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Singular Perturbation and Homogenization Problems in Control Theory, Differential Games and fully nonlinear Partial Differential Equations

Ph.D. Thesis

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Abstract

In this thesis we address different topics related to homogenization of first and second order fully nonlinear PDEs, essentially of Hamilton–Jacobi type, and more generally to singular perturbation in optimal control problems and differential games, in the light of the viscosity solution theory.

We take into account a singularly perturbed control systems (*i.e.* a system where the state variables evolve with two different time scales), both in the deterministic and in the stochastic setting, and the related first and second order Hamilton–Jacobi equations. A first part of the work is devoted to *order reduction procedures*: the goal of such procedures is to obtain, as the perturbation parameter tends to zero, a system where only the slow variables appear. The construction of the limit dynamics relies on the asymptotic behavior of the fast variables of the original system. We use *limiting relaxed controls*, *i.e.* suitably defined Radon probability measures to average the fast part of the controlled dynamics. We give - both in the deterministic and in the stochastic framework - representation formulae for the effective Hamiltonian in terms of limiting relaxed controls. This allow a control interpretation of the limiting dynamics. As an application of these reduction procedures, we study the propagation of fronts moving with normal velocity depending on the position and undergoing fast oscillations.

In the second part of the work we study asymptotic controllability properties of a deterministic singularly perturbed systems and of the limit system. We prove first that, under suitable assumptions, the weak lower semilimit of Lyapunov functions of a singularly perturbed system is a lower semicontinuous Lyapunov function for the limiting system. Furthermore, we also prove that the asymptotic controllability to the origin of the (smaller) limit system is enough to infer asymptotic controllability of the slow part of the (larger) perturbed system. More precisely, perturbing a Lyapunov pair for the limit dynamics, we construct a Lyapunov pair for the original system.

The third and last part of the thesis concerns homogenization of non-coercive Hamilton-Jacobi equations with oscillating Hamiltonian and initial data. We take into account a rather general class of Hamiltonians convex in some gradient variables and concave with respect to the others. In particular it is shown that for some of these equations homogenization does not take place, in contrast with the usual coercive case. Sufficient conditions for homogenization are provided involving the structure of the running cost and the initial data.

Riassunto

In questa tesi vengono trattati argomenti inerenti l'omogeneizzazione di equazioni differenziali alle derivate parziali completamente non lineari del primo e del second'ordine, essenzialmente di tipo Hamilton-Jacobi. Più in generale si studiano problemi di controllo ottimo e giochi differenziali singolarmente perturbati, nell'ambito della teoria delle soluzioni di viscosità.

Si considerano, sia nel caso deterministico che stocastico, sistemi controllati singolarmente perturbati (*i.e.* sistemi in cui le variabili di stato evolvono lungo due differenti scale temporali), e le equazioni di Hamilton-Jacobi del primo e del second'ordine ad essi associate. Una prima parte del lavoro consiste nella determinazione di *procedure di riduzione dell'ordine*. Viene costruito un sistema limite per le sole variabili lente, tenendo conto del comportamento asintotico delle variabili veloci del sistema originario. Si fa uso di *limiting relaxed controls*, cioè misure di Radon di probabilità opportunamente definite allo scopo per mediare la parte veloce della dinamica. Vengono fornite delle formule di rappresentazione dell'Hamiltoniana effettiva in termini di controlli rilassati; questo permette di interpretare il sistema limite come un sistema di controllo. Queste procedure vengono poi applicate allo studio del moto di fronti in presenza di forti oscillazioni.

Nella seconda parte del lavoro si affronta lo studio della controllabilità di sistemi singolarmente perturbati. Dapprima si prova che il semilimite debole inferiore di funzioni di Lyapunov è una funzione di Lyapunov per la dinamica limite. Poi, si prova che la controllabilità asintotica all'origine del sistema limite (più piccolo) è sufficiente a dedurre la controllabilità asintotica all'origine della dinamica lenta del sistema perturbato (più grande). Più precisamente, perturbando opportunamente una coppia di Lyapunov per la dinamica limite, si costruisce una coppia di Lyapunov per il sistema originario.

La terza e ultima parte della tesi concerne l'omogeneizzazione di equazioni di Hamilton-Jacobi non coercive con Hamiltoniana e dati iniziali oscillanti. Si prendono in esame alcune classi di Hamiltoniane convesse in alcune variabili del gradiente e concave rispetto alle altre. Viene provato che per alcune di queste equazioni l'omogeneizzazione non ha luogo, contrariamente a quanto accade nell'usuale caso di Hamiltoniane coercive. Vengono fornite condizioni sufficienti per l'omogeneizzazione che coinvolgono la struttura del costo corrente e del dato iniziale.

Contents

Notations

$\mathbb{R}^N$	the Euclidean N-dimensional space
$\mathbb{T}^N$	the N-dimensional flat torus, *i.e.* the quotient $\mathbb{R}^N/\mathbb{Z}^N$
$\mathbb{Z}^N$	the set of $z \equiv (z_1, \ldots, z_N)$, $z_i \in \mathbb{Z}$
$\mathbb{M}^{n\times k}$	the set of $n \times k$ matrices
$\mathbb{S}^n$	the set of $n \times n$ symmetric matrices
$x \cdot y$	the scalar product in $\mathbb{R}^N$
$\|x\|$	the Euclidean norm of $x \in \mathbb{R}^N$
$B(x_0, r)$	the open ball $\{x \in \mathbb{R}^N : \|x - x_0\| < r\}$
$\bar{B}(x_0, r)$	the closed ball $\{x \in \mathbb{R}^N : \|x - x_0\| \leq r\}$
∂S	the boundary of the set S
$\bar{S}$	the closure of the set S
$\overline{\text{co}} S$	the closure of the convex hull of the set S
$\mathcal{B}(X)$	the Borel σ-algebra of the space X
$\mathcal{L}^N$	the N-dimensional Lebesgue measure
$d_H(S, T)$	the Hausdorff distance between the sets S and T
$\pi(\mu_1, \mu_2)$	the Prohorov distance between the measures μ_1 and μ_2
$\arg\min \varphi$	the set of minimum point of the function φ
$\|\|\varphi\|\|_\infty$	the supremum norm of a function $\varphi : X \to \mathbb{R}$, $\sup_{x \in X} \|\varphi(x)\|$
$\mathbf{1}_X$	the characteristic (or indicator) function of the set X
ω	a modulus, *i.e.* a function $\omega : [0, +\infty) \to [0, +\infty)$ continuous, nondecreasing, and such that $\omega(0) = 0$
$\mathcal{K}$	the class of comparison functions $\beta : [0, +\infty) \to [0, +\infty)$ continuous and strictly increasing, with $\lim_{t\to 0^+} \beta(t) = 0$ and $\lim_{t\to+\infty} \beta(t) = +\infty$
$\mathcal{KL}$	the class of comparison functions $\eta : [0, +\infty)^2 \to [0, +\infty)$ continuous, strictly increasing in the first variable, strictly decreasing in the second variable and satisfying $\eta(0, t) = 0$ for any $t \geq 0$ and $\lim_{t\to+\infty} \eta(r, t) = 0$ for any $r \geq 0$
$D\varphi(x)$	the gradient if the function φ at x
$\partial_{x_j}\varphi(x)$	the partial derivative with respect to the x_j variable of the function φ at $x = (x_1, \ldots, x_N)$
$J^+\varphi(x)$, $J^-\varphi(x)$	the super- and subdifferential of φ at x
$\varphi^*(x)$, $\varphi_*(x)$	the upper and lower weak semilimit of the function φ
$\mathcal{A}$	the set of control functions, *i.e.* the set of Lebesgue measurable functions $a : [0, +\infty) \to A$, where A is a given compact metric space
A^r	the set of Radon probability measures on A
$\mathcal{A}^r$	the set of relaxed controls

$t_x(a)$	the first time the solution of a control problem enters a given closed set $\mathcal{T}$
$C(X)$	the space of continuous function $\varphi : X \to \mathbb{R}$
$C^k(\Omega)$	for $k \geq 1$ and $\Omega \subset \mathbb{R}^N$ open, the subspace of $C(\Omega)$ of functions with continuous partial derivatives in Ω, up to order k
$BUC(X)$	the space of bounded and uniformly continuous functions $\varphi : X \to \mathbb{R}$
$\mathrm{Lip}(\varphi)$	the Lipschitz constant of the Lipschitz-continuous function $\varphi : X \to \mathbb{R}$
$\mathrm{Lip}_S(\varphi)$	the Lipschitz constant on the compact subset $S \subset X$ of the locally-Lipschitz continuous function $\varphi : X \to \mathbb{R}$
$USC(X)$, $LSC(X)$	the spaces of upper and lower semicontinuous functions $\varphi : X \to \mathbb{R}$
Δ, Γ	nonanticipating strategies for the first and second player of a differential game

Introduction

The theory of viscosity solutions of fully nonlinear partial differential equations has given a large number of contributions in deterministic and stochastic optimal control theory and differential games. Connections between optimal control problems and viscosity solutions of Hamilton–Jacobi–Bellman equations are accurately detailed in the books of Bardi and Capuzzo Dolcetta [22], Barles [25], Fleming and Soner [50].

The study of deterministic and stochastic singularly perturbed control systems is motivated by many problems coming from chemistry, physics and engineering. We consider a system where some state variables evolve at a much faster time scale than the others: this is modelled by a small positive parameter ε appearing in front of the time derivative of such *fast* variables. The prototype of a singularly perturbed control system is

$$\begin{aligned} dx_t &= f(x_t, y_t, a_t)ds + \sigma(x_t, y_t, a_t)dW_t, & x \in \mathbb{R}^N \\ dy_t &= \tfrac{1}{\varepsilon} g(x_t, y_t, a_t)ds + \tfrac{1}{\sqrt{\varepsilon}}\tau(x_t, y_t, a_t)dW_t, & y \in \mathbb{R}^M \end{aligned} \qquad (\varepsilon > 0)$$

where $a_\cdot$ is a control function taking values in a certain compact set A. Our singular perturbation problem is passing to the limit as ε vanishes. The result is the reduction of the original $(N+M)$–dimensional system to an N–dimensional system keeping some informations on the fast part of the dynamics, but involving only the slow variables.

The first approach in order reduction procedures for ODEs goes back to the works of Levinson and Tichonov and their students in the fifties; such approach was extended to deterministic control system by several Authors. See Kokotović *et al.* [60], Bensoussan [29], Dontchev and Zolezzi [43], Veliov [77] and the references therein.

The Levinson-Tichonov theory took place originally in the deterministic uncontrolled framework; the singularly perturbed initial value problem

$$\begin{aligned} \dot{x}_t &= f(x_t, y_t), & x_0 = x \\ \varepsilon \dot{y}_t &= g(x_t, y_t), & y_0 = y \end{aligned}$$

is considered. A *reduced* problem is obtained by setting $\varepsilon = 0$

$$\dot{x}_t = f(x_t, y_t), \quad 0 = g(x_t, y_t), \quad x_0 = x$$

Such differential–algebraic system supplies the correct expression of the limiting system provided that for any fixed x the problem

$$\dot{z}_t = g(x, z_t)$$

is *boundary layer stable*, namely if it has a bounded solution z_t defined for any $t > 0$, z_t has a limit at infinity and this limit is a solution of the algebraic equation $g(x, y) = 0$.

Even if many important problems can be putted in this reduced form, setting the perturbation parameter to zero fails to give the correct approximation when the fast variables oscillate for large times instead to converging to a steady state. Then, averaging methods have been developed by Artstein, Gaitsgory, Leizarowitz and others, in the spirit of Krylov-Bogolyubov theory of invariant measures of ODEs.

Our approach involves the fully nonlinear partial differential equations related to the optimal control problem. We consider a given cost functional $J(t, x, y, a)$ defined on each time interval $[0, t]$, and the value function

$$u^\varepsilon(t, x, y) := \inf\{J(t, x, y, a) : a \text{ admissible control function}\}$$

which satisfies, in viscosity sense, a certain Hamilton–Jacobi–Bellman equation. The approach to the singular perturbation problem we follow goes back to Lions [63], Jensen and Lions [59], Artstein and Gaitsgory [16] and is deeply inspired by some recent works of Bardi and Alvarez [2], [3], [4] and [5]. It consists in studying the limit as $\varepsilon \to 0^+$ of the value functions and characterize such limit as the unique viscosity solution of some *limiting* partial differential equation. One expects that the limit $u(t, x)$ of such value functions does not depend on the fast variables y and satisfies a limit PDE involving an effective operator, the *effective Hamiltonian* $\bar{H}$. In connection with the ergodic control theory, the effective Hamiltonian is found as the value of an M–dimensional ergodic control problem for the fast subsystem, obtained by freezing the slow variable x and setting $\varepsilon = 1$. [2] and [3] studied the asymptotic behavior of solutions of general degenerate parabolic equations

$$\partial_t u^\varepsilon + H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}, D_{xx} u^\varepsilon, \frac{D_{yy} u^\varepsilon}{\varepsilon}, \frac{D_{xy} u^\varepsilon}{\sqrt{\varepsilon}}\right) = 0$$

with H satisfying some natural structural assumption, and with initial condition

$$u^\varepsilon(0, x, y) = h(x, y).$$

Two properties for the pair (H, h), called *ergodicity* and *stabilization to a constant*, have been pointed out to be crucial in proving the convergence of u^ε's. Such properties pertain with the possibility of defining the effective operator $\bar{H}$ and an effective initial datum $\bar{h}$, and are related to the limit behavior of solutions of degenerate parabolic PDE's connected with the fast subsystem. The main result in [3] affirms that, if the pair (H, h) enjoys these two properties, then the upper and lower weak semilimit of u^ε are respectively viscosity sub and supersolution of the effective problem

$$\partial_t u + \bar{H}(x, D_x u, D_{xx} u) = 0, \qquad u(0, x) = \bar{h}(x).$$

This is our starting point; now we describe the contributions of this thesis. Let us focus our attention on the deterministic case, *i.e.* when $\sigma \equiv 0$ and $\tau \equiv 0$:

$$\begin{aligned} \dot{x}_t &= f(x_t, y_t, a_t), \quad x_0 = x \\ \varepsilon \dot{y}_t &= g(x_t, y_t, a_t), \quad y_0 = y \end{aligned} \tag{0.1}$$

As mentioned before, the limit behavior of a singularly perturbed system can be described using invariant measures; more precisely, the limit as $\varepsilon \to 0$ of the slow flow can be portrayed by solutions of a differential inclusion generated by invariant measures of the fast motion. See Artstein [12], [13], Artstein and Gaitsgory [15], [16] and Gaitsgory and Leizarowitz [55]. In particular in [55] the Authors take into account a set of limit occupational measures of the *fast subsystem*

$$\dot{y}_t = g(x, y_t, a_t), \quad y_0 = y, \quad x \text{ fixed} \tag{0.2}$$

and show the existence of the limit set

$$\lim_{s \to \infty} \bigcup_a \{\mu(s,a), \text{ occupational measure of (0.2) on } [0,s]\}. \tag{0.3}$$

The set (0.3) is used to describe the limiting dynamics. In our terminology, an *occupational measure* indicates the proportion of time spent by the solution of the subsystem (0.2) on a given set. A similar approach was proposed in [2] by Bardi and Alvarez: instead of the set (0.3), it is taken into account, for any x, the set of *limiting relaxed controls*, *i.e.* weak-star limits of occupational measures. Observe that considering the fast sub-system (0.2) permits to decouple the dynamics, and to analyze first, for any frozen x, the faster system, and then to define the effective dynamics.

The purpose of the first Chapter is twofold. First we prove that, under suitable controllability assumptions, the set of limit occupational measure and the set of limiting relaxed controls coincide. This permits to infer the compactness and the convexity of the limiting relaxed control set and, then, to regard it as an appropriate candidate set of controls for a *limit control problem.* The use of relaxed controls (whose theory goes back to Warga [78]) is a way to convexify the problem enlarging the set of admissible control functions, and permits, in this context, to provide a control interpretation of the limiting dynamics of the singularly perturbed problem; this is in fact the second issue addressed in the first Chapter. Such limit control problem is governed by a differential inclusion

$$\dot{x}_t \in F^r(x_t), \quad x_0 = x \tag{0.4}$$

The multivalued function F^r is obtained relaxing the original slow dynamics with limiting relaxed controls related the fast subsystem (0.2):

$$F^r(x) := \left\{ \int f(x,y,a)\mu(dy,da) \,\middle|\, \mu \text{ limiting relaxed control measure} \right\}$$

A crucial point is to show that the effective Hamiltonian can be expressed in the Bellman form $\max_\mu L^r(x,p,\mu)$, for a certain generator L^r, and then that the value function corresponding to the limit control problem (0.4) solves the effective PDE. Such a representation for the effective Hamiltonian was already pointed out in [2] in the deterministic setting. We prove here analogous representations for stochastic control problems and, with some additional assumption, for differential games. This suggests that a control interpretation of the asymptotic behavior of singularly perturbed control systems and differential games can be provided also in the stochastic setting.

In the concluding section of the first Chapter, the order reduction method based upon limiting relaxed controls are applied to the study of the propagation of fronts moving with normal velocity depending on the position and undergoing fast oscillations. The limit control problem underlying the effective evolution is exhibited, and the effective moving front is described as the zero level set of the value function of such control problem. This allows an interpretation of the effective behavior of the front in terms of generalized characteristics. The case where the normal velocity changes its sign is also treated.

In the second part of the work we focus our attention on controllability questions. More precisely, we study, by means of Lyapunov functions, the asymptotic controllability properties of a deterministic singularly perturbed systems and the one of the limit system. We show that the property of being a Lyapunov function is stable with respect to the weak semilimit: under suitable assumptions, the lower semilimit of Lyapunov functions for the singularly perturbed system is a lower semicontinuous Lyapunov function for the limit system. In particular, the monotonicity property of a Lyapunov function, *i.e.* its decrease along trajectories of the dynamics driving the system to a certain target, is characterized by a suitable Hamilton–Jacobi first order differential inequality

$$H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}\right) \geq 0 \tag{0.5}$$

interpreted in viscosity sense. We apply a result in [3] regarding the convergence of weak semilimits of viscosity sub- and supersolutions of ergodic Hamilton-Jacobi equation to viscosity sub- and supersolutions of the effective equation.

Furthermore, we also prove that assuming asymptotic controllability to the origin of the limit system in a certain basin of attraction, is enough to infer local asymptotic controllability of the slow part of the original system. A global version of the result is also established.

The method is inspired by the asymptotic expansions in multiple scale problems (see Bensoussan *et al.* [30]) and use the perturbed test function method (see Evans [44]). We suppose to posses a Lyapunov function for the limiting system, *i.e.* a supersolution u of the effective Hamiltonian, and try to obtain a supersolution of (0.5) as a first order perturbation in ε of u:

$$u^\varepsilon(x, y) := u(x) + \varepsilon\chi(y). \tag{0.6}$$

where the function $\chi(y)$ is the solution of the so called *cell problem*

$$H(\bar{x}, y, Du(\bar{x}), D_y\chi) = \bar{H}(\bar{x}, Du(\bar{x})), \quad \bar{x} \text{ fixed}$$

Such arguments have been rigorously developed in [2] and [3] to establish the local uniform convergence of solutions of first and second order Hamilton–Jacobi equations to the solution of the effective equation. A subtle issue arises here. The function χ depends in fact not only on y but also on x. Therefore, in proving that u^ε solves (0.5), a contribution of χ should appear in place of $D_x u^\varepsilon$. Unfortunately, the dependence on x of χ is not clear, and remains still an open question. We eliminate such a dependence by introducing the auxiliary Hamiltonian

$$K_{r,R}(y, q) := \inf\{H(x, y, p, q) :\ r \leq |x| \leq R,\ p \in J^- V(x)\}$$

Our strategy consists in showing, for any $R > r > 0$, the existence of a continuous function $\chi(y)$, independent of x, satisfying

$$K_{r,R}(y, D\chi) > \gamma > 0.$$

The main result in the second Chapter asserts that the function u^ε obtained perturbing the function u as in (0.6) is a supersolution of (0.5), and therefore a Lyapunov function for the singularly perturbed system. We conclude that the asymptotic controllability of the limiting (smaller) system is enough to infer asymptotic controllability of the slow dynamics of the singularly perturbed system, to each neighborhood of the origin. Our contribution is compared with an analogous result proved by Artstein in [12] for uncontrolled systems.

The third and last part of the thesis concerns homogenization of non-coercive Hamilton-Jacobi equations with oscillating initial data. We consider a two-players zero-sum singularly perturbed deterministic differential game with a running cost $l(x, y, a, b)$ and a terminal cost $h(x, y)$. Consequently, a rather general class of min-max Isaacs Hamiltonians is taken into account. Several sufficient condition for ergodicity of such operators have been given in Alvarez and Bardi [4], [5] in Bardi [21], in terms of asymptotic controllability of the fast part of the game with respect to certain targets. These conditions apply in many different situations, namely if the fast part of the game is bounded time controllable, or if the running cost is independent by the controls and has a saddle point. Other conditions apply to games in *splitted form*, *i.e.* when a player controls a group of state variables and the other controls the remaining variables.

Our goal is to apply the general convergence results to provide homogenization theorems for non coercive Hamiltonians, more precisely Hamiltonians that are convex with respect to some gradient variables, and concave with respect to the others. We show with a certain number of examples that in this case homogenization may fail, in contrast with the usual case of coercive Hamiltonian. A general negative result, proved in the beginning of the chapter, is particularly interesting. It applies to the following homogenization problem:

$$\begin{cases} \partial_t u^\varepsilon + H_1(x, \frac{x}{\varepsilon}, D_{x_A} u^\varepsilon) - H_2(x, \frac{x}{\varepsilon}, D_{x_B} u^\varepsilon) = l(\frac{x}{\varepsilon}) \\ u^\varepsilon(0, x) = 0 \end{cases} \qquad x \equiv (x_A, x_B)$$

with H_1 and H_2 coercive. The negative result asserts that we can always find an analytic cost l such that the equation does not homogenize.

These negative cases make apparent the most relevant difference with the coercive case: not only conditions on the structure of the Hamiltonian, but also on the running cost and the initial data have to be required. We single out two rather general classes of convex-concave homogenizing Hamiltonian. In both cases we suppose l to be independent from the controls, in order to use the known ergodicity results.

Homogenization holds true in two completely different and opposite situation. In a first class of results, we suppose the problem to be splitted between the two competitors, in the dynamics and in the costs. Under this assumption the ergodic problem is also splitted, and we get rather easily sufficient conditions for stabilization and then for homogenization. Our results in particular applies to the so called *convex-concave eikonal equation*:

$$\partial_t u^\varepsilon + g(x, \tfrac{x_A}{\varepsilon})|D_{x_A} u^\varepsilon| - \gamma g(x, \tfrac{x_B}{\varepsilon})|D_{x_B} u^\varepsilon| = l_1(x, \tfrac{x_A}{\varepsilon}) + l_2(x, \tfrac{x_B}{\varepsilon})$$

$$u^\varepsilon(0, x) = h(x, \tfrac{x}{\varepsilon})$$

$$x \equiv (x_A, x_B)$$

where g is a nonnegative function and $\gamma > 0$ a parameter representing who of the two competitors has the possibility to drive the system faster than the other. The effective Hamiltonian, and the effective initial datum are explicitly computed for this equation in the 2D case.

The second class of problems where we show that homogenization holds, concerns the case in which the dependence of the dynamics and the cost on the fast states is expressed only in terms of the difference of the fast variables:

$$\partial_t u^\varepsilon + H_1(x, \tfrac{x_A - x_B}{\varepsilon}, D_{x_A} u^\varepsilon) - H_2(x, \tfrac{x_A - x_B}{\varepsilon}, D_{x_B} u^\varepsilon) = 0$$

$$u(0, x, y) = h(x, \tfrac{x_A - x_B}{\varepsilon})$$

$$x \equiv (x_A, x_B), \quad x_A, x_B \in \mathbb{R}^{N/2}$$

We show that the ergodicity and stabilizing properties can be derived looking at a new Hamiltonian, for which the usual sufficient conditions applies.

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Chapter 1

Singular perturbation of optimal control problems

This first Chapter is devoted to the study of the limit behavior of singularly perturbed optimal control problems. Instead of the classical Levinson–Tikonov order reduction method, we use relaxed controls to define the limiting dynamics. Such approach is actually close to the one adopted by Artstein [12], [14] and by Gaitsgory and Leizarowitz [55] for deterministic controlled dynamics, and by Borkar and Gaitsgory [32] in the stochastic setting, an approach based upon the use of invariant measures, and of the set of limits of occupational measures.

A *limiting relaxed control* is a Radon probability measure which is a limit, in the weak-star topology, of occupational measures for the fast part of the controlled dynamics; see Definition 1.8 below. The starting point of this Chapter is the proof of the fact that, under suitable controllability assumptions, the set of limit occupational measures (in the sense of Gaitsgory) coincides with the set of limiting relaxed controls. The first advantage in considering limiting relaxed controls is the possibility to recognize the limiting dynamics as an appropriate control problem, in fact the set of limiting relaxed controls turns out to be convex and compact in the weak-star topology.

The study of such limit control problem involves only PDE and viscosity solutions methods, instead of techniques related to invariant measures. We first observe that using relaxed controls, the effective Hamiltonian can be represented as a Bellman Hamiltonian; analogous representation formulae are proved also in the stochastic framework. Then, using the dynamic programming principle, we show that, under reasonable controllability assumptions, the value function related to the limit control problem solves the effective Hamiltonian, and therefore it is its unique solution. This permits to regard the limit control problem as the correct control problem describing the asymptotic behavior, as ε vanishes, of the singularly perturbed control problem.

The Chapter is organized as follows. After introducing the subject and some preliminary results about the study of singularly perturbation problems via PDE methods in a rather general setting, in Section 1.2.1 we recall some basic concepts regarding weak-star convergence of measures and relaxed control. Then we introduce the concept of limiting relaxed control both in the deterministic and in the stochastic setting and compare the set of limiting relaxed controls with the set of limit occupational measures. Section 1.3 is devoted to the study of the limit control problem of deterministic singularly perturbed control problems. Such limit is explicitly computed in an example. In Section 1.4 representation formulae of the effective Hamiltonian in terms of relaxed controls are provided in many different situations, for deterministic and stochastic control problem, and for a stochastic differential game. Finally, in Section 1.5 we study, as an application of the previous theory, the evolution of fronts with normal velocity depending on the position and undergoing fast oscillation. Results contained in the first sections permits to write the effective front as a zero level set of the value function of a certain control problem and consequently to describe it by means of generalized characteristics. Also the case where the normal velocity changes its sign is taken in account.

1.1 Terminology, preliminaries and assumptions

Consider, for $t > 0$, the following singularly perturbed stochastic optimal control problem

$$\begin{array}{ll} dx_t = f(x_t, y_t, a_t)dt + \sigma(x_t)dW_t & x_0 = x \\ dy_t = \frac{1}{\varepsilon} g(x_t, y_t, a_t)dt + \frac{1}{\sqrt{\varepsilon}}\tau(x_t, y_t)dW_t & y_0 = y \end{array} \tag{1.1}$$

with running cost $l(x, y, a)$ and terminal cost $h(x, y)$. We will refer to the small positive parameter ε as to a *perturbation parameter* representing, physically, the action of many fast influencences. As a consequence of the presence of the perturbation parameter, the state variables are divided in two groups: a first group of *slow* variables x, belonging to $\mathbb{R}^N$, evolving on a $O(1)$ time scale, and a second group of *fast* variables y, lying on $\mathbb{R}^M$ and evolving on a faster time scale.

In this first Chapter we will consider both the stochastic control system (1.1) and its deterministic version (obtained from (1.1) by putting $\sigma \equiv 0$ and $\tau \equiv 0$). For sake of completeness, we choose to expose the following preliminaries in the general stochastic framework. We denote by W_t a D–dimensional Brownian motion and by $a.$ an admissible control, *i.e.* a process valued in a compact set A that we will define next.

We denote by $\Omega_t := \{\omega \in C([0, t]; \mathbb{R}^D) : \omega_0 = 0\}$ a sample space, by $\mathcal{F}_s$ the σ–algebra generated by the paths of the Brownian motion up to time t and by P_t the Wiener measure. An admissible control on $[0, t]$ is a $\mathcal{F}_s$–progressively measurable process $a.$ taking values in a compact metric space A. The space of admissible controls on $[0, t]$ is denoted by $\mathcal{A}(t)$.

Now let us list the assumptions on the data we will suppose to hold throughout the work without any further mention; we will refer to such assumptions as to the *standing assumptions*:

- the functions f, g, σ, τ and l are bounded and uniformly continuous in $\mathbb{R}^N \times \mathbb{R}^M \times A$, with values, respectively in $\mathbb{R}^N$, $\mathbb{R}^M$, $\mathbb{M}^{N\times D}$, $\mathbb{M}^{M\times D}$ and $\mathbb{R}$;

- the function h is bounded uniformly continuous from $\mathbb{R}^N \times \mathbb{R}^M$ to $\mathbb{R}$;

- the functions $f(\cdot, \cdot, a)$ and $g(\cdot, \cdot, a)$ are Lipschitz–continuous in x, y, uniformly with respect to a; the functions σ and τ are Lipschitz–continuous;

- all the data are periodic with respect to y: $\varphi(x,y)=\varphi(x,y+k)$ for any $k\in\mathbb{Z}^M$ and $\varphi=f(\cdot,\cdot,a),g(\cdot,\cdot,a),l(\cdot,\cdot,a),\tau,h$.

The value function and the related Cauchy problem. The *value function* of our optimal control problem is defined as the infimum of a certain cost functional among all the admissible trajectories of the dynamics (1.1), namely

$$u^\varepsilon(t,x,y)=\inf_{a\in\mathcal{A}(t)}\mathbb{E}_{(x,y)}\left[\int_0^t l(x_s,y_s,a_s)ds+h(x_t,y_t)\right]$$

where $\mathbb{E}$ stands for the mathematical expectation. The PDE approach to the issue, based on the Hamilton–Jacobi–Bellman equations, was started in Lions [63] and Jensen and Lions [59], and consists in associating to the value function of the control problem a fully nonlinear PDE.

Consider the following *diffusion matrices*:

$$\mathbf{A}(x):=\frac{1}{2}\sigma(x)\sigma^T(x),\quad \mathbf{B}(x,y):=\frac{1}{2}\tau(x,y)\tau^T(x,y),\quad \mathbf{C}(x,y)=\frac{1}{2}\tau(x,y)\sigma^T(x)$$

where $\cdot^T$ denotes the transpose. Let us indicate by '$\cdot$' both the scalar product of vectors and the scalar product of matrices defined, for $M_1,M_2\in\mathbb{M}^{n\times k}$, as

$$M_1\cdot M_2:=\mathrm{tr}M_1M_2^T=\mathrm{tr}M_2^TM_1$$

For $X\in\mathbb{S}^N$, $Y\in\mathbb{S}^M$, $Z\in\mathbb{M}^{N\times M}$, $x,p\in\mathbb{R}^N$, $y,q\in\mathbb{R}^M$ and $a\in A$ define:

$$\begin{aligned}L(x,y,p,q,X,Y,Z,a):=&-\mathbf{A}(x)\cdot X-\mathbf{B}(x,y)\cdot Y-2\mathbf{C}(x,y)\cdot Z\\&-p\cdot f(x,y,a)-q\cdot g(x,y,a)-l(x,y,a)\end{aligned}$$

The Lagrangian L is the *infinitesimal generator of the diffusion process* (1.1), and we use it to define the following second–order Hamiltonian:

$$H(x,y,p,q,X,Y,Z):=\max_{a\in A}L(x,y,p,q,X,Y,Z,a)\tag{1.2}$$

The Cauchy problem for the value function $u^\varepsilon(t,x,y)$ is then

$$\begin{cases}\partial_t u^\varepsilon+H\left(x,y,D_xu^\varepsilon,\dfrac{D_yu^\varepsilon}{\varepsilon},D_{xx}u^\varepsilon,\dfrac{D_{yy}u^\varepsilon}{\varepsilon},\dfrac{D_{xy}u^\varepsilon}{\sqrt{\varepsilon}}\right)=0\\ \qquad\qquad\text{in }(0,+\infty)\times\mathbb{R}^N\times\mathbb{R}^M\\ u^\varepsilon(0,x,y)=h(x,y)\text{ for any }x\in\mathbb{R}^N,y\in\mathbb{R}^M\end{cases}\tag{1.3}$$

The current assumptions on the data ensure that H is continuous and degenerate elliptic, that is

$$H(x,y,p,q,X,Y,Z)\le H(x,y,p,q,X',Y',Z')$$
$$\text{whenever}\begin{pmatrix}X&Z\\Z^T&Y\end{pmatrix}\ge\begin{pmatrix}X'&Z'\\Z'^T&Y'\end{pmatrix}$$

so, the Hamilton–Jacobi–Bellman equation in (1.3) is degenerate parabolic. Furthermore, H satisfies the *structure condition* (see the User's guide [39]), a regularity property implying the Comparison Principle between bounded viscosity sub and supersolutions of (1.3). It follows that (1.3) possesses at most one bounded continuous viscosity solution. The following result, due to Lions [64] asserts that (1.3) is the correct PDE associated to the value function u^ε. It goes back to Evans and Souganidis [49] in the deterministic framework, and has been extended to min max Bellman–Isaac's Hamiltonians by Fleming and Souganidis [51]. See Fleming and Soner [50] for a general treatment.

Proposition 1.1. *Under the standing assumption the value function u^ε is the unique bounded continuous viscosity solution of* (1.3)

Ergodicity, stabilization and the effective Cauchy problem. One expects that the value functions $u^\varepsilon(t, x, y)$ converge, as ε tends to zero, to a function $u(t, x)$ where the dependence on the fast states disappeared, and that it solves in viscosity sense a certain limiting equation

$$\partial_t u + \bar{H}(x, D_x u, D^2_{xx} u) = 0$$

The operator $\bar{H}$ is called *effective Hamiltonian.* The proof of the existence of such an operator, and possibly its explicit expression, constitute a wide line of research, going back to the firsts pioneering works on homogenization of PDE, in particular to the famous unpublished preprint by Lions, Papanicolaou, and Varadhan [66].

Recently, two crucial properties about the convergence of the u^ε have been singled out in [3]. The first is an *ergodicity* property of the operator, and pertains with the definition of the effective Hamiltonian; the second property regards the possibility to define an effective initial datum for the effective Cauchy problem.

