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# SEPARATELY CR FUNCTIONS AND PEAK INTERPOLATION MANIFOLDS


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# Abstract


In this thesis two different themes are analysed: the first concerns the theory of separately CR functions on CR manifolds and the second deals with the problem of characterizing peak interpolation manifolds on boundaries of pseudoconvex domains of $\mathbb{C}^n$.

For the first theme, let $M$ be a CR manifold of $\mathbb{C}^n$, with boundary $N$, foliated by a family $\{L_\lambda\}_{\lambda\in\Lambda}$ of CR manifolds of CR-dimension 1, issued from $N$, such that the following transversality condition is satisfied at any common point of $N \cap L_\lambda$: $T^{\mathbb{C}}L_\lambda\big|_{N\cap L_\lambda} + TN\big|_{N\cap L_\lambda} = TM\big|_{N\cap L_\lambda}$. By using the approximation of CR functions by polynomials, we prove that if $f \in C^0(M) \cap CR(N)$ and $f$ is CR and $C^1$ along each leaf $L_\lambda$, then $f$ is CR in a neighbourhood of the boundary $N$ in $M$. Assuming $M$ to be connected, we also prove that the function $f$ comes to be CR all over $M$, thus reaching the global result.

The use of the technique of polynomial approximation by integration with the heat kernel enables us to reprove a result by Henkin and Tumanov; moreover, a generalization of their result is given for foliations by CR manifolds of CR-dimension 1 instead of foliations by complex curves.

Our problem, concerning separately CR functions, reminds the well-known result by Hartogs on separately holomorphic functions. We present a simplified proof of Hartogs Theorem, using a "propagation" argument; further applications and variations of this theorem, using different techniques, are also proved in the present work.

For the second theme of the thesis, let $D$ be a bounded domain in $\mathbb{C}^n$ with smooth boundary $S = \partial D$; we denote by $A(D)$ the algebra of continuous functions on $\bar{D}$, that are holomorphic in $D$. A submanifold $M$ of $S$ is an interpolation manifold for $A(D)$ if, for every $f \in C^0(M)$ and every compact set $K \subset M$, there exists a function $F \in A(D)$ such that $F|_K = f|_K$, while $M$ is a peak manifold for $A(D)$ if, for every compact set $K \subset M$, there exists a function $F \in A(D)$ such that $F|_K = 1$ and $|F| < 1$ on $\bar{D} \setminus K$. The problem of characterizing peak interpolation manifolds on boundaries of strictly pseudoconvex domains has been completely solved by Henkin and Tumanov, as well as by Rudin, with different techniques: it turns out that, in order for a smooth submanifold $M$ of $S$ to be a peak interpolation manifold, it is necessary and sufficient that $M$ satisfies a certain directional condition, which is the one of being complex tangential ($TM \subset T^{\mathbb{C}}S$).

For a complex tangential submanifold $M$ of $S$, with $S = \partial D$ strictly pseudo-

convex, we present a geometric and easy proof of the property of being totally real; then, we generalize such a result, proving that, also in the case of a (weakly) pseudoconvex domain $D$ of finite type, a complex tangential submanifold $M$ of $S$ is totally real.

We analyse in details the techniques of Henkin-Tumanov and Rudin for strictly pseudoconvex domains, with the aim at extending their characterization to weakly pseudoconvex domains.

Following the first of these technique, we generalize some steps of Henkin-Tumanov proof and get some conclusions for the case of (weakly) pseudoconvex domains of type 4 in $\mathbb{C}^n$.

The second technique, based on the construction of suitable integrals and an application of a Theorem by Bishop, admits a generalization, proposed by Bharali, for weakly convex domains having real analytic boundary. Analysing the main tools of Bharali's proof, we focus our attention on his local stratification for submanifolds of the boundary; we present a different technique to stratify real analytic boundaries of weakly pseudoconvex domains, such that on each strata the Levi form is non degenerate.

Finally, we add a remark on the idea of extending the notion of peaking: even if the natural generalization of holomorphic functions is given by $\bar{\partial}$-closed complex differential forms, it turns out that these forms always peak inside the domain of definition, so the notion of peaking does not serve any purpose.

# Riassunto

In questa tesi vengono analizzati due diversi temi: il primo riguarda la teoria delle funzioni separatamente CR e il secondo tratta il problema della caratterizzazione di varietà cosiddette "peak interpolating" su frontiere di domini pseudoconvessi di $\mathbb{C}^n$.

Per il primo tema, sia $M$ una varietà CR di $\mathbb{C}^n$, con frontiera $N$, che ammette una fogliazione $\{L_\lambda\}_{\lambda\in\Lambda}$ di varietà CR di dimensione CR 1, che intersecano $N$, tali che in ogni punto di $N \cap L_\lambda$ sia soddisfatta la seguente ipotesi di trasversalità : $T^{\mathbb{C}}L_\lambda\big|_{N\cap L_\lambda} + TN\big|_{N\cap L_\lambda} = TM\big|_{N\cap L_\lambda}$. Approssimando le funzioni CR tramite polinomi, dimostriamo che se $f \in C^0(M)\cap \mathrm{CR}(N)$ ed $f$ è CR e $C^1$ lungo ogni foglia $L_\lambda$, allora $f$ è CR in un intorno della frontiera $N$ in $M$. Assumendo $M$ connesso, dimostriamo anche che la funzione $f$ risulta essere CR su tutto $M$, e in questo modo raggiungiamo il risultato globale.

L'uso della tecnica di approssimazione polinomiale tramite integrazione con il nucleo del calore ci consente di riottenere un risultato di Henkin e Tumanov; d'altra parte, siamo in grado di fornire una generalizzazione del loro risultato, presentando l'enunciato per fogliazioni mediante varietà CR di dimensione CR 1, e non curve complesse.

Il nostro problema, riguardante funzioni separatamente CR, è strettamente legato al noto Teorema di Hartogs sulle funzioni separatamente olomorfe. Presentiamo una dimostrazione semplificata del Teorema di Hartogs, utilizzando un argomento di "propagazione"; ulteriori applicazioni e variazioni di tale risultato sono dimostrate, con differenti tecniche, nel presente lavoro.

Per quanto riguarda il secondo tema sviluppato nella tesi, sia $D$ un dominio limitato di $\mathbb{C}^n$ con frontiera $S = \partial D \in C^\infty$; indichiamo con $A(D)$ l'algebra delle funzioni continue su $\bar{D}$ e olomorfe in $D$. Una sottovarietà $M$ di $S$ è "interpolating manifold" per $A(D)$ se, per ogni $f \in C^0(M)$ e ogni insieme compatto $K \subset M$, esiste una funzione $F \in A(D)$ tale che $F|_K = f|_K$, mentre $M$ è "peaking manifold" per $A(D)$ se, per ogni insieme compatto $K \subset M$, esiste una funzione $F \in A(D)$ tale che $F|_K = 1$ e $|F| < 1$ su $\bar{D} \setminus K$. Il problema della caratterizzazione delle varietà "peak interpolating" su frontiere di domini strettamente pseudoconvessi è stato completamente risolto da Henkin e Tumanov, così come da Rudin, seppur attraverso tecniche molto diverse: condizione necessaria e sufficiente affinchè una sottovarietà $M$ di $S$ di classe $C^\infty$ sia "peak interpolating ", è che $M$ soddisfi un'ipotesi sulle direzioni tangenti, definita tangenza complessa ($TM \subset T^{\mathbb{C}}S$).

Per una sottovarietà tangente complessa $M$ di $S$, con $S = \partial D$ strettamente

pseudoconvessa, proponiamo una dimostrazione semplice e puramente geometrica del fatto che tali varietà sono totalmente reali; quindi, generalizziamo tale risultato, dimostrando che, anche nel caso di un dominio $D$ debolmente pseudoconvesso di tipo finito, una sottovarietà tangente complessa $M$ di $S$ è totalmente reale.

Analizziamo in dettaglio le tecniche di Henkin-Tumanov e Rudin per domini strettamente pseudoconvessi, con l'obiettivo di estendere tale caratterizzazione a domini debolmente pseudoconvessi.

Seguendo la prima di queste tecniche, arriviamo a generalizzare alcune tappe della dimostrazione di Henkin-Tumanov e ad ottenere alcune conclusioni nel caso di domini (debolmente) pseudoconvessi di tipo 4 in $\mathbb{C}^n$.

La seconda tecnica, basata sulla costruzione di opportuni integrali e su un'applicazione di un Teorema di Bishop, consente una generalizzazione, proposta da Bharali, al caso di domini debolmente convessi con frontiera reale analitica. Analizzando i principali passi della dimostrazione di Bharali, ci siamo soffermati sulla stratificazione locale che egli ottiene su sottovarietà della frontiera; presentiamo dunque una tecnica diversa per ottenere una stratificazione sulla frontiera reale analitica di domini debolmente pseudoconvessi, tale che su ogni strato la forma di Levi è non degenere.

Concludiamo con un'osservazione sull'idea di estendere la nozione di "peaking": anche se la generalizzazione naturale delle funzioni olomorfe è data dalle forme differenziali complesse $\bar{\partial}$-chiuse, risulta che tali forme soddisfano sempre la proprietà di "peaking" all'interno del dominio di definizione, per cui in questo contesto tale nozione è destituita di ogni interesse.

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# Contents

# Chapter 1

# Introduction

The aim of this Ph.D. thesis is to describe and develop two different themes, concerning separately CR functions (Chapter 2) and peak interpolation manifolds (Chapter 3). For this reason, the first paragraph of the introduction, containing the background and the statements of the main results of the thesis, is divided in two subsections, corresponding to the second and the third chapters. Another paragraph follows, to describe in details how the dissertation is organized.

## 1.1 Background and statements of the results

**§. Second Chapter.** Let $\Omega$ be a domain of $\mathbb{C}^n$; the result of Hartogs of 1906 states that any function $f : \Omega \to \mathbb{C}$, which is separately holomorphic in each variable, turns to be jointly holomorphic, without any assumption on the initial regularity of $f$: this is $C^\infty$, and even analytic (in particular it is $C^1$; cf. Definition 2.1.1), as a consequence. The main difficulty in proving the previous result concerns the ability to prove the local boundedness of the function.

To simplify notations, we can consider a bidisc $\bar{\Delta} \times \bar{\Delta} \subset\subset \Omega$ in $\mathbb{C}^2$ and remind that, by Hartogs, a function on the bidisc, which is holomorphic along the lines $z_1 \equiv \text{const}$ and $z_2 \equiv \text{const}$, is jointly holomorphic on the full bidisc. The statement, in fact, is local and it is possible to pass from $\mathbb{C}^2$ to $\mathbb{C}^n$ adding, one by one, the directions of separate analyticity.

It turns out that the proof of Hartogs reduces, by an application of Baire Theorem, to the following statement: if $f$ is a function on the bidisc, which is separately holomorphic along the lines $z_2 = \text{const}$ and jointly holomorphic in a neighbourhood of $z_1 = 0$, then it is in fact holomorphic. We have presented the proof of this simplified statement of Hartogs result in Section 2.3.

Many questions have been raised on separately holomorphic functions, assuming some changes in the setting of Hartogs statement; considering the problem in $\mathbb{C}^2$, the first question is if the same conclusion holds with the holomorphic foliation $z_2 \equiv \text{const}$ replaced by a foliation of complex curves. If we assume $f \in C^1$, or even

$C^0$, the result can be proved in several ways: we have showed it in Section 2.4, as well as in our paper [43]. In case the foliation is real analytic, we have

**Theorem 1.1.1** ([6]). *Let $\Omega$ be a domain in $\mathbb{C}^2$, foliated by complex holomorphic 1-dimensional leaves $\{\Gamma_t\}_t$, depending in a real analytic way on $t \in \mathbb{R}^2$ and let $M \subset \Omega$ be a 2-dimensional real submanifold transversal to the leaves. If $f$ is a complex function on $\Omega$, such that $f$ is separately holomorphic along each leaf $\Gamma_t$ and $f$ is jointly holomorphic in a neighbourhood of $M$, then $f$ is holomorphic in $\Omega$.*

A proof of this result can be found in [6]. Furthermore, it has an immediate generalization from $\mathbb{C}^2$ to $\mathbb{C}^n$ and, in case of a holomorphic foliation, it is Hartogs statement.

Note that when the leaves are complex curves it is a problem of propagation so the proofs require arguments which are completely different from the classical ones. In full generality, that is for a smooth foliation of complex curves, the problem has been recently solved by E. M. Chirka [16]; in his paper he has presented the following two problems on separately holomorphic functions and the first of them is exactly what we are speaking about.

(1) Suppose that $\Omega \subseteq \mathbb{C}^n$ is foliated by $n$ families of complex curves $\{S^j_\tau\}_{j=1,\dots,n}$ such that for every point in $\Omega$ the tangent vectors to these curves generate $\mathbb{C}^n$. Is it true that any function $f$ on $\Omega$, holomorphic on all such curves, is holomorphic on $\Omega$?

(2) Suppose that $D$ is a domain in $\mathbb{C}^m$, $0 \in D$, and $\Omega \subset D \times \mathbb{C}^k$ is foliated by graphs $\{S_\tau\}$ of holomorphic mappings from $D$ to $C^k$ (that is there exists a continuous mapping $D \times G \ni (z,t) \to (z, \Phi(z,t)) \in \Omega$, holomorphic in $z$). Is it true that any function $f$ on $\Omega$, holomorphic on all graphs and in a neighbourhood of $\Omega_0 = 0 \times G = \{(0, \Phi(0,t)), t \in G\}$, is holomorphic on $\Omega$?

Chirka has proved that the first problem has an affirmative answer when $n = 2$ or when $n$ is arbitrary and $f$ is bounded; for $n > 2$ the question remains open. Chirka has also proved that the answer to the second problem is always positive when the function $\Phi$ is Lipschitz; furthermore, for $k = 1$, the assumption that the foliation $\{S_\tau\}$ is Lipschitzian can be omitted.

These statements are proved using uniform approximations of general holomorphic motions by smooth holomorphic motions and by estimates of the width of holomorphic envelopes of domains obtained through holomorphic motions.

Coming back to the setting of complex curves in $\mathbb{C}^2$, we have to remember that the extremal case in which $f$ is no more analytic in a neighbourhood of $z_1 = 0$ but just CR on a hypersurface transversal to the foliation has been treated by L. Baracco and G. Zampieri in [6] with the following Theorem

**Theorem 1.1.2** ([6]). *Let $\Omega$ be a domain of $\mathbb{C}^2$, foliated by a $C^\omega$ family of holomorphic curves $\{\Gamma_t\}_t$, $t \in \mathbb{R}^2$, transversal to a real hypersurface $M$. Let $f$ be a complex function on $\Omega$ which is separately holomorphic along each $\Gamma_t$ and CR on $M$. Then, $f$ is holomorphic on $\Omega$.*

If we remove the hypothesis of real analyticity for the foliation, but we still consider CR manifolds, we can raise the following question: is it possible to regain a similar result for CR functions?

The first remark is the necessity for the function $f$ to be continuous. For CR functions on CR manifolds admitting foliation by holomorphic curves, Henkin and Tumanov had already obtained in 1983 the following Theorem

**Theorem 1.1.3** ([33]). *Let $M$ be a smooth CR manifold in $\mathbb{C}^n$, that admits a foliation by complex curves $\{\gamma_\lambda\}$, $\lambda \in \Lambda$; in addition, suppose that on $\partial M$ there is a smooth CR manifold $N$, such that each complex curve is transversal to $N$ at any common point of $\gamma_\lambda \cap N$. Then, any function $f \in C^0(M)$ which is CR on $N$ and holomorphic along the $\gamma_\lambda$'s, is also CR on $M$.*

By using the approximation of CR functions by polynomials, we have regained the result of Henkin and Tumanov by a simple and expressive proof and, at the same time, we have admitted a generalization of their statement, replacing the foliation $\{\gamma_\lambda\}$ of complex curves by a foliation $\{L_\lambda\}$ of CR manifolds of CR dimension 1. We have proved the following final statement

**Theorem 1.1.4** ([42]). *Let $M$ be a CR connected manifold of $\mathbb{C}^n$ with boundary $N$, foliated by a family $\{L_\lambda\}$ of CR manifolds of CR dimension 1 issued from $N$, with $T^{\mathbb{C}}L_\lambda$ transversal to $TN$ at any common point of $L_\lambda \cap N$. Let $f$ be a $C^0$ function on $M$, which is CR along $N$, CR and $C^1$ along each $L_\lambda$. Then, $f$ is CR all over $M$.*

For a more detailed account of notations, basic statements and remarks on the setting of separately holomorphic functions and separately CR functions, we refer to Sections 2.1 and 2.2 of the thesis.

**§. Third Chapter.** Let $D$ be a bounded domain in $\mathbb{C}^n$, let $\bar{D}$ denote its closure and $S = \partial D$ its boundary. Let $A(D)$ be the algebra of functions continuous on $\bar{D}$ and holomorphic in $D$. Let's assume that $S$ is a compact hypersurface of $\mathbb{C}^n$. A submanifold $M$ of $S$ is called an interpolation manifold for $A(D)$ if, for every $f \in C^0(M)$ and every compact set $K \subset M$, there exists a function $F \in A(D)$ such that $F|_K = f|_K$, while a submanifold $M$ of $S$ is called a peak manifold for $A(D)$ if, for every compact set $K \subset M$, there exists a function $F \in A(D)$ such that $F|_K = 1$ and $|F| < 1$ on $\bar{D} \setminus K$.

We are interested in determining when a sufficiently smooth submanifold $M \subset \partial D$ is a peak interpolation manifold (or even set) for $A(D)$. When $D$ is a strictly pseudoconvex domain having $C^3$ boundary and $M$ is of class $C^3$, the situation is very well understood through the work of Henkin and Tumanov [32]. It is required to know that $M$ is said to be complex tangential if $TM \subset T^{\mathbb{C}}S$. This is the main result of Henkin-Tumanov

**Theorem 1.1.5** ([32]). *Le $S = \partial D$ be a strictly pseudoconvex compact hypersurface of $\mathbb{C}^n$ of class $C^3$ and let $M$ be a complex tangential submanifold of $S$ of class $C^3$ and of real dimension $\leq n-1$. Then, $M$ is a peak-interpolation manifold for $A(S)$.*

Note that this Theorem yields a sufficient condition for $M$ to be a peak interpolation manifold, that is the one of being complex tangential; Henkin and Tumanov have proved that such a condition is also necessary (for the details cf. [32] while a simplified proof in the case of peaking functions of class $C^1$ is presented at the beginning of Section 3.3).

For the sufficiency, they first show that if $M \subset \partial D$ is complex tangential then it has to be totally real; then, they construct a smooth function $f$ which is almost holomorphic with respect to $M$ (that is $f$ has the property that $\frac{\partial f}{\partial \bar{z}_j}$, $j = 1, \dots, n$, vanish up to specified order on $M$) and locally peaks on $M$. This is the main part of the proof. To pass from the local construction to the global construction, the function $f$ is used to set up a certain $\bar{\partial}$-equation, from which it is possible to show that $M$ "peak-interpolates". All the details of the proof are presented in Section 3.3.

Independently, in his paper [44], Nagel has reached the same results showing that if $D \subset\subset \mathbb{C}^n$ has a $C^3$ strictly pseudoconvex boundary and $M$ is a complex tangential submanifold of $\partial D$ of class $C^3$, then $M$ is a peak-interpolation manifold for $A(D)$. He has showed that, given $f \in C^0(M)$, the interpolating function can be obtained by integrating $f$ with respect to a suitable complex measure on $M$.

W. Rudin, in his paper of 1978 (cf. [48]), has taken Nagel technique of exhibiting appropriate functions in $A(D)$ by means of integrals and has showed that any complex tangential manifold $M \subset \partial D$ of class $C^1$, with $\partial D$ strictly pseudoconvex of class $C^2$, is a peak-interpolation manifold for $A(D)$. The result is achieved as an application of Bishop Theorem (cf. [11]) which is a generalization of Rudin-Carleson Theorem (cf. Section 3.1 for further details) and provides a measure-theoretic characterization for peak interpolation sets for $A(D)$. Here is the statement

**Theorem 1.1.6** ([11])**.** *Let $D$ be a bounded domain in $\mathbb{C}^n$. A compact set $K \subset \partial D$ is a peak interpolation set for $A(D)$ if and only if $\mu(K_0) = 0$ for every compact $K_0 \subset K$ and for every complex Borel measure $\mu$ on $\partial D$ such that $\mu \perp A(D)$, that is $\int f d\mu = 0 \ \forall f \in A(D)$.*

The following is Rudin Theorem, and it concerns strictly convex domains; we have analysed the proof in details in Section 3.4. Rudin has proved the result for strictly pseudoconvex domains, using an embedding Theorem by Fornaess.

**Theorem 1.1.7** ([48])**.** *Let $D$ be a bounded strictly convex domain in $\mathbb{C}^n$, with $C^2$ boundary $S = \partial D$ and let $M$ be a submanifold of $S$ parametrized by a non singular $C^1$-mapping mapping $\Phi : \Omega \to \partial D$, where $\Omega$ is an open set of $\mathbb{R}^m$. Assume that $M$ is complex tangential, that means for $M$ to satisfy the orthogonality condition*

$$\langle \Phi'(x)v, \bar{\partial}\rho(\Phi(x)) \rangle = 0 \qquad \textit{for all } x \in \Omega, v \in \mathbb{R}^m,$$

*where $\rho$ is the defining function for $S$. If $K$ is a compact subset of $\Omega$, then $\Phi(K)$ is a peak-interpolation set for $A(D)$.*

Very little is known, however, when $D$ is a weakly pseudoconvex domain of finite type (there are many notions of type, for the definitions we refer to Section 3.1). By a result of Nagel and Rudin [45], it is still necessary for a peak interpolation submanifold of $\partial D$ to be complex tangential; for sufficient conditions the problem in its full generality, for arbitrary pseudoconvex domains of finite type in $\mathbb{C}^n$, is very difficult, even if we try to prove it only for boundary points (which are the simplest examples of complex tangential compact submanifolds of $\partial D$).

Following the technique of Rudin, G. Bharali has proved this result for a certain class of weakly pseudoconvex domains

**Theorem 1.1.8** ([10])**.** *Let $D$ be a bounded (weakly) convex domain in $\mathbb{C}^n$, $n \geq 2$, having real-analytic boundary $S = \partial D$, and let $M$ be a real-analytic submanifold of $S$. If $M$ is complex tangential, then $M$ is a peak-interpolation manifold for $A(D)$.*

After having described in details the setting of peak-interpolation manifolds in Section 3.1, we have proposed in this thesis an easier and geometric proof of the property of being totally real, for a complex tangential submanifold $M$ of $S$, with $S$ strictly pseudoconvex. Moreover, using the same technique, we have generalized the previous property, proving the following result, which is contained in Section 3.2.

**Theorem 1.1.9.** *Let $S = \partial D$ be a pseudoconvex hypersurface of $\mathbb{C}^n$ of type $k = 2m$, $m \in \mathbb{N}$ and let $M \subset S$ be a complex tangential submanifold of $S$; then, $M$ is totally real, that is*

$$T^{\mathbb{C}}_{z_0} M = \{0\}, \qquad \forall z_0 \in M.$$

After having analysed, in Sections 3.3 and 3.4, the proofs of Henkin-Tumanov, Rudin and Bharali, we have generalized some steps of Henkin-Tumanov technique, obtaining some conclusions in $\mathbb{C}^2$ for pseudoconvex domains of type 4. Here is our statement

**Theorem 1.1.10.** *Let $\gamma \subset \widetilde{M} \subset S \subset \mathbb{C}^2$, where $S$ is a real hypersurface of $\mathbb{C}^2$ with defining function $\rho$ ($dim_{\mathbb{R}} S = 3$), $\gamma$ is a complex tangential curve of $S$ ($dim_{\mathbb{R}} \gamma = 1$) and $\widetilde{M}$ is a totally real manifold of real dimension 2, with $\tau(z) = \mathcal{J}\, grad\, \rho \in T_z \widetilde{M}$. We also define $\xi \in T_z \gamma$, $\eta = \mathcal{J}\xi$, $\chi = grad\, \rho$, so that $T\widetilde{M} = Span\{\xi, \tau\}$. Let $f = u + iv \in C^\infty(\widetilde{M})$ such that*

$$\begin{cases} u_{|_{\widetilde{M}}} \equiv 0 & (1.1a) \\ v_{|_\gamma} \equiv 0 & (1.1b) \\ \tau v_{|_\gamma} < 0 & (1.1c) \end{cases}$$

*and extends as holomorphic, thus satisfying on $\widetilde{M}$*

$$\begin{cases} \xi u = \eta v \\ \eta u = -\xi v \end{cases} \quad (1.2) \qquad\qquad \begin{cases} \chi u = \tau v \\ \tau u = -\chi v. \end{cases} \quad (1.3)$$

*Assume that each point of $\gamma$ is of type four, that is, for each point of $\gamma$, the following hold*

$$[\eta,\xi] \in Span\{\xi,\eta\} \tag{1.4}$$
$$[\eta\,[\eta,\xi]\,] \in Span\{\xi,\eta\} \tag{1.5}$$
$$[\eta\,[\eta\,[\eta,\xi]\,]\,] = \tau \neq 0, \quad \tau \in \frac{\mathbb{C}\otimes TS}{\mathbb{C}\otimes T^{\mathbb{C}}S}. \tag{1.6}$$

*Then, we have on $\gamma$*

$$\begin{cases} \eta u = 0 \\ \eta v = 0 \end{cases} \text{(i)} \qquad \begin{cases} \eta^2 u = 0 \\ \eta^2 v = 0 \end{cases} \text{(ii)} \qquad \begin{cases} \eta^3 u = 0 \\ \eta^3 v = 0 \end{cases} \text{(iii)} \qquad \eta^4 u > 0 \ \text{(iv)}$$

We are still studying to find a good setting, among the class of weakly pseudoconvex domains of type 4 in $\mathbb{C}^2$, where also the remaining part of Henkin-Tumanov technique turns to be true. However, we have realized that the problem of peak-interpolation manifolds for pseudoconvex domains of finite type in $\mathbb{C}^2$ has been solved by Gautam Bharali in 2005 with a different technique; the result is contained in the following Theorem

**Theorem 1.1.11** ([9])**.** *Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^2$ and let $M \subset \partial D$ be a smooth curve.*

*(i) Let $\partial D$ be of class $C^\infty$ and $D$ of finite type. If $M$ is complex tangential and if $\partial D$ is of constant type along $M$, then each compact subset of $M$ is a peak-interpolation set for $A(D)$.*

*(ii) Let $\partial D$ be real-analytic and let $M \subset \partial D$ be a real-analytic, complex tangential curve; then, each compact subset of $M$ is a peak-interpolation set for $A(D)$*

A similar problem had been previously treated by A. Noell in $A^\infty(D)$ - the algebra of smooth functions on $\bar{D}$ that are holomorphic on $D$; here are his main results

**Theorem 1.1.12** ([46])**.** *Let $D$ be a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$ and let $M \subset \partial D$ be a smooth complex tangential curve.*

*(i) If $\partial D$ is of constant type along $M$, then $M$ is locally a peak set for $A^\infty(D)$.*

*(ii) If $\partial D$ is of constant type along $M$, then every compact subset of $M$ is an interpolation set for $A^\infty(D)$.*

*(iii) If $\partial D$ and $M$ are real-analytic, then for each $p \in M$ there exists a neighbourhood $V$ of $p$ such that every compact subset of $M \cap V$ is an interpolation set for $A^\infty(D \cap V)$*

Note that the proof of *(ii)* depends on *(i)*, after an application of a Theorem proved by Noell in [47]. Also note that the regularity obtained for the interpolating function is better than the regularity obtained by Bharali in [9], but the property of peaking is only local in $A^\infty(D)$.

In Section 3.4 of the thesis, after the analysis of the proof of Bharali for weakly convex domains with $C^\omega$ boundary, we have tried to generalize his local stratification for submanifolds of the boundary; with this aim, we have presented a technique to stratify $C^\omega$ boundaries of weakly pseudoconvex domains, such that on each strata the Levi form is non degenerate.

We have concluded the present dissertation with a remark on the idea of extending the notion of peaking to $\bar{\partial}$-closed complex differential forms (which are the natural generalization of holomorphic functions); by the fact that these forms always peak inside the domain, the notion of peaking looses any interest.

## 1.2 Structure of the thesis

The thesis is divided into three chapters

- The first chapter is the Introduction.
- The second chapter is divided into five sections: the first and the second one are an overview of notations and properties for the setting of separately holomorphic and CR functions. The third presents our proof of Hartogs Theorem, using a propagation argument, while variations and applications of it are contained in Section 2.4 (and have been published in [43]). Section 2.5 presents our main Theorem on separately CR functions, divided into Theorem A, which uses polynomial approximation to get a local CR extension, and Theorem B, where we get the global CR extension (this result has been published in [42]).
- The third chapter is divided into seven sections: the first one is a survey on convex and pseudoconvex domains, the notion of finite type and the setting of peak-interpolation manifolds. The second one describes the properties of complex tangential manifolds: here we prove our results on the necessity of being totally real for complex tangential submanifolds of strictly pseudoconvex hypersurfaces and for complex tangential submanifolds of (weakly) pseudoconvex hypersurfaces of finite type. In Section 3.3 we present Henkin-Tumanov technique to characterize peak interpolation manifolds on strictly pseudoconvex domains as, in Section 3.4, we present the technique of Rudin. In Section 3.4 we also specify how the proof of Rudin reduces in the case of the sphere in $\mathbb{C}^2$, we present the Theorem of Bharali and the basic steps of his proof, and finally we add our technique to stratify real analytic boundaries of weakly pseudoconvex domains. In Section 3.5 we present our partial generalization of Henkin-Tumanov Theorem for weakly pseudoconvex domains of type 4. Section 3.6 is a brief survey on standard forms for defining functions of pseudoconvex domains of finite type and describes their relation with the existence of local peak functions in $A^\omega(D)$, by a result of Bloom [12]. Finally, the remark on the idea of extending the notion of peaking to $\bar{\partial}$-closed forms is contained in Section 3.7.

# Chapter 2

# Separately CR functions

## 2.1 Basic definitions and remarks

We will consider functions $f : \mathbb{C}^n \to \mathbb{C}$ and use $(z, \bar{z})$ as coordinates in $\mathbb{C}^n$ by the identification of $\mathbb{R}^{2n}$ with the diagonal of $\mathbb{C}^n \times \bar{\mathbb{C}}^n$

$$(x, y) \mapsto (z, \bar{z}) = (x + iy, x - iy) \tag{2.1}$$

$$(z, \bar{z}) \mapsto (x, y) = \left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) \tag{2.2}$$

that induces the following correspondence of derivatives

$$\begin{cases} \partial_x = \partial_z + \partial_{\bar{z}} \\ \partial_y = i(\partial_z - \partial_{\bar{z}}) \end{cases} \tag{2.3}$$

$$\begin{cases} \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \\ \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \end{cases} \tag{2.4}$$

and the following correspondence of differentials

$$\begin{cases} dx = \frac{dz+d\bar{z}}{2} \\ dy = \frac{dz-d\bar{z}}{2i} \end{cases} \tag{2.5}$$

$$\begin{cases} dz = dx + idy \\ d\bar{z} = dx - idy. \end{cases} \tag{2.6}$$

In view of these correspondences, the differential of a complex function $f$ whose domain is in $\mathbb{C}^n$ can be expressed by

$$df = \sum_j (\partial_{x_j} f dx_j + \partial_{y_j} f dy_j) = \sum_j (\partial_{z_j} f dz_j + \partial_{\bar{z}_j} f d\bar{z}_j) = \partial f + \bar{\partial} f,$$

where we will denote by $\partial f$ the component of $df$ of type (1,0) and by $\bar{\partial} f$ the component of $df$ of type (0,1); in fact, in general, we say that differential forms are of type (1,0) when they are combinations of the $dz_j$'s and differential forms are of type (0,1) when they are combinations of the $d\bar{z}_j$'s.

Let's define holomorphic and separately holomorphic functions: these, in fact, are the only notions we will need for the next two sections of the chapter.

**Definition 2.1.1.** *A function $f$, defined on a domain $\Omega$ in $\mathbb{C}^n$, is holomorphic if it is $C^1$ and satisfies the differential Cauchy Riemann system $\partial_{\bar{z}_j} f = 0$, $\forall j = 1, ..., n$.*

**Definition 2.1.2.** *A function $f$, defined on a domain $\Omega$ in $\mathbb{C}^n$, is separately holomorphic if it satisfies the differential Cauchy Riemann system $\partial_{\bar{z}_j} f = 0$, $\forall j = 1, ..., n$.*

We will denote the space of holomorphic functions on $\Omega$ with $\mathcal{O}(\Omega)$ or $hol(\Omega)$. For holomorphic functions, $df$, which is in general $\mathbb{R}$-linear, becomes $\mathbb{C}$-linear and, if we write $f = \mathsf{Re}\, f + i\mathsf{Im}\, f$, then the Cauchy Riemann system becomes a system of 2n real equations in this way

$$\partial_{\bar{z}_j} f = 0 \quad \Rightarrow \quad \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})(\mathsf{Re}\, f + i\mathsf{Im}\, f) \quad \Rightarrow \quad \begin{cases} \partial_{x_j}\mathsf{Re}\, f - \partial_{y_j}\mathsf{Im}\, f = 0 \\ \partial_{y_j}\mathsf{Re}\, f + \partial_{x_j}\mathsf{Im}\, f = 0. \end{cases}$$

It is obvious that a holomorphic function is separately holomorphic; what will be more interesting is to show that the hypothesis of separate analyticity is sufficient to conclude that the function is $C^1$, so jointly holomorphic in all variables.

To have the basic notations for the setting of separately CR functions, that will be the theme of this chapter, let's recall what we mean by complex structure on $T\mathbb{C}^n$ and then, let's recall the definition of CR manifold of $\mathbb{C}^n$.

Note that, given any complex manifold $X$, it is possible to define a complex structure $\mathcal{J}$ on $TX$, just considering for the real underlying manifold $X^{\mathbb{R}}$ the morphism induced on $TX^{\mathbb{R}}$ by the multiplication by $i$ on $TX$. For $X = \mathbb{C}^n$ the definition is the following

**Definition 2.1.3.** *A complex structure on $T\mathbb{C}^n$ is the morphism $\mathcal{J}$ induced on $T\mathbb{R}^{2n}$ by the multiplication by $i$ on $T\mathbb{C}^n$ (that implies $\mathcal{J}^2 = -Id$, as the real counterpart of $i^2 = -1$)*

$$\begin{array}{ccc} T\mathcal{X} & \xrightarrow{i} & T\mathcal{X} \\ || & & || \\ T\mathcal{X}^{\mathbb{R}} & \xrightarrow{\mathcal{J}} & T\mathcal{X}^{\mathbb{R}}. \end{array}$$

In local coordinates on $T\mathbb{R}^{2n}$ it is defined by

$$\mathcal{J}\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad \mathcal{J}\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, \qquad 1 \le j \le n$$

and $\mathcal{J}$ extends as a $\mathbb{C}$-linear operator on the complexification of $T\mathbb{R}^{2n}$ (that is $T\mathbb{C}^n$)

$$\mathcal{J}\left(\frac{\partial}{\partial z_j}\right) = i\frac{\partial}{\partial z_j}, \quad \mathcal{J}\left(\frac{\partial}{\partial \bar{z}_j}\right) = -i\frac{\partial}{\partial \bar{z}_j}, \qquad 1 \le j \le n$$

having eigenvalues $+i$ and $-i$ with corresponding eigenspaces denoted

$$T^{1,0}(\mathbb{C}^n) := Span\{\partial_{z_j}\} \quad \text{and} \quad T^{0,1}(\mathbb{C}^n) := Span\{\partial_{\bar{z}_j}\}$$

$$T^{1,0}(\mathbb{C}^n) \cap T^{0,1}(\mathbb{C}^n) = \{0\}, \quad T^{1,0}(\mathbb{C}^n) \oplus T^{0,1}(\mathbb{C}^n) = (T\mathbb{R}^{2n}) \otimes \mathbb{C}.$$

Given a smooth submanifold $M$ of $\mathbb{C}^n$ and its real tangent space $T_zM$ at a point $z \in M$, it is evident that $T_zM$ is not invariant, in general, under the complex structure $\mathcal{J}$ for $T_z(\mathbb{C}^n)$, so it has meaning to look for the largest $\mathcal{J}$-invariant subspace of $T_zM$.

**Definition 2.1.4.** *For a point $z \in M$, the complex tangent space of $M$ at $z$ is the vector space $T_z^{\mathbb{C}}M = T_zM \cap \mathcal{J}T_zM$.*

**Remark 2.1.1.** Note that the real dimension of $T_z^{\mathbb{C}}M$ must be even because

$$\mathcal{J} \circ \mathcal{J}_{|_{T_z^{\mathbb{C}}M}} = -\mathrm{Id}$$

which implies

$$\left[\det \mathcal{J}_{|_{T_z^{\mathbb{C}}M}}\right]^2 = (-1)^m, \qquad \text{for } m = \dim_{\mathbb{R}} T_z^{\mathbb{C}}M;$$

then $m$ has to be even by the positivity of the left hand side of the last equality.

**Remark 2.1.2.** If $M$ is a real submanifold of $\mathbb{C}^n$ with $\dim_{\mathbb{R}} M = 2n - d$, then

$$2n - 2d \overset{(i)}{\leq} \dim_{\mathbb{R}} T_z^{\mathbb{C}}M \overset{(ii)}{\leq} 2n - d;$$

in fact, (ii) is given by the obvious inclusion $T_z^{\mathbb{C}}M \subset T_zM$, while, for (i), we note that

$$T_zM + \mathcal{J}T_zM \subset T_z\mathbb{R}^{2n} \quad \Rightarrow \quad \dim_{\mathbb{R}}(T_zM + \mathcal{J}T_zM) \leq 2n$$

and linear algebra gives $\dim_{\mathbb{R}}(T_zM \cap \mathcal{J}T_zM) = \dim_{\mathbb{R}} T_zM + \dim_{\mathbb{R}} \mathcal{J}T_zM - \dim_{\mathbb{R}}(T_zM + \mathcal{J}T_zM)$; then, being $\mathcal{J}$ an isometry,

$$\begin{aligned} \dim_{\mathbb{R}} T_z^{\mathbb{C}}M &\geq (2n - d) + (2n - d) - 2n \\ &\geq 2n - 2d. \end{aligned}$$

While $TM$ is a fiber bundle, not always this happens for $T^{\mathbb{C}}M$: when this is true, we get a CR manifold.

