Università degli Studi di Padova
Dipartimento di Matematica
Scuola di Dottorato di Ricerca in Scienze Matematiche
Indirizzo: Matematica
XXVII Ciclo

Université de Bordeaux
École Doctorale de Mathématiques et Informatique
Spécialité: Mathématiques Pures

Darmon Cycles and the Kohnen - Shintani lifting

Direttore della Scuola:
Ch.mo Prof. Pierpaolo Soravia

Coordinatore dell'indirizzo:
Ch.mo Prof. Franco Cardin

Supervisori:
Ch.mo Prof. Denis Benois
Ch.mo Prof. Adrian Iovita
Ch.mo Prof. Matteo Longo

Dottorando:

Guhanvenkat Harikumar

To Mom, Daddy and Karthik

Contents

Abstract

Let $\mathbf{f}(q)$ be a *Coleman family* of cusp forms of tame level N. Let k_0 be the p-new classical weight of the Coleman family $\mathbf{f}(q)$. By the Kohnen-Shintani correspondence, we associate to every even classical weight k a half-integral weight form (for $k \neq k_0$) $g_k = \sum_{D>0} c(D,k)q^D \in S_{\frac{k+1}{2}}(\Gamma_0(4N))$ and $g_{k_0} = \sum_{D>0} c(D,k)q^D \in S_{\frac{k+1}{2}}(\Gamma_0(4Np))$.
We first prove that the Fourier coefficients $c(D,k)$ for $k \in 2\mathbb{Z}_{>0}$ can be interpolated by a p-adic analytic function $\tilde{c}(D,\kappa)$ with κ varying in a neighbourhood of k_0 in the p-adic weight space.

Based on the eigenvalue of the Atkin-Lehner operator at p, we partition the discriminants D appearing in the Fourier expansion, $\sum_{D>0} c(D,k)q^D$, into two types (Type I and Type II). For any Type II discriminant D, we show that the derivative along the weight at k_0, $\frac{d}{d\kappa}[\widetilde{c}(D,\kappa)]_{k=k_0}$, is related to certain algebraic cycles associated to the motive $\mathcal{M}_{k_0}$ attached to the space of cusp forms of weight k_0 on $\Gamma_0(Np)$. These algebraic cycles appear in the theory of Darmon cycles.

Contents

Acknowledgements

First and foremost, I would like to thank my supervisors Professors Denis Benois (U. of Bordeaux), Adrian Iovita and Matteo Longo (U. of Padova) for their immense support and encouragement without which this thesis would not have been possible. They have been extremely helpful and patient throughout and have helped me many a time when I was stuck or was slow in my progress.

I improved the exposition of my thesis vastly in the last couple of weeks leading to my defence. I would like to thank all the members of my Jury, especially the referees Professors Mladen Dimitrov (U. of Lille 1) and Matthew Greenberg (U. of Calgary) for carefully reading my draft and making very useful remarks and suggestions.

I would also like to thank all the members of the Number Theory and Arithmetic Geometry groups at Bordeaux and Padova for creating a vibrant atmosphere. I have benefited tremendously from the various activities and discussions within the group. I consider myself very lucky to have had very close friends both within Mathematics and outside. In particular I relish the time I spent in the company of Velibor Bojkovic, Farhad Babaee, Stefano Tolomeo, Taoufiq Damir, Vincent Mineo, Genaro Hernandez Mada and many other others, both academically and personally. I will always hold these memories close to my heart.

There were a lot of bureaucratic hardships that I would have encountered as an international student registered in two European universities. Thanks to the wonderful team of people at the ALGANT working committee, particularly Christopher Niesen, Chiara Penon, Deborah Didio and Virginie Bielenda, I faced none of these.

My interest in Number Theory goes back to my BS - MS days at IISER Pune, India. I am extremely grateful to Professors A. Raghuram, Baskar Balasubramanyam, Chandan Singh Dalawat, Sujatha Ramdorai and Anupam Kumar Singh for giving me the opportunity to learn the subject.

I would have been completely lost in life if not for the unconditional love, guidance and support my family has showered up on me. They coped up with my tantrums when I was stressed and helped me cross the line. Without saying this thesis is dedicated to my mom, brother and to the memory of my father. Last but not the least, I thank the almighty for everything.

Chapter 1

Introduction

Let $N \geq 1$ be a positive integer. We denote by $S_k(\Gamma_0(N))$ (resp. $S_{\frac{k+1}{2}}(\Gamma_0(4N))$) the space of cusp forms of weight k (resp. $\frac{k+1}{2}$) on $\Gamma_0(N)$ (resp. $\Gamma_0(4N)$).

1.1 The Kohnen-Shintani correspondence.

One of the major themes in the study of automorphic forms is Langlands' *principle of functoriality* which describes the existence of correspondence between automorphic forms on different reductive groups. The Shimura-Shintani correspondence between integral weight modular forms (automorphic forms on $GL_2(\mathbb{Q})$) and half integral weight modular forms (automorphic forms on the metaplectic cover of $SL_2(\mathbb{Q})$) is one of the earliest examples of Langlands' functoriality. Shimura initiated the study of half integral weight modular forms in [39] in which he defined suitable Hecke operators and constructed a Hecke-equivariant correspondence between integral weight and half integral weight modular forms. Later, in [42], Shintani constructed the inverse correspondence using *theta lifts*. He showed the existence of a Hecke-equivariant

$\mathbb{C}$-linear isomorphism

$$\theta_k : S_k(\Gamma_0(N)) \rightarrow S_{\frac{k+1}{2}}(\Gamma_0(4N))$$

for $k \geq 2$ even.

When N is odd square free, W. Kohnen showed the existence of a Hecke equivariant isomorphism (denoted as D-th Shintani liftings) in [22] between

$$\theta_{N,k} : S_k(\Gamma_0(N))^{new} \rightarrow S^{new}_{\frac{k+1}{2}}(\Gamma_0(4N))^+$$

where + denotes the Kohnen + space of newforms of weight $\frac{k+1}{2}$, i.e. $g = \theta_{D,k}(f) \in S^{new}_{\frac{k+1}{2}}(\Gamma_0(4N))^+ \implies g(z)$ has a Fourier expansion, $g(z) = \sum_{D>0} c(D)q^D$ where $c(D) = 0$ unless $D^* := (-1)^{k/2}D \equiv 0, 1(\text{mod}4)$.

The arithmetic significance of the Kohnen-Shintani lifting is given by the following Waldspurger type formula :-(See Theorem 1 of [44] and Corollary 1 of [22]) .
Let D be a fundamental discriminant such that $(D, N) = 1$. Then

$$c(D)^2 = \lambda_g D^{\frac{k-1}{2}} L(f, D^*, k/2) \qquad \text{if} \qquad \left(\frac{D^*}{\ell}\right) = w_l \ \forall \ l|N$$

where

- $L(f, D^*, s) := \sum_n a(n)\chi_{D^*}(n)n^{-s}$ is the twisted L-function attached to $f(z) = \sum a(n)q^n$ and the Dirichlet character $\chi_{D^*}(n) := \left(\frac{D^*}{n}\right)$.
- λ_g is a non-zero complex number which depends only on the choice of g.
- $w_l \in (\pm 1)$ are the eigenvalues of the Atkin-Lehner involution W_l acting on f.

The twisted L-function admits a functional equation relating the values at s and $k-s$ (See Section 2.2 of Chapter 2). The sign that appears in this functional equation is given by

$$w(f, D^*) := (-1)^{k/2}\chi_{D^*}(-N)w_N$$

where $w_N := \Pi_{l|N} w_l$ and $f \in S_k(\Gamma_0(N))$.

In particular, the central critical value $L(f, D^*, k/2)$ vanishes when $w(f, D^*) = -1$. This also forces $c(D)$ to vanish and one is naturally interested in studying the central critical derivative $L'(f, D^*, k/2)$. We study the p-adic variation of this phenomenon and the results obtained are a higher weight / finite slope analogue of the recent work of Henri Darmon and Gonzalo Tornaria in [13].

1.2 The Results

Let p be an odd prime integer. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and embeddings $\sigma_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $\sigma_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Let N be an odd square free integer such that $p \nmid N$ and let U be an open affinoid of the p-adic weight space $\mathcal{X}$. A p-adic analytic family of cuspidal eigenforms over U is a formal q-expansion

$$\mathbf{f}(q) := \sum_{n\geq 1} \mathbf{a}_n q^n \in \mathcal{O}(U)[[q]]$$

such that for all $k \in U^{\mathrm{cl}} := \{n \in 2\mathbb{Z} : n \geq 0\} \cap U$,

$$f_k(q) = \sum_{n\geq 1} \mathbf{a}_n(k) q^n \in S_k(\Gamma_0(Np), \overline{\mathbb{Q}})$$

The p-adic valuation of $\mathbf{a}_p(k)$ is a constant called the slope of $\mathbf{f}(q)$. We will assume that we are in the finite non-ordinary case (i.e. $\mathbf{a}_p(k) \neq 0$ and $v_p(\mathbf{a}_p(k)) > 0$) and also that f_k is N-new for all $k \in U^{\mathrm{cl}}$. Since the slope of $\mathbf{f}$ is constant, there is at most one $k_0 \in U^{\mathrm{cl}}$ such that f_{k_0} is p-new. This happens exactly when $\mathbf{a}_p(k_0) = \pm p^{k_0/2-1}$.

For each $k \neq k_0 \in U^{\mathrm{cl}}$, there is a newform $f_k^{\#} \in S_k(\Gamma_0(N))$ such that f_k is the p-stabilization of $f_k^{\#}$, i.e.

$$f_k(q) = f_k^{\#}(q) - \frac{p^{k-1}}{\mathbf{a}_p(k)} f_k^{\#}(q^p)$$

In particular, the eigenvalues of the Hecke operators, T_l for all $l \nmid N$, of f_k and $f_k^{\#}$ coincide. For convenience, we denote $f_{k_0}^{\#} = f_{k_0} \in S_{k_0}(\Gamma_0(Np))$.