The ergodicity of H can be expressed in different equivalent manners; its definition is based on the asymptotic behavior of solutions of certain cell problems. Fix $(\bar{x}, \bar{p}, \bar{X})$ and consider, for $\delta > 0$ the following *δ–cell problem*

$$\delta w_\delta + H(\bar{x}, y, \bar{p}, Dw_\delta, \bar{X}, D^2 w_\delta, 0) = 0 \text{ in } \mathbb{R}^M, \quad w_\delta \text{ periodic} \tag{1.4}$$

Under the standing assumptions such a problem has a unique viscosity solution $w_\delta(y; \bar{x}, \bar{p}, \bar{X})$.

The Hamiltonian (or the operator) H is said to be (uniformly, or uniquely) *ergodic* at $(\bar{x}, \bar{p}, \bar{X})$ if

$$\delta w_\delta(y; \bar{x}, \bar{p}, \bar{X}) \to \text{const}, \quad \text{as } \delta \to 0^+, \text{ uniformly in } y \tag{1.5}$$

We say that it is ergodic at $\bar{x}$ if it is ergodic at $(\bar{x}, \bar{p}, \bar{X})$ for any $(\bar{p}, \bar{X})$, and that it is ergodic if it is ergodic at any $\bar{x}$.

Equivalently, one can consider the evolutive *t–cell problem*

$$\begin{aligned} &\partial_t w + H(\bar{x}, y, \bar{p}, D_y w, \bar{X}, D^2_{yy} w, 0) = 0 \text{ in } (0, +\infty) \times \mathbb{R}^M, \\ &w(0, y) = 0, \quad w \text{ periodic} \end{aligned} \tag{1.6}$$

Denoted by $w(t, y; \bar{x}, \bar{p}, \bar{X})$ its solution, the Hamiltonian is said to be ergodic at $(\bar{x}, \bar{p}, \bar{X})$ if and only if

$$\frac{w(t, y; \bar{x}, \bar{p}, \bar{X})}{t} \to \text{const}, \quad \text{as } t \to +\infty, \text{ uniformly in } y \tag{1.7}$$

Finally, the property of H of being ergodic can be characterized with the *true* cell problem:

$$\lambda + H(\bar{x}, y, \bar{p}, D\chi, \bar{X}, D^2\chi, 0) = 0 \text{ in } \mathbb{R}^M, \quad \chi \text{ periodic}$$

for some constant λ. It has been shown that there exists at most one constant λ such that the true cell problem has a continuous solution χ, defined up to an additive constant, called *corrector*; in this case H is said to be ergodic at $(\bar{x}, \bar{p}, \bar{X})$.

The definition of ergodicity given with the true cell problem is the most diffused in the literature; anyway it can be proved (see the Abelian–Tauberian Theorem 4 in [3]) that these three definitions are equivalent, and the constants appearing in the limits (1.5) and (1.7) coincides with λ and one defines $\bar{H}(\bar{x}, \bar{p}, \bar{X})$ equal to this constant. Furthermore, thanks to existing representation formulae for the solutions $w_\delta(y)$ and $w(t, y)$ of the cell problems (1.4) and (1.6), also representation formulae for $\bar{H}(\bar{x}, \bar{p}, \bar{X})$ are available, as we will mention in the sequel.

With respect to the regularity of the effective Hamiltonian, Proposition 3 in [3] affirms that it is continuous in $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N$ and degenerate elliptic. More regularity for $\bar{H}$ can be proved in the deterministic framework ($\sigma, \tau \equiv 0$ in (1.1))provided the dynamics satisfies an additional controllability assumption:

Proposition 1.2. [2, Proposition 4]. *Assume that the problem*

$$\begin{aligned} \dot{x}_t &= f(x_t, y_t, a_t) \quad & x_0 = x \\ \varepsilon \dot{y}_t &= g(x_t, y_t, a_t) \quad & y_0 = y \end{aligned}$$

is controllable in y, i.e. *that there exists $r > 0$ such that*

$$B(0, r) \subset \overline{\text{co}}\{g(x, y, a) : a \in A\}$$

Then $\bar{H}(\cdot, p)$ is Lipschitz–continuous, and its Lipschitz constant depends only on the data f, g and l, on their Lipschitz constants, and on the radius r.

Let now pass to the second of the two properties ensuring the local uniform convergence of the u^ε's: the *stabilization (to a constant)* of the pair (H, h). The definition of this property takes in account an auxiliary operator. We say that H has a recession function in a neighborhood of $\bar{x}$ if a function $H'(x, y, q, Y)$, positively 1–homogeneous in (q, Y) there exists such that

$$\begin{gathered} \text{for any } \bar{p} \in \mathbb{R}^N, \bar{X} \in \mathbb{S}^N \text{ there is a constant } C \text{ such that} \\ |H(x, y, p, q, X, Y, 0) - H'(x, y, q, Y)| \le C \text{ for all } (y, q, Y) \in \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{S}^M \\ \text{for every } (x, p, X) \text{ in a neighborhood of } (\bar{x}, \bar{p}, \bar{X}) \end{gathered}$$

Such H' is named *recession* or *homogeneous part* of H; since it satisfies

$$H'(x, y, q, Y) = \lim_{\lambda \to +\infty} \frac{1}{\lambda} H(x, y, 0, \lambda q, 0, \lambda Y, 0), \text{ uniformly}$$

it is continuous and degenerate elliptic. In our context, the recessive Hamiltonian of the Bellman operator (1.2) is given by

$$H'(x, y, q, Y) = \max_{a \in A}\{-\mathbf{B} \cdot Y - q \cdot g(x, y, a)\}$$

Fix $\bar{x}$, and consider the Cauchy problem for the recessive Hamiltonian:

$$\partial_t w + H'(\bar{x}, y, D_y w, D^2_{yy} w) = 0, \text{ in } (0, +\infty) \times \mathbb{R}^M$$
$$w(0, y) = h(\bar{x}, y), w \text{ periodic}$$

Since $h(\bar{x}, \cdot)$ is continuous and $\mathbb{Z}^M$–periodic, such a problem has a unique bounded viscosity solution $w(t, y; \bar{x})$. The pair (H, h) is said to be *stabilizing* (to a constant) at $\bar{x}$ if

$$w(t, y; \bar{x}) \to \text{const}, \quad \text{as } t \to +\infty, \text{ uniformly in } y$$

In this case we set

$$\bar{h}(\bar{x}) := \lim_{t \to +\infty} w(t, y; \bar{x})$$

In the recent paper [4] by Alvarez and Bardi, singular perturbation problems are studied with PDE methods, in the generality of stochastic differential games, and a large number of sufficient conditions for ergodicity and stabilization have been singled out.

These condition are essentially of three types. The first is a non-degeneracy (or uniform ellipticity) of the Hamiltonian H as an operator on the fast variables y:

$$\begin{gathered}\text{for any } (\bar{x}, \bar{p}, \bar{X}) \text{ there exist } \nu, \nu' > 0 \text{ such that:}\\ \nu \operatorname{tr} W \le H(\bar{x}, y, \bar{p}, q, \bar{X}, Y, 0) - H(\bar{x}, y, \bar{p}, q, \bar{X}, Y + W, 0) \le \nu' \operatorname{tr} W\\ \text{for any } W \in \mathbb{S}^M,\ W \ge 0, \text{ and all } y, q, Y.\end{gathered}$$

The second is a coercivity assumption on the $q = D_y u$ variables:

$$\begin{gathered}\text{for any } (\bar{x}, \bar{p}, \bar{X}) \text{ there exist } \nu > 0, C \text{ such that:}\\ H(\bar{x}, y, \bar{p}, q, \bar{X}, Y, 0) - H(\bar{x}, y, \bar{p}, 0, \bar{X}, 0, 0) \ge \nu |q| - C\\ \text{for all } y, q, Y.\end{gathered}$$

These conditions are rather classical (see also [22], [45], [66]). The coercivity assumption can be expressed in terms of controllability of the fast flow.

A third sufficient condition for ergodicity is a non-resonance condition related to the classical theorem of Jacobi on dynamical systems on the torus. A thorough discussion presentation of ergodicity of Hamilton-Jacobi-Isaac's equations can be also find in [10].

Convergence. In [3] it has been proved that, whenever the Hamiltonian is ergodic and stabilizing in the fast variables, u^ε converge to the solution of the effective Hamiltonian. More precisely, since the effective Hamiltonian is not continuous in general, the comparison principle may fail to hold. Then the convergence result is proved first for upper and lower semilimits of u^ε. In fact, under the standing assumptions, the family u^ε is equibounded, and semilimits can be therefore defined; the lower semilimit $u_*(t, x)$ of u^ε is the following bounded lower semicontinuous function:

$$u_*(t, x) = \liminf_{\substack{z \to x \\ \varepsilon \to 0^+}} \inf_y u^\varepsilon(t, z, y)$$

the upper semilimit $u^*(t, x)$ of u^ε is analogously defined.

Proposition 1.3. [3, Theorem 1]. *Assume that the Hamiltonian defined in* (1.2) *is ergodic and stabilizing. Then the upper and the lower semilimits of* u^ε *are respectively viscosity subsolution and supersolution of the effective Cauchy problem*

$$\begin{array}{ll} \partial_t u + \bar{H}(x, D_x u, D^2_{xx} u) = 0 & \textit{in } (0,+\infty)\times\mathbb{R}^N \\ u(0,x) = \bar{h}(x) & \textit{for any } x \end{array} \tag{1.8}$$

We furthermore know that the local uniform convergence of u^ε on compacta of $(0,T)\times\mathbb{R}^N\times\mathbb{R}^M$, $T>0$, can be proved if we assume, besides the current assumptions, the *Comparison Principle* for the limit problem (1.8), *i.e.*

if u is a bounded u.s.c. subsolution of (1.8)
and v is a bounded l.s.c. supersolution
then $u \le v$ on $[0,T)\times\mathbb{R}^N$

Proposition 1.4. [3, Corollary 2]. *Assume H is ergodic, (H,h) is stabilizing and $\bar{H}$ satisfies the comparison principle. Then u^ε converges uniformly on compact subsets of* $(0,T)\times\mathbb{R}^N\times\mathbb{R}^M$ *to the unique solution of* (1.8).

1.2 Limiting Relaxed Controls

1.2.1 Preliminaries: Relaxed Control functions, Prohorov distance and weak–star convergence of measures.

In the next sections we deal with relaxed controls, *i.e.* suitably defined Radon probability measures; we shall use such controls to average the fast dynamics and to define the limit problem of a singularly perturbed control problem. In this paragraph we recall some basic concepts about relaxed control function and weak–star convergence of measures, and introduce a metric on the space of probability measures defined on the Borel σ–algebra of a certain given compact metric space.

Let Y be a given compact metric space. The class of *relaxed* or *chattering* controls on Y is defined as

$$\mathcal{Y}^r := \{ \text{ measurable functions } [0,+\infty) \to Y^r \}$$

where

$$Y^r := \{ \text{ Radon probability measures on } Y \}.$$

The theory of relaxed controls has been introduced by J.Warga in 70's and an important reference is his book on optimal control [78].

Thanks to a representation theorem due to Riesz (see [78, Theorem I.5.8]), there exists an algebraic isomorphism $\mathcal{I}$ of the space $\mathcal{M}(Y)$ of *all* Radon measures on Y onto the space $C(Y)^*$ (the dual space of the space of continuous functions on Y); the isomorphism is given by

$$\mathcal{I}[\mu](\varphi) := \int \varphi(y)\mu(dy) \qquad (\mu \in \mathcal{M}(Y),\ \varphi \in C(Y))$$

Then we can also endow Y^r with a suitable topology, the *weak–star topology* induced by $C(Y)^*$. We say that a sequence of Radon probability measures μ_n converges weak–star to μ if, for any continuous function φ, one has

$$\int \varphi(y)\mu_n(dy) \to \int \varphi(y)\mu(dy)$$

Furthermore Y^r with the weak–star topology is metrizable (*i.e.* there exists the *weak–star norm* $|\cdot|_w$, see [78, Theorem I.3.11]) and the corresponding topology coincides with the weak–star topology of $C(Y)^*$. The crucial properties are the following:

- The normed vector space $(\mathcal{M}(Y), |\cdot|_w)$ is separable and its subset Y^r is compact (see [78, Theorem IV.1.4]);
- The set $\mathcal{Y}^r$ is convex, compact and sequentially compact (see [78, Theorem IV.2.1]).

Consider relaxed controls is a way of *convexifying* the problem; the advantage in considering convex (or convexified) control problems is clear. First of all observe that if the Hamiltonian related to two different control problems coincide, then the value function related to the two problems must coincide. Now, it is easy to prove that the Hamiltonian obtained convexifying all the given functions f, g and l, coincides with the original H (this is a consequence of a classical Carathéodory theorem), therefore the value functions of the original and of the convexified problem are the same. In Section 1.3 we will observe that an analogous property holds if we consider the Hamiltonian obtained *relaxing* all the given functions by means of relaxed controls. Recall also that convex control problems are interesting because they enjoy some special properties, as the fact that (if A is compact) the set of their trajectories is closed on bounded time intervals, with respect to the uniform convergence. Also relaxed controls are interesting because, for example, a relaxed control problem has usually an optimal control, whereas the original problem does not, in general.

By the previous discussion, Y^r can be treated as a compact metric space, and we endow it with the *Prohorov distance* $\pi(\cdot,\cdot)$ which is consistent with the weak–star topology. See [31]. There are many ways to define π; in the present work the Prohorov distance of two probability measures on Y, μ_1 and μ_2, is defined as follows:

$$\pi(\mu_1,\mu_2) := \inf\{\varepsilon > 0 : \ \mu_1(Q) \le \mu_2(Q+\varepsilon B)+\varepsilon \ \text{ for any measurable } Q\} \quad (1.9)$$

If M is a set of probability measures, and μ is a probability measure as well, we define in the usual way $\pi(\mu, M)$ as the infimum of $\{\pi(\mu,\nu) |\, \nu \in M\}$. The Hausdorff distance will be also used be to estimate distances of sets of probability measures; in this case the Hausdorff distance is defined in the usual way, using the Prohorov distance π instead of the Euclidean one. To avoid ambiguities we denote by d_H the usual Hausdorff distance, the one obtained with the Euclidean distance d, defined for any pair of compact subsets S_1 and S_2 in $\mathbb{R}^N$ as

$$d_H(S_1,S_2) := \max\left\{\sup_{s\in S_1} d(s,S_2),\ \sup_{s\in S_2} d(s,S_1)\right\}$$

and by π_H the Hausdorff distance obtained with $\pi(\cdot,\cdot)$ in place of $d(\cdot,\cdot)$.

Remark 1.5. Note that the definition (1.9) is symmetric, i.e. $\pi(\mu_1,\mu_2) = \pi(\mu_2,\mu_1)$ for any μ_1, μ_2. Suppose that $\pi(\mu_1,\mu_2) = \varepsilon$ and that, by contradiction, $\pi(\mu_2,\mu_1) > \varepsilon$. Then a measurable A does exist such that $\mu_2(A) > \mu_1(A+\varepsilon B)+\varepsilon$. Equivalently we have for the complement

$$\mu_1(A+\varepsilon B)^c > \mu_2(A^c)+\varepsilon \ge \mu_2((A+\varepsilon B)^c+\varepsilon B)+\varepsilon$$

since $A^c \supseteq (A+\varepsilon B)^c + \varepsilon B$. Now, if we consider the measurable set $B := (A+\varepsilon B)^c$ we obtain a contradiction: $\mu_1(B) > \mu_2(B+\varepsilon B) + \varepsilon$. $\square$

The following lemma gives a useful characterization of the weak star convergence of measures.

Lemma 1.6. *Let $\{\mu_n\}_n$ and μ be probability measures on a given compact metric space Y. The following statements are equivalent:*

i. $\mu_n \to \mu$ *weak star, as* $n \to +\infty$*;*

ii. $\pi(\mu_n, \mu) \to 0$*, as* $n \to +\infty$*;*

iii. If $\{f_j : Y \to \mathbb{R},\ j \in \mathbf{N}\}$ *is a dense sequence in the unit ball of* $C(Y)$*, then*

$$\lim_n \sum_{j=1}^{\infty} 2^{-j} \left| \int f_j \mathrm{d}\mu_n - \int f_j \mathrm{d}\mu \right| = 0$$

In the sequel of the section we will mention some results from [55] and [32] where the Prohorov distance is defined by means of the series appearing in the third statement of the Lemma; the Lemma has also the scope to show that the definition of π is consistent with the weak–star topology. Moreover, the definition of Prohorov distance given by (iii.) makes apparent the triangle inequality for $\pi(\cdot,\cdot)$.

Proof of Lemma 1.6. The equivalence between the first two statements is standard, and can be found for example in [31]. Let us prove that i. is equivalent to iii. From the definition of weak star convergence, if i. holds

$$\lim_n \left| \int f_j \mathrm{d}\mu_n - \int f_j \mathrm{d}\mu \right| = 0 \quad \text{for all } j$$

So, for any ε if n is large enough, one has

$$\left| \int f_j \mathrm{d}\mu_n - \int f_j \mathrm{d}\mu \right| < \varepsilon/2$$

and therefore

$$\sum_{j=1}^{\infty} 2^{-j} \left| \int f_j \mathrm{d}\mu_n - \int f_j \mathrm{d}\mu \right| < \varepsilon$$

Conversely, if iii. holds, for any $\varepsilon > 0$, for a large enough n one has

$$0 \le \sum_{j=1}^{\infty} 2^{-j} \left| \int f_j \mathrm{d}\mu_n - \int f_j \mathrm{d}\mu \right| < \varepsilon$$

then in particular for any j

$$\left| \int f_j \mathrm{d}\mu_n - \int f_j \mathrm{d}\mu \right| < \varepsilon$$

and since $\{f_j\}_j$ is dense in $C(Y)$ this implies the weak star convergence of μ_n to μ.

$\square$

1.2.2 Limiting relaxed controls set and limit occupational measures set.

The *order reduction method* is one of the first attempt to construct the limit of the singularly perturbed system. It was used in the deterministic setting (*i.e.* $\sigma, \tau \equiv 0$ in (1.1)) to detect, under suitable assumption, the limit dynamics of a singularly perturbed control system. Following the classical Levinson–Tichonov approach, one consider the natural candidate for the limit, *i.e.* the system we obtain setting $\varepsilon = 0$ in the singularly perturbed system. The result is an ordinary differential equation combined with an algebraic equation. One expects the limiting dynamics to be given by the differential inclusion

$$\dot{x}_s = f(x_s, y_s, a_s), \quad (y_s, a_s) \in Z(x_s)$$

where

$$Z(x) := \{(y, a) \in \mathbb{R}^M \times A \text{ such that } g(x, y, a) = 0\}$$

Under certain conditions, this approach gives the appropriate limit in several situations; anyway, many systems fail to verify such assumptions. The most important restriction that have to be satisfied is the convergence of the fast dynamics to a stationary point. In the literature many examples are available where this procedures fails to give the correct limit system; see [15],[16]. [20] also analyzes a case where the order reduction does not apply.

Let us observe that any couple (y, a) belonging to $Z(x)$ can be interpreted as a Dirac mass concentrated at (y, a), *i.e.* as a probability measure on the product space $\mathbb{R}^M \times A$. In this section we introduce the concept of relaxed control. These controls are Radon probability measures on $\mathbb{R}^M \times A$, and are used to average the fast dynamic, in order to reduce the dimension of the singularly perturbed system, and to provide an N–dimensional control problem representing the limit behavior of the singularly perturbed control system.

We associate with (1.1) the following M–dim system,

$$\begin{aligned} dy_t &= g(x, y_t, a_t))dt + \tau(x, y_t)dW_t \\ y_0 &= y \end{aligned} \tag{1.10}$$

where the slow state x is frozen and considered as a parameter, and take into account a family of *occupational measures* μ_s, *i.e.* suitably defined Radon probability measures on $\mathcal{B}(\mathbb{R}^M \times A)$ (the Borel σ–algebra of $\mathbb{R}^M \times A$); the precise definition will be given in formula (1.13) below. We call *limiting relaxed control* a weak star limit μ of a sequence of such occupational measures.

The goal of the following sections is the following: investigate the properties of the set of the limiting relaxed controls, use such controls to obtain a lower–order system, and prove that this system is in fact the limiting system of the singularly perturbed one.

There are two natural ways to define the relaxed controls. The first consists in considering the set of all occupational measures of solutions of (1.10) up to time s, and then take the limit as $s \to \infty$ of this set, with respect to π_H. The second consists in considering directly the set of the weak star limits of occupational measures of solutions of (1.10), for some diverging sequence t_n, and some control function a. In this section we will prove that, under certain controllability assumptions, both in the deterministic and in the stochastic framework, the two approaches provide the same limit set of measures. As a consequence of this, we will get for the set of limiting relaxed measures some topological properties that allow us to regard it as a set of controls.

We conclude this Section with the following abstract Lemma we will exploit later. We record before the definition of limsup of a multivalued function Φ:

$$\limsup_{s \to +\infty} \Phi(s) := \{\varphi : \liminf_{s \to +\infty} \pi(\varphi, \Phi(s)) = 0\}$$

Lemma 1.7. *Let $\mathcal{M}$ be a set of probability measures defined on the σ–algebra of a certain given compact metric space. Let A be a compact metric space, and $\{\mu(s,a)\}$ be a nonempty subset of $\mathcal{M}$ parametrized by $a \in A$ and $s \in [0, +\infty)$. Put*

$$Z := \{\mu \in \mathcal{M} \,|\, \exists t_n \to +\infty \text{ and } a \in A \text{ such that } \mu(t_n, a) \to \mu \text{ weak star}\}$$

and, for any $s > 0$,

$$\Phi(s) := \bigcup_{a \in A} \{\mu(s,a)\}.$$

Then

$$\limsup_{s \to +\infty} \Phi(s) = Z$$

Proof. Let us prove first that

$$Z \subseteq \limsup_{s \to +\infty} \Phi(s).$$

If $\mu \in Z$ then a sequence $t_n \to +\infty$ and $a \in A$ exist, such that

$$\mu = \lim_n \mu(t_n, a) \text{ weak star.}$$

Since $\mu(t_n, a) \in \Phi(t_n)$, we have

$$0 \le \liminf_{s \to +\infty} \pi(\mu, \Phi(s)) = \liminf_{s \to +\infty} \inf_{\psi \in \Phi(s)} \pi(\mu, \psi) \le \lim_n \pi(\mu, \mu(t_n, a)) = 0,$$

hence

$$\mu \in \limsup_{s \to +\infty} \Phi(s).$$

To prove the converse inclusion take $\mu \in \limsup_{s \to +\infty} \Phi(s)$; then, for a certain sequence $s_n \to +\infty$ one has

$$0 = \liminf_{s \to +\infty} \pi(\mu, \Phi(s)) = \lim_{n \to +\infty} \pi(\mu, \Phi(s_n)) = \lim_{n \to +\infty} \inf_{\psi \in \Phi(s_n)} \pi(\mu, \psi)$$

Since $\Phi(s)$ are nonempty and A is compact, for any n there exists $a_n \in A$ such that $\psi_n := \mu(s_n, a_n) \in \Phi(s_n)$ satisfies

$$\inf_{\psi \in \Phi(s_n)} \pi(\mu, \psi) = \pi(\mu, \psi_n)$$

then,

$$\lim_{n \to +\infty} \pi(\mu, \psi_n) = 0$$

this means that ψ_n converges weak-star to μ. By the compactness of A, there exists a subsequence a_{n_k} converging to some $\bar{a} \in A$. We finally check that

$\bar{\psi}_{n_k} := \mu(s_{n_k}, \bar{a}) \in \Phi(s_{n_k})$ also converges to μ weak-star, *i.e.* μ belongs to Z. In fact, by the triangle inequality one has

$$\pi(\mu, \bar{\psi}_{n_k}) \leq \pi(\mu, \psi_{n_k}) + \pi(\psi_{n_k}, \bar{\psi}_{n_k})$$

and both the summands in the left hand side tend to zero as $n \to +\infty$.

□

1.2.3 Limiting relaxed controls for deterministic systems

In this subsection we consider a deterministic singular perturbation problem

$$\begin{aligned} \dot{x}_t &= f(x_t, y_t, a_t) \quad x_0 = x \\ \varepsilon \dot{y}_t &= g(x_t, y_t, a_t) \quad y_0 = y \end{aligned} \tag{1.11}$$

(*i.e.* $\sigma \equiv 0$ and $\tau \equiv 0$ in (1.1)), and investigate the set of limiting relaxed controls some topological properties of it; the analysis is done showing the connection between limiting relaxed control sets and the *limit occupational measures sets* studied by Gaitsgory and Leizarowitz in [55].

The fast subsystem associated to (1.11) is the M–dimensional system

$$\dot{y}_t = g(x, y_t, a_t), \quad y_0 = y \tag{1.12}$$

where x is frozen and considered as a parameter. An *occupational measures* μ_s for (1.12) is a Radon probability measures on $\mathcal{B}(\mathbb{R}^M \times A)$ (the Borel σ–algebra of $\mathbb{R}^M \times A$) defined by:

$$\mu_s := \frac{1}{s} \int_0^s \delta_{(y_t, a_t)} dt \tag{1.13}$$

where $\delta_{(y_t, a_t)}$ is the Dirac mass concentrated on (y_t, a_t). The motivation of the name is apparent, in fact the evaluation of such a measure on a Borel set Q, gives the proportion of time spent by the trajectory in the Borel set; in other words, the following representation holds:

$$\mu_s(Q) = \frac{1}{s} \mathcal{L}^1 \{t \in [0, s] : (y_t, a_t) \in Q\} \tag{1.14}$$

where $\mathcal{L}^1$ stands for the 1–dim Lebesgue measure. It is apparent that μ_s depends also on the choice of the control a, on the initial point y and on x. The occupational measures are used in the following definition of limiting relaxed controls.

Definition 1.8. [2]. *A measure μ on $\mathcal{B}(\mathbb{R}^M \times A)$ is a* limiting relaxed control *if there exist a control function $a \in \mathcal{A}$, an initial position y, and a diverging sequence t_n such that the occupational measure μ_{t_n} of the corresponding solution of* (1.12) *converges weak star to μ. We will denote by $Z_l(x)$ the set of all limiting relaxed measures related to* (1.12).

In [55] the Authors show that the set of probability measures

$$\Phi(y, s; x) := \bigcup_{a \in \mathcal{A}} \{\mu_s\},$$

i.e. the collection of all occupational measures given by (1.12), has a limit for $s \to +\infty$ with respect to the Hausdorff distance π_H, say $\Phi(x)$, and that this limit is convex and compact in the weak star topology. Furthermore, under suitable controllability assumptions, $\Phi(x)$ is independent of the initial point y.

In the sequel $y_t(y, a)$, or y_t, will stand for the solution of (1.12) at time t, starting at y and using the control function a. We omit to write the dependence of such solution on x, if no ambiguities can arise.

In the next result we will exploit the property of (1.12) of being bounded time controllable. We record below the definition of such type of controllability.

Definition 1.9. *The system* (1.12) *is* bounded time controllable *if there exists a* $T > 0$ *such that, for any* y^1, y^2 *there exists a control* $a \in \mathcal{A}$ *such that* $y_t(y^1, a) = y^2$ *for some* $t \leq T$.

Now we can state the main result of this section:

Theorem 1.10. *Under the standing assumptions, if the system* (1.12) *is bounded time controllable, then*

$$Z_l(x) = \Phi(x) \quad \textit{for any } x.$$

We need to introduce some technical tools and some preliminary lemmata. Let us $\varphi = \varphi(y, a)$ be any continuous function from $\mathbb{R}^M \times A$ to $\mathbb{R}^K$ $(K \geq 1)$, periodic with respect to y. Define the following subsets of $\mathbb{R}^K$:

$$Y_\varphi(s, y; x) := \bigcup_{a \in \mathcal{A}} \left\{ \frac{1}{s} \int_0^s \varphi(y_t, a_t) \mathrm{d}t \right\}$$

These sets are collection of means of a continuous function evaluated along trajectories y_t of (1.12) obtained for a certain choice of the control a. The sets $Y_\varphi(s, y; x)$ have been used in [55] to give a characterization of the set of limit occupational measures $\Phi(x)$. More precisely their main result is based upon the possibility to find a convex compact subset Y_φ of $\mathbb{R}^N$ such that

$$\lim_{s \to +\infty} d_H(Y_\varphi(s, y; x), Y_\varphi(x)) = 0 \tag{1.15}$$

The assertion is the following

Proposition 1.11. [55, Theorem 3.1] *Let x be fixed. Assume that for any continuous function φ there exists a convex compact set $Y_\varphi(x)$ such that* (1.15) *holds. Then there exists $\Phi(x)$, a set of probability measures on $\mathbb{R}^M \times A$, such that*

$$\lim_{s \to +\infty} \pi_H(\Phi(s, y; x), \Phi(x)) = 0$$

for all y. In addition $\Phi(x)$ is convex and compact with respect to the weak star topology.

On the other hand, the validity of (1.15) follows by the next result, that is in Grammel [57].

Lemma 1.12. [57, Proposition 3.2] *Let x be fixed. Suppose that*

$$d_H(Y_\varphi(s, y^1; x), Y_\varphi(s, y^2; x)) \leq \nu(s), \textit{ for any } y^1, y^2 \textit{ and any } s > 0 \tag{1.16}$$

for a suitable function ν, with $\nu(s) \to 0$ as $s \to +\infty$. Then a convex compact set $Y_\varphi(x) \subset \mathbb{R}^K$ does exist, such that (1.15) *holds for any initial state y.*

So, the point is to prove (1.16). In [55] it has been proved that this estimation holds true under the following stability condition

$$\begin{array}{c}\text{For any couple of controls and any couple of initial states } y^1, y^2 \\ \text{the corresponding trajectories } y^1_t \text{ and } y^2_t \text{ of (1.12) satisfy} \\ |y^1_t - y^2_t| \le \nu(t)|y^1 - y^2|, \text{ where } \nu(t) \to 0 \text{ as } t \to +\infty.\end{array} \tag{1.17}$$

In the next result, we will prove that the condition (1.16) is verified also under the bounded time controllability condition, a condition we will exploit several times in the work, in order to get more regularity for the effective Hamiltonian. The fact that bounded time controllability is sufficient for (1.16) was already mentioned in [55]; we provide a complete proof for the sake of completeness.

Lemma 1.13. *If* (1.12) *is bounded time controllable, then there exists a constant $C > 0$ such that for any $s > 0$ the estimate*

$$d_H(Y_\varphi(s, y^1; x); Y_\varphi(s, y^2; x)) \le Cs^{-1}$$

holds for every y^1,y^2 in $\mathbb{R}^M$.

Proof. Let be $Y_i := Y_\varphi(s, y^i; x)$, y^1 and y^2 fixed. The assertion follows if we prove that for any $v_2 \in Y_2$, $d(v_2, Y_1) \le Cs^{-1}$. If $v_2 \in Y_2$ there exists a control a^2 such that

$$v_2 = \frac{1}{s} \int_0^s \varphi(y^2_t, a^2_t) dt$$

with

$$y^2_t = y_t(y^2, a^2)$$

Since (1.12) is bounded time controllable, there is a $T > 0$ such that for our y^1 and y^2 a control a^0 does exist such that $y_{t_0}(y^1, a^0) = y^2$ for some $t_0 \le T$.

Let us define

$$a^1_t := \begin{cases} a^0_t & \text{if } t \le t_0 \\ a^2_{t-t_0} & \text{if } t > t_0 \end{cases}$$

and

$$y^1_t := y_t(y^1, a^1)$$

and finally the following element of Y_1:

$$v_1 := \frac{1}{s} \int_0^s \varphi(y^1_t, a^1_t) dt$$

Then

$$\begin{aligned} d(v_2, Y_1) &\le d(v_2, v_1) = \frac{1}{s} \left| \int_0^s \varphi(y^1_t, a^1_t) dt - \int_0^s \varphi(y^2_t, a^2_t) dt \right| \\ &= \frac{1}{s} \left| \int_0^{t_0} \varphi(y^1_t, a^1_t) dt + \int_{t_0}^s \varphi(y^1_t, a^1_t) dt - \int_0^s \varphi(y^2_t, a^2_t) dt \right| \\ &\le \frac{1}{s} \left(\int_0^{t_0} |\varphi(y^1_t, a^1_t)| dt - \int_{s-t_0}^s |\varphi(y^2_t, a^2_t)| dt \right) \\ &\le \frac{2t_0 M}{s} \le \frac{2TM}{s} \end{aligned}$$

where $M := \max|\varphi|$ is finite, thanks to the compactness of A and the fact that φ is continuous and periodic with respect to y. The desired estimation is then established.

□

We can therefore infer the existence of the limit occupational measure set $\Phi(x)$, as in [55]. Finally, we can prove Theorem 1.10 showing the connection between this set and the set of limiting relaxed control $Z_l(x)$.

Proof of Theorem 1.10. Observe first that for any fixed x and any initial point y, and any a, the system (1.12) admits a solution y_t. Therefore $(y_t, a_t)\big|_{[0,s]}$ can be used in the definition (1.13) in order to obtain an element of $\Phi(s, y; x)$. So for any $s > 0$, $\Phi(s, y; x)$ is nonempty. Thanks to Proposition 1.11 a limit $\Phi(x)$ for $\Phi(s, y; x)$ exists and is independent of the initial state y, and by Lemma 1.7 we get the conclusion.

□

1.2.4 Limiting relaxed controls for stochastic systems

We perform now for a stochastic control system, the analysis done before in the deterministic framework. A recent reference in this context is a paper by Vivek Borkar and Vladimir Gaitsgory [32]. It offers a stochastic version of the results contained in [55], and the approach is an adaptation of that one adopted therein. In particular, their main result consists in the stochastic counterpart of Proposition 1.11, *i.e.* the existence of a limit set for the set of occupational measures is shown. On the other hand, we construct again the set Z_l of the limiting relaxed controls, being this controls suitably defined for stochastic control systems. As in the previous section, the goal is to establish connections between the limit occupational measures set and the set of limiting relaxed controls.

