$

$$T_{1,j} := \inf \left\{ t > \sum_{k=0}^{j-1} (T_{1,k} + T_{c,k}) \mid \sigma_i(t) = (-1)^{j+1} \text{ for some } i \in \Lambda_N \right\} - \sum_{k=0}^{j-1} (T_{1,k} + T_{c,k}),$$

$$T_{c,j} := \inf \left\{ t > T_{1,j} + \sum_{k=0}^{j-1} (T_{1,k} + T_{c,k}) \mid \sigma_i(t) = (-1)^{j+1} \text{ for all } i \in \Lambda_N \right\} - \sum_{k=0}^{j-1} (T_{1,k} + T_{c,k})$$

with $T_{1,0} = T_{c,0} = 0$. Let $\{Y_i\}_{i=1}^n$ be a sequence of i.i.d. random variables distributed according to

$$P(Y_1 > y) = \exp(-e^y), \quad \forall y \in \mathbb{R}.$$

Suppose $\beta, N \uparrow +\infty$ with the condition

$$\lim_{\beta, N} \frac{\log N}{\beta} = c \in [0, 1[.$$

Then, for any $j = 1, \dots, n$,

$$\alpha \log N (T_{1,j} - t(\gamma, N)) \xrightarrow[\gamma, N \uparrow +\infty]{d} X_j$$

$$\frac{T_{c,j}}{\frac{N^2}{2\alpha \log N} e^{-2\beta}} \xrightarrow[\beta, N \uparrow +\infty]{d} Z_j,$$

where $t(\gamma, N)$ is defined as in (3.2.3), $\{X_j\}_{j=0}^n$ is the discrete Markov process defined by

$$X_0 = 0, \quad X_j = Y_j + \frac{c}{4 - c} X_{j-1},$$

and, for any j , Z_j distributed as e^{-X_j} .

Now we want to prove that at macroscopic level the system presents an oscillating behavior. More formally, we show that the total magnetization converges to a process which regularly oscillates between -1 and $+1$: to do so, we have to choose a proper time-scaling since the two phenomena analysed (the first spin flip and the covering of the box) occur on very different time scales. In fact, taking

$$\gamma = 4\beta - \log N + \log \log N + \log \alpha$$

and letting $\beta, N \uparrow +\infty$ under the condition $\frac{\log N}{\beta} \rightarrow c \in [0, 1[$ implies

$$\lim_{\beta, N \uparrow +\infty} t(\gamma, N) = \begin{cases} +\infty, & \text{if } c = 0, \\ \frac{1}{\alpha} \log \left(\frac{4-c}{c} \right), & \text{otherwise,} \end{cases} \quad \text{and} \quad \lim_{\beta, N \uparrow +\infty} \frac{N^2 e^{-2\beta}}{2\alpha \log N} = 0.$$

Therefore, given the total magnetization process

$$m_N(t) = \frac{1}{N} \sum_{i \in \Lambda_N} \sigma_i(t),$$

consider the time changing process

$$\theta_N(t) = \int_0^t t(\gamma, N) \mathbf{1}_{\{|m_N(s)|=1\}} + \frac{N^2 e^{-2\beta}}{2\alpha \log N} \mathbf{1}_{\{|m_N(s)|<1\}} ds, \quad (3.4.1)$$

which "speeds up" time whenever all the spins are equal and we are waiting for the following flip and "slows down" time whenever we are observing the very fast invasion of a droplet of spins opposite sign. Then, we define a time-scaled version of total magnetization process by

$$\tilde{m}_N(t) := m_N(\theta_N(t)). \quad (3.4.2)$$

By Theorem 3.4.1 and the analysis performed in the proof of Theorem 3.3.1, we expect that the process \tilde{m}_N converges to a stochastic process \tilde{x} with the following behavior: $\tilde{x}(0) = -1$ then it does not move for a unit of time, then it takes a random time Z_1 to linearly grow from -1 to $+1$; after reaching $+1$, it does not move for a unit of time, then it takes a random time Z_2 to linearly decrease from $+1$ to -1 and so on, where the random variables Z_1, Z_2, \dots are given in Theorem 3.4.1. We expect a linear profile between -1 and $+1$ and viceversa since in the proof of Theorem 3.3.1 we saw that during the growth of the droplet each step occurs essentially at the same time. A graphical intuition on the behavior of the limiting dynamics is given in Figure 3.1. Let us give a formal definition of the limiting process \tilde{x} : consider the deterministic trajectory $x(t)$ such that

$$x(t) = \begin{cases} -1 & \text{for } t \in [0, 1], \\ 2t - 3 & \text{for } t \in [1, 2], \\ +1 & \text{for } t \in [2, 3], \\ -2t + 7 & \text{for } t \in [3, 4], \end{cases}$$

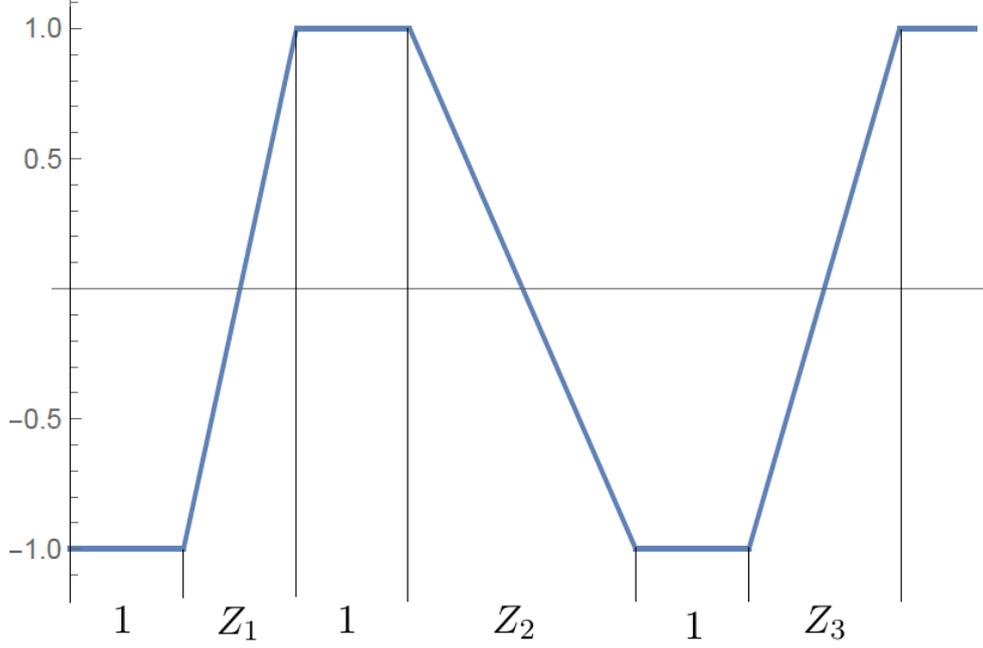


Figure 3.1: Oscillating behavior of the limiting process $\tilde{x}(t)$. Z_1, Z_2, Z_3 represent random variables distributed according to what stated in Theorem 3.4.1.

and then extended periodically on \mathbb{R}_+ for $t \geq 4$. Then, consider the family of random variables $\{Z_i\}_{i \geq 1}$ defined in Theorem 3.4.1 and define the following time-changing process:

$$\phi(t) = \int_0^t \mathbf{1}_{\{|x(s)|=1\}} + \sum_{i \geq 1} Z_i^{-1} \mathbf{1}_{\{s \in [2i-1, 2i]\}} ds. \quad (3.4.3)$$

Finally, the limiting process is defined as

$$\tilde{x}(t) = x(\phi(t)). \quad (3.4.4)$$

Theorem 3.4.2. *Let γ and $\{Z_i\}_{i \geq q}$ as in Theorem 3.4.1. Suppose $\beta, N \uparrow +\infty$ with the condition $\frac{\log N}{\beta} = c \in [0, 1[$. Then, for any $T > 0$, the process $(\tilde{m}_N(t))_{t \in [0, T]}$ defined by (3.4.1)-(3.4.2) converges, in sense of weak convergence of stochastic processes, to $(\tilde{x}(t))_{t \in [0, T]}$ defined by (3.4.3)-(3.4.4).*

Proof. Let us start with some preliminary considerations: define the stopping times

$$\tau_{\tilde{m}} = \inf\{t > 0 \mid \tilde{m}_N(t) = +1\}, \quad \tau_{\tilde{x}} = \inf\{t > 0 \mid \tilde{x}(t) = +1\}.$$

For simplicity of notations and readability of the proof we only prove that the process $(\tilde{m}_N(t \wedge \tau_{\tilde{m}}))_{t \in [0, T]}$ converges to $(\tilde{x}(t \wedge \tau_{\tilde{x}}))_{t \in [0, T]}$ as $\beta, N \uparrow +\infty$: the result can be extended for any finite number of iterations (the same idea as Theorem 3.4.1) and this is sufficient since

$$P\left(\sum_{i=1}^{+\infty} Z_i < T\right) = 0.$$

Moreover, consider the process $(\hat{\underline{\sigma}}(t), \hat{\underline{\lambda}}(t))_{t \geq 0}$ defined at the beginning of Step 2 in proof of Theorem 3.3.1, for which all "undesired" flips are suppressed: coupling this process with the original one $(\underline{\sigma}(t), \underline{\lambda}(t))_{t \geq 0}$ via the graphical construction implies that

$$\lim_{\beta, N \uparrow +\infty} P(\hat{\sigma}_i(t) = \sigma_i(t) \text{ for } i \in \Lambda_N, t \in [0, T_1 + T_c]) = 1,$$

hence we can prove the result using the total magnetization corresponding to the process $\hat{\underline{\sigma}}$ instead of $\underline{\sigma}$. Now we are ready for the proof.

Since the amplitude of jumps of the process $(\tilde{m}_N(t))_{t \in [0, T]}$ converges to zero, then, denoting with $(\tilde{m}(t))_{t \in [0, T]}$ its limit, it holds $P(\tilde{m} \in \mathcal{C}([0, T], \mathbb{R})) = 1$ (see [6], Theorem 13.4). This implies that the convergence can be studied on the space $\mathcal{D}([0, T], \mathbb{R})$ endowed with the uniform metric and topology (see for example Lemma 1.6.4 in [52]). With this choice, on $\mathcal{M}_1(\mathcal{D}([0, T], \mathbb{R}))$ the Wasserstein distance \mathcal{W}_1 (see Definition 1.3.1) reads

$$\mathcal{W}_1(\mu, \nu) = \inf_{\gamma \in \Gamma(x, y)} \int_{\mathcal{D} \times \mathcal{D}} \|x - y\|_{\infty} d\gamma(x, y) \quad (3.4.5)$$

where $\Gamma(\mu, \nu)$ is the set of all possible couplings of μ and ν .

Consider the events

$$A_N = \{-\log N \leq \lambda_i(T_1^-) \leq -\log N + 2 \log \log N : i = 1, \dots, N\},$$

$$B_N = \left\{ \tau_n - \tau_{n-1} > \frac{e^{-d\beta}}{N} : n = 1, \dots, N-1 \right\}$$

introduced while proving Theorem 3.3.1. The strategy of the proof is to use the graphical construction to couple $(\tilde{m}_N(t))_{t \in [0, T]}$ with two processes $(\tilde{m}_N^+(t))_{t \in [0, T]}$ and $(\tilde{m}_N^-(t))_{t \in [0, T]}$, both converging to $(\tilde{x}(t))_{t \in [0, T]}$, in such a way under $A_N \cap B_N$ it holds

$$\tilde{m}_N^-(t) \leq \tilde{m}_N(t) \leq \tilde{m}_N^+(t) \quad \text{for any } t \in [0, T \wedge \tau_{\tilde{m}}].$$

Roughly speaking, \tilde{m}_N^+ (respectively \tilde{m}_N^-) represents the time-scaled magnetization of a spin system $\underline{\eta}_N^+$ (resp. $\underline{\eta}_N^-$) in which all undesired spins are suppressed and, after time T_1 , each flip is performed with a higher (resp. lower) rate with respect to $\hat{\underline{\sigma}}$. Moreover, the rates for $\underline{\eta}_N^+$ and $\underline{\eta}_N^-$ have to be chosen in such a way both \tilde{m}_N^+ and \tilde{m}_N^- converge to \tilde{x} .

On the probability space already defined for the graphical construction, define the spin processes $\underline{\eta}^+$ and $\underline{\eta}^-$ in the following way:

$$\eta_i^+(t) = \eta_i^-(t) = \sigma_i(t), \quad t \in [0, T_1],$$

while, after T_1 , the local fields for $\underline{\eta}^+$ and $\underline{\eta}^-$ are defined as:

$$\lambda_i^+(t) = \begin{cases} 2\beta + L_N e^{-\alpha e^{-d\beta}} & \text{if } \eta_i^+(t) = -1, \text{ and } \eta_j^+(t) = 1 \text{ for at least one } j \in \{i-1, i+1\} \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_i^-(t) = \begin{cases} 2\beta e^{-\alpha e^{-d\beta}} + L_N & \text{if } \eta_i^-(t) = -1, \text{ and } \eta_j^-(t) = 1 \text{ for at least one } j \in \{i-1, i+1\} \\ 0 & \text{otherwise} \end{cases}$$

where, as in the proof of Theorem 3.3.1,

$$L_N = \lambda_i(T_1^-) = -\log N + \log \log N + \log \alpha + X_N + o(1).$$

Of course, by construction, after time T_1 a random point τ is accepted for $\underline{\eta}^+$ (respectively $\underline{\eta}^-$) if and only if

$$\frac{\exp[-\eta_i^+(\tau)\lambda_i^+(\tau)]}{e^{4\beta}} > U_\tau, \quad \left(\text{respectively } \frac{\exp[-\eta_i^-(\tau)\lambda_i^-(\tau)]}{e^{4\beta}} > U_\tau \right)$$

where U_τ is uniform random variable associated with τ .

Notice that, under the event $A_N \cap B_N$ (see again the proof of Theorem 3.3.1), it holds that

$$\exp[-\eta_i^-(t)\lambda_i^-(t)] \leq \exp[-\sigma_i(t)\lambda_i(t)] \leq \exp[-\eta_i^+(t)\lambda_i^+(t)] \quad (3.4.6)$$

for any t up to $T_1 + T_c$ which means that any point which is accepted for a spin flip for $\underline{\eta}^-$ is also accepted for $\hat{\sigma}$, and any point which is accepted for a spin flip for $\hat{\sigma}$ is also accepted for $\underline{\eta}^+$, therefore, since we constructed a *monotone coupling* (see [39]), it holds

$$\eta_i^-(t) \leq \hat{\sigma}_i(t) \leq \eta_i^+(t), \quad i \in \Lambda_N, \quad (3.4.7)$$

for any t up to $T_1 + T_c$. Actually, (3.4.6) and (3.4.7) are true only up to the second to last flip of $\hat{\sigma}$, since its last flip occurs with rate of order $e^{4\beta+L_N}$: anyway this is not so important. In fact, if we denote by m_N^+ (respectively m_N^-) the total magnetization associated with $\underline{\eta}^+$ (respectively $\underline{\eta}^-$), our goal is to give bounds on m_N by means of m_N^+ and m_N^- : it is true that, up to the second to last flip of $\hat{\sigma}$, by (3.4.7) it holds that

$$m_N^-(t) \leq m_N(t) \leq m_N^+(t);$$

to extend the bounds on the whole interval $[0, T_1 + T_c]$ it is sufficient to add a term $+\frac{2}{N}$ to the upper bound. These bounds are true also passing to the time-scaled processes $\tilde{m}_N^\pm(t) = m_N^\pm(\theta_N(t))$: to sum up, with this construction, under the event $A_N \cap B_N$, it holds

$$\tilde{m}_N^-(t) \leq \tilde{m}_N(t) \leq \tilde{m}_N^+(t) + \frac{2}{N} \quad \text{for any } t \in [0, \tau_{\tilde{m}}]$$

which implies, again under $A_N \cap B_N$,

$$\max\{\|\tilde{m}_N - \tilde{m}_N^-\|_\infty, \|\tilde{m}_N - (\tilde{m}_N^+ + \frac{2}{N})\|_\infty\} \leq \|\tilde{m}_N^+ - \tilde{m}_N^-\|_\infty + \frac{2}{N}, \quad (3.4.8)$$

where $\|\cdot\|_\infty$ denotes the uniform norm on $\mathcal{D}([0, T \wedge \tau_{\tilde{m}}], \mathbb{R})$. Notice that by the graphical construction and the definition of \tilde{m}_N^+ and \tilde{m}_N^- , one gets that

$$\|\tilde{m}_N^+ - \tilde{m}_N^-\|_\infty \rightarrow 0 \quad \text{in probability as } \beta, N \uparrow +\infty.$$

Hence, since $P(A_N \cap B_N) \rightarrow 0$, thanks to (3.4.8) we also obtain that

$$\|\tilde{m}_N - \tilde{m}_N^-\|_\infty \rightarrow 0 \quad \text{in } L^1 \text{ as } \beta, N \uparrow +\infty, \quad (3.4.9)$$

where the convergence in L^1 follows by convergence in probability and uniform integrability, the latter due to the fact that $\|\tilde{m}_N - \tilde{m}_N^-\|_\infty \leq 2$ for any N .

Observe that $\{\tilde{m}_N^-\}_N$ (but also $\{\tilde{m}_N^+\}_N$), stopped as soon as it reaches $+1$, converges to the process $\tilde{x}(t \wedge \tau_{\tilde{x}})$. This can be seen by Proposition 3.2.1, the definition of the time scaling (3.4.1) and by the fact that after the first spin flip, keeping in mind the time scaling, \tilde{m}_N^\pm essentially become Poisson processes rescaled by $\frac{2}{N}$ with random intensity Ne^{-X_N} .

