UNIVERSITÀ DEGLI STUDI DI PADOVA

UNIVERSITÀ DEGLI STUDI DI PADOVA
Sede amministrativa: Università degli Studi di Padova
Dipartimento di Matematica Pura e Applicata

SCUOLA DI DOTTORATO DI RICERCA IN: MATEMATICA
INDIRIZZO: MATEMATICA
CICLO: XXIV

Some new results on reaction-diffusion equations and geometric flows

Direttore della Scuola: Ch.mo Prof. PAOLO DAI PRA

Coordinatore d'indirizzo: Ch.mo Prof. FRANCO CARDIN

Supervisore: Ch.mo Prof. PIERPAOLO SORAVIA

Dottorando: CECILIA DE ZAN

Abstract

In this thesis we discuss the asymptotic behavior of the solutions of scaled reaction-diffusion equations in the unbounded domain $\mathbb{R}^n \times (0 + \infty)$, in the cases when such a behavior is described in terms of moving interfaces.

As first class of asymptotic problems we consider the singular limit of bistable reaction-diffusion equations in the case when the velocity of the traveling wave equation depends on the space variable, i.e. $c^\varepsilon = c^\varepsilon(x)$, and it satisfies, in some suitable sense, $c^\varepsilon/\varepsilon^\tau \to \alpha$, as $\varepsilon \to 0^+$, where α is a discontinuous function and τ is an integer that can be equal to 0 or 1.

The second part of the thesis concerns semilinear reaction-diffusion equations with diffusion term of type $\mathrm{tr}(A_\varepsilon(x)D^2u_\varepsilon)$, where tr denotes the trace operator, $A_\varepsilon = \sigma_\varepsilon\sigma_\varepsilon^t$ for some matrix map $\sigma_\varepsilon : \mathbb{R}^n \to \mathbb{R}^{n\times(m+n)}$ and A_ε converges to a degenerate matrix.

In order to establish such results rigorously, we modify and adapt to our problems the ”geometric approach” introduced by G. Barles and P. E. Souganidis for solving problems in $\mathbb{R}^n$, and then partially revisited by G. Barles and F. Da Lio for reaction-diffusion equations in bounded domains. When it is possible we always consider the question of the well posedness of the Cauchy problems governing the motion of the fronts that describe the asymptotics we consider.

Sommario

In questa tesi discutiamo il comportamento asintotico delle soluzioni di equazioni di reazione-diffusione nel dominio illimitato $\mathbb{R}^n \times (0, +\infty)$ nei casi in cui tale comportamento sia descritto da un'interfaccia in movimento.

Come primo tipo di problemi asintotici consideriamo il limite singolare di equazioni di reazione-diffusione bistabili nel caso in cui la velocità dell'onda viaggiante dipenda dalla variabile di stato, cioè $c^\varepsilon = c^\varepsilon(x)$, e sia soddisfatto, al tendere di ε a zero e in qualche modo opportuno, $c^\varepsilon/\varepsilon^\tau \to \alpha$, laddove α è una funzione discontinua e τ è un intero che può essere uguale a 0 o a 1.

La seconda parte della tesi riguarda equazioni di reazione-diffusione semilineari e aventi termini di diffusione del tipo $\mathrm{tr}(A_\varepsilon(x)D^2u_\varepsilon)$, laddove tr denota l'operatore traccia, $A_\varepsilon = \sigma_\varepsilon\sigma_\varepsilon^t$ per qualche funzione $\sigma_\varepsilon : \mathbb{R}^n \to \mathbb{R}^{n\times(m+n)}$ e A_ε converge ad una matrice degenere.

Al fine di provare tali risultati in modo rigoroso, abbiamo modificato e adattato ”l'approccio geometrico” introdotto da G. Barles e P. E. Souganidis per risolvere problemi in $\mathbb{R}^n$ e in seguito parzialmente rivisto dallo stesso G. Barles assieme a F. Da Lio per equazioni di reazione-diffusione in domini limitati. Laddove possibile abbiamo sempre considerato la questione della buona posizione dei problemi di Cauchy che governano il moto dei fronti che descrivono le asintotiche da noi considerate.

To my cats

Contents

Introduction

Interfacial phenomena are commonplace in physics, chemistry and biology. They occur whenever a continuum is present that can exists in at least two different chemical or physical ”states”, and there is some mechanism that generates or enforces a spatial separation between these states. The separation boundary is what we call an **interface**.

Some examples of physical processes where we can observe the generation of an interface are:

- the so called **phase transition** that occurs whenever there is a double-well potential that drives a substance into one of two possible phases, such as solid or liquid;
- the **electrophoresis phenomenum** that is the motion of ions in a fluid under the influence of an electric fields;
- in the **combustion phenomena** two different temperatures establish two different zones in the flame profile: the preheat zone, where the temperature is low enough so that no chemical reaction has yet occurred and the burned zone where the gas has attained its final state.

In mathematics interfaces appear in the study of the asymptotic limits of evolving systems, like **reaction-diffusion equations**, whose solution, often an *order parameter*, is expected to approach for large times the equilibria of the system. When there is more than one equilibrium, interfaces separate regions where the parameter tends to the different equilibria, called phases for instance in phase transition models.

To fix the ideas assume we have a smooth state variable u, a function of space and time depending also on a small parameter $\varepsilon > 0$,

$$u^\varepsilon(x,t) = u(x,t;\varepsilon), \quad x \in \mathbb{R}^n, t \in [0,+\infty).$$

Typically u^ε is the solution of a semilinear reaction-diffusion equation

$$u_t^\varepsilon + \mathfrak{L}^\varepsilon(u^\varepsilon, x) = 0$$

satisfying an initial condition

$$u^\varepsilon(x,0)=g(x),$$

where g is a continuous bounded function. If such a solution u^ε exists for any $\varepsilon>0$ one can try to look at the behavior of the family $(u^\varepsilon)_\varepsilon$ as $\varepsilon\to 0^+$. A famous example of semilinear reaction-diffusion equation is the so called **Allen-Cahn equation**

$$u_t^\varepsilon(x,t)-\Delta u^\varepsilon(x,t)+\frac{f(u^\varepsilon(x,t))}{\varepsilon^2}=0,\quad (x,t)\in\mathbb{R}^n\times(0,+\infty), \tag{1}$$

where the reaction term f is the derivative of a double well potential W, $f=W'$. In particular $f:\mathbb{R}\to\mathbb{R}$ is a cubic function with zeroes $m_-<m_0<m_+$, with structure conditions modeled on the following main example

$$f(q)=2(q-m_-)(q-m_0)(q-m_+). \tag{2}$$

It is known in the literature that, if the wells of W have the same depth, i.e.

$$W(m_+)=W(m_-),$$

and the initial condition g represents a sharp interface across the unstable equilibrium m_0, then the asymptotics is governed by the mean curvature equation

$$\begin{cases} \text{(i)} \quad u_t(x,t)-\operatorname{tr}\Big[\Big(I-\dfrac{Du(x,t)}{|Du(x,t)|}\otimes\dfrac{Du(x,t)}{|Du(x,t)|}\Big)D^2u(x,t)\Big]=0, \quad \text{in } \mathbb{R}^n\times(0,+\infty),\\ \text{(ii)} \quad u(x,0)=u_o(x), \quad \text{in } \mathbb{R}^n, \end{cases} \tag{3}$$

where the initial condition $u_o\in C(\mathbb{R}^n)$ is chosen in such a way that the initial front $\Gamma_o=\{x\in\mathbb{R}^n: u_o(x)=0\}=\{x\in\mathbb{R}^n: g(x)=m_0\}$ is a nonempty and closed set (ideally an hypersurface). Moreover $u_o(x)>0$ (resp. $u_o(x)<0$) if $g(x)>m_0$ (resp. $g(x)<m_0$). Indeed one proves that the convergence occurs locally uniformly off the moving front determined by (3) to the stable equilibria of the reaction-diffusion equation, namely

$$u^\varepsilon(x,t)\to\begin{cases} m_+ & \text{if } u(x,t)>0,\\ m_- & \text{if } u(x,t)<0,\end{cases}$$

where u is the solution of (3). Equivalently we have that the asymptotic behavior of the solutions u^ε of the Cauchy problem for the Allen-Cahn equation is described by moving interfaces $t\longmapsto\Gamma_t$, with $\Gamma_t=\{x: u(x,t)=0\}$.

Equation (1) was proposed by Allen and Cahn [1] as a phase transition model for a moving interface with normal velocity being the mean curvature of the front. The first formal study of the asymptotics of the Allen-Cahn equation is by Fife [28], Caginalp [11], [12] and Keller Rubinstein and Sternberg [35]. In [16] Chen gives a rigorous local in time proof of the generation and propagation of a smooth interface in (1) under the assumption that the mean curvature motion is smooth. The first rigorous and global in time proof of the asymptotics, not assuming particular conditions on the regularity of the mean curvature motion, is due to Evans, Soner and Souganidis in [26]. A general study of the asymptotics of equations of the form (1) for bistable nonlinearities and the propagation of interfaces was done by Barles, Soner and Souganidis [7] based on the theory of weak front evolutions. The application of these methods to study the asymptotic behavior of FitzHugh-Nagumo type systems can be found in the paper of Soravia and Souganidis [40]. The particular case of the asymptotics of the Allen-Cahn equation has been the object of even more study. For an exhaustive treatment of the argument we refer to the work of Souganidis [41] where the author also presents three different approaches to the study of the limiting behavior of the solutions of (1). Finally we observe that the mean curvature equation in (3) is a quasilinear parabolic degenerate equation with a singularity for $Du = 0$. To be solved globally in time, the Cauchy problem (3) has to be meant in the sense of viscosity solutions, see Chen, Giga and Goto [18] or Evans and Spruck [27], and it turns out that it has a unique continuous solution $u \in C(\mathbb{R}^n \times [0, +\infty))$ for any $u_o \in C(\mathbb{R}^n)$.

The study of front propagation is a classical problem and it can be formulated in the following way. Let Ω_0 be an open subset of $\mathbb{R}^n$, study the evolution of the interfaces $t \longmapsto \Gamma_t$ moving with normal velocity

$$V = v(Dn(x), n(x), x, t) \tag{4}$$

and starting at time $t = 0$ from $\Gamma_0 = \partial\Omega_0$. Interfaces with normal velocity as in (4) satisfy monotonicity property, i.e., loosely speaking, if two fronts moving with velocity as in (4) are separated at some time, then they remain separated.

The main issues of interface dynamics as in (4) are the development of singularities in finite time, independently of the smoothness of the initial surface Γ_0. Classical examples in this directions are the evolutions by mean curvature of ”bar-bells” and ”tori” in $\mathbb{R}^n$. To overcome this geometric difficulty and interpret the evolution past the singularities it was necessary to develop some weak (generalized) notions of evolving fronts. In [41] Souganidis summarizes some different approaches to the problem and shows that they turn out to be equivalent when there is no-fattening phenomenon, i.e when the interface Γ_t has empty interior for any $t > 0$.

An equivalent way to formulate problem (4) is the so-called *level set approach*: if $u : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ is a function such that, for any $t > 0$, $\Omega_t = \{x : u(x,t) > 0\}$, $\Gamma_t = \{x : u(x,t) = 0\}$, and $|Du(x,t)| \neq 0$ for any $x \in \Gamma_t$, then problem (4) becomes

$$u_t(x,t) + F(x, t, Du(x,t), D^2u(x,t)) = 0 \tag{5}$$

where the function F is related to V by,

$$F(x,t,p,X) = -|p|v(-\frac{1}{|p|}\Big(I - \frac{p\otimes p}{|p|^2}\Big)X, -\frac{p}{|p|}, x, t), \quad X \in \mathcal{S}^n,\, p, x \in \mathbb{R}^n,\, t \in (0,+\infty).$$

Geometric equations, as for example equation (5), are particular pde's in which one can observe that if u is a solution of (5) and $\psi : \mathbb{R} \to \mathbb{R}$ is smooth and increasing, then also $\psi(u)$ is a solution of (5). As a consequence, it is easy to see that if u_o^1 and u_o^2 are two initial conditions such that

$$\{x : u_o^1(x) = 0\} = \{x : u_o^2(x) = 0\},$$

and u^1, u^2 are the solutions of the corresponding Cauchy problem for (5), then one has

$$\{x : u^1(x,t) = 0\} = \Gamma_t = \{x : u^2(x,t) = 0\},$$

for all $t > 0$. One can therefore *define* the family of closed sets $(\Gamma_t)_t$ to be the geometric flow of the front or interface Γ_0 with normal velocity $-F$. In general, a geometric flow describes the motion of fronts with prescribed normal velocity, possibly depending upon position, time, normal direction and principal curvatures. It is easy to realize that these motions exibit many interesting qualitative properties, for instance they may develop singularities, change topology or become extinct in finite time, and one needs to define a weak front propagation via viscosity solutions in order to have a well defined flow globally in time. The level set approach was proposed by Osher and Sethian [36] for numerical computations of solutions of geometric equations of type (5) while the rigorous theory started with the work by Evans and Spruck [27] for the mean curvature flow and by Chen Giga and Goto [18] for more general geometric flows. For the mathematical analysis of the level set method via viscosity solutions, the reader is referred to the book by Giga [31], where the approach is discussed in detail. Among others, one of the most striking applications of the theory of weak front propagation is the fact that it allows to rigorously determine the asymptotics of reaction-diffusion equations and systems which model phase transitions. For instance, we described above that the Allen-Cahn equation (1) as $\varepsilon \to 0$ gives rise to an interface moving according (3).

More recently Barles and Souganidis in [8], and subsequently Barles and Da Lio in [5], proposed a new and more geometric approach to study singular limits giving rise to moving interfaces, which allows to include in the analysis both the generation and propagation phenomena. The approach is more flexible to describe geometric flows also in KPP-type systems, equations with oscillating coefficients, nonlocal terms or appearing in the study of interacting particle systems, see again Souganidis [41] where the reader can find many more references. The approach in [8] is based on a new definition of generalized propagation of fronts in $\mathbb{R}^n$, which turns out to be equivalent to the level set approach when there is no fattening phenomenon, and leads to a general method for establishing the asymptotic limit of a large class of reaction-diffusion equations. Since a family of moving hypersurfaces $(\Gamma_t)_t$ separates, at any $t > 0$, two open and disjoint subsets of $\mathbb{R}^n$, the idea of Barles and Souganidis is to consider the evolution of open subsets of $\mathbb{R}^n$ instead of $t \longmapsto \Gamma_t$ itself. This can be done through a "local monotonicity test". Roughly speaking, the localized monotonicity property claims that, if $(\Omega^1_s)_{s\in(a,b)}$, $(\Omega^2_s)_{s\in(a,b)}$ are two families of open subsets evolving with the same normal velocity and if, for some $r > 0$ and $t \in (a, b)$,

$$\Omega^1_t \cap B(x, r) \subset \Omega^2_t \cap B(x, r),$$

and if, for all $t \leq s < b$, we have

$$\Omega^1_s \cap \partial B(x, r) \subset \Omega^2_s \cap \partial B(x, r),$$

then,

$$\Omega^1_s \cap B(x, r) \subset \Omega^2_s \cap B(x, r) \quad \text{for any } s \in [t, b).$$

The new definition of generalized propagation of fronts in [5] uses the local monotonicity property above to study the evolution of a family of open subsets of $\mathbb{R}^n$ through the comparison with smooth evolutions: roughly speaking one may say that a family $(\Omega^2_s)_{s\in(a,b)}$ is a generalized flow with normal velocity greater than V if it satisfies the local monotonicity property when tested on a suitable class of families $(\Omega^1_s)_{s\in(a,b)}$ evolving smoothly and with normal velocity less than V. Since a family of open subsets $(\Omega_s)_{s\in(a,b)}$ has normal velocity smaller than V if and only if the family $(\Omega^c_s)_{s\in(a,b)}$ has normal velocity grater than $-V$, a notion of generalized motion with normal velocity smaller than V can be defined analogously. The main issues of this new definition of generalized propagation of fronts are that

1. it is enough to make the monotonicity test against families of open subsets evolving *smoothly*,
2. the test can be done *locally* in space and only on small time intervals,
3. as said above, one can use families whose normal velocity is smaller or grater than the normal velocity considered.

Barles and Souganidis use this new definition to study the asymptotics of reaction-diffusion equations. They revisit with their new approach the results in [26] and [7] about the limiting behavior of the solution of the Allen-Cahn equation (1) and they present some new results regarding the asymptotics of semilinear reaction-diffusion equations, reaction-diffusion equations with oscillating coefficients and nonlocal fully nonlinear integral-differential equations. In [5] Barles and Da Lio slightly modify the definition in [8] to study the asymptotic behavior of the solution of semilinear and quasilinear Allen-Cahn equations with (x, t)-dependent cubic function f in a bounded domain with Neumann boundary conditions. The proofs in [5] extend the idea in [8] to define a family of open subsets $(\Omega_s)_{s\in(a,b)}$ and to prove that they move with a certain normal velocity; these open subsets are, roughly speaking, the interiors of the sets where u_ε converges to the stable equilibria of the equation with an $o(\varepsilon^\tau)$ rate of convergence, where τ depends on the problem.

In our thesis we study two families of reaction-diffusion equations. Chapter 3 is about the singular limit of bistable reaction-diffusion equations in the case when the velocity of the traveling wave solution c^ε depends on the space variable x and satisfies the following

$$\frac{c^\varepsilon(x)}{\varepsilon^\tau} \xrightarrow[\varepsilon\to 0^+]{} \alpha(x), \quad \text{locally uniformly off an hypersurface } \tilde{\Gamma} \subset \mathbb{R}^n, \tag{6}$$

where α is only piecewise continuous with discontinuity set $\tilde{\Gamma}$ and τ is an integer that can be equal only to 0 or 1. To be more precise we assume that $\alpha : \mathbb{R}^n \to [\rho, +\infty)$ is a bounded measurable function which is piecewise continuous across an oriented, closed hypersurface $\tilde{\Gamma} \subset \mathbb{R}^n$ and satisfies

$$\alpha(x) \in \begin{cases} \{n_1(x)\} & \text{if } \tilde{d}(x) < 0, \\ \{n_2(x)\} & \text{if } \tilde{d}(x) > 0, \\ [n_1(x), n_2(x)] & \text{if } \tilde{d}(x) = 0, \end{cases}$$

where $\tilde{d}$ is the signed distance function from $\tilde{\Gamma}$ and n_1 and n_2 are two bounded and locally Lipschitz continuous functions such that $n_1(x) < n_2(x)$, for all $x \in \mathbb{R}^n$. The reaction-diffusion equations we consider in Chapter 3 are

$$u_t^\varepsilon(x,t) - \varepsilon\Delta u^\varepsilon(x,t) + \varepsilon^{-1} f^\varepsilon(u^\varepsilon, x) = 0, \quad \text{in } \mathbb{R}^n \times (0, +\infty) \tag{7}$$

and the nonlinear Allen-Cahn equation

$$u_t^\varepsilon(x,t) - \Delta u^\varepsilon(x,t) + \varepsilon^{-2} f^\varepsilon(u^\varepsilon, x) = 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty) \tag{8}$$

where $f^\varepsilon : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is of bistable type, with structure conditions modeled on the following main example

$$f^\varepsilon(q,x) := 2\Big(q - \frac{c^\varepsilon(x)}{2}\Big)(q^2 - 1) \tag{9}$$

with $-1 < c^\varepsilon(x)/2 < 1$. In both cases we ask the functions u^ε to satisfy a Cauchy condition at time zero, i.e.

$$u^\varepsilon(x,0) = g(x), \tag{10}$$

where g is a continuous real function which takes values in the interval $[-1, 1]$. As it regards c^ε, we will assume that it satisfies assumption (6) with $\tau = 0$ when we study the limit behavior of the solutions of the equation (7) and $\tau = 1$ when we consider (8).

We will show that the family u^ε of the solutions of (7) converges to the stable equilibria of f^ε off the evolving interface which moves with normal velocity $-\alpha$ and is determined by the geometric equation

$$\begin{cases} u_t(x,t) + \alpha(x)|Du(x,t)| = 0, & (x,t) \in \mathbb{R}^n \times (0,+\infty), \\ u(x,0) = u_o(x), & x \in \mathbb{R}^n, \end{cases} \tag{11}$$

once we initialize it by setting, in the case (9),

$$\Gamma_o = \{x \in \mathbb{R}^n : u_o(x) = 0\} = \{x \in \mathbb{R}^n : \frac{\alpha_*(x)}{2} \leq g(x) \leq \frac{\alpha^*(x)}{2}\},$$

where α_*, α^* indicate the lower and upper semicontinuous envelopes of α, respectively. Moreover $u_o(x) > 0$ (resp. $u_o(x) < 0$) if $g(x) > \dfrac{\alpha(x)}{2}$ (resp. $g(x) < \dfrac{\alpha(x)}{2}$). We notice that Γ_o may contain relatively open subsets of the hypersurface of discontinuity of α where $\dfrac{\alpha_*(x)}{2} < g(x) < \dfrac{\alpha^*(x)}{2}$. The initial front that controls the convergence of the solutions of reaction-diffusion equation (7) to the stable equilibria moves in this case with a velocity which is discontinuous in space. In geometric optics, discontinuous coefficients α in the propagation equation (11) arise in the refraction phenomenon and $1/\alpha$ is then the discontinuous refraction index.

The main steps to apply the new definition of generalized propagation of fronts in [8] and [5] to the study of the asymptotics of our problems are the following: (i) prove the well-posedness of the Cauchy problem that governs the motion of the moving hypersurface describing the limiting behavior of the u^ε's; (ii) define two collections of open subsets of $\mathbb{R}^n$ $(\Omega^1_s)_{s\in(0,T)}$ and $(\Omega^2_s)_{s\in(0,T)}$ so that $u^\varepsilon(x,s)$ converges to m_- if $x \in \Omega^1_s$ and to m_+ if $x \in \Omega^2_s$; (iii) find suitable traces of these families at time zero, Ω^1_0 and Ω^2_0, and prove that $\{u_o > 0\} \subseteq \Omega^1_0$ and $\{u_o < 0\} \subseteq \Omega^2_0$; (iv) prove that $(\Omega^1_s)_{s\in(0,T)}$ (respectively $(\Omega^2_s)_{s\in(0,T)}$) moves (according the new definition above) with normal velocity smaller (resp. greater) than the velocity of the front that describes the limiting behavior of the u^ε's.

The novelty of our study is that the function α is only piecewise continuous. In our master thesis [21] and then in [22] we proved that the problem (11) is well-posed, and a comparison principle holds for viscosity solutions as defined by Ishii [33] when α has constant sign and is piecewise continuous across an hypersurface. Actually in [21], [22] we consider the well-posedness of the Cauchy problem for noncoercive and more general Hamilton-Jacobi equations. The particular equation in (11) is coercive and a uniqueness result for it was also previously proved by Camilli [14].

When α is piecewise continuous and we study the asymptotics of (7) we have to allow the sequence c^ε to converge to α only almost everywhere off the hypersurface. Moreover the norms of the gradients and of the Laplacians, $\|Dc^\varepsilon\|_\infty$ and $\|\Delta c^\varepsilon\|_\infty$, may blow up as $\varepsilon \to 0$. We have to link the blow up rate of $\|Dc^\varepsilon\|_\infty$ (respectively $\|\Delta c^\varepsilon\|_\infty$) to the parameter ε and require that it is not faster than $\varepsilon^{-1/2}$ (resp. ε^{-1}). A further difficulty is to prove that a family of open (resp. close) subsets of $\mathbb{R}^n$ is a generalized superflow (resp. subflow) with a discontinuous normal velocity α. To do this we have to approximate the definition of super- and subflow by using suitable families of continuous velocities and we construct these families starting from the smooth functions c^ε's.

With similar methods we prove that, when $c^\varepsilon/\varepsilon \to \alpha$ locally uniformly off $\tilde{\Gamma}$, the limiting behavior of the solutions of the Allen-Cahn equation (8) satisfying (10) gives rise to an interface moving with normal velocity $\mathcal{K} - \alpha$, where $\mathcal{K}$ indicates the mean curvature of the interface, i.e. according to the geometric equation

$$\begin{cases} u_t(x,t) + \alpha(x)|Du(x,t)| + F(Du(x,t), D^2u(x,t)) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,+\infty), \\ u(x,0) = u_o(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{12}$$

where $F : \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ is defined as

$$F(p,X) = -\operatorname{tr}\Big[\Big(I - \frac{p}{|\,p|} \otimes \frac{p}{|\,p|}\Big)X\Big]. \tag{13}$$

In this second case the front is initialized in the standard way by the initial front

$$\Gamma_o = \{x \in \mathbb{R}^n : u_o(x) = 0\} = \{x \in \mathbb{R}^n : g(x) = 0\}$$

and choosing the initial condition $u_o \in C(\mathbb{R}^n)$ such that $u_o(x) > 0$, (respectively $u_o(x) < 0$) if and only if $g(x) > 0$ (resp. $g(x) < 0$). A comparison principle for (12) when α is only piecewise continuous does not yet appear in the literature and we prove it in Theorem 3.3.8 under some particular assumptions on the set $\tilde{\Gamma}$ of discontinuity of α. For example $\tilde{\Gamma}$ has to be the global graph of a Lipschitz continuous function with Lipschitz constant smaller than $1/\rho$. In the comparison result in Theorem 3.3.8 we ask the subsolution u (or to the supersolution v) to be continuous along a suitable direction η transversal to $\tilde{\Gamma}$. This assumption turns out to be crucial in the proof of our Theorem and recovers the ideas used by Soravia in his works about the uniqueness of viscosity solutions for some kinds of discontinuous Hamilton-Jacobi equations (see [38], [37], [39] and the references therein). The existence of a continuous solution of (12) remains an open problem. We try to solve it with Perron's method but we obtain only an upper semicontinuous viscosity solution of (12).

In [8], [5] the authors study the asymptotics of solutions to the initial-value problem for a semilinear reaction-diffusion equations of type

$$\begin{cases} u^\varepsilon_t(x,t) - \operatorname{tr}(A(x)D^2u^\varepsilon(x,t)) + \varepsilon^{-2} f(u^\varepsilon(x,t)) = 0 \quad (x,t) \in \mathbb{R}^n \times (0,+\infty), \\ u^\varepsilon(x,0) = g(x) \quad x \in \mathbb{R}^n \end{cases} \tag{14}$$

where $A(\cdot)$ is a matrix map that takes values in the space of the $n \times n$ symmetric matrices such that

$$A(x)p \cdot p \geq \mu |\, p|^2,$$

for some $\mu > 0$ and for any $x, p \in \mathbb{R}^n$. Barles, Souganidis (and then Barles and Da Lio for the same problem in a bounded open set with Neumann boundary condition) proved that, when $c = 2m_0 - m_+ - m_- = 0$, the asymptotics of the solutions of (14) as $\varepsilon \to 0^+$ is described by the the following Cauchy problem

$$\begin{cases} \text{(i)} & u_t + G(x, Du, D^2u) = 0 \\ \text{(ii)} & u(x,0) = u_o(x), \end{cases} \tag{15}$$

where the function $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ is defined as

$$G(x,p,X) = -\operatorname{tr}(A(x)X) + \operatorname{tr}\left(\left(A(x)X + \sigma^t(x)D\sigma^t(x)p\right)\frac{\sigma^t(x)p}{|\,\sigma^t(x)p|} \otimes \frac{\sigma^t(x)p}{|\,\sigma^t(x)p|}\right)$$

and $\sigma : \mathbb{R}^n \to \mathcal{S}^n$ is the square root matrix of A, i.e. $A(x) = \sigma(x)\sigma^t(x)$, for any $x \in \mathbb{R}^n$. Since A is positive definite the quadratic form $\langle A(x)^{-1}\xi, \xi\rangle$ is a Riemannian tensor $\mathfrak{g}$ and the set of vector fields $\{\sigma^{(i)}(x), i = 1, \ldots, n\}$ spans $\mathbb{R}^n$ for any $x \in \mathbb{R}^n$. We observe that the equation in (15-i) differs from the mean curvature equation in the metric $\mathfrak{g}$ for the term $-\operatorname{tr}(\sigma^t(x)D\sigma^t(x)p)$. In the last chapter of this thesis we look for the right reaction-diffusion equation that gives rise to a front moving according to the mean curvature equation in a sub-Riemannian metric. Indeed we assume that the matrix map $\sigma(\cdot)$ takes values in the space of the $n \times m$ real matrices, $m < n$, and thus at any point $x \in \mathbb{R}^n$ the matrix $A(x) = \sigma(x)\sigma^t(x)$ is only semi-positive definite. If we define, for any $x \in \mathbb{R}^n$ and for any $\varepsilon > 0$, a matrix

$$\sigma_\varepsilon(x) = \left[\sigma(x)\ \varepsilon^k I_n\right] \in \mathbb{R}^{n\times(m+n)},$$

where $k > 0$ and I_n denotes the $n \times n$ identity matrix, we have that the matrix map

$$A_\varepsilon \equiv \sigma_\varepsilon \sigma_\varepsilon^t$$

is a Riemannian approximation of A. We prove that, under some particular assumptions on the matrix σ, the asymptotic behavior of the solutions of the semilinear reaction-diffusion equation

$$u_t^\varepsilon(x,t) - \operatorname{tr}(A_\varepsilon(x)D^2u^\varepsilon(x,t) + \sigma_\varepsilon(x)^t D\sigma_\varepsilon(x)^t Du^\varepsilon(x,t)) + \varepsilon^{-2}f(u^\varepsilon(x,t)) = 0, \tag{16}$$

satisfying the initial condition (10), is governed by the solution of the Cauchy problem for the mean curvature equation in the sub-Riemannian metric A,

$$\begin{cases} \text{(i)} \quad u_t - \operatorname{tr}\Big(\big(A(x)D^2u + \sigma^t(x)D\sigma^t(x)Du\big)\big(I - \dfrac{\sigma^t(x)Du}{|\,\sigma^t(x)Du|} \otimes \dfrac{\sigma^t(x)Du}{|\,\sigma^t(x)Du|}\big)\Big) = 0, \\ \text{(ii)} \quad u(x,0) = u_0(x). \end{cases} \tag{17}$$

A crucial property in the theory of the mean curvature equation (3-i) is the correspondence between the points where this equation degenerates, the zero set of the gradient and of the Hessian matrix of suitable powers of the Euclidean norm. To be more precise, if $|\,x|$ denotes the standard Euclidean norm, $|\,x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$, then

$$D(|\,x|^4) = 0 \quad \text{if and only if} \quad D^2(|\,x|^4) = 0 \quad \text{if and only if} \quad x = 0.$$

Since equation (17-i) becomes degenerate in a set bigger than the one of (3-i), we have to use a different norm. If the m vector fields $X_1 = \sigma^{(1)} \cdot \nabla, \dots, X_m = \sigma^{(m)} \cdot \nabla$, generate a Carnot group of step two $G = (\mathbb{R}^n, \circ, \delta_\lambda)$, where $\circ$ is a composition law on $\mathbb{R}^n$ and $\{\delta_\lambda\}_{\lambda>0}$ is a family of dilatations, then there exists an homogeneous norm[1] on G defined as

$$\|\,x\|_G = [|\,x_H|^4 + |\,x_V|^2]^{1/4}, \qquad x = (x_H, x_V) \in \mathbb{R}^m \times \mathbb{R}^{n-m}.$$

If we put $N(x) = \|\,x\|_G^4$ we have that the function N satisfies a crucial property, in fact we have

$$|\,XN(x)| = 0 \quad \text{if and only if} \quad |\,X^2N(x)| = 0 \quad \text{if and only if} \quad x_H = 0. \tag{18}$$

In other words $\|\cdot\|_G$ plays in $\mathfrak{g}$ the role that the Euclidean norm has in the Euclidean metric. Property (18) is crucial since it allows us to restrict the family of test functions in the definition of viscosity solution. To be more precise we have that, if u is a viscosity subsolution of (17) and $\varphi \in C^2(\mathbb{R}^n \times (0, +\infty))$ is a test function so that $u - \varphi$ has a maximum in (x,t), then there are two possibilities, either $X\varphi(x,t) \neq 0$, or $X\varphi(x,t) = 0$ and $X^2\varphi(x,t) = 0$. This new characterization of the test functions for viscosity solutions allows us to use the existence and uniqueness result for (17) obtained by Capogna and Citti in [15] in the framework of Carnot Groups and for some particular initial data u_o. The definition of weak solution they use is the sub-elliptic analogous of the viscosity solution in the formulation stated by Evans and Spruck in [27]. In the Euclidean case it is clear that the definition of viscosity solution formulated in [27] is indeed the same of Crandall, Ishii and Lions

[1]We call *homogeneous norm* on the Carnot group G, every continuous function $d : G \to [0, \infty)$ such that

1. $d(\delta_\lambda(x)) = \lambda d(x)$ for every $\lambda > 0$ and $x \in G$;
2. $d(x) = 0$ iff $x = 0$.

(see [20]). In the sub-elliptic case this is not obvious. In Chapter 4 we use norm $\|\cdot\|_G$ to prove it for Carnot Groups of step two. We just point out that in [25] Dirr, Dragoni and von Renesse exhibit another existence result for (17), in particular they show that the value function of suitable family of stochastic control problems is a viscosity solution of (17).

As it regards the study of the asymptotic behavior of the solution of (16), our proof of the generation of the front that describes the asymptotics of (16) works without any particular assumption on the matrix σ, i.e. on the sub-Riemannian metric $\mathfrak{g}$. For the proof of the propagation of such a front we need to restrict again to Carnot groups of step two in order to use property (18). As we said before Barles, Souganidis and Da Lio in their papers [8], [5] defined a family of open sets $(\Omega_t)_{t\in(0,T)}$ to be a generalized flow with normal velocity $-F$ if it satisfies a (local in space and time) *monotonicity test* against families of open subsets evolving smoothly with normal velocity smaller or bigger than $-F$. We use again $\|\cdot\|_G$ and its property (18) to modify the definition of generalized flow when we consider as F the function

$$F(x,p,X) = -\operatorname{tr}\Big(\big(A(x)X + \sigma^t(x)D\sigma^t(x)p\big)\big(I - \frac{\sigma^t(x)p}{|\,\sigma^t(x)p|}\otimes\frac{\sigma^t(x)p}{|\,\sigma^t(x)p|}\big)\Big). \tag{19}$$

In our definition we avoid the monotonicity test against families of open subsets where the velocity $-F$ becomes degenerate.

In Chapter 4 we also consider the asymptotic behavior of the solution of a semilinear reaction-diffusion equation with a rescaling different from the one in (16). In fact at the beginning of Chapter 4 we consider the asymptotics as $\varepsilon \to 0$ of the solutions of the Cauchy problems for the equations

$$u_t^\varepsilon(x,t) - \varepsilon\operatorname{tr}(A_\varepsilon(x)D^2u^\varepsilon(x,t) + \sigma_\varepsilon(x)^tD\sigma_\varepsilon(x)^tDu^\varepsilon(x,t)) + \varepsilon^{-1}f^\varepsilon(u^\varepsilon(x,t)) = 0 \tag{20}$$

and we prove that, when $c = 2m_0 - m_+ - m_- \neq 0$, it is described by the evolutional eikonal equation

$$u_t(x,t) + c|\,\sigma^t(x)Du(x,t)| = 0. \tag{21}$$

In the study of this asymptotics we succeed in the proof of the entire result without assuming any particular condition on the matrix σ. Obviously the front evolving according to the equation (21) can have some points x in which the standard Euclidean normal vector is well-defined but it is orthogonal to the vector fields $\sigma^{(1)}(x),\dots,\sigma^{(m)}(x)$. We are able to prove our result also in these points by choosing the exponent k that appears in the definition of σ_ε in a suitable way.

The thesis is organized as follows. In Chapter 1 we recall the definition of viscosity solution as defined in [20] and we give the proof of some results that will turn out to be useful later. A complete treatment of the subject can be found in the User's guide [20] and in the book of Bardi and Capuzzo Dolcetta [3]. Chapter 2 is a collection of definitions and results about front propagation. We talk about the level-set approach to the problem (see also the book of Giga [31]) and the geometrical approach introduced by Barles and Souganidis in [8] and then recovered and partially revisited by Barles and Da Lio in [5]. See also [41]. Chapter 3 and 4 are the product of a joint work of the author and her advisor Soravia. In Chapter 3 we treat bistable reaction-diffusion equations when the velocity of the traveling wave solution c^ε depends on the space variable and c^ε/τ, $\tau \in \{0, 1\}$, converges in a suitable sense to a discontinuous function. Moreover we consider the well-posedness of the geometric equation that describes the asymptotic behavior of these equations. These results can be found in [22], [23], [24]. Finally Chapter 4 is about the limiting behavior of two particular semilinear reaction-diffusion equations with x-dependent diffusion term that gives rise to some interesting pde's in the framework of sub-Riemannian geometry. On the contents of this last section two papers are in preparation.

Chapter 1

Viscosity Solutions

In this preliminary chapter we just want to present the definition of viscosity solution together with some classic results that we will use in the following chapters. For a complete treatise of the argument we remand to [19] and [20].

We consider parabolic equations of the form

$$u_t(x,t) + F(x, Du(x,t), D^2u(x,t)) = 0, \tag{1.1}$$

where $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ and $\mathcal{S}^n$ is the set of symmetric $n \times n$ matrices. One of the main virtues of this theory is that it allows also to discontinuous functions to be solutions of fully nonlinear equations of second order.

We recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *upper semicontinuous* if for any point $x \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(y) \leq f(x) + \varepsilon,$$

for all $y \in [x-\delta, x+\delta]$. Similarly, f is *lower semicontinuous* if for any point $x \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(y) \geq f(x) - \varepsilon,$$

for all $y \in [x-\delta, x+\delta]$.

If $f : \mathbb{R}^n \to \mathbb{R}$ is a generic discontinuous function we defined the *upper semicontinuous envelope* of f

$$f^*(x) = \lim_{r\to 0^+} \sup_{|\,y-x|<r} f(y) = \inf_{r>0} \sup_{|\,y-x|<r} f(y),$$

and the *lower semicontinuous envelope* of f

$$f_*(x) = \lim_{r\to 0^+} \inf_{|\,y-x|<r} f(y) = \sup_{r>0} \inf_{|\,y-x|<r} f(y).$$

Definition 1.0.1. (i) An upper semicontinuous function $u : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R}$ is a *viscosity subsolution* of the equation (1.1) if and only if for every $\varphi \in C^2(\mathbb{R}^n \times (0, +\infty))$ and for every local maximum point $(x, t) \in \mathbb{R}^n \times (0, +\infty)$ of $u - \varphi$, we have

$$\varphi_t(x, t) + F_*(x, D\varphi(x, t), D^2\varphi(x, t)) \leq 0.$$

We call φ a *test function* at (x, t) for the subsolution u.
(ii) Similarly a lower semicontinuous function $u : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R}$ is a *viscosity supersolution* of the equation (1.1) if and only if for every $\varphi \in C^2(\mathbb{R}^n \times (0, +\infty))$ and for every local minimum point $(x, t) \in \mathbb{R}^n \times (0, +\infty)$ of $u - \varphi$, we have

$$\varphi_t(x, t) + F^*(x, D\varphi(x, t), D^2\varphi(x, t)) \geq 0.$$

We call φ a *test function* at (x, t) for the supersolution v.
(iii) A function $u : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R}$ is a *viscosity solution* of the equation (1.1) if and only if u^* is a viscosity subsolution of (1.1) and u_* is a viscosity subsolution of (1.1)

Remark 1.0.2. Using a density argument it can be easily shown that an equivalent definition of viscosity solution can be given by using $C^\infty(\mathbb{R}^n \times (0, +\infty))$ instead of $C^2(\mathbb{R}^n \times (0, +\infty))$ as "test function space".

Sometimes it is useful to consider a fully nonlinear parabolic equation like (1.1) only in bounded time interval, i.e. we consider (1.1) in $\mathbb{R}^n \times (0, T)$ with $T < +\infty$. In the following proposition we show that any viscosity subsolution (resp. supersolution) of (1.1) in $\mathbb{R}^n \times (0, T)$ can be extended in a suitable way to a viscosity subsolution (supersolution) of (1.1) in $\mathbb{R}^n \times (0, T]$. The proof is well-known in the literature, see for example [3].

Proposition 1.0.3. Let $T \in (0, +\infty)$, and $\Omega \subseteq \mathbb{R}^n$ open. We consider a Borel measurable function $F : \Omega \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ and we assume that u (resp. v): $\Omega \times (0, T) \to \mathbb{R}$ is an upper semicontinuous viscosity subsolution (resp. lower semicontinuous supersolution) of

$$u_t(x, t) + F(x, Du(x, t), D^2u(x, t)) = 0, \quad (x, t) \in \Omega \times (0, T), \tag{1.2}$$

and that

$$u(x, T) := \limsup_{(y,s) \to (x, T^-)} u(y, s) \quad (\text{resp. } v(x, T) := \liminf_{(y,s) \to (x, T^-)} v(y, s)).$$

Then u (resp. v) is a viscosity subsolution (resp. supersolution) of

$$u_t(x, t) + F(x, Du(x, t), D^2u(x, t)) = 0, \quad (x, t) \in \Omega \times (0, T]. \tag{1.3}$$

Proof. We prove the result only for the subsolution u. To do this we consider a pair $(\hat{x}, T) \in \Omega \times \{t = T\}$ and a function $\varphi \in C^2((0,T] \times \mathbb{R}^n)$ such that $(\hat{x}, T)$ is a strict local maximum for $\psi := u - \varphi$. Let $r > 0$ so that

$$\psi(\hat{x}, T) > \psi(x,t), \quad \text{for any } (x,t) \in B(\hat{x}, r) \times (T - r, T].$$

Let (x_n, t_n) be a maximum point of

$$\psi^n(x,t) := u(x,t) - \Big(\varphi(x,t) + \frac{1}{n(T-t)}\Big), \ n \in \mathbb{N}$$

in $B(\hat{x}, r] \times [T - r, T[$. We want to prove that

$$(x_n, t_n) \longrightarrow (\hat{x}, T). \tag{1.4}$$

Up to some subsequence, $(x_n, t_n) \longrightarrow (\tilde{x}, \tilde{t}) \in B(\hat{x}, r] \times [T - r, T]$, and thus, to get (1.4) we have to prove that $(\hat{x}, T) = (\tilde{x}, \tilde{t})$. Let's consider a sequence $(\bar{x}_n, \bar{t}_n) \in \Omega \times [0, T)$ such that

$$(\bar{x}_n, \bar{t}_n) \longrightarrow (\hat{x}, T), \qquad u(\bar{x}_n, \bar{t}_n) \longrightarrow u(\hat{x}, T) \tag{1.5}$$

and

$$\frac{1}{n(T - \bar{t}_n)} \longrightarrow 0. \tag{1.6}$$

Since $(\bar{x}_n, \bar{t}_n) \in B(\hat{x}, r] \times [T - r, T[$ for n big enough, we get

$$\psi^n(\bar{x}_n, \bar{t}_n) \leq \psi^n(x_n, t_n) = \psi(x_n, t_n) - \frac{1}{n(T - t_n)} \leq \psi(x_n, t_n),$$

and then, by taking the $\limsup$ as $n \to +\infty$,

$$\psi(\hat{x}, T) \leq \limsup_{n \to \infty} \psi(x_n, t_n) \leq \psi(\tilde{x}, \tilde{t}).$$

Since $(\hat{x}, T)$ is a strict maximum point for ψ in $B(\hat{x}, r] \times [T - r, T]$ we immediately get $(\hat{x}, T) = (\tilde{x}, \tilde{t})$. Therefore (1.4) is proved.

By the definition of subsolution we get

$$\begin{aligned} 0 &\geq \varphi_t(x_n, t_n) + \frac{1}{n(T - t_n)^2} + F_*(x_n, D\varphi(x_n, t_n), D^2\varphi(x_n, t_n)) \\ &\geq \varphi_t(x_n, t_n) + F_*(x_n, D\varphi(x_n, t_n), D^2\varphi(x_n, t_n)), \end{aligned}$$

and thus, by letting $n \to +\infty$,

$$\begin{aligned} 0 &\geq \liminf_{n\to+\infty}(\varphi_t(x_n,t_n) + F_*(x_n, D\varphi(x_n,t_n), D^2\varphi(x_n,t_n))) \\ &\geq \varphi_t(\hat{x},T) + F_*(\hat{x}, D\varphi(\hat{x},T), D^2\varphi(\hat{x},T)). \end{aligned}$$

□

Another important property of viscosity solutions is the so-called *stability property*, i.e. the construction of a limit subsolution (resp. supersolution) form an arbitrary sequence of subsolutions (resp. supersolutions) of approximate problems without any control of the derivatives.

We recall that if $f_\varepsilon : E \to \mathbb{R}$, $E \subseteq \mathbb{R}^n$ and $\varepsilon > 0$ is a family of real functions, we can define the *lower weak limit* at the point $(x,t) \in E \times (0,+\infty)$ as

$$\begin{aligned} \underline{u}(x,t) = \liminf_{\varepsilon\to0^+}{}_* \, u_\varepsilon(x,t) &:= \liminf_{(y,s,\varepsilon)\to(x,t,0^+)} u_\varepsilon(y,s) \\ &= \sup_{\delta>0} \inf\{u_\varepsilon(y,s) : |\, x-y|, |\, t-s| < \delta, 0 < \varepsilon < \delta\}, \end{aligned}$$

and the *upper weak limit*

$$\begin{aligned} \overline{u}(x,t) = \limsup_{\varepsilon\to0^+}{}^* \, u_\varepsilon(x,t) &:= \limsup_{(y,s,\varepsilon)\to(x,t,0^+)} u_\varepsilon(y,s) \\ &= \inf_{\delta>0} \sup\{u_\varepsilon(y,s) : |\, x-y|, |\, t-s| < \delta, 0 < \varepsilon < \delta\}. \end{aligned}$$

Obviously $\underline{u}$ is a lower semicontinuous function and $\overline{u}$ is an upper semicontinuous function.

Again, the proof of the stability result stated in the next theorem, when F is a continuous function, can be found also in the book of Bardi and Capuzzo Dolcetta [5].

Theorem 1.0.4. Let $\Omega \subseteq \mathbb{R}^n$ open and $F_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$, $\varepsilon > 0$ be a family of continuous function such that

$$F^*(x,p,X) = \limsup_{\varepsilon\to0^+}{}^* \, F_\varepsilon(x,p,X),$$

$$F_*(x,p,X) = \liminf_{\varepsilon\to0^+}{}_* \, F_\varepsilon(x,p,X),$$

for any $(x,p,X) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n$. We consider a family of functions $u_\varepsilon : \mathbb{R}^n \times (0,+\infty) \to \mathbb{R}$, $\varepsilon > 0$ such that, for any compactum $\mathcal{K} \subset \mathbb{R}^n \times (0,+\infty)$ there exists a positive constant $C_\mathcal{K}$ such that

$$\sup_{\mathcal{K}} |\, u_\varepsilon| \leq C_\mathcal{K}, \quad \text{for any } \varepsilon > 0.$$

(i) If u_ε is an upper semicontinuous viscosity subsolution of

$$u_t(x,t) + F_\varepsilon(x, Du(x,t), D^2u(x,t)) = 0, \quad (x,t) \in \Omega \times (0,+\infty) \tag{1.7}$$

for all $\varepsilon > 0$, then the upper weak limit $\overline{u}$ is a viscosity subsolution of (1.1) in $\Omega \times (0, +\infty)$;
(ii) If u_ε is a lower semicontinuous viscosity supersolution of

$$u_t(x,t) + F_\varepsilon(x, Du(x,t), D^2u(x,t)) = 0, \quad (x,t) \in \Omega \times (0, +\infty)$$

for all $\varepsilon > 0$, then the lower weak limit $\underline{u}$ is a viscosity supersolution of (1.1) in $\Omega \times (0, +\infty)$.

Proof. (i) Let $\varphi \in C^2(\Omega \times (0, +\infty))$ and $(x,t) \in \Omega \times (0, +\infty)$ be a local maximum point for $\overline{u} - \varphi$. It is not restrictive to assume that (x,t) is a strict local maximum point for $\overline{u} - \varphi$. Let $r > 0$ so that

$$\overline{u}(x,t) - \varphi(x,t) > \overline{u}(y,s) - \varphi(y,s), \quad |(y-x, t-s)| \leq r, \ (y,s) \neq (x,t).$$

We want to prove that

$$\varphi_t(x,t) + F_*(x, D\varphi(x,t), D^2\varphi(x,t)) \leq 0. \tag{1.8}$$

We define $B := B((x,t), r]$ and we divide the proof into two steps.
STEP 1. There exists a sequence of points $(x_n, t_n) \in B$ and $\epsilon_n \to 0^+$ so that (x_n, t_n) is a maximum point for $u_{\epsilon_n} - \varphi$ in B and $(x_n, t_n) \longrightarrow (x,t)$, $u_{\epsilon_n}(x_n, t_n) \longrightarrow \overline{u}(x,t)$.
To prove this first claim we consider $\epsilon_n \to 0^+$ and a sequence of points $(x^{(n)}, t^{(n)})$ such that

$$(x^{(n)}, t^{(n)}) \to (x,t), \quad u_{\epsilon_n}(x^{(n)}, t^{(n)}) \to \overline{u}(x,t), \qquad \text{if } n \to \infty.$$

Let (x_n, t_n) be a maximum point for $u_{\epsilon_n} - \varphi$ in B. Since (x_n, t_n) and $u_{\epsilon_n}(x_n, t_n)$ are two bounded sequences, we can extract two subsequences, that we still denote with index n, such that

$$(x_n, t_n) \longrightarrow (\bar{x}, \bar{t}) \in B, \quad u_{\epsilon_n}(x_n, t_n) \longrightarrow s.$$

If we send $n \to +\infty$ in $u_{\epsilon_n}(x_n, t_n) - \varphi(x_n, t_n) \geq u_{\epsilon_n}(x^{(n)}, t^{(n)}) - \varphi(x^{(n)}, t^{(n)})$ we obtain

$$s - \varphi(\bar{x}, \bar{t}) \geq \overline{u}(x,t) - \varphi(x,t).$$

By the definition of $\overline{u}$ we get $s \leq \overline{u}(\bar{x}, \bar{t})$ and so

$$\overline{u}(\bar{x}, \bar{t}) - \varphi(\bar{x}, \bar{t}) \geq s - \varphi(\bar{x}, \bar{t}) \geq \overline{u}(x,t) - \varphi(x,t).$$

Thus, since (x,t) is the unique maximum point for $\overline{u} - \varphi$ in B we can conclude that $(x,t) = (\bar{x}, \bar{t})$ and $\overline{u}(\bar{x}, \bar{t}) = s = \lim u_{\epsilon_n}(x_n, t_n)$.
STEP 2. The inequality in (1.8) holds. Let $(x_n, t_n) \in B$, $\epsilon_n \to 0^+$ be the sequences of Step 1. Since

u_{ϵ_n} is a viscosity subsolution of (1.7), with $\varepsilon = \varepsilon_n$, we get

$$\varphi_t(x_n, t_n) + F_{\varepsilon_n}(x_n, D\varphi(x_n, t_n), D^2\varphi(x_n, t_n)) \leq 0$$

and thus

$$\begin{aligned} 0 &\geq \liminf_{n\to+\infty} \big(\varphi_t(x_n, t_n) + F_{\varepsilon_n}(x_n, D\varphi(x_n, t_n), D^2\varphi(x_n, t_n))\big) \\ &\geq \varphi_t(x, t) + F_*(x, D\varphi(x, t), D^2\varphi(x, t)). \end{aligned}$$

(ii) The proof of this second statement is close to the first one and we omit it. □

We now state and prove a nice proposition which allow us to restrict the choice of the test functions in Definition 1.0.1. The first formulation of the result is given in [6] for the mean curvature equation (3) and it is due to Barles and Georgelin. The same proof works also for more general equation of type (1.1). Here we generalized the result of Barles and Georgelin and we repeat the proof they give in [6].

Proposition 1.0.5. Let $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ be a locally bounded function so that for any $x \in \mathbb{R}^n$

$$F^*(x, 0, 0) = F_*(x, 0, 0) = 0. \tag{1.9}$$

Suppose moreover that F satisfies the *ellipticity condition*, i.e. for any $(x, t) \in \mathbb{R}^n \times (0, +\infty)$, $p \in \mathbb{R}^n$ and $X, Y \in \mathcal{S}^n$

$$F(x, t, p, X) \leq F(x, t, p, Y), \quad \text{if } X \geq Y. \tag{1.10}$$

An upper (respectively lower) semicontinuous function u is a viscosity subsolution (respectively supersolution) of (1.1) if and only if for any $\phi \in C^2(\mathbb{R}^n \times (0, +\infty))$, if $(x, t) \in \mathbb{R}^n \times (0, +\infty)$ is a local maximum (respectively minimum) point for $u - \phi$, one has

$$\phi_t(x, t) + F_*\big(x, D\phi(x, t), D^2\phi(x, t)\big) \leq 0 \quad \text{if } D\phi(x, t) \neq 0 \tag{1.11}$$

and

$$\phi_t(x, t) \leq 0 \quad \text{if} \quad D\phi(x, t) = 0 \text{ and } D^2\phi(x, t) = 0, \tag{1.12}$$

(respectively

$$\phi_t(x, t) + F^*\big(x, D\phi(x, t), D^2\phi(x, t)\big) \geq 0 \quad \text{if } D\phi(x, t) \neq 0$$

and

$$\phi_t(x, t) \geq 0 \quad \text{if} \quad D\phi(x, t) = 0 \text{ and } D^2\phi(x, t) = 0. \tag{1.13}$$

Proof. We treat only the subsolution case since the other one is completely analogous. Let u be an upper semicontinuous function that satisfies (1.11) and (1.12). We want to show that u is a viscosity subsolution of (1.1). Let ϕ be a C^2-function on $\mathbb{R}^n \times (0, +\infty)$ and (x,t) be a strict local maximum point of $u - \phi$. The only difficult is when $D\phi(x,t) = 0$ and $D^2\phi(x,t) \neq 0$. In this case we consider the function

$$\psi_\varepsilon(x,y,t) = \frac{| x - y|^4}{\varepsilon} + \phi(y,t),$$

where $\varepsilon > 0$. Since (x,t) is a strict maximum point of $u - \phi$, one can prove that there is a sequence $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ of local maximum points of $u - \psi_\varepsilon$ converging to (x,x,t) as $\varepsilon \to 0^+$. We have in the y variable the classical properties of a local maximum point

$$D\phi(y_\varepsilon, t_\varepsilon) = \frac{4(x_\varepsilon - y_\varepsilon)| x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \tag{1.14}$$

and

$$D^2\phi(y_\varepsilon, t_\varepsilon) \geq -\frac{4| x_\varepsilon - y_\varepsilon|^2}{\varepsilon} I_n - \frac{8(x_\varepsilon - y_\varepsilon) \otimes (x_\varepsilon - y_\varepsilon)}{\varepsilon}.$$

Two cases now may occur.

If $D\phi(y_\varepsilon, t_\varepsilon) = 0$ the equality in (1.14) implies that $x_\varepsilon = y_\varepsilon$. If we fix $y = y_\varepsilon$ the function $(x,t) \longmapsto u(x,t) - \psi_\varepsilon(x, y_\varepsilon, t)$ has a maximum at $(x_\varepsilon, t_\varepsilon)$. Therefore, using that u satisfies (1.12) we get

$$\phi_t(y_\varepsilon, t_\varepsilon) \leq 0.$$

Since $D^2\phi(y_\varepsilon, t_\varepsilon) \geq 0$ and assumptions (1.9) and (1.10) on F, this implies

$$\phi_t(y_\varepsilon, t_\varepsilon) + F_*(y_\varepsilon, D\phi(y_\varepsilon, t_\varepsilon), D^2\phi(y_\varepsilon, t_\varepsilon)) \leq 0.$$

We conclude by letting $\varepsilon \to 0^+$. Indeed we obtain

$$\begin{aligned} 0 \quad &\geq \liminf_{\varepsilon \to 0^+} \left(\phi_t(y_\varepsilon, t_\varepsilon) + F_*(y_\varepsilon, D\phi(y_\varepsilon, t_\varepsilon), D^2\phi(y_\varepsilon, t_\varepsilon)) \right) \\ &\geq \phi_t(x,t) + F_*(x, D\phi(x,t), D^2\phi(x,t)). \end{aligned}$$

On the contrary if $D\phi(y_\varepsilon, t_\varepsilon) \neq 0$ we observe that $(x_\varepsilon, t_\varepsilon)$ is a maximum point of

$$(x,t) \longmapsto u(x,t) - \psi_\varepsilon(x, x - (x_\varepsilon - y_\varepsilon), t) = u(x,t) - \frac{| x_\varepsilon - y_\varepsilon|^4}{\varepsilon} - \phi(x - (x_\varepsilon - y_\varepsilon), t)$$

and using (1.11), we obtain

$$\phi_t(y_\varepsilon, t_\varepsilon) + F_*(x_\varepsilon, D\phi(y_\varepsilon, t_\varepsilon), D^2\phi(y_\varepsilon, t_\varepsilon)) \leq 0.$$

Again we conclude by letting $\varepsilon \to 0^+$. □

Remark 1.0.6. From this last proposition it immediately follows that we can restrict the family of test functions in the definition of viscosity sub and supersolution. Indeed if $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ is a locally bounded function that satisfies assumptions (1.9) and (1.10) it is not restrictive to assume in Definition 1.0.1 that, if u (respectively v) is an upper semicontinuous subsolution (resp. a lower semicontinuous supersolution) of equation (1.1) and $\varphi \in C^2(\mathbb{R}^n \times (0, +\infty)$ is a test function for u (resp. for v) at the point (x, t), then

$$D\varphi(x, t) = 0,$$

implies

$$D^2\varphi(x, t) = 0.$$

Moreover by looking at the proof of Proposition 1.0.5 we observe that we can also assume that, if $D\varphi(x, t) = 0$ then $D\varphi(y, s) \neq 0$ for any $(y, s) \neq (x, t)$.

With some simple modifications of the proof of Proposition 1.0.5 we obtain the following Corollary in which we further restrict the possibility of choice for test functions in Definition 1.0.1.