**Definition 2.1.5.** *$M$ is called a CR submanifold of $\mathbb{C}^n$ when the rank of $T_z^{\mathbb{C}}M$ is constant, not depending on the point $z \in M$ (that is when $T^{\mathbb{C}}M$ is a bundle).*

We refer to the $\mathrm{rank}_{\mathbb{C}}$ of $T^{\mathbb{C}}M$ as the CR dimension of $M$ and we will denote it by $\dim_{CR} M$. Two different situations can be distinguished among the class of CR manifolds

**Definition 2.1.6.** *A submanifold $M$ of $\mathbb{C}^n$ is called totally real if $T_z^{\mathbb{C}}M = \{0\}$, for every $z \in M$.*

**Definition 2.1.7.** *A submanifold $M$ of $\mathbb{C}^n$ is called generic if $T_zM + \mathcal{J}T_zM = T_z\mathbb{C}^n$, for every $z \in M$.*

Let's recall some equivalent characterizations of a generic manifold of $\mathbb{C}^n$ and provide good examples for all these definitions

**Proposition 2.1.1.** *Let $M \subset \mathbb{C}^n$ be a CR manifold with $dim_{\mathbb{R}} M = 2n-d$, $0 \le d \le n$. The following are equivalent*

*(i)* *$M$ is generic*

*(ii)* *$dim_{\mathbb{R}} T_z^{\mathbb{C}} M$ is minimal, that is $2n-2d$, for $z \in M$*

*(iii)* *there is a system of equations for $M$, $\rho_1 = 0, \ldots, \rho_d = 0$, such that $\partial\rho_1 \wedge \ldots \wedge \partial\rho_d \neq 0$ on $M$*

*(iv)* *$dz_{1|_M}, \ldots, dz_{n|_M}$ are independent*

*(v)* *there are local coordinates $z = x + iy$, $z = (z', z'')$, $z' = (z_1, \ldots, z_d)$ such that $z_0 = 0$ and $M$ is graphed over $x', z''$ by*

$$y_j = h_j(x', z'') \quad \textit{with } h_j(0) = 0,\ dh_j(0) = 0,\ \textit{for } j = 1, \ldots, d$$

*(vi)* *there is $N \subset M$ totally real maximal.*

**Examples 2.1.1** (CR manifolds)**.** In $\mathbb{C}^n$, any complex submanifold is a CR submanifold, because, for a complex submanifold $M$, the real tangent space is already $\mathcal{J}$-invariant, so $T_z^{\mathbb{C}} M \equiv T_z M$. Note that from the opposite side than complex manifolds, but always remaining in the class of CR manifolds, we find totally real manifolds, for which $T_z^{\mathbb{C}} M \equiv \{0\}$, $\forall z \in M$.

Every real hypersurface $S$ of $\mathbb{C}^n$ is a CR submanifold: this follows immediately by Remark 2.1.1 and 2.1.2, because $\dim_{\mathbb{R}} T_z^{\mathbb{C}} S$ has to be an even number between $2n-2d$ and $2n-d$, and $d = 1$ for a real hypersurface, which implies, as the only possibility, $\dim_{\mathbb{R}} T_z^{\mathbb{C}} S = 2n-2$, independently from the point $z \in S$.

Let's consider the manifold $M = \{z \in \mathbb{C}^n : |z| = 1, \mathsf{Im}\, z_1 = 0\}$, which is the equator of the unit sphere in $\mathbb{C}^n$: just rewriting it in real coordinates as $M = \{(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n} : \rho_1(x,y) = x_1^2 + y_1^2 + \ldots + x_n^2 + y_n^2 - 1 = 0,\ \rho_2(x,y) = y_1 = 0\}$, we immediately calculate its real tangent space (of real codimension 2) at $p$

$$\begin{aligned} T_p M &= \{v = (a_1, b_1, \ldots, a_n, b_n) \in T\mathbb{R}^{2n} : \langle \partial\rho_1(p), v\rangle = 0, \langle \partial\rho_2(p), v\rangle = 0\} \\ &= \{(a_1, b_1, \ldots, a_n, b_n) : 2x_{1|_p} a_1 + 2y_{1|_p} b_1 + \ldots + 2x_{n|_p} a_n + 2y_{n|_p} b_n = 0; b_1 = 0\}. \end{aligned}$$

For $p = (\pm 1, 0, \ldots, 0) \in M$

$$\begin{aligned} T_p M =& \{v \in T\mathbb{R}^{2n} : a_1 = 0, b_1 = 0\} \\ =& \mathrm{Span}_{\mathbb{R}}\{\partial_{x_2}, \partial_{y_2}, \ldots, \partial_{x_n}, \partial_{y_n}\} \end{aligned}$$

and this subspace is $\mathcal{J}$-invariant, so $\dim_{\mathbb{R}} T_p^{\mathbb{C}} M = \dim_{\mathbb{R}} T_p M = 2n-2$. Instead, when $p \neq (\pm 1, 0, \ldots, 0)$, the dimension of $T_p^{\mathbb{C}} M$ decreases. Take for instance $p = (0, 0, x_2 = 1, 0, \ldots, 0) \in M$; then,

$$\begin{aligned} T_p M =& \{v \in T\mathbb{R}^{2n} : a_2 = 0, b_1 = 0\} \\ =& \mathrm{Span}_{\mathbb{R}} \{\partial_{x_1}, \partial_{y_2}, \partial_{x_3}, \partial_{y_3}, \ldots, \partial_{x_n}, \partial_{y_n}\} \end{aligned}$$

and, applying the map $\mathcal{J}$, $T_p M$ is no more $\mathcal{J}$-invariant because

$$\begin{aligned} \mathcal{J}(\partial_{x_1}) =& \partial_{y_1} \notin T_p M, \\ \mathcal{J}(\partial_{y_2}) =& -\partial_{x_2} \notin T_p M; \end{aligned}$$

$\partial_{x_1}$ and $\partial_{y_2} \notin T_p^{\mathbb{C}} M$ and $\dim_{\mathbb{R}} T_p^{\mathbb{C}} M = 2n-4$, so the equator of the unit sphere is not a CR submanifold of $\mathbb{C}^n$.

**Examples 2.1.2** (Totally real and generic manifolds)**.** In $\mathbb{C}^2$: $\mathbb{R} \times \mathbb{R}$ is a totally real submanifold (as any smooth graph over this copy of $\mathbb{R}^2$) and it is generic, $\mathbb{R} \times \{0\}$ is totally real (not generic), $\mathbb{R} \times \mathbb{C}$ and $\mathbb{C} \times \mathbb{C}$ are generic (not totally real), while $\mathbb{C} \times \{0\}$ is complex non-generic.

In $\mathbb{C}^n$ a real hypersurface $S$ is always generic (because we have already noticed, in the previous list of examples, that $\dim_{\mathbb{R}} T_z^{\mathbb{C}} S$ has to be equal to $2n-2$, which means the minimal dimension for $T_z^{\mathbb{C}} S$, and then $S$ is generic by $(ii)$ of Proposition 2.1.1). Using again $(ii)$ of Proposition 2.1.1, we can also deduce that a generic CR submanifold of $\mathbb{C}^n$, with real codimension $d$ at least $n$, must be totally real.

Given a CR manifold, we define the fiber tangent bundles of type $(1,0)$ and $(0,1)$ in this way

$$\begin{aligned} T^{1,0} M :=& T^{1,0} \mathbb{C}^n \cap (\mathbb{C} \otimes TM), \\ T^{0,1} M :=& T^{0,1} \mathbb{C}^n \cap (\mathbb{C} \otimes TM). \end{aligned}$$

Let $M$ be a generic CR submanifold of $\mathbb{C}^n$, defined by a system $\rho_1 = 0, \ldots, \rho_d = 0$ of independent equations. We introduce the notion of CR function on $M$ as solution of the tangential Cauchy Riemann system, which provides these two equivalent definitions

**Definition 2.1.8.** *A $C^1$ function $f : M \to \mathbb{C}$ is CR if $\bar{L} f = 0$, for every $\bar{L} \in T^{0,1} M$.*

**Definition 2.1.9.** *A $C^1$ function $f : M \to \mathbb{C}$ is CR if $\bar{\partial} \widetilde{f} \wedge \bar{\partial} \rho_1 \wedge \ldots \wedge \bar{\partial} \rho_d = 0$ on $M$, where $\widetilde{f} : \mathbb{C}^n \to \mathbb{C}$ is any $C^1$ extension of $f$.*

CR functions on CR manifolds are analogous to holomorphic functions on complex manifolds, though there are relevant differences, as the fact that CR functions are not always smooth. For the analogies, the first is that the restriction of a holomorphic function to a CR submanifold is a CR function; in particular, if $M = \mathbb{C}^n$, holomorphic functions are CR, while, for the converse, we need $M$ and $f$ to be $C^\omega$. In general, the class of CR functions is strictly larger than the class of restrictions of holomorphic functions; in other terms, CR functions do not always extend as holomorphic functions.

**Examples 2.1.3.** If we consider in $\mathbb{C}^2$ the manifold $M = \mathbb{R} \times \mathbb{R}$ (totally real and generic), then $T^{0,1}M = \{0\}$, so all the functions $f(x_1, x_2)$ of class $C^1$ on $M$ are CR. Taking $M = \mathbb{R} \times \mathbb{C}$ as a generic submanifold of $\mathbb{C}^2$, the dimension of $T^{0,1}M$ increases, in fact it is spanned by the vector field $\partial_{\bar{z}_2}$. Therefore every function $f(x_1, z_2)$ of class $C^1$ on $M$, which satisfies $\partial_{\bar{z}_2} f = 0$, is CR. These functions are separately holomorphic in $z_2$ and the holomorphic extension needs $f$ to be $C^\omega$. Finally, for $M = \mathbb{C} \times \mathbb{C}$ (complex) in $\mathbb{C}^2$, $T^{0,1}M$ is spanned by $\partial_{\bar{z}_1}$ and $\partial_{\bar{z}_2}$, thus in this case CR functions and holomorphic functions coincide.

## 2.2 Properties of holomorphic and separately holomorphic functions

There are equivalent characterizations for holomorphic functions defined on open sets of $\mathbb{C}^n$; for example, they can be represented locally as sums of convergent power series. It is obvious that a holomorphic function of several complex variables is separately holomorphic in each variable. Just reasoning on separated variables, a lot of the well-known properties of holomorphic functions of one complex variable, as integral Cauchy formula, have a corresponding version in several complex variables; for this technique of separation of variables, the function needs to be continuous.

In this section we present some properties of holomorphic functions of several complex variables and get two first results on separate analyticity with the hypothesis of continuity and boundedness on compacts.

Let us begin with Cauchy integral formula on polydiscs; this is the generalization of Cauchy formula on the complex plane, which is a consequence of Stokes formula. In fact, for $f \in C^1(\bar{\Omega})$, where $\Omega$ is a bounded open set of $\mathbb{C}$ with piecewise $C^1$ boundary (that is $\partial\Omega$ consists of a finite number of $C^1$ Jordan curves), Stokes formula says that

$$\int_{\partial\Omega} f d\zeta = \int\int_{\Omega} df \wedge d\zeta = \int\int_{\Omega} \frac{\partial f}{\partial \bar{\zeta}} d\bar{\zeta} \wedge d\zeta.$$

If $f$ is $C^1(\bar{\Omega})$ and $f$ is analytic in $\Omega$, then $\int_{\partial\Omega} f d\zeta = 0$. On the other hand, Stokes formula applied to the function $f(\zeta)/(\zeta - z)$, after considering a little ball in $\Omega$ centered in $z$, yields for $f \in C^1(\bar{\Omega})$

$$f(z) = (2\pi i)^{-1} \left\{ \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta + \int\int_{\Omega} \frac{\partial_{\bar{\zeta}} f}{\zeta - z} d\zeta \wedge d\bar{\zeta} \right\},$$

which is precisely Cauchy integral formula on the complex plane. Here we present the generalization on polydiscs: the $C^1$-regularity in each variable is needed for Stokes formula and is guaranteed by the hypothesis of separate analyticity, while the joint $C^0$-regularity is needed for Fubini Theorem.

**Theorem 2.2.1** (Cauchy integral formula on polydiscs)**.** *Let $f$ be a continuous function on the closure of a polydisc*

$$P = P(z_0, r) = \prod_{j=1}^{n} \{z_j \in \mathbb{C} : |z_j - z_{0,j}| < r_j\} \subset \mathbb{C}^n \qquad \text{for } z_0 \in \mathbb{C}^n, r \in (\mathbb{R}^+)^n;$$

*let $f$ be, for any $j$, a holomorphic function of $z_j$, when the other variables $z_k$, for $k \neq j$, are kept fixed. Then, we have*

$$f(z) = (2\pi i)^{-n} \int_{\partial_0 P} \frac{f(\zeta)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \wedge ... \wedge d\zeta_n \qquad \text{for } z \in P, \tag{2.7}$$

*where $\partial_0 P(z_0, r) = \{\zeta \in \mathbb{C}^n : |\zeta_j - z_{0,j}| = r_j, 1 \leq j \leq n\}$.*

**Remark 2.2.1.** Note that, in case $n > 1$, the region of integration $\partial_0 P$ in (2.7) is strictly smaller than the topological boundary $\partial P$ of $P$, which is stratified by strata of different dimensions, depending on the number of equations of type $|\zeta_j - z_{0,j}|$ we want to consider. The stratum of codimension $n$ is $\partial_0 P$, also called the distinguished or Shilov boundary of $P$, which plays in many situations the same role of the unit circle in one complex variable (for example in maximum principle).

*Proof of Theorem 2.2.1.* To simplify notation we may assume that $z_{0,j} = 0$ and $r_j = 1\ \forall j$, so that $P = D_1 \times \ldots \times D_n$ and $\partial_0 P = \partial D_1 \times \ldots \times \partial D_n$, where $D_j$ are the standard unit discs, $\forall j = 1, \ldots, n$.

We prove the result by induction on the dimension $n$ of the space. For $n = 1$ it is Cauchy formula on the complex plane; we suppose the result true for holomorphic functions of $n - 1$ variables and prove it in the case of $n$ variables. Given $P' = D_1 \times \ldots \times D_{n-1}$ and $z_n \in D_n$, we consider the function $(z_1, \ldots, z_{n-1}) \longmapsto f(z_1, \ldots, z_{n-1}, z_n)$ and use the inductive hypothesis

$$f(z_1, \ldots, z_n) = (2\pi i)^{-(n-1)} \int_{\partial_0 P'} \frac{f(\zeta_1, \ldots, \zeta_{n-1}, z_n)}{(\zeta_1 - z_1) \cdots (\zeta_{n-1} - z_{n-1})} d\zeta_1 \wedge \ldots \wedge d\zeta_{n-1}.$$

We apply Cauchy integral formula to the function at the numerator

$$f(\zeta_1, \ldots, \zeta_{n-1}, z_n) = (2\pi i)^{-1} \int_{\partial D_n} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_n - z_n)} d\zeta_n$$

and substitute it in the previous expression. The function $f$ is jointly continuous so it is possible to apply Fubini's Theorem and the expected conclusion holds. $\square$

As a corollary of Cauchy integral formula on polydiscs, we get the first result on separate analyticity with the hypothesis of continuity.

**Corollary 2.2.1.** *If $f$ is $C^0(\Omega)$ and separately holomorphic in each $z_j$, when the other variables are fixed, then $f$ is $C^\infty(\Omega)$. (In particular $f$ is $C^1$ and then holomorphic on $\Omega$.)*

*Proof.* It suffices to consider Cauchy formula on a polydisc contained in $\Omega$; the integrand

$$\frac{f(\zeta)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}$$

is a $C^\infty$ function when $(\zeta, z) \in \partial_0 P \times P$ (because the denominator does not vanish) and it is analytic in $z \in P$ (because it is $C^1$ in $z$ and separately holomorphic in each $z_j$); therefore, we can derive under the integral sign as often as needed, and then $f \in C^\infty(\Omega)$, because all the derivatives of $f(z)$ exist and are continuous. $\square$

Note that the proof shows how a holomorphic function is always of class $C^\infty$ and says that all the derivatives of holomorphic functions are holomorphic functions; this fact was already known in the setting of one complex variable.

For a holomorphic function of one and several complex variables, a relevant property - direct consequence of Cauchy integral formula - consists in the possibility to estimate the derivatives in terms of the function itself; these are the well-known Cauchy inequalities.

We introduce first the notions of multiindices, multipowers and multiderivatives: for a multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we put $\alpha! = \alpha_1! \cdots \alpha_n!$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\alpha + 1 = (\alpha_1 + 1, \dots, \alpha_n + 1)$; we also define multipowers by $r^\alpha = r_1^{\alpha_1} \cdots r_n^{\alpha_n}$ and multiderivatives by $f^{(\alpha)} = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n}$.

**Theorem 2.2.2** (Cauchy inequalities)**.** *Let $f$ be holomorphic on $P = P(z_0, r)$ and continuous on $\bar{P}$. Then,*

$$|f^{(\alpha)}(z_0)| \le \frac{\alpha!}{r^\alpha} \sup_{\partial_0 P(z_0,r)} |f| \tag{2.8a}$$

$$|f^{(\alpha)}(z_0)| \le C_{\alpha,r} \|f\|_{L^1(P(z_0,r))} \tag{2.8b}$$

*Proof.* We fix $0 < \rho < r$, apply Cauchy formula to $f$ on $P(z_0, \rho) \subset\subset P(z_0, r)$ and derive under integral sign, to get the expression for the derivatives of $f$

$$f^{(\alpha)}(z_0) = \frac{\alpha!}{(2\pi i)^n} \int_{\partial_0 P(z_0,\rho)} \frac{f(\zeta)}{(\zeta - z_0)^{\alpha+1}} d\zeta_1 \wedge \dots \wedge d\zeta_n. \tag{2.9}$$

We estimate the absolute value of $f^{(\alpha)}(z_0)$, after the change to polar coordinates $\zeta_j = z_{0,j} + \rho_j \, e^{i\,\theta_j}$,

$$\begin{aligned} |f^{(\alpha)}(z_0)| &\le \frac{\alpha!}{(2\pi i)^n} \int_{[0,2\pi]^n} \left| \frac{f(z_0 + \rho\, e^{i\theta})}{(\rho\, e^{i\theta})^{\alpha+1}} i^n \rho\, e^{i\theta} \right| d\theta_1 \wedge \dots \wedge d\theta_n \\ &\le \frac{\alpha!}{(2\pi)^n \rho^\alpha} \int_{[0,2\pi]^n} |f(z_0 + \rho\, e^{i\theta})| d\theta_1 \wedge \dots \wedge d\theta_n. \end{aligned} \tag{2.10}$$

Taking the limit, for $\rho \to r$, (2.8a) follows; to get (2.8b), we need to integrate (2.10) over $0 \le \rho_j \le r_j$, for $1 \le j \le n$, and to transform that integral, expressed in polar coordinates, into a volume integral. □

**Remark 2.2.2.** The previous proof lets us estimate the derivatives of $f$ also for every point $z \in \overline{P(z_0, r - \rho)}$, for $0 < \rho < r$, in this way

$$|f^{(\alpha)}(z)| \le \frac{r\alpha!}{\rho^{\alpha+1}} \sup_{\partial_0 P(z_0,r)} |f|$$

because, using (2.9) to define $f^{(\alpha)}(z)$, it is sufficient to note that $|\zeta - z| \ge \rho$, for $z \in \overline{P(z_0, r - \rho)}$.

**Remark 2.2.3.** We can also obtain, by the previous theorem, the following useful estimate: for every compact set $K \subset \Omega$ and every open neighbourhood $U \subset \Omega$ of $K$, there are constants $c_\alpha$ such that, for every holomorphic function $f$ on $\Omega$,

$$\sup_{z \in K} |f^{(\alpha)}(z)| \le c_\alpha \|f\|_{L^1(U)}$$

in fact, taking $d = \frac{1}{2}\min_{z\in K} \operatorname{dist}(z, \partial U)$, we can say that $d > 0$ because the distance from a closed set is a continuous function and $\overline{P(z_0, d)} \subset U$, for $z_0 \in K$, so Cauchy estimates provide

$$|f^{(\alpha)}(z_0)| \leq \frac{\alpha!}{d^\alpha} \sup_{\partial_0 P(z_0,d)} |f| \leq \frac{\alpha!}{d^\alpha} \sup_U |f|;$$

passing to the sup, for $z_0 \in K$, the result follows.

The previous theorems and remarks are useful to study the convergence of sequences of holomorphic functions. The first result, due originally to Weierstrass, if referred in topological terms, says that the space of holomorphic functions is closed in the space of continuous functions on an open set $\Omega$ of $\mathbb{C}^n$, with the topology of uniform convergence on compacts.

**Proposition 2.2.1.** *Let $\{f_n\}$ be a sequence of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}^n$, which is uniformly convergent, on compact sets of $\Omega$, to a function $f$; then, $f$ is holomorphic and the sequences $\{\partial^\alpha f_n\}$ converge uniformly on compact sets of $\Omega$ to $\partial^\alpha f$, $\forall \alpha \in \mathbb{N}^n$.*

*Proof.* An application of Remark 2.2.3 to $f_n - f_m$ shows that $\partial_{z_j} f_n$ converges uniformly on compact sets; being $\{f_n\}$ holomorphic, and using the correspondences of derivatives $\partial_{x_j} = \partial_{z_j} + \partial_{\bar{z}_j}$, $\partial_{y_j} = i(\partial_{z_j} - \partial_{\bar{z}_j})$, we get that $\{\partial_{x_j} f_n\}$ and $\{\partial_{y_j} f_n\}$ also converge uniformly on compact sets. We also know that $\{f_n\} \in C^1(\Omega)$ because $\{f_n\} \in \mathcal{O}(\Omega)$. Using the theorem of passage of the limit under derivative sign, we conclude that $f \in C^1(\Omega)$ and $\partial_{\bar{z}_j} f = \partial_{\bar{z}_j} \lim_n f_n = \lim_n \partial_{\bar{z}_j} f_n = 0$, so $f \in \mathcal{O}(\Omega)$. □

Let's remember that a sequence $\{f_n\}$ is uniformly bounded on $\Omega$ if there exists a constant $M$ such that, $\forall n$ and $\forall z \in \Omega$, $|f_n(z)| \leq M$, while $\{f_n\}$ is equicontinuous if $\forall \epsilon > 0$ $\exists \delta > 0$ such that, if $z, w \in \Omega$ and $|z - w| < \delta$, then, for every $n$, $|f_n(z) - f_n(w)| < \epsilon$. The well-known Ascoli-Arzelà Theorem connects these two definitions.

**Theorem 2.2.3** (Ascoli-Arzelà ). *Let $\{f_n\}$ be a sequence of complex functions defined on $\Omega \subseteq \mathbb{C}^n$; if $\{f_n\}$ is uniformly bounded and equicontinuous, then there exists a subsequence $\{f_{n_k}\}$ converging to a continuous function $f$.*

To apply it to the analysis of sequences of holomorphic functions, we need the following lemma whose proof is immediate

**Lemma 2.2.1.** *Let $\{f_n\}$ be a sequence of complex functions on $\Omega \subseteq \mathbb{C}^n$ such that $\{\partial^\alpha f_n\}$ is uniformly bounded on every compact set of $\Omega$. Then, $\{f_n\}$ is equicontinuous on compact sets of $\Omega$.*

*Proof.* It is an argument of compactness, jointed to

$$|f_n(z) - f_n(w)| \leq |z - w| \sup_K |\partial^\alpha f_n| \leq M|z - w|, \qquad z, w \in K.$$

□

**Theorem 2.2.4** (Stieltjes-Vitali). *Let $\{f_n\}$ be a sequence of holomorphic functions on $\Omega \subseteq \mathbb{C}^n$, uniformly bounded on compact sets of $\Omega$; then, there exists a subsequence $\{f_{n_k}\}$ uniformly convergent on compact sets of $\Omega$, and its limit is holomorphic.*

*Proof.* Applying Remark 2.2.3, we get that, if $\{f_n\}$ is uniformly bounded on compact sets of $\Omega$, also $\{\partial^\alpha f_n\}$ has the same property; then, using Lemma 2.2.1, $\{f_n\}$ is equicontinuous on compact sets of $\Omega$ and the result follows by Ascoli-Arzelà Theorem. □

We denote by "normal family" a subset $\mathcal{F} \subset C^0(\Omega)$ such that every sequence $\{f_n\} \subset \mathcal{F}$ admits a subsequence uniformly convergent on compact sets to a function $f \in C^0(\Omega)$ (equivalently $\mathcal{F}$ is a "normal family" if and only if the closure of $\mathcal{F}$ in $C^0(\Omega)$ is compact with the topology of uniform convergence on compacts).

Stieltjes-Vitali Theorem provides the following characterization of normal families of holomorphic functions

**Proposition 2.2.2.** *A family $\mathcal{F}$ of holomorphic functions, defined on an open set $\Omega$ of $\mathbb{C}^n$ is a "normal family" if and only if it is uniformly bounded on compact sets of $\Omega$.*

Now we are able to present the second result on separate analyticity with the hypothesis of boundedness on compact sets of $\Omega$ (that relaxes the requirement of continuity of Corollary 2.2.1 for the function $f$).

**Theorem 2.2.5.** *If $f$ is separately holomorphic on $\Omega$ and bounded on compact sets of $\Omega$, then $f$ is holomorphic on $\Omega$.*

*Proof.* We want to prove that $f$ is continuous; after considering a point $z_0 \in \Omega$ and $z = (z_1, z')$ as coordinates in $\mathbb{C}^n$, let $z'_j$ be a sequence in $\mathbb{C}^{n-1}$ converging to $z'_0$ and let $z_1$ move near $z_1^0$ so that $\operatorname{dist}((z_1, z'_j), \partial\Omega) > r$, for $r \in \mathbb{R}^+$.

It is clear that in the $r-$neighbourhood of those points $f$ is uniformly bounded. Let's define

$$F_j(z_1) := f(z_1, z'_j) - f(z_1, z'_0).$$

As a function of the single variable $z_1$, $F_j$ is holomorphic because, by hypothesis, $f$ is separately holomorphic. $\{F_j\}$ is uniformly bounded on compact sets of $\Omega$ by the assumptions on $f$; then, by Stieltjes-Vitali Theorem (which is an application of Remark 2.2.3, Lemma 2.2.1 and Ascoli-Arzelà Theorem), there exists a subsequence $\{F_{j_k}\}$, uniformly convergent on compact sets of $\Omega$, whose limit is holomorphic

$$F_{j_k}(z_1) \to g(z_1) \in \mathcal{O}(\Omega);$$

$g(z_1) \equiv 0$ because there is a pointwise convergence to 0 by the separate continuity of $f$. If we take $\{z_1\}_j$ convergent to $z_1^0$, we finally have

$$\lim_j f((z_1)_j, z'_j) = f(z_1^0, z'_0),$$

which gives the continuity of $f$. □

The space of holomorphic functions on an open set $\Omega$ of $\mathbb{C}^n$ can also be defined in terms of power series: we want to explain and motivate this characterization, in order to appreciate the second part of Hartogs Theorem in the fourth section. Before stating the main result, we recall the definition of normal convergence. We will use the previously defined notions of multiindices, multipowers and multiderivatives.

**Definition 2.2.1.** *The power series* $\sum_{\alpha\in\mathbb{N}^n} c_\alpha(z-z_0)^\alpha$ *is normally convergent on an open set* $\Omega$ *if, given* $K \subset\subset \Omega$ *and* $\epsilon > 0$ *arbitrary, there exists* $\alpha_0$ *such that for* $\alpha \geq \alpha_0$ *the power series* $\sum_{\alpha\geq\alpha_0} \sup_{\alpha\in\mathbb{N}^n} |c_\alpha(z-z_0)^\alpha|$ *is convergent.*

**Remark 2.2.4.** The domain of convergence $D$ of the power series $\sum_{\alpha\in\mathbb{N}^n} c_\alpha(z-z_0)^\alpha$ is the interior of the set $\{w \in \mathbb{C}^n : \sup_{\alpha\in\mathbb{N}^n} |c_\alpha(w-z_0)|^\alpha = M < \infty\}$ and, as an application of Abel's Lemma, the convergence is normal in $D$.

**Theorem 2.2.6.** *The following results hold:*

(a) *A power series* $f(z) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha(z-z_0)^\alpha$ *with nonempty domain of convergence* $D$ *defines a holomorphic function* $f \in \mathcal{O}(D)$

(b) *Let* $f \in \mathcal{O}(\Omega)$*, for* $\Omega$ *an open set of* $\mathbb{C}^n$*; then, for every* $z_0 \in \Omega$*, the Taylor series of* $f$ *at* $z_0$ *converges to* $f$ *on each polydisc* $P(z_0, r) \subset\subset \Omega$*.*

*Proof.* The first statement is just a remark but has to be noted.

(a) A function defined by a uniformly convergent complex power series is of course holomorphic because such a series is the limit, uniformly on compact sets, of its partial sums (which are plainly holomorphic) and we can apply Proposition 2.2.1.

(b) The Taylor series of $f$ at $z_0 = (z_1^0, \ldots, z_n^0)$ is

$$\sum_{\alpha\in\mathbb{N}^n} \frac{f^{(\alpha)}(z_0)}{\alpha!}(z-z_0)^\alpha; \tag{2.11}$$

we want to prove that, for each polydisc, $f$ coincides with (2.11). In Cauchy integral formula applied to $z \in P(z_0,\rho) \subset\subset P(z_0, r)$, we expand $(\zeta - z)^{-1} = (\zeta - z_0 - (z-z_0))^{-1} = (\zeta_1 - z_1^0 - (z_1 - z_1^0))^{-1} \cdots (\zeta_n - z_n^0 - (z_n - z_n^0))^{-1}$ into a multiple geometric series

$$\begin{aligned}
\frac{1}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} &= \frac{1}{(\zeta_1 - z_1^0 - (z_1 - z_1^0))\cdots(\zeta_n - z_n^0 - (z_n - z_n^0))} \\
&= \frac{1}{(\zeta_1 - z_1^0)\cdots(\zeta_n - z_n^0)\left(1 - \frac{z_1 - z_1^0}{\zeta_1 - z_1^0}\right)\cdots\left(1 - \frac{z_n - z_n^0}{\zeta_n - z_n^0}\right)} \\
&= \sum_{\alpha\in\mathbb{N}^n} \frac{(z-z_0)^\alpha}{(\zeta - z_0)^{\alpha+1}}
\end{aligned} \tag{2.12}$$

that converges normally for $\zeta \in \partial_0 P(z_0,\rho)$, since

$$\frac{|z_j - z_{0,j}|}{|\zeta_j - z_{0,j}|} = \frac{|z_j - z_{0,j}|}{\rho_j} < 1, \qquad z \in P(z_0,\rho),\ 1 \le j \le n$$

(an obvious application of the general $\sum\limits_k z^k = \frac{1}{1-z}$ if $|z|<1$). We are therefore legitimated to substitute the expression (2.12) in Cauchy integral formula

$$\begin{aligned} f(z) &= \frac{1}{(2\pi i)^n}\int_{\partial_0 P} \frac{f(\zeta)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} d\zeta_1 \wedge ... \wedge d\zeta_n \qquad z \in P(z_0,\rho), \\ &= \frac{1}{(2\pi i)^n}\int_{\partial_0 P} \sum_{\alpha\in\mathbb{N}^n} \frac{(z-z_0)^\alpha}{(\zeta - z_0)^{\alpha+1}} f(\zeta) d\zeta_1 \wedge ... \wedge d\zeta_n \\ &= \sum_{\alpha\in\mathbb{N}^n} \left( \frac{1}{(2\pi i)^n}\int_{\partial_0 P} \frac{f(\zeta)}{(\zeta - z_0)^{\alpha+1}} d\zeta_1 \wedge ... \wedge d\zeta_n \right) (z-z_0)^\alpha \\ &= \sum_{\alpha\in\mathbb{N}^n} \frac{f^{(\alpha)}(z_0)}{\alpha!}(z-z_0)^\alpha. \end{aligned}$$

□

## 2.3 Hartogs Theorem

The aim of this section is to present a remarkable phenomenon of complex analysis, discovered by F. Hartogs in 1906 (cf. [28]): any function $f : \Omega \to \mathbb{C}$, which is separately holomorphic in each variable, is jointly holomorphic. This shows that the requirement that $f \in C^1(\Omega)$, as well as the relaxed hypothesis of continuity or boundedness on compacts for the function $f$, presented in the previous section, can be dropped in the definition of holomorphic function.

After what has been proved before, it is obvious that the main difficulty of Hartogs proof is to check that a separately holomorphic function is locally bounded.

Hartogs original proof, using a lemma of Osgood and some subharmonic function theory, was really ingenious and a bit difficult to understand deeply. No essentially simpler argument has been found, during all these years.

**Remark 2.3.1.** An old conjecture of Hervé was that a separately subharmonic function is subharmonic. This implication would give an easy and natural way to see that a separately holomorphic function is locally bounded, and then holomorphic. Unfortunately, it was discovered by Wiegerinck that Hervé conjecture is false.

Just keeping Hartogs sketch of proof, we have simplified it using a "propagation" argument, so that the way in which we prove Hartogs Theorem in this section becomes different from the classical proof. Our argument, which is contained in the paper [43], is useful for some applications and helps us to connect them to the main theorem of the fifth section, in the setting of separately CR functions.

**Remark 2.3.2.** In order to appreciate the strength of Hartogs Theorem, note that a corresponding result for real analytic functions is false, as the following example shows: the function

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto f(x,y) = \begin{cases} xy/(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases} \end{aligned}$$

is separately $C^\infty$, and even $C^\omega$, in $x$ (when $y$ is fixed) and in $y$ (when $x$ is fixed), $f$ is bounded on the plane because

$$(|x|-|y|)^2 \geq 0 \quad \Rightarrow \quad x^2+y^2-2|x||y| \geq 0 \quad \Rightarrow \quad \frac{|xy|}{x^2+y^2} \leq \frac{1}{2}$$

but $f$ is not continuous at (0,0), because the limit of $f$ along straight lines through the origin, as $(x,y) \to (0,0)$, does not exist.

Before our proof of Hartogs result, we briefly remind the basic elements of subharmonic function theory. We state the following definitions and properties for one complex variable, which is strictly the setting for this theory; then, we consider an open set $\Omega$ of $\mathbb{C}$.

**Definition 2.3.1.** *A real function $f \in C^2(\Omega)$ is harmonic if $\partial_z \partial_{\bar{z}} f = 0$.*

Note that $\partial_z \partial_{\bar{z}}$ coincides, up to a constant factor, with the well-known Laplace operator by

$$\partial_z \partial_{\bar{z}} = \frac{1}{4}(\partial_x^2 + \partial_y^2)$$

**Definition 2.3.2.** *A real function $f : \Omega \to [-\infty, +\infty)$ is upper semicontinuous if for every $z_0 \in \Omega$*

$$\limsup_{z \to z_0} f(z) \leq f(z_0)$$

*or, equivalently, if $\{z \in \Omega : f(z) < c\}$ is open, for every $c \in \mathbb{R}$.*

**Definition 2.3.3.** *A real function $f : \Omega \to [-\infty, +\infty)$ is subharmonic when*

*(i) $f$ is upper semicontinuous*

*(ii) for every $K \subset\subset \Omega$ and for every $\varphi$ continuous on $K$ and harmonic on $\overset{\circ}{K}$*

$$f|_{\partial K} \leq \varphi|_{\partial K} \quad \Rightarrow \quad f|_K \leq \varphi|_K.$$

The following proposition characterizes subharmonic functions and shows that subharmonicity is a local property.

**Proposition 2.3.1.** *Let $f : \Omega \to [-\infty, +\infty)$ be a upper semicontinuous function. The following are equivalent:*

*(i) $f$ is subharmonic*

*(ii) for any disc $\Delta \subset\subset \Omega$ and for any polynomial $P = P(z)$:*

$$f|_{\partial\Delta} \leq \mathsf{Re}\, P|_{\partial\Delta} \quad \Rightarrow \quad f|_{\bar{\Delta}} \leq \mathsf{Re}\, P|_{\bar{\Delta}}$$

*(iii) (1-dimensional submean) for any $z_0 \in \Omega$ there exists $0 < \widetilde{r} < d(z_0, \partial\Omega)$ such that, if $r < \widetilde{r}$, then*

$$f(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

*(iv) (2-dimensional submean) for any $z_0 \in \Omega$ there exists $0 < \widetilde{r} < d(z_0, \partial\Omega)$ such that, if $r < \widetilde{r}$, then*

$$f(z_0) \leq \frac{1}{\pi r^2} \int_0^{2\pi} \int_0^r \rho f(z_0 + \rho e^{i\theta}) d\theta d\rho.$$

In differential terms, subharmonicity can be described in terms of the Laplacian operator, for $f \in C^2(\Omega)$.

**Proposition 2.3.2.** *Let $f \in C^2(\Omega)$; then, $f$ is subharmonic if and only if $\partial_z \partial_{\bar{z}} f \geq 0$.*

The following results will be crucial in the proof of Hartogs Lemma.

**Proposition 2.3.3.** *If $f \in \mathcal{O}(\Omega)$, then* $\log |f|$ *is subharmonic on $\Omega$.*

*Proof.* Let $K$ be a compact of $\Omega$ and $P$ a polynomial such that

$$\log |f|_{|\partial K} \leq \mathsf{Re}\, P_{|\partial K}.$$

Applying "exp" and noting that $e^{\mathsf{Re}\, P} = |e^P|$ we get

$$|f|_{|\partial K} \leq |e^P|_{|\partial K} \quad \Rightarrow \quad \left| \frac{f}{e^P} \right|_{|\partial K} \leq 1.$$

By maximum principle for holomorphic functions, we can pass from $\partial K$ to $K$ in the previous estimate, from which, coming back with "log"

$$\log |f|_{|K} \leq \mathsf{Re}\, P_{|K};$$

the subharmonicity of $\log |f|$ derives from Proposition 2.3.1. □

**Proposition 2.3.4.** *If $f \in C^2(\Omega)$ is subharmonic and $\varphi : \mathbb{R} \to \mathbb{R}$ is $C^2(\mathbb{R})$, with $\dot{\varphi} \geq 0$ and $\ddot{\varphi} \geq 0$, then $\varphi \circ f$ is subharmonic.*

*Proof.* An easy calculation shows

$$\partial_z \partial_{\bar{z}} (\varphi \circ f) = \ddot{\varphi} \partial_z f \partial_{\bar{z}} f + \dot{\varphi} \partial_z \partial_{\bar{z}} f \geq 0.$$

□

**Proposition 2.3.5.** *Let $f \in \mathcal{O}(\Omega)$ and $\alpha > 0$; then, $|f|^\alpha$ is subharmonic.*

*Proof.* $|f|^\alpha = \exp(\log |f|^\alpha) = \exp(\alpha \log |f|)$; by Proposition 2.3.3, $\alpha \log |f|$ is subharmonic in the open set where $f \neq 0$ and the result follows by Proposition 2.3.4. Where $f = 0$, we get the result by submean property, being 0 the minimum of $|f|^\alpha$. □

We are ready to state Hartogs Lemma and Hartogs Theorem. For them, we come back to domains in $\mathbb{C}^n$.