Now denote with μ_N the law of \tilde{m}_N on $\mathcal{D}([0, T], \mathbb{R})$. Let also μ_x be the law of the limiting process \tilde{x} . To show the weak convergence of \tilde{m}_N to \tilde{x} it is enough to show that $\mathcal{W}_1(\mu_N, \mu_x)$ converges to 0 (see [58]). Let μ_N^- be the law of the process \tilde{m}_N^- . Since μ_N and μ_N^- can be coupled via the graphical construction of \tilde{m}_N and \tilde{m}_N^- , by (3.4.9) and the definition of the Wasserstein distance, it holds

$$\mathcal{W}_1(\mu_N, \mu_N^-) \leq E [\|\tilde{m}_N - \tilde{m}_N^-\|_\infty] \rightarrow 0 \quad \text{as } \beta, N \uparrow +\infty.$$

Therefore, by the fact that \tilde{m}_N^- weakly converges to \tilde{x} ,

$$\mathcal{W}_1(\mu_N, \mu_x) \leq \mathcal{W}_1(\mu_N, \mu_N^-) + \mathcal{W}_1(\mu_N^-, \mu_x) \rightarrow 0 \quad \text{as } \beta, N \uparrow +\infty,$$

which proves the weak convergence of \tilde{m}_N to \tilde{x} . □

Part II

The dynamics of critical fluctuations in asymmetric Curie-Weiss models

Chapter 4

Critical fluctuations at a Hopf bifurcation

In this chapter we continue the analysis of spin-flip mean field models presenting the emergence of collective periodic behavior. Theorem 1.2.2 and Theorem 2.1.5 respectively establish the macroscopic law of the Curie-Weiss model with dissipation and of the bi-populated Curie-Weiss model in terms of a Law of Large Number (i.e. the weak convergence of the empirical measure flow). In this chapter, we aim to study the fluctuations around their limits. As one could expect, standard fluctuations (obtained by rescaling the empirical flow by a factor \sqrt{N}) obey to a Central Limit Theorem, as they converge to a Gaussian process. This result holds for any choice of the parameters but, as first shown in [16, 22] for reversible mean field dynamics with ferromagnetic interaction, fluctuations at the critical point present some peculiarities, with an anomalous space-time scaling and, possibly, a non-Gaussian limit.

We study here the role of the presence of a Hopf bifurcation. The nature of the bifurcation is relevant, as the dynamics of fluctuations is deeply related to the linearization of the McKean-Vlasov equation. In both examples considered in this chapter, the analysis leads to the study of the evolution of a two-dimensional order parameter, which have a peculiar feature due to the Hopf bifurcation. After a change of variables, we can identify a fast variable (evolving on a scale of order \sqrt{N}) and a slow variable (evolving on a scale of order 1). The convergence to the limit process is achieved by noticing that in the natural time scale the fast component averages out, producing a limiting dynamics for the slow variable via an averaging principle. We also show that the critical fluctuations for both models belong to the same class of universality.

4.1 Fluctuations for the CW model with dissipation

In this section we study standard and critical fluctuations for the Curie-Weiss model with dissipation presented in Section 1.2. The proof of the main theorem (concerning the critical fluctuation) is comprised by a relevant number of technical steps, so it is postponed to

Section 4.2.

4.1.1 Dynamics of normal fluctuations

Consider again the spin-flip Curie-Weiss model with dissipation (see Section 1.2): we sum up here basic notation and assumptions for convenience. Let $(\underline{\sigma}(t), \lambda_N(t))_{t \in [0, T]}$ be the Markov process with infinitesimal generator (1.2.2) and initial condition such that

$$\text{Law}(\underline{\sigma}(0), \lambda_N(0)) = \nu_0^{\otimes N} \otimes \delta_{\lambda_0} \quad (4.1.1)$$

where ν_0 is a probability measure on $\mathcal{S} = \{-1, +1\}$ and δ_{λ_0} is a Dirac delta centered in $\lambda_0 \in \mathbb{R}$. For any $t \in [0, T]$, let $m_N(t)$ represent the magnetization

$$m_N(t) = \frac{1}{N} \sum_{i=1}^N \sigma_i(t),$$

and $\mu_N(t)$ the marginal empirical distribution at time t

$$\mu_N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{\{\sigma_i(t), \lambda_N(t)\}}.$$

In Section 1.2.2 we have shown a Law of Large Number in the form (1.2.10) through a weak convergence of the order parameter of the system (Theorem 1.2.2). It is natural to wonder whether a Central Limit Theorem holds as well. Let $(\mu(t))_{t \in [0, T]}$ denote the limiting dynamics of $(\mu_N(t))_{t \in [0, T]}$. In this paragraph we study the fluctuation flow $(\tilde{\mu}_N(t))_{t \in [0, T]}$ where

$$\tilde{\mu}_N(t) = N^{\frac{1}{2}}(\mu_N(t) - \mu(t)) \quad (4.1.2)$$

for any $t \in [0, T]$. As analysed in Section 1.2.2, $(\mu(t))_{t \in [0, T]}$ is described by $(m(t), \lambda(t))_{t \in [0, T]}$, solution of (1.2.13), and $(m_N(t), \lambda_N(t))_{t \in [0, T]}$ is an order parameter which fully describes $(\mu_N(t))_{t \in [0, T]}$. Therefore, the flow of fluctuations (4.1.2) is fully described by the process $(\tilde{m}_N(t), \tilde{\lambda}_N(t))_{t \in [0, T]}$ where

$$\tilde{m}_N(t) = N^{\frac{1}{2}}(m_N(t) - m(t)), \quad \tilde{\lambda}_N(t) = N^{\frac{1}{2}}(\lambda_N(t) - \lambda(t)). \quad (4.1.3)$$

As one may expect, this order parameter converges to a Gaussian bi-dimensional process, as stated in the following theorem.

Theorem 4.1.1. *For any $\alpha, \beta > 0$ and $T > 0$, under initial conditions (4.1.1), the process $(\tilde{m}_N(t), \tilde{\lambda}_N(t))_{t \in [0, T]}$ defined by (4.1.3) converges, in sense of weak convergence of stochastic processes, as $N \uparrow +\infty$ to the Gaussian process $(\tilde{m}(t), \tilde{\lambda}(t))_{t \in [0, T]}$, unique solution of the linear time-inhomogenous stochastic differential equation*

$$d \begin{pmatrix} \tilde{m}(t) \\ \tilde{\lambda}(t) \end{pmatrix} = A(t) \begin{pmatrix} \tilde{m}(t) \\ \tilde{\lambda}(t) \end{pmatrix} dt + \sqrt{1 - m(t) \tanh(\lambda(t))} \begin{pmatrix} 2 \\ 2\beta \end{pmatrix} dB(t) \quad (4.1.4)$$

with $(m(t), \lambda(t))_{t \in [0, T]}$ solution of (1.2.13),

$$A(t) = \begin{pmatrix} -2 & 2(1 + \tanh(\lambda(t))) \\ -2\beta(1 - \tanh(\lambda(t))) & 2\beta - \alpha \end{pmatrix},$$

$B(t)$ one-dimensional standard Brownian Motion and

$$\text{Law}(\tilde{m}(0), \tilde{\lambda}(0)) = \mathcal{N}(0, V(\nu_0)) \otimes \delta_0$$

where $\mathcal{N}(0, V(\nu_0))$ denotes a zero-mean Gaussian distribution with the same variance of ν_0 .

Proof. The argument of the proof is again the weak convergence of initial conditions plus the uniform convergence of infinitesimal generators. When considering initial condition (4.1.1), the weak convergence of $(\tilde{m}_N(0), \tilde{\lambda}_N(0))$ to $(\tilde{m}(0), \tilde{\lambda}(0))$ with

$$\text{Law}(\tilde{m}(0), \tilde{\lambda}(0)) = \mathcal{N}(0, V(\nu_0)) \otimes \delta_0$$

is a straightforward application of the classical Central Limit Theorem.

Let I_t be the infinitesimal generator of the solution of 4.1.4, namely:

$$\begin{aligned} I_t f(\tilde{m}, \tilde{\lambda}) &= (1 - m(t) \tanh(\lambda(t))) (2f_{\tilde{m}\tilde{m}} + 2\beta^2 f_{\tilde{\lambda}\tilde{\lambda}} + 4\beta f_{\tilde{m}\tilde{\lambda}}) + \\ &+ 2((1 + \tanh(\lambda(t)))\tilde{\lambda} - \tilde{m})f_{\tilde{m}} + ((2\beta - \alpha)\tilde{\lambda} - 2\beta(1 - \tanh(\lambda(t)))\tilde{m})f_{\tilde{\lambda}}, \end{aligned}$$

where $(m(t), \lambda(t))_{t \in [0, T]}$ is the deterministic solution of (1.2.13). Let K_N be the infinitesimal generator of the process $(m_N(t), \lambda_N(t))$ (which has been calculated in (1.2.12)) and consider the time-dependent, linear and invertible transformation

$$(\tilde{m}, \tilde{\lambda}) = g_t(m, \lambda) = (N^{\frac{1}{2}}(m - m(t)), N^{\frac{1}{2}}(\lambda - \lambda(t))) :$$

applying K_N to a function $f(g_t(m, \lambda))$, by simple computations and Lemma 1.2.1 we can check that $\tilde{I}_{t, N}$, the infinitesimal generator of the time-inhomogeneous Markov process $(\tilde{m}_N(t), \tilde{\lambda}_N(t))$ reads:

$$\begin{aligned} \tilde{I}_{t, N} f(\tilde{m}, \tilde{\lambda}) &= K_N f(g_t(m, \lambda)) = \\ &= \frac{N(1 + m(t)) + N^{\frac{1}{2}}\tilde{m}}{2} \left(1 - \tanh(N^{-\frac{1}{2}}\tilde{\lambda} + \lambda(t))\right) \left(f\left(\tilde{m} - \frac{2}{N^{\frac{1}{2}}}, \tilde{\lambda} - \frac{2\beta}{N^{\frac{1}{2}}}\right) - f(\tilde{m}, \tilde{\lambda})\right) + \\ &+ \frac{N(1 - m(t)) - N^{\frac{1}{2}}\tilde{m}}{2} \left(1 + \tanh(N^{-\frac{1}{2}}\tilde{\lambda} + \lambda(t))\right) \left(f\left(\tilde{m} + \frac{2}{N^{\frac{1}{2}}}, \tilde{\lambda} + \frac{2\beta}{N^{\frac{1}{2}}}\right) - f(\tilde{m}, \tilde{\lambda})\right) + \\ &- \alpha(\tilde{\lambda} + N^{\frac{1}{2}}\lambda(t))f_{\tilde{\lambda}}(\tilde{m}, \tilde{\lambda}) - N^{\frac{1}{2}}\dot{m}(t)f_{\tilde{m}} - N^{\frac{1}{2}}\dot{\lambda}(t)f_{\tilde{\lambda}}. \end{aligned}$$

Taking $f \in \mathcal{C}_b^3(\mathbb{R}^2)$ and performing a first order Taylor expansion of $\tanh(N^{-\frac{1}{2}}\tilde{\lambda} + \lambda(t))$ around $\lambda(t)$ and a second order Taylor expansion of f around $(\tilde{m}, \tilde{\lambda})$ in the expression for $\tilde{I}_{N, t} f(\tilde{m}, \tilde{\lambda})$, one obtains that

$$\tilde{I}_{N, t} f(\tilde{m}, \tilde{\lambda}) = I_t f(\tilde{m}, \tilde{\lambda}) + o(1)$$

with $o(1)$ denoting a quantity uniformly converging to 0 as $N \uparrow +\infty$ for any $t \in [0, T]$. Notice that we are dealing with time-dependent infinitesimal generators, so, in order to apply Lemma 1.2.3, we have to overcome this aspect. This can be easily done by introducing the (deterministic, hence Markov) process $\tau(t) = t$. In this way, $\tilde{I}_{t,N}f(\tilde{m}, \tilde{\lambda}) = \tilde{I}_N f(\tilde{m}, \tilde{\lambda}, \tau)$, $\tilde{I}_t f(\tilde{m}, \tilde{\lambda}) = \tilde{I}f(\tilde{m}, \tilde{\lambda}, \tau)$ and it holds that

$$\lim_{N \rightarrow \infty} \sup_{(\tilde{m}, \tilde{\lambda}, \tau) \in \mathbb{R}^2 \times [0, T]} |\tilde{I}_N f(\tilde{m}, \tilde{\lambda}, \tau) - I f(\tilde{m}, \tilde{\lambda}, \tau)| = 0,$$

which concludes the proof. \square

4.1.2 Dynamics of critical fluctuations

The result of Theorem 4.1.1 holds for any regime, but our main goal is to study more closely the long-time behaviour of fluctuations at the critical point, since typically they display some peculiar features (see [13], [16] and [22]). From now on, we will always take the parameters α and β in such a way $\beta = \frac{\alpha}{2} + 1$, which represents the critical point for our model (see Theorem 1.2.3).

Let us consider the critical fluctuation flow for $t \in [0, T]$:

$$\hat{\mu}(t) = N^{\frac{1}{4}}(\mu_N(N^{\frac{1}{2}}t) - \mu_0^*),$$

where μ_0^* denotes the stationary solution correspondent to $(0, 0)$, the equilibrium point of (1.2.13). In this way, we are assuming that the process starts in local equilibrium, which simplifies the proof of our result, but it should not be too difficult to extend it to a general initial condition. Notice also that we are employing the usual space-time scaling involved in critical fluctuations. The flow $(\hat{\mu}_N(t))_{t \in [0, T]}$ can be fully described by the order parameter:

$$\hat{m}_N(t) = N^{\frac{1}{4}}m_N(N^{\frac{1}{2}}t), \quad \hat{\lambda}_N(t) = N^{\frac{1}{4}}\lambda_N(N^{\frac{1}{2}}t). \quad (4.1.5)$$

After having performed the change of variable (see Section 4.2.1 for motivation)

$$\begin{cases} z_N(t) = \hat{\lambda}_N(t), \\ u_N(t) = \frac{\beta \hat{m}_N(t) - \hat{\lambda}_N(t)}{\sqrt{\beta-1}}, \end{cases} \quad (4.1.6)$$

consider the process $\kappa_N(t) = z_N^2(t) + u_N^2(t)$. Since we are studying fluctuations around $(0, 0)$, it is reasonable to assume initial conditions such that $(m_N(0), \lambda_N(0)) \rightarrow (0, 0)$ as $N \uparrow +\infty$. We shall also consider the following assumption on $\lambda_N(0)$:

(H1) $N^{\frac{1}{4}}\lambda_N(0) \rightarrow \bar{\lambda} \in \mathbb{R} \setminus \{0\}$ in probability, as $N \rightarrow +\infty$.

Remark 4.1.1. The condition $\bar{\lambda} \neq 0$ in Hypothesis **(H1)** is of pure technical nature: the change of variable in the proof of Theorem 4.1.2 is singular in the origin, so we require the process involved does not start in the origin. We believe this assumption could be avoided by approximation arguments that we have not succeed to complete.

Theorem 4.1.2. *As $N \uparrow +\infty$, if $(m_N(0), \lambda_N(0)) \rightarrow (0, 0)$ and **(H1)** holds, the process $(\kappa_N(t))_{t \in [0, T]}$ converges, in sense of weak convergence of stochastic processes, to $(\kappa(t))_{t \in [0, T]}$, unique solution of the stochastic differential equation*

$$d\kappa(t) = \left(4\beta^2 - \frac{\beta}{2}\kappa^2(t) \right) dt + 2\beta\sqrt{2\kappa(t)}dB(t) \quad (4.1.7)$$

with initial condition $\kappa(0) = \frac{\beta}{\beta-1}\bar{\lambda}^2$.

4.2 Proof of the main theorem

In this section we give the proof for Theorem 4.1.2.

Let us try to sketch the idea of the proof before going into the details. We describe the behavior of the pair $(z_N(t), u_N(t))$ through the polar coordinates $(\kappa_N(t), \theta_N(t))$ such that

$$z_N(t) = \sqrt{\kappa_N(t)} \cos \theta_N(t), \quad u_N(t) = \sqrt{\kappa_N(t)} \sin \theta_N(t).$$

The dominant part of the angular variable θ_N can be approximated by a deterministic drift of order $N^{\frac{1}{2}}$. So, as $N \uparrow +\infty$, the angular variable averages out, producing the limiting dynamics (4.1.7) via an averaging principle. This is the reason for which we get convergence only of the radial variable $\kappa_N(t)$. Actually, this convergence will be proved through a localization argument. First of all, we analyse the convergence of the process $\kappa_N(t)$ stopped when it becomes too large or too small. By means of these stopping times (in particular the one related to the lower bound) we are able to avoid technical problems due to the singularity of the polar coordinates in the origin. Secondly, we characterize the limit of the stopped process as the solution of a *stopped* martingale problem related to the infinitesimal generator of the solution of (4.1.7). Finally, we will exploit this characterization of the limit of the stopped process to get the thesis of Theorem 4.1.2.

4.2.1 Preliminary computations

In this subsection we perform some preliminary computations which will be useful in the following. We will also prove existence and uniqueness of the limiting process (4.1.7).