Corollary 1.0.7. Let k be a fixed natural number and $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ a locally bounded function that satisfies assumptions (1.9) and (1.10). It is not restrictive to assume in Definition 1.0.1 that, if u (respectively v) is an upper semicontinuous subsolution (resp. a lower semicontinuous supersolution) of equation (1.1) and $\varphi \in C^k(\mathbb{R}^n \times (0, +\infty))$ is a test function for u (resp. for v) at the point (x, t), then

$$D\varphi(x, t) = 0,$$

implies

$$D\varphi(y, s) \neq 0, \qquad \text{whenever} (y, s) \neq (x, t),$$

and

$$\partial_{x_{i_1}}\varphi(x, t) = \partial_{x_{i_1}}\partial_{x_{i_2}}\varphi(x, t) = \cdots = \partial_{x_{i_1}}\partial_{x_{i_2}} \ldots \partial_{x_{i_k}}\varphi(x, t) = 0,$$

for any $i_1, \ldots, i_k \in \{1, \ldots, n\}$.

Proof. To get this Corollary one has to repeat the proof of Proposition 1.0.5 with the function ψ_ε defined as

$$\psi_\varepsilon(x, y, t) = \frac{| x - y|^{k+1}}{\varepsilon} + \phi(y, t)$$

if k is odd, or

$$\psi_\varepsilon(x, y, t) = \frac{| x - y|^{k+2}}{\varepsilon} + \phi(y, t)$$

if k is even. □

Chapter 2

Front Propagation

In this chapter we briefly talk about the evolution of interfaces (for example fronts or surfaces). For a summary about the argument we remand also to the work of Souganidis [41] and the references therein.

We start with the formulation of the problem. Let Γ_t be a generical interface at time $t > 0$, we suppose that Γ_t is the topological boundary of an open subset of $\mathbb{R}^n$ Ω_t, i.e. $\Gamma_t = \partial\Omega_t \subset \mathbb{R}^n$. Assume moreover that, for any point $x \in \mathbb{R}^n$, the exterior normal vector $\mathbf{n}(x)$ at Ω_t in x is well defined and that x evolves with normal velocity

$$V = v(x, t, \mathbf{n}(x), D\mathbf{n}(x))$$

where v is a continuous function of its argument.

A classical problem to study is the following one. Let Ω_0 be an open subset of $\mathbb{R}^n$, study the evolution of the interfaces $t \longmapsto \Gamma_t$ moving with normal velocity

$$V = v(x, t, D\mathbf{n}(x), \mathbf{n}(x)) \tag{2.1}$$

and starting at time $t = 0$ from $\Gamma_0 = \partial\Omega_0$.

As we mentioned in the Introduction one of the main problems of interface dynamics as in (2.1) is the development of singularities in finite time, independently of the smoothness of the initial surface Γ_0. To interpret the evolution past the singularities it is necessary to use some weak notions of evolving fronts. In this chapter we present two different approaches to the problem above. These approaches turn out to be equivalent when the interface Γ_t has empty interior for any $t > 0$ (*no-fattening phenomenon*).

2.1 The classical level set approach

We start with the so called level set approach. This approach, which is based on viscosity solutions, was first developed independently by Evans and Spruck in [27] and by Chen, Giga and Goto in [18] for more general geometric motions. These works were later extended by Ishii and Souganidis in [34] and by Goto in [32] for more general motion and more general initial surfaces, see also the work of Barles, Soner and Souganidis in [7]. For a detailed analysis of the approach we referred to the book of Giga [31].

Problem (2.1) can be formulated in an equivalent way. Assume that there exists a smooth function $u : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ such that

$$\Gamma_t = \{x \in \mathbb{R}^n : u(x,t) = 0\}, \quad \Omega_t = \{x \in \mathbb{R}^n : u(x,t) > 0\} \text{ and } Du \neq 0 \text{ on } \Gamma_t$$

it can be easily seen that

$$V = \frac{u_t}{|Du|}, \quad \mathbf{n} = -\frac{Du}{|Du|} \quad \text{and} \quad D\mathbf{n} = -\frac{1}{|Du|}\Big(I - \frac{Du \otimes Du}{|Du|^2}\Big)D^2 u$$

and so the equation (2.1) becomes

$$u_t = F(x, t, Du, D^2u) \tag{2.2}$$

with F defined as

$$F(x,t,p,X) = |p| v\Big(x, t, -\frac{p}{|p|}, -\frac{1}{|p|}(I - \frac{p \otimes p}{|p|^2})X\Big), \qquad (x,t,p,X) \in \mathbb{R}^n \times (0 + \infty) \times \mathbb{R}^n \times \mathcal{S}^n,$$

where $\mathcal{S}^n$ denote the space of the $n \times n$ symmetric matrices. Obviously this means that F is smooth as v with possible discontinuity at $p = 0$ and that F is geometric, i.e. it satisfies, for any $(x,t) \in \mathbb{R}^n \times (0, +\infty)$, $p \in \mathbb{R}^n$ and $X \in \mathcal{S}^n$,

$$F(x, t, \lambda p, \lambda X + \mu(p \otimes p)) = \lambda F(x, t, p, X) \quad \text{for all } \lambda > 0 \text{ and } \mu \in \mathbb{R}.$$

In this formal reasoning we have derived equation (2.2) from (2.1). In the so called *level-set approach* one wants to solve the interface evolution equation (2.1) starting at a given $\Gamma_0 = \partial\Omega_0$ looking at the (viscosity) solutions of (2.2). To do this we take an auxiliary function $u_0 : \mathbb{R}^n \to \mathbb{R}$, at least continuous and such that

$$\Gamma_0 = \{x \in \mathbb{R}^n : u_0(x) = 0\}, \quad \Omega_0 = \{x \in \mathbb{R}^n : u_0(x) > 0\}.$$

The assumption that u_0 is positive in Ω_0 gives the orientation of Γ_0; in fact, with this choice of u_0 the normal unit vector to Γ_0, outward to Ω_0 is given by $\mathbf{n} = Du_0/|Du_0|$.
Once function u_0 is chosen one solve (2.2) with initial data $u(x,0) = u_0(x)$ and define for any $t > 0$,

$$\Gamma_t = \{x \in \mathbb{R}^n : u(x,t) = 0\}, \qquad \Omega_t = \{x \in \mathbb{R}^n : u(x,t) > 0\}.$$

In order to consider the collection of pair $(\Gamma_t, \Omega_t)_{t\geq 0}$ as a kind of generalized solution of our evolution problem with initial data (Γ_0, Ω_0) it is necessary to prove that, under some suitable hypotesis, $(\Gamma_t, \Omega_t)_{t\geq 0}$ depends only on (Γ_0, Ω_0) and not on the particular function u_0.

In conclusion the main issues to follow the approach above will be:

- the well-posedness of the Cauchy problem

$$\begin{cases} u_t(x,t) = F(x,t,Du(x,t),D^2u(x,t)), & (x,t) \in \mathbb{R}^n \times (0,+\infty), \\ u(x,0) = u_o(x), & x \in \mathbb{R}^n; \end{cases} \tag{2.3}$$

- the uniqueness of the *generalized evolution* $(\Gamma_t, \Omega_t, \bar{\Omega}_t^c)_{t>0}$ once $(\Gamma_0, \Omega_0, \bar{\Omega}_0^c)$ is given.

The first issue will be developed under particular assumptions on F in the next chapters. In this section we want to treat the second topic, i.e. we want to discuss whether Γ_t depends only on Γ_o and not on the particular choice of the function u_o. The issue was settled in [7], [41] and in the book of Giga [31]. We start with a rigorous definition.

Definition 2.1.1. Consider an open set $\Omega_o \in \mathbb{R}^n$. A collection of pair $(\Gamma_t, \Omega_t)_{t\geq 0}$ is a *level set evolution* of $(\Gamma_o = \partial\Omega_o, \Omega_o)$ with normal velocity $V = v$ if there exists a viscosity solution of (2.2) such that, for any $t \geq 0$,

$$\Gamma_t = \{x \in \mathbb{R}^n : u(x,t) = 0\}, \quad \Omega_t = \{x \in \mathbb{R}^n : u(x,t) > 0\},$$

and

$$\Gamma_0 = \Gamma_o, \qquad \Omega_0 = \Omega_o.$$

To prove that this is a good definition we will assume that the function F satisfies the following assumptions.

(F1) F is a real-valued, locally bounded function on $\mathbb{R}^n \times (0,+\infty) \times \mathbb{R}^n \times \mathcal{S}^n$ and satisfying

$$F^*(x,t,0,0) = F_*(x,t,0,0), \qquad \text{for all } (x,t) \in \mathbb{R}^n \times (0,+\infty); \tag{2.4}$$

(F2) F satisfies the *ellipticity condition*, i.e. for any $(x,t) \in \mathbb{R}^n \times (0,+\infty)$, $p \in \mathbb{R}^n$ and $X, Y \in \mathcal{S}^n$

$$F(x,t,p,X) \leq F(x,t,p,Y), \quad \text{if } X \geq Y; \tag{2.5}$$

(F3) finally F has to be *geometric*, i.e., as already said,

$$F(x,t,\lambda p,\lambda X+\mu(p\otimes p))=\lambda F(x,t,p,X)\quad \text{for all } \lambda>0 \text{ and } \mu\in\mathbb{R} \tag{2.6}$$

for every $(x,t)\in\mathbb{R}^n\times(0,+\infty),\ p\in\mathbb{R}^n$ and $X\in\mathcal{S}^n$.

Moreover we suppose that a comparison result holds for (2.2) that is

(CP) if there exist two functions u and v, respectively a viscosity sub- and super-solution of (2.2) in $\mathbb{R}^n\times(0,T)$ so that $u(x,0)\le v(x,0)$ for any $x\in\mathbb{R}^n$, then $u(x,t)\le v(x,t)$, for any $(x,t)\in\mathbb{R}^n\times[0,T]$.

We state a Proposition in which we claim that, if F is geometric, then equation (2.2) is invariant by nondecreasing changes of variable $u\mapsto\theta(u)$.

Proposition 2.1.2. Assume that F satisfies assumptions **(F1)**, **(F2)** and **(F3)** above and let $\theta:\mathbb{R}\to\mathbb{R}$ be a continuous and nondecreasing function. If u is a viscosity subsolution (resp. supersolution) then $\theta(u)$ is also a viscosity subsolution (resp. supersolution).

Proof. We omit it. Anyway this proof can be found in [18]. □

The invariance by nondecreasing changes of variable of geometric equations is a crucial point to prove the following theorem in which the uniqueness of the level set evolution of a pair (Γ_o,Ω_o) becomes clear.

Theorem 2.1.3. Assume that the comparison principle (**CP**) holds for the equation (2.2). If u and v are two continuous solutions of the equation (2.2) in $\mathbb{R}^n\times[0,+\infty)$ so that

$$\{x:u(x,0)>0\}=\{x:v(x,0)>0\},\ \{x:u(x,0)<0\}=\{x:v(x,0)<0\},$$

$$\{x:u(x,0)=0\}=\{x:v(x,0)=0\}$$

and

$$\lim_{|x|\to+\infty}|u(x,0)|,\quad \lim_{|x|\to+\infty}|v(x,0)|>0.$$

Then, for all $t>0$,

$$\{x:u(x,t)>0\}=\{x:v(x,t)>0\},\ \{x:u(x,t)<0\}=\{x:v(x,t)<0\},$$

and $\{x:u(x,t)=0\}=\{x:v(x,t)=0\}$.

Proof. For the proof of this result we refer to [18] or to the book of Giga [31].

□

The properties and the regularity of the level set evolution have been the object of extensive study. One of the more basic question is whether the so-called *fattening phenomenon* occurs, i.e. whether the set $\bigcup_{t>0} \Gamma_t \times \{t\}$ has an interior. We give a more precise definition.

Definition 2.1.4. Let $(\Gamma_t, \Omega_t)_t$ be the level set evolution of $(\Gamma_o = \partial\Omega_o, \Omega_o)$. We say that the *no-interior condition* holds for $(\Gamma_t)_{t\geq 0}$ if

$$\begin{aligned} \mathrm{cl}\{(x,t) : u(x,t) > 0\} = \{(x,t) : u(x,t) \geq 0\} \text{ and} \\ \mathrm{int}\{(x,t) : u(x,t) \geq 0\} = \{(x,t) : u(x,t) > 0\}, \end{aligned} \tag{2.7}$$

where u is a solution of the level set pde (2.2) null in Γ_o and strictly positive in Ω_o.

Clearly if the no-interior condition in (2.7) holds then the set $\cup_{t\geq 0}(\Gamma_t \times \{t\})$ has an empty interior in $\mathbb{R}^n \times [0, +\infty)$. In most examples it can be proved that this is equivalent to prove that Γ_t has an empty interior in $\mathbb{R}^n$ for any $t \geq 0$. From now on we always denote with $(\Gamma_t, \Omega_t^+)_t$ the level set evolution of (Γ_o, Ω_o). Moreover for any $t > 0$ we put $\Omega_t^- = \overline{\Gamma_t \cup \Omega_t}^c$.

Theorem 2.1.5. Suppose that F satisfies (2.4) and (2.6).

(i) The two functions $\overline{\chi}(x,t) = \mathbb{1}_{\Omega_t^+ \cup \Gamma_t}(x) - \mathbb{1}_{\Omega_t^-}(x)$, $\underline{\chi}(x,t) = \mathbb{1}_{\Omega_t^+}(x) - \mathbb{1}_{\Omega_t^- \cup \Gamma_t}(x)$ are viscosity solutions of (2.2) associated respectively with the discontinuous initial data $w_o = \mathbb{1}_{\Omega_o^+ \cup \Gamma_o} - \mathbb{1}_{\Omega_o^-}$ and $w_o = \mathbb{1}_{\Omega_o^+} - \mathbb{1}_{\Omega_o^- \cup \Gamma_o}$.

(ii) Suppose that Γ_o has an empty interior and that a comparison result (**CP**) holds for the equation (2.2). Then the Cauchy problem for (2.2) associated with the initial data $w_o = \mathbb{1}_{\Omega_o^+} - \mathbb{1}_{\Omega_o^-}$ has a unique discontinuous solution if and only if the no-interior condition (2.7) holds, and this solution is given by the function

$$\chi(x,t) = \mathbb{1}_{\Omega_t^+}(x) - \mathbb{1}_{\Omega_t^-}(x). \tag{2.8}$$

Remark 2.1.6. In the above statement, uniqueness of discontinuous solutions is meant in the sense that u, w are locally bounded, $u(x,0) = w(x,0) = w_o(x)$ and u, w are continuous at $(x,0) \in \Omega_o^+ \cap \Omega_o^-$ and $u^* = w^*$, $u_* = w_*$ in $\mathbb{R}^n \times [0, +\infty)$.

Proof. In [7, 41], Barles, Soner and Souganidis proves this result when F is a function continuous in $\mathbb{R}^n \times (0, +\infty) \times \mathbb{R}^n \backslash \{0\} \times \mathcal{S}^n$. Here we slightly modify their proof to obtain the result for a generic geometric function F satisfying (2.4).
(i) The first statement of the theorem follows from the stability of viscosity solutions which holds for discontinuous equations as well (see Theorem 1.0.4). To prove that the function $\underline{\chi}(x,t)$ is a solution of (2.2) associated with the initial datum $w_o = \mathbb{1}_{\Omega_o^+} - \mathbb{1}_{\Omega_o^- \cup \Gamma_o}$, we consider the change of

variables $\psi^\epsilon(r) = \tanh\big(\frac{r-\sqrt{\epsilon}}{\epsilon}\big)$. Since for every $\epsilon > 0$ the function ψ^ϵ in strictly increasing we also have that every $u^\epsilon(x,t) = \psi^\epsilon(u(x,t))$ is a continuous viscosity solution of (2.2) associated with the initial datum $\psi^\epsilon(u_o)$. Moreover we can easily see that $\underline{\chi}^*(x,t) = \limsup_{\varepsilon\to 0^+}{}^* \ u^\epsilon(x,t)$, $\underline{\chi}(x,t) = \underline{\chi}_*(x,t) = \liminf_{\varepsilon\to 0^+}{}_* \ u^\epsilon(x,t)$ and hence, by the stability property of viscosity sub/supersolutions, $\underline{\chi}$ is a discontinuous viscosity solution of (2.2).

(ii) Now we assume that Γ_o has empty interior. If the set $\{u = 0\}$ doesn't satisfy (2.7) by point (i) we have that $\overline{\chi}$ and $\underline{\chi}$ have different semicontinuous envelopes and are both solutions of the Cauchy problem. To deal with the other implication, assume on the contrary that condition (2.7) holds and let χ as in (2.8). Then $\chi^* = \overline{\chi}$, $\chi_* = \underline{\chi}$ and so, by (i), χ is a solution of (2.2). The proof of uniqueness follows the argument in [7, 41] once we have assume that a comparison result holds for the equation (2.2). Indeed if w is a discontinuous solution of (2.2) with discontinuous initial condition $w_o = \mathbb{1}_{\Omega_o^+} - \mathbb{1}_{\Omega_o^-}$, then by comparison principle $-1 \leq w \leq 1$ in $\mathbb{R}^n \times [0,+\infty)$. Consider now a family of increasing smooth functions $\{\psi_n\}_n$ such that $-1 \leq \psi_n \leq 1,\ \psi_n(r) = 1$ if $r \geq 0$ and $\inf_n \psi_n = -1$ in $(-\infty, 0)$. By the comparison principle, we obtain that for all n, $w \leq w^* \leq \psi_n(u)$, where u is the solution of (2.3) and thus $w = -1$ in Ω_t^-. Similarly one proves that $w = 1$ in Ω_t^+ and we conclude by the no-interior condition. $\square$

2.2 Generalized propagation of fronts

An equivalent way to study the evolution of a collection of hypersurfaces is the so called *generalized propagation of fronts* introduced for the first time by Barles and Souganidis in [8], see also [41], and then reformulated (and partially revisited) by Barles and Da Lio in [5] for bounded domains with Neumann boundary condition.

In this section we denote with $(\Omega_t)_{t\in(0,T)}$ a family of open subsets of $\mathbb{R}^n$ and we set $\Gamma_t = \partial\Omega_t$, for any $t \geq 0$. The signed distance function $d(x,t)$ from x to Γ_t is defined by

$$d(x,t) = \begin{cases} d(x,\Gamma_t) & \text{if } x \in \Omega_t, \\ -d(x,\Gamma_t) & \text{otherwise}, \end{cases}$$

where $d(x,\Gamma_t)$ denotes the usual non negative distance from $x \in \mathbb{R}^n$ to Γ_t. If Γ_t is a smooth hypersurface, then d is a smooth function in a neighborhood of Γ_t, and for $x \in \Gamma_t$, $\mathbf{n}(x,t) = -Dd(x,t)$ is the unit normal to Γ_t pointing away from Ω_t. Again we assume that the function F satisfies assumptions **(F1), (F2)** and **(F3)**.

We recall here the definition of generalized flow in $\mathbb{R}^n$ but in the formulation given in [5].

Definition 2.2.1. Let F be a real-valued, locally bounded function on $\mathbb{R}^n \times [0,+\infty) \times \mathbb{R}^n \times \mathcal{S}^n$. A family $(\Omega_t)_{t\in(0,T)}$ (resp. $(\mathcal{F}_t)_{t\in(0,T)}$) of open (resp. close) subsets of $\mathbb{R}^n$ is called a *generalized* *superflow* (resp. *subflow*) with normal velocity $-F(x,t,Dd,D^2d)$ if, for any $x_0 \in \mathbb{R}^n$, $t \in (0,T)$, $r > 0$, $h > 0$ so that $t+h < T$ and for any smooth function $\phi : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ such that:

(i) $\partial\phi(x,s)/\partial t + F^*(x,s,D\phi(x,s),D^2\phi(x,s)) < 0$ in $B(x_0,r] \times [t,t+h]$ (resp. $\partial\phi(x,s)/\partial t + F_*(x,s,D\phi(x,s),D^2\phi(x,s)) > 0$ in $B(x_0,r] \times [t,t+h]$),

(ii) for any $s \in [t,t+h]$, $\{x \in B(x_0,r] : \phi(x,s) = 0\} \neq \emptyset$ and

$$|D\phi(x,s)| \neq 0 \text{ on } \{(x,s) \in B(x_0,r] \times [t,t+h] : \phi(x,s) = 0\},$$

(iii) $\{x \in B(x_0,r] : \phi(x,t) \geq 0\} \subset \Omega_t$ (resp. $\{x \in B(x_0,r] : \phi(x,t) \leq 0\} \subset \mathcal{F}_t^c$),

(iv) for all $s \in [t,t+h]$, $\{x \in \partial B(x_0,r] : \phi(x,s) \geq 0\} \subset \Omega_s$ (resp. $\{x \in \partial B(x_0,r] : \phi(x,s) \leq 0\} \subset \mathcal{F}_s^c$),

then we have

$$\{x \in B(x_0,r] : \phi(x,s) > 0\} \subset \Omega_s, \quad \text{(resp. } \{x \in B(x_0,r] : \phi(x,s) < 0\} \subset \mathcal{F}_s^c,)$$

for every $s \in (t,t+h)$.

A family $(\Omega_t)_{t\in(0,T)}$ of open subsets of $\mathbb{R}^n$ is called a *generalized flow* with normal velocity $-F$ if $(\Omega_t)_{t\in(0,T)}$ is a superflow and $(\overline{\Omega}_t)_{t\in(0,T)}$ is a subflow.

Remark. In smooth classical flows, if we change orientation, i.e., if we consider $-d$ instead of d and we look at the motion of $(\Omega_t^c)_{t\in(0,T)}$ instead of $(\Omega_t)_{t\in(0,T)}$, then we have to consider as prescribed normal velocity $V = -v(x,t,-\mathbf{n},-D\mathbf{n})$ instead of $V = v(x,t,\mathbf{n},D\mathbf{n})$. This elementary fact still holds in Definition 2.2.1. Indeed it can be easily shown that $(\Omega_t)_{t\in(0,T)}$ is a generalized superflow with normal velocity $-F(x,t,Dd,D^2d)$ if and only if $(\Omega_t^c)_{t\in(0,T)}$ is a generalized subflow with normal velocity $F(x,t,-Dd,-D^2d)$.

In the study of the evolution of a family of hypersurfaces the two approaches described, the level set approach and the notion of generalized sub- and superflow, turn out to be equivalent, as we claim in the following results.

Theorem 2.2.2. **(i)** Let $(\Omega_t)_{t\in(0,T)}$ be a family of open subsets of $\mathbb{R}^n$ such that the set Ω, $\Omega := \bigcup_{t\in(0,T)} \Omega_t \times \{t\}$, is open in $\mathbb{R}^n \times [0,T]$. Then $(\Omega_t)_{t\in(0,T)}$ is a generalized superflow with normal velocity $-F$ if and only if the function $\chi = \mathbb{1}_\Omega - \mathbb{1}_{\Omega^c}$ is a viscosity supersolution of (2.2).

(ii) Let $(\mathcal{F}_t)_{t\in(0,T)}$ be a family of close subsets of $\mathbb{R}^n$ such that the set $\mathcal{F} := \bigcup_{t\in(0,T)} \mathcal{F}_t \times \{t\}$ is closed in $\mathbb{R}^n \times [0,T]$. Then $(\mathcal{F}_t)_{t\in(0,T)}$ is a generalized subflow with normal velocity $-F$ if and only if the function $\overline{\chi} = \mathbb{1}_{\mathcal{F}} - \mathbb{1}_{\mathcal{F}^c}$ is a viscosity subsolution of (2.2).

Proof. The proof follows the one in [5] although here the function F may be discontinuous in the x variable. We give here in detail the superflow/supersolution case.

We first assume that $\chi = \mathbb{1}_\Omega - \mathbb{1}_{\Omega^c}$ is a supersolution of (2.2) and we show that $(\Omega_t)_{t\in(0,T)}$ is a generalized superflow. To do this we consider a smooth function ϕ, a point $(x_0, t) \in \mathbb{R}^n \times (0,T)$ and $r, h > 0$ satisfying conditions (i), (ii), (iii), (iv) in Definition 2.2.1. We assume that $\phi \le 1$ in $B(x_0, r] \times [t, t+h]$(otherwise we change ϕ with $\eta\phi$ for $\eta > 0$ small enough and we use the assumption (2.6) on F). We consider

$$m := \min\{\chi(x,s) - \phi(x,s) : (x,s) \in B(x_0, r] \times [t, t+h]\}$$

Since ϕ satisfies condition (i) and χ is a supersolution of equation (2.2) in $B(x_0,r) \times (t, t+h)$ and, by Proposition 1.0.3 also in $B(x_0, r) \times (t, t+h]$, the minimum m cannot be attained in $B(x_0, r) \times (t, t+h]$. Therefore it has to be attained either in $\partial B(x_0, r)$ or at time t.

We observe that, for any $(x,s) \in (\partial B(x_0,r) \times [t, t+h]) \cup (B(x_0, r] \times \{t\})$,

if $x \in \Omega_s$, then $\chi(x,s) = 1$ and $(\chi - \phi)(x,s) \ge 0$ because $\phi \le 1$ in $B(x_0, r] \times [t, t+h]$,

if $x \notin \Omega_s$, then $\chi(x,s) = -1$ and, by (iii) and (iv), $(\chi - \phi)(x,s) \ge -1 + \delta$ with $\delta > 0$.

Anyway we can conclude that $m \ge -1 + \delta$ and so if $(y,s) \in B(x_0, r] \times [t, t+h]$ and $y \notin \Omega_s$, we have

$$\chi(y,s) - \phi(y,s) \ge -1 + \delta,$$

i.e. $\phi(y,s) \le -\delta$. This means that for every $s \in [t, t+h]$,

$$\{y \in B(x_0, r] : \phi(y,s) \ge 0\} \cap \Omega_s^c = \emptyset,$$

which implies that $(\Omega_t)_{t\in(0,T)}$ is a generalized superflow with normal velocity $-F$.

Conversely, we assume that $(\Omega_t)_{t\in(0,T)}$ is a generalized superflow and we show that χ is a supersolution of the equation (2.2) in $\mathbb{R}^n \times (0,T)$. To this aim we consider a point $(x,t) \in \mathbb{R}^n \times (0,T)$ and a function $\phi \in C^\infty(\mathbb{R}^n \times [0,T])$ so that (x,t) is a strict local minimum point of $\chi - \phi$. Changing ϕ to $\phi - \phi(x,t)$, if necessary, we may assume that $\phi(x,t) = 0$. We have to show that

$$\frac{\partial \phi}{\partial t}(x,t) + F^*(x, t, D\phi(x,t), D^2\phi(x,t)) \ge 0. \tag{2.9}$$

If (x,t) is in the interior of either $\{\chi = 1\}$ or $\{\chi = -1\}$ then χ is constant in a neighborhood of (x,t) and therefore $\partial_t \phi(x,t) = 0$, $D\phi(x,t) = 0$ and $D^2\phi(x,t) \le 0$. Thanks to assumptions (2.4) and (2.5) on F the inequality in (2.9) thus follows.

Assume that $(x,t) \in \partial\{\chi = 1\} \cap \partial\{\chi = -1\}$. The lower semicontinuity of χ yields $\chi(x,t) = -1$. We suppose by contradiction that the inequality in (2.9) doesn't hold; therefore, for some $\alpha > 0$, we have

$$\frac{\partial \phi}{\partial t}(x,t) + F^*(x,t,D\phi(x,t),D^2\phi(x,t)) < -\alpha.$$

Since ϕ is smooth and F^* is upper semicontinuous, we can find $r, h > 0$ such that for all $(y,s) \in B(x,r] \times [t-h, t+h]$,

$$\frac{\partial \phi}{\partial t}(y,s) + F^*(y,s,D\phi(y,s),D^2\phi(y,s)) < -\frac{\alpha}{2}. \tag{2.10}$$

Moreover, since (x,t) is a strict local minimum point of $\chi - \phi$, by taking smaller r and h if necessary, we can assume also, for $(y,s) \in B(x,r] \times [t-h,t+h]$ and $(y,s) \neq (x,t)$,

$$\chi(x,t) - \phi(x,t) = -1 < \chi(y,s) - \phi(y,s). \tag{2.11}$$

We first consider the case $|D\phi(x,t)| \neq 0$. For $0 < \delta \ll 1$, we introduce the test function $\phi_\delta(y,s) := \phi(y,s) + \delta(s - (t-h))$. Since $\phi(x,t) = 0$ and $D\phi(x,t) \neq 0$, it is easy to see that if h and δ are small enough then, for any $t-h \leq s \leq t+h$, the set $\{y \in B(x,r) : \phi_\delta(y,s) = 0\}$ is non empty. Moreover choosing smaller r, h and δ, we may also assume that $|D\phi| \neq 0$ in $B(x,r] \times [t-h, t+h]$. We observe that, for $\delta > 0$ small enough, because of (2.10) and (2.11), we have both

$$\phi_\delta(y,s) - 1 < \chi(y,s) \tag{2.12}$$

for all $(y,s) \in (B(x,r) \times \{t-h\}) \cup (\partial B(x,r) \times [t-h,t+h])$ and

$$\frac{\partial \phi}{\partial t}(y,s) + F^*(y,s,D\phi(y,s),D^2\phi(y,s)) < -\frac{\alpha}{4}$$

for all $(y,s) \in B(x,r] \times [t-h,t+h]$. The inequality in (2.12) implies that

$$\{y \in B(x,r] : \phi_\delta(y,t-h) \geq 0\} \subset \Omega_{t-h},$$

and for all $s \in [t-h,t+h]$,

$$\{y \in \partial B(x,r) : \phi_\delta(y,s) \geq 0\} \subset \Omega_s.$$

By the definition of superflow this yield

$$\{y \in B(x,r] : \phi_\delta(y,s) > 0\} \subset \Omega_s,$$

for every $s \in (t-h, t+h)$. But, since $\phi_\delta(x,t) = \delta h > 0$, we deduce that $x \in \Omega_t$, and this is a contradiction.

Now we turn to the case when $|\,D\phi(x,t)| = 0$. We can assume without loss of generality that $D^2\phi(x,t) = 0$ as well, see Proposition 1.0.5. Thus, to prove (2.9), we have to show that

$$\frac{\partial \phi}{\partial t}(x,t) \geq 0.$$

Suppose by contradiction that $a := \dfrac{\partial \phi}{\partial t}(x,t) < 0$. We have

$$\phi(y,s) = \underbrace{\phi(x,t)}_{=0} + \frac{\partial \phi}{\partial t}(x,t)(s-t) + o(|\,s-t|) + o(|\,y-x|^2) \quad \text{as } s \to t,\ |\,y-x| \to 0.$$

Thus, for all $\varepsilon > 0$, there exist $r = r_\varepsilon, h = h_\varepsilon, h' = h'_\varepsilon > 0$ such that

$$h < -\frac{\varepsilon r^2}{a}$$

and, for any $(y,s) \in B(x,r] \times [t-h, t+h']$

$$\begin{aligned} \phi(y,s) &\geq a(s-t) + \frac{a}{2}|\,s-t| - \varepsilon|\,y-x|^2 \\ &= \frac{a}{2}(s-t) + a(s-t)^+ - \varepsilon|\,y-x|^2 \\ &\geq \frac{a}{2}(s-t) - \varepsilon|\,y-x|^2 + ah'. \end{aligned}$$

By (2.11) we can take $\beta > 0$ such that

$$2\beta + \phi(y,s) - 1 < \chi(y,s)$$

for all $(y,s) \in (B(x,r] \times \{t-h\}) \cup (\partial B(x,r) \times (t-h, t+h'))$. By taking β smaller we may also suppose $\beta < \varepsilon r^2/2$. We consider the function $\psi_\beta(y,s) = (a/2)(s-t) - \varepsilon|\,y-x|^2 + \beta$. Since we can take h' smaller we assume from now on that $h' \leq -\beta/a$. Combining the last two inequalities and the assumptions on β, h, h' and r we get

$$\psi_\beta(y,s) - 1 < \chi(y,s) \tag{2.13}$$

for all $(y,s) \in (B(x,r] \times \{t-h\}) \cup (\partial B(x,r) \times [t-h, t+h'])$. Furthermore consider a fixed $s \in [t-h, t+h']$. We have $\psi_\beta(x,s) = a(s-t)/2 + \beta \geq ah'/2 + \beta > 0$ and

$$\psi_\beta(y,s) = \frac{a}{2}(s-t) - \varepsilon r^2 + \beta \leq -\frac{ah + \varepsilon r^2}{2} \leq 0$$

for $|\,y-x| = r$. Thus the set $\{y \in B(x,r] : \psi_\beta(y,s) = 0\}$ is non empty and $|\,D\psi_\beta(y,s)| \neq 0$ for every $(y,s) \in \{B(x,r] \times [t-h, t+h'] : \psi_\beta(y,s) = 0\}$.

Since F^* is upper semicontinuous and $F^*(y,s,0,0) = 0$ for any y and s, for small ε we have

$$\frac{a}{2} + F^*(y,s,-2\varepsilon(y-x), -2\varepsilon I) < 0 \quad \text{on } B(x,r] \times [t-h, t+h'].$$

Finally by (2.13) we get

$$\{y \in B(x,r] : \psi_\beta(y, t-h) \geq 0\} \Subset \Omega_{t-h},$$

and for all $s \in [t-h, t+h']$,

$$\{y \in \partial B(x,r) : \psi_\beta(y,s) \geq 0\} \Subset \Omega_s.$$

Thus, since $(\Omega_t)_t$ is a generalized superflow, we have

$$\{y \in B(x,r] : \psi_\beta(y,s) > 0\} \Subset \Omega_s$$

for any $s \in (t-h, t+h')$. But again $\psi_\beta(x,t) = \beta > 0$, and this means $x \in \Omega_t$, which is a contradiction.

□

Corollary 2.2.3. Assume to have two families of open subsets of $\mathbb{R}^n$, $(\Omega_t^1)_{t\in[0,T)}$ and $(\Omega_t^2)_{t\in[0,T)}$ such that $(\Omega_t^1)_{t\in(0,T)}$ and $((\Omega_t^2)^c)_{t\in(0,T)}$ are respectively super- and subflows with normal velocity $-F$ and also $\cup_{t\in(0,T)}\Omega_t^1 \times \{t\}$, $\cup_{t\in(0,T)}\Omega_t^2 \times \{t\}$ are open and disjoint. Suppose moreover that there exists $(\partial\Omega_0^+, \Omega_0^+, \Omega_0^-) \in \mathcal{E}$ such that $\Omega_0^+ \subseteq \Omega_0^1$ and $\Omega_0^- \subseteq \Omega_0^2$. Then, if we denote with $(\Gamma_t, \Omega_t^+, \Omega_t^-)_{t\in(0,T)}$ the level set evolution of $(\partial\Omega_0^+, \Omega_0^+, \Omega_0^-)$, we have:

(i) for all $t \in [0,T)$,

$$\Omega_t^+ \subset \Omega_t^1 \subset \Omega_t^+ \cup \Gamma_t, \qquad \Omega_t^- \subset \Omega_t^2 \subset \Omega_t^- \cup \Gamma_t,$$

(ii) if $\bigcup_t \Gamma_t \times \{t\}$ satisfies the no-interior condition (2.7), then for all $t \in [0,T)$,

$$\Omega_t^+ = \Omega_t^1, \qquad \Omega_t^- = \Omega_t^2.$$

Proof. The proof of this theorem follows by combining the results in Theorem 2.1.5 and in Theorem 2.2.2. □

2.3 Applications to the asymptotics of reaction-diffusion equations. The abstract method

In this section we present an abstract method to study the asymptotic behavior of solutions to semilinear reaction-diffusion equations by means of generalized sub- and superflows. We do this following the ideas explained in [8] and in [5].

Consider a given family $(u_\varepsilon)_\varepsilon$ of bounded functions on $\mathbb{R}^n \times [0,T]$, typically the solution of a Cauchy problem for a reaction-diffusion equation with small parameter $\varepsilon > 0$. Our aim will be to prove that, for any $t \in [0,T]$, there exist two regions Ω_t^1 and Ω_t^2 where the $u^\varepsilon(\cdot,t)$'s are close to two different values, and to study the evolution of the moving front $t \mapsto \Gamma_t$ that separates Ω_t^1 and Ω_t^2. The key point will be to prove that $(\Omega_t^1)_{t\in(0,T)}$ and $((\Omega_t^2)^c)_{t\in(0,T)}$ are respectively a super and a subflow with normal velocity determined by the data of the problem. In other words we will prove that there exists a generalized flow $(\Gamma_t, \Omega_t^1, \Omega_t^2)_{t\in[0,T)}$ such that, as $\varepsilon \to 0$,

$$
\begin{aligned}
u^\varepsilon(x,t) \to b(x,t) \quad &\text{if } (x,t) \in \Omega^1 := \bigcup_{t\in(0,T)} \Omega_t^1 \times \{t\},\\
u^\varepsilon(x,t) \to a(x,t) \quad &\text{if } (x,t) \in \Omega^2 := \bigcup_{t\in(0,T)} \Omega_t^2 \times \{t\},
\end{aligned}
$$

where, for all $(x,t) \in \mathbb{R}^n \times [0,T]$, $a(x,t), b(x,t) \in \mathbb{R}$ can be interpreted as the stable equilibria of the system. To be more precise we first assume that there exist sequences $(a_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon$ of real-valued functions defined in $\mathbb{R}^n \times [0,T]$ such that

$$
a_\varepsilon(x,t) \le u^\varepsilon(x,t) \le b_\varepsilon(x,t), \qquad \text{for any } (x,t) \in \mathbb{R}^n \times [0,T],
$$

and $a_\varepsilon \to a$, $b_\varepsilon \to b$ uniformly in $\mathbb{R}^n \times [0,T]$ as $\varepsilon \to 0$. Then we define two open sets Ω^1 and Ω^2, and two families $(\Omega_t^1)_{t\in(0,T)}$ and $(\Omega_t^2)_{t\in(0,T)}$ by putting, for some suitable $\tau \ge 0$,

$$
\begin{aligned}
\Omega^1 &= \mathrm{Int}\left\{(x,t) \in \mathbb{R}^n \times [0,T] : \liminf_{\varepsilon\to 0^+}{}_* \left[\frac{u^\varepsilon - b_\varepsilon}{\varepsilon^\tau}\right](x,t) \ge 0\right\}\\
\Omega^2 &= \mathrm{Int}\left\{(x,t) \in \mathbb{R}^n \times [0,T] : \limsup_{\varepsilon\to 0^+}{}^* \left[\frac{u^\varepsilon - a_\varepsilon}{\varepsilon^\tau}\right](x,t) \le 0\right\},
\end{aligned}
\tag{2.14}
$$

and, for all $t \in (0,T)$,

$$
\begin{aligned}
\Omega_t^1 &= \{x \in \mathbb{R}^n : (x,t) \in \Omega^1\} = \mathrm{pr}_{\mathbb{R}^n}(\Omega^1 \cap (\mathbb{R}^n \times \{t\})),\\
\Omega_t^2 &= \{x \in \mathbb{R}^n : (x,t) \in \Omega^2\}.
\end{aligned}
\tag{2.15}
$$

Obviously $\Omega^1 = \cup_{t\in(0,T)} \Omega_t^1 \times \{t\}$, $\Omega^2 = \cup_{t\in(0,T)} \Omega_t^2 \times \{t\}$. Since Ω^1 and Ω^2 are open and disjoint subsets of $\mathbb{R}^n \times (0,T)$ the two step functions χ and $\overline{\chi}$, defined as

$$\chi(x,t) = \mathbb{1}_{\Omega^1} - \mathbb{1}_{(\Omega^1)^c}, \quad \overline{\chi}(x,t) = \mathbb{1}_{(\Omega^2)^c} - \mathbb{1}_{\Omega^2} \tag{2.16}$$

are respectively lower and upper semicontinuous in $\mathbb{R}^n \times (0,T)$. Finally we observe that χ, $\overline{\chi}$ can be extended by lower and upper semicontinuity to the whole of $\mathbb{R}^n \times [0,T]$. For simplicity of notation we'll call again χ and $\overline{\chi}$ these extensions.

The proof of the asymptotics result for our functions u^ε can be divided into the following three main steps.

1. Initialization: we define the traces Ω_0^1 and Ω_0^2 of Ω^1 and Ω^2 for $t = 0$ as

$$\Omega_0^1 = \{x \in \mathbb{R}^n : \chi(x,0) = 1\}, \qquad \Omega_0^2 = \{x \in \mathbb{R}^n : \overline{\chi}(x,0) = -1\}. \tag{2.17}$$

We do this by constructing suitable sub and supersolutions of the u^ε's equation in the set $\mathbb{R}^n \times (0,\bar{h})$, with $\bar{h}$ small enough.

2. Propagation: we show that $(\Omega_t^1)_{t\in(0,T)}$ and $((\Omega_t^2)^c)_{t\in(0,T)}$ are respectively a generalized subflow and a generalized superflow with normal velocity $-F$. To do this we have to construct suitable smooth sub- and supersolutions to the Cauchy problem satisfied by u^ε in sets of the form $B(x,r] \times [t, t+h]$ with Dirichlet boundary conditions on $\partial B(x,r) \times [t, t+h]$. We just notice that, unlike the first step, we build these sub- and supersolutions locally in space but not in time since h is not suppose to be small.

3. Conclusion: we conclude our proof by applying Corollary 2.2.3 to $(\Omega_t^1)_{t\in[0,T)}$ and $((\Omega_t^2)^c)_{t\in[0,T)}$.

Chapter 3

Discontinuous velocities

In their paper [7], Barles, Soner and Souganidis consider reaction-diffusion equations of the form

$$\phi_t - \Delta\phi + f^\varepsilon(x,t,\phi) = 0 \quad \text{in } \mathbb{R}^n \times (0,+\infty)$$

with the two scalings $(x/\varepsilon, t/\varepsilon)$ and $(x/\varepsilon, t/\varepsilon^2)$. These give rise to two different singular perturbation problems

$$u_t^\varepsilon - \varepsilon\Delta u^\varepsilon + \frac{f^\varepsilon(x,t,u^\varepsilon)}{\varepsilon} = 0 \quad \text{in } \mathbb{R}^n \times (0,+\infty), \tag{3.1}$$

and

$$u_t^\varepsilon - \Delta u^\varepsilon + \frac{f^\varepsilon(x,t,u^\varepsilon)}{\varepsilon^2} = 0 \quad \text{in } \mathbb{R}^n \times (0,+\infty), \tag{3.2}$$

with initial datum

$$u^\varepsilon(x,0) = g(x),$$

where g is a given function that initializes the front and $u \longmapsto f^\varepsilon(x,t,u)$ is a cubic type function, i.e., it has two stable and one unstable equilibria. Classical examples of f^ε are

$$f^\varepsilon(x,t,q) = 2(q-\alpha(x,t))(q^2-1) + \varepsilon\theta \tag{3.3}$$

and

$$f^\varepsilon(x,t,q) = 2(q-\varepsilon\alpha(x,t))(q^2-1), \tag{3.4}$$

where $\alpha \in W^{1,\infty}(\mathbb{R}^n \times [0,+\infty)) \cap C(\mathbb{R}^n \times [0,+\infty))$, $\theta \in \mathbb{R}$ are given and α takes values in $(-1,1)$. If, as in (3.3), the unstable equilibrium of f^ε converges locally uniformly as $\varepsilon \to 0^+$ to some continuum function α then the fronts that described the asymptotics of the Cauchy problem for (3.1) propagate with normal velocity

$$V = \alpha.$$

On the contrary in (3.4) the unstable equilibrium of f^ε goes uniformly to zero when $\varepsilon \to 0^+$ and one has to go to the following scaling $(x/\varepsilon, t/\varepsilon^2)$, i.e. one has to consider the equation (3.2). The interface that describes the limiting behavior of the solutions of the Cauchy problem (3.2) moves with normal velocity

$$V = \alpha + \frac{1}{t}\kappa.$$

In this chapter we want to generalize these results allowing to the function $\alpha : \mathbb{R}^n \to [-1, 1]$ to be a bounded measurable function, piecewise continuous across an oriented, closed and Lipshitz hypersurface $\tilde{\Gamma} \subset \mathbb{R}^n$. Indeed we consider reaction terms of type $f^\varepsilon : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$

$$f^\varepsilon(q, x) := 2\Big(q - \frac{c^\varepsilon(x)}{2}\Big)(q^2 - 1)$$

with $-1 < c^\varepsilon/2 < 1$. When the unstable equilibrium satisfies

$$\frac{c^\varepsilon(x)}{2} \xrightarrow[\varepsilon \to 0^+]{} \alpha(x), \quad \text{locally uniformly off the hypersurface } \tilde{\Gamma} \subset \mathbb{R}^n,$$

and the initial condition g represents a sharp interface across $c^\varepsilon/2$ we consider the Cauchy problem for the reaction-diffusion equation (3.1) and we prove that the limiting behavior of its solutions is governed by the following first order Hamilton-Jacobi equation

$$\begin{cases} (i) \quad u_t(x, t) + \alpha(x)|Du(x, t)| = 0, \quad (x, t) \in \mathbb{R}^n \times (0, +\infty), \\ (ii) \quad u(x, 0) = u_o(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{3.5}$$

The initial condition u_o is chosen in such a way that the initial front $\Gamma_o = \{x \in \mathbb{R}^n : u_o(x) = 0\} = \{x \in \mathbb{R}^n : g(x) = \dfrac{\alpha(x)}{2}\}$ is a nonempty and closed set (ideally an hypersurface). Moreover $u_o(x) > 0$ (resp. $u_o(x) < 0$) if $g(x) > \dfrac{\alpha(x)}{2}$ (resp. $g(x) < \dfrac{\alpha(x)}{2}$).

On the contrary if

$$\frac{c^\varepsilon(x)}{\varepsilon} \xrightarrow[\varepsilon \to 0^+]{} \alpha(x), \quad \text{locally uniformly off } \tilde{\Gamma} \subset \mathbb{R}^n,$$

the equilibrium c^ε goes to zero uniformly on $\mathbb{R}^n$ and the asymptotic of the Cauchy problem for (3.2) is described by the geometric Hamilton-Jacobi equation

$$u_t + \alpha(x)|Du| - \operatorname{tr}\Big[\Big(I - \frac{Du}{|Du|} \otimes \frac{Du}{|Du|}\Big) D^2 u\Big] = 0, \quad \text{in } \mathbb{R}^n \times (0, +\infty). \tag{3.6}$$

The technique used to study the asymptotics we are looking at follows the ideas of Barles and Souganidis in [8] and then partially revisited by Barles and Da Lio in [5]. We briefly described this approach in Chapter 2. As we said in the Introduction the novelty of the problem we consider here is that now the convergence of the sequence $c^\varepsilon/\varepsilon^\tau$, $\tau \in \{0,1\}$, occurs off the discontinuity set of α. This compel us to approximate the standard definition of generalized sub- and superflows with normal velocity $-\alpha$ by means of continuous velocities. We prove that this is possible in Propositions 3.1.6 and 3.3.4. Finally we point out that in our relaxed assumptions the terms $\|Dc^\varepsilon\|_\infty$ and $\|\Delta c^\varepsilon\|_\infty$ may blow up as $\varepsilon \to 0^+$. To overcome this difficulty we have to link their blow up rate to the parameter ε.

3.1 Asymptotics of reaction diffusion equation

3.1.1 Main assumptions

We are now ready to study the asymptotic behavior of the solutions of the Cauchy problem

$$\begin{cases} \text{(i)} \quad u^\varepsilon_t(x,t) - \varepsilon\Delta u^\varepsilon(x,t) + \varepsilon^{-1} f^\varepsilon(u^\varepsilon, x) = 0 & \text{in } \mathbb{R}^n \times (0,+\infty), \\ \text{(ii)} \quad u^\varepsilon(x,0) = g(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{3.7}$$

Throughout this section we will suppose that $g \in C(\mathbb{R}^n)$, $-1 \le g \le 1$ while $f^\varepsilon \in C^2(\mathbb{R} \times \mathbb{R}^n)$, satisfies the following properties, where $\gamma, \rho \in (0,1)$:

$$\begin{cases} \text{for any } x \in \mathbb{R}^n, \\ f^\varepsilon(\cdot,x) \text{ has exactly three zeroes } -1, m^\varepsilon_o(x), 1, \text{ with } 0 < \rho < m^\varepsilon_o(x) < 1-\rho, \\ f^\varepsilon(\cdot,x) > 0 \text{ in } (-1, m^\varepsilon_o(x)) \cup (1,+\infty) \text{ and } f^\varepsilon(\cdot,x) < 0 \text{ in } (-\infty,-1) \cup (m^\varepsilon_o(x),1), \\ f^\varepsilon_q(q,x) \ge \gamma \text{ for all } q \in (-\infty,-1+\gamma] \cup [1-\gamma,+\infty), \\ f^\varepsilon_{qq}(-1,x) < 0 \text{ and } f^\varepsilon_{qq}(1,x) > 0, \end{cases} \tag{3.8}$$

and also, if $k \in [0, \frac{1}{2}]$,

$$\begin{cases} \text{for every compact } K \subset \mathbb{R} \text{ there exist } C = C(K), C_i = C_i(K) > 0, \ i = 1,2 \\ \text{such that, for all } (q,x) \in K \times \mathbb{R}^n, \ 1 \le i,j \le n, \\ |f^\varepsilon_q(q,x)|, |f^\varepsilon_{qq}(q,x)| \le C, \ |f^\varepsilon_{x_i}(q,x)|, |f^\varepsilon_{x_i q}(q,x)| \le \dfrac{C_1}{\varepsilon^k}, \ |f^\varepsilon_{x_i x_j}(q,x)| \le \dfrac{C_2}{\varepsilon^{2k}}. \end{cases} \tag{3.9}$$

Below we denote with $\overline{m}(x) = \limsup^*_{\varepsilon\to 0^+} m^\varepsilon_o(x)$, $\underline{m}(x) = \liminf_{*\,\varepsilon\to 0^+} m^\varepsilon_o(x)$ the upper semicontinuous and, respectively, lower semicontinuous half relaxed limits of the family $\{m^\varepsilon_o(\cdot) : \varepsilon > 0\}$. We also assume on f that:

for every compact $K_1 \subset \mathbb{R}^n$ and $m_1 > \sup_{x\in K_1} \overline{m}(x)$, $m_2 < \inf_{x\in K_1} \underline{m}$, there are two functions

$$\begin{gathered} \bar{f},\ \underline{f} \in C^2(\mathbb{R}) \text{ satisfying } (3.8), (3.9) \\ \text{with zeroes in } \{-1, m_1, 1\}, \{-1, m_2, 1\} \text{ respectively,} \\ \text{and } \underline{f} \leq f^\varepsilon \leq \overline{f},\ \text{for all } x \in K_1,\ q \in [-1,1],\ \varepsilon > 0 \text{ sufficiently small.} \end{gathered} \tag{3.10}$$

As we said before a typical example for the function f^ε is

$$f^\varepsilon(q,x) := 2\Big(q - \frac{c^\varepsilon(x)}{2}\Big)(q^2-1). \tag{3.11}$$

It satisfies all the assumptions listed above with $m_o^\varepsilon(x) = c^\varepsilon(x)/2$, provided that

$$\begin{aligned} &c^\varepsilon \in C^2(\mathbb{R}^n), 0 < \rho < c^\varepsilon(x)/2 < 1-\rho, \\ &|\,\partial_{x_i} c^\varepsilon(x)| \leq \frac{C_1}{\varepsilon^k},\ |\,\partial^2_{x_i x_j} c^\varepsilon(x)| \leq \frac{C_2}{\varepsilon^{2k}},\ \forall x \in \mathbb{R}^n,\ i,j \in \{1,\dots n\}, \end{aligned} \tag{3.12}$$

and in (3.10) we can choose $\overline{f}(q) := 2(q-m_1)(q^2-1)$, $\underline{f}(q) := 2(q-m_2)(q^2-1)$.

Thanks to these properties of f^ε, as proven by Aronsson-Weinberger [2] and Fife-McLeod [30], for all $x \in \mathbb{R}^n$ there is a unique pair $(q^\varepsilon(\cdot,x), c^\varepsilon(x))$, solution of the traveling wave equation

$$q^\varepsilon_{rr}(r,x) + c^\varepsilon(x) q^\varepsilon_r(r,x) = f^\varepsilon(q^\varepsilon(r,x),x), \qquad (r,x) \in \mathbb{R}\times\mathbb{R}^n, \tag{3.13}$$

subject to the following conditions

$$q^\varepsilon(-\infty,x) = -1,\ q^\varepsilon(+\infty,x) = 1,\ q^\varepsilon(0,x) = m_o^\varepsilon(x)$$

and we have that $q^\varepsilon_r > 0$. We will assume that the pair $(q^\varepsilon(\cdot), c^\varepsilon(x))$ satisfies a series of properties. There are $a, b > 0$ such that

$$\inf_{x\in\mathbb{R}^n} q^\varepsilon(r,x) \geq 1 - ae^{-br} \text{ as } r \to +\infty, \quad \sup_{x\in\mathbb{R}^n} q^\varepsilon(r,x) \leq -1 + ae^{br} \text{ as } r \to -\infty, \tag{3.14}$$

and moreover

$$\begin{gathered} q^\varepsilon_r(r,x) \geq K(x,\bar{r}) > 0, \quad \text{for } x \in \mathbb{R}^n,\ |r| \leq \bar{r}, \\ \sup_{(r,x)\in\mathbb{R}\times\mathbb{R}^n} [(1+|r|)q^\varepsilon_r(r,x) + (1+|r|^2)q^\varepsilon_{rr}(r,x)] < +\infty. \end{gathered} \tag{3.15}$$

For any compact $K_1 \subset \mathbb{R}^n$ there exist constants $M_1, M_2 > 0$ such that

$$|Dq^\varepsilon(r,x)|,\ |Dq^\varepsilon_r(r,x)| \leq \frac{M_1}{\varepsilon^k},\ |D^2 q^\varepsilon(r,x)| \leq \frac{M_2}{\varepsilon^{2k}},\ \text{for all } x \in K_1,\ r \in \mathbb{R}. \tag{3.16}$$

For instance in the case (3.11) easy explicit calculations are possible. It turns out that the traveling wave equation admits as unique solution the function

$$q^\varepsilon(r,x) = \tanh(r + r^\varepsilon(x)), \tag{3.17}$$

where $r^\varepsilon(x) = \dfrac{1}{2}\ln\left(\dfrac{2+c^\varepsilon(x)}{2-c^\varepsilon(x)}\right)$ and the velocity of the traveling wave is precisely $c^\varepsilon(x)$ of (3.11). Some simple computations, using the properties of c^ε, show that properties (3.14), (3.15) and (3.16) are satisfied for each $\varepsilon > 0$.

We also notice that there exists a $\bar\delta$ such that, for all $\delta \in [-\bar\delta, \bar\delta]$ the function $f^{\varepsilon,\delta} = f^\varepsilon + \delta$ satisfies similar properties to those of f^ε, (3.8) (3.9) and (3.10), and it has exactly three zeroes in $m_-^{\varepsilon,\delta}(x) < m_o^{\varepsilon,\delta}(x) < m_+^{\varepsilon,\delta}(x)$. In particular, for each $\delta \in [-\bar\delta, \bar\delta]$, there exists a unique pair $(q^{\varepsilon,\delta}(\cdot),\ c^{\varepsilon,\delta})$ which solves the traveling wave equation

$$q_{rr}^{\varepsilon,\delta}(r,x) + c^{\varepsilon,\delta}(x) q_r^{\varepsilon,\delta}(r,x) = f^{\varepsilon,\delta}(q^{\varepsilon,\delta}(r,x),x), \qquad (r,x) \in \mathbb{R}\times\mathbb{R}^n,$$

subject to

$$q^{\varepsilon,\delta}(-\infty,x) = -1,\ q^{\varepsilon,\delta}(+\infty,x) = 1,\ q^{\varepsilon,\delta}(0,x) = m_o^{\varepsilon,\delta}(x)$$

and such that $q_r^{\varepsilon,\delta} > 0$. The pair moreover satisfies (3.14), (3.15), (3.16) uniformly in δ and we suppose that there is a constant $M > 0$ independent of ε such that

$$\sup_{x\in\mathbb{R}^n}\Big[|c^\varepsilon(x) - c^{\varepsilon,\delta}(x)| + |\,1 - m_+^{\varepsilon,\delta}(x)| + |\,1 + m_-^{\varepsilon,\delta}(x)|\Big] \le M|\,\delta| \le \gamma, \tag{3.18}$$

for $\bar\delta$ small enough. In the case (3.11), one can explicitly compute

$$c^{\varepsilon,\delta}(x) = 2m_o^{\varepsilon,\delta}(x) - m_+^{\varepsilon,\delta}(x) - m_-^{\varepsilon,\delta}(x)$$

and therefore the estimate (3.18) is an easy consequence of an uniform estimate of the derivative $|f_q^\varepsilon(q,x)| \ge \gamma > 0$, for all $x \in \mathbb{R}^n$ and q in a neighborhood of the three zeroes, which follows from (3.8).

Now for the asymptotics of the velocity of the traveling waves, we suppose that there are two bounded and locally Lipschitz continuous functions $n_1, n_2 : \mathbb{R}^n \to [\rho, +\infty)$ and an oriented, closed, Lipschitz hypersurface $\tilde\Gamma$ that satisfy

$$\begin{aligned} &0 < 2\rho \le n_1(x) < c^\varepsilon(x) < n_2(x) \le 2(1-\rho), \quad \text{for any } x \in \mathbb{R}^n,\\ &c^\varepsilon \longrightarrow \alpha, \quad \text{locally uniformly off } \tilde\Gamma, \end{aligned} \tag{3.19}$$

where $\alpha : \mathbb{R}^n \to [\rho, +\infty)$ is a bounded measurable function which is piecewise continuous across

$\tilde{\Gamma}$ in the following way. Let $\tilde{d}$ be the signed distance function from $\tilde{\Gamma}$, α has to satisfies

$$\alpha(x) \in \begin{cases} \{n_1(x)\} & \text{if } \tilde{d}(x) < 0, \\ \{n_2(x)\} & \text{if } \tilde{d}(x) > 0, \\ [n_1(x), n_2(x)] & \text{if } \tilde{d}(x) = 0. \end{cases} \tag{3.20}$$

We observe that in (3.11), $\overline{m} = \dfrac{\alpha^*}{2}$ and $\underline{m} = \dfrac{\alpha_*}{2}$. Finally in the case (3.11), we can explicitly choose the family of velocities c^ε satisfying the assumptions above. As for instance if

$$c^\varepsilon(x) = \frac{n_1(x)}{2}\Big(1 - \tanh\big(\frac{\bar{d}(x)}{\varepsilon^k}\big)\Big) + \frac{n_2(x)}{2}\Big(1 + \tanh\big(\frac{\bar{d}(x)}{\varepsilon^k}\big)\Big), \tag{3.21}$$

where $\bar{d} \in C^2(\mathbb{R}^n)$ and coincides with a signed distance function from $\tilde{\Gamma}$ in a neighborhood of it.