**Lemma 2.3.1** (Hartogs lemma)**.** *Let $v_k$ be a sequence of subharmonic functions, defined on a domain $\Omega \subseteq \mathbb{C}^n$, which are uniformly bounded on any compact subset of $\Omega$, that means*

$$\limsup_{k \to \infty} \sup_{z \in K} v_k(z) \leq M, \tag{2.13}$$

*and such that the following pointwise estimate holds*

$$\limsup_{k \to \infty} v_k(z) \leq C, \qquad \forall z \in \Omega; \tag{2.14}$$

*then, $\forall \epsilon > 0$ and $\forall K \subset\subset \Omega$, there is $k_0$ such that*

$$\sup_{z \in K} v_k(z) \leq C + \epsilon, \qquad \forall k \geq k_0. \tag{2.15}$$

**Remark 2.3.3.** The proof is an application of Fatou's lemma in the integrals used in the submean property. For sequences of functions which admit integral representations, or estimates by integrals like submeans, in case they have a uniform bound, then the pointwise "lim sup" enters into the integrals and becomes "uniform".

*Proof of Lemma 2.3.1.* We fix $z_0 \in K$; for $|z - z_0| < \delta$ and $r < d(z_0, \partial\Omega) - \delta$ we have, by Proposition 2.3.1 *(iv)* applied to each $\{v_k\}$, that

$$\begin{aligned} v_k(z) &\leq \frac{1}{\pi r^2} \int_0^{2\pi} \int_0^r \rho v_k(z + \rho e^{i\theta}) d\theta d\rho \\ &\leq \frac{1}{\pi(r+\delta)^2} \int_0^{2\pi} \int_0^{r+\delta} \rho v_k(z_0 + \rho e^{i\theta}) d\theta d\rho + O(\delta). \end{aligned}$$

The last one is given by the uniform boundedness on compacts of the $v_k$; this lets us control the error term $O(\delta)$ when we pass from $B(z_0, r)$ to $B(z_0, r + \delta)$.

Note that subharmonic functions, not identically equal to $-\infty$, defined on a domain $\Omega$, are locally integrable: to prove it it's enough to use the 2-dimensional submean property and the fact that upper semicontinuous functions are bounded above on compacts.

Then, using the local integrability of subharmonic functions, after having considered the "lim sup" of the previous inequality, we can apply Fatou Lemma

$$\begin{aligned} \limsup_{k\to\infty} \sup_{z\in B(z_0, r+\delta)} v_k(z) &\leq \frac{1}{\pi(r+\delta)^2} \limsup_{k\to\infty} \int_0^{2\pi} \int_0^{r+\delta} \rho v_k(z_0 + \rho e^{i\theta}) d\theta d\rho + O(\delta) \\ &\leq \frac{1}{\pi(r+\delta)^2} \int_0^{2\pi} \int_0^{r+\delta} \rho C d\theta d\rho + O(\delta) \\ &\leq \frac{2\pi C}{\pi(r+\delta)^2} \frac{\rho^2}{2}\Big|_0^{r+\delta} + O(\delta) \\ &= C + O(\delta). \end{aligned}$$

Using a finite covering argument for $K$ and by the arbitrary of $\delta$, we get the conclusion, that is

$$\limsup_{k\to\infty} \sup_{z\in K} v_k(z) \leq C$$

$\square$

**Remark 2.3.4.** Note that the final bound is uniform on all compact sets and coincides with the pointwise "lim sup".

**Remark 2.3.5.** To have 1-dimensional submean integrals, the only possibility is to use Poisson kernel; otherwise, Fatou Lemma cannot be applied because what is unknown is the estimate of "lim sup" of the function $\theta \longmapsto \sup_{z\in K} v_k(z + re^{i\theta})$.

The proof of Hartogs Theorem can be divided into two parts: the first is an application of Baire Theorem, the second uses Hartogs lemma.

**Theorem 2.3.1** (Hartogs, 1906). *Let $\Omega$ be a domain of $\mathbb{C}^n$; if $f : \Omega \to \mathbb{C}$ is separately holomorphic, then it is holomorphic.*

*Proof.* The statement is local and can be proved by adding, one by one, the directions of separate analyticity: so we can consider the bidisc $\bar{\Delta} \times \bar{\Delta} \subset\subset \Omega$ in $\mathbb{C}^2$ (where $\Delta$ is the standard unit disc of $\mathbb{C}$: $\Delta = \{z \in \mathbb{C} : |z| < 1\}$) and prove these two steps

- *[Step 1] - Analyticity on* $\Delta_\epsilon \times \Delta = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < \epsilon, |z_2| < 1\}$
- *[Step 2] - Analyticity on* $\Delta \times \Delta = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$

*Proof of [Step 1]* We prove that $f$, which is separately holomorphic in $z_1$ and $z_2$ on the bidisc, is holomorphic on the strip $\Delta_\epsilon \times \Delta$. Let's define, for large $l > 0$,

$$E_l := \{z_1 \in \Delta : \sup_{z_2 \in \Delta} |f(z_1, z_2)| \leq l\}.$$

$E_l$ is closed because $f$ is separately holomorphic, and then separately continuous, in $z_1$ for $z_2$ fixed and $\underset{l}{\cup} E_l = \Delta$, again for the separate continuity, and then boundedness, of $f$, for all $z_1 \in \Delta$. By Baire Theorem, there exists $l_0$ such that $\overset{\circ}{E}_{l_0} \neq \varnothing$: at this point we have a function $f$, which is separately holomorphic and bounded on $\overset{\circ}{E}_{l_0} \times \Delta$: by Theorem 2.2.5, $f$ is holomorphic on $\overset{\circ}{E}_{l_0} \times \Delta$ and, repeating the same construction with different sets $E_l$ on any open subset of $\Delta$, we can say that $f$ is holomorphic on $B \times \Delta$, for an open dense subset $B \subset \Delta$. Also, we can assume, without loss of generality, that $0 \in B$, because otherwise we arbitrarily shrink a bit the radius of the set of analyticity. Therefore, considering a disc $\Delta_\epsilon$, centered at 0 and of radius $\epsilon$, all contained in $B$, we can say that $f$ is holomorphic on the strip $\Delta_\epsilon \times \Delta$.

*Proof of [Step 2]* Note that, at this point, we can even forget that $f$ is separately holomorphic in $z_2$, when $z_1$ is outside $\Delta_\epsilon$. We want to prove that, given the function $f$, which is holomorphic on $\Delta_\epsilon \times \Delta$ and separately holomorphic in $z_1 \in \Delta$, for $z_2$ fixed, we get the joint analyticity of $f$ on the whole bidisc $\Delta \times \Delta$.
We consider the Taylor series of $f$ with respect to $z_1$, at $z_1 = 0$:

$$f(z_1, z_2) = \sum_k \frac{\partial^k_{z_1} f(0, z_2)}{k!} z_1^k; \tag{2.16}$$

by Proposition 2.2.6 [$b$], it converges normally for $(z_1, z_2) \in \Delta_\epsilon \times \Delta$, that means absolute convergence of that series, uniformly on each compact subset of $\Delta_\epsilon \times \Delta$.
We define the functions

$$v_k(z_2) := \left( \frac{|\partial^k_{z_1} f(0, z_2)|}{k!} \right)^{\frac{1}{k}},$$

that are subharmonic, because, being $f$ separately holomorphic in $z_2$, for $z_1 \in \Delta_\epsilon$, also the derivatives $\partial^k_{z_1} f(0, z_2)$ are separately holomorphic in $z_2$ for $z_1 = 0 \in \Delta_\epsilon$

fixed and the $\left(\frac{1}{k}\right)$-powers of their absolute values are subharmonic by Proposition 2.3.5.
By our assumption of separate analyticity in $z_1$, we can apply Cauchy inequalities for any fixed $z_2$ and the analyticity on $\Delta_\epsilon \times \Delta$ provides the rest of the following estimates

$$\begin{aligned}
&|\partial_{z_1}^k f(0, z_2)| \leq \frac{k!}{\epsilon^k} \sup_{\partial\Delta_\epsilon} |f| \\
\implies &\frac{|\partial_{z_1}^k f(0, z_2)|}{k!} \leq \frac{L}{\epsilon^k} \\
\implies &\left(\frac{|\partial_{z_1}^k f(0, z_2)|}{k!}\right)^{1/k} \leq \frac{L^{1/k}}{\epsilon} \\
\implies &\sup_{z_2 \in \Delta} v_k(z_2) \leq L^{1/k} \epsilon^{-1} \\
\implies &\limsup_{k \to \infty} \sup_{z_2 \in \Delta} v_k(z_2) \leq \epsilon^{-1}.
\end{aligned}$$

Then, $\{v_k\}$ are uniformly bounded. On the other hand, Cauchy-Hadamard principle on the radius $R$ of convergence of the power series $\sum\limits_k a_k z^k$ says that

$$R = \frac{1}{\limsup\limits_k \sqrt[k]{a_k}}$$

and the separate analiticity in $z_1$ gives $R \geq 1$, therefore

$$\begin{aligned}
\frac{1}{\limsup\limits_k v_k(z_2)} &= \frac{1}{\limsup\limits_k \left(\frac{|\partial_{z_1}^k f(0,z_2)|}{k!}\right)^{1/k}} \\
&= \frac{1}{\limsup\limits_k (a_k)^{1/k}} \\
&= R \geq 1
\end{aligned}$$

from which we get the pointwise estimate for the $\{v_k\}$

$$\limsup_k v_k(z_2) \leq 1.$$

Applying Hartogs Lemma, the pointwise estimate becomes uniform

$$\limsup_k \sup_\Delta v_k \leq 1,$$

and gives, for any $r < 1$,

$$\begin{aligned} &\sup_{|z_2|\le r} v_k(z_2) \le \frac{1}{r} \\ \Longrightarrow &\sup_{|z_2|\le r} \left|\frac{\partial^k_{z_1} f(0,z_2)}{k!}\right| \le \left(\frac{1}{r}\right)^k \\ \Longrightarrow &\sup_{|z_2|\le r} \left|\frac{\partial^k_{z_1} f(0,z_2)}{k!}\right| r^k \le 1. \end{aligned}$$

The last inequality gives the normal convergence of (2.16), whose terms are holomorphic, in $\Delta_r \times \Delta_r$ and, by the arbitrary of $r$, the power series converges normally for $(z_1, z_2) \in \Delta \times \Delta$. Then, $f$ is holomorphic on $\Delta \times \Delta$. □

**Remark 2.3.6.** Note that Hartogs Theorem consists only in Step 2; Step 1 is just an application of Baire's principle. In the second step we prove two important results: the uniformity in $\Delta$ and the propagation. When we say that "we can even forget that $f$ is separately holomorphic in $z_2$, when $z_1$ is outside $\Delta_\epsilon$", even if we suppose that $f$ is continuous, it is not easy to prove the analyticity on the bidisc: the problem has become a problem of propagation.

Figure 2.1 is a useful mnemonic to distinguish the two steps and appreciate the "propagation" argument of the proof.

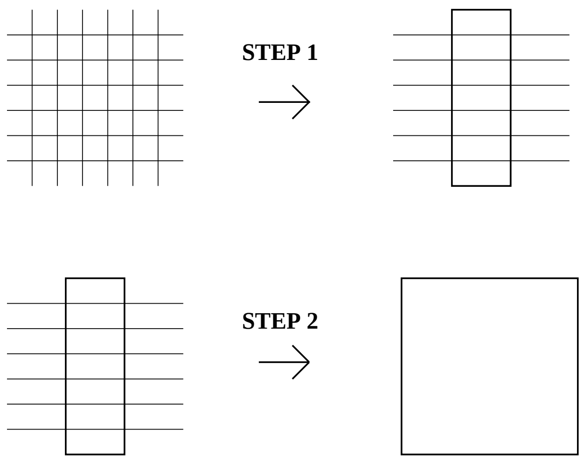



Figure 2.1: Hartogs Theorem

## 2.4 Variations from Hartogs Theorem

Starting from the result of Hartogs, we propose in this section some applications and variations of his statement: the aim is to pass, step by step, from the main Theorem for separately holomorphic functions to our Theorem in the setting of separately CR functions. The basic results of this section are contained in our paper [43].

First of all, an application of Hartogs Theorem can be easily proved just iterating the technique of "doubling" the radius of convergence. To keep the contact with the previous section, we consider the following propositions again on the bidisc $\Delta \times \Delta = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$.

**Proposition 2.4.1.** *Let $f$ be separately holomorphic in $z_1$ on $\Delta^+ \times \Delta = \{(z_1, z_2) \in \Delta \times \Delta : \mathrm{Re}\, z_1 > 0\}$ and holomorphic on $\Delta^+_\epsilon \times \Delta = \{(z_1, z_2) \in \Delta^+ \times \Delta : |z_1| < \epsilon\}$; then, $f$ is holomorphic on $\Delta^+ \times \Delta$.*

**Remark 2.4.1.** Note that, comparing this result with the second step of Hartogs Theorem, the effect of the proposition is to move the strip of holomorphy to the boundary of the domain where $f$ is separately holomorphic. Also note that with a conformal mapping of $\Delta^+$ to $\Delta$ it is possible to regain the classical Hartogs statement.

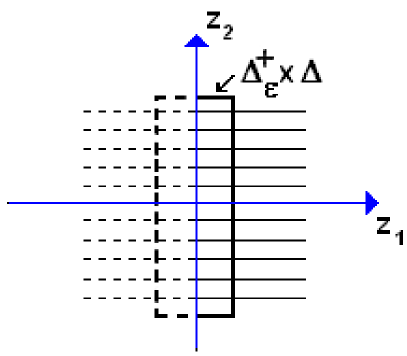



Figure 2.2: First application for a strip on the boundary

*Proof of Proposition 2.4.1.* We consider along the strip $\Delta^+_\epsilon \times \Delta$ a smaller strip of width $\delta$ as in figure 2.3: we call it $\Delta^+\left(\epsilon - \frac{\delta}{2}, \frac{\delta}{2}\right) \times \Delta$ by the fact that its center, along $z_1$-complex axis, is $(\epsilon - \delta) + \frac{\delta}{2} = \epsilon - \frac{\delta}{2}$ and its radius is $\frac{\delta}{2}$, so it is precisely equal to the set $\{(z_1, z_2) \in \Delta^+ \times \Delta : \left|z_1 - \left(\epsilon - \frac{\delta}{2}\right)\right| < \frac{\delta}{2}\}$.

The function $f$ is holomorphic on $\Delta^+\left(\epsilon - \frac{\delta}{2}, \frac{\delta}{2}\right) \times \Delta$ and separately holomorphic on the bidisc $\Delta^+\left(\epsilon - \frac{\delta}{2}, \epsilon - \frac{\delta}{2}\right) \times \Delta$ of $z_1$-center $\epsilon - \frac{\delta}{2}$ and radius $\epsilon - \frac{\delta}{2}$, contained in $\Delta^+ \times \Delta$: by Hartogs Theorem $f$ is holomorphic on $\Delta^+\left(\epsilon - \frac{\delta}{2}, \epsilon - \frac{\delta}{2}\right) \times \Delta$, which is a strip of total width $2\left(\epsilon - \frac{\delta}{2}\right) = 2\epsilon - \delta$, because the technique of the proof has the effect of doubling the $z_1$-radius of convergence of the Taylor series of $f$. Repeating the same argument with other strips of radius $\delta$ on the "right" borderline of the new set of analyticity, we are able to get the analyticity on the whole $\Delta^+ \times \Delta$. $\square$

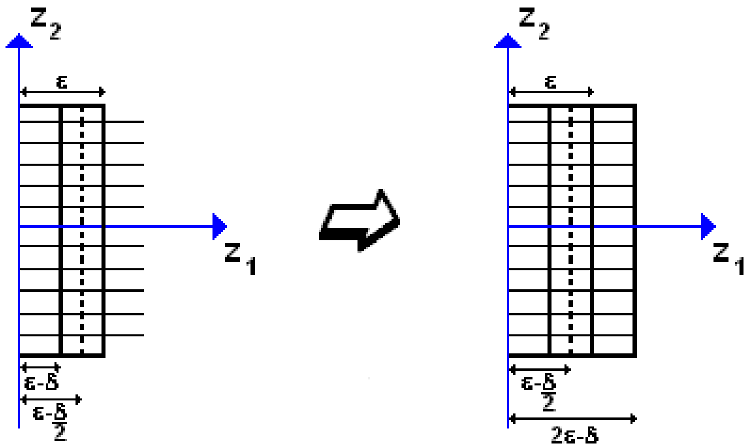



Figure 2.3: Doubling of the radius of convergence

When the leaves of the foliation are complex curves, the problem changes: it is a problem of propagation, as *[Step 2]* was, but this time for a non-holomorphic foliation. In full generality, the validity of the statement has been proved, only recently, by Chirka (cf. [16] and Section 1.1 for a brief survey of his results).

If we assume in addition that $f \in C^1$, or even $f \in C^0$, the result can be proved in several way. Remaining in the setting of the bidisc $\Delta \times \Delta$ in $\mathbb{C}^2$, this is the statement, for which we propose two different proofs.

**Proposition 2.4.2.** *Let $\{\gamma_\lambda\}_{\lambda\in\Lambda}$, for $\Lambda$ an open set of $\mathbb{R}^2 \approx \mathbb{C}$, be a smooth foliation of $\Delta^+ \times \Delta$ by complex curves, such that $\gamma_\lambda \cap (\Delta^+_\epsilon \times \Delta) \neq \varnothing$, $\forall \lambda \in \Lambda$. Let $f$ be a $C^0$ function on $\Delta^+ \times \Delta$, such that $f$ is holomorphic on $\Delta^+_\epsilon \times \Delta$ and $f|_{\gamma_\lambda}$ is holomorphic, $\forall \lambda \in \Lambda$; then, $f$ is holomorphic on $\Delta^+ \times \Delta$.*

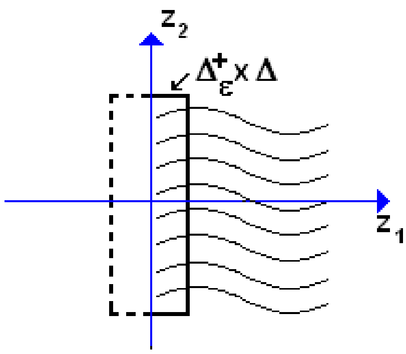



Figure 2.4: Foliation by complex curves

*First proof of Proposition 2.4.2.* Assume for instance that $f \in C^1(\Delta^+ \times \Delta)$; then, let the foliation be described by a mapping

$$\begin{aligned} \Phi : \mathbb{C} \times \Lambda &\to \mathbb{C}^2 \\ (\tau, \lambda) &\mapsto \Phi(\tau, \lambda), \end{aligned}$$

which is $C^\infty$ in both its arguments $\tau$ and $\lambda$, and is holomorphic in $\tau$, so that the choice of different values of $\lambda \in \Lambda$ defines the complex leaves

$$\gamma_\lambda := \{\Phi(\tau, \lambda) \in \mathbb{C}^2, \forall \tau \in \mathbb{C}\}.$$

Since the statement is local with respect to $\lambda$, we can assume that in a neighbourhood of a fixed value $\widetilde{\lambda}$ the set $\Delta_\epsilon^+ \times \Delta$ contains the image under $\Phi$ of a neighbourhood of $\tau = 0$; in other terms, for $\delta_1$, $\delta_2$ small enough, $\{\Phi(\tau, \lambda) \in \mathbb{C}^2 : |\tau| < \delta_1, |\lambda - \widetilde{\lambda}| < \delta_2, \widetilde{\lambda} \in \Lambda\} \subseteq \Delta_\epsilon^+ \times \Delta$.

Then, we consider the form

$$\Phi^*(df \wedge dz_1 \wedge dz_2)$$

whose coefficients are precisely

$$\frac{\partial f}{\partial \bar{z}_1}(\Phi(\tau, \lambda)) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}_2}(\Phi(\tau, \lambda)) \tag{2.17}$$

as we show now

$$\begin{aligned} &\Phi^*(df \wedge dz_1 \wedge dz_2) = \\ &\quad = \Phi^*((\partial_{z_1} f dz_1 + \partial_{z_2} f dz_2 + \partial_{\bar{z}_1} f d\bar{z}_1 + \partial_{\bar{z}_2} f d\bar{z}_2) \wedge dz_1 \wedge dz_2) \\ &\quad = \Phi^*(\partial_{\bar{z}_1} f d\bar{z}_1 \wedge dz_1 \wedge dz_2 + \partial_{\bar{z}_2} f d\bar{z}_2 \wedge dz_1 \wedge dz_2) \\ &\quad = \partial_{\bar{z}_1} f(\Phi(\tau, \lambda)) \Phi^*(d\bar{z}_1 \wedge dz_1 \wedge dz_2) + \partial_{\bar{z}_2} f(\Phi(\tau, \lambda)) \Phi^*(d\bar{z}_2 \wedge dz_1 \wedge dz_2) \\ &\quad = \partial_{\bar{z}_1} f(\Phi(\tau, \lambda)) d\bar{\Phi}_1 \wedge d\Phi_1 \wedge d\Phi_2 + \partial_{\bar{z}_2} f(\Phi(\tau, \lambda)) d\bar{\Phi}_2 \wedge d\Phi_1 \wedge d\Phi_2. \end{aligned} \tag{2.18}$$

The coefficients (2.17) have the following properties:

- they are $\equiv 0$ for $|\tau| < \delta_1$, $|\lambda - \widetilde{\lambda}| < \delta_2$ because the function $f$ is assumed to be holomorphic on $\Delta_\epsilon^+ \times \Delta$ and the set $\{\Phi(\tau, \lambda) \in \mathbb{C}^2 : |\tau| < \delta_1, |\lambda - \widetilde{\lambda}| < \delta_2\}_{\widetilde{\lambda} \in \Lambda} \subseteq \Delta_\epsilon^+ \times \Delta$;
- they are holomorphic with respect to $\tau$ for $\lambda \equiv$ cost, because they are the coefficients of $(df \circ \Phi) \wedge d\Phi_1 \wedge d\Phi_2$ that are holomorphic in $\tau$.

Hence the above form is $\equiv 0$ for any $\tau$ by the uniqueness of analytic continuation and, by (2.18), $f$ is holomorphic on $\Delta^+ \times \Delta$.

We can handle the case in which $f$ is for instance $C^0$ by putting the above argument in a "weak sense", that means considering, instead of a function, a distribution $f$, continuous or even bounded (in such a case we need a "normal family" argument to prove continuity). Using the inverse of a $C^\infty$ change of coordinates $\Phi$, which corresponds to the straightening of the curves $\gamma_\lambda$, we can define

$$g(\tau) = \langle f(\Phi(\tau, \cdot)), \Phi^* d\psi(\cdot)\rangle$$

where $\psi$ is any (2,1)-form in $\lambda$ which is constant along the $\gamma_\lambda$ (so that $\Phi^*\psi$ has coefficients only depending on $\lambda$) and $\langle \cdot, \cdot \rangle$ is the pairing between (0,0) and (2,2)-forms. The function $g$ is holomorphic by the initial assumptions on the analyticity of $f|_{\gamma_\lambda}$ and $g \equiv 0$ for $|\tau| < \delta_1$ because $f$ is holomorphic on $\Delta^+_\epsilon \times \Delta$. Therefore, $f$ is holomorphic on $\Delta^+ \times \Delta$. □

Another approach to the proof takes advantage from the celebrated Theorem of Hanges and Treves [27]

**Theorem 2.4.1** (Hanges-Treves, 1983). *Let $M$ be a hypersurface of $\mathbb{C}^n$, $\Omega$ one side of $M$, $\gamma$ a complex curve of $M$, $z_o$ a point of $\gamma$, $f$ a holomorphic function on $\Omega$, such that $|f(z)| \le Cdist(z, \partial\Omega)^{-N}$ for suitable $C$ and $N$. Then, if $f$ extends across $M$ at $z_o$, it also extends at any other point $z_1 \in \gamma$.*

**Remark 2.4.2.** Note that the hypothesis $|f(z)| \le Cdist(z, \partial\Omega)^{-N}$ says that the holomorphic function $f$ has tempered growth near the boundary $M$.

**Remark 2.4.3.** Theorem 2.4.1 describes a phenomenon of propagation of holomorphic extendibility which is really sofisticated because not a full foliation of complex curves but only a single leaf is needed. This leaf has the property of "being a propagator of analyticity". (We refer to [3] and [4] for a more detailed account of propagation of CR extendibility; note that the techniques of [3] and [4] apply also to a function $f$ which is not tempered at $\partial\Omega$.)

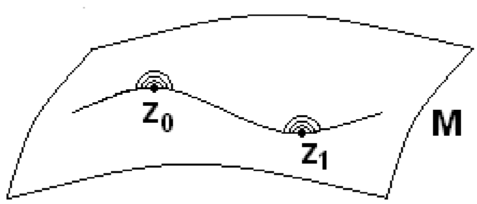



Figure 2.5: Hanges-Treves Theorem

*Second proof of Proposition 2.4.2.* We consider a real curve $\psi_t$, $t \in \mathbb{R}$, contained in $\Delta^+_\epsilon \times \Delta$ and transverse to all the complex curves $\gamma_\lambda, \lambda \in \Lambda$. Let $N$ be the union of the $\gamma_\lambda$ such that $\gamma_\lambda \cap \psi_t \neq \varnothing$. Note that, for a fixed $z \in \psi_t$, if we denote by $\gamma_z$ the curve containing $z$, we have that $T^{\mathbb{C}}_z N = T^{\mathbb{C}}\gamma_z$. Moreover, the restriction of $f$ to $N$ comes to be CR (in fact: $f$ is CR($N$) if and only if $\bar{L}f=0$ $\forall \bar{L} \in T^{0,1}N$, but $T^{0,1}N\big|_\gamma = T^{0,1}\gamma$). Now, we consider $z_0 \in \psi_t \cap \gamma_\lambda$ for some $\lambda \in \Lambda$; $z_0 \in \Delta^+_\epsilon \times \Delta$, where the function $f$ is holomorphic. Then, $f$ extends holomorphically at $z_0$ to a neighbourhood of $z_0$ in $\Delta^+_\epsilon \times \Delta \subset \Delta^+ \times \Delta$.

We know that $\gamma_\lambda \hookrightarrow N$ and $f \in \mathrm{CR}(N)$. For the result proved in [4], that generalizes the statement of Hanges-Treves Theorem, $f$ extends holomorphically at any other point of $\gamma_\lambda$. Repeating the same argument with points of other curves issued from $\psi_t$ and in case moving $\psi_t$, we get a result of propagation along any complex curve. Now we have to show that the extension we get coincides with the initial definition of $f$: this is true for identity principle. So we have the required extension. $\square$

Obviously, the previous proposition can be presented in a general setting, not only for the bidisc in $\mathbb{C}^2$: this would be the general statement

**Proposition 2.4.3.** *Let $U$ and $V$ be open subsets of $\mathbb{C}^n$, with $U \subset V$ and $V$ connected; let $\{\gamma_\lambda\}_{\lambda \in \Lambda}$ be a foliation of $V$ by complex curves, such that $\gamma_\lambda \cap U \neq \varnothing, \forall \lambda \in \Lambda$. Let $f$ be a $C^0$ function defined on $V$, such that $f$ is holomorphic on $U$ and $f|_{\gamma_\lambda}$ is holomorphic, $\forall \lambda \in \Lambda$; then, $f$ is holomorphic on $V$.*

The former statement can be generalized in many directions: first, by replacing the open sets $U$ and $V$ (as well as $\Delta^+_\epsilon \times \Delta$ and $\Delta^+ \times \Delta$) by CR manifolds $M$ and $N$, and second, in replacing the foliation $\{\gamma_\lambda\}$ of complex curves by a foliation $\{L_\lambda\}$ of CR manifolds of CR dimension 1.

The first is a result contained in a paper by Henkin and Tumanov [33] of 1983, the second is our main result and we will prove it in the following section.

The paper of Henkin and Tumanov deals with local characterization of holomorphic automorphisms and proves the existence of biholomorphic maps between Siegel domains. With this aim, Henkin and Tumanov introduce a lemma, that they define as "a sufficiently general assertion about CR functions on CR manifolds admitting foliation by holomorphic curves". Here is their statement

**Theorem 2.4.2** (Henkin-Tumanov, 1983)**.** *Let $M$ be a smooth CR manifold in $\mathbb{C}^n$, that admits a foliation by complex curves $\{\gamma_\lambda\}$, $\lambda \in \Lambda$; in addition, suppose that on $\partial M$ there is a smooth CR manifold $N$, such that each complex curve is transversal to $N$ at any common point of $\gamma_\lambda \cap N$. Then, any function $f \in C^0(M)$ which is CR on $N$ and holomorphic along the $\gamma_\lambda$'s, is also CR on $M$.*

## 2.5 The main Theorem for separately CR functions

In this section we state our main result, which in contained in the paper [42], and present its proof in details, providing suitable remarks and applications.

The result is achieved through two fundamental steps: the first one corresponds to Theorem A and uses the technique of polynomial approximation to get a local CR extension, while the second one corresponds to Theorem B, where a repeated use of the previous technique and a connectedness argument yield the global CR extension.

### 2.5.1 Theorem A

We start with

**Theorem A.** *Let $M$ be a CR manifold of $\mathbb{C}^n$, with boundary $N$ and let $\{L_\lambda\}_{\lambda\in\Lambda}$ be a foliation of $M$ with the following properties:*

*(i)* *every $L_\lambda$ is a CR manifold of CR-dimension* $1$*;*

*(ii)* $L_\lambda \cap N \neq \varnothing, \forall\lambda \in \Lambda$*;*

*(iii)* $T^{\mathbb{C}}L_\lambda\big|_{N\cap L_\lambda} + TN\big|_{N\cap L_\lambda} = TM\big|_{N\cap L_\lambda}$.

*Let $f$ be a $C^0$ function on $M$, which is CR along $N$, CR and $C^1$ along each leaf $L_\lambda$; then, $f$ is CR in a neighbourhood of $N$ in $M$.*

**Remark 2.5.1.** Note that $M$ is not required to be compact and it is not necessary that $codim_M N = 1$.

**Remark 2.5.2.** We will refer to hypothesis *(iii)* just saying that $T^{\mathbb{C}}L_\lambda$ has to be *transversal* to $TN$ at any common point of $L_\lambda \cap N$. This is exactly the same type of condition used for the complex curves $\{\gamma_\lambda\}_{\lambda\in\Lambda}$ by Henkin and Tumanov: they said that the $\gamma_\lambda$ had to be transversal to $N$ at any common point of $\gamma_\lambda \cap N$, that is $T\gamma_\lambda\big|_{N\cap\gamma_\lambda} + TN\big|_{N\cap\gamma_\lambda} = TM\big|_{N\cap\gamma_\lambda}$ but, being $T\gamma_\lambda \equiv T^{\mathbb{C}}\gamma_\lambda$ because the curves are complex, it follows that $T^{\mathbb{C}}\gamma_\lambda\big|_{N\cap\gamma_\lambda} + TN\big|_{N\cap\gamma_\lambda} = TM\big|_{N\cap\gamma_\lambda}$.

**Remark 2.5.3.** Since $M$ is CR, then $N$ is also CR. In fact, its CR codimension (we mean by "CR codimension of $N$" the dimension $\dim_{\mathbb{C}} T^{\mathbb{C}}M/T^{\mathbb{C}}N$) is in this case exactly equal to 1 because we have a foliation by leaves whose complex structure is transversal to $N$. This is due to the fact that

$$T^{\mathbb{C}}L_\lambda + TN = TM \quad \Longrightarrow \quad T^{\mathbb{C}}L_\lambda \oplus T^{\mathbb{C}}N = T^{\mathbb{C}}M \tag{2.19}$$

from which, being the $\{L_\lambda\}$ CR manifolds of CR-dimension 1 ($\dim_{\mathbb{C}} T^{\mathbb{C}}L_\lambda = 1$) and being $M$ CR, also $N$ is CR. We prove (2.19).

*Proof of (2.19)*: $N$ is the boundary of $M$, that is $\dim_{\mathbb{R}} TN = \dim_{\mathbb{R}} TM - 1$; in particular, $\dim_{\mathbb{R}} T^{\mathbb{C}}N \leq \dim_{\mathbb{R}} T^{\mathbb{C}}M$.

First we prove that equality cannot hold: if, for instance, we assume that $\dim_{\mathbb{R}} T^{\mathbb{C}}N = \dim_{\mathbb{R}} T^{\mathbb{C}}M$, by the fact that $T^{\mathbb{C}}L_\lambda \subseteq T^{\mathbb{C}}M$, we would have $T^{\mathbb{C}}L_\lambda \subseteq T^{\mathbb{C}}M = T^{\mathbb{C}}N \subseteq TN$, from which $TN + T^{\mathbb{C}}L_\lambda = TN$ which is an absurd, because we are assuming that $TN + T^{\mathbb{C}}L_\lambda = TM \neq TN$.

Now, we prove that the difference between the complex dimensions of $T^{\mathbb{C}}N$ and $T^{\mathbb{C}}M$ is exactly 1, that is

$$\dim_{\mathbb{C}} T^{\mathbb{C}}N = \dim_{\mathbb{C}} T^{\mathbb{C}}M - 1$$

from which we get the conclusion $T^{\mathbb{C}}L_\lambda \oplus T^{\mathbb{C}}N = T^{\mathbb{C}}M$, because $\dim_{\mathbb{C}} T^{\mathbb{C}}L_\lambda \equiv 1$. If $d = \operatorname{codim}_{\mathbb{R}} M$, we have

$$\begin{aligned}
\dim_{\mathbb{R}} T^{\mathbb{C}}N &= \dim_{\mathbb{R}}(TN \cap \mathcal{J}TN) \\
&= 2\dim_{\mathbb{R}} TN - \dim_{\mathbb{R}}(TN + \mathcal{J}TN) \\
&= 2(\dim_{\mathbb{R}} TM - 1) - \dim_{\mathbb{R}}(TN + \mathcal{J}TN) \\
&= 2(2n - d - 1) - \dim_{\mathbb{R}}(TN + \mathcal{J}TN) \\
&= 4n - 2d - 2 - \dim_{\mathbb{R}}(TN + \mathcal{J}TN).
\end{aligned}$$

If $N$ is generic, then $\dim_{\mathbb{R}}(TN + \mathcal{J}TN) = 2n$, from which

$$\begin{aligned}
\dim_{\mathbb{R}} T^{\mathbb{C}}N &= 2n - 2d - 2 \\
&= \dim_{\mathbb{R}} T^{\mathbb{C}}M - 2,
\end{aligned}$$

where the last equality is given by the fact that $M$ is CR, so $M$ can be assumed to be generic (see the following remark for details). Otherwise, if $N$ is not generic $\dim_{\mathbb{R}}(TN + \mathcal{J}TN) < 2n$, which implies

$$\dim_{\mathbb{R}} T^{\mathbb{C}}N > 2n - 2d - 2;$$

$\dim_{\mathbb{R}} T^{\mathbb{C}}N$ cannot be $2n - 2d - 1$ because it has to be even and if we assume, for instance, that $\dim_{\mathbb{R}} T^{\mathbb{C}}N \geq 2n - 2d = \dim_{\mathbb{R}} T^{\mathbb{C}}M$ we get an absurd. Then, $N$ has to be generic, from which the desired conclusion holds.

*Proof of Theorem A.* We want to prove that $f$ is, in a neighbourhood of each point of $N$, a limit of polynomials: this is enough to say that $f$ is CR in a neighbourhood $M_\epsilon$ of $N$ in $M$ in fact, being $f$ the limit of sequences of entire functions, it's possible to apply Proposition 2.2.1; then, $f$ is holomorphic and, when restricted to the CR manifold $M$, $f$ becomes CR, as desidered.

We can appreciate the role of the local CR extension through Figure 2.6, which is also useful to remember the effects of both Theorem A and Theorem B.

We fix a point $z_0 \in N$ and prove the conclusion in a neighbourhood of $z_0$. The manifold $M$ is CR and, without loss of generality, we can assume that $M$ is generic because $M$ comes to be CR diffeomorphic to a generic submanifold of some subspace $\mathbb{C}^{n'}$, $n' \leq n$, just considering the complex transversal projection

$$j : \mathbb{C}^n \twoheadrightarrow T_{z_0}M + \mathcal{J}T_{z_0}M$$

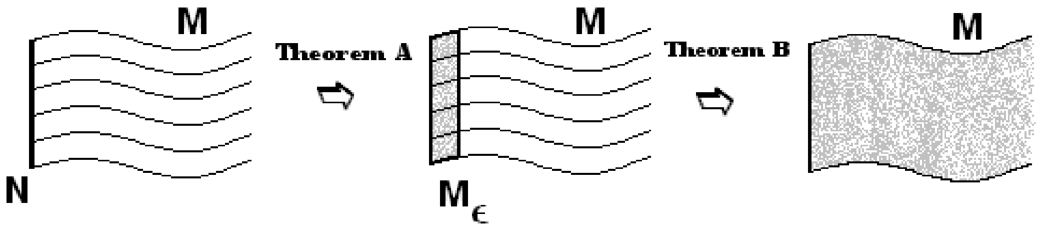



Figure 2.6: CR extension from $N$ to $M_\epsilon$ and global extension

where we call $Y$ the complex plane $T_{z_0}M + \mathcal{J}T_{z_0}M$, identified, through a suitable choice of coordinates, to a plane of $\mathbb{C}^n$ and where $j$ has the following properties

(i) $j$ is a CR diffeomorphism between $M$ and $M' := jM$,

(ii) $TY\big|_{M'} = TM' + \mathcal{J}TM'$. (That is $M'$ is generic in $Y$.)

**Remark 2.5.4.** The existence of such a map $j$ and its properties are guaranteed by the fact that

(a) $j$ is bjective over $M$ since the fibers of the projection are transversal to $M$,

(b) $j$ is a CR mapping between $M$ and $M'$ since it is the restriction of a complex projection,

(c) $j^{-1}$ is also CR as a consequence of the fact that $\dim_{\mathrm{CR}} M = \dim_{\mathrm{CR}} M'$ and $j : M \to M'$ is a CR and $C^1$ diffeomorphism.