Let $g_k = \sum\limits_{D>0} c(D,k) q^D \in S_{\frac{k+1}{2}}(\Gamma_0(4N))$ be the Shintani lift of $f_k^{\#}$ for all $k \neq k_0 \in U^{\mathrm{cl}}$ and let $g_{k_0} = \sum\limits_{D>0} c(D,k_0) q^D \in S_{\frac{k_0+1}{2}}(\Gamma_0(4Np))$ correspond to the lift of f_{k_0}.
The values of D for which $c(D,k)$ need not necessarily vanish for $k \neq k_0 \in U^{cl}$ can be classified in two types :
(I) All $D > 0$ such that $\chi_{D^*}(p) = w_p$.
(II) All $D > 0$ such that $\chi_{D^*}(p) = -w_p$.
Note that for Type II discriminants D, we have $w(f_{k_0}, D^*) = -1$ and hence $L(f_{k_0}, D^*, k_0/2) = 0$ (therefore $c(D,k_0) = 0$).

By making a suitable normalization of the Fourier coefficients $c(D,k)$, the function $k \to c(D,k)$ extends to a p-adic analytic function $\widetilde{c}(D,\kappa)$ in a neighbourhood of k_0. For D_2 a Type II discriminant and D_1 a Type I discriminant such that $c(D_1,k_0) \neq 0$, let $D := D_1^*.D_2^*$ and K be the real quadratic field $\mathbb{Q}(\sqrt{D})$ (Note that $D > 0$).

Let V_p^{Np} be the base change to $\overline{\mathbb{Q}_p}$ of the p-adic Galois reprsentation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ attached to $S_{k_0}(\Gamma_0(Np),\mathbb{Q})^{p-new}$. Let $\mathcal{M}_{k_0}$ be the motive over $\mathbb{Q}$ associated to $S_{k_0}(\Gamma_0(Np))$ constructed in [35]. For L any number field, let $CH_0^{k_0/2}(\mathcal{M}_{k_0}\otimes L)$ be the Chow group of algebraic cycles of co-dimension $k_0/2$ on $\mathcal{M}_{k_0}$ base change to L that are homologous to the null cycle. We have a *global p-adic Abel-Jacobi* map

$$\mathrm{cl}^{\mathrm{k_0/2}}_{\mathrm{0,L}} : \mathrm{CH}_0^{k_0/2}(\mathcal{M}_{k_0}\otimes L)\to \mathrm{Sel}_{st}(L, V_p^{Np}(k_0/2))$$

See Sections 1 - 4 of [30] for a detailed discussion on the Abel - Jacobi map.

The main theorem we prove is

Theorem 1. *There exists a global cycle*

$$d_{k_0}^{\chi_{D_2^*}}\in CH_0^{k_0/2}(\mathcal{M}_{k_0}\otimes\mathbb{Q}(\sqrt{D_2^*}))^{\chi_{D_2^*}}\subset(\mathcal{M}_{k_0}\otimes\mathbb{Q}(\sqrt{D_2^*},\sqrt{D_1^*}))$$

and a constant $s_f\in K^\times_{f_{k_0}}$ *such that*

$$\frac{d}{dk}[\widetilde{c}(D_2,k)]_{k=k_0}=\frac{|D_1|^{\frac{k_0-2}{4}}}{|D_2|^{\frac{k_0-2}{4}}}.s_f.\mathrm{exp_{BK}}^{-1}(\mathrm{res}_p(\mathrm{cl}^{k_0/2}_{0,H_K^+}(d_{k_0}^{\chi_{D_2^*}})))(\phi_{k_0})$$

where ϕ_{k_0} is the modular symbol attached to f_{k_0} and $\mathrm{res_p}:\mathrm{Sel}_{st}(H_K^+,V_p^{Np}(k_0/2))\to H^1_{st}(K_p,V_p^{Np}(k_0/2))$ is the restriction at p (See Section 3.5 of Chapter 3).
Sketch of the proof:-

- Use Kohnen's formula to interpret $\widetilde{c}(D_2,k)$ as a p-adic variant of Shintani periods and relate it to a certain p-adic L-function defined by Seveso in [Sev].
- Use this interpretation to relate the derivative in the weight direction of $\widetilde{c}(D_2,k)$ to the image of *Darmon cycles* under a p-adic Abel-Jacobi map.
- Use the Rationality theorem of Seveso to show that these Darmon cycles

are in fact restriction of global cycles.

1.3 Outline of the thesis

Chapter 2. - In this chapter we recall the basic theory of half-integral weight forms and then go on to discuss the Kohnen - Shintani lifting and their arithmetic significance.

Chapter 3. - We recall the theory of Darmon cycles and the construction of a cohomological p-adic Abel-Jacobi map. The construction is due to Victor Rotger and Marco Seveso ([33]) and can be thought of as a heigher weight analogue of *Stark - Heegner* points.

Chapter 4. - Since the proof of the main theorem involves a systematic study of various p-adic L-functions, we devote this chapter completely to describe the construction and properties of these p-adic L-functions.

Chapter 5. - This is the final chapter of the thesis where we prove our main theorem.

Appendix A. - In this expository appendix, we recall some of the relations between quadratic forms and quadratic fields.

All the results in Chapters 2, 3 and 4 exist in the literature and we have provided explicit references to the results that have been recalled in this text.

Chapter 2

The Kohnen-Shintani correspondence

2.1 Half-integral weight modular forms

In this section, we recall the basic theory of half-integer weight modular forms. The exposition is standard and all the material discussed here can be found in the fundamental papers on the subject - for example [39], [42], [22] and [23].

We first recall the definition of three congruence subgroups of $SL_2(\mathbb{Z})$.

Definition 1. *Let $N \geq 1$ be an integer. Then*

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv \pm 1, b \equiv c \equiv 0 \bmod(N) \right\}.$$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1, c \equiv 0 \bmod(N) \right\}.$$

and

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \bmod (N) \right\}.$$

For p an odd prime, recall that an integer c is called a quadratic residue modulo p if there exists an integer x such that $c \equiv x^2 \bmod p$. Recall the Legendre Symbol given by

$$\left(\frac{c}{p}\right) = \begin{cases} +1 & \text{if } c \text{ is a quadratic residue modulo } p \text{ and } p \nmid c \\ -1 & \text{if } c \text{ is not a quadratic residue modulo } p \\ 0 & \text{if } p \mid c \end{cases}$$

and for $p = 2$,

$$\left(\frac{c}{2}\right) = \begin{cases} +1 & \text{if } c \equiv \pm 1 \bmod (8) \\ -1 & \text{if } c \equiv \pm 3 \bmod (8) \\ 0 & \text{if } c \text{ is even} \end{cases}$$

For n any integer with prime factorization $n = up_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$ where $u = \pm 1$ and p_i's are primes, the Kronecker symbol is defined as

$$\left(\frac{a}{n}\right) := \left(\frac{a}{u}\right) \prod_{i=1}^{\ell} \left(\frac{a}{p_i}\right)^{e_i}$$

where

$$\left(\frac{a}{1}\right) = 1$$

and

$$\left(\frac{a}{-1}\right) = \begin{cases} +1 & \text{if } a \geq 0 \\ -1 & \text{if } a < 0 \end{cases}$$

When d is an odd integer, define

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \bmod (4) \\ \sqrt{-1} & \text{if } d \equiv 3 \bmod (4) \end{cases}$$

For all $z \in \mathbb{C}$, we will define by $\sqrt{z}$ to be the square-root function with argument in the interval $(-\pi/2, \pi/2]$.

Let $\mathcal{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the complex upper half plane and let G be the group of pairs

$$G := \{(\alpha, \phi(z)) \mid \alpha \in GL_2^+(\mathbb{Q}), \phi(z) : \mathcal{H} \to \mathbb{C} \text{ is holomorphic}, \phi(z)^2 = \pm \frac{cz+d}{\sqrt{\det\alpha}}\}$$

The group composition on G is given by

$$(\alpha_1, \phi_1(z)).(\alpha_2, \phi_2(z)) := (\alpha_1\alpha_2, \phi_1(\alpha_2.z)\phi_2(z))$$

where $\alpha_2.z$ is the usual fractional liner transformation (See (2.1) below).

Let $\Pi : G \to GL_2^+(\mathbb{Q})$ be the canonical projection. We can define an action of G on the space of all complex valued functions on $\mathcal{H}$ as follows

$$f \mid [g]_{k/2}(z) := f(\alpha.z)\phi(z)^{-k} \tag{2.1}$$

where $\alpha.z = \frac{\alpha_{11}z+\alpha_{12}}{\alpha_{21}z+\alpha_{22}}$ for $\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ and $g = (\alpha, \phi(z)) \in G$ and $f : \mathcal{H} \to \mathbb{C}$.

Fix an integer M divisible by 4. For Γ a congruence subgroup of $\Gamma_0(M)$, we consider the following subgroup of G

$$\Delta := \{\tilde{\gamma} := (\gamma, \phi_\gamma(z)) \mid \gamma \in \Gamma\}$$

where $\phi_\gamma(z) = \left(\frac{c}{d}\right)\epsilon_d^{-1}\sqrt{cz+d}$ associated to $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\epsilon_d = 1$ (resp. $\sqrt{-1}$) when $d \equiv 1 \bmod (4)$ (resp. $d \equiv 3 \bmod (4)$). When Γ is $\Gamma_0(M)$ (resp. $\Gamma(M)$ or $\Gamma_1(M)$), we denote the corresponding subgroup of G by $\Delta_0(M)$ (resp. $\Delta(M)$ or $\Delta_1(M)$). Let $k \geq 0$ be odd and χ a Dirichlet character mod M.