Remark. It is clear that the case (3.11) is cleaner and only needs (3.12), (3.19) and (3.20) in order to have the whole set of assumptions satisfied. Many technical assumptions may thus be avoided, in particular due to the direct relationship between the unstable equilibrium and the velocity of the approximating front provided by the traveling waves.

3.1.2 The result

We now state one of the two main results of this chapter. In the following theorem the asymptotic behavior of the solutions of the Cauchy problem (3.7) is totally described.

Theorem 3.1.1. Assume (3.8), (3.9), (3.10), (3.14), (3.15), (3.16), (3.18), (3.19) and (3.20). Let u^ε be the unique (and continuous) solution of (3.7), where $g : \mathbb{R}^n \to [-1, 1]$ is a continuous function such that the sets $\Gamma_o = \{x : \underline{m}(x) \le g(x) \le \overline{m}(x)\}$, $\Omega_o^+ = \{x : g(x) > \overline{m}(x)\}$, $\Omega_o^- = \{x : g(x) < \overline{m}(x)\}$ are nonempty and mutually disjoint subsets of $\mathbb{R}^n$. Then, as $\varepsilon \to 0^+$,

$$u^\varepsilon(x,t) \longrightarrow \begin{cases} 1, & \text{if } u(x,t) > 0, \\ -1, & \text{if } u(x,t) < 0, \end{cases}$$

locally uniformly, where u is the unique viscosity solution of

$$\begin{cases} u_t(x,t) + \alpha(x)|Du(x,t)| = 0 \text{ in } \mathbb{R}^n \times (0, +\infty), \\ u(x,0) = d_o(x), \end{cases} \tag{3.22}$$

and d_o is the signed distance to Γ_o which is positive in Ω_o^+ and negative in Ω_o^-. If in addition the no-interior condition (2.7) for the set $\{u = 0\}$ holds, then, as $\varepsilon \to 0^+$,

$$u^\varepsilon(x,t) \longrightarrow \begin{cases} 1 & \text{if } (x,t) \in \{u > 0\}, \\ -1 & \text{if } (x,t) \in \overline{\{u > 0\}}^c, \end{cases}$$

locally uniformly.

Remark 3.1.2. The existence (and uniqueness) of the viscosity solution u of the Cauchy problem (3.22) will be treat in the next section.

Remark 3.1.3. The results of the theorem are more elegant in the case that the initialized front Γ_o has empty interior. In the open sets where $\overline{m} = \underline{m} = m$, then the family $\{m_o^\varepsilon\}$ converges locally uniformly, and Γ_o is determined by the equation $g = m$. If this is not the case, Γ_o may contain relatively open subsets of $\{x : \overline{m}(x) > \underline{m}(x)\}$. Notice also that in the case (3.11) then $\overline{m} = \dfrac{\alpha^*}{2}$ and $\underline{m} = \dfrac{\alpha_*}{2}$, therefore even in that case it is preferable to have a set of discontinuities of α with empty interior.

Proof. For the proof of this Theorem we follow the abstract method described in section 2.3 and we define two families of open sets of $\mathbb{R}^n$, $(\Omega_t^1)_{t\in(0,T)}$ and $(\Omega_t^2)_{t\in(0,T)}$ as in (2.14), (2.15) and two further sets Ω_0^1, Ω_0^2 as in (2.17). We recall that by maximum principle $-1 \le u^\varepsilon \le 1$. We are now ready for the first step of the proof.

First step: initialization. We want to show that

$$\Omega_0^+ = \{d_o > 0\} \subseteq \Omega_0^1, \qquad \Omega_0^- = \{d_o < 0\} \subseteq \Omega_0^2.$$

Since the proofs of these two inclusions are similar we only show the first one. Consider $\hat{x} \in \{d_o > 0\}$, then we have that $g(\hat{x}) > \overline{m}(\hat{x})$ and so, by the continuity of g, upper semicontinuity and definition of $\overline{m}$, we can find an $r, \sigma > 0$ such that

$$g(x) \ge \sup_{B(\hat{x},r)} \overline{m} + \sigma \ge m_o^\varepsilon(x) + \frac{3}{4}\sigma,$$

for all $x \in B(\hat{x}, r)$ and ε sufficiently small. This means that

$$u^\varepsilon(x,0) = g(x) \ge \Big(\sup_{B(\hat{x},r)} \overline{m} + \sigma\Big)\mathbb{1}_{B(\hat{x},r)}(x) - \mathbb{1}_{B(\hat{x},r)^c}(x). \tag{3.23}$$

Now we introduce the function $\Phi : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ defined by

$$\Phi(x,t) = r^2 - |x - \hat{x}|^2 - Ct, \tag{3.24}$$

with $C > 0$ a constant that will be chosen later. We denote by $d(\cdot,t)$ the signed distance to the set $\{\Phi(\cdot,t)=0\}$ defined in such a way to have the same sign of Φ. Explicitly $d(x,t)=\sqrt{(r^2-Ct)^+}-|x-\hat{x}|$. Note in particular that $d(x,0)\geq\beta$ if and only if $x\in B(\hat{x},r-\beta]$.

To prove the first step we need the two following lemmas.

Lemma 3.1.4. Under the assumptions of Theorem 3.1.1 we have that for any $\beta > 0$ there exist $\tau=\tau(\beta)>0$ and $\bar{\varepsilon}=\bar{\varepsilon}(\beta)$ such that, for all $0<\varepsilon\leq\bar{\varepsilon}$, we have

$$u^\varepsilon(x,t_\varepsilon)\geq(1-\beta)\mathbb{1}_{\{d(\cdot,0)\geq\beta\}}(x)-\mathbb{1}_{\{d(\cdot,0)<\beta\}}(x),\quad x\in\mathbb{R}^n,$$

where $t_\varepsilon=\tau\varepsilon$ and $d(x,t)=\sqrt{(r^2-Ct)^+}-|x-\hat{x}|$.

Lemma 3.1.5. There exist $\bar{h}=\bar{h}(r,\hat{x})>0$, $\bar{\beta}=\bar{\beta}(r,\hat{x})$ independent of ε such that if $\beta\leq\bar{\beta}$, $\beta>0$, and $\varepsilon\leq\bar{\varepsilon}(\beta)$, then there is a subsolution $\omega^{\varepsilon,\beta}$ of (3.7-i) in $\mathbb{R}^n\times(0,\bar{h})$ that satisfies

$$\omega^{\varepsilon,\beta}(x,0)\leq(1-\beta)\mathbb{1}_{\{d(\cdot,0)\geq\beta\}}(x)-\mathbb{1}_{\{d(\cdot,0)<\beta\}}(x),\quad x\in\mathbb{R}^n.$$

If moreover $(x,t)\in B(\hat{x},r)\times(0,\bar{h})$ and $d(x,t)>3\beta$, then

$$\liminf_{\varepsilon\to0^+}{}_*\,\omega^{\varepsilon,\beta}(x,t)\geq1-3\beta.$$

Before proving Lemmas 3.1.4 and 3.1.5 we give the short conclusion of the first step which follows [5]. To do this, we first notice that, combining these two Lemmas, we get the existence of a (viscosity) subsolution $\omega^{\varepsilon,\beta}$ of (3.7-i) in $\mathbb{R}^n\times(0,\bar{h})$ such that

$$\omega^{\varepsilon,\beta}(x,0)\leq u^\varepsilon(x,t_\varepsilon),\quad\text{for all } x\in\mathbb{R}^n,$$

and so, by the maximum principle,

$$\omega^{\varepsilon,\beta}(x,s)\leq u^\varepsilon(x,s+t_\varepsilon),\quad\text{for all } (x,s)\in\mathbb{R}^n\times[0,\bar{h}].$$

Therefore, using the second part of Lemma 3.1.5, we get that for all $(x,s)\in B(\hat{x},r)\times(0,\bar{h})$, $d(x,s)>3\beta$,

$$\liminf_{\varepsilon\to0^+}{}_*\,u^\varepsilon(x,s)\geq1-3\beta.$$

Since β is arbitrary and does not depend on $\bar{h}$ we can send it to zero in order to obtain that, for all $(x,s)\in B(\hat{x},r)\times(0,\bar{h})$, $d(x,s)>0$,

$$\liminf_{\varepsilon\to0^+}{}_*\,u^\varepsilon(x,s)\geq1,$$

i.e. $x \in \Omega_s^1$ by definition. According to the definition of d, it follows that there exist $\bar{\eta} < r$, $\bar{t} < \bar{h}$ so that $B(\hat{x}, \bar{\eta}) \subset \{d(\cdot, t) > 0\}$ for any $0 < t < \bar{t}$. This implies that $B(\hat{x}, \bar{\eta}) \subset \Omega_t^1$ for any $0 < t < \bar{t}$ and therefore $\chi(\hat{x}, 0) = 1$ and $\hat{x} \in \Omega_0^1$.

Proof of Lemma 3.1.4. For the proof of this lemma we follow the ideas of Chen [16, 17], based on the fact that for ε small in the reaction-diffusion equation the diffusion term is negligible for short time, and of Barles-Da Lio [5]. This lemma is a local short time generation of the interface. The corresponding proof in [16] is more precise since there the time needed to generate the interface is precisely determined. Let $\beta > 0$ be fixed. Due to the maximum principle we just need to show that $u^\varepsilon(x, t_\varepsilon) \geq 1 - \beta$ if $d(x, 0) \geq \beta$.

1. We denote by $\chi = \chi(\tau, \xi; x) \in C^2([0, +\infty) \times \mathbb{R} \times \mathbb{R}^n)$ the solution of

$$\begin{cases} \dot{\chi}(\tau, \xi; x) + f^\varepsilon(\chi(\tau, \xi; x), x) = 0, & \tau > 0, \\ \chi(0, \xi; x) = \xi. \end{cases} \tag{3.25}$$

It is then simple to see, by the properties of ordinary differential equations, that χ satisfies the following properties

$$\chi_\xi(\tau, \xi; x) > 0, \qquad \text{for any } (\tau, \xi; x) \in [0, +\infty) \times \mathbb{R} \times \mathbb{R}^n, \tag{$\chi 1$}$$

and there exists $\tau_o = \tau_o(\beta) > 0$ such that, for all $\tau \geq \tau_o$

$$\chi(\tau, \xi; x) \geq 1 - \beta, \quad \forall\, \xi \geq \sup_{B(\hat{x}, r)} \overline{m} + \frac{\sigma}{2}. \tag{$\chi 2$}$$

(Regarding the proof of the estimate in (χ2), which is independent of ε and x, we just notice that we can choose a cubic-like function $\overline{f}$ as in (3.10) with $K = B(\hat{x}, r],\ m_1 = \sup_{B(\hat{x}, r)} \overline{m} + \frac{\sigma}{4}$ such that

$$\overline{f}(q) \geq f^\varepsilon(q, x),$$

for all $x \in B(\hat{x}, r)$, $q \in [-1, 1]$, and ε sufficiently small.)

Moreover, since for any $C > 1$ we have that $\chi(\tau, \xi, x) \in [-C, C]$ for all $\xi \in [-C, C]$, $\tau \geq 0$, $x \in \mathbb{R}^n$, it also holds that for any $C > 1$, $\tau > 0$ there exists a constant $M_{C,\tau} > 0$ such that

$$\begin{aligned} &|\chi_{\xi\xi}(\tau, \xi; x)| \leq M_{C,\tau} \chi_\xi(\tau, \xi; x), \quad |\chi_{x_i}(\tau, \xi; x)|, \leq \frac{M_{C,\tau}}{\varepsilon^k} \\ &|\chi_{\xi x_i}(\tau, \xi; x)| \leq \frac{M_{C,\tau}}{\varepsilon^k} \chi_\xi(\tau, \xi; x), \quad |\chi_{x_i x_i}(\tau, \xi; x)| \leq \frac{M_{C,\tau}}{\varepsilon^{2k}} \chi_\xi(\tau, \xi; x), \end{aligned} \tag{$\chi 3$}$$

for any $\xi \in [-C, C],\ x \in \mathbb{R}^n$, $i \in \{1, 2, \cdots, n\}$ and ε small enough.

2. Let ψ be a nondecreasing smooth function in $\mathbb{R}$ such that

$$\psi(z) = \begin{cases} -1 & \text{if } z \le 0, \\ \sup\limits_{B(\hat{x},r)} \overline{m} + \sigma & \text{if } z \ge \beta \wedge \dfrac{\sigma}{2}. \end{cases}$$

We can define a function $\underline{u}^\varepsilon$ in $\mathbb{R}^n \times [0,T]$ as

$$\underline{u}^\varepsilon(x,t) = \chi\Big(\frac{t}{\varepsilon}, \psi(d(x,0)) - Kt; x\Big),$$

for K a constant to be decided later. Thanks to a computation similar to those in [8] one can prove that, if K is large enough, $\underline{u}^\varepsilon$ is a subsolution of (3.7-i) in $\mathbb{R}^n \times (0, \tau_o\varepsilon)$, with τ_o as in (χ2). In fact, since χ satisfies (3.25) and ψ' has compact support, we obtain

$$\begin{aligned} \underline{u}^\varepsilon_t - \varepsilon\Delta\underline{u}^\varepsilon + \frac{f^\varepsilon(\underline{u}^\varepsilon,x)}{\varepsilon} &= \frac{\dot{\chi}}{\varepsilon} - K\chi_\xi - \varepsilon\Big[\chi_{\xi\xi}\Big|\,\psi' Dd(x,0)\Big|^2 + \chi_\xi\Big(\psi'' + \psi'\Delta d(x,0)\Big) \\ &\quad + \Delta\chi + 2D\chi_\xi \cdot \Big(\psi' Dd(x,0)\Big)\Big] + \frac{f^\varepsilon(\chi,x)}{\varepsilon} \\ &\le -K\chi_\xi + \varepsilon[M_1|\,\chi_{\xi\xi}| + M_2\chi_\xi + |\,\Delta\chi| + M_3|\,D\chi_\xi|]. \end{aligned} \tag{3.26}$$

Now we want to use properties (χ1) and (χ3) in order to get an estimate for the terms $|\,\chi_{\xi\xi}|$, $|\,D\chi_\xi|$, $|\,\Delta\chi|$. Indeed since $\psi(d(x,0)) \in I = [-1, 1+\sigma]$ for all $x \in \mathbb{R}^n$, by evaluating (3.26) at a point of $\mathbb{R}^n \times (0, \tau_o\varepsilon)$ we obtain

$$\underline{u}^\varepsilon_t - \varepsilon\Delta\underline{u}^\varepsilon + \frac{f^\varepsilon(\underline{u}^\varepsilon,x)}{\varepsilon} \le -\chi_\xi\left(K - \varepsilon M_2 - \varepsilon M_{2,\tau_o}\left(M_1 + \frac{M_3}{\varepsilon^k} + \frac{1}{\varepsilon^{2k}}\right)\right) \le 0,$$

for K large enough. Moreover,

$$\begin{aligned} \underline{u}^\varepsilon(x,0) = \psi(d(x,0)) &\le \Big(\sup_{B(\hat{x},r)} \overline{m} + \sigma\Big)\mathbb{1}_{\{d(x,0)>0\}}(x) - \mathbb{1}_{\{d(x,0)\le 0\}}(x) \\ &= \Big(\sup_{B(\hat{x},r)} \overline{m} + \sigma\Big)\mathbb{1}_{B(\hat{x},r)}(x) - \mathbb{1}_{B(\hat{x},r)^c}. \end{aligned}$$

Therefore combining the last inequality with (3.23) we get

$$\underline{u}^\varepsilon(x,0) \le u^\varepsilon(x,0), \text{ for all } x \text{ in } \mathbb{R}^n.$$

Thus, by the maximum principle,

$$\underline{u}^\varepsilon(x,t) \le u^\varepsilon(x,t) \text{ in } \mathbb{R}^n \times [0, \tau_o\varepsilon].$$

Now if we evaluate the last inequality for $x \in \{d(\cdot,0) \geq \beta \wedge \sigma/2\}$ and $t = t_\varepsilon = \tau_o \varepsilon$, we get

$$u^\varepsilon(x,t_\varepsilon) \geq \chi\big(\tau_o, \sup_{B(\hat{x},r)} \overline{m} + \sigma - K\tau_o\varepsilon, x\big) \geq \chi\big(\tau_o, \sup_{B(\hat{x},r)} \overline{m} + \frac{\sigma}{2}, x\big),$$

for $\varepsilon \leq \dfrac{\sigma}{2K\tau_o}$. Therefore by ($\chi 2$) and we obtain

$$u^\varepsilon(x,t_\varepsilon) \geq 1-\beta,$$

for all $x \in \{d(\cdot,0) \geq \beta\}$. $\square$

Proof of Lemma 3.1.5. The proof follows with some modifications the ideas in [5] and [8]. First of all we consider the smooth function Φ defined in (3.24) where now C is fixed and satisfies

$$C \geq 8r. \tag{3.27}$$

Since $D\Phi(x,t) \neq 0$ if $\Phi(x,t) = 0$, there exist γ, $\bar{h} > 0$ such that $\bar{h} < r^2/C$, d is smooth in the set $Q_{\gamma,\bar{h}} = \{(x,t) : |\,(d(x,t))| \leq \gamma,\ |x-\hat{x}| \geq \gamma,\ 0 \leq t \leq \bar{h}\}$, and $D\Phi(x,t) \neq 0$ in $Q_{\gamma,\bar{h}}$. Now we construct a subsolution by steps.

1. We first define a smooth function v^ε in $Q_{\gamma,\bar{h}}$ as

$$v^\varepsilon(x,t) = q^{\varepsilon,\delta}\Big(\frac{d(x,t)-2\beta}{\varepsilon}, x\Big) - 2\beta,$$

with $\delta \in [0,\bar{\delta}]$ to be chosen later. Using the definition of d, the assumption (3.27) on C and the properties (3.18), (3.19) satisfied by $c^{\varepsilon,\delta}$ and c^ε, we can see that in $Q_{\gamma,\bar{h}}$,

$$\begin{aligned} v_t^\varepsilon - \varepsilon\Delta v^\varepsilon + \frac{f^\varepsilon(v^\varepsilon,x)}{\varepsilon} &= \frac{q_r^{\varepsilon,\delta} d_t}{\varepsilon} - \frac{q_{rr}^{\varepsilon,\delta}}{\varepsilon} - 2Dq_r^{\varepsilon,\delta}\cdot Dd - q_r\Delta d - \varepsilon\Delta q^{\varepsilon,\delta} + \frac{f^\varepsilon(q^{\varepsilon,\delta}-2\beta,x)}{\varepsilon} \\ &\leq \frac{q_r^{\varepsilon,\delta}}{\varepsilon}\Big(\frac{-C}{2\sqrt{r^2-Ct}} + c^{\varepsilon,\delta}(x) + \varepsilon\frac{n-1}{|\,x-\hat{x}|}\Big) - \frac{\delta}{\varepsilon} - 2Dq_r^{\varepsilon,\delta}\cdot Dd - \varepsilon\Delta q^{\varepsilon,\delta} \\ &\qquad - \frac{2\beta f_q^\varepsilon(q^{\varepsilon,\delta},x)}{\varepsilon} + \frac{2\beta^2\|\,f_{qq}^\varepsilon\|_\infty}{\varepsilon} \\ &\leq \frac{1}{\varepsilon}\Big[-q_r^{\varepsilon,\delta} - 2\beta f_q^\varepsilon(q^{\varepsilon,\delta},x) + 2\beta^2\|\,f_{qq}^\varepsilon\|_\infty\Big] + \Big[-\frac{\delta}{\varepsilon} + 2|\,Dq_r^{\varepsilon,\delta}| + \varepsilon|\,\Delta q^{\varepsilon,\delta}|\Big], \end{aligned}$$

for ε and $|\,\delta|$ small enough. Since for any $x \in \mathbb{R}^n$, $\delta \in [0,\bar{\delta}]$,

$$q^{\varepsilon,\delta}(\cdot,x) \in [m_-^{\varepsilon,\delta}(x), m_+^{\varepsilon,\delta}(x)] \subseteq [-1-\delta, 1+\delta],$$

here and below the L^∞ norm of the derivatives of f^ε are taken for its first argument q in the compact set $[-1-\bar{\delta}, 1+\bar{\delta}]$. To prove that v^ε is a subsolution of (3.7-i) it remains to see that the right hand side of the last inequality above is non positive. For the right bracket we use the properties (3.16) satisfied by $q^{\varepsilon,\delta}$ and we compute

$$-\frac{\delta}{\varepsilon} + 2|\, Dq_r^{\varepsilon,\delta}| + \varepsilon|\, \Delta q^{\varepsilon,\delta}| \le -\frac{\delta}{\varepsilon} + \frac{2M_1}{\varepsilon^k} + \varepsilon\frac{M_2}{\varepsilon^{2k}} \le -\frac{\delta}{2\varepsilon}$$

when $\delta > 0$ is fixed and ε is small enough. For the left bracket, we recall that, combining (3.8) and (3.18), $f_q^\varepsilon(m_\pm^{\varepsilon,\delta}(x), x) \ge \gamma > 0$ and $q^{\varepsilon,\delta}(r,x) \to m_\pm^{\varepsilon,\delta}(x)$ if $r \to \pm\infty$ exponentially fast, uniformly for $x \in \mathbb{R}^n$. This means that we may suppose that there exists an $\bar{r} > 0$ such that

$$f_q^\varepsilon(q^{\varepsilon,\delta}(r,x), x) \ge \frac{\gamma}{2}, \quad \text{for any } |r| \ge \bar{r},$$

and we can choose β small enough, independent of ε, δ, in order to get

$$\beta\|\, f_{qq}^\varepsilon\|_\infty = \beta \sup\{|\, f_{qq}^\varepsilon(q,x)| : (q,x) \in [-1-\bar{\delta}, 1+\bar{\delta}] \times \mathbb{R}^n\} \le \frac{\gamma}{2}.$$

Thus we consider two cases. If $|d(x,t) - 2\beta| \ge \varepsilon\bar{r}$, we have that

$$v_t^\varepsilon - \varepsilon\Delta v^\varepsilon + \frac{f^\varepsilon(v^\varepsilon, x)}{\varepsilon} \le -\frac{q_r^{\varepsilon,\delta}}{\varepsilon} - \frac{\delta}{2\varepsilon} < 0$$

for ε small enough. If, on the other hand, $|d(x,t) - 2\beta| < \varepsilon\bar{r}$ and we denote with K a strictly positive constant (which depends on $\bar{r}$) so that $q_r^{\varepsilon,\delta}(r,x) \ge K > 0$ for any $|r| \le \bar{r}$, $x \in \mathbb{R}^n$, we get that, for β small compared to K,

$$v_t^\varepsilon - \varepsilon\Delta v^\varepsilon + \frac{f^\varepsilon(v^\varepsilon, x)}{\varepsilon} \le \frac{1}{\varepsilon}(-K + 2\beta(\|\, f_q^\varepsilon\|_\infty + 2\beta\|\, f_{qq}^\varepsilon\|_\infty)) - \frac{\delta}{2\varepsilon} < 0.$$

2. We define for each $(x,t) \in \{(x,t) \in \mathbb{R}^n \times [0,\bar{h}] : d(x,t) \le \gamma\}$,

$$\bar{v}^\varepsilon(x,t) = \begin{cases} \sup(v^\varepsilon(x,t), -1) & \text{if} \quad -\gamma < d(x,t) \le \gamma, \\ -1 & \text{if} \quad d(x,t) \le -\gamma. \end{cases}$$

$\bar{v}^\varepsilon$ is a continuous viscosity subsolution of (3.7-i) in $\{(x,t) \in \mathbb{R}^n \times [0,\bar{h}] : d(x,t) \le \gamma\}$, for ε sufficiently small. This in obvious in the set $\{|d| \le \gamma\}$ since $\bar{v}^\varepsilon$ is the supremum of two subsolutions. Consider a point (x,t) such that $d(x,t) \le -\gamma/2$; by properties (3.14) we have that

$$v^\varepsilon(x,t) \le q^{\varepsilon,\delta}\Big(-\frac{\gamma+4\beta}{2\varepsilon}, x\Big) - 2\beta \le m_-^{\varepsilon,\delta}(x) + ae^{-\frac{b(\gamma+4\beta)}{2\varepsilon}} - 2\beta \le m_-^{\varepsilon,\delta}(x) \le -1$$

and $\bar{v}^\varepsilon(x,t) = -1$. Therefore $\bar{v}^\varepsilon$ is a subsolution of (3.7-i) in $\{(x,t) \in \mathbb{R}^n \times [0,\bar{h}] : d(x,t) \le \gamma\}$.

3. We finally define our function $\omega^{\varepsilon,\beta} : \mathbb{R}^n \times [0, \bar{h}] \to \mathbb{R}$ as

$$\omega^{\varepsilon,\beta}(x,t) = \begin{cases} \psi(d(x,t))\bar{v}^{\varepsilon}(x,t) + (1-\psi(d(x,t)))(1-\beta) & \text{if } d(x,t) < \gamma, \\ 1-\beta & \text{if } d(x,t) \geq \gamma, \end{cases}$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\psi' \leq 0$ in $\mathbb{R}$, $\psi = 1$ in $(-\infty, \gamma/2]$, $0 < \psi < 1$ in $(\gamma/2, 3\gamma/4)$ and $\psi = 0$ in $[3\gamma/4, +\infty)$. The only subset of $\mathbb{R}^n \times (0, \bar{h})$ in which we have to check that $\omega^{\varepsilon,\beta}$ is a subsolution is $\{(x,t) \in \mathbb{R}^n \times (0, \bar{h}) : \gamma/2 \leq d(x,t) \leq 3\gamma/4\}$. Since $|\,Dd| = 1$

$$\begin{aligned} \omega_t^{\varepsilon,\beta} - \varepsilon\Delta\omega^{\varepsilon,\beta} + \frac{f^{\varepsilon}(\omega^{\varepsilon,\beta},x)}{\varepsilon} =& \psi(\bar{v}_t^{\varepsilon} - \varepsilon\Delta\bar{v}^{\varepsilon}) - 2\varepsilon\psi' Dd \cdot D\bar{v}^{\varepsilon} \\ &+ (\psi' d_t - \varepsilon\psi'\Delta d - \varepsilon\psi'')(\bar{v}^{\varepsilon} - (1-\beta)) + \frac{f^{\varepsilon}(\omega^{\varepsilon,\beta},x)}{\varepsilon}. \end{aligned} \tag{3.28}$$

If we take $2\beta < \gamma/4$ we obtain that

$$\begin{aligned} v^{\varepsilon}(x,t) &\geq q^{\varepsilon,\delta}\Big(\frac{\gamma}{4\varepsilon}, x\Big) - 2\beta \\ &\geq m_+^{\varepsilon,\delta}(x) - ae^{-\frac{b\gamma}{4\varepsilon}} - 2\beta \geq 1 - M\delta - ae^{-\frac{b\gamma}{4\varepsilon}} - 2\beta \end{aligned}$$

and so for $\varepsilon, \beta, \delta$ small $\bar{v}^{\varepsilon}(x,t) = v^{\varepsilon}(x,t)$ and $\bar{v}^{\varepsilon}(x,t) - (1-\beta) \leq -\beta$. Moreover, since $f_{qq}^{\varepsilon}(1,x) > 0$, $f^{\varepsilon}(\omega^{\varepsilon,\beta},x) \leq \psi f^{\varepsilon}(v^{\varepsilon},x) + (1-\psi)f^{\varepsilon}(1-\beta,x)$. Thus (3.28) becomes

$$\begin{aligned} \omega_t^{\varepsilon,\beta} - \varepsilon\Delta\omega^{\varepsilon,\beta} + \frac{f^{\varepsilon}(\omega^{\varepsilon,\beta},x)}{\varepsilon} &\leq -\psi\frac{\delta}{2\varepsilon} - 2\psi' q_r^{\varepsilon,\delta} + 2\varepsilon^{1-k}M_1 \\ &\quad + \psi' d_t(v^{\varepsilon} - (1-\beta)) + (1-\psi)\frac{f^{\varepsilon}(1-\beta,x)}{\varepsilon} + O(\varepsilon) \\ &\leq -\frac{1}{\varepsilon}\left(\psi\frac{\delta}{2} + (1-\psi)(-f^{\varepsilon}(1-\beta,x))\right) + \tilde{M}_3 + o_{\varepsilon}(1) \leq 0, \end{aligned}$$

for ε small enough. To get the last inequality, we also used the fact that $d_t \leq 0$ and $\sup_{x\in\mathbb{R}^n} f^{\varepsilon}(1-\beta,x) < 0$ for β small enough.

4. Now we observe that, if $d(x,t) < \beta$, then $v^{\varepsilon}(x,t) \leq q^{\varepsilon,\delta}(-\frac{\beta}{\varepsilon}, x) - 2\beta \leq m_-^{\varepsilon,\delta}(x) + ae^{-\frac{b\beta}{\varepsilon}} - 2\beta \leq m_-^{\varepsilon,\delta}(x) \leq -1$ for ε small enough (and β fixed). This means that, for ε small enough

$$v^{\varepsilon}(x,t) \leq (1-\beta)\mathbb{1}_{\{d\geq\beta\}}(x,t) - \mathbb{1}_{\{d<\beta\}}(x,t).$$

By definition of $\bar{v}^{\varepsilon}$ and of $\omega^{\varepsilon,\beta}$ the last inequality still holds for $\bar{v}^{\varepsilon}$ and $\omega^{\varepsilon,\beta}$ (we just point out that if $d(x,t) \geq \beta$ then $\omega^{\varepsilon,\beta}(x,t)$ is equal to $1-\beta$ or to a convex linear combination of elements of $(-\infty, 1-\beta]$). If we consider $t = 0$ we have proved the second part of our Lemma.

5. Finally we just remark that, with a reasoning similar to the one in point 4. one can prove that

if $(x,t)\in B(\hat{x},r)\times(0,\bar{h})$ and $d(x,t)>3\beta$, then

$$v^\varepsilon(x,t)\ge q^{\varepsilon,\delta}(\frac{\beta}{\varepsilon},x)-2\beta\ge 1-ae^{-\frac{b\beta}{\varepsilon}}-2\beta-M\delta.$$

Hence $\liminf_{\varepsilon\to 0^+}{}_* \,\omega^{\varepsilon,\beta}(x,t)\ge 1-3\beta$, for $\beta\ge M\delta$.

∎

Second step: propagation. In this step we show that $(\Omega_t^1)_{t\in(0,T)}$ and $((\Omega_t^2)^c)_{t\in(0,T)}$ are respectively super and subflows with normal velocity $-\alpha$. Since the two proofs are similar we only show that $(\Omega_t^1)_{t\in(0,T)}$ is a superflow. One of the difficulties here is due to the fact that we want to approximate the definition of super- and subflow by using continuous velocities. We do that by means of the smooth functions c^ε to approximate our discontinuous velocity α. We consider the following modified families of continuous functions and define:

$$\overline{c}^\varepsilon(x):=\eta^\varepsilon(x)n_2(x)+(1-\eta^\varepsilon(x))c^\varepsilon(x),\quad \underline{c}^\varepsilon(x):=\xi^\varepsilon(x)n_1(x)+(1-\xi^\varepsilon(x))c^\varepsilon(x),$$

where $\eta^\varepsilon,\xi^\varepsilon\in C^2(\mathbb{R}^n)$, $\eta^\varepsilon(x),\ \xi^\varepsilon(x)\in[0,1]$,

$$\eta^\varepsilon(x):=\begin{cases}1 & \text{if}\quad \tilde{d}(x)\ge-\varepsilon\\ 0 & \text{if}\quad \tilde{d}(x)\le-2\varepsilon\end{cases};\quad \xi^\varepsilon(x):=\begin{cases}1 & \text{if}\quad \tilde{d}(x)\le\varepsilon\\ 0 & \text{if}\quad \tilde{d}(x)\ge 2\varepsilon\end{cases}.$$

Notice that

$$\underline{n}_1\le\underline{c}^\varepsilon\le c^\varepsilon\le\overline{c}^\varepsilon\le n_2,\quad \underline{c}^\varepsilon\le\alpha_*\le\alpha^*\le\overline{c}^\varepsilon$$

and $\limsup_{\varepsilon\to0^+}{}^*\ \overline{c}^\varepsilon(x)=\alpha^*(x)$, $\liminf_{\varepsilon\to0^+}{}_*\ \underline{c}^\varepsilon(x)=\alpha_*(x)$ We denote below as $\overline{\mathcal{F}}=\{\overline{c}^\varepsilon,\,\varepsilon>0\}$, $\underline{\mathcal{F}}=\{\underline{c}^\varepsilon,\,\varepsilon>0\}$.

Proposition 3.1.6. (i) A family $(\Omega_t)_{t\in(0,T)}$ of open subsets of $\mathbb{R}^n$ such that the set $\Omega:=\cup_{t\in(0,T)}\Omega_t\times\{t\}$ is open in $\mathbb{R}^n\times[0,T]$, is a *generalized superflow* with normal velocity $-\alpha$ if and only if it is a generalized superflow with normal velocity $-\overline{c}\in C(\mathbb{R}^n)$, for all $\bar{c}\in\overline{\mathcal{F}}$;
(ii) A family $(\mathcal{F}_t)_{t\in(0,T)}$ of close subsets of $\mathbb{R}^n$ such that the set $\mathcal{F}:=\cup_{t\in(0,T)}\mathcal{F}_t\times\{t\}$ is closed in $\mathbb{R}^n\times[0,T]$ is a *generalized subflow* with normal velocity $-\alpha$ if and only if it is a generalized subflow with normal velocity $-\underline{c}$, for all $\underline{c}\in\underline{\mathcal{F}}$.

Proof. (i) In view of Theorem 2.2.2, in order to prove this proposition we have to prove that the function $\chi=\mathbb{1}_\Omega-\mathbb{1}_{\Omega^c}$ is a viscosity supersolution of (3.5-i) if and only if it is a viscosity supersolution of

$$\chi_t(x,t)+\overline{c}^\varepsilon(x)|D\chi(x,t)|=0,\quad (x,t)\in\mathbb{R}^n\times(0,T),\tag{3.29}$$

for all $\varepsilon>0$. We start assuming that for every continuous function $\overline{c}^\varepsilon$, χ is a viscosity supersolution of (3.29). The conclusion follows from the stability of viscosity supersolutions and the fact that $\alpha^\star = \limsup^*_{\varepsilon\to 0^+} \overline{c}^\varepsilon$. Therefore χ is a supersolution also of (3.5-i). Since $\overline{c}^\varepsilon \geq \alpha^*$, the other implication is trivial.

(ii) The proof concerning the subflow is similar and we omit it. $\square$

Next we want to show that $(\Omega^1_t)_{t\in(0,T)}$ is a superflow with normal velocity $-\overline{c}$, for any $\overline{c} \in \overline{\mathcal{F}}$.

Proposition 3.1.7. Let $\overline{c} \in \overline{\mathcal{F}}$ be fixed and let $x_0 \in \mathbb{R}^n$, $t \in (0,T)$, $r > 0$, $h > 0$ so that $t + h < T$. Suppose that $\phi : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is a smooth function such that, for a suitable $\tilde{C} > 0$,

(i) $\phi_t(x,s) + \overline{c}(x)|D\phi(x,s)| \leq -\tilde{C} < 0$, for all $(x,s) \in B(x_0,r) \times [t,t+h]$,

(ii) for any $s \in [t,t+h]$, $\{x \in B(x_0,r) : \phi(x,s) = 0\} \neq \emptyset$ and

$$|D\phi(x,s)| \neq 0 \text{ on } \{(x,s) \in B(x_0,r) \times [t,t+h] : \phi(x,s) = 0\},$$

(iii) $\{x \in B(x_0,r) : \phi(x,t) \geq 0\} \subset \Omega^1_t$,

(iv) for all $s \in [t,t+h]$, $\{x \in \partial B(x_0,r) : \phi(x,s) \geq 0\} \subset \Omega^1_s$.

Then, for every $s \in (t,t+h)$,

$$\{x \in B(x_0,r) : \phi(x,s) > 0\} \subset \Omega^1_s.$$

Proof. Using the assumptions and the definition of $(\Omega^1_t)_{t\in(0,T)}$ we need to prove that for all $x \in B(x_0,r)$, $s \in (t,t+h)$ such that $\phi(x,s) > 0$, then we have

$$\liminf_{\varepsilon\to 0^+}{}_* \, u^\varepsilon(y,\tau) \geq 1$$

for (y,τ) in a neighborhood of (x,s). This proof proceeds like the one of the first step with the difference that here we have to construct a subsolution of (3.7-i) only in the ball $B(x_0,r)$ and not in the whole space $\mathbb{R}^n$. We will need to use an extra boundary condition coming from (iv). In fact to prove this result it is enough to prove the following lemma which plays the role of Lemma 3.1.5 in the first step. We denote below with $d(\cdot,s)$ the signed distance function to the set $\{\phi(\cdot,s) = 0\}$ which has the same sign of ϕ.

Lemma 3.1.8. Let the assumptions of Proposition 3.1.7 hold true. There exists $\bar{\beta}$ small enough such that, if $\beta \leq \bar{\beta}$ and $\varepsilon \leq \bar{\varepsilon}(\beta)$ then there is a viscosity subsolution $\omega^{\varepsilon,\beta}$ of (3.7-i) in $B(x_0,r) \times (t,t+h)$ that satisfies,

1. $\omega^{\varepsilon,\beta}(x,t) \leq (1-\beta)\mathbb{1}_{\{d(\cdot,t)\geq\beta\}}(x) - \mathbb{1}_{\{d(\cdot,t)<\beta\}}(x), \quad$ for all $x \in B(x_0,r]$,
2. $\omega^{\varepsilon,\beta}(x,s) \leq (1-\beta)\mathbb{1}_{\{d(\cdot,s)\geq\beta\}}(x) - \mathbb{1}_{\{d(\cdot,s)<\beta\}}(x), \quad$ for all $x \in \partial B(x_0,r]$, $s \in [t,t+h]$.

3. if $(x,s) \in B(x_0,r] \times [t,t+h]$ satisfies $d(x,s) > 3\beta$, then

$$\liminf_{\varepsilon\to 0^+}{}_* \, \omega^{\varepsilon,\beta}(x,s) \geq 1-\beta.$$

If we assume for the moment that Lemma 3.1.8 holds true then we can prove Proposition 3.1.7 as a direct consequence (see also [5]). In fact, if $d(x,t) \geq \beta > 0$, then also $\phi(x,t) > 0$ and so, by property (iii) of ϕ, $x \in \Omega_t^1$. By definition of $(\Omega_t^1)_{t\in(0,T)}$ this means that $\liminf_{\varepsilon\to 0^+}{}_* \, u^\varepsilon(x,t) \geq 1 > 1-\beta$ and so there exists an $\varepsilon_{x,t} > 0$ such that, for all $\varepsilon \leq \varepsilon_{x,t}$, $(y,\tau) \in B(x,\varepsilon_{x,t}) \times (t-\varepsilon_{x,t}, t+\varepsilon_{x,t})$, we have $u^\varepsilon(y,\tau) \geq 1-\beta$. Thus, by the compactness of $\{x \in B(x_o,r] : \phi(x,t) \geq 0\}$ we can select an $\bar\varepsilon > 0$, possibly depending only on β, so that, for all $\varepsilon \leq \bar\varepsilon$, and $x \in \{y \in B(x_0,r] : d(y,t) \geq \beta\}$, we have $u^\varepsilon(x,t) \geq 1-\beta$. Therefore

$$u^\varepsilon(x,t) \geq (1-\beta)\mathbb{1}_{\{d(\cdot,t)\geq\beta\}}(x) - \mathbb{1}_{\{d(\cdot,t)<\beta\}}(x).$$

for all $\varepsilon \leq \bar\varepsilon$, $x \in B(x_0,r]$. In a similar way we can also obtain that, for ε small enough,

$$u^\varepsilon(x,s) \geq (1-\beta)\mathbb{1}_{\{d(\cdot,s)\geq\beta\}}(x) - \mathbb{1}_{\{d(\cdot,s)<\beta\}}(x),$$

for any $(x,s) \in \partial B(x_0,r] \times [t,t+h]$. Combining these inequalities with those in 1. and 2. in the statement of Lemma 3.1.8, by the maximum principle we can conclude that

$$\omega^{\varepsilon,\beta}(x,s) \leq u^\varepsilon(x,s), \quad \text{for all } (x,s) \in B(x_0,r] \times [t,t+h].$$

By 3. in Lemma 3.1.8, $\liminf_{\varepsilon\to 0^+}{}_* \, u^\varepsilon(x,s) \geq 1-\beta$ for every $(x,s) \in B(x_0,r] \times [t,t+h]$ such that $d(x,s) > 3\beta$. Since β is arbitrary we can now send β to zero in order to obtain that

$$\liminf_{\varepsilon\to 0^+}{}_* \, u^\varepsilon(x,s) \geq 1$$

if $(x,s) \in B(x_0,r] \times [t,t+h]$ and $\phi(x,s) > 0$. Finally we remark that, if $s \in (t,t+h)$, $x \in B(x_0,r)$ are given and $\phi(x,s) > 0$, we have that $\phi(y,\tau) > 0$ in a neighborhood of (x,s) and therefore $\liminf_{\varepsilon\to 0^+}{}_* \, u^\varepsilon(y,\tau) \geq 1$ for (y,τ) in a neighborhood of (x,s) in $B(x_0,r) \times (t,t+h)$. Thus $x \in \Omega_s^1$. $\square$

Proof of Lemma 3.1.8. This proof is similar to the one of Lemma 3.1.5, although with a different and not explicit function Φ, and therefore we just give a sketch. First of all we observe that since ϕ satisfies property (ii) of Proposition 3.1.7 there exists $\gamma > 0$ such that d is smooth in the set $Q_\gamma = \{(x,s) \in B(x_0,r] \times [t,t+h] : |d(x,s)| \leq \gamma\}$, $|D\phi(x,s)| \neq 0$ in Q_γ. Since $Dd = \dfrac{D\phi}{|D\phi|}$ and

$d_t = \dfrac{\phi_t}{|D\phi|}$ on $\{\phi = 0\}$, and using (i), we may also suppose that

$$d_t(x,s) + \bar{c}(x) \leq -\frac{\tilde{C}}{4|D\phi(x,s)|} \quad \text{for all } (x,s) \in Q_\gamma. \tag{3.30}$$

We notice that for every ε sufficiently small we have that $c^\varepsilon \leq \overline{c}$ and will restrict to such values of ε in the reaction-diffusion equation.

As in Lemma 3.1.5 we first define a function v^ε in Q_γ as $v^\varepsilon(x,t) = q^{\varepsilon,\delta}\Big(\dfrac{d(x,t) - 2\beta}{\varepsilon}, x\Big) - 2\beta$, with a suitable auxiliary parameter $\delta \in (0, \bar{\delta}]$. Thanks to inequality (3.30), the traveling wave equation and (3.18), we can see that for $(x,t) \in Q_\gamma$,

$$\begin{aligned} v_t^\varepsilon - \varepsilon\Delta v^\varepsilon + \frac{f^\varepsilon(v^\varepsilon, x)}{\varepsilon} &\leq \frac{q_r^{\varepsilon,\delta}}{\varepsilon}\Big(- \bar{c}(x) - \frac{\tilde{C}}{4|D\phi(x,s)|} + c^{\varepsilon,\delta}(x) - \varepsilon\Delta d\Big) - \frac{\delta}{\varepsilon} + \\ &\qquad + 2|\, Dq_r^{\varepsilon,\delta}| + \varepsilon|\, \Delta q^{\varepsilon,\delta}| - \frac{2\beta f_q^\varepsilon(q^{\varepsilon,\delta}, x)}{\varepsilon} + \frac{2\beta^2\|\, f_{qq}^\varepsilon\|_\infty}{\varepsilon} \\ &\leq \frac{1}{\varepsilon}\Big[q_r^{\varepsilon,\delta}\big(M\delta - \frac{\tilde{C}}{4\|\, D\phi_{|Q_\gamma}\|_\infty} + \varepsilon|\, \Delta d|\big) - 2\beta f_q^\varepsilon(q^{\varepsilon,\delta}, x) + 2\beta^2\|\, f_{qq}^\varepsilon\|_\infty\Big] \\ &\qquad - \frac{\delta}{\varepsilon} + \frac{2M_1}{\varepsilon^k} + \varepsilon\frac{M_2}{\varepsilon^{2k}} \\ &\leq \frac{1}{\varepsilon}\Big[- \frac{\tilde{C}}{16\|\, D\phi_{|Q_\gamma}\|_\infty} q_r^{\varepsilon,\delta} - 2\beta f_q^\varepsilon(q^{\varepsilon,\delta}, x) + 2\beta^2\|\, f_{qq}^\varepsilon\|_\infty\Big] - \frac{\delta}{2\varepsilon}, \end{aligned}$$

for $\delta > 0$ (independent of β) and then ε small enough. As in Lemma 3.1.5 it can be easily seen that, if we choose β small enough and independent of δ, the sum of the terms inside the square brackets is non positive and so v^ε is a strict subsolution in Q_γ. From now on the extension to a global subsolution $\omega^{\varepsilon,\beta}$ in $B(x_o, r] \times [t, t+h]$ and the proof that such a function satisfies 1, 2, 3, is similar to that of Lemma 3.1.5 and we omit it. □

The proof of Theorem 3.1.1 is now easy. In fact, since $(\{u(\cdot,t) = 0\}, \{u(\cdot,t) > 0\}, \{u(\cdot,t) < 0\})_{t>0}$ is the *generalized evolution* (or the level-set evolution) of $(\{d_o = 0\}, \{d_o > 0\}, \{d_o < 0\})$, Corollary 2.2.3 and the previous two steps hold

$$\begin{aligned} \{u(\cdot,t) > 0\} \subset \Omega_t^1 \subset \{u(\cdot,t) \geq 0\} \\ \{u(\cdot,t) < 0\} \subset \Omega_t^2 \subset \{u(\cdot,t) \leq 0\}, \end{aligned}$$

for any $t \geq 0$. Thus by the definition of Ω_t^1 and Ω_t^2 Theorem 3.1.1 is proved. □

3.2 Well-posedness of the Cauchy problem

We now study the well-posedness of the Cauchy problem (3.5).

In [22] we prove that there exists a unique continuous viscosity solution of the Cauchy problem

$$\begin{cases} u_t(x,t) + \alpha(x) H(x, Du(x,t)) = 0, & \mathbb{R}^n \times (0, +\infty), \\ u(x,0) = u_o(x) \in C(\mathbb{R}^n), \end{cases} \tag{3.31}$$

where the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ is continuous and positively 1-homogeneous and α satisfies the following assumptions

$$\begin{gathered} \alpha \text{ is bounded, piecewise Lipschitz continuous across Lipschitz hypersurfaces and} \\ \alpha(x) \geq \rho > 0 \text{ for any } x \in \mathbb{R}^n. \end{gathered} \tag{3.32}$$

Since in (3.5) we are considering a particular coercive hamiltonian $H(x,p) = |\,p|$ a uniqueness result for (3.5) was also previously proved by Camilli in [14]. Anyway the comparison principle, that we are going to prove here, follows the ideas we developed in [22] for a more general H.

3.2.1 Comparison principle for the HJ equation

We start with a precise definition of *piecewise Lipschitz continuous function*.

Definition 3.2.1. We say that $\alpha : \mathbb{R}^n \to [0, +\infty)$ is a piecewise Lipschitz continuous function if its discontinuity set $\Gamma \subset \mathbb{R}^n$ is a finite union of Lipschitz hypersurfaces with the following properties. For any $\hat{x} \in \Gamma$ there is $\hat{r} > 0$ such that: we can partition

$$B(\hat{x}, \hat{r}) = \Omega^+_{\hat{x}} \cup \Omega^-_{\hat{x}} \cup (\Gamma \cap B(\hat{x}, \hat{r})),$$

where $\Omega^{\pm}_{\hat{x}}$ are nonempty, open, connected (locally, the two sides of Γ). Moreover, $\inf_{\Omega^+_{\hat{x}}} \alpha > \sup_{\Omega^-_{\hat{x}}} \alpha$; α is locally Lipschitz continuous in $\mathbb{R}^n \backslash \Gamma$; α has a Lipschitz continuous extension in $\Omega^+_{\hat{x}} \cup (\Gamma \cap B(\hat{x}, \hat{r}))$ (i.e. α^*), and in $\Omega^-_{\hat{x}} \cup (\Gamma \cap B(\hat{x}, \hat{r}))$ (α_*); for all $x \in \Gamma$ we have $\alpha(x) \in [\alpha_*(x), \alpha^*(x)]$.
Below, we usually drop the subscript $\hat{x}$ in $\Omega^{\pm}_{\hat{x}}$.

Remark 3.2.2. When α is piecewise continuous and Γ is the union of disjoint Lipschitz hypersurfaces, at every $\hat{x} \in \Gamma$ we can always find unit vectors $\eta^+, \eta^- \in \mathbb{R}^n$ inward Ω^+, Ω^- respectively (transversal to Γ). This means that for some $c, h > 0$ we have $B(y + t\eta^+, tc) \subset \Omega^+$ for all $y \in B(\hat{x}, h) \cap \overline{\Omega}^+$ and $t \in (0, c)$, see [4]. Similarly for η^-.

We now state the comparison principle for solutions of (3.5) in finite time-interval. Since the term α is discontinuous, it requires some a-priori continuity of the functions to be compared.

Theorem 3.2.3. Let $T \in (0, +\infty]$ and assume that α satisfies (3.32). Let $u, v : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ be respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution of the HJ equation

$$w_t(x,t) + \alpha(x)|\,Dw(x,t)| = 0, \quad (x,t) \in \mathbb{R}^n \times (0,T) \tag{3.33}$$

such that $u(x,0) \le v(x,0)$ and, if $T < +\infty$,

$$u(x,T) = \limsup_{(y,s)\to(x,T^-)} u(y,s), \quad v(x,T) = \liminf_{(y,s)\to(x,T^-)} v(y,s).$$

Suppose moreover that for all $(\hat{x}, \hat{t}) \in \Gamma \times (0, T]$ we can find sequences $\varepsilon_k \to 0^+$, $\sigma_{\varepsilon_k} \to 0$, $p^{\varepsilon_k} \in \mathbb{R}^n$, $|p^{\varepsilon_k}| \to 0^+$ such that $\sigma_{\varepsilon_k} \le 0$ if $\hat{t} = T$, and either

$$\lim_{k\to+\infty} u(\hat{x} + \varepsilon_k \eta^+ + \varepsilon_k p^{\varepsilon_k}, \hat{t} + \sigma_{\varepsilon_k}) = u(\hat{x}, \hat{t})$$

or

$$\lim_{k\to+\infty} v(\hat{x} + \varepsilon_k \eta^- + \varepsilon_k p^{\varepsilon_k}, \hat{t} + \sigma_{\varepsilon_k}) = v(\hat{x}, \hat{t}),$$

where η^+, η^- are inward unit vectors to Ω^+, Ω^-, respectively with the notation of Definition 3.2.1 and Remark 3.2.2.
Then $u \le v$ in $\mathbb{R}^n \times [0, T]$.

Remark 3.2.4. The coefficient α has as discontinuity set $\Gamma \times [0, T]$, in the space (x, t). In this sense the continuity of the functions u, v is required along families of points $(x_\varepsilon, t_\varepsilon) = (\hat{x} + \varepsilon\eta^\pm + \varepsilon p^\varepsilon, \hat{t} + \sigma_\varepsilon)$ such that $\dfrac{x_\varepsilon - \hat{x}}{\varepsilon} \to \eta^\pm$ (transversal to Γ), but the way σ_ε tends to 0 is not prescribed. For example, if $\varepsilon = o(\sigma_\varepsilon)$, then $(x_\varepsilon, t_\varepsilon)$ tend to $(\hat{x}, \hat{t}) \in \Gamma \times [0, T]$ in a tangential fashion. For this reason the comparison principle above is not a direct consequence of the general result of Soravia in [39], although the method of proof we use is similar.

Proof. Since our equation is invariant by an increasing change of the dependent variable, it is not restrictive to suppose that u, v are bounded.
Assume now by contradiction that there is $(x_o, t_o) \in \mathbb{R}^n \times (0, T]$ such that

$$u(x_o, t_o) - v(x_o, t_o) = A > 0.$$

We set by simplicity of notation $x_o = 0$. For any $\beta, \delta > 0$, $0 < m < 1$ sufficiently small, let $(\hat{x}, \hat{t})$ be the maximum of $\Phi(x,t) := u(x,t) - v(x,t) - \beta\langle x\rangle^m - \delta t$ in $\mathbb{R}^n \times [0, T]$, where $\langle x\rangle = (1 + |x|^2)^{1/2}$. In the rest of the proof we will suppose that $\beta + \delta t_o < A$. Therefore $\Phi(x_o, t_o) > 0$ and thus $\hat{t} \ne 0$.

Moreover from $\Phi(\hat{x},\hat{t}) \geq \Phi(x_o,t_o)$ we get

$$u(\hat{x},\hat{t}) - v(\hat{x},\hat{t}) > 2\gamma > 0.$$

We have $(\hat{x},\hat{t}) \in \Gamma \times (0,T]$. Thus by the assumption, we suppose that we can find sequences $\varepsilon_k \to 0^+$, $\sigma_{\varepsilon_k} \to 0$, $p^{\varepsilon_k} \in \mathbb{R}^n$, $|p^{\varepsilon_k}| \to 0^+$, and $\sigma_{\varepsilon_k} \leq 0$ if $\hat{t} = T$, such that $\lim_{k\to+\infty} v(\hat{x} + \varepsilon_k \eta^- + \varepsilon_k p^{\varepsilon_k}, \hat{t} + \sigma_{\varepsilon_k}) = v(\hat{x},\hat{t})$. We drop the index k from now on.

Notice that as $\varepsilon \to 0^+$ we can always find p^ε, $|p^\varepsilon| \to 0$ and $\sigma_\varepsilon \to 0$ such that

$$\lim_{\varepsilon\to 0^+} v(\hat{x} + \varepsilon(p^\varepsilon + \eta), \hat{t} + \sigma_\varepsilon) = v(\hat{x},\hat{t}),$$

where

$$\begin{cases} \eta = 0 & \text{if} \quad \hat{x} \notin \Gamma \\ \eta = \eta^- & \text{if} \quad \hat{x} \in \Gamma. \end{cases}$$

We now define

$$\omega^\varepsilon(x,y,t,s) := u(x,t) - v(y,s) - \frac{\gamma}{2}\Big(\Big|\frac{x-y}{\varepsilon} + \eta\Big|^2 + \Big|\frac{t-s}{\sqrt{|\sigma_\varepsilon|}}\Big|^2\Big) - \frac{r}{2}(|x-\hat{x}|^2 + |t-\hat{t}|^2) - \beta\langle x\rangle^m - \delta t$$

and consider $(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) \in \mathbb{R}^{2n} \times [0,T]^2$ such that

$$\omega^\varepsilon(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) = \max\{\omega^\varepsilon(x,y,t,s) : (x,y,t,s) \in \mathbb{R}^{2n} \times [0,T]^2\}.$$

By definition $\omega^\varepsilon(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) \geq \omega^\varepsilon(\hat{x},\hat{x},\hat{t},\hat{t}) = \Phi(\hat{x},\hat{t}) - \frac{\gamma}{2}|\eta|^2 > 0$ for a sufficiently small γ. From here the sequences x_ε, y_ε are bounded and $|x_\varepsilon - y_\varepsilon| \leq (C+1)\varepsilon$, $|t_\varepsilon - s_\varepsilon| \leq C\sqrt{|\sigma_\varepsilon|}$, for some $C > 0$. We therefore get that

$$\lim_{\varepsilon\to 0^+} (x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) = (\bar{x}, \bar{x}, \bar{t}, \bar{t}) \in \mathbb{R}^{2n} \times [0,T]^2.$$

By semicontinuity of u, v we compute

$$\begin{aligned}
\Phi(\bar{x},\bar{t}) &= u(\bar{x},\bar{t}) - v(\bar{x},\bar{t}) - \beta\langle \bar{x}\rangle^m - \delta\bar{t} \\
&\geq \limsup_{\varepsilon\to 0^+}(u(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon,s_\varepsilon) - \beta\langle x_\varepsilon\rangle^m - \delta t_\varepsilon) \\
&\geq \liminf_{\varepsilon\to 0^+}(\omega^\varepsilon(x_\varepsilon,y_\varepsilon,t_\varepsilon,s_\varepsilon) + \frac{r}{2}(|x_\varepsilon - \hat{x}|^2 + |t_\varepsilon - \hat{t}|^2)) \\
&\geq \liminf_{\varepsilon\to 0^+}(\omega^\varepsilon(\hat{x},\hat{x}+\varepsilon(p^\varepsilon+\eta),\hat{t},\hat{t}+\sigma_\varepsilon) + \frac{r}{2}(|x_\varepsilon - \hat{x}|^2 + |t_\varepsilon - \hat{t}|^2)) \\
&= \liminf_{\varepsilon\to 0^+}(u(\hat{x},\hat{t}) - v(\hat{x}+\varepsilon(p^\varepsilon+\eta),\hat{t}+\sigma_\varepsilon) - \frac{\gamma}{2}(|p^\varepsilon|^2 + |\sqrt{|\,\sigma_\varepsilon|}|^2) \\
&\qquad\qquad - \beta\langle\hat{x}\rangle^m - \delta\hat{t} + \frac{r}{2}(|x_\varepsilon - \hat{x}|^2 + |t_\varepsilon - \hat{t}|^2)) \\
&= \Phi(\hat{x},\hat{t}) + \frac{r}{2}(|\bar{x} - \hat{x}|^2 + |\bar{t} - \hat{t}|^2).
\end{aligned}$$

From here, as $(\hat{x},\hat{t})$ is a maximum of Φ in $\mathbb{R}^n \times [0,T]$, we obtain $\bar{x} = \hat{x}$, $\bar{t} = \hat{t}$ and

$$\lim_{\varepsilon\to 0^+} u(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon,s_\varepsilon) = u(\hat{x},\hat{t}) - v(\hat{x},\hat{t}).$$

We make this information more precise by observing that

$$\begin{aligned}
u(\hat{x},\hat{t}) &\geq \limsup_{\varepsilon\to 0^+} u(x_\varepsilon,t_\varepsilon) \geq \liminf_{\varepsilon\to 0^+} u(x_\varepsilon,t_\varepsilon) = \liminf_{\varepsilon\to 0^+}((u(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon,s_\varepsilon)) + v(y_\varepsilon,s_\varepsilon)) \\
&\geq (u(\hat{x},\hat{t}) - v(\hat{x},\hat{t})) + v(\hat{x},\hat{t}) = u(\hat{x},\hat{t}),
\end{aligned}$$

and then

$$\lim_{\varepsilon\to 0^+} u(x_\varepsilon,t_\varepsilon) = u(\hat{x},\hat{t}), \quad \lim_{\varepsilon\to 0^+} v(y_\varepsilon,s_\varepsilon) = v(\hat{x},\hat{t}). \tag{3.34}$$

Again from

$$\begin{aligned}
&\omega^\varepsilon(\hat{x},\hat{x}+\varepsilon(p^\varepsilon+\eta),\hat{t},\hat{t}+\sigma_\varepsilon) \leq \omega^\varepsilon(x_\varepsilon,y_\varepsilon,t_\varepsilon,s_\varepsilon) \leq u(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon,s_\varepsilon) \\
&-\frac{\gamma}{2}\Big(\Big|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \eta\Big|^2 + \Big|\frac{t_\varepsilon - s_\varepsilon}{\sqrt{|\,\sigma_\varepsilon|}}\Big|^2\Big) - \beta\langle x_\varepsilon\rangle^m - \delta t_\varepsilon
\end{aligned}$$

we obtain

$$\lim_{\varepsilon\to 0^+}\Big|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \eta\Big| = 0, \qquad \lim_{\varepsilon\to 0^+}\Big|\frac{t_\varepsilon - s_\varepsilon}{\sqrt{|\,\sigma_\varepsilon|}}\Big| = 0; \tag{3.35}$$

and hence for ε sufficiently small

$$|x_\varepsilon - y_\varepsilon + \varepsilon\eta| \leq c\varepsilon \tag{3.36}$$

where $c > 0$ appears in Remark 3.2.2. In particular if $\hat{x} \in \Gamma$ and $x_\varepsilon \in \Omega^- \cup \Gamma$, then $y_\varepsilon \in \Omega^-$ which is something that we keep in mind for later.