**Remark 2.5.5.** This type of construction takes its origin from a more general discussion about the existence, given a CR submanifold $M$ of a complex manifold $X$, of an intermediate complex manifold $Y$, with $M \subset Y \subset X$, such that $M$ is generic in $Y$. If $M \in C^\omega$ the complex manifold $Y$ exists and $M$ is embedded as generic in $Y$. $Y$ is the so-called "partial complexification" of $M$. However, if $M$ is not $C^\omega$ this is not true in general: we have to use a projection instead of an embedding to get an analogous result for manifolds which are not necessarily $C^\omega$.

Repeating the same argument for $N$, which is CR by Remark 2.5.3, and using the characterization for generic submanifolds of $\mathbb{C}^n$, presented in Proposition 2.1.1, we know that there always exists a totally real maximal submanifold of $N$; we select it with the property of being invariant under the foliation $\{L_\lambda \cap N\}$ and call it $E_0$.

Let's define on $M$ a sequence of polynomials (then, entire functions) $\{f_\alpha\}$ by means of the convolution with the heat kernel

$$f_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \int\limits_{+E_o} f(\xi) e^{-\alpha(z-\xi)^2} d\xi_1 \wedge \ldots \wedge d\xi_n, \tag{2.20}$$

where $(z-\xi)^2 = (z_1 - \xi_1)^2 + \ldots + (z_n - \xi_n)^2$. Our eventual goal is to show that this sequence of entire functions provides a uniform approximation of $f$ in a full neighbourhood of $E_0$ in $M$, by deforming the manifold $E_0$ in a suitable way; then, by using different $E_0$'s we will prove the convergence of $f_\alpha$ to $f$ in a neighbourhood of $N$ in $M$.

Note that, in the above expression for $f_\alpha$, it is possible to introduce the symbol of integration by the fact that $f$ is $C^0(M)$, then locally integrable on subsets of $M$ where $f$ has compact support, and by the fact that $E_0$ has been chosen totally real; otherwise, for $f \in \mathcal{E}'(M)$, the space of distributions on $M$ which is the dual of $C^\infty(M)$, the definition of $f_\alpha$ would be

$$\begin{aligned} f_\alpha(z) &= \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \langle f(\xi)|_{E_0}, e^{-\alpha(z-\xi)^2}\rangle \\ &\equiv f * K_\alpha(z), \end{aligned}$$

where $K_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} e^{-\alpha z^2}$ is the heat kernel. We are able to integrate with respect to all $n$ variables because $M$ has been supposed generic.

There is a classical result, concerning the theory of approximation of CR functions by polynomials on generic submanifolds of $\mathbb{C}^n$, due to Baouendi and Treves [2], which provides these first two results:

*(i)* $f \in CR(N)$, *when restricted to the totally real maximal submanifold* $E_0 \subset N$, *is uniformly approximated by* $\{f_\alpha\}$

*(ii)* $f$ *is uniformly approximated by* $\{f_\alpha\}$ *at a neighbourhood of each point of* $E_0$ *in* $N$.

Note that *(i)* and *(ii)* can be referred just saying that a CR function on a generic submanifold of $\mathbb{C}^n$ can be uniformly approximated by polynomials in controlled neighbourhoods of any point.

We briefly motivate *(i)* and *(ii)*:

*Proof of (i).* $N$ is generic so there are local coordinates $z = (z', z'')$, $z' = (z_1, \dots, z_{d+1})$, $z'' = (z_{d+2}, \dots, z_n)$, for $d = \operatorname{codim}_{\mathbb{R}} M$, such that at $z_0 = 0$ the equations of $N$ and $E_0$ can be normalized as follows

$$\begin{aligned} &N : y'_j = h_j(x', z'') && j = 1, \dots, d+1 \\ &E_0 : \begin{cases} y'_j = h_j(x', z'') & \\ y''_k = 0 & \end{cases} && \begin{matrix} j = 1, \dots, d+1 \\ k = d+2, \dots, n \end{matrix} \\ &h_j(0) = 0, \quad \partial h_j(0) = 0 \end{aligned}$$

This allows us to say that there exist suitable neighbourhoods of $z_0$ on $N$ and $E_0$ such that

$$|\mathsf{Im}\, z| < \epsilon;$$

then, shrinking $E_0$ if it necessary, there exists $0 < k < 1$ such that for all $z, \xi \in E_0, z \neq \xi$, the following estimate holds

$$|\mathsf{Im}\,(z - \xi)| \leq k|\mathsf{Re}\,(z - \xi)|.$$

Without loss of generality, we may assume that $E_0 = \mathbb{R}^n$; otherwise, $E_0$ may be considered a small perturbation of $\mathbb{R}^n$ and it is sufficient to use its local parametrization $\Phi$ over $\mathbb{R}^n$: $\mathbb{R}^n \xrightarrow{\Phi} E_0$, $\xi \mapsto \Phi(\xi)$, which is a diffeomorphism, to get corresponding results. For $E_0 = \mathbb{R}^n$, let's remember the identity

$$\left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{-\alpha\xi^2} d\xi = 1,$$

given by the value of Gauss integral. We want to prove that $f_\alpha \rightrightarrows f$ on compact sets of $E_0$ (where $\rightrightarrows$ denotes the uniform convergence); using first of all the previous identity and then the change of coordinates $s := \sqrt{\alpha}(x - \xi)$, for $x \in \mathbb{R}^n$, we get

$$\begin{aligned} f_\alpha(x) - f(x) &= \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{-\alpha(x-\xi)^2} \left[f(\xi) - f(x)\right] d\xi \\ &= \left(\frac{1}{\pi}\right)^{\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{-s^2} \left[f\left(x - \frac{s}{\sqrt{\alpha}}\right) - f(x)\right] ds \end{aligned}$$

where the integrand, for $\alpha \to \infty$, converges to 0 and its absolute value is dominated by $e^{-s^2}$; applying Lebesgue dominated convergence Theorem, the integral converges to 0 for fixed $x$. The convergence is uniform in fact

$$\begin{aligned} |f_\alpha(x) - f(x)| &\leq \int\limits_{\mathbb{R}^n} e^{-s^2} \left| f\left(x - \frac{s}{\sqrt{\alpha}}\right) - f(x) \right| ds \\ &\approx \int\limits_{|s| \geq R} \cdot \ + \int\limits_{|s| \leq R} \cdot \\ &= \epsilon(R^{-1}) + \epsilon_R(\alpha^{-1}) \end{aligned}$$

because $e^{-s^2}$ is an infinitesimal, uniformly in $\alpha$, when $|s| \geq R$ and is bounded for $|s| \leq R$, while $\left|f\left(x - \frac{s}{\sqrt{\alpha}}\right) - f(x)\right|$ is bounded uniformly in $\alpha$ for $|s| \geq R$ and goes as $\alpha^{-1}$ for $|s| \leq R$; passing to the limit, as $\alpha \to \infty$, the second term disappears and we get the desired uniformity. □

*Proof of (ii).* To prove that it is possible to move the uniform approximation of $f$ at a neighbourhood of each point $z_0 \in E_0$ in $N$, it is sufficient to show that $f_\alpha$ does not depend of $E_0$.

Let $\widetilde{E_0}$ be a $C^1$ small deformation of $E_0$ such that $E_0$ is unchanged outside a compact $N'' \subset\subset N'$, where $N'$ is a neighbourhood of $z_0$ in $N$; let's consider a submanifold $S$ of $N$ of dimension $n + 1$, satisfying $+\partial S = (+E_0) \cup (-\widetilde{E_0})$. The situation is well represented by Figure 2.7

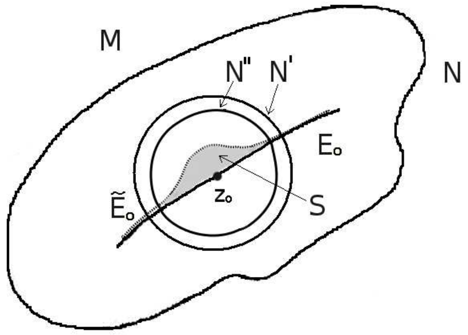



Figure 2.7: Deformation of the totally real submanifold $E_0$ in $N$

If we use Stokes formula and the hypothesis that $f$ is CR, we get, for $\bar{L}$ a generator of CR fields on $S$, that

$$\begin{aligned}\int_{+E_0} f(\xi)e^{-\alpha(z-\xi)^2}d\xi - \int_{+\widetilde{E_0}} f(\xi)e^{-\alpha(z-\xi)^2}d\xi &= \int_{+\partial S} f(\xi)e^{-\alpha(z-\xi)^2}d\xi \\ &\overset{\text{Stokes}}{=} \int_{+S} \bar{L}\left(f(\xi)e^{-\alpha(z-\xi)^2}\right)d\xi \\ &\overset{f\text{ is CR}}{=} \int_{+S} f(\xi)\bar{L}\left(e^{-\alpha(z-\xi)^2}\right)d\xi \\ &= 0.\end{aligned}$$

□

Our result is more than statements *(i)* and *(ii)* on CR approximation because we want to go beyond the manifold $N$ where $f$ is CR. We will prove that

*(iii)* *$f$ is uniformly approximated by $\{f_\alpha\}$ at a neighbourhood of each point of $E_0$ in $M$.*

*(iv)* *$f$ is uniformly approximated by $\{f_\alpha\}$ at a neighbourhood of $N$ in $M$.*

*Proof of (iii).* Without loss of generality, we can assume, apart from a change of coordinates, that $E_0$ is a small perturbation of $\mathbb{R}^n$. We insert $E_0$ into a foliation $\{E\}$ by totally real maximal submanifolds on $N$ invariant under the manifolds $\{L_\lambda \cap N\}$. We denote by $\{\Sigma\}$ the family of the unions of the $L_\lambda$'s issued from each $E$; this provides a foliation of $M$. Therefore, each point of $M$ belongs to a unique leaf $\Sigma$.

Given $z_0 \in E_0$, we consider $z$ next to $z_0$, $z \in M\backslash E_0$, such that $z_0$ is the projection of $z$ on $E_0$ along the unique leaf $\Sigma_z$ that contains $z$.

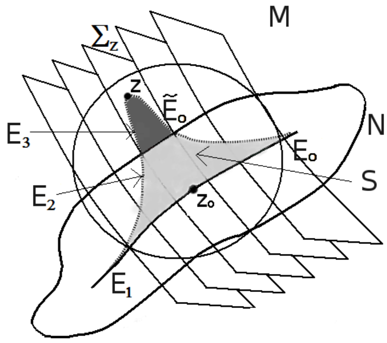



Figure 2.8: Deformation of the totally real submanifold $E_0$ along the leaves

We take a deformation $\widetilde{E_0}$ of $E_0$ which contains $z$ and of the type $\widetilde{E_0} = E_1 \cup E_2 \cup E_3$, where $E_2 \subset N$, $E_3 \subset \Sigma_z$ and $E_1$ is the piece of $E_0$ outside a neighborhood of $z_0$ in $M$ containing $z$. Figure 2.8 represents the previous description.

We require that the deformation is small so that the following condition is fulfilled for some $0 < k < 1$:

$$|\mathsf{Im}\,(z_0 - \xi)| \leq k|\mathsf{Re}\,(z_0 - \xi)| \qquad \forall \xi \in \widetilde{E_0}. \tag{2.21}$$

($\widetilde{E_0}$ needs possibly to be shrunk here.) We denote by $S$ a piecewise smooth manifold contained in $N \cup \Sigma_z$, with boundary $+\partial S = (+E_0) \cup (-\widetilde{E_0})$. We define

$$\widetilde{f_\alpha}(z) = \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \int\limits_{+\widetilde{E_0}} f(\xi) e^{-\alpha(z-\xi)^2} d\xi_1 \wedge ... \wedge d\xi_n. \tag{2.22}$$

The estimate (2.21) guarantees that $\widetilde{f_\alpha}(z)$ converges uniformly to $f(z)$, $\forall z \in \widetilde{E_0}$. We show that $\widetilde{f_\alpha}(z) = f(z)$ $\forall z \in M$. $E_0$ and $\widetilde{E_0}$ delimit a submanifold of $M$, that we have denoted by $S$; therefore we can apply Stokes Theorem

$$\begin{aligned} f_\alpha(z) - \widetilde{f_\alpha}(z) = {} & \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \Big[ \int\limits_{+E_0} f(\xi) e^{-\alpha(z-\xi)^2} d\xi - \int\limits_{+\widetilde{E_0}} f(\xi) e^{-\alpha(z-\xi)^2} d\xi \Big] \\ = {} & \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \int\limits_{+\partial S} f(\xi) e^{-\alpha(z-\xi)^2} d\xi_1 \wedge \ldots \wedge d\xi_n \\ \overset{\text{Stokes}}{=} {} & \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \int\limits_{+S} d_S \Big( f(\xi) e^{-\alpha(z-\xi)^2} d\xi_1 \wedge \ldots \wedge d\xi_n \Big). \end{aligned} \tag{2.23}$$

Note here that if $f$ is only $C^0$ and not $C^1$, as "formally" required by Stokes Theorem, nonetheless the form $d_S(\cdot)$ has $C^0$ coefficients and the above conclusion holds.

Now let's remember the decomposition of the differential on $S$ into its holomorphic and antiholomorphic parts tangentially to $S$: $d_S = \partial + \bar{\partial}_S$; let's consider the term on the right in (2.23): since $e^{-\alpha(z-\xi)^2}$ is holomorphic (in $\mathbb{C}^n$) and $f$ is CR on $S$, we have that the product is a CR function on $S$, so $\bar{\partial}_S(f(\xi)e^{-\alpha(\xi-z)^2}) = 0$. On the other hand, $\partial(f(\xi)e^{-\alpha(\xi-z)^2}d\xi_1 \wedge ... \wedge d\xi_n) = 0$, because the expression is full in the holomorphic differentials.

Then, $f_\alpha$ coincides with $\widetilde{f_\alpha}$, which provides the uniform convergence of $f_\alpha$ to $f$ on any perturbation $\widetilde{E_0}$ of $E_0$. □

*Proof of (iv).* Let's remember that, due to our construction, $N$ is foliated by a family $\{E\}$ of totally real maximal submanifolds on $N$ invariant under the manifolds $\{L_\lambda \cap N\}$ and $M$ is foliated by a family $\{\Sigma\}$, where each $\Sigma$ is the union of $L_\lambda$'s issued from a fixed $E$. Considering different $E_0$ in $\{E\}$ and repeating the same argument of *(iii)* for each $E_0$, we provide the uniform approximation of $f$ at a neighbourhood of $N$ in $M$. □

This completes the proof of Theorem A.

□

**Remark 2.5.6.** Our proof takes its origin from the approximation Theorem due to Baouendi and Treves [2], for which, given a smooth CR submanifold $M$ of $\mathbb{C}^n$ and $z_0 \in M$, each function $f \in C^0(M) \cap \mathrm{CR}(M)$ is approximated by a sequence of polynomials. At the same time, our proof uses the technique of Baouendi-Treves but diverges from their method because we don't have, as hypothesis, a CR function but we want to prove, by approximation, that a given function is CR.

The same method was first exploited by Tumanov in [51]: in his article, Tumanov "goes beyond" the totally real maximal manifold, just proving that $f_\alpha \rightrightarrows f$ on boundaries of suitable analytic discs, called "thin discs"; these are discs stretched along complex tangential directions to $M$ and Tumanov proves that they fill a neighbourhood of 0 in $M$.

### 2.5.2 Theorem B

A repeated use of Theorem A and a connectedness argument yield the following global theorem. This is substantially due to Henkin-Tumanov [33] but it allows general foliations by manifolds of CR dimension 1 instead of complex curves. The proof is also far different.

**Theorem B.** *Let $M$ be a CR connected manifold of $\mathbb{C}^n$ with boundary $N$, foliated by a family $\{L_\lambda\}$ of CR manifolds of CR dimension 1 issued from $N$, with $T^{\mathbb{C}}L_\lambda$ transversal to $TN$ at any common point of $L_\lambda \cap N$. Let $f$ be a $C^0$ function on $M$, which is CR along $N$, CR and $C^1$ along each $L_\lambda$. Then, $f$ is CR all over $M$.*

*Proof of Theorem B.* According to Theorem A, $f$ is CR in a neighbourhood $M_\epsilon$ of $N$ in $M$. Let $z$ be a point of $M$; we consider a family of domains $\Omega_\nu \subset M$, $\nu \in [0\,1]$, with $C^1$ and CR boundary $M_\nu = \partial\Omega_\nu$, such that:

- $\Omega_0 \subset M_\epsilon$
- $T^{\mathbb{C}}\partial\Omega_\nu$ is transversal to $T^{\mathbb{C}}L_\lambda$ at any point of $\partial\Omega_\nu \cap L_\lambda$
- $\partial\Omega_\nu \setminus M_\epsilon \subset\subset M$
- $\Omega_\nu \subset \Omega_\mu$ if $\mu > \nu$
- $\bigcup\limits_{\nu<\mu} \Omega_\nu = \Omega_\mu$
- $\overline{\Omega_\mu} = \bigcap\limits_{\nu>\mu} \Omega_\nu$
- $\Omega_1 \ni z$.

We claim that $f$ is CR on $\Omega_1$, which concludes the proof. In fact, if, by absurd, $\nu_0 < 1$ is the maximal index for which $f$ is CR on one of the $\Omega_\nu$, we can apply Theorem A, for $N$ replaced by $\partial\Omega_{\nu_0}$. Note here that, since $f$ is CR on $\Omega_{\nu_0}$, then its boundary value on $\partial\Omega_{\nu_0}$ is also CR, by the initial assumption of continuity for $f$.

We get that $f$ is CR in a neighbourhood of $\partial\Omega_{\nu_0}$ in $M$. Since $\partial\Omega_{\nu_0} \setminus M_\epsilon$ is compact, we have, by a finite covering argument, that $f$ is CR in some $\Omega_\mu$, for $\mu > \nu_0$; this is a contradiction for the maximality of $\nu_0$. □

**Remark 2.5.7.** As we have already noticed in the course of the proof of Theorem A, by slicing $S$ by means of the leaves $L_\lambda$ and by applying Fubini's Theorem, we need not to assume that $f$ is $C^1$ and we can just require that the restrictions $f|_{L_\lambda}$ are $C^1$. In particular, when the $L_\lambda$'s are replaced by complex curves, this comes as a consequence of the hypothesis that $f$ is holomorphic along these curves.

Theorem B applies in particular if we replace $M$ by an open subset $V$ of $\mathbb{C}^n$ and the manifolds $L_\lambda$ by complex curves $\gamma_\lambda$, and yields the proof of the following

**Corollary 2.5.1.** *Let $V$ be an open domain of $\mathbb{C}^n$, $N$ a part of its boundary, $\{\gamma_\lambda\}$ a foliation of $V$ transversal to $N$ and let $f$ be continuous on $V$, CR on $N$ and holomorphic on each $\gamma_\lambda$. Then $f$ is holomorphic all over $V$.*

Here is another relevant application of our results.

**Example 2.5.1.** We present an application of Theorems A and B which gives relevance to our result, because it is a case in which the Theorem by Henkin and Tumanov cannot be used, but our Theorem does.

We want that $N$ is not open and that the $L_\lambda$ are not foliated by complex curves. To get the second requirement, we can choose pseudoconvex manifolds $L_\lambda$, in fact, if $L_\lambda : r = 0$ is pseudoconvex, and $\gamma$ is a complex curve, $\gamma \subseteq \{r = 0\}$, we have that $r \circ \gamma(\tau) = 0$ and $\overline{\partial}_\tau \partial_\tau r \circ \gamma(\tau) = L(r)(\gamma'(\tau), \overline{\gamma'(\tau)}) = 0$: an absurd.

In $\mathbb{C}^4$ let $M = \{z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : y_1 \geq -(|z_2|^2 + |z_3|^2 + |z_4|^2)\}$, with boundary $N = \{z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : y_1 = -(|z_2|^2 + |z_3|^2 + |z_4|^2)\}$. Let $\{L_{a,b,c}\}_{a,b,c\in\mathbb{R}}$ be manifolds in $\mathbb{C}^4$ defined by

$$\begin{cases} y_2 = |z_1|^2 + a \\ y_3 = |z_1|^2 + b \\ y_4 = |z_1|^2 + c. \end{cases}$$

$M$ and $N$ are CR manifolds ($M$ is open and $N$ is a hypersurface); the $L_{a,b,c}$ are CR manifolds of CR-dimension 1 and they provide a foliation of $M$; then, the intersection of $L_{a,b,c}$ with $N$ is not empty and gives a manifold of dimension 4.

We consider $T^{\mathbb{C}}L_{a,b,c}$ as $\mathsf{Re}\, T^{1,0}L_{a,b,c}$; we get that $T^{1,0}L_{a,b,c}$ is a complex bundle of dimension 1 generated by the vector $\partial_{z_1} + 2i\overline{z_1}(\partial_{z_2} + \partial_{z_3} + \partial_{z_4})$, so its real part is a (real) space of dimension 2. We have

$$\begin{aligned} T^{\mathbb{C}}L_{a,b,c} = \mathsf{Re}\, T^{1,0}L_{a,b,c} = \\ &= \Big\langle \frac{1}{2}\partial_{x_1} + y_1\partial_{x_2} + x_1\partial_{y_2} + y_1\partial_{x_3} + x_1\partial_{y_3} + y_1\partial_{x_4} + x_1\partial_{y_4}, \\ &\frac{1}{2}\partial_{y_1} + y_1\partial_{y_2} - x_1\partial_{x_2} + y_1\partial_{y_3} - x_1\partial_{x_3} + y_1\partial_{x_4} - x_1\partial_{x_4} \Big\rangle \end{aligned}$$

The tangent space of $M$ is $\mathbb{C}^4$, while $TN$ is a (real) space of dimension 7 defined by the equation:

$$Y_1 - 2x_2X_2 - 2y_2Y_2 - 2x_3X_3 - 2y_3Y_3 - 2x_4X_4 - 2y_4Y_4 = 0.$$

Note that, near $N$, all the variables, except $y_1$, are not relevant; the direction of $y_1$, not present in $TN$, comes from the second vector of the basis of $T^{\mathbb{C}}L_{a,b,c}$; so the condition $T^{\mathbb{C}}L_{a,b,c}\big|_{N\cap L_{a,b,c}} + TN\big|_{N\cap L_{a,b,c}} = TM\big|_{N\cap L_{a,b,c}}$ holds.

If we consider a small perturbation of this setting, the conclusions are the same. The setting of this example is good for our Theorem; then, if $f \in C^0(M) \cap CR(N) \cap CR(L_{a,b,c})$, for $a, b, c \in \mathbb{R}$, we conclude that $f$ is CR on $M$. The use of Henkin-Tumanov Theorem is not possible, because the $L_{a,b,c}$ are strictly pseudoconvex, so they cannot be foliated by complex curves.

# Chapter 3

# Peak interpolation manifolds

## 3.1 Basic definitions and remarks

### §. Convexity and pseudoconvexity

We start the introduction to Chapter 3 with the basic notions of convexity and pseudoconvexity, described from a geometrical and analytical point of view. The first definitions are in $\mathbb{R}^n$, but we will briefly pass to the complex-analytic analogous of these notions.

Let $D \subseteq \mathbb{R}^n$ be a domain with $C^1$ boundary and let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a $C^1$ defining function for $D$, that means $D = \{x \in \mathbb{R}^n : \rho(x) < 0\}$, $\partial D = \{x \in \mathbb{R}^n : \rho(x) = 0\}$ and $\nabla\rho(x) \neq 0$, $\forall x \in \partial D$. Given a boundary point $p$, we denote by $\nu_p$ the unit outward normal to $\partial D$ at $p$. The geometric definition of tangent vector, for a vector $w \in \mathbb{R}^n$, requires $w \perp \nu_p$ at $p$, that is $w \cdot \nu_p = 0$, but it is well known from calculus that $\nabla\rho$ is the normal $\nu_p$ and that the normal is uniquely determined and independent of the choice of $\rho$, so $w \in T_p(\partial D)$ if

$$\sum_{j=1}^{n} \left(\frac{\partial \rho}{\partial x_j}\right)(p) \cdot w_j = 0.$$

The geometric notion of convexity is well known and classical

**Definition 3.1.1.** *A domain $D \subseteq \mathbb{R}^n$ is said to be convex if, whenever the points $p$, $q \in D$ and $0 \leq \lambda \leq 1$, then $(1-\lambda)p + \lambda q \in D$.*

Being this notion nonquantitative and non local, it is difficult to use it, so we need to express it in an analytic way. Here is the first definition

**Definition 3.1.2.** *Let $D \subset\subset \mathbb{R}^n$ be a domain with $C^2$ boundary, $\rho$ a defining function for $D$ and $p \in \partial D$. $\partial D$ is (weakly) convex at $p$ if*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(p) w_j w_k \geq 0, \qquad \forall w \in T_p(\partial D); \tag{3.1}$$

*$\partial D$ is strongly (or strictly) convex at $p$ if the inequality is strict whenever $w \neq 0$, $w \in T_p(\partial D)$. The domain itself is said to be convex if, for every boundary point, $\partial D$ is convex.*

Note that the matrix

$$\left(\frac{\partial^2 \rho}{\partial x_j \partial x_k}(p)\right)_{j,k} \tag{3.2}$$

is called the "real Hessian" of the function $\rho$, while the quadratic form

$$T_p(\partial D) \ni w \longmapsto \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(p) w_j w_k, \qquad p \in \partial D$$

(which is positive semi-definite at each point $p \in \partial D$ if $D$ is convex) is also known as the second fundamental form for $\partial D$.

**Remark 3.1.1.** We will use for the rest of the thesis the word "strict" referred to strict inequality in condition (3.1) of the previous definition, as most of the texts and papers do in this setting. It is common to use either of the words "strong" or "strict" with this meaning, but there is a technical difference, as it is referred and explained by Lempert in his paper [40]. A domain $D$ is strictly convex if, given $z_1, z_2 \in \bar{D}$, the internal points of the line segment connecting $z_1$ and $z_2$ belong to $\overset{\circ}{D}$ (that is equivalent to say that the boundary of $D$ does not contain any line segment), while a domain $D$ is strongly convex if it is bounded, with $C^2$ boundary and all the curvatures, normal to $\partial D$, are positive. Note that the matrix (3.2) is positive definite at $p \in \partial D$ if and only if all curvatures are positive at $p$, because one may change coordinates at $p$ so that $\partial D$ agrees with a ball up to and including second order at $p$. This says that the right word for Definition 3.1.2 in the case of strict inequality should be "strong", not "strict".

It is easy to show that if a domain (or even a set) is strongly convex, then it is also strictly convex in the geometric sense; the converse does not hold in general. It is also true, for domains having $C^2$ boundary, that $D$ is weakly convex (in the analytic definition) if and only if $D$ is geometrically convex (in the Definition 3.1.1).

For an interesting discussion about these questions, we refer to a paper by Dalla and Hatziofratis [20], where a a result for bounded domains with real analytic boundary is given.

Now we show the complex-analytic analogous of convexity in $\mathbb{R}^n$. First of all, if $D \subset\subset \mathbb{C}^n$ with $C^2$ boundary, a vector $w \in \mathbb{C}^n$ belongs to $T_p(\partial D)$ if

$$\mathsf{Re}\left(\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p) w_j\right) = 0$$

while $w \in T^{\mathbb{C}}(\partial D)$ if

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p) w_j = 0,$$

and the last equation is closed under multiplication by $i$ of $w$. When we write in complex coordinates the condition of convexity on tangent vectors, as defined in (3.1), we get the following inequality

$$\mathsf{Re}\left(\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(p) w_j w_k\right) + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq 0. \tag{3.3}$$

To simplify notations we will use sometimes the following definition, also good for Section 3.4.

**Definition 3.1.3.** *If $U$ is an open set of $\mathbb{C}^n$ and $\rho : U \to \mathbb{R}$ is a function of class $C^2$, we denote by*

$$\begin{aligned}
\mathcal{P}_p(\zeta) &:= \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(p) \zeta_j \zeta_k, && \zeta \in \mathbb{C}^n, p \in U, \\
\mathcal{L}_p(\zeta, \eta) &:= \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) \zeta_j \bar{\eta}_k, && \zeta, \eta \in \mathbb{C}^n, p \in U, \\
Q_p(\zeta) &:= \mathcal{P}_p(\zeta) + \mathcal{L}_p(\zeta, \zeta), && \zeta \in \mathbb{C}^n, p \in U.
\end{aligned}$$

With the new notation, (3.3) can be written as

$$\mathsf{Re}\, Q_p(w) = \mathsf{Re}\, \mathcal{P}_p(w) + \mathcal{L}_p(w, w) \geq 0$$

which says, as before, that the real Hessian is positive semidefinite at $p \in \partial D$. In complex coordinates, the real Hessian decomposes into two expressions: the first, denoted by $\mathsf{Re}\, \mathcal{P}_p(w)$ is called Levi polynomial of $\rho$ at $p$, while the second one $\mathcal{L}_p(w, w)$ is the Levi form of the function $\rho$ and is the complex Hessian of $\rho$, because it is the only part of the real Hessian preserved under biholomorphic mappings. Note that the Levi form is real valued because $\overline{\mathcal{L}_p(w, w)} = \mathcal{L}_p(w, w)$ and its image is contained in the totally real part of the real tangent space of $\partial D$ at $p$. We are ready for the definitions of pseudoconvex and strictly pseudoconvex domains.

**Definition 3.1.4.** *Let $D \subseteq \mathbb{C}^n$ be a domain with $C^2$ boundary and let $p \in \partial D$. Let $\rho$ be a $C^2$ defining function for $D$. $\partial D$ is pseudoconvex at $p$ if*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq 0, \qquad \forall w \in T_p^{\mathbb{C}}(\partial D);$$

*$\partial D$ is strictly pseudoconvex at $p$ if the inequality is strict whenever $w \neq 0, w \in T_p^{\mathbb{C}}(\partial D)$. The domain itself is said to be pseudoconvex (or strictly pseudoconvex) if, for every boundary point, $\partial D$ is pseudoconvex (or strictly pseudoconvex).*

Note that the collection of pseudoconvex domains is, in a "local sense", the smallest class of domains that contains the convex domains and is closed under biholomorphic mappings, as the following proposition states.

**Proposition 3.1.1.** *If $D \subset\subset \mathbb{C}^n$ is a domain with $C^2$ boundary, then every point $p \in \partial D$ where $\partial D$ is convex is also a point where $\partial D$ is pseudoconvex.*

It is extremely difficult to give an elementary geometric description of weakly pseudoconvex points in terms of convexity; in 1972 it was conjectured that a (weakly) pseudoconvex point $p \in \partial D$ had the property that there is a holomorphic change of coordinates $\Phi$ on a neighbourhood $\Omega$ of $p$ such that $\Phi(\Omega \cap \partial D)$ is convex, but this conjecture is false, as it has been showed through the famous example of Kohn-Nirenberg. Reserchers are still working to understand which pseudoconvex boundary points are "convexifiable" and what this implies for the domain under consideration (see for example the paper of Martin Kolář [38]).

Much more is known in the case of strictly pseudoconvex points, for which the geometric description is easier, as the following proposition of Narasimhan clarifies.

**Proposition 3.1.2.** *Let $D \subset\subset \mathbb{C}^n$ be a domain with $C^2$ boundary and $p \in \partial D$ a point of strict pseudoconvexity. Then, there is a neighbourhood $\Omega \subseteq \mathbb{C}^n$ of $p$ and a biholomorphic mapping $\Phi$ on $\Omega$ such that $\Phi(\Omega \cap \partial D)$ is strictly convex.*

The previous statement was refined by Fornaess in 1974 with the following embedding Theorem

**Theorem 3.1.1** ([22]). *Let $D \subset\subset \mathbb{C}^n$ be a strictly pseudoconvex domain with $C^2$ boundary. Then, there is an integer $n' > n$, a strictly convex domain $D' \subseteq \mathbb{C}^{n'}$, a neighbourhood $\widetilde{D}$ of $\overline{D}$ and a one-to-one embedding $\Phi : \widetilde{D} \to \mathbb{C}^{n'}$ such that*

1. $\Phi(D) \subseteq D'$
2. $\Phi(\partial D) \subseteq \partial D'$
3. $\Phi(\widetilde{D} \setminus \overline{D}) \subseteq \mathbb{C}^{n'} \setminus \overline{D'}$
4. *$\Phi(\widetilde{D})$ is transversal to $\partial D'$ (that means $T_p\Phi(\widetilde{D}) + T_p(\partial D') = T_p\mathbb{C}^{n'}$, $\forall p \in \Phi(\widetilde{D}) \cap T_p(\partial D')$)*

Fornaess asserted that a strictly pseudoconvex domain (which in general is not strictly convex) embeds properly into a high dimensional strictly convex domain: we will use this Theorem in Section 3.4 to extend the result of Rudin for strictly convex domains to the strictly pseudoconvex case. Moreover, it is possible to have proper embeddings of strictly pseudoconvex domains into balls and polydiscs: Forstnerič in 1986 proved the following Theorem

**Theorem 3.1.2** ([26]). *If $X$ is a Stein space and $D$ a relatively compact strictly pseudoconvex domain in $X$ whose boundary is of class $C^2$ and is contained in the set of smooth points of $X$, then $D$ can be mapped biholomorphically onto a closed complex subvariety of a ball $\mathbb{B}^N$.*

In particular, every bounded strictly pseudoconvex domain $D$ of class $C^2$ in $\mathbb{C}^n$ can be embedded properly into a high dimensional ball. The same result was proved simultaneously by Løw who showed that the embedding can be made continuous on $\bar{D}$. He also showed that every such domain can be embedded into a polydisc (cf. [41])

To conclude, we present an example of pseudoconvex domain.

**Example 3.1.1.** Let $D = \{z \in \mathbb{C}^2 : |z_1|^2 + |z_2|^4 < 1\}$; the Levi form of the defining function $\rho(z_1, z_2) = |z_1|^2 + |z_2|^4 - 1$, when applied to $(w_1, w_2)$ is $\mathcal{L}_z(w, w) = |w_1|^2 + 4|z_2|^2|w_2|^2$. This calculus shows immediately that $\partial D$ is strictly pseudoconvex except when the boundary points $z$ satisfy $|z_2|^2 = 0$ and when the tangent vectors $w$ satisfy $w_1 = 0$: the boundary points where the domain is weakly pseudoconvex are of the type $(e^{i\theta}, 0)$.

### §. Finite type

The theory of weakly pseudoconvex domains of $\mathbb{C}^n$ often involves finite type conditions. If $D$ is a domain of $\mathbb{C}^n$ with smooth boundary $S = \partial D$, such that $S$ has a degenerate Levi form, it is possible to introduce several distinct definitions that go by the name of "points of finite type". These notions have arisen since 1972, when Kohn first defined this concept for points on the boundaries of smoothly bounded pseudoconvex domains in $\mathbb{C}^2$; he established that this notion was a sufficient condition for subelliptic estimates in the $\bar{\partial}$-Neumann problem.

The algebraic-geometric definition is due to D'Angelo and deals with the order of contact with complex varieties and their intersection theory. The complete work has appeared in [19] but has taken its origin with [18].

For domains in $\mathbb{C}^2$ D'Angelo definition of finite type becomes easier

**Definition 3.1.5.** *Let $D \subset \mathbb{C}^2$ be a bounded domain having a smooth boundary and let $p \in \partial D$. The type of $p$, denoted by $\tau(p)$, is the maximum order of contact that a 1-dimensional complex subvariety (of some open neighbourhood of $p$) can have with $\partial D$ at $p$. The point $p$ is said to be of finite type if $\tau(p) < \infty$. The domain $D$ is said to be of finite type if there is $n \in \mathbb{N}$ such that $\tau(p) \leq n$ for each $p \in \partial D$.*

We are interested in "brackets" or "Bloom-Graham" finite type, which is different from D'Angelo finite type. To introduce it, let's start considering a smooth manifold $S$, its holomorphic and antiholomorphic vector bundles $T^{1,0}S$ and $T^{0,1}S$ and its complex tangent space $T^{\mathbb{C}}S = TS \cap \mathcal{J}TS$. Note that $\mathbb{C} \otimes_{\mathbb{R}} TS$ is integrable (that is closed under Lie brackets), but $\mathbb{C} \otimes_{\mathbb{R}} T^{\mathbb{C}}S = T^{1,0}S \oplus T^{0,1}S$ is not, in general. We set $\mathscr{L}^1 = \mathbb{C} \otimes_{\mathbb{R}} T^{\mathbb{C}}S$ and denote by $\mathscr{L}^j$ the distribution of vector spaces spanned by Lie brackets of holomorphic and antiholomorphic vector fields of length $\leq j$. If, at a point $p_0$, there is an integer $m_1 \geq 2$ such that

$$\mathscr{L}^j_{p_0} = \mathscr{L}^1_{p_0} \qquad \forall j \leq m_1 - 1, \qquad \mathscr{L}^{m_1}_{p_0} \supsetneqq \mathscr{L}^1_{p_0} \tag{3.4}$$

we refer to $m_1$ as the first Hörmander number of $S$ at $p_0$. Otherwise, if $\mathscr{L}^j_{p_0} = \mathscr{L}^1_{p_0}, \forall j$, we set $m_1 = +\infty$. This process can continue by looking for $m_2 > m_1$ such that

$$\mathscr{L}^j_{p_0} = \mathscr{L}^{m_1}_{p_0} \qquad \forall j < m_2, \qquad \mathscr{L}^{m_2}_{p_0} \neq \mathscr{L}^{m_1}_{p_0}.$$

We arrive at the following definition

**Definition 3.1.6.** *$S$ is of finite type at $p_0$ when commutators span the full $\mathbb{C}\otimes_{\mathbb{R}} T^{\mathbb{C}}_{p_0}S$, that is when the above process ends with a number $m_r < +\infty$.*

By linearity of commutators, the condition of finite type for $S$ is equivalent to the following condition

$$\begin{aligned} &[X_1, [X_2, \ldots, [X_{j-1}, X_j] \ldots]] \in \mathscr{L}^1 \ \ \forall X_i \in \mathscr{L}^1, \forall j \leq m_1 - 1 \\ &[X_0^{\epsilon_1}, [X_0^{\epsilon_2}, \ldots, [X_0^{\epsilon_{m_1-2}}, [X_0, \bar{X}_0]] \ldots]] \notin \mathscr{L}^1 \ \text{ for some } X_0 \in \Gamma(T^{1,0}S), \end{aligned} \tag{3.5}$$

where $\Gamma(T^{1,0}S)$ is the space of sections of the fiber bundle $T^{1,0}S$, $\epsilon = (\epsilon_1, \ldots, \epsilon_{m_1-2})$ and $\epsilon_i = +1$ or $\epsilon_i = -1$, with the convention that $X_0^1 = X_0$ and $X_0^{-1} = \bar{X}_0$.