Definition 2. *A modular form of weight $k/2$, level $\Delta_0(M)$ and Nebentypus χ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that*

- $f \mid [\tilde{\gamma}]_{k/2}(z) = \chi(d) f(z)\ \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$.
- *f is holomorphic at the cusps of $\Gamma_0(M)$.*

See Pages 2-3 of [45] for a discussion about holomorphicity at the cusps of a congruence subgroup. We denote the space of modular forms of weight $k/2$ on $\Delta_0(M)$ and Nebentypus χ by $M_{k/2}(\Gamma_0(M), \chi)$. If $f(z)$ vanishes at the cusps, we call it a cusp form and denote the subspace of cuspforms by $S_{k/2}(\Gamma_0(M), \chi) \subset M_{k/2}(\Gamma_0(M), \chi)$.

Example 1. *Let $\theta : \mathcal{H} \to \mathbb{C}$ be given by $\theta(z) := \sum_{n\in\mathbb{Z}} q^{n^2}$ where $q = e^{2\pi i z}$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, we have*

$$\left(\frac{\theta(\gamma.z)}{\theta(z)}\right)^2 = \left(\frac{-1}{d}\right)(cz+d)$$

Hence $\theta(z) \in M_{1/2}(\Gamma_0(4), \chi_{-1})$.

2.1.1 Hecke Operators

Similar to the integral weight case, one can define Hecke operators acting on $M_{k/2}(\Gamma)$.

Definition 3. *Recall that two subgroups $H, H' \subseteq GL_2^+(\mathbb{Q})$ are said to be commensurable if*

$$[H : H \cap H'] < \infty \qquad [H' : H \cap H'] < \infty$$

Let $\gamma \in GL_2^+(\mathbb{Q})$ Then $\Delta_1(M)$ and $\gamma^{-1}\Delta_1(M)\gamma$ are commensurable[1]. Write $\Delta_1(M)\gamma\Delta_1(M) = \bigcup_\alpha \Delta_1(M)\gamma_\alpha$. Define the operator $\mid [\Delta_1(M)\gamma\Delta_1(M)]_{k/2}$ on the space $M_{k/2}(\Gamma_1(M))$ as follows

$$f \mid [\Delta_1(M)\gamma\Delta_1(M)]_{k/2} := \det(\gamma)^{k/4-1} \sum_\alpha f \mid [\gamma_\alpha]_{k/2}$$

where $f \mid [\gamma_\alpha]_{k/2}$ is the action defined in (2.1). For m positive, the Hecke operator T_m is defined to be the restriction of $\mid [\Delta_1(M)\gamma\Delta_1(M)]_{k/2}$ to $M_{k/2}(\Gamma_0(M), \chi)$ for $\gamma = \left(\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, m^{1/4} \right)$. By Proposition 1.0 of [39], the Hecke operator T_m is zero whenever m is not a square.

Now let $\Delta_0(M)\gamma\Delta_0(M) = \bigcup_\alpha \Delta_0(M)\gamma_\alpha$. Then $T_{n^2} \in \mathrm{End}(M_{k/2}(\Gamma_0(M), \chi))$ is the operator given by

$$T_{n^2}(f) := n^{k/2-2} \sum_\alpha \chi(a_\alpha) f \mid [\gamma_\alpha]_{k/2}$$

where a_α is the upper left entry of γ_α.

Theorem 2 (Shimura, Theorem 1.7, [39]). *Let $f(z) = \sum_n a_n q^n \in M_{k/2}(\Gamma_0(M), \chi)$.*

[1]We would like to thank Professor Mladen Dimitrov for pointing out this

For a prime p, $T_{p^2}(f)(z) = \sum_n b_n q^n$ where

$$b_n = a_{p^2 n} + \chi(p)\Big(\frac{-1}{p}\Big)^{\frac{k-1}{2}}\Big(\frac{n}{p}\Big)p^{\frac{k-3}{2}}a_n + \chi(p^2)p^{k-2}a_{n/p^2}$$

where $a_{n/p^2} = 0$ whenever $p^2 \nmid n$.

When $p \mid M$, the Hecke operator will be denoted $U(p^2)$. When $(m, n) = 1$, $T_{m^2n^2} = T_{m^2}T_{n^2} = T_{n^2}T_{m^2}$ (See Proposition 1.6 of [39]).

2.1.2 New forms of half-integral weight

For each prime $p \mid M$, we recall the 'Atkin-Lehner' type involution $W(p)$, defined by Kohnen in [23], given by

$$W(p) := \left(\begin{bmatrix} p & a \\ 4M & pb \end{bmatrix}, \Big(\frac{-4}{p}\Big)^{-k-1/2} p^{-k/2-1/4}(4Mz + pb)^{k+1/2}\right)$$

where $p^2b - 4Ma = p$. The operator $W(p)$ induces an isomorphism between $S_{k+1/2}(\Gamma_0(M), \chi)$ and $S_{k+1/2}\Big(\Gamma_0(M), \Big(\frac{\cdot}{p}\Big)\chi\Big)$. Let μ be the conductor of χ - the Dirichlet character modulo M. Since the conductor μ is a divisor of M, we consider the Hecke operator $U(\mu^2)$. Then:

Proposition 1 (Proposition 3, [23]). *$S_{k+1/2}(\Gamma_0(M)) \xrightarrow{U(\mu^2)} S_{k+1/2}(\Gamma_0(M), \chi)$ is an isomorphism.*

For each $p \mid M$, denote by

$$w^N_{p,k+1/2} := p^{-k/2+1/4}U(p^2)W(p)$$

and

$$w^N_{p,k+1/2,\chi} := U(\mu^2)^{-1}w^N_{p,k+1/2}U(\mu^2)$$

Further, for each $d \mid M$, let $S_{k+1/2}(\Gamma_0(d), \chi) := S_{k+1/2}(\Gamma_0(d)) \mid U(\mu^2)$. The space of *old forms* $S_{k+1/2}(\Gamma_0(M), \chi)^{\text{old}}$ is then defined as

$$\sum_{d \mid M, d < M} \left(S_{k+1/2}(\Gamma_0(d), \chi) + S_{k+1/2}(\Gamma_0(d), \chi) \mid U(M^2/d^2) \right)$$

We have a 'Petersson inner product' on $S_{k+1/2}(\Gamma_0 M)$ given by

$$< f, g >:= \frac{1}{[SL_2(\mathbb{Z}) : \Gamma_0(M)]} \int_{\Gamma_0(M)/\mathcal{H}} f(z)\overline{g(z)} y^{k/2-2} dx dy$$

where $z = x + iy$. The space of *newforms* $S^{\text{new}}_{k+1/2}(\Gamma_0(M), \chi)$ is defined as the orthogonal complement of the space of oldforms w.r.t the Petersson inner product. We will now recall some fundamental results about the space of newforms.

Theorem 3 (W. Kohnen)**.** *For $p \mid M$, the endomorphisms $U(p^2)$ and $w^N_{p,k+1/2,\chi}$ preserve the space $S^{new}_{k+1/2}(\Gamma_0(M), \chi)$. Further, $U(p^2) = -p^{k-1} w^N_{p,k+1/2,\chi}$. $S^{new}_{k+1/2}(\Gamma_0(M), \chi)$ has an orthogonal basis (with respect to the Petersson inner product) of simultaneous eigenforms for the Hecke operators $T(p^2)$ for all $p \nmid M$ and $U(p^2)$ for all $p \mid M$. The eigenvalues for $U(p^2)$ are given by $\pm p^{k-1}$.*

Proof. See Theorem 1 and Theorem 2, Section 5 of [23]. □

2.2 The Kohnen-Shintani Lifting

Fix $f \in S_k(\Gamma_0(N))$, a cusp form of weight k on $\Gamma_0(N)$. Let $Q(x, y) = ax^2 + bxy + cy^2$ be an integral binary quadratic form (i.e $a, b, c \in \mathbb{Z}$). The group $SL_2(\mathbb{Z})$ acts on the right on the space of integral quadratic forms by

$$(Q \mid \epsilon)(x, y) := Q(\delta x - \gamma y, -\beta x + \alpha y)$$

for $\epsilon = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Let $D > 0$ be an integer such that $D^* := (-1)^{k/2}D$ is congruent to $0, 1 (\bmod 4)$ and D^* divides $\Delta = b^2 - 4ac$, the discriminant of $Q(x, y)$. Let $\Delta = D^* D'^*$. Define

$$\omega_{D^*,D'^*}(Q) := \begin{cases} \left(\frac{D'^*}{Q(m,n)}\right) & \text{when } \gcd(D'^*, Q(m,n)) = 1 \\ \left(\frac{D^*}{Q(m,n)}\right) & \text{when } \gcd(D^*, Q(m,n)) = 1 \end{cases}.$$

If there exist $r, s \in \mathbb{Z}$ such that $(D^*, Q(r, s)) = 1$, then $\left(\frac{D^*}{Q(r,s)}\right) = \left(\frac{D^*}{Q(m,n)}\right)$ (See Lemma 8 in Appendix A). Hence ω_{D^*,D'^*} is well-defined. Genus theory shows that ω_{D^*,D'^*} is a quadratic character of the class group of integral binary quadratic forms of discriminant Δ. This character cuts out the bi-quadratic extension $\mathbb{Q}(\sqrt{D^*}, \sqrt{D'^*})$. See Appendix A for a discussion of these results on *genus characters*.