Since $t_\varepsilon, s_\varepsilon \in (0,T]$ we can use the definition of sub and supersolution (and Proposition 1.0.3) and compute, respectively,

$$0 \geq \frac{\gamma}{\sqrt{|\,\sigma_\varepsilon|}}\Big(\frac{t_\varepsilon - s_\varepsilon}{\sqrt{|\,\sigma_\varepsilon|}}\Big) + r(t_\varepsilon - \hat{t}) + \delta + \alpha_*(x_\varepsilon)\Big|\,\frac{\gamma}{\varepsilon}\Big(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \eta\Big) + r(x_\varepsilon - \hat{x}) + \beta m x_\varepsilon \langle x_\varepsilon\rangle^{m-2}\Big|,$$

and

$$0 \leq \frac{\gamma}{\sqrt{|\,\sigma_\varepsilon|}}\Big(\frac{t_\varepsilon - s_\varepsilon}{\sqrt{|\,\sigma_\varepsilon|}}\Big) + \alpha^*(y_\varepsilon)\Big|\,\frac{\gamma}{\varepsilon}\Big(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \eta\Big)\Big|.$$

Combining the two inequalities we obtain

$$\begin{aligned} r(t_\varepsilon - \hat{t}) + \delta &\leq \alpha^*(y_\varepsilon)\Big|\,\frac{\gamma}{\varepsilon}\Big(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \eta\Big)\Big| - \alpha_*(x_\varepsilon)\Big|\,\frac{\gamma}{\varepsilon}\Big(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \eta\Big) + r(x_\varepsilon - \hat{x}) + \beta m x_\varepsilon \langle x_\varepsilon\rangle^{m-2}\Big| \\ &\leq \gamma\frac{|\,\alpha^*(y_\varepsilon) - \alpha_*(x_\varepsilon)|}{\varepsilon}\Big|\,\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \eta\Big| + \|\alpha\|_\infty\Big|\,r(x_\varepsilon - \hat{x}) + \beta m x_\varepsilon \langle x_\varepsilon\rangle^{m-2}\Big|, \end{aligned}$$

and hence taking the $\limsup$ as $\varepsilon \to 0^+$,

$$\delta \leq \gamma \limsup_{\varepsilon\to 0^+}\Big(\frac{|\,\alpha^*(y_\varepsilon) - \alpha_*(x_\varepsilon)|}{\varepsilon}\Big|\,\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \eta\Big|\Big) + \|\alpha\|_\infty \beta m|\,\hat{x}|\langle\hat{x}\rangle^{m-2}. \tag{3.37}$$

Now we compute the limsup in the right hand side of (3.37). We start with the cases $\hat{x} \notin \Gamma$, or else $\hat{x} \in \Gamma$, $x_\varepsilon \in \Omega^- \cup \Gamma$ and thus $y_\varepsilon \in \Omega^-$, for all ε small enough. By (3.35) we get

$$\delta \leq \gamma \limsup_{\varepsilon\to 0^+}\frac{L_\alpha|\,y_\varepsilon - x_\varepsilon|}{\varepsilon}\Big|\,\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \eta\Big| = 0 + \|\alpha\|_\infty \beta m|\,\hat{x}|\langle\hat{x}\rangle^{m-2}.$$

If instead $\hat{x} \in \Gamma$ and along a subsequence $x_\varepsilon \in \Omega^+$, we have two further cases: either for ε small $y_\varepsilon \in \Omega^+ \cup \Gamma$ and we proceed again as above, or $y_\varepsilon \in \Omega^-$ on a subsequence. In the latter situation (3.37) becomes

$$\begin{aligned} \delta &\leq \gamma \limsup_{\varepsilon\to 0^+}\frac{|\,\alpha(y_\varepsilon) - \alpha_(x_\varepsilon)|}{\varepsilon}\Big|\,\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \eta\Big| + \|\alpha\|_\infty \beta m|\,\hat{x}|\langle\hat{x}\rangle^{m-2} \\ &\leq \gamma\underbrace{[\alpha_*(\hat{x}) - \alpha^*(\hat{x})]}_{<0}\limsup_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\Big|\,\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \eta\Big| + \|\alpha\|_\infty \beta m|\,\hat{x}|\langle\hat{x}\rangle^{m-2} \\ &\leq \|\alpha\|_\infty \beta m|\,\hat{x}|\langle\hat{x}\rangle^{m-2} \end{aligned}$$

In any case we thus obtain $\delta \leq \|\alpha\|_\infty \beta m|\,\hat{x}|\langle\hat{x}\rangle^{m-2}$, hence a contradiction for a sufficiently small m and given δ, β. □

The following uniqueness result is an immediate consequence of Theorem 3.2.3.

Corollary 3.2.5. Assume the same hypotesis of Theorem 3.2.3. A viscosity solution $u \in C(\mathbb{R}^n \times [0, +\infty))$ of (3.5) is unique within the class of discontinuous solutions.

3.2.2 Existence of (the) continuous viscosity solution

We now construct the (unique) continuous viscosity solution for the Cauchy problem (3.5). Once more time we recall that in [22] we made a similar construction to obtain a continuous viscosity solution for the more general Cauchy problem in (3.31).

In order to use the control theoretic interpretation of solutions and avoid dealing with discontinuous vector fields, we rather look at (3.5) as the following Hamilton-Jacobi-Bellman equation

$$\begin{cases} (i) \quad \dfrac{u_t(x,t)}{\alpha(x)} + \max_{a \in A}\{-a \cdot Du(x,t)\} = 0, & (x,t) \in \mathbb{R}^n \times (0,+\infty), \\ (ii) \quad u(x,0) = u_o(x), & x \in \mathbb{R}^n, \end{cases} \tag{3.38}$$

where $A = \{a \in \mathbb{R}^n : |\, a| \leq 1\}$, and construct the corresponding value function.

We consider the following control system

$$\begin{cases} \dot{x}(s) = a(s), & x(0) = x_o, \\ \text{"}t(s) = t_o - \displaystyle\int_o^s \frac{1}{\alpha(x(r))} dr\text{"}, & t_o > 0, \end{cases} \tag{3.39}$$

where we will make precise the second equation by using the semicontinuous envelopes of f. As control set we will consider the set $\mathcal{A}$ defined as

$$\mathcal{A} = \{a : [0, +\infty) \to A,\ a(\cdot) \text{ measurable function}\}.$$

From now on $x(\cdot) = x(\cdot; x_o, a)$ will be a trajectory of the first equation in (3.39) corresponding to the control function $a \in \mathcal{A}$. We have two possible candidate value functions. If we set $\hat{t}(s) = t_o - \displaystyle\int_o^s \frac{1}{\alpha^*(x(s))} ds$, and $\hat{\tau}_{x_o,t_o}$ is such that $\hat{t}(\hat{\tau}_{x_o,t_o}) = 0$, then we define

$$\hat{v}(x_o, t_o) = \inf_{a \in \mathcal{A}} u_o(x(\hat{\tau}_{x_o,t_o}; x_o, a)),$$

If on the other hand $\check{t}(s) = t_o - \displaystyle\int_0^s \frac{1}{\alpha_*(x(s))} ds$, and $\check{\tau}_{x_o,t_o}$ is such that $\check{t}(\check{\tau}_{x_o,t_o}) = 0$, then we define

$$\check{v}(x_o, t_o) = \inf_{a \in \mathcal{A}} u_o(x(\check{\tau}_{x_o,t_o}; x_o, a)). \tag{3.40}$$

We claim that under suitable assumptions $\hat{v}$ and $\check{v}$ coincide and are the unique continuous solution of the Cauchy problem (3.5) (and of (3.38)).

Remark 3.2.6. Suppose that $\alpha_1(x) \geq \alpha_2(x) \geq \rho > 0$, then for $(x,t) \in \mathbb{R}^n \times (0,+\infty)$ and all controls $a \in \mathcal{A}$ we define $\tau^i = \tau^i(a)$, $i = 1,2$, by setting

$$t = \int_0^{\tau^i} \frac{1}{\alpha_i(x(s))} ds.$$

Each τ^i is well defined since f is strictly positive. We have that $\tau^1 \geq \tau^2$ since

$$\int_0^{\tau^1} \frac{1}{\alpha_2(x(s))} ds \geq \int_0^{\tau^1} \frac{1}{\alpha_1(x(s))} ds = \int_0^{\tau^2} \frac{1}{\alpha_2(x(s))} ds.$$

Now for given $x \in \mathbb{R}^n$ we modify the control by setting

$$\tilde{a}(s) := \begin{cases} a(s) & s \leq \tau^2, \\ 0, & s > \tau^2 \end{cases}$$

and we obtain

$$x(\tau^2(a); x, a) = x(\tau^2(\tilde{a}); x, \tilde{a}) = x(\tau^1(\tilde{a}); x, \tilde{a}).$$

From here the corresponding value functions satisfy the relationship

$$u_1(x,t) = \inf_{a \in \mathcal{A}} u_o(x(\tau^1)) \leq \inf_{a \in \mathcal{A}} u_o(x(\tau^2)) = u_2(x,t).$$

A first consequence of Remark 3.2.6 is that $\hat{v} \leq \check{v}$.

We show that the Hamilton-Jacobi-Bellman equation (3.38) is satisfied by approximation with problems without discontinuities. We prove the following general result.

Theorem 3.2.7. Suppose that the function α satisfies assumptions (3.32).
(i) The function $\hat{v}$ is lower semicontinuous in $\mathbb{R}^n \times [0,+\infty)$, continuous at the points of $\{(x,0) : x \in \mathbb{R}^n\}$.
(ii) If we approximate α^* (from above) by the family of Lipschitz continuous functions

$$\alpha^\varepsilon(x) = \sup_y \{\alpha^*(y) - \frac{|x-y|^2}{2\varepsilon^2}\},$$

then $\hat{v}(x,t) = \sup_{\varepsilon \downarrow 0} v^\varepsilon(x,t)$, where $v^\varepsilon \in C(\mathbb{R}^n \times [0,+\infty))$ solves the HJ equation in (3.38-i) with α replaced by α^ε.
(iii) $\hat{v}$ is the minimal viscosity solution of (3.38-i).

Proof. We start by observing that the sequence α^ε has uniform bounds, since $\rho \leq \alpha^* \leq \alpha^\varepsilon \leq \|\alpha\|_\infty$.

Moreover by well known results, each sup-convolution α^ε is Lipschitz continuous, $\alpha^\varepsilon \downarrow \alpha^*$, and thus

$$\begin{aligned} \alpha^*(x) = \inf_\varepsilon \alpha^\varepsilon(x) = \lim_{r\to 0^+} \sup_{|y-x|<r, 0<\varepsilon<r} \alpha^\varepsilon(y) \ (=: \limsup^*_{\varepsilon\to 0^+} \alpha^\varepsilon(x)), \\ \alpha_*(x) = \lim_{r\to 0^+} \inf_{|y-x|<r, 0<\varepsilon<r} \alpha^\varepsilon(y) \ (=: \liminf{}_{*\varepsilon\to 0^+} \alpha^\varepsilon(x)). \end{aligned} \tag{3.41}$$

We consider the approximating Cauchy problem

$$\begin{cases} u_t(x,t) + \alpha^\varepsilon(x)| \, Du(x,t)| = 0, \\ u(x,0) = u_o(x), \end{cases} \tag{3.42}$$

and for any given $(x_o, t_o) \in \mathbb{R}^n \times [0, +\infty)$ and controls $a(\cdot)$, define the unique $\hat{\tau}^\varepsilon_{x_o,t_o} \geq 0$ such that $0 = t_o - \int_o^{\hat{\tau}^\varepsilon_{x_o,t_o}} \frac{1}{\alpha^\varepsilon(x(s))} ds$, where $x(\cdot) = x(\cdot; x_o, a)$. Notice that $t_o\rho \leq \hat{\tau}^\varepsilon_{x_o,t_o} \leq t_o\|\alpha\|_\infty$. In particular every value function

$$v^\varepsilon(x,t) = \inf_{a\in\mathcal{A}} u_o(x(\hat{\tau}^\varepsilon_{x,t}))$$

is continuous at the points of $\{(x,0) : x \in \mathbb{R}^n\}$. Indeed this follows from

$$|u_o(x(\hat{\tau}^\varepsilon_{x_o,t_o})) - u_o(x)| \leq \omega_x(|x(\hat{\tau}^\varepsilon_{x_o,t_o}) - x_o| + |x - x_o|) \leq \omega_x(\|\alpha\|_\infty t_o + |x - x_o|),$$

where ω_x is a local modulus of continuity for u_o. Thus

$$|v^\varepsilon(x_o, t_o) - u_o(x)| \leq \omega_x(\|\alpha\|_\infty t_o + |x - x_o|).$$

We pause to observe that for the same reason this fact holds also for $\hat{v}$ and $\check{v}$. By classical results v^ε is therefore the unique continuous viscosity solution of (3.42), see e.g. [3].

Observe now that, by Remark 3.2.6, the family $\{v^\varepsilon\}$ increases as $\varepsilon \to 0^+$. Therefore we can define the lower semicontinuous function

$$\underline{v}(x,t) = \sup_\varepsilon v^\varepsilon(x,t) = \liminf_{\varepsilon\to 0^+}{}_* \ v^\varepsilon(x,t),$$

which also satisfies

$$\underline{v}^*(x,t) = \limsup_{\varepsilon\to 0^+}{}^* \ v^\varepsilon(x,t).$$

By stability of viscosity solutions, see Theorem 1.0.4, it is then well known that $\underline{v}$ is a viscosity solution of the HJ equation in (3.38).

We now show that $\hat{v} = \underline{v}$. By Remark 3.2.6, it is clear that $v^\varepsilon \leq \hat{v}$ and then $\underline{v} \leq \hat{v}$. Now we suppose by contradiction that $\underline{v}(x,t) + 2\delta \leq \hat{v}(x,t)$ for some $(x,t) \in \mathbb{R}^n \times (0, +\infty)$ and $\delta > 0$. By definition, for all $\varepsilon > 0$ sufficiently small, we can choose a strategy a_ε such that,

$$u_o(x(\hat{\tau}^{\varepsilon}_{x,t}; x, a_\varepsilon)) + \delta \leq \hat{v}(x,t). \tag{3.43}$$

We will find a control function $a \in \mathcal{A}$ such that, at least for a subsequence,

$$\lim_{\varepsilon \to 0} x(\hat{\tau}^{\varepsilon}_{x,t}; x, a_\varepsilon) = x(\hat{\tau}_{x,t}; x, a). \tag{3.44}$$

This will give a contradiction in (3.43) by continuity of u_o and definition of $\hat{v}$.

To prove (3.44) we consider the subsequence $\varepsilon_n = 1/n$. By viewing $a_{1/n}$ as an element of the space

$$L^\infty((0, t\|\alpha\|; A) = (L^1((0, t\|\alpha\|), A))^*$$

which is compact the weak star topology we can find a subsequence $\{n_k\}$ and an element $\tilde{a} \in \mathcal{A}$ such that

$$a_{\varepsilon_{n_k}} \overset{*}{\rightharpoonup} \tilde{a}.$$

From the definition of weak convergence and Lemma 3.2.8, that we postpone after the end of the proof, we know that

$$\begin{gathered} x(s; x, a_{\varepsilon_{n_k}}) \to x(s; x, \tilde{a}), \quad \text{for all } s \in [0, +\infty) \\ \hat{\tau}_{x,t}(\tilde{a}) \geq \tilde{T} := \limsup_{\varepsilon_{n_k} \to 0^+} \hat{\tau}^{\varepsilon_{n_k}}_{x,t}(a_{\varepsilon_{n_k}}). \end{gathered}$$

We then restrict ourselves to a further subsequence that we simply denote a_{ε_n} such that $\hat{\tau}^{\varepsilon_n}_{x,t}(a_{\varepsilon_n}) \to \tilde{T}$ and get

$$\begin{aligned} \hat{x}(\hat{\tau}^{\varepsilon_n}_{x,t}; x, a_{\varepsilon_n}) &= x + \int_0^{\hat{\tau}^{\varepsilon_n}_{x,t}} a_{\varepsilon_n}(s) ds \\ &= x + \int_0^{\tilde{T}} a_{\varepsilon_n}(s) ds + \int_{\tilde{T}}^{\hat{\tau}^{\varepsilon_n}_{x,t}} a_{\varepsilon_n}(s) ds \\ &\longrightarrow x + \int_0^{\tilde{T}} \tilde{a}(s) = x(\tilde{T}; x, \tilde{a}), \qquad n \to \infty. \end{aligned}$$

We now modify the strategy $\tilde{a}$ by setting

$$a^{\#}(s) := \begin{cases} \tilde{a}(s) & s \leq \tilde{T} \\ 0, & s > \tilde{T} \end{cases},$$

and we obtain (3.44) as desired.

Concerning the fact that $\hat{v}$ is the minimal viscosity solution, observe that since $\alpha^\varepsilon \geq \alpha^* \geq \alpha_*$, then v^ε is a continuous viscosity subsolution of (3.38). Thus by the comparison principle of the previous section, for any viscosity solution v of (3.38) we have $v^\varepsilon \le v_*$ and therefore $\hat{v} \le v_*$. □

We are left to show the claimed Lemma.

Lemma 3.2.8. $\hat{\tau}_{x,t}(\tilde{a}) \ge \tilde{T} := \limsup_{\varepsilon\to 0^+} \hat{\tau}_{x,t}^{\varepsilon_{n_k}}(a_{\varepsilon_{n_k}})$.

Proof. We restrict ourselves to a subsequence, that for simplicity we indicate with ε_n such that $\tilde{T} := \lim_{\varepsilon\to 0^+} \hat{\tau}_{x,t}^{\varepsilon_n}(a_{\varepsilon_n})$. By definition

$$\int_0^{\hat{\tau}_{x,t}(\tilde{a})} \frac{1}{\alpha^*(x(s;x,\tilde{a}))}\,ds = t = \int_0^{\hat{\tau}_{x,t}^{\varepsilon_n}(a_{\varepsilon_n})} \frac{1}{\alpha_n^\varepsilon(x(s;x,a_{\varepsilon_n}))}\,ds,$$

and hence from Fatou's Lemma and the approximating properties of α^ε in (3.41) we have that

$$t \ge \int_0^{\tilde{T}} \frac{1}{\alpha^*(x(s;x,\tilde{a}))}\,ds.$$

The conclusion then holds. □

The following is the corresponding statement for the approximation of the solution of our problem from above and it has an identical proof.

Theorem 3.2.9. Suppose that the function α satisfies the assumptions in (3.32). Given the inf-convolutions

$$\alpha_\varepsilon(x) = \inf_y\{\alpha_*(y) + \frac{|x-y|^2}{2\varepsilon^2}\},$$

let $\overline{v}(x,t) = \inf_{\varepsilon\downarrow 0} v_\varepsilon(x,t)$, where $t = \int_0^{\check{\tau}_{x,t}^\varepsilon(a)} \frac{1}{\alpha_\varepsilon(x(s;x,a))}ds$ and

$$v_\varepsilon(x,t) = \inf_{a\in\mathcal{A}} u_o(x(\check{\tau}_{x,t}^\varepsilon(a))),$$

$v_\varepsilon \in C(\mathbb{R}^n \times [0,+\infty))$ solves the HJ equation in (3.38) with α replaced by α_ε. Then $\check{v} = \overline{v}$ is upper semicontinuous in $\mathbb{R}^n \times [0,+\infty)$, continuous at the points of $\{(x,0) : x \in \mathbb{R}^n\}$ and the maximal viscosity solution of (3.38).

Proof. The proof is identical to the one of the previous Theorem except for the identity $\check{v} = \overline{v}$. Therefore we now show that

$$\overline{v}(x,t) = \check{v}(x,t) = \inf_{a\in\mathcal{A}} u_o(x(\check{\tau}_{x,t};x,a)).$$

Suppose on the contrary that for some $\delta > 0$, $(x,t) \in \mathbb{R}^n \times (0,+\infty)$

$$\overline{v}(x,t) \geq \check{v}(x,t) + 2\delta.$$

Therefore we can find $\check{a} \in \mathcal{A}$ such that

$$u_o(x(\check{\tau}_{x,t}; x, \check{a})) + \delta \leq \overline{v}(x,t). \tag{3.45}$$

If we verify that

$$\lim_{\varepsilon \to 0^+} \check{\tau}^{\varepsilon}_{x,t}(\check{a}) = \check{\tau}_{x,t}(\check{a}), \tag{3.46}$$

where

$$\int_0^{\check{\tau}^{\varepsilon}_{x,t}(\check{a})} \frac{1}{\alpha_\varepsilon(x(s;x,\check{a}))} ds = t,$$

then we reach a contradiction in (3.45) for ε sufficiently small, because

$$\overline{v}(x,t) \leq v_\varepsilon(x,t) \leq u_o(x(\check{\tau}^{\varepsilon}_{x,t}; x, \check{a})).$$

Now we prove (3.46). Since $\check{\tau}^{\varepsilon}_{x,t}$ is bounded, we take any converging subsequence $\check{\tau}^{\varepsilon_n}_{x,t}(\check{a}) \longrightarrow \tilde{T}$. Then we pass to the limit in

$$\int_0^{\check{\tau}_{x,t}(\check{a})} \frac{1}{\alpha_*(x(s;x,\check{a}))} ds = t = \int_0^{\check{\tau}^{\varepsilon_n}_{x,t}(\check{a})} \frac{1}{\alpha_{\varepsilon_n}(x(s;x,\check{a}))} ds$$

and obtain

$$\int_0^{\check{\tau}_{x,t}(\check{a})} \frac{1}{\alpha_*(x(s;x,\check{a}))} ds = \int_0^{\tilde{T}} \frac{1}{\alpha_*(x(s;x,\check{a}))} ds,$$

hence $\check{\tau}_{x,t}(\check{a}) = \tilde{T}$. □

Remark 3.2.10. By the classical dynamic programming principle we could show directly that $\check{v}$ is a viscosity solution of (3.38). This is a matter that we skip.

We have reached the following point.

Corollary 3.2.11. Suppose that the function α satisfies the assumptions in (3.32). Then

$$\hat{v} \leq \check{v},$$

and $\hat{v}$, $\check{v}$ are lower and upper semicontinuous, respectively, and viscosity solutions of (3.5).

We plan now to prove that indeed under appropriate assumptions $\check{v} \leq \hat{v}$ by using the comparison principle in Theorem 3.2.3, so there is a unique continuous solution of (3.38) which is the uniform limit of suitable approximation of continuous problems. Therefore we now discuss how to obtain the extra continuity properties of a value function that we need in order to apply that theorem. We define the following property of the the trajectories of the control system.

Definition 3.2.12. We say that condition (T_x) holds at $x \in \Gamma$ if there are sequences $s_n \downarrow 0$, $a_n \in \mathcal{A}$, an inward vector η^+ to Ω^+ and $k > 0$ such that

$$\begin{aligned} &x_n = x(s_n; x, a_n) = x + (s_n)^k \eta^+ + o((s_n)^k), \\ &x(0; x, a_n) = x. \end{aligned} \tag{3.47}$$

We have the following consequence.

Proposition 3.2.13. If condition (T_x) holds at $x \in \Gamma$ then for any $t > 0$, at (x, t) we have that

$$\check{v}(x,t) = \lim_{n \to +\infty} \check{v}(x_n, t_n),$$

where $t_n = \check{t}(s_n) = t - \displaystyle\int_o^{s_n} \frac{1}{\alpha_*(x(s))} ds$, $x_n = x(s_n; a_n) = x + (s_n)^k \eta^+ + o((s_n)^k))$, η^+ is inward unit vector to Ω^+.

Proof. The result is a consequence of the dynamic programming principle. Indeed for any $s > 0$ we have

$$\check{v}(x,t) = \inf_{a \in \mathcal{A}} \check{v}(x(s \wedge \check{\tau}_{x,t}), t(s \wedge \check{\tau}_{x,t})).$$

Therefore we immediately obtain by choosing $s = s_n$ and n sufficiently large

$$\check{v}(x,t) \leq \check{v}(x_n, t_n)$$

and the conclusion follows since we already know from Theorem 3.2.9 that $\check{v}$ is an upper semicontinuous function. ☐

Proposition 3.2.14. The trajectories of the control system (3.39) satisfies the condition (T_x) for all $x \in \Gamma$.

Proof. For any $x \in \Gamma$ consider a $a \in A$ such that $\eta_x^+ \cdot a > 0$ with η_x^+ as in Remark 3.2.2. Then we can choose $a_n \equiv a$, $s_n = 1/n$, $\eta = a$, to reach condition (T_x) with $k = 1$. ☐

We have obtained the following result.

Theorem 3.2.15. Suppose that the function α satisfies assumptions (3.32). Then the Cauchy problem (3.38) has a unique viscosity solution $v = \hat{v} = \check{v} \in C(\mathbb{R}^n \times [0, +\infty))$.

Proof. By Proposition 3.2.13 and Proposition 3.2.14 we can apply Theorem 3.2.3 to the lower semicontinuous subsolution $\check{v}$ and the upper semicontinuous supersolution $\hat{v}$. We thus obtain that $\check{v} \leq \hat{v}$ and $v = \hat{v} = \check{v}$ is a continuous solution of (3.5). Then Corollary 3.2.5 states its uniqueness within discontinuous solutions. □

3.2.3 The no-interior condition

In this section we want to prove that, since the velocity α has a constant sign, the zero level set $\{x : v(x,t) = 0\}$ of the (unique) viscosity solution v of the Cauchy problem (3.5) has an empty interior provided so does the zero level set of the initial condition $\{x : u_o(x) = 0\}$. To be more precise we will use the representation formula for v in (3.40) to prove that condition (2.7) is fullfilled by v if we assume that the initial datum u_0 satisfies

$$\begin{gathered} \{u_0 > 0\} \neq \emptyset, \quad \{u_0 < 0\} \neq \emptyset, \\ \Gamma_0 = \{u_0 = 0\} = \partial\{u_0 > 0\} = \partial\{u_0 < 0\} \end{gathered} \tag{3.48}$$

Theorem 3.2.16. Suppose that α satisfies assumptions (3.32). If the initial datum $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a continuous function so that (3.48) holds, then the zero level set $\{(x,t) : v(x,t) = 0\}$ satisfies the no-interior condition in (2.7).

Proof. For all $(\hat{x}, \hat{t}) \in \mathbb{R}^n \times (0, +\infty)$ we define the (bounded) set of reachable points from $(\hat{x}, \hat{t})$ as

$$\mathcal{R}_{\hat{x},\hat{t}} := \{x(\check{\tau}_{\hat{x},\hat{t}}(a); \hat{x}, a) \ : a \ \in \mathcal{A}\}.$$

First of all observe that $B(\hat{x}, \rho\hat{t}] \subseteq \mathcal{R}_{\hat{x},\hat{t}}$. In fact if $x \in B(\hat{x}, \rho\hat{t}]$ $x \neq \hat{x}$, then $x = \hat{x} + m|\, x - \hat{x}|$, with $m = \dfrac{x - \hat{x}}{|\, x - \hat{x}|}$. We consider the control

$$\tilde{a}(s) = \begin{cases} \dfrac{x - \hat{x}}{|x - \hat{x}|}, & \text{if } s \leq |\, x - \hat{x}|, \\ 0 & \text{if } s > |\, x - \hat{x}|. \end{cases}$$

We have that $\check{\tau}_{\hat{x},\hat{t}}(\tilde{a}) \geq \rho\hat{t} \geq |\, x - \hat{x}|$ and $x(\check{\tau}_{\hat{x},\hat{t}}(\tilde{a}); \hat{x}, \tilde{a}) = x(|\, x - \hat{x}|; \hat{x}, \tilde{a}) = x$, i.e. $x \in \mathcal{R}_{\hat{x},\hat{t}}$. Using this inclusion and concatenation of control functions, one can then easily show that for every $h \in (0, \hat{t})$

$$\overline{\mathcal{R}_{\hat{x},\hat{t}-h}} \subseteq \overline{\bigcup_{x \in \mathcal{R}_{\hat{x},\hat{t}-h}} B(x, \rho\frac{h}{2})} \subseteq \bigcup_{x \in \mathcal{R}_{\hat{x},\hat{t}-h}} B(x, \rho h) \subseteq \bigcup_{x \in \mathcal{R}_{\hat{x},\hat{t}-h}} \mathcal{R}_{x,h} \subseteq \mathcal{R}_{\hat{x},\hat{t}},$$

and so

$$\overline{\mathcal{R}_{\hat{x},\hat{t}-h}} \subseteq \overset{\circ}{\mathcal{R}}_{\hat{x},\hat{t}} \quad \text{for all } (\hat{x},\hat{t}) \in \mathbb{R}^n \times [0,+\infty),\ h > 0. \tag{3.49}$$

Next we claim that if $v(\hat{x},\hat{t}) = 0$ then $v(\hat{x},\hat{t}-h) > 0$ for every $h > 0$, thus $(\hat{x},\hat{t}) \notin \mathrm{Int}\{(x,t) : v(x,t) = 0\}$. Indeed suppose that $v(\hat{x},\hat{t}) = 0$ and $h > 0$. By (3.49) and the representation formula (3.40) for v we have that $v(\hat{x},\hat{t}-h) = \inf\{u_0(y) : y \in \mathcal{R}_{\hat{x},\hat{t}-h}\} \geq 0$. Assume by contradiction that $v(\hat{x},\hat{t}-h) = 0$, i.e. there exists $\hat{y} \in \overline{\mathcal{R}_{\hat{x},\hat{t}-h}}$ such that $u_0(\hat{y}) = 0$. Let $r > 0$ be such that $B(\hat{y},r) \subseteq \overset{\circ}{\mathcal{R}}_{\hat{x},\hat{t}}$; by (3.48) we have that there exist $y_1, y_2 \in B(\hat{y},r)$ such that $u_0(y_1) < 0$ and $u_0(y_2) > 0$. Again, this means that

$$v(\hat{x},\hat{t}) = \inf_{y \in \mathcal{R}_{\hat{x},\hat{t}}} u_0(y) \leq u_0(y_1) < 0,$$

and we get a contradiction since $v(\hat{x},\hat{t}) = 0$.

Assuming the claim, our Theorem immediately follows since we have that, for any $(\hat{x},\hat{t}) \in \mathbb{R}^n \times (0,+\infty)$, $h > 0$ sufficiently small,

$$\text{if } v(\hat{x},\hat{t}) = 0, \text{ then } \quad v(\hat{x},\hat{t}-h) > 0 \text{ and } v(\hat{x},\hat{t}+h) < 0.$$

□

3.3 Another asymptotic problem

In this last section of the chapter we want to briefly discuss a different scaling with respect to the reaction-diffusion equation (3.7) , namely we will consider

$$\begin{cases} \text{(i)} & u^\varepsilon_t(x,t) - \Delta u^\varepsilon(x,t) + \varepsilon^{-2} f^\varepsilon(u^\varepsilon, x) = 0 & \text{in } \mathbb{R}^n \times (0,+\infty), \\ \text{(ii)} & u^\varepsilon(x,0) = g(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{3.50}$$

Under different assumptions to those in Theorem 3.1.1 on the cubic function f^ε, we will prove that the front which describes the asymptotic behavior of these u^ε as $\varepsilon \to 0^+$ has normal velocity given by $\mathcal{K} - \alpha$, where $\mathcal{K}$ is the mean curvature of the front. To be more precise we will prove that the front that "separates" the two regions where the solutions of (3.50) converges to the stable equilibria of the system evolves according to the geometric pde

$$u_t(x,t) + F(Du(x,t), D^2u(x,t)) + \alpha(x)|Du(x,t)| = 0, \quad (x,t) \in \mathbb{R}^n \times (0,+\infty), \tag{3.51}$$

where the term α satisfies condition (3.20) and $F : \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ is defined as

$$F(p, X) = -\operatorname{tr}\Big[\Big(I - \frac{p}{|\,p|} \otimes \frac{p}{|\,p|}\Big) X\Big]. \tag{3.52}$$

Unlucky we are not able to claim that the Cauchy problem for (3.51) is well-posed (even if we conjectured this). We are able to prove a comparison principle **(CP)** for viscosity sub and super-solutions of the equation (3.51) that tells us that, if a continuous viscosity solution of (3.51) exists, then it is unique. What we are not able to do it to construct such a continuous solution.

3.3.1 The result

We now modify some of the assumptions of section 3.1.1 in order to study the asymptotic behavior of the solutions of (3.50).

We have a cubic function f^ε with the same structure as in section 3.1.1 but with (3.9) replaced by the stronger condition, this time for some $k \in [0, 1)$,

$$\begin{cases} \text{for every compact } K \subset \mathbb{R} \text{ there exists a constant } C = C(K) > 0 \\ \text{such that, for all } (q, x) \in K \times \mathbb{R}^n,\ 1 \le i, j \le n, \\ |f^\varepsilon_q(q, x)|, |f^\varepsilon_{qq}(q, x)| \le C,\ |f^\varepsilon_{x_i}(q, x)|, |f^\varepsilon_{x_i q}(q, x)| \le \dfrac{C_1}{\varepsilon^{k-1}},\ |f^\varepsilon_{x_i x_j}(q, x)| \le \dfrac{C_2}{\varepsilon^{2k-1}}. \end{cases} \tag{3.53}$$

Moreover we assume that

$$m^\varepsilon_o \longrightarrow 0^+ \qquad \text{uniformly in } \mathbb{R}^n, \tag{3.54}$$

i.e. for any $\sigma > 0$ we can find an $\varepsilon_\sigma > 0$ such that $m^\varepsilon_o(x) \in (0, \sigma]$ for all $\varepsilon \le \varepsilon_\sigma$, $x \in \mathbb{R}^n$. This means that instead of (3.10) we will assume that for any $\sigma > 0$ there exists two functions

$$\begin{gathered} \bar f,\ \underline{f} \in C^2(\mathbb{R} \times \mathbb{R}^n) \text{ satisfying } (3.8), (3.53) \\ \text{with zeroes in } \{-1, \sigma, 1\}, \{-1, 0, 1\} \text{ respectively,} \\ \text{and } \underline{f} \le f^\varepsilon \le \overline{f},\ \text{for all } x \in \mathbb{R}^n,\ q \in [-1, 1],\ 0 < \varepsilon \le \varepsilon_\sigma. \end{gathered} \tag{3.55}$$

Consequently we adapt the growth rate in (3.16) as

$$|Dq^\varepsilon(r, x)|,\ |Dq^\varepsilon_r(r, x)| \le \frac{M_1}{\varepsilon^{k-1}},\ |D^2 q^\varepsilon(r, x)| \le \frac{M_2}{\varepsilon^{2k-1}},\ \text{for all } x \in \mathbb{R}^n,\ r \in \mathbb{R}. \tag{3.56}$$

During the proofs we also need to modify the cubic-like function f^ε as $f^{\varepsilon,\delta} = f^\varepsilon + \varepsilon\delta$, for $\delta \in [-\overline{\delta}, \overline{\delta}]$ and modify accordingly the notations for the properties of $f^{\varepsilon,\delta}$. Moreover we assume that there is a constant $M > 0$ independent of ε, δ such that

$$\sup_{x\in\mathbb{R}^n} \left[|c^\varepsilon(x) - c^{\varepsilon,\delta}(x)| + |1 - m_+^{\varepsilon,\delta}(x)| + |1 + m_-^{\varepsilon,\delta}(x)| \right] \le M\delta\varepsilon. \tag{3.57}$$

As for the asymptotics of the velocity of the traveling wave solutions, we replace (3.19) by

$$\begin{gathered} 0 < 2\rho \le n_1(x) < \frac{c^\varepsilon(x)}{\varepsilon} < n_2(x) \le 2(1-\rho), \quad \text{for any } x \in \mathbb{R}^n, \\ \frac{c^\varepsilon}{\varepsilon} \longrightarrow \alpha, \quad \text{locally uniformly off } \tilde{\Gamma}, \end{gathered} \tag{3.58}$$

where the functions α, n_1, n_2 are assumed as in (3.20) and $\tilde{\Gamma}$ is a Lipshitz hypersurface.

We formalize the asymptotic result for (3.50) in the following theorem.

Theorem 3.3.1. Assume (3.8), (3.53), (3.55), (3.14), (3.15), (3.56), (3.57), (3.58) and (3.20). Let u^ε be the unique solution of (3.50), where $g : \mathbb{R}^n \to [-1, 1]$ is a continuous function such that the sets $\Gamma_o = \{x : g(x) = 0\}$, $\Omega_o^+ = \{x : g(x) > 0\}$, $\Omega_o^- = \{x : g(x) < 0\}$ are nonempty and mutually disjoint subsets of $\mathbb{R}^n$.
We suppose that the (unique) continuous viscosity solution u of the Cauchy problem

$$\begin{cases} u_t(x,t) + F(Du(x,t), D^2u(x,t)) + \alpha(x)|Du(x,t)| = 0 \text{ in } \mathbb{R}^n \times (0, +\infty), \\ u(x,0) = d_o(x), \end{cases} \tag{3.59}$$

exists, where F is as in (3.52) and d_o is the signed distance to Γ_o which is positive in Ω_o^+ and negative in Ω_o^-.

Then

$$u^\varepsilon(x,t) \longrightarrow \begin{cases} 1 & \text{in} \quad \{(x,t) : u(x,t) > 0\}, \\ -1 & \text{in} \quad \{(x,t) : u(x,t) < 0\}, \end{cases}$$

locally uniformly as $\varepsilon \to 0$, If in addition the no-interior condition (2.7) for the set $\{u = 0\}$ holds, then, as $\varepsilon \to 0$,

$$u^\varepsilon(x,t) \longrightarrow \begin{cases} 1 & \text{in} \quad \{u > 0\}, \\ -1 & \text{in} \quad \overline{\{u > 0\}}^c, \end{cases}$$

locally uniformly.

Proof. The proof follows the same steps as the one of Theorem 3.1.1, so we just point out the main changes. Consider two regions Ω^1 and Ω^2 as in (2.14) with $\tau = 1$. Define two families of open sets of $\mathbb{R}^n$, $(\Omega_t^1)_{t\in[0,T)}$ and $(\Omega_t^2)_{t\in[0,T)}$, as in (2.15) and (2.17). By the maximum principle $-1 \le u^\varepsilon \le 1$.

First step: initialization. We want to show that $\Omega_0^+ = \{d_o > 0\} \subseteq \Omega_0^1$ and $\Omega_0^- = \{d_o < 0\} \subseteq \Omega_0^2$.

For the first inclusion we consider $\hat{x} \in \{x : d_o(x) > 0\}$ and find $r, \sigma > 0$ such that

$$\begin{aligned} g(x) &\geq 5\sigma && \text{for all } x \in B(\hat{x}, r) \\ &\geq c^\varepsilon(x) + 4\sigma && \text{for all } x \in B(\hat{x}, r),\ \varepsilon \leq \varepsilon_\sigma \end{aligned}$$

and

$$m_o^\varepsilon(x) \in (0, \sigma], \quad \text{for all } x \in \mathbb{R}^n,\ \varepsilon \leq \varepsilon_\sigma.$$

This means in particular that

$$u^\varepsilon(x, 0) = g(x) \geq 5\sigma \mathbb{1}_{B(\hat{x},r)}(x) - \mathbb{1}_{B(\hat{x},r)^c}(x). \tag{3.60}$$

As in (3.24) we define the function $\Phi : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ as $\Phi(x, t) = r^2 - |x - \hat{x}|^2 - Ct$ with $C > 0$ a constant that will be chosen later.

Now we state the analogous of Lemma 3.1.4 and of Lemma 3.1.5

Lemma 3.3.2. Under the same assumptions of Theorem 3.3.1 we have that for any $\beta > 0$ there exist $\tau = \tau(\beta) > 0$ and $\bar{\varepsilon} = \bar{\varepsilon}(\beta)$ such that, for all $0 < \varepsilon \leq \bar{\varepsilon}$, we have

$$u^\varepsilon(x, t_\varepsilon) \geq (1 - \beta\varepsilon)\mathbb{1}_{\{d(\cdot,0) \geq \beta\}}(x) - \mathbb{1}_{\{d(\cdot,0) < \beta\}}(x), \quad x \in \mathbb{R}^n,$$

where $t_\varepsilon = \tau\varepsilon^2 |\lg \varepsilon|$ and $d(x, t) = \sqrt{(r^2 - Ct)^+} - |x - \hat{x}|$ is the signed distance to the set $\{x : \Phi(x, t) = 0\}$.

Lemma 3.3.3. There exist $\bar{h} = \bar{h}(r, \hat{x})$, $\bar{\beta} = \bar{\beta}(r, \hat{x}) > 0$ independent of ε such that if $\beta \leq \bar{\beta}$ and $\varepsilon \leq \bar{\varepsilon}(\beta)$, then there exists a subsolution $\omega^{\varepsilon,\beta}$ of (3.50-i) in $\mathbb{R}^n \times (0, \bar{h})$ that satisfies

$$\omega^{\varepsilon,\beta}(x, 0) \leq (1 - \beta\varepsilon)\mathbb{1}_{\{d(\cdot,0) \geq \beta\}}(x) - \mathbb{1}_{\{d(\cdot,0) < \beta\}}(x), \quad x \in \mathbb{R}^n.$$

If moreover $(x, t) \in B(\hat{x}, r) \times (0, \bar{h})$ and $d(x, t) > 3\beta$, then

$$\liminf_{\varepsilon \to 0^+}{}_* \frac{\omega^{\varepsilon,\beta}(x, t) - 1}{\varepsilon} \geq -2\beta.$$

Proof of Lemma 3.3.2. Let $\beta > 0$ fixed. From now on we restrict ε to $\varepsilon \leq \varepsilon_\sigma$. To prove our thesis we have to modify the function f^ε as in [16, 5]. Let $\bar{f} \in C^2(\mathbb{R} \times \mathbb{R}^n)$ be a function as in (3.55) with $m_2 = 2\sigma$. Consider a smooth cut-off $\rho \in C_0^\infty(\mathbb{R})$ such that $0 \leq \rho \leq 1$, $\rho(s) = 1$ if $|s| \leq 1$ and $\rho(s) = 0$ if $|s| \geq 2$. Assume moreover that ρ satisfies $-2 \leq s\rho'(s) \leq 0$ and $|\rho''(s)| \leq 4$ for all $s \in \mathbb{R}$. Now define two further smooth functions $\rho_1, \rho_2 : \mathbb{R} \to [0, 1]$ as

$$\rho_1(q) = \rho\Big(\frac{q - 2\sigma}{\sigma}\Big) \qquad \rho_2(q) = \rho\Big(\frac{q - 2\sigma}{\frac{\sigma}{4}}\Big)$$

and set

$$\bar{f}^\varepsilon(q,x) = (1-\rho_1(q))f^\varepsilon(q,x) + \rho_1(q)\bar{f}(q)$$

and

$$\tilde{f}^\varepsilon(q,x) = (1-\rho_2(q))\bar{f}^\varepsilon(q,x) + \rho_2(q)\frac{2\sigma - q}{|\lg\varepsilon|}.$$

Notice that for any $x \in \mathbb{R}^n$, $\tilde{f}^\varepsilon(\cdot,x)$ has $\{-1, 2\sigma, 1\}$ as zeros and satisfies properties similar to f^ε. Moreover $\tilde{f}^\varepsilon$ does not depend on x for all $q \in [\sigma, 3\sigma]$ and $f^\varepsilon \le \min\{\bar{f}^\varepsilon, \tilde{f}^\varepsilon\}$.

1. As in Chen [16], if we denote by $\chi = \chi(\tau,\xi;x) \in C^2([0,+\infty)\times\mathbb{R}\times\mathbb{R}^n)$ the solution of

$$\begin{cases} \dot{\chi}(\tau,\xi;x) + \tilde{f}^\varepsilon(\chi(\tau,\xi;x),x) = 0, \quad \tau > 0, \\ \chi(0,\xi;x) = \xi, \end{cases} \tag{3.61}$$

it follows that χ satisfies property ($\chi 1$) in the proof of Lemma 3.1.4 while properties ($\chi 2$) and ($\chi 3$) are replaced by the following: for all $\beta,\ \sigma > 0$ there exist $\tau_o = \tau_o(\beta,\sigma), \varepsilon_o = \varepsilon_o(\beta,\sigma) > 0$ such that, for all $\tau \ge \tau_o|\log\varepsilon|$ and $\varepsilon \le \varepsilon_o$

$$\chi(\tau,\xi;x) \ge 1 - \beta\varepsilon \quad \forall\, \xi \ge 4\sigma. \tag{$\tilde{\chi}2$}$$

Moreover, since for any $C > 1$ we have that $\chi(\tau,\xi,x) \in [-C,C]$ for all $\xi \in [-C,C]$, $\tau \ge 0$, $x \in \mathbb{R}^n$, it also holds that for any $C > 1$, $a > 0$ there exists a constant $M_{C,a} > 0$ such that

$$\begin{aligned} &|\chi_{\xi\xi}(\tau,\xi;x)| \le \frac{M_{C,a}}{\varepsilon}\chi_\xi(\tau,\xi;x), \quad |\chi_{x_i}(\tau,\xi;x)|, \le \frac{M_{C,a}}{\varepsilon^{k-1}} \\ &|\chi_{\xi x_i}(\tau,\xi;x)| \le \frac{M_{C,a}}{\varepsilon^{k-1}}\chi_\xi(\tau,\xi;x), \quad |\chi_{x_i x_i}(\tau,\xi;x)| \le \frac{M_{C,a}}{\varepsilon^{2k-1}}\chi_\xi(\tau,\xi;x), \end{aligned} \tag{$\tilde{\chi}3$}$$

for any $\tau \le a|\ln\varepsilon|$, $\xi \in [-C,C]$, $x \in \mathbb{R}^n$, $i \in \{1,2,\cdots,n\}$ and ε small enough.

2. Consider a smooth nondecreasing function ψ such that $\psi(z) = -1$ if $z \le 0$ and $\psi(z) = 5\sigma$ if $z \ge \beta \wedge \dfrac{\sigma}{2}$. Similarly as before, the function

$$\underline{u}^\varepsilon(x,t) = \chi\Big(\frac{t}{\varepsilon^2}, \psi(d(x,0)) - \frac{Kt}{\varepsilon}, x\Big)$$

satisfies $\underline{u}^\varepsilon(x,0) \le u^\varepsilon(x,0)$. Moreover $\underline{u}^\varepsilon$ is a subsolution of (3.50-i) in $\mathbb{R}^n \times (0, \tau_o\varepsilon^2|\lg\varepsilon|)$. Indeed we can compute by ($\tilde{\chi}3$),

$$\begin{aligned}
\underline{u}^\varepsilon_t - \Delta \underline{u}^\varepsilon + \frac{f^\varepsilon(\underline{u}^\varepsilon, x)}{\varepsilon^2} &= \frac{\dot{\chi} + f^\varepsilon(\chi, x)}{\varepsilon^2} - K\frac{\chi_\xi}{\varepsilon} - \chi_{\xi\xi}(\psi')^2 - \chi_\xi(\psi'' + \psi'\Delta d) \\
&\quad +2\psi' D\chi_\xi \cdot Dd + \Delta\chi \\
&= \frac{f^\varepsilon(\chi, x) - \tilde{f}^\varepsilon(\chi, x)}{\varepsilon^2} + \frac{\chi_\xi}{\varepsilon}[-K - \varepsilon(\psi'' + \psi'\Delta d) + \\
&\quad +M_{2,\tau_0}((\psi')^2 + \varepsilon^{2-k}\psi') + \varepsilon^{2-2k}] \\
&\leq -\frac{\chi_\xi}{\varepsilon}\big(K - M_{2,\tau_0}\|\,\psi'\|^2_\infty + o_\varepsilon(1)\big) \leq 0,
\end{aligned}$$

for K large enough. Therefore using the maximum principle and property ($\tilde{\chi}2$) we can prove that $u^\varepsilon(x, t_\varepsilon) \geq 1 - \beta\varepsilon$ if $t_\varepsilon = \tau_o\varepsilon^2|\lg\varepsilon|$ and $d(x, 0) \geq \beta$ (from which Lemma 3.3.2 follows). □

Proof of Lemma 3.3.3*.* The construction of a subsolution that satisfies this Lemma is very similar to the one in Lemma 3.1.5. Let Φ, d and $Q_{\gamma,\bar{h}}$ defined as in (3.24) where now the fixed constant C satisfies

$$C \geq 2r\Big[\frac{n-1}{\gamma} + 4\Big].$$

The construction of our subsolution $\omega^{\varepsilon,\beta}$ follows the usual steps. We first define for any $(x, t) \in Q_{\gamma,\bar{h}}$

$$v^\varepsilon(x, t) = q^{\varepsilon,\delta}\Big(\frac{d(x, t) - 2\beta}{\varepsilon}, x\Big) - 2\beta\varepsilon,$$

where $q^{\varepsilon,\delta}$ is the solution of the travelling wave equation (3.13) with f^ε replaced by $f^{\varepsilon,\delta} = f^\varepsilon + \varepsilon\delta$. The function v^ε is a subsolution of (3.50-i) in $Q_{\gamma,\bar{h}}$. Indeed,

$$\begin{aligned}
v^\varepsilon_t - \Delta v^\varepsilon + \frac{f^\varepsilon(v^\varepsilon, x)}{\varepsilon^2} &= \frac{q^{\varepsilon,\delta}_r d_t}{\varepsilon} - \frac{q^{\varepsilon,\delta}_{rr}}{\varepsilon} - \frac{2}{\varepsilon}Dq^{\varepsilon,\delta}_r \cdot Dd - \frac{q^{\varepsilon,\delta}_r}{\varepsilon}\Delta d - \Delta q^{\varepsilon,\delta} + \frac{f^\varepsilon(q^{\varepsilon,\delta} - 2\beta, x)}{\varepsilon^2} \\
&\quad -\frac{2\beta}{\varepsilon}f^\varepsilon_q(q^{\varepsilon,\delta}, x) + 2\beta^2\varepsilon\|\,f^\varepsilon_{qq}\|_\infty \\
&\leq \frac{1}{\varepsilon}\Big[-q^{\varepsilon,\delta}_r - 2\beta f^\varepsilon_q(q^{\varepsilon,\delta}, x) + 2\beta^2\varepsilon\|\,f^\varepsilon_{qq}\|_\infty\Big] + \Big[-\frac{\delta}{\varepsilon} + 2\frac{M_1}{\varepsilon^k} + \frac{M_2}{\varepsilon^{2k-1}}\Big],
\end{aligned}$$

and then we conclude as before. The extension of v^ε to a subsolution in the entire strip $\mathbb{R}^n \times [0, \bar{h}]$ proceed now similarly to the one in Lemma 3.1.5. We first prove that the function $\bar{v}^\varepsilon : \{(x, t) \in \mathbb{R}^n \times [0, \bar{h}] : d(x, t) \leq \gamma\} \to \mathbb{R}$, defined as

$$\bar{v}^\varepsilon(x, t) = \begin{cases} \sup(v^\varepsilon(x, t), -1) & \text{if} \quad -\gamma < d(x, t) \leq \gamma, \\ -1 & \text{if} \quad d(x, t) \leq -\gamma, \end{cases}$$

is a subsolution of (3.50-i). Eventually we define our subsolution $\omega^{\varepsilon,\beta}$ as

$$\omega^{\varepsilon,\beta}(x,t) = \begin{cases} \psi(d(x,t))\bar{v}^{\varepsilon}(x,t) + (1-\psi(d(x,t)))(1-\varepsilon\beta) & \text{if } d(x,t) < \gamma, \\ 1-\varepsilon\beta & \text{if } d(x,t) \geq \gamma. \end{cases}$$

for $(x,t) \in \mathbb{R}^n \times [0,\bar{h}]$. Anyway since these proofs does not contain any new ideas with respect to the ones in Lemma 3.1.5 we omit them. □

Second step: propagation. The proof of the fact that $(\Omega^1_t)_{t\in(0,T)}$ and $((\Omega^2_t)^c)_{t\in(0,T)}$ are respectively super and subflows with normal velocity $\mathcal{K}-\alpha$, where $\mathcal{K}$ is the mean curvature of the level set, is very close to the one in Theorem 3.1.1. We sketch the proof for $(\Omega^1_t)_{t\in(0,T)}$. Here we approximate our discontinuous limit velocity α with the following continuous functions:

$$\hat{c}^{\varepsilon}(x) := \eta^{\varepsilon}(x)n_2(x) + (1-\eta^{\varepsilon}(x))\frac{c^{\varepsilon}(x)}{\varepsilon}, \quad \check{c}^{\varepsilon}(x) := \xi^{\varepsilon}(x)n_1(x) + (1-\xi^{\varepsilon}(x))\frac{c^{\varepsilon}(x)}{\varepsilon},$$

with η^{ε} and ξ^{ε} as in Theorem 3.1.1. If we put $\hat{\mathcal{F}} = \{\hat{c}^{\varepsilon},\, \varepsilon > 0\}$, $\check{\mathcal{F}} = \{\check{c}^{\varepsilon},\, \varepsilon > 0\}$, then Proposition 3.1.6 takes the following form.

Proposition 3.3.4. Let $F : \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ be defined as in (3.52).

(i) A family $(\Omega_t)_{t\in(0,T)}$ of open subsets of $\mathbb{R}^n$ such that the set $\Omega := \cup_{t\in(0,T)}\Omega_t \times \{t\}$ is open in $\mathbb{R}^n \times [0,T]$ is a *generalized superflow* with normal velocity $-F-\alpha$ if and only if it is a generalized superflow with normal velocity $-F-\hat{c} \in C(\mathbb{R}^n)$, for all $\hat{c} \in \hat{\mathcal{F}}$;

(ii) A family $(\mathcal{F}_t)_{t\in(0,T)}$ of close subsets of $\mathbb{R}^n$ such that the set $\mathcal{F} := \cup_{t\in(0,T)}\mathcal{F}_t \times \{t\}$ is closed in $\mathbb{R}^n \times [0,T]$ is a *generalized subflow* with normal velocity $-F-\alpha$ if and only if it is a generalized subflow with normal velocity $-F-\check{c}$, for all $\check{c} \in \check{\mathcal{F}}$.

Thus, by this proposition, to prove that $(\Omega^1_t)_{t\in(0,T)}$ is a superflow with normal velocity $\mathcal{K}-\alpha$ we have to prove that $(\Omega^1_t)_{t\in(0,T)}$ is a superflow with normal velocity $\mathcal{K}-\bar{c}$ for any $\bar{c} \in \mathcal{F}$, i.e. we have to prove the following proposition

Proposition 3.3.5. Let $\overline{c} \in \overline{\mathcal{F}}$ be fixed and let $x_0 \in \mathbb{R}^n$, $t \in (0,T)$, $r > 0$, $h > 0$ so that $t+h < T$. Suppose that $\phi : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is a smooth function such that, for a suitable $\tilde{C} > 0$,

(i) $\phi_t(x,s) + F^*(D\phi(x,s), D^2\phi(x,s)) + \overline{c}(x)|D\phi(x,s)| \leq -\tilde{C} < 0$, for all $(x,s) \in B(x_0,r] \times [t,t+h]$,

(ii) for any $s \in [t,t+h]$, $\{x \in B(x_0,r] : \phi(x,s) = 0\} \neq \emptyset$ and

$$|D\phi(x,s)| \neq 0 \text{ on } \{(x,s) \in B(x_0,r] \times [t,t+h] : \phi(x,s) = 0\},$$

(iii) $\{x \in B(x_0,r] : \phi(x,t) \geq 0\} \subset \Omega^1_t$,

(iv) for all $s \in [t, t+h]$, $\{x \in \partial B(x_0, r] : \phi(x,s) \geq 0\} \subset \Omega^1_s$.