Note that the requirement (3.5) in the definition of finite type throught commutators can be strengthened asking for *every* $X \in \Gamma(T^{1,0}S)$ that the corresponding commutator $[X^{\epsilon_1}, [X^{\epsilon_2}, \ldots, [X^{\epsilon_{m_1-2}}, [X, \bar{X}]] \ldots]]$ of length $m_1 - 1$ (that is a total of $m_1$ $X$'s and $\bar{X}$'s) lies out of $\mathscr{L}^1 = \mathbb{C} \otimes_{\mathbb{R}} T^{\mathbb{C}}S$. This will be our choice in Section 3.2.1 in proving a property of pseudoconvex domains through Lie brackets.

Note also that iterated commutators of the type of (3.5) are strictly connected with the derivatives of the Levi form in the directions defined by $X$ and its conjugate $\bar{X}$. The precise formula is contained in a paper of D'Angelo [17].

### §. Peak interpolation manifolds

Let $D$ be a bounded domain in $\mathbb{C}^n$ with smooth boundary $S = \partial D$. For $0 \leq \alpha \leq \infty$, we denote by $A^\alpha(D)$ the algebra of all functions $f \in C^\alpha(\bar{D})$ that are holomorphic in $D$. We also denote by $A^\omega(D)$ the algebra of all functions holomorphic in a neighbourhood of $\bar{D}$. We will write $A(D)$ for $A^0(D)$ and this will be our setting in most of the following work; in particular we will always refer to $A(D)$ when we will describe the techniques of Henkin-Tumanov and Rudin. Note that the obvious relation holds: $A^\omega(D) \subset A^\infty(D) \subset \ldots \subset A^k(D) \subset A^{k-1}(D) \subset \ldots \subset A^1(D) \subset A(D)$.

Here are the basic definitions of the chapter; we assume for instance that $S = \partial D$ is a compact hypersurface of $\mathbb{C}^n$.

**Definition 3.1.7.** *A submanifold $M$ of $S$ is called an interpolation manifold for $A^\alpha(D)$ if, for every $f \in C^\alpha(M)$ and every compact set $K \subset M$, there exists a function $F \in A^\alpha(D)$ such that $F|_K = f|_K$.*

**Definition 3.1.8.** *A submanifold $M$ of $S$ is called a peak manifold for $A^\alpha(D)$ if, for every compact set $K \subset M$, there exists a function $F \in A^\alpha(D)$ such that $F|_K = 1$ and $|F| < 1$ on $\bar{D} \setminus K$.*

If a manifold $M$ of $S$ is a peak interpolation manifold for $A^\alpha(D)$ for some $\alpha$, we will use sometimes the notation "PI manifold".

**Remark 3.1.2.** Compactness is essential in the previous definition; for this reason we have assumed that $S$ is a compact hypersurface of $\mathbb{C}^n$; if not, the property of "peaking" has to be referred to the compact submanifolds (or even subsets) of $M$. In particular, we need $M$ (or its subsets) to be closed because otherwise there would be a point $p$ of accumulation for $M$ that lies out of $M$ and extension with continuity would be impossible for a function defined on $D$: this is, evidently, in contrast with the definition of peak manifold that requires $f \in C(\bar{D})$. Note that it has been proved in [53] and [31] that for the algebra $A(S)$, with $S$ a strictly pseudoconvex hypersurface of $\mathbb{C}^n$, the problems of characterizing compact interpolation sets, compact peak sets and compact peak interpolation sets are equivalent.

**Definition 3.1.9.** *A submanifold $M$ of $S$ is complex-tangential (or is an integral manifold of $T^{\mathbb{C}}S$) if $TM \subset T^{\mathbb{C}}S$.*

The definition of complex tangential manifold is essential for our purpose: in fact, for $S$ a strictly pseudoconvex compact hypersurface, we will prove in Section 3.3 that a smooth submanifold $M$ of $S$ is a PI manifold if and only if it is a complex tangential submanifold of $S$. Note that the easiest example of complex tangential manifold is the point and we refer to Section 3.2 for the description of the properties of complex tangential manifolds in suitable settings.

The property of "peaking" can also be referred to a single point of the boundary, as well as to a single function, so we have to add the following definition.

**Definition 3.1.10.** *A point $p \in S = \partial D$ is a peak point for $A^\alpha(D)$ if there exists a function $F \in A^\alpha(D)$ such that $F(p) = 1$ and $|F| < 1$ on $\bar{D}\backslash\{p\}$. We call $F$ a (global) peak function at $p \in \partial D$, or, equivalently, we say that $F$ peaks (globally) at $p$. When this property is referred to a function $F \in A(D \cap \Omega)$, for some neighbourhood $\Omega$ of $p$, $F$ is said to be a local peak function at $p$.*

The fact that the point $p$ in the above definition has been chosen on $\partial D$ is forced: in fact, as it is well known, the maximum principle for holomorphic functions says that if $f \in \mathcal{O}(D) \cap C(\bar{D})$ then

$$|f(z)| \leq \sup_{\partial D}|f|, \qquad z \in D$$

and the inequality is strict unless $f$ is constant; then, by maximum principle, any peak point for any subalgebra of $A(D)$ must lie on the boundary of $D$.

The problem of proving the existence of peaking functions on various subalgebras of $A(D)$ is strictly related to the converse of maximum principle and gives important informations on the structure of the given subalgebra of $A(D)$: peak functions are used for the construction of solution operators of $\bar{\partial}$ which satisfy $L^\infty$ or Hölder estimates, for the estimate of Caratheodory metric near the boundary and for the embedding problem for abstract CR manifolds.

The existence of a peak function at $p \in \partial D$ is equivalent to the existence of a strong support function at $p$, that is defined as follows.

**Definition 3.1.11.** *A function $g \in A^\alpha(D)$ is a strong support function at $p \in \partial D$ if $g(p) = 0$ and $\mathsf{Re}\, g > 0$ on $\bar{D} \setminus \{p\}$.*

The equivalence mentioned above is due to the following two remarks: from one side, if $f$ is a peak function at $p$, then $f(p) = 1$ and the property $|f| < 1$ implies $|\mathsf{Re}\, f| < 1$ in $U_p$, so it is sufficient to define locally $g := 1 - f$ to have $g(p) = 1 - f(p) = 0$ and $\mathsf{Re}\, g = 1 - \mathsf{Re}\, f > 1 - |\mathsf{Re}\, f| > 0$; from the other side, if $g$ is a strong support function at $p$, that is $g(p) = 0$ and $\mathsf{Re}\, g > 0$, we can choose, as a peak function at $p$, the function $f := e^{-\mathsf{Re}\, g}$, but also $f := \frac{1-g}{1+g}$; for the second one, note for instance, that calculation gives

$$\left|\frac{1-g}{1+g}\right|^2 = \frac{1 - 2\mathsf{Re}\, g + |g|^2}{1 + 2\mathsf{Re}\, g + |g|^2} < 1,$$

in fact $f$ is the image of $g$ through the mapping

$$\begin{aligned} \psi : &\mathbb{C} \longrightarrow \mathbb{C} \\ &z \longmapsto \frac{1-z}{1+z} \end{aligned}$$

in which the image of the real half plane of $\mathbb{C}$ is the unit complex disc.

There is a notion of support function, which is weaker than the previous one and whose existence is easily ensured in the case of boundary points of strict convexity and strict pseudoconvexity.

**Definition 3.1.12.** *A point $p \in \partial D$ has a holomorphic support function for $D$ if there is a neighbourhood $U_p$ of $p$ and a holomorphic function $g_p : U_p \to \mathbb{C}$ such that $\{z \in U_p : g_p(z) = 0\} \cap \bar{D} = \{p\}$.*

**Proposition 3.1.3.** *If $D \subseteq \mathbb{C}^n$ is a domain and $p \in \partial D$ is a point of strict convexity, then there exists a holomorphic support function for $D$ at $p$.*

*Proof.* We can assume $p = 0$ and, being $p$ a point of strict convexity, there exists a neighbourhood $U_p$ of $p$ such that $T_p(\partial D) \cap \bar{D} \cap U_p = \{p\}$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is the unit outward normal to $\partial D$ at $p$, we may always identify $T_p(\partial D)$ with

$$\{(z_1, \ldots, z_n) : \mathsf{Re} \sum_{j=1}^n \bar{\alpha}_j z_j = 0\}.$$

Then, the function $f(z) := \sum_{j=1}^n \bar{\alpha}_j z_j \in \mathcal{O}(\mathbb{C}^n)$ is the required holomorphic support function for $D$ at $p$, since the zero set of $f$ lies in $T_p(\partial D)$. $\square$

**Proposition 3.1.4.** *If $D \subseteq \mathbb{C}^n$ is a domain and $p \in \partial D$ is a point of strict pseudoconvexity, then there exists a holomorphic support function for $D$ at $p$.*

*Proof.* The result follows from Narasimhan's Proposition but it is also possible to construct explicitely the holomorphic support function in the following way: if $\rho$ is the defining function for $D$, we write the Taylor expansion of $\rho$ at $p$ using Definition 3.1.3 and the usual inner product in $\mathbb{C}^n$ $\langle\cdot,\cdot\rangle$, where $\langle z,w\rangle = \sum_{j=1}^{n} z_j\bar{w}_j$,

$$\rho(z) = 2\mathsf{Re}\,\langle z-p, \bar{\partial}\rho(p)\rangle + \mathsf{Re}\,\mathcal{P}_p(z-p) + \mathcal{L}_p(z-p,z-p) + o(|z-p|^2)$$

By strict pseudoconvexity at $p$

$$\mathcal{L}_p(z-p,z-p) \geq C|z-p|^2 \qquad \text{for } z \in U_p; \tag{3.6}$$

note that (3.6) is motivated by the fact that there is no loss in generality if we add $C\rho^2$ to the defining function $\rho$, to have the Levi form of $\rho + C\rho^2$ strictly positive also in the normal direction, and not only in the directions of $T^{\mathbb{C}}(\partial D)$. If we define

$$f(z) := \langle z-p, \bar{\partial}\rho(p)\rangle + \frac{1}{2}\mathcal{P}_p(z-p)$$

and if $z$ is a point for which $f(z) = 0$ we get

$$\begin{aligned}\rho(z) &= \mathcal{L}_p(z-p,z-p) + o(|z-p|^2)\\ &\geq C|z-p|^2 + o(|z-p|^2)\\ &\geq \frac{C}{2}|z-p|^2 \qquad\qquad \text{if } z \approx p;\end{aligned}$$

Then, if $z \in U_p$ and $f(z) = 0$, we only have two possibilities: $\rho(z) > 0$ (that means $z \notin \bar{D}$) or $z = p$, which ensures that $f$ is the required holomorphic support function at $p$. □

Note that Kohn-Nirenberg example [37] has showed that at a point of weak pseudoconvexity it is no more ensured the presence of a strong, or even weak, holomorphic support function.

Now we provide some useful remarks; the first justifies the choice of pseudoconvex domains as the natural setting where look for peaking points (and functions).

**Remark 3.1.3.** If $f$ is a peak function in $A^\alpha(D)$ at $p \in \partial D$, then $\frac{1}{1-f}$ is a holomorphic function on $D$ with no holomorphic extension past $p$; therefore, if every point of $\partial D$ is a peak point, $D$ is a domain of holomorphy. It has been proved that these domains are exactly pseudoconvex domains (by the solution of the Levi problem), so it has sense to restrict our research of peaking points to pseudoconvex domains.

Another natural requirement in the setting of peaking points on the boundary of pseudoconvex domains is the condition of finite type.

**Remark 3.1.4.** If we assume for instance that a complex disc lies on the boundary of our domain $D$, by maximum principle there would not be peak points in the interior of the disc, so the obvious requirement is a finite upper bound of the order of contact of complex analytic varieties with $\partial D$ at $p$, that is exactly the definition of finite type, given by D'Angelo.

The following proposition allows us to avoid any sort of research on pseudoconvex domains of odd type; in fact, it turns out, through an easy application of Boggess-Pitts Theorem [14], that a pseudoconvex domain can only have even type at its boundary points.

**Proposition 3.1.5.** *Let $D$ be a bounded pseudoconvex domain of $\mathbb{C}^n$ of type $k$, $k \in \mathbb{N}$, at $p \in S = \partial D$; then, $k = 2m$, $m \in \mathbb{N}$.*

*Proof.* We remind the definition of k-th Levi form; if $X_p \in T_p^{1,0}S$, then

$$\begin{aligned} \mathcal{L}_p^k : T_p^{1,0}S &\longrightarrow \frac{T_p^{1,0}S \oplus T_p^{0,1}S \oplus \left(\frac{T_pS}{T_p^{\mathbb{C}}S} \otimes_{\mathbb{R}} \mathbb{C}\right)}{\mathscr{L}_p^k(S)} \\ \mathcal{L}_p^k(X_p) &:= \frac{1}{2i}\pi_p \left\{ \sum_{\epsilon_1 \ldots \epsilon_{k-2}} C_\epsilon [X^{\epsilon_1}, [X^{\epsilon_2}, \ldots, [X^{\epsilon_{k-2}}, [X, \bar{X}]] \ldots]]_p \right\} \end{aligned} \tag{3.7}$$

where $X \in \Gamma(T^{1,0}S)$ is a vector field extension of $X_p$,

$$\pi_p : T_p^{1,0}S \oplus T_p^{0,1}S \oplus \left(\frac{T_pS}{T_p^{\mathbb{C}}S} \otimes_{\mathbb{R}} \mathbb{C}\right) \longrightarrow \frac{T_p^{1,0}S \oplus T_p^{0,1}S \oplus \left(\frac{T_pS}{T_p^{\mathbb{C}}S} \otimes_{\mathbb{R}} \mathbb{C}\right)}{\mathscr{L}_p^k(S)}$$

is the natural projection and the sum is taken over all $\epsilon = (\epsilon_1, \ldots, \epsilon_{k-2})$ for $\epsilon_i = +1$ or $\epsilon_i = -1$, with the convention that $X^1 = X$ and $X^{-1} = \bar{X}$.

We say that a point $p$ is of type $k$ if and only if $\mathcal{L}_p^{k-1} = 0$ and $\mathcal{L}_p^k \neq 0$ (for some $X_p \in T_p^{1,0}S$).

Let $N_p(S)$ be the space of vectors at $p$ which are normal to $T_pS$; the usual complex structure $\mathcal{J}$ on $\mathbb{R}^{2n}$ induces an isometry $\frac{T_pS}{T_p^{\mathbb{C}}S} \to N_p(S)$, so that if $p$ is a point of type $k$, the k-th extrinsic Levi form $\tilde{\mathcal{L}}_p^k : T^{1,0}(S) \to N_p(S)$ is defined by $\tilde{\mathcal{L}}_p^k = \mathcal{J} \circ \mathcal{L}_p^k$.

Being $S$ a real hypersurface and $\mathcal{L}_p^k \neq 0$, because $p$ is of type $k$, it is clear from (3.7) that image $[\tilde{\mathcal{L}}_p^k] = N_p(S)$ if $k$ is odd and image $[\tilde{\mathcal{L}}_p^k]$ is at least a ray if $k$ is even.

Therefore, applying Boggess-Pitts extension Theorem, CR functions on $S$ near $p$ extend to holomorphic functions on an open set $\tilde{\omega}$ in $\mathbb{C}^n$; if $k$ is odd, then $\tilde{\omega}$ contains $p$, that is $\tilde{\omega}$ lies on both side of $S$, while, if $k$ is even, then $\tilde{\omega}$ lies at least to the one side of $S$ given by image$[\tilde{\mathcal{L}}_p^k]$.

Assuming for instance that $k = 2m + 1$, we get extension on the opposite side of $D$, which is in contrast with the fact that $D$ is pseudoconvex. □

To conclude, we mention Bishop Theorem [11] which provides a measure-theoretic characterization for peak interpolation sets for $A(D)$, when $D \subset \mathbb{C}^n$. For $n = 1$, a previous characterization for the disc algebra is due to Rudin and Carleson and can be summarized by the following

**Theorem 3.1.3** (Rudin-Carleson). *Let $\Delta$ be the open unit disc in $\mathbb{C}$. The interpolation, peak, peak interpolation sets for $A(\Delta)$ coincide and are precisely the subsets of $\partial\Delta$ having Lebesgue measure 0 (relative to $\partial\Delta$).*

Recall that $A(D)$, for $D \subseteq \mathbb{C}^n$, is a proper closed subspace of $C(\bar{D})$, thus we can consider those bounded linear functionals of $C(\bar{D})$, which are precisely the regular, complex Borel measures on $\bar{D}$, that annihilate $A(D) \subset C(\bar{D})$. We say that a regular, complex Borel measure $\mu$ on $\bar{D}$ is an annihilating measure if, viewed as a bounded linear functional of $C(\bar{D})$, it annihilates $A(D)$, that is $\int f d\mu = 0$, $\forall f \in A(D)$; we write $\mu \perp A(D)$ for this property. Here is the statement of Bishop's Theorem, which will be useful in Section 3.4 in applying Rudin's technique.

**Theorem 3.1.4** (Bishop). *Let $D$ be a bounded domain in $\mathbb{C}^n$. A compact set $K \subset \partial D$ is a peak interpolation set for $A(D)$ if and only if $\mu(K_0) = 0$ for every annihilating measure $\mu$ $(\mu \perp A(D))$ and for every compact set $K_0 \subset K$.*

**Remark 3.1.5.** Note that Bishop's Theorem applies in a more general context than our setting of pseudoconvex domains. The original statement in [11] is given for $X$ a compact Hausdorff space, taking the uniformly-normed Banach space $C(X)$, a closed subspace $B \subset C(X)$ and $B^{\perp}$ as the space of all measures $\mu$ on $X$ such that $\mu \perp f$, $\forall f \in B$.

## 3.2 Properties of complex tangential manifolds

We will say in the next section that the necessary and sufficient condition, for a smooth submanifold of a strictly pseudoconvex hypersurface, to be a peak interpolation manifold is the property of being complex tangential; before proving the result, our interest is to investigate the setting of complex tangential submanifolds of $\mathbb{C}^n$.

For strictly pseudoconvex domains in $\mathbb{C}^n$, complex tangential manifolds satisfy a relevant property, which has been presented by Henkin and Tumanov in [32] and Rudin in [48]: they are totally real, which means that the complex tangent space is the null space.

The aim of the first subsection is to propose an easier proof of this property and to show how the same technique leads us to generalize the statement to pseudoconvex domains of finite type.

Another property concerns the dimension of a complex tangential manifold (or, equivalently, for strictly pseudoconvex domains, the dimension of a peak interpolation manifold): this cannot exceed $n-1$. In the second subsection we prove this statement and refer to a paper by Stout ([50]) to get the same result for a class of domains more extensive than the class of strictly pseudoconvex domains. We consider the dimension of peak interpolation sets not only in a topological sense but also in metric sense and we show that no results of this type can be stated for metric dimension, through the examples of Tumanov [52] and Stensønes [49].

### 3.2.1 Necessity of being totally real

#### §. Strictly pseudoconvex case

Let $D$ be a domain of $\mathbb{C}^n$ with smooth boundary and $\rho$ a defining function for $D$; here is the statement of the first property when $S = \partial D$ is a strictly pseudoconvex hypersurface of $\mathbb{C}^n$.

**Theorem 3.2.1.** *Let $S = \partial D$ be a strictly pseudoconvex hypersurface of $\mathbb{C}^n$ and let $M \subset S$ be a complex tangential submanifold of $S$ (that means $TM \subset T^{\mathbb{C}}S$); then, $M$ is totally real, that is*

$$T^{\mathbb{C}}_{z_0}M = \{0\}, \qquad \forall z_0 \in M.$$

*Proof.* Assume that $T^{\mathbb{C}}_{z_0}M \neq \{0\}$; by definition, it means that there exists $X \in \Gamma(TM)$, the space of sections of the fiber bundle $TM$, such that both $X(z_0)$ and $\mathcal{J}X(z_0)$ belong to $T_{z_0}M$, where $\mathcal{J}$ is the complex structure on $TM$. Then, we can extend $\mathcal{J}X(z_0)$ to a section $Y \in \Gamma(TS)$ (for which obviously $Y(z_0) = \mathcal{J}X(z_0)$); we are able to prove that

(i) $2i[X, Y] \in \Gamma(\mathbb{C} \otimes T^{\mathbb{C}}S)$

proof: if $X$ and $Y$ are in $\Gamma(TM)$, also $[X,Y] \in \Gamma(TM)$, from which $2i[X,Y] \in \Gamma(\mathbb{C} \otimes TM)$; the property of being complex tangential for $M$, that is $TM \subset T^{\mathbb{C}}S$, lets us conclude that $2i[X,Y] \in \Gamma(\mathbb{C} \otimes T^{\mathbb{C}}S)$;

(ii)

$$\begin{cases} 2i[X,Y] = [X - iY, X + iY] \\ [X - iY, X + iY](z_0) = [X - i\mathcal{J}X, X + i\mathcal{J}X](z_0) \quad \text{modulo } T^{\mathbb{C}}S \end{cases}$$

proof: the first follows immediately by the definition of commutator of vector fields. For the second, if we define $X_0 = X - i\mathcal{J}X$, we note that $X_0$ is a section of $T^{1,0}S$ and $[X - i\mathcal{J}X, X + i\mathcal{J}X](z_0) = 2i\mathcal{L}_S(z_0)(X_0, \bar{X}_0)$ modulo $T^{\mathbb{C}}S$, where we define by $\mathcal{L}_S$ the Levi form of $S$; then, by the fact that the Levi form of a hypersurface is "functorial", that is invariant under biholomorphic changes of coordinates, we can interchange $[X - i\mathcal{J}X, X + i\mathcal{J}X](z_0)$ with $[X - iY, X + iY](z_0)$ through the change $Y = \mathcal{J}X$. By (i) and (ii) we have obtained that there exists a section $X_0$ of $T^{1,0}S$, such that the Levi form of it at $z_0$ is in $\mathbb{C} \otimes T^{\mathbb{C}}S$.

(iii) $\langle [X - i\mathcal{J}X, X + i\mathcal{J}X], \partial\rho \rangle > 0 \quad \text{in } \Gamma\left(\frac{\mathbb{C}\otimes TS}{\mathbb{C}\otimes T^{\mathbb{C}}S}\right)$

proof: the statement follows by the fact that $S$ is strictly pseudoconvex.

The existence of a section $X_0$ of $T^{1,0}S$, such that the Levi form of it at $z_0$ is in $\mathbb{C} \otimes T^{\mathbb{C}}S$ is in contradiction with the fact that, by (iii), its Levi form at $z_0$ is also in $\frac{\mathbb{C}\otimes TS}{\mathbb{C}\otimes T^{\mathbb{C}}S}$. Then, $M$ has to be totally real. □

### §. Pseudoconvex finite type case

Our Theorem for (weakly) pseudoconvex domains of type $k = 2m$, $m \in \mathbb{N}$, requires the following definition of finite type domains.

**Definition 3.2.1.** *The hypersurface $S$ of $\mathbb{C}^n$ is of type $k = 2m$ if for every $X \in \Gamma\left(T^{1,0}S\right)$ we have*

$$\langle [X, [X, \ldots, [X, \bar{X}] \ldots]], \partial\rho \rangle \neq 0 \ \text{ in } \Gamma\left(\frac{\mathbb{C} \otimes TS}{\mathbb{C} \otimes T^{\mathbb{C}}S}\right)$$

*for every commutator of length $2m - 1$ (that is a total of $2m$ $X$'s and $\bar{X}$'s).*

**Theorem 3.2.2.** *Let $S = \partial D$ be a pseudoconvex hypersurface of $\mathbb{C}^n$ of type $k = 2m$, $m \in \mathbb{N}$, and let $M \subset S$ be a complex tangential submanifold of $S$ (that means $TM \subset T^{\mathbb{C}}S$); then, $M$ is totally real, that is*

$$T^{\mathbb{C}}_{z_0}M = \{0\}, \qquad \forall z_0 \in M.$$

*Proof.* Assume that $T^{\mathbb{C}}_{z_0}M \neq \{0\}$; by definition, it means that there exists $X \in \Gamma(TM)$, the space of sections of the fiber bundle $TM$, such that both $X(z_0)$ and $\mathcal{J}X(z_0)$ belong to $T_{z_0}M$, where $\mathcal{J}$ is the complex structure on $TM$. Then, we can extend $\mathcal{J}X(z_0)$ to a section $Y \in \Gamma(TS)$ (for which obviously $Y(z_0) = \mathcal{J}X(z_0)$); we are able to prove that

(i) $[X - iY, [X - iY, \dots, [X - iY, X + iY] \dots]] =$

$= 2i[X, [X, \dots, [X, Y] \dots]] + \sum_{\epsilon_1 \dots \epsilon_{k-2}} C_\epsilon [Z^{\epsilon_1}, [Z^{\epsilon_2}, \dots, [Z^{\epsilon_{k-2}}, [X, Y]] \dots]] \in \Gamma(\mathbb{C} \otimes T^{\mathbb{C}}S)$,

for some $C_\epsilon \in \mathbb{C}$, where the sum is taken over all $\epsilon = (\epsilon_1, \dots, \epsilon_{k-2})$ for $\epsilon_i = +1$ or $\epsilon_i = -1$, with the convention that $Z^1 = X$ and $Z^{-1} = Y$, and where the commutators are of length $k - 1$;

proof: first of all, the equality is given by iterating the definition of Lie brackets; then, for the right hand side, note that if $X$ and $Y$ are in $\Gamma(TM)$, also $[X, Y] \in \Gamma(TM)$, and we can take again the commutator of it with $X$ or $Y$, that are in $\Gamma(TM)$, from which $2i[X, [X, \dots, [X, Y] \dots]] \in \Gamma(\mathbb{C} \otimes TM)$; the property of being complex tangential for $M$ lets us assert that $2i[X, [X, \dots, [X, Y] \dots]] \in \Gamma(\mathbb{C} \otimes T^{\mathbb{C}}S)$.

Reasoning in the same way for $\sum_{\epsilon_1 \dots \epsilon_{k-2}} C_\epsilon [Z^{\epsilon_1}, [Z^{\epsilon_2}, \dots, [Z^{\epsilon_{k-2}}, [X, Y]] \dots]]$, which is given by iterated Lie brackets of elements of $\Gamma(\mathbb{C} \otimes TM) \subset \Gamma(\mathbb{C} \otimes T^{\mathbb{C}}S)$, we get that $[X - iY, [X - iY, \dots, [X - iY, X + iY] \dots]] \in \Gamma(\mathbb{C} \otimes T^{\mathbb{C}}S)$.

(ii) $[X - iY, [X - iY, \dots, [X - iY, X + iY] \dots]](z_0) = [X - i\mathcal{J}X, [X - i\mathcal{J}X, \dots, [X - i\mathcal{J}X, X + i\mathcal{J}X] \dots]](z_0)$ modulo $T^{\mathbb{C}}S$, where the commutators are of length $k - 1$;

proof: the higher order Levi form of a hypersurface is "functorial", that is invariant under biholomorphic changes of coordinates, so we can use the change $Y = \mathcal{J}X$, to get the previous expression. By (i) and (ii) we have obtained that there exists a section $X_0 = X - i\mathcal{J}X$ of $T^{1,0}S$, such that the higher order Levi form of it at $z_0$ is in $\mathbb{C} \otimes T^{\mathbb{C}}S$.

(iii) $\langle [X - i\mathcal{J}X, [X - i\mathcal{J}X, \dots, [X - i\mathcal{J}X, X + i\mathcal{J}X] \dots]], \partial\rho \rangle \neq 0 \quad \text{in } \Gamma\left(\frac{\mathbb{C} \otimes TS}{\mathbb{C} \otimes T^{\mathbb{C}}S}\right)$, where the commutators are of length $k - 1$;

proof: the statement follows by the definition of finite type given before.

The existence of a section $X_0$ of $T^{1,0}S$, such that the higher order Levi form of it at $z_0$ is in $\mathbb{C} \otimes T^{\mathbb{C}}S$ is in contradiction with the fact that, by (iii), its higher order Levi form at $z_0$ is $\neq 0$ in $\frac{\mathbb{C} \otimes TS}{\mathbb{C} \otimes T^{\mathbb{C}}S}$. Then, $M$ has to be totally real. $\square$

### 3.2.2 The dimension of peak interpolation sets

If we consider a bounded strictly pseudoconvex domain $D$ in $\mathbb{C}^n$ with boundary $S = \partial D$, the problem of characterizing peak interpolation manifolds $M \subset S$ is completely solved by the condition for $M$ to be complex tangential. From this condition, it follows immediately that the dimension of a peak interpolation manifold $M$ cannot exceed $n-1$, as the following proposition shows.

**Proposition 3.2.1.** *Let $S = \partial D$ be a strictly pseudoconvex hypersurface of $\mathbb{C}^n$ or a pseudoconvex hypersurface of $\mathbb{C}^n$ of type $k = 2m$, $m \in \mathbb{N}$; let $M \subset S$ be a complex tangential submanifold of $S$. Then, the dimension of $M$ cannot exceed $n-1$.*

*Proof.* $S$ is a hypersurface of $\mathbb{C}^n$ defined by $\rho(z) = 0$, where $\rho(z)$ is a real-valued smooth function, so it has $\dim_{\mathbb{R}} = 2n-1$; $T^{\mathbb{C}}S$ is a real hyperplane in the tangent space $TS$, so its $\dim_{\mathbb{R}}$ is $\leq 2n-1$. Being $M$ a complex tangential manifold, $TM \subset T^{\mathbb{C}}S$, but also $\mathcal{J}(TM) \subset T^{\mathbb{C}}S$, because $T^{\mathbb{C}}S$ is $\mathcal{J}$-invariant; also, the submanifolds $TM$ and $\mathcal{J}(TM)$ have the same dimension, because the action of $\mathcal{J}$ is an isomorphism.

Adding the hypothesis that $M$ is totally real, which follows by Theorem 3.2.1 if $S$ is strictly pseudoconvex, or by Theorem 3.2.2 if $S$ is pseudoconvex of finite type, we know that in $T^{\mathbb{C}}S$ there have to be two submanifolds of equal dimension that have no common intersection. Then, their maximal dimension on $\mathbb{R}$ has to be $n-1$ and we conclude asserting that $\dim_{\mathbb{R}} M \leq n-1$. □

**Remark 3.2.1.** In the setting of strictly pseudoconvex domains the previous proposition can, equivalently, be referred just saying that the dimension of a peak interpolation manifold $M$ of $S$ cannot exceed $n-1$; in the setting of pseudoconvex domains of finite type the previous statement is true for complex tangential manifolds, that are totally real by Theorem 3.2.2, but cannot be reformulated in terms of peak interpolation manifolds. Note that for strictly pseudoconvex domains the problems of characterizing compact interpolation sets, compact peak sets and compact peak interpolation sets are equivalent, so we can always say that the dimension of an interpolation set, or the dimension of a peak set, cannot exceed $n-1$.

The problem of finding the correct bound for the (topological) dimension of a peak interpolation set for a strictly pseudoconvex domain in $\mathbb{C}^n$ was raised in 1978 by Rudin in [48], who conjectured $n-1$ as upper bound. The characterization of Henkin-Tumanov in [32] and an argument of the type of Proposition 3.2.1 yield a proof of Rudin's conjecture. Stout in a paper of 1982 (cf. [50]) obtains the same result for a class of domains larger than the class of strictly pseudoconvex domains. To formulate it, let's recall that a point $p$ in the boundary of a convex domain $D$ is said to be strongly exposed if there are neighbourhoods of $p$ in $\partial D$ of arbitrarily small diameter and of the form $\{z \in \partial D : L(z) < 0\}$, where $L$ is a real-valued, real affine functional on $\mathbb{C}^n$ with $L(p) = 0$. Each point in the boundary of the ball has this property, as does each point in the distinguished boundary of the polydisc. Here is Stout result.

**Theorem 3.2.3** ([50]). *Let $D$ be a bounded open convex set in $\mathbb{C}^n$. If $N \subset \partial D$ is a peak set that consists entirely of strongly exposed points, then dim $N \leq n-1$.*

Stout Theorem implies the corresponding result for smoothly bounded, strictly pseudoconvex domains, for the problem is local and in a neighbourhood of each boundary point of a smoothly bounded strictly pseudoconvex domain $D$, $\partial D$ is strictly convex with respect to some set of local holomorphic coordinates (see Narasimhan Theorem in Section 3.1). The proof of Theorem 3.2.3 uses a Theorem of H. Alexander concerning polynomially convex sets on the boundary of convex domains.

Note that here "dimension" means, of course, topological dimension. There is no such result for metric dimension, as it is showed by the example of Tumanov in [52] and by the result of Stensønes in [49]. Remember that, for a separable metric space $X$, Hausdorff dimension $\geq$ topological dimension and $\inf\limits_{Y} \dim_{\mathrm{Hausd}}(Y) = \dim_{\mathrm{top}}(X)$, for $Y$ varying among the metric spaces omeomorphic to $X$.

Tumanov shows how to construct on the sphere $\mathbb{S}^2$ in $\mathbb{C}^2$ (which is a strictly pseudoconvex domain in $\mathbb{C}^2$ with real-analytic boundary) a set $E$ of metric dimension 2.5 (in the sense of Hausdorff-Besicovitch) which is even a peak set, so there exists a function $f$ holomorphic in $D$ and continuous in $\bar{D}$ such that $f(z) = 1$ on $E$ and $|f(z)| < 1$ for $z \in \mathbb{S}^2 \setminus E$. A more definitive example, which improves the one of Tumanov, has been obtained by Stensønes. She has constructed a peak interpolation set of Hausdorff dimension $2n-1$ in the boundary $\partial D$ of a smoothly domain $D$ in $\mathbb{C}^n$. Note that this is a set of maximal Hausdorff dimension because the Hausdorff dimension of $\partial D$ is exactly $2n-1$.

What do the examples of Tumanov and Stensønes suggest? Peak interpolation sets are really "a lot" in a measure-theoretic sense and, also, it is not easy to provide simple geometric characterizations of all peak interpolation sets for bounded domains in $\mathbb{C}^n$, $n \geq 2$.

### 3.2.3 Complex tangential submanifolds of the sphere in $\mathbb{C}^2$

Let's consider the unit sphere $\mathbb{S}^2$ in $\mathbb{C}^2$, which is the classical case of strictly pseudoconvex domain; the aim of this subsection is to find examples of complex tangential and non-complex tangential submanifolds on the sphere, to make this notion concrete and geometrically more evident.

The unit sphere is

$$\begin{aligned}\mathbb{S}^2 &= \{|z_1|^2 + |z_2|^2 = 1\} \subseteq \mathbb{C}^2 \\ &\simeq \{x_1^2 + y_1^2 + x_2^2 + y_2^2\} \subseteq \mathbb{R}^4\end{aligned}$$

and has real dimension 3; it is a hypersurface, so it is generic, as it has been noticed in Section 2.1 (cf. Examples 2.1.2), and applying Proposition 2.1.1 we have that $\dim_{\mathbb{R}} T^{\mathbb{C}}\mathbb{S}^2 = 2$.

Moreover, being $\rho(z_1, z_2) = z_1\bar{z}_1 + z_2\bar{z}_2 - 1$ the defining function of $\mathbb{S}^2$,

$$\begin{aligned} T^{\mathbb{C}}\mathbb{S}^2 \simeq T^{1,0}\mathbb{S}^2 &= \{v \in T^{1,0}\mathbb{C}^n : \langle v, \partial\rho\rangle = 0\} \\ &= \{\alpha\partial_{z_1} + \beta\partial_{z_2} : \alpha\bar{z}_1 + \beta\bar{z}_2 = 0\}. \end{aligned} \tag{3.8}$$

If we look for complex tangential manifolds $M$ on $\mathbb{S}^2$, that means $TM \subset T^{\mathbb{C}}\mathbb{S}^2$, and we exclude the case in which equality $TM = T^{\mathbb{C}}\mathbb{S}^2$ holds, we are looking for real curves (real dimension 1) on $\mathbb{S}^2$.

Let's consider the maximal circles on the sphere. If $\mathscr{C}$ is a maximal circle passing through the point $p = (1, 0, 0, 0)$ such that $T_p^{\mathbb{C}}\mathscr{C} \equiv T_p^{\mathbb{C}}\mathbb{S}^2$, we have, by equation (3.8), that

$$\begin{aligned} v = \alpha\partial_{z_1} + \beta\partial_{z_2} \in T_p^{\mathbb{C}}\mathbb{S}^2 &\Leftrightarrow \alpha\bar{z}_1 + \beta\bar{z}_2 = 0 \\ &\Leftrightarrow \alpha \cdot 1 + \beta \cdot 0 = 0 \\ &\Leftrightarrow \alpha = 0 \end{aligned}$$

from which a complex tangent vector for $\mathbb{S}^2$ at $p$ is of the type $\beta\partial_{z_2}$ in $\mathbb{C}^2$ or (equivalently) of the type $(0, 0, a, b)$ in $\mathbb{R}^4$; the maximal circles by $p$ having as tangent vector at $p$ a complex tangent vector for $\mathbb{S}^2$ are given by the following intersection

$$\mathbb{S}^2 \cap \langle (1, 0, 0, 0), (0, 0, a, b)\rangle.$$

A parametrization for big circles of this type is given by

$$\begin{aligned} \theta \longmapsto \mathscr{C}(\theta) &= (\cos\theta, 0, a\sin\theta, b\sin\theta) \in \mathbb{R}^4 \qquad a, b \in \mathbb{R} \\ &\simeq (\cos\theta, (a + ib)\sin\theta) \in \mathbb{C}^2 \end{aligned} \tag{3.9}$$

with the additional condition $a^2 + b^2 = 1$. The tangent space for the curve $\mathscr{C}$ is given by

$$\begin{aligned} \mathscr{C}'(\theta) &= (-\sin\theta, 0, a\cos\theta, b\cos\theta) \\ &\simeq (-\sin\theta, (a + ib)\cos\theta). \end{aligned}$$

We want to see if these circles are complex tangential curves for $T^{\mathbb{C}}\mathbb{S}^2$, so let's control if, given any $p \in \mathscr{C}(\theta)$, $T_p\mathscr{C} \subset T_p^{\mathbb{C}}\mathbb{S}^2$, that is

$$\langle \mathscr{C}'(\theta), \partial\rho(p)\rangle = 0.$$

First of all, for $p \in \mathscr{C}(\theta)$, $\partial\rho(p) = (\cos\theta, (a - ib)\sin\theta)$, from which

$$\begin{aligned} \langle \mathscr{C}'(\theta), \partial\rho(p)\rangle &= -\sin\theta\cos\theta + (a^2 + b^2)\cos\theta\sin\theta \\ &= -\sin\theta\cos\theta + \cos\theta\sin\theta \\ &= 0. \end{aligned}$$

Then, the family of circles given by

$$\{(\cos\theta, 0, a\sin\theta, b\sin\theta)\}_{a,b\in\mathbb{R}, a^2+b^2=1} \tag{3.10}$$

is a family of complex tangential curves of $\mathbb{S}^2$.