Let δ be a positive integer such that $\delta^2 \equiv \Delta (\bmod 4N)$. We will call a primitive binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ a *Heegner form* of level N if

$$N \mid a, \quad \text{and} \quad b \equiv \delta (\bmod N).$$

We will denote the set of Heegner forms of discriminant Δ by $\mathcal{F}_\Delta$. Assume Δ is not a perfect square and let $r + s\sqrt{\Delta}$ be the totally positive (i.e. $r, s > 0$) fundamental unit in the order $\mathcal{O}_\Delta := \mathbb{Z}[\frac{\Delta+\sqrt{\Delta}}{2}]$. Let

$$\gamma_Q := \begin{pmatrix} r + sb & 2cs \\ -2as & r - sb \end{pmatrix} \in \Gamma_0(N)$$

be the generator of the cyclic subgroup Γ_Q - the stabilizer of Q in $\Gamma_0(N)$. For any point $\tau \in \mathcal{H}$, let C_Q be the image in $\Gamma_0(N)/\mathcal{H}$ of the geodesic in $\mathcal{H}$ of complex numbers $z = x + iy$ such that

$$a|z|^2 + bx + c = 0.$$

Let $\tau \in \mathcal{H}$ be any base point. In our case (i.e Δ is not a perfect square), C_Q is equivalent to the geodesic joining τ and $\gamma_Q \tau$. To each $Q \in \mathcal{F}_\Delta$, we associate the Shintani period given by

$$r(f, Q) := \int_{C_Q} f(z) Q(z, 1)^{k-1} dz$$

Let $\Delta = D^*.D'^*$ be the factorization such that $D, D' > 0$ and $D^*, D'^* = (-1)^{k/2} D_i \equiv 0, 1(\text{mod}4)$ for $i = 1, 2$. Consider the liner combination

$$r_{k,N}(f, D^*, D'^*) = \sum_{Q \in \mathcal{F}_\Delta / \Gamma_0(N)} \omega_{D^*, D'^*}(Q) r(f, Q)$$

Let $\mu(n)$ denote the Mobius function which is defined as the sum of the primitive n-th roots of unity. Then $\mu(n) \in \{-1, 0, 1\}$. Let $S^+_{\frac{k+1}{2}}(\Gamma_0(4N))$ denote the Kohnen '+' space of half integral weight cusp forms, i.e. forms that have a Fourier expansion of the form

$$g(\tau) = \sum_{\substack{D \geq 1 \\ D^* \equiv 0,1(\text{mod}4)}} c(D) q^D \in S_{k+1/2}(\Gamma_0(4N)).$$

For m a fudamental discriminant such that $m^* = (-1)^{k/2} m > 0$, the m-th Shintani lifting, $g(\tau)$, of $f \in S_k(\Gamma_0(N))$ is defined as

$$\Theta_{k,N,m}(f)(q) := \sum_{\substack{D \geq 1 \\ D^* \equiv 0,1(\text{mod}4)}} \Big(\sum_{t|N} \mu(t) \Big(\frac{m}{t}\Big) t^{\frac{k-1}{2}} r_{k,Nt}(f, m, (-1)^{k/2} D t^2) \Big) q^D$$

Theorem 4. *For every m as above, $\Theta_{k,N,m} : S_k(\Gamma_0(N)) \to S^+_{\frac{k+1}{2}}(\Gamma_0(4N))$ is* *an isomorphism. Further if N is odd square free, then $\Theta_{k,N,m}$ maps $S_k^{new}(\Gamma_0(N))$*[2] *isomorphically onto $S^{+,new}_{\frac{k+1}{2}}(\Gamma_0(4N))$.*

Proof. See Theorem 2 of [23]. □

We will now recall a formula of Kohnen which relates the Fourier coefficients of a Shintani lifting to the Shintani periods.

For any m as above and $f \in S_k(\Gamma_0(N))$, let the Fourier expansion of the m-th Shintani lifting - $g(\tau) := \Theta_{k,N,m}(f)(z)$ be given by $g(\tau) = \sum_{\substack{D\geq 1 \\ D^*\equiv 0,1 (\mathrm{mod} 4)}} c(D)q^D$ (i.e. $c(D) = \sum_{t|N} \mu(t)\left(\frac{m}{t}\right) t^{\frac{k-1}{2}} r_{k,Nt}(f, m, (-1)^{k/2}Dt^2)$).

Theorem 5. *Then*

$$\frac{c(D_1)\overline{c(D_2)}}{< g, g >} = \frac{(-2i)^{k/2}2^{\nu(N)}}{< f, f >} r_{k,N}(f, D_1^*, D_2^*)$$

where $\nu(N)$ is the number of distinct prime divisors of N.

Proof. This is Theorem 3 of [23][3]. □

Let $D > 0$ be an integer such that $D^* \equiv 0, 1 \bmod(4)$. Recall the twisted L-series of f:

$$L(f, D^*, s) := \sum_{n\geq 1} \left(\frac{D^*}{n}\right) a(n)n^{-s}; \qquad \mathrm{Re}(s) >> 0$$

where $f(z) = \sum_{n\geq 1} a(n)q^n \in S_k(\Gamma_0(N))$ and $\left(\frac{D^*}{\cdot}\right)$ is the quadratic Dirichlet character. This twisted L-function admits a holomorphic continuation to $\mathbb{C}$ given by

$$\Lambda(f, D^*, s) = (2\pi)^{-s}(ND^{*,2})^{s/2}\Gamma(s)L(f, D^*, s)$$

[2] See [3] for the theory of newforms of integral weight cusp forms.

[3] There is an extra factor of $2^{\nu(N)}$ that appears here due to the choice of embeddings. See Theorem 2.3 of [25]

and admits a functional equation

$$\Lambda(f, D^*, s) = (-1)^{k/2}\Big(\frac{D^*}{-N}\Big) w_N \Lambda(f, D^*, k-s)$$

where $w_N := \prod_{\ell | N} w_\ell \in \{\pm 1\}$ is the product of the Atkin-Lehner eigenvalues indexed by the primes dividing the level. See Theorem 7.7 of [20] for a discussion about analytic continuation of Automorphic L-functions. At $s = k/2$, the critical L-value , $L(f, D^*, k)$, vanishes when $(-1)^{k/2}\chi_{D^*}(-N) = -w_N$.

We end this chapter with the following result which can be derived from Theorem 5 above

Corollary 1. *Let D be a above such that $\left(\frac{D^*}{\ell}\right) = w_\ell$, for all primes $\ell \mid N$, we have*

$$\frac{|c(|D|)|^2}{< g, g >} = 2^{\nu(N)} \frac{(k/2-1)!}{\pi^{k/2}} |D|^{\frac{k-1}{2}} \frac{L(f, D^*, k/2)}{< f, f >}$$

with $\nu(N)$ being the number of distinct prime divisors of N.

Proof. See Corollary 1 of [22]. □

Remark 1. By the above Proposition, we know that the vanishing of $c(D)$ is equivalent to the vanishing of $L(f, D^*, k/2)$.

Chapter 3

Darmon Cycles

Let p be an odd prime and let N be a square free integer such that $p \nmid N$. We fix a real quadratic extension $K/\mathbb{Q}$ such that

- All primes dividing N are split in K
- p is inert in K.

Let D_K be the discriminant of K. Recall the fixed embeddings

$$\sigma_\infty : \overline{\mathbb{Q}} \to \mathbb{C} \qquad\qquad \sigma_p : \overline{\mathbb{Q}} \to \mathbb{C}_p.$$

Let $\Gamma_0 := \Gamma_0(Np)$ and $\Gamma := \Gamma_0(N)$ be the congruence subgroups of level Np and N respectively. Let $\widetilde{\Gamma} := \Gamma[\frac{1}{p}]$ and $W := \mathbb{Q}_p^2 - (0,0)$. Denote by $P_{k-2}(E)$ to be the set of homogeneous polynomials of degree $k-2$ in two variables over a field E. Let $V_{k-2}(E)$ be the E-dual of $P_{k-2}(E)$.

3.1 Bruhat-Tits trees and p-adic upper half planes

3.1.1 Bruhat-Tits trees

Let $\mathcal{T}$ denote the Bruhat-Tits tree of $\mathbb{Q}_p$ whose vertices are given by homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$. We denote the set of vertices (resp. edges) of $\mathcal{T}$ by $\mathcal{V}$ (resp. $\mathcal{E}$). We will denote a vertex v by $[L]$ where $[L]$ stands for the homothety class of lattices equivalent to some lattice $L \subset \mathbb{Q}_p^2$ (i.e $L' \in [L]$ if and only if there exists $\alpha \in \mathbb{C}^\times$ such that $L' = \alpha L$). There is an edge e between two vertices v_1 and $v_2 \in \mathcal{V}$ if for some lattices $L_1, L_2 \subset \mathbb{Q}_p^2$ such that $v_1 = [L_1]$ and $v_2 = [L_2]$;

$$L_1 \supset L_2 \supset pL_1$$

If $L_1 \supset L_2 \supset pL_1$ then $L_2 \supset pL_1 \supset pL_2$ and since $[pL_1] = [L_1]$ we see that $\mathcal{T}$ is an undirected graph (i.e. we identify the edges $v_1 \to v_2$ and $v_2 \to v_1$). In fact, $\mathcal{T}$ is a tree with each vertex $v \in \mathcal{V}$ having degree $p+1$ (See Proposition 1.3.2 of [14]).

We write $< \ell_1, \ell_2 >$ to denote the lattice L generated by ℓ_1 and ℓ_2, i.e. $L = \mathbb{Z}_p\ell_1 + \mathbb{Z}_p\ell_2$. We have a natural left $GL_2(\mathbb{Q}_p)$-action on $\mathcal{T}$ as follows : Let $v = [L] \in \mathcal{V}$ and let $\gamma \in GL_2(\mathbb{Q}_p)$. Then :

$$\gamma \cdot v = [\gamma.L :=< \gamma\ell_1, \gamma\ell_2 >].$$

Here we view ℓ_i as column vectors in $\mathbb{Q}_p^2$ and $\gamma\ell_i$ is the usual matrix multiplication. For $\lambda \in \mathbb{Q}_p^\times$, we know that $\gamma.\lambda L = \lambda\gamma.L$ and hence $L' \sim L \Longrightarrow \gamma.L' \sim \gamma.L$. This shows that the action is well-defined.

Denote the distinguished vertex $v_* := [L_*]$ where $L_* := \mathbb{Z}_p^2$ and by $\mathcal{V}^+$ (respectively $\mathcal{V}^-$) the set of vertices at even (respectively odd) distance from v_*. We can define an orientation on $\mathcal{T}$ as follows - for every $e \in \mathcal{E}(\mathcal{T})$, denote by $s(e)$

the source vertex of e and $t(e)$ the target vertex of e. This assigns a direction to each edge thus making $\mathcal{T}$ into a directed graph. Denote by $\bar{e}$ to be the edge such that $s(\bar{e}) = t(e)$ and $t(\bar{e}) = s(e)$.