Then, for every $s \in (t, t+h)$,

$$\{x \in B(x_0, r] : \phi(x,s) > 0\} \subset \Omega^1_s.$$

Proof. We denote with $d(\cdot, s)$ the signed distance function to the set $\{\phi(\cdot, s) = 0\}$ which has the same sign of ϕ. Using the definition of $(\Omega^1_t)_{t\in(0,T)}$ we need to prove that for all $x \in B(x_o, r)$, $s \in (t, t+h)$ such that $\phi(x,s) > 0$, then we have

$$\liminf_{\varepsilon\to 0^+}{}_* \left(\frac{u^\varepsilon(y,\tau) - 1}{\varepsilon}\right) \geq 0$$

for (y, τ) in a suitable neighborhood of (x, s). To see this we have to construct a subsolution of (3.50-i) in $B(x_0, r) \times (t, t+h)$ as we claim in the following lemma.

Lemma 3.3.6. Let the assumptions of Proposition 3.3.5 hold true. There exists $\bar\beta$ small enough such that, if $\beta \leq \bar\beta$ and $\varepsilon \leq \bar\varepsilon(\beta)$ then there is a viscosity subsolution $\omega^{\varepsilon,\beta}$ of (3.50-i) in $B(x_0, r) \times (t, t+h)$ that satisfies,

1. $\omega^{\varepsilon,\beta}(x,t) \leq (1 - \beta\varepsilon)\mathbb{1}_{\{d(\cdot,t)\geq\beta\}}(x) - \mathbb{1}_{\{d(\cdot,t)<\beta\}}(x)$,   for all $x \in B(x_0, r]$,
2. $\omega^{\varepsilon,\beta}(x,s) \leq (1 - \beta\varepsilon)\mathbb{1}_{\{d(\cdot,s)\geq\beta\}}(x) - \mathbb{1}_{\{d(\cdot,s)<\beta\}}(x)$,   for all $x \in \partial B(x_0, r]$, $s \in [t, t+h]$
3. if $(x,s) \in B(x_0, r] \times [t, t+h]$ satisfies $d(x,s) > 3\beta$, then

$$\liminf_{\varepsilon\to 0^+}{}_* \left(\frac{\omega^{\varepsilon,\beta}(x,s) - 1}{\varepsilon}\right) \geq -\beta.$$

The proof of this lemma follows combining the ideas in the proofs of Lemma 3.1.8, Lemma 3.1.5 and Lemma 3.3.3. □

□

3.3.2 Comparison principle for the second order equation

We conclude the Chapter with a comparison result for the equation

$$u_t(x,t) + \alpha(x)|Du(x,t)| + F(Du(x,t), D^2u(x,t)) = 0, \tag{3.62}$$

where $F : \mathbb{R}^n \times \mathcal{S}^n$ is the standard mean curvature term

$$F(p, X) = -\operatorname{tr}[(I - -\frac{p}{|p|} \otimes \frac{p}{|p|})X]$$

and the first order term α satisfies the following assumptions.

(α**1**) $\alpha : \mathbb{R}^n \to [\rho_\alpha, +\infty)$ is a bounded measurable function, piecewise Lipschitz continuous across an hypersurface $\tilde{\Gamma}$ that partitions $\mathbb{R}^n$ as $\mathbb{R}^n = \Omega^- \dot{\cup} \Gamma \dot{\cup} \Omega^+$ with Ω^+ and Ω^- open and disjoint. Moreover there exist $c, h > 0$ and a unit vector η so that

$$B(y \pm t\eta, ct) \subset \Omega^\pm, \text{for any } y \in B(\hat{x}, h) \cap \overline{\Omega}^\pm, \, t > 0 \text{ and } \hat{x} \in \tilde{\Gamma}. \tag{3.63}$$

Finally α is locally Lipschitz continuous in $\mathbb{R}^n \backslash \tilde{\Gamma}$ with Lipschitz continuous extension in $\overline{\Omega^+} = \Omega^+ \cup \tilde{\Gamma}$ (called α^*) and in $\overline{\Omega^-} = \Omega^- \cup \tilde{\Gamma}$ (called α_*) and it satisfies.

$$\inf_{\Omega^+} \alpha > \sup_{\Omega^-} \alpha$$

and

$$\alpha(x) \in [\alpha_*(x), \alpha^*(x)], \quad \text{for all } x \in \tilde{\Gamma}.$$

Remark 3.3.7. We just notice that the assumption (3.63) on $\tilde{\Gamma}$ is satisfied when $\tilde{\Gamma}$ is the global graph of a Lipschitz continuous function with Lipshitz constant smaller than $1/c$.

We now state and prove our comparison result for the equation (3.62) in $\Omega \times (0, T)$, with Ω an unbounded subset of $\mathbb{R}^n$. The techniques used in this proof are a mixed between the ideas of Chen, Giga and Goto to treat the singularity of the mean curvature term in $p = 0$ (see [18] and [31]) and the techniques developed by Soravia to prove the uniqueness of viscosity solutions for discontinuous Hamilton-Jacobi equations.

We denote with Ω_T the open set $\Omega \times (0, T)$, and with $\partial_p \Omega_T$ its parabolic boundary, i.e. $\partial_p \Omega_T = \Omega \times \{t = 0\} \cup \partial\Omega \times [0, T]$.

Theorem 3.3.8. Let Ω be an unbounded open subset of $\mathbb{R}^n$. Fix $T > 0$ and suppose that α satisfies all the assumptions in (α**1**). Consider two functions $u, v : \mathbb{R}^n \times [0, T]$, respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution of (3.62) in Ω_T. Assume moreover that u is continuous in the direction η, i.e. for any point $(\hat{x}, \hat{t}) \in \mathbb{R}^n \times (0, T)$ there are two sequences $x_\varepsilon = \hat{x} + \varepsilon\eta + o(\varepsilon)$ and $t_\delta = \hat{t} + o(\delta)$ so that

$$\lim_{(\varepsilon,\delta)\to(0,0)} u(x_\varepsilon, t_\delta) = u(\hat{x}, \hat{t}), \tag{I1}$$

and that there exists a function $\bar{\omega} : [0, +\infty) \to [0, +\infty)$ such that $\bar{\omega}(r) \to 0$ if $r \to 0^+$ and

$$\sup\{u(x, t) - u(y, s), |\, x - y| < \varepsilon, \, |\, t - s| < \delta, \, |\, x| > 1/\rho\} \leq \bar{\omega}(\varepsilon + \delta + \rho) \tag{I2}$$

for any ε, δ and $\rho > 0$. Finally $u > -\infty$, $v < \infty$ on $\partial_p \Omega_T$ and they satisfy

$$\begin{aligned}\lim_{\delta\to 0^+} \sup \quad &\{\, u(x,t) - v(y,s) : |\, x-y| \le \delta, |\, t-s| \le \delta, \mathrm{dist}((x,t), \partial_p\Omega_T) \le \delta, \\ &\mathrm{dist}((y,s), \partial_p\Omega_T) \le \delta, (x,t),(y,s) \in \overline{\Omega} \times [0,T']\,\} \le 0\end{aligned} \tag{3.64}$$

for each $T' \in (0,T)$. Then

$$\lim_{\delta\to 0^+} \sup\{u(x,t) - v(y,s) : |\, x-y| \le \delta, |\, t-s| \le \delta, (x,t),(y,s) \in \overline{\Omega} \times [0,T']\} \le 0 \tag{3.65}$$

for each $T' \in (0,T)$.

Remark 3.3.9. The hypotesis in (I1) and in (I2) can be replaced by the analogous ones for v with the only difference that v has to be continuous in the direction $-\eta$ (that is in (I1) we have to replace η with $-\eta$)

Proof. First of all we notice that thanks to Proposition 1.0.3 we can assume that $T' = T$ with u and v respectively a viscosity sub and supersolution of (3.62) in $\Omega \times (0,T]$. We may assume that u and v are bounded in $\overline{\Omega}_T$ and that $\| u\|_\infty, \| v\|_\infty \le c^4/2$, with c as in $(\alpha\mathbf{1})$. Indeed, since u and $-v$ are upper semicontinuous in $\overline{\Omega}_T$ they are bounded from above. Moreover since the function $\psi : \mathbb{R} \to [-c^4/2, c^4/2]$, $\psi(z) = c^4/2\tanh(z)$ is strictly increasing and the equation (3.62) is geometric also $\tilde{u} = \psi(u)$ and $\tilde{v} = \psi(v)$ are respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution of (3.62) in Ω_T with $\| \tilde{u}\|_\infty, \| \tilde{v}\|_\infty \le c^4/2$. Thus, since $\tilde{u}$ still satisfies (I1) and (I2), it is possible to consider $\tilde{u}$ and $\tilde{v}$ instead of u and v or, equivalently, to assume that $\| u\|_\infty, \| v\|_\infty \le c^4/2$. We assume that $\| u\|_\infty, \| v\|_\infty \le c^4/2$. Suppose that the conclusion is false and that

$$\theta_0 := \lim_{r\to 0^+} \sup\{u(x,t) - v(y,s) : (x,t,y,s) \in \overline{\Omega}_T \times \overline{\Omega}_T, |\,(x-y, t-s)| < r\} > 0;$$

thus also $M := \sup\{u(x,t) - v(y,s) : (x,t),(y,s) \in \overline{\Omega}_T \times \overline{\Omega}_T\} \ge \theta_0 > 0$, $M \le c^4$. Then for $a > 0$ sufficiently small we have that

$$\mu_0 := \lim_{r\to 0^+} \sup\{u(x,t) - v(y,s) - at : (x,t,y,s) \in \overline{\Omega}_T \times \overline{\Omega}_T, |\,(x-y,t-s)| < r\} > 0.$$

We fixed $a > 0$ in such a way to have $\mu_0 > 0$ and we consider another positive constant $\beta > 0$ so that $\mu_0 \ge \beta > 0$. By the definition of μ_0 this means that, for any $r > 0$,

$$\sup\{u(x,t) - v(y,s) - at : (x,t,y,s) \in \overline{\Omega}_T \times \overline{\Omega}_T, |\,(x-y,t-s)| < r\} \ge \beta > 0$$

and thus also

$$\sup\{u(x,t)-v(y,s)-at:(x,t,y,s)\in\overline{\Omega}_T\times\overline{\Omega}_T, |\,(x-y-\varepsilon\eta,t-s)|<r\}\geq\beta>0$$

for any $r>0$, $\varepsilon>0$. Therefore we can conclude that

$$\tilde{\mu}_0:=\lim_{r\to0^+}\sup\{u(x,t)-v(y,s)-at:(x,t,y,s)\in\overline{\Omega}_T\times\overline{\Omega}_T, |\,(x-y-\varepsilon\eta,t-s)|<r\}\geq\beta>0.$$

For any pair of fixed parameters $\varepsilon,\,\delta>0$ we now define in $\overline{\Omega}_T\times\overline{\Omega}_T$ a function ω^σ as

$$\omega^\sigma(x,y,t,s):=u(x,t)-v(y,s)-\frac{1}{4}\Big|\frac{x-y}{\varepsilon}-\eta\Big|^4-\frac{1}{2}\Big|\frac{t-s}{\delta}\Big|^2-at,\quad\sigma=(\varepsilon,\delta),\,\varepsilon,\delta>0,$$

and, for any $r>0$,

$$\begin{aligned}\mu_1(r)\ &:=\ \sup\{\omega^\sigma(x,t,y,s):(x,t,y,s)\in\overline{\Omega}_T\times\overline{\Omega}_T, |\,x-y-\varepsilon\eta|\leq r\}\\ &\leq\ \sup_{\overline{\Omega}_T\times\overline{\Omega}_T}\omega^\sigma=:\vartheta.\end{aligned}$$

Observe that μ_1 and ϑ depend on σ and that $\lim_{r\to0^+}\mu_1(r)\geq\tilde{\mu}_0$ uniformly in σ. This means that there exists $r_1>0$ independent of σ such that

$$\mu_1(r)\geq\frac{3\tilde{\mu}_0}{4},\ \text{for } r\in(0,r_1).$$

By the hypothesis (3.64) on the values of u and v on the boundary there is $r_*>0$, $r_*<r_1$ such that

$$\sup\{\omega^\sigma(x,t,y,s):(x,t,y,s)\in\partial_p\overline{\Omega}_T\times\overline{\Omega}_T\cup\overline{\Omega}_T\times\partial_p\overline{\Omega}_T, |\,x-y,t-s|\leq r_*\}\leq\frac{\tilde{\mu}_0}{2}.$$

Since $\omega^\sigma(x,t,y,s)>0$ implies

$$|\,x-y-\varepsilon\eta|\leq M^{1/4}\varepsilon\leq c\varepsilon,\quad |\,t-s|\leq M^{1/2}\delta,$$

we can find a pair $\varepsilon_o,\delta_o>0$ so that, for any $\varepsilon\leq\varepsilon_o$, $\delta\leq\delta_o$, if $\omega^\sigma(x,t,y,s)>0$ then $|(x-y,t-s)|\leq r_*$. Therefore we observe that, by the choice of r_*, if $\omega^\sigma(x,t,y,s)>\tilde{\mu}_0/2$ and $\varepsilon\leq\varepsilon_0,\delta\leq\delta_0$, then $(x,t),(y,s)\in\Omega\times(0,T]$.

Case 1. There exists a sequence $\varepsilon_j\to0^+$, $\varepsilon_j\leq\varepsilon_0$ such that for each $j>0$ and $r\in(0,r_*)$ there is a $\delta_r=\delta_r(j)$ so that $\vartheta(\varepsilon_j,\delta_r)=\mu_1(r;\varepsilon_j,\delta_r)$ and $\delta_r\to0^+$ as $r\to0^+$. We first fix $\varepsilon_j\leq\varepsilon_0$ and

$r \in (0, r_*)$. By the definition of μ_1 there exists a sequence $\{(x_m, t_m, y_m, s_m)\}$ such that

$$\begin{cases} |\, x_m - y_m - \varepsilon_j \eta| \le r \\ \omega^\sigma(x_m, t_m, y_m, s_m) \ge \vartheta(\varepsilon_j, \delta_r) - \dfrac{1}{m}. \end{cases}$$

Therefore there exists a subsequence $\{(x_{m_k}, t_{m_k}, y_{m_k}, s_{m_k})\}_k$ so that $(t_{m_k}, s_{m_k}) \to (\hat{t}, \hat{s})$ and $x_{m_k} - y_{m_k} + \varepsilon_j \eta \to \omega_j$ as $k \to +\infty$, with $|\omega_j| \le r$.We omit from now on the subindex k. We now consider the functions

$$\begin{aligned} \psi^+(x,t) &= u(x,t) - \varphi^+(x,t) \\ \varphi^+(x,t) &= \left| \frac{x - y_m}{\varepsilon_j} - \eta \right|^4 + \left| \frac{t - s_m}{\delta_r} \right|^2 + at + (t - \hat{t})^2 + |\, x - y_m - \varepsilon_j \eta - \omega_j|^4 \end{aligned}$$

Let (ξ_m, τ_m) be a maximum point for ψ^+ in $\overline{\Omega}_T$. This implies

$$\psi^+(x_m, t_m) \le \psi^+(\xi_m, \tau_m)$$

and thus, if we subtract $v(y_m, s_m)$ from both sides we obtain

$$\begin{aligned} &\omega^\sigma(x_m, t_m, y_m, s_m) - (t_m - \hat{t})^2 - |\, x_m - y_m - \varepsilon_j \eta - \omega_j|^4 \le \\ &\qquad \le \omega^\sigma(\xi_m, \tau_m, y_m, s_m) - (\tau_m - \hat{t})^2 - |\, \xi_m - y_m - \varepsilon_j \eta - \omega_j|^4. \quad (3.66) \end{aligned}$$

Since by the definition of ϑ, $\omega^\sigma(\xi_m, \tau_m, y_m, s_m) \le \vartheta(\varepsilon_j, \delta_r)$ this becomes

$$\begin{aligned} |\, \xi_m - y_m - \varepsilon_j \eta - \omega_j|^4 + (\tau_m - \hat{t})^2 &\le \vartheta(\varepsilon_j, \delta_r) - \omega^\sigma(x_m, t_m, y_m, s_m) + \\ &\qquad + (t_m - \hat{t})^2 + |\, x_m - y_m - \varepsilon_j \eta - \omega_j|^4 \\ &\le \frac{1}{m} + (t_m - \hat{t})^2 + |\, x_m - y_m - \varepsilon_j \eta - \omega_j|^4 \end{aligned}$$

and so also $\xi_m - y_m - \varepsilon_j \eta \to \omega_j$, $\tau_m \to \hat{t}$ as $m \to +\infty$. Moreover since $r < r_* < r_1$ and $\vartheta(\varepsilon_j, \delta_r) = \mu_1(r; \varepsilon_j, \delta_r) \ge 3\tilde{\mu}_o/4$ the inequality in (3.66) becomes

$$\omega^\sigma(\xi_m, \tau_m, y_m, s_m) + (t_m - \hat{t})^2 + |\, x_m - y_m - \varepsilon_j \eta - \omega_j|^4 \ge \vartheta(\varepsilon_j, \delta_r) - \frac{1}{m} \ge \frac{3\tilde{\mu}_o}{4} - \frac{1}{m}.$$

Thus for m large enough $\omega^\sigma(\xi_m, \tau_m, y_m, s_m) > \tilde{\mu}_o/2$ and so we get $(\xi_m, \tau_m), (y_m, s_m) \in \Omega \times (0, T]$. Using the fact that u is a viscosity subsolution of (3.62) in $\Omega \times (0, T]$ we have

$$\varphi^+_t(\xi_m, \tau_m) = \frac{\tau_m - s_m}{\delta_r^2} + 2(\tau_m - \hat{t}) + a \le 0$$

if, up to some subsequence $\xi_m - y_m + \varepsilon_j \eta = 0$, and

$$\varphi_t^+(\xi_m, \tau_m) + \alpha_*(\xi_m)|\, D\varphi^+(\xi_m, \tau_m)| + F(D\varphi^+(\xi_m, \tau_m), D^2\varphi^+(\xi_m, \tau_m)) \leq 0$$

if $\xi_m - y_m - \varepsilon_j \eta \neq 0$ for m large enough. We send m to infinity to get

$$\frac{\hat{t} - \hat{s}}{\delta_r^2} + a \leq 0 \tag{3.67}$$

if $\omega_j = 0$, and

$$\begin{aligned} 0 \ &\geq \frac{\hat{t} - \hat{s}}{\delta_r^2} + a + \limsup_{m \to +\infty} (\underbrace{\alpha_*(\xi_m)}_{\geq \rho_\alpha > 0} |\, D\varphi^+(\xi_m, \tau_m)|) + \frac{|\,\omega_j|^2}{\varepsilon_j^4} F(\omega_j, I_n) \\ &\geq \frac{\hat{t} - \hat{s}}{\delta_r^2} + a + \rho_\alpha \frac{|\,\omega_j|^3}{\varepsilon_j^4} + (1 - n) \frac{|\,\omega_j|^2}{\varepsilon_j^4} \end{aligned} \tag{3.68}$$

if $\omega_j \neq 0$. Similarly, if we consider the functions

$$\begin{aligned} \psi^-(x, t) \ &= -v(y, s) + \varphi^-(x, t) \\ \varphi^-(x, t) \ &= -\Big|\, \frac{x_m - y}{\varepsilon_j} - \eta \Big|^4 - \Big|\, \frac{t_m - s}{\delta_r} \Big|^2 - (s - \hat{s})^2 - |\, x_m - y - \varepsilon_j \eta - \omega_j|^4 \end{aligned}$$

and we denote with (ζ_m, σ_m) a maximum point for ψ^- in $\Omega \times (0, T]$, with an argument similar to the one for ψ^+ we get $x_m - \zeta_m - \varepsilon_j \eta \to \omega_j$, $\sigma_m \to \hat{s}$ and $(\zeta_m, \sigma_m) \in \Omega \times (0, T]$. Since v is a supersolution we obtain

$$\varphi_t^-(\zeta_m, \sigma_m) = \frac{t_m - \sigma_m}{\delta_r^2} + 2(\hat{s} - \sigma_m) \geq 0$$

if, up to some subsequence $x_m - \zeta_m - \varepsilon_j \eta = 0$, and

$$\varphi_t^-(\zeta_m, \sigma_m) + \alpha^*(\zeta_m)|\, D\varphi^-(\zeta_m, \sigma_m)| + F(D\varphi^-(\zeta_m, \sigma_m), D^2\varphi^-(\zeta_m, \sigma_m)) \geq 0$$

if $x_m - \zeta_m - \varepsilon_j \eta \neq 0$ for m large enough. As m goes to infinity we find

$$0 \leq \frac{\hat{t} - \hat{s}}{\delta_r^2} \tag{3.69}$$

if $\omega_j = 0$, and

$$0 \leq \frac{\hat{t} - \hat{s}}{\delta_r^2} + \|\,\alpha\|_\infty \frac{|\,\omega_j|^3}{\varepsilon_j^4} + (n - 1) \frac{|\,\omega_j|^2}{\varepsilon_j^4} \tag{3.70}$$

if $\omega_j \neq 0$. If $\omega_j = 0$ the inequalities in (3.67) and (3.69) immediately yield a contradiction $a \leq 0$.

If instead $\omega_j \neq 0$ we have to combine (3.68) and (3.70) to obtain

$$a \leq \frac{|\,\omega_j|^3}{\varepsilon_j^4}(\|\,\alpha\|_\infty - \rho_\alpha) + 2(n-1)\frac{|\,\omega_j|^2}{\varepsilon_j^4}.$$

To conclude we notice that ω_j depends on r; more precisely $|\,\omega_j| \leq r$ and so we send $r \to 0^+$ to get again the contradiction $a \leq 0$.

Case 2. For sufficiently small ε, let's say $\varepsilon \leq \tilde{\varepsilon}$, there are $r_\varepsilon \in (0, r_*)$ and $\delta_\varepsilon > 0$ such that $\vartheta(\varepsilon,\delta) > \mu_1(r_\varepsilon;\varepsilon,\delta)$ for any $\delta \leq \delta_\varepsilon$. We define a function $\Psi_{\sigma\rho}$ in $\overline{\Omega}_T \times \overline{\Omega}_T$ as

$$\Psi_{\sigma\rho}(x,t,y,s) = \omega^\sigma(x,t,y,s) - \rho(|\,x|^2 + |\,y|^2)$$

with $\rho > 0$. Obviously $\Psi_{\sigma\rho}$ attains a maximum at some point $(x^{\sigma\rho}, t^{\sigma\rho}, y^{\sigma\rho}, s^{\sigma\rho}) \in \overline{\Omega}_T \times \overline{\Omega}_T$. Since $\sup\limits_{\overline{\Omega}_T\times\overline{\Omega}_T} \Psi_{\sigma,\rho} \uparrow \vartheta(\sigma)$ as $\rho \to 0^+$ we can find a $\rho_0 = \rho_0(\sigma) > 0$ so that

$$\sup_{\overline{\Omega}_T\times\overline{\Omega}_T} \Psi_{\sigma\rho} > \mu_2 := \mu_1(r_\varepsilon;\varepsilon,\delta).$$

for any $\rho \leq \rho_0$. Thus $\Psi_{\sigma\rho}(x^{\sigma\rho}, t^{\sigma\rho}, y^{\sigma\rho}, s^{\sigma\rho}) > \mu_2 \geq 3\tilde{\mu}_0/4$ and so $(x^{\sigma\rho}, t^{\sigma\rho}), (y^{\sigma\rho}, s^{\sigma\rho}) \in \Omega \times (0,T]$. Moreover since

$$\omega^\sigma(x^{\sigma\rho}, t^{\sigma\rho}, y^{\sigma\rho}, s^{\sigma\rho}) \geq \Psi_{\sigma\rho}(x^{\sigma\rho}, t^{\sigma\rho}, y^{\sigma\rho}, s^{\sigma\rho}) > \mu_1(r_\varepsilon;\varepsilon,\delta) > 0 \tag{3.71}$$

we get, by the definition of μ_1,

$$|\,x^{\sigma\rho} - y^{\sigma\rho} - \varepsilon\eta| > r_\varepsilon$$

and, by the strict positivity of μ_1,

$$|x^{\sigma\rho} - y^{\sigma\rho} - \varepsilon\eta| \leq (\|u\|_\infty + \|v\|_\infty)^{1/4}\varepsilon \leq c\varepsilon, \qquad |t^{\sigma\rho} - s^{\sigma\rho}| \leq (\|u\|_\infty + \|v\|_\infty)^{1/2}\delta \leq c^2\delta. \tag{3.72}$$

Finally we observe that, since $0 < \Psi_{\sigma\rho}(x^{\sigma\rho}, t^{\sigma\rho}, y^{\sigma\rho}, s^{\sigma\rho}) \leq \|\,u\|_\infty + \|\,v\|_\infty - \rho(|\,x^{\sigma\rho}|^2 + |\,y^{\sigma\rho}|^2)$, then

$$\rho(|\,x^{\sigma\rho}| + |\,y^{\sigma\rho}|) \to 0 \quad \text{as } \rho \to 0^+. \tag{3.73}$$

We denote with ξ and ζ the pairs (x,t) and (y,s) respectively, and we define the function φ as

$$\varphi(x,t,y,s) = \frac{1}{4}\Big|\frac{x-y}{\varepsilon} - \eta\Big|^4 + \frac{1}{2}\Big|\frac{t-s}{\delta}\Big|^2.$$

Since the map $(x,t,y,s) \longmapsto u(x,t) - v(y,s) - at - \varphi(x,t,y,s) - \rho(|x|^2 + |y|^2)$ takes its maximum over $\overline{\Omega}_T \times \overline{\Omega}_T$ at $(x^{\sigma\rho}, t^{\sigma\rho}, y^{\sigma\rho}, s^{\sigma\rho}) \in \Omega \times (0,T] \times \Omega \times (0,T]$, we get

$$\left(\begin{pmatrix} D_\xi\varphi \\ D_\zeta\varphi \end{pmatrix}, A\right) \in J^{2,+}\big(u(\xi^{\sigma\rho}) - v(\zeta^{\sigma\rho}) - at^{\sigma\rho} - \rho(|\, x^{\sigma\rho}|^2 + |\, y^{\sigma\rho}|^2)\big),$$

where

$$A = \begin{pmatrix} D^2_{\xi\xi}\varphi & D^2_{\xi\zeta}\varphi \\ D^2_{\zeta\xi}\varphi & D^2_{\zeta\zeta}\varphi, \end{pmatrix},$$

$D_\xi\varphi = D_\xi\varphi(\xi^{\sigma\rho}, \zeta^{\sigma\rho})$, $D_\zeta\varphi = D_\zeta\varphi(\xi^{\sigma\rho}, \zeta^{\sigma\rho})$ and so on. Now we apply the well-known Theorem on Sum (see [20] and [19]) that tells us that for every $\lambda > 0$ there exist two matrices $X^{\sigma\rho}, Y^{\sigma\rho} \in \mathcal{S}^n$ such that

$$(a + \varphi_t(\xi^{\sigma\rho}, \zeta^{\sigma\rho}), D_x\varphi(\xi^{\sigma\rho}, \zeta^{\sigma\rho}) + 2\rho x^{\sigma\rho}, X^{\sigma\rho} + 2\rho I_n) \in \overline{\mathcal{P}}^{2,+}u(x^{\sigma\rho}, t^{\sigma\rho})$$

$$(-\varphi_s(\xi^{\sigma\rho}, \zeta^{\sigma\rho}), -D_y\varphi(\xi^{\sigma\rho}, \zeta^{\sigma\rho}) - 2\rho y^{\sigma\rho}, Y^{\sigma\rho} - 2\rho I_n) \in \overline{\mathcal{P}}^{2,-}v(y^{\sigma\rho}, s^{\sigma\rho})$$

and

$$-\Big(\frac{1}{\lambda} + \|\, A_0\|\Big)I \leq \begin{pmatrix} X^{\sigma\rho} & O \\ O & -Y^{\sigma\rho} \end{pmatrix} \leq A_0 + \lambda A_0^2, \tag{3.74}$$

where

$$A_0 = \begin{pmatrix} D^2_{xx}\varphi & D^2_{xy}\varphi \\ D^2_{yx}\varphi & D^2_{yy}\varphi \end{pmatrix} = \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

$$B = \frac{|\, p^{\sigma\rho}|^2}{\varepsilon^2} I + \frac{2}{\varepsilon^2} p^{\sigma\rho} \otimes p^{\sigma\rho}, \qquad p^{\sigma\rho} = \frac{x^{\sigma\rho} - y^{\sigma\rho}}{\varepsilon} - \eta.$$

Moreover if we compute explicitly the derivatives of φ with respect to x and y we see that

$$D_x\varphi(\xi^{\sigma\rho}, \zeta^{\sigma\rho}) = -D_y\varphi(\xi^{\sigma\rho}, \zeta^{\sigma\rho}) = \frac{1}{\varepsilon}|\, p^{\sigma\rho}|^2 p^{\sigma\rho}.$$

Now we use the hypothesis that u and v are respectively a sub and a supersolution to obtain

$$a + \Big(\frac{t^{\sigma\rho} - s^{\sigma\rho}}{\delta^2}\Big) + \alpha_*(x^{\sigma\rho})\Big|\, \frac{|\, p^{\sigma\rho}|^2 p^{\sigma\rho}}{\varepsilon} + 2\rho x^{\sigma\rho}\Big| + F\Big(\frac{|\, p^{\sigma\rho}|^2 p^{\sigma\rho}}{\varepsilon} + 2\rho x^{\sigma\rho}, X^{\sigma\rho} + 2\rho I_n\Big) \leq 0,$$

$$\Big(\frac{t^{\sigma\rho} - s^{\sigma\rho}}{\delta^2}\Big) + \alpha^*(y^{\sigma\rho})\Big|\, \frac{|\, p^{\sigma\rho}|^2 p^{\sigma\rho}}{\varepsilon} - 2\rho y^{\sigma\rho}\Big| + F\Big(\frac{|\, p^{\sigma\rho}|^2 p^{\sigma\rho}}{\varepsilon} - 2\rho y^{\sigma\rho}, Y^{\sigma\rho} - 2\rho I_n\Big) \geq 0.$$

Since the hamiltonian F is elliptic and (3.74) implies $X^{\sigma\rho} \le Y^{\sigma\rho}$ we thus get

$$a \le \underbrace{\alpha^*(y^{\sigma\rho})\Big|\, \frac{|\, p^{\sigma\rho}|^2 p^{\sigma\rho}}{\varepsilon} - 2\rho y^{\sigma\rho}\Big| - \alpha_*(x^{\sigma\rho})\Big|\, \frac{|\, p^{\sigma\rho}|^2 p^{\sigma\rho}}{\varepsilon} + 2\rho x^{\sigma\rho}\Big|}_{\mathcal{A}} + \underbrace{F\Big(\frac{|\, p^{\sigma\rho}|^2 p^{\sigma\rho}}{\varepsilon} - 2\rho y^{\sigma\rho}, X^{\sigma\rho} - 2\rho I_n\Big) - F\Big(\frac{|\, p^{\sigma\rho}|^2 p^{\sigma\rho}}{\varepsilon} + 2\rho x^{\sigma\rho}, X^{\sigma\rho} + 2\rho I_n\Big)}_{\mathcal{B}}. \quad (3.75)$$

To get our contradiction (and to conclude the proof of our Theorem) we want to send ρ to zero. We start with the analysis of the second term $\mathcal{B}$. First of all we observe that, since $r_\varepsilon/\varepsilon < |\, p^{\sigma\rho}| \le c$, then

$$p^{\sigma\rho} \to \overline{p}^{\sigma} \quad \text{as } \rho \to 0^+, \quad (3.76)$$

with $r_\varepsilon/\varepsilon \le |\, \overline{p}^{\sigma}| \le c$. As it concerns the matrix $X^{\sigma\rho}$, (3.74) implies that $\|\, X^{\sigma\rho}\| \le \|\, A_0\| + \lambda\|\, A_0^2\| + 1/\lambda$; therefore to get a bound for $X^{\sigma\rho}$ we need a bound on A_0. Some easy computations show that

$$\|\, A_0\| \le 2\|\, B\| \le \frac{6}{\varepsilon^2}|\, p^{\sigma\rho}|^2 \le \frac{6}{\varepsilon^2}$$

and thus

$$\|\, X^{\sigma\rho}\| \le \frac{6}{\varepsilon^2} + \lambda\frac{36}{\varepsilon^4} + \frac{1}{\lambda}.$$

Since we have obtained an estimate for $\|X^{\sigma\rho}\|$ independent from ρ we have that there exists a matrix $\overline{X}^{\sigma}$ so that $\|\, \overline{X}^{\sigma}\| \le 6/\varepsilon^2 + \lambda 36/\varepsilon^4 + 1/\lambda$ and

$$X^{\sigma\rho} \to \overline{X}^{\sigma}, \quad \text{as } \rho \to 0^+. \quad (3.77)$$

Using (3.73), (3.76) and (3.77) we can claim that $\mathcal{B}$ goes to zero as $\rho \to 0^+$. As it concerns the behaviour of tems in $\mathcal{A}$ as ρ goes to zero we notice that

$$\begin{aligned}
\limsup_{\rho\to 0^+} \mathcal{A} \; &= \limsup_{\rho\to 0^+} [(\alpha^*(y^{\sigma\rho}) - \alpha_*(x^{\sigma\rho}))|\, \frac{|\, p^{\sigma\rho}|^2 p^{\sigma\rho}}{\varepsilon} - 2\rho y^{\sigma\rho}| \\
&\qquad + \alpha_*(x^{\sigma\rho})(|\, \frac{|\, p^{\sigma\rho}|^2 p^{\sigma\rho}}{\varepsilon} - 2\rho y^{\sigma\rho}| - |\, \frac{|\, p^{\sigma\rho}|^2 p^{\sigma\rho}}{\varepsilon} - 2\rho x^{\sigma\rho}|)] \\
&\le \limsup_{\rho\to 0^+} [(\frac{\alpha^*(y^{\sigma\rho}) - \alpha_*(x^{\sigma\rho})}{\varepsilon})|\, p^{\sigma\rho}|^3 + 2\|\, \alpha\||\, \rho(3|\, y^{\sigma\rho}| + |\, x^{\sigma\rho}|)|] \\
&\le \limsup_{\rho\to 0^+} (\frac{\alpha^*(y^{\sigma\rho}) - \alpha_*(x^{\sigma\rho})}{\varepsilon})|\, p^{\sigma\rho}|^3.
\end{aligned}$$

To conclude we notice that the usual bound for $|p^{\sigma\rho}|$, $|p^{\sigma\rho}| \le c$, implies $y^{\sigma\rho} \in B(x^{\sigma\rho} - \varepsilon\eta, c\varepsilon)$. Thus by the definition of η and for ε small enough, we have that, if $x^{\sigma\rho} \in \Omega^- \cup \Gamma$ then also $y^{\sigma\rho} \in \Omega^-$. This means that, if $x^{\sigma\rho} \in \Omega^- \cup \Gamma$ we can use the Lipschitz continuity of α_* in $\Omega^- \cup \Gamma$ to get

$$\limsup_{\rho\to 0^+} \mathcal{A} \le \limsup_{\rho\to 0^+} L_\alpha \frac{|\,y^{\sigma\rho} - x^{\sigma\rho}|}{\varepsilon} |\,p^{\sigma\rho}|^3 = \limsup_{\rho\to 0^+} L_\alpha |\,p^{\sigma\rho} + \eta||\,p^{\sigma\rho}|^3 \tag{3.78}$$

If $x^{\sigma\rho} \in \Omega^+$ we have to distinguish whatever $y^{\sigma\rho} \in \Omega^+ \cup \Gamma$ or $y^{\sigma\rho} \in \Omega^-$. In fact in the latter situation we have $\alpha^*(y^{\sigma\rho}) = \alpha(y^{\sigma\rho}) \le \alpha(x^{\sigma\rho}) = \alpha_*(x^{\sigma\rho})$ that immediately gives us a contradiction in (3.75) as $\rho \to 0$ since it implies $\limsup_{\rho\to 0^+} \mathcal{A} \le 0$. Instead if $y^{\sigma\rho} \in \Omega^+ \cup \Gamma$ we use the Lipschitz continuity of α^* in $\overline{\Omega}^+ = \Omega^+ \cup \Gamma$ to obtain again the inequality in (3.78). Therefore to get a contradiction in (3.75) in the cases $x^{\sigma\rho} \in \Omega^- \cup \Gamma$, $y^{\sigma\rho} \in \Omega^-$ and $x^{\sigma\rho} \in \Omega^+$, $y^{\sigma\rho} \in \Omega^+ \cup \Gamma$ we need to prove that the right hand side of the inequality in (3.78) goes to zero as $\varepsilon \to 0^+$; to do this it is enough to prove that

$$\lim_{\sigma\to 0^+} \lim_{\rho\to 0^+} |\,p^{\sigma\rho}| = 0. \tag{3.79}$$

We have to differentiate three cases.

CASE 2.A There exists a constant $C > 0$ so that $|\,x^{\sigma\rho}| \le C$ for σ and ρ small enough. This means that for any $\sigma > 0$ there exists a sequence $\rho_j = \rho_j(\sigma) \to 0^+$ so that $(x^{\sigma\rho_j}, t^{\sigma\rho_j}, y^{\sigma\rho_j}, s^{\sigma\rho_j}) \to (x^\sigma, t^\sigma, y^\sigma, s^\sigma)$ as $j \to +\infty$. Since $|\,x^\sigma| \le C$, $|\,x^\sigma - y^\sigma - \varepsilon\eta| \le \varepsilon$ and $|\,t^\sigma - s^\sigma| \le \delta$ we can find another sequence $\sigma_k = (\varepsilon_k, \delta_k) \to (0^+, 0^+)$ so that $(x^{\sigma_k}, t^{\sigma_k}, y^{\sigma_k}, s^{\sigma_k}) \to (\hat{x}, \hat{t}, \hat{x}, \hat{t}) \in \overline{\Omega}_T \times \overline{\Omega}_T$ as $k \to +\infty$. We omit from now on the subindexes j and k. By the definition of $(x^{\sigma\rho}, t^{\sigma\rho}, y^{\sigma\rho}, s^{\sigma\rho})$,

$$\Psi_{\sigma\rho}(x^{\sigma\rho}, t^{\sigma\rho}, y^{\sigma\rho}, s^{\sigma\rho}) \ge \Psi_{\sigma\rho}(\hat{x} + \varepsilon\eta, \hat{t}, \hat{x}, \hat{t}),$$

and thus

$$\begin{aligned}\lim_{\sigma\to 0^+} \limsup_{\rho\to 0^+} \frac{|\,p^{\sigma\rho}|^4}{4} \le \lim_{\sigma\to 0^+} \limsup_{\rho\to 0^+} \big(u(x^{\sigma\rho}, t^{\sigma\rho}) - u(\hat{x} + \varepsilon\eta, \hat{t}) + \\ + v(\hat{x}, \hat{t}) - v(y^{\sigma,\rho}, s^{\sigma,\rho}) + a(\hat{t} - t^{\sigma\rho}) + \rho(|\,\hat{x}|^2 + |\,\hat{x} + \varepsilon\eta|^2)\big).\end{aligned}$$

Since $\alpha(\hat{t} - t^{\sigma\rho})$ and $\rho(|\,\hat{x}|^2 - |\,\hat{x} + \varepsilon\eta|^2)$ goes to zero as ρ and $\sigma \to 0^+$ we get

$$\begin{aligned}\lim_{\sigma\to 0^+} \limsup_{\rho\to 0^+} \frac{|\,p^{\sigma\rho}|^4}{4} \le \lim_{\sigma\to 0^+} \limsup_{\rho\to 0^+} \Big(\underbrace{u(x^{\sigma\rho}, t^{\sigma\rho}) - u(\hat{x}, \hat{t})}_{(a)} + \\ + \underbrace{u(\hat{x}, \hat{t}) - u(\hat{x} + \varepsilon\eta, \hat{t})}_{(b)} + \underbrace{v(\hat{x}, \hat{t}) - v(y^{\sigma\rho}, s^{\sigma\rho})}_{(c)}\Big).\end{aligned}$$

Using the upper semicontinuity of u in the term (a), the lower semicontinuity of v in (c) and (I1) in

(b) we can obtain (3.79) from this last inequality.

CASE 2.B For any $\sigma > 0$ there exist two positive constants $C_1(\sigma), C_2(\sigma)$ so that $C_1(\sigma), C_2(\sigma) \to +\infty$ as $\sigma \to 0^+$ and $C_1(\sigma) \le |\, x^{\sigma\rho}| \le C_2(\sigma)$ for any $\rho > 0$ small enough. This means that there exist two infinitesimal sequences $\rho_j = \rho_j(\sigma)$ and σ_k so that for any $\sigma > 0$ $(x^{\sigma\rho_j}, t^{\sigma\rho_j}, y^{\sigma\rho_j}, s^{\sigma\rho_j}) \to (x^\sigma, t^\sigma, y^\sigma, s^\sigma))$ as $j \to +\infty$ and $|\, x^{\sigma_k}|, |\, y^{\sigma_k}| \to +\infty,\ t^{\sigma_k}, s^{\sigma_k} \to \hat{t}$ as $k \to +\infty$. Again we omit the subindexes j and k and we use the definition of $(x^{\sigma\rho}, t^{\sigma\rho}, y^{\sigma\rho}, s^{\sigma\rho})$ in order to obtain

$$\Psi_{\sigma\rho}(x^{\sigma\rho}, t^{\sigma\rho}, y^{\sigma\rho}, s^{\sigma\rho}) \ge \Psi_{\sigma\rho}(y^\sigma + \varepsilon\eta, s^\sigma, y^\sigma, s^\sigma)$$

and

$$\frac{|\, p^{\sigma\rho}|^4}{4} \le u(x^{\sigma\rho}, t^{\sigma\rho}) - u(y^\sigma + \varepsilon\eta, s^\sigma) + v(y^\sigma, s^\sigma) - v(y^{\sigma\rho}, s^{\sigma\rho}) + \\ + \alpha(s^\sigma - t^{\sigma\rho}) + \rho(|\, y^\sigma + \varepsilon\eta|^2 + |\, y^\sigma|^2).$$

If we add $\pm u(x^\sigma, t^\sigma)$ to the right hand side this implies that

$$\begin{aligned} \lim_{\rho\to 0^+} \frac{|\, p^{\sigma\rho}|^4}{4} &\le \limsup_{\rho\to 0^+} \Big(u(x^{\sigma\rho}, t^{\sigma\rho}) - u(x^\sigma, t^\sigma) + u(x^\sigma, t^\sigma) - u(y^\sigma + \varepsilon\eta, s^\sigma) + \\ &\qquad v(y^\sigma, s^\sigma) - v(y^{\sigma\rho}, s^{\sigma\rho}) + \alpha(s^\sigma - t^{\sigma\rho}) + \rho(|\, y^\sigma + \varepsilon\eta|^2 + |\, y^\sigma|^2 \Big) \\ &\le \limsup_{\rho\to 0^+} \big(u(x^\sigma, t^\sigma) - u(y^\sigma + \varepsilon\eta, s^\sigma) \big) \\ &\le \bar{\omega}(\varepsilon + \delta + \frac{1}{C_1(\sigma)}) \end{aligned}$$

where to obtain the last inequality we have used (I2). Taking the limit as $\sigma \to 0^+$ we obtain (3.79).

CASE 2.C There exists a sequence $(\sigma_i)_i$ so that $\sigma_i \to 0^+$ as $i \to +\infty$ and for any $\sigma_i, \rho > 0$ we can find a constant $C_{\sigma_i}(\rho)$ such that $|x^{\sigma_i\rho}| \ge C_{\sigma_i}(\rho)$ and $C_{\sigma_i}(\rho) \to +\infty$ as $\rho \to 0$. Obviously this means that for any $i > 0$ there exists another sequence $(\rho_j^{(i)})_j$ such that $\rho_j^{(i)} \to 0^+$ and $|\, x^{\sigma_i\rho_j^{(i)}}| \to +\infty$ as $j \to +\infty$. By (3.72) we can assume that $|\, s^{\sigma_i\rho_j} - t^{\sigma_i\rho_j}| \to 0$ if $i, j \to +\infty$. We omit the indexes i, j and we obtain, as in the previous cases,

$$\Psi_{\sigma\rho}(x^{\sigma\rho}, t^{\sigma\rho}, y^{\sigma\rho}, s^{\sigma\rho}) \ge \Psi_{\sigma\rho}(y^{\sigma\rho} + \varepsilon\eta, s^{\sigma\rho}, y^{\sigma\rho}, s^{\sigma\rho})$$

and so, since u satisfies (I2)

$$\begin{aligned} \frac{|\, p^{\sigma\rho}|^4}{4} &\le u(x^{\sigma\rho}, t^{\sigma_\rho}) - u(y^{\sigma\rho} + \varepsilon\eta, s^{\sigma\rho}) + \alpha(s^{\sigma\rho} - t^{\sigma\rho}) + \rho(|\, y^{\sigma\rho} + \varepsilon\eta|^2 - |\, x^{\sigma\rho}|^2) \\ &\le \bar{\omega}(\varepsilon + \delta + \frac{1}{C_\sigma(\rho)}) + \alpha(s^{\sigma\rho} - t^{\sigma\rho}) + \rho(|\, y^{\sigma\rho} + \varepsilon\eta|^2 - |\, x^{\sigma\rho}|^2). \end{aligned}$$

We send $\sigma, \rho \to 0^+$ and we obtain (3.79) also in this case. $\square$

Example of solution that satisfies (I2)

Assume that $u_0 \in UC_\beta(\mathbb{R}^n)$ for some $\beta \in \mathbb{R}$, i.e. $u_0 - \beta$ is a uniformly continuous function with compact support in $\mathbb{R}^n$, and that $u_0(x) \geq \beta$ for any $x \in \mathbb{R}^n$. Let $R > 0$ so that $u_0(x) = \beta$ for any $|\,x| \geq R$. In this section we want to show that, if u is a viscosity solution of (3.62) such that $u(x,0) = u_0(x)$, then

$$u(x,t) \equiv \beta, \quad \text{for any } (x,t) \in \mathbb{R}^n \times [0,+\infty),\ |\,x| \geq R$$

and thus u satisfies (I2).
Consider the function $\omega : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$ defined as

$$\omega(x,t) = \psi\Big(\frac{M}{2(n-1)}(R^2 - |\,x|^2) - Mt + \beta\Big)$$

where $\psi : \mathbb{R} \to [\beta, +\infty)$, $\psi(r) := r \vee \beta$ and $M \geq 1$ is a constant that we will choose in a right way. A simple computation shows that ω is a solution of the mean curvature equation and then a supersolution of our equation (3.62). Since $\omega(x,t) = \beta$ if $|\,x| \geq R$ we can choose M small enough to obtain $\omega(x,0) \geq u_0(x)$ for any $x \in \mathbb{R}^n$. Obviously also the function $v \equiv \beta$ is a solution of (3.62) and $v(x,0) \leq u_0(x)$ for every $x \in \mathbb{R}^n$. Since v and ω satisfy the hypotesis of Theorem 3.3.8 we obtain

$$\beta = v(x,t) \leq u_*(x,t)$$

and

$$u^*(x,t) \leq \omega(x,t).$$

Combining this two inequalities we can conclude that $u(x,t) \equiv \beta$ for any $|\,x| \geq R$, $t \geq 0$.

Chapter 4

Degenerate Asymptotics

Let $\sigma^{(1)}, \dots, \sigma^{(m)} \in C^2(\mathbb{R}^n, \mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ be a family of m vector fields on $\mathbb{R}^n$ with $m < n$. We define the so-called *horizontal gradient* of a C^1-function h as

$$D_H h(x) := \sigma^t(x) Dh(x) = \begin{pmatrix} \sigma^{(1)}(x) \cdot Dh(x) \\ \vdots \\ \sigma^{(m)}(x) \cdot Dh(x) \end{pmatrix} \in \mathbb{R}^m, \qquad x \in \mathbb{R}^n,$$

where $\sigma = (\sigma_{ik})_{i,k} \in C^2(\mathbb{R}^n, \mathbb{R}^{n\times m}) \cap W^{2,\infty}(\mathbb{R}^n, \mathbb{R}^{n\times m})$ denote the matrix map with $\sigma^{(1)}, \dots, \sigma^{(m)}$ as columns, i.e.

$$\sigma \equiv \left[\sigma^{(1)}, \dots, \sigma^{(m)}\right].$$

Similarly we define also the *horizontal Hessian matrix* of h, $\tilde{D}^2_H h(x)$, and the *symmetrized horizontal Hessian matrix* of h $D^2_H h(x)$ by putting

$$\begin{aligned} \tilde{D}^2_H h(x) := D_H(D_H h(x)) &= \left(\sigma^{(i)}(x) D(\sigma^{(j)}(x) Dh(x))\right)_{i,j=1,\dots,m} \\ &= \begin{pmatrix} \sigma^{(1)}(x) D(\sigma^{(1)}(x) Dh(x)) & \cdots & \sigma^{(1)}(x) D(\sigma^{(m)}(x) Dh(x)) \\ \vdots & \ddots & \vdots \\ \sigma^{(m)}(x) D(\sigma^{(1)}(x) Dh(x)) & \cdots & \sigma^{(m)}(x) D(\sigma^{(m)}(x) Dh(x)) \end{pmatrix} \\ &= \sigma^t(x) D^2 h(x) \sigma(x) + \sigma^t(x) D(\sigma^t(x)) Dh(x) \end{aligned}$$

and

$$D^2_H h(x) = (\tilde{D}^2_H h(x))^* = \frac{\tilde{D}^2_H h(x) + \tilde{D}^2_H h(x)^t}{2}$$

where, $\sigma^t(x)D(\sigma^t(x))Dh(x)$ denotes the following (non symmetric) matrix

$$\begin{aligned}\sigma^t(x)D(\sigma^t(x))Dh(x) &= \Big(\sigma^{(i)}(x)D(\sigma^{(j)}(x))Dh(x)\Big)_{i,j=1,\dots,m}\\ &= \Big(\sum_{k,l=1}^{n}\sigma_k^{(i)}(x)(\partial_{x_k}\sigma_l^{(j)}(x))\partial_{x_l}h(x)\Big)_{i,j=1,\dots,m}.\end{aligned}$$

We define a matrix map $A(\cdot)\in C^2(\mathbb{R}^n,\mathcal{S}^n)\cap W^{2,\infty}(\mathbb{R}^n,\mathcal{S}^n)$ as

$$A(x)=\sigma^t(x)\sigma(x),\qquad \text{for any } x\in\mathbb{R}^n.$$

Clearly A satisfies the following conditions,

$$\text{for all } i,j,k\in\{1,\dots,n\},\ a_{ij},a_{ij,x_k} \text{ are bounded and continuous on } \mathbb{R}^n, \tag{4.1}$$

$$A(x)q\cdot q\geq 0 \text{ for any } (x,q)\in\mathbb{R}^n\times\mathbb{R}^n. \tag{4.2}$$

Moreover

$$\mathrm{tr}(D_H^2h(x))=\mathrm{tr}(\tilde{D}_H^2h(x))=\mathrm{tr}(A(x)D^2h(x))+\mathrm{tr}(\sigma^t(x)D(\sigma^t(x))Dh(x)).$$

Reaction-diffusion equations of the forms

$$\phi_t(x,t)-\Delta\phi(x,t)+f(\phi(x,t))=0,\quad (x,t)\in\mathbb{R}^n\times(0,+\infty) \tag{4.3}$$

arise naturally in a lot of mathematical models, such as phase transition, flame propagation, etc. In most of these applications one can observe, for large times, the development of fronts as the boundaries of the regions where the solution ϕ of (4.3) converges to the stable equilibria of the cubic f. Formal results of Fife [28] and Caginalp [11], [12] and [13] show that the fronts propagate with normal velocity

$$V=\alpha+\frac{1}{t}\kappa+O\Big(\frac{1}{t^2}\Big),\qquad t\gg 1, \tag{4.4}$$

where $\alpha=2m_0-m_+-m_-$, with $m_-<m_0<m_+$ zeroes of f, and κ denotes the curvature of the front. Barles, Soner and Souganidis give a rigorous justification of this result when the cubic function depend also by the variables x,t (i.e. $f=f(q,x,t)$). To obtain the first term in (4.4) one has to consider the equation in (4.3) with the scaling $(\varepsilon^{-1}x,\varepsilon^{-1}t)$. If $\alpha=0$ one has to go to the next time-scaling $(\varepsilon^{-1}x,\varepsilon^{-2}t)$ in order to obtain the second term in the expansion of the velocity V in (4.4). This means that one has to consider the equation

$$\phi_t^\varepsilon(x,t)-\varepsilon\Delta\phi^\varepsilon(x,t)+\frac{f(\phi(x,t))}{\varepsilon}=0,$$

when $\alpha \neq 0$, and

$$\phi_t^\varepsilon(x,t) - \Delta\phi^\varepsilon(x,t) + \frac{f(\phi(x,t))}{\varepsilon^2} = 0,$$

when $\alpha = 0$.

In this chapter we would like to replace the standard Euclidean derivatives with the derivatives along the m vector fields $\sigma_1, \dots, \sigma_m$, that is we would like to study the asymptotic behavior of the solutions u^ε of the Cauchy problems for the following reaction-diffusion equations

$$u_t^\varepsilon(x,t) - \varepsilon\,\mathrm{tr}(D_H^2 u^\varepsilon(x,t)) + \frac{f(u^\varepsilon(x,t))}{\varepsilon} = 0 \quad \text{in } \mathbb{R}^n \times (0,+\infty)$$

and

$$u_t^\varepsilon(x,t) - \mathrm{tr}(D_H^2 u^\varepsilon(x,t)) + \frac{f(u^\varepsilon(x,t))}{\varepsilon^2} = 0 \quad \text{in } \mathbb{R}^n \times (0,+\infty),$$

where the map $q \mapsto f(q)$ is a cubic-type linearity such that

$$\begin{aligned}&f \in C^2(\mathbb{R}) \text{ has exactly three zeroes } m_- < m_o < m_+,\\ &f(s) > 0 \text{ in } (m_-, m_o) \text{ and } f(s) < 0 \text{ in } (m_o, m_+),\\ &f'(m_\pm) \geq d > 0, f'(m_o) < -d < 0, f''(m_-) < 0 \text{ and } f''(m_+) > 0.\end{aligned} \tag{4.5}$$

Writing explicitly the trace of $D_H^2 u^\varepsilon$ we have that the equations above can be rewritten respectively as

$$u_t^\varepsilon(x,t) - \varepsilon\,\mathrm{tr}(A(x)D^2u^\varepsilon(x,t)) - \varepsilon\,\mathrm{tr}((\sigma^t(x)D(\sigma^t(x))Du^\varepsilon(x,t))) + \varepsilon^{-1}f(u^\varepsilon) = 0, \tag{4.6}$$

and

$$u_t^\varepsilon(x,t) - \mathrm{tr}(A(x)D^2u^\varepsilon(x,t)) - \mathrm{tr}((\sigma^t(x)D(\sigma^t(x))Du^\varepsilon(x,t))) + \varepsilon^{-2}f(u^\varepsilon) = 0. \tag{4.7}$$

Since, by assumption (4.2), the matrix $A(x)$ is only semi-positive definite the equations in (4.6) and (4.7) do not satisfy a uniform ellipticity condition. Thus we can't apply the classical theory of elliptic equations to get the existence of a smooth solution of the Cauchy problems for (4.6) and (4.7).

We introduce a Riemannian approximation of the matrix A. We start by defining a matrix map $\sigma_\varepsilon \in C^2(\mathbb{R}^n, \mathbb{R}^{n\times(m+n)}) \cap W^{2,\infty}(\mathbb{R}^n, \mathbb{R}^{n\times(m+n)})$, $\varepsilon > 0$ as

$$\sigma_\varepsilon(\cdot) = [\sigma(\cdot)\ \varepsilon^k I_n], \tag{4.8}$$

where $k > 0$ and I_n denotes the $n \times n$ identity matrix. As Riemannian approximation of A we consider

$$A_\varepsilon(\cdot) = \sigma_\varepsilon(\cdot)\sigma_\varepsilon^t(\cdot) = A(\cdot) + \varepsilon^{2k} I_n.$$

For any $x \in \mathbb{R}^n$ the matrix $A_\varepsilon(x) \in \mathbb{R}^{n\times n}$ is strictly positive definite, in fact

$$A_\varepsilon(x)q \cdot q = A(x)q \cdot q + \varepsilon^{2k}|\, q|^2 \geq \varepsilon^{2k}|\, q|^2. \tag{4.9}$$

Moreover

$$A_\varepsilon \to A \text{ uniformly as } \varepsilon \to 0^+.$$

If we put

$$D_{H,\varepsilon} = \sigma_\varepsilon^t(x) D,$$

we have that

$$D_{H,\varepsilon}h(x) = \sigma_\varepsilon^t(x)Dh(x) = \begin{pmatrix} \sigma^t(x)Dh(x) \\ \varepsilon^k Dh(x) \end{pmatrix} = \begin{pmatrix} D_H h(x) \\ \varepsilon^k Dh(x) \end{pmatrix},$$

$$D^2_{H,\varepsilon}h(x) = \left(\begin{array}{c|c} D^2_H h(x) & \varepsilon^k \sigma^t(x) D^2 h(x) \\ \hline \varepsilon^k D(\sigma^t(x)Dh(x)) & \varepsilon^{2k} D^2 h(x) \end{array}\right)$$

and

$$D^2_{H,\varepsilon}h(x) = (D^2_{H,\varepsilon}h(x))^* = \frac{D^2_{H,\varepsilon}h(x) + D^2_{H,\varepsilon}h(x)^t}{2}.$$

The inequality in (4.9) leads us to consider instead of (4.6), (4.7) the equations

$$u_t^\varepsilon(x,t) - \varepsilon\, \mathrm{tr}(D^2_{H,\varepsilon}u^\varepsilon(x,t)) + \varepsilon^{-1} f(u^\varepsilon) = 0 \tag{4.10}$$

and

$$u_t^\varepsilon(x,t) - \mathrm{tr}(D^2_{H,\varepsilon}u^\varepsilon(x,t)) + \varepsilon^{-2} f(u^\varepsilon) = 0. \tag{4.11}$$

We prove that the front that describes the asymptotics of (4.10) is governed by

$$u_t(x,t) + c|D_H u(x,t)| = 0.$$

As it regards equation (4.11) we show, without adding any assumptions on the vector fields $\sigma_1, \ldots \sigma_m$, that the asymptotic behavior of the solutions of (4.11) generates a front at time $t = 0$. To prove that that this front evolves according to the ”degenerate mean curvature equation”

$$u_t(x,t) - \mathrm{tr}\left[\left(I - \frac{D_H u(x,t)}{|\, D_H u(x,t)|} \otimes \frac{D_H u(x,t)}{|\, D_H u(x,t)|}\right) D^2_H u(x,t)\right] = 0, \tag{4.12}$$

we need to restrict to the Carnot group of step two.