It's easy now to provide examples of big circles that are not complex tangential submanifolds of $\mathbb{S}^2$. For example, the family of curves on $\mathbb{S}^2$ given by

$$\{(a\cos\theta, b\sin\theta, 0, 0)\}_{a,b\in\mathbb{R}, a^2+b^2=1} \tag{3.11}$$

as well as the family of curves given by

$$\{(a\sin\theta, b\cos\theta, 0, 0)\}_{a,b\in\mathbb{R}, a^2+b^2=1} \tag{3.12}$$

are both families of big circles on the sphere that are not complex tangential for $\mathbb{S}^2$. In fact, from a geometric point of view, they correspond to the intersection

$$\mathbb{S}^2 \cap \{z_2 = 0\},$$

and, by the fact that the complex curve $\{z_2 = 0\}$ is transversal to the sphere, its tangent space cannot be in $T^{\mathbb{C}}\mathbb{S}^2$. In any case, it is also easy to prove it by calculus: for example (3.11) is not a complex tangential submanifold of $\mathbb{S}^2$ because if we take a vector $v$ tangent to $\mathscr{C}(\theta)$

$$\begin{aligned} v &= (-a\sin\theta, b\cos\theta, 0, 0) \\ \mathcal{J}v &= (-b\cos\theta, -a\sin\theta, 0, 0) \end{aligned}$$

but $\mathcal{J}v \notin T\mathbb{S}^2$ because, at $p \in \mathscr{C}(\theta)$

$$\begin{aligned} \langle v, \partial\rho(p)\rangle &= \langle(-b\cos\theta, -a\sin\theta, 0, 0), (a\cos\theta, b\sin\theta, 0, 0)\rangle \\ &= -ab\cos^2\theta - ab\sin^2\theta \\ &= -ab \neq 0 \qquad \text{if } a \neq 0 \text{ and } b \neq 0. \end{aligned}$$

## 3.3 The technique of Henkin and Tumanov

We present a first technique to construct peak interpolation manifolds for $A(D)$, exploited and described by Henkin and Tumanov in 1974 ([32]). The aim of their paper was at proving the following result.

*In order that a smooth submanifold $M$ of a strictly pseudoconvex hypersurface $S$ be a peak interpolation manifold, it is necessary and sufficient that it be a complex tangential submanifold of $S$.*

For the necessity, it is easy to give a proof in the case the peaking function is of class $C^1$ (for $f \in A(D)$, then only continuous, cf. [32] for the details). In fact, assume for instance that $M \subseteq S = \partial D$ is a peak interpolation manifold for $A^1(D)$; then, there is a function $f \in C^1(\bar{D})$ such that $f|_M = 1$ and $|f| < 1$ on $\bar{D} \setminus M$. Considering $u = \mathsf{Re}\, f$, $u$ obtains its maximum on $M$, being $v = \mathsf{Im}\, f = 0$ on $M$. If, by absurd, we assume that $TM \nsubseteq T^{\mathbb{C}}S$, then there exists $X \in \Gamma(TM \setminus T^{\mathbb{C}}S)$; there is no loss of generality if we assume that $\mathcal{J}X$ points toward the interior of $D$, because $\mathcal{J}X$ cannot lie in $TM$. By Hopf Lemma, the derivative of $u = \mathsf{Re}\, f$ in the direction of the inward normal is positive, so $(\mathcal{J}X)u > 0$. By Cauchy-Riemann equations, $(\mathcal{J}X)u = -Xv$, but $Xv = 0$, because $v$ is constant on $M$, from which the absurd.

We are particularly interested to the sufficient condition, for which Henkin and Tumanov construct, in an explicit way, the "almost analytic" peaking function on a complex tangential submanifold $M$ of $S$. The construction is local for a neighbourhood $\Omega \subseteq \mathbb{C}^n$ of every compact set $K \subset M$ and then they apply an argument of globalization, which is the second step of their proof.

We present the local and global construction in details, because these techniques let us obtain partial results also for (weakly) pseudoconvex domains of finite type of $\mathbb{C}^n$; this will be the theme of Section 3.5.

### 3.3.1 Local construction of a peak interpolating function

Here is the first theorem, proved under the assumption for $M$ and $S$ to be of class $C^2$.

**Theorem 3.3.1.** *Let $S = \partial D$ be a strictly pseudoconvex $C^2$ hypersurface of $\mathbb{C}^n$ and let $M$ be a complex tangential $C^2$ submanifold of $S$ of real dimension $n-1$. Then, for every compact set $K \subset M$, there exist a neighbourhood $\Omega$ of $K$ in $\mathbb{C}^n$, a function $f \in C^2(\Omega)$ and a constant $\gamma > 0$, such that*

$$f_{|\Omega \cap M} = 1, \quad \bar{\partial} f_{|\Omega \cap M} = 0 \quad \textit{together with the first derivatives,}$$

$$|f(z)| \leq 1 - \gamma d^2(z, M) \quad \textit{for } z \in \Omega \cap \bar{D}.$$

**Remark 3.3.1.** The theorem holds also if $\dim_{\mathbb{R}} M \leq n-1$.

**Remark 3.3.2.** The theorem holds also when $M$ and $S$ are smooth of class $C^k$ (producing a function $f$ of class $C^k$) and when $M$ and $S$ are $C^\omega$ (producing a holomorphic function $f$).

*Proof.* First of all, we notice that the requirement, for a submanifold $M$ of $S = \partial D$, to be a peak manifold for $A(D)$, can be expressed in two equivalent ways:

(1) there exists $f \in A(D)$, $f = 1$ on $M$ and $|f| < 1$ on $\bar{D} \setminus M$;

(2) there exists $f \in A(D)$, $f = 0$ on $M$ and $\mathsf{Re}\, f > 0$ on $\bar{D} \setminus M$.

We will prove the second one, in a neighbourhood of the compact set $K \subset M$, just noting that, if (2) holds, also (1) is satisfied by the function $\frac{1-f}{1+f}$, or by the function $e^{-\mathsf{Re}\, f}$.
For $z \in \partial D$ we denote:

$$\xi(z) \in T_z^{\mathbb{C}}(\partial D), \quad \eta(z) := \mathcal{J}\xi(z), \quad \chi(z) := \mathrm{grad}\rho(z), \quad \tau(z) := \mathcal{J}\chi(z),$$

where $\rho$ is the defining function for $D$; the first two vectors belong to

$$T_z^{\mathbb{C}}(\partial D) = \{\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n : \sum_{j=1}^{n} \xi_j \frac{\partial \rho}{\partial z_j}(z) = 0\},$$

the maximal complex subspace of $T_z(\partial D)$ of complex dimension $n - 1$.

**Remark 3.3.3.** $\tau(z)$ belongs to $T_z(\partial D)$;
in fact $\chi(z) = \mathrm{grad}\rho(z)$ is orthogonal to $T_z(\partial D)$ but the action of the complex structure $\mathcal{J}$ on the real vectors is

$$\mathcal{J}\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad \mathcal{J}\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, \quad 1 \leq j \leq n$$

and $\mathcal{J}$ extends as a $\mathbb{C}$-linear operator on $T_z(\mathbb{C}^n)$, so that

$$\mathcal{J}\left(\frac{\partial}{\partial z_j}\right) = i\frac{\partial}{\partial z_j}, \quad \mathcal{J}\left(\frac{\partial}{\partial \bar{z}_j}\right) = -i\frac{\partial}{\partial \bar{z}_j}, \quad 1 \leq j \leq n;$$

we conclude noting that $\tau(z) = \mathcal{J}\mathrm{grad}\rho(z)$ is in $T_z(\partial D)$ because an easy calculation shows that $\langle \mathcal{J}\mathrm{grad}\rho, \mathrm{grad}\rho \rangle = 0$.

**Remark 3.3.4.** $T_z(\partial D)$, which has real dimension $2n-1$, admits the following real orthogonal decomposition

$$T_z(\partial D) = \mathbb{R}[\tau(z)] \oplus T_z^{\mathbb{C}}(\partial D)$$

where we have used the notation $\mathbb{R}[\tau(z)]$ for the real vector subspace generated by $\tau(z)$. This lets us characterize each vector $\xi \in T_z(\partial D)$ that is in $T_z^{\mathbb{C}}(\partial D)$ as the one for which $\langle \xi, \tau(z) \rangle = 0$.

Denoting by $\mathbb{C}T_z(\partial D)$ the complexification $T_z(\partial D) \otimes \mathbb{C}$, and by $T_z^{1,0}(\partial D)$ and $T_z^{0,1}(\partial D)$ the subspaces of $\mathbb{C}T_z(\partial D)$ of complex dimension $n-1$ defined by

$$T_z^{1,0}(\partial D) = \{X \in \mathbb{C}T_z(\mathbb{C}^n) : X = \sum_{j=1}^n a_j \frac{\partial}{\partial z_j}, a_j \in \mathbb{C}, \sum_{j=1}^n a_j \frac{\partial \rho}{\partial z_j}(z) = 0\},$$

$$T_z^{0,1}(\partial D) = \{X \in \mathbb{C}T_z(\mathbb{C}^n) : X = \sum_{j=1}^n a_j \frac{\partial}{\partial \bar{z}_j}, a_j \in \mathbb{C}, \sum_{j=1}^n a_j \frac{\partial \rho}{\partial \bar{z}_j}(z) = 0\},$$

we remember the natural identification

$$T_z^{\mathbb{C}}(\partial D) = \{\mathsf{Re}\, X, X \in T_z^{1,0}(\partial D)\}$$

(the inclusion is evident and the real dimensions are equal). If $X \in T_z^{1,0}(\partial D)$, then $\bar{X} \in T_z^{0,1}(\partial D)$, so for every vector $\xi \in T_z^{\mathbb{C}}(\partial D)$ we can write

$$\xi = X + \bar{X}, \quad X \in T_z^{1,0}(\partial D), \quad \bar{X} \in T_z^{0,1}(\partial D).$$

**Remark 3.3.5.** Let $X$ and $Y$ be two vector fields, tangent to $\partial D$, and let $\xi$ and $\eta$ be two smooth vector fields, sections of $T^{\mathbb{C}}(\partial D)$; we define the Lie brackets

$$[X, Y] = XY - YX;$$

we define on $T_z^{\mathbb{C}}(\partial D)$ the hermitian form

$$L_z(\xi, \eta) = \langle [X, \bar{Y}](z), \tau(z)\rangle$$

for $\xi = X + \bar{X} = \sum_j (a_j \partial_{z_j} + \bar{a}_j \partial_{\bar{z}_j})$ and $\eta = Y + \bar{Y} = \sum_j (b_j \partial_{z_j} + \bar{b}_j \partial_{\bar{z}_j})$ in $T^{\mathbb{C}}(\partial D)$. An easy calculation gives

$$L_z(\xi, \eta) = -2i \sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) a_j \bar{b}_k,$$

where we recognize, apart from the coefficient, the Levi form of $\rho$ at $z$.

**Remark 3.3.6.** For $\xi$ and $\eta$ in $T^{\mathbb{C}}(\partial D)$, the definitions

$$F_z(\xi, \eta) = \langle [\mathcal{J}\xi, \eta](z), \tau(z)\rangle$$
$$G_z(\xi, \eta) = \langle [\xi, \eta](z), \tau(z)\rangle$$

give

$$F_z(\xi, \eta) + iG_z(\xi, \eta) = 2iL_z(\xi, \eta)$$

in fact,

$$\begin{aligned} F_z(\xi, \eta) + iG_z(\xi, \eta) &= \langle ([\mathcal{J}\xi, \eta] + i[\xi, \eta])(z), \tau(z)\rangle \\ &= \langle ([iX - i\bar{X}, Y + \bar{Y}] + i[X + \bar{X}, Y + \bar{Y}])(z), \tau(z)\rangle \\ &= \langle (2[iX, Y] + 2[iX, \bar{Y}] - [i\bar{X}, Y] - [i\bar{X}, \bar{Y}] + [i\bar{X}, Y] + \\ &\quad +[i\bar{X}, \bar{Y}])(z), \tau(z)\rangle \\ &= \langle 2i[X, Y](z), \tau(z)\rangle + \langle 2i[X, \bar{Y}](z), \tau(z)\rangle \\ &= 2iL_z(\xi, \eta). \end{aligned}$$

where $\langle 2i[X,Y](z), \tau(z)\rangle = 0$ because $X$ and $Y$ are in $T^{1,0}(\partial D)$, which is closed under Lie brackets, so $[X,Y] \in T^{1,0}(\partial D)$, which is orthogonal to $\tau$ by the previous real orthogonal decomposition.

**Remark 3.3.7.** • The real part of the Levi form $\langle([\mathcal{J}\xi,\eta])(z), \tau(z)\rangle$ defines on $T_z^{\mathbb{C}}(\partial D)$ a real scalar product that we denote by $\langle\cdot,\cdot\rangle_L$. It can be extended to a scalar product on $T_z(\mathbb{C}^n)$ if we set, for $\xi(z) \in T_z^{\mathbb{C}}(\partial D)$,

$$\langle \tau(z), \xi(z)\rangle_L = 0, \quad \langle \chi(z), \xi(z)\rangle_L = 0,$$

$$\langle \tau(z), \chi(z)\rangle_L = 0, \quad \langle \tau(z), \tau(z)\rangle_L = 1, \quad \langle \chi(z), \chi(z)\rangle_L = 1.$$

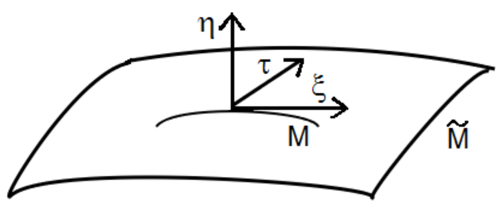
- If $M$ is complex tangential, that is $T_zM \subset T_z^{\mathbb{C}}(\partial D)$, then $T_zM$ and $\mathcal{J}T_zM$ are $L$-orthogonal

  in fact, $T_zM \subset T_z^{\mathbb{C}}(\partial D)$ and $\mathcal{J}T_zM \subset T_z^{\mathbb{C}}(\partial D)$ because $T_z^{\mathbb{C}}(\partial D)$ is $\mathcal{J}$-invariant; taking $\xi(z)$ a vector of $\mathcal{J}T_zM$ and $\eta(z)$ a vector of $T_zM$, there exist locally two vector fields $\xi'$ and $\eta$ of $T(\partial D)$ tangent to $M$ with $\xi(z) = \mathcal{J}\xi'(z)$. We have $\langle[\xi',\eta](z), \tau(z)\rangle = 0$, because the Lie bracket of two vectors tangent to $M$ is tangent to $M$, $M$ is complex tangential so $T_zM \subset T_z^{\mathbb{C}}(\partial D)$ and $T^{\mathbb{C}}(\partial D)$ is orthogonal to $\tau(z)$. This implies $\langle \xi(z), \eta(z)\rangle_L = 0$.

- The last remarks give the following $L$-orthogonal decomposition of $T_z(\partial D)$

$$T_z(\partial D) = \mathbb{R}[\tau(z)] \oplus T_zM \oplus \mathcal{J}T_zM.$$

$M$ is totally real, as we have noticed in the previous section, and $\dim_{\mathbb{R}} M = n-1$; then, it is possible to find in $S$ a manifold $\widetilde{M}$ of $\dim_{\mathbb{R}} = n$ which is again totally real and such that $\tau(z) \in T_z\widetilde{M}$. (By the real orthogonal decomposition it will be sufficient to construct the vector subspace generated by $M$ and $\tau(z)$; to have a concrete idea of the situation we propose the following figure, where $n = 2$, $M$ has $\dim_{\mathbb{R}} = 1$ so $M \simeq \mathbb{R}$, being $M$ a curve, and $\widetilde{M} \simeq \mathbb{R}^2$).




In general, in a neighbourhood $\Omega \subseteq \mathbb{C}^n$ of $z_0 \in M$, there exist vector fields $(\xi_i)$, $i = 1, \ldots, n-1$, tangent to $\partial D$ such that

(a) $\forall z \in M \cap \Omega$, $\{(\xi_i), i = 1, \ldots, n-1\}$ is an $L$-orthogonal basis of $T_z M$;

(b) $\forall z \in \widetilde{M} \cap \Omega$, $\{\tau, (\xi_i), i = 1, \ldots, n-1\}$ is an $L$-orthogonal basis of $T_z \widetilde{M}$;

(c) $\forall z \in \partial D \cap \Omega$, $\{\tau, (\xi_i), (\mathcal{J}\xi_i), i = 1, \ldots, n-1\}$ is an $L$-orthogonal basis of $T_z(\partial D)$.

The expression $\langle [\xi, \tau](z), \tau(z) \rangle$ defines a linear form in $\xi \in T_z^{\mathbb{C}}(\partial D)$, for $z$ fixed, by the properties of scalar product and Lie brackets, so there exists a costant $C > 0$ such that

$$\langle [\xi, \tau](z), \tau(z) \rangle \ \leq \ C\|\xi(z)\|$$

$\Rightarrow$ there exists $\alpha > 0$ such that $C = \alpha\|\tau(z)\|^2$ and

$$\langle [\xi, \tau](z), \tau(z) \rangle \ \leq \ \alpha\|\tau(z)\|^2\|\xi(z)\|.$$

By the previous remarks

$$L_z(\eta, \eta) = -2i\sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) b_j \bar{b}_k, \tag{3.13}$$

and

$$F_z(\eta, \eta) + iG_z(\eta, \eta) = 2iL_z(\eta, \eta); \tag{3.14}$$

by putting (3.13) into (3.14) we get on the right a real term $\Rightarrow G_z(\eta, \eta) = 0 \Rightarrow F_z(\eta, \eta) = 4\sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) b_j \bar{b}_k$, which is the Levi form at $z \in \partial D$. By hypothesis, $S = \partial D$ is strictly pseudoconvex $\Rightarrow$ the last term is $> 0 \Rightarrow$ there exists $\gamma > 0$ such that

$$F_z(\eta, \eta) \geq 2\gamma\|\tau(z)\|^2\|\eta(z)\|^2.$$

We set

$$\beta(z) = \gamma\|\tau(z)\|^2 + \frac{\alpha^2}{\gamma}.$$

Let $f_0 = u_0 + iv_0 \in C^2(\widetilde{M})$ such that for $z \in M$

$$\begin{cases} f_0(z) = 0 \\ (\tau v_0)(z) = -1 < 0 \\ (\tau u_0)(z) = 0 \\ (\tau^2 u_0)(z) = \beta(z) > 0 \end{cases}$$

We use the following proposition due to Hörmander and Wermer [36](note that there exists an equivalent statement for functions of class $C^\infty$, due to Harvey and Wells [29]; this result is used by J. Chaumat and A. M. Chollet in their paper “Ensembles pics pour $A^\infty(D)$” of 1979 [15] to get the same conclusion of Henkin and Tumanov for functions in $A^\infty(D)$, the class of functions that are analytic in $D$ and such that all the derivatives are continuous in $\bar{D}$)

**Proposition 3.3.1** (Hörmander and Wermer). *Let $N$ be a totally real $C^k$-manifold defined on an open set $V$ of $\mathbb{C}^n$ and let $f_0 \in C^k(N)$. Then, there exists a function $f \in C^k(V)$ such that*

*(1) $f = f_0$ on $N$*

*(2) $\bar{\partial} f = 0$ on $N$ of order $k$, that is all derivatives of $\bar{\partial} f$, up to order $k-1$ inclusive, annihilate on $N$.*

Then, taking $\widetilde{M}$ as the totally real manifold of the proposition, there exists a function $f = u + iv \in C^2(\widetilde{\Omega})$, $\widetilde{\Omega} \subseteq \mathbb{C}^n$, $\widetilde{M} \subseteq \widetilde{\Omega}$, such that $f_{|_{\widetilde{M}}} = f_0$ and $\bar{\partial} f_{|_{\widetilde{M}}} = 0$ with the first derivatives.
If $\Omega$ is a neighbourhood of a compact set $K$ of $M$, we can say that, for $z \in \widetilde{M} \cap \Omega$, $\bar{\partial} f = 0$, that means for $f$ to satisfy the Cauchy-Riemann equations

$$\begin{cases} \xi u = \eta v \\ \eta u = -\xi v \end{cases} \qquad\qquad \begin{cases} \chi u = \tau v \\ \tau u = -\chi v. \end{cases}$$

It is immediate to notice that

$$\xi u = \xi v = 0 \quad \text{for } z \in M$$

because $f$ is null on $M$ and $\xi \in T^{\mathbb{C}} \partial D$ is a vector field tangent to $M$. We calculate grad $u$ and grad $v$ for $z \in M$ (it is not a loss of generality to suppose $\|\chi\| = 1$ and $\|\tau\| = 1$)

$$\begin{aligned} \text{grad } u &= (\xi u, \eta u, \chi u, \tau u) \\ &= (0, -\xi v, \tau v, 0) \\ &= (0, 0, -1, 0) \\ &= -\frac{\chi}{\|\chi\|^2}. \end{aligned} \qquad\qquad \begin{aligned} \text{grad } v &= (\xi v, \eta v, \chi v, \tau v) \\ &= (0, \xi u, -\tau u, -1) \\ &= (0, 0, 0, -1) \\ &= -\frac{\tau}{\|\tau\|^2}. \end{aligned}$$

This lets us say that grad $u$ has the same direction of $\chi = \text{grad } \rho$; towards the interior of $D$, given by the normal $\overrightarrow{N}$, $u = \mathsf{Re}\, f \approx d(z, M)$ because $|\partial_t u(z + t\overrightarrow{N})| = |\nabla u \cdot \overrightarrow{N}| = |\text{grad } \rho \cdot \overrightarrow{N}| = 1$; then, $\mathsf{Re}\, f$ has a linear growing which, in a neighbourhood of $z \in \partial D$, is bigger than $\gamma d^2(z, M)$ and we can even forget to prove the estimate $\mathsf{Re}\, f(z) \geq \gamma d^2(z, M)$ for $z \in \Omega \cap \bar{D}$: we only need to prove it for $z \in \Omega \cap \partial D$. For the last one it is enough to verify that for $z \in M$ the second derivatives of $\mathsf{Re}\, f$ are positive in all directions in $T_z S$ orthogonal to $T_z M$, that are $\eta$ and $\tau$. (Note that $\xi \in T^{\mathbb{C}} S$ but is tangent to $M$, not orthogonal, while $\chi = \text{grad } \rho$ is not in $T_z S$.)

We will use the following definition

$$X := \eta + t\tau, \qquad t \in \mathbb{R}.$$

Our aim will be to prove that $X^2 u \geq \gamma \|X\|^2$ for $z \in M$ (where we use $X^2$ to denote the second derivatives). Making explicit the inequality in terms of $\eta$ and $\tau$, the expression becomes of degree two in $t$

$$(\eta^2 u - \gamma\|\eta\|^2) + 2t\eta\tau u + t^2(\tau^2 u - \gamma\|\tau\|^2) \geq 0$$

so an equivalent requirement is to have, for $z \in M$,

$$\begin{cases} \tau^2 u - \gamma\|\tau\|^2 > 0 & \textit{(1)} \\ (\eta\tau u)^2 - (\eta^2 u - \gamma\|\eta\|^2)(\tau^2 u - \gamma\|\tau\|^2) \leq 0. & \textit{(2)} \end{cases}$$

*Proof of (1)*

$$\beta(z) = \gamma\|\tau(z)\|^2 + \frac{\alpha^2}{\gamma} \quad \text{and} \quad (\tau^2 u)(z) = \beta(z) \quad \text{for } z \in M$$

$$\Rightarrow \tau^2 u(z) - \gamma\|\tau(z)\|^2 = \beta(z) - \gamma\|\tau(z)\|^2 = \frac{\alpha^2}{\gamma} > 0 \quad \text{for } z \in M.$$

*Proof of (2)*

We compute $\eta\tau u$ and $\eta^2 u$ on $M$ using the hypothesis on $M$, the Cauchy-Riemann equations and the consequences of the fact that $\xi(z) \in T_z M$ ($\xi u = \xi v = 0$)

$$\begin{aligned} \eta\tau u &= \tau\eta u = \tau(-\xi v) = -\tau\xi v = [\xi, \tau]v - \xi\tau v \\ \eta^2 u &= \eta\eta u = \eta(-\xi v) = -\eta\xi v = [\xi, \eta]v - \xi\eta v \end{aligned}$$

We also know that

$$\begin{aligned} \xi\tau v &= \xi(-1) = 0 \\ \xi\eta v &= \xi\xi u = \xi 0 = 0 \end{aligned}$$

from which, using the previous expressions for grad $u$ and grad $v$ and the fact that $\xi = -\mathcal{J}\eta$,

$$\begin{aligned} \eta\tau u &= [\xi, \tau]v = ([\xi, \tau], \mathrm{grad} v) = -\frac{([\xi, \tau], \tau)}{\|\tau\|^2} \\ \eta^2 u &= [\xi, \eta]v = ([\xi, \eta], \mathrm{grad} v) = -\frac{([\xi, \eta], \tau)}{\|\tau\|^2} = -\frac{([-\mathcal{J}\eta, \eta], \tau)}{\|\tau\|^2} = \frac{F_z(\eta, \eta)}{\|\tau\|^2} \end{aligned}$$

We are able to estimate them

$$\begin{aligned} |\eta\tau u| &= \left| -\frac{([\xi, \tau], \tau)}{\|\tau\|^2} \right| \leq \frac{\alpha\|\tau\|^2\|\xi\|}{\|\tau\|^2} \leq \alpha\|\xi\| = \alpha\| - \mathcal{J}\eta\| = \alpha\|\eta\| \\ |\eta^2 u| &= \left| \frac{F_z(\eta, \eta)}{\|\tau\|^2} \right| \geq \frac{2\gamma\|\tau\|^2\|\eta\|^2}{\|\tau\|^2} = 2\gamma\|\eta\|^2 \end{aligned}$$

for $z \in M$. The last step is a substitution

$$(\eta\tau u)^2 - (\eta^2 u - \gamma\|\eta\|^2)(\tau^2 u - \gamma\|\tau\|^2) \leq \alpha^2\|\eta\|^2 - \gamma\|\eta\|^2 \cdot \frac{\alpha^2}{\gamma} = 0.$$

The theorem is proved. □

### 3.3.2 End of the proof of Henkin-Tumanov Theorem

Here is the second part of the theorem, to get the global construction of the peak interpolation manifold; it starts from the local construction at a neighbourhood of any compact set of the complex tangential submanifold of the boundary, given in the first part of the theorem. It is an argument of "globalization" and needs the use of a uniform estimate for the solution of the $\bar{\partial}$-equation in bounded strictly pseudoconvex domains of $\mathbb{C}^n$, contained in a paper of G. M. Henkin of 1970 [30].

Note that, although it is possible to get the formula for the solution of the $\bar{\partial}$-problem for an arbitrary pseudoconvex domain, a uniform estimate is (apparently) possible only for a strictly pseudoconvex domain. The estimate is contained in the following theorem

**Theorem 3.3.2** (Henkin). *Let $D$ be a bounded strictly pseudoconvex domain of $\mathbb{C}^n$ with smooth boundary $\partial D$. There exists a solution $\varphi(z) \in C^\infty(D)$ for the system of differential equations*

$$\frac{\partial \varphi}{\partial \bar{z}_k} = F_k \quad (k = 1, \dots, n)$$

*admitting the estimate*

$$\|\varphi(z)\|_{C(D)} \leq \gamma(D)\|F(z)\|_{C(D)} \tag{3.15}$$

*where*

$$\|\varphi(z)\|_{C(D)} = \max_{z \in \bar{D}}|\varphi(z)|, \qquad \|F(z)\|_{C(D)} = \sum_{k=1}^{n}\|F_k(z)\|_{C(D)}.$$

**Remark 3.3.8.** The theorem obtains an explicit formula for the solution of the $\bar{\partial}$-problem using an integral representation called Leray-Stokes formula, derived by the well-known Bochner and Martinelli formula. Note that the constant $\gamma(D)$ in inequality (3.15) depends not only on the diameter of the domain $D$, as was the case in the $L^2$-estimate of Hörmander [35], but also on other parameters of the domain; in particular, $\gamma(D)$ substantially depends on the parameter characterizing the strict pseudoconvexity of $D$.

**Remark 3.3.9.** Necessary conditions for the solvability of the $\bar{\partial}$-equation are the condition of "consistency" of its right-hand side: $\partial_{\bar{z}_j} F_k = \partial_{\bar{z}_k} F_j$ $(k, j = 1, \dots, n)$ and, in addition, the property of "pseudoconvexity" of the domain. The classical way of solving the $\bar{\partial}$-equation in $\mathbb{C}^n$, originated by Oka, starts with the explicit solution (Grothendieck's lemma) for the polydiscs $\{w \in \mathbb{C}^n : |w_k| < 1, k = 1, \dots, n\}$, continues with the propagation of the result onto the analytic polytopes $\{z \in \mathbb{C}^n : |w_k(z)| < 1, k = 1, \dots, n\}$, and concludes through approximation of an arbitrary pseudoconvex domain by analytic polytopes. This technique makes difficult to keep track of the bounds in some metric, even in the case of such a simple domain as a ball in $\mathbb{C}^n$.

An entirely different approach to the $\bar{\partial}$-problem was given by Morrey, Kohn and Hörmander, whose methods allow us to prove directly the existence of a solution in an arbitrary pseudoconvex domain with $L^2$-estimate. All these methods don't obtain solutions in a uniform metric, while the Leray-Stokes method allows us to write out the solution of the $\bar{\partial}$-equation in the form of an explicit formula, from which the necessary uniform bounds are obtained.

Here is the theorem for the global construction of the peak interpolating function

**Theorem 3.3.3.** *Le $S$ be a strictly pseudoconvex $C^3$ manifold of $\mathbb{C}^n$ and let $M$ be a complex tangential $C^3$ submanifold of $S$ of real dimension $n-1$. Then, any compact set $K \subset M$ is a peak interpolation set for $A(S)$.*

*Proof.* We apply Theorem 3.3.1 to prove that every compact set $K$ of $M$ is a peak interpolation manifold locally, at $\Omega \cap S$, where $\Omega$ is a neighbourhood of $K$ in $\mathbb{C}^n$. We call $f$ the function with the properties of peaking for $K$. Let $U$ be a neighbourhood of $K$ such that $\bar{U} \subset \Omega$; we choose a nonnegative $C^3$ function $\chi$ with support in $\Omega$, such that $\chi = 1$ on $\Omega$, and define the functions

$$\begin{aligned} g_m(z) &:= \begin{cases} f^m(z) \cdot \chi(z) & \text{for } z \in \Omega \\ 0 & \text{for } z \in \mathbb{C}^n \setminus \bar{\Omega}, \end{cases} \\ F_m(z) &:= \bar{\partial} g_m(z). \end{aligned}$$

By the properties of $f$, we are able to prove that

$$\|F_m\|_{C(\bar{D})} = \sup_{z \in \bar{D}} |F_m(z)| \to 0$$

where $|F_m(z)|$ is the maximum modulus of the coefficients of the form $F_m$ at the point $z$. In fact, for $z \in \mathbb{C}^n \setminus \bar{\Omega}$ it is obvious, due to the fact that $g_m = 0$, while for $z \in \Omega$

$$\begin{aligned} F_m &= \bar{\partial} g_m \\ &= \bar{\partial}(f^m \cdot \chi) \\ &= m \cdot \ f^{m-1} \cdot \bar{\partial} f \cdot \chi + f^m \cdot \bar{\partial}\chi. \end{aligned}$$

We remember that $\bar{\partial} f = 0$ on $M$, but the $C^3$-regularity allows us to use Whitney extension theorem, to have on $\bar{D}$

$$\begin{aligned} &|\bar{\partial} f| = o(r^2) \quad \text{for } r = d(z, M) \\ &\frac{o(r^2)}{r^2} \to 0 \quad \text{as } r^2 \to 0; \end{aligned}$$

we also have the property of peaking

$$|f(z)| \le 1 - \gamma r^2;$$

substitutions give

$$|F_m(z)| \le m(1-\gamma r^2)^{m-1} o(r^2) + (1-\gamma r^2)^m |\bar{\partial}\chi(z)|$$

and then

$$\|F_m\| \le m \cdot a^{m-1} \cdot o(r^2) + a^m \cdot |\bar{\partial}\chi(z)| \quad \text{for } 0 < \alpha < 1$$

from which $\|F_m\| \to 0$, also because $|\bar{\partial}\chi(z)|$ is bounded.
We consider the $\bar{\partial}$-equation

$$\bar{\partial}\varphi_m = F_m, \quad z \in D;$$

by Theorem 3.3.2 there exists a solution $\varphi_m$ of the $\bar{\partial}$-problem, with the uniform estimate

$$\|\varphi_m\|_{C(\bar{D})} \le \gamma(D)\|F_m\|_{C(\bar{D})},$$

from which immediately follows that

$$\|\varphi_m\|_{C(\bar{D})} \to 0 \quad \text{as } m \to \infty.$$

Given the sequence of functions $h_m = g_m - \varphi_m$, we easily note that $h_m \in A(S)$, they are uniformly bounded by the previous inequalities and their limit is equal to 0 out of $M$ and equal to 1 on $K$. The result follows by the following proposition, as an application of Bishop Theorem. □

**Proposition 3.3.2.** *A sufficient condition for a compact set $K \subset M$ to be a peak interpolation manifold for $A(\Omega)$ is that there exists a uniformly bounded sequence of functions $f_k \in A(\Omega)$, such that*

$$\lim_{k\to\infty} f_k(z) = 0 \quad \textit{for} \;\; z \in \Omega \tag{3.16}$$

$$\lim_{k\to\infty} f_k(z) = 1 \quad \textit{for} \;\; z \in K \tag{3.17}$$

*Proof.* Given a measure $\mu$ on $M$ ortogonal to $A(\Omega)$, and defining $\nu(K') = |\mu|(K' \cap K)$ for every Borel subset $K' \subset M$, we observe that $\nu$ is absolutely continuous with respect to $\mu$ and, by (3.16),

$$\lim_{k\to\infty} \int f_k(z)d\nu = 0 \quad \text{for} \;\; z \in K.$$

If, at the same time, $\{f_k\}$ satisfies (3.17), we also have

$$\lim_{k\to\infty} \int f_k(z)d\nu = \lim_{k\to\infty} \int_K f_k(z)d|\mu| = |\mu|(K)$$

and the result follows by Bishop theorem. □

## 3.4 The technique of Rudin

Given a bounded strictly pseudoconvex domain $D$ of $\mathbb{C}^n$ and a submanifold $M$ of the boundary $S = \partial D$ with the property of being complex tangential, the main result we can obtain is that, under some assumptions of regularity for $\partial D$ and $M$, every compact subset of $M$ is a peak interpolation set for $A(D)$.

Independently, Henkin-Tumanov [32] and Nagel [44] reached the same conclusions assuming $\partial D$ and $M$ of class $C^3$, while Walter Rudin, in his paper of 1978 (cf. [48]), was able to weaken these regularity hypothesis proving the sufficient condition for peak interpolation manifolds with $\partial D$ of class $C^2$ and $M$ of class $C^1$. It has to be said that Rudin technique is really different from the one of Henkin and Tumanov; it takes from the proof of Nagel the basic idea of exhibiting appropriate functions in $A(D)$ by means of integrals and reaches the result first considering the case of strictly convex domains and then applying an embedding theorem by Fornaess to pass to the case of strictly pseudoconvex domains: this is the reason for which Rudin's proof is simpler and lets him require less differentiability.

We have analysed Rudin theorem for strictly convex domains and have reformulated some of its steps: this will be the theme of the first part of the Section; only few words will be given for the use of the embedding Theorem by Fornaess, while great relevance will be reserved to the way in which Bishop theorem is used in the construction of the integrals.

For the notations, we will use Definition 3.1.3 of Section 3.1, so that if $U$ is an open set of $\mathbb{C}^n$ and $\rho : U \to \mathbb{R}$, $\rho \in C^2$, is the defining function for $D$, we denote by

$$\begin{aligned}
\mathcal{P}_{z_0}(\zeta) &:= \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(z_0)\zeta_j\zeta_k, && \zeta \in \mathbb{C}^n, z_0 \in U,\\
\mathcal{L}_{z_0}(\zeta,\eta) &:= \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z_0)\zeta_j\bar{\eta}_k, && \zeta,\eta \in \mathbb{C}^n, z_0 \in U,\\
Q_{z_0}(\zeta) &:= \mathcal{P}_{z_0}(\zeta) + \mathcal{L}_{z_0}(\zeta,\zeta), && \zeta \in \mathbb{C}^n, z_0 \in U.
\end{aligned}$$

$\mathcal{L}_{z_0}(\zeta,\eta)$ is the Levi form of $\rho$ at $z_0$, while $\mathsf{Re}\, Q_{z_0}(\zeta)$ is the real Hessian of $\rho$ at $z_0$; being the domain $D$ strictly convex, the real Hessian of $\rho$ is positive definite at each $z_0 \in \partial D$. Considering the Taylor expansion of $\rho$ at $z_0 \in U$, we get

$$\rho(z) = \rho(z_0) + 2\mathsf{Re}\,\langle z - z_0, \bar{\partial}\rho(z_0)\rangle + \mathsf{Re}\, Q_{z_0}(z - z_0) + o(|z - z_0|^2)$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{C}^n$ defined by $\langle z, w\rangle = \sum_{j=1}^{n} z_j\bar{w}_j$ and $|z| = \langle z, z\rangle^{\frac{1}{2}}$.

For $\Omega$ an open set of $\mathbb{R}^m$, we will consider the non singular $C^1$-mapping $\Phi : \Omega \to \partial D$, with the aim of parametrizing $M \subset S$ by $\Phi$ (Rudin Theorem holds also if $\Phi(\Omega)$ is not a manifold because $\Phi$ is not assumed to be bjective on $\Omega$).