3.1.2 The p-adic upper half plane

Definition 4. *The p-adic upper half plane $\mathcal{H}_p$ is the rigid analytic variety over $\mathbb{Q}_p$ whose E-rational points, for E a finite extension of $\mathbb{Q}_p$, are given by $\mathcal{H}_p(E) := \mathbb{P}^1(E) - \mathbb{P}^1(\mathbb{Q}_p)$.*

Admissible coverings of $\mathcal{H}_p$. To recall the rigid analytic structure on $\mathcal{H}_p$, we need to define an admissible covering. We will construct a family of affinoid subdomains obtained by deleting balls around $\mathbb{P}^1(\mathbb{Q}_p)$ [1].

Let $t = [t_0 : t_1] \in \mathbb{P}^1(\mathbb{Q}_p)$ be such that the homogeneous coordinates $[t_0 : t_1]$ are unimodular - i.e. both $t_i \in \mathbb{Z}_p$ but p does not divide at least one of the t_i's. For $r \in \mathbb{R}^+$, let

$$B(t,r) := \{s \in \mathbb{P}^1(\mathbb{C}_p) | \operatorname{ord}_p(s_0t_1 - s_1t_0) \geq r\},$$

and

$$B^-(t,r) := \{s \in \mathbb{P}^1(\mathbb{C}_p) | \operatorname{ord}_p(s_0t_1 - s_1t_0) > r\}.$$

where $[s_0 : s_1]$ is a unimodular representative of $s \in \mathbb{P}^1(\mathbb{C}_p)$.

Since we choose unimodular homogeneous coordinates for points in $\mathbb{P}^1(\mathbb{Q}_p)$, we can consider the reduction modulo p^n, $n \geq 1$. Let $\mathcal{P}_n$ be a set of representatives of $\mathbb{P}^1(\mathbb{Q}_p)$ modulo p^n.

[1] For a thorough treatment of affinoids, admissible coverings and rigid spaces, see Chapters 3 and 4 of [16]

Definition 5. *For each $n \geq 1$, let $\mathcal{H}_n$ be the set*

$$\mathcal{H}_n := \mathbb{P}^1(\mathbb{C}_p) \setminus \bigcup_{t \in \mathcal{P}_n} B(t, n)$$

and $\mathcal{H}_n^- \subset \mathcal{H}_n$ be

$$\mathcal{H}_n^- := \mathbb{P}^1(\mathbb{C}_p) \setminus \bigcup_{t \in \mathcal{P}_n} B^-(t, n-1)$$

Then

$$\mathcal{H}_p = \bigcup_n \mathcal{H}_n = \bigcup_n \mathcal{H}_n^-$$

Proposition 2. *$\{\mathcal{H}_n\}_{n=1}^{\infty}$ and $\{\mathcal{H}_n^-\}_{n=1}^{\infty}$ are admissible coverings of $\mathcal{H}_p$. In fact, $\{\mathcal{H}_n^-\}_{n=1}^{\infty}$ is an admissible cover of affinoid subdomains.*

Proof. This has been discussed below Lemma 3 in [34]. □

We will denote by $\mathcal{H}_p^{ur}$ for $\mathcal{H}_p(\mathbb{Q}_p^{ur}) = \mathbb{P}^1(\mathbb{Q}_p^{ur}) - \mathbb{P}^1(\mathbb{Q}_p)$. $\mathcal{H}_p^{ur}$ has a natural left action of $GL_2(\mathbb{Q}_p)$ via fractional linear transformation. Consider the unique $GL_2(\mathbb{Q}_p)$-equivariant reduction map (See Proposition 5.1 of [12])

$$r : \mathcal{H}_p^{ur} \longrightarrow \mathcal{T}.$$

Let $\mathcal{H}_{p,\pm} := r^{-1}(\mathcal{V}^{\pm})$ and $\mathcal{H}_{p,v} := r^{-1}(v)$ for $v \in \mathcal{V}$.

Definition 6. *Let $([L_1], [L_2], \ldots, [L_i], \ldots)$ be an infinite, non-retracing sequence of adjacent vertices in $\mathcal{T}$. By non-retracing, we mean that $\nexists\, n \in \mathbb{N}$ such that $[L_i] = [L_{i+n}]$ for all $i \in \mathbb{N}$. We interpret such a sequence as a ray starting from the vertex $v = [L_1]$ and heading off to ∞. We introduce an equivalence relation on the set of all such sequences given by :*

$$([L_1], [L_2], \ldots, [L_i], \ldots) \sim ([L_1'], [L_2'], \ldots, [L_i'], \ldots)$$

if there exists a fixed $m \in \mathbb{Z}$ such that $[L_n] = [L'_{n+m}]$ for all $n \in \mathbb{N}$. We call such an equivalence class to be an end in $\mathcal{T}$.

The compact open subsets of $\mathbb{P}^1(\mathbb{Q}_p)$ are in one-one correspondence with the ends in $\mathcal{E}$ (See Theorem 5.9 of Chapter 5, [12]). For $e \in \mathcal{E}$, we denote by U_e to be the compact open subset under this correspondence.

For more details and proofs about the p-adic upper half plane and its connection to the *Bruhat-Tits* tree, see Chapter 5 of [12] and Section 1 of [14].

3.2 p-adic Abel-Jacobi maps : Darmon's setting

Let us denote by K_p to be the completion of the image of the embedding $\sigma_p : K \hookrightarrow \mathbb{C}_p$. By the hypothesis that p is inert in K, we know that K_p is isomorphic to the unramified quadratic extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$. For $*$ either empty, $\pm$ or $v \in \mathcal{V}$, denote by $\Delta_* := (\mathrm{Div}(\mathcal{H}_{p,*}^{ur}))^{G_{K_p^{ur}/K_p}}$ and $\Delta_*^0 := (\mathrm{Div}^0(\mathcal{H}_{p,*}^{ur}))^{G_{K_p^{ur}/K_p}}$ where $G_{K_p^{ur}/K_p} = \mathrm{Gal}(K_p^{ur}/K_p)$ and $\mathrm{Div}(\mathcal{H}_{p,*}^{ur})$ (respectively $\mathrm{Div}^0(\mathcal{H}_{p,*}^{ur})$) denotes the set of divisors (respectively set of zero divisors) on $\mathcal{H}_{p,*}^{ur}$.

We can consider $\Delta_*(P_{k-2}) := \Delta_* \otimes_{\mathbb{Z}} P_{k-2}$ and $\Delta_*^0(P_{k-2}) := \Delta_*^0 \otimes_{\mathbb{Z}} P_{k-2}$ as left $GL_2(\mathbb{Q}_p)$-modules (resp. left $GL(L)$-modules) when $*$ is empty (resp. $* = v = [L]$) via the usual tensor product action. We have the following exact sequence:

$$0 \to \Delta_*^0(P_{k-2}) \to \Delta_*(P_{k-2}) \xrightarrow{deg} P_{k-2} \to 0 \tag{3.1}$$

Recall the set of vertices $\mathcal{V}$ and edges $\mathcal{E}$ of the Bruhat - Tits tree $\mathcal{T}$. Denote by $\mathcal{C}(\mathcal{E}, V_{k-2})$ the set of all maps $c : \mathcal{E} \to V_{k-2}$.

Definition 7. *A harmonic cocycle is an element in $\mathcal{C}(\mathcal{E}, V_{k-2})$ such that $c(\bar{e}) = -c(e)$ for all $e \in \mathcal{E}$ and $\sum_{s(e)=v} c(e) = 0$ for every $v \in \mathcal{V}$. The space of harmonic* *cocycles is denoted by* $\mathcal{C}_{har}(\mathcal{E}, V_{k-2}) \subseteq \mathcal{C}(\mathcal{E}, V_{k-2})$.

Recall that $W = \mathbb{Q}_p^2 - (0,0)$.

Definition 8. *We say that a function* $f : W \to K_p$ *is locally analytic if for all* $w \in W$, *there exists an open neighbourhood* $V \ni w$ *such that* $f|_V(x,y) = \sum_{i,j\geq 0} a_{ij}x^i y^j$ *for* $a_{ij} \in K_p$. *Further, we say that* f *is homogeneous of degree* $k-2$ *under multiplication by* $\mathbb{Q}_p^\times$ *if for all* $t \in \mathbb{Q}_p^\times$, $f(t(x,y)) = t^{k-2}f(x,y)$.

Denote by $\mathcal{A}(W)_{k-2}$ be the space of such K_p-valued locally analytic functions on W that are homogeneous of degree $k-2$. Let $\mathcal{D}(W)_{k-2}$ be the continuous K_p-dual of $\mathcal{A}(W)_{k-2}$ equipped with the strong topology. Note that $P_{k-2}(K_p) \subset \mathcal{A}(W)_{k-2}$. Further, denote by $\mathcal{D}(W)^0_{k-2}$ the subspace of distributions that are zero on $P_{k-2}(K_p)$. Consider

$$\theta_\ell^{\tau_2-\tau_1,P} : W \to \mathbb{C}_p, \qquad \theta_\ell^{\tau_2-\tau_1,P}(x,y) := \ell\left(\frac{y+\tau_2 x}{y+\tau_1 x}\right)P(x,y)$$

where $\ell = \log < . >$ - the Iwasawa logarithm [2] or ord_p ; $\tau_1, \tau_2 \in \mathcal{H}^{ur}_{p,*}$ and $P \in P_{k-2}(K_p)$. Since any $d \in \Delta^0_*$ is a linear combination of divisors of the form $\tau_2 - \tau_1$, we can extend by linearity to define $\theta_l^{d,P}$ for any $d \in \Delta^0_*$. For every $t \in \mathbb{Q}_p^\times$; $\theta_l^{d,P}(t(x,y)) = t^{k-2}\theta_l^{d,P}(x,y)$ and hence we have $\theta_l^{d,P} \in \mathcal{A}(W)_{k-2}$. We denote :

$$I_l^0(\mu, d\otimes P) \in K_p := \mu(\theta_l^{d,P}), \ \ \mu \in \mathcal{D}(W)_{k-2}.$$

Lemma 1 ([17], Lemma 6.1)**.** *The pairing*

$$I_l^0 : \mathcal{D}(W)^0_{k-2} \times \Delta^0_*(P_{k-2}) \to K_p$$

is invariant for the $GL_2(\mathbb{Q}_p)$*-action (resp.* $GL(L)$*-action) when* $*$ *is empty*

[2] We write $w \in \mathbb{C}_p^\times$ as $w = p^r\zeta.z$ where r is a rational number, ζ a root of unity and $|z-1|_p < 1$. Then the Iwasawa logarithm $\log < w >$ is defined as the p-adic logarithm $\log_p(z)$

(resp. $* = v = [L]$*).*

Let $\pi : W \to \mathbb{P}^1(\mathbb{Q}_p)$ be the projection $\pi(x, y) := y/x$. Note that π is well defined since x and y can not be both simultaneously 0.