We consider the following singularly perturbed control system

$$\begin{array}{ll} dx_t = f(x_t, y_t, a_t)dt + \sigma(x_t)dW_t, & x_0 = x \\ dy_t = \frac{1}{\varepsilon} g(x_t, y_t, a_t)dt + \frac{1}{\sqrt{\varepsilon}}\tau(x_t, y_t)dW_t, & y_0 = y \end{array} \tag{1.18}$$

and the fast subsystem associated to it, that is

$$dy_t = g(x, y_t, a_t)dt + \tau(x, y_t)dW_t, \quad y_0 = y, \quad x \text{ fixed.} \tag{1.19}$$

Being the solution of this stochastic differential equation a stochastic process, it is natural to define the occupational measure via the following mathematical expectation

$$\mu_s := \mathbb{E}_y\left[\frac{1}{s}\int_0^s \delta_{(y_t, a_t)} dt\right] \tag{1.20}$$

As before we say that a measure μ on $\mathcal{B}(\mathbb{R}^M \times A)$ is a *limiting relaxed control* if it is a weak star limit of such a μ_{t_n}, for some $t_n \to +\infty$, and some control a.

Remark 1.14. The definition (1.20) gives (1.13) in the deterministic case, *i.e.* whenever Ω is reduced to a singleton. Note also that in the deterministic framework, the equivalence between the Definition 1.13 of occupational measure, and the formula (1.14) is evident. Also in the stochastic setting, an analogous representation holds. In fact, put

$$\widetilde{\mu}_s(Q) := \frac{1}{s}\mathcal{L}^1(\{t \in [0,s] : (y_t, a_t) \in Q\}) = \frac{1}{s}\int_0^s \delta_{(y_t,a_t)}(Q)dt$$

and note that, since y_t is a random variable, $\widetilde{\mu}_s$ is a probability–measure–valued random variable defined on the Borel subsets of $\mathbb{R}^M \times A$. Then, the occupational measure defined in (1.20) admits the representation

$$\mu_s = \mathbb{E}_y\left[\widetilde{\mu}_s\right]$$

□

Remark 1.15. The previous Remark permits also to recognize that μ_s defined in (1.20) is actually a measure on $\mathcal{B}(\mathbb{R}^M \times A)$. In fact, trivially $\mu_s(\emptyset) = 0$, and for any countable family $\{Q_i\}_i \subset \mathcal{B}(\mathbb{R}^M \times A)$ pairwise disjoint, we have

$$\mu_s\left(\bigcup_i Q_i\right) = \mathbb{E}\left[\frac{1}{s}\sum_i \mathcal{L}^1(\{t \in [0,s] : (y_t, a_t) \in Q_i\})\right]$$

$$= \sum_i \mathbb{E}\left[\frac{1}{s}\mathcal{L}^1(\{t \in [0,s] : (y_t, a_t) \in Q_i\})\right] = \sum_i \mu_s(Q_i)$$

□

In analogy with the previous section we define

$$\Phi(s, y; x) := \bigcup_{a\in\mathcal{A}} \{\mu_s\}$$

and, for any continuous function $\varphi = \varphi(x, a)$ from $\mathbb{R}^M \times A$ to $\mathbb{R}^K$, $(K \geq 1)$, periodic in y, the following subsets of $\mathbb{R}^K$:

$$Y_\varphi(s, y; x) := \bigcup_{a\in\mathcal{A}} \mathbb{E}_y\left[\frac{1}{s}\int_0^s \varphi(y_t, a_t)dt\right] \tag{1.21}$$

where y_t is the solution of (1.19) using the control a.

In [32] the Authors show that under a certain condition, called *weak approximation condition* is shown to be sufficient for the existence of the limit occupational measure set, provided that the system (1.19) satisfies a suitable stability condition. The precise statement is the following

Proposition 1.16. [32, Theorem 3.3] *Let $\{f_j(y, a)\}$ be a sequence of Lipschitz–continuous functions which is dense in the unit ball of $C(\mathbb{R}^M \times A)$ and assume that for any j, and for any $\varphi(y, a) := (f_1(y, a), \ldots, f_j(y, a))$ the following condition is satisfied*

$$\begin{array}{c}\text{There exists } C > 0 \text{ such that, for any } y^1, y^2, \text{ and any } s > 0 \\ d_H(Y_\varphi(s, y^1; x); Y_\varphi(s, y^2; x)) \leq C\nu(s)\end{array} \tag{1.22}$$

for some function ν converging to zero as s goes to $+\infty$. Assume furthermore that the following stability condition holds:

$$\begin{array}{c}\text{There exists } \alpha > 0 \text{ and } C > 0 \text{ such that} \\ \text{any solution of (1.19) obtained by an admissible control satisfies} \\ \sup_{t,a.} \mathbb{E}[|y_t|^\alpha] \leq C(\mathbb{E}[|y_0|^\alpha] + 1)\end{array} \tag{1.23}$$

Then, there exists a limit occupational measure set, namely a convex compact set $\Phi(x)$ contained in $(\mathbb{R}^M \times A)^r$ such that, for any initial condition y having a certain probability distribution, one has

$$\pi_H(\Phi(s, y; x), \Phi(x)) \leq \nu(s)$$

for some function ν converging to zero as s goes to $+\infty$. The function ν depends on the probability distribution of the initial condition.

A stochastic version of Theorem 1.10 is obtained under the assumptions ensuring the existence of the limit occupational measure set. In fact, if such limit set exists, by Lemma 1.7, it must coincide with the limiting relaxed control set $Z_l(x)$. In this case we also know that $Z_l(x)$ is a convex and compact subset of $(\mathbb{R}^M \times A)^r$, with respect to the weak–star topology.

Theorem 1.17. *Under the standing assumption, if hypotheses* (1.22) *and* (1.23) *of Proposition 1.16 are satisfied, then*

$$Z_l(x) = \Phi(x) \quad \text{for any } x$$

REMARK 1.18. In the deterministic case, Theorem 1.10 holds under the bounded time controllability assumption on the fast subsystem. Such assumption allows to prove Lemma 1.13, and consequently, the estimation (1.15) giving, by Proposition 1.11, the existence of the limit set $\Phi(x)$.

In the stochastic framework, some subtle issues arise in proving that a suitable notion of bounded time controllability implies the estimation of Lemma 1.13 for the sets (1.21), that is the condition (1.22). Such difficulties arise with the possibility to piece together admissible controls to define a new admissible control for the stochastic differential equation governing the fast subsystem. We postpone to future investigation the study of such issue. □

1.3 The limit optimal control problem of deterministic singularly perturbed systems

In this section we consider again a singularly perturbed deterministic control system

$$\begin{aligned}\dot{x}_t &= f(x_t, y_t, a_t) \\ \varepsilon \dot{y}_t &= g(x_t, y_t, a_t)\end{aligned} \tag{1.24}$$

with running cost $l(x, y, a)$ and terminal cost $h(x, y)$, and the related Hamiltonian:

$$H(x, y, p, q) := \max_{a \in A}\{-p \cdot f(x, y, a) - q \cdot g(x, y, a) - l(x, y, a)\} \tag{1.25}$$

We want to show that the limiting system as $\varepsilon \to 0$ can be represented as a suitable optimal control problem. To this goal we introduce a certain *relaxed optimal control problem*; the term *relaxed* emphasizes the fact that this control problem is obtained relaxing the data using the limiting relaxed controls μ, as below:

$$\psi^r(x,\mu) := \int_{\mathbb{R}^M \times A} \psi(x,y,a)d\mu(y,a). \tag{1.26}$$

Here the symbol ψ is used in place of functions f, g and l.

Let us discuss some effects of the relaxation. One of the most relevant is a convexification of the dynamics. In fact one can prove (see [22, (2.32) in Chapter III] and [22, Ex.2.10 in Chapter III])

$$\text{for any } x,\ \overline{\text{co}}\psi(x,y,A) = \{\psi^r(x,\mu) : \mu \text{ Radon measure on } \mathbb{R}^M \times A\}$$

Furthermore, by the definition (1.26), we see that the relaxed function is affine in the control variable μ; this permits to get the property below, that we will exploit several times in the following

$$\psi^r\left(x, \frac{1}{t}\int_0^t \delta_{(y_s,a_s)}\mathrm{d}s\right) = \frac{1}{t}\int_0^t \psi^r(x,\delta_{(y_s,a_s)})ds = \frac{1}{t}\int_0^t \psi(x,y_s,a_s)ds$$

It is also important to recall that if $\psi(x,y,a)$ is continuous (resp. bounded, Lipschitz–continuous in (x,y) uniformly with respect to a, uniformly continuous), then $\psi^r(x,\mu)$ is continuous (resp. bounded, Lipschitz–continuous in x uniformly with respect to μ, uniformly continuous). See [22, Lemma III.2.20].

We proceed as following. First, we recall that the effective Hamiltonian related to (1.25) admits a representation as a control Hamiltonian; more precisely, it coincides with the Hamiltonian related to the relaxed control problem. We prove that the value function of the relaxed problem solves the effective Cauchy problem and then, provided that the solution of such problem is unique, it coincides with the limit of the value functions of the singular perturbed system, as $\varepsilon \to 0$. The importance of such assertion is twofold. The first interesting information is that the local uniform limit of value functions is as well a value function of an optimal control problem. Moreover, we can regard at the relaxed control system as to the system capturing the effective limit behavior of the singularly perturbed system, as $\varepsilon \to 0$. At the and we present an example where the computations are explicitly made.

1.3.1 The limit control problem

Before considering the relaxed control problem, let us recall that, thanks to Theorem 1.10, the set of limiting relaxed controls $Z_l(x)$ is convex and compact (with respect to weak star topology) for any x, and that the effective Hamiltonian $\bar{H}(x,p)$ of the Hamiltonian (1.25) can be represented as a control Hamiltonian:

Proposition 1.19. [2, Theorem 7] *The effective Hamiltonian admits the following representation:*

$$\bar{H}(x,p) = \max_{\mu \in Z_l(x)} L^r(x,p,\mu) =: H^r(x,p) \tag{1.27}$$

where $L^r(x,p,\mu) := -p \cdot f^r(x,\mu) - l^r(x,\mu)$.

Analogous representations for the effective Hamiltonian will be established in Section 1.4, for stochastic singularly perturbed systems.

Recall the definition of f^r, g^r and l^r given in (1.26). Consider also the following problem:

$$\dot{x}_t \in F^r(x_t), \quad x_0 = x \tag{1.28}$$

where

$$F^r(x) := f^r(x, Z_l(x))$$

We will refer to (1.28) as the *limiting control system.* It is a control system with with a state-constraint on the controls. Observe that, thanks to Theorem 1.10 and Proposition 1.11 $Z_l(\cdot)$ is convex and compact valued. Moreover the space of Radon measures, equipped with the Prohorov metric, is compact metric space.

In the sequel we assume that the function f satisfies the following growth condition.

Assumption 1.20. There exists $C > 0$ such that

$$|f(x, y, u)| \leq C(1 + |x| + |(y, u)|) \tag{1.29}$$

The value function associated to the relaxed control problem (1.28), with terminal cost $\bar{h}(x) := \inf_y h(x, y)$, and running cost $l^r(x, \mu)$ is the function

$$u^r(t, x) := \inf \left\{ \int_0^t l^r(x_s, \mu_s) ds + \bar{h}(x_t) \right\} \tag{1.30}$$

where the infimum is taken among all solutions of (1.28). The core of this section is the following result

Theorem 1.21. *Under the standing assumptions, if H is ergodic, and the growth condition* (1.29) *is satisfied by the slow flow, then the value function $u^r(t, x)$ is a viscosity solution of the effective Cauchy problem*

$$\begin{array}{ll} \partial_t u + \bar{H}(x, D_x u) = 0 & \textit{in } (0, +\infty) \times \mathbb{R}^N \\ u(0, x) = \bar{h}(x) & \textit{on } \mathbb{R}^N \end{array}$$

Corollary 1.22. *If $\bar{H}$ satisfies the comparison principle, the value functions u^ε converge locally uniformly, to the value function u^r.*

Remark 1.23. In Proposition (1.19) we have recalled that $\bar{H}$ can be represented as the Hamiltonian associated to the limiting control system. Now, in order to produce a value function for the relaxed control problem, we deal with the minimization of the functional

$$J^r(t, x, \mu) := \int_0^t l^r(x_s, \mu_s) ds + \bar{h}(x_t)$$

Recall (see [22, p.147]) that such a Bolza problem can be converted into a Mayer problem, if l^r is Lipschitz continuous with respect to the state variables (and this is the case, if we make this assumption on the original running cost $l(x, y, a)$). This conversion can be done by adding an $(N+1)$–th scalar variable, subjected to the dynamics

$$(x_s^{N+1})' = l^r(x_s, \mu_s), \quad x_0^{N+1} = 0$$

A new terminal cost $\psi(x_t^{N+1} + \bar{h}(x_t))$ should be considered, being ψ any bounded strictly increasing function. □

Taking into account Proposition 1.19 and the previous Remark hereafter we assume, without loss of generality, that

$$\bar{H}(x,p) = \max_{\mu \in Z_l(x)} -p \cdot f^r(x,\mu) \tag{1.31}$$

The assertion of Theorem 1.21 is equivalent to the fact that the value function u^r solves, in viscosity sense, the problem

$$\begin{array}{l} \partial_t u + H^r(x, D_x u) = 0 \\ u(0,x) = \bar{h}(x) \end{array}$$

Such a property of the value function directly follows by the dynamic programming principle (see, for example, [22], [50, Theorem II.7.1], and for system with state-dependent constraint on the controls [33], [52]), once we prove that u^r is continuous on its domain, and that the growth condition

$$|F^r(x)| := \sup\{|y| : y \in F^r(x)\} \le C(1+|x|), \quad \text{for any } x \tag{1.32}$$

is satisfied. This condition easily follows from the similar assumption we made on $|f|$ in Assumption 1.20. Another crucial property of $F^r(x)$ is given in the following Proposition.

Proposition 1.24. *Under the standing assumptions, if H is ergodic, the multivalued function $x \mapsto F^r(x)$ is upper semicontinuous.*

Proof. As mentioned in the introduction of this section, the relaxed dynamics results to be convex with respect to the controls. Since $Z_l(x)$ is convex and compact valued, with respect to the Prohorov metric, $F^r(x)$ is compact and convex for any x. Then the support function of the set $-F^r(x) := \{-q : q \in F^r(x)\}$, *i.e.*

$$\sigma_{-F^r(x)}(p) := \max_{q \in -F^r(x)} q \cdot p = \bar{H}(x,p) \tag{1.33}$$

can be used to characterize the convex closed sets $-F^r(x)$:

$$-F^r(x) = \{q \in \mathbb{R}^N \,|\, p \cdot q \le \sigma_{-F^r(x)}(p) \text{ for any } p \in \mathbb{R}^N\} \tag{1.34}$$

This is a classical result in convex analysis (see [18, p.30] or [70, p.112]) based on the fact that any closed convex subset of $\mathbb{R}^N$ is the intersection of the closed half–spaces which contain it. The support function of a closed convex set describes precisely these closed half–spaces.

Proposition 3 in [3] affirms that if H is ergodic, then $\bar{H}(x,p)$ is automatically continuous in $\mathbb{R}^N \times \mathbb{R}^N$. Then we get the conclusion, in fact: for any x_0, any $x_n \to x_0$ take a sequence $q_n \in F^r(x_n)$ converging to some q_0. By (1.33),(1.34), for any $p \in \mathbb{R}^N$ one has,

$$p \cdot (-q_0) = \lim_n p \cdot (-q_n) \le \lim_n \bar{H}(x_n,p) = \bar{H}(x_0,p)$$

Then $q_0 \in F^r(x_0)$.

□

Thanks to the previous Proposition and to the growth condition (1.32) we infer the existence of a.c. solutions for the differential inclusion (1.28) (See, for example [42, Theorem 5.2]). Moreover, the following qualitative property of the set of solutions of (1.28) holds:

Proposition 1.25. [42, Theorem 7.1] *For any $T > 0$ the multivalued map*

$$\xi \mapsto \mathcal{S}_T(\xi) := \{x : \ x \text{ is a solution of } (1.28) \text{ in } (0,T), \text{ with } x_0 = \xi\}$$

is upper semicontinuous with respect to the sup-norm.

Proposition 1.26. *The value function $u^r(t,x)$ is continuous on its domain.*

Proof. Let us denote by $\mathcal{S}_T(x)$ the set of solutions of (1.28) in $(0,T)$. Show first the continuity with respect to t. Fix t, pick any $\delta > 0$ and, for any $t' \in I_\delta := [t - \delta/2, t + \delta/2]$, estimate

$$u^r(t',x) - u^r(t,x) =$$
$$\inf\left\{\int_0^{t'} l^r(x_s,\mu_s)ds + \bar{h}(x_{t'})\right\} - \inf\left\{\int_0^{t} l^r(x_s,\mu_s)ds + \bar{h}(x_t)\right\}$$

where both infima are taken over the set of solutions of the relaxed control problem (1.28).

For any fixed $\gamma > 0$ let $(\bar{x},\bar{\mu})$ be such that

$$\inf\left\{\int_0^{t} l^r(x_s,\mu_s)d\tau + \bar{h}(x_t)\right\} = \int_0^t l^r(\bar{x}_s,\bar{\mu}_s)ds + \bar{h}(\bar{x}_t) - \gamma$$

Since f^r is bounded (because f is), $|\bar{x}_{t'} - \bar{x}_t| \le K\delta$ for a suitable constant K. Then

$$\begin{aligned} |u^r(t',x) - u^r(t,x)| &\le \int_t^{t'} |l^r(\bar{x}_s,\bar{\mu}_s)|ds + |\bar{h}(\bar{x}_{t'}) - \bar{h}(\bar{x}_t)| \\ &\le \delta \max_{s\in I_\delta} l^r(\bar{x}_s,\bar{\mu}_s) + \omega(K\delta) + \gamma \end{aligned}$$

where $\omega(\cdot)$ is the modulus of uniform continuity of $\bar{h}$; this sum tends to 0 as δ goes to 0, in force of the assumption on the functions l and f. We get the conclusion since γ is arbitrarily small.

Let us now verify the continuity with respect to x. Fix t, an initial point x_0. By the upper semicontinuity of $\mathcal{S}_t(\cdot)$ (Proposition 1.25),

for any $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that
if $x \in B(x_0,\delta_\varepsilon)$ then $\mathcal{S}_t(x_0) + \varepsilon B \supseteq \mathcal{S}_t(x)$

i.e.

for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that, if $x \in B(x_0,\delta_\varepsilon)$ then
for any $\bar{x}_\cdot \in \mathcal{S}_t(x)$ there is $\widetilde{x}_\cdot \in \mathcal{S}_t(x_0)$ such that
$|\bar{x}_s - \widetilde{x}_s| < \varepsilon$ for any $s \in [0,t]$.

Now take any $x \in B(x_0,\delta_\varepsilon)$ and, in order to estimate $u^r(t,x_0) - u^r(t,x)$, fix $\gamma > 0$ and let $\bar{x}_\cdot$ be the trajectory of $\mathcal{S}_t(x)$ such that

$$\inf_{x\in\mathcal{S}_t(x)}\left\{\int_0^{t} l^r(x_s,\mu_s)ds + \bar{h}(x_t)\right\} = \int_0^t l^r(\bar{x}_s,\bar{\mu}_s)ds + \bar{h}(\bar{x}_t) - \gamma$$

then,

$$u^r(t,x_0) - u^r(t,x) =$$

$$\inf_{x\in\mathcal{S}_t(x_0)}\left\{\int_0^t l^r(x_s,\mu_s)ds+\bar{h}(x_t)\right\}-\int_0^t l^r(\bar{x}_s,\bar{\mu}_s)ds+\bar{h}(\bar{x}_t)-\gamma$$

and

$$\begin{aligned}
|u^r(t,x_0)-u^r(t,x)| &\leq \int_0^t |l^r(\widetilde{x}_s,\widetilde{\mu}_s)-l^r(\bar{x}_s,\bar{\mu}_s)|ds+|\bar{h}(\widetilde{x}_t)-\bar{h}(\bar{x}_t)|+\gamma\\
&\leq \int_0^t \omega_{l^r}(|\widetilde{x}_s-\bar{x}_s|)ds+\omega_{\bar{h}}(|\widetilde{x}_t-\bar{x}_t|)+\gamma\\
&< t\omega_{l^r}(\varepsilon)+\omega_{\bar{h}}(\varepsilon)+\gamma
\end{aligned}$$

This completes the proof because of the arbitrariness of ε and γ.

□

1.3.2 An Example

The following example is devoted to better understand how the occupational measure are employed to detect the limit control problem. It take place in the state–constrains framework, *i.e.* we assume that

$$\begin{array}{c}\text{a connected and compact set } Y \text{ does exist, such that}\\ y(t)\in Y \text{ for any } t\geq 0.\end{array}$$

In fact, the y–periodicity made in the previous sections on all given functions, can be replaced with the state–constraint for the fast states.

Consider the unforced problem

$$\begin{aligned}\dot{x}_t &= f(x_t,y_t)\\ \varepsilon\dot{y}_t &= g(x_t,y_t)\end{aligned}\tag{1.35}$$

with f and g given by

$$f:\mathbb{R}^3\to\mathbb{R}\qquad f(x,y_1,y_2)=x(y_1+y_2)$$

$$g:\mathbb{R}^3\to\mathbb{R}^2\quad g(x,y_1,y_2)=\begin{pmatrix}-xy_2+y_1(1-y_1^2-y_2^2)\\ xy_1+y_2(1-y_1^2-y_2^2)\end{pmatrix}$$

The fast subsystem for the fast flow is $y(t)=(y_1(t),y_2(t))$ is

$$\dot{y}(t)=g(x,y(t))\tag{1.36}$$

where x is supposed to be fixed, *i.e.*

$$\begin{aligned}\dot{y}_1 &= -xy_2+y_1(1-y_1^2-y_2^2)\\ \dot{y}_2 &= xy_1+y_2(1-y_1^2-y_2^2)\end{aligned}$$

The motion takes place in the region $R:=\{y_1^2+y_2^2\leq 1\}$ of $\mathbb{R}^2$, then R is also an invariant set for the dynamics. Passing in polar coordinates (r,θ), this system assumes the following expression:

$$\begin{aligned}\dot{r} &= r(1-r^2)\\ \dot{\theta} &= x\end{aligned}$$

It is apparent that for any $x \in \mathbb{R}$, the point $r = 0$ is stable; furthermore, if $x \neq 0$ the cycle $\{r = 1\}$ is a limit cycle for the dynamics. This means that for any initial value (r_0, θ_0), the trajectory $(r(t), \theta(t))$ approaches the cycle $\{r = 1\}$ as t goes to $+\infty$. If $x = 0$, the motion take place on the radius $\{\theta = \theta_0\}$, and all trajectories tends to $(1, \theta_0)$ along this radius.

Now we detect the set $Z_l(x)$ of the limiting relaxed measures. Let us recall that a measure μ belongs to $Z_l(x)$ when it is a weak–star limit of occupational measures

$$\mu_\tau := \frac{1}{\tau} \int_0^\tau \delta_{y(t)} dt$$

as $\tau \to +\infty$, where $\delta_{y(t)}$ is the Dirac's mass concentrated in $y(t)$, the solution of (1.36).

Let φ be any continuous function on R, i.e. 2π–periodic with respect to θ. μ_τ converges weak-star to μ if and only if

$$\begin{aligned} \int_R \varphi(p)\mu_\tau(dp) &= \frac{1}{\tau} \int_0^\tau dt \int_R \varphi(p)\delta_{y(t)}(dp) \\ &= \frac{1}{\tau} \int_0^\tau \varphi(y(t))dt \longrightarrow \int_R \varphi(p)\mu(dp) \qquad (1.37) \end{aligned}$$

Let us suppose, initially, $x \neq 0$, and study the limit for $\tau \to +\infty$ of the integrals $\frac{1}{\tau}\int_0^\tau \varphi(r(t), \theta(t))dt$. Take $\tau = 2\pi n$, $n \in \mathbf{N}$; one has

$$\begin{aligned} \frac{1}{\tau} \int_0^\tau \varphi(r(t), \theta(t))dt &= \frac{1}{2\pi n} \sum_{k=0}^{n-1} \int_{2k\pi}^{2(k+1)\pi} \varphi(r(t), \theta(t))dt \\ &= \frac{1}{2\pi n} \sum_{k=0}^{n-1} \int_0^{2\pi} \varphi(r(t+2k\pi), \theta(t+2k\pi))dt \end{aligned}$$

As detailed before $r(t + 2k\pi) \to 1$, as $k \to +\infty$, and if $x \neq 0$, $\theta(t) = \theta_0 + xt$; then

$$\begin{aligned} \lim_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau \varphi(r(t), \theta(t))dt &= \frac{1}{2\pi} \int_0^{2\pi} \varphi(1, \theta_0 + xt)dt \\ &= \frac{1}{2\pi x} \int_{\theta_0}^{\theta_0 + 2\pi x} \varphi(1, \theta)d\theta \\ &= \frac{1}{2\pi x} \int_0^{2\pi x} \varphi(1, \theta)d\theta \qquad (1.38) \end{aligned}$$

The last equality is due to the periodicity of φ with respect to θ. Note that, since φ is continuous and so bounded on R, the limit we are studying exists and does not depend on the particular way in which τ approaches $+\infty$. Moreover, in (1.38) we see that this limit is also independent on the initial data (r_0, θ_0). So, if $x \neq 0$, $Z_l(x)$ consists of exactly two measures: δ_0, the Dirac's mass concentrated at the origin, and

$$\mu_x := \lim_{\tau \to +\infty} \mu_\tau \text{ weak star.}$$

From (1.38) we can also get an explicit expression for μ_x; in fact (1.38) represents a functional equality defined for any continuous function φ; then it can be extended to any measurable bounded function.[1] So, let Q be any measurable Borel subset or R, and $\mathbf{1}_Q(r,\theta)$ its characteristic function. From (1.38) we get

$$\begin{aligned} \mu_x Q = \int_R \mathbf{1}_Q(r,\theta) d\mu_x &= \frac{1}{2\pi x}\int_0^{2\pi x} \mathbf{1}_Q(r,\theta)d\theta \\ &= \frac{1}{2\pi x}\mathcal{L}^1(\{\theta \in [0,2\pi x] : (1,\theta)\in Q\}) \end{aligned}$$

In particular we see that $\mu_x(\{r=1\})=1$.

If $x=0$, the limiting measures in $Z_l(0)$ are δ_0 and $\delta_{(1,\theta_0)}$, the Dirac's masses concentrated at the end points of the radius $\{\theta=\theta_0\}$. In fact, in order to determine the limit measure μ_0, we came back to (1.37); for any continuous function φ one has

$$\begin{aligned} \int_R \varphi(r,\theta)d\mu_0 &= \lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^{\tau}\varphi(r(t),\theta(t))dt \\ &= \lim_n \frac{1}{2\pi n}\sum_{k=0}^{n-1}\int_{2k\pi}^{2(k+1)\pi}\varphi(r(t),\theta(t))dt \\ &= \lim_n \frac{1}{2\pi n}\sum_{k=0}^{n-1}\int_0^{2\pi}\varphi(r(t+2k\pi),\theta(t+2k\pi))dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\varphi(1,\theta_0)dt = \varphi(1,\theta_0) \end{aligned} \tag{1.39}$$

The previous equality is defined for continuous functions; once again, it can be extended in a single–valued way to measurable bounded functions. So, if Q is a Borel set of R, (1.39) tells us:

$$\mu_0 Q = \mathbf{1}_Q(1,\theta_0) = \begin{cases} 1 & \text{if } (1,\theta_0)\in Q \\ 0 & \text{if } (1,\theta_0)\notin Q \end{cases}$$

that means

$$\mu_0 = \delta_{(1,\theta_0)}.$$

Let us summarize:

$$Z_l(x) = \begin{cases} \{\delta_0,\mu_x\}, & \text{if } x\neq 0 \\ \{\delta_0,\delta_{(1,\theta_0)}\}, & \text{if } x=0 \end{cases}$$

Recalling that $f^r(x,\mu) := \int_R f(x,y)d\mu(y)$, from the definition of f we immediately get, for $x=0$

$$f^r(0,\delta_0) = f^r\left(0,\delta_{(1,\theta_0)}\right) = 0.$$

For $x\neq 0$ we have

$$f^r(x,\delta_0) = f(x,0) = 0$$

[1] In fact, if A is a positive linear functional defined for continuous functions, and if $\mathbf{1}_Q$ is the characteristic function of an open set Q, we can put $A\mathbf{1}_Q := \lim_n Af_n$, whenever f_n is a nondecreasing sequence of continuous functions such that $\mathbf{1}_Q = \lim_n f_n$ It can be showed that this definition $A\mathbf{1}_Q$ is independent of the sequence f_n.

and

$$\begin{aligned} f^r(x,\mu_x) &= \int_R f(x,y)d\mu_x(y) = \lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^\tau f(x,y(t))dt \\ &= \frac{1}{2\pi x}\int_0^{2\pi x} f(x,\cos\theta,\sin\theta)d\theta \\ &= \frac{1}{2\pi x}\int_0^{2\pi x} x(\cos\theta+\sin\theta)d\theta \\ &= \frac{1}{2\pi}(\sin(2\pi x)-\cos(2\pi x)+1) \end{aligned}$$

Conclusion. The limiting dynamics for $\varepsilon \to 0^+$ in (1.35) is governed by the differential inclusion

$$\dot{x} \in f^r(x, Z_l(x)) = \begin{cases} \{0\}, & \text{if } x = 0 \\ \{0, f^r(x,\mu_x)\}, & \text{if } x \neq 0 \end{cases}, \qquad x_0 = x$$

The effective dynamics take place on the real line and is periodic in $[0,1]$; if $x_0 = 0$ or $x_0 = \frac{3}{4}$ the dynamics admits only the steady solution; for any other initial value, the point $\bar{x} = \frac{3}{4}$ is a limit point for the multivalued dynamics.

1.4 Relaxed representation formulae of the effective Hamiltonian for stochastic systems

In section 1.2.4 we considered the set of limiting relaxed controls for a stochastic control system, and proved that under suitable controllability conditions it coincides with the set of limiting occupational measures.

In this section, following the ideas of [2, Theorem 7] (that has been quoted in Proposition 1.19), we show that, even in the stochastic framework, it is possible to represent the effective Hamiltonian of a singularly perturbed system, in terms of a relaxed Hamiltonian, *i.e.* the Hamiltonian obtained via the relaxed data. At the end of the section, we will provide a similar result for a stochastic differential game, where the competitors play together only the slow dynamics, while the fast dynamics is governed only by the second player.

The relaxation procedure is similar to that proposed in the deterministic context in Section 1.3 and we refer to this section for any remark on the properties of the relaxed functions.

1.4.1 Relaxed representation of the effective Hamiltonian for stochastic optimal control problems.

Consider the following stochastic singularly perturbed problem

$$\begin{aligned} dx_s &= f(x_s,y_s,a_s)ds + \sigma(x_s)dW_s, & x_0 = x \\ dy_s &= \tfrac{1}{\varepsilon}g(x_s,y_s,a_s)ds + \tfrac{1}{\sqrt{\varepsilon}}\tau(x_s,y_s)dW_s, & y_0 = y \end{aligned} \tag{1.40}$$

Case 1. Deterministic fast dynamics.
Consider first the intermediate case in which $\tau \equiv 0$ and $\sigma \not\equiv 0$ in (1.40), *i.e.* the

slow variables are governed by a stochastic differential equation, and the fast one are deterministic. The system under investigation is then

$$\begin{aligned} dx_s &= f(x_s, y_s, a_s)ds + \sigma(x_s)dW_s \\ \dot{y}_s &= \tfrac{1}{\varepsilon} g(x_s, y_s, a_s) \end{aligned} \tag{1.41}$$

The associated 2nd order Hamiltonian is

$$H(x, y, p, q, X) = \max_a \{-\mathrm{tr}\mathbf{A}(x) \cdot X - p \cdot f(x, y, a) - q \cdot g(x, y, a) - l(x, y, a)\}$$

where we denote by $\mathbf{A}$ the matrix $\sigma\sigma^T/2$. The corresponding effective Hamiltonian is represented via the formula

$$\bar{H}(x, p, X) = \lim_{t \to +\infty} \sup_a \frac{1}{t} \int_0^t L(y_s, a_s; x, p, X) ds \tag{1.42}$$

where

$$L(y, a; x, p, X) := -\mathrm{tr}\mathbf{A}(x) \cdot X - p \cdot f(x, y, a) - l(x, y, a)$$

and y_s is the solution of the (deterministic) fast subsystem

$$\dot{y}_s = g(x, y_s, a_s), \quad y_0 = y \tag{1.43}$$

where x plays the role of a fixed parameter.

In this case the set of limiting relaxed controls is the set

$$Z_l(x) := \left\{ \mu = \lim \frac{1}{t_n} \int_0^{t_n} \delta_{(y_s, a_s)} ds \text{ weak star, for some } t_n \to +\infty,\, a \in \mathcal{A} \right\}$$

where y_s is the solution of (1.43) corresponding to a_s. It is convex and compact with respect to the weak star topology, by Theorem 1.10 and Proposition 1.11.

In accordance with formula (1.26) we perform the relaxation of the data and define

$$L^r(x, \mu, p, X) := -\mathrm{tr}\mathbf{A} \cdot X - p \cdot f^r(x, \mu) - l^r(x, \mu). \tag{1.44}$$

Proposition 1.27. *The effective Hamiltonian admits the following representation*

$$\bar{H}(x, p, X) = \max_{\mu \in Z_l(x)} L^r(x, \mu, p, X). \tag{1.45}$$

Proof. We argue as in [2, Section 1.4 and Theorem 7]. Let us denote by $H^r(x, p, X)$ the right hand side of (1.45), and put

$$G(x, a, p, X) := \limsup_{t \to +\infty} \frac{1}{t} \int_0^t L(y_s, a_s; x, p, X) ds$$

then, by (1.42),

$$\bar{H}(x, p, X) = \sup_{a \in \mathcal{A}} G(x, a, p, X) \tag{1.46}$$

Note that if μ is generated by $t_n \to +\infty$ and $(y_\cdot, a_\cdot)$, then

$$G(x, a, p, X) = L^r(x, \mu, p, X). \tag{1.47}$$

So, let $\bar{\mu} \in Z_l(x)$, generated by $(\bar{y}_\cdot, \bar{a}_\cdot)$, be such that

$$H^r(x, p, X) = L^r(x, \bar{\mu}, p, X)$$

then

$$H^r(x,p,X) = G(x,\bar{a},p,X) \le \sup_{a\in A} G(x,a,p,X) = \bar{H}(x,p,X).$$

Conversely, fix $\bar{a}.$ and an initial point y for the fast subsystem (1.43), and let $t_n \to +\infty$ be a sequence such that

$$G(x,\bar{a},p,X) := \lim_n \frac{1}{t_n}\int_0^{t_n} L(y_s,\bar{a}_s;x,p,X)ds$$

Finally, let $\bar{\mu}$ be the measure generated, up to subsequences, by $(\bar{y}.,\bar{a}.)$ and t_n. One has

$$G(x,\bar{a},p,X) = L^r(x,\bar{\mu},p,X) \le \sup_{\mu\in Z_l(x)} L^r(x,\mu,p,X) = H^r(x,p,X)$$

The assertion follows by (1.46) and the arbitrariness of $\bar{a}.$.