Note that, in terms of $\Phi$, the "geometric" requirement for $M$ to be complex tangential is realized by the "analytic" orthogonality condition

$$\langle \Phi'(x)v, \bar{\partial}\rho(\Phi(x))\rangle = 0, \quad \text{for all } x \in \Omega, v \in \mathbb{R}^m, \tag{3.18}$$

because the tangent vectors are represented by $\Phi'(x)v$ and the inner product is in $\mathbb{C}^n$. Here are the basic lemmas used in the theorem

**Lemma 3.4.1.** *If $D$ is strictly convex, then the following estimate holds*

$$2\mathsf{Re}\,\langle w - z, \bar{\partial}\rho(w)\rangle \geq \alpha|w - z|^2$$

*for all $w \in \partial D$, $z \in \bar{D}$ and $\alpha > 0$.*

*Proof.* We define $h(t) := \rho((1-t)w + tz)$, for $t \in \mathbb{R}$, and observe that $h(0) = \rho(w) = 0$, because $w \in \partial D$, while $h(1) = \rho(z) \leq 0$, because $z \in \bar{D}$. We want to use Taylor formula, so we first calculate

$$\begin{aligned} h'(0) &= \sum_j \partial_j\rho(w)(z - w) + \sum_j \bar{\partial}_j\rho(w)\overline{(z - w)} \\ &= 2\mathsf{Re}\,\langle z - w, \bar{\partial}\rho(w)\rangle \end{aligned}$$

and

$$\begin{aligned} h''(t) &= \frac{d}{dt}(\sum_j \partial_j\rho((1-t)w + tz)(z - w) + \sum_j \bar{\partial}_j\rho((1-t)w + tz)\overline{(z - w)}) \\ &= 2\mathsf{Re}\,Q_{(1-t)w+tz}(z - w) \\ &\geq 2\alpha|z - w|^2. \end{aligned}$$

where the last one is given by the strict convexity of $D$. Just writing Taylor formula in this way

$$h(1) = h(0) + h'(0) + \frac{1}{2}h''(t) \qquad \text{for some } t \in (0, 1),$$

we immediately get

$$h'(0) < 0 \Rightarrow 2\mathsf{Re}\,\langle w - z, \bar{\partial}\rho(w)\rangle \geq \alpha|w - z|^2.$$

□

**Lemma 3.4.2.** *If $\rho$ is of class $C^2$ and $\Phi$ is of class $C^1$, the following holds*

$$\langle \Phi'(x)v, \frac{d}{dx}[\bar{\partial}\rho(\Phi(x))]v\rangle = Q_{\Phi(x)}(\Phi'(x)v)$$

*Proof.* It is an immediate calculus

$$\begin{aligned} \langle \Phi'(x)v, \frac{d}{dx}[\bar{\partial}\rho(\Phi(x))]v\rangle &= \langle \Phi'(x)v, [\bar{\partial}^2\rho(\Phi(x)) \cdot \Phi'(x) + \partial\bar{\partial}\rho(\Phi(x)) \cdot \overline{\Phi'(x)}]v\rangle \\ &= \partial^2\rho(\Phi(x)) \cdot \Phi'(x)v\Phi'(x)v + \bar{\partial}\partial\rho(\Phi(x)) \cdot \Phi'(x)v\overline{\Phi'(x)}v \\ &= Q_{\Phi(x)}(\Phi'(x)v). \end{aligned}$$

□

Note that only in the following lemma we use the hypothesis that $M$ is complex tangential; as in the previous lemma, also in the following one we have completely reformulated and simplified the proof.

**Lemma 3.4.3.** *If $\rho$ is of class $C^2$, $\Phi$ is of class $C^1$ and $M$, parametrized by the mapping $\Phi : \Omega \to \partial D$, is complex tangential, that means*

$$\langle \Phi'(x)v, \bar{\partial}\rho(\Phi(x))\rangle = 0 \qquad \textit{for all } x \in \Omega, v \in \mathbb{R}^m,$$

*then, for $y \in \Omega, v \in \mathbb{R}^m$, we have*

$$\left\langle \frac{\Phi(y+\delta v) - \phi(y)}{\delta^2}, \bar{\partial}\rho(\Phi(y+\delta v))\right\rangle \overset{as\ \delta\to 0}{\longrightarrow} \frac{1}{2}Q_{\Phi(y)}(\Phi'(y)v).$$

*Proof.* We only need few passages, if we apply Hopital Theorem

$$\lim_{\delta\to 0}\frac{\left\langle \Phi(y+\delta v) - \phi(y), \bar{\partial}\rho(\Phi(y+\delta v))\right\rangle}{\delta^2} = \lim_{\delta\to 0}\left[\frac{\left\langle \Phi'(y+\delta v)v, \bar{\partial}\rho(\Phi(y+\delta v))\right\rangle}{2\delta} + \right.$$

$$\left. + \frac{\langle \Phi(y+\delta v) - \Phi(y), \bar{\partial}^2\rho(\Phi(y+\delta v))\Phi'(y+\delta v)v + \partial\bar{\partial}\rho(\Phi(y+\delta v))\overline{\Phi'(y+\delta v)}v\rangle}{2\delta}\right].$$

The first part annihilates because $M$ is complex tangential, while, for the second part, it becomes

$$\left\langle \frac{[\Phi(y+\delta v) - \Phi(y)]v}{\delta v}, \frac{1}{2}\left[\bar{\partial}^2\rho(\Phi(y+\delta v))\Phi'(y+\delta v)v + \partial\bar{\partial}\rho(\Phi(y+\delta v))\overline{\Phi'(y+\delta v)}v\right]\right\rangle$$

whose limit, as $\delta \to 0$, is

$$\begin{aligned}
&\langle \Phi'(y)v, \frac{1}{2}\left[\bar{\partial}^2\rho(\Phi(y))\Phi'(y)v + \partial\bar{\partial}\rho(\Phi(y))\overline{\Phi'(y)}v\right]\rangle = \\
=&\frac{1}{2}\left[\partial^2\rho(\Phi(y))\Phi'(y)v\Phi'(y)v + \bar{\partial}\partial\rho(\Phi(y))\Phi'(y)v\overline{\Phi'(y)}v\right] = \\
=&\frac{1}{2}Q_{\Phi(y)}(\Phi'(y)v).
\end{aligned}$$

The lemma is proved. ∎

**Lemma 3.4.4.** *If $F : \mathbb{R}^m \to \mathbb{C}$ is a homogeneous polynomial of degree 2, such that $\operatorname{Re} F(x) > 0$, for $x \neq 0$, then*

$$\int_{\mathbb{R}^m} \frac{dx}{[1+F(x)]^m} \neq 0.$$

*Proof.* Through the hypothesis $\mathsf{Re}\, F(x) \geq c|x|^2, c > 0$, for $x \neq 0$, and the obvious relation $|F(x)| \geq |\mathsf{Re}\, F(x)|$, we immediately get

$$\frac{1}{|1+F(x)|^m} \leq \frac{1}{|F(x)|^m} \leq \frac{1}{|\mathsf{Re}\, F(x)|^m} \leq \frac{1}{c|x|^{2m}},$$

so the integrand is in $L^1(\mathbb{R}^m)$ (because $2m > m$: remember that $\int_{\mathbb{R}^m} \frac{1}{x^\alpha}$ converges if and only if $\alpha > m$ out of a little ball centered at the origin). $F$ is a homogeneous polynomial of degree 2, so it can be expressed as

$$F(x) = \sum_{j,k=1}^{m} c_{jk} x_j x_k, \qquad c_{jk} \in \mathbb{C}, c_{jk} = c_{kj}.$$

If we denote by $(c_{jk})$ the matrix associated to $F$ and $a_{jk} = \mathsf{Re}\, c_{jk}$, the hypothesis $\mathsf{Re}\, F(x) > 0$ becomes a condition of strict positivity for all the eigenvalues $\lambda_1, \ldots, \lambda_m$ of $(a_{jk})$. We define

$$J = \int_{\mathbb{R}^m} \frac{dx}{[1+|x|^2]^m} > 0$$

and prove that, if $(a_{jk})$ is strictly positive-definite, then

$$\det(c_{jk})\{\int_{\mathbb{R}^m} \frac{dx}{[1+\sum c_{jk}x_j x_k]^m}\} = J^2 \tag{3.19}$$

which implies

$$\int_{\mathbb{R}^m} \frac{dx}{[1+F(x)]^m} = \frac{J^2}{\det(c_{jk})} \neq 0$$

*Proof of (3.19).* We first suppose that $c_{jk} \in \mathbb{R}$, i.e. $c_{jk} = a_{jk}$. By an orthogonal transformation of $\mathbb{R}^m$, it is possible to diagonalize $(c_{jk}) = (a_{jk})$, so that the integrand becomes $[1+\sum \lambda_j x_j^2]^{-m}$ and the determinant of $c_{jk}$ becomes $\prod \lambda_j$. (3.19) follows by the change of variables $y_j = \sqrt{\lambda_j} x_j$.

For the general case, that is $c_{jk} \in \mathbb{C}$, we can consider the symmetric matrices $(c_{jk})$ as points of $\mathbb{C}^N$, where $N = \frac{m(m+1)}{2}$. Let $T = \{(c_{jk}) : (\mathsf{Re}\, c_{jk}) = (a_{jk})$ is strictly positive-definite$\} \subset \mathbb{C}^n$. Our aim is to prove that (3.19) holds for all $(c_{jk}) \in T$. We have already proved that (3.19) holds for $(c_{jk}) \in T \cap \mathbb{R}^N$; by the fact that the integral of (3.19) is a holomorphic function of $(c_{jk})$ in $T$, the result follows for all $(c_{jk}) \in T$. □

We are ready to prove the main theorem, given by Rudin to have peak interpolation sets for $A(D)$. The setting we will present is the one of strictly convex domains, because the passage to strictly pseudoconvex domains is a technical application of Fornaess embedding Theorem [22], to which we do not want to give great relevance.

**Theorem 3.4.1.** *Let $D$ be a bounded strictly convex domain in $\mathbb{C}^n$, with $C^2$ boundary $S = \partial D$ and let $M$ be a submanifold of $S$ parametrized by a non singular*

*$C^1$-mapping $\Phi : \Omega \to \partial D$, where $\Omega$ is an open set of $\mathbb{R}^m$. Assume that $M$ is complex tangential, that means for $M$ to satisfy the orthogonality condition*

$$\langle \Phi'(x)v, \bar{\partial}\rho(\Phi(x))\rangle = 0 \qquad \text{for all } x \in \Omega, v \in \mathbb{R}^m,$$

*where $\rho$ is the defining function for $S$. If $K$ is a compact subset of $\Omega$, then $\Phi(K)$ is a peak interpolation set for $A(D)$.*

*Proof.* We want to prove that for every $p \in \Omega$, there is a neighbourhood $\Omega_p$ such that $\mu(\Phi(K)) = 0$, for every compact subset $K$ of $\Omega_p$ and for every complex Borel measure $\mu$ on $\partial D$, $\mu \perp A(D)$; then, the result follows by Bishop Theorem and by the properties of the Borel complex measures. We localize the problem around a point $p \in \Omega$. The mapping $\phi$ is non singular, so $\Phi'(x)$ has rank $m$, for all $x \in \Omega$, and $\Phi'$ is continuous (because $\Phi$ is $C^1$); these two facts imply, by maximum principle, the existence of a constant $c > 0$ and a ball $B = B(p, r)$, with $\bar{B} \subset \Omega$, such that

$$|\Phi'(x)v| \geq c|v| \qquad \text{for all } x \in \bar{B} \text{ and } v \in \mathbb{R}^n, \tag{3.20}$$

and

$$|\Phi(x) - \Phi(y)| \geq c|x - y| \qquad \text{for all } x, y \in \bar{B}. \tag{3.21}$$

We will prove the result for $\Omega_p = B$. By the property of strict convexity of $D$, there exists $\alpha > 0$ such that

$$\begin{aligned} \operatorname{Re} Q_{\Phi(y)}(\Phi'(y)v) &\geq \alpha|\Phi'(y)v|^2 \\ &\overset{(3.20)}{\geq} \alpha c^2 v^2 \qquad \text{for } y \in \bar{B} \text{ and } v \in \mathbb{R}^n, \end{aligned} \tag{3.22}$$

where the last inequality is given by maximum principle.
Now we define the following integral

$$g(y) = \int_{\mathbb{R}^m} \frac{dv}{\{1 + \frac{1}{2}Q_{\Phi(y)}(\Phi'(y)v)\}^m} \qquad \text{for } y \in \bar{B}.$$

We need to prove that

(i) this integral converges

(ii) $g(y) \neq 0$

to define, for $\delta > 0$, the functions

$$h_\delta(z) = \int_B \frac{\delta^m (f/g)(x)dx}{\{\delta^2 + \langle \Phi(x) - z, \bar{\partial}\rho(\Phi(x))\rangle\}^m},$$

where $f : \mathbb{R}^m \to \mathbb{C}$, $f \in C^0$ and $\operatorname{supp}(f) \subset B$. First, we prove (i)

$$\begin{aligned}
\frac{1}{|\{1+\frac{1}{2}Q_{\Phi(y)}(\Phi'(y)v)\}|^m} &\leq \frac{1}{|\{1+\frac{1}{2}\operatorname{Re} Q_{\Phi(y)}(\Phi'(y)v)\}|^m} \\
&= \frac{1}{\{1+\frac{1}{2}\operatorname{Re} Q_{\Phi(y)}(\Phi'(y)v)\}^m} \\
&\leq \frac{1}{(1+\frac{1}{2}\alpha c^2|v|^2)^m} \\
&\approx \frac{1}{(\alpha^m c^{2m}|v|^{2m})}
\end{aligned}$$

where the last function is in $L^1(\mathbb{R}^m)$ because $2m > m$, and the approximation is good for $|v|$ big. Also (ii) holds, by Lemma (3.4.4), because $Q(\zeta)$ is a homogeneous polynomial of degree 2, whose real part is strictly bigger than 0, for $\zeta \neq 0$, by the hypothesis of strict convexity of $D$. This lets us say that the functions $h_\delta \in A(D)$. Our aim is to prove that

(iii) $\{h_\delta\}$ are uniformly bounded on $\bar{D}$

(iv) $\lim\limits_{\delta\to 0} h_\delta(z) = 0$ for all $z \in \bar{D} \setminus \Phi(\bar{B})$

(v) $\lim\limits_{\delta\to 0} h_\delta(\Phi(y)) = f(y)$ for all $y \in \bar{B}$,

in fact, for every compact set $K \subset B$, taking, as $f$, the characteristic function on the image of $K$ through $\Phi$ ($f \equiv \chi_{|\Phi(K)}$) and using a compactness argument, we have by (iii), (iv), (v)

$$\int_{\Phi(K)} h_\delta(\Phi(y))dy \longrightarrow \int_{\Phi(K)} f(y)dy \equiv \int_{\Phi(K)} \chi(y)dy = \mu(\Phi(K));$$

but we also know that $\int h_\delta\, d\mu = 0$ on $D$, for the definition of complex Borel measure on $\partial D$ orthogonal to $A(D)$ ($\mu \perp A(D)$ if $\int f\, d\mu = 0$ for all $f \in A(D)$) and because we have proved that $h_\delta$ are functions in $A(D)$. Then, $\mu(\Phi(K)) = 0$ for every compact $K \subset B$, which means, by Bishop Theorem, that every $\Phi(K)$ is a peak interpolation set for $A(D)$. So, it remains to prove (iii), (iv) and (v).

*Proof of* (iii).
We fix $z \in \bar{D}$ and choose $y \in \bar{B}$ such that

$$\operatorname{dist}(z, \partial D) = |\Phi(y) - z| \leq |\Phi(x) - z| \qquad \text{for all } x \in B. \tag{3.23}$$

In this way, we get

$$\begin{aligned}
|\Phi(x) - \Phi(y)| &= |\Phi(x) - z + z - \Phi(y)| \\
&\leq |\Phi(x) - z| + |z - \Phi(y)| \\
&\leq 2|\Phi(x) - z|.
\end{aligned} \tag{3.24}$$

It is always possible to define the integrands of the $h_\delta$ to be 0 when $x \notin B$, so that the integrals on $B$ become integrals on $\mathbb{R}^m$. Then, using the change of variables $x = y + \delta v$, for which $|J| = \delta^m$, the definitions of the $h_\delta$ become

$$\begin{aligned} h_\delta(z) &= \int_{\mathbb{R}^m} \frac{\delta^{2m}(f/g)(y+\delta v)dv}{\{\delta^2(1+\delta^{-2}\langle \Phi(y+\delta v)-z, \bar\partial\rho(\Phi(y+\delta v))\rangle)\}^m} \\ &= \int_{\mathbb{R}^m} \frac{(f/g)(y+\delta v)dv}{\{1+\delta^{-2}\langle \Phi(y+\delta v)-z, \bar\partial\rho(\Phi(y+\delta v))\rangle\}^m} \end{aligned}$$

If $y + \delta v \in B$, we can use Lemma 3.4.1 and the previous estimates to get for all $z \in \bar D$

$$\begin{aligned} \mathsf{Re}\,\langle \Phi(y+\delta v)-z, \bar\partial\rho(\Phi(y+\delta v))\rangle &\overset{\text{(Lemma 3.4.1)}}{\geq} \frac{\alpha}{2}|\Phi(y+\delta v)-z|^2 \\ &\overset{(3.24)}{\geq} \frac{\alpha}{8}|\Phi(y+\delta v)-\Phi(y)|^2 \\ &\overset{(3.21)}{\geq} \frac{\alpha}{8}c^2\delta^2|v|^2; \end{aligned}$$

for the absolute value of the integrands, we first get

$$\begin{aligned} \left|\frac{1}{\delta^2}\langle \Phi(y+\delta v)-z, \bar\partial\rho(\Phi(y+\delta v))\rangle\right| &\geq \left|\frac{1}{\delta^2}\mathsf{Re}\,\langle \Phi(y+\delta v)-z, \bar\partial\rho(\Phi(y+\delta v))\rangle\right| \\ &= \frac{1}{\delta^2}\mathsf{Re}\,\langle \Phi(y+\delta v)-z, \bar\partial\rho(\Phi(y+\delta v))\rangle \\ &\geq \frac{\alpha}{8}c^2\delta^2|v|^2 \end{aligned}$$

and then

$$\left|\frac{(f/g)(y+\delta v)}{\{1+\delta^{-2}\langle \Phi(y+\delta v)-z, \bar\partial\rho(\Phi(y+\delta v))\rangle\}^m}\right| \leq \frac{\|f/g\|}{\{1+\frac{\alpha}{8}c^2\delta^2|v|^2\}^m}$$

where the term on the right is in $L^1(\mathbb{R}^m)$, for $|v|$ big, because $2m > m$ and the bound does not depend on $z \in \bar D, \delta > 0$, so we get (iii).

*Proof of* (iv).
We fix $z \in \bar D \setminus \Phi(\bar B)$ and choose $y$ satisfing (3.23); if $y + \delta v \in B$

$$\begin{aligned} 2\mathsf{Re}\,\langle \Phi(y+\delta v)-z, \bar\partial\rho(\Phi(y+\delta v))\rangle &\overset{\text{(Lemma 3.4.1)}}{\geq} \alpha|\Phi(y+\delta v)-z|^2 \\ &\overset{(3.23)}{\geq} \alpha|\Phi(y)-z|^2 > 0, \end{aligned}$$

where the last term cannot be zero because $z \notin \Phi(\bar B) \Rightarrow z \neq \Phi(x), \forall x \in \bar B \Rightarrow z \neq \Phi(y)$. The result follows applying the dominated convergence Theorem.

*Proof of* (v).
We evaluate the functions $h_\delta$ for $z = \Phi(y)$

$$h_\delta(\Phi(y)) = \int_B \frac{\delta^m (f/g)(x)dx}{\{\delta^2 + \langle \Phi(x) - \Phi(y), \bar\partial\rho(\Phi(x))\rangle\}^m}$$

and apply the previous change of variables $x = y + \delta v$, with $|J| = \delta^m$,

$$h_\delta(\Phi(y)) = \int_{\mathbb{R}^m} \frac{(f/g)(y+\delta v)dv}{\{1 + \delta^{-2}\langle \Phi(y+\delta v) - \Phi(y), \bar\partial\rho(\Phi(y+\delta v))\rangle\}^m}.$$

The first step now is given by dominated convergence Theorem, which lets us pass the sign of $\lim_{\delta\to 0}$ into the integrals; then, the numerator of the integrands of $h_\delta$ tends to $f(y)/g(y)$, as $\delta \to 0$, while, for the denominator, we apply Lemma 3.4.3. To conclude, it is sufficient to remember the definition of $g(y)$ and (v) follows.

□

An application of Fornaess embedding Theorem [22] yields to the extension of this result to the strictly pseudoconvex case. We have presented the statement of that theorem in Section 3.1, so we refer to that part of the thesis; we also refer to the proof of Rudin for further details.

### 3.4.1 Example of the sphere in $\mathbb{C}^2$

Here we show how the proof of Rudin reduces in the case of the sphere $\mathbb{S}^2$ in $\mathbb{C}^2$.

If $p \in \partial\mathbb{S}^2 \subseteq \mathbb{C}^2$, $p = (z_1, z_2) \approx (x_1, y_1, x_2, y_2)$; we denote by $\nu_p$ the corresponding vector $(x_1, y_1, x_2, y_2) \approx (x_1 + iy_1, x_2 + iy_2)$ and by $\eta_p$ the vector $(-y_1, x_1, -y_2, x_2) \approx (-y_1 + ix_1, -y_2 + ix_2)$. By the fact that the dimension of a complex tangential manifold on the sphere in $\mathbb{C}^n$ cannot exceed $n-1$, the only complex tangential manifolds in $\mathbb{C}^2$ are curves $\gamma : [0,1] \to \partial\mathbb{S}^2$, $\gamma \in C^1$, satisfying

$$\langle \gamma(t), \eta_{\gamma(t)}\rangle \equiv \sum_{j=1}^n \dot\gamma_j(t)(\overline{\eta_{\gamma(t)}}) = 0 \qquad \forall t \in [0,1]. \tag{3.25}$$

Note that if we replace $\nu_p$ by $(\partial_{\bar z_1}\rho(p), \dots, \partial_{\bar z_n}\rho(p))$ and $\eta_p$ by $i\nu_p$, we regain condition (3.18) just multiplying by $i$ the previous (3.25). With an abuse of notation, let $\gamma$ denote also $\{\gamma(t) : 0 \le t \le 1\}$.

Let $f : \gamma \to \mathbb{C}$ be continuous; after reparametrization, we can always suppose that $|\dot\gamma(t)| = 1$, $\forall t$, so let's define

$$\begin{aligned} g(t) &:= \int_{-\infty}^{+\infty} \left(1 + \frac{1}{2}\left|\dot\gamma(t)s\right|^2\right)^{-1} ds, \qquad 0 \le t \le 1 \\ &= \frac{\pi\sqrt{2}}{|\dot\gamma(t)|} \\ &= \pi\sqrt{2}. \end{aligned}$$

Noting that $\mathsf{Re}\,\langle\gamma(t)-z,\eta_{\gamma(t)}\rangle \geq 0$, we also define, for $\delta > 0$,

$$h_\delta(z) = \frac{1}{\pi\sqrt{2}}\int_0^1 \frac{\delta f(t)dt}{\delta^2 + \langle\gamma(t)-z,\eta_{\gamma(t)}\rangle}.$$

In this special case, it is easy to prove that

(i) $|h_\delta(z)| \leq C$, for all $0 < \delta < 1$, $z \in \bar{\mathbb{S}}^2$,

(ii) $h_\delta \in A(\mathbb{S}^2)$, because $g(t) = \pi\sqrt{2} \neq 0$,

(iii) $\lim\limits_{\delta\to 0^+} h_\delta(z) = 0$, $z \in \bar{\mathbb{S}}^2 \setminus \gamma$

(iv) $\lim\limits_{\delta\to 0^+} h_\delta(\gamma(s)) = f(s)$, for all $0 \leq s \leq 1$.

If $\mu$ is a Borel measure on $\partial\mathbb{S}^2$, $\mu \perp A(\mathbb{S}^2)$, we get, by the previous results, that $\mu|_\gamma = 0$ and applying Bishop's peak interpolation Theorem, we can conclude that there is a function $F \in A(\mathbb{S}^2)$ such that $F|_\gamma = f$ and $|F(z)| \leq \sup_\gamma |f|$, for all $z \in \bar{\mathbb{S}}^2 \setminus \gamma$. For the case $f \equiv 1$ on $\gamma$, we have just exhibited a function in $A(\mathbb{S}^2)$ that is $\equiv 1$ on $\gamma$ and has modulus $< 1$ elsewhere, thus proving that $\gamma$ is a peak set.

### 3.4.2 A Theorem of Bharali for weakly convex domains with $C^\omega$ boundary

The approach of Rudin, based on the construction of suitable integrals that yield to peaking functions in strictly convex domains by Bishop Theorem, is adapted by Gautam Bharali to certain cases of (weakly) convex domains; in particular, he considers bounded (weakly) convex domains having real-analytic boundary.

The main theorem of Bharali in [10], which is achieved through a preliminar result in $\mathbb{C}^2$ (cf. [9]), is the following

**Theorem 3.4.2.** *Let $D$ be a bounded (weakly) convex domain in $\mathbb{C}^n$, $n \geq 2$, having real-analytic boundary $S = \partial D$, and let $M$ be a real-analytic submanifold of $S$. If $M$ is complex tangential, then $M$ is a peak interpolation manifold for $A(D)$.*

When we pass from strictly convex domains to (weakly) convex domains, the second fundamental form can degenerate, and in particular there may be submanifolds of $M \subset \partial D$ along which the real Hessian annihilates; reducing to the case in which $\partial D$ and $M$ are real-analytic, Bharali can apply a structure theorem by Lojasiewicz, which is fundamental for his proof; here is the statement

**Theorem 3.4.3** (Lojasiewicz)**.** *Let $F$ be a non-constant real-analytic function defined in a neighbourhood of $0 \in \mathbb{R}^n$, and assume that $V(F) = F^{-1}\{0\} \ni 0$. Then, there is a small neighbourhood $U \ni 0$ such that $V(F) \cap U$ has the decomposition*

$$V(F) \cap U = \bigcup_{j=0}^{n-1} S_j,$$

*where each $S_j$ is a finite, disjoint union of (not necessarily closed) $j$-dimensional real-analytic submanifolds contained in $U$, such that each connected component of $S_j$ is a closed real-analytic submanifold of $U \setminus \left(\bigcup_{j=0}^{n-1} S_j\right)$, $j = 1, \dots, n-1$.*

The proof technique of Bharali can be divided into three fundamental steps:
*Step 1)* Since $M$ is real-analytic, for each $p \in M$, $M$ has the following local stratification in a neighbourhood $U_p$ of $p$

$$M \cap U_p = \bigcup_{j=0}^{\dim M} M_j, \tag{3.26}$$

where

- each $M_j$ is a disjoint union of finitely many $j$-dimensional (not necessarily closed) real-analytic submanifolds of $\partial D$;
- each connected component of $M_j$ is a closed real-analytic submanifold of $U_p \setminus \left(\bigcup_{k=0}^{j-1} S_j\right)$, $j = 1, \dots, \dim M$;
- if $\mathscr{M}$ is a stratum of positive dimension (using also the weak convexity of $\partial D$), the second fundamental form of $\partial D$ is strictly positive definite on the tangent space of $\mathscr{M}$.

*Step 2)* Each stratum $\mathscr{M}$ of $M \cap U_p$ of positive dimension is a countable union of peak interpolation (compact) sets for $A(D)$.
*Step 3)* Each point of $M_0$ is a peak point for $A(D)$.

Let's first remember the definition of second fundamental form for $\partial D$ and its null space: if $\rho$ is the defining function for $\partial D$, $\mathcal{H}\rho$ is the real Hessian of $\rho$, $p \in \partial D$ and $v \in T_p(\partial D)$, then the second fundamental form for $\partial D$ at $p$ is the quadratic form

$$v \longmapsto \sum_{j,k} (\mathcal{H}\rho)_{j,k}(p) v_j v_k$$

and its null space at $p$ is

$$\mathcal{N}_p = \{v \in T_p(\partial D) : \sum_{j,k} (\mathcal{H}\rho)_{j,k}(p) v_j v_k = 0\} \subseteq T_p(\partial D).$$

Therefore, when we speak about strata where the real Hessian is strictly positive, we refer to the following requirement for the connected components $M_{j,\alpha}$ of $M_j$ (if $D$ is convex)

$$\mathcal{N}_\zeta \cap T_\zeta(M_{j,\alpha}) \equiv \{0\}, \qquad \forall \zeta \in M_{j,\alpha} \tag{3.27}$$

We want to give the basic ideas of the proofs of the previous steps.

For *Step 1)*, the following remark is the starting point.

**Remark 3.4.1.** If $D$ is a bounded domain with real analytic boundary $\partial D$, then $\partial D$ contains no line segments.

*Proof.* Assume, by absurd, that $\partial D$ contains a line segment; without loss of generality, we may consider, as its end points, $0 \in \mathbb{R}^n$ and $\xi_0 = (\xi', \xi_n) \in \mathbb{R}^n$. The boundary of $D$ can be described in a neighbourhood $U$ of the origin as follows

$$\partial D \cap U = \{x : x_n = f(x_1, \dots, x_{n-1})\}$$

where $f$ is real-analytic and convex. If we consider the function

$$g : t \longmapsto f(\xi_0' t),$$

then $g$ is a real-analytic function that is $\equiv 0$ on an interval that has $t = 0$ as an end-point. By (real) analytic continuation principle, $g$ has to vanish on a full neighbourhood of $t = 0$: a contradiction. $\square$

Two preliminary lemmas about convex domains having smooth boundary in $\mathbb{R}^n$ are needed (note that we are not requiring real analyticity on the boundary):

(1) If there is no line segment in the boundary, then there cannot be a boundary curve always pointing in the direction of $\mathcal{N}$ (that means $\nexists\, \sigma : \mathbb{R} \supseteq I \to \partial D$, with $\sigma'(t) \in \mathcal{N}_{\sigma(t)}$, $\forall t \in I$)

(2) (A consequence of Lemma (1)) If there is no line segment in the boundary, then the set
$$\{p \in M : T_pM \cap \mathcal{N}_p = \{0\}\}$$
is open and dense in $M$.

Let's consider a real-analytic parametrization of $M$ near $p \in M$

$$\gamma : (B_d(0, \epsilon), 0) \longrightarrow (M, p), \qquad d = \mathrm{dim} M$$

such that $\mathrm{rank}_{\mathbb{R}}[d\gamma(x)]$ is maximal $\forall x$ and let's take the pull back of the real Hessian $\mathcal{H}\rho$ by $\gamma$, which is the function

$$\begin{aligned} \mathcal{F} : B_d(0, \epsilon) &\longrightarrow \mathbb{R} \\ x &\longmapsto \det\, [d\gamma(x)^T (\mathcal{H}\rho)(\gamma(x)) d\gamma(x)]. \end{aligned}$$

By the hypothesis of real analyticity of $\partial D$, we can apply Lemma (2); we get that $\{p \in M : T_pM \cap \mathcal{N}_p = \{0\}\}$ is open and dense, thus $\mathcal{F} \neq 0$. We may also assume, without loss of generality, that

$$\mathcal{F}^{-1}\{0\} \neq \varnothing \quad \Rightarrow \quad \mathcal{F}^{-1}\{0\} \ni 0$$

that lets us apply Lojasiewicz's Theorem. Through an inductive argument and a repeated use of Lemma (2) on each connected component of the strata of positive dimension, we get condition (3.27) on each connected component and, then, *Step 1)* is proved.

*Step 2)* involves a construction of integrals of the type of Rudin's article and the following well-known property of complex measures (concerning countable unions of sets)

$$\mu\left(\bigcup_j M_j\right) = \sum_j \mu(M_j),$$

to apply Bishop Theorem and get unions of peak interpolation manifolds. It has become sufficient for $M$ to be a countable union of compact sets $K_j$, for which $|\mu|(K) = 0$, $\forall \mu \perp A(\Omega)$. The construction of the integrals, used by Bharali, requires first, for $D \subseteq \mathbb{C}^n$, to be a bounded, weakly convex domain having a smooth boundary that *contains no line segments*, and, secondly, for $\gamma : B_d(0,R) \to \partial D$, to be a smooth imbedding (not necessarily real-analytic) whose image is complex tangential. Note that the previous parametrization has been used for each connected component of the local stratification of $M$ given by (3.26). In fact, $d$ denotes the dimension of the submanifold of $\partial D$ for which the following condition has to hold

$$d\gamma(x)(\mathbb{R}^d) \cap \mathcal{N}_{\gamma(x)} = \{0\} \qquad \forall x,$$

which is the third requirement for the construction of the integrals and is obtained in *Step 1)* for the connected components of the strata of the decomposition (3.26).

Then, using the previous three conditions, it can be proved that there exists $r > 0$ such that, if $f \in C_c[B_d(0,r);\mathbb{C}]$, we can define

$$h_\delta(z) = \int_{B_d(0,r)} \frac{\delta^d f(x)/G(x)dx}{\{\delta^2 + \sum_{j=1}^n \partial_j \rho(\gamma(x))[\gamma_j(x) - z_j]\}^d}, \qquad z \in \bar{D}$$

$$G(x) = \int_{\mathbb{R}^d} \{1 + \sum_{j,k=1}^n (\bar{\partial}^2_{jk}\rho(\gamma(x))(d\gamma(x)v)_j \overline{(d\gamma(x)v)_k} + \\ + \partial^2_{jk}\rho(\gamma(x))(d\gamma(x)v)_j (d\gamma(x)v)_k)\}^{-d} dv$$

and the following statements hold (as in the proof of Rudin)

(i) $\{h_\delta\}$ is uniformly bounded on $\bar{D}$

(ii) $h_\delta \in A(D)$

(iii) $\lim_{\delta\to 0} h_\delta(z) = 0$ if $z \in \bar{D} \setminus \gamma(B_d(0,r))$

(iv) $\lim_{\delta\to 0} h_\delta(\gamma(s)) = f(s)$ $\forall s \in B_d(0,r)$.

The images by $\gamma$ of the $d$-dimensional balls of $\mathbb{R}^n$, denoted above by $\gamma(B_d(0,r))$, represent the open subsets of a stratum of positive dimension in the local stratification (3.26) of $M$ near $p$. Then, the compact sets required by *Step 2)* are the images by $\gamma$ of compact sets $K \subset B_d(0,r)$. Using a shrinking family $\{K_\nu\}_\nu$ of compact subsets of $K$ such that $K_\nu \subset B_d(0,r)$, $K_{\nu+1} \subset \overset{\circ}{K}_\nu$, $\bigcap_{\nu\in\mathbb{N}} K_\nu = K$, it is easy to prove that $\mu(\gamma(K)) = 0$, $\forall \mu \perp A(D)$.

*Step 3)* is easy because we are in the case of a smoothly bounded convex domain whose boundary contains no line segments. In fact, if we define, for $\zeta \in \partial D$ and $z \in \mathbb{C}^n$, the function

$$G(\zeta,\cdot) : \bar{D} \longrightarrow \mathbb{C} \qquad z \longmapsto G(\zeta,z) = \sum_{j=1}^{n} \partial_j \rho(\zeta)(\zeta_j - z_j),$$

then,

- for a fixed $\zeta \in \partial D$, the equation $G(\zeta,z) = 0$ defines $T_\zeta^{\mathbb{C}}(\partial D)$ (viewed as an affine $\mathbb{C}$-hyperplane in $\mathbb{C}^n$)
- $\mathsf{Re}\, G(\zeta,z)$ is the perpendicular distance of $z$ from $T_\zeta(\partial D)$
- by the fact that $D$ is convex, if $z \in \bar{D}$, we have $\mathsf{Re}\, G(\zeta,z) \geq 0$ and $G(\zeta,z) = 0 \iff z = \zeta$ (this is a standard fact due to the geometry of the domain).

The last one takes us immediately to the conclusion because if we define

$$F(\zeta,\cdot) : \bar{D} \longrightarrow \mathbb{C} \qquad z \longmapsto F(\zeta,z) = \exp^{-G(\zeta,z)},$$

$F(\zeta,\cdot)$ peaks at $\zeta$ and $\{F(\zeta,\cdot)\}_{\zeta\in\partial D}$ comes to be a smoothly varying family of peak functions for $A(D)$.

**Remark 3.4.2.** Fornaess and Sibony in [24] have proved a result in $\mathbb{C}^2$ which is more general than *Step 3)*: if $D$ is a bounded pseudoconvex finite type domain in $\mathbb{C}^2$ having $C^\infty$ boundary, every $\zeta \in \partial D$ is a peak point for $A(D)$. Note that in the general setting of smoothly bounded weakly pseudoconvex domains in $\mathbb{C}^n$ of finite type, not only it is a hard problem to investigate if and when every complex tangential submanifold of the boundary is a PI manifold for $A(D)$, but it is also very difficult to prove that every point of $\partial D$ is a peak point for $A(D)$.

### 3.4.3 Stratification for $C^\omega$ boundaries of pseudoconvex domains

In this section we want to present a technique to stratify real analytic boundaries of weakly pseudoconvex domains, which turns to be really different, but easier and geometrically more evident, than the technique of Bharali.

It is based on two theorems, denoted by Theorem 1 and Theorem 2. The first one, which is valid in the more general setting of smooth generic manifolds, shows that if the Levi form of a smooth CR manifold is degenerate in some conormal direction next to a fixed one, in a neighbourhood of a fixed point, then the manifold is locally foliated by complex curves; the second one, as a consequence, shows that every real analytic boundary of finite D'angelo type can be stratified in suitable way.

It turns out that each stratum of the previous stratification is locally contained in a Levi non degenerate hypersurface.