To conclude we just remark that, for any $h \in C^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $\varepsilon > 0$,

$$| D_{H,\varepsilon}h(x)|^2 = | D_H h(x)|^2 + \varepsilon^{2k}| Dh(x)|^2 \geq \varepsilon^{2k}| Dh(x)|^2,$$

$$| D_{H,\varepsilon}h(x)| \longrightarrow | D_H h(x)|, \quad \text{as } \varepsilon \to 0^+$$

and

$$\{x \in \mathbb{R}^n : D_H h(x) = 0\} \supsetneq \{x \in \mathbb{R}^n : Dh(x) = 0\} = \{x \in \mathbb{R}^n : D_{H,\varepsilon}h(x) = 0\}$$

Therefore in the study of the evolution of a front $t \longmapsto \{\phi(\cdot, t) = 0\}$ there can exist some point $x \in \{\phi(\cdot, t) = 0\}$ in which the normal vector is well-defined (i.e $D\phi(x, t) \neq 0$) but $D_H\phi(x, t) = 0$. Moreover in these points it holds

$$0 \neq D_{H,\varepsilon}\phi(x, t) \xrightarrow[\varepsilon \to 0^+]{} 0$$

and so, when we study of the asymptotic behavior of the solutions of the equations (4.10) and (4.11), we have to choose the exponent k that appears in the Riemannian approximation of A in a suitable way in order to obtain the right rate of convergence.

4.0.3 The traveling wave equation

Thanks to the properties of f in (4.5) it can be shown (see for example [2] and [30]) that there exists a unique pair $(q(\cdot), c)$, solution of the traveling wave equation

$$\ddot{q} + c\dot{q} = f(q) \tag{4.13}$$

with

$$\lim_{r \to \pm\infty} q(r) = m_\pm, \qquad q(0) = m_o.$$

Moreover we assume that q satisfies the following properties

$$\begin{aligned} &\sup_{r \in \mathbb{R}} [(1 + |r|)\dot{q}(r) + (|r| + |r|^2)| \ddot{q}(r)|] < +\infty \\ &\exists\, a, b > 0 \text{ such that } \begin{array}{l} q(r) \geq m_+ - ae^{-br} \text{ as } r \to +\infty, \\ q(r) \leq m_- + ae^{-br} \text{ as } r \to -\infty, \end{array} \end{aligned} \tag{4.14}$$

and

$$\text{there exists a constant } N > 0 \text{ such that } |\ddot{q}| \leq N\dot{q}. \tag{4.15}$$

A typical example for the function f is

$$f(q) = 2(q - m_-)(q - m_o)(q - m_+); \tag{4.16}$$

in this case we have

$$c = 2m_o - m_+ - m_-, \qquad q(r) = m_- + \frac{m_+ - m_-}{1 + e^{-(m_+ - m_-)(r+\bar{r})}},$$

where $\bar{r} = \dfrac{1}{m_+ - m_-} \ln\big(\dfrac{m_o - m_-}{m_+ - m_o}\big)$. Moreover the constant N in (4.15) will be smaller than $m_+ - m_-$.

If we linearize around q the equation satisfied by the traveling wave we obtain

$$\ddot{p} + c\dot{p} = f(p) = f(q) + f'(q)(p - q)$$

and so, since q satisfies (4.13),

$$\ddot{p} - \ddot{q} + c(\dot{p} - \dot{q}) = f'(q)(p - q).$$

This leads us to consider the linear operator $\mathcal{A} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$\mathcal{A}p := -\ddot{p} - c\dot{p} + f'(q)p, \qquad p \in L^2(\mathbb{R}),$$

and, for suitable functions $\chi \in L^2(\mathbb{R})$, the inhomogeneous equation

$$\mathcal{A}p = \chi. \tag{4.17}$$

Some simple computations show that if $c = 0$ the operator $\mathcal{A}$ is self-adjoint and $\dot{q} \in \ker \mathcal{A} = \ker \mathcal{A}^*$, while if $c \neq 0$ then $\dot{q} \in \ker \mathcal{A}$ and $\dot{\tilde{q}} \in \ker \mathcal{A}^*$ with $\tilde{q}(s) = q(-s)$ for any $s \in \mathbb{R}$. In the sequel we will assume that

$$\ker \mathcal{A} = \ker \mathcal{A}^* = \dot{q}\mathbb{R}$$

if $c = 0$, and

$$\ker \mathcal{A} = \dot{q}\mathbb{R}, \qquad \ker \mathcal{A}^* = \dot{\tilde{q}}\mathbb{R}$$

if $c \neq 0$.

Moreover we also assume that for any $\chi \in L^2(\mathbb{R})$ such that $\chi \in (\ker \mathcal{A}^*)^\perp$, i.e.

$$\int_{-\infty}^{+\infty} \chi(s)\dot{q}(s)ds = 0, \qquad \text{if } c = 0, \tag{4.18}$$

or,

$$\int_{-\infty}^{+\infty} \chi(s)\dot{q}(-s)ds = 0, \qquad \text{if } c \neq 0, \tag{4.19}$$

there exists a unique $p \in C^2(\mathbb{R}) \cap H^1(\mathbb{R})$, solution of the equation (4.17) and such that

$$p(s) \to 0, \qquad \text{as } |\, s| \to +\infty \tag{4.20}$$

and

$$\sup_{s\in\mathbb{R}}(|\,\dot{p}(s)| + (1 + |\,s|)|\,\ddot{p}(s)|) < +\infty. \tag{4.21}$$

In the special case

$$f(q) = 2q(q^2 - 1)$$

the traveling wave equation (4.13) becomes

$$\ddot{q} = f(q)$$

and its solution q is

$$q(s) = \tanh(s).$$

Moreover in this special case the solution p of the inhomogeneous equation (4.17) has the form

$$p(s) = \dot{q}(s)\int_{-\infty}^{s} \frac{1}{(\dot{q}(\tau))^2}\Big(\int_{-\infty}^{\tau} \chi(\eta)\dot{q}(\eta)d\eta\Big)d\tau.$$

4.1 The first asymptotic problem

As we said at the beginning of the chapter, the first asymptotic problem we will consider is the Cauchy problem for the equation (4.10). We will prove that, if

$$c = 2m_0 - m_+ - m_- \neq 0,$$

then the evolution of the front associated with this problem has a first order normal velocity and it is governed by the following geometric pde

$$u_t(x,t) + F(x, Du(x,t)) = 0, \qquad (x,t) \in \mathbb{R}^n \times (0, +\infty), \tag{4.22}$$

where $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$F(x,p) = c|\,\sigma^t(x)p|.$$

It is well known that the Cauchy problem for this equation is well-posed. In fact since we supposed that $\sigma = (\sigma_{ik})_{i,k} \in C^2(\mathbb{R}^n, \mathbb{R}^{n\times m}) \cap W^{2,\infty}(\mathbb{R}^n, \mathbb{R}^{n\times m})$ we get that there exists a positive constant L_σ so that

$$| F(x,p) - F(y,p)| \leq cL_\sigma | x-y|| p|.$$

This assures us that a a comparison result **(CP)** for sub and supersolution of the equation (4.22) holds (see for example [20]). As it regards the existence of a viscosity solution we observe that equation (4.22) can be rewritten as

$$\max_{b\in B}\{-(u_t(x,t), Du(x,t)^t)\cdot(-1, c\sigma(x)b)\} = 0, \qquad (x,t)\in\mathbb{R}^n\times(0,+\infty),$$

if $c > 0$, or

$$\min_{b\in B}\{-(u_t(x,t), Du(x,t)^t)\cdot(-1, c\sigma(x)b)\} = 0, \qquad (x,t)\in\mathbb{R}^n\times(0,+\infty),$$

if $c < 0$, where the control set B is the set $\{b \in \mathbb{R}^m : | b| \leq 1\}$. This leads us to consider, for any control function $\beta \in \mathcal{B} = \{\beta : [0,+\infty) \to B$ measurable function$\}$ and for any $x \in \mathbb{R}^n$, the Caratheodory solution $y_x(\cdot,\beta)$ of the dynamical system

$$\begin{cases} \dot{y}(s) = c\sigma(y(s))\beta(s) \\ y(0) = x. \end{cases}$$

By regularity assumptions on σ and classical results in control theory (see for example [3] Proposition 3.12 ch.IV) one can prove that the value function $v : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$ defined as

$$v(x,t) = \inf_{\beta\in\mathcal{B}} u_0(y_x(t;\beta)), \tag{4.23}$$

if $c > 0$, and

$$v(x,t) = \sup_{\beta\in\mathcal{B}} u_0(y_x(t;\beta)), \tag{4.24}$$

if $c < 0$, is (the unique) continuous viscosity solution of (4.22) that satisfies the initial condition $v(\cdot,0) = u_0(\cdot) \in UC(\mathbb{R}^n)$.

Now we are ready to study the asymptotic behavior of the solution of the Cauchy problem for (4.10).

Theorem 4.1.1. Assume that the matrix map A and the function f satisfy (4.1), (4.2) (4.5), (4.14) and (4.15). Let u^ε be the unique solution of

$$\begin{cases} \text{(i)} \quad u^\varepsilon_t(x,t) - \varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}u^\varepsilon(x,t)) + \varepsilon^{-1} f^\varepsilon(u^\varepsilon) = 0 \quad (x,t)\in\mathbb{R}^n\times(0,+\infty) \\ \text{(ii)} \quad u^\varepsilon(x,0) = g(x), \quad x\in\mathbb{R}^n, \end{cases} \tag{4.25}$$

where $g : \mathbb{R}^n \to [m_-, m_+]$ is a continuous function such that the sets $\Gamma_o = \{x : g(x) = m_o\}$, $\Omega_o^+ = \{x : g(x) > m_0\}$, $\Omega_o^- = \{x : g(x) < m_0\}$ are nonempty and mutually disjoint subsets of $\mathbb{R}^n$. Moreover we assume that the exponent k in (4.8) satisfies $0 < k < 1$. Then,

$$u^\varepsilon(x,t) \longrightarrow \begin{cases} m_+ & \text{in} \quad \{u > 0\}, \\ m_- & \text{in} \quad \{u < 0\}, \end{cases}$$

locally uniformly as $\varepsilon \to 0$, where u is the unique viscosity solution of

$$\begin{cases} \text{(i)} & u_t(x,t) + c|\sigma^t(x) Du(x,t)| = 0, \quad (x,t) \in \mathbb{R}^n \times (0, +\infty), \\ \text{(ii)} & u(x,0) = d_o(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{4.26}$$

and d_o is the signed distance to Γ_o which is positive in Ω_o^+ and negative in Ω_o^-. If in addition the no-interior condition (2.7) for the set $\{u = 0\}$ holds, then, as $\varepsilon \to 0$,

$$u^\varepsilon(x,t) \longrightarrow \begin{cases} m_+ & \text{in} \quad \{u > 0\}, \\ m_- & \text{in} \quad \overline{\{u > 0\}}^c, \end{cases}$$

locally uniformly.

Remark 4.1.2. For what we said at the beginning of the section we have that the function u is just the value function v defined in (4.23) if $c > 0$, or in (4.24) if $c < 0$, with u_0 replaced by d_o.

Before proving this theorem we stop for some preliminary remarks. In fact in our proof it will be necessary to construct, for any $\varepsilon > 0$, some suitable sub and supersolutions of the equation (4.25-i) in a neighborhood of a front $\{\phi(\cdot,t) = 0\}$ where $D\phi \neq 0$. Following the idea of Barles, Soner and Souganidis in [7] we look for subsolutions (supersolutions) Φ^ε of the type $\Phi^\varepsilon(x,t) = q(\dfrac{z^\varepsilon(x,t)}{\varepsilon})$ where q is the solution of (4.13) and z^ε is a function to be chosen in a right way. In order to understand the conditions that z^ε has to satisfy we ask Φ^ε to be a subsolution of (4.25-i) in a neighborhood of $\{\phi = 0\}$ and we obtain

$$\frac{\dot q}{\varepsilon}[z_t - \varepsilon \operatorname{tr}(D^2_{H,\varepsilon} z) + c] + \frac{\ddot q}{\varepsilon^2}[1 - |\sigma^t_\varepsilon(x) Dz|^2] \leq 0.$$

Obviously this last inequality is satisfied if it holds

$$\begin{gathered} z_t(x,t) - \varepsilon \operatorname{tr}(D^2_{H,\varepsilon} z(x,t)) + c \leq 0 \\ |D_{H,\varepsilon} z(x,t)| = |\sigma^t_\varepsilon(x) Dz(x,t)| = 1. \end{gathered} \tag{4.27}$$

Let's observe the second condition required in (4.27). It means that, for any $t \in (0, +\infty)$ the function $z^\varepsilon(\cdot, t)$ has to be a solution of the equation

$$A^\varepsilon(x)Dp(x) \cdot Dp(x) = |\sigma^t_\varepsilon(x)Dp(x)|^2 = |\sigma^t(x)Dp(x)|^2 + \varepsilon^{2k}|Dp(x)|^2 = 1 \tag{4.28}$$

in a neighborhood of $\{\phi = 0\}$. Since $\{\phi = 0\}$ is compact, $D\phi \neq 0$ on $\{\phi = 0\}$ and

$$A^\varepsilon(x)p \cdot p = A(x)p \cdot p + \varepsilon^{2k}|\,p|^2 \geq \varepsilon^{2k}|\,p|^2 \qquad \text{for any } x \in \mathbb{R}^n,$$

we can apply the method of characteristic to get the existence of a classical solution ρ_ε of (4.28) having the same sign of ϕ. By the definition of A_ε, $A_\varepsilon \to A$ uniformly as $\varepsilon \to 0^+$ and thus one could expect that the sequence $(\rho_\varepsilon)_\varepsilon$ converges (in some sense) to a solution of the equation

$$A(x)Dp(x) \cdot Dp(x) = |\sigma^t_H(x)Dp(x)|^2 = 1.$$

Since A can be degenerate we are not able to claim the existence of a classical solution of this last equation providing a problem in the study of the behavior of ρ_ε for small ε. This suggests us to modify slightly the idea of Barles and Souganidis in [8] and to consider as z^ε not ρ_ε but a function of type $\phi|\,D_{H,\varepsilon}\phi|^{-1}$. This choice of z^ε create additional terms in the computations; in fact

$$D_{H,\varepsilon}\big(\phi|\,D_{H,\varepsilon}\phi|^{-1}\big) = \frac{D_{H,\varepsilon}\phi}{|\,D_{H,\varepsilon}\phi|} + \phi D_{H,\varepsilon}\big(|\,D_{H\varepsilon}\phi|^{-1}\big) = \pm 1 + \phi D_{H,\varepsilon}\big(|\,D_{H\varepsilon}\phi|^{-1}\big).$$

To estimate these additional terms for small ε it will be crucial the choice of the exponent k.

Now we are ready to prove Theorem 4.1.1.

Proof. Following the abstract method described in Chapter 2 we define two families of open sets of $\mathbb{R}^n$, $(\Omega^1_t)_{t\in(0,T)}$ and $(\Omega^2_t)_{t\in(0,T)}$ as in (2.14), (2.15) and (2.17) with $a_\varepsilon, a = m_-$, $b_\varepsilon, b = m_+$ and $\tau = 0$. We observe that by the maximum principle $m_- \leq u^\varepsilon \leq m_+$. The proof will be divided into the usual three steps, initialization, propagation and conclusion.

First step: initialization. We want to show that $\Omega^+_0 = \{d_o > 0\} \subseteq \Omega^1_0$ and $\Omega^-_0 = \{d_o < 0\} \subseteq \Omega^2_0$. Since the proofs of these two inclusions are similar we only show the first one. Consider $\hat{x} \in \{d_o > 0\}$ and find $r, \sigma > 0$ so that $g(x) \geq m_o + \sigma$ for any $x \in B(\hat{x}, r)$. This means that

$$u^\varepsilon(x, 0) = g(x) \geq (m_o + \sigma)\mathbb{1}_{B(\hat{x},r)}(x) + m_-\mathbb{1}_{B(\hat{x},r)^c}(x). \tag{4.29}$$

Now we introduce the function $\Phi : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ defined as

$$\Phi(x, t) = r^2 - |x - \hat{x}|^2 - Ct, \tag{4.30}$$

with $C > 0$ a suitable constant that will be chosen later. Using the following two lemmas it is possible to conclude that $\hat{x} \in \Omega_0^1$ (We omit this proof since it is close to the one in Theorem 3.1.1).

Lemma 4.1.3. Under the assumption of Theorem 4.1.1 we have that for any $\beta > 0$ there exist $\tau = \tau(\beta) > 0$ and $\bar{\varepsilon} = \bar{\varepsilon}(\beta)$ such that, for all $0 < \varepsilon \leq \bar{\varepsilon}$, we have

$$u^\varepsilon(x, t_\varepsilon) \geq (m_+ - \beta)\mathbb{1}_{\{\Phi(\cdot,0)\geq\beta\}}(x) + m_-\mathbb{1}_{\{\Phi(\cdot,0)<\beta\}}(x), \quad x \in \mathbb{R}^n,$$

where $t_\varepsilon = \tau\varepsilon$.

Lemma 4.1.4. There exist $\bar{h} = \bar{h}(r, \hat{x})$, $\bar{\beta} = \bar{\beta}(r, \hat{x}) > 0$ independent of ε such that if $\beta \leq \bar{\beta}$ and $\varepsilon \leq \bar{\varepsilon}(\beta) \wedge 1$, then there exists a subsolution $\omega^{\varepsilon,\beta}$ of (4.25-i) in $\mathbb{R}^n \times (0, \bar{h})$ that satisfies

$$\omega^{\varepsilon,\beta}(x, 0) \leq (m_+ - \beta)\mathbb{1}_{\{\Phi(\cdot,0)\geq\beta\}}(x) + m_-\mathbb{1}_{\{\Phi(\cdot,0)<\beta\}}(x), \quad x \in \mathbb{R}^n.$$

If moreover $(x, t) \in B(\hat{x}, r) \times (0, \bar{h})$ and $\Phi(x, t) > 3\beta$, then

$$\liminf_{\varepsilon\to 0^+}{}_* \, \omega^{\varepsilon,\beta}(x, t) \geq m_+ - 2\beta.$$

Proof of Lemma 4.1.3. Since the proof of this first lemma is very close to the ones in Chapter 3 and in [5] we only give a sketch. The main idea of this proof is that for ε small enough and for short time the diffusion term in (4.25-i) is negligible, see also Chen [16, 17]. Let $\chi = \chi(\tau, \xi) \in C^2([0, +\infty) \times \mathbb{R})$ be the solution of

$$\begin{cases} \dot{\chi}(\tau, \xi) + f(\chi(\tau, \xi)) = 0, & \tau > 0, \\ \chi(0, \xi) = \xi. \end{cases}$$

With some simple computations one can see that χ satisfies the following properties

$$\chi_\xi(\tau, \xi) > 0, \quad \text{in } [0, +\infty) \times \mathbb{R}; \tag{$\chi1$}$$

there exists $\tau_o = \tau_o(\beta) > 0$ such that, for all $\tau \geq \tau_o$

$$\chi(\tau, \xi) \geq m_+ - \beta, \quad \forall\, \xi \geq m_o + \frac{\sigma}{2}; \tag{$\chi2$}$$

finally, since for any $C > \max\{|m_- - m_0|, |m_+ - m_0|\}$ we have that $\chi(\tau, \xi) \in [m_0 - C, m_0 + C]$ for all $\xi \in [-C, C]$, $\tau \geq 0$, it also holds that there exists a constant $M_{C,\tau} > 0$ such that, for any $\xi \in [-C, C]$ and ε small enough,

$$|\chi_{\xi\xi}(\tau, \xi)| \leq M_{C,\tau}\chi_\xi(\tau, \xi). \tag{$\chi3$}$$

Now we can define a function $\underline{u}^\varepsilon$ in $\mathbb{R}^n \times [0,T]$ as

$$\underline{u}^\varepsilon(x,t) = \chi\Big(\frac{t}{\varepsilon}, \psi(\Phi(x,0)) - Kt\Big),$$

where K is a constant to be chosen later and ψ is a nondecreasing smooth function in $\mathbb{R}$ such that

$$\psi(z) = \begin{cases} m_- & \text{if } z \le 0, \\ m_o + \sigma & \text{if } z \ge \beta \wedge \dfrac{\sigma}{2}. \end{cases}$$

With some simple computation it can be easily shown that for a suitable choice of the constant K and ε small enough $\underline{u}^\varepsilon$ is a subsolution of (4.25-i) in $\mathbb{R}^n \times (0, \tau_o\varepsilon)$. Indeed, by ($\chi 1$) and ($\chi 3$) we obtain

$$\begin{aligned} \underline{u}^\varepsilon_t - \varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}\underline{u}^\varepsilon) + \frac{f(\underline{u}^\varepsilon)}{\varepsilon} &= -K\chi_\xi - \varepsilon\big[(\chi_{\xi\xi}(\psi')^2 + \chi_\xi\psi'')|\,D_{H,\varepsilon}\Phi|^2 + \chi_\xi\psi'\,\mathrm{tr}(A^\varepsilon(x)D^2\Phi)\big] \\ &\qquad -\varepsilon\chi_\xi\psi'\,\mathrm{tr}(\sigma^t(x)D\sigma^t(x)D\Phi) \\ &\le \chi_\xi(-K + O(\varepsilon)) \le 0 \end{aligned}$$

for ε small enough. Moreover, since $\underline{u}^\varepsilon(\cdot,0) = \psi(\Phi(\cdot,0)) \le u^\varepsilon(\cdot,0)$ we can apply the maximum principle in order to obtain $\underline{u}^\varepsilon(x,t) \le u^\varepsilon(x,t)$ for any $(x,t) \in \mathbb{R}^n \times [0, \tau_o\varepsilon]$. In particular

$$\begin{aligned} u^\varepsilon(x,\tau_o\varepsilon) &\ge \underline{u}^\varepsilon(x,\tau_o\varepsilon) = \chi(\tau_o, \psi(\Phi(x,0)) - K\tau_o\varepsilon) \\ &\ge \chi(\tau_o, m_o + \sigma - K\tau_o\varepsilon) \qquad \text{if } \Phi(x,0) \ge \beta. \end{aligned}$$

From this last inequality and property ($\chi 2$) of χ we get $u^\varepsilon(x,\tau_o\varepsilon) \ge m_+ - \beta$, for ε small enough. □

Proof of Lemma 4.1.4. For the proof of this second lemma it will be very useful the formal reasoning we made above. Let Φ be the smooth function defined in (4.30) where now C is a fixed constant that satisfies

$$C \ge (c+1)\sup_{Q_{\gamma,\bar h}} \big(|D_H\Phi|^2 + |D\Phi|^2\big)^{1/2} > 0. \tag{4.31}$$

Since $D\Phi(x,t) \ne 0$ if $\Phi(x,t) = 0$, $t < r^2/C$, there exist γ, $\bar h > 0$ such that $\bar h < r^2/C$ and $D\Phi(x,t) \ne 0$ in the set $Q_{\gamma,\bar h} = \{|\Phi(x,t)| \le \gamma,\ 0 \le t \le \bar h\}$. Obviously this also means that

$$D_{H,\varepsilon}\Phi(x,t) \ne 0 \qquad \text{for any } (x,t) \in Q_{\gamma,\bar h},\ \varepsilon > 0.$$

Using the ideas in [8] and [5] we now construct a subsolution of (4.25-i) by steps.

1. Construction of a strict subsolution of (4.25-i) in the set $Q_{\gamma,\bar h}$.

We define a smooth function v^ε in $Q_{\gamma,\bar h}$ as

$$v^\varepsilon(x,t) = Q\Big(\frac{\Phi(x,t)-2\beta}{\varepsilon}, x, t\Big) - 2\beta,$$

with Q a suitable function in $C^2(\mathbb{R}\times\mathbb{R}^n\times[0,+\infty);\mathbb{R})$ to be chosen later. If we put v^ε inside (4.25-i) we obtain

$$\begin{aligned} v^\varepsilon_t - \varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}v^\varepsilon) + \frac{f(v^\varepsilon)}{\varepsilon} \quad &= \frac{\dot Q\Phi_t}{\varepsilon} + Q_t - \frac{\ddot Q}{\varepsilon}|D_{H,\varepsilon}\Phi|^2 - 2\langle D_{H,\varepsilon}\dot Q, D_{H,\varepsilon}\Phi\rangle - \varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}Q) \\ &-\dot Q\,\mathrm{tr}(D^2_{H,\varepsilon}\Phi) + \frac{f(Q-2\beta)}{\varepsilon} \\ &\le \frac{I_\varepsilon}{\varepsilon} + \frac{II_\varepsilon}{\varepsilon} + III_\varepsilon \end{aligned}$$

where

$$\begin{aligned} I_\varepsilon &= -\ddot Q|D_{H,\varepsilon}\Phi|^2 + f(Q) \\ II_\varepsilon &= \dot Q\big(\Phi_t - \varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}\Phi)\big) - 2\beta f'(Q) + 2\|\, f''_{|[m_-,m_+]}\|_\infty\beta^2 \\ III_\varepsilon &= Q_t - \varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}Q) - 2\langle D_{H,\varepsilon}\dot Q, D_{H,\varepsilon}\Phi\rangle \end{aligned} \tag{4.32}$$

If we set

$$Q(s,x,t) = q\Big(\frac{s}{|D_{H,\varepsilon}\Phi(x,t)|}\Big), \tag{4.33}$$

with q the solution of the traveling wave equation (4.13) we get $I_\varepsilon = c\dot q(\frac{s}{|\,D_{H,\varepsilon}\Phi|})$, with $s = \frac{\Phi-2\beta}{\varepsilon}$,

$$\begin{aligned} II_\varepsilon \quad &= \frac{\dot q}{|D_{H,\varepsilon}\Phi|}[-C + 2\varepsilon\,\mathrm{tr}(A^\varepsilon(x) + \sigma^t(x)D\sigma^t(x)(x-\hat x))] - 2\beta f'(q) + 2\|\, f''_{|[m_-,m_+]}\|_\infty\beta^2 \\ &\le \frac{\dot q}{|D_{H,\varepsilon}\Phi|}(-C + O(\varepsilon)) - 2\beta f'(q) + 2\|\, f''_{|[m_-,m_+]}\|_\infty\beta^2 \end{aligned}$$

and

$$\begin{aligned} III_\varepsilon = 0 - \varepsilon\dot q s\,\mathrm{tr}\left(D^2_{H,\varepsilon}(\frac{1}{|\,D_{H,\varepsilon}\Phi|})\right) - \varepsilon\ddot q s^2\Big|D_{H,\varepsilon}(\frac{1}{|\,D_{H,\varepsilon}\Phi|})\Big|^2 \\ - 2\Big(\frac{\ddot q s}{|\,D_{H,\varepsilon}\Phi|} + \dot q\Big)\langle D_{H,\varepsilon}(\frac{1}{|\,D_{H,\varepsilon}\Phi|}), D_{H,\varepsilon}\Phi\rangle. \end{aligned}$$

With a simple computation we obtain

$$\langle D_{H,\varepsilon}(\frac{1}{|\,D_{H,\varepsilon}\Phi|}), D_{H,\varepsilon}\Phi\rangle = -\frac{\langle D^2_{H,\varepsilon}\Phi D_{H,\varepsilon}\Phi, D_{H,\varepsilon}\Phi\rangle}{|\,D_{H,\varepsilon}\Phi|^3},$$

and we estimate III_ε in the following way

$$III_\varepsilon \le -\varepsilon\dot q s\,\mathrm{tr}\,\big(D^2_{H,\varepsilon}(\frac{1}{|\,D_{H,\varepsilon}\Phi|})\big)+\varepsilon\frac{|\,\ddot q|s^2}{|\,D_{H,\varepsilon}\Phi|^4}\|\,D^2_{H,\varepsilon}\Phi\|^2+2\Big(\frac{|\,\ddot q||\,s|}{|\,D_{H,\varepsilon}\Phi|}+\dot q\Big)\frac{\|\,D^2_{H,\varepsilon}\Phi\|}{|\,D_{H,\varepsilon}\Phi|}.$$

By (4.14), $\dot q|\,s||\,D_{H,\varepsilon}\Phi|^{-1}+|\,\ddot q|(|\,s||\,D_{H,\varepsilon}\Phi|^{-1}+s^2|\,D_{H,\varepsilon}\Phi|^{-2})=O(1)$ and thus

$$\begin{aligned}III_\varepsilon &\le -\varepsilon\dot q s\,\mathrm{tr}\,\big(D^2_{H,\varepsilon}\Big(\frac{1}{|\,D_{H,\varepsilon}\Phi|}\Big)\big)+O(\frac{\varepsilon}{|\,D_{H,\varepsilon}\Phi|^2}+\frac{1}{|\,D_{H,\varepsilon}\Phi|})\\ &\le -\varepsilon\dot q s\,\mathrm{tr}\,\big(D^2_{H,\varepsilon}\Big(\frac{1}{|\,D_{H,\varepsilon}\Phi|}\Big)\big)+O(\varepsilon^{1-2k}+\varepsilon^{-k}).\end{aligned}$$

Since

$$\begin{aligned}D^2_{H,\varepsilon}\Big(\frac{1}{|\,D_{H,\varepsilon}\Phi|}\Big)=&-\frac{(D^2_{H,\varepsilon}\Phi)^2}{|\,D_{H,\varepsilon}\Phi|^3}-\frac{A^\varepsilon(x)D^2(D_{H,\varepsilon}\Phi)\cdot D_{H,\varepsilon}\Phi}{|\,D_{H,\varepsilon}\Phi|^3}\\ &+3\frac{(D^2_{H,\varepsilon}\Phi D_{H,\varepsilon}\Phi)\otimes(D^2_{H,\varepsilon}\Phi D_{H,\varepsilon}\Phi)}{|\,D_{H,\varepsilon}\Phi|^5}-\frac{\sigma^t(x)D\sigma^t(x)D(D_{H,\varepsilon}\Phi)\cdot D_{H,\varepsilon}\Phi}{|\,D_{H,\varepsilon}\Phi|^3},\end{aligned}$$

we get

$$\begin{aligned}\mathrm{tr}\,\big(D^2_{H,\varepsilon}\Big(\frac{1}{|\,D_{H,\varepsilon}\Phi|}\Big)\big)&\le(n+3)\frac{\|\,D^2_{H,\varepsilon}\Phi\|^2}{|\,D_{H,\varepsilon}\Phi|^3}+\frac{\mathrm{tr}(A^\varepsilon(x)\lfloor D^2(D_{H,\varepsilon}\Phi)\rfloor+\lfloor\sigma^t(x)D\sigma^t(x)D(D_{H,\varepsilon}\Phi)\rfloor)}{|\,D_{H,\varepsilon}\Phi|^2}\\ &\le O\Big(\frac{1}{|\,D_{H,\varepsilon}\Phi|^3}+\frac{1}{|\,D_{H,\varepsilon}\Phi|^2}\Big)\end{aligned}$$

where $\|\cdot\|$ denotes a generical matrix norm and $\lfloor D^2(D_{H,\varepsilon}\Phi)\rfloor$, $\lfloor\sigma^tD\sigma^tD(D_{H,\varepsilon}\Phi)\rfloor$ are two quadratic matrices defined as

$$\lfloor D^2(D_{H,\varepsilon}\Phi)\rfloor=(|\,(D_{H,\varepsilon}\Phi)_{x_ix_j}|)_{i,j=1,\dots n}=(\Big[\sum_{l=1}^{n+m}((\sigma^{(l)}_\varepsilon(x)D\Phi(x,t))_{x_ix_j})^2\Big]^{1/2})_{i,j=1,\dots n}$$

$$\begin{aligned}\lfloor\sigma^tD\sigma^tD(D_{H,\varepsilon}\Phi)\rfloor&=(|\,\sigma^{(i)}D\sigma^{(j)}D(D_{H,\varepsilon}\Phi)|)_{i,j=1,\dots m}\\ &=([\sum_{l=1}^{n+m}(\sigma^{(i)}(x)D\sigma^{(j)}(x)D(\sigma^{(l)}_\varepsilon(x)D\Phi(x,t)))^2]^{1/2})_{i,j=1,\dots m}\end{aligned}.$$

Again by (4.14),

$$\varepsilon\dot q s\,\mathrm{tr}\,\big(D^2_{H,\varepsilon}\Big(\frac{1}{|\,D_{H,\varepsilon}\Phi|}\Big)\big)=O(\varepsilon^{1-2k}),$$

and thus, since $0<k<1$, we obtain $III_\varepsilon=O(\varepsilon^{-k})$ and $\varepsilon III_\varepsilon=o_\varepsilon(1)$. Combining all the estimates obtained above and using the assumption on C in (4.31) we get

$$\begin{aligned} v_t^\varepsilon - \varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon} v^\varepsilon) + \frac{f(v^\varepsilon)}{\varepsilon} &\le \frac{1}{\varepsilon}\Big[\dot q\Big(c - \frac{C}{|D_{H,\varepsilon}\Phi|} + o_\varepsilon(1)\Big) - 2\beta f'(q) + 2\beta^2 \| f''_{|[m_-,m_+]}\|_\infty + o_\varepsilon(1)\Big] \\ &\le \frac{1}{\varepsilon}\Big[\dot q(-1 + o_\varepsilon(1)) - 2\beta f'(q) + 2\beta^2\| f''_{|[m_-,m_+]}\|_\infty + o_\varepsilon(1)\Big], \end{aligned}$$

and thus, as $\varepsilon \to 0^+$,

$$v_t^\varepsilon - \varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon} v^\varepsilon) + \frac{f(v^\varepsilon)}{\varepsilon} \le \frac{1}{\varepsilon}\Big[-\frac{\dot q}{2} - 2\beta f'(q) + 2\beta^2\| f''_{|[m_-,m_+]}\|_\infty + o_\varepsilon(1)\Big]. \tag{4.34}$$

To prove that v^ε is a subsolution of (4.25-i) it remains to see that the right hand side of (4.34) is non positive. To do this we recall that $f'(m_\pm) \ge d > 0$ and $q(r) \to m_\pm$ if $r \to \pm\infty$. This means that there exists an $M > 0$ such that

$$f'(q(r)) \ge \frac{d}{2}, \quad \text{for any } |r| \ge M;$$

moreover we can choose β small enough in order to get

$$4\beta \| f''_{|[m_-,m_+]}\|_\infty \le d.$$

Therefore if $\dfrac{|\Phi - 2\beta|}{\varepsilon|D_{H,\varepsilon}\Phi|} \ge M$, we have that

$$v_t^\varepsilon - \varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon} v^\varepsilon) + \frac{f(v^\varepsilon)}{\varepsilon} \le \frac{1}{\varepsilon}\Big[-\frac{\dot q}{2} - \frac{\beta d}{2} + o_\varepsilon(1)\Big] \le \frac{1}{\varepsilon}\Big[-\frac{\beta d}{2} + o_\varepsilon(1)\Big] \le -\frac{\beta d}{4\varepsilon} < 0$$

for ε small enough. Now we consider the case $\dfrac{|\Phi - 2\beta|}{\varepsilon|D_{H,\varepsilon}\Phi|} < M$; if we denote with K a strictly positive constant (which depends by M) so that $\dot q(r) \ge K$ for any $|r| \le M$ we get that there exists a $\mu > 0$ so that, for β small compared to K,

$$v_t^\varepsilon - \varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon} v^\varepsilon) + \frac{f(v^\varepsilon)}{\varepsilon} \le \frac{1}{\varepsilon}[-\frac{K}{2} + 2\beta\| f'\| + 2\beta^2\| f''\| + o_\varepsilon(1)] \le -\frac{\mu}{\varepsilon} < 0.$$

for β small compared with K and ε small enough.

2. Construction of a subsolution of (4.25-i) in the set $\{(x,t) \in \mathbb{R}^n \times [0,\bar h] : \Phi(x,t) \le \gamma\}$. Once we have proved that v^ε is a strict subsolution of (4.25-i) in $Q_{\gamma,\bar h}$ we define, for each $(x,t) \in \{(x,t) \in \mathbb{R}^n \times [0,\bar h] : \Phi(x,t) \le \gamma\}$,

$$\bar v^\varepsilon(x,t) = \begin{cases} \sup(v^\varepsilon(x,t), m_-) & \text{if } \ -\gamma < \Phi(x,t) \le \gamma, \\ m_- & \text{if } \ \Phi(x,t) \le -\gamma. \end{cases}$$

$\bar v^\varepsilon$ is a continuous viscosity subsolution of (4.25-i) in $\{(x,t)\in\mathbb{R}^n\times[0,\bar h]:\Phi(x,t)\le\gamma\}$, for ε sufficiently small. This is obvious in the set $\{|\Phi|\le\gamma\}$ since $\bar v^\varepsilon$ is the supremum of two subsolutions. Consider a point (x,t) such that $\Phi(x,t)\le-\gamma/2$; by properties (4.14) we have that

$$v^\varepsilon(x,t)\le q\Big(-\frac{\gamma+4\beta}{2\varepsilon|\,D_{H,\varepsilon}\Phi|}\Big)-2\beta\le m_-+ae^{-\frac{b(\gamma+4\beta)}{2\varepsilon|\,D_{H,\varepsilon}\Phi|}}-2\beta\le m_-$$

and $\bar v^\varepsilon(x,t)=m_-$. Therefore $\bar v^\varepsilon$ is a subsolution of (4.25-i) in $\{(x,t)\in\mathbb{R}^n\times[0,\bar h]:\Phi(x,t)\le\gamma\}$.

3. Construction of the subsolution $\omega^{\varepsilon,\beta}$ of (4.25-i) in $\mathbb{R}^n\times[0,\bar h]$.

Finally we define our function $\omega^{\varepsilon,\beta}:\mathbb{R}^n\times[0,\bar h]\to\mathbb{R}$ as

$$\omega^{\varepsilon,\beta}(x,t)=\begin{cases}\psi(\Phi(x,t))\bar v^\varepsilon(x,t)+(1-\psi(\Phi(x,t)))(m_+-\beta) & \text{if } \Phi(x,t)<\gamma,\\ m_+-\beta & \text{if } \Phi(x,t)\ge\gamma,\end{cases}$$

where $\psi:\mathbb{R}\to\mathbb{R}$ is a smooth function such that $\psi'\le 0$ in $\mathbb{R}$, $\psi=1$ in $(-\infty,\gamma/2]$, $0<\psi<1$ in $(\gamma/2,3\gamma/4)$ and $\psi=0$ in $[3\gamma/4,+\infty)$. We show that $\omega^{\varepsilon,\beta}$ is a subsolution of (4.25-i) in $\mathbb{R}^n\times[0,\bar h]$ and for ε small. The only subset of $\mathbb{R}^n\times(0,\bar h)$ in which we have to check that $\omega^{\varepsilon,\beta}$ is a subsolution is $\{(x,t)\in\mathbb{R}^n\times(0,\bar h):\gamma/2\le\Phi(x,t)\le 3\gamma/4\}$. If we take β so that $2\beta<\gamma/4$ we have that $\Phi(x,t)-2\beta>\gamma/4$, $v^\varepsilon(x,t)\ge m_+-ae^{-\frac{b\gamma}{4\varepsilon|\,D_{H,\varepsilon}\Phi|}}-2\beta$ and $\bar v^\varepsilon(x,t)=v^\varepsilon(x,t)$. We obtain

$$\begin{aligned}\omega^{\varepsilon,\beta}_t-\varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}\omega^{\varepsilon,\beta})+\frac{f(\omega^{\varepsilon,\beta})}{\varepsilon}&\le[\psi'(\Phi_t-\varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}\Phi))-\varepsilon\psi''|D_{H,\varepsilon}\Phi|^2](v^\varepsilon-(m_+-\beta))\\&\quad+\psi(v^\varepsilon_t-\varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}v^\varepsilon))-2\varepsilon\psi'\langle D_{H,\varepsilon}\Phi,D_{H,\varepsilon}v^\varepsilon\rangle+\frac{f(\omega^{\varepsilon,\beta})}{\varepsilon}\end{aligned}\tag{4.35}$$

Since by definition, $v^\varepsilon-(m_+-\beta)\le-\beta$, f is convex in a neighborhood of m_+ and v^ε is a strict subsolution of (4.25-i) in $Q_{\gamma,\bar h}$, the inequality in (4.35) becomes

$$\begin{aligned}\omega^{\varepsilon,\beta}_t-\varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}\omega^{\varepsilon,\beta})+\frac{f^\varepsilon(\omega^{\varepsilon,\beta})}{\varepsilon}&\le-C\psi'(v^\varepsilon-(m_+-\beta))+\psi\Big(v^\varepsilon_t-\varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}v^\varepsilon)+\frac{f(v^\varepsilon)}{\varepsilon}\Big)\\&\qquad+O(\varepsilon)+(1-\psi)\frac{f(m_+-\beta)}{\varepsilon}\\&\le-\frac{\psi\bar\mu(\beta)}{\varepsilon}+O(\varepsilon)+(1-\psi)\frac{f(m_+-\beta)}{\varepsilon}\\&=-\frac{1}{\varepsilon}\Big(\psi\bar\mu(\beta)+(1-\psi)(-f(m_+-\beta))\Big)+O(\varepsilon)\\&=-\frac{\hat\alpha_\beta}{\varepsilon}+O(\varepsilon)\le 0,\quad\text{ for }\beta\text{ and then }\varepsilon\text{ small enough}.\end{aligned}$$

In the last chain of inequalities $\bar\mu(\beta)=\mu\wedge\beta d/4$ and in the last line we have used the fact that the term inside the bracket is strictly posive since it is a convex combination of strictly positive terms.

4. Proof of the estimates for $\omega^{\varepsilon,\beta}(x,0)$, $x\in\mathbb{R}^n$. Now we observe that if $(x,t)\in Q_{\gamma,\bar h}$ satisfies

$\Phi(x,t) < \beta$, then

$$v^\varepsilon(x,t) \le q\Big(\frac{-\beta}{\varepsilon|\,D_{H,\varepsilon}\Phi(x,t)|}\Big) - 2\beta \le m_- + ae^{-\frac{b\beta}{\varepsilon|\,D_{H,\varepsilon}\Phi|}} - 2\beta \le m_-$$

for ε small enough. This means that for any $(x,t) \in Q_{\gamma,\bar{h}}$ it holds

$$v^\varepsilon(x,t) \le (m_+ - \beta)\mathbb{1}_{\{\Phi\ge\beta\}}(x,t) + m_-\mathbb{1}_{\{\Phi<\beta\}}(x,t),$$

for ε small enough. By definition of $\bar{v}^\varepsilon$ and of $\omega^{\varepsilon,\beta}$ this inequality still holds for $\bar{v}^\varepsilon$ and $\omega^{\varepsilon,\beta}$ in their domain of definition. If we take $t = 0$ we have proved the second part of the lemma.

5. Finally we just remark that, with a reasoning similar to the one in point 4., one can prove that if $(x,t) \in B(\hat{x}, r) \times (0, \bar{h})$ and $\gamma \ge \Phi(x,t) > 3\beta$, then

$$v^\varepsilon(x,t) \ge q(\frac{\beta}{\varepsilon|\,D_{H,\varepsilon}\Phi|}) - 2\beta \ge m_+ - ae^{-\frac{b\beta}{\varepsilon|\,D_{H,\varepsilon}\Phi|}} - 2\beta.$$

Hence $\liminf_{\varepsilon\to0^+}{}_* \,\omega^{\varepsilon,\beta}(x,t) \ge m_+ - 2\beta$ for any $(x,t) \in Q_{\gamma,\bar{h}}$. $\square$

Second step: propagation. In this step we want to show that $(\Omega^1_t)_{t\in(0,T)}$ and $((\Omega^2_t)^c)_{t\in(0,T)}$ are respectively super and subflow with normal velocity $-F$ with F defined as

$$F(x,p) = c|\sigma^t(x)p|^2.$$

Since the two proofs are similar we'll do only the one for $(\Omega^1_t)_{t\in(0,T)}$.
Let $x_0 \in \mathbb{R}^n$, $t \in (0,T)$, $r > 0$, $h > 0$ so that $t + h < T$. Suppose that $\phi : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is a smooth function such that, for a suitable $\tilde{C} > 0$,

(i) $\phi_t(x,s) + c|\sigma^t(x)D\phi(x,s)| \le -\tilde{C} < 0$, for all $(x,s) \in B(x_0,r] \times [t,t+h]$,

(ii) for any $s \in [t,t+h]$, $\{x \in B(x_0,r] : \phi(x,s) = 0\} \ne \emptyset$ and

$$|D\phi(x,s)| \ne 0 \text{ on } \{(x,s) \in B(x_0,r] \times [t,t+h] : \phi(x,s) = 0\},$$

(iii) $\{x \in B(x_0,r] : \phi(x,t) \ge 0\} \subset \Omega^1_t$,

(iv) for all $s \in [t,t+h]$, $\{x \in \partial B(x_0,r] : \phi(x,s) \ge 0\} \subset \Omega^1_s$.

We have to show that for every $s \in (t,t+h)$,

$$\{x \in B(x_0,r] : \phi(x,s) > 0\} \subset \Omega^1_s.$$

Using the assumptions and the definition of $(\Omega^1_t)_{t\in(0,T)}$ this is equivalent to prove that for any $x \in$

$B(x_0, r)$, $s \in (t, t+h)$ such that $\phi(x, s) > 0$, we have

$$\liminf_{\varepsilon \to 0^+}{}_* \, u^\varepsilon(y, \tau) \geq m_+ \tag{4.36}$$

for (y, τ) in a neighborhood of (x, s) where, for any $\varepsilon > 0$, u^ε is the solution of the Cauchy problem (4.25). This proof proceeds like the one of the first step with the difference that here we have to construct a subsolution of (4.25-i) only in the ball $B(x_0, r)$ and not in the whole space $\mathbb{R}^n$. In fact to prove this result it is enough to prove the following lemma which plays the role of Lemma 4.1.4 in the first step.

Lemma 4.1.5. Let ϕ be a smooth function as above. There exists $\bar{\beta}$ small enough such that, if $\beta \leq \bar{\beta}$ and $\varepsilon \leq \bar{\varepsilon}(\beta)$ then there is a viscosity subsolution $\omega^{\varepsilon,\beta}$ of (4.25-i) in $B(x_0, r) \times (t, t+h)$ that satisfies,

1. $\omega^{\varepsilon,\beta}(x, t) \leq (m_+ - \beta)\mathbb{1}_{\{\phi(\cdot,t) \geq \beta\}}(x) + m_- \mathbb{1}_{\{\phi(\cdot,t) < \beta\}}(x)$, for all $x \in B(x_0, r]$,
2. $\omega^{\varepsilon,\beta}(x, s) \leq (m_+ - \beta)\mathbb{1}_{\{\phi(\cdot,s) \geq \beta\}}(x) + m_- \mathbb{1}_{\{\phi(\cdot,s) < \beta\}}(x)$, for all $x \in \partial B(x_0, r]$, $s \in [t, t+h]$,
3. if $(x, s) \in B(x_0, r] \times [t, t+h]$ satisfies $\phi(x, s) > 3\beta$, then

$$\liminf_{\varepsilon \to 0^+}{}_* \, \omega^{\varepsilon,\beta}(x, s) \geq m_+ - 3\beta.$$

We assume for the moment that Lemma 4.1.5 holds and we prove (4.36). In fact if $\phi(x, t) \geq \beta > 0$ then, by property (iii) of ϕ, $x \in \Omega_t^1$, i.e.

$$\liminf_{\varepsilon \to 0^+}{}_* \, u^\varepsilon(x, t) > m_+ - \beta.$$

This means that there exists an $\varepsilon_{x,t} = \varepsilon_{x,t}(\beta) > 0$ such that, for all $\varepsilon \leq \varepsilon_{x,t}$, $(y, \tau) \in B(x, \varepsilon_{x,t}) \times (t - \varepsilon_{x,t}, t + \varepsilon_{x,t})$, we have $u^\varepsilon(y, \tau) \geq m_+ - \beta$. Thus, by the compactness of $\{x \in B(x_o, r] : \phi \geq 0\}$ we can select an $\bar{\varepsilon} > 0$, possibly depending only on β, so that, for all $\varepsilon \leq \bar{\varepsilon}$, and $x \in \{x \in B(x_0, r] : \phi(\cdot, t) \geq \beta\}$ we have $u^\varepsilon(x, t) \geq m_+ - \beta$. Therefore

$$u^\varepsilon(x, t) \geq (m_+ - \beta)\mathbb{1}_{\{\phi(\cdot,t) \geq \beta\}}(x) + m_- \mathbb{1}_{\{\phi(\cdot,t) < \beta\}}(x).$$

for all $\varepsilon \leq \bar{\varepsilon}$, $x \in B(x_0, r]$. In the same way we can also obtain that, for ε small enough,

$$u^\varepsilon(x, s) \geq (m_+ - \beta)\mathbb{1}_{\{\phi(\cdot,s) \geq \beta\}}(x) + m_- \mathbb{1}_{\{\phi(\cdot,s) < \beta\}}(x),$$

for any $(x, s) \in \partial B(x_0, r] \times [t, t+h]$. Combining these inequalities with those in 1. and 2. in the statement of Lemma 4.1.5 we can conclude, by the maximum principle, that

$$\omega^{\varepsilon,\beta}(x, s) \leq u^\varepsilon(x, s), \quad \text{for all } (x, s) \in B(x_0, r] \times [t, t+h].$$

Moreover by property 3. of $\omega^{\varepsilon,\beta}$ in Lemma 4.1.5,

$$\liminf_{\varepsilon\to 0^+}{}_* \, u^\varepsilon(x,s) \geq m_+ - 3\beta$$

for every $(x,s) \in B(x_0,r] \times [t,t+h]$ such that $\phi(x,s) > 3\beta$. Since β is arbitrary we can now send β to zero in order to obtain that

$$\liminf_{\varepsilon\to 0^+}{}_* \, u^\varepsilon(x,s) \geq m_+$$

if $(x,s) \in B(x_0,r] \times [t,t+h]$ and $\phi(x,s) > 0$. Finally we remark that, if $s \in (t,t+h)$, $x \in B(x_0,r)$ and $\phi(x,s) > 0$ we have that $\phi(y,\tau) > 0$ in a neighborhood of (x,s) and thus (4.36) is proved.

Proof of Lemma 4.1.5. The proof is similar to the one of Lemma 4.1.4 and we just give point out the main changes. First of all we observe that since ϕ satisfies property (ii) above we have that there exists $\gamma > 0$ such that $|D\phi(x,s)| \neq 0$ in the set $Q_\gamma = \{(x,s) \in B(x_0,r] \times [t,t+h] : |\phi(x,s)| \leq \gamma\}$. Obviously this also means $|D_{H,\varepsilon}\phi(x,s)| \neq 0$ for any $(x,s) \in Q_\gamma$, $\varepsilon > 0$. As in Lemma 4.1.4 we construct our subsolution by steps and to do this we first define a function v^ε in Q_γ as $v^\varepsilon(x,s) = Q\Big(\dfrac{\phi(x,s) - 2\beta}{\varepsilon}, x, s\Big) - 2\beta$. Let $(x,s) \in Q_\gamma$; with the usual computations it turns out that

$$v_t^\varepsilon(x,s) - \varepsilon \operatorname{tr}(D^2_{H,\varepsilon} v^\varepsilon(x,s)) + \frac{f(v^\varepsilon(x,s))}{\varepsilon} = \frac{I_\varepsilon}{\varepsilon} + \frac{II_\varepsilon}{\varepsilon} + III_\varepsilon,$$

where I_ε, II_ε, III_ε are exactly the same terms defined in (4.32) with Φ replaced by ϕ. We put

$$Q(a,x,s) = q\Big(\frac{a}{|D_{H,\varepsilon}\phi(x,s)|}\Big)$$

in (4.32) and we get

$$I_\varepsilon = c\dot{q}$$

and

$$II_\varepsilon \leq \frac{\dot{q}}{|D_{H,\varepsilon}\phi|}[-\tilde{C} - c|D_H\phi| - \varepsilon \operatorname{tr}(D^2_{H,\varepsilon}\phi)] - 2\beta f'(q) + 2\| f''\|\beta^2,$$

where $\tilde{C}$ is the constant that appears in (i). As far it concerns the terms in III_ε we proceed as in Lemma 4.1.4 with the only difference that now we can't claim that Q_s is null. Anyway,

$$Q_s = \dot{q}a\Big(-\frac{D_{H,\varepsilon}\phi \cdot D_{H,\varepsilon}\phi_s}{| D_{H,\varepsilon}\phi|^3}\Big) = O\Big(\frac{1}{| D_{H,\varepsilon}\phi|}\Big) = O(\varepsilon^{-k}),$$

and thus also in this case

$$III_\varepsilon = O(\varepsilon\Big(\frac{1}{| D_{H,\varepsilon}\phi|^2} + \frac{1}{| D_{H,\varepsilon}\phi|}\Big)) + O\Big(\frac{1}{| D_{H,\varepsilon}\phi|}\Big) = O(\varepsilon^{1-2k}) + O(\varepsilon^{-k}) = O(\varepsilon^{-k}).$$

Since $|\, D_{H,\varepsilon}\phi| \to |\, D_H\phi|$ uniformly in Q_γ, we have

$$\begin{aligned} v_t^\varepsilon - \varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}v^\varepsilon) + \frac{f(v^\varepsilon)}{\varepsilon} &\le \frac{1}{\varepsilon}\Big[\frac{\dot q}{|D_{H,\varepsilon}\phi|}\Big(-\tilde C + c\,\underbrace{(|D_{H,\varepsilon}\phi| - |D_H\phi|)}_{o_\varepsilon(1)} - \\ &\qquad -\varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}\phi)\Big) - 2\beta f'(q) + 2\|\, f''\|\beta^2 + o_\varepsilon(1)\Big] \qquad (4.37) \\ &\le \frac{1}{\varepsilon}\Big[-\frac{\dot q\tilde C}{2|\, D_{H,\varepsilon}\phi|} - 2\beta f'(q) + 2\beta^2\|\, f''\| + o_\varepsilon(1)\Big] \end{aligned}$$

for ε small enough. To prove that the right hand side of (4.37) is strictly negative we proceed like in the proof of Lemma 4.1.4. In fact, since $f'(m_\pm) \ge d > 0$ and $q(r) \to m_\pm$ as $r \to \pm\infty$, there exists an $\bar r > 0$ so that

$$f'(q(r)) \ge \frac{d}{2}, \quad \text{for any } |\, r| \ge \bar r.$$

Thus if $|\phi(x,s) - 2\beta| \ge \bar r\varepsilon |D_{H,\varepsilon}\phi(x,s)|$ and we choose β small enough in order to get $4\beta\|\, f''\| \le d$, we have

$$v_t^\varepsilon - \varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}v^\varepsilon) + \frac{f(v^\varepsilon)}{\varepsilon} \le \frac{1}{\varepsilon}\Big(-\frac{\dot q\tilde C}{2|\, D_{H,\varepsilon}\phi|} - \frac{\beta d}{2} + o_\varepsilon(1)\Big) \le \frac{1}{\varepsilon}\big(-\frac{\beta d}{2} + o_\varepsilon(1)\big) \le -\frac{\beta d}{4\varepsilon}$$

for ε small enough.

If instead $|\,\phi(x,s) - 2\beta| < \bar r\varepsilon|\, D_{H,\varepsilon}\phi|$, we denote with $K = K(\bar r)$ a strictly positive constant so that $\dot q(r) \ge K > 0$ for any $|\, r| \le \bar r$. If we assume that β satisfies

$$2\beta\big(\|\, f'\| + \beta\|\, f''\| + 1\big) \sup_{(x,s)\in Q_\gamma} [|\, D_H\phi(x,s)|^2 + |\, D\phi(x,s)|^2]^{1/2} \le \frac{K\tilde C}{2},$$

then the inequality in (4.37) becomes

$$\begin{aligned} v_t^\varepsilon - \varepsilon\,\mathrm{tr}(D^2_{H,\varepsilon}v^\varepsilon) + \frac{f(v^\varepsilon)}{\varepsilon} &\le \frac{1}{\varepsilon}\Big[-\frac{K\tilde C}{2|\, D_{H,\varepsilon}\phi|} + 2\beta\|\, f'\| + 2\beta^2\|\, f''\| + o_\varepsilon(1)\Big] \\ &\le \frac{1}{\varepsilon}(-2\beta + o_\varepsilon(1)) \le -\frac{\beta}{\varepsilon}, \end{aligned}$$

for ε small enough compared with β fixed. Now that we have proved v^ε is a strict subsolution of (4.25-i), the extension of v^ε to a global subsolution $\omega^{\varepsilon,\beta}$ in $B(x_o, r] \times [t, t+h]$ and the proof that such a function satisfies 1, 2, 3, is similar to that of Lemma 4.1.4 and we omit it. □

Once we have proved the first two steps (initialization and propagation of the front) the proof of Theorem 4.1.1 follows immediately using Corollary 2.2.3

□

4.2 The second asymptotic problem: the degenerate Allen-Cahn equation

In this second section we consider the asymptotics of the solutions of the Cauchy problem for the equation (4.11). To be more precise we consider the behavior (as $\varepsilon \to 0$) of the solution of the Cauchy problem

$$\begin{cases} \text{(i)} \quad u_t^\varepsilon - \operatorname{tr}\left(A^\varepsilon(x)D^2u^\varepsilon + \sigma^t(x)D\sigma^t(x)Du^\varepsilon\right) + \dfrac{f(u^\varepsilon)}{\varepsilon^2} = 0 \quad \text{in } \mathbb{R}^n \times (0,+\infty), \\ \text{(ii)} \quad u^\varepsilon(x,0) = g(x) \quad \text{in } \mathbb{R}^n, \end{cases} \tag{4.38}$$

where the exponent k that appears in (4.8) is a fixed real number so that $0 < k < 1/3$ and g is a continuous real function in $\mathbb{R}^n$ which takes values in $[m_-, m_+]$. Moreover we suppose that the zeroes of f satisfy $c = 2m_o - m_+ - m_- = 0$.