□

Case 2. Noise affecting all variables.
We deal now with the fully stochastic system (1.40). The 2nd–order Hamiltonian related to this problem is

$$\begin{aligned} H(x,y,p,q,X,Y,Z) = \max_a \{&-\mathrm{tr}(\mathbf{A}(x)\cdot X) - \mathrm{tr}(\mathbf{B}(x,y)\cdot Y) - 2\mathrm{tr}(\mathbf{C}(x,y)\cdot Z) \\ &-p\cdot f(x,y,a) - q\cdot g(x,y,a) - l(x,y,a)\} \end{aligned}$$

where

$$\mathbf{A}(x) := \frac{1}{2}\sigma(x)\sigma^T(x), \quad \mathbf{B}(x,y) := \frac{1}{2}\tau(x,y)\tau^T(x,y), \quad \mathbf{C}(x,y) = \frac{1}{2}\tau(x,y)\sigma^T(x) \tag{1.48}$$

and the effective Hamiltonian has the representation

$$\bar{H}(x,p,X) = \lim_{t\to+\infty} \sup_a \mathbb{E}_y\left[\frac{1}{t}\int_0^t L(y_s,a_s;x,p,X)ds\right]$$

where $L(y,a;x,p,X) := -\mathrm{tr}\mathbf{A}(x)\cdot X - p\cdot f(x,y,a) - l(x,y,a)$, and y_s is the solution, at x fixed, of the fast subsystem

$$dy_s = g(x,y_s,a_s)ds + \tau(x,y_s)dW_s, \quad y_0 = y \tag{1.49}$$

As pointed out in Section 1.2.4, the main difference with the previous case is that, being the fast subsystem stochastic, the occupational measures have to be defined via the mean

$$\mu_{t_n} := \mathbb{E}_y\left[\frac{1}{t_n}\int_0^{t_n} \delta_{(y_s,a_s)}ds\right]$$

A limiting relaxed control is a weak star limit of such a μ_{t_n}, for some $t_n \to +\infty$, and some control a. The set of the limiting relaxed controls related to the fast subsystem (1.49) is denoted, as in the deterministic case, by $Z_l(x)$. Recall the definition of L^r given in (1.44). Our goal is to prove the following

Proposition 1.28. *The effective Hamiltonian admits the following representation*

$$\bar{H}(x,p,X) = \sup_{\mu\in Z_l(x)} L^r(x,\mu,p,X). \tag{1.50}$$

Remark 1.29. As pointed out in the preliminary Section, we know from the general theory of relaxed controls that the space of Radon probability measures $(\mathbb{R}^M \times A)^r$ is convex and compact with respect to the weak–star convergence. Furthermore, in the deterministic setting under bounded time controllability, also the set of limiting relaxed controls $Z_l(x)$ is compact. This allows us to take the maximum in the formulae (1.27) and (1.45).

In the stochastic setting sufficient conditions for $Z_l(x)$ to be compact are (1.22) and (1.23), so the compactness of $Z_l(x)$ is not guaranteed under the current assumptions. This is the reason why we take the supremum in (1.50) instead of the maximum. Anyway the compactness of $(\mathbb{R}^M \times A)^r$ is sufficient to prove such relaxed expression for the effective Hamiltonian. $\square$

We need some preliminary lemmata. The first lemma is an abstract result about the exchange of the order of integration.

Lemma 1.30. *Let Y be a compact metric space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Let μ a probability–measure–valued random variable, that is, a jointly measurable function $\mu : \Omega \times \mathcal{B}(Y) \to \mathbb{R}^+$ such that, for any $\omega \in \Omega$, $\mu(\omega, \cdot)$ is a probability measure on $\mathcal{B}(Y)$. Finally define on $\mathcal{B}(Y)$ the following measure*

$$\mathbb{E}[\mu](Q) := \mathbb{E}[\mu(\cdot, Q)] := \int_\Omega \mu(\omega, Q)\mathbb{P}(d\omega), \quad Q \in \mathcal{B}(Y).$$

Let $\psi : Y \to \mathbb{R}$ a continuous function and let ψ^r be the function defined as

$$\psi^r(\nu) := \int_Y \psi(y)\nu(dy), \quad \nu \text{ probability measure on } \mathcal{B}(Y)$$

Then

$$\mathbb{E}[\psi^r(\mu)] = \psi^r(\mathbb{E}[\mu]).$$

Proof. Suppose initially that ψ is non negative, and let

$$\psi_n(y) := \sum_{k=1}^n \beta_k \chi_{Q_k}(y), \quad \beta_k \geq 0, \quad Q_k \in \mathcal{B}(Y) \text{ for any } k$$

a piecewise constant function such that $\psi_n \nearrow \psi$ uniformly on Y. Then for any $\omega \in \Omega$

$$\left| \int_Y \psi_n \mu(\omega, dy) \right| \leq \int_Y \psi_n \mu(\omega, dy) \leq \int_Y \psi(\omega, dy) \leq \sup_Y \psi < +\infty$$

then, applying twice the dominated convergence theorem, one has

$$\mathbb{E}\left[\int_Y \psi(y)\mu(\cdot, dy)\right] = \mathbb{E}\left[\lim_n \int_Y \psi_n(y)\mu(\cdot, dy)\right] = \lim_n \mathbb{E}\left[\int_Y \psi_n(y)\mu(\cdot, dy)\right] \tag{1.51}$$

Let us denote temporarily by Λ the measure $\mathbb{E}[\mu]$. We have for any n

$$\begin{aligned}
\mathbb{E}\left[\int_Y \psi_n(y)\mu(\cdot, dy)\right] &= \mathbb{E}\left[\int_Y \sum_{k=1}^n \beta_k \chi_{Q_k}(y)\mu(\cdot, dy)\right] = \mathbb{E}\left[\sum_{k=1}^n \beta_k \mu(\cdot, Q_k)\right] \\
&= \sum_{k=1}^n \beta_k \mathbb{E}[\mu(\cdot, Q_k)] = \sum_{k=1}^n \beta_k \int_\Omega \mu(\omega, Q_k)\mathbb{P}(d\omega) \\
&= \sum_{k=1}^n \beta_k \Lambda(Q_k) = \int_Y \sum_{k=1}^n \beta_k \chi_{Q_k}(y)\Lambda(dy) \\
&= \int_Y \psi_n(y)\Lambda(dy)
\end{aligned}$$

By the dominated convergence theorem, and taking into account (1.51), we get

$$\mathbb{E}[\psi^r(\mu)] = \mathbb{E}\left[\int_Y \psi(y)\mu(\cdot, dy)\right]$$

$$= \lim_n \int_Y \psi_n(y)\Lambda(dy) = \int_Y \psi(y)\Lambda(dy) = \psi^r(\mathbb{E}[\mu]).$$

For a general $\psi = \psi^+ - \psi^-$ we get the conclusion by the linearity of the mathematical expectation. In fact, since

$$\psi^r(\mu) = (\psi^+)^r(\mu) - (\psi^-)^r(\mu)$$

then

$$\mathbb{E}[\psi^r(\mu)] = \mathbb{E}[(\psi^+)^r(\mu) - (\psi^-)^r(\mu)] = \mathbb{E}[(\psi^+)^r(\mu)] - \mathbb{E}[(\psi^-)^r(\mu)]$$

$$= (\psi^+)^r(\mathbb{E}[\mu]) - (\psi^-)^r(\mathbb{E}[\mu]) = \psi^r(\mathbb{E}[\mu])$$

□

Consider now the function

$$G(x, a, p, X) := \limsup_{t\to+\infty} \mathbb{E}_y \left[\frac{1}{t}\int_0^t L(y_s, a_s; x, p, X)ds\right]$$

In analogy with (1.47), we have the following

Lemma 1.31. *For any (x, p, X), if $\mu \in Z_l(x)$ is generated by $(y., a.)$ and $t_n \to +\infty$, then*

$$G(x, a, p, X) = L^r(x, \mu, p, X)$$

where L^r has been defined in (1.44).

Proof. Let $\mu \in Z_l(x)$ be generated by $(y., a.)$ and $t_n \to +\infty$. Applying Lemma 1.30 we get

$$\mathbb{E}[f^r(x, \delta_{(y_s,a_s)})] = f^r(x, \mathbb{E}[\delta_{(y_s,a_s)}])$$

then

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{t}\int_0^t f(x,y_s,a_s)ds\right] &= \mathbb{E}\left[\frac{1}{t}\int_0^t f^r(x,\delta_{(y_s,a_s)})ds\right] \\
&= \frac{1}{t}\int_0^t \mathbb{E}[f^r(x,\delta_{(y_s,a_s)})]ds \\
&= \frac{1}{t}\int_0^t f^r(x,\mathbb{E}[\delta_{(y_s,a_s)}]) \\
&= f^r\left(x,\frac{1}{t}\int_0^t \mathbb{E}[\delta_{(y_s,a_s)}]\right)
\end{aligned}$$

Analogously, for l we have

$$\mathbb{E}\left[\frac{1}{t}\int_0^t l(x,y_s,a_s)ds\right] = l^r\left(x,\frac{1}{t}\int_0^t \mathbb{E}[\delta_{(y_s,a_s)}]\right)$$

therefore

$$L^r\left(x,\mathbb{E}_y\left[\frac{1}{t_n}\int_0^{t_n}\delta_{(y_s,a_s)}ds\right],p,X\right) = \mathbb{E}_y\left[\frac{1}{t_n}\int_0^{t_n} L(y_s,a_s;x,p,X)ds\right]$$

The statement follows observing that the former member tends to $L^r(x,\mu,p,X)$ as $n\to+\infty$, and the latter to $G(x,a,p,X)$.

□

Proof of Proposition 1.28. Thanks to Lemma 1.31, the proof goes as in Proposition 1.27. Consider (x,p,X) fixed, and let $\bar{\mu}\in Z_l(x)$ be generated by $(\bar{y}.,\bar{a}.)$. Then, by Lemma 1.31,

$$L^r(x,\bar{\mu},p,X) = G(x,\bar{a},p,X) \le \sup_{a\in\mathcal{A}} G(x,a,p,X) = \bar{H}(x,p,X).$$

Passing to the supremum among all $\mu\in Z_l(x)$ we get

$$H^r(x,p) \le \bar{H}(x,p)$$

Conversely, fix $\bar{a}\in\mathcal{A}$ and an initial point y for the fast subsystem (1.49) and let $\bar{y}$ be the corresponding solution, and $t_n\to+\infty$ a sequence such that

$$G(x,\bar{a},p,X) := \lim_n \mathbb{E}_y\left[\frac{1}{t_n}\int_0^{t_n} L(\bar{y}_s,\bar{a}_s;x,p,X)ds\right]$$

Since $(\mathbb{R}^M\times A)^r$ is compact with respect to the weak-star convergence, there exists a diverging sequence t_n such that $\mathbb{E}[\frac{1}{t_n}\int_0^{t_n}\delta_{(\bar{y}_s,\bar{a}_s)}ds]$ converges to some Radon probability measure $\bar{\mu}$; furthermore $\bar{\mu}$ belongs to $Z_l(x)$ by definition. Then

$$G(x,\bar{a},p,X) = L^r(x,\bar{\mu},p,X) \le \max_{\mu\in Z_l(x)} L^r(x,\mu,p,X) = H^r(x,p,X)$$

and we get the conclusion by the arbitrariness of $\bar{a}$.

□

1.4.2 Relaxed representation of the effective Hamiltonian for stochastic differential games.

Finally we provide a version of the previous results for the following stochastic differential game:

$$\begin{array}{ll} dx_s = f(x_s, y_s, a_s, b_s)ds + \sigma(x_s)dW_s, & x_0 = x \\ dy_s = \frac{1}{\varepsilon} g(x_s, y_s, b_s)ds + \frac{1}{\sqrt{\varepsilon}} \tau(x_s, y_s)dW_s, & y_0 = y \end{array} \tag{1.52}$$

The Bellman-Isaacs Hamiltonian related to this problem is

$$H(x, y, p, q, X, Y, Z) =$$

$$\min_{b \in B} \max_{a \in A} \{ -\mathrm{tr}(\mathbf{A}(x) \cdot X) - \mathrm{tr}(\mathbf{B}(x, y) \cdot Y) - 2\mathrm{tr}(\mathbf{C}(x, y) \cdot Z)$$

$$-p \cdot f(x, y, a, b) - q \cdot g(x, y, b) - l(x, y, a, b) \}$$

where the diffusion matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are defined as in (1.48).

The effective Hamiltonian has the following representation:

$$\bar{H}(x, p, X) = \lim_{t \to +\infty} \inf_{\alpha \in \Gamma(t)} \sup_{b \in \mathcal{B}(t)} \mathbb{E}_y \left[\frac{1}{t} \int_0^t L(y_s, \alpha[b]_s, b_s; x, p, X) ds \right]$$

where $L(y, a, b; x, p, X) := -\mathrm{tr}(\mathbf{A}(x, y) \cdot X) - p \cdot f(x, y, a, b) - l(x, y, a, b)$, and where $\Gamma(t)$ is the set of nonanticipating strategy $\alpha : \mathcal{A}(t) \to \mathcal{B}(t)$ for the first player in $[0, t]$ and $\Delta(t)$ is the set of nonanticipating strategy $\beta : \mathcal{B}(t) \to \mathcal{A}(t)$ for the second player in $[0, t]$. $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are the sets of admissible controls for the first and the second player in $[0, t]$. See [51].

For any fixed x the set of limiting relaxed controls can be defined in this context, like in section 1.2.4, as the following set

$$Z_l(x) := \left\{ \mu = \lim_n \mathbb{E} \frac{1}{t_n} \int_0^{t_n} \delta_{(y_s, b_s)} ds \text{ weak star, for some } t_n \to +\infty,\ b. \in \mathcal{B} \right\}$$

where y_s is the solution of the fast subsystem

$$dy_s = g(x, y_s, b_s)ds + \tau(x, y_s)dW_s \tag{1.53}$$

As before, we are interested in comparing this function with the relaxed Hamiltonian, defined in this case via the formula

$$H^r(x, p, X) := \max_{\mu \in Z_l(x)} \min_{a \in A} L^r(a, \mu; x, p, X)$$

where $L^r(a, \mu; x, p, X) := -\mathrm{tr}\mathbf{A}(x) \cdot X - p \cdot f^r(x, a, \mu) - l^r(x, a, \mu)$. In [4, Th. 9.11] has been proved that, under suitable assumptions,

$$\bar{H}(x, p, X) = H_0(x, p, X) := \min_{\substack{(y,b): \\ g(x,y,b)=0}} \max_{a \in A} L(y, a, b; x, p, X)$$

This is actually inspired by the order reduction method, explained in section 1.2.2. Following the same spirit of this Theorem, and under very similar assumptions we shall prove that

Proposition 1.32. *For any* (x,p,X)*,* $\bar{H}(x,p,X) \geq H^r(x,p,X)$*. Furthermore, if the following assumptions are satisfied:*

1. *for any* $\bar{\alpha} \in \Gamma$*, there are sequences* $t_n \to +\infty$ *and* $b^n \in \mathcal{B}$ *such that*

$$\sup_b \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} L(y_s, \bar{\alpha}[b]_s, b_s) ds \right] + o(t_n) = \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} L(y_s^{b^n}, \bar{\alpha}[b^n]_s, b_s^n) ds \right]$$

   and there exists (y^*, b^*) *such that*

$$\lim_n \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} \left(|y_s^{b_n} - y^*| + |\bar{\alpha}[b^n]_s - \bar{\alpha}[b^*]| + |b_s^n - b^*| \right) ds \right] = 0$$

2. *(Isaacs type condition)* $H^r(x,p,X) = \min_{a \in A} \max_{\mu \in Z_l(x)} L^r(a, \mu; x, p, X)$*.*

3. $\bigcup_{\alpha \in \Gamma} \{\alpha(\mathcal{B})\} = \mathcal{A}$

then, $\bar{H}(x,p,X) \leq H^r(x,p,X)$

Proof. Consider (x,p,X) fixed, and write $L(y,a,b)$ and $L^r(a,\mu)$ instead of $L(y,a,b;x,p,X)$ and $L^r(a,\mu;x,p,X)$. Fix $\mu \in Z_l(x)$, generated by $t_n \to +\infty$ and $(\bar{y}_\cdot, \bar{b}_\cdot)$. For any strategy α arguing as in Lemma 1.31 one has

$$\begin{aligned}
& \sup_{b \in \mathcal{B}} \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} L(y_s, \alpha[b]_s, b_s) ds \right] \\
= \; & \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} L(\bar{y}_s, \alpha[\bar{b}]_s, \bar{b}_s) ds \right] = \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} L^r(\alpha[\bar{b}]_s, \delta_{(\bar{y}_s, \bar{b}_s)}) ds \right] \\
= \; & \mathbb{E}_y \left[L^r \left(\frac{1}{t_n} \int_0^{t_n} \alpha[\bar{b}]_s ds, \frac{1}{t_n} \int_0^{t_n} \delta_{(\bar{y}_s, \bar{b}_s)} ds \right) ds \right] \\
= \; & L^r \left(\frac{1}{t_n} \int_0^{t_n} \alpha[\bar{b}]_s ds, \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} \delta_{(\bar{y}_s, \bar{b}_s)} ds \right] \right)
\end{aligned}$$

Since A is compact, there exists a subsequence of t_n that we do not relabel, such that $\lim_n \frac{1}{t_n} \int_0^{t_n} \alpha[\bar{b}]_s ds$ belongs to A. So, for n large enought

$$\inf_{\alpha \in \Gamma} L^r \left(\frac{1}{t_n} \int_0^{t_n} \alpha[\bar{b}]_s ds, \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} \delta_{(\bar{y}_s, \bar{b}_s)} ds \right] \right) \geq$$

$$\min_{a \in A} L^r \left(a, \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} \delta_{(\bar{y}_s, \bar{b}_s)} ds \right] \right)$$

then one has

$$\min_{a \in A} L^r \left(a, \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} \delta_{(\bar{y}_s, \bar{b}_s)} ds \right] \right) \leq \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} L(y_s, \alpha[b]_s, b_s) ds \right]$$

taking the limit for $n \to +\infty$ one has

$$\min_{a \in A} L^r(a, \bar{\mu}) \leq H(x,p,X)$$

and the conclusion follows by the arbitrariness of μ.

To prove the opposite inequality, fix $\bar{\alpha} \in \Gamma$, and let b^n, t_n and (y^*, b^*) such that the additional assumption 1 of the statement is fulfilled. Then

$$\begin{aligned} & \inf_{\alpha\in\Gamma}\sup_{b\in\mathcal{B}} \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} L(y_s, \alpha[b]_s, b_s) ds \right] + o(1) \\ \leq\ & \sup_{b\in\mathcal{B}} \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} L(y_s, \bar{\alpha}[b]_s, b_s) ds \right] + o(1) \\ =\ & \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} L(y_s^{b^n}, \bar{\alpha}[b^n]_s, b_s^n) ds \right] \end{aligned}$$

The last term tends to $L(y^*, \bar{\alpha}[b^*], b^*)$ as $n \to +\infty$; in fact by Jensen's inequality, we get

$$\begin{aligned} & \mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} |L(y_s^{b^n}, \bar{\alpha}[b^n]_s, b_s^n) - L(y^*, \bar{\alpha}[b^*], b^*)| ds \right] \\ \leq\ & \omega_L \left(\mathbb{E}_y \left[\frac{1}{t_n} \int_0^{t_n} \left(|y_s^{b_n} - y^*| + |\bar{\alpha}[b^n]_s - \bar{\alpha}[b^*]| + |b_s^n - b^*| \right) ds \right] \right) \end{aligned}$$

being ω_L a concave modulus of continuity of the function $L(\cdot, \cdot, \cdot; x, p, X)$ with respect to the arguments (y, a, b).

Observing that $\delta_{(y^*, b^*)}$ belongs to $Z_l(x)$, being generate by the constant solution $y_s \equiv y^*$ of (1.53), corresponding to $b_s \equiv b^*$, and passing to the limit in the previous calculations, we have that for any $\bar{\alpha} \in \Gamma$, there exists b^* such that

$$\bar{H}(x, p, X) \leq L(y^*, \bar{\alpha}[b^*], b^*) = L^r(\bar{\alpha}[b^*], \delta_{(y^*, b^*)}) \leq \max_{\mu \in Z_l(x)} L^r(\bar{\alpha}[b^*], \mu)$$

i.e. , for any $a \in \widetilde{A}$

$$\bar{H}(x, p, X) \leq \max_{\mu \in Z_l(x)} L^r(a, \mu)$$

so, by the assumptions 2. and 3.,

$$\begin{aligned} \bar{H}(x, p, X) &\leq \min_{a \in \widetilde{A}} \max_{\mu \in Z_l(x)} L^r(a, \mu) \\ &= \min_{a \in A} \max_{\mu \in Z_l(x)} L^r(a, \mu) = H^r(x, p, X) \end{aligned}$$

□

1.5 Homogenization of motion of interfaces

As an application of the previous theory, we study the evolution of a front propagating in $\mathbb{R}^N$ with normal velocity φ depending on position and undergoing fast periodic oscillations.

1.5.1 Level–set method.

The level–set formulation is an efficient tool in studying evolution of interfaces, or propagating fronts with normal velocity depending on the position. Such problems is motivated, for example, by the study of phase transformations. Here we are supposing that the medium where the evolution occurs is rapidly oscillating: therefore the detailed evolution of the front is complicated. Homogenization results apply to this topic, offering the possibility to define an *effective front* averaging the detailed evolution.

In the sequel we shall represent the normal velocity of the propagating front using a function $\varphi = \varphi(x, y) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, periodic with respect to the y variables.

Assumption 1.33. We assume φ to satisfy the following requirements:

(h1) φ is continuous and bounded;

(h2) φ is locally Lipschitz;

(h3) $\varphi > 0$.

In order to describe the periodicity of the medium where the front is moving we shall consider $\varphi(x, \frac{x}{\varepsilon})$, being ε a parameter representing the size of the periodic structure. We are interested in the limit behavior of the front as $\varepsilon \to 0$.

As we will recall below, if φ depends also on time, and changes its sign, some pathologies – like the presence of fronts with non–empty interior – may occur. Consequently, it is impossible to get an effective front. In our setting, we will find an effective propagating front even when φ is neither strictly positive nor strictly negative. In particular we shall require, for any fixed x the sign of $\varphi(x, y)$ to be constant. Anyway we initially require the sign of φ to be constant (say positive), in order to explain our approach in the simplest possible framework.

The propagating front is represented as a level–set. More precisely, we suppose that the initial front is given by

$$\Gamma_0 = \{x \in \mathbb{R}^N : u_0(x, y) = 0\}$$

where u_0 is a continuous bounded function, and that the position of the front at time t is given by

$$\Gamma_t^\varepsilon = \{x \in \mathbb{R}^N : u^\varepsilon(t, x) = 0\}$$

where $u^\varepsilon(t, x)$ is the solution of the Hamilton–Jacobi equation

$$\begin{aligned} &u_t^\varepsilon + H\left(x, \tfrac{x}{\varepsilon}, Du^\varepsilon\right) = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^N \\ &u^\varepsilon(x, 0) = u_0\left(x, \tfrac{x}{\varepsilon}\right) \end{aligned} \tag{1.54}$$

being H the function defined by

$$H(x, y, p) = \varphi(x, y)|p| = \max_{|a| \le 1} \varphi(x, y) p \cdot a$$

The control functions a are valued in the closed unit ball $\overline{B}(0, 1)$ hereafter denoted by B.

We want to capture the behavior of the front when ε tends to zero, *i.e.* to find a front Γ_t, we will call *effective front* or *homogenized front*, such that

$$\text{"}\Gamma_t^\varepsilon \to \Gamma_t \quad \text{as } \varepsilon \to 0\text{"}$$

using the local uniform convergence of solution of (1.54) to the solution of an effective equation. The main contribution of this section is the fact that the effective front can be described from a control point of view, as a level–set of a certain value function. This is possible because the limit of value functions u^ε is the value function related to the limiting optimal control problem, as explained in the previous sections.

1.5.2 Limit control problem for the homogenized front.

The problem (1.54) is solved by the value function of the control system associated to it, *i.e.*

$$\begin{aligned} &\dot{x}_t = \varphi\left(x_t, \tfrac{x_t}{\varepsilon}\right) a_t \qquad |a_t| \le 1 \\ &x_0 = x \end{aligned} \tag{1.55}$$

Such function is given by

$$u^\varepsilon(t,x) = \inf\left\{u_0\left(x_t, \frac{x_t}{\varepsilon}\right) : x_\cdot \text{ solves (1.55) in } [0,t]\right\} \tag{1.56}$$

We replace the system (1.55) with the equivalent two–scale system

$$\begin{aligned} \dot{x}_t &= \varphi(x_t, y_t) a_t \\ \varepsilon \dot{y}_t &= \varphi(x_t, y_t) a_t \end{aligned} \tag{1.57}$$

with the initial conditions

$$x_0 = x, \quad y_0 = \frac{x}{\varepsilon}$$

As explained in the previous section, we use the limiting relaxed controls to obtain the limiting control problem. So, consider for any fixed x the occupational measures μ_s $(s > 0)$ of the fast subsystem

$$\dot{y}_t = \varphi(x, y_t) a_t, \quad y_0 = y \tag{1.58}$$

and the set of the limiting relaxed controls $Z_l(x)$ (see Definition 1.8) *i.e.*

$$Z_l(x) = \{\mu : \mu = \lim_n \mu_{t_n} \text{ weak star, for some } t_n \to \infty, \text{ some } a \in B\}.$$

We record that the map $x \mapsto Z_l(x)$ is valued into the subsets of the Radon probability measures on $\mathcal{B}(\mathbb{R}^N \times B)$. These measures are used to relax the dynamics as described in Section 1.3, according with (1.26):

$$\varphi^r(x,\mu) := \int_{\mathbb{R}^N \times B} \varphi(x,y) a \, \mu(dy, da) \tag{1.59}$$

In the following theorem we describe the effective front in terms of the value function of the relaxed control problem.

Theorem 1.34. *Under Assumptions 1.33 the homogenized front is given by*

$$\Gamma_t = \{x \in \mathbb{R}^N : \bar{u}(t,x) = 0\}$$

where

$$\begin{aligned} \bar{u}(t,x) &= \inf\{\bar{u}_0(x_t)\} \qquad (1.60) \\ \bar{u}_0(x) &= \inf_y u_0(x,y) \end{aligned}$$

and the infimum in (1.60) *is taken over the solution of the differential inclusion*

$$\begin{aligned} &\dot{x}_t \in \varphi^r(x_t, Z_l(x_t)) \\ &x_0 = x \end{aligned} \tag{1.61}$$

An interesting aspect of the result is an interpretation of the asymptotic behavior of the front in terms of *generalized characteristics*: the effective front at time t is described by trajectories of the effective control problem (1.61) at time t.

Since $H(x, y, p)$ is Lipschitz continuous, y–periodic and p–coercive, for any (x, p) there exists $\bar{H}(x, p) \in \mathbb{R}$ such that the cell problem

$$H(x, y, Dv + p) = \bar{H}(x, p) \tag{1.62}$$

has a periodic viscosity solution v; see [45]. Furthermore, thanks to the very expression of the dynamics in (1.57), we easily see that

$$\begin{gathered}\text{for any } (x, y) \text{ one has} \\ \overline{\text{co}}\{\varphi(x, y)a \mid a \in B\} = \bar{B}(0, \varphi(x, y))\end{gathered} \tag{1.63}$$

Such a strong controllability condition permits to establish that the effective Hamiltonian $\bar{H}$ is Lipschitz–continuous with respect to x (see [2, Proposition 4]). In the next Proposition we give a characterization of the effective Hamiltonian using relaxed controls. Given $p \in \mathbb{R}^N$, Let us denote by $\hat{p}$ the unit vector $p/|p|$.

Proposition 1.35. *For any (x, p) the effective Hamiltonian enjoys the following formula*

$$\bar{H}(x, p) = \bar{\varphi}(x, \hat{p})|p|$$

with

$$\bar{\varphi}(x, \hat{p}) := -\max_{\mu \in Z_l(x)} \hat{p} \cdot \int_{\mathbb{R}^N \times B} \varphi(x, y)a\, \mu(dy, da). \tag{1.64}$$

Furthermore, for any x, p,

$$\bar{\varphi}(x, \hat{p}) \geq \min_y \varphi(x, y)$$

then $\bar{H}$ is coercive.

Proof. By applying Proposition 1.19 we see that $\bar{H}$ is 1-homogeneous in the p variables:

$$\bar{H}(x, p) = \max_{\mu \in Z_l(x)} -p \cdot \varphi^r(x, \mu) =: \bar{\varphi}(x, \hat{p})|p| \tag{1.65}$$

and the maximum is attained at some $\mu_x \in Z_l(x)$, being $Z_l(x)$ compact.

The coercivity of $\bar{H}$ follows by standard calculations involving the comparison principle. Here we propose an alternative proof of this fact, using only the representation formula (1.65) for $\bar{H}$.

Fix x, p and choose $\bar{a} = -\hat{p} \in \bar{B}(0, 1)$. Then consider the solution y_t of the constant controlled equation

$$\dot{y}_s = \varphi(x, y_s)\bar{a}, \quad y_0 = y.$$

Such trajectory generates for some diverging sequence t_n a limiting relaxed control measure we denote by $\bar{\mu}$:

$$\bar{\mu} := \lim_n \frac{1}{t_n} \int_0^{t_n} \delta_{(y_s, \bar{a})} ds, \quad \text{weak star.}$$

Then

$$\lim_n \frac{1}{t_n}\int_0^{t_n} \varphi(x,y_s)\bar{a}ds = \lim_n \varphi^r\left(x, \frac{1}{t_n}\int_0^{t_n} \delta_{(y_s,\bar{a})}ds\right) = \varphi^r(x,\bar{\mu}).$$

We finally get

$$\begin{aligned}\bar{\varphi}(x,\hat{p}) &\geq -\hat{p}\cdot\varphi^r(x,\bar{\mu}) = -\lim_n \hat{p}\cdot\frac{1}{t_n}\int_0^{t_n}\varphi(x,y_s)\bar{a}ds \\ &= \lim_n \frac{1}{t_n}\int_0^{t_n}\varphi(x,y_s)ds \geq \min_y \varphi(x,y) > 0\end{aligned}$$

□

Proof of Theorem 1.34. Record that u_0 is uniformly continuous and bounded and observe that the function $\varphi(x,y)a$ is Lipschitz since φ is so. Then the standing assumptions listed in Section 1.1 are satisfied. Thanks to the particular form of the control system (1.57), bounded time controllability, and then ergodicity, is guaranteed by (1.63). Furthermore, since u_0 is bounded and uniformly continuous, since H is uniformly continuous in p, and $H(y,p,0)=0$ for any (x,y), then (see [3, Proposition 1]) the functions u^ε defined in (1.56) are equibounded. So, by [3, Theorem 1] the upper and lower semilimits of u^ε are respectively subsolution and supersolution of

$$\begin{cases}\partial_t u + \bar{H}(x,Du) = 0 \\ u(0,x) = \bar{u}_0(x)\end{cases} \tag{1.66}$$

Moreover a comparison principle can be established for this problem, because the following condition holds:

$$\begin{gathered}\text{for any } R>0 \text{ there exists a modulus } \omega_R(\cdot) \text{ such that} \\ \text{for any } k>0, \text{ and any } |x|,|x'|<R \\ H(x',y,k(x-x')) \leq H(x,y,k(x-x')) + \omega_R(|x'-x| + k|x'-x|^2)\end{gathered}$$

In fact

$$\begin{aligned}&H(x',y,k(x-x')) - H(x,y,k(x-x')) = \\ &\qquad \max_{a\in A}\varphi(x',y)a\cdot k(x-x') - \max_{a\in A}\varphi(x,y)a\cdot k(x-x') \\ &\quad\leq \varphi(x',y)a'\cdot k(x-x') - \varphi(x,y)a'\cdot k(x-x') \\ &\quad= a'\cdot k(x-x')\left[\varphi(x',y)-\varphi(x,y)\right] \leq k\mathrm{Lip}_R(\varphi)|x-x'|^2\end{aligned}$$

for some $a' \in B$. Then u^ε converges locally uniformly on $(0,T)\times\mathbb{R}^N\times\mathbb{R}^N$ as $\varepsilon\to 0^+$ to the unique solution of (1.66).

The statement follows by Theorem 1.21 provided that the growth condition (1.29) is fulfilled. Since $|a|\leq 1$,

$$|f(x,y,a)| \leq |\varphi(x,y)| \leq |\varphi(0,0)| + \mathrm{Lip}(\varphi)(|x|+|y|) \leq C(1+|x|+|y|)$$

if C is chosen to be $\max\{|\varphi(0,0)|, \mathrm{Lip}(\varphi)\}$.

□

1.5.3 Sign variations in the normal velocity.

When sign variations are allowed in the normal velocity, the problem may become considerably more complicated. In this situation many pathological phenomena may occur, depending on the fact that, in some regions, the front may invert the direction in which it is propagating. For example regions with non–empty interior may appear in the front. In [24, Theorem 4.1], the Authors prove that if the initial front Γ_0 has not interior, and if the normal velocity does not depend on time, then the front Γ_t^ε has empty interior for all t, *i.e.* such *fat* level sets do not appear. Furthermore, in [28] the Authors showed that, in the case in which $\varphi = \varphi(y)$ depends only on the fast states, in order to get a non–stationary homogenized front it is essential to assume φ strictly positive or strictly negative.