We denote by $T^*(\mathbb{C}^n)$ the real cotangent bundle of $\mathbb{C}^n$ and, for a smooth generic manifold $M \subset \mathbb{C}^n$ defined by $\rho$, we also denote by $N^*(M)$ the conormal bundle of $M$ in $\mathbb{C}^n$, whose fiber at $p \in M$ is $N^*_p(M) = \{\xi \in T^*_p(\mathbb{C}^n) : \mathsf{Re}\,\xi|_{T_pM} = 0\}$. The forms $\{\partial\rho_j\}$ give a basis of $N^*_p(M)$, so every $\xi \in N^*_p(M)$ can be written as $\xi = \sum c_j \partial\rho_j$, for c a real vector. We will denote the Levi form of $M$ and of a selected conormal $\xi$ as $\mathcal{L}^\xi_M$, so that if $\xi = \sum c_j\partial\rho_j \in N^*_p(M)$, then $\mathcal{L}^\xi_M(p)(X,\bar{Y}) = \sum c_j \partial\bar{\partial}\rho_j(p)(X,\bar{Y})$, for $X, Y \in T^{1,0}_p(M)$. As for the Levi form, the contraction between $\xi$ and the matrix $(\partial\rho)_{i,j}$ is denoted by $\partial\rho^\xi$. We will sometimes refer to the rank of the kernel of the Levi form $L_M$ as to the "Levi rank".

**Theorem 1.** *Let $M \subset \mathbb{C}^n$ be a smooth generic submanifold, with defining function $\rho$; assume that, given a conormal $\xi_0 \in N^*(M)$ and a point $z_0 \in M$, locally for $(z,\xi) \sim (z_0,\xi_0)$*

$$\begin{cases} \mathcal{L}^\xi_M \textit{is degenerate}, \\ \textit{rank } (\textit{Ker } \mathcal{L}^\xi_M) \equiv k \textit{ constant.} \end{cases}$$

*Then, there exists a foliation of $M$ by integral leaves of $\mathcal{L}^\xi_M$ of constant dimension $k$.*

*Proof of Theorem 1.* We first select a conormal $\xi \in N^*(M)$, $\xi \sim \xi_0$, that will give the direction along which evaluate the Levi form of $M$. It will be sufficient to prove that $\operatorname{Ker}\mathcal{L}^\xi_M$ is involutive, and then integrability will follow by Frobenious Theorem. Let $L_1$, $L_2 \in \operatorname{Ker}\mathcal{L}^\xi_M$; this means

$$\begin{cases} \langle \partial\rho^\xi, [L_1, \bar{L}]\rangle = 0 \qquad \forall L \in T^{\mathbb{C}}M & (3.28a) \\ \langle \partial\rho^\xi, [L_2, \bar{L}]\rangle = 0 \qquad \forall L \in T^{\mathbb{C}}M. & (3.28b) \end{cases}$$

It follows, by Jacobi identity $[X,[W,Z]] + [W,[Z,X]] + [Z,[X,W]] = 0$ and the property on commutators $[X,W] = -[W,X]$, that

$$\begin{aligned} \langle \partial\rho^\xi, [[L_1,L_2],\bar{L}]\rangle &= \langle \partial\rho^\xi, [L_1,[L_2,\bar{L}]]\rangle + \langle \partial\rho^\xi, [L_2,[\bar{L},L_1]]\rangle \qquad \forall L \in T^{\mathbb{C}}M \\ &= 0 \end{aligned}$$

because the first term on the right hand side annihiliates for (3.28a) and the second term annihilates for (3.28b). Then, $\operatorname{Ker} \mathcal{L}_M^\xi$ is involutive, and Frobenious Theorem yields to the required integrability. □

**Theorem 2.** *Let $D \subset \mathbb{C}^n$ be a bounded domain with $S = \partial D$, such that*

$$\begin{cases} S \in C^\omega, \\ S \ \textit{ is of finite D'Angelo type.} \end{cases}$$

*Then, $S$ is stratified by strata where the Levi form is non degenerate.*

*Proof of Theorem 2.* We denote by

$$\begin{aligned} S_N &:= \{z \in S : \mathcal{L}_S(z) \text{ is non degenerate}\} \\ \mathscr{S}_N &:= \{z \in S : \mathcal{L}_S(z) \text{ is degenerate}\}. \end{aligned}$$

We have that $\mathscr{S}_N$ is stratified because the hypothesis of real analyticity of the boundary $S$ lets us apply Lojasiewicz Theorem.

We can refine the stratification in such a way that each stratum has constant rank for the kernel of the Levi form (or equivalently, has constant Levi rank): in fact, let $S_{N-1}$ be the maximal stratum inside $\mathscr{S}_N$ (that exists by Lojasiewicz Theorem) with respect to the property that $\mathcal{L}_S^{\partial\rho}(z)|_{T^{\mathbb{C}} S_{N-1}}$ is non degenerate. It must be

$$\dim S_{N-1} < \dim S_N,$$

otherwise, near a point of maximal Levi rank in $\mathscr{S}_N$, by Theorem 1, $S$ would be foliated by integral leaves of $\operatorname{Ker} \mathcal{L}_M$ of dimension $\geq 1$, which is evidently in contrast with the hypothesis of finite D'Angelo type for $S$.

As a next step, let's define

$$\mathscr{S}_{N-1} := \{z \in S_{N-1} : \mathcal{L}_S^{\partial\rho}(z)|_{T^{\mathbb{C}} S_{N-1}} \text{ is degenerate}\};$$

again, $\mathscr{S}_{N-1}$ is stratified by real analyticity (through Lojasiewicz Theorem) and $\mathscr{S}_{N-1}$ contains a stratum $S_{N-2}$ which is maximal with respect to the property that $\mathcal{L}_S^{\partial\rho}(z)|_{T^{\mathbb{C}} S_{N-2}}$ is non degenerate; it must be

$$\dim S_{N-2} < \dim S_{N-1},$$

by Theorem 1 and the hypothesis of finite D'Angelo type for $S$. An iteration of the previous argument yields to the collection of maximal strata $\{S_j\}_{j=1}^N$. Then, taking for each stratum $S_j$ the subset $S_j \setminus \mathscr{S}_j$, we get the stratification

$$S = \bigcup_{j=1}^{N} (S_j \setminus \mathscr{S}_j),$$

where for each stratum the (restriction of the) Levi form is non degenerate. □

Note that

- if $S$ is an arbitrary compact real-analytic variety in $\mathbb{C}^n$, then $S$ does not contain any nontrivial germs of complex analytic varieties (this is a result by Diederich and Fornaess [21]);
- for a real analytic manifold $M$, finite D'Angelo type means that $M$ does not contain complex analytic curves.

With the previous remarks, if we consider a bounded domain $D$ with real analytic boundary $S$, it follows that $S$ is of D'Angelo finite type. Being $S \in C^\omega$ and of finite D'Angelo type, we can apply Theorem 2, which provides a stratification of $S$ by strata $\{S_j \setminus \mathscr{S}_j\}$, where the Levi form is non degenerate.

**§. Open problem**: We are still investigating on the possibility to apply the previous stratification to the setting of pseudoconvex domains $D$ of $\mathbb{C}^n$ with $C^\omega$ boundary, with the aim at obtaining results on the existence of peak interpolation manifolds on $\partial D$.

## 3.5 Some results for pseudoconvex domains of type four in $\mathbb{C}^2$

After the results of Henkin-Tumanov and Rudin, the open problem is the following: what happens if we remove the hypothesis of strict pseudoconvexity of the domain, substituting it with the (weaker) property of pseudoconvexity of finite type?

We have already proved that, for (weakly) pseudoconvex domains $S$ of type $k = 2m, m \in \mathbb{N}$, every complex tangential submanifold $M$ of $S$ is totally real. Now, we consider the case in which $S$ is pseudoconvex of type four in $\mathbb{C}^2$, to have one of the simplest cases of finite type domains of $\mathbb{C}^n$; the aim of what will follow is to generalize the technique of Tumanov and Henkin, described in Section 3.3, to the case in which there is no more strict pseudoconvexity for the domain under consideration. We have obtained partial results, while the complete answer to the problem remains unsolved with this technique, for the inability to treat the linear combinations of the derivatives of $\mathsf{Re}\, f$ in the directions of $T_zS$ orthogonal to $T_zM$.

We first state two lemmas; the first has been presented in Section 3.3 for $C^k$ functions (Hormander and Wermer [36]) and we give here the counterpart for $C^\infty$ functions (due to Harvey and Wells [29], as we have already noticed), while the second one is for $C^\omega$ functions.

**Lemma 3.5.1** (Harvey and Wells). *Let $M$ be a totally real $C^\infty$ manifold defined on an open set $V$ of $\mathbb{C}^n$ and let $f_0 \in C^\infty(M)$. Then, there exists a function $f \in C^\infty(V)$ such that*

*(1) $f = f_0$ on $M$*

*(2) $\bar\partial f = 0$ on $M$ of infinite order, that is, for all $\alpha$, the derivatives of $\bar\partial f$ annihilate on $M$.*

*Proof.* Cf. [29]

If we consider $\gamma \subset S \subset \mathbb{C}^2$, where $S = \partial D$ is a real hypersurface of $\mathbb{C}^2$ ($\dim_{\mathbb{R}} S = 3$) and $\gamma$ is a complex tangential curve of $S$ ($\dim_{\mathbb{R}}\gamma = 1$), we can define (as in the theorem of Henkin and Tumanov) $\xi \in T_z\gamma$ and $\eta = \mathcal{J}\xi$. Then, for $f \in C^\omega(S)$ satisfying Cauchy Riemann equations on $S$, the holomorphic extension is immediate applying Hartogs-Bochner Theorem.

**Lemma 3.5.2.** *Let $\gamma \subset S \subset \mathbb{C}^2$, as before, and $\xi \in T_z\gamma$, $\eta = \mathcal{J}\xi$; then, given $f = u + iv \in C^\omega(S)$ such that on $S$*

$$\begin{cases} \xi u = \eta v \\ \eta u = -\xi v, \end{cases}$$

*we can extend $f$ to a function $F \in hol(\bar D)$.*

Here we present our result, generalizing some parts of the proof of Henkin and Tumanov.

**Theorem 3.5.1.** *Let $\gamma \subset \widetilde{M} \subset S \subset \mathbb{C}^2$, where $S$ is a real hypersurface of $\mathbb{C}^2$ with defining function $\rho$ ($dim_{\mathbb{R}} S = 3$), $\gamma$ is a complex tangential curve of $S$ ($dim_{\mathbb{R}}\gamma = 1$) and $\widetilde{M}$ is a totally real manifold of real dimension 2, with $\tau(z) = \mathcal{J} grad\,\rho \in T_z\widetilde{M}$. We also define $\xi \in T_z\gamma$, $\eta = \mathcal{J}\xi$, $\chi = grad\,\rho$, so that $T\widetilde{M} = Span\{\xi, \tau\}$. Let $f = u + iv \in C^\infty(\widetilde{M})$ such that*

$$\begin{cases} u_{|_{\widetilde{M}}} \equiv 0 & (3.29a) \\ v_{|_\gamma} \equiv 0 & (3.29b) \\ \tau v_{|_\gamma} < 0 & (3.29c) \end{cases}$$

*and extends as holomorphic, thus satisfying on $\widetilde{M}$*

$$\begin{cases} \xi u = \eta v \\ \eta u = -\xi v \end{cases} \quad (3.30) \qquad \begin{cases} \chi u = \tau v \\ \tau u = -\chi v. \end{cases} \quad (3.31)$$

*Assume that each point of $\gamma$ is of type four, that is, for each point of $\gamma$, the following hold*

$$[\eta, \xi] \in Span\{\xi, \eta\} \quad (3.32)$$

$$[\eta\,[\eta, \xi]\,] \in Span\{\xi, \eta\} \quad (3.33)$$

$$[\eta\,[\eta\,[\eta, \xi]\,]\,] = \tau \neq 0, \quad \tau \in \frac{\mathbb{C} \otimes TS}{\mathbb{C} \otimes T^{\mathbb{C}}S}. \quad (3.34)$$

*Then, we have on $\gamma$*

$$\begin{cases} \eta u = 0 \\ \eta v = 0 \end{cases} \text{(i)} \qquad \begin{cases} \eta^2 u = 0 \\ \eta^2 v = 0 \end{cases} \text{(ii)} \qquad \begin{cases} \eta^3 u = 0 \\ \eta^3 v = 0 \end{cases} \text{(iii)} \qquad \eta^4 u > 0 \quad \text{(iv)}$$

*Proof.* (i) By Cauchy-Riemann equations, the hypothesis (3.29a) and (3.29b) and the fact that $\xi$ is tangent to $\gamma$, we have on $\gamma$ that

$$\begin{cases} \eta u = -\xi v \equiv 0 \\ \eta v = \xi u \equiv 0. \end{cases}$$

(ii) Adding the definition of Lie brackets, what we have already proved in (i), and hypothesis (3.32), that lets us write $[\eta, \xi]$ as a linear combination of $\xi$ and $\eta$ in this way $[\eta, \xi] = a\xi + b\eta, a, b \in \mathbb{C}$, we get

$$\begin{aligned} \eta^2 u &= \eta(\eta u) \overset{(3.30)}{=} -\eta\xi v = -[\eta, \xi]v - \xi\eta v \\ &= -(a\xi + b\eta)v - \xi^2 u \\ &= 0 \end{aligned}$$

$$\begin{aligned} \eta^2 v &= \eta(\eta v) \overset{(3.30)}{=} \eta\xi u = -[\xi, \eta]u + \xi\eta u \\ &= -(a\xi + b\eta)u - \xi^2 v \\ &= 0 \end{aligned}$$

(iii) We first use the definition of commutator of length 2

$$\begin{aligned}[][\eta\,[\eta,\xi]\,]v &= \eta[\eta,\xi]v - [\eta,\xi]\eta v \\ &= \eta(\eta\xi - \xi\eta)v - [\eta,\xi]\eta v \\ &= \eta^2\xi v - \eta\xi\eta v - [\eta,\xi]\eta v \end{aligned} \tag{3.38}$$

Then we get

$$\begin{aligned} \eta^3 u \; = \; \eta^2(\eta u) \; &= \; -\eta^2\xi v \\ &\overset{(3.38)}{=} -[\eta[\eta,\xi]]v - \eta\xi\eta v - [\eta,\xi]\eta v \\ &\overset{(3.33)}{=} -(a\xi + b\eta)v - \eta\xi\eta v - [\eta,\xi]\eta v \\ &\overset{(\mathrm{i})}{=} -\eta\xi\eta v - [\eta,\xi]\eta v. \end{aligned}$$

We first treat the second term $[\eta,\xi]\eta v$, just remembering that the first order commutator of $\eta$ and $\xi$ is in the $\mathrm{Span}\{\xi,\eta\}$ and that $\xi$ is tangent to $M$

$$\begin{cases} [\eta,\xi]\eta v = (a'\xi + b'\eta)\eta v = a'\xi\eta v + b'\eta^2 v \\ \eta^2 v \overset{(\mathrm{ii})}{=} 0 \\ \xi\eta v \overset{(3.30)}{=} \xi^2 u \overset{(3.29a)}{=} 0, \end{cases}$$

then, the second term is 0. For the first one, we note that also $[\xi,\eta] \in \mathrm{Span}\{\xi,\eta\}$, so we have

$$\begin{cases} \eta\xi\eta v = -[\xi,\eta]\eta v + \xi\eta^2 v \\ [\xi,\eta]\eta v = (a''\xi + b''\eta)\eta v = a''\xi\eta v + b''\eta^2 v \\ \xi\eta v \overset{(3.30)}{=} \xi^2 u \overset{(3.29a)}{=} 0 \\ \eta^2 v \overset{(\mathrm{ii})}{=} 0 \Rightarrow \text{also } \xi\eta^2 v = 0, \text{ being } \xi \text{ tangential to } \gamma; \end{cases}$$

then, $\eta\xi\eta v \equiv 0$, and this lets us conclude that $\eta^3 u = 0$ on $\gamma$. Repeating as before,

$$\begin{aligned} \eta^3 v \; = \; \eta^2(\eta v) \; &= \; \eta^2\xi u \\ &\overset{(3.38)}{=} [\eta[\eta,\xi]]u + \eta\xi\eta u + [\eta,\xi]\eta u \\ &\overset{(3.33)}{=} \eta\xi\eta u + [\eta,\xi]\eta u \end{aligned}$$

For the second term

$$\begin{cases} [\eta,\xi]\eta u = (a'\xi + b'\eta)\eta u = a'\xi\eta u + b'\eta^2 u \\ \eta^2 u \overset{(\mathrm{ii})}{=} 0 \\ \xi\eta u \overset{(3.30)}{=} -\xi^2 v \overset{(3.29a)}{=} 0, \end{cases}$$

and for the first term

$$\begin{cases} \eta\xi\eta u = -[\xi,\eta]\eta u + \xi\eta^2 u \\ [\xi,\eta]\eta u = (a''\xi + b''\eta)\eta u = a''\xi\eta u + b''\eta^2 u \\ \xi\eta u \overset{(3.30)}{=} -\xi^2 v \overset{(3.29a)}{=} 0 \\ \eta^2 u \overset{\text{(ii)}}{=} 0 \Rightarrow \text{also } \xi\eta^2 u = 0, \text{ being } \xi \text{ tangential to } \gamma; \end{cases}$$

then, $\eta^3 v \equiv 0$ on $\gamma$.

(iv) Finally let's use the definition of commutator of length 3:

$$\begin{aligned}
[\eta\,[\eta\,[\eta,\xi]\,]\,]v &= \eta\,[\eta\,[\eta,\xi]\,]v - [\eta\,[\eta,\xi]\,]\eta v \\
&\overset{(3.38)}{=} \eta(\eta^2\xi v - \eta\xi\eta v - [\eta,\xi]\eta v) - (\eta^2\xi - \eta\xi\eta - [\eta,\xi]\eta)\eta v \\
&= \eta^3\xi v - \eta^2\xi\eta v - \eta(\eta\xi - \xi\eta)\eta v - \eta^2\xi\eta v + \eta\xi\eta^2 v + (\eta\xi - \xi\eta)\eta^2 v \\
&= \eta^3\xi v - 3\eta^2\xi\eta v + 3\eta\xi\eta^2 v - \xi\eta^3 v
\end{aligned}$$

from which we get

$$\begin{aligned}
\eta^4 u &= \eta^3(\eta u) \overset{(3.30)}{=} -\eta^3\xi v \\
&= -[\eta\,[\eta\,[\eta,\xi]\,]\,]v - 3\eta^2\xi\eta v + 3\eta\xi\eta^2 v - \xi\eta^3 v.
\end{aligned}$$

The last term $\xi\eta^3 v \equiv 0$, because, by (iii), $\eta^3 v \equiv 0$ on $\gamma$ and $\xi$ is tangential to $\gamma$; for the second and the third terms

$$\begin{aligned}
\eta^2\xi\eta v &= [\eta\,[\eta,\xi]\,]\eta v + (\eta\xi\eta)\eta v + [\eta,\xi]\eta^2 v \\
&= [\eta\,[\eta,\xi]\,]\eta v + (\eta\xi\eta)\eta v + (\eta\xi\eta)\eta v - \xi\eta^3 v \\
&= [\eta\,[\eta,\xi]\,]\eta v + 2(\eta\xi\eta)\eta v \qquad (\text{as before } \xi\eta^3 v = 0) \\
&= [\eta\,[\eta,\xi]\,]\eta v + 2([\eta,\xi]\eta^2 v + \xi\eta^3 v) \\
&= [\eta\,[\eta,\xi]\,]\eta v + 2[\eta,\xi]\eta^2 \qquad (\text{as before } \xi\eta^3 v = 0) \\
&\overset{(3.33)-(3.32)}{=} (a'''\xi + b'''\eta)\eta v + 2(a'\xi + b'\eta)\eta^2 v \\
&= a'''\xi(\eta v) + b'''\eta^2 v + 2a'\xi(\eta^2 v) + b'\eta^3 v \\
&\overset{(i)-(ii)-(iii)}{=} 0.
\end{aligned}$$

$$\begin{aligned}
\eta\xi\eta^2 v &= [\eta,\xi]\eta^2 v + \xi\eta^3 v \\
&\overset{(3.32)}{=} (a'\xi + b'\eta)\eta^2 v \qquad (\text{as before } \xi\eta^3 v = 0) \\
&= a'\xi\eta^2 v + b'\eta^3 v \\
&\overset{(ii)-(iii)}{=} 0.
\end{aligned}$$

then, the second and third terms of $\eta^4 u$ are 0 on $\gamma$; by the initial hypothesis $\tau v_{|\gamma} < 0$, the first term of $\eta^4$ is strictly greater than 0, so we have proved the final step of the theorem, which is $\eta^4 u > 0$ on $\gamma$.

□

## 3.6 Standard form for defining functions of domains in $\mathbb{C}^n$ and relation with local peak functions

Let's first consider a domain $D$ in $\mathbb{C}^n$ with $C^\infty$ boundary and $p \in \partial D$. We choose local holomorphic coordinates $(z, w)$, where $z = z_1, \dots, z_{n-1}$ and $w = u + iv$, such that $p = 0$ and $u$ points in the outward normal direction to $\partial D$ at $p$; then, using the implicit function Theorem, the defining function of $\partial D$ has the form

$$\rho(z, w) = u + R(z, v) \tag{3.39}$$

where $R$ vanishes to order $\geq 2$ at 0 and is independent of $u = \mathsf{Re}\, w$. Considering the Taylor expansion of $R(z, v)$ up to order 2 in $v$, we have

$$R(z, v) = A(z) + vB(z) + v^2 C(z) + f$$

where

(i) $A$ vanishes to order $\geq 2$ at $0 \in \mathbb{C}^{n-1}$

(ii) $B$ vanishes to order $\geq 1$ at $0 \in \mathbb{C}^{n-1}$

(iii) $f = o(v^2)$

Note that $A$ and $B$ depend on the choice of coordinates. Necessary and sufficient conditions for a local peak function have been given in terms of $A$ and $B$ by a general result due to Bloom (cf. [12]) that is expressed by

**Proposition 3.6.1.** *Let $D$ be a domain in $\mathbb{C}^n$ with $C^\infty$ boundary and $p \in \partial D$. Then, $p$ is a local peak point for $A^\omega(D)$ if and only if the following condition holds*

(*) *there exist local holomorphic coordinates in which for some $T > 0$*

$$B^2(z) < TA(z),$$

*if $z \neq 0$ is sufficiently small.*

*Proof.* We will prove only the sufficient condition, to which we are interested; for the necessity cf. [12]. Assume (*) is satisfied for $\rho$ of the type of (3.39); our aim is to show that there exists a neighbouhood $V$ of $p$ and a function $F$ holomorphic on $V$ such that $F(p) = 1$ and $|f| < 1$ on $V \cap \bar{D} \setminus \{p\}$. From (*) it follows that

$$A + Bv + Tv^2 > 0 \qquad \text{for } (z, v) \neq (0, 0) \text{ sufficiently small.} \tag{3.40}$$

We take $F = \exp(w + Kw^2)$, for $K$ a real positive constant such that

$$2C(0) - 1 + K \geq 2T. \tag{3.41}$$

Now $|F| = \exp(\mathsf{Re}\,(w + Kw^2)) = \exp(u + Ku^2 - Kv^2)$. If $q$ is a point in $V \cap \bar{D} \setminus \{p\}$ where $u(q) \leq 0$ and $|u(q)|$ is small, then $|F(q)| < 1$ for $q \neq p$. Otherwise, if $u(q) > 0$ and $|u(q)|$ is small enough, then

$$|F(q)| < \exp(2u - Kv^2)$$

and, being $u = \rho(z, w) - A(z) - vB(z) - v^2C(z) - f$ and $\rho(q) \leq 0$ we have

$$\begin{aligned}|F(q)| &= \exp[2\rho - 2(A + Bv + Cv^2 + f) - Kv^2]\\ &\leq \exp[-2(A + Bv + Cv^2 + f) - Kv^2]\\ &< 1\end{aligned}$$

by (3.40) and(3.41) for $(z, v) \neq (0, 0)$ sufficiently small. □

Now we restrict our attention to pseudoconvex domains in $\mathbb{C}^2$. Note that the previous result does not depend on the boundary being pseudoconvex, even if pseudoconvexity is a natural assumption in this problem. We want to show what restrictions are imposed on the function $A$ by the pseudoconvexity of $\partial D$, studying its Levi form. We summarize all our remarks in the following

**Proposition 3.6.2.** *Let $D \subseteq \mathbb{C}^2$ be a pseudoconvex domain of type $m$ at $p \in \partial D$. For the standard form (3.39), we have that*

(i) $A \neq 0$

(ii) $A(z) = P_m(z) + O(|z|^{m+1})$, *where $P_m$ is a homogeneous polynomial of degree $m \geq 2$, $P_m$ is subharmonic and $m = 2k$*

(iii) *$D$ is of finite type at $p$ if and only if there exists coordinates in which $P_m$ is not harmonic; the degree of $P_m$ is exactly the type of $D$ at $p$.*

*Proof.* (i) If, in standard form (3.39), for some coordinates, we have $A \equiv 0$, then the complex manifold $\{w = 0\}$ lies in the boundary, and this contradicts the hypothesis of finite type at $p$.

(ii) We can assume $p = 0$. By Taylor expansion we can write

$$A(z) = \sum_{i+j=m} a_{ij} z^i \bar{z}^j + O(|z|^{m+1}) = P_m(z) + O(|z|^{m+1})$$

for $P_m$ a homogeneous polynomial of degree $m \geq 2$. Note that, in $\mathbb{C}^2$, $T^{1,0}\partial D = \langle(\rho_w, -\rho_z)\rangle$ so the Levi form of $\rho$ at a neighbourhood of 0 is

$$\begin{aligned}\mathcal{L}\rho &= \rho_{z\bar{z}}|\rho_w|^2 - \rho_{z\bar{w}}\rho_w\rho_{\bar{z}} - \rho_{\bar{z}w}\rho_{\bar{w}}\rho_z + \rho_{w\bar{w}}|\rho_z|^2\\ &= \rho_{z\bar{z}}|\rho_w|^2 - 2\mathsf{Re}\,(\rho_{z\bar{w}}\rho_w\rho_{\bar{z}}) + \rho_{w\bar{w}}|\rho_z|^2.\end{aligned} \tag{3.42}$$

Note that the first term in (3.42) gives the laplacian of $P_m(z)$, while the other terms annihilate when we derive for $\bar{w}$. Rewriting the defining function as

$$\rho(z, w) = u + P_m(z) + O(v^2, vz, z^{m+1}),$$

we get that the Levi form of $\rho$ is

$$\mathcal{L}\rho = \frac{1}{4}(\Delta P_m) + O(v^2, vz, z^{m+1}).$$

By the hypothesis of pseudoconvexity of the domain $D$ at $p = 0$, we get $\Delta P_m \geq 0$, that means for $P_m$ to be subharmonic.

It is also obvious that $m = 2k$, $k \in \mathbb{N}$, because if we assume for instance that the degree $m$ of the polynomial is odd, also its laplacian is a polynomial of odd degree and it is impossible for it to be $\geq 0$ at a neighbourhood of 0.

(iii) If $P_m$ is harmonic, we use the change of coordinates

$$\begin{cases} \widetilde{z} = z \\ \widetilde{w} = w + 2a_{m,0}z^m \end{cases}$$

to have

$$\rho(\widetilde{z}, \widetilde{w}) = \mathsf{Re}\, w + \sum_{i+j=m+1} \widetilde{a}_{ij}\widetilde{z}^i\widetilde{z}^j + O(\widetilde{v}^j, \widetilde{v}\widetilde{z}, \widetilde{z}^{m+2}).$$

If we repeat this process ad infinitum, 0 becomes a point of infinite type because there are complex manifolds $\{w = 0\}$ tangent to $\partial D$ to arbitrarily high order. For the converse, if $P_m$ is not harmonic, it is evident that complex manifolds can be tangent to $\partial D$ up to order $m$, but not to higher order; $k$ is also uniquely determined.

$\square$

It turns out (for the details cf. [12]) that pseudoconvexity and an additional property, denoted as Condition (2) by Bloom, are sufficient to show that the order of vanishing at the origin of $B$ is related to the order of vanishing of $A$, from which it can be deduced that condition (*) is satisfied if $A$ is positive definite.

## 3.7 The notion of peaking form

Let $D$ be a bounded domain of $\mathbb{C}^n$; we denote by $\Lambda^1(D)$ the set $T^*D$ of 1-forms defined on $D$, which has dimension $2n$ over $\mathbb{C}$. If $\omega \in \Lambda^1(D)$, then it has the following representation

$$\omega = \sum_{j=1}^{n} a_j dz_j + \sum_{j=1}^{n} b_j d\bar{z}_j,$$

where each $a_j$ or $b_j : D \to \mathbb{C}$ is an element of $C^\infty(D)$, the space of infinitely differentiable complex-valued functions on $D$. We also denote by $\Lambda^{1,0}(D)$ the space of complex differential 1-forms containing only $dz$'s and by $\Lambda^{0,1}(D)$ the space of complex differential 1-forms containing only $d\bar{z}$'s.

Let $p$ and $q$ be a pair of non-negative integers $\leq n$. The space $\Lambda^{p,q}(D)$ of $(p,q)$-forms is defined by taking linear combinations of the wedge products of $p$ elements from $\Lambda^{1,0}(D)$ and $q$ elements from $\Lambda^{0,1}(D)$, so that if $\Lambda^k(D)$ is the space of all complex differential forms of total degree $k$, then each element $\omega$ of $\Lambda^k(D)$ can be expressed in a unique way as a linear combination of elements of $\Lambda^{p,q}(D)$, with $p+q=k$, in this way

$$\omega = \sum_{I,J} a_{IJ} dz_I \wedge d\bar{z}_J,$$

where $I = (i_1, i_2, \ldots, i_p)$, $1 \leq i_r \leq n$, for all $r$, $J = (j_1, j_2, \ldots, j_q)$, $1 \leq j_s \leq n$, for all $s$, $dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}$, $d\bar{z}_J = d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}$, each $a_{IJ} : D \to \mathbb{C}$ is an element of $C^\infty(D)$ and the previous sum is extended over all $I$, $J$, with $|I| = p$ and $|J| = q$.

We have raised the following questions:

- is it possible to define for complex differential $\bar{\partial}$-closed $k$-forms the property of "peaking", as for holomorphic functions?
- can it be interesting to look for peaking forms, as it happens for functions?

To give an answer to these questions, we have first formulated our notion of peaking forms and, then, we have made some relevant remarks. Here are our definitions

**Definition 3.7.1.** *The norm of a differential $k$-form $\omega = \sum_{I,J} a_{IJ} dz_I \wedge d\bar{z}_J$ is defined as*

$$\|\omega\| := \sum_{I,J} |a_{IJ}|^2,$$

*where*

$$\sum_{I,J} |a_{IJ}|^2 : D \to \mathbb{R}, \qquad z \longmapsto \sum_{I,J} |a_{IJ}(z)|^2.$$

For a differential 1-form $\omega$, as previously defined, the norm becomes

$$\|\omega\| := \sum_{j=1}^{n} |a_j|^2 + \sum_{j=1}^{n} |b_j|^2,$$

where

$$\sum_{j=1}^{n} |a_j|^2 + \sum_{j=1}^{n} |b_j|^2 : D \to \mathbb{R}, \qquad z \longmapsto \sum_{j=1}^{n} |a_j(z)|^2 + \sum_{j=1}^{n} |b_j(z)|^2$$

**Definition 3.7.2.** *Let $D$ be a bounded domain of $\mathbb{C}^n$; we say that a differential $k$-form $\omega = \sum_{I,J} a_{IJ} dz_I \wedge d\bar{z}_J$, with $a_{IJ} \in C^\infty(D)$, locally peaks at $z_0 \in \overline{D}$ if there exists a neighbourhood $U$ of $z_0$ in $\mathbb{C}^n$ such that $\bar{\partial}\omega_{|_U} = 0$, $\|\omega\|(z_0) = 1$ and $\|\omega\|(z) < 1$, for $z \in U \cap D$.*

The basic point of discussion is the following: is it possible, for a complex differential form, to peak in the interior of the domain $D$?

If it is not possible, then it has meaning and becomes of interest the research of conditions to have peaking forms on the boundary of the domain of definition. Comparing with the well-known setting of functions, let's remember that the definition of peaking function requires $f \in C(\bar{D}) \cap \mathcal{O}(D)$; then, maximum principle ensures that $f$ can only peak on the boundary, where maximum is attained. By the fact that analyticity for the peaking function is substituted in our definition by the requirement, for the differential form $\omega$, to be $\bar{\partial}$-closed, it can happen that $\omega$ peaks in the interior of the domain.

Also note that, for functions, the existence of a peaking function $f$ at a point $z_0 \in \partial D$ is equivalent to the existence of a strong support function $g$ (that means $g(z_0) = 0$ and $\mathsf{Re}\, g > 0$ on $\bar{D} \setminus \{z_0\}$); this is not true for forms, because we do not have the analogous for forms of functions like $\frac{1-g}{1+g}$ or $e^{-\mathsf{Re}\, g}$.

We have analysed the problem for differential forms in $\mathbb{C}$ and in $\mathbb{C}^2$ and we have come to the conclusions contained in the following Proposition.

**Proposition 3.7.1.**
- *If $D$ is a domain in $\mathbb{C}$, there exist complex differential forms that peak in the interior of $D$.*
- *If $D$ is a domain in $\mathbb{C}^2$ and $\omega = \bar{\partial}u$, where $u$ is a cubic function in $z$, $\bar{z}$, then there exist complex differential forms that peak in the interior of $D$.*

*Proof.* For $D$ a domain of $\mathbb{C}$, we consider a complex differential 1-form of the type

$$\omega = a d\bar{z}$$

which has norm

$$\|\omega\| = |a|^2 \geq 0.$$

It is obvious that $\omega$ is $\bar{\partial}$-closed, because it has maximum degree in $\bar{z}$, so if we choose

$$\omega = e^{-|z|^2} d\bar{z},$$

it peaks at 0, which can always be considered an interior point for $D$.

If $D$ is a domain in $\mathbb{C}^2$ and $\omega = \bar{\partial} u$, it is clear that $\bar{\partial}\omega = \bar{\partial}\bar{\partial} u = 0$. As a first step, let's consider a quadratic function $u(z, \bar{z})$ in $\mathbb{C}^2$; it will be sufficient to have

$$u(z, \bar{z}) = \sum_{i,j=1}^{2} (a_{ij} z_i \bar{z}_j + b_{ij} \bar{z}_i \bar{z}_j) + \sum_{i=1}^{2} c_i \bar{z}_i$$

because the other terms, not depending on $\bar{z}_j$ would be annihilated by $\bar{\partial}$. Then

$$\bar{\partial} u(z, \bar{z}) = \sum_i \left( \sum_j (a_{ji} z_j) + 2 b_{ii} \bar{z}_i + \sum_{j \neq i} (b_{ij} + b_{ji}) \bar{z}_j + c_i \right) d\bar{z}_i$$

and its norm $\|\bar{\partial} u(z)\|$ is a polynomial $P(z, \bar{z}) \geq 0 \; \forall z$, with degree 2 in $z, \bar{z}$. If $P(z, \bar{z})$ has the "peak" property at $0 \in D$, that means $P(0, 0) = 1$ and $P(z, \bar{z}) < 1$ in a little ball around 0, then the derivative of $P(z, \bar{z})$ at $(0, 0)$ is 0 and, by Taylor expansion, we would have $P(z, \bar{z}) = P(0, 0) + A(z, \bar{z}) = 1 + A(z, \bar{z}) \geq 0$, where $A$ is quadratic and $A$ can be $\geq 0$ in a neighbourhood of $0 \in D$. We have obtained $P(z, \bar{z}) \geq 1$, which contradicts the peak property. Then, there are no peaking forms of the type $\omega = \bar{\partial} u$, if $u$ is a quadratic function.

The previous step suggests to look for peaking forms of the type $\bar{\partial} u$, where $u$ has higher degree. Taking forms $u$ of that type and with degree 3 in $z, \bar{z}$, we have easily found a complex differential form in $\mathbb{C}^2$ that peaks in the interior of $D$. The construction is the following: take the cubic function in $z, \bar{z}$ in $\mathbb{C}^2$

$$u(z, \bar{z}) = \frac{\sqrt{2}}{2} \bar{z}_1 + \frac{\sqrt{2}}{2} \bar{z}_2 - \frac{1}{2} \bar{z}_1^2 z_1 - \frac{1}{2} \bar{z}_2^2 z_2$$

and consider the form $\omega$ given by $\bar{\partial} u$ (which is obviously $\bar{\partial}$-closed)

$$\omega = \bar{\partial} u = \left( \frac{\sqrt{2}}{2} - |z_1|^2 \right) d\bar{z}_1 + \left( \frac{\sqrt{2}}{2} - |z_2|^2 \right) d\bar{z}_2;$$

the norm of $\omega$ is

$$\begin{aligned} \|\omega\| &= \left( \frac{\sqrt{2}}{2} - |z_1|^2 \right)^2 + \left( \frac{\sqrt{2}}{2} - |z_2|^2 \right)^2 \\ &= 1 - \sqrt{2}|z_1|^2 - \sqrt{2}|z_2|^2 + |z_1|^4 + |z_2|^4. \end{aligned}$$

It is evident that we have constructed a complex differential form that peaks at $0 \in D$, because $\|\omega\|(0, 0) = 1$ and $\|\omega\|(z, \bar{z}) < 1$, for $(z, \bar{z})$ next to $(0, 0)$. □

**§. Open problem**: At this point we have to exclude an easy application of the notion of "peaking" for $\bar{\partial}$-closed forms, because we have exhibited examples of complex differential forms that peak in the interior of the domain. Nevertheless, we are still investigating on the possibility to introduce the "peaking" notion for $q$-holomorphic functions (cf. the paper of Basener [7] for definitions), where it is possible to regain maximum principle (as for holomorphic functions).

# Conclusions

There are surely some possible directions we would like to pursue in the future. While the theme of the second Chapter turns to be concluded for the moment, much work can be done for some aspects of the theme of the third Chapter; in particular we are already investigating on

- the possibility to apply our stratification in Subsection 3.4.3 for pseudoconvex domains of finite type with $C^\omega$ boundary with the aim at obtaining results on the existence of peak interpolation manifolds, first for convex domains with $C^\omega$ boundary and then for other types of weakly pseudoconvex domains;

- the possibility to get a stratification of the type of Bharali Theorem for convex domains with only smooth boundary; with this aim we want to analyse a way of stratifying smooth manifolds in the jet-place, contained in a paper of M. S. Baouendi, L. P. Rothschild and D. Zaitsev [1];

- the possibility to find special cases in which it is possible to conclude the generalization of Henkin-Tumanov Theorem, presented in Section 3.5, to get sufficient conditions for the existence of peak interpolation manifolds in some weakly pseudoconvex domains;

- the possibility to introduce the notion of "peaking" for $q$-holomorphic functions, as defined in the paper of Basener [7]. In fact, for them, it is possible to regain maximum principle, as for holomorphic functions.

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