Change of variables

First of all, we want to give a motivation for the change of variable (4.1.6). Let \mathcal{L}_N be the infinitesimal generator of the process $(\hat{m}_N(t), \hat{\lambda}_N(t))$ defined in (4.1.5). By expanding the generator in a similar fashion to what is done in Lemma 4.2.2, one can check that, for a function f regular enough,

$$\mathcal{L}_N f(\hat{m}, \hat{\lambda}) = N^{\frac{1}{2}} \mathcal{L}_1 f(\hat{m}, \hat{\lambda}) + \mathcal{L}_2 f(\hat{m}, \hat{\lambda}) + o(1). \quad (4.2.1)$$

In particular, the dominant part of order $N^{\frac{1}{2}}$ is given by

$$\mathcal{L}_1 f(\hat{m}, \hat{\lambda}) = (2\hat{\lambda} - 2\hat{m})\partial_{\hat{m}} f(\hat{m}, \hat{\lambda}) + (2\hat{\lambda} - 2\beta\hat{m})\partial_{\hat{\lambda}} f(\hat{m}, \hat{\lambda}) = (\nabla f(\hat{m}, \hat{\lambda}))^\top A \begin{pmatrix} \hat{m} \\ \hat{\lambda} \end{pmatrix},$$

where

$$A = \begin{pmatrix} -2 & 2 \\ -2\beta & 2 \end{pmatrix}.$$

Notice that A corresponds to the Jacobian matrix in $(0, 0)$ for system (1.2.13) at critical point $\beta = \frac{\alpha}{2} + 1$ and its eigenvalues are $\lambda_{1,2} = \pm i2\sqrt{\beta - 1}$. Consider an invertible matrix C such that

$$CAC^{-1} = \begin{pmatrix} 0 & -2\sqrt{\beta - 1} \\ 2\sqrt{\beta - 1} & 0 \end{pmatrix}$$

and take the change of variables

$$\begin{pmatrix} z \\ u \end{pmatrix} = C \begin{pmatrix} \hat{m} \\ \hat{\lambda} \end{pmatrix}.$$

Without pretending to be formal here (see Lemmas 4.2.1 and 4.2.2 for the formal computations), one gets

$$\mathcal{L}_1 f(z, u) = (\nabla f(z, u))^\top CAC^{-1} \begin{pmatrix} z \\ u \end{pmatrix} = -2\sqrt{\beta - 1}u\partial_z f(z, u) + 2\sqrt{\beta - 1}z\partial_u f(z, u),$$

then, passing to the polar coordinates $\kappa = (z)^2 + (u)^2$, $\theta = \arctan(u/z)$,

$$\mathcal{L}_1 f(\kappa, \theta) = 2\sqrt{\beta - 1}\partial_\theta f(\kappa, \theta).$$

It follows that \mathcal{L}_1 , which according to (4.2.1) is the fast component of the generator \mathcal{L}_N , involves only the derivative with respect to θ , which therefore plays the role of fast variable compared to the evolution of the radial variable κ . This suggests to derive the asymptotic evolution of κ by an averaging principle. One can easily check that a suitable choice for C is given by

$$C = \begin{pmatrix} 0 & 1 \\ \frac{\beta}{\sqrt{\beta - 1}} & -\frac{1}{\sqrt{\beta - 1}} \end{pmatrix},$$

which justifies the change of variable (4.1.6), i.e.

$$\begin{cases} z = \hat{\lambda}, \\ u = \frac{\beta\hat{m} - \hat{\lambda}}{\sqrt{\beta - 1}}. \end{cases}$$

Remark 4.2.1. We can give a more intuitive idea on the argument for this change of variable. As stated before, one may expect that the "dominant" behaviour of $(w_N(t), z_N(t))$ should be driven by the linear system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}. \quad (4.2.2)$$

So, studying the solutions of (4.2.2): we look for a quadratic function

$$F(x, y) = (x, y)Q \begin{pmatrix} x \\ y \end{pmatrix}$$

which is a first integral for (4.2.2). Let Q be a symmetric matrix and let $X(t) = (x(t), y(t))^\top$: then $F(X(t))$ is a first integral if and only if

$$\frac{d}{dt}F(X(t)) = 0 \Leftrightarrow (\dot{X}(t))^\top Q X(t) + (X(t))^\top Q \dot{X}(t) = 0 \Leftrightarrow A^\top Q + QA = 0.$$

It can be easily checked that

$$Q = \begin{pmatrix} \beta & -1 \\ -1 & 1 \end{pmatrix}$$

satisfies $A^\top Q + QA = 0$, hence, for suitable $c \in \mathbb{R}$, the equation

$$(x, y)Q \begin{pmatrix} x \\ y \end{pmatrix} = c \Leftrightarrow \beta x^2 - 2xy + y^2 = c$$

identifies an ellipse which is an orbit of the linearized system. Hence, in order to use polar coordinates, we want to transform this ellipse into a circle and this transformation is equivalent to the change of variables described above.

Expansion of infinitesimal generators

In this paragraph, we study the asymptotic expansion of the infinitesimal generators of the processes involved in the proof of the main result.

Lemma 4.2.1. *In the critical case $\beta = \frac{\alpha}{2} + 1$, the infinitesimal generator G_N of the Markov pair $(z_N(t), u_N(t))$ is given by:*

$$\begin{aligned} G_N f(z, u) &= \\ &= N^{\frac{1}{2}} \frac{N(1 + N^{-\frac{1}{4}}\beta^{-1}(\sqrt{\beta-1}u + z))}{2} \left(1 - \tanh(N^{-\frac{1}{4}}z)\right) \left(f\left(z - \frac{2\beta}{N^{\frac{3}{4}}}, u\right) - f(z, u)\right) + \\ &+ N^{\frac{1}{2}} \frac{N(1 - N^{-\frac{1}{4}}\beta^{-1}(\sqrt{\beta-1}u + z))}{2} \left(1 + \tanh(N^{-\frac{1}{4}}z)\right) \left(f\left(z + \frac{2\beta}{N^{\frac{3}{4}}}, u\right) - f(z, u)\right) + \\ &+ N^{\frac{1}{2}} \left(-2(\beta-1)zf_z(z, u) + 2\sqrt{\beta-1}zf_u(z, u)\right). \end{aligned}$$

Proof. Define the following processes, for $t \in [0, T]$:

$$\tilde{w}_N(t) = N^{\frac{1}{4}}m_N(t), \quad \tilde{z}_N(t) = N^{\frac{1}{4}}\lambda_N(t), \quad \tilde{u}_N(t) = \frac{\beta\tilde{w}_N(t) - \tilde{z}_N(t)}{\sqrt{\beta-1}}.$$

We want to identify the infinitesimal generator of $(\tilde{z}_N(t), \tilde{u}_N(t))$ applying Lemma 1.2.1. Consider the function

$$g(m, \lambda) = \left(N^{\frac{1}{4}}\lambda, N^{\frac{1}{4}}\frac{\beta m - \lambda}{\sqrt{\beta-1}}\right)$$

and evaluate the infinitesimal generator K_N defined in (1.2.12) on the function $(f \circ g)$. It is just a matter of computation to notice that we get the infinitesimal generator \tilde{G}_N defined by

$$\begin{aligned} \tilde{G}_N f(\tilde{z}, \tilde{u}) &= K_N(f \circ g)(m, \lambda) = \\ &= \frac{N(1 + N^{-\frac{1}{4}}\beta^{-1}(\sqrt{\beta-1}\tilde{u} + \tilde{z}))}{2} \left(1 - \tanh(N^{-\frac{1}{4}}\tilde{z})\right) \left(f\left(\tilde{z} - \frac{2\beta}{N^{\frac{3}{4}}}, \tilde{u}\right) - f(\tilde{z}, \tilde{u})\right) + \\ &+ \frac{N(1 - N^{-\frac{1}{4}}\beta^{-1}(\sqrt{\beta-1}\tilde{u} + \tilde{z}))}{2} \left(1 + \tanh(N^{-\frac{1}{4}}\tilde{z})\right) \left(f\left(\tilde{z} + \frac{2\beta}{N^{\frac{3}{4}}}, \tilde{u}\right) - f(\tilde{z}, \tilde{u})\right) + \\ &+ \left(-2(\beta-1)\tilde{z}f_{\tilde{z}}(\tilde{z}, \tilde{u}) + 2\sqrt{\beta-1}\tilde{z}f_{\tilde{u}}(\tilde{z}, \tilde{u})\right). \end{aligned}$$

Finally, we obtain $(z_N(t), u_N(t))$ from $(\tilde{z}_N(t), \tilde{u}_N(t))$ rescaling time by a factor $N^{\frac{1}{2}}$: the infinitesimal generator G_N of $(z_N(t), u_N(t))$ is obtained by

$$G_N f(z, u) = N^{\frac{1}{2}} \tilde{G}_N f(\tilde{z}, \tilde{u}).$$

□

Lemma 4.2.2. Fix any $r, R > 0$ such that $r < \frac{\beta}{\beta-1}\bar{\lambda}^2 < R$ and let

$$\tau_{r,R}^N := \inf\{t \in [0, T] \mid z_N^2(t) + u_N^2(t) \notin]r, R[\}.$$

For any $t \in [0, T]$ define the polar coordinates process $(\kappa_N(t), \theta_N(t))$ by

$$\begin{aligned} \kappa_N(t) &= z_N^2(t) + u_N^2(t), \\ \theta_N(t) &= \arctan\left(\frac{u_N(t)}{z_N(t)}\right). \end{aligned}$$

Then, the stopped process $(\kappa_N(t \wedge \tau_{r,R}^N), \theta_N(t \wedge \tau_{r,R}^N))$ has an infinitesimal generator $H_N^{r,R}$ which, for a function $f : [r, R] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \mathbb{R}$, $f \in \mathcal{C}^3$, satisfies

$$\begin{aligned} H_N^{r,R} f(\kappa, \theta) &= \mathbb{1}_{]r,R[}(\kappa) \left[8\beta^2 \kappa \cos^2 \theta f_{\kappa\kappa}(\kappa, \theta) - 8\beta^2 \cos \theta \sin \theta f_{\kappa\theta}(\kappa, \theta) + \frac{2\beta^2 \sin^2 \theta}{\kappa} f_{\theta\theta}(\kappa, \theta) + \right. \\ &+ \left(N^{\frac{1}{2}} 2\sqrt{\beta-1} + \frac{4\beta^2 \cos \theta \sin \theta}{\kappa} + \frac{2\beta}{3} \kappa \cos^3 \theta \sin \theta + 2\sqrt{\beta-1} \sin \theta \right) f_{\theta}(\kappa, \theta) + \\ &\left. + \left(4\beta^2 - \frac{4\beta}{3} \kappa^2 \cos^4 \theta \right) f_{\kappa}(\kappa, \theta) + o_{r,R}(1) \right], \end{aligned}$$

where the remainders $o_{r,R}(1)$ can be uniformly dominated by a term of order $o(1)$ on $[r, R] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Remark 4.2.2. The process $\theta_N(t)$ defined in Lemma 4.2.2 is almost surely well-defined for $t \in [0, T]$. In fact we have that

$$P\left(\exists t \in [0, T] \text{ s.t. } z_N(t) = 0\right) = 0 :$$

from (3.1.2), we can see that λ_N (hence z_N) can hit 0 only when a jump occurs. Let $(\tau_n)_n$ be a sequence of jump times for λ_N : for any n , we have that

$$P(\lambda_N(\tau_n) = 0 \mid \mathcal{F}_{\tau_{n-1}}) \leq P(\tau_n \in A_{n-1} \mid \mathcal{F}_{\tau_{n-1}})$$

where A_{n-1} is an aleatory set such that $|A_{n-1}| \leq 1$. In fact, intuitively, since the jump size of λ_N is fixed and the trajectories of λ_N are strictly increasing or strictly decreasing between two jumps, then there exists at most one point in which the jump leading to 0 can occur. Moreover, the distribution of the jump times is absolutely continuous with respect to the Lebesgue measure over \mathbb{R} , hence

$$P(\tau_n \in A_{n-1} \mid \mathcal{F}_{\tau_{n-1}}) = 0.$$

In conclusion,

$$P\left(\exists t \in [0, T] \text{ s.t. } z_N(t) = 0\right) \leq \sum_n E \left[P(\tau_n \in A_{n-1} \mid \mathcal{F}_{\tau_{n-1}}) \right] = 0.$$

Proof. Consider the function

$$g(z, u) = \left(z^2 + u^2, \arctan\left(\frac{u}{z}\right) \right)$$

and apply generator G_N defined in Lemma 4.2.1 to $(f \circ g)(z, u)$ with $f \in \mathcal{C}^3([r, R] \times [-\frac{\pi}{2}, \frac{\pi}{2}])$.

By standard Taylor expansions we get:

$$\begin{aligned}
G_N(f \circ g)(z, u) &= \\
&= \frac{N^{\frac{3}{2}}(\beta + N^{-\frac{1}{4}}(\sqrt{\beta-1}u + z))}{2\beta} \left(1 - \tanh(N^{-\frac{1}{4}}z)\right) \left((f \circ g)\left(z - \frac{2\beta}{N^{\frac{3}{4}}}, u\right) - (f \circ g)(z, u)\right) + \\
&+ \frac{N^{\frac{3}{2}}(\beta - N^{-\frac{1}{4}}(\sqrt{\beta-1}u + z))}{2\beta} \left(1 + \tanh(N^{-\frac{1}{4}}z)\right) \left((f \circ g)\left(z + \frac{2\beta}{N^{\frac{3}{4}}}, u\right) - (f \circ g)(z, u)\right) + \\
&+ N^{\frac{1}{2}} \left(-2(\beta-1)z(f \circ g)_z + 2\sqrt{\beta-1}z(f \circ g)_u\right) = \\
&= \frac{\beta N^{\frac{3}{2}} + N^{\frac{5}{4}}(\sqrt{\beta-1}u + z)}{2\beta} \left(1 - \frac{z}{N^{\frac{1}{4}}} + \frac{z^3}{3N^{\frac{3}{4}}} + o\left(\frac{1}{N^{\frac{3}{4}}}\right)\right) \cdot \\
&\quad \cdot \left(-\frac{2\beta}{N^{\frac{3}{4}}}(f \circ g)_z + \frac{2\beta^2}{N^{\frac{3}{2}}}(f \circ g)_{zz} + o\left(\frac{1}{N^{\frac{3}{2}}}\right)\right) + \\
&+ \frac{\beta N^{\frac{3}{2}} - N^{\frac{5}{4}}(\sqrt{\beta-1}u + z)}{2\beta} \left(1 + \frac{z}{N^{\frac{1}{4}}} - \frac{z^3}{3N^{\frac{3}{4}}} + o\left(\frac{1}{N^{\frac{3}{4}}}\right)\right) \cdot \\
&\quad \cdot \left(\frac{2\beta}{N^{\frac{3}{4}}}(f \circ g)_z - \frac{2\beta^2}{N^{\frac{3}{2}}}(f \circ g)_{zz} + o\left(\frac{1}{N^{\frac{3}{2}}}\right)\right) + \\
&+ N^{\frac{1}{2}} \left(-2(\beta-1)z(f \circ g)_z + 2\sqrt{\beta-1}z(f \circ g)_u\right) = \\
&= 2\beta^2(f \circ g)_{zz} - \left(\frac{2\beta z^3}{3} + N^{\frac{1}{2}}2\beta\sqrt{\beta-1}u\right) (f \circ g)_z + N^{\frac{1}{2}}2\sqrt{\beta-1}z(f \circ g)_u + o(1).
\end{aligned}$$

Now, observe that:

$$(f \circ g)_z = 2zf_\kappa - \frac{u}{z^2 + u^2}f_\theta, \quad (f \circ g)_u = 2uf_\kappa + \frac{z}{z^2 + u^2}f_\theta,$$

$$(f \circ g)_{zz} = 4z^2f_{\kappa\kappa} - \frac{4zu}{z^2 + u^2}f_{\kappa\theta} + \frac{u^2}{(z^2 + u^2)^2}f_{\theta\theta} + 2f_\kappa + \frac{2uz}{(z^2 + u^2)^2}f_\theta,$$

hence

$$\begin{aligned}
G_N(f \circ g)(z, u) &= 8\beta^2z^2f_{\kappa\kappa} - \frac{8\beta^2zu}{z^2 + u^2}f_{\kappa\theta} + \frac{2\beta^2u^2}{(z^2 + u^2)^2}f_{\theta\theta} + \\
&+ \left(N^{\frac{1}{2}}2\sqrt{\beta-1} + \frac{4\beta^2zu}{(z^2 + u^2)^2} + \frac{2\beta z^3u}{3(z^2 + u^2)}\right) f_\theta + \\
&+ \left(4\beta^2 - \frac{4\beta z^4}{3}\right) f_\kappa + o(1).
\end{aligned}$$

Thanks to Lemma 1.2.1, the infinitesimal generator H_N of $(\kappa_N(t), \theta_N(t))$ satisfies:

$$\begin{aligned} H_N f(\kappa, \theta) &= 8\beta^2 \kappa \cos^2 \theta f_{\kappa\kappa}(\kappa, \theta) - 8\beta^2 \cos \theta \sin \theta f_{\kappa\theta}(\kappa, \theta) + \frac{2\beta^2 \sin^2 \theta}{\kappa} f_{\theta\theta}(\kappa, \theta) + \\ &+ \left(N^{\frac{1}{2}} 2\sqrt{\beta-1} + \frac{4\beta^2 \cos \theta \sin \theta}{\kappa} + \frac{2\beta}{3} \kappa \cos^3 \theta \sin \theta + 2\sqrt{\beta-1} \sin \theta \right) f_\theta(\kappa, \theta) + \\ &+ \left(4\beta^2 - \frac{4\beta}{3} \kappa^2 \cos^4 \theta \right) f_\kappa(\kappa, \theta) + o(1). \end{aligned}$$

Therefore, the infinitesimal generator $H_N^{r,R}$ of the stopped process $(\kappa_N(t \wedge \tau_{r,R}^N), \theta(t \wedge \tau_{r,R}^N))$ satisfies, for a function $f \in \mathcal{C}^3([r, R] \times [-\frac{\pi}{2}, \frac{\pi}{2}])$:

$$\begin{aligned} H_N^{r,R} f(\kappa, \theta) &= \mathbb{1}_{]r,R[}(\kappa) \left[8\beta^2 \kappa \cos^2 \theta f_{\kappa\kappa}(\kappa, \theta) - 8\beta^2 \cos \theta \sin \theta f_{\kappa\theta}(\kappa, \theta) + \frac{2\beta^2 \sin^2 \theta}{\kappa} f_{\theta\theta}(\kappa, \theta) + \right. \\ &+ \left(N^{\frac{1}{2}} 2\sqrt{\beta-1} + \frac{4\beta^2 \cos \theta \sin \theta}{\kappa} + \frac{2\beta}{3} \kappa \cos^3 \theta \sin \theta + 2\sqrt{\beta-1} \sin \theta \right) f_\theta(\kappa, \theta) + \\ &\left. + \left(4\beta^2 - \frac{4\beta}{3} \kappa^2 \cos^4 \theta \right) f_\kappa(\kappa, \theta) + o_{r,R}(1) \right], \end{aligned}$$

Notice that the remainders of the expansion of \tanh and $(f \circ g)$ are continuous functions of (κ, θ) on the compact set $[r, R] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, so $o_{r,R}(1)$ can be uniformly dominated by a term of order $o(1)$. \square

Existence and uniqueness of limit process

In this paragraph we prove the well-posedness of the SDE (4.1.7).