Recently, a weak convergence result has been pointed out by Cardaliaguet, Lions and Souganidis in [35] concerning the homogenization of non coercive geometric equations, with some interesting applications to the homogenization of motion of interfaces. They consider the equation

$$\begin{array}{ll} \partial_t u^\varepsilon + v\left(\frac{x}{\varepsilon}\right)|Du^\varepsilon| = 0 & \text{in } (0,+\infty)\times\mathbb{R}^N \\ u^\varepsilon(0,x) = u_0(x) & \text{on } \mathbb{R}^N \end{array}$$

where

$$\begin{array}{c} v:\mathbb{R}^N \to \mathbb{R} \text{ is Lipschitz continuous,} \\ \text{periodic in the unit cube } [0,1]^N \text{ and changes its sign} \end{array}$$

After dividing the unit cube in regions Z_0 where v is null, and $\{Z_i\}_{i\in I}$ the connected components where $v \not\equiv 0$, they prove first that in each Z_i there exists a Lipschitz continuous geometric $\bar{F}_i$ such that, for any $p \in \mathbb{R}^N$, the cell problem

$$v(y)|D\chi(y) + p| = \bar{F}_i(p), \quad \text{in } Z_i$$

has a continuous periodic solution χ. Then, they show that, fixed a uniformly continuous initial value u_0,

$$u^\varepsilon \rightharpoonup \bar{u} := \theta_0 u_0 + \sum_{i\in I}\theta_i \bar{u}_i, \ \textit{weak star} \text{ in } L^\infty_{loc}((0,+\infty)\times\mathbb{R}^N)$$

where

$$\theta_i := \left|Z_i \cap [0,1]^N\right|$$

and $\bar{u}_i$ is the solution of

$$\begin{array}{ll} \partial_t \bar{u}_i + \bar{F}_i(D\bar{u}_i) = 0 & \text{in } (0,+\infty)\times\mathbb{R}^N \\ \bar{u}_i(0,x) = u_0(x) & \text{on } \mathbb{R}^N \end{array}$$

In this section we allow φ to change its sign only on the macroscopic scale and prove the convergence of the moving front to an homogenized interface, and give an interpretation of such effective front in terms of trajectories of an optimal control problem. So, we begin weakening the hypothesis (h3) in Assumption 1.33 in the following

(h3') For any x the function $y \mapsto \varphi(x,y)$ has constant sign.

In this case φ splits $\mathbb{R}^N$ into connected subsets where the sign of φ is constant

$$\Omega^+ = \{x \in \mathbb{R}^N : \varphi(x,y) > 0 \text{ for any } y\}$$

and

$$\Omega^- = \{x \in \mathbb{R}^N : \varphi(x,y) < 0 \text{ for any } y\}$$

Define

$$\mathcal{H}(x,y,p) = \varphi(x,y)|p| = \begin{cases} \max_{|a|\le 1} \varphi(x,y)p \cdot a & \text{in } \Omega^+ \\ \\ -\min_{|a|\le 1} \varphi(x,y)p \cdot a & \text{in } \Omega^- \end{cases}$$

$\mathcal{H}$ (respectively $-\mathcal{H}$) is coercive with respect to p in Ω^+ (respectively in Ω^-); furthermore for any x such that $\varphi(x,y) = 0$, $u^\varepsilon(t,x) = u_0(x)$ for any t. We are interested in the homogenization of the equation

$$u_t + \mathcal{H}\left(x, \frac{x}{\varepsilon}, Du^\varepsilon\right) = 0 \tag{1.67}$$

An important step consists in observing that, loosely speaking, each trajectory of the control system associated to (1.67) starting in Ω^+ (or in Ω^-), stays always in Ω^+ (or in Ω^-); this will be proved in Lemma 1.37. This allows us to treat the phenomenon separately in each connected component of Ω^+ or Ω^-. Observe also that the bounded time controllability condition (1.63) is valid in each of such components, because $\text{co}\{\varphi(x,y)a \,|\, a \in B\} = B(0, |\varphi(x,y)|)$. Then as in the previous section, it is possible to give a control interpretation of the limiting dynamics underlying the motion of the front:

$$\dot{x} \in \varphi^r(x, Z_l(x)), \quad x_0 = x \tag{1.68}$$

φ^r being defined as in (1.59). We pass to the limit in each connected component of Ω^+ and Ω^-, where the effective Hamiltonian $\bar{\mathcal{H}}$ is defined and whose form has been given in Proposition 1.35. The following statement summarizes the previous considerations.

Theorem 1.36. *Let Assumption 1.33 with (h3) replaced by (h3') be satisfied. Then the effective front is represented as the zero level set of the function*

$$\lim_{\varepsilon \to 0^+} u^\varepsilon(t,x) = \bar{u}(t,x) = \begin{cases} \inf \bar{u}_0(x_t) & \text{if } x \in \Omega^+ \\ \sup \bar{u}_0(x_t) & \text{if } x \in \Omega^- \end{cases}$$

where the infimum and the supremum are taken over the solutions of (1.68) *and*

$$\bar{u}_0(x) = \inf_y u_0(x,y).$$

Theorem 1.36 directly follows by Theorem 1.34 provided that the following two lemmata are satisfied. First of all let us show that any trajectory of the control system associated to (1.67) starting in Ω^+, or in Ω^- must remain in Ω^+, or, respectively in Ω^-.

Lemma 1.37. *If* (h3') *holds, then each trajectory of the controlled system associated to* (1.67) *starting in Ω^+ (or Ω^-) cannot reach the boundary of Ω^+ (or Ω^-) in finite time.*

Example 1.38. Let φ be the identic function in $\mathbb{R}$: $\varphi(x) = x$, and consider the system

$$\begin{cases} \dot{x}_t = -x \\ x_0 = 1 \end{cases}$$

It is apparent that if the solution $x_t = e^{-t}$ starts in $\Omega^+ = \{x > 0\}$, or in $\Omega^- = \{x < 0\}$, then $x(t) \neq 0$ for any finite t. So x_t remains in Ω^+ or in Ω^-. □

Proof of Lemma 1.37. Suppose, without loss of generality, that φ splits $\mathbb{R}^N$ in exactly three connected components:

$$\mathbb{R}^N = \Omega^+ \cup \Omega^0 \cup \Omega^-$$

where φ is, respectively, positive, null, and negative. Furthermore, φ is Lipschitz–continuous, so a constant $L = \text{Lip}(\varphi)$ does exist such that

$$|\varphi(x_1, y_1) - \varphi(x_0, y_0)| \leq L(|x_1 - x_0| + |y_1 - y_0|) \tag{1.69}$$

for any (x_0, y_0), (x_1, y_1).

If $x_0 \in \Omega^0$, one has $\varphi(x_0, y) = 0$ for any y, thanks to (h3'). This allows us to obtain by (1.69) an upper bound for the modulus of the normal velocity:

$$|\varphi(x, y)| = |\varphi(x, y) - \varphi(x_0, y)| \leq L|x - x_0| \quad \forall y, \forall x_0 \in \Omega^0 \tag{1.70}$$

Let us denote by $x(t)$ the trajectory we want to observe. So, suppose that $x(\cdot)$ starts for example in Ω^+, and consider the following sequence of open subsets of Ω^+:

$$\Omega_n^+ := \left\{ x \in \Omega^+ : \frac{1}{n} < d(x, \Omega^0) < \frac{1}{n-1} \right\}, \quad n \geq 2$$

Suppose that there exists $N_0 \in \mathbf{N}$ such that,

$$\begin{array}{c} \text{for any } n \geq N_0,\ x(t) \in \Omega_n^+ \\ \text{for } t \text{ belonging to a certain nonempty interval.} \end{array} \tag{1.71}$$

We can prove that the velocity allowed in each Ω_n^+ decreases as n tends to infinity. For any $x \in \Omega^+$, let $\hat{x} := \text{proj}_{\Omega^0} x$ be the point of Ω^0 such that $|\hat{x} - x| = d(x, \Omega^0)$. By (1.70) follows

$$|\varphi(x, y)| \leq L|\hat{x} - x|, \quad \forall y.$$

Since if $x \in \Omega_n^+$ then $|\hat{x} - x| < \frac{1}{n-1}$, we get

$$|\varphi(x, y)| \leq \frac{L}{n-1} \quad \forall x \in \Omega_n^+. \tag{1.72}$$

Finally, let T_n be the time spent by $x(\cdot)$ to pass from Ω_n^+ in Ω_{n+1}^+, *i.e.*

$$T_n := \sup\{t > 0 : x(t) \in \Omega_n^+\} - \sup\{t > 0 : x(t) \in \Omega_{n-1}^+\};$$

observe that this times are well defined, by (1.71). By (1.72) we get the following estimation:

$$T_n \geq \frac{\frac{1}{n-1} - \frac{1}{n}}{\frac{L}{n-1}} = \frac{1}{Ln}$$

We conclude that the time T needed to reach Ω^0 starting from Ω^+ is

$$T > \sum_{n=N_0}^{+\infty} T_n \geq \frac{1}{L} \sum_{n=N_0}^{+\infty} \frac{1}{n} = +\infty$$

and the assertion in proved.

□

We finally check that the effective Hamiltonian is locally Lipschitz in the whole $\mathbb{R}^N$.

Lemma 1.39. *For any $R > 0$ there exists $C_R > 0$ such that for any $|x_1|, |x_2| < R$*

$$|\bar{\mathcal{H}}(x_1, p) - \bar{\mathcal{H}}(x_2, p)| \leq C_R |p| |x_1 - x_2|$$

for any p.

Proof. If $B(0, R) \subset \Omega^+$ or $B(0, R) \subset \Omega^-$ the assertion follows by the Lipschitz continuity of $\bar{\mathcal{H}}$ in Ω^+ and Ω^+. We focus our attention on the case $B(0, R) \cap \Omega^+ \neq B(0, R)$; so there are $x^+ \in \Omega^+$ and $x^- \in \Omega^-$ with $|x^+|, |x^-| < R$. In force of Proposition 1.35 one has,

$$|\bar{\mathcal{H}}(x^+, p) - \bar{\mathcal{H}}(x^-, p)| = \left|\bar{\varphi}(x^+, \hat{p}) - \bar{\varphi}(x^-, \hat{p})\right| |p|$$

and

$$\bar{\varphi}(x^\pm, \hat{p}) = -\int_{\mathbb{R}^N \times B} \varphi(x^\pm, y) \hat{p} \cdot a \mathrm{d}\mu^\pm(y, a)$$

for some $\mu^\pm \in Z_l(x^\pm)$. Therefore

$$\begin{aligned}
\left|\bar{\varphi}(x^+, \hat{p}) - \bar{\varphi}(x^-, \hat{p})\right| &\leq \int_{\mathbb{R}^N \times B} \left|\varphi(x^+, y)\right| d\mu^+ + \int_{\mathbb{R}^N \times B} \left|\varphi(x^-, y)\right| d\mu^- \\
&= \int_{\mathbb{R}^N \times B} \varphi(x^+, y) d\mu^+ + \int_{\mathbb{R}^N \times B} -\varphi(x^-, y) d\mu^- \\
&\leq \int_{\mathbb{R}^N \times B} \left(\varphi(x^+, y) - \varphi(x^-, y)\right) d\mu^+ \\
&\quad + \int_{\mathbb{R}^N \times B} \left(\varphi(x^+, y) - \varphi(x^-, y)\right) d\mu^- \\
&= \int_{\mathbb{R}^N \times B} \left|\varphi(x^+, y) - \varphi(x^-, y)\right| d\mu^+ \\
&\quad + \int_{\mathbb{R}^N \times B} \left|\varphi(x^+, y) - \varphi(x^-, y)\right| d\mu^- \\
&\leq 2 \operatorname{Lip}_R(\varphi) |x^+ - x^-|
\end{aligned}$$

where $\operatorname{Lip}_R(\varphi)$ is the Lipschitz constant of φ on $B(0, R)$. The assertion follows.

□

Chapter 2

Asymptotic Controllability of singularly perturbed control systems

This Chapter is devoted to controllability questions. We concentrate on the following singularly perturbed system

$$\begin{aligned} \dot{x}_t &= f(x_t, y_t, a_t), \quad x_0 = x \\ \varepsilon \dot{y}_t &= g(x_t, y_t, a_t), \quad y_0 = y \end{aligned} \tag{2.1}$$

Let us recall here for reader convenience the standing assumption:

– $x \in \mathbb{R}^N$ and y belongs to the flat torus $\mathbb{T}^M \simeq \mathbb{R}^M/\mathbb{Z}^M$;

– A is a compact metric space, and the control functions are measurable functions $a : \mathbb{R}^+ \to A$;

– the functions f and g are Lipschitz–continuous from $\mathbb{R}^N \times \mathbb{T}^M \times A$ to $\mathbb{R}^N$ and $\mathbb{T}^M$ respectively;

– the problem is controllable in the fast variables y, in the following sense:

$$\exists r > 0 \text{ such that } B(0, r) \subset \overline{\text{co}}\{g(x, y, a) : a \in A\} \tag{2.2}$$

Note that in this section we require y to lie on the flat torus, instead of the periodicity assumption on the given functions.

Besides (2.1) we take into account the limiting system

$$\dot{x}_t \in F^r(x_t), \qquad x_0 = x \tag{2.3}$$

where

$$F^r(x) := f^r(x, Z_l(x))$$

$Z_l(x)$ is the set of limiting relaxed controls, *i.e.* the set of weak–star limits of occupational measures of the fast subsystem of (2.1); see Sections 1.2.2 and 1.2.3. Recall that f^r is obtained by *relaxing* f as explained in Section 1.3. We previously investigated relations between these two system, and concluded that (2.3) captures the limit behavior of (2.1), as ε tends to zero; see Section 1.3. Now we want to establish if stability properties can be deducted for the limit system, assuming that the singularly perturbed one is stable and, more significantly, vice versa.

Stability of such systems is studied by means of Lyapunov pairs, *i.e.* a *Lyapunov function* and another *rate function* estimating the infinitesimal decrease of the Lyapunov function along the integral trajectories of the dynamics driving to a certain target, that we suppose for simplicity to be the origin. We characterize such a monotonicity property with a suitable differential inequality interpreted in viscosity sense.

Let us write for later use the Hamiltonian related to the control problem (2.1) with running cost $W(x)$, that is

$$H(x, y, p, q) := \max_{a \in A}\{-p \cdot f(x, y, a) - q \cdot g(x, y, a)\} - W(x) \tag{2.4}$$

and recall that the effective Hamiltonian related to H admits the following *relaxed* representation (see Proposition 1.19)

$$\bar{H}(x, p) = \max_{\mu \in Z_l(x)}\{-p \cdot f^r(x, \mu)\} - W(x), \quad \text{for any } x, p \tag{2.5}$$

i.e. it coincides with the Hamiltonian associated to the limit control problem (2.3).

We proceed as follows. In the next Section 2.1 we show that the property of being a Lyapunov function is stable with respect to weak semilimits. More precisely we prove that the lower semilimit of Lyapunov functions for (2.1) is a lower semicontinuous Lyapunov function for (2.3). To establish such a result we need to ensure, under suitable assumptions, some properties for the lower semilimit of viscosity supersolutions, and to show its decrease along trajectories of the limiting dynamics. To this scope we exploit the convergence result [3, Theorem 1].

In Section 2.2 we invert such implication: we shall prove that the asymptotic stability to the origin of the limiting system implies the asymptotic stability of the slow dynamics of the singularly perturbed system *near* the origin, in a sense that will be clear in Definition 2.21. Similar results have been already pointed out in the light of the Levinson–Tikonov order reduction theory; see the book by Kokotović *et al.* [60] for uncontrolled systems, and, for singularly perturbed systems, Christofides and Teel [40], and Teel, Moreau and Nešić [76].

A result in this direction was also established by Artstein in [12] for uncontrolled systems. In that work, the limit system is detected as the so-called *chattering limit*, that is, the differential inclusion

$$\dot{x} \in F(x)$$

with

$$F(x) := \left\{ \int f(x, y) d\mu(y) : \mu \in I(x) \right\}$$

and $I(x)$ is the set of the invariant probability measures of the differential equation $\dot{y} = g(x, y)$ where x is frozen. The main result in [12] asserts that if the equilibrium is asymptotically stable for the limit differential inclusion, then the singularly perturbed system is asymptotically stable *near* the origin. Even if the way to obtain the limiting system proposed in [12] is actually close to the one adopted here, the arguments used are completely different: techniques connected to the theory of invariant measures and occupational measures are used in [12], while we will exploit techniques from viscosity solution theory. More precisely, we prove that it is possible to construct a Lyapunov function for the singularly perturbed system as an ε–perturbation of a given Lyapunov function for the limiting system. Such a result has also an intrinsic theoretical interest, in fact the construction of such a Lyapunov function, pertains to the possibility of constructing local supersolutions of Hamilton–Jacobi equations, starting from a supersolution of the effective equation.

2.1 LSC Control Lyapunov Functions for the Limit Control Problem

The goal of this Section is to prove that the lower semilimit of control Lyapunov functions for (2.1) is as well a Lyapunov function for the limit system. To this scope we assume the system (2.7) to be asymptotically controllable, or equivalently, that a continuous *Lyapunov pair* for it does exist, for any ε. Before to define Lyapunov pairs, let us recall first the following definition.

Definition 2.1. *A* Control Lyapunov function *for the control system* (2.1) *with respect to the target* $(0,0)$ *is a continuous function* $V^\varepsilon : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^+$ *satisfying the following properties:*

1. Positive Definiteness. $V^\varepsilon(x,y) \geq 0$ *for any* (x,y)*, and* $V^\varepsilon(x,y) = 0$ *if and only if* $(x,y) = (0,y^*)$ *for a certain* $y^* \in \mathbb{T}^M$*;*
2. Properness. *The sublevel sets* $\{(x,y) : V^\varepsilon(x,y) \leq \lambda\}$ *are bounded for any positive* λ*, or, equivalently,*
$$V^\varepsilon(x,y) \to +\infty \text{ as } |x| \to +\infty, \quad \text{for any } y, \text{ any } \varepsilon \tag{2.6}$$
3. Infinitesimal decrease.
$$\begin{gathered} \text{for any } x \neq 0 \text{ and any } (p,q) \in J^- V^\varepsilon(x,y) \\ \min_{a \in A}\{p \cdot f(x,y,a) + q \cdot \varepsilon^{-1} g(x,y,a)\} \leq 0 \end{gathered} \tag{2.7}$$

Definition 2.2. *A* Control Lyapunov pair *for the control system* (2.1) *with respect to the target* $(0,0)$ *consists in a couple of functions* $(V^\varepsilon, W^\varepsilon)$*, where*

1. V^ε *is a Control Lyapunov function;*
2. Rate function. $W^\varepsilon : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^+$ *is a continuous and positive definite function such that*
$$\begin{gathered} \text{for any } x \neq 0 \text{ and any } (p,q) \in J^- V^\varepsilon(x,y) \\ \min_{a \in A}\{p \cdot f(x,y,a) + q \cdot \varepsilon^{-1} g(x,y,a)\} \leq -W^\varepsilon(x,y) \end{gathered} \tag{2.8}$$

Definition 2.3. *A* LSC Control Lyapunov Function *for the control system* (2.3) *with respect to the target* $x = 0$ *is a lower semicontinuous function* $V : \mathbb{R}^N \to \mathbb{R}^+$ *satisfying the following properties:*

1. *V is continuous at 0 and positive definite, that is $V(0) = 0$ and $V(x) > 0$ otherwise;*

2. *V is proper, that is $V(x)$ tends to $+\infty$ as $|x| \to +\infty$ or, equivalently, the sublevel sets $\{x : V(x) \leq \lambda\}$ are bounded for any positive λ;*

3. *for any $x \neq 0$, and any $p \in J^- V(x)$*

$$\min_{\mu \in Z_l(x)} p \cdot f^r(x, \mu) \leq 0$$

Definition 2.4. *A* LSC Control Lyapunov pair *for the control system* (2.3) *with respect to the target $x = 0$ consists in a couple of functions (V, W), where*

1. *V is a LSC Control Lyapunov function;*

2. Rate function. *$W : \mathbb{R}^N \to \mathbb{R}^+$ is a continuous and positive definite such that for any $x \neq 0$, and any $p \in J^- V(x)$*

$$\min_{\mu \in Z_l(x)} p \cdot f^r(x, \mu) \leq -W(x)$$

We assume, for any $\varepsilon > 0$ the existence of a control Lyapunov pair for the singular perturbation system (2.1), satisfying the following assumptions:

Assumption 2.5. i. The functions V^ε are locally equibounded, equicontinuous in $x = 0$ uniformly with respect to y;

ii. A continuous and positive definite function $W(x)$ does exist such that, for any x,
$$\inf_\varepsilon W^\varepsilon(x, y) \geq W(x) \text{ for any } y.$$

iii. A continuous positive definite and proper function $V(x)$ does exist such that, for any x,
$$\inf_\varepsilon \inf_y V^\varepsilon(x, y) \geq V(x)$$

Remark 2.6. The previous hypotheses express a sort of *equi-stability* of the perturbed system. In the next section we will construct a control Lyapunov pair for the system (2.1) as a perturbation of a control Lyapunov pair for the limiting dynamics, assuming it to be controllable to the origin. It is useful to observe now that such control Lyapunov pair will turns out to satisfy the conditions in Assumption 2.5. In Corollary 2.17 we will summarize the relationship between equi-stable singularly perturbed system and its dynamics limit. □

Let us recall the definition and some property of lower semilimits.

Definition 2.7. *Let be $\Omega \subset \mathbb{R}^N$ and $\varepsilon > 0$. For the functions $\varphi_\varepsilon : \Omega \to \mathbb{R}$, the* lower semilimit *in Ω as $\varepsilon \to 0^+$ is the function $\varphi_* : \Omega \to \mathbb{R}^N$ defined as*

$$\varphi_*(x) := \sup_{\delta > 0} \inf\{\varphi_\varepsilon(z) : z \in \Omega, |x - z| < \delta \text{ and } 0 < \varepsilon < \delta\}$$

or, more concisely

$$\varphi_*(x) = \liminf_{\substack{z \to x \\ \varepsilon \to 0^+}} \varphi_\varepsilon(z)$$

From the definition directly follows (see [22, Lemma 1.5 in Chapter V]) that the lower semilimit is lower semicontinuous. The following Proposition is a straightforward PDE interpretation of the decreasing property (2.8).

Proposition 2.8. *Under Assumption 2.5, the control Lyapunov functions V^ε are viscosity supersolutions of*

$$H(x, y, D_x V^\varepsilon(x, y), \varepsilon^{-1} D_y V^\varepsilon(x, y)) \geq 0 \quad \text{in } \mathbb{R}^N \times \mathbb{T}^M$$

where H is defined in (2.4).

The following statements provide some properties of the lower semi-limit; these properties permit to recognize that V_* is a LSC control Lyapunov function for the limiting system with respect to the origin. The first of these properties follows from the general convergence result for viscosity supersolutions of Hamilton–Jacobi equations [3, Theorem 1], and by the relaxed representation we recalled before in (2.5).

Proposition 2.9. *The lower semilimit $V_*(x)$ of $V^\varepsilon(x, y)$,* i.e. *the function*

$$V_*(x) = \liminf_{\substack{z \to x \\ \varepsilon \to 0^+}} \inf_y V^\varepsilon(z, y)$$

satisfies

$$\bar{H}(x, D_x V_*) \geq 0 \quad \text{in } \mathbb{R}^N$$

in viscosity sense, i.e. *it satisfies the following inequality:*

$$\min_{\mu \in Z_l(x)} p \cdot f^r(x, \mu) \leq -W(x) \quad \text{for any } p \in J^- V_*(x)$$

Other properties of V_* are detailed in the following Proposition.

Proposition 2.10. *Under Assumption 2.5 the lower semilimit V_* is lower semicontinuous, positive definite, proper and is continuous in 0.*

Proof. The lower semicontinuity of the lower semilimit is a straightforward consequence of the definition; a detailed proof can be found for instance in [22, Lemma 1.5 in Chapter V]. The continuity in $x = 0$ easily follows by virtue of the equi-continuity assumption on V^ε. Therefore only the positive definiteness and the properness remain to be proved. The short notation $\widetilde{V}^\varepsilon(x)$ shall be used in the follows to denote $\inf_y V^\varepsilon(x, y)$.

Let's establish first the positive definiteness. Observe preliminarily that, since $V^\varepsilon(x, y) \geq 0$, then $\widetilde{V}^\varepsilon(x) \geq 0$ for any x. So

$$V_*(x) = \sup_{\delta > 0} \inf\{\widetilde{V}^\varepsilon(x') : |x' - x| < \delta,\ 0 < \varepsilon < \delta\} \geq 0$$

Moreover $V_*(x) = 0$ iff $x = 0$. In fact, for any $\beta > 0$ there exists $\bar{\delta} > 0$ such that

$$\begin{aligned} 0 \leq V_*(0) &= \sup_{\delta > 0} \inf\{\widetilde{V}^\varepsilon(x') : |x'| < \delta,\ 0 < \varepsilon < \delta\} \\ &\leq \inf\{\widetilde{V}^\varepsilon(x') : |x'| < \bar{\delta},\ 0 < \varepsilon < \bar{\delta}\} - \beta = -\beta \end{aligned}$$

then $V_*(0) = 0$ by the arbitrariness of β.

Suppose now that there exists $x' \neq 0$ such that $V_*(x') = 0$, and set $\bar{\delta} := \frac{1}{2}|x'|$. We obtain a contradiction, in fact, by Assumption 2.5:

$$\begin{aligned} 0 = V_*(x') &= \sup_{\delta>0} \inf\{\widetilde{V}^\varepsilon(z) : |z - x'| < \delta,\, 0 < \varepsilon < \delta\} \\ &\geq \inf\{\widetilde{V}^\varepsilon(z) : |z - x'| < \bar{\delta},\, 0 < \varepsilon < \bar{\delta}\} > 0. \end{aligned}$$

In order to prove the properness of $V_*(x)$ suppose by contradiction that

$$\liminf_{|x|\to+\infty} V_*(x) < M < +\infty \tag{2.9}$$

and let x_n be a diverging sequence such that

$$\liminf_{|x|\to+\infty} V_*(x) = \lim_n V_*(x_n) \tag{2.10}$$

For any n one has

$$V_*(x_n) = \liminf_{\substack{z\to x_n \\ \varepsilon\to 0^+}} \widetilde{V}^\varepsilon(z) = \lim_k \widetilde{V}^{\varepsilon_k}(z_{k,n})$$

for a certain sequence $z_{k,n}$ converging to x_n as k goes to $+\infty$. Take the diagonal subsequence $\xi_k := z_{k,k}$; from (2.9) and (2.10) we get

$$\lim_k \widetilde{V}^{\varepsilon_k}(\xi_k) = \liminf_{|x|\to+\infty} V_*(x) < M < \infty$$

then for k large enough, $\widetilde{V}^{\varepsilon_k}(\xi_k) < 2M$, and this violates (2.6). In fact, one has

$$\widetilde{V}^{\varepsilon_k}(\xi_k) = \inf_y V^{\varepsilon_k}(\xi_k, y) = V^{\varepsilon_k}(\xi_k, y_k)$$

for some y_k, and $V^{\varepsilon_k}(\xi_k, y_k)$ tends to $+\infty$ as $k \to +\infty$, since $|\xi_k| \to +\infty$.

□

In conclusion, we can summarize the results of the preceding Propositions in the following

Theorem 2.11. *Let* $(V^\varepsilon, W^\varepsilon)$ *be a control Lyapunov pair for the singularly perturbed system* (2.1) *satisfying Assumption 2.5. Then the lower semilimit* V_* *is a LSC control Lyapunov function for the limit problem* (2.3)*, with rate function* W *(appearing in Assumption 2.5).*

2.2 Controllability of the ε–problem

In this section we prove that around the equilibrium, it is enough to look at the (smaller) limit system, to infer the controllability of the slow flow of the original perturbed system.

Heuristics and formal expansions. The ideas presented here arise with asymptotic expansions in multiple scale problems (see [30]) and deal with the perturbed test function method (see [44]).

In order to explain our ideas in an informal manner, suppose to posses a *smooth* (say C^1) solution $\bar{u}$ of the effective Hamiltonian. Then

$$\bar{H}(x, D\bar{u}(x)) = 0 \text{ in } \mathbb{R}^N$$

For any fixed point $\bar{x}$, let $\chi(y)$ be the *first corrector*, *i.e.* the solution of the *cell problem*:

$$H(\bar{x}, y, Du(\bar{x}), D_y\chi) = \bar{H}(\bar{x}, Du(\bar{x}))$$

and set

$$\varphi^\varepsilon(x, y) := \bar{u}(x) + \varepsilon\chi(y).$$

This function is a first order perturbation in ε of the effective solution, and should solve the original Hamilton equation:

$$H(x, y, D_x\varphi^\varepsilon, \varepsilon^{-1}D_y\varphi^\varepsilon) = H(x, y, Du, D_y\chi) = \bar{H}(\bar{x}, Du(\bar{x})) = 0$$

at $x = \bar{x}$.

Such arguments have been rigorously developed in [2] and [3] to establish the local uniform convergence of solutions of first and second order Hamilton–Jacobi equations to the solution of the effective equation. Now we want to *reverse* the construction, and some difficulties arise, more or less evidently. An intrinsic obstruction is the fact that the function χ depends not only on y but also on x. Therefore in the previous computation, in place of $D_x\varphi^\varepsilon$ should appear also a contribution of χ, that we are not able to manage, being the dependence on x of χ not clear.

Rigorous calculations. First of all we show in the next Lemma that the an homogeneous Hamiltonian remains ergodic if add to it a continuous cost independent by the fast flow.

Set

$$H_0(x, y, p, q) := \max_{a \in A}\{-p \cdot f(x, y, a) - q \cdot g(x, y, a)\}$$

(it is the homogeneous part of $H(x, y, p, q)$ defined in (2.4)). As usual in our terminology, we say that H_0 is ergodic at $x = \bar{x}$, $p = \bar{p}$, if

$$\begin{aligned} \delta w_\delta + H_0(\bar{x}, y, \bar{p}, Dw_\delta) = 0 \text{ in } \mathbb{R}^M \quad &\text{implies} \\ \delta w_\delta(y) \to \mathit{const} =: -\bar{H}_0(\bar{x}, \bar{p}), \quad \text{as } \delta \to 0^+, \quad &\text{uniformly in } y. \end{aligned} \tag{2.11}$$

Notation. With the symbol $H_W(x, y, p, q)$ we will denote the Hamiltonian obtained by H_0 via the following formula:

$$H_W(x, y, p, q) := H_0(x, y, p, q) - W(x)$$

Lemma 2.12. *Let $W : \mathbb{R}^N \to \mathbb{R}^+$ be a continuous function. If H_0 is ergodic then $H_W(x, y, p, q)$ is ergodic at any (x, p) and $\bar{H}_W(x, p) = \bar{H}_0(x, p) - W(x)$.*

Proof. Since H_0 is ergodic (2.11) holds. Let w'_δ be a solution of

$$0 = \delta w'_\delta + H_W(x, y, p, Dw'_\delta) = \delta\left(w'_\delta - \frac{W(x)}{\delta}\right) + H_0(x, y, p, Dw'_\delta)$$

By uniqueness,

$$w'_\delta - \frac{W(x)}{\delta} = w_\delta$$

where w_δ is the function appearing in (2.11). So

$$\delta w'_\delta - W(x) = \delta w_\delta \to -\bar{H}_0(x,p), \quad \text{as } \delta \to 0^+.$$

□

Definition 2.13. *We say that the perturbed control system* (2.1) has an ergodic fast subsystem *if* $H_0(x,y,p,q)$ *is ergodic,* i.e. *if* (2.11) *holds.*

Such a property holds for example if the fast variables are bounded time controllable, in the sense of (2.2). Ergodicity holds true also under coercivity assumption for the Hamiltonian, or under non–resonance assumptions. (See [3], [10])

Recall that, by Proposition 1.19, that is Theorem 7 in [2], the effective Hamiltonian $\bar{H}_0$ admits the following representation:

$$\bar{H}_0(x,p) = \max_{\mu \in Z_l(x)} \{-p \cdot f^r(x,\mu)\} \tag{2.12}$$

We suppose that the limit system, governed by the differential inclusion (2.3), is *asymptotically controllable* to the state $x = 0$, in a certain open bounded set $\mathcal{O} \subset \mathbb{R}^N$ containing the origin. We know that this implies the existence of a control Lyapunov pair (V, W) (see, for example, [37]). Moreover, by Rifford [69], we can assume without loss of generality V locally Lipschitz.

Assumption 2.14. There exists a Control Lyapunov pair (V, W) for the limit system (2.3) with V Lipschitz–continuous in $\bar{\mathcal{O}}$.

Definition 2.15. *Let $\rho > 0$ be fixed. A couple $(V^\varepsilon, W^\varepsilon)$ of functions $V^\varepsilon(x,y)$: $\bar{\mathcal{O}} \times \mathbb{T}^M \to \mathbb{R}^+$, $W^\varepsilon(x,y) : \bar{\mathcal{O}} \times \mathbb{T}^M \to \mathbb{R}^+$ is called a* (local) Control Lyapunov pair *for the system* (2.1) *in the region*
$\Omega := (\mathcal{O} \setminus B(0,\rho)) \times \mathbb{T}^M$ with respect to the target $\{0\} \times \mathbb{T}^M$ if:

1. *V^ε is a Control Lyapunov function;*
2. *it enjoys the differential inequality*

$$H_0(x,y,D_x V^\varepsilon(x,y), \varepsilon^{-1} D_y V^\varepsilon(x,y)) \geq W^\varepsilon(x,y) \quad \text{in } \Omega \tag{2.13}$$

   in viscosity sense.