Lemma 2. *The image of the* $GL_2(\mathbb{Q}_p)$*-equivariant map*

$$R : \mathcal{D}(W)^0_{k-2} \to \mathcal{C}(\mathcal{E}, V_{k-2})$$

given by $R(\mu)(e)(P) := \mu(P.\chi_{W_e})$*, is contained in* $\mathcal{C}_{har}(\mathcal{E}, V_{k-2})$*.*

Proof. Note that $W = W_e \cup W_{\bar{e}}$. Hence we can write $P(x, y) = P.\chi_{W_e} + P.\chi_{W_{\bar{e}}}$. Since $\mu \in \mathcal{D}(W)^0_{k-2}$, we have $\mu(P(x, y)) = 0$ and hence $R(\mu)(\bar{e})(P) = -R(\mu)(e)(P)$. Now, for every $v \in \mathcal{V}$ we have $\bigcup_{s(e)=v} W_e = W$. We write $P(x, y) = \sum_{s(e)=v} P.\chi_{W_e}$ and hence $\mu(\sum_{s(e)=v} P.\chi_{W_e}) = 0$ which implies that $\sum_{s(e)=v} R(\mu)(e) = 0$. Hence $R(\mu) \in \mathcal{C}_{har}(\mathcal{E}, V_{k-2})$. □

For $e \in \mathcal{E}$, denote by $\rho_e : \mathcal{C}_{har}(\mathcal{E}, V_{k-2}) \to V_{k-2}$ the evaluation map. By Lemma 2, we have:

$$R_e : \mathcal{D}(W)^0_{k-2} \xrightarrow{R} \mathcal{C}_{har}(\mathcal{E}, V_{k-2}) \xrightarrow{\rho_e} V_{k-2}$$

The action of $GL_2(\mathbb{Q}_p)$ on the vertices, $\mathcal{V}(\mathcal{T})$ induces an action on the edges, $\mathcal{E}(\mathcal{T})$. We choose an edge $\bar{e} \in \mathcal{E}$ such that its stabilizer in $\widetilde{\Gamma}$ is $\Gamma_0 = \Gamma_0(Np)$.

Definition 9. *We say that a distribution* $\mu \in \mathcal{D}(W)^0_{k-2}$ *is h-admissible if for all* $j \to \infty$*,* $i \geq 0$ *and all* $a \in \mathbb{Z}_p$*, we have*

$$|\mu((x - a)^i | a + p^j \mathbb{Z}_p)| = o(p^{j(h-i)}).$$

for all $i = 0, 1, \ldots, h - 1$*.*

Denote by $\mathcal{D}(W)^{0,h}_{k-2} \subset \mathcal{D}(W)^{0}_{k-2}$ to be the set of such *h-admissible* distributions.

Lemma 3 (Lemma 6.2, [17]). *Passing on to cohomology, $R_{\bar{e}}$ induces*

$$R_{\bar{e}} : H^1\left(\tilde{\Gamma}, \mathcal{D}(W)^{0,h}_{k-2}\right) \cong H^1(\Gamma_0(Np), V_{k-2})^{p-new}.$$

See Definition 2.7 of [33] for the definition of $H^1(\Gamma_0(Np), V_{k-2})^{p-new}$.

Denote by $\mathbb{T}^p_{Np}$ the Hecke algebra over $\mathbb{Q}_p$ generated by the Hecke operators T_ℓ for $\ell \nmid Np$ and U_ℓ for $\ell \mid Np$.

Definition 10. *We say that a Hecke module M admits an Eisenstein/cuspidal decomposition if we can write $M = M_e \oplus M_c$ and there exists a Hecke operator T_l for $l \nmid Np$ such that $t_l := T_l - l^{k-1} - 1$ is nilpotent on M_e and is invertible on M_c. We call M_e (resp. M_c) to be the Eisenstein (resp. cuspidal) part of M.*

Let V be a $\Gamma_0(Np)$-module and denote by $\Gamma_{0,c}$ the stabilizer in $\Gamma_0(Np)$ for c a $\Gamma_0(Np)$-equivalence class of cusps. We can then define the parabolic cohomology group to be

$$H^1_{\text{par}}(\Gamma_0(Np), V) := \ker\left(H^1(\Gamma_0(Np), V) \xrightarrow{res} \bigoplus_{\text{cusps c}} H^1(\Gamma_{0,c}, V)\right)$$

The Hecke module $H^1(\Gamma_0(Np), V_{k-2})^{p-new}$ admits an Eisenstein/cuspidal decomposition with the cuspidal part given by $H^1_{\text{par}}(\Gamma_0(Np), V_{k-2})^{p-new}$. which for brevity we denote by $\mathbb{H}_k$.

The isomorphism of Lemma 3 induces

$$R_{\bar{e},c} : H^1\left(\widetilde{\Gamma}, \mathcal{D}(W)^{0,h}_{k-2}\right)_c \cong \mathbb{H}_k$$

By taking the $\widetilde{\Gamma}$-homology of (3.1), we get

$$\ldots \to H_2(\widetilde{\Gamma}, P_{k-2}) \xrightarrow{\delta} H_1(\widetilde{\Gamma}, \Delta^0(P_{k-2}) \xrightarrow{i} H_1(\widetilde{\Gamma}, \Delta(P_{k-2}) \to H_1(\widetilde{\Gamma}, P_{k-2}) \to \ldots$$

Now by Lemma 3.10, [33] we know that $H_1(\widetilde{\Gamma}, P_{k-2}) = 0$ and hence we have the following isomorphism

$$\bar{i} : H_1(\widetilde{\Gamma}, \Delta^0(P_{k-2}))/\mathrm{im}(\delta) \cong H_1(\widetilde{\Gamma}, \Delta(P_{k-2})).$$

Consider the cap product

$$H_1(\widetilde{\Gamma}, \Delta^0(P_{k-2})) \otimes H^1(\widetilde{\Gamma}, \mathcal{D}(W)^{0,h}_{k-2}) \to H_0\Big(\widetilde{\Gamma}, \Delta^0(P_{k-2}) \otimes \mathcal{D}(W)^{0,h}_{k-2}\Big)$$

where $\mathcal{D}(W)^{0,h}_{k-2}$ is the set of h-admissible distributions (See Definition 9). We know that $H_0\Big(\widetilde{\Gamma}, \Delta^0(P_{k-2}) \otimes \mathcal{D}(W)^{0,h}_{k-2}\Big) = (\Delta^0(P_{k-2}) \otimes \mathcal{D}(W)^{0,h}_{k-2})_{\widetilde{\Gamma}}$ (the set of $\widetilde{\Gamma}$ co-invariants.) which is obtained from $\Delta^0(P_{k-2}) \otimes \mathcal{D}(W)^{0,h}_{k-2}$ by introducing the relations $\gamma.(\tau \otimes P(x,y)) \otimes \mu = (\tau \otimes P(x,y)) \otimes \mu.\gamma$ for $\tau \in \Delta^0, P(x,y) \in P_{k-2}$ and $\mu \in \mathcal{D}(W)^{0,h}_{k-2}$. By Lemma 1, the pairing I^0_l is in particular $\widetilde{\Gamma}$-invariant and hence it extends to a pairing on the cap product, i.e. we have

$$\tilde{I}^0_l : H_1(\widetilde{\Gamma}, \Delta^0(P_{k-2})) \otimes H^1(\widetilde{\Gamma}, \mathcal{D}(W)^{0,h}_{k-2}) \to K_p$$

We now define

$$\mathrm{AJ}^0_l : H_1(\widetilde{\Gamma}, \Delta^0(P_{k-2})) \xrightarrow{\tilde{I}^0_l} H^1\Big(\widetilde{\Gamma}, \mathcal{D}(W)^{0,h}_{k-2}\Big)^\vee \xrightarrow{pr_c \circ R^{-1}_{\bar{\varepsilon}}} \mathbb{H}^{\pm,\vee}_k$$

where pr_c denotes the projection onto the cuspidal part, $\mathbb{H}^{\pm}_k$ denotes the direct summand of $\mathbb{H}_k$ on which $W_\infty = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ acts with eigenvalue ± 1 and '$\vee$' denotes K_p-dual.