Lemma 4.2.3. *There exists a unique solution of the limiting equation (4.1.7).*

Proof. Let $(X(t), Y(t))_{t \geq 0}$ be the unique and globally existent solution of

$$\begin{cases} dX(t) = -\frac{\beta^3}{2} X(t)(X^2(t) + Y^2(t))dt + dB_1(t) \\ dY(t) = -\frac{\beta^3}{2} Y(t)(X^2(t) + Y^2(t))dt + dB_2(t) \end{cases} \quad (4.2.3)$$

with B_1, B_2 independent Brownian motion and $X(0) = Y(0) = \frac{\bar{\lambda}}{2\sqrt{\beta(\beta-1)}}$. Now, let us define

$$\kappa(t \wedge \tau_{r,R}) = 2\beta^2(X^2(t \wedge \tau_{r,R}) + Y^2(t \wedge \tau_{r,R})) \quad (4.2.4)$$

with

$$\tau_{r,R} := \inf_{t \in [0, T]} \{t \geq 0 \mid 2\beta^2(X^2(t) + Y^2(t)) \notin]r, R[\}.$$

Applying Ito's formula to (4.2.4) for $t \in [0, T \wedge \tau_{r,R}]$:

$$d\kappa(t) = \left(4\beta^2 - \frac{\beta}{2} \kappa^2(t) \right) dt + 2\beta\sqrt{2\kappa(t)} \left(\frac{X(t)}{\sqrt{X^2(t) + Y^2(t)}} dB_1(t) + \frac{Y(t)}{\sqrt{X^2(t) + Y^2(t)}} dB_2(t) \right).$$

Notice that, for $t \in [0, T \wedge \tau_{r,R}]$, the process

$$\int_0^t \frac{X(s)}{\sqrt{X^2(s) + Y^2(s)}} dB_1(s) + \frac{Y(s)}{\sqrt{X^2(s) + Y^2(s)}} dB_2(s)$$

is a zero-mean martingale with quadratic variation dt , hence it is a Brownian motion. Therefore, the process $\kappa(t)$ defined in (4.2.4), is a solution of (4.1.7) on the time interval $[0, T \wedge \tau_{r,R}]$ and existence is proved. On the same time interval, also uniqueness holds for the solution of (4.1.7) since the coefficients

$$b(\kappa) = 4\beta^2 - \frac{\beta}{2}\kappa, \quad \sigma(\kappa) = 2\beta\sqrt{2\kappa}$$

are Lipschitz-continuous functions over $[r, R]$. To conclude the proof, we want to show that

$$P(\tau_{r,R} \leq T) \rightarrow 0, \quad \text{as } r \rightarrow 0^+, R \rightarrow +\infty. \quad (4.2.5)$$

By the global existence of $(X(t), Y(t))_{t \geq 0}$,

$$P\left(\sup_{t \in [0, T]} 2\beta^2(X^2(t) + Y^2(t)) \geq R\right) \rightarrow 0$$

as $R \rightarrow \infty$. On the other hand, take a sequence $(r_n)_{n \geq 1}$ of positive numbers converging monotonically to zero. For any $n \geq 1$, define the event A_n as

$$A_n := \left\{ \inf_{t \in [0, T]} 2\beta^2(X^2(t) + Y^2(t)) \leq r_n \right\}$$

and notice that $(A_n)_{n \geq 1}$ is a decreasing sequence of events converging to

$$\bar{A} := \bigcap_n A_n = \left\{ \inf_{t \in [0, T]} 2\beta^2(X^2(t) + Y^2(t)) = 0 \right\} = \left\{ \exists t \in [0, T] \text{ s.t. } X(t) = Y(t) = 0 \right\}.$$

Finally, recall that $(X(t), Y(t))$, being a bidimensional diffusion, is absolutely continuous with respect to a bidimensional Brownian motion. Since a bidimensional Brownian motion never visits the origin a.s., we conclude by

$$P\left(\exists t \in [0, T] \text{ s.t. } X(t) = Y(t) = 0\right) = 0.$$

□

4.2.2 Tightness of the sequence of stopped processes

In this section, we want to prove that the sequence of stopped processes $(\kappa_N(t \wedge \tau_{r,R}^N))_{N \geq 1}$ defined in Lemma 4.2.2 is tight.

Remark 4.2.3. Let P_N be the law of $(\kappa_N(t))_{t \in [0, T]}$ on $\mathcal{D}([0, T], \mathbb{R})$, endowed with the Skorohod topology. For any $x \in \mathcal{D}([0, T], \mathbb{R})$ and any $r, R > 0$ consider

$$\tau_{r,R} := \inf\{t \in [0, T] \mid x(t) \notin]r, R[\}$$

and define the map $\varphi_{r,R} : \mathcal{D}([0, T], \mathbb{R}) \rightarrow \mathcal{D}([0, T], \mathbb{R})$ as

$$\varphi_{r,R}(x(\cdot)) = x(\cdot \wedge \tau_{r,R}). \quad (4.2.6)$$

Formally speaking, "the sequence of stopped processes $(\kappa_N(t \wedge \tau_{r,R}^N))_{N \geq 1}$ is tight" means "the sequence of probability measures $(P_N \circ \varphi_{r,R}^{-1})_{N \geq 1}$ is tight".

Before proving the tightness, let us recall the following results on jump processes (see [36]).

Proposition 4.2.1. *Let (Ω, \mathcal{F}, P) be a complete probability space with a right-continuous increasing family $(\mathcal{F}_t)_{t \geq 0}$ of sub- σ -fields of \mathcal{F} each containing all P -null sets. Let X be a semi-martingale of the form*

$$X(t) = X(0) + \int_0^t \int_Y f(s, y) \Lambda(ds, dy)$$

where $\Lambda(ds, dy)$ is a point process of intensity $A(s, dy)ds$ on $[0, T] \times Y$, with $T > 0$ and Y measurable space, and $X(0)$ is an \mathcal{F}_0 -measurable random variable. If f is (\mathcal{F}_t) -predictable and for every $t \in [0, T]$

$$\int_0^t \int_Y |f(s, y)| \Lambda(ds, dy) < \infty \quad a.s.$$

and $F \in \mathcal{C}(\mathbb{R})$ then

$$F(X(t)) = F(X(0)) + \int_0^t \int_Y [F(X(s) + f(s, y)) - F(X(s))] \Lambda(ds, dy).$$

Proposition 4.2.2. *Let (Ω, \mathcal{F}, P) , $(\mathcal{F}_t)_{t \geq 0}$, $\Lambda(ds, dy)$, $A(s, dy)$, Y and $X(0)$ as in Proposition 4.2.1. Let X be a martingale of the form*

$$X(t) = X(0) + \int_0^t \int_Y f(s, y) \tilde{\Lambda}(ds, dy)$$

where $\tilde{\Lambda}(ds, dy)$ is the compensated point process

$$\tilde{\Lambda}(ds, dy) = \Lambda(ds, dy) - A(s, dy)ds.$$

If f is (\mathcal{F}_t) -predictable and, for every $t \in [0, T]$

$$E \left[\int_0^t \int_Y |f(s, y)| A(s, dy) ds \right] < \infty, \quad E \left[\int_0^t \int_Y |f(s, y)|^2 A(s, dy) ds \right] < \infty$$

then

$$E [X^2(t)] = E \left[\left(\int_0^t \int_Y f(s, y) \tilde{\Lambda}(ds, dy) \right)^2 \right] = E \left[\int_0^t \int_Y f^2(s, y) A(s, dy) ds \right].$$

Proposition 4.2.3. *For any $N \geq 1$, $t \in [0, T]$ let $\kappa_N(t \wedge \tau_{r,R}^N)$ be the process defined in Lemma 4.2.2. Then, the sequence of processes $(\kappa_N(t \wedge \tau_{r,R}^N))_{N \geq 1}$ is tight.*

Proof. First of all, for $j \in \mathcal{S}$, $t > 0$, consider the set $A_N(j, t)$ of spins equal to j at time t (see also (1.2.11)): we have

$$|A_N(j, t)| = \frac{N(1 + jm_N(t))}{2}. \quad (4.2.7)$$

We can write the jump part of the process $z_N(t)$ in the following way, for $t \in [0, T]$:

$$\int_0^t \sum_{j \in \mathcal{S}} \left[-\frac{j2\beta}{N^{\frac{3}{4}}} \right] \Lambda_N(j, ds), \quad (4.2.8)$$

where $\Lambda(j, ds)$ indicates a point process of intensity

$$N^{\frac{1}{2}} |A_N(j, N^{\frac{1}{2}}s)| (1 - j \tanh \lambda_N(N^{\frac{1}{2}}s)) dt.$$

From (1.2.3) and (4.1.6), it is easy to see that $u_N(t)$ has continuous trajectories, so for any $t \in [0, T]$,

$$\kappa_N(t) - \kappa_N(t-) = z_N^2(t) - z_N^2(t-).$$

Let us study the jumps of the process $z_N^2(t)$. Using the generalized Itô's formula provided by Proposition 4.2.1, we can write the jump part of $z_N^2(t)$ as

$$\int_0^t \sum_{j \in \mathcal{S}} \left[\left(z_N(s) - \frac{j2\beta}{N^{\frac{3}{4}}} \right)^2 - (z_N(s))^2 \right] \Lambda_N(j, dt) = \int_0^t \sum_{j \in \mathcal{S}} \left[\frac{4\beta^2}{N^{\frac{3}{2}}} - \frac{j2\beta z_N(s)}{N^{\frac{3}{4}}} \right] \Lambda_N(j, dt). \quad (4.2.9)$$

The stopped process $\kappa_N(t \wedge \tau_{r,R}^N)$ can be decomposed in the following way:

$$\kappa_N(t \wedge \tau_{r,R}^N) = \kappa_N(0) + \int_0^{t \wedge \tau_{r,R}^N} H_N^{r,R} \kappa_N(s) ds + \mathcal{M}_{N,\kappa}^{t \wedge \tau_{r,R}^N} \quad (4.2.10)$$

where $H_N^{r,R} \kappa$ indicates the infinitesimal generator of Lemma 4.2.2 evaluated on the function $f(\kappa, \theta) = \kappa$, while $\mathcal{M}_{N,\kappa}$ is the local martingale given by

$$\int_0^{t \wedge \tau_{r,R}^N} \sum_{j \in \mathcal{S}} \left[\frac{4\beta^2}{N^{\frac{3}{2}}} - \frac{j2\beta z_N(s)}{N^{\frac{3}{4}}} \right] \tilde{\Lambda}_N(j, ds), \quad (4.2.11)$$

with $\tilde{\Lambda}_N(j, ds)$ representing the compensated version of $\Lambda_N(j, ds)$ (see the notation introduced in Proposition 4.2.1). We will use the *Aldous' tightness criterion* (see [6]). A sequence of processes $\{\xi_N(t)\}_{N \geq 1}$ is tight if:

1. for every $\varepsilon > 0$ there exists $M > 0$ such that

$$\sup_N P \left(\sup_{t \in [0, T]} |\xi_N(t)| \geq M \right) \leq \varepsilon, \quad (4.2.12)$$

2. for every $\varepsilon > 0$ and $\alpha > 0$ there exists $\delta > 0$ such that

$$\sup_N \sup_{0 \leq \tau_1 \leq \tau_2 \leq (\tau_1 + \delta) \wedge T} P(|\xi_N(\tau_2) - \xi_N(\tau_1)| \geq \alpha) \leq \varepsilon, \quad (4.2.13)$$

where the second sup is taken over stopping times τ_1 and τ_2 , adapted to the filtration generated by process ξ_N .

Notice that, for any $t \in [0, T]$, $|\kappa_N(t)| = \kappa_N(t)$ and $\kappa_N(t \wedge \tau_{r,R}^N) \leq R$ hence the uniform boundedness (4.2.12) trivially holds.

Let now τ_1, τ_2 be two stopping times adapted to the the filtration generated by κ_N , such that, for $\delta > 0$, $\tau_1 \leq \tau_2 \leq (\tau_1 + \delta) \wedge T$ a.s.. By decomposition (4.2.10), we have that

$$\begin{aligned} |\kappa_N(\tau_2 \wedge \tau_{r,R}^N) - \kappa_N(\tau_1 \wedge \tau_{r,R}^N)| &= \left| \int_{\tau_1 \wedge \tau_{r,R}^N}^{\tau_2 \wedge \tau_{r,R}^N} H_N^{r,R} \kappa_N(s) ds + \mathcal{M}_{N,\kappa}^{(\tau_1, \tau_2) \wedge \tau_{r,R}^N} \right| \leq \\ &\leq \int_{\tau_1 \wedge \tau_{r,R}^N}^{\tau_2 \wedge \tau_{r,R}^N} |H_N^{r,R} \kappa_N(s)| ds + \left| \mathcal{M}_{N,\kappa}^{(\tau_1, \tau_2) \wedge \tau_{r,R}^N} \right| \end{aligned}$$

where

$$\mathcal{M}_{N,\kappa}^{(\tau_1, \tau_2) \wedge \tau_{r,R}^N} := \int_{\tau_1 \wedge \tau_{r,R}^N}^{\tau_2 \wedge \tau_{r,R}^N} \sum_{j \in \mathcal{S}} \left[\frac{4\beta^2}{N^{\frac{3}{2}}} - \frac{j2\beta z_N(s)}{N^{\frac{3}{4}}} \right] \tilde{\Lambda}_N(j, ds).$$

Let's give some estimations: if $f(\kappa, \theta) = k$, by Lemma 4.2.2

$$H_N^{r,R} f(\kappa, \theta) = \mathbb{1}_{]r,R[}(\kappa) \left[4\beta^2 - \frac{4\beta}{3} \kappa^2 \cos^4 \theta + o_{r,R}(1) \right]$$

so

$$\begin{aligned} \int_{\tau_1 \wedge \tau_{r,R}^N}^{\tau_2 \wedge \tau_{r,R}^N} |H_N^{r,R} \kappa_N(s)| ds &= \int_{\tau_1 \wedge \tau_{r,R}^N}^{\tau_2 \wedge \tau_{r,R}^N} \left| 4\beta^2 - \frac{4\beta}{3} (\kappa_N(s))^2 \cos^4 \theta_N(s) + o_{r,R}(1) \right| ds \leq \\ &\leq \int_{\tau_1 \wedge \tau_{r,R}^N}^{\tau_2 \wedge \tau_{r,R}^N} 4\beta^2 + \frac{4\beta}{3} R^2 + o_{r,R}(1) ds \leq \int_{\tau_1 \wedge \tau_{r,R}^N}^{\tau_2 \wedge \tau_{r,R}^N} 4\beta^2 + \frac{4\beta}{3} R^2 + \frac{C_1(R)}{N^\gamma} ds, \end{aligned}$$

where $\gamma > 0$ and $C_1(R)$ is a constant which depends on R . Then, taking $C_2(R) := 4\beta^2 + \frac{4\beta}{3}R^2 + C_1(R)$,

$$\int_{\tau_1 \wedge \tau_{r,R}^N}^{\tau_2 \wedge \tau_{r,R}^N} |H_N^{r,R} \kappa_N(s)| ds \leq C_2(R)\delta. \quad (4.2.14)$$

Moreover, if we fix $t \in [0, T]$, we can study the expected value of $\left(\mathcal{M}_{N,\kappa}^{t \wedge \tau_{r,R}^N}\right)^2$ applying the Itô isometry for stochastic integrals with respect to point processes (Proposition 4.2.1):

$$\begin{aligned} & E \left[\left(\mathcal{M}_{N,\kappa}^{t \wedge \tau_{r,R}^N}\right)^2 \right] = \\ & E \left[\int_0^{t \wedge \tau_{r,R}^N} \sum_{j \in \mathcal{S}} \left[\frac{4\beta^2}{N^{\frac{3}{2}}} - \frac{j2\beta z_N(s)}{N^{\frac{3}{4}}} \right]^2 N^{\frac{1}{2}} |A_N(j, N^{\frac{1}{2}}s)| (1 - j \tanh \lambda_N(N^{\frac{1}{2}}s)) ds \right] \leq \\ & E \left[\int_0^{t \wedge \tau_{r,R}^N} \sum_{j \in \mathcal{S}} 2 \left[\frac{16\beta^4}{N^3} + \frac{4\beta^2 z_N^2(s)}{N^{\frac{3}{2}}} \right] N^{\frac{1}{2}} |A_N(j, N^{\frac{1}{2}}s)| 2ds \right] \leq \\ & E \left[\int_0^{t \wedge \tau_{r,R}^N} 8 \left[\frac{16\beta^4}{N^3} + \frac{4\beta^2 R}{N^{\frac{3}{2}}} \right] N^{\frac{1}{2}} \sup_{j \in \mathcal{S}} |A_N(j, N^{\frac{1}{2}}s)| ds \right] \leq \\ & E \left[\int_0^{t \wedge \tau_{r,R}^N} 32\beta^2 \left[\frac{4\beta^2}{N^{\frac{3}{2}}} + R \right] ds \right]. \end{aligned}$$

By the Optional Sampling Theorem,

$$\begin{aligned} & E \left[\left(\mathcal{M}_{N,\kappa}^{(\tau_1, \tau_2) \wedge \tau_{r,R}^N}\right)^2 \right] = E \left[\left(\mathcal{M}_{N,\kappa}^{\tau_2 \wedge \tau_{r,R}^N}\right)^2 - \left(\mathcal{M}_{N,\kappa}^{\tau_1 \wedge \tau_{r,R}^N}\right)^2 \right] \leq \\ & \leq E \left[32\beta^2(4\beta^2 + R) [(\tau_2 \wedge \tau_{r,R}^N) - (\tau_1 \wedge \tau_{r,R}^N)] \right] \leq 32\beta^2(4\beta^2 + R)\delta =: C_3(R)\delta. \end{aligned} \quad (4.2.15)$$

Now, given $\varepsilon, \alpha > 0$, take $\delta > 0$ such that $\alpha - C_2(R)\delta > 0$ and $C_3(R)\delta/(\alpha - C_2(R)\delta)^2 < \varepsilon$: by (4.2.14), Chebyscev Inequality and (4.2.15) it holds

$$\begin{aligned} & \sup_N \sup_{0 \leq \tau_1 \leq \tau_2 \leq (\tau_1 + \delta) \wedge T} P \left(|\kappa_N(\tau_2 \wedge \tau_{r,R}^N) - \kappa_N(\tau_1 \wedge \tau_{r,R}^N)| \geq \alpha \right) \leq \\ & \sup_N \sup_{0 \leq \tau_1 \leq \tau_2 \leq (\tau_1 + \delta) \wedge T} P \left(\int_{\tau_1 \wedge \tau_{r,R}^N}^{\tau_2 \wedge \tau_{r,R}^N} |H_N^{r,R} \kappa_N(s)| ds + \left| \mathcal{M}_{N,\kappa}^{(\tau_1, \tau_2) \wedge \tau_{r,R}^N} \right| \geq \alpha \right) \leq \\ & \sup_N \sup_{0 \leq \tau_1 \leq \tau_2 \leq (\tau_1 + \delta) \wedge T} P \left(\left| \mathcal{M}_{N,\kappa}^{(\tau_1, \tau_2) \wedge \tau_{r,R}^N} \right| \geq \alpha - C_2(R)\delta \right) \leq \\ & \sup_N \sup_{0 \leq \tau_1 \leq \tau_2 \leq (\tau_1 + \delta) \wedge T} \frac{E \left[\left| \mathcal{M}_{N,\kappa}^{(\tau_1, \tau_2) \wedge \tau_{r,R}^N} \right|^2 \right]}{(\alpha - C_2(R)\delta)^2} \leq \frac{C_3(R)\delta}{(\alpha - C_2(R)\delta)^2} < \varepsilon, \end{aligned}$$

which proves (4.2.13). \square

4.2.3 Averaging principle

In the this paragraph we prove an elementary averaging principle that is built *ad hoc* for our purpose.