The main result of the Section is the following

Theorem 2.16. *Assume that* (2.1) *has an ergodic fast subsystem, and that Assumption 2.14 holds. Then, for any $\rho > 0$ and any $\alpha \in (0,1)$ there exists a continuous function $\chi : \mathbb{T}^M \to \mathbb{R}$ such that $(V(x) + \varepsilon\chi(y), \alpha W(x))$ is a control Lyapunov pair for the singularly perturbed system* (2.1)*, in $\Omega := (\mathcal{O} \setminus B(0,\rho)) \times \mathbb{T}^M$, with respect to the target $\{0\} \times \mathbb{T}^M$.*

Note that, as announced in Remark 2.6, the control Lyapunov pair $(V^\varepsilon, W^\varepsilon)$ $:= (V(x) + \varepsilon\chi(y), \alpha W(x))$ satisfies the hypotheses of Assumption 2.5. Joining together the statements of Theorem 2.11 and Theorem 2.16 we derive the following

Corollary 2.17. *(i) If there exists a control Lyapunov pair (V, W) for the limit system* (2.3)*, with V locally Lipschitz, then there exists a control Lyapunov pair $(V^\varepsilon, W^\varepsilon)$ for the system* (2.1) *satisfying Assumption 2.5.*

(ii) If there exists a control Lyapunov pair $(V^\varepsilon, W^\varepsilon)$ for the system (2.1) *satisfying Assumption 2.5, then there exists a control Lyapunov pair (V, W) for the limit system* (2.3)*.*

Let us consider, for any $\rho > 0$ and any $\alpha \in (0, 1)$ the following auxiliary Hamiltonian:

$$K_\rho(y, q) := \inf\{H_0(x, y, p, q) - \alpha W(x) : \\ x \in \mathcal{O} \setminus B(0, \rho),\, p \in J^- V(x)\} \tag{2.14}$$

Remark 2.18. Observe that the infimum in (2.14) is over a nonempty set, because $\{x : J^- V(x) \neq \emptyset\}$ is dense in $\mathbb{R}^N$ (more precisely, under the current assumptions DV exists almost everywhere) and $\mathcal{O}$ is open. Furthermore

$$\begin{aligned} H_0(x, y, p, q) - \alpha W(x) &\geq -\alpha \max_{\bar{\mathcal{O}}} W - |q| \max_{\bar{\mathcal{O}} \times \mathbb{T}^M \times A} |g| - |p| \max_{\bar{\mathcal{O}} \times \mathbb{T}^M \times A} |f| \\ &\geq C > -\infty \quad \text{for any fixed } q \end{aligned}$$

because $|p| \leq \mathrm{Lip}_{\bar{\mathcal{O}}}(V)$. Then $-\infty < K_\rho(y, q) < +\infty$ for any (y, q). □

As announced in the Introduction, the proof of the Theorem 2.16 is based essentially on the construction of a strict supersolution for the auxiliary Hamiltonian K_ρ. By the very definition of K_ρ such a function will not depend on x.

Proposition 2.19. *For any $\rho > 0$ there exist $\gamma > 0$ and a continuous function $\chi : \mathbb{T}^M \to \mathbb{R}$ such that*

$$K_\rho(y, D\chi(y)) > \gamma \text{ in } \mathbb{T}^M$$

in viscosity sense.

The proof of Proposition 2.19 uses the following Lemma; we will prove it at the end of the section.

Lemma 2.20. *For any $\rho > 0$, and any $\delta > 0$ there exists a unique viscosity solution w_δ of the problem*

$$\delta w_\delta + K_\rho(y, Dw_\delta) = 0 \text{ in } \mathbb{T}^M. \tag{2.15}$$

Proof of Proposition 2.19. The idea is that the desired function is w_δ (the solution of (2.15)), for δ small enough. Fix $\alpha \in (0, 1)$ and

$$0 < \beta < \frac{(1 - \alpha)}{4} \min_{\bar{\mathcal{O}} \setminus B(0,\rho)} W \tag{2.16}$$

and find $\bar{x} \in \bar{\mathcal{O}}$ with $|\bar{x}| \geq \rho$ and $\bar{p} \in J^- V(\bar{x})$ such that

$$H_0(\bar{x}, y, \bar{p}, q) - \alpha W(\bar{x}) \leq K_\rho(y, q) + \beta, \quad \text{for any } y, q.$$

Then, in $\mathbb{T}^M$, one has

$$0 = \delta w_\delta + K_\rho(y, Dw_\delta) \geq \delta w_\delta + H_0(\bar{x}, y, \bar{p}, Dw_\delta) - \alpha W(\bar{x}) - \beta.$$

Let now v_δ be a solution of

$$\delta v_\delta + H_{\alpha W}(\bar{x}, y, \bar{p}, Dv_\delta) - \beta = 0.$$

By comparison principle, being w_δ a subsolution of the latter equation, we have $w_\delta \leq v_\delta$ in $\mathbb{T}^M$. Moreover, by definition,

$$\lim_{\delta \to 0^+} \delta v_\delta - \beta = -\bar{H}_{\alpha W}(\bar{x}, \bar{p}).$$

Then, if δ is small enough, say less than a certain $\bar{\delta} > 0$, we have

$$\delta w_\delta \leq \delta v_\delta \leq 2\beta - \bar{H}_{\alpha W}(\bar{x}, \bar{p}). \tag{2.17}$$

Since (V, W) is a control Lyapunov pair for (2.3) we have

$$\bar{H}_0(x, p) - W(x) \geq 0, \quad \text{for any } x \in \bar{\mathcal{O}}, \text{ any } p \in J^- V(x).$$

So, by Lemma 2.12, for $\bar{x}$, $\bar{p} \in J^- V(\bar{x})$

$$\begin{aligned} \bar{H}_{\alpha W}(\bar{x}, \bar{p}) &= \bar{H}_0(\bar{x}, \bar{p}) - \alpha W(\bar{x}) \\ &\geq (1-\alpha) W(\bar{x}) \geq (1-\alpha) \min_{\bar{\mathcal{O}} \setminus B(0,\rho)} W =: 2\gamma \end{aligned}$$

In conclusion, we have by (2.16) $\beta < \frac{\gamma}{2}$, and by (2.17), for $\delta < \bar{\delta}$, $\delta w_\delta \leq 2\beta - 2\gamma < -\gamma$, that is

$$K_\rho(y, Dw_\delta) = -\delta w_\delta > \gamma$$

as desired.

□

Proof of Theorem 2.16. Let $\chi(\cdot)$ and γ be the continuous function and the positive parameter provided by Proposition 2.19. In view of the regularity of V and χ, $V^\varepsilon(x, y) := V(x) + \varepsilon \chi(y)$ is continuous with respect to y on $\mathbb{T}^M$, and locally Lipschitz with respect to x. The properness also follows from the one of V.

The function χ is defined up to an additive constant. Then we can substitute it with the function $\widetilde{\chi} := \chi - \chi(y^*)$, with $y^* \in \arg\min \chi$. In this way, the function $V^\varepsilon(x, y) := V(x) + \varepsilon \widetilde{\chi}(y)$ is positive for any (x, y), and is zero at $(0, y^*)$; this guarantees the positive definiteness of V^ε for any ε.

Finally, we have to prove that V^ε satisfies

$$\begin{gathered} H_0(x, y, p, \varepsilon^{-1} q) \geq \alpha W(x) \\ \text{for any } (x, y) \in (\mathcal{O} \setminus B(0, \rho)) \times \mathbb{T}^M, \text{ and any } (p, q) \in J^- V^\varepsilon(x, y) \end{gathered} \tag{2.18}$$

More precisely, recalling that $J^- V^\varepsilon(x, y) \subseteq J^- V(x) \times \varepsilon J^- \chi(y)$, we prove that the latter inequality holds for any $p \in J^- V(x)$ and any $q \in \varepsilon J^- \chi(y)$. In fact, for any $(x, y) \in (\mathcal{O} \setminus B(0, \rho)) \times \mathbb{T}^M$ and any $p \in J^- V(x)$ we have

$$H_0(x, y, p, \varepsilon^{-1} q) - \alpha W(x) \geq K_\rho(y, \varepsilon^{-1} q) > \gamma > 0, \quad \text{for any } q \in \varepsilon J^- \chi(y)$$

since χ satisfies, in viscosity sense, the differential inequality $K_\rho(y, D\chi) > \gamma$. □

We complete this section whit the proof of Lemma 2.20; it follows by standard arguments as comparison principles and stability for viscosity solutions.

Proof of Lemma 2.20. (*Uniqueness*). For any ρ, K_ρ inherits from H the regularity required for the comparison (see User's guide [39, Theorem 3.3]). More precisely, the inequality

$$K_\rho(y_1, k(y_2 - y_1)) - K_\rho(y_2, k(y_2 - y_1)) \leq \omega(|y_2 - y_1| + k|y_2 - y_1|^2) \qquad (2.19)$$

is valid for any $y_1, y_2 \in \mathbb{T}^M$, and any $k > 0$, using the same modulus ω ensuring a similar inequality for H. In fact, for any $\beta > 0$ there exist $\bar{x} \in (\mathcal{O} \setminus B(0, \rho)) \times \mathbb{T}^M$ and $\bar{p} \in J^- V(\bar{x})$ such that

$$K_\rho(y_2, k(y_2 - y_1)) + \beta \geq H_0(\bar{x}, y_2, \bar{p}, k(y_2 - y_1)) - \alpha W(\bar{x})$$

Then

$$\begin{aligned}
& K_\rho(y_1, k(y_2 - y_1)) - K_\rho(y_2, k(y_2 - y_1)) \\
\leq \; & K_\rho(y_1, k(y_2 - y_1)) - H_0(\bar{x}, y_2, \bar{p}, k(y_2 - y_1)) + \alpha W(\bar{x}) + \beta \\
\leq \; & H_0(\bar{x}, y_1, \bar{p}, k(y_2 - y_1)) - H_0(\bar{x}, y_2, \bar{p}, k(y_2 - y_1)) + \beta \\
\leq \; & \omega(|y_2 - y_1| + k|y_2 - y_1|^2) + \beta
\end{aligned}$$

the last inequality holds thanks to the expression of H_0, and to hypothesis made on the data f and g. (2.19) follows by the arbitrariness of β.

2. (*Existence*). The existence can be showed using the Perron's method. Since the fast variables lie on the flat torus, $|K_\rho(y, 0)|_\infty \leq C$. Then the constant functions $-C/\delta$ and C/δ are respectively sub– and super–solution of (2.15). Let's consider the function w_δ defined on $\mathbb{T}^M$ via the formula

$$w_\delta(\cdot) := \{-C/\delta \leq w(\cdot) \leq C/\delta : \; w^* \text{ is a subsolution of (2.15)}\}$$

where w^* stands for the upper semicontinuous envelop of w, that is defined by

$$w^*(y) := \limsup_{\eta \to 0^+} \{w(z) : \; |y - z| \leq \eta\}$$

the lower semicontinuous envelop w_* is defined analogously. The Perron's method showed in the User's Guide [39, Theorem 4.1] ensures that $w_\delta(\cdot)$ is a solution.

On the other hand w_δ^* and $w_{\delta *}$ are are still sub– and super–solution of the equation in (2.15), as proved in [39, Lemma 4.2]. Therefore, by comparison

$$w_\delta^* \leq w_{\delta *};$$

the opposite inequality is apparent:

$$w_\delta^* \geq w_\delta \geq w_{\delta *};$$

then w_δ is continuous and solves the equation, and so is the unique viscosity solution.

□

Dynamical counterpart. Let us denote by $(x_t^\varepsilon(x,y,a), y_t^\varepsilon(x,y,a))$ the solution of the singularly perturbed problem (2.1) with initial state (x_0, y_0). We will write also simply (x_t, y_t), when no ambiguity can arise.

Since the functions V^ε constructed in Theorem 2.16 do not satisfy (2.13) in a whole neighborhood of $\{0\} \times \mathbb{T}^M$, controllability properties for the ε–problem cannot be directly derived by Lyapunov theorems. Anyway a sort of asymptotic controllability *near* the origin for the slow flow can be established, as a consequence of Theorem 2.16. In the sequel we will consider as a target for the singular perturbation system, the following closed neighborhood of $\{0\} \times \mathbb{T}^M$:

$$\mathcal{T}_\rho := \bar{B}(0,\rho) \times \mathbb{T}^M, \quad \rho > 0 \text{ fixed}$$

We will take into account also the value function $T^\rho(x,y)$ of the problem with exit time, that is

$$T^\rho(x,y) := \inf_{a \in \mathcal{A}} \int_0^{t^\rho_{x,y}(a)} (\alpha W(x_s) + \gamma) ds$$

where α and γ are like in Theorem 2.16 and Proposition 2.19. With $t^\rho_{x,y}(a)$ we denote the first time the trajectory (x_t, y_t) starting at (x,y) and using the control function a hits the target $\mathcal{T}_\rho$:

$$t^\rho_{x,y}(a) := \min\{t : (x_t(x,y,a), y_t(x,y,a) \in \mathcal{T}_\rho\}$$

The notion of controllability we consider is described in the following definition.

Definition 2.21. *We say that the state $x = 0$ is* nearly asymptotically controllable *for the singularly perturbed system* (2.1)*, with basin of attraction $\Omega \subseteq \mathbb{R}^N$ if, for any $R > 0$ there exist $\varepsilon_0 > 0$, $r > 0$ and $T_0 > 0$ such that the following holds:*

1. (Lyapunov Stability) *For any $x \in \Omega$, with $|x| < r$ there exists a control $\bar{a} \in \mathcal{A}$ such that $|x_t^\varepsilon(x,y,\bar{a})| < R$ for any $t > 0$, any $\varepsilon < \varepsilon_0$ and any $y \in \mathbb{T}^M$;*

2. (Attractiveness) *For any $x \in \Omega$ there exists $\bar{a} \in \mathcal{A}$ such that $|x_t^\varepsilon(x,y,\bar{a})| < R$ for any $t > T_0$, any $\varepsilon < \varepsilon_0$ and any $y \in \mathbb{T}^M$.*

The main result of the present Section is the following

Theorem 2.22. *Suppose that the state $x = 0$ is asymptotically controllable for the limit control problem* (2.3)*. Then there exists a basin of attraction $\Omega \subseteq \mathbb{R}^N$ containing the origin, such that the same state is nearly asymptotically controllable for the singularly perturbed system* (2.1) *in Ω.*

Remark 2.23. The basin of attraction Ω mentioned in the statement of Theorem 2.22 can be explicitly described by means of the basin of attraction $\mathcal{O}$ of the state $x = 0$ for the limit control problem. In fact, we will point out that there exists $u_0 > 0$, depending only on $\mathcal{O}$, such that for every $c \in (0, u_0)$, if ε is small enough,

$$\Omega \supseteq \{x : V(x) < c\} \times \mathbb{T}^M.$$

See the following Lemma 2.26, below. □

Before proving Theorem 2.22, let us make some remarks on Definition 2.21. This definition is actually very similar to the definition of *near asymptotic stability* given in [12] and showed to be valid for uncontrolled singularly perturbed systems whose limit dynamics is asymptotically stable to the origin. More precisely, in [12] the Author considers the uncontrolled system

$$\begin{aligned} \dot{x}_t &= f(x_t, y_t), \quad x_0 = x \in \mathbb{R}^N \\ \varepsilon \dot{y}_t &= g(x_t, y_t), \quad y_0 = y \in \mathbb{R}^M \end{aligned} \tag{2.20}$$

and gives the following notion of stability.

Definition 2.24. *The state $x = 0$ is* near asymptotically stable *with respect to the system* (2.20) *if the following holds. There exists a bound $B > 0$ such that, for every $b > 0$ and $\rho > 0$ there exist ε_0, $\beta > 0$ and T_0 with the following properties:*

1. *if $\varepsilon < \varepsilon_0$, $|x| \leq \beta$ and $|y| \leq b$ then $|x_t^\varepsilon(x, y)| \leq \rho$ for any $t \geq 0$;*
2. *if $\varepsilon \leq \varepsilon_0$, $|x| \leq B$ and $|y| \leq b$ then $|x_t^\varepsilon(x, y)| \leq \rho$ for any $t \geq T_0$.*

The main result in [12] affirms that if the state $x = 0$ is asymptotically stable for the limit differential inclusion, then the same state is near asymptotically stable for original problem.

The two properties of the Definition 2.24 can be easily rewritten in terms of Definition 2.21. Moreover, since in our case the fast variables lies on the flat torus $\mathbb{T}^M$, no bounds on the initial state of the fast flow are required.

As mentioned before, we know that asymptotic controllability of the state $x = 0$ for (2.3) is equivalent to Assumption 2.14. We then assume to have a Control Lyapunov function V for the limiting dynamics, which is Lipschitz-continuous in a certain open set $\mathcal{O} \subset \mathbb{R}^N$. Theorem 2.16 shows how to construct, starting from V, for any positive ρ, a control Lyapunov function V^ε for the perturbed dynamics, satisfying the monotonicity property in $(\mathcal{O} \setminus B(0, \rho)) \times \mathbb{T}^M$. Theorem 2.22 provides the dynamical interpretation of this result, and clarifies the sort of stability we are able to deduce from the existence of such V^ε.

The proof of Theorem 2.22 uses a couple of preliminary results. Let us first of all fix some notations. For any positive c, we indicate with $S_V(c)$ and $S_{V^\varepsilon}(c)$ the following sublevel sets:

$$S_V(c) := \{x \in \mathcal{O} : V(x) < c\}, \quad S_{V^\varepsilon}(c) := \{(x, y) \in \mathcal{O} \times \mathbb{T}^M : V^\varepsilon(x, y) < c\}$$

The first preliminary result concerns the comparison of supersolutions of Hamilton–Jacobi equations with the value function of the problem with exit time. (See [22], [74]).

Proposition 2.25. [74, Theorem 2.5]. *Let $\rho > 0$ be fixed. Let $\Omega \subseteq \mathbb{R}^N \times \mathbb{T}^M$ be an open set and g be a nonnegative function, continuous on $\mathcal{T}_\rho$. Assume that there exists a constant $u_0 \in \mathbb{R} \cup \{+\infty\}$ and a function $U \in C(\Omega)$ such that*

$$\begin{aligned} H_{\alpha W}(x, y, D_x U, \varepsilon^{-1} D_y U) &\geq 0 \quad \text{in } \Omega \setminus \mathcal{T}_\rho \\ U &\geq g \quad \text{on } \Omega \cap \mathcal{T}_\rho \end{aligned}$$

and the following boundary condition is satisfied:

$$\begin{aligned} \lim_{(x,y) \to (x_0, y_0)} U(x, y) &= u_0 \quad \text{for any } (x_0, y_0) \in \partial\Omega, \\ U(x, y) &< u_0 \quad \text{for any } (x, y) \in \Omega. \end{aligned}$$

Then

$$U(x,y) \geq T^\rho(x,y) \quad \text{for any } (x,y) \in \Omega.$$

Lemma 2.26. *There exists $u_0 > 0$ such that, for any $\rho > 0$ such that $\bar{B}(0,\rho) \subset \mathcal{O}$, the following holds.*

i. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$

$$\Omega^\varepsilon := \{(x,y) \in \mathcal{O} \times \mathbb{T}^M : V^\varepsilon(x,y) < u_0\} \supseteq \mathcal{T}_\rho$$

ii. For any $(x,y) \in \Omega^\varepsilon$ there exists $\bar{a} \in \mathcal{A}$ such that

$$t^\rho_{x,y}(\bar{a}) < C < +\infty$$

iii. For any $c < u_0$ there exists $\varepsilon_c > 0$ such that if $\varepsilon < \varepsilon_c$ then

$$\mathcal{T}_\rho \subseteq S_{V^\varepsilon}(c) \subseteq \Omega^\varepsilon;$$

furthermore, the level sets $S_{V^\varepsilon}(c)$ are viable, i.e.

$$\begin{array}{l} \text{for any } (x,y) \in S_{V^\varepsilon}(c) \text{ exists } \bar{a} \in \mathcal{A} \text{ such that} \\ (x^\varepsilon_t(x,y,\bar{a}), y^\varepsilon_t(x,y,\bar{a})) \in S_{V^\varepsilon}(c) \text{ for any } t \geq 0. \end{array}$$

Proof. Fix $\rho > 0$ and consider the control Lyapunov function $V^\varepsilon(x,y) := V(x) + \varepsilon\chi(y)$ provided by Theorem 2.16; such a function satisfies the following differential inequality:

$$\max_{a\in A}\{-f(x,y,a)\cdot D_xV^\varepsilon - \varepsilon^{-1}g(x,y,a)\cdot D_yV^\varepsilon - \alpha W(x) - \gamma\} \geq 0 \qquad \text{in } (\mathcal{O} \setminus B(0,\rho)) \times \mathbb{T}^M$$

The open set $\mathcal{O}$ and the parameters α and γ are defined in Assumption 2.14, Theorem 2.16 and Proposition 2.19, respectively. To ease notations, we drop the subscript ρ in T^ρ, $t^\rho_{x,y}$ and $\mathcal{T}_\rho$.

Since V is proper, we assume without loss of generality, that

$$\max_{|x|=\rho} V(x) < \min_{x\in\partial\mathcal{O}} V(x) =: u_0/2$$

If $\varepsilon < \varepsilon_0 := u_0/(2\max\chi)$ then

$$\max_{|x|=\rho,\, y\in\mathbb{T}^M} V^\varepsilon(x,y) = \max_{|x|=\rho} V(x) + \varepsilon\max\chi < u_0/2 + \varepsilon\max\chi < u_0$$

which proves *i.* The same computation shows that, as announced in Remark 2.23, for any $c < u_0$ and $\varepsilon < (u_0 - c)/\max\chi$, $S_V(c) \times \mathbb{T}^M$ is contained in Ω^ε.

By applying Proposition 2.25 in Ω^ε we obtain:

$$T(x,y) := \inf_{a\in\mathcal{A}} \int_0^{t_{x,y}(a)} (\alpha W(x_s) + \gamma)ds \leq V^\varepsilon(x,y). \tag{2.21}$$

By (2.21) we derive that the target $\mathcal{T}$ is reached in finite time by a trajectory of the perturbed dynamics starting in Ω^ε. Indeed, fix any $(x,y) \in \Omega^\varepsilon$; then for any positive δ there exists a control $\bar{a} \in \mathcal{A}$ such that

$$\delta + T(x,y) \geq \int_0^{t_{x,y}(\bar{a})} (\alpha W(x_s) + \gamma)ds.$$

In particular we have

$$\gamma t_{x,y}(\bar{a}) < \int_0^{t_{x,y}(\bar{a})} (\alpha W(x_s) + \gamma) ds \le T(x,y) + \delta \le V^\varepsilon(x,y) + \delta.$$

Furthermore, using the definition of Ω^ε we can estimate $t_{x,y}(\bar{a})$:

$$t_{x,y}(\bar{a}) \le \frac{V^\varepsilon(x,y) + \delta}{\gamma} < \frac{u_0 + \delta}{\gamma}.$$

This gives *ii.*

Now, observe that the inclusion $S_{V^\varepsilon}(c) \subseteq \Omega^\varepsilon$ is obvious. Furthermore, if $\varepsilon < \varepsilon_c := \frac{c - u_0/2}{\max \chi}$, for any x with $|x| = \rho$ we have

$$V^\varepsilon(x,y) \le u_0/2 + \varepsilon \max \chi < c$$

To prove the viability of the sublevel sets $S_{V^\varepsilon}(c)$ we use the following super-optimality principle (see Proposition 4.1 and Remark 4.2 in Soravia [75]): the supersolution $V^\varepsilon(x,y)$ satisfies, for any (x,y) in Ω^ε,

$$V^\varepsilon(x,y) = \inf_{a \in \mathcal{A}} \sup_{t \in [0, t_{x,y}(a))} \left\{ V^\varepsilon(x_t, y_t) + \int_0^t (\alpha W(x_s) + \gamma) ds \right\}$$

Fix $c < u_0$, $(x,y) \in S_{V^\varepsilon}(c)$ and $|x| > \rho$, then choose β such that

$$0 < \beta < c - V^\varepsilon(x,y).$$

Then there exists $\bar{a} \in \mathcal{A}$ such that the corresponding trajectory of (2.1) satisfy

$$V^\varepsilon(x_t, y_t) < c - \int_0^t (\alpha W(x_s) + \gamma) ds < c$$

for any $t \in [0, t_{x,y}(\bar{a}))$. Then, in the same time interval, (x_t, y_t) belongs to $S_{V^\varepsilon}(c)$.

Now assume that a certain $t' > t_{x,y}(\bar{a})$ does exist, such that $(x_{t'}, y_{t'}) \in S_{V^\varepsilon}(c)$, $|x_{t'}| > \rho$ and $(x_t, y_t) \in S_{V^\varepsilon}(c)$ for any $t \in [t_{x,y}(\bar{a}), t']$. Now, consider the solution of the control problem (2.1) starting at $(x', y') := (x_{t'}, y_{t'})$. The previous computation shows that another control $a' \in \mathcal{A}$ does exist, such that $(x'_t, y'_t) := (x_t(x', y', a'), y_t(x', y', a'))$ satisfies

$$V^\varepsilon(x'_t, y'_t) < c, \qquad \text{for any } t \in [0, t_{x',y'}(a')).$$

Then the solution $(\widetilde{x}_t, \widetilde{y}_t)$ obtained piecing together the trajectory (x_t, y_t) linking (x,y) to (x', y') and (x'_t, y'_t) using the control function defined by

$$\widetilde{a}_t := \begin{cases} \bar{a}_t & \text{for } t < t' \\ \\ a'_{t-t'} & \text{for } t \ge t' \end{cases}$$

satisfies $V^\varepsilon(\widetilde{x}_t, \widetilde{y}_t) < c$ for any $t \in [0, t' + t_{x',y'}(a'))$. By iterating the argument, we obtain a trajectory of the perturbed dynamics which remains in $S_{V^\varepsilon}(c)$ for any $t \ge 0$. □

Proof of Theorem 2.22. Fix $R > 0$ and find $c, \delta > 0$ such that

$$c + \delta < u_0, \qquad S_V(c) \subseteq S_V(c+\delta) \subseteq \bar{B}(0,R)$$

with u_0 as in Lemma (2.26). Find also $r > 0$ such that

$$S_V(c) \supseteq \bar{B}(0,r)$$

Now fix $\rho < r/2$, and consider the Lyapunov function V^ε constructed in Theorem 2.16. If

$$\varepsilon < \min\left\{\frac{c+\delta-u_0/2}{\max\chi}, \frac{\delta}{2\max\chi}\right\} =: \varepsilon_0$$

the following inclusions follow by Lemma 2.26:

$$\mathcal{T}_\rho \subseteq S_{V^\varepsilon}(c+\delta) \subseteq \Omega^\varepsilon, \qquad S_V(c) \times \mathbb{T}^M \subseteq S_{V^\varepsilon}(c+\delta) \tag{2.22}$$

Let $|x| \leq r$; then $x \in S_V(c)$ and, by (2.22), $(x,y) \in S_{V^\varepsilon}(c+\delta)$ for any $y \in \mathbb{T}^M$. The set $S_{V^\varepsilon}(c+\delta)$ is viable by Lemma 2.26 (iii.), then a control function $\bar{a} \in \mathcal{A}$ does exist, such that $(x_t^\varepsilon, y_t^\varepsilon) \in S_{V^\varepsilon}(c+\delta)$ for any $t \geq 0$. Since $V(x_t^\varepsilon) < V^\varepsilon(x_t^\varepsilon, y_t^\varepsilon)$, we have $x_t^\varepsilon \in S_V(c+\delta)$, and then $|x_t^\varepsilon| < R$ for any $t \geq 0$. This proves the Lyapunov stability (part (i.) in Definition 2.21).

We pass now to prove the attractiveness (part (ii.) in Definition 2.21. Take $(x,y) \in \Omega^\varepsilon$ with $|x| > \rho$. By Lemma 2.26 (ii.) there is a control $\bar{a}$ such that the solution of the perturbed dynamics starting at (x,y) and driven by $\bar{a}$ hits $\mathcal{T}_\rho$ the first time at $t_{x,y}^\rho(\bar{a}) < +\infty$. A fortiori, such a trajectory enters in finite time the set $S_V(c) \times \mathbb{T}^M$, which is contained in the viable set $S_{V^\varepsilon}(c+\delta)$. Let T_0 be the first time such that $x_t^\varepsilon \in S_V(c)$. By Lemma 2.26 (iii.), we can obtain a new trajectory (which we continue to denote $x_t^\varepsilon, y_t^\varepsilon$), starting at (x,y) and satisfying $(x_t^\varepsilon, y_t^\varepsilon) \in S_{V^\varepsilon}(c+\delta)$ for any $t \geq T_0$. As before, this implies $|x_t^\varepsilon| \leq R$ for any $t \geq T_0$.

□

Global asymptotic controllability of the ε-problem. Theorem 2.22 has a local nature: the basin of attraction of the state $x = 0$ for the perturbed system is described starting from the basin of attraction of the same state for the limit system.

In this final part we adapt the construction described before in order to obtain a global version (*i.e.* valid in the whole $\mathbb{R}^N$) of the nearly asymptotic controllability. More precisely, we do not construct a global control Lyapunov function for the perturbed system, but, starting from a locally Lipschitz global control Lyapunov function for the limit dynamics, we construct, *for any* $R > 0$, a control Lyapunov function V_R^ε exhibiting the near asymptotic stability in the ball $B(0,R)$.

Definition 2.27. *We say that the state* $x = 0$ *is* nearly globally asymptotically controllable *for the perturbed system* (2.1) *if it is nearly asymptotically controllable with basin of attraction* $\Omega = \mathbb{R}^N$.

We assume that the limit control problem (2.3) is globally asymptotically controllable, then there exists a control Lyapunov pair (V,W) defined in the whole $\mathbb{R}^N$ with V locally Lipschitz in $\mathbb{R}^N$.

Consider, for any $0 < \rho < R$ and any $\alpha \in (0,1)$ the following auxiliary Hamiltonian:

$$K_{\rho,R}(y,q) := \inf\{H_0(x,y,p,q) - \alpha W(x) : \rho \le |x| \le R,\, p \in J^- V(x)\};$$

(compare with (2.14)). Arguing as in Proposition 2.19 we can construct, for any ρ, R a function $\chi_{\rho,R} : \mathbb{T}^M \to \mathbb{R}$ such that

$$K_{\rho,R}(y, D\chi_{\rho,R}) > \gamma, \qquad \text{in } \mathbb{T}^M$$

in viscosity sense, for a certain constant $\gamma > 0$ depending on ρ and R. It follows that the function

$$V^\varepsilon_{\rho,R}(x,y) := V(x) + \varepsilon\chi_{\rho,R}(y)$$

satisfies in viscosity sense

$$H_0(x,y,D_xV^\varepsilon_{\rho,R}(x,y), \varepsilon^{-1}D_yV^\varepsilon_{\rho,R}(x,y)) \ge \alpha W(x) \quad \text{in } (B(0,R)\setminus B(0,\rho)) \times \mathbb{T}^M.$$

The next Lemma adapts to this new context the statement of Lemma 2.26.

Lemma 2.28. *For any $R > 0$ there exists $u_R > 0$ such that, for any $\rho > 0$ the following conditions hold.*

i. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$

$$\Omega^\varepsilon_R := \{(x,y) \in \mathbb{R}^N \times \mathbb{T}^M : V^\varepsilon(x,y) < u_R\} \supseteq \mathcal{T}_\rho$$

ii. For any $(x,y) \in \Omega^\varepsilon_R$ there exists $\bar{a} \in \mathcal{A}$ such that

$$t^\rho_{x,y}(\bar{a}) < C < +\infty$$

iii. For any $c < u_R$ there exists $\varepsilon_c > 0$ such that if $\varepsilon < \varepsilon_c$ then

$$\mathcal{T}_\rho \subseteq S_{V^\varepsilon}(c) \subseteq \Omega^\varepsilon_R;$$

furthermore, the level sets $S_{V^\varepsilon}(c)$ are viable, i.e.

for any $(x,y) \in S_{V^\varepsilon}(c)$ exists $\bar{a} \in \mathcal{A}$ such that $(x^\varepsilon_t(x,y,\bar{a}), y^\varepsilon_t(x,y,\bar{a})) \in S_{V^\varepsilon}(c)$ for any $t \ge 0$.

Proof. The proof is the same as in the Lemma 2.26, provided u_0 is substituted by u_R, which is defined as

$$u_R/2 := \min_{x \in \partial B(0,R)} V(x).$$

Note that, since V is proper, it is not restrictive to suppose

$$u_R > 2 \max_{x \in \partial B(0,\rho)} V(x).$$

□

The key tool we will apply to prove the global controllability is stated in the following lemma.

Lemma 2.29. *For any $R' > 0$ and any $\varepsilon > 0$ there exists $R > 0$ such that*

$$B(0, R') \subseteq \Omega_R^\varepsilon.$$

Proof. Since V is a control Lyapunov function, it can be estimated by means of two comparison functions $\beta_1, \beta_2 \in \mathcal{K}$:

$$\beta_1(|x|) \leq V(x) \leq \beta_2(|x|), \tag{2.23}$$

then

$$\beta_1(R) \leq \min_{x \in \partial B(0,R)} V(x) \leq \beta_2(R)$$

Fix now $\bar{x}$, with $|\bar{x}| = R'$. To prove the desired inclusion it is enough to choose

$$R > \beta_1^{-1} \left(\beta_2(R') + \varepsilon \max \chi_{\rho,R} \right).$$

Indeed, since $V(\bar{x}) \leq \beta_2(R')$ by (2.23), one has for any y

$$V_{\rho,R}^\varepsilon(\bar{x}, y) \leq \beta_2(R') + \varepsilon \max \chi_{\rho,R}(y) < \min_{x \in \partial B(0,R)} V(x).$$

□

Then, for any fixed $\rho > 0$ and for any $x \in \mathbb{R}^N \setminus B(0, \rho)$ we can construct $V_{\rho,R}^\varepsilon$, where R is provided by Lemma 2.29 for $R' := |x|$. Such a function satisfies the properties of Lemma 2.28 in the set $\Omega_R^\varepsilon \supset B(0, R')$. Then the dynamical properties of Lyapunov stability and attractiveness of the Definition 2.21 can be proved in (any) ball centered at the origin using Lemma 2.28, as done in the proof of Theorem 2.22. We finally get the following

Theorem 2.30. *Suppose the state $x = 0$ is globally asymptotically controllable for the limit control problem* (2.3)*. Then the same state is nearly globally asymptotically controllable for the perturbed system* (2.1)*.*

Chapter 3

Homogenization of non–coercive HJ equations

The main contributions of this section concern homogenization of min-max type Hamilton-Jacobi-Isaacs equations. We start recalling the setup, some terminology and some preliminary results. Consider the following deterministic two-person zero-sum singularly perturbed differential game

$$\begin{aligned} \dot{x}_s &= f(x_s, y_s, a_s, b_s), \quad x_0 = x \\ \varepsilon \dot{y}_s &= g(x_s, y_s, a_s, b_s), \quad y_0 = y \end{aligned} \tag{3.1}$$

with a running cost $l(x, y, a, b)$ and a terminal cost $h(x, y)$. The current assumptions are listed below:

- $x \in \mathbb{R}^N$, and $y \in \mathbb{R}^M$;
- the *controls* of the two players are measurable functions a and b from $(0, +\infty)$ to compact metric spaces A and B. The spaces of such measurable functions are denoted by $\mathcal{A}$ and $\mathcal{B}$ respectively;
- the functions f, g and l are bounded uniformly continuous in $\mathbb{R}^N \times \mathbb{R}^M \times A \times B$, with values, respectively in $\mathbb{R}^N$, $\mathbb{R}^M$ and $\mathbb{R}$;
- the function h is bounded uniformly continuous from $\mathbb{R}^N \times \mathbb{R}^M$ to $\mathbb{R}$;
- the functions f and g are Lipschitz–continuous in x, y, uniformly with respect to the controls a, b;
- all the data are periodic with respect to y.