We study the limiting behavior as $\varepsilon \to 0^+$ of the solutions u^ε of (4.38) only in the framework of Carnot groups. To be more precise we will show that if $\mathbb{R}^n$ can be endowed with a particular group law so that $(\mathbb{R}^n, \circ)$ is isomorphic to a Carnot group of step two and we define two regions Ω^1 and Ω^2 as in (2.14) with $a_\varepsilon = a = m_-$, $b_\varepsilon = b = m_+$ and $\tau = 1$, then the front that separates Ω^1 and Ω^2 evolves according to the geometric pde

$$u_t(x,t) + F(x, Du(x,t), D^2u(x,t)) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,+\infty), \tag{4.39}$$

where $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ is defined as

$$F(x,p,X) = -\operatorname{tr}\left[\left(\sigma^t(x)X\sigma(x) + \sigma^t(x)D_x\sigma^t(x)p\right)\left(I - \frac{\sigma^t(x)p \otimes \sigma^t(x)p}{|\,\sigma^t(x)p|^2}\right)\right]. \tag{4.40}$$

We start the section with some preliminary definitions and results about Carnot groups.

4.2.1 Carnot groups

Let $\circ$ be a given group law on $\mathbb{R}^n$ and suppose that the maps

$$\mathbb{R}^n \times \mathbb{R}^n \ni (x,y) \mapsto x \circ y \in \mathbb{R}^n$$

and

$$\mathbb{R}^n \ni x \mapsto x^{-1} \in \mathbb{R}^n$$

are smooth. Then $G := (\mathbb{R}^n, \circ)$ is called a *Lie group on* $\mathbb{R}^n$. To make the notation simpler we shall assume that the origin 0 of $\mathbb{R}^n$ is the identity of G.

We say that the Lie group $G = (\mathbb{R}^n, \circ)$ is a *homogeneous (Lie) group on* $\mathbb{R}^n$ if there exists an $n-$uple of real numbers $\sigma = (\sigma_1, \ldots, \sigma_n)$, with $1 \le \sigma_1 \le \cdots \le \sigma_n$, so that the dilatation $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$, $\delta_\lambda(x_1, \ldots, x_n) := (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_n} x_n)$ is an automorphism of the group G for every $\lambda > 0$. We shall denote by $G = (\mathbb{R}^n, \circ, \delta_\lambda)$ the homogeneous Lie group on $\mathbb{R}^n$ with composition law $\circ$ and dilatation $\{\delta_\lambda\}_{\lambda>0}$.

Let $\alpha \in G$, we denote by $\tau_\alpha : G \to G$, $\tau_\alpha(x) := \alpha \circ x$ the left translation by α on G. A vector field X on $\mathbb{R}^n$ is called *left-invariant* on G if

$$X(\varphi(\tau_\alpha(x))) = (X\varphi)(\tau_\alpha(x))$$

for every $x \in \mathbb{R}^n$, $\alpha \in G$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ smooth function. We denote by $\mathfrak{g}$ the set of the left-invariant vector fields on G. Since for any $X, Y \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{R}$ we have $\lambda X + \mu Y \in \mathfrak{g}$ and $[X, Y] \in \mathfrak{g}$, then $\mathfrak{g}$ is a Lie algebra of vector fields. It is called the *Lie algebra* of G.

A non -identically-vanishing linear differential operator X is called $\delta_\lambda-$*homogeneous* of degree $m \in \mathbb{R}$ if and only if, for every $\varphi \in C^\infty(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $\lambda > 0$, it holds

$$X(\varphi(\delta_\lambda(x))) = \lambda^m (X\varphi)(\delta_\lambda(x)).$$

Definition 4.2.1. We say that a Lie group on $\mathbb{R}^n$, $G = (\mathbb{R}^n, \circ)$, is a *(homogeneous) Carnot group* or a *(homogenous) stratified group*, if the following properties hold:

(C1) $\mathbb{R}^n$ can be split as $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}$, and the dilatation $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, \ldots, x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)}), \quad x^{(i)} \in \mathbb{R}^{n_i},$$

is an automorphism of the group G for every $\lambda > 0$.

(C2) if $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}_1$ is the linear subspace of $\mathfrak{g}$ of the left-invariant vector fields which are δ_λ-homogeneous of degree 1, then

$$\mathrm{Lie}\{\mathfrak{g}_1\} = \mathfrak{g}.$$ [1]

We say that G has step r and $n_1 = \dim(\mathfrak{g}_1)$ generators.

---

[1] If $U \subseteq T(\mathbb{R}^n)$ is a set of smooth vector fields on $\mathbb{R}^n$ and we set

$$U_1 := \mathrm{span}\{U\}, \quad U_n := \mathrm{span}\{[u, v] : u \in U, v \in U_{n-1}\}, \ \ n \ge 2,$$

then

$$\mathrm{Lie}\{U\} = \mathrm{span}\{U_n : n \in \mathbb{N}\}.$$

We denote with $\mathcal{J}_{\tau_x}(y)$ the Jacobian matrix at point y of the map τ_x. If $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$, we define the *Jacobian basis* of $\mathfrak{g}$, $\{Z_1, \ldots, Z_n\}$ as

$$Z_j I(x) = \mathcal{J}_{\tau_x}(0) \cdot e_j = j - \text{th column of } \mathcal{J}_{\tau_x}(0) \quad \forall x \in \mathbb{R}^n.$$ [2]

Since $\mathcal{J}_{\tau_0}(y) \equiv \mathbb{I}_n$, we have $Z_j I(0) = e_j$ and thus

$$Z_j(0) = \frac{\partial}{\partial x_j}_{|0}, \quad \text{for } j = 1, \ldots, n.$$

With these notations condition **(C2)** means that

$$\text{rank}(\text{Lie}\{Z_1, \ldots, Z_{n_1}\}(x))^3 = n, \quad \text{for any } x \in \mathbb{R}^n$$

and

$$\text{Lie}\{Z_1, \ldots, Z_{n_1}\} = \mathfrak{g}.$$

From a set of Vector Fields to a Carnot Group

We now want to see when a set of smooth vector fields on $\mathbb{R}^n$, $\{X_1, \ldots, X_m\}$, $m < n$, is a basis for the Lie algebra of a Carnot group on $\mathbb{R}^n$. For a complete treatment of the subject see the book of Bonfiglioli, Lanconelli and Uguzzoni [9], section 4.2.

We put, for every $k \in \mathbb{N}$,

$$W^{(k)} = \text{span}\{X_J | J \in \{1, \ldots, m\}^k\},$$

[2] $I(x) = (I_1(x), \ldots, I_n(x)) = (x_1, \ldots, x_n)$, $x \in \mathbb{R}^n$. If X is a linear differential operator with the notation XI we mean the action of X upon the components of I, i.e.

$$XI(x) = \begin{pmatrix} XI_1(x) \\ \vdots \\ XI_n(x) \end{pmatrix}$$

Therefore if $X = \sum_{i=1}^{n} a_j \partial_j$, then

$$XI(x) = \begin{pmatrix} a_1(x) \\ \vdots \\ a_n(x) \end{pmatrix}$$

[3] If $U \subseteq T(\mathbb{R}^n)$ is a set of smooth vector fields on $\mathbb{R}^n$ we define

$$\text{rank}(\text{Lie}\, U(x)) = \dim_{\mathbb{R}}\{ZI(x) : Z \in \text{Lie}\{U\}\}$$

for any $x \in \mathbb{R}^n$.

where, if $J = (j_1, \dots, j_k)$,

$$X_{(j_1,\dots,j_k)} = [X_{j_1}, [X_{j_2}, \dots, [X_{j_{k-1}}, X_{j_k}]] \dots].$$

We assume that the vector fields X_j's satisfy the following conditions:

(H0) $X_1, \dots, X_m$ are linearly independent and δ_λ-homogeneous of degree one with respect to a suitable family of dilatations $\{\delta_\lambda\}_{\lambda>0}$ of the following type

$$\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n, \quad \delta_\lambda(x) = \delta_\lambda(x^{(1)}, \dots, x^{(r)}) := (\lambda x^{(1)}, \dots, \lambda^r x^{(r)}),$$

where $r \geq 1$ is an integer, $x^{(i)} \in \mathbb{R}^{n_i}$ for $i = 1, \dots, r$, $n_1 = m$ and $n_1 + \dots + n_r = n$;

(H1) $\dim(W^{(k)}) = \dim\{XI(0) : X \in W^{(k)}\}$ for every $k = 1, \dots, r$;

(H2) $\dim(\mathrm{Lie}\{X_1, \dots, X_m\}I(0)) = n$.

It can be shown that if $X_1, \dots, X_m$ satisfy these assumptions , then they also satisfy

(H1)* $\dim(W^{(k)}I(x)) = \dim(W^{(k)})$ for any $k \leq r$, $x \in \mathbb{R}^n$,

(H2)* $\dim(\mathrm{Lie}\{X_1, \dots, X_m\}I(x)) = n$ for any $x \in \mathbb{R}^n$.

For every $k = 1, \dots, r$ we consider a fixed basis for $W^{(k)}$. We know that

$$\{Z_1, \dots, Z_n\} := \{Z_1^{(1)}, \dots, Z_{n_1}^{(1)}, \dots, Z_1^{(r)}, \dots, Z_{n_r}^{(r)}\}$$

is a basis of $\mathfrak{a} = \mathrm{Lie}\{X_1, \dots, X_m\}$. Therefore $\mathfrak{a} = \{\xi \cdot Z = \sum_{j=1}^{n} \xi_j Z_j : \xi \in \mathbb{R}^n\}$.

It can be shown that the map $(x, t) \longmapsto \exp(t\xi \cdot Z)(x)$ is well defined for every $(x, t) \in \mathbb{R}^n \times \mathbb{R}$. Furthermore,

$$\mathrm{Exp} : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathrm{Exp}(\xi) := \exp(\xi \cdot Z)(0)$$

is a global diffeomorphism. We denote with Log its inverse function and we set

$$x \circ y := \exp(\mathrm{Log}(y) \cdot Z)(0), \qquad x, y \in \mathbb{R}^n. \tag{4.41}$$

We now state a Theorem that characterizes the Carnot group $G = (\mathbb{R}^n, \circ, \delta_\lambda)$ whose Lie algebra is generated by the vector field $\{X_1, \dots, X_m\}$. For the proof we refer to [9].

Theorem 4.2.2. Let $\{X_1, \dots, X_m\}$ be smooth vector fields in $\mathbb{R}^n$ satisfying hypothesis **(H0)**, **(H1)** and **(H2)**. Let $\{\delta_\lambda\}_{\lambda>0}$ be the family of dilatations defined in **(H0)**. Finally, let $\circ$ be the composition law on $\mathbb{R}^n$ introduced in (4.41). Then

$$G = (\mathbb{R}^n, \circ, \delta_\lambda)$$

is a homogeneous Carnot group of step r with m generators whose Lie algebra $\mathfrak{g}$ is Lie-generated by $\{X_1, \dots, X_m\}$, i.e.

$$\mathfrak{g} = \text{Lie}\{X_1, \dots, X_m\}.$$

4.2.2 The mean curvature equation in Carnot groups

We want to consider equation (4.39) in a Carnot group on $\mathbb{R}^n$, $G = (\mathbb{R}^n, \circ, \delta_\lambda)$. With the notations above we call $\mathfrak{g}_1$ the linear subspace of $\mathfrak{g}$ of the left-invariant vector fields which are δ_λ-homogeneous of degree one and we put

$$m = \dim \mathfrak{g}_1.$$

We set, for a generical point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$x_H = (x_1, \dots, x_m) \quad \text{and} \quad x_V = (x_{m+1}, \dots, x_n).$$

If $\{X_1, \dots, X_m\}$ is an orthonormal basis of $\mathfrak{g}_1$ by property **(C2)** we have

$$\text{Lie}\{X_1, \dots, X_m\} = \mathfrak{g}.$$

We assume that

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix} = \sigma^t(x) \cdot \nabla, \tag{4.42}$$

where $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times m}$. We point out that if we consider as orthonormal basis the Jacobian basis $\{Z_1, \dots, Z_n\}$ the matrix σ is just the matrix obtained by taking the first m columns of $\mathcal{J}_{\tau_x}(0)$. With this notations the equation in (4.39) can be rewritten as

$$u_t(x,t) - \sum_{i,j=1}^{m} \left(\delta_{ij} - \frac{X_i u(x,t) X_j u(x,t)}{\sum_{i=1}^m (X_i u(x,t))^2} \right) X_i X_j u(x,t) = 0, \tag{4.43}$$

or

$$u_t(x,t) + \tilde{F}(Xu(x,t), X^2 u(x,t)) = 0$$

where $\tilde{F} : \mathbb{R}^m \times \mathcal{S}^m \to \mathbb{R}$ is defined as

$$\tilde{F}(p, X) = -\operatorname{tr}[\big(I - \frac{p}{|\,p|} \otimes \frac{p}{|\,p|}\big)X], \qquad (p, X) \in \mathbb{R}^m \times \mathcal{S}^m. \tag{4.44}$$

This last equivalent formulation explicitly shows us that equation (4.39) is indeed the mean curvature equation in the sub-Riemannian metric $\mathfrak{g}$.

In [15] Capogna and Citti prove that, if $G = (\mathbb{R}^n, \circ, \delta_\lambda)$ is an homogeneous Carnot group, then the Cauchy problem for the equation (4.43),

$$\begin{cases} u_t(x,t) + \tilde{F}(Xu(x,t), X^2u(x,t)) = 0, & (x,t) \in \mathbb{R}^n \times (0, +\infty), \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{4.45}$$

is well-posed under some particular assumptions on the initial datum u_0. To treat the discontinuity that appears in (4.43) for $Xu(x,t) = 0$ they use the following definition of *weak solution*.

Definition 4.2.3. A function $u \in C(G \times [0, +\infty)$ is a *weak subsolution (supersolution)* of (4.43) in $G \times (0, +\infty)$ if for any $(x,t) \in G \times (0, +\infty)$ and any function $\phi \in C^2(G \times (0, +\infty))$ such that $u - \phi$ has a local maximum (minimum) at (x,t) then

$$\frac{\partial}{\partial_t}\phi \leq (\geq) \begin{cases} \displaystyle\sum_{i,j=1}^{m} \Big(\delta_{ij} - \frac{X_i\phi X_j\phi}{|\,X\phi|^2}\Big) X_iX_j\phi, & \text{if } |\,X\phi| \neq 0 \\ \displaystyle\sum_{i,j=1}^{m} (\delta_{ij} - p_ip_j) X_iX_j\phi \text{ for some } p \in \mathbb{R}^m, |\,p| \leq 1, & \text{if } |\,X\phi| = 0. \end{cases}$$

A *weak solution* of (4.43) is a function u which is both a weak subsolution and a weak supersolution.

This definition of weak solution is quite different from the classical definition of viscosity solution. Anyway, as we now show, the two definitions turn out to be equivalent, at least for homogeneous Carnot groups of step two.

Carnot group of step two

Consider an homogeneous Carnot group of step two $G = (\mathbb{R}^n, \circ, \delta_\lambda)$ where the dilatation δ_λ is defined as

$$\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n, \qquad \delta_\lambda(x) = (\lambda x_H, \lambda^2 x_V). \tag{4.46}$$

By Theorem 1.3.15 on page 39 in [9] the composition law $\circ$ takes the form

$$x \circ y = (x_H, x_V) \circ (y_H, y_V) = (x_H + y_H, x_V + y_V + \langle Bx_H, y_H \rangle). \tag{4.47}$$

where $\langle Bx_H, y_H\rangle$ denotes the $(n-m)$-tuple

$$\big(\langle B^{(1)}x_H, y_H\rangle, \dots, \langle B^{(n-m)}x_H, y_H\rangle\big),$$

$\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^m$ and $B^{(1)}, \dots B^{(n-m)}$ is a suitable $(n-m)$-tuple of $m\times m$ matrices with real entries. Obviously the identity element for the composition law defined in (4.47) is the origin 0 and the inverse of a generic element $x\in\mathbb{R}^n$ is given by $x^{-1} = (-x_H, -x_V + \langle Bx_H, x_H\rangle)$. We point out that the inverse x^{-1} is equal to $-x$ if and only if $\langle B^{(k)}z, z\rangle = 0$ for any $x\in\mathbb{R}^m$ and $k = 1, \dots, n-m$, i.e. if and only if the matrices $B^{(k)}$ are skew-symmetric. The Jacobian matrix at the origin 0 of the left translation by x, τ_x, is the following matrix block,

$$\mathcal{J}_{\tau_{\mathrm{x}}}(0) = \left(\begin{array}{c|c} \mathbb{I}_m & \mathbb{O}_{m\times(n-m)} \\ \hline & \\ Bx_H & \mathbb{I}_{n-m}, \end{array}\right)$$

where Bx_H denotes the $n-m\times m$ matrix

$$\begin{pmatrix} (B^{(1)}x_H)^T \\ \vdots \\ (B^{(n-m)}x_H)^T \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^m B^{(1)}_{1,j}x_j & \sum_{j=1}^m B^{(1)}_{2,j}x_j & \dots & \sum_{j=1}^m B^{(1)}_{m,j}x_j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^m B^{(n-m)}_{1,j}x_j & \sum_{j=1}^m B^{(n-m)}_{2,j}x_j & \dots & \sum_{j=1}^m B^{(n-m)}_{m,j}x_j \end{pmatrix}.$$

This means that the Jacobian basis of the Lie algebra of G is

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i} + \sum_{k=m+1}^{n}\Big(\sum_{j=1}^{m} B^{(k)}_{i,j}x_j\Big)\frac{\partial}{\partial x_k}, \quad i = 1, \dots, m, \\ T_i &= \frac{\partial}{\partial x_{m+i}}, \quad i = 1, \dots, n-m. \end{aligned}$$

A simple computation shows that, for any $i, j\in\{1, \dots, m\}$,

$$[X_j, X_i] = \sum_{k=1}^{n-m} C^{(k)}_{i,j}T_k,$$

where $C^{(k)}$ is the skew-symmetric part of $B^{(k)}$, i.e. $C^{(k)} = (B^{(k)} - (B^{(k)})^T)/2$. If $C^{(1)}, \ldots, C^{(n-m)}$ are linearly independent this implies that

$$\mathrm{span}\{[X_j, X_i] : i, j = 1, \ldots, m\} = \mathrm{span}\{T_1, \ldots, T_{n-m}\}$$

and therefore

$$\mathrm{rank}(\mathrm{Lie}\{X_1, \ldots, X_m\}(0,0)) = \dim(\mathrm{span}\{\partial_{x_1}, \ldots, \partial_{x_n}\}) = n.$$

This shows that G is a Carnot group of step two and Jacobian generators $X_1, \ldots, X_m$. Since the condition of the linear independence of the matrices $C^{(1)}, \ldots, C^{(n-m)}$ is also necessary for G to be a Carnot group we can conclude that $G = (\mathbb{R}^n, \circ, \delta_\lambda)$, with $\circ$ as in (4.47) and δ_λ as in (4.46) is a Carnot group of step two and generators $X_1, \ldots, X_m$ if and only if $C^{(1)}, \ldots, C^{(n-m)}$ are linearly independent.

We now observe that any Carnot group $G = (\mathbb{R}^n, \circ, \delta_\lambda)$ is isomorphic, by means of the group isomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi(x) = \varphi(x_H, x_V) = (x_H, x_V - \langle \frac{B}{2} x_H, x_H \rangle)$, to the Carnot group $\tilde{G} = (\mathbb{R}^n, \tilde{\circ}, \delta_\lambda)$ where the group operation $\tilde{\circ}$ is defined as in (4.47) with the matrices $B^{(1)}, \ldots, B^{(n-m)}$ replaced by their skew-symmetric part $C^{(1)}, \ldots, C^{(n-m)}$. Therefore we can assume, without loss of generality, that the matrices $B^{(k)}$, $k = 1, \ldots, n-m$, that appear in the definition of the composition law (4.47) are all skew-symmetric.

Example 4.2.4 (The Heisenberg group)**.** The most famous example of Carnot group of step 2 is the so called Heisenberg group. Let us consider in $\mathbb{C}^n \times \mathbb{R}$ the following composition law

$$(\omega, z) \circ (\omega', z') = (\omega + \omega, z + z' + 2\mathrm{Im}(\omega \cdot \overline{\omega'})).$$

If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and we denote the points of $\mathbb{R}^{2n+1}$ as (x, y, z), with $x, y \in \mathbb{R}^n$, $z \in \mathbb{R}$, the composition $\circ$ can be written explicitly as

$$(x, y, z) \circ (x', y', z') = (x + x', y + y', z + z' + 2(\langle y, x' \rangle - \langle x, y' \rangle)) \tag{4.48}$$

The Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ)$ is a Lie group homogeneous with respect to the family of dilatations $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$, $\delta_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^2 z)$, $\lambda > 0$.

The Jacobian matrix of the left translation $\tau_{(x,y,z)}$ at zero is

$$\mathcal{J}_{\tau_{(x,y,z)}}(0) = \begin{pmatrix} \mathbb{I}_n & \mathbb{O}_{n\times n} & 0 \\ \mathbb{O}_{n\times n} & \mathbb{I}_n & 0 \\ 2y^T & -2x^T & 1 \end{pmatrix}$$

and so the Jacobian basis of the lie algebra of $\mathbb{H}$ is given by the 2n vectors

$$X_i(x) = \partial_{x_i} + 2y_i\partial_z, \quad Y_i(x) = \partial_{y_i} - 2x_i\partial_z, \quad i = 1, \dots, n.$$

Since $[X_i, Y_i] = -4\partial_z$ for any $i = 1, \dots, n$ we can conclude that the Heisenberg group is an homogeneous Carnot group of step two. Moreover following the notations above we have that the matrix B is a $2n \times 2n$ skew-symmetric matrix defined as

$$B = \begin{pmatrix} \mathbb{O}_{n\times n} & 2\mathbb{I}_n \\ -2\mathbb{I}_n & \mathbb{O}_{n\times n} \end{pmatrix}$$

and in fact $(x, y, z)^{-1} = (-x, -y, -z)$.

Following the notations above we define the horizontal gradient and the horizontal (symmetric) Hessian matrix of a twice differentiable function $f : G \to \mathbb{R}$ as

$$Xf(x) = \begin{pmatrix} X_1 f(x) \\ \vdots \\ X_m f(x) \end{pmatrix} = \sigma^t(x) Df(x), \tag{4.49}$$

$$X^2 f(x) = \left(\left(\frac{X_i X_j f(x) + X_j X_i f(x)}{2} \right)_{i,j=1,\dots,m} \right) = \sigma^t(x) D^2 f(x) \sigma(x).$$

where the matrix σ is obtained by taking the first m columns of $\mathcal{J}_{\tau_x}(0)$,

$$\sigma(x) := \begin{pmatrix} \mathbb{I}_m \\ Bx_H \end{pmatrix} \tag{4.50}$$

We just point out that the horizontal (symmetric) Hessian matrix does not contain first order terms because of the skew-symmetry of the matrices $B^{(1)}, \cdots, B^{(n-m)}$.

When it will be necessary to emphasize the variable x in which we are computing the vector fields X_i (and with respect to we are computing the derivatives), we will denote the horizontal gradient and the horizontal Hessian matrix as X_x and X_x^2 . For example if $g = g(x, y)$ is a C^2 function defined in $G \times G$ and (x_o, y_o) is a generic point of $G \times G$ we will denote with $X_x f(x_o, y_o)$ the horizontal gradient of f with respect to the variable x and with $X_y f(x_o, y_o)$ the horizontal gradient of f with respect to y, both computed in the point (x_o, y_o). Analogous definitions hold for $X_x^2 f(x_o, y_o)$ and $X_y^2 f(x_o, y_o)$.

We consider an homogeneous (with respect to any dilatation δ_λ, $\lambda > 0$) norm on G,

$$\| x\|_G = [|x_H|^4 + |x_V|^2]^{1/4}, \tag{4.51}$$

and we define a left invariant metric $d_G : G \times G \to [0, +\infty)$ as

$$\begin{aligned} d_G(x,y) &= \| x^{-1} \circ y\|_G \\ &= [| y_H - x_H|^4 + | y_V - x_V - \langle Bx_H, y_H\rangle|^2]^{1/4}. \end{aligned} \tag{4.52}$$

We now prove a nice property of the homogeneous metric d_G defined in (4.52).

Lemma 4.2.5. Put $N(x) = \| x\|_G^4$ for any $x \in \mathbb{R}^N$.

(i) $\{x \in G : |XN(x)| = 0\} = \{x \in G : X^2N(x) = \mathbb{O}\} = \{x \in G : x_H = 0_m\}$.

(ii) $|X_x d_G^4(x,y)| = |X_y d_G^4(x,y)|$ and $X_x^2 d_G^4(x,y) = X_y^2 d_G^4(x,y)$ for any $x, y \in G$; moreover they all have as zero-set the set $\{(x,y) \in G \times G : x_H = y_H\}$.

Proof. (i) The proof of the first point follows immediately by some simple computations. In fact since

$$XN(x) = 4| x_H|^2 x_H + 2\sum_{k=1}^{n-m} (x_V)_k B^{(k)} x_H$$

we have

$$|XN(x)|^2 = 16|x_H|^6 + 4\sum_{k,l}(x_V)_k(x_V)_l\langle B^{(k)}x_H, B^{(l)}x_H\rangle$$

(we recall that, since the matrices $B^{(k)}$ are all skew symmetric the mixed products are all null). Moreover

$$X^2N(x) = 4|x_H|^2 I_m + 8x_H \otimes x_H + 2(Bx_H)^T Bx_H.$$

(ii) First of all we observe that, since the vector fields X_i are invariant by left composition of the operation law $\circ$, we have

$$\begin{aligned} X_y d_G^4(x,y) &= X_y N(x^{-1} \circ y) = XN(x^{-1} \circ y) \\ X_y^2 d_G^4(x,y) &= X^2 N(x^{-1} \circ y) \end{aligned}$$

and so by point (i) $X_y d_G^4(x,y)$ and $X_y^2 d_G^4(x,y)$ are null if and only if $(x^{-1} \circ y)^{(1)} = 0$, i.e. $y_H = x_H$. To compute the horizontal gradient and the horizontal Hessian matrix with respect the x variable we observe that, since $N(x^{-1}) = N(x)$, it holds $d_G^4(x,y) = N(x^{-1} \circ y) = N(y^{-1} \circ x)$ and

$$X_x d_G^4(x,y) = XN(y^{-1} \circ x)$$
$$X_x^2 d_G^4(x,y) = X^2 N(y^{-1} \circ x),$$

and again $X_x d_G^4(x,y)$ and $X_x^2 d_G^4(x,y)$ are null exactly when $y_H = x_H$.

Finally we observe that $| X_y d_G^4(x,y)|^2 = | X_x d_G^4(x,y)|^2$ and $X_y^2 d_G^4(x,y) = X_x^2 d_G^4(x,y)$ □

We use this Lemma to prove the equivalence between the definition of *weak solution* in Definition 4.2.3 and the usual definition of *viscosity solution* for the equation (4.39). It immediately follows from the following property of viscosity solutions of equation (4.39). The analogous result in the Euclidean case can be found in [6] and in the first Chapter of our thesis.

Proposition 4.2.6. An upper (respectively lower) semicontinuous function u is a viscosity subsolution (respectively supersolution) of (4.39) if and only if for any $\phi \in C^2(\mathbb{R}^n \times (0,+\infty))$, if $(x,t) \in \mathbb{R}^n \times (0,+\infty)$ is a local maximum (respectively minimum) point for $u - \phi$, one has

$$\frac{\partial \phi(x,t)}{\partial t} - \mathrm{tr}\left[\left(I - \frac{X\phi(x,t) \otimes X\phi(x,t)}{|X\phi(x,t)|^2}\right) X^2\phi(x,t)\right] \leq 0 \quad \text{if } X\phi(x,t) \neq 0 \tag{4.53}$$

and

$$\frac{\partial \phi(x,t)}{\partial t} \leq 0 \quad \text{if} \quad X\phi(x,t) = 0 \text{ and } \mathrm{X}^2\phi(x,t) = 0, \tag{4.54}$$

(respectively

$$\frac{\partial \phi(x,t)}{\partial t} - \mathrm{tr}\left[\left(I - \frac{X\phi(x,t) \otimes X\phi(x,t)}{|X\phi(x,t)|^2}\right) X^2\phi(x,t)\right] \geq 0 \quad \text{if } X\phi(x,t) \neq 0$$

and

$$\frac{\partial \phi(x,t)}{\partial t} \geq 0 \quad \text{if} \quad X\phi(x,t) = 0 \text{ and } \mathrm{X}^2\phi(x,t) = 0). \tag{4.55}$$

Proof. Let u be an upper semicontinuous function which satisfies (4.53) and (4.54). Consider $\phi \in C^2(\mathbb{R}^n \times (0,+\infty))$ and $(\hat{x},\hat{t}) \in \mathbb{R}^n \times (0,+\infty)$ a local maximum point for $u - \phi$ such that $X\phi(\hat{x},\hat{t}) = 0$ and $X^2\phi(\hat{x},\hat{t}) \neq 0$. Without loss of generality we can assume that u is bounded and $(\hat{x},\hat{t})$ is a strict local maximum point for $u - \phi$. If we prove that

$$\frac{\partial \phi(\hat{x},\hat{t})}{\partial t} + \tilde{F}_*(X\phi(\hat{x},\hat{t}), X^2\phi(\hat{x},\hat{t})) \leq 0, \tag{4.56}$$

with $\tilde{F}$ as in (4.44), we have that u is a viscosity subsolution of (4.43). For any $\varepsilon > 0$ we consider a function

$$\psi_\varepsilon(x,y,t) = u(x,t) - \frac{d_G^4(x,y)}{\varepsilon} - \phi(y,t)$$

and we denote with $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ the maximum point of ψ_ε in $\mathbb{R}^n \times (0,+\infty)$. With some classical computations one easily proves that $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ converges to $(\hat{x}, \hat{x}, \hat{t})$.Moreover since the function $y \mapsto \psi_\varepsilon(x_\varepsilon, y, t_\varepsilon)$ has a local maximum in y_ε we have

$$D\phi(y_\varepsilon, t_\varepsilon) = -\frac{D_y d_G^4(x_\varepsilon, y_\varepsilon)}{\varepsilon},$$

$$D^2\phi(y_\varepsilon, t_\varepsilon) \geq -\frac{D_y^2 d_G^4(x_\varepsilon, y_\varepsilon)}{\varepsilon};$$

thus

$$X\phi(y_\varepsilon, t_\varepsilon) = -\frac{X_y d_G^4(x_\varepsilon, y_\varepsilon)}{\varepsilon} \tag{4.57}$$

$$X^2\phi(y_\varepsilon, t_\varepsilon) \geq -\frac{X_y^2 d_G^4(x_\varepsilon, y_\varepsilon)}{\varepsilon}, \tag{4.58}$$

Two cases now may occur.

1. $X\phi(y_\varepsilon, t_\varepsilon) = 0$. This means that $X_y d_G^4(x_\varepsilon, y_\varepsilon) = 0$ and by the previous Lemma $(x_\varepsilon)_H = (y_\varepsilon)_H$. Since the map $(x,t) \mapsto u(x,t) - \varphi(x,t)$, with $\varphi(x,t) = \dfrac{d_G^4(x, y_\varepsilon)}{\varepsilon} + \phi(y_\varepsilon, t)$ attains a maximum on $(x_\varepsilon, t_\varepsilon)$ and

$$X\varphi(x,t) = 0 \Leftrightarrow x_H = (y_\varepsilon)_H \Leftrightarrow X^2\varphi(x,t) = 0,$$

by (4.54) we get

$$\frac{\partial \varphi}{\partial t}(x_\varepsilon, t_\varepsilon) = \frac{\partial \phi}{\partial t}(y_\varepsilon, t_\varepsilon) \leq 0.$$

Moreover, since in this case (4.58) becomes $X^2\phi(y_\varepsilon, t_\varepsilon) \geq \mathbb{O}_{m\times m}$, using the ellipticity of F_* it holds

$$\frac{\partial \phi}{\partial t}(y_\varepsilon, t_\varepsilon) + \tilde{F}_*(\underbrace{X\phi(y_\varepsilon, t_\varepsilon)}_{=0}, X^2\phi(y_\varepsilon, t_\varepsilon)) \leq \frac{\partial \phi}{\partial t}(y_\varepsilon, t_\varepsilon) + \tilde{F}_*(0,0) = \frac{\partial \phi}{\partial t}(y_\varepsilon, t_\varepsilon) \leq 0$$

and we conclude by letting ε go to 0.

2. $X\phi(y_\varepsilon, t_\varepsilon) \neq 0$; using (4.57) and the previous Lemma this means $(y_\varepsilon)_H \neq (x_\varepsilon)_H$. The point $(x_\varepsilon, t_\varepsilon)$ is a maximum for

$$\begin{aligned}(x,t) \mapsto \psi_\varepsilon(x, x \circ x_\varepsilon^{-1} \circ y_\varepsilon, t) &= u(x,t) - \frac{d_G^4(x_\varepsilon, y_\varepsilon)}{\varepsilon} - \phi(x \circ x_\varepsilon^{-1} \circ y_\varepsilon, t) \\ &= u(x,t) - \varphi(x,t)\end{aligned}.$$

Let $\tilde{\tau}_\alpha(x) = x \circ \alpha$ be the right translation by α and $\mathcal{J}_{\tilde{\tau}_\alpha}(x) \equiv \mathcal{J}_{\tilde{\tau}_\alpha}$ its Jacobian matrix; a simple computation shows that $\mathcal{J}_{\tilde{\tau}_\alpha}$ has the form

$$\mathcal{J}_{\tilde{\tau}_\alpha} = \left(\begin{array}{c|c} \mathbb{I}_m & \mathbb{O}_{m\times n} \\ \hline ((B^{(1)})^T\alpha_H)^T & \\ \vdots & \mathbb{I}_n \\ ((B^{(n)})^T\alpha_H)^T & \end{array}\right) = \left(\begin{array}{c|c} \mathbb{I}_m & \mathbb{O}_{m\times n} \\ \hline (-B^{(1)}\alpha_H)^T & \\ \vdots & \mathbb{I}_n \\ (-B^{(n)}\alpha_H)^T & \end{array}\right)$$

$$= \left(\begin{array}{c|c} \mathbb{I}_m & \mathbb{O}_{m\times n} \\ \hline -B\alpha_H & \mathbb{I}_n, \end{array}\right).$$

By the chain rule we get

$$\begin{aligned} X\varphi(x_\varepsilon, t_\varepsilon) &= \sigma(x_\varepsilon)^T \mathcal{J}^T_{\tilde{\tau}_{x_\varepsilon^{-1}\circ y_\varepsilon}} D\phi(\tilde{\tau}_{x_\varepsilon^{-1}\circ y_\varepsilon}(x_\varepsilon), t_\varepsilon) \\ &= \left(\mathcal{J}_{\tilde{\tau}_{x_\varepsilon^{-1}\circ y_\varepsilon}} \sigma(x_\varepsilon)\right)^T D\phi(y_\varepsilon, t_\varepsilon) \\ &= \sigma(2x_\varepsilon - y_\varepsilon)^T D\phi(y_\varepsilon, t_\varepsilon) \\ &\longrightarrow X\phi(\hat{x}, \hat{t}) = 0, \quad \text{as } \varepsilon \to 0 \end{aligned}$$

and

$$\begin{aligned} X^2\varphi(x_\varepsilon, t_\varepsilon) &= \sigma(x_\varepsilon)^T \mathcal{J}^T_{\tilde{\tau}_{x_\varepsilon^{-1}\circ y_\varepsilon}} D^2\phi(\tilde{\tau}_{x_\varepsilon^{-1}\circ y_\varepsilon}(x_\varepsilon), t_\varepsilon) \mathcal{J}_{\tilde{\tau}_{x_\varepsilon^{-1}\circ y_\varepsilon}} \sigma(x_\varepsilon) \\ &= \sigma(2x_\varepsilon - y_\varepsilon)^T D^2\phi(y_\varepsilon, t_\varepsilon)\sigma(2x_\varepsilon - y_\varepsilon) \\ &\longrightarrow X^2\phi(\hat{x}, \hat{t}) \neq 0, \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Moreover $X\varphi(x_\varepsilon, t_\varepsilon) \neq 0$; in fact,

$$\begin{aligned} X\varphi(x_\varepsilon, t_\varepsilon) &= \sigma(2x_\varepsilon - y_\varepsilon)^T D\phi(y_\varepsilon, t_\varepsilon) = -\varepsilon^{-1}\sigma(2x_\varepsilon - y_\varepsilon)^T D_y d_G^4(x_\varepsilon, y_\varepsilon) \\ &= -\varepsilon^{-1}\sigma(2x_\varepsilon - y_\varepsilon)^T \mathcal{J}^T_{\tau_{x_\varepsilon^{-1}}} DN(x_\varepsilon^{-1} \circ y_\varepsilon) \\ &= -\varepsilon^{-1}\sigma(x_\varepsilon - y_\varepsilon)^T DN(x_\varepsilon^{-1} \circ y_\varepsilon) = \varepsilon^{-1}\sigma(x_\varepsilon - y_\varepsilon)^T DN(y_\varepsilon^{-1} \circ x_\varepsilon) \\ &= \varepsilon^{-1} XN(y_\varepsilon^{-1} \circ x_\varepsilon) \end{aligned}$$

and by the previous Lemma this is null if and only if $(y_\varepsilon)_H = (x_\varepsilon)_H$. Thus by (4.53) it holds

$$\frac{\partial\varphi}{\partial t}(x_\varepsilon, t_\varepsilon) + \tilde{F}(X\varphi(x_\varepsilon, t_\varepsilon), X^2\varphi(x_\varepsilon, t_\varepsilon)) \leq 0$$

and we conclude by letting $\varepsilon \to 0$,

$$\begin{aligned} 0 &\geq \liminf_{\varepsilon\to 0} \left(\frac{\partial\varphi}{\partial t}(x_\varepsilon, t_\varepsilon) + \tilde{F}(X\varphi(x_\varepsilon, t_\varepsilon), X^2\varphi(x_\varepsilon, t_\varepsilon))\right) \\ &\geq \frac{\partial\phi}{\partial t}(\hat{x}, \hat{t}) + \tilde{F}_*(X\phi(\hat{x}, \hat{t}), X^2\phi(\hat{x}, \hat{t})). \end{aligned}$$

□

As for Euclidean derivatives, from this last proposition we immediately get the following characterization of viscosity sub and supersolution of equation (4.39).

Remark 4.2.7. It is not restrictive to assume in Definition 1.0.1 that, if u (respectively v) is an upper semicontinuous subsolution (respectively a lower semicontinuous supersolution) of equation (4.39) and $\varphi \in C^2(\mathbb{R}^n \times (0, +\infty))$ is a test function for u (resp. for v) at the point (x, t), then

$$X\varphi(x, t) = 0$$

implies

$$X^2\varphi(x, t) = 0.$$

Moreover we can assume that, at any point (y, s) in a neighborhood of (x, t) so that

$$X\varphi(y, s) = X\varphi(x, t) = 0,$$

it holds

$$X^2\varphi(y, s) = 0.$$

We conclude the section with the comparison principle proved by Capogna and Citti in [15]. It holds for weak sub- and supersolutions and therefore, thanks to the proposition above, also for viscosity sub- and supersolutions if the group G is an homogeneous Carnot group of step two.

Theorem 4.2.8. Consider an homogeneous Carnot group $G = (\mathbb{R}^n, \circ, \delta_\lambda)$. Assume that u is a bounded weak subsolution and v is a bounded weak supersolution of (4.39). Suppose further:
(i) for any pair $(x_H, x_V),\ (x_H, y_V) \in G$, $u(x_H, x_V, 0) \leq v(x_H, y_V, 0)$;
(ii) either u or v is uniformly continuous when restricted to $G \times \{t = 0\}$.
Then $u(x, t) \leq v(x, t)$ for all $x \in G$ and $t \geq 0$.

Using this theorem we immediately get the uniqueness of a continuous weak subsolution of (4.45) but only for a particular type of uniformly continuous initial data u_0.

Corollary 4.2.9. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a bounded uniformly continuous function so that, for any fixed $x_H \in \mathbb{R}^m$, the map

$$\mathbb{R}^{n-m} \ni y \longmapsto u_0(x_H, y)$$

is a constant map. A bounded weak solution of (4.45) $u \in C(G \times [0, +\infty))$ is unique.

As last result Capogna and Citti proved the existence of a weak solution of our Cauchy problem.

Theorem 4.2.10. For any bounded continuous function $u_0 : \mathbb{R}^n \to \mathbb{R}$ there exists a weak solution $u \in C(G \times [0, +\infty))$ of (4.45).

4.2.3 The degenerate Allen-Cahn equation

We now study the limiting behavior, as ε goes to zero, of the solutions of the "degenerate Allen-Cahn equation" (4.7) when $G = (\mathbb{R}^n, \circ, \delta_\lambda)$ is an homogenous Carnot group of step two (for the precise definition see the previous section). We denote with $\sigma^{(1)}(\cdot), \dots, \sigma^{(m)}(\cdot)$ the columns of the matrix $\sigma(\cdot)$ and we define m vector fields $X_1, \dots, X_m$ as

$$X_i = \sigma^{(i)}(x) \cdot \nabla, \qquad x \in \mathbb{R}^n,\ i = 1, \dots m, \tag{4.59}$$

i.e.

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix} = \sigma^t(x) \cdot \nabla.$$

We recall that, by Theorem 4.2.2, if $X_1, \dots, X_m$ satisfy the hypothesis (**H0**), (**H1**) and (**H2**) with $r = 2$, then they generate an homogeneous Carnot group of step two.

Theorem 4.2.11. Assume that the matrix map $A \equiv \sigma^t(x)\sigma(x)$ and the vector fields $X_1, \dots, X_m$ defined in (4.59) satisfy (4.1), (4.2), (**H0**), (**H1**) and (**H2**) with $r = 2$. Moreover condition (4.5) holds with $c = 2m_o - m_+ - m_- = 0$. Finally we suppose that the functions q and p, solutions respectively of the travelling wave equation (4.13) and of (4.17), satisfy (4.14), (4.15), (4.20) and (4.21).

Let u^ε be the unique solution of the Cauchy problem (4.38), with $k \in (0, 1/3)$ and $g : \mathbb{R}^n \to [m_-, m_+]$ a continuous function such that the sets $\Gamma_o = \{x : g(x) = m_o\}$, $\Omega_o^+ = \{x : g(x) > m_o\}$, $\Omega_o^- = \{x : g(x) < m_o\}$ are nonempty and $(\Gamma_o, \Omega_o^+, \Omega_o^-) \in \mathcal{E}$. Then

$$u^\varepsilon(x,t) \longrightarrow \begin{cases} m_+ & \text{in} \quad \{u > 0\}, \\ m_- & \text{in} \quad \{u < 0\}, \end{cases}$$

locally uniformly as $\varepsilon \to 0$, where u is the unique viscosity solution of

$$\begin{cases} u_t(x,t) + F(x, Du(x,t), D^2u(x,t)) = 0 \text{ in } \mathbb{R}^n \times (0, +\infty), \\ u(x,0) = d_o(x), \end{cases} \tag{4.60}$$

where F is as in (4.40) and d_o is the signed distance to Γ_o which is positive in Ω_o^+ and negative in Ω_o^-. If in addition the no-interior condition (2.7) for the set $\{u = 0\}$ holds, then, as $\varepsilon \to 0$,

$$u^\varepsilon(x,t) \longrightarrow \begin{cases} m_+ & \text{in } \{u > 0\}, \\ m_- & \text{in } \overline{\{u > 0\}}^c, \end{cases}$$

locally uniformly.

Remark. As said before conditions (**H0**), (**H1**) and (**H2**) with $r = 2$ guarantee us that $X_1, \dots, X_m$ generate a Carnot group of step two $G = (\mathbb{R}^n, \circ, \delta_\lambda)$, where $\circ$ is the composition law on $\mathbb{R}^n$ introduced in (4.41) and $\{\delta_\lambda\}_{\lambda>0}$ is the family of dilatations defined in (**H0**). This allows us to use, in the following definition and proofs, the nice properties we prove in Lemma 4.2.5 and in Proposition 4.2.6

Before proving Theorem 4.2.11 we give a different definition of generalized super- and subflow with normal velocity F defined in (4.40) and we prove that it turns out to be equivalent to the usual Definition in 2.2.1.

Definition 4.2.12. Let F be the real-valued, locally bounded function on $\mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n$ defined in (4.40). A family $(\Omega_t)_{t\in(0,T)}$ (resp. $(\mathcal{F}_t)_{t\in(0,T)}$) of open (resp. close) subsets of $\mathbb{R}^n$ is called a *generalized superflow* (resp. *subflow*) with normal velocity $-F(x, Dd, D^2d)$ if, for any $x_0 \in \mathbb{R}^n$, $t \in (0,T)$, $r > 0$, $h > 0$ so that $t + h < T$ and for any smooth function $\phi : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ such that:

(i) $\partial\phi(x,s)/\partial t + F^*(x, D\phi(x,s), D^2\phi(x,s)) < 0$ in $B(x_0,r] \times [t, t+h]$ (resp. $\partial\phi(x,s)/\partial t + F_*(x, D\phi(x,s), D^2\phi(x,s)) > 0$ in $B(x_0,r] \times [t,t+h]$),

(ii) for any $s \in [t,t+h]$, $\{x \in B(x_0,r] : \phi(x,s) = 0\} \neq \emptyset$ and

$$|D\phi(x,s)| \neq 0 \text{ on } \{(x,s) \in B(x_0,r] \times [t,t+h] : \phi(x,s) = 0\},$$

(iii) if there exists a pair $(x,s) \in B(x_0,r] \times [t,t+h]$ so that $|\,D_H\phi(x,s)| = 0$, then it holds also $|\,D_H^2\phi(x,s)| = 0$,

(iv) $\{x \in B(x_0,r] : \phi(x,t) \geq 0\} \subset \Omega_t$ (resp. $\{x \in B(x_0,r] : \phi(x,t) \leq 0\} \subset \mathcal{F}_t^c$),

(v) for all $s \in [t,t+h]$, $\{x \in \partial B(x_0,r] : \phi(x,s) \geq 0\} \subset \Omega_s$ (resp. $\{x \in \partial B(x_0,r] : \phi(x,s) \leq 0\} \subset \mathcal{F}_s^c$),

then we have

$$\{x \in B(x_0,r] : \phi(x,s) > 0\} \subset \Omega_s, \quad \text{(resp. } \{x \in B(x_0,r] : \phi(x,s) < 0\} \subset \mathcal{F}_s^c,)$$

for every $s \in (t, t+h)$.

A family $(\Omega_t)_{t\in(0,T)}$ of open subsets of $\mathbb{R}^n$ is called a *generalized flow* with normal velocity $-F$ if $(\Omega_t)_{t\in(0,T)}$ is a superflow and $(\overline{\Omega}_t)_{t\in(0,T)}$ is a subflow.

We now state and prove the analogous result of Theorem 2.2.2.

Theorem 4.2.13. (i) Let $(\Omega_t)_{t\in(0,T)}$ be a family of open subsets of $\mathbb{R}^n$ such that the set $\Omega := \bigcup_{t\in(0,T)} \Omega_t \times \{t\}$ is open in $\mathbb{R}^n \times [0,T]$. Then $(\Omega_t)_{t\in(0,T)}$ is a generalized superflow with normal velocity $-F$, with F defined as in (4.40), if and only if the function $\chi = \mathbb{1}_\Omega - \mathbb{1}_{\Omega^c}$ is a viscosity supersolution of (4.39).

(ii) Let $(\mathcal{F}_t)_{t\in(0,T)}$ be a family of close subsets of $\mathbb{R}^n$ such that the set $\mathcal{F} := \bigcup_{t\in(0,T)} \mathcal{F}_t \times \{t\}$ is closed in $\mathbb{R}^n \times [0,T]$. Then $(\mathcal{F}_t)_{t\in(0,T)}$ is a generalized subflow with normal velocity $-F$, with F as in (4.40), if and only if the function $\overline{\chi} = \mathbb{1}_\mathcal{F} - \mathbb{1}_{\mathcal{F}^c}$ is a viscosity subsolution of (4.39).

Proof. As the proof of Theorem 2.2.2 also this one follows the ideas in [5]. Here we point out the main changes in the superflow/supersolution case. Since the new Definition 4.2.12 of generalized superflow restricts the set of the test function ϕ the proof that $(\Omega_t)_{t\in(0,T)}$ is a a generalized superflow when χ is a viscosity supersolution of the equation in (4.39) follows immediately from Theorem 2.2.2. Conversely, we assume that $(\Omega_t)_{t\in(0,T)}$ is a generalized superflow and we show that χ is a supersolution of the equation (4.39) in $\mathbb{R}^n \times (0,T)$. As in the proof of Theorem 2.2.2 we consider a point $(x,t) \in \mathbb{R}^n \times (0,T)$ and a function $\phi \in C^\infty(\mathbb{R}^n \times [0,T])$ so that (x,t) is a strict local minimum point of $\chi - \phi$ and $\phi(x,t) = 0$ and we show that

$$\frac{\partial \phi}{\partial t}(x,t) + F^*(x, D\phi(x,t), D^2\phi(x,t)) \geq 0. \tag{4.61}$$

When (x,t) is in the interior of either $\{\chi = 1\}$ or $\{\chi = -1\}$ then χ is constant in a neighborhood of (x,t) and therefore $\partial_t\phi(x,t) = 0$, $D\phi(x,t) = 0$ and $D^2\phi(x,t) \leq 0$. Since F satisfies the ellipticity condition in (2.5) and

$$F^*(x,0,0) = F_*(x,0,0) = 0$$

the inequality in (4.61) is true. Assume that $(x,t) \in \partial\{\chi = 1\} \cap \partial\{\chi = -1\}$. Thus, by the lower semicontinuity of χ, $\chi(x,t) = -1$. We suppose by contradiction that there exists an $\alpha > 0$ so that we have

$$\frac{\partial \phi}{\partial t}(x,t) + F^*(x, D\phi(x,t), D^2\phi(x,t)) < -\alpha.$$

As in Theorem 2.2.2 we can find $r, h > 0$ such that for all $(y,s) \in B(x,r] \times [t-h, t+h]$,

$$\frac{\partial \phi}{\partial t}(y,s) + F^*(y, D\phi(y,s), D^2\phi(y,s)) < -\frac{\alpha}{2}. \tag{4.62}$$

and

$$\chi(x,t)-\phi(x,t)=-1<\chi(y,s)-\phi(y,s), \quad (y,s)\neq(x,t). \tag{4.63}$$

We consider the case $|D\phi(x,t)|\neq 0$ and we introduce the test function $\phi_\delta(y,s):=\phi(y,s)+\delta(s-(t-h))$, for $0<\delta\ll 1$. In the proof of Theorem 2.2.2 we showed that ϕ_δ satisfies conditions (i), (ii), (iv) and (v) in Definition 4.2.12. We want to show that also assumption (iii) holds. Indeed, if $|D_H\phi(x,t)|\neq 0$, choosing smaller r and h, we may assume that $|D_H\phi|\neq 0$ in $B(x,r]\times[t-h,t+h]$. On the contrary if $|D_H\phi(x,t)|=0$ by Proposition 4.2.6 and Remark 4.2.7 we may also assume that $|D^2_H\phi(x,t)|=0$ and, in general, $|D^2_H\phi(y,s)|=0$ whenever $|D_H\phi(y,s)|=0$, $(y,s)\in B(x,r]\times[t-h,t+h]$. Thus ϕ_δ satisfies also assumption (iii). By the Definition 4.2.12 of superflow this yields

$$\{y\in B(x,r]:\phi_\delta(y,s)>0\}\subset\Omega_s,$$

for every $s\in(t-h,t+h)$. But, since $\phi_\delta(x,t)=\delta h>0$, we get $x\in\Omega_t$, and this is a contradiction.

Now we turn to the case when $|\,D\phi(x,t)|=0$. By Corollary 1.0.7 we have restricted the test functions so that

$$\frac{\partial^2\phi}{\partial_{x_i}\partial_{x_j}}(x,t)=\frac{\partial^3\phi}{\partial_{x_i}\partial_{x_j}\partial_{x_k}}(x,t)=\frac{\partial^4\phi}{\partial_{x_i}\partial_{x_j}\partial_{x_k}\partial_{x_l}}(x,t)=0$$

holds for any $i,j,k,l\in\{1,\dots,n\}$. To prove (4.61), we have to show that

$$\frac{\partial\phi}{\partial t}(x,t)\geq 0.$$

Suppose by contradiction that $a:=\dfrac{\partial\phi}{\partial t}(x,t)<0$. We have

$$\phi(y,s)=\underbrace{\phi(x,t)}_{=0}+\frac{\partial\phi}{\partial t}(x,t)(s-t)+o(|\,s-t|+|\,y-x|^4) \quad \text{as } s\to t,\ |\,y-x|\to 0.$$

Thus, for all $\varepsilon>0$, there exist $r=r_\varepsilon, h=h_\varepsilon, h'=h'_\varepsilon>0$ such that

$$h<-\frac{\varepsilon r^2}{a}$$

and, for any $(y,s)\in B(x,r]\times[t-h,t+h']$

$$\begin{aligned}\phi(y,s)&\geq a(s-t)+\frac{a}{2}|\,s-t|-\varepsilon|\,y-x|^4\\&=\frac{a}{2}(s-t)+a(s-t)^+-\varepsilon|\,y-x|^4\\&\geq\frac{a}{2}(s-t)-\varepsilon|\,y-x|^4+ah'.\end{aligned}$$

Let d_G be the distance function defined in (4.52). For any compact set $K \subset \mathbb{R}^n$ there exists a positive constant $C_K > 0$ so that

$$\frac{|\, x-y|}{C_K} \le d_G(x,y) \le C_K|\, x-y|^{1/2},$$

for any $x, y \in K$, (see, for example, Proposition 5.15.1 in [9]). Thus, if we put $C_r = (C_{B(x,r]})^4$, we get

$$\frac{|\, x-y|^4}{C_r} \le d_G(x,y)^4 \le C_r|\, x-y|^2$$

and

$$\phi(y,s) \ge \frac{a}{2}(s-t) - \varepsilon C_r d_G(x,y)^4 + ah'$$

for any $(y,s) \in B(x,r] \times [t-h, t+h']$. By (4.63) we can take $\beta > 0$ such that

$$2\beta + \phi(y,s) - 1 < \chi(y,s)$$

for all $(y,s) \in (B(x,r] \times \{t-h\}) \cup (\partial B(x,r) \times (t-h, t+h'))$. By taking β smaller we may also suppose $\beta < \varepsilon r^4/2$. We consider the function $\psi_\beta(y,s) = (a/2)(s-t) - \varepsilon C_r d_G(x,y)^4 + \beta$. Since we can take h' smaller we assume from now on that $h' \le -\beta/a$. Combining the last two inequalities and the assumptions on β, h, h' and r we get

$$\psi_\beta(y,s) - 1 < \chi(y,s) \tag{4.64}$$

for all $(y,s) \in (B(x,r] \times \{t-h\}) \cup (\partial B(x,r) \times [t-h, t+h'])$. Thus, with a reasoning similar to the one in Theorem 2.2.2, it is possible to prove that ψ_β satisfies conditions (iv) and (v) in Definition 4.2.12. Furthermore we consider a fixed $s \in [t-h, t+h']$. We have $\psi_\beta(x,s) = a(s-t)/2 + \beta \ge ah'/2 + \beta > 0$ and

$$\begin{aligned} \psi_\beta(y,s) &= \frac{a}{2}(s-t) - \varepsilon C_r d_G(x,y)^4 + \beta \le \frac{a}{2}(s-t) - \varepsilon|\, y-x|^4 + \beta \\ &\le -\frac{ah}{2} - \varepsilon r^4 + \beta \le -\frac{ah + \varepsilon r^4}{2} \le 0 \end{aligned}$$

for $|\, y-x| = r$. Thus the set $\{y \in B(x,r] : \psi_\beta(y,s) = 0\}$ is non empty. Let $y \in B(x,r]$, we compute $D\psi_\beta(y,s)$,

$$D\psi_\beta(y,s) = -\varepsilon C_r \left(\begin{array}{c} 4|\, y_H - x_H|^2(y_H - x_H) - 2\sum_{i=1}^{n-m}(y_{m+i} - x_{m+i} - \langle B^{(i)}x_H, y_H\rangle)B^{(i)}x_H \\ 2(y_V - x_V - \langle Bx_H, y_H\rangle). \end{array} \right)$$

Thus, since the matrices $B^{(i)}$ are skew-symmetric, $D\psi_\beta(y,s)=0$ if and only if $y=x$ and therefore $|\,D\psi_\beta(y,s)|\neq 0$ for every $(y,s)\in\{B(x,r]\times[t-h,t+h']:\psi_\beta(y,s)=0\}$. This proves that ψ_β satisfies (ii) in Definition 4.2.12. Moreover it satisfies also (iii) since, by Lemma 4.2.5,

$$|\,D_H\psi_\beta(y,s)|=0\Leftrightarrow y_H=x_H\Leftrightarrow|\,D_H^2\psi_\beta(y,s)|=0.$$

It remains to prove that (i) holds. Since $\tilde{F}^*$ is upper semicontinuous and $\tilde{F}^*(0,0)=0$, we have that

$$\frac{\partial\psi_\beta}{\partial s}(y,s)+\tilde{F}^*(X\psi_\beta(y,s),X^2\psi_\beta(y,s))=\frac{a}{2}+\tilde{F}^*(-\varepsilon C_r X_y d_G(x,y)^4,-\varepsilon C_r X_y^2 d_G(x,y)^4)<0.$$

for $(y,s)\in B(x,r]\times[t-h,t+h']$ and ε small enough.