Proposition 4.2.4. *Consider $\phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ a locally Lipschitz continuous function, 2π -periodic in the second variable. Let $\{(x_N(t), y_N(t))_{t \in [0, T]}\}_{N \geq 1}$ be a family of cadlag Markov processes such that:*

1. *as $N \rightarrow \infty$, $(x_N(t))_{t \in [0, T]}$ converges, in sense of weak convergence of stochastic processes, to a process $(\bar{x}(t))_{t \in [0, T]}$. Assume also that there exists a compact set $K \subset \mathbb{R}$ such that, for any $t \in [0, T]$ and $N \geq 1$, $x_N(t) \in K$ and $\bar{x}(t) \in K$ and that condition (4.2.13) holds true for the sequence $(x_N(t))_{N \geq 1}$;*
2. *for any $\gamma > 0$ there exist $h' > 0$ and $\bar{N} \geq 1$, such that*

$$\sup_{0 \leq h \leq h'} E [|y_N(t+h) - y_N(t)|] \leq \gamma \quad (4.2.16)$$

for any $N \geq \bar{N}$ and $t \in [0, T]$.

Then, for any $c > 0$ and $\xi > 0$, the following averaging principle holds:

$$\int_0^T \phi(x_N(s), cN^\xi s + y_N(s)) ds \xrightarrow{\text{weakly}} \int_0^T \bar{\phi}(\bar{x}(s)) ds, \quad \text{as } N \rightarrow \infty \quad (4.2.17)$$

where $\bar{\phi}$ is the averaged function defined by

$$\bar{\phi}(x) = \frac{1}{2\pi} \int_0^{2\pi} \phi(x, \theta) d\theta.$$

Before proving Proposition 4.2.4, recall the Skorohod's Representation Theorem (see [6]).

Theorem 4.2.1. *Suppose that the sequence of probability measures $(P_N)_{N \geq 1}$ weakly converges to P and P has a separable support. Then there exist random elements $(X_N)_{N \geq 1}$ and X , defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$, such that $\mathcal{L}(X_N) = P_N$ for any $N \geq 1$, $\mathcal{L}(X) = P$ and $X_N(\omega) \rightarrow X(\omega)$ for every ω .*

Proof of Proposition 4.2.4. Let $P_N = \mathcal{L}(x_N)$, $\tilde{P}_N = \mathcal{L}((x_N, y_N))$ for any $N \geq 1$ and $P = \mathcal{L}(\bar{x})$. Then, P_N and P are probability measures over the set of cadlag trajectories $\mathcal{D}([0, T], \mathbb{R})$, which, endowed with the Skorohod topology, is a complete and separable metric space (see [29]). Therefore, by the Skorohod's Representation Theorem, there exists a probability space on which are defined $\mathcal{D}([0, T], \mathbb{R})$ -valued random variables x_N with distribution P_N , for $N \geq 1$, and \bar{x} with distribution P such that x_N converges to \bar{x} a.s.. Notice that on this common probability space (or at least enlarging it) we can also define, for any $N \geq 1$, a random variable y_N such that the joint distribution of (x_N, y_N) is \tilde{P}_N . In the following, we will prove that, on this common probability space,

$$\int_0^T \phi(x_N(s), cN^\xi s + y_N(s)) ds \xrightarrow{L^1} \int_0^T \bar{\phi}(\bar{x}(s)) ds, \quad \text{as } N \rightarrow \infty$$

which implies (4.2.17).

First of all, we have that

$$\begin{aligned} & E \left| \int_0^T \phi(x_N(s), cN^\xi s + y_N(s)) ds - \int_0^T \bar{\phi}(\bar{x}(s)) ds \right| \leq \\ & \leq E \left| \phi(x_N(s), cN^\xi s + y_N(s)) ds - \int_0^T \bar{\phi}(x_N(s)) ds \right| + \end{aligned} \quad (4.2.18)$$

$$+ E \left| \int_0^T \bar{\phi}(x_N(s)) ds - \int_0^T \bar{\phi}(\bar{x}(s)) ds \right|. \quad (4.2.19)$$

Notice that, since $x_N \rightarrow \bar{x}$ a.s., than also

$$\int_0^T \bar{\phi}(x_N(s)) ds \rightarrow \int_0^T \bar{\phi}(\bar{x}(s)) ds \quad a.s..$$

Since there exists a compact set such that, for any $s \in [0, T]$, $x_N(s) \in K$ and $\bar{\phi}$ is continuous, the quantities above can be uniformly dominated by a constant hence, by the Dominated Convergence Theorem, we also have convergence in L^1 sense so the quantity in (4.2.19) converges to 0.

We have to study (4.2.18): for any $n \geq 1$ consider the partition \mathcal{P}_n of $[0, T]$ defined as

$$0 = t_0 < t_1 < \dots < t_n \leq t_{n+1} = T$$

where

$$n = \left\lfloor \frac{cN^\xi}{2\pi} T \right\rfloor, \quad |t_i - t_{i-1}| = \frac{2\pi}{cN^\xi} \quad \forall i = 1, \dots, n, \quad |T - t_n| < \frac{2\pi}{cN^\xi}.$$

Then,

$$\begin{aligned} & E \left| \int_0^T \phi(x_N(s), cN^\xi s + y_N(s)) ds - \int_0^T \bar{\phi}(x_N(s)) ds \right| \leq \\ & \leq E \left| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \phi(x_N(s), cN^\xi s + y_N(s)) ds - \int_0^T \bar{\phi}(x_N(s)) ds \right| + \end{aligned} \quad (4.2.20)$$

$$+ E \int_{t_n}^T |\phi(x_N(s), cN^\xi s + y_N(s))| ds \quad (4.2.21)$$

The term in (4.2.21) converge to 0: since $x_N(s) \in K$ with K for all $s \in [0, T]$, and ϕ is continuous and periodic in its second variable, there exists $C_1 > 0$ independent of N such that

$$\max_{s \in [0, T]} |\phi(x_N(s), cN^\xi s + y_N(s))| \leq C_1.$$

Hence,

$$E \left[\int_{t_n}^T |\phi(x_N(s), cN^\xi s + y_N(s))| ds \right] \leq C_1(T - t_n) \rightarrow 0$$

as $n \rightarrow \infty$, so we can deal with the term in (4.2.20) only.

$$E \left| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \phi(x_N(s), cN^\xi s + y_N(s)) ds - \int_0^T \bar{\phi}(x_N(s)) ds \right| \leq$$

$$E \left| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} [\phi(x_N(s), cN^\xi s + y_N(s)) - \phi(x_N(t_i), cN^\xi s + y_N(t_i))] ds \right| + \quad (4.2.22)$$

$$E \left| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \phi(x_N(t_i), cN^\xi s + y_N(t_i)) ds - \int_0^T \bar{\phi}(x_N(s)) ds \right|. \quad (4.2.23)$$

Notice that, for any $i = 0, \dots, n-1$,

$$\int_{t_i}^{t_{i+1}} \phi(x_N(t_i), cN^\xi s + y_N(t_i)) ds = \int_{t_i}^{t_i + \frac{2\pi}{cN^\xi}} \phi(x_N(t_i), cN^\xi s + y_N(t_i)) ds \stackrel{\dagger}{=} \\ \stackrel{\dagger}{=} \frac{1}{cN^\xi} \int_0^{2\pi} \phi(x_N(t_i), \sigma + cN^\xi t_i + y_N(t_i)) d\sigma = \frac{2\pi}{cN^\xi} \bar{\phi}(x_N(t_i)) = \bar{\phi}(x_N(t_i)) (t_{i+1} - t_i)$$

where in \dagger we used the change of variable $\sigma = cN^\xi(s - t_i)$. The quantity in (4.2.23) can therefore be written as

$$E \left| \sum_{i=0}^{n-1} \bar{\phi}(x_N(t_i)) (t_{i+1} - t_i) - \int_0^T \bar{\phi}(x_N(s)) ds \right|$$

which clearly converges to 0 as $n \rightarrow \infty$.

So we are only left to prove convergence of the term in (4.2.22):

$$E \left| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} [\phi(x_N(s), cN^\xi s + y_N(s)) - \phi(x_N(t_i), cN^\xi s + y_N(t_i))] ds \right| \leq$$

$$= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} E |\phi(x_N(s), cN^\xi s + y_N(s)) - \phi(x_N(t_i), cN^\xi s + y_N(s))| ds + \quad (4.2.24)$$

$$+ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} E |\phi(x_N(t_i), cN^\xi s + y_N(s)) - \phi(x_N(t_i), cN^\xi s + y_N(t_i))| ds. \quad (4.2.25)$$

Notice that the function ϕ is locally Lipschitz continuous and periodic in the second variable, then it is also locally Lipschitz continuous in the first variable uniformly in the second variable. Since $x_N(s) \in K$ for all $s \in [0, T]$, there exists $L_K > 0$ such that

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} E |\phi(x_N(s), cN^\xi s + y_N(s)) - \phi(x_N(t_i), cN^\xi s + y_N(s))| ds \leq$$

$$\leq L_K \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} E |x_N(s) - x_N(t_i)| ds.$$

Since (4.2.13) holds for the sequence $(x_N)_{N \geq 1}$, for any $\varepsilon, \alpha > 0$ there exists $\delta > 0$ such that

$$\sup_N \sup_{0 \leq \tau_1 \leq \tau_2 \leq (\tau_1 + \delta) \wedge T} P(|x_N(\tau_2) - x_N(\tau_1)| \geq \alpha) \leq \varepsilon.$$

Then, for any N such that $\frac{2\pi}{cN^\xi} < \delta$, for any $i = 0, \dots, n-1$ and $s \in [t_i, t_{i+1}]$ we have

$$\begin{aligned} & E |x_N(s) - x_N(t_i)| \leq \\ & \leq E [|x_N(s) - x_N(t_i)| \mathbb{1}_{\{|x_N(s) - x_N(t_i)| \geq \alpha\}}] + E [|x_N(s) - x_N(t_i)| \mathbb{1}_{\{|x_N(s) - x_N(t_i)| < \alpha\}}] \leq \\ & \leq \max_{x, y \in K} |x - y| P(|x_N(s) - x_N(t_i)| \geq \alpha) + \alpha \leq \bar{C}\varepsilon + \alpha \end{aligned}$$

where $\bar{C} := \max_{x, y \in K} |x - y|$. So, for any $\varepsilon, \alpha > 0$ and N large enough,

$$\begin{aligned} & \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} E |\phi(x_N(s), cN^\xi s + y_N(s)) - \phi(x_N(t_i), cN^\xi s + y_N(s))| ds \leq \\ & \leq L_K \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \bar{C}\varepsilon + \alpha ds \leq L_K(\bar{C}\varepsilon + \alpha) \left[\frac{cN^\xi}{2\pi} T \right] \frac{2\pi}{cN^\xi} \leq L_K(\bar{C}\varepsilon + \alpha)T. \end{aligned}$$

Therefore, the quantity in (4.2.24) converges to 0.

Finally, we are left with (4.2.25): for any $\gamma > 0$ and N large enough,

$$\begin{aligned} & \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} E |\phi(x_N(t_i), cN^\xi s + y_N(s)) - \phi(x_N(t_i), cN^\xi s + y_N(t_i))| ds \leq \\ & \leq L_K \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} E |y_N(s) - y_N(t_i)| ds \leq \\ & \leq L_K \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \sup_{0 \leq h \leq \frac{2\pi}{cN^\xi}} E |y_N(t_i + h) - y_N(t_i)| ds \leq \\ & \leq L_K \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \gamma ds \leq \left[\frac{cN^\xi}{2\pi} T \right] \frac{2\pi}{cN^\xi} L_K \gamma \leq L_K T \gamma \end{aligned}$$

where we used Lipschitz continuity and condition (4.2.16). Since γ is arbitrary, the proof is completed. \square

4.2.4 Analysis of the fast component

Consider again the process $(\kappa_N(t), \theta_N(t))_{t \in [0, T]}$ defined in Lemma 4.2.2 and define the process $(\eta_N(t))_{t \in [0, T]}$ by

$$\eta_N(t) := \theta_N(t) - N^{\frac{1}{2}} 2\sqrt{\beta - 1}t. \quad (4.2.26)$$

The following result proves that, given the stopping time

$$\tau_{r,R}^N := \inf\{t \in [0, T] \mid \kappa_N(t) \notin]r, R[\},$$

the stopped process $\eta_N(t \wedge \tau_{r,R}^N)$ satisfies (a stronger version of) condition (4.2.16).

Proposition 4.2.5. *Let $\eta_N(t)$ be the Markov process defined by (4.2.26). Then, for $h' > 0$, there exist $C > 0$ and $\bar{N} \geq 1$ such that*

$$\sup_{0 \leq h \leq h'} E \left[\left| \eta_N((t+h) \wedge \tau_{r,R}^N) - \eta_N(t \wedge \tau_{r,R}^N) \right| \right] \leq C\sqrt{h'} \quad (4.2.27)$$

for all $N \geq \bar{N}$.

Proof. Notice that $(\kappa_N(t), \eta_N(t))$ is a time-dependent invertible transformation of $(z_N(t), u_N(t))$: so, $(\kappa_N(t), \eta_N(t))$ is itself a (time inhomogeneous) Markov process. We want to find an expression for its generator $J_{N,t}$. Actually, in order to overcome time-dependence, we let time play the role of additional variable. Let $(y_N(t))_{t \in [0, T]}$ be the process

$$dy_N(t) = N^{\frac{1}{2}} 2\sqrt{\beta - 1}dt$$

and consider the transformation

$$\begin{aligned} \kappa_N(t) &= z_N^2(t) + u_N^2(t), \\ \eta_N(t) &= \arctan\left(\frac{u_N(t)}{z_N(t)}\right) - y_N(t), \\ \xi_N(t) &= y_N(t). \end{aligned}$$

Recall generator G_N introduced in Lemma 4.2.1: then, infinitesimal generator \bar{G}_N associated with $(z_N(t), u_N(t), y_N(t))$ is given by

$$\bar{G}_N f(z, u, y) = G_N f(z, u) + N^{\frac{1}{2}} 2\sqrt{\beta - 1}f_y.$$

Let g be the function

$$g(z, u, y) = \left(z^2 + u^2, \arctan\left(\frac{u}{z}\right) - y, y \right)$$

and compute $\bar{G}_N(f \circ g)(z, u, y)$, with $f \in \mathcal{C}_b^3([r, R] \times \mathbb{R}^2)$. Very similar computations to the one performed in the proof of Lemma 4.2.2 yield us

$$\begin{aligned} \bar{G}_N(f \circ g)(z, u, y) &= 2\beta^2(f \circ g)_{zz} - \left(\frac{2\beta z^3}{3} + N^{\frac{1}{2}} 2\beta\sqrt{\beta - 1}u \right) (f \circ g)_z + \\ &\quad + N^{\frac{1}{2}} 2\sqrt{\beta - 1}z(f \circ g)_u + N^{\frac{1}{2}} 2\sqrt{\beta - 1}f_y + o(1). \end{aligned}$$

Observe that:

$$(f \circ g)_z = 2zf_\kappa - \frac{u}{z^2 + u^2}f_\eta, \quad (f \circ g)_u = 2uf_\kappa + \frac{z}{z^2 + u^2}f_\eta, \quad (f \circ g)_y = -f_\eta + f_\xi,$$

$$(f \circ g)_{zz} = 4z^2 f_{\kappa\kappa} - \frac{4zu}{z^2 + u^2}f_{\kappa\eta} + \frac{u^2}{(z^2 + u^2)^2}f_{\eta\eta} + 2f_\kappa + \frac{2uz}{(z^2 + u^2)^2}f_\eta,$$

hence

$$\begin{aligned} \bar{G}_N(f \circ g)(z, u, y) &= 8\beta^2 z^2 f_{\kappa\kappa} - \frac{8\beta^2 zu}{z^2 + u^2}f_{\kappa\eta} + \frac{2\beta^2 u^2}{(z^2 + u^2)^2}f_{\eta\eta} + \\ &+ \left(\frac{4\beta^2 zu}{(z^2 + u^2)^2} + \frac{2\beta z^3 u}{3(z^2 + u^2)} \right) f_\eta + \\ &+ \left(4\beta^2 - \frac{4\beta z^4}{3} \right) f_\kappa + N^{\frac{1}{2}} 2\sqrt{\beta - 1}f_\xi + o(1), \end{aligned}$$

which finally yields that the infinitesimal generator J_N of $(\kappa_N(t), \eta_N(t), \xi_N(t))$ in the form (see Lemma 1.2.1):

$$\begin{aligned} J_N f(\kappa, \eta, \xi) &= 8\beta^2 \kappa \cos^2(\eta + \xi) f_{\kappa\kappa} - 8\beta^2 \cos(\eta + \xi) \sin(\eta + \xi) f_{\kappa\eta} + \frac{2\beta^2 \sin^2(\eta + \xi)}{\kappa} f_{\eta\eta} + \\ &+ \left(\frac{4\beta^2 \cos(\eta + \xi) \sin(\eta + \xi)}{\kappa} + \frac{2\beta}{3} \kappa \cos^3(\eta + \xi) \sin(\eta + \xi) + 2\sqrt{\beta - 1} \sin(\eta + \xi) \right) f_\eta + \\ &+ \left(4\beta^2 - \frac{4\beta}{3} \kappa^2 \cos^4(\eta + \xi) \right) f_\kappa + N^{\frac{1}{2}} 2\sqrt{\beta - 1}f_\xi + o(1). \end{aligned}$$

So, the infinitesimal generator $J_N^{r,R}$ of the stopped process $(\kappa_N(t \wedge \tau_{r,R}^N), \eta_N(t \wedge \tau_{r,R}^N), \xi_N(t \wedge \tau_{r,R}^N))$ will satisfy for $f \in \mathcal{C}_b^3([r, R] \times \mathbb{R}^2)$:

$$\begin{aligned} J_N^{r,R} f(\kappa, \eta, \xi) &= \\ &= \mathbb{1}_{]r, R[}(\kappa) \left[8\beta^2 \kappa \cos^2(\eta + \xi) f_{\kappa\kappa} - 8\beta^2 \cos(\eta + \xi) \sin(\eta + \xi) f_{\kappa\eta} + \frac{2\beta^2 \sin^2(\eta + \xi)}{\kappa} f_{\eta\eta} + \right. \\ &+ \left(\frac{4\beta^2 \cos(\eta + \xi) \sin(\eta + \xi)}{\kappa} + \frac{2\beta}{3} \kappa \cos^3(\eta + \xi) \sin(\eta + \xi) + 2\sqrt{\beta - 1} \sin(\eta + \xi) \right) f_\eta + \\ &\left. + \left(4\beta^2 - \frac{4\beta}{3} \kappa^2 \cos^4(\eta + \xi) \right) f_\kappa + N^{\frac{1}{2}} 2\sqrt{\beta - 1}f_\xi + o_{r,R}(1) \right] \end{aligned}$$

where, as in Lemma 4.2.2, one can easily check that the remainders $o_{r,R}(1)$ can be uniformly dominated by a term of order $o(1)$ on $[r, R] \times \mathbb{R}^2$.