A strategy for the first player (respectively for the second player) is a map $\alpha : \mathcal{A} \to \mathcal{B}$ (resp. $\beta : \mathcal{B} \to \mathcal{A}$). A strategy α for the first player is said to be *nonanticipating* if for any $t > 0$ and any $b, b' \in \mathcal{B}$, $b_s = b'_s$ for any $0 \leq s \leq t$ implies $\alpha[b]_s = \alpha[b']_s$ for any $0 \leq s \leq t$. We define

$$\Gamma := \{\alpha : \mathcal{A} \to \mathcal{B} \text{ nonanticipating strategy for the first player}\}$$

$$\Delta := \{\beta : \mathcal{B} \to \mathcal{A} \text{ nonanticipating strategy for the second player}\}$$

The symbols $\mathcal{A}(t)$, $\mathcal{B}(t)$, $\Gamma(t)$ and $\Delta(t)$ are used to indicate subsets of $\mathcal{A}$, $\mathcal{B}$, Γ and Δ respectively, of controls and strategies defined on $[0,t]$.

The cost functional (on the time interval $[0,t]$) is given by

$$J(t,x,y,a,b) := \int_0^t l(x_s,y_s,a_s,b_s)ds + h(x_t,y_t)$$

and we assume that the first player wants to minimize the cost, meanwhile the second wants to maximize it. Therefore we take into account the *lower value function*

$$u^\varepsilon(t,x,y) = \inf_{\alpha\in\Gamma(t)} \sup_{b\in\mathcal{B}(t)} J(t,x,y,\alpha[b],b).$$

and the Cauchy problem for the Bellman–Isaacs equation associated to the game

$$\begin{aligned} &u_t^\varepsilon + H(x,y,D_x u^\varepsilon, \varepsilon^{-1} D_y u^\varepsilon) = 0 && \text{in } (0,+\infty)\times\mathbb{R}^N\times\mathbb{R}^N \\ &u^\varepsilon(0,x,y) = h(x,y) && \text{on } \mathbb{R}^N\times\mathbb{R}^M \end{aligned} \tag{3.2}$$

where H is given by the following min–max formula:

$$H(x,y,p,q) = \min_{b\in B}\max_{a\in A}\{-p\cdot f(x,y,a,b) - q\cdot g(x,y,a,b) - l(x,y,a,b)\} \tag{3.3}$$

Under the standing assumptions H is continuous and degenerate elliptic, therefore the equation in (3.2) is degenerate parabolic; it also satisfies conditions implying the Comparison Principle for bounded viscosity sub- and super-solutions. It follows that (3.2) possesses at most one bounded uniformly continuous viscosity solution. Moreover the following result, due to Fleming and Souganidis [51] holds.

Theorem 3.1. *Under the current assumptions the lower value function u^ε is the unique bounded uniformly continuous viscosity solution of* (3.2).

The property of being *ergodic* of an Hamiltonian, has to do with the possibility to define an effective differential operator, the *effective Hamiltonian*, and the possibility to *stabilize to a constant* the oscillating initial datum h allows to define an *effective initial datum.* Our final goal is again to apply the convergence results of [3]; to this scope, in this section, we investigate ergodicity and stabilization properties of the Hamilton–Jacobi–Isaacs operator defined in (3.3). We recall first some definitions.

Consider, for later use, the M–dimensional controlled system obtained from (3.1), fixing $\bar{x}$ and setting ε equals to 1

$$\dot{y}_s = g(\bar{x},y_s,a_s,b_s), \quad y_0 = y \tag{3.4}$$

There are other ways to define the ergodicity of H (see the Remark below). Here we adopt the following definition, based on the following evolutive cell problem. For any $(\bar{x},\bar{p})$ consider

$$w_t + H(\bar{x},y,\bar{p},Dw) = 0, \quad w(0,y) = 0 \tag{3.5}$$

Under the standing assumptions, such problem has a unique viscosity solution $w(t,y)$.

Definition 3.2. *We say that H is* ergodic *at $(\bar{x}, \bar{p})$ if*

$$\lim_{t \to +\infty} \frac{w(t, y)}{t} = \text{const} \quad \textit{uniformly in } y$$

and in this case we set $-\bar{H}(\bar{x}, \bar{p})$ equals to this constant.

Remark 3.3. This definition of ergodicity is connected with the classical ergodic theory, and motivates the name of such property. The following *equivalent definitions* are based on other stationary cell problems:

i. We say that H is ergodic at $(\bar{x}, \bar{p})$ if the unique periodic viscosity solution of the δ–problem ($\delta > 0$)

$$\delta w_\delta + H(\bar{x}, y, \bar{p}, Dw_\delta) = 0 \quad \text{in } \mathbb{R}^M$$

satisfies

$$\lim_{\delta \to 0^+} \delta w_\delta(y; \bar{x}, \bar{p}) = \text{const} \quad \text{uniformly in } y$$

ii. Consider the *true* cell problem

$$\lambda + H(\bar{x}, y, \bar{p}, D\chi) = 0 \quad \text{in } \mathbb{R}^M \tag{3.6}$$

It is well known that there exists at most one $\lambda \in \mathbb{R}$ such that (3.6) admits a continuous periodic viscosity solution χ; in this case we put $\bar{H}(\bar{x}, \bar{p}) = -\lambda$. The function χ is called *corrector* and is non-unique, being defined up to an additive constant. H is ergodic at $(\bar{x}, \bar{p})$ if and only if

$$\lambda_1 := \sup\{\lambda \,|\, \exists \text{ an u.s.c. subsolution of (3.6)}\}$$

$$= \lambda_2 := \inf\{\lambda \,|\, \exists \text{ a l.s.c. supersolution of (3.6)}\}$$

and in this case we put $\bar{H}(\bar{x}, \bar{p}) = -\lambda_1 = -\lambda_2$. □

Consider now for any $\bar{x}$ the Cauchy problem

$$\begin{cases} v_t + H'(\bar{x}, y, Dv) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^M \\ v(0, y) = h(\bar{x}, y) \\ v \text{ periodic} \end{cases} \tag{3.7}$$

where

$$H'(x, y, q) := \min_{b \in B} \max_{a \in A} \{-q \cdot g(x, y, a, b)\}$$

is called *recession function*, or *homogeneous part* of H. The problem (3.7) has a unique bounded viscosity solution $v(t, y)$.

Definition 3.4. *We say that the pair (H, h) is* stabilizing *at $\bar{x}$ if*

$$\lim_{t \to +\infty} v(t, y) = \text{const} \quad \textit{uniformly in } y$$

and in this case we set $\bar{h}(\bar{x})$ equals to this constant.

Ergodicity and Stabilization can be characterized (see Propositions 3.3 and 4.2 in [4]) using strong maximum principle for the problems (3.5) and (3.7), and equicontinuity of the value functions. Moreover the effective Hamiltonian and the effective initial datum can be represented by means of viscosity solutions of the problems (3.5) and (3.7). More precisely, since the solution of (3.5) is the lower value function of the differential game, with finite horizon cost functional, we have

$$\bar{H}(\bar{x},\bar{p}) = -\lim_{t\to+\infty} \inf_{\alpha\in\Gamma(t)} \sup_{b\in\mathcal{B}(t)} \left\{ \frac{1}{t}\int_0^t \bar{p}\cdot f(\bar{x},y_s,\alpha[b]_s,b_s) + l(\bar{x},y_s,\alpha[b]_s,b_s)\right\} \tag{3.8}$$

where $y.$ denotes a solution of (3.4). Similarly, the unique solution of (3.7) is the lower value function of a finite horizon differential game (see [51]):

$$v(t,y;\bar{x}) = \inf_{\alpha\in\Gamma(t)} \sup_{b\in\mathcal{B}(t)} h(\bar{x},y_t)$$

Then the following representation formula for the effective initial datum holds:

$$\bar{h}(\bar{x}) = \lim_{t\to+\infty} \inf_{\alpha\in\Gamma(t)} \sup_{b\in\mathcal{B}(t)} h(\bar{x},y_s) \tag{3.9}$$

Exploiting these representation formulae, we will give some sufficient condition for the validity of the ergodicity and stabilization properties.

Failure of homogenization for non-coercive Hamiltonians. For a control Hamiltonian like

$$\max_{a\in A}\{-f(x,y,a)\cdot p - g(x,y,a)\cdot q - l(x,y,a)\}$$

the property of being ergodic and stabilizing, is related only to the dynamic part of it. The known sufficient conditions for ergodicity regard coercivity with respect to the gradient variables p, q (that are multiplied by the dynamic flows), controllability in bounded time of the fast variables, non-resonance, etc. If one of these assumptions is satisfied, homogenization follows without assuming any other conditions, neither on the running cost l nor on the final cost h.

In what follows, we will consider a rather general class of non-coercive Hamiltonians. More precisely, we will take into account Hamiltonians that are convex with respect to some gradient variables, and concave with respect to the other, a very special case of non-coercive Hamiltonians. Nevertheless, this class of Hamiltonians permits to easily observe the most interesting difference with the coercive case: as we will show in the two examples below, in the non-coercive case homogenization may fail to hold if no assumptions are made on the initial datum h and on the running cost l.

Example 3.5. The problem

$$\begin{cases} \partial_t u^\varepsilon + |u^\varepsilon_x| - \gamma|u^\varepsilon_y| = \cos\left(\frac{x-y\gamma^{-1}}{\varepsilon}\right) & x,y\in\mathbb{R} \quad (\gamma>0) \\ u^\varepsilon(0,x,y) = 0 \end{cases} \tag{3.10}$$

does not homogenize if $x \neq \frac{y}{\gamma}$. In fact, an explicit solution of (3.10) is given by

$$u^\varepsilon(t,x,y) = t\cos\left(\frac{x-y\gamma^{-1}}{\varepsilon}\right)$$

and u^ε has no limit for $\varepsilon \to 0$. □

Example 3.6. The problem

$$\begin{array}{ll} \partial_t u^\varepsilon + |u^\varepsilon_x| - \gamma |u^\varepsilon_y| = 0 & x, y \in \mathbb{R} \quad (\gamma > 0) \\ u^\varepsilon(0,x,y) = \cos\left(\frac{x - y\gamma^{-1}}{\varepsilon}\right) & \end{array} \tag{3.11}$$

does not homogenize if $x \neq \frac{y}{\gamma}$. In fact, an explicit solution of (3.10) is given by the steady solution

$$u^\varepsilon(t,x,y) = \cos\left(\frac{x - y\gamma^{-1}}{\varepsilon}\right)$$

and u^ε does not converge to any function as ε vanishes. □

We conclude this introduction exhibiting a rather large class of Hamiltonians for which homogenization does not hold.

Proposition 3.7. *Consider the following problem*

$$\begin{array}{l} \partial_t u^\varepsilon + H_1(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, D_x u^\varepsilon) - H_2(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, D_y u^\varepsilon) = l(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}) \\ u^\varepsilon(0,x,y) = 0 \end{array} \tag{3.12}$$

and assume that constants $\alpha_1, \alpha_2, \beta > 0$ and $C_1, C_2 > 0$ exist, such that

$$\begin{array}{l} \left|H_1(x,y,\xi,\eta,p_x) - \alpha_1 |p_x|^\beta\right| \leq C_1 \\ \left|H_2(x,y,\xi,\eta,p_x) - \alpha_2 |p_y|^\beta\right| \leq C_2 \end{array} \tag{3.13}$$

Then, if either $\beta = 1$ or $\alpha_1 = \alpha_2$ there exists $l : \mathbb{T}^N \to \mathbb{R}$ analytic such that (3.12) *does not homogenize. More precisely, for any $\delta > 0$*

$$\limsup_{\varepsilon \to 0^+} u^\varepsilon \geq \delta t, \qquad \liminf_{\varepsilon \to 0^+} u^\varepsilon \leq -\delta t$$

for any $t > 0$ and any x, y.

The statement of this negative result will become sharper, once completed, at the end of the chapter, by Proposition 3.28. In fact, we will prove that for a special class of Hamiltonians of this type, if $\alpha_1 \neq \alpha_2$ homogenization holds.

Before proving Proposition 3.7 we prove the following Lemma.

Lemma 3.8. *If either $\beta = 1$ or $\alpha_1 = \alpha_2$, for any $c, c' \in \mathbb{R}$ then there is no homogenization for the problem*

$$\begin{array}{l} \partial_t v^\varepsilon + \alpha_1 |D_x v^\varepsilon|^\beta - \alpha_2 |D_y u^\varepsilon|^\beta = c + c' \cos\left(\frac{1}{\varepsilon}\left(\frac{x}{\alpha_1} - \frac{y}{\alpha_2}\right)\right) \\ v^\varepsilon(0,x,y) = 0 \end{array} \tag{3.14}$$

Proof. For the problem (3.14) we can write an explicit solution. We distinguish two cases.

Case I. Put

$$v^\varepsilon(t,x,y) := tc + tc' \cos\left(\frac{1}{\varepsilon}\left(\frac{x}{\alpha_1^{1/\beta}} - \frac{y}{\alpha_2^{1/\beta}}\right)\right)$$

and observe that v^ε readily solves the problem (3.14) if $\beta = 1$.
Case II. Define

$$v^\varepsilon(t,x,y) := tc + tc' \cos\left(\frac{1}{\varepsilon}\left(\frac{x}{\alpha_1} - \frac{y}{\alpha_2}\right)\right)$$

then

$$\partial_t v^\varepsilon + \alpha_1 |D_x v^\varepsilon|^\beta - \alpha_2 |D_y v^\varepsilon|^\beta = c + c' \cos\left(\frac{1}{\varepsilon}\left(\frac{x}{\alpha_1} - \frac{y}{\alpha_2}\right)\right)$$
$$+ \frac{tc'}{\varepsilon^\beta \alpha_1^{\beta-1}} \left|\sin\left(\frac{1}{\varepsilon}\left(\frac{x}{\alpha_1} - \frac{y}{\alpha_2}\right)\right)\right|^\beta - \frac{tc'}{\varepsilon^\beta \alpha_2^{\beta-1}} \left|\sin\left(\frac{1}{\varepsilon}\left(\frac{x}{\alpha_1} - \frac{y}{\alpha_2}\right)\right)\right|^\beta$$

and the last two summands cancel out if $\alpha_1 = \alpha_2$.

In conclusion, in both cases I. and II. one has

$$\liminf_{\varepsilon\to 0^+} v^\varepsilon(t,x,y) = t(c + c')$$

and

$$\limsup_{\varepsilon\to 0^+} v^\varepsilon(t,x,y) = t(c - c')$$

and consequently in general homogenization does not take place.

□

Proof of Proposition 3.7. We argue by comparison with supersolutions and subsolutions of the auxiliary problem (3.14). For a fixed $\delta > 0$ define the running cost to be

$$l\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) := -(C_1 + C_2 + \delta)\cos\left(\frac{1}{\varepsilon}\left(\frac{x}{\alpha_1} - \frac{y}{\alpha_2}\right)\right)$$

We have

$$\begin{aligned} 0 &= \partial_t u^\varepsilon + H_1\left(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, D_x u^\varepsilon\right) - H_2\left(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, D_y u^\varepsilon\right) - l\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \\ &\leq \partial_t u^\varepsilon + \alpha_1 |D_x u^\varepsilon|^\beta - \alpha_2 |D_y u^\varepsilon|^\beta + C_1 + C_2 - l\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \end{aligned}$$

Put $c := -(C_1 + C_2)$ and $c' := C_1 + C_2 + \delta$, and let v^ε be the solution of (3.14) corresponding to this choice of c, c'; the previous computation shows that u^ε turns out to be a supersolution of the same problem. Therefore, if either $\beta = 1$ or $\alpha_1 = \alpha_2$, by comparison we get

$$\limsup_{\varepsilon\to 0^+} u^\varepsilon \geq \limsup_{\varepsilon\to 0^+} v^\varepsilon = t(c + c') = \delta t$$

Analogously, we have

$$\begin{aligned} 0 &= \partial_t u^\varepsilon + H_1\left(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, D_x u^\varepsilon\right) - H_2\left(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, D_y u^\varepsilon\right) - l\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \\ &\geq \partial_t u^\varepsilon + \alpha_1 |D_x u^\varepsilon|^\beta - \alpha_2 |D_y u^\varepsilon|^\beta - C_1 - C_2 - l\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \end{aligned}$$

and, by comparison with the solution $\widetilde{v}^\varepsilon$ of the problem (3.14) with c, c' replaced by $\widetilde{c} := C_1 + C_2$ and $\widetilde{c}' := -(C_1 + C_2 + \delta)$ respectively we obtain

$$\liminf_{\varepsilon\to 0^+} u^\varepsilon \leq \liminf_{\varepsilon\to 0^+} \widetilde{v}^\varepsilon = t(\widetilde{c} - \widetilde{c}') = -\delta t$$

as desired.

□

The previous result and examples convinces us about the importance to pose conditions on the costs l and h - besides to structural conditions on the Hamiltonian - to get sufficient conditions for ergodicity and stabilization, and then for homogenization.

3.1 Ergodicity and Stabilization to a constant in convex–concave HJ Equations

The coercivity of H with respect to the gradient variables is, besides to other controllability and non resonance conditions, a very classical assumption in homogenization results. Recently, several sufficient condition for ergodicity are known thanks to recent works of Alvarez and Bardi; see [4], [5], [21]. These conditions are given in terms of asymptotic controllability of the fast subsystem (3.4) with respect to certain targets. The scope of this section is to provide sufficient conditions for stabilization, if possible of the same type of those ensuring the ergodicity. This will permit to apply the general convergence results and to provide homogenization theorems.

We begin with some definitions.

Definition 3.9. *We say that the fast subsystem* (3.4) *is* asymptotically controllable *by the first player to a closed target* $\mathcal{T} \subset \mathbb{R}^M$ *if there exists a function* η*, with* $\eta(t) \to 0$ *as* $t \to +\infty$*, such that, for any initial state* $y \in \mathbb{R}^M$ *there is a strategy* $\widetilde{\alpha} \in \Gamma$ *such that*

$$d(y_t, \mathcal{T}) \leq \eta(t) \quad \text{for any } b \in \mathcal{B}, \text{ any } t > 0$$

where y_s *is the solution of*

$$\dot{y}_s = g(\bar{x}, y_s, \widetilde{\alpha}[b]_s, b_s), \quad y_0 = y$$

Notation. Whenever we need to emphasize the presence of two groups of fast variables, we write the fast subsystem (3.4) in *splitted form*, *i.e.* we write (3.4) as

$$\begin{array}{ll} \dot{y}_t^A = g^A(x, y_t, a, b), & y_0^A = y^A \\ \dot{y}_t^B = g^B(x, y_t, a, b), & y_0^B = y^B \end{array} \qquad y_t = (y_t^A, y_t^B) \tag{3.15}$$

with M_A and M_B such that $M_A + M_B = M$.

Definition 3.10. *Suppose that the fast subsystem is in splitted form. We say that* the y^A variables are asymptotically controllable by the first player *if there exists a function* η*, with* $\eta(t) \to 0$ *as* $t \to +\infty$*, such that, for any initial state* $y \in \mathbb{R}^M$*, and any* $\bar{y}^A \in \mathbb{R}^{M_A}$*, there exists a strategy* $\widetilde{\alpha} \in \Gamma$ *such that*

$$d(y_t^A, \bar{y}^A) \leq \eta(t) \quad \text{for any } b \in \mathcal{B}, \text{ any } t > 0$$

where y_t^A *is the* A*–part of the solution of*

$$\dot{y}_s = g(\bar{x}, y_s, \widetilde{\alpha}[b]_s, b_s), \quad y_0 = y$$

with g *in the form* (3.15)*. A similar definition can be given for the asymptotic controllability of the* y^B *variables by the second player.*

Let us summarize in the following proposition some sufficient condition for ergodicity. As mentioned before, such conditions are quoted from recent papers [4],[5] and [21], and apply in different situation, both if the fast subsystem (3.4) is in splitted form, and if it is not so.

Proposition 3.11. *i.* [4, Theorem 9.1] *If the fast subsystem* (3.4) *is bounded time controllable at* $x = \bar{x}$ *then the Hamiltonian* H *is ergodic at* $\bar{x}$*.*

ii. [21, Proposition 1] *Assume that the running cost is uniformly continuous and independent of the controls. If the fast subsystem* (3.4) *is uniformly asymptotically controllable in the mean by the first (resp., second) player, with respect to the target* $\mathcal{T} = \arg\min l(\bar{x}, \cdot)$ *(resp.,* $\mathcal{T} = \arg\max l(\bar{x}, \cdot)$*)*[1]*, then* H *is ergodic at* $\bar{x}$*.*

iii. [5, Proposition 2.2] *Assume the running cost to be independent of the controls and the system* (3.4) *is in splitted form. If the* y^A *variables are asymptotically controllable by the first player, the* y^B *variables are asymptotically controllable by the second player, and a saddle point for the running cost does exists, then* H *is ergodic at* $\bar{x}$*.*

iv. [21, Proposition 2] *Assume the running cost to be independent of the controls and the system* (3.4) *is in splitted form. If* (3.4) *is asymptotically controllable in the mean with respect to a certain target* $\mathcal{T}^*$*, and the* y^B *variables are asymptotically controllable in the mean with respect to another target* $\mathcal{T}^B$*, then* H *is ergodic at* $\bar{x}$ *(existence of saddle points in the running cost is not required).*

The following propositions give sufficient conditions for stabilization. They are given in the same spirit of condition quoted in the previous Proposition. The proofs also follow the same strategies used in proving the ergodicity.

We say that the fast subsystem (3.4) is *stoppable by the first player*, for $x = \bar{x}$, if

$$\text{for any } \widetilde{y}, \text{ and any } b \in B$$
$$\text{exists } a_b \in A \text{ such that } g(\bar{x}, \widetilde{y}, a_b, b) = 0.$$

Proposition 3.12. [4, Proposition 9.5-i]. *Suppose that the fast subsystem* (3.4) *is bounded time controllable and stoppable by the first player. Then, for any* $h \in BUC(\mathbb{R}^N \times \mathbb{R}^M)$*, the pair* (H, h) *is stabilizing at* $\bar{x}$*, and* $\bar{h}(\bar{x}) = \min_y h(\bar{x}, y)$*.*

Proposition 3.13. *Suppose the fast subsystem* (3.4) *is asymptotically controllable by the first player (resp. the second player), with respect to the target* $\mathcal{T} = \arg\min h(\bar{x}, \cdot)$ *(resp.* $\mathcal{T} = \arg\max h(\bar{x}, \cdot)$*). Then, for any* $h \in BUC(\mathbb{R}^N \times \mathbb{R}^M)$*, the pair* (H, h) *is stabilizing at* $\bar{x}$*, and* $\bar{h}(\bar{x}) = \min_y h(\bar{x}, y)$ *(resp.* $\bar{h}(\bar{x}) = \max_y h(\bar{x}, y)$*).*

[1] We say that the system (3.4) is *uniformly asymptotically controllable in the mean* by the first player, with respect to a closed target $\mathcal{T} \subset \mathbb{R}^M$ if there exists a function $\eta \in \mathcal{KL}$ and for any initial state y there is a strategy $\widetilde{\alpha} \in \Gamma$ such that

$$\frac{1}{t}\int_0^t \operatorname{dist}(y_s, \mathcal{T})ds \le \eta(||y||, t), \text{ for any } b \in \mathcal{B}$$

Proof. Fix an initial state for the fast subsystem (3.4), and consider the strategy $\widetilde{\alpha}$ and the solution y_s as in Definition 3.9. Let z_s be the projection of this solution on the target $\mathcal{T}$, *i.e.*

$$|z_s - y_s| = d(y_s, \mathcal{T}).$$

The following inequalities hold:

$$\begin{aligned} h(\bar{x}, y_s) &\leq \omega(|y_s - z_s|) + h(\bar{x}, z_s) \\ &= \omega(d(y_s, \mathcal{T})) + \min_y h(\bar{x}, \cdot) \end{aligned} \tag{3.16}$$

where $\omega(\cdot)$ is a continuity modulus of the function $h(\bar{x}, \cdot)$. Then

$$h(\bar{x}, y_t) \leq \omega(\eta(t)) + \min h(\bar{x}, \cdot) \quad \text{for any } b \in \mathcal{B}$$

By the representation formula (3.9), we have that the solution $v(t, y; \bar{x})$ of the homogeneous Cauchy problem for H' satisfies:

$$\min h(\bar{x}, \cdot) \leq v(t, y; \bar{x}) \leq \sup_{b \in \mathcal{B}} h(\bar{x}, y_t) \leq \omega(\eta(t)) + \min h(\bar{x}, \cdot)$$

We conclude that

$$\lim_{t \to +\infty} v(t, y; \bar{x}) = \min_y h(\bar{x}, y)$$

□

Proposition 3.14. *Suppose the fast subsystem* (3.4) *is in splitted form* (y^A, y^B)*, and that the* y^A *variables are uniformly asymptotically controllable by the first player, and the* y^B *by the second player. Suppose also that the function* $h(\bar{x}, \cdot)$ *has a saddle point. Then the pair* (H, h) *is stabilizing at* $\bar{x}$*, and* $\bar{h}(\bar{x})$ *is the value of* $h(\bar{x}, \cdot)$ *at the saddle point.*

Proof. Let us denote by $\omega(\cdot)$ a modulus of continuity of the function $h(\bar{x}, \cdot)$, and observe that being continuous and periodic it is bounded. Fix an initial state, and a point $\bar{y}^A$ on $\mathbb{R}^{M_A}$, as in Definition 3.10; so there exists a strategy $\widetilde{\alpha}$ such that, for the corresponding path $\widetilde{y}_s$, one has as in the preceding proof,

$$h(\bar{x}, \widetilde{y}_s^A, \widetilde{y}_s^B) \leq \omega(|\widetilde{y}_s^A - \bar{y}^A|) + h(\bar{x}, \bar{y}^A, \widetilde{y}_s^B)$$

and therefore, for any $b \in \mathcal{B}$,

$$h(\bar{x}, \widetilde{y}_t^A, \widetilde{y}_t^B) \leq \omega(\eta(t)) + \max_{y^B} h(\bar{x}, \bar{y}^A, y^B)$$

Then

$$\inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} h(\bar{x}, y_t) \leq \sup_{b \in \mathcal{B}} h(\bar{x}, \widetilde{y}_t^A, \widetilde{y}_t^B) \leq \omega(\eta(t)) + \max_{y^B} h(\bar{x}, \bar{y}^A, y^B)$$

Taking the lim sup for $t \to +\infty$, and the minimum over all $\bar{y}^A$ we get

$$\limsup_{t \to +\infty} \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} h(\bar{x}, y_t^A, y_t^B) \leq \min_{y^A} \max_{y^B} h(\bar{x}, y^A, y^B)$$

Similarly we obtain the converse inequality:

$$\liminf_{t\to+\infty} \sup_{\beta\in\Delta} \inf_{a\in\mathcal{A}} h(\bar{x}, y_t^A, y_t^B) \geq \max_{y^B} \min_{y^A} h(\bar{x}, y^A, y^B)$$

Since $h(\bar{x}, \cdot)$ has a saddle point, say $(y_\sharp^A, y_\sharp^B)$, *i.e.*

$$h(\bar{x}, y_\sharp^A, y_\sharp^B) = \max_{y^B} \min_{y^A} h(\bar{x}, y^A, y^B) = \min_{y^A} \max_{y^B} h(\bar{x}, y^A, y^B)$$

one has

$$\limsup_{t\to+\infty} \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} h(\bar{x}, y_t^A, y_t^B) \leq h(\bar{x}, y_\sharp^A, y_\sharp^B) \leq \liminf_{t\to+\infty} \sup_{\beta\in\Delta} \inf_{a\in\mathcal{A}} h(\bar{x}, y_t^A, y_t^B)$$

therefore, by the representation formula (3.9) for the solution of the recessive Cauchy problem, one has

$$\lim_{t\to+\infty} v(t, y; \bar{x}) = h(\bar{x}, y_\sharp^A, y_\sharp^B)$$

□

Proposition 3.15. *Suppose the fast subsystem* (3.4) *is in splitted form* (y^A, y^B)*. Assume also that, at* $x = \bar{x}$*:*

i. The fast subsystem (3.4) *is asymptotically controllable by the first player with respect to the target*

$$\mathcal{T}^*(\bar{x}) := \left\{ (z^A, z^B) : \; h(\bar{x}, z^A, z^B) \leq \max_{y^B} \min_{y^A} h(\bar{x}, y^A, y^B) \right\}$$

ii. The y^B *variables are asymptotically controllable by the second player with respect to the target*

$$\mathcal{T}^B(\bar{x}) := \arg\max \min_{y^A} h(\bar{x}, y^A, \cdot)$$

Then, the pair (H, h) *is stabilizing at* $x = \bar{x}$*, and*

$$\bar{h}(\bar{x}) = \max_{y^B} \min_{y^A} h(\bar{x}, y^A, y^B)$$

Proof. By the assumptions, there are functions η, ν such that, for any initial state y^0, there is a strategy $\widetilde{\alpha} \in \Gamma$ such that

$$d(\widetilde{y}_t, \mathcal{T}^*(\bar{x})) \leq \eta(t) \quad \forall t > 0, \, \forall b \in \mathcal{B}$$

where $\widetilde{y}_\cdot$ is a solution of

$$\dot{y}_s = g(\bar{x}, y_s, \widetilde{\alpha}[b]_s, b_s), \quad y_0 = y^0 \tag{3.17}$$

and there is $\widetilde{\beta} \in \Delta$ such that

$$d(\widetilde{y}_t^B, \mathcal{T}^B(\bar{x})) \leq \nu(t) \quad \forall t > 0, \, \forall a \in \mathcal{A}$$

where $\widetilde{y}^B_\cdot$ is the B–part of the solution $\widetilde{y}_\cdot$ of

$$\dot{y}_s = g(\bar{x}, y_s, a_s, \widetilde{\beta}[a]_s), \quad y_0 = y^0 \tag{3.18}$$

Fix y^0, and let $\widetilde{y}_s$ be the solution of (3.17). Denote by $z_s = (z^A_s, z^B_s)$ the projection of $\widetilde{y}_s$ on $\mathcal{T}^*(\bar{x})$, *i.e.*

$$h(\bar{x}, z^A_s, z^B_s) \leq \max_{y^B} \min_{y^A} h(\bar{x}, y^A, y^B) \quad \text{for any } s > 0$$

We have

$$h(\bar{x}, \widetilde{y}_s) \leq \omega(|\widetilde{y}_s - z_s|) + \max_{y^B} \min_{y^A} h(\bar{x}, y^A, y^B) \tag{3.19}$$

where we have denoted with $\omega(\cdot)$ a modulus of continuity of the function $h(\bar{x}, \cdot)$. It follows that

$$h(\bar{x}, \widetilde{y}_t) \leq \omega(\eta(t)) + \max_{y^B} \min_{y^A} h(\bar{x}, y^A, y^B), \quad \forall t > 0, \forall b \in \mathcal{B} \tag{3.20}$$

Now denote again by $\widetilde{y}_s = (\widetilde{y}^A_s, \widetilde{y}^B_s)$ the solution of (3.18), and consider its projection on the target $\mathcal{T}^B(\bar{x})$

$$z^B_s := \text{proj}_{\mathcal{T}^B(\bar{x})}(\widetilde{y}^B_s)$$

By the definition of $\mathcal{T}^B(\bar{x})$ we have:

$$h(\bar{x}, \widetilde{y}^A_s, \widetilde{y}^B_s) \geq -\omega(d(\widetilde{y}^B_s, \mathcal{T}^B)) + \max_{y^B} \min_{y^A} h(\bar{x}, y^A, y^B) \tag{3.21}$$

and therefore

$$h(\bar{x}, \widetilde{y}^A_t, \widetilde{y}^B_t) \geq -\omega(\nu(t)) + \max_{y^B} \min_{y^A} h(\bar{x}, y^A, y^B), \quad \forall t > 0, \forall a \in \mathcal{A} \tag{3.22}$$

So, (3.20) yields

$$\inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} h(\bar{x}, y_t) \leq \sup_{b \in \mathcal{B}} h(\bar{x}, \widetilde{y}_t) \leq \omega(\eta(t)) + \max_{y^B} \min_{y^A} h(\bar{x}, y^A, y^B)$$

and therefore

$$\limsup_{t \to +\infty} \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} h(\bar{x}, y_t) \leq \max_{y^B} \min_{y^A} h(\bar{x}, y^A, y^B) \tag{3.23}$$

on the other hand, (3.22) yields

$$\sup_{\beta \in \Delta} \inf_{a \in \mathcal{A}} h(\bar{x}, y_t) \quad \geq \quad \inf_{a \in \mathcal{A}} h(\bar{x}, \widetilde{y}_t) \geq -\omega(\nu(t)) + \max_{y^B} \min_{y^A} h(\bar{x}, y^A, y^B)$$

and therefore

$$\liminf_{t \to +\infty} \sup_{\beta \in \Delta} \inf_{a \in \mathcal{A}} h(\bar{x}, y_t) \geq \max_{y^B} \min_{y^A} h(\bar{x}, y^A, y^B) \tag{3.24}$$

By (3.23) and (3.24) we finally get

$$\lim_{t \to +\infty} \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} h(\bar{x}, y_t) = \max_{y^B} \min_{y^A} h(\bar{x}, y^A, y^B)$$

and then, by the representation formula (3.9), the assertion is proved.

□

Example with non-stabilizing initial datum. We conclude this section providing an example in which the stabilizing property does not hold. The example is constructed considering a game where each player controls one of the two groups in which the state variables are divided. If the two controllers use the same dynamics, no players have the possibility to drive the system faster than the other.