By the isomorphism of Lemma 3, $H^1\left(\widetilde{\Gamma}, \mathcal{D}(W)^{0,h}_{k-2}\right)$ inherits an action of $\mathbb{T}^p_{Np}$. We have

Theorem 6 (Corollary 3.13, [33]). *There exists a unique $\mathcal{L} \in \mathbb{T}^p_{Np}$ such that $\tilde{I}^0_{log} - \mathcal{L}\tilde{I}^0_{ord}$ annihilates* $\mathrm{im}(\delta)$

Remark 2. *The unique element $\mathcal{L} \in \mathbb{T}^p_{Np}$ is called the $\mathcal{L}$-invariant associated to $S_k(\Gamma_0(Np))$ - the space of weight k cusp forms on $\Gamma_0(Np)$.*

Let $\log \mathrm{AJ}^0 := \mathrm{AJ}^0_{\log} - \mathcal{L}\mathrm{AJ}^0_{\mathrm{ord}}$. By the above Theorem, we know that $\log \mathrm{AJ}^0$ factors through $H_1(\widetilde{\Gamma}, \Delta^0(P_{k-2}))/\mathrm{im}(\delta)$. We then define the *cohomological Abel-Jacobi map* to be

$$\log \mathrm{AJ} : H_1(\widetilde{\Gamma}, \Delta(P_{k-2}) \xrightarrow{\bar{i}^{-1}} H_1(\widetilde{\Gamma}, \Delta^0(P_{k-2}))/\mathrm{im}(\delta) \xrightarrow{\log \mathrm{AJ}^0} \mathbb{H}_k^{\pm,\vee}$$

3.3 Darmon Cycles

We can view K as a subfield of both $\mathbb{R}$ and $\mathbb{C}_p$ via the fixed embeddings σ and σ_p respectively. For $\tau \in K$, we denote by $\overline{\tau}$ the image of the non-trivial automorphism $\gamma \in \mathrm{Gal}(K/\mathbb{Q})$. We can think of the positive square root $\sqrt{D_K}$ as an element in K_p. Consider the set of all $\mathbb{Q}$-algebra embeddings of K into $M_2(\mathbb{Q})$, denoted by Emb := $\mathrm{Emb}(K, M_2(\mathbb{Q}))$. Let $\mathcal{R}$ be the $\mathbb{Z}[\frac{1}{p}]$-order in $M_2(\mathbb{Q})$ which consists of matrices upper triangular modulo N. Note that $\widetilde{\Gamma} = \mathcal{R}_1^\times$ (the set of invertible matrices of $\mathcal{R}$ with determinant 1). For $\mathcal{O}$ a $\mathbb{Z}[\frac{1}{p}]$-order of conductor c such that $(c, D_K Np) = 1$, denote by $\mathrm{Emb}(\mathcal{O}, \mathcal{R})$ the set of $\mathbb{Z}[\frac{1}{p}]$-embeddings of $\mathcal{O}$ into $\mathcal{R}$. We can attach the following data to every $\psi \in \mathrm{Emb}(\mathcal{O}, \mathcal{R})$

- the fixed points τ_ψ and $\overline{\tau}_\psi \in \mathcal{H}_p$ for the action of $\psi(K^\times)$ on $\mathcal{H}_p(K)$[3].
- the fixed vertex $v_\psi \in \mathcal{V}$ in the Bruhat-Tits tree for the action of $\psi(K^\times)$ on

[3]Since $\psi(K^\times) \subseteq M_2(\mathbb{Q})$, it acts on $\mathcal{H}_p(K)$ by fractional linear transformation

$\mathcal{V}$.

• the unique quadratic form

$$P_\psi(x,y) := cx^2 + (d-a)xy + by^2 \in P_2(K)$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \psi(\sqrt{D_K})$.

• for $u \in \mathcal{O}^\times$, the fundamental unit (i.e $\sigma(u) > 1$) of K, let $\gamma_\psi := \psi(u)$ and Γ_ψ be the cyclic group generated by γ_ψ which is also the stabilizer of ψ in $\widetilde{\Gamma}$. In particular Γ_ψ also fixes $P_\psi(x,y)$. Note that $\Gamma_\psi = \psi(K^\times) \cap \widetilde{\Gamma}$.

We say that $\tau \in \mathcal{H}_p$ has positive orientation if $\mathrm{red}_p(\tau) \in \mathcal{V}^+$. Denote by $\mathcal{H}_p^+$ the set of elements of $\mathcal{H}_p$ with positive orientation. Say $\psi \in \mathrm{Emb}(\mathcal{O},\mathcal{R})$ has positive orientation if $\tau_\psi, \overline{\tau}_\psi \in \mathcal{H}_p^+$. Since $\mathcal{V} = \mathcal{V}^+ \sqcup \mathcal{V}^-$, we have

$$\mathrm{Emb}(\mathcal{O},\mathcal{R}) = \mathrm{Emb}_+(\mathcal{O},\mathcal{R}) \sqcup \mathrm{Emb}_-(\mathcal{O},\mathcal{R})$$

The group $\widetilde{\Gamma}$ acts on $\mathrm{Emb}(\mathcal{O},\mathcal{R})$ by conjugation. Since Γ_ψ is infinite cyclic, we have

$$H_1(\Gamma_\psi, \Delta(P_{k-2})) = H^0(\Gamma_\psi, \Delta(P_{k-2})) := (\Delta(P_{k-2}))^{\Gamma_\psi}$$

(See Example 1, Chapter 3, Page 58 of [7]). Since Γ_ψ acts trivially on $\tau_\psi \otimes D_K^{\frac{k-2}{4}} P_\psi^{\frac{k-2}{2}}(x,y)$, we have

$$D_{\psi,k} := \psi(u) \otimes (\tau_\psi \otimes D_K^{\frac{k-2}{4}} P_\psi^{\frac{k-2}{2}}(x,y)) \in H_1(\Gamma_\psi, \Delta(P_{k-2}))$$

The inclusion $\Gamma_\psi \subset \widetilde{\Gamma}$ induces a co-restriction

$$H_1(\Gamma_\psi, \Delta(P_{k-2})) \to H_1(\widetilde{\Gamma}, \Delta(P_{k-2}))$$

Hence, we can consider $D_{\psi,k}$ as an element in $H_1(\widetilde{\Gamma}, \Delta(P_{k-2}))$.

Lemma 4. *The cycle $D_{\psi,k}$ does not depend on the representative of the conjugacy class of $\psi \in Emb(\mathcal{O}, \mathcal{R})$. Hence we have a well defined map*

$$D_k : \widetilde{\Gamma} \setminus Emb(\mathcal{O}, \mathcal{R}) \to H_1(\widetilde{\Gamma}, \Delta(P_{k-2}))$$

given by $D_k([\psi]) := D_{k,[\psi]}$, where $[\psi]$ denotes the conjugacy class of ψ.

Proof. See Lemma 2.19 of [36]. □

Definition 11. *The Darmon cycle associated to the $\widetilde{\Gamma}$-conjugacy class $[\psi]$ is the element*

$$D_{k,[\psi]} := \psi(u) \otimes (\tau_\psi \otimes D_K^{\frac{k}{4}} P_\psi^{\frac{k-2}{2}}(x, y)) \in H_1(\widetilde{\Gamma}, \Delta(P_{k-2}))$$

Given an embedding $\psi \in \mathrm{Emb}(\mathcal{O}, \mathcal{R})$, we define $\overline{\psi}$ to be the embedding given by $\overline{\psi}(\tau) := \psi(\overline{\tau})$ for $\tau \in K$.

Lemma 5. *We have*

$$(\tau_{\overline{\psi}}, P_{\overline{\psi}}, \gamma_{\overline{\psi}}) = (\overline{\tau}_\psi, -P_\psi, \gamma_\psi^{-1})$$

Proof. By definition of $\overline{\psi}$, we have $\tau_{\overline{\psi}} = \overline{\tau}_\psi$. Now $\overline{\psi}(\sqrt{D_K}) = \psi(-\sqrt{D_K})$. Hence $\overline{\psi}(\sqrt{D_K}) = \left(\begin{smallmatrix} -a & -b \\ -c & -d \end{smallmatrix}\right)$ and we get that $P_{\overline{\psi}}(x, y) = -P_\psi(x, y)$. Now since u is a fundamental unit, we have that $\overline{u} = u^{-1}$. Thus $\gamma_{\overline{\psi}} = \overline{\psi}(u) = \psi(u^{-1}) = \gamma_\psi^{-1}$. □

Recall the *cohomological Abel-Jacobi map* we defined earlier :

$$\log \mathrm{AJ} \ : \ H_1(\widetilde{\Gamma}, \Delta(P_{k-2})) \to \mathbb{H}_k^{\pm,\vee}$$

Definition 12. *The Darmon cohomology class associated to* $[\psi] \in \widetilde{\Gamma}/Emb(\mathcal{O}, \mathcal{R})$ *is* $[j_\psi] := log\ AJ(D_{k,[\psi]}) \in \mathbb{H}_k^{\pm,\vee}$

To every $I \subset K$, a free rank two $\mathbb{Z}[\frac{1}{p}]$-submodule, we associate the order

$$\mathcal{O}_I := \{\lambda \in K \mid \lambda.I \subset I\}$$

Definition 13. *A fractional* $\mathcal{O}$*-ideal is a free rank two* $\mathbb{Z}[\frac{1}{p}]$*-submodule whose assoicated order is* $\mathcal{O}$*. Two such fractional ideals are said to be strictly equivalent if* $I_1 = \alpha I_2$ *for some* $\alpha \in K^\times$ *with positive norm.*

Denote by $\mathrm{Pic}^+(\mathcal{O})$ the *narrow Picard group* of strict equivalence class of fractional $\mathcal{O}$-ideals. By class field theory (See Theorem 4.2 of [31]), we have the reciprocity isomorphism

$$rec : \mathrm{Pic}^+(\mathcal{O}) \cong \mathrm{Gal}(H_{\mathcal{O}}^+/K)$$

where $H_{\mathcal{O}}^+$ is the narrow ring class field of K associated to the order $\mathcal{O}$.

Proposition 3 (Proposition 5.8, [11])**.** *The sets* $\widetilde{\Gamma}/Emb(\mathcal{O}, \mathcal{R})$ *and* $Pic^+(\mathcal{O})$ *are in bijection.*

Hence we have a natural action of $\mathrm{Pic}^+(\mathcal{O})$ on $\widetilde{\Gamma}/\mathrm{Emb}(\mathcal{O}, \mathcal{R})$. By the reciprocity isomorphism, we can consider the action of $\mathrm{Gal}(H_{\mathcal{O}}^+/K)$ on $\widetilde{\Gamma}/\mathrm{Emb}(\mathcal{O}, \mathcal{R})$.