We will use the decomposition

$$\eta_N(t \wedge \tau_{r,R}^N) = \eta_N(0) + \int_0^{t \wedge \tau_{r,R}^N} J_N^{r,R} \eta_N(s) ds + \mathcal{M}_{N,\eta}^{t \wedge \tau_{r,R}^N} \quad (4.2.28)$$

where $J_N^{r,R} \eta$ indicates the infinitesimal generator $J_N^{r,R}$ evaluated on the function $f(\kappa, \eta, \xi) = \eta$ and

$$\mathcal{M}_{N,\eta}^{t \wedge \tau_{r,R}^N} = \int_0^{t \wedge \tau_{r,R}^N} \sum_{j \in \mathcal{S}} \Delta \eta_N(s) \tilde{\Lambda}_N(j, ds)$$

where $\tilde{\Lambda}(j, ds)$ is the same compensated point process introduced in Proposition 4.2.3 and $\Delta \eta_N(s)$ will be estimated in the following.

Let $h > 0$ and study the quantity

$$|\eta_N((t+h) \wedge \tau_{r,R}^N) - \eta_N(t \wedge \tau_{r,R}^N)| \leq \int_{t \wedge \tau_{r,R}^N}^{(t+h) \wedge \tau_{r,R}^N} |J_N^{r,R} \eta_N(s)| ds + \left| \mathcal{M}_{N,\eta}^{(t+h) \wedge \tau_{r,R}^N} - \mathcal{M}_{N,\eta}^{t \wedge \tau_{r,R}^N} \right|.$$

We have

$$J_N^{r,R} \eta = \frac{4\beta^2 \cos(\eta + \xi) \sin(\eta + \xi)}{\kappa} + \frac{2\beta}{3} \cos^3(\eta + \xi) \sin(\eta + \xi) + 2\sqrt{\beta - 1} \sin(\eta + \xi) + o_{r,R}(1),$$

therefore

$$\int_{t \wedge \tau_{r,R}^N}^{(t+h) \wedge \tau_{r,R}^N} |J_N^{r,R} \eta_N(s)| ds \leq \int_{t \wedge \tau_{r,R}^N}^{(t+h) \wedge \tau_{r,R}^N} \frac{4\beta^2}{r} + 2\beta R + 2\sqrt{\beta - 1} + o_{r,R}(1) ds \leq C(r, R)h. \quad (4.2.29)$$

On the other hand, using the same arguments of the proof of Proposition 4.2.3,

$$\begin{aligned} E \left[\left(\mathcal{M}_{N,\eta}^{(t+h) \wedge \tau_{r,R}^N} - \mathcal{M}_{N,\eta}^{t \wedge \tau_{r,R}^N} \right)^2 \right] &= E \left[\left(\mathcal{M}_{N,\eta}^{(t+h) \wedge \tau_{r,R}^N} \right)^2 - \left(\mathcal{M}_{N,\eta}^{t \wedge \tau_{r,R}^N} \right)^2 \right] = \\ &= E \left[\int_{t \wedge \tau_{r,R}^N}^{(t+h) \wedge \tau_{r,R}^N} \sum_{j \in \mathcal{S}} (\Delta \eta_N(s))^2 N^{\frac{1}{2}} |A_N(j, N^{\frac{1}{2}}s)| \left(1 - j \tanh(\lambda_N(N^{\frac{1}{2}}s)) \right) ds \right]. \end{aligned}$$

We want to estimate the term $(\Delta \eta_N(t))^2$ for $t \in [0, T \wedge \tau_{r,R}^N]$. Recall (4.2.26), so that

$$\eta_N(t) - \eta_N(t-) = \theta_N(t) - \theta_N(t-) = \arctan \left(\frac{u_N(t)}{z_N(t)} \right) - \arctan \left(\frac{u_N(t-)}{z_N(t-)} \right).$$

As previously remarked, $u_N(t) = u_N(t-)$ for all $t \in [0, T]$ while, if $\tau \in [0, T \wedge \tau_{r,R}^N]$ is a jump time for z_N ,

$$|z_N(\tau) - z_N(\tau-)| = \frac{2\beta}{N^{\frac{3}{4}}}.$$

Since $|\frac{d}{dx} \arctan(x)| \leq 1$ for all $x \in \mathbb{R}$,

$$|\eta_N(\tau) - \eta_N(\tau-)| \leq \left| \frac{u_N(\tau)}{z_N(\tau)} - \frac{u_N(\tau-)}{z_N(\tau-)} \right| = \left| u_N(\tau) \frac{z_N(\tau) - z_N(\tau-)}{z_N(\tau)z_N(\tau-)} \right|. \quad (4.2.30)$$

Notice also that using the well-known property of arctan

$$\left| \arctan(x) + \arctan\left(\frac{1}{x}\right) \right| = \frac{\pi}{2} \quad \forall x \neq 0,$$

we have that

$$\left| \arctan\left(\frac{u_N(\tau)}{z_N(\tau)}\right) - \arctan\left(\frac{u_N(\tau-)}{z_N(\tau-)}\right) \right| = \left| \arctan\left(\frac{z_N(\tau)}{u_N(\tau)}\right) - \arctan\left(\frac{z_N(\tau-)}{u_N(\tau-)}\right) \right|$$

and we get a second estimate

$$|\eta_N(\tau) - \eta_N(\tau-)| \leq \left| \frac{z_N(\tau)}{u_N(\tau)} - \frac{z_N(\tau-)}{u_N(\tau-)} \right| = \left| \frac{z_N(\tau) - z_N(\tau-)}{u_N(\tau)} \right|. \quad (4.2.31)$$

Let now fix $\varepsilon > 0$ and $\bar{N} \geq 1$ such that $2\beta\bar{N}^{-\frac{3}{4}} < \varepsilon$ and $4\varepsilon^2 < r$; so, for $N > \bar{N}$, if $|z_N(\tau)z_N(\tau-)| \geq \varepsilon^2$ it holds, by (4.2.30),

$$|\eta_N(\tau) - \eta_N(\tau-)| \leq \frac{\sqrt{R} 2\beta^2}{\varepsilon^2 N^{\frac{3}{4}}}.$$

Otherwise, consider the case $|z_N(\tau)z_N(\tau-)| < \varepsilon^2$: this implies $|z_N(\tau)| < 2\varepsilon$ hence

$$|u_N(\tau)| \geq \sqrt{r - 4\varepsilon^2}.$$

Therefore, by (4.2.31),

$$|\eta_N(\tau) - \eta_N(\tau-)| \leq \frac{1}{\sqrt{r - 4\varepsilon^2}} \frac{2\beta}{N^{\frac{3}{4}}}.$$

In conclusion, having fixed ε and \bar{N} as above, set $\bar{C} = \max\left\{\frac{1}{\sqrt{r-4\varepsilon^2}}, \frac{\sqrt{R}}{\varepsilon^2}\right\}$ we have

$$|\Delta\eta_N(t)| \leq \bar{C} \frac{2\beta}{N^{\frac{3}{4}}} \quad (4.2.32)$$

for any $t \in [0, T \wedge \tau_{r,R}^N]$ and $N \geq \bar{N}$. By means of (4.2.32)

$$\begin{aligned} & E \left[\left(\mathcal{M}_{N,\eta}^{(t+h) \wedge \tau_{r,R}^N} - \mathcal{M}_{N,\eta}^{t \wedge \tau_{r,R}^N} \right)^2 \right] = \\ &= E \left[\int_{t \wedge \tau_{r,R}^N}^{(t+h) \wedge \tau_{r,R}^N} \sum_{j \in \mathcal{S}} (\Delta\eta_N(s))^2 N^{\frac{1}{2}} |A_N(j, N^{\frac{1}{2}}s)| \left(1 - j \tanh(\lambda_N(N^{\frac{1}{2}}s))\right) ds \right] \leq \\ &\leq E \left[\int_{t \wedge \tau_{r,R}^N}^{(t+h) \wedge \tau_{r,R}^N} \sum_{j \in \mathcal{S}} \bar{C}^2 \frac{4\beta^2}{N^{\frac{3}{2}}} N^{\frac{1}{2}} \sup_{j \in \mathcal{S}} |A_N(j, N^{\frac{1}{2}}s)| 2 ds \right] \leq 16\beta^2 \bar{C}^2 h, \end{aligned}$$

which, by Hölder inequality, implies

$$E \left[\left| \mathcal{M}_{N,\eta}^{(t+h) \wedge \tau_{r,R}^N} - \mathcal{M}_{N,\eta}^{t \wedge \tau_{r,R}^N} \right| \right] \leq \sqrt{E \left[\left(\mathcal{M}_{N,\eta}^{(t+h) \wedge \tau_{r,R}^N} - \mathcal{M}_{N,\eta}^{t \wedge \tau_{r,R}^N} \right)^2 \right]} \leq 4\beta\bar{C}\sqrt{h}. \quad (4.2.33)$$

In conclusion, using (4.2.29) and (4.2.33), for a given $h' > 0$ there exists $\bar{N} \geq 1$ and $\bar{C} > 0$ such that

$$\begin{aligned} & \sup_{0 \leq h \leq h'} E \left[\left| \eta_N((t+h) \wedge \tau_{r,R}^N) - \eta_N(t \wedge \tau_{r,R}^N) \right| \right] \leq \\ & \leq \sup_{0 \leq h \leq h'} E \int_{t \wedge \tau_{r,R}^N}^{(t+h) \wedge \tau_{r,R}^N} |J_N \eta_N(s)| ds + \sup_{0 \leq h \leq h'} E \left| \mathcal{M}_{N,\eta}^{(t+h) \wedge \tau_{r,R}^N} - \mathcal{M}_{N,\eta}^{t \wedge \tau_{r,R}^N} \right| \leq \\ & \leq \sup_{0 \leq h \leq h'} C(r, R)h + \sup_{0 \leq h \leq h'} 4\beta\bar{C}\sqrt{h} \leq C(r, R)h' + 4\beta\bar{C}\sqrt{h'}, \end{aligned}$$

for any $N \geq \bar{N}$. Being $h' \ll \sqrt{h'}$ the proof is completed. \square

4.2.5 Characterization of the limit of the sequence of stopped processes

By Proposition 4.2.3, for any choice of r, R such that $0 < r < \frac{\beta}{\beta-1}\bar{\lambda} < R$, the sequence $(P_N \circ \varphi_{r,R}^{-1})_{N \geq 1}$ admits a converging subsequence $(P_n \circ \varphi_{r,R}^{-1})_{n \geq 1}$, where $P_N = \mathcal{L}(\kappa_N)$ and $\varphi_{r,R}$ defined by (4.2.6) (see Remark 4.2.3). Let $\bar{P}_{r,R}$ be the limit of $(P_n \circ \varphi_{r,R}^{-1})_{n \geq 1}$, which identifies a *cadlag* stochastic process $(\bar{\kappa}_{r,R}(t))_{t \in [0, T]}$. Defining the stopping time

$$\tau_{r,R}^n := \inf\{t \in [0, T] \mid \kappa_n(t) \notin]r, R[\},$$

we have that the sequence of processes $(\kappa_n(t \wedge \tau_{r,R}^n))_{n \geq 1}$ weakly converges to the stochastic process $\bar{\kappa}_{r,R}(t)$. Let now L denote the infinitesimal generator of the solution of the stochastic differential equation

$$d\kappa(t) = \left(4\beta^2 - \frac{\beta}{2}\kappa^2(t) \right) dt + 2\beta\sqrt{2\kappa(t)}dB(t),$$

namely, for any $f \in \text{dom}(L)$,

$$Lf(\kappa) = 4\beta^2\kappa f''(\kappa) + \left(4\beta^2 - \frac{\beta}{2}\kappa^2 \right) f'(\kappa). \quad (4.2.34)$$

The process $\bar{\kappa}_{r,R}(t)$ satisfies the following property.

Proposition 4.2.6. *For any $f \in \mathcal{C}_c^\infty([r, R])$ the stochastic process*

$$\bar{\mathcal{N}}_{r,R}^f(t) := f(\bar{\kappa}_{r,R}(t)) - f(\bar{\kappa}_{r,R}(0)) - \int_0^t Lf(\bar{\kappa}_{r,R}(s))ds$$

is a martingale.

Proof. Let $f \in \mathcal{C}_c^\infty([r, R])$ and fix the following notations:

$$A_f(\kappa, \theta) := 8\beta^2 \kappa \cos^2 \theta f''(\kappa) + \left(4\beta^2 - \frac{4\beta}{3} \kappa^2 \cos^4 \theta\right) f'(\kappa),$$

$$\bar{A}_f(\kappa) := \frac{1}{2\pi} \int_0^{2\pi} A_f(\kappa, \theta) d\theta.$$

Notice that it holds that

$$H_n^{r,R} f(\kappa, \theta) = A_f(\kappa, \theta) + o_{r,R}(1)$$

where $H_n^{r,R}$ is the infinitesimal generator of the process $\kappa_n(t \wedge \tau_{r,R}^n)$ and $\bar{A}_f(\kappa) = Lf(\kappa)$. Define the following processes, for $t \in [0, T]$:

$$\mathcal{M}_n^f(t) = f(\kappa_n(t \wedge \tau_{r,R}^n)) - f(\kappa_n(0)) - \int_0^{t \wedge \tau_{r,R}^n} H_n^{r,R} f(\kappa_n(s)) ds,$$

$$\mathcal{N}_n^f(t) = f(\kappa_n(t \wedge \tau_{r,R}^n)) - f(\kappa_n(0)) - \int_0^{t \wedge \tau_{r,R}^n} A_f(\kappa_n(s), \theta_n(s)) ds;$$

moreover it holds that

$$\bar{\mathcal{N}}_{r,R}^f(t) = f(\bar{\kappa}_{r,R}(t)) - f(\bar{\kappa}_{r,R}(0)) - \int_0^t \bar{A}_f(\bar{\kappa}_{r,R}(s)) ds.$$

We want to show that $\bar{\mathcal{N}}_{r,R}^f(t)$ is a martingale. Fix $m \geq 1$, g_1, \dots, g_m continuous and bounded functions and $0 \leq t_1 \leq \dots \leq t_m \leq s \leq t \leq T$. Since \mathcal{M}_n^f is a martingale,

$$E \left[\left(\mathcal{M}_n^f(t) - \mathcal{M}_n^f(s) \right) g_1(\kappa_n(t_1 \wedge \tau_{r,R}^n)) \cdots g_m(\kappa_n(t_m \wedge \tau_{r,R}^n)) \right] = 0,$$

which implies

$$E \left[\left(\mathcal{N}_n^f(t) - \mathcal{N}_n^f(s) \right) g_1(\kappa_n(t_1 \wedge \tau_{r,R}^n)) \cdots g_m(\kappa_n(t_m \wedge \tau_{r,R}^n)) \right] = o_{r,R}(1).$$

The last equivalence can be written as

$$\begin{aligned} & E \left[\left(f(\kappa_n(t \wedge \tau_{r,R}^n)) - f(\kappa_n(s \wedge \tau_{r,R}^n)) \right) g_1(\kappa_n(t_1 \wedge \tau_{r,R}^n)) \cdots g_m(\kappa_n(t_m \wedge \tau_{r,R}^n)) \right] + \\ & E \left[- \left(\int_{s \wedge \tau_{r,R}^n}^{t \wedge \tau_{r,R}^n} A_f(\kappa_n(\sigma), \theta_n(\sigma)) d\sigma \right) g_1(\kappa_n(t_1 \wedge \tau_{r,R}^n)) \cdots g_m(\kappa_n(t_m \wedge \tau_{r,R}^n)) \right] = \\ & = o_{r,R}(1). \end{aligned} \tag{4.2.35}$$