Suppose that the fast sub-system of the singularly perturbed differential game is given by

$$\begin{aligned} \dot{\xi}_t &= g_1(\bar{x}, \bar{y}, \xi_t, \eta_t, a_t) \\ \dot{\eta}_t &= g_2(\bar{x}, \bar{y}, \xi_t, \eta_t, b_t) \end{aligned} \tag{3.25}$$

where (x, y) are the slow variables and (ξ, η) the corresponding fast variables and g^A, g^B are Lipschitz–continuous function. Suppose also that the running cost is $l \equiv 0$ and consider an initial datum of class C^1, enjoying the following form:

$$h = h(\bar{x}, \bar{y}, \xi - \eta)$$

The Isaac's Hamiltonian related to the game is

$$H(x, y, \xi, \eta, p_x, p_y) = \min_{b \in B} \max_{a \in A} \{-p_x \cdot g_1(x, y, \xi, \eta, a) - p_y \cdot g_2(x, y, \xi, \eta, b)\}$$

$$= \max_{a \in A} \{-p_x \cdot g(x, y, \xi, \eta, a)\} + \min_{b \in B} \{-p_y \cdot g(x, y, \xi, \eta, b)\}$$

Let us assume $A = B$, $M_A = M_B = M/2$, $g_1 = g_2 = g$ and put

$$H_0(x, y, \xi, \eta, p) := \max_{a \in A} \{-p \cdot g(x, y, \xi, \eta, a)\}.$$

Then

$$H(x, y, \xi, \eta, p_x, p_y) = H_0(x, y, \xi, \eta, p_x) - H_0(x, y, \xi, \eta, -p_y)$$

The Hamiltonian G related to the homogenization problem is

$$G(x, y, \xi, \eta, p_x, p_y, q_\xi, q_\eta) := H(x, y, \xi, \eta, p_x + q_\xi, p_y + q_\eta)$$

Let us compute its recession function G':

$$G'(x, y, \xi, \eta, q_\xi, q_\eta) = \lim_{\lambda \to 0^+} \frac{1}{\lambda} G(x, y, \xi, \eta, 0, 0, \lambda q_\xi, \lambda q_\eta) = H(x, y, \xi, \eta, q_\xi, q_\eta)$$

We are interested in the asymptotic behavior of the solution $u(t, \xi, \eta; \bar{x}, \bar{y})$ of the problem

$$\begin{aligned} &\partial_t u + G'(\bar{x}, \bar{y}, \xi, \eta, D_\xi u, D_\eta u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^{M/2} \times \mathbb{R}^{M/2} \\ &u(0, \xi, \eta) = h(\bar{x}, \bar{y}, \xi - \eta) \end{aligned}$$

for fixed $(\bar{x}, \bar{y})$.

Since $h \in C^1$, the steady solution $u \equiv h$ solves the problem. The fact that h is not a constant shows that the solution cannot converge to any constant as $t \to +\infty$.

3.2 Homogenization of Convex-Concave HJ Equations

In the present section we will provide homogenization results for a class of non–coercive Hamilton–Jacobi equations. We will concentrate our attention on the cases in which the dynamics is in *splitted form.* This means that a player controls a group of variables, and the other player controls the remaining variables. Under this assumption the ergodic problem also is decoupled. Anyway, we do not require such a decoupling in the oscillating initial datum h.

Notation. We divide the state variables in two groups, say $z = (x, y)$; with the Greek letters $\zeta = (\xi, \eta)$ we will denote the vector of the fast variables.

The problem under investigation is then the following:

$$\begin{array}{l} \partial_t v^\varepsilon + H(z, \frac{z}{\varepsilon}, D_z v^\varepsilon) = l(z, \frac{z}{\varepsilon}) \\ v^\varepsilon(0, z) = h(z, \frac{z}{\varepsilon}) \end{array} \tag{3.26}$$

We assume for H the following form:

$$H(z, \zeta, p) = \max_{|a| \le 1} \min_{|b| \le \gamma} \{-p \cdot f(z, \zeta, a, b) - l(z, \zeta)\} \tag{3.27}$$

where γ is a positive parameter, and

$$f(z, \zeta, a, b) = (f_1(x, y, \xi, a), f_2(x, y, \eta, b)) \tag{3.28}$$

$$l(z, \zeta) = l_1(x, y, \xi) + l_2(x, y, \eta) \tag{3.29}$$

Then, defining the following Hamiltonians H_1 and H_2

$$H_1(x, y, \xi, p) = \max_{a \in A} -p \cdot f_1(x, y, \xi, a)$$

$$H_2(x, y, \eta, q) = \max_{b \in B} -q \cdot f_2(x, y, \eta, b)$$

we can write the problem (3.26) as the following:

$$\begin{array}{l} \partial_t u^\varepsilon + H_1(x, y, \frac{x}{\varepsilon}, D_x u^\varepsilon) - H_2(x, y, \frac{y}{\varepsilon}, D_y u^\varepsilon) = l_1(x, y, \frac{x}{\varepsilon}) + l_2(x, y, \frac{y}{\varepsilon}) \\ u^\varepsilon(0, x, y) = h(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}) \end{array} \tag{3.30}$$

The known ergodicity results in the convex–concave setting require the system to be in splitted form, and l to be independent of the controls; see [5], [21]. This is the reason why we made such assumptions.

Under the current assumptions, the game associated to (3.26) is

$$\begin{array}{l} \dot{z} = f(z, \zeta, a, b) \\ \varepsilon\dot{\zeta} = f(z, \zeta, a, b) \end{array}$$

with f as in (3.28). Then, the fast subsystem, for $(\bar{x}, \bar{y})$ frozen, is

$$\begin{array}{ll} \dot{\xi}_t = f_1(\bar{x}, \bar{y}, \xi_t, a_t), & \xi_0 = \xi \\ \dot{\eta}_t = f_2(\bar{x}, \bar{y}, \eta_t, b_t), & \eta_0 = \eta \end{array} \tag{3.31}$$

We require the system (3.31) to satisfy the following

Assumption 3.16. Suppose that for any $(\bar{x}, \bar{y})$ the ξ variables are Bounded Time Controllable by the first player, and that the η variables are Bounded Time Controllable by the second player; *i.e.*

$$\begin{gathered}\text{for any } (\bar{x}, \bar{y}) \text{ exist } T, T' > 0 \text{ such that}\\ \text{for any } \xi, \widetilde{\xi} \text{ exists } a \in \mathcal{A}, \text{ and for any } \eta, \widetilde{\eta} \text{ exists } b \in \mathcal{B} \text{ such that}\\ \text{the solution } \xi_t \text{ in (3.31) satisfies, } \xi_{\bar{t}} = \widetilde{\xi} \text{ for some } \bar{t} \leq T\\ \text{and the solution } \eta_t \text{ in (3.31) satisfies } \eta_{\bar{t}} = \widetilde{\eta} \text{ for some } \bar{t} \leq T'\end{gathered}$$

Under these standing assumptions, the ergodic problem is decoupled, and there exist an effective Hamiltonian $\bar{H}_1$ associated to H_1 with the running cost l_1, and an effective Hamiltonian $\bar{H}_2$ associated to H_2 with running cost l_2. Hence it is possible to associate to the problem (3.30) the effective Hamiltonian $\bar{H} := \bar{H}_1 - \bar{H}_2$.

We need more controllability in order to establish the comparison principle for $\bar{H}$. A sufficient condition (see [2, Proposition 4]) for $\bar{H}$ to be Lipschitz–continuous is the following

Assumption 3.17. $\exists r_1, r_2 > 0$ such that $B(0, r_1) \subset \overline{\text{co}}\{f_1(x, y, \xi, a) : a \in A\}$ and $B(0, r_2) \subset \overline{\text{co}}\{f_2(x, y, \eta, b) : b \in B\}$.

In conclusion, we consider the following setting.

Assumption 3.18. At least one of the following conditions holds:

i. For each $(\bar{x}, \bar{y})$, the function $h(\bar{x}, \bar{y}, \xi, \eta)$ has a saddle point in (ξ, η);

ii. The function h enjoys the form $h(x, y, \xi, \eta) = h_1(x, y, \xi - \eta) + h_3(x, y, \eta)$, Assumption 3.16 holds and the system (3.31) is asymptotically controllable by the first player;

iii. The function h enjoys the form $h(x, y, \xi, \eta) = h_1(x, y, \xi - \eta) + h_2(x, y, \xi)$, Assumption 3.16 holds and the system (3.31) is asymptotically controllable by the second player.

Assumption 3.18 is a list of sufficient conditions for the stabilizing property. The fact that Assumption 3.18(i.) is sufficient for the stabilization of the initial datum has been proved in Proposition 3.14. The next result, based upon Proposition 3.15, shows that also Assumption 3.18(ii.) and (iii.) are sufficient conditions for the stabilizing property.

Proposition 3.19. *Under Assumption 3.18(ii.), the pair (H, h) is stabilizing, and for any (x, y),*

$$\bar{h}(x, y) = \max_{\eta} \min_{\xi} \{h_1(x, y, \xi - \eta) + h_3(x, y, \eta)\} \tag{3.32}$$

Similarly, under Assumption 3.18(iii.), the pair (H, h) is stabilizing, and for any (x, y),

$$\bar{h}(x, y) = \max_{\eta} \min_{\xi} \{h_1(x, y, \xi - \eta) + h_2(x, y, \xi)\} \tag{3.33}$$

Proof. We prove the assertion only under Assumption 3.18(ii.), being the proof under Assumption 3.18(iii.) very similar.

The system is in particular asymptotically controllable by the first player with respect to the target

$$\mathcal{T}_1(\bar{x}, \bar{y}) := \{(\xi, \eta) : \xi - \eta \in \arg\min h_1\}$$

If $h_3 \equiv 0$, then the assertion follows by Proposition 3.13. If $h_3 \not\equiv 0$, observe that since

$$\begin{aligned} \max_\eta \min_\xi h &= \max_\eta \min_\xi \{h_1(\bar{x}, \bar{y}, \xi - \eta) + h_3(\bar{x}, \bar{y}, \eta)\} \\ &= \min_{\xi,\eta} h_1(\bar{x}, \bar{y}, \xi - \eta) + \max_\eta h_3(\bar{x}, \bar{y}, \eta) \end{aligned}$$

then, if $(\bar{\xi}, \bar{\eta}) \in \mathcal{T}_1(\bar{x}, \bar{y})$,

$$\begin{aligned} h(\bar{x}, \bar{y}, \bar{\xi}, \bar{\eta}) &= \min_{\xi,\eta} h_1(\bar{x}, \bar{y}, \xi - \eta) + h_3(\bar{x}, \bar{y}, \bar{\eta}) \\ &\leq \min_{\xi,\eta} h_1(\bar{x}, \bar{y}, \xi - \eta) + \max_\eta h_3(\bar{x}, \bar{y}, \eta) \end{aligned}$$

This shows that $\mathcal{T}_1(\bar{x}, \bar{y})$ is contained in the set

$$\mathcal{T}^*(\bar{x}, \bar{y}) := \{(\xi, \eta) : h(\bar{x}, \bar{y}, \bar{\xi}, \bar{\eta}) \leq \max_\eta \min_\xi h(\bar{x}, \bar{y}, \xi, \eta)\}$$

and therefore, the system is asymptotically controllable on this set, being controllable on a subset of it, by the first player. Moreover, the η variables are controllable in bounded time by the second player, then they are asymptotically controllable by the second player on the target

$$\mathcal{T}_2(\bar{x}, \bar{y}) := \arg\max \min_\xi h(\bar{x}, \bar{y}, \xi, \cdot)$$

Then the conclusion and (3.32) follow by Proposition 3.15.

□

By the local uniform convergence of $u^\varepsilon(t, z, \zeta)$ to the solution $u(t, z)$ of the effective problem, developed in the singular perturbation context, we derive the following homogenization result

Theorem 3.20. *Suppose Assumption 3.16 and Assumption 3.18 hold. Then H is ergodic, the pair (H, h) is stabilizing, and there exist $\bar{H}$ and $\bar{h}$ such that the upper and lower semilimits of u^ε are respectively a subsolution and a supersolution of the effective problem*

$$\begin{cases} \partial_t u + \bar{H}(x, y, D_x u, D_y u) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^N \\ u(0, x, y) = \bar{h}(x, y) & \text{on } \{0\} \times \mathbb{R}^N \times \mathbb{R}^N \end{cases} \tag{3.34}$$

where:

- *if Assumption 3.18(i.) holds, then $\bar{h}(x, y)$ is given by the value of $h(x, y, \cdot, \cdot)$ at the saddle point;*
- *if Assumption 3.18 (ii.) (resp. Assumption 3.18 (iii.)) holds, then $\bar{h}(x, y)$ is given by formula* (3.32) *(resp.* (3.32)*).*

Furthermore, if Assumption 3.17 holds, $\bar{H}$ is Lipschitz–continuous in x and then u^ε converge, locally uniformly on the compact subset of $(0, T) \times \mathbb{R}^N$, $T > 0$, as $\varepsilon \to 0^+$, to the unique solution of (3.34).

Proof. Ergodicity properties come from Assumption 3.16 and stabilization properties are guaranteed by Assumption 3.18 thanks to Proposition 3.14 and Proposition 3.19. The conclusion then follows by the convergence results contained in [3]. The regularity properties of $\bar{H}$ follow by [2, Proposition 4].

□

3.2.1 The convex–concave Eikonal equation

In the light of the example of non stabilizing initial datum given before, we concentrate our attention on the following problem. Suppose that the fast subsystem to be

$$\begin{array}{ll} \dot{\xi}_t = g(x, y, \xi_t)a_t & \\ \dot{\eta}_t = g(x, y, \eta_t)b_t & \end{array} \qquad \xi_t, \eta_t \in \mathbb{R}^N$$

where x, y are fixed, g is a function such that $g \geq g_0 > 0$, and for a fixed parameter $\gamma > 0$, the control functions a_t and b_t satisfy

$$|a_t| \leq 1, \qquad |b_t| \leq \gamma$$

The role of γ is clear: being the dynamics for the two group of variables essentially the same, the parameter γ allows a player to drive the system faster or slower than the other.

We continue the description of the model. Accordingly with the previous section, the running cost is supposed to be in the form

$$l(x, y, \xi, \eta) = l_1(x, y, \xi) + l_2(x, y, \eta) \tag{3.35}$$

The min − max Hamiltonian related to this game is given by

$$H(x, y, \xi, \eta, p_x, p_y) = H_0(x, y, \xi, p_x) - \gamma H_0(x, y, \eta, -p_y) - l(x, y, \xi, \eta)$$

where

$$H_0(x, y, \xi, p) := \max_{|a| \leq 1}\{-g(x, y, \xi)a \cdot p\} = g(x, y, \xi)|p|$$

Then we deal with the homogenization of

$$\begin{array}{l} \partial_t u^\varepsilon + g\left(x, y, \frac{x}{\varepsilon}\right)|D_x u^\varepsilon| - \gamma g\left(x, y, \frac{y}{\varepsilon}\right)|D_y u^\varepsilon| = l\left(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \\ u^\varepsilon(0, x, y) = h\left(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \end{array} \tag{3.36}$$

where the motivation of the name *convex–concave eikonal equation* is apparent.

Being g strictly positive, and the control sets the closed balls $B(0, 1)$ and $B(0, \gamma)$, the ξ variables are bounded time controllable by the first player and the η variables are bounded time controllable by the second player. In order to apply Theorem 3.20, we made the following requirements on the system:

Assumption 3.21. For any (x, y)

i. The running cost l is in the form (3.35).

ii. Either the function $h(x, y, \cdot, \cdot)$ has a saddle point,
or, $h = h_1(x, y, \xi - \eta) + h_3(x, y, \eta)$ and $\gamma < 1$
or, $h = h_1(x, y, \xi - \eta) + h_2(x, y, \xi)$ and $\gamma > 1$

Assumption 3.21 (ii.) translates in the current setting the Assumption 3.18. In fact, if for example $\gamma < 1$, since the dynamics of the ξ and the η variables is the same, but the first player can drive ξ at an higher speed, for any fixed $\eta' \in \mathbb{R}^{M/2}$ the first player can drive the system from any initial position to $\xi = \eta + \eta'$ in finite time, for all controls of the second player. Furthermore, thanks to the periodicity assumption on the fast variables, this can be done in a uniformly bounded time. Then, in particular, the fast subsystem is asymptotically controllable by the first player. We directly derive from Theorem 3.20 the following result for the convex–concave eikonal equation.

Theorem 3.22. *Under Assumption 3.21, there exist $\bar{H}$ and $\bar{h}$ such that the solutions u^ε of* (3.36) *converge as $\varepsilon \to 0^+$, to the solution of*

$$\begin{aligned} &\partial_t u + \bar{H}(x, y, D_x u, D_y u) = 0 \\ &u(0, x, y) = \bar{h}(x, y) \end{aligned}$$

3.2.2 The 2D case

This section is devoted to specialize in $\mathbb{R}^2$ the preceding result for the convex-concave eikonal equation. In fact, explicit representation formulae for the effective Hamiltonian are known since [66] in the one-dimensional case.

Let us simplify the setting of the preceding section requiring the dimension N to be equal 1, and $g \equiv 1$. So the dynamics for the fast states is the following:

$$\begin{aligned} &\dot{\xi}_t = a_t \quad \text{with } |a_t| \le 1 \\ &\dot{\eta}_t = b_t \quad \text{with } |b_t| \le \gamma \end{aligned} \tag{3.37}$$

where γ, as before, is a fixed positive parameter.

Then we deal with the following PDE:

$$\begin{aligned} &\partial_t u^\varepsilon + |\partial_x u^\varepsilon| - \gamma |\partial_y u^\varepsilon| = l_1(\tfrac{x}{\varepsilon}) + l_2(\tfrac{y}{\varepsilon}) && \text{in } (0, +\infty) \times \mathbb{R} \times \mathbb{R} \\ &u^\varepsilon(0, x, y) = h(x, y, \tfrac{x}{\varepsilon}, \tfrac{y}{\varepsilon}) && \text{on } \{0\} \times \mathbb{R} \times \mathbb{R} \end{aligned} \tag{3.38}$$

The cell problem, for fixed (x, y), (p_x, p_y) is the following

$$\begin{aligned} &\partial_t v + |p_x + \partial_\xi v| - \gamma |p_y + \partial_\eta v| = l_1(\xi) + l_2(\eta) && \text{in } (0, +\infty) \times \mathbb{R} \times \mathbb{R} \\ &v(0, \xi, \eta) = 0 && \text{on } \{0\} \times \mathbb{R} \times \mathbb{R} \end{aligned} \tag{3.39}$$

In order to solve the cell problem, we can split (3.39) in two problems, and seek for two functions $v^1 = v^1(t, \xi)$ and $v^2 = v^2(t, \eta)$ such that

$$\partial_t v^1 + |p_x + \partial_\xi v^1| = l_1(\xi) \qquad v^1(0, \xi) = 0 \tag{3.40}$$

and

$$\partial_t v^2 - \gamma |p_y + \partial_\eta v^2| = l_2(\eta) \qquad v^2(0, \eta) = 0 \tag{3.41}$$

For these two ergodic problems there exist explicit formulae for the effective Hamiltonian, that we briefly recall below.

General facts. Let us recall some general facts about homogenization in dimension 1; cf. [66]. We consider the equation

$$|\chi'(x) + p| - l(x) = \lambda \tag{3.42}$$

and, for any fixed p we look for the unique value λ such that (3.42) admits a periodic solution. Of course this solution will be non unique, being defined up to an additive constant.

Let us suppose, for simplicity, that $\min_{[0,1]} l(x) = l(x_0) = 0$, and denote by $\langle l \rangle$ the average of l on a period, $\int_0^1 l(x)dx$. We claim that

$$\lambda = (|p| - \langle l \rangle)^+ \tag{3.43}$$

where $(\cdot)^+$ stands for the positive part. To prove the claim, we show that, under the position (3.43) it is possible to exhibit a viscosity solution of (3.42).

If $|p| \leq \langle l \rangle$, then a solution of (3.42) is given by the function

$$\chi(x) = \begin{cases} \int_{x_0}^{x} (l(s) - p)ds & \text{if } x_0 \leq x \leq \bar{x} \\ \\ \int_{x}^{x_0+1} (l(s) + p)ds & \text{if } \bar{x} \leq x \leq x_0 + 1 \end{cases}$$

extended by periodicity to the whole $\mathbb{R}$. In the definition of χ the point $\bar{x}$ is such that

$$\int_{x_0}^{\bar{x}} (l(s) - p)ds = \int_{\bar{x}}^{x_0+1} (l(s) + p)ds$$

Such a $\bar{x}$ exists, in fact if we put

$$\varphi_1(x) := \int_{x_0}^{x} (l(s) - p)ds, \qquad \varphi_2(x) := \int_{x}^{x_0+1} (l(s) + p)ds$$

and observe that

$$\varphi_1(x_0) = 0, \qquad \varphi_1(x_0 + 1) = \langle l \rangle - p \geq 0$$

$$\varphi_2(x_0) = \langle l \rangle + p \geq 0, \qquad \varphi_2(x_0 + 1) = 0$$

the existence of $\bar{x}$ follows by the intermediate value Theorem. Then the function χ is well defined, and is a viscosity solution of (3.42). In fact it is continuous in $[x_0, x_0 + 1] \setminus \{\bar{x}\}$, since l is, and in $\bar{x}$ by the very definition of $\bar{x}$. Moreover

$$\chi'(x) = \begin{cases} l(x) - p & \text{if } x_0 \leq x \leq \bar{x} \\ \\ -l(x) - p & \text{if } \bar{x} \leq x \leq x_0 + 1 \end{cases}$$

then, in $\bar{x}$

$$\lim_{x \to \bar{x}^-} \chi'(x) = l(\bar{x}) - p \geq -l(\bar{x}) - p = \lim_{x \to \bar{x}^+} \chi'(x)$$

and, if $x \neq \bar{x}$, (3.42) holds.

If $|p| \geq \langle l \rangle$, then a solution of (3.42) with λ given by (3.43), is

$$\chi(x) = \int_{\hat{x}}^{x} (l(s) + \lambda - p)ds, \quad \text{for any } x \in [x_0, x_0 + 1]$$

extended on $\mathbb{R}$ by periodicity. In this case, $\hat{x}$ is such that $l(\hat{x}) = \langle l \rangle$. Such a point exists by the mean value theorem.

Back to (3.39)**.** Let us adapt this computations first for (3.40). In what follows we allow also the general case, in which $\min l \neq 0$. One has:

$$\bar{H}_1(p_x) = (|p_x| - \langle l_1 \rangle)^+ + \min_{[0,1]} l_1(x) \tag{3.44}$$

This is a straightforward derivation of the preceding calculations.

In an analogous way we find, for the problem (3.41),

$$\bar{H}_2(p_y) = -(\gamma|p_y| - \langle l_2 \rangle)^+ - \max_{[0,1]} l_2(x) \tag{3.45}$$

In this case, in fact, if $\gamma|p_y| \leq \langle l_2 \rangle$ a solution for the cell problem

$$-\gamma|\chi'(x) + p_y| - l_2(x) = \lambda \tag{3.46}$$

at fixed p_y, with λ given by the position (3.45), is given by the periodic extension on $\mathbb{R}$ of the function

$$\chi(x) = \begin{cases} \int_{x_0}^{x} \left(\frac{l(s)}{\gamma} - p_y\right) ds & \text{if } x_0 \leq x \leq \bar{x} \\ \\ \int_{x}^{x_0+1} \left(\frac{l(s)}{\gamma} + p_y\right) ds & \text{if } \bar{x} \leq x \leq x_0 + 1 \end{cases}$$

where $x_0 \in [0,1]$ is such that $l(x_0) = \max_{[0,1]} l_2$, and $\bar{x}$ is defined as before, by the relation

$$\int_{x_0}^{\bar{x}} (l_2(s) - p_y) ds = \int_{\bar{x}}^{x_0+1} (l_2(s) + p_y) ds$$

If $\gamma|p_y| \geq \langle l_2 \rangle$, a solution of the cell problem (3.46) whit λ provided by the position (3.45), is given by

$$\chi(x) = \int_{x_0}^{x} \left(\frac{l_2(s)}{\gamma} + \lambda - p\right) ds, \quad \text{for any } x \in [x_0, x_0+1]$$

where x_0 is such that $l_2(x_0) = \langle l_2 \rangle$.

Finally, by (3.44) and (3.45), we can write the following explicit expression for the effective Hamiltonian

$$\bar{H}(p_x, p_y) = \min_{[0,1]} l_1(x) - \max_{[0,1]} l_2(x) + (|p_x| - \langle l_1 \rangle)^+ - (\gamma|p_y| - \langle l_2 \rangle)^+ \tag{3.47}$$

In conclusion, with some additional assumption, we can establish the stabilizing property, and formulae for the effective initial datum. So, the effective problem for the 2D convex–concave eikonal equation can be explicitly written. The additional assumptions we require, are those used in Theorem 3.22

Theorem 3.23. *Suppose that, for any $(x,y) \in \mathbb{R}^2$ there exists $(\xi_{x,y}, \eta_{x,y}) \in \mathbb{R}^2$ saddle point of the function $(\xi, \eta) \mapsto h(x, y, \xi, \eta)$. Then the solutions u^ε of* (3.38) *converge as $\varepsilon \to 0^+$ to the solution of the effective problem*

$$\begin{cases} \partial_t u + \bar{H}(D_x u, D_y u) = 0 & \textit{in } (0, +\infty) \times \mathbb{R} \times \mathbb{R} \\ u(0, x, y) = h(x, y, \xi_{x,y}, \eta_{x,y}) & \textit{on } \{0\} \times \mathbb{R} \times \mathbb{R} \end{cases}$$

where $\bar{H}(p_x, p_y)$ is given by (3.47)*.*

Theorem 3.24. *Suppose that either*

$$h(x,y,\xi,\eta) = h_1(x,y,\xi-\eta) + h_3(x,y,\eta), \qquad \textit{and } \gamma < 1$$

or

$$h(x,y,\xi,\eta) = h(x,y,\xi,\eta) = h_1(x,y,\xi-\eta) + h_2(x,y,\xi), \qquad \textit{and } \gamma > 1$$

Then the solutions u^ε of (3.38) converge as $\varepsilon \to 0^+$ to the solution of the effective problem

$$\begin{array}{ll} \partial_t u + \bar{H}(D_x u, D_y u) = 0 & \textit{in } (0,+\infty)\times\mathbb{R}\times\mathbb{R} \\ u(0,x,y) = \max_\eta \min_\xi h(x,y,\xi,\eta) & \textit{on } \{0\}\times\mathbb{R}\times\mathbb{R} \end{array}$$

where $\bar{H}(p_x,p_y)$ is given by (3.47).

3.3 Homogenization with strongly dependent fast variables

In the previous section, we studied the case in which the two competitors control a group of variables, and the ergodic problem is splitted in two sub-problems. The purpose of this section is to study a complete different case. Such strong interaction is apparent in the expression of the Isaacs' equation.

Let us denote by $x, y \in \mathbb{R}^{N/2}$ two groups of variables. We consider the following homogenization problem

$$\begin{array}{ll} \partial_t u^\varepsilon + H\left(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},D_x u^\varepsilon, D_y u^\varepsilon\right) = 0 & \text{in } (0,+\infty)\times\mathbb{R}^N\times\mathbb{R}^N \\ u(0,x,y) = h\left(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right) & \text{on } \{0\}\times\mathbb{R}^N\times\mathbb{R}^N \end{array} \tag{3.48}$$

assuming H and h have the following form:

$$H(x,y,\xi,\eta,p_x,p_y) = H_1(x,y,\xi-\eta,p_x) - H_2(x,y,\xi-\eta,p_y) \tag{3.49}$$

$$h = h(x,y,\xi-\eta) \tag{3.50}$$

The Greek letters ξ and η are used to indicate the fast variables associated to x and y, respectively.

We want to prove ergodicity and stabilization properties for H, h, accordingly with the definitions given in Section 1.1. To establish ergodicity for H, we fix x, y and p_x, p_y and deal with the following cell problem:

$$\partial_t v + H_1(\bar{x},\bar{y},\xi-\eta,D_\xi v+\bar{p}_x) - H_2(\bar{x},\bar{y},\xi-\eta,D_\eta v+\bar{p}_y) = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N \tag{3.51}$$

with initial condition $v(0,\bar{x},\bar{y},\xi,\eta) = 0$. Recall that ergodicity holds if

$$\lim_{t\to\infty} \frac{v(t,\bar{x},\bar{y},\xi,\eta)}{t} = \text{const}, \text{ uniformly in } \xi,\eta \tag{3.52}$$

and that stabilization holds if the unique bounded viscosity solution $w(t,\bar{x},\bar{y},\xi,\eta)$ of the cell problem for the homogeneous part of H associated to (3.51) with initial condition $w(0,\bar{x},\bar{y},\xi,\eta) = h(\bar{x},\bar{y},\xi-\eta)$ satisfies

$$\lim_{t\to\infty} w(t,\bar{x},\bar{y},\xi,\eta) = \text{const}, \text{ uniformly in } \xi,\eta.$$

Let us concentrate on the problem (3.51). Motivated by the way in which H and h depend on the fast variables, we set $\xi - \eta =: \zeta$, and guess the solution v has the following form:

$$v(t, \xi, \eta) = u(t, \xi - \eta).$$

Hereafter, to ease notations, we omit to write the frozen variables $\bar{x}$, $\bar{y}$. The function $u(t, \zeta)$ solves the following problem:

$$\begin{array}{ll} \partial_t u + H_1(\zeta, D_\zeta u + p_x) - H_2(z, -D_\zeta u + p_y) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N \\ u(0, \zeta) = h(\zeta) & \text{on } \{0\} \times \mathbb{R}^N \end{array}$$

If the operator

$$\mathcal{H}(\zeta, P) := H_1(\zeta, P) - H_2(\zeta, -P)$$

is ergodic, than $u(t, \zeta)$ converges, as t goes to infinity, to a constant, which is uniform in ζ; consequently (3.52) holds. Let us summarize this simple remark in the following statement.

Lemma 3.25. *Assume H and h satisfy* (3.49) *and* (3.50)*. If $\mathcal{H}$ is ergodic, then H is ergodic. Similarly, the pair (H, h) is stabilizing if $(\mathcal{H}, h)$ is so.*

Example 3.26. Let γ be a positive fixed parameter. Consider

$$H(\xi, \eta, p_x, p_y) = |p_x| - \gamma |p_y| - l(\xi - \eta)$$

(the fixed variables $\bar{x}$ and $\bar{y}$ are dropped in the notations). Then we can write H in the form (3.49) with

$$H_1(\xi, \eta, p_x) = |p_x| - \frac{l(\xi - \eta)}{1 - \gamma}, \qquad H_2(\xi, \eta, p_y) = \gamma |p_y| - \frac{\gamma l(\xi - \eta)}{1 - \gamma}$$

In this case the cell problem (3.51) is

$$\partial_t v + |D_\xi v + p_x| - \gamma |D_\eta v + p_y| = l(\xi - \eta)$$

Applying the change of variables $\zeta = \xi - \eta$, it is apparent that, if $\gamma \neq 1$ the operator

$$\mathcal{H}(z, P) := H_1(\zeta, P) - H_2(\zeta, -P) = (1 - \gamma)|P| - l(\zeta)$$

is coercive, *i.e.*

$$\lim_{|P| \to +\infty} \mathcal{H}(z, P) = +\infty$$

and then it is ergodic. □

The previous simple example can be easily generalized to a class of convex–concave eiconal equations which is disjoint by the one considered in Theorem 3.22.

Proposition 3.27. *Let γ be a positive fixed parameter, and g a function such that $g \geq g_0 > 0$. If $\gamma \neq 1$ then there exist $\bar{H}$ and $\bar{h}$ such that the solutions u^ε of*

$$\begin{array}{l} \partial_t u^\varepsilon + g\left(x, y, \frac{x-y}{\varepsilon}\right) |D_x u^\varepsilon| - \gamma g\left(x, y, \frac{x-y}{\varepsilon}\right) |D_y u^\varepsilon| = l\left(x, y, \frac{x-y}{\varepsilon}\right) \\ u^\varepsilon(0, x, y) = h\left(x, y, \frac{x-y}{\varepsilon}\right) \end{array}$$

converge as $\varepsilon \to 0^+$*, to the solution of*

$$\begin{aligned} &\partial_t u + \bar{H}(x, y, D_x u, D_y u) = 0 \\ &u(0, x, y) = \bar{h}(x, y) \end{aligned}$$

Proposition 3.7 shows that, even if the two Hamiltonians H_1 and H_2 are coercive, but they do not enjoy the form (3.49), then homogenization may not hold. More precisely, we can always exhibit a cost l for which homogenization fails. The next Proposition completes the description and generalize the statement of Proposition 3.27.

Proposition 3.28. *Assume* H *and* h *satisfy* (3.49) *and* (3.50) *respectively, and that suitable positive constants* $\alpha_1, \alpha_2, \beta, C_1, C_2$ *do exist such that*

$$\begin{aligned} &\left|H_1(x, y, \xi - \eta, p_x) - \alpha_1 |p_x|^\beta\right| \le C_1 \\ &\left|H_2(x, y, \xi - \eta, p_x) - \alpha_2 |p_y|^\beta\right| \le C_2 \end{aligned} \tag{3.53}$$

If $\alpha_1 \neq \alpha_2$ *then homogenization holds,* i.e. *there exist* $\bar{H}$ *and* $\bar{h}$ *such that the solutions* u^ε *of* (3.48) *converge as* $\varepsilon \to 0^+$*, to the solution of*

$$\begin{aligned} &\partial_t u + \bar{H}(x, y, D_x u, D_y u) = 0 \\ &u(0, x, y) = \bar{h}(x, y) \end{aligned}$$

Proof. Let us omit in the notation the variables x, y. In the light of the comments made in the beginning of the section, it is enough to prove that is ergodic the operator $\mathcal{H}(z, P) := H_1(z, P) - H_2(z, -P)$, where we posed $z = \xi - \eta$.

Now, if (3.53) holds, we easily check that

$$H_1(z, P) - H_2(z, -P) \ge (\alpha_1 - \alpha_2)|P|^\beta + C$$

being C a certain constant. Then, if $\alpha_1 > \alpha_2$ (respectively, $<$) then $\mathcal{H}$ (resp, $-\mathcal{H}$) is coercive in the P variables, and then ergodic. The conclusion follows by Lemma 3.25.

□

Thanks to Proposition 3.7 and Proposition 3.28 the analysis of non-coercive Hamiltonians of the form (3.49) satisfying condition (3.53) becomes very sharp.

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