Thus, since $(\Omega_t)_{t\in(0,T)}$ is a generalized superflow, we have

$$\{y\in B(x,r]:\psi_\beta(y,s)>0\}\in\Omega_s$$

for any $s\in(t-h,t+h')$. But again $\psi_\beta(x,t)=\beta>0$, and this means $x\in\Omega_t$, which is a contradiction. □

We are now ready to prove Theorem 4.2.11.

Proof. As in the proof of Theorem 3.3.1 we define two families of open sets of $\mathbb{R}^n$ $(\Omega_t^1)_{t\in[0,T)}$ and $(\Omega_t^2)_{t\in[0,T)}$ as in (2.14), (2.15), (2.17), with $a_\varepsilon=a=m_-$, $b_\varepsilon=b=m_+$ and $\tau=1$. By the maximum principle $m_-\leq u^\varepsilon\leq m_+$. The proof will be divided into the three usual steps, initialization, propagation and conclusion.

First step: initialization. In this first part we want to show that

$$\Omega_o^+=\{d_o>0\}\subseteq\Omega_o^1,\qquad\Omega_o^-=\{d_o<0\}\subseteq\Omega_o^2.$$

For the proof of the first inclusion (the proof of the second is very similar) we consider a point $\hat{x}$ such that $d_o(\hat{x})>0$, i.e. $g(\hat{x})>m_o$. By the continuity of g we can find an $r,\sigma>0$ such that $g(x)\geq m_o+4\sigma$, for any $x\in B(\hat{x},r)$. This means that

$$u^\varepsilon(x,0)=g(x)\geq(m_o+4\sigma)\mathbb{1}_{B(\hat{x},r)}(x)+m_-\mathbb{1}_{B(\hat{x},r)^c}(x).\tag{4.65}$$

We consider the same function $\Phi:\mathbb{R}^n\times[0,T]\to\mathbb{R}$ defined (4.30) and we state the analogous of Lemma 4.1.3 and Lemma 4.1.4.

Lemma 4.2.14. Under the same assumptions of Theorem 4.2.11 we have that for any $\beta>0$ there exist $\tau = \tau(\beta) > 0$ and $\bar{\varepsilon} = \bar{\varepsilon}(\beta) > 0$ such that, for all $0 < \varepsilon \leq \bar{\varepsilon}$, we have

$$u^\varepsilon(x, t_\varepsilon) \geq (m_+ - \beta\varepsilon)\mathbb{1}_{\{\Phi(\cdot,0)\geq\beta\}}(x) + m_-\mathbb{1}_{\{\Phi(\cdot,0)<\beta\}}(x), \quad x \in \mathbb{R}^n,$$

where $t_\varepsilon = \tau\varepsilon^2|\lg\varepsilon|$.

Lemma 4.2.15. There exist $\bar{h} = \bar{h}(r, \hat{x})$, $\bar{\beta} = \bar{\beta}(r, \hat{x}) > 0$ independent of ε such that if $\beta \leq \bar{\beta}$ and $\varepsilon \leq \bar{\varepsilon}(\beta) \wedge 1$, then there exists a subsolution $\omega^{\varepsilon,\beta}$ of (4.38-i) in $\mathbb{R}^n \times (0, \bar{h})$ that satisfies

$$\omega^{\varepsilon,\beta}(x, 0) \leq (m_+ - \beta\varepsilon)\mathbb{1}_{\{\Phi(\cdot,0)\geq\beta\}}(x) + m_-\mathbb{1}_{\{\Phi(\cdot,0)<\beta\}}(x), \quad x \in \mathbb{R}^n.$$

If moreover $(x, t) \in B(\hat{x}, r) \times (0, \bar{h})$ and $\Phi(x, t) > 3\beta$, then

$$\liminf_{\varepsilon\to 0^+}{}_* \frac{\omega^{\varepsilon,\beta}(x, t) - m_+}{\varepsilon} \geq -3\beta.$$

Proof of Lemma 4.2.14. Let $\beta > 0$ fixed. To prove our thesis we have to modify the function f. Consider a smooth cut-off $\rho \in C_0^\infty$ such that $0 \leq \rho \leq 1$ in $\mathbb{R}$, $\rho(s) = 1$ if $|s| \leq 1$ and $\rho(s) = 0$ if $|s| \geq 2$; moreover ρ satisfies $-2 \leq s\rho'(s) \leq 0$ and $|\rho''(s)| \leq 4$ for all $s \in \mathbb{R}$. Now define two smooth functions $\rho_1, \rho_2 : \mathbb{R} \to [0, 1]$ as

$$\rho_1(q) = \rho\Big(\frac{q - m_o - \sigma/2}{\sigma/2}\Big) \qquad \rho_2(q) = \rho\Big(\frac{q - m_o - \sigma/2}{\sigma/4}\Big)$$

and set

$$\bar{f}(q) = (1 - \rho_1(q))f(q) + \rho_1(q)f(q - \sigma/2)$$

and

$$\tilde{f}^\varepsilon(q) = (1 - \rho_2(q))\bar{f}(q) + \rho_2(q)\frac{\frac{\sigma}{2} + m_o - q}{|\lg\varepsilon|}.$$

Notice that $\tilde{f}^\varepsilon$ satisfies properties similar to those of f with zeroes $\{m_-, m_o + \sigma/2, m_+\}$. Moreover $f \leq \tilde{f}^\varepsilon$.

1. If we denote by $\chi = \chi(\tau, \xi) \in C^2([0, +\infty) \times \mathbb{R}^n)$ the solution of

$$\begin{cases} \dot{\chi}(\tau, \xi) + \tilde{f}^\varepsilon(\chi(\tau, \xi)) = 0, \quad \tau > 0, \\ \chi(0, \xi) = \xi, \end{cases}$$

it follows that χ satisfies property ($\chi 1$) (stated in the proof of Lemma 4.1.4), while properties ($\chi 2$) and ($\chi 3$) are replaced by the following ones:

there exists $\tau_o = \tau_o(\beta,\sigma), \varepsilon_o = \varepsilon_o(\beta,\sigma) > 0$ such that, for all $\tau \geq \tau_o|\lg\varepsilon|$ and $\varepsilon \leq \varepsilon_o$

$$\chi(\tau,\xi) \geq m_+ - \beta\varepsilon \quad \forall\,\xi \geq m_o + \frac{3}{2}\sigma. \tag{$\tilde{\chi}2$}$$

Moreover for any $C \geq \max\{|\,m_-|,|\,m_+|\}$ we have $\chi(\tau,\xi) \in [-C,C]$ for all $\xi \in [-C,C]$, $\tau \geq 0$. Thus, for any $C \geq \max\{|\,m_-|,|\,m_+|\}$, $a > 0$ there exists a constant $M_{C,a} > 0$ such that

$$|\chi_{\xi\xi}(\tau,\xi)| \leq \frac{M_{C,a}}{\varepsilon}\chi_\xi(\tau,\xi), \tag{$\tilde{\chi}3$}$$

for any $\tau \leq a|\ln\varepsilon|$, $\xi \in [-C,C]$, $i \in \{1,2,\cdots,n\}$ and ε small enough.

2. Consider a smooth nondecreasing function ψ such that $\psi(z) = m_-$ if $z \leq 0$ and $\psi(z) = m_o + 4\sigma$ if $z \geq \beta \wedge \frac{\sigma}{2}$. The function $\underline{u}^\varepsilon$ defined by

$$\underline{u}^\varepsilon(x,t) = \chi\Big(\frac{t}{\varepsilon^2}, \psi(\Phi(x,0)) - \frac{Kt}{\varepsilon}\Big)$$

is a subsolution of (4.38-i) in $\mathbb{R}^n \times (0, \tau_o\varepsilon^2|\lg\varepsilon|)$ and it satisfies $\underline{u}^\varepsilon(x,0) \leq u^\varepsilon(x,0)$. In fact, since $f \leq \tilde{f}^\varepsilon$,

$$\begin{aligned} \underline{u}^\varepsilon_t - \mathrm{tr}(D^2_{H,\varepsilon}\underline{u}^\varepsilon) + \frac{f(u^\varepsilon)}{\varepsilon^2} &\leq \frac{\dot{\chi} + f(\chi)}{\varepsilon^2} + \frac{\chi_\xi}{\varepsilon}(-K + O(\varepsilon)) \\ &\leq \frac{\chi_\xi}{\varepsilon}(-K + O(\varepsilon)) \leq 0, \end{aligned}$$

for ε small enough. Thus using the maximum principle and property ($\tilde{\chi}2$) we can prove that $u^\varepsilon(x,t_\varepsilon) \geq 1 - \beta\varepsilon$ if $t_\varepsilon = \tau_o\varepsilon^2|\lg\varepsilon|$ and $\Phi(x,0) \geq \beta$ (from which Lemma 4.2.14 follows). □

Proof of Lemma 4.2.15. As in Lemma 4.1.4 we have that there exist γ, $\bar{h} > 0$ such that $\bar{h} < r^2/C$ and

$$D\Phi(x,t) \neq 0,\ D_{H,\varepsilon}\Phi(x,t) \neq 0,$$

for any $(x,t) \in Q_{\gamma,\bar{h}} = \{|\Phi(x,t)| \leq \gamma,\, 0 \leq t \leq \bar{h}\}$, $\varepsilon > 0$. We assume from now on that the constant C that appears in the definition of Φ in (4.30) satisfies

$$\begin{aligned} C \geq 2(n+1)\sup\{\big[\|\,D^2_H\Phi(x,t)\| + \|\,\sigma^t(x)D^2\Phi(x,t)\| + \|\,D(\sigma^t(x)D\Phi(x,t))\| \\ + \|\,D^2\Phi(x,t)\|\big], \quad (x,t) \in Q_{\gamma,\bar{h}}\,\}, \end{aligned} \tag{4.66}$$

where $\|\cdot\|$ denotes a matrix norm. We now construct a subsolution of (4.38-i).

1. For any $(x,t) \in Q_{\gamma,\bar{h}}$ we define a smooth function $v^\varepsilon(x,t)$ as

$$v^\varepsilon(x,t) = Q\Big(\frac{\Phi(x,t) - 2\beta}{\varepsilon}, x, t\Big) + \varepsilon\Big[P\Big(\frac{\Phi(x,t) - 2\beta}{\varepsilon}, x, t\Big) - 2\beta\Big],$$

with $Q, P \in C^2(\mathbb{R} \times \mathbb{R}^n \times [0,+\infty);\mathbb{R})$ two suitable functions that we will choose later. If we put v^ε inside (4.38-i) we obtain

$$v_t^\varepsilon(x,t) - \mathrm{tr}(D^2_{H,\varepsilon} v^\varepsilon(x,t)) + \frac{f(v^\varepsilon(x,t))}{\varepsilon^2} = \frac{I_\varepsilon}{\varepsilon^2} + \frac{II_\varepsilon}{\varepsilon} + III_\varepsilon$$

where

$$I_\varepsilon = -\ddot{Q}|D_{H,\varepsilon}\Phi|^2 + f(Q), \tag{4.67}$$

$$II_\varepsilon = \dot{Q}\big(\Phi_t - \mathrm{tr}(D^2_{H,\varepsilon}\Phi)\big) - 2\langle D_{H,\varepsilon}\dot{Q}, D_{H,\varepsilon}\Phi\rangle - \ddot{P}|D_{H,\varepsilon}\Phi|^2 + f'(Q)(P-2\beta), \tag{4.68}$$

and

$$\begin{aligned} III_\varepsilon = Q_t - \mathrm{tr}(D^2_{H,\varepsilon}Q) + \varepsilon[P_t - \mathrm{tr}(D^2_{H,\varepsilon}P)] + \dot{P}\big(\Phi_t - \mathrm{tr}(D^2_{H,\varepsilon}\Phi)\big) \\ - 2\langle D_{H,\varepsilon}\dot{P}, D_{H,\varepsilon}\Phi\rangle + \frac{\| f''_{|[m_-,m_+]}\|_\infty}{2}(P-2\beta)^2. \end{aligned} \tag{4.69}$$

If we set

$$Q(s,x,t) = q\Big(\frac{s}{|D_{H,\varepsilon}\Phi(x,t)|}\Big) \tag{4.70}$$

where q is the solution of the traveling wave equation (4.13) with $c=0$, we obtain

$$I_\varepsilon = 0,$$

$$II_\varepsilon = \overline{II}_\varepsilon - 2\beta f'(q(\lambda)),$$

with $\lambda = \dfrac{\Phi(x,t) - 2\beta}{\varepsilon| D_{H,\varepsilon}\Phi(x,t)|}$ and

$$\begin{aligned} \overline{II}_\varepsilon = \frac{\dot{q}(\lambda)}{| D_{H,\varepsilon}\Phi|}\big(\Phi_t - \mathrm{tr}(D^2_{H,\varepsilon}\Phi)\big) - 2\Big(\ddot{q}(\lambda)\lambda + \dot{q}(\lambda)\Big)\langle D_{H,\varepsilon}\Phi, D_{H,\varepsilon}\big(\frac{1}{| D_{H,\varepsilon}\Phi|}\big)\rangle \\ - \ddot{P}|D_{H,\varepsilon}\Phi|^2 + f'(q(\lambda))P. \end{aligned}$$

If we put $s = \dfrac{\Phi(x,t) - 2\beta}{\varepsilon}$ and

$$\tilde{\chi}(s,x,t) = \Big(\frac{2\ddot{q}\big(\frac{s}{| D_{H,\varepsilon}\Phi(x,t)|}\big)s}{| D_{H,\varepsilon}\Phi(x,t)|} + \dot{q}\big(\frac{s}{| D_{H,\varepsilon}\Phi(x,t)|}\big)\Big)\frac{\langle (D^2_{H,\varepsilon}\Phi(x,t))D_{H,\varepsilon}\Phi(x,t), D_{H,\varepsilon}\Phi(x,t)\rangle}{| D_{H,\varepsilon}\Phi(x,t)|^3},$$

then

$$\overline{II}_\varepsilon = \frac{\dot q(\lambda)}{|\, D_{H,\varepsilon}\Phi(x,t)|}\Big[\Phi_t(x,t) - \mathrm{tr}(D^2_{H,\varepsilon}\Phi(x,t)) + \frac{\langle D^2_{H,\varepsilon}\Phi(x,t)D_{H,\varepsilon}\Phi(x,t), D_{H,\varepsilon}\Phi(x,t)\rangle}{|\, D_{H,\varepsilon}\Phi(x,t)|^2}\Big]+$$
$$+\,\tilde\chi\Big(\frac{\Phi(x,t)-2\beta}{\varepsilon}, x, t\Big) - \ddot P|D_{H,\varepsilon}\Phi(x,t)|^2 + f'(q)P.$$

With a simple computation we can see that

$$\int_{-\infty}^{+\infty}\tilde\chi(s,x,t)\dot q\Big(\frac{s}{|\, D_{H,\varepsilon}\Phi(x,t)|}\Big)ds = 0.$$

Therefore by (4.18) there exists a unique $p \in C^2(\mathbb{R})$ so that $-\ddot p + f'(q(\frac{s}{|\, D_{H,\varepsilon}\Phi(x,t)|}))p = -\tilde\chi(s,x,t)$. We put

$$P(s,x,t) = p\Big(\frac{s}{|\, D_{H,\varepsilon}\Phi(x,t)|}\Big)$$

inside $\overline{II}_\varepsilon$ and we obtain, by assumption (4.66) on C,

$$\begin{aligned}\overline{II}_\varepsilon &= \frac{\dot q(\lambda)}{|\, D_{H,\varepsilon}\Phi|}\Big(-C - \mathrm{tr}(D^2_{H,\varepsilon}\Phi) + \frac{\langle (D^2_{H,\varepsilon}\Phi)D_{H,\varepsilon}\Phi, D_{H,\varepsilon}\Phi\rangle}{|\, D_{H,\varepsilon}\Phi|^2}\Big)\\ &\le \frac{\dot q(\lambda)}{|\, D_{H,\varepsilon}\Phi|}\Big(-C + (n+1)\|\, D^2_{H,\varepsilon}\Phi\|\Big) \le -\frac{C\dot q(\lambda)}{2|\, D_{H,\varepsilon}\Phi|}.\end{aligned}$$

Furthermore with this choice of Q and P, III_ε takes the form

$$III_\varepsilon = (\dot q + \varepsilon\dot p)s\Big(\underbrace{\frac{\partial}{\partial t}\Big(\frac{1}{|\, D_{H,\varepsilon}\Phi|}\Big)}_{=0} - \mathrm{tr}(D^2_{H,\varepsilon}\Big(\frac{1}{|\, D_{H,\varepsilon}\Phi|}\Big))\Big) - (\ddot q + \varepsilon\ddot p)s^2|\, D_{H,\varepsilon}\Big(\frac{1}{|\, D_{H,\varepsilon}\Phi|}\Big)|^2$$

$$+\frac{\dot p}{|\, D_{H,\varepsilon}\Phi|}(\Phi_t - \mathrm{tr}(D^2_{H,\varepsilon}\Phi)) - 2\Big(\frac{\ddot p s}{|\, D_{H,\varepsilon}\Phi|} + \dot p\Big)\langle D_{H,\varepsilon}\Big(\frac{1}{|\, D_{H,\varepsilon}\Phi|}\Big), D_{H,\varepsilon}\Phi\rangle + \frac{\|\, f''_{|[m_-,m_+]}\|_\infty}{2}(p-2\beta)^2,$$

and then, using the same notations as those of Lemma 4.1.4 and properties (4.14), (4.21) of q and p,

$$\begin{aligned}III_\varepsilon &\le (\dot q + \varepsilon|\,\dot p|)|\, s|\Big[\frac{\mathrm{tr}((D^2_{H,\varepsilon}\Phi)^2)}{|\, D_{H,\varepsilon}\Phi|^3} + \frac{\mathrm{tr}(A^\varepsilon(x)\lfloor D^2(D_{H,\varepsilon}\Phi)\rfloor)}{|\, D_{H,\varepsilon}\Phi|^2} + \frac{\mathrm{tr}(\lfloor\sigma^t(x)D\sigma^t(x)D(D_{H,\varepsilon}\Phi)\rfloor)}{|\, D_{H,\varepsilon}\Phi|^2}\Big]\\ &+2|\,\ddot q + \varepsilon\ddot p|s^2\frac{\|\, D^2_{H,\varepsilon}\Phi\|^2}{|\, D_{H,\varepsilon}\Phi|^4} + \frac{\dot p}{|\, D_{H,\varepsilon}\Phi|}(\Phi_t - \mathrm{tr}(D^2_{H,\varepsilon}\Phi)) + 2\Big(\frac{|\,\ddot p||\, s|}{|\, D_{H,\varepsilon}\Phi|} + \dot p\Big)\frac{\|\, D^2_{H,\varepsilon}\Phi\|}{|\, D_{H,\varepsilon}\Phi|} + O(1)\\ &\le O\Big(\frac{1}{|\, D_{H,\varepsilon}\Phi|^3} + \frac{1}{|\, D_{H,\varepsilon}\Phi|^2} + \frac{1}{|\, D_{H,\varepsilon}\Phi|}\Big) + O(1) = O\Big(\frac{1}{|\, D_{H,\varepsilon}\Phi|^3}\Big) + O(1)\\ &\le O\big(\varepsilon^{-3k}\big) + O(1).\end{aligned}$$

Therefore, since $k < 1/3$,

$$\begin{aligned} v_t^\varepsilon - \mathrm{tr}(D^2_{H,\varepsilon} v^\varepsilon) + \frac{f(v^\varepsilon)}{\varepsilon^2} &= \frac{I_\varepsilon}{\varepsilon^2} + \frac{II_\varepsilon}{\varepsilon} + III_\varepsilon \\ &\leq \frac{1}{\varepsilon}\Big[- \frac{C}{2}\frac{\dot q(\lambda)}{|\, D_{H,\varepsilon}\Phi|} - 2\beta f'(q(\lambda))\Big] + O\big(\varepsilon^{-3k}\big) + O(1) \\ &= \frac{1}{\varepsilon}\Big[- \frac{C}{2}\frac{\dot q(\lambda)}{|\, D_{H,\varepsilon}\Phi|} - 2\beta f'(q(\lambda)) + o_\varepsilon(1)\Big]. \end{aligned}$$

To prove that v^ε is a subsolution of (4.38-i) in $Q_{\gamma,\bar h}$ it remains to see that the right hand side of the inequality above is non positive. As in Lemma 4.1.4 we recall that $f'(m_\pm) > 0$ and $q(r) \to m_\pm$ if $r \to \pm\infty$; let $M > 0$ be a positive constant so that

$$f'(q(r)) \geq \frac{d}{2}, \quad \text{for any } |r| \geq M.$$

Therefore, if $\dfrac{|\Phi - 2\beta|}{\varepsilon|D_{H,\varepsilon}\Phi|} \geq M$, we have that

$$v_t^\varepsilon(x,t) - \mathrm{tr}(D^2_{H,\varepsilon} v^\varepsilon(x,t)) + \frac{f(v^\varepsilon(x,t))}{\varepsilon^2} \leq \frac{1}{\varepsilon}\Big[- \beta d + o_\varepsilon(1)\Big] < -\frac{\beta d}{2\varepsilon}$$

for ε small enough. If on the contrary $\dfrac{|\Phi - 2\beta|}{\varepsilon|D_{H,\varepsilon}\Phi|} < M$ and we denote with K a strictly positive constant (which depends by M) so that $\dot q(r) \geq K$ for any $|r| \leq M$, we get that there exists a $\mu = \mu(\beta)$, $0 < \mu \leq \beta d/2$ so that, for β small compared to K,

$$\begin{aligned} v_t^\varepsilon - \mathrm{tr}(D^2_{H,\varepsilon} v^\varepsilon) + \frac{f(v^\varepsilon)}{\varepsilon^2} &\leq \frac{1}{\varepsilon}[-\frac{CK}{2|\, D_{H,\varepsilon}\Phi|} + 2\beta\|\, f'_{|[m_-,m_+]}\| + o_\varepsilon(1)] \\ &\leq \frac{1}{\varepsilon}[-2\mu(\beta) + o_\varepsilon(1)] \leq -\frac{\mu}{\varepsilon}. \end{aligned}$$

for small ε.

2. Once we have proved that v^ε is a strict subsolution of (4.38-i) in $Q_{\gamma,\bar h}$ we define, for each $(x,t) \in \{(x,t) \in \mathbb{R}^n \times [0,\bar h] : \Phi(x,t) \leq \gamma\}$,

$$\bar v^\varepsilon(x,t) = \begin{cases} \sup(v^\varepsilon(x,t), m_-) & \text{if } \; -\gamma < \Phi(x,t) \leq \gamma, \\ m_- & \text{if } \; \Phi(x,t) \leq -\gamma. \end{cases}$$

We prove that $\bar v^\varepsilon$ is a continuous viscosity subsolution of equation (4.38-i) in $\{(x,t) \in \mathbb{R}^n \times [0,\bar h] : \Phi(x,t) \leq \gamma\}$, for ε sufficiently small. This in obvious in the set $\{|\,\Phi| \leq \gamma\}$ since $\bar v^\varepsilon$ is the supremum of two subsolutions. Consider a point (x,t) such that $\Phi(x,t) \leq -\gamma/2$; by properties (4.14) and

(4.20) we have that

$$v^\varepsilon(x,t) \le m_- + ae^{-\frac{b(\gamma+4\beta)}{2\varepsilon|\,D_{H,\varepsilon}\Phi|}} + \varepsilon(o_\varepsilon(1) - 2\beta) \le m_-$$

for small ε and thus $\bar v^\varepsilon(x,t) = m_-$. Therefore $\bar v^\varepsilon$ is a subsolution of (4.38-i) in $\{(x,t) \in \mathbb{R}^n \times [0,\bar h] : \Phi(x,t) \le \gamma\}$.

3. Finally we can define our function $\omega^{\varepsilon,\beta} : \mathbb{R}^n \times [0,\bar h] \to \mathbb{R}$ as

$$\omega^{\varepsilon,\beta}(x,t) = \begin{cases} \psi(\Phi(x,t))\bar v^\varepsilon(x,t) + (1-\psi(\Phi(x,t)))(m_+ - \beta\varepsilon) & \text{if } \Phi(x,t) < \gamma, \\ m_+ - \beta\varepsilon & \text{if } \Phi(x,t) \ge \gamma, \end{cases}$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\psi' \le 0$ in $\mathbb{R}$, $\psi = 1$ in $(-\infty, \gamma/2]$, $0 < \psi < 1$ in $(\gamma/2, 3\gamma/4)$ and $\psi = 0$ in $[3\gamma/4, +\infty)$. The only subset of $\mathbb{R}^n \times (0,\bar h)$ in which we have to check that $\omega^{\varepsilon,\beta}$ is a subsolution of (4.38-i) is $\{(x,t) \in \mathbb{R}^n \times (0,\bar h) : \gamma/2 \le \Phi(x,t) \le 3\gamma/4\}$. We have

$$\begin{aligned} \omega_t^{\varepsilon,\beta} - \mathrm{tr}(D^2_{H,\varepsilon}\omega^{\varepsilon,\beta}) + \frac{f(\omega^{\varepsilon,\beta})}{\varepsilon^2} &= [\psi'(\Phi_t - \mathrm{tr}(D^2_{H,\varepsilon}\Phi)) - \psi''|\,D_{H,\varepsilon}\Phi|^2](\bar v^\varepsilon - (m_+ - \beta\varepsilon)) \\ &\quad + \psi(\bar v^\varepsilon_t - \mathrm{tr}(D^2_{H,\varepsilon}\bar v^\varepsilon)) - 2\psi'\langle D_{H,\varepsilon}\Phi, D_{H,\varepsilon}\bar v^\varepsilon\rangle + \frac{f(\omega^{\varepsilon,\beta})}{\varepsilon^2} \end{aligned} \tag{4.71}$$

If we take $2\beta < \gamma/4$ we have $\Phi(x,t) - 2\beta > \gamma/4$ and $v^\varepsilon(x,t) \ge m_+ - ae^{\frac{-b\gamma}{4\varepsilon|\,D_{H,\varepsilon}\Phi|}} + \varepsilon(o_\varepsilon(1) - 2\beta)$; thus, for small ε, $\bar v^\varepsilon(x,t) = v^\varepsilon(x,t)$ and $v^\varepsilon(x,t) - (m_+ - \beta\varepsilon) \le \varepsilon(o_\varepsilon(1) - \beta) \le 0$. Moreover by (4.5) f is convex in a neighborhood of m_+ and $f(\omega^{\varepsilon,\beta}) \le \psi f(v^\varepsilon) + (1-\psi)f(m_+ - \beta\varepsilon)$. The equality (4.71) thus becomes

$$\begin{aligned} \omega_t^{\varepsilon,\beta} - \mathrm{tr}(D^2_{H,\varepsilon}\omega^{\varepsilon,\beta}) + \frac{f(\omega^{\varepsilon,\beta})}{\varepsilon^2} &\le [\psi'(-C - \mathrm{tr}(D^2_{H,\varepsilon}\Phi)) - \psi''|\,D_{H,\varepsilon}\Phi|^2](v^\varepsilon - (m_+ - \beta\varepsilon)) \\ &\qquad -\psi\frac{\mu}{\varepsilon} - 2\psi'\langle D_{H,\varepsilon}\Phi, D_{H,\varepsilon}v^\varepsilon\rangle + (1-\psi)\frac{f(m_+ - \beta\varepsilon)}{\varepsilon^2} \\ &\le -\psi\frac{\mu}{\varepsilon} + (1-\psi)\frac{f(m_+ - \beta\varepsilon)}{\varepsilon^2} \\ &\qquad -\psi''|\,D_{H,\varepsilon}\Phi|^2(v^\varepsilon - (m_+ - \beta\varepsilon)) - 2\psi'\langle D_{H,\varepsilon}\Phi, D_{H,\varepsilon}v^\varepsilon\rangle \\ &= -\psi\frac{\mu}{\varepsilon} + (1-\psi)\frac{f(m_+ - \beta\varepsilon)}{\varepsilon^2} + O(1) + O(\varepsilon^{k-1}). \end{aligned}$$

Finally we observe that for ε small and by assumption (4.5) on f,

$$\frac{f(m_+ - \beta\varepsilon)}{\varepsilon^2} = \frac{f(m_+ - \beta\varepsilon) - f(m_+)}{\varepsilon^2} \le -\frac{d\beta}{\varepsilon} < 0.$$

Thus

$$\omega_t^{\varepsilon,\beta} - \mathrm{tr}(D^2_{H,\varepsilon}\omega^{\varepsilon,\beta}) + \frac{f(\omega^{\varepsilon,\beta})}{\varepsilon^2} \le \frac{1}{\varepsilon}[-\psi\mu - (1-\psi)d\beta + O(\varepsilon + \varepsilon^k)] < 0$$

for ε small enough.

4. Now we want to examine $\omega^{\varepsilon,\beta}(\cdot,0)$. To this end we first observe that, by (4.20), there exists a $\bar{c}>0$ such that, if $|\,u|\geq\bar{c}$, then

$$|\,p(u)|\leq\beta;$$

in particular if $(x,t)\in Q_{\gamma,\bar{h}}$ is such that $|\,\Phi(x,t)-2\beta|\geq\bar{c}\varepsilon|\,D_{H,\varepsilon}\Phi(x,t)|$, then

$$v^{\varepsilon}(x,t)\leq q\Big(\frac{\Phi(x,t)-2\beta}{\varepsilon|\,D_{H,\varepsilon}\Phi(x,t)|}\Big)-\beta\varepsilon\leq m_{+}-\beta\varepsilon. \tag{4.72}$$

We consider the case $|\,\Phi(x,t)-2\beta|\leq\bar{c}\varepsilon|\,D_{H,\varepsilon}\Phi(x,t)|$, $(x,t)\in Q_{\gamma,\bar{h}}$. Let $\nu(\bar{c})>0$ be a positive constant so that, for any $|\,u|\leq\bar{c}$,

$$q(u)\leq m_{+}-\nu(\bar{c}).$$

Therefore, for ε small enough,

$$v^{\varepsilon}(x,t)\leq m_{+}-\nu(\bar{c})+\varepsilon(\|\,p\|_{\infty}-2\beta)\leq m_{+}-2\beta\varepsilon. \tag{4.73}$$

Combining the estimates in (4.72) and (4.73) for v^{ε} we can conclude that $\omega^{\varepsilon,\beta}(x,t)\leq m_{+}-\beta\varepsilon$ for any $(x,t)\in\mathbb{R}^{n}\times[0,\bar{h}]$ and in particular for $t=0$. If we also assume that ε is such that

$$\sup_{(x,t)\in Q_{\gamma,\bar{h}}}\bar{c}\varepsilon|\,D_{H,\varepsilon}\Phi(x,t)|\leq\beta,$$

we get that for any $(x,t)\in Q_{\gamma,\bar{h}}$ so that $\Phi(x,t)<\beta$ it holds $\Phi(x,t)-2\beta<-\beta\leq-\bar{c}\varepsilon|D_{H,\varepsilon}\Phi(x,t)|$ and

$$v^{\varepsilon}(x,t)\leq q\Big(\frac{-\beta}{\varepsilon|\,D_{H,\varepsilon}\Phi(x,t)|}\Big)-\beta\varepsilon\leq m_{-}+ae^{-\frac{b\beta}{\varepsilon|\,D_{H,\varepsilon}\Phi|}}-\beta\varepsilon\leq m_{-}.$$

Therefore if $\Phi(x,t)<\beta$ then $\bar{v}^{\varepsilon}(x,t)=m_{-}$ and, since we have assumed $\beta<\gamma/8$, $\omega^{\varepsilon,\beta}(x,t)=m_{-}$. We conclude that

$$\omega^{\varepsilon,\beta}(x,t)\leq(m_{+}-\beta\varepsilon)\mathbb{1}_{\{\Phi\geq\beta\}}(x,t)+m_{-}\mathbb{1}_{\{\Phi<\beta\}}(x,t),$$

for any $(x,t)\in\mathbb{R}^{n}\times[0,\bar{h}]$, and in particular for $t=0$.

5. Finally we just remark that, with a reasoning similar to the one in point 4., one can prove that if $(x,t)\in B(\hat{x},r)\times(0,\bar{h})$, $\gamma\geq\Phi(x,t)>3\beta$, and ε is so small to have $\bar{c}\varepsilon\max\limits_{B(\hat{x},r)}|\,D_{H,\varepsilon}\Phi|\leq\beta$, then

$$v^{\varepsilon}(x,t)\geq q(\frac{\beta}{\varepsilon|\,D_{H,\varepsilon}\Phi|})-3\beta\varepsilon\geq m_{+}-ae^{-\frac{b\beta}{\varepsilon|\,D_{H,\varepsilon}\Phi|}}-3\beta\varepsilon.$$

Hence $\liminf_{\varepsilon\to0^+}{}_* \dfrac{\omega^{\varepsilon,\beta}(x,t)-m_+}{\varepsilon} \geq -3\beta$ for any $(x,t)\in B(\hat{x},r)\times(0,\bar{h})$ with $\Phi(x,t)>3\beta$. □

Second step: propagation. In this step we show that $(\Omega_t^1)_{t\in(0,T)}$ and $((\Omega_t^2)^c)_{t\in(0,T)}$ are respectively super and subflow with normal velocity $-F$ with F defined as in (4.40). Since the two proofs are similar we only show that $(\Omega_t^1)_{t\in(0,T)}$ is a superflow. Let $x_0\in\mathbb{R}^n$, $t\in(0,T)$, $r>0$, $h>0$ so that $t+h<T$. Suppose that $\phi:\mathbb{R}^n\times[0,T]\to\mathbb{R}$ is a smooth function such that, for a suitable $\tilde{C}>0$,

(i) $\phi_t(x,s)+F^*(x,D\phi(x,s),D^2\phi(x,s))\leq-\tilde{C}<0$, for all $(x,s)\in B(x_0,r]\times[t,t+h]$,

(ii) for any $s\in[t,t+h]$, $\{x\in B(x_0,r]:\phi(x,s)=0\}\neq\emptyset$ and

$$|D\phi(x,s)|\neq0 \text{ on } \{(x,s)\in B(x_0,r]\times[t,t+h]:\phi(x,s)=0\},$$

(iii) if there exists $(x,s)\in B(x_0,r]\times[t,t+h]$ so that $|\,D_H\phi(x,s)|=0$, then it holds also $|\,D_H^2\phi(x,s)|=0$,

(iv) $\{x\in B(x_0,r]:\phi(x,t)\geq0\}\subset\Omega_t^1$,

(v) for all $s\in[t,t+h]$, $\{x\in\partial B(x_0,r]:\phi(x,s)\geq0\}\subset\Omega_s^1$.

We have to show that for every $s\in(t,t+h)$,

$$\{x\in B(x_0,r]:\phi(x,s)>0\}\subset\Omega_s^1.$$

Using the assumptions and the definition of $(\Omega_t^1)_{t\in(0,T)}$ this is equivalent to prove that for all $x\in B(x_0,r)$, $s\in(t,t+h)$ such that $\phi(x,s)>0$, we have

$$\liminf_{\varepsilon\to0^+}{}_*\left(\frac{u^\varepsilon(y,\tau)-m_+}{\varepsilon}\right)\geq0 \tag{4.74}$$

for (y,τ) in a neighborhood of (x,s). As in Theorem 3.1.1 to prove this result it is enough to prove the following lemma.

Lemma 4.2.16. Let ϕ be a smooth function as above. There exists $\bar{\beta}$ small enough such that, if $\beta\leq\bar{\beta}$ and $\varepsilon\leq\bar{\varepsilon}(\beta)$ then there is a viscosity subsolution $\omega^{\varepsilon,\beta}$ of (4.38-i) in $B(x_0,r)\times(t,t+h)$ that satisfies,

1. $\omega^{\varepsilon,\beta}(x,t)\leq(m_+-\beta\varepsilon)\mathbb{1}_{\{\phi(\cdot,t)\geq\beta\}}(x)+m_-\mathbb{1}_{\{\phi(\cdot,t)<\beta\}}(x)$, for all $x\in B(x_0,r]$,
2. $\omega^{\varepsilon,\beta}(x,s)\leq(m_+-\beta\varepsilon)\mathbb{1}_{\{\phi(\cdot,s)\geq\beta\}}(x)+m_-\mathbb{1}_{\{\phi(\cdot,s)<\beta\}}(x)$, for all $x\in\partial B(x_0,r]$, $s\in[t,t+h]$
3. if $(x,s)\in B(x_0,r]\times[t,t+h]$ satisfies $\phi(x,s)>3\beta$, then

$$\liminf_{\varepsilon\to0^+}{}_*\frac{\omega^{\varepsilon,\beta}(x,s)-m_+}{\varepsilon}\geq-3\beta.$$

Proof of Lemma 4.2.16. This proof is similar to the one of Lemma 4.2.15 and we just point out the main changes. First of all we observe that since ϕ satisfies property (ii) above we have that there exists $\gamma > 0$ such that $|D\phi(x,s)| \neq 0$ in the set $Q_\gamma = \{(x,s) \in B(x_0,r] \times [t,t+h] : |\phi(x,s)| \leq \gamma\}$. Obviously this also means $|D_{H,\varepsilon}\phi(x,s)| \neq 0$ for any $(x,s) \in Q_\gamma$, $\varepsilon > 0$. As in Lemma 4.2.15 we construct our subsolution by steps and to do this we first define a function v^ε in Q_γ as

$$v^\varepsilon(x,s) = Q\Big(\frac{\phi(x,s) - 2\beta}{\varepsilon}, x, s\Big) + \varepsilon\Big[P\Big(\frac{\phi(x,s) - 2\beta}{\varepsilon}, x, s\Big) - 2\beta\Big].$$

Let $(x,s) \in Q_\gamma$. With the usual computations it turns out that

$$v^\varepsilon_t(x,s) - \mathrm{tr}(D^2_{H,\varepsilon} v^\varepsilon(x,s)) + \frac{f(v^\varepsilon(x,s))}{\varepsilon^2} = \frac{I_\varepsilon}{\varepsilon^2} + \frac{II_\varepsilon}{\varepsilon} + III_\varepsilon,$$

where I_ε, II_ε, III_ε are exactly the same terms defined in (4.67), (4.68) and (4.69) with the function Φ replaced by ϕ. We put

$$Q(a,x,s) = q\Big(\frac{a}{|D_{H,\varepsilon}\phi(x,s)|}\Big),$$
$$P(a,x,s) = p\Big(\frac{a}{|D_{H,\varepsilon}\phi(x,s)|}\Big)$$

in (4.67), (4.68) and we get $I_\varepsilon = \ddot{q}(\lambda) + f(q(\lambda)) = 0$ and

$$\begin{aligned} II_\varepsilon &= \overline{II}_\varepsilon - 2\beta f'(q(\lambda)) \\ &= \frac{\dot{q}(\lambda)}{|D_{H,\varepsilon}\phi|}\Big(\phi_t - \mathrm{tr}(D^2_{H,\varepsilon}\phi) + \frac{\langle (D^2_{H,\varepsilon}\phi) D_{H,\varepsilon}\phi, D_{H,\varepsilon}\phi\rangle}{|D_{H,\varepsilon}\phi|^2}\Big) - 2\beta f'(q(\lambda)), \end{aligned}$$

with $\lambda = \dfrac{\phi(x,s) - 2\beta}{\varepsilon|D_{H,\varepsilon}\phi(x,s)|}$. By property (i) of ϕ, $\phi_t \leq -\tilde{C} - F^*(x, D\phi, D^2\phi)$ and thus

$$\begin{aligned} \overline{II}_\varepsilon &= \frac{\dot{q}(\lambda)}{|D_{H,\varepsilon}\phi|}\Big(\phi_t - \mathrm{tr}(A^\varepsilon(x)D^2\phi + \sigma^t(x)D\sigma^t(x)D\phi) + \frac{\langle (D^2_{H,\varepsilon}\phi) D_{H,\varepsilon}\phi, D_{H,\varepsilon}\phi\rangle}{|D_{H,\varepsilon}\phi|^2}\Big) \\ &\leq \frac{\dot{q}(\lambda)}{|D_{H,\varepsilon}\phi|}\Big(-\tilde{C} - \langle (D^2_H\phi)\frac{D_H\phi}{|D_H\phi|}, \frac{D_H\phi}{|D_H\phi|}\rangle^* - \varepsilon^{2k}\,\mathrm{tr}(D^2\phi) + \frac{\langle (D^2_{H,\varepsilon}\phi) D_{H,\varepsilon}\phi, D_{H,\varepsilon}\phi\rangle}{|D_{H,\varepsilon}\phi|^2}\Big) \end{aligned} \tag{4.75}$$

Since

$$\begin{aligned} \frac{\langle (D^2_{H,\varepsilon}\phi) D_{H,\varepsilon}\phi, D_{H,\varepsilon}\phi\rangle}{|D_{H,\varepsilon}\phi|^2} &= \frac{\langle (D^2_H\phi) D_H\phi, D_H\phi\rangle}{|D_{H,\varepsilon}\phi|^2} + 2\varepsilon^{2k}\frac{\langle (\sigma^t(x)D^2\phi) D\phi, D_H\phi\rangle}{|D_{H,\varepsilon}\phi|^2} \\ &\quad + \varepsilon^{2k}\frac{\langle (D\sigma^t(x)D\phi) D_H\phi, D\phi\rangle}{|D_{H,\varepsilon}\phi|^2} + \varepsilon^{4k}\frac{\langle (D^2\phi) D\phi, D\phi\rangle}{|D_{H,\varepsilon}\phi|^2} \\ &\leq \frac{\langle (D^2_H\phi) D_H\phi, D_H\phi\rangle}{|D_{H,\varepsilon}\phi|^2} + \underbrace{\varepsilon^k\|\sigma^t(x)D^2\phi\| + \varepsilon^k\|D\sigma^t(x)D\phi\| + \varepsilon^{2k}\|D^2\phi\|}_{o_\varepsilon(1)} \end{aligned}$$

the inequality in (4.75) becomes

$$\overline{II}_\varepsilon \leq \frac{\dot{q}(\lambda)}{|D_{H,\varepsilon}\phi|}\Big(- \tilde{C} - \langle (D^2_H\phi)\frac{D_H\phi}{|\, D_H\phi|}, \frac{D_H\phi}{|\, D_H\phi|}\rangle^* + \frac{\langle (D^2_H\phi)D_H\phi, D_H\phi\rangle}{|\, D_{H,\varepsilon}\phi|^2} + o_\varepsilon(1)\Big) \qquad (4.76)$$

where $D_H\phi$, $D_{H,\varepsilon}\phi$ and $D^2_H\phi$ are computed in (x,s).

We first consider the case $|\, D_H\phi(x,s)| = 0$. By assumption (iii) on ϕ we have $|\, D^2_H\phi(x,s)| = 0$. Thus we obtain

$$\overline{II}_\varepsilon \leq \frac{\dot{q}(\lambda)}{|D_{H,\varepsilon}\phi|}\big(- \tilde{C} + o_\varepsilon(1)\big) \leq -\frac{\tilde{C}\dot{q}(\lambda)}{2|D_{H,\varepsilon}\phi|}$$

for ε small enough.

If $|\, D_H\phi(x,t)| \neq 0$ the inequality in (4.76) becomes

$$\begin{aligned} \overline{II}_\varepsilon &\leq \frac{\dot{q}(\lambda)}{|D_{H,\varepsilon}\phi|}\Big(- \tilde{C} + \langle (D^2_H\phi)D_H\phi, D_H\phi\rangle\big(\frac{1}{|\, D_{H,\varepsilon}\phi|^2} - \frac{1}{|\, D_H\phi|^2}\big) + o_\varepsilon(1)\Big) \\ &= \frac{\dot{q}(\lambda)}{|D_{H,\varepsilon}\phi|}\Big(- \tilde{C} - \langle (D^2_H\phi)D_H\phi, D_H\phi\rangle \frac{\varepsilon^{2k}|\, D\phi|^2}{|\, D_H\phi|^2|\, D_{H,\varepsilon}\phi|^2} + o_\varepsilon(1)\Big) \\ &\leq \frac{\dot{q}(\lambda)}{|D_{H,\varepsilon}\phi|}\Big(- \tilde{C} + \|\, D^2_H\phi\| \frac{\varepsilon^{2k}|\, D\phi|^2}{|\, D_{H,\varepsilon}\phi|^2} + o_\varepsilon(1)\Big) \end{aligned} \qquad (4.77)$$

To conclude also in this case that, for ε small enough, $\overline{II}_\varepsilon \leq -\dfrac{\tilde{C}\dot{q}}{2|D_{H,\varepsilon}\phi|}$ we have to prove that

$$\|\, D^2_H\phi(x,s)\|\frac{\varepsilon^{2k}|\, D\phi(x,s)|^2}{|\, D_{H,\varepsilon}\phi(x,s)|^2} = \|\, D^2_H\phi(x,s)\|\frac{\varepsilon^{2k}|\, D\phi(x,s)|^2}{|\, D_H\phi(x,s)|^2 + \varepsilon^{2k}|\, D\phi(x,s)|^2} = o_\varepsilon(1). \qquad (4.78)$$

To do this we distinguish two cases. If $|\, D_H\phi(x,s)|^2 > \varepsilon^k$ we get

$$\|\, D^2_H\phi(x,s)\|\frac{\varepsilon^{2k}|\, D\phi(x,s)|^2}{|\, D_{H,\varepsilon}\phi(x,s)|^2} < \varepsilon^k\|\, D^2_H\phi(x,s)\|\|\, D\phi(x,s)|^2 = O(\varepsilon^k).$$

On the contrary if $|\, D_H\phi(x,s)|^2 \leq \varepsilon^k$ we observe that

$$\begin{aligned} \|\, D^2_H\phi(x,s)\|\frac{\varepsilon^{2k}|\, D\phi(x,s)|^2}{|\, D_{H,\varepsilon}\phi(x,s)|^2} &\leq \|\, D^2_H\phi(x,s)\| \\ &\leq \sup\{\|\, D^2_H\phi(x,s)\| : (x,s) \in Q_\gamma,\ |\, D_H\phi(x,s)|^2 \leq \varepsilon^k\} \end{aligned}$$

To get (4.78) also in this case it remains to prove that the right hand side of this last inequality goes to zero as $\varepsilon \to 0^+$. This immediately follows using the following Lemma and property (iii) of ϕ.

Lemma 4.2.17. Let $K \subset \mathbb{R}^n$ and $f, g : K \to [0, +\infty)$ be a compact set and two continuous functions such that

$$\text{if } f(x)=0 \text{ for some } x\in K, \text{ then } g(x)=0.$$

Then, it holds

$$\lim_{\varepsilon\to 0^+}\sup_{x\in K}\{g(x): f(x)\le \varepsilon\}=0$$

Proof. We suppose by contradiction that there exists a $c>0$ such that for any $\varepsilon>0$ there exists $x_\varepsilon=x_\varepsilon(c)\in K$ so that

$$g(x_\varepsilon)\ge c \qquad \text{and} \qquad f(x_\varepsilon)\le\varepsilon. \tag{4.79}$$

Since K is compact there exists a subsequence x_{ε_n} so that $x_{\varepsilon_n}\to\hat{x}\in K$ for some $\hat{x}\in K$. Passing to the limit as $\varepsilon\to 0^+$ in (4.79) we obtain the contradiction $g(\hat{x})\ge c>0$ and $f(\hat{x})=0$. □

Thus we have proved that

$$\overline{II}_\varepsilon \le -\frac{\tilde{C}\dot{q}(\lambda)}{2|\,D_{H,\varepsilon}\phi(x,s)|}$$

for any $(x,s)\in Q_\gamma$ and for ε small enough. As far it concerns the terms in III_ε one can prove, with exactly the same reasoning of Lemma 4.1.5, that

$$III_\varepsilon = O\Big(1+\frac{1}{|\,D_{H,\varepsilon}\phi|^3}\Big)=O(\varepsilon^{-3k}).$$

As in the proof of Lemma 4.1.5 the only difference with the analogous result in step one is that now we can't claim that the term Q_t and P_t are null. Anyway by (4.14), (4.21),

$$\begin{aligned} Q_t+\varepsilon P_t \;&= -(\dot{q}(\lambda)+\varepsilon\dot{p}(\lambda))a\frac{D_{H,\varepsilon}\phi\cdot D_{H,\varepsilon}\phi_t}{|\,D_{H,\varepsilon}\phi|^3} \\ &\le \Big(\frac{|\,\dot{q}(\lambda)||\,a|}{|\,D_{H,\varepsilon}\phi|^2}+\frac{|\,\dot{p}(\lambda)||\,\phi-2\beta|}{|\,D_{H,\varepsilon}\phi|^2}\Big)|\,D_{H,\varepsilon}\phi_t| = O\Big(\frac{1}{|\,D_{H,\varepsilon}\phi|}+\frac{1}{|\,D_{H,\varepsilon}\phi|^2}\Big)=O(\varepsilon^{-2k}), \end{aligned}$$

with, as usual, $\lambda=\dfrac{a}{|D_{H,\varepsilon}\phi(x,s)|}=\dfrac{\phi(x,s)-2\beta}{\varepsilon|D_{H,\varepsilon}\phi(x,s)|}$.
To conclude, since $0<k<1/3$, we get

$$\begin{aligned} v^\varepsilon_t(x,s)-\mathrm{tr}(D^2_{H,\varepsilon}v^\varepsilon(x,s))+\frac{f(v^\varepsilon(x,s))}{\varepsilon^2} \;&\le \frac{1}{\varepsilon}\Big[-\frac{\tilde{C}\dot{q}(\lambda)}{2|\,D_{H,\varepsilon}\phi|}-2\beta f'(q(\lambda))\Big]+O(\varepsilon^{-3k}) \\ &= \frac{1}{\varepsilon}\Big[-\frac{\tilde{C}\dot{q}(\lambda)}{2|\,D_{H,\varepsilon}\phi|}-2\beta f'(q(\lambda))+o_\varepsilon(1)\Big]. \end{aligned}$$

With the same reasoning in Lemma 4.2.15 one can see that the right hand side of this last inequality is non negative for any $(x,s)\in Q_\gamma$ and ε small enough.

The proof of the extension of v^ε to a global subsolution of (4.38-i) $\omega^{\varepsilon,\beta}$ in $B(x_o, r] \times [t, t+h]$ and the proof that such a function satisfies 1, 2, 3, is similar to that of Lemma 4.2.15 and we omit it. $\square$

Once we have proved the first two steps (initialization and propagation of the front) the proof of Theorem 4.2.11 follows immediately using Corollary 2.2.3. $\square$

Bibliography

[1] ALLEN, S. M. , CAHN, J. W., *A macroscopic theory for antiphase boundary motion and its application to antiphase domain coarsening*, Acta Metal. **27** (1979), 1085–1095.

[2] ARONSON, D. G. , WEINBERGER, H., *Multidimensional nonlinear diffusion arising in population genetics*, Adv. Math. **30** (1978), 33–76.

[3] M. BARDI, I. CAPUZZO-DOLCETTA, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser, 1997.

[4] BARDI, M.; SORAVIA, P., A comparison result for Hamilton-Jacobi equations and applications to some differential games lacking controllability, *Funkcial. Ekvac.* **37** (1994), 19–43.

[5] BARLES, G.; DA LIO, F., *A geometrical approach to front propagation problems in bounded domains with Neumann-type boundary conditions*, Interfaces Free Bound. **5** (2003), 239–274.

[6] BARLES, G.; GEORGELIN, C., *A simple proof of convergence for an approximation scheme for computing motions by mean curvature*, SIAM J. Numer. Anal. **32** (1995), no. 2, 484–500.

[7] BARLES, G., SONER, H. M., SOUGANIDIS, P. E., *Front propagation and phase field theory*, SIAM J. Control. Optim. **31** (1993), 489–469.

[8] BARLES, G., SOUGANIDIS, P. E., *A new approach to front propagation problems: theory and applications*, Arch. Ration. Mech. Anal. **141** (1998), 237–296.

[9] BONFIGLIOLI, A., LANCONELLI, U., UGUZZONI, F., *Stratified lie groups and potential theory for their sub-laplacians*, Spinger, (2007).

[10] BREZIS, H., *Analisi Funzionale*, Liguori Editore, (1986).

[11] CAGINALP, G., *An analysis of a phase field model of a free boundary*, Arch. Rational Mech. Anal. **92** (1986) 205-245.

[12] CAGINALP, G., *Mathematical models of phase boundaries*, Material instabilities in Continuum Mechanics and Related Mathematical Problems, (Ed. J. Ball), Oxford Science Publications, (1988) 35-52.

[13] CAGINALP, G., *Stefan and Hele-Shaw type models as asymptotic limits of the phase-field equations*, Phys. Rev. A **39** (1989) 5887-5896.

[14] CAMILLI, F., *An Hopf-Lax formula for a class of measurable Hamilton-Jacobi equations*, Nonlinear Anal. **57** (2004), no. 2, 265–286.

[15] CAPOGNA, L.; CITTI, G., *Generalized mean curvature flow in Carnot groups*, Comm. Partial Differential Equations**34** (2009), no7-9 937–956.

[16] CHEN, X., *Generation and propagation of interfaces in reaction-diffusion equations*, J. Differential Equations **96** (1992), 116–141.

[17] CHEN, X., *Generation and propagation of interfaces in reaction-diffusion systems,* Transactions of the American Mathematical Society. **334**, n. 2 (1992), 877–913.

[18] CHEN, Y.-G., GIGA, Y., GOTO, S., *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom. **33** (1991), 749–786.

[19] CRANDALL, M.G., *Viscosity solutions: a primer,* Viscosity Solutions and Applications (Montecatini Terme, 1995), I. Capuzzo Dolcetta et al. (eds.), Lecture Notes in Math. **1660**, Springer Verlag 1997, 1–43.

[20] CRANDALL, M. G., ISHII, H., LIONS, P.-L., *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1992), 1–67.

[21] DE ZAN, C., *Problemi di Cauchy per equazioni di Hamilton-Jacobi con termini discontinui*, Master's Thesis.

[22] DE ZAN, C.; SORAVIA, P., *Cauchy problems for noncoercive Hamilton-Jacobi-Isaacs equations with discontinuous coefficients*, Interfaces Free Bound. **12** (2010), 347–368.

[23] DE ZAN, C., SORAVIA, P., *Singular limits of reaction diffusion equations and geometric flows with discontinuous velocity*, Work in progress.

[24] DE ZAN, C.; SORAVIA, P., *Geometric flows with discontinuous velocity: a comparison principle*, in preparation.

[25] DIRR, N.; DRAGONI, F; VON RENESSE, M.;, *Evolution by mean curvature flow in sub-Riemannian geometries*, Comm. Pure Appl. Math. **9** (2010), 307–326.

[26] EVANS, L. C., SONER, H. M., SOUGANIDIS, P. E., *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math. **45** (1992), 1097–1123.

[27] EVANS, L. C. , SPRUCK, J., *Motion of level sets by mean curvature I*, J. Differential Geom. **33** (1991), 635–681.

[28] FIFE, P.C., *Dynamics of internal layers and diffusive interfaces*, CBMS-NSF Regional Conf. Ser. in Appl. Math. **53**, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1988.

[29] FIFE, P.C., *Nonlinear diffusive equations*, CBMS-NSF Regional Conf. Ser. in Appl. Math. **53**, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1988.

[30] FIFE, P. C. , MCLEOD, B., *The approach of solutions of nonlinear differential diffusion equations to traveling solutions*, Arch. Ration. Mech. Anal. **65** (1977), 335–361.

[31] GIGA, Y., *Surface evolution equations. A level set approach*, Monographs in Mathematics, **99**, Birkhäuser Verlag, Basel, 2006.

[32] GOTO, G., *Generalized motion of hypersurfaces whose growth speed depends superlinearly o curvature tensor*, J. Diff. Int. Eq. **7** (1994), 323-343.

[33] ISHII, H., *Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open sets* Bull. Fac. Sci. Engrg. Chuo Univ. **28** (1985), 33–77.

[34] ISHII, H., SOUGANIDIS, P. E., *Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor* Tohoku Math. J. **47** (1995), 227–250.

[35] KELLER, J. B., RUBINSTEIN, J., STERNBERG, P., *Fast reaction, slow diffusion and curve shortening*, SIAM J. Appl. Math. **49** (1989), 116–133.

[36] OSHER, S. , SETHIAN, J. A., *Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations*, J. Comput. Phys. **79** (1988), 12–49.

[37] SORAVIA, P., *Degenerate eikonal equations with discontinuous refraction index, ESAIM Control Optim. Calc. Var.* **12** (2006), 216–230.

[38] SORAVIA, P., *Boundary value problems for Hamilton-Jacobi equations with discontinuous Lagrangian, Indiana Univ. Math. J.* **51** (2002), 451–476.

[39] SORAVIA, P., *Uniqueness results for viscosity solutions of fully nonlinear, degenerate elliptic equations with discontinuous coefficients, Commun. Pure Appl. Anal.* **5** (2006), 213–240.

[40] SORAVIA, P.; SOUGANIDIS, P. E., *Phase field theory for FitzHugh-Nagumo type systems*, SIAM J. Math. Anal. **42** (1996), 1341–1359.

[41] SOUGANIDIS, P. E., *Front propagation: theory and applications*, Viscosity Solutions and Applications (Montecatini Terme, 1995), I. Capuzzo Dolcetta et al. (eds.), Lecture Notes in Math. **1660**, Springer Verlag 1997, 186–242.