Notice that, since $(\kappa_n(t \wedge \tau_{r,R}^n))_{n \geq 1}$ weakly converges to $\bar{\kappa}_{r,R}(t)$, the term in (4.2.35) converges to

$$E \left[\left(f(\bar{\kappa}_{r,R}(t)) - f(\bar{\kappa}_{r,R}(s)) \right) g_1(\bar{\kappa}_{r,R}(t_1)) \cdots g_m(\bar{\kappa}_{r,R}(t_m)) \right]$$

and $o_{r,R}(1)$ converges to 0, in order to show that $\bar{\mathcal{N}}_{r,R}^f(t)$ is a martingale we only have to prove that the term

$$E \left[\left(\int_{s \wedge \tau_{r,R}^n}^{t \wedge \tau_{r,R}^n} A_f(\kappa_n(\sigma), \theta_n(\sigma)) d\sigma \right) g_1(\kappa_n(t_1 \wedge \tau_{r,R}^n)) \cdots g_m(\kappa_n(t_m \wedge \tau_{r,R}^n)) \right]$$

converges to

$$E \left[\int_s^t \bar{A}_f(\bar{\kappa}_{r,R}(\sigma)) d\sigma g_1(\bar{\kappa}_{r,R}(t_1)) \cdots g_m(\bar{\kappa}_{r,R}(t_m)) \right].$$

Notice that, since $f \in \mathcal{C}_c^\infty([r, R])$, it holds that

$$\int_{s \wedge \tau_{r,R}^n}^{t \wedge \tau_{r,R}^n} A_f(\kappa_n(\sigma), \theta_n(\sigma)) d\sigma = \int_s^t A_f(\kappa_n(\sigma \wedge \tau_{r,R}^n), \theta_n(\sigma \wedge \tau_{r,R}^n)) d\sigma,$$

so we have all the elements to apply the averaging principle of Proposition 4.2.4: we can take $x_N(t) = \kappa_N(t \wedge \tau_{r,R}^N)$, $\bar{x}(t) = \bar{\kappa}_{r,R}(t)$, $y_N(t) = \eta_N(t \wedge \tau_{r,R}^N)$ with $\eta_N(t)$ defined by (4.2.26), $c = 2\sqrt{\beta - 1}$ and $\xi = \frac{1}{2}$. Thanks to Proposition 4.2.5 we can apply the averaging principle, from which

$$E \left[\int_s^t A_f(\kappa_n(\sigma \wedge \tau_{r,R}^n), \theta_n(\sigma \wedge \tau_{r,R}^n)) d\sigma \right] \rightarrow E \left[\int_s^t \bar{A}_f(\bar{\kappa}_{r,R}(\sigma)) d\sigma \right]$$

as $n \rightarrow \infty$. Finally, since

$$\int_s^t A_f(\kappa_n(\sigma \wedge \tau_{r,R}^n), \theta_n(\sigma \wedge \tau_{r,R}^n)) d\sigma$$

can be bounded by a constant uniformly in n ,

$$E \left[\left(\int_s^t A_f(\kappa_n(\sigma \wedge \tau_{r,R}^n), \theta_n(\sigma \wedge \tau_{r,R}^n)) d\sigma \right) g_1(\kappa_n(t_1 \wedge \tau_{r,R}^n)) \cdots g_m(\kappa_n(t_m \wedge \tau_{r,R}^n)) \right]$$

↓

$$E \left[\left(\int_s^t \bar{A}_f(\bar{\kappa}_{r,R}(\sigma)) d\sigma \right) g_1(\bar{\kappa}_{r,R}(t_1)) \cdots g_m(\bar{\kappa}_{r,R}(t_m)) \right]$$

as $n \rightarrow \infty$ and the proof is concluded. □

Let us recall the definition of *stopped martingale problem* and a very useful theorem related to it.

Definition 4.2.1. Let E be a metric space, U an open subset of E and $(X(t))_{t \in [0, T]}$ be a stochastic process with sample paths in $\mathcal{D}([0, T], E)$. Define the stopping time

$$\tau = \inf\{t \in [0, T] \mid X(t) \notin U \text{ or } X(t-) \notin U\}.$$

Let L be a linear operator $L : \text{dom}(L) \subset B(E) \rightarrow B(E)$ with $B(E)$ the Banach space of bounded measurable function $E \rightarrow \mathbb{R}$. Then, $X(\cdot \wedge \tau)$ is a solution of the stopped martingale problem for (L, U) if

$$f(X(t \wedge \tau)) - f(X(0)) - \int_0^{t \wedge \tau} Lf(X(s))ds$$

is a martingale for all $f \in \text{dom}(L)$.

Theorem 4.2.2 ([29], Thm 6.1 p.216). *Let (E, d) be a complete and separable metric space and let L be a linear operator $L : \mathcal{C}_b(E) \rightarrow B(E)$. If the $\mathcal{D}([0, T], E)$ martingale problem for L is well-posed, then for any open set $U \subset E$ there exists a unique solution of the stopped martingale problem for (L, U) .*

Proposition 4.2.7. *Fix $\varepsilon > 0$: let $\bar{\kappa}_{r,R}(t)$ be the weak limit of the sequence of stopped processes $(\kappa_n(t \wedge \tau_{r,R}^n))_{n \geq 1}$ and define the stopping time*

$$\bar{\tau}_{r,R}^\varepsilon = \inf\{t \in [0, T] \mid \bar{\kappa}_{r,R}(t) \notin]r + \varepsilon, R - \varepsilon[\},$$

with r, R such that $0 < r < \frac{\beta}{\beta-1} \bar{\lambda}^2 < R$. Let $\kappa(t)$ be the unique solution of the stochastic differential equation

$$d\kappa(t) = \left(4\beta^2 - \frac{\beta}{2}\kappa^2(t)\right) dt + 2\beta\sqrt{2\kappa(t)}dB(t)$$

with $\kappa(0) = \frac{\beta}{\beta-1} \bar{\lambda}^2$ and define

$$\tau_{r,R}^\varepsilon = \inf\{t \in [0, T] \mid \kappa(t) \notin]r + \varepsilon, R - \varepsilon[\}.$$

Then, the processes $\bar{\kappa}_{r,R}(t \wedge \bar{\tau}_{r,R}^\varepsilon)$ and $\kappa(t \wedge \tau_{r,R}^\varepsilon)$ have the same distribution.

Proof. First of all, we have that $\kappa_N(0)$ converges in distribution to $\kappa(0)$. Notice that, given $g \in \mathcal{C}_0^\infty([0, +\infty[)$, for any $\varepsilon > 0$ there exists a function $f \in \mathcal{C}_c^\infty([r, R])$ such that $f(x) = g(x)$ for any $x \in]r + \varepsilon, R - \varepsilon[$. Then, by Proposition 4.2.6, for any $g \in \mathcal{C}_0^\infty([0, +\infty[)$, the process

$$g(\bar{\kappa}_{r,R}(t \wedge \bar{\tau}_{r,R}^\varepsilon)) - g(\bar{\kappa}_{r,R}(0)) - \int_0^{t \wedge \bar{\tau}_{r,R}^\varepsilon} Lg(\bar{\kappa}_{r,R}(s))ds$$

is a martingale.

On the other hand, the process $\kappa(t \wedge \tau_{r,R}^\varepsilon)$ obviously solves the stopped martingale problem for $(L,]r + \varepsilon, R - \varepsilon[)$ where L is given by (4.2.34). By Lemma 4.2.3, the martingale problem for L is well-posed hence for Theorem 4.2.2 the stopped martingale problem for $(L,]r + \varepsilon, R - \varepsilon[)$ has a unique solution. But, since $\mathcal{C}_0^\infty([0, +\infty[)$ is measure-determining and for all $g \in \mathcal{C}_0^\infty([0, +\infty[)$ the process

$$g(\bar{\kappa}_{r,R}(t \wedge \bar{\tau}_{r,R}^\varepsilon)) - g(\bar{\kappa}_{r,R}(0)) - \int_0^{t \wedge \bar{\tau}_{r,R}^\varepsilon} Lg(\bar{\kappa}_{r,R}(s))ds$$

is a martingale, the processes $\kappa(t \wedge \tau_{r,R}^\varepsilon)$ and $\bar{\kappa}_{r,R}(t \wedge \bar{\tau}_{r,R}^\varepsilon)$ must have the same distribution. \square

4.2.6 From localization to the proof of Theorem 4.1.2

In this paragraph we exploit the localization argument to conclude the proof of Theorem 4.1.2.

Proposition 4.2.8. *Given Proposition 4.2.7 and Lemma 4.2.3, the sequence of processes $(\kappa_N(t))_{N \geq 1}$ weakly converges to the unique solution of (4.1.7).*

Proof. Let us fix some notation: for any $m \geq 1$ define

$$U_m =]r_m + \varepsilon_m, R_m - \varepsilon_m[$$

such that $0 < r_m < r_m + \varepsilon_m < \frac{\beta}{\beta-1} \bar{\lambda}^2 < R_m - \varepsilon_m < R_m$ for all $m \geq 1$, $U_1 \subset U_2 \subset \dots$ and

$$\lim_{m \rightarrow \infty} r_m + \varepsilon_m = 0, \quad \lim_{m \rightarrow \infty} R_m - \varepsilon_m = +\infty.$$

For any m , let $\bar{\kappa}_m(t) := \bar{\kappa}_{r_m, R_m}(t)$ be the weak limit of $\kappa_N(t \wedge \tau_{r_m, R_m}^N)$ and let $\kappa(t)$ be the solution of the limiting equation (4.1.7). Moreover, we define the stopping times

$$\begin{aligned} \tau_{U_m}^N &= \inf\{t \in [0, T] \mid \kappa_N(t) \notin U_m\}, \\ \bar{\tau}_m &= \inf\{t \in [0, T] \mid \bar{\kappa}_m(t) \notin U_m\}, \\ \tau_m &= \inf\{t \in [0, T] \mid \kappa(t) \notin U_m\}. \end{aligned}$$

By Proposition 4.2.7, the process $\bar{\kappa}_m(t)$ is continuous a.s., hence the weak convergence of $(\kappa_N(t \wedge \tau_{r_m, R_m}^N))_{N \geq 1}$ to $\bar{\kappa}_m(t)$ also holds endowing the space $\mathcal{D}([0, T], \mathbb{R})$ with the uniform topology (see for example Lemma 1.6.4 in [52]). For any $m \geq 1$ consider the set

$$A_m = \{x \in \mathcal{D}([0, T], \mathbb{R}) \mid x(t) \in U_m \quad \forall t \in [0, T]\};$$

since A_m^c is a closed set in the uniform topology of $\mathcal{D}([0, T], \mathbb{R})$ for any m , it holds

$$\limsup_N P(\kappa_N(\cdot \wedge \tau_{r_m, R_m}^N) \in A_m^c) \leq P(\bar{\kappa}_m(\cdot) \in A_m^c), \quad \forall m \geq 1 \quad (4.2.36)$$

by the Portmanteau Theorem. By Proposition 4.2.7 $\bar{\kappa}_m(t \wedge \bar{\tau}_m)$ and $\kappa(t \wedge \tau_m)$ have the same distribution, so observe that

$$P(\bar{\kappa}_m(\cdot) \in A_m^c) = P(\bar{\tau}_m \leq T) = P(\tau_m \leq T).$$

Moreover, since $U_m \subset]r_m, R_m[$,

$$P(\kappa_N(\cdot \wedge \tau_{r_m, R_m}^N) \in A_m^c) = P(\tau_{U_m}^N \leq T) \geq P(\tau_{r_m, R_m}^N \leq T),$$

hence, by (4.2.36),

$$\limsup_N P(\tau_{r_m, R_m}^N \leq T) \leq P(\tau_m \leq T), \quad \forall m \geq 1. \quad (4.2.37)$$

Let $f : \mathcal{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$ be a continuous and bounded function: we want to show that

$$|E[f(\kappa_N(\cdot))] - E[f(\kappa(\cdot))]| \rightarrow 0$$

as $N \rightarrow \infty$. We can see that, for any $m, N \geq 1$,

$$\begin{aligned} |E[f(\kappa_N(\cdot))] - E[f(\kappa(\cdot))]| &\leq |E[f(\kappa_N(\cdot))] - E[f(\kappa_N(\cdot \wedge \tau_{r_m, R_m}^N))]| + \\ &\quad + |E[f(\kappa_N(\cdot \wedge \tau_{r_m, R_m}^N))] - E[f(\bar{\kappa}_m(\cdot))]| + \\ &\quad + |E[f(\bar{\kappa}_m(\cdot))] - E[f(\bar{\kappa}_m(\cdot \wedge \bar{\tau}_m))]| + \\ &\quad + |E[f(\bar{\kappa}_m(\cdot \wedge \bar{\tau}_m))] - E[f(\kappa(\cdot \wedge \tau_m))]| + \\ &\quad + |E[f(\kappa(\cdot \wedge \tau_m)) - E[f(\kappa(\cdot))]|. \end{aligned}$$

By Proposition 4.2.7, the processes $\bar{\kappa}_m(t \wedge \bar{\tau}_m)$ and $\kappa(t \wedge \tau_m)$ have the same distribution, so

$$|E[f(\bar{\kappa}_m(\cdot \wedge \bar{\tau}_m))] - E[f(\kappa(\cdot \wedge \tau_m))]| = 0$$

and

$$|E[f(\bar{\kappa}_m(\cdot))] - E[f(\bar{\kappa}_m(\cdot \wedge \bar{\tau}_m))]| + |E[f(\kappa(\cdot \wedge \tau_m)) - E[f(\kappa(\cdot))]| \leq 2\|f\|_\infty P(\tau_m \leq T).$$

Notice also that

$$|E[f(\kappa_N(\cdot))] - E[f(\kappa_N(\cdot \wedge \tau_{r_m, R_m}^N))]| \leq \|f\|_\infty P(\tau_{r_m, R_m}^N \leq T),$$

so we get, for any $m, N \geq 1$,

$$\begin{aligned} |E[f(\kappa_N(\cdot))] - E[f(\kappa(\cdot))]| &\leq \|f\|_\infty P(\tau_{r_m, R_m}^N \leq T) + \\ &\quad + |E[f(\kappa_N(\cdot \wedge \tau_{r_m, R_m}^N))] - E[f(\bar{\kappa}_m(\cdot))]| + \\ &\quad + 2\|f\|_\infty P(\tau_m \leq T). \end{aligned}$$

Fix $\varepsilon > 0$: by Lemma 4.2.3 we have that there exists m large enough such that

$$P(\tau_m \leq T) < \varepsilon.$$

After having chosen such m , by (4.2.37) and the convergence of $\kappa_N(t \wedge \tau_{r_m, R_m}^N)$ to $\bar{\kappa}_m(t)$, we can choose N large enough such that $P(\tau_{r_m, R_m}^N \leq T) \leq 2\varepsilon$ and

$$|E[f(\kappa_N(\cdot \wedge \tau_{r_m, R_m}^N))] - E[f(\bar{\kappa}_m(\cdot))]| < \varepsilon.$$

So, for any $\varepsilon > 0$, there exist m and N large enough such that

$$|E[f(\kappa_N(\cdot))] - E[f(\kappa(\cdot))]| \leq \varepsilon + 4\varepsilon\|f\|_\infty$$

which concludes the proof. \square

4.3 Critical dynamics for the bi-populated CW model

In this section we briefly analyse critical fluctuations in presence of a Hopf bifurcation for the bi-populated Curie-Weiss model without delay (see Section 2.1). We will see that they belong to the same class of universality given by Theorem 4.1.2.

Let $(\underline{\sigma}(t))_{t \in [0, T]}$ be a Markov process with infinitesimal generator (2.2.3). Recall that an order parameter of the system is given by the pair

$$m_{1,N}(t) = \frac{1}{N} \sum_{i \in I_1} \sigma_i(t), \quad m_{2,N}(t) = \frac{1}{N} \sum_{i \in I_2} \sigma_i(t).$$

In Theorem 2.1.1 we studied the macroscopic dynamics of the model, through the convergence of its order parameter. As pointed out in Section 2.3.1, the parameters γ , J_{11} , J_{12} , J_{21} , J_{22} can be adjusted to create an Hopf bifurcation at the origin for the limiting dynamics (2.1.5): it is sufficient to impose that

$$\gamma J_{11} - 1 = -((1 - \gamma)J_{22} - 1), \quad (4.3.1)$$

$$\Gamma := (\gamma J_{11} - 1)^2 + \gamma(1 - \gamma)J_{12}J_{21} < 0. \quad (4.3.2)$$

In this case, the matrix obtained linearizing (2.1.5) around $(0, 0)$ will be

$$A_{cr} = \begin{pmatrix} 2(\gamma J_{11} - 1) & 2\gamma J_{12} \\ 2(1 - \gamma)J_{21} & -2(\gamma J_{11} - 1) \end{pmatrix}$$

and its eigenvalues are the purely imaginary numbers $\lambda_{1,2} = \pm 2i\sqrt{|\Gamma|}$.

Notice that a result concerning standard fluctuations similar to Theorem 4.1.1 can be stated but we focus on the critical fluctuations of the process $(m_{1,N}(t), m_{2,N}(t))$ when (4.3.1) and (4.3.2) hold, hence in presence of a Hopf bifurcation. The critical fluctuations flow, when starting from the local equilibrium, is described by the two-dimensional process

$$x_N(t) = N^{\frac{1}{4}} m_{1,N}(N^{\frac{1}{2}}t), \quad y_N(t) = N^{\frac{1}{4}} m_{2,N}(N^{\frac{1}{2}}t). \quad (4.3.3)$$

Consider the change of variables

$$w_N(t) = \frac{y_N(t)}{(1 - \gamma)J_{21}}, \quad v_N(t) = \frac{1}{\sqrt{|\Gamma|}} \left(-x_N(t) + \frac{(\gamma J_{11} - 1)}{(1 - \gamma)J_{21}} y_N(t) \right). \quad (4.3.4)$$

and define

$$\kappa_N(t) = w_N(t)^2 + v_N(t)^2.$$

Similarly to the case with dissipation, our technique to prove the convergence for the process $(\kappa_N(t))_{t \in [0, T]}$ requires to fix initial conditions such that

$$(m_{1,N}(0), m_{2,N}(0)) \xrightarrow[N \rightarrow +\infty]{w} (0, 0), \quad \kappa_N(0) \xrightarrow[N \rightarrow +\infty]{w} \bar{\kappa} \quad (4.3.5)$$

with $\bar{\kappa} > 0$. In this case, to obtain initial conditions which verify (4.3.5), one can take a small asymmetry in the initial distribution for the spins. For simplicity, we introduce this small asymmetry only in one family of spins. As noticed in Remark 4.1.1, we believe this asymmetry could be avoided.

(H2) the initial spins $\{\sigma_i(0)\}_{i=1,\dots,N}$ constitute a family of independent random variable with the distributions satisfying the following conditions:

- if $i \in I_1$, then

$$\lim_{N \rightarrow +\infty} N^{\frac{1}{4}} \left[P(\sigma_i(0) = +1) - \frac{1}{2} \right] = \epsilon \neq 0;$$

- if $i \in I_2$, then

$$\lim_{N \rightarrow +\infty} N^{\frac{1}{4}} \left[P(\sigma_i(0) = +1) - \frac{1}{2} \right] = 0;$$

If (H2) holds, then $x_N(0) \rightarrow 2\epsilon\gamma$ and $y_N(0) \rightarrow 0$ in distribution, hence $\kappa_N(0)$ weakly converges to $4\epsilon^2\gamma^2|\Gamma|^{-1}$.

Theorem 4.3.1. *Assume (H2) holds. Take $\gamma \in]0, 1[$, $(J_{11}, J_{12}, J_{21}, J_{22}) \in \mathbb{R}^4$ such that (4.3.1)-(4.3.2) are verified and such that $Z_2(\gamma, J_{11}, J_{12}, J_{21}) < 0$, where*

$$\begin{aligned} Z_2(\gamma, J_{11}, J_{12}, J_{21}) &= -2J_{11}^2|\Gamma| - 2J_{21}^2 + (\gamma J_{11} - 1)|\Gamma|(J_{11}^2 - J_{21}^2) + (\gamma J_{11} - 1)J_{21}^2 \\ &\quad + (\gamma J_{11} - 1)(J_{11}(\gamma J_{11} - 1) + (1 - \gamma)J_{12}J_{21})^2 + \\ &\quad - J_{11}(\gamma J_{11} - 1)(J_{11}(\gamma J_{11} - 1) + (1 - \gamma)J_{12}J_{21}). \end{aligned}$$

Then, as $N \uparrow +\infty$ in such a way γ remains constant, the process $(\kappa_N(t))_{t \in [0, T]}$ converges, in sense of weak convergence of stochastic processes, to $(\kappa(t))_{t \in [0, T]}$, unique solution of the stochastic differential equation

$$\begin{aligned} d\kappa(t) &= \left(4Z_1(\gamma, J_{11}, J_{12}, J_{21}) + \frac{1}{4}Z_2(\gamma, J_{11}, J_{12}, J_{21})\kappa^2(t) \right) dt \\ &\quad + 2\sqrt{2Z_1(\gamma, J_{11}, J_{12}, J_{21})\kappa(t)}dB(t) \end{aligned} \tag{4.3.6}$$

with

$$Z_1(\gamma, J_{11}, J_{12}, J_{21}) = \frac{|\Gamma| + \gamma(1 - \gamma)J_{21}^2 + (\gamma J_{11} - 1)^2}{(1 - \gamma)J_{21}^2|\Gamma|}$$

and $\kappa(0) = 4\epsilon^2\gamma^2|\Gamma|^{-1}$.

Remark 4.3.1. Notice that requiring $Z_2(\gamma, J_{11}, J_{12}, J_{21}) < 0$ guarantees global existence and uniqueness of the solution of (4.3.6). It is important to state this assumption formulating Theorem 4.3.1, since it is easy to find choices for $\gamma, J_{11}, J_{12}, J_{21}, J_{22}$ which satisfy (4.3.1)-(4.3.2) but not $Z_2(\gamma, J_{11}, J_{12}, J_{21}) < 0$: for example, one can check it with $\gamma = 0.6$, $J_{11} = -10$, $J_{12} = 20$, $J_{21} = -15$.

Remark 4.3.2. The dynamics of critical fluctuations in presence of a Hopf bifurcation of the two model analysed belong to the same class of universality. Indeed, the stochastic differential equations (4.1.7) and (4.3.6), which describe the limit of the critical dynamics of the two models, have the same structure, i.e.

$$d\kappa(t) = (C_1 - C_2\kappa^2(t))dt + \sqrt{C_3\kappa(t)}dB(t),$$

with $C_1, C_2, C_3 > 0$.

4.3.1 Sketch of proof

The proof of Theorem 4.3.1 presents the same technical steps of the proof of Theorem 4.1.2. Therefore, we only sketch the computations sufficient to identify the correct change of variables and the limiting equation (4.3.6).

As pointed out in the proof of Theorem 4.1.2, we are looking for a change of variables

$$\begin{pmatrix} w_N(t) \\ v_N(t) \end{pmatrix} = C \begin{pmatrix} x_N(t) \\ y_N(t) \end{pmatrix}$$

where C has to be such that

$$CAC^{-1} = \begin{pmatrix} 0 & -2\sqrt{|\Gamma|} \\ 2\sqrt{|\Gamma|} & 0 \end{pmatrix}.$$

We can take

$$C = \begin{pmatrix} 0 & \frac{1}{(1-\gamma)J_{21}} \\ -\frac{1}{\sqrt{|\Gamma|}} & \frac{(\gamma J_{11}-1)}{(1-\gamma)J_{21}\sqrt{|\Gamma|}} \end{pmatrix},$$

which provides

$$w_N(t) = \frac{y_N(t)}{(1-\gamma)J_{21}}, \quad v_N(t) = \frac{1}{\sqrt{|\Gamma|}} \left(-x_N(t) + \frac{(\gamma J_{11}-1)}{(1-\gamma)J_{21}} y_N(t) \right). \quad (4.3.7)$$

Using the same tools of Subsection 4.2.1, the infinitesimal generator G_N of the process $(w_N(t), v_N(t))$ satisfies, for $f \in \mathcal{C}^3$,

$$\begin{aligned} G_N f(w, v) &= \\ &= \frac{2}{(1-\gamma)J_{21}^2} f_{ww} + \frac{4(\gamma J_{11}-1)}{(1-\gamma)J_{21}^2\sqrt{|\Gamma|}} f_{wv} + \frac{2(\gamma(1-\gamma)J_{21}^2 + (\gamma J_{11}-1)^2)}{(1-\gamma)J_{21}^2|\Gamma|} f_{vv} + \\ &+ \frac{1}{(1-\gamma)J_{21}} \left(\frac{(1-\gamma)}{3} \mathcal{R}_2^3(x, y) - y\mathcal{R}_2^2(x, y) \right) f_w + \\ &+ \left(\frac{\gamma J_{11}-1}{(1-\gamma)J_{21}\sqrt{|\Gamma|}} \left(\frac{(1-\gamma)}{3} \mathcal{R}_2^3(x, y) - y\mathcal{R}_2^2(x, y) \right) - \frac{1}{\sqrt{|\Gamma|}} \left(\frac{\gamma}{3} \mathcal{R}_1^3(x, y) - x\mathcal{R}_1^2(x, y) \right) \right) f_v + \\ &+ N^{\frac{1}{2}} \left(-2\sqrt{|\Gamma|} v f_w + 2\sqrt{|\Gamma|} w f_v \right) + o(1), \end{aligned}$$

where $x = (\gamma J_{11} - 1)w - \sqrt{|\Gamma|}v$, $y = (1 - \gamma)J_{21}w$. This means that for $0 < r < R$ and

$$\tau_{r,R}^N := \inf\{t \in [0, T] \mid \kappa_N(t) \notin]r, R[\},$$

the stopped process $(\kappa_N(t \wedge \tau_{r,R}^N), \theta_N(t \wedge \tau_{r,R}^N))$ has an infinitesimal generator $H_N^{r,R}$ which, for $f \in \mathcal{C}^3$, satisfies

$$Hf_N^{r,R}f(\kappa, \theta) = \mathcal{H}_f^{r,R}(\kappa, \theta) + N^{\frac{1}{2}}2\sqrt{|\Gamma|}f_\theta(\kappa, \theta) + o_{r,R}(1)$$

where $\mathcal{H}_f^{r,R}(\kappa, \theta)$ is composed by terms of order 1. If we apply $\mathcal{H}^{r,R}$ to a function of the type $f(\kappa, \theta) = f(\kappa)$ we get $\mathcal{H}_f^{r,R}(\kappa) = \mathbb{1}_{]r,R[}(\kappa)\mathcal{A}_f(\kappa, \theta)$ with

$$\begin{aligned} \mathcal{A}_f(\kappa, \theta) &= \\ &= \left(\frac{8}{(1-\gamma)J_{21}^2}w^2 + \frac{16(\gamma J_{11} - 1)}{(1-\gamma)J_{21}^2\sqrt{|\Gamma|}}wv + \frac{8(\gamma(1-\gamma)J_{21}^2 + (\gamma J_{11} - 1)^2)}{(1-\gamma)J_{21}^2|\Gamma|}v^2 \right) f''(\kappa) + \\ &+ 4 \frac{|\Gamma| + \gamma(1-\gamma)J_{21}^2 + (\gamma J_{11} - 1)^2}{(1-\gamma)J_{21}^2|\Gamma|} f'(\kappa) + \\ &+ \left(\frac{2w}{(1-\gamma)J_{21}} + \frac{2(\gamma J_{11} - 1)v}{(1-\gamma)J_{21}\sqrt{|\Gamma|}} \right) \left(\frac{(1-\gamma)}{3}\mathcal{R}_2^3(x, y) - y\mathcal{R}_2^2(x, y) \right) f'(\kappa) + \\ &- \frac{2v}{\sqrt{|\Gamma|}} \left(\frac{\gamma}{3}\mathcal{R}_1^3(x, y) - x\mathcal{R}_1^2(x, y) \right) f'(\kappa), \end{aligned}$$

where $x = (\gamma J_{11} - 1)w - \sqrt{|\Gamma|}v$, $y = (1 - \gamma)J_{21}w$, $w = \sqrt{\kappa} \cos \theta$ and $v = \sqrt{\kappa} \sin \theta$. Now we have to calculate $\bar{\mathcal{A}}_f(\kappa) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{A}_f(\kappa, \theta) d\theta$. Notice that the averaging will make any term with odd power in w or v disappear since

$$\int_0^{2\pi} \cos^n \theta \sin^m \theta d\theta = 0$$

if at least one between n and m is odd. Then, one gets

$$\bar{\mathcal{A}}_f(\kappa) = 4Z_1(\gamma, J_{11}, J_{12}, J_{21}) (\kappa f''(\kappa) + f'(\kappa)) + \frac{1}{4}Z_2(\gamma, J_{11}, J_{12}, J_{21}) \kappa^2 f'(\kappa)$$

where

$$Z_1(\gamma, J_{11}, J_{12}, J_{21}) = \frac{|\Gamma| + \gamma(1-\gamma)J_{21}^2 + (\gamma J_{11} - 1)^2}{(1-\gamma)J_{21}^2|\Gamma|}$$

and

$$\begin{aligned} Z_2(\gamma, J_{11}, J_{12}, J_{21}) &= -2J_{11}^2|\Gamma| - 2J_{21}^2 + (\gamma J_{11} - 1)|\Gamma|(J_{11}^2 - J_{21}^2) + (\gamma J_{11} - 1)J_{21}^2 \\ &+ (\gamma J_{11} - 1)(J_{11}(\gamma J_{11} - 1) + (1 - \gamma)J_{12}J_{21})^2 + \\ &- J_{11}(\gamma J_{11} - 1)(J_{11}(\gamma J_{11} - 1) + (1 - \gamma)J_{12}J_{21}). \end{aligned}$$

Then, as long as the parameters are chosen in a way that $Z_2(\gamma, J_{11}, J_{12}, J_{21}) < 0$, $\bar{\mathcal{A}}_f(\kappa)$ corresponds to the infinitesimal generator of the process $(\kappa(t))_{t \in [0, T]}$, unique solution of (4.3.6). The arguments to be used are the same of the proof of Theorem 4.1.2: we fix r, R such that $0 < r < 4\gamma^2|\Gamma|^{-1} < R$ and we characterize the weak limit for the stopped process $(\kappa_N(t \wedge \tau_{r, R}^N))_{t \in [0, T]}$ using the averaging principle; then, as in Proposition 4.2.8, one proves that it also implies that $(\kappa_N(t))_{t \in [0, T]}$ converges, in sense of weak convergence of stochastic processes, to $(\kappa(t))_{t \in [0, T]}$.

Conclusions and future perspectives

In the first part of this thesis we explored two interaction mechanisms that enhance self-sustained periodic behavior in the macroscopic dynamics of complex systems.

In Chapter 1 we study the role of dissipation in the evolution of the interaction potential of mean field cooperative systems. Collective periodicity induced by this feature is not a novelty in literature ([19], [20], [14]) and this chapter provides a generalization of these results. Adopting the approach described in [14], we introduce dissipation in the Langevin dynamics of the generalized Curie-Weiss model. We derive the limiting dynamics in a general framework adopting a *propagation of chaos* technique, however, to obtain a complete picture of the evolution of the macroscopic law we restrict to the Gaussian case. This allows to describe the macroscopic dynamics as the solution of a Liénard system, for which a number of results are available. This analysis permits us to depict a rich behavior of the limiting dynamics: depending on choices of the model's details, we can observe patterns which were not present in previous works, including the coexistence of multiple stable limit cycles in the phase space.

The connection between dissipation and periodicity is by now well-understood in the mean field context, but some aspects may be worth further investigations. One possible outline is to consider more general symmetric single site distributions rather than restricting to the Gaussian one. We believe that this does not rule out macroscopic oscillations but we also expect the formal analysis of the limiting dynamics to be dramatically harder, since it would not longer possible to recover a finite-dimensional problem. This aspect has been partially overcome in [14] by considering a small-noise Gaussian approximation of the macroscopic process and this tool may be helpful in studying more general cases.

In Chapter 2 we focus on the emergence of self-sustained periodic behavior in a bi-populated Curie-Weiss model. While periodicity has already been observed in multi-populated systems with delayed interactions [23, 55], here we show that delay is not an essential ingredient to produce this phenomenon, which can appear simply considering a Glauber dynamics in our model. The key feature seems rather to be the presence of a frustration in the interaction network. However, we also show that delay can enhance the appearance of macroscopic rhythm in configurations in which it was otherwise absent.

Future works may explore what happens when more than two populations are present: we expect to observe periodicity again, granted that there exists "enough" frustration in the

system. Depending on the topology of the interaction network and the strength of each interaction, more general behaviors could be possible (for example having some populations which oscillate and others which relax to an equilibrium). However, a formal analysis in this direction is still lacking.

More complicated generalizations may include stochastic dynamics also for the parameters which tune the strength of interactions: in this case, the limiting dynamics could display a much richer behavior.

In Chapter 3 we come back to the mechanism of dissipation, this time outside the context of mean field systems. We consider a short-range interacting system, obtained by introducing dissipation in a Glauber dynamics for the classical 1-dimensional Ising model. We prove that we can obtain regular oscillations in a suitable zero-temperature infinite-volume limit when a proper time scaling is performed. This result is achieved through a deep study of the distributions of the time at which the first spin flip occurs and the successive time taken by the newborn droplet to cover all the space.

To the best of our knowledge, this is the first result connecting periodicity and dissipation in systems with local interactions. Therefore, there exists a huge number of intriguing questions that will hopefully be addressed in future works: here we list some of them.

Recall that our analysis we took the limit for $\beta, N \uparrow +\infty$ in such a way

$$\frac{\log N}{\beta} \rightarrow c \in [0, 1[,$$

a choice that allows us to rule out the birth of a second droplet during the covering process. What does it happen when $c \geq 1$? Some naive computations tell us that as soon as

$$\frac{\log N}{\beta} \rightarrow c = 1 + \varepsilon, \quad \text{for some } \varepsilon > 0$$

the situation changes dramatically. In this case, the space is too big to be covered by a single droplet and actually we expect the birth of an infinite number of separate droplet which eventually coalesce and cover all the space. A promising approach to this case is to divide the space in boxes of size $e^{(1-\varepsilon)\beta}$ in order to exploit the study performed in this thesis.

The choice of the initial conditions can also be discussed. For simplicity, in Chapter 3 we chose all the spins to be equal and a constant profile (plus some small inhomogeneities) for the local fields at the beginning. However, there is not any obvious reason to believe that this is the unique configuration which allows the system to converge to some form of oscillating behavior. This aspect certainly deserves further investigations.

Finally, we would like to mention that the phenomenon described this chapter should also hold in higher dimensions. We studied the 1-dimensional case, in which the growth of the droplet is relatively easy to analyse, since it can only grow along one direction. For the d -dimensional Ising model with $d \geq 2$, the droplet is allowed to grow along d directions, which makes the estimate of scale of the covering time a challenging task.

The second part of the thesis is dedicated to the study of critical fluctuations of mean field spin systems exhibiting a Hopf bifurcation in the dynamical system describing the evolution of the macroscopic law.

It is known that dynamics of fluctuations at critical point in mean field models typically present an anomalous space-time scaling and, possibly, a non-Gaussian limit [16, 22, 13, 35]. In Chapter 4, we extend the analysis of critical fluctuations to the case in which a Hopf bifurcation is present. As we discussed, the fluctuations are deeply connected to the linearization of the limiting dynamics, therefore the kind of bifurcation has a relevant impact on the critical fluctuations. Our result is not obtained in full generality, but we focus on the Curie-Weiss model with dissipation [19] and on the two-population Curie-Weiss model studied in Chapter 2. We identify a slow and a fast variable, respectively corresponding to the radius and the angle of a polar coordinate description: in the "natural" time scale, the fast variable averages out, producing a limiting dynamics for the slow variable via an averaging principle. The critical fluctuations for both models belong to the same class of universality, which we conjecture to be independent of the microscopic dynamical details of the model, but rather strongly associated with the presence of the Hopf bifurcation in the macroscopic dynamics.

Future works may attempt to investigate further this class of universality, by considering more complicated models displaying a Hopf bifurcation, in particular continuous models such as the Kuramoto model with disorder.

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