Let $\chi : \mathrm{Gal}(H_{\mathcal{O}}^+/K) \to \mathbb{C}^\times$ be a character. We will consider the following liner combination

$$D_k^\chi := \sum_{\sigma \in \mathrm{Gal}(H_{\mathcal{O}}^+/K)} \chi^{-1}(\sigma) D_{\sigma.[\psi],k} \in H_1(\widetilde{\Gamma}, \Delta(P_{k-2}))^\chi$$

3.4 Monodromy modules

Recall that p is inert in K and hence K_p is isomorphic to the unramified quadratic extension of $\mathbb{Q}_p$. Let $\sigma_{Frob} : K_p \to K_p$ denote the absolute Frobenius automorphism of K_p. Let $\mathbb{T}_p$ be a finite dimensional commutative semisimple $\mathbb{Q}_p$-algebra. We write $\mathbb{T}_{K_p} := \mathbb{T}_p \otimes_{\mathbb{Q}_p} K_p$ and $\sigma_{K_p} := \mathrm{Id} \otimes \sigma_{Frob}$ on $\mathbb{T}_{K_p}$.

Definition 14. *A $\mathbb{T}_p$-monodromy module is a four tuple (D, ϕ, N, F) where D is a free rank two $\mathbb{T}_{K_p}$-module, $\phi : D \to D$ is a σ_{Frob}-linear endomorphism (the Frobenius of D) and $N : D \to D$ a $\mathbb{T}_{K_p}$-linear nilpotent endomorphism (the monodromy operator) such that*

- *As a two dimensional K_p-vector space, D is endowed with a filtration F*

$$0 = F^k \subset F^{k-1} \subset \ldots \subset F^0 = D$$

where F^{k-1} is a free rank-one $\mathbb{T}_{K_p}$-module.
- $D \xrightarrow{N} D \xrightarrow{N} D$ *is exact.*
- $D = F^{k-1} \oplus \ker(N(D))$ *as a $\mathbb{T}_{K_p}$-module.*
- $N \circ \phi = p\phi \circ N$ *and* $\forall\, T \in \mathbb{T}_{K_p}$, $\phi \circ T = \sigma_{K_p}(T) \circ \phi$.

The integer k which occurs in the filtration F is called the *weight* of the monodromy module D. Usually, we denote the monodromy module simply by D. When we forget the action of $\mathbb{T}_p$, D is called a filtered Frobenius monodromy module or a filtered (ϕ, N)-module over K_p. Let $\mathrm{MF}_{K_p}(\phi, N)$ denote the category of filtered (ϕ, N)-modules over K_p. The morphisms in $\mathrm{MF}_{K_p}(\phi, N)$ are K_p-module homomorphisms which preserve the filtrations and commute with both the Frobenius and monodromy endomorphisms.

Let $D \in \mathrm{MF}_{K_p}(\phi, N)$. By [46], D admits a canonical *slope decomposition*:

$$D = \bigoplus_{\alpha \in \mathbb{Q}} D^{\alpha}$$

where for $\alpha = \frac{r}{s}$ with $r, s \in \mathbb{Z}$ such that $s > 0$, $D^{\alpha} \subset D$ is the largest K_p-subspace of D which has an $\mathcal{O}_{K_p}$-stable lattice $D^{\alpha,0}$ such that $\phi^s(D^{\alpha,0}) = p^r D^{\alpha,0}$. When $D^{\alpha} \neq 0$, we call α a *slope* of D. (D, ϕ) is called isotypical of slope α_0 if α_0 is the only slope of D. Further $N(D^{\alpha+1}) \subseteq D^{\alpha}$ (See Section 2 of [19]).

Let V be a p-adic Galois representation of G_{K_p}. By Fontaine theory, one can attach to V a filtered (ϕ, N) module defined by

$$D_{st}(V) := (V \otimes B_{st})^{G_{K_p}}$$

where B_{st} is Fontaine's semi-stable ring of periods. $D_{st}(V)$ inherits the structure of a filtered (ϕ, N) module from B_{st}. For the definition and properties of B_{st}, see [15].

Definition 15. *V is called a semi-stable p-adic representation if the canonical $B_{st}[G_{K_p}]$-homomorphism*

$$\alpha_{st}(V) : B_{st} \otimes_{K_p} D_{st}(V) \to B_{st} \otimes_{\mathbb{Q}_p} V; \qquad \lambda \otimes x \mapsto \lambda x$$

is an isomorphism.

Definition 16. *$D \in MF_{K_p}(\phi, N)$ is called admissible if there exists some semi-stable p-adic Galois representation V of G_{K_p} such that*

$$D \cong D_{st}(V)$$

as filtered (ϕ, N) *modules.*

Let $\mathrm{MF}_{K_p}^{ad} \subset \mathrm{MF}_{K_p}$ be the full subcategory of *admissible* filtered (ϕ, N)-modules over K_p and let $\mathrm{Rep}_{st}(G_{K_p})$ be the subcategory of semi-stable p-adic Galois representations of G_{K_p}.

Remark 3. *The functor*

$$D_{st} : \mathrm{Rep}_{st}(G_{K_p}) \to \mathrm{MF}_{K_p}^{ad}$$

is an equivalence of categories (See [10]).

Now let D be a $\mathbb{T}_p$-monodromy module of rank two. By Lemma 2.3 of [19], we have a decomposition

$$D = D_{(1)} \oplus D_{(2)}$$

where each $D_{(i)}$ is a free rank one $\mathbb{T}_{K_p}$-submodule stable under the Frobenius endomorphism ϕ. The monodromy endomorphism $N : D \to D$ restricts to an isomorphism $N |_{D_{(2)}}: D_{(2)} \to D_{(1)}$. Such a decomposition is uniquely determined by these properties. Hence we write $D_{(2)} \cong \mathbb{T}_{K_p}.e$ and $D_{(1)} \cong \mathbb{T}_{K_p}.N(e)$for some $e \neq 0 \in D_{(2)}$.

We will now recall the definitions of two important invariants associated to a $\mathbb{T}_{K_p}$-monodromy module D.

Definition 17. *The* $\mathcal{L}$*-invariant of a* $\mathbb{T}_p$*-monodromy module* (D, F, ϕ, N) *is the unique element* $\mathcal{L}_D$ *in* $\mathbb{T}_{K_p}$ *such that*

$$x - \mathcal{L}_D N(x) \in F^{k-1} \ \ \forall x \in D_{(2)}$$

For the existence and uniqueness (and other properties) of the $\mathcal{L}$-invariant, see Pierre Colmez's survey [9].

Definition 18. *U_D is defined to be the element in $\mathbb{T}_{K_p}$ such that $\phi N(e) = U_D N(e)$. U_D exists and is well-defined since $D_{(1)}$ is stabe under ϕ and $D_{(1)} = \mathbb{T}_{K_p} N(e)$.*

A two-dimensional $\mathbb{T}_p$-monodromy module is completely determined up to isomorphism by the invariants $\mathcal{L}_D, U_D$ and the weight $k = k_D$.

Proposition 4. *Given an integer $k \in \mathbb{Z}$ and elements $\mathcal{L}_D, U_D \in \mathbb{T}_{K_p}$, there exists D, a $\mathbb{T}_p$-monodromy module of dimension two over K_p, such that k is its weight, $\mathcal{L}_D$ its $\mathcal{L}$-invariant and U_D its U-invariant*

Proof. See Proposition 4.6 of [33]. □

Let $\mathbb{T} := \mathbb{T}^{p-new}_{\Gamma_0(Np)}$ be the quotient of the Hecke algebra over $\mathbb{Q}$ of level Np that acts faithfully on the space $S_{k_0}(\Gamma_0(Np))^{p-new}$ and let $\mathbb{T}_p := \mathbb{T} \otimes \mathbb{Q}_p$. Let $V_{k_0}(Np)^{p-new}$ be the p-adic Galois representation of $G_{\mathbb{Q}}$ associated to $S_k(\Gamma_0(Np))^{p-new}$. Let V_p^{Np} denote the restriction of $V_{k_0}(Np)^{p-new}$ to a decomposition group at p.

Proposition 5. *The local p-adic Galois representation V_p^{Np} is semistable but not crystalline.*

Proof. This is Corollary 7.5 of [19]. □

Fontaine and Mazur attach to V_p^{Np} the admissible monodromy module

$$D^{\mathrm{FM}} := D_{st}(V_p^{Np})$$

Let $\mathcal{L}_{\mathrm{FM}}$ be the $\mathcal{L}$-invariant of D^{FM}.

Recall the $\mathbb{T}_p$-module $\mathbb{H}_k^{\pm} = H^1_{\text{par}}(\Gamma_0(Np), V_{k-2}(\mathbb{Q}_p)^{p\text{-new},\pm}$ which is free of rank one over $\mathbb{T}_p$. Let

$$\mathbb{D}_k := \mathbb{H}_k^{\pm,\vee} \oplus \mathbb{H}_k^{\pm,\vee}$$

We define a filtration $F_{\mathbb{D}_k}$ on $\mathbb{D}_k$ as follows :-

$$0 = F^k \subset F^{k-1} \subset \ldots \subset F^0 = \mathbb{D}_k$$

where $F^i = \{(-\mathcal{L}_{\text{FM}}x, x) : x \in \mathbb{H}_k^{\pm,\vee}\}$ for all $1 \leq i \leq k-1$. By Proposition 5, $\mathbb{D}_k$ along with the filtration $F_{\mathbb{D}_k}$ is a $\mathbb{T}_p$-monodromy module over $\mathbb{Q}_p$.

Theorem 7. *We have a $\mathbb{T}_p$-monodromy module isomorphism*

$$D^{FM} \cong \mathbb{D}_{k_0} := \mathbb{H}_{k_0}^{\pm,\vee} \oplus \mathbb{H}_{k_0}^{\pm,\vee}$$

Further the isomorphism is stable under base change to K_p and the following diagram commutes

$$\begin{array}{ccc} \mathbb{H}_{k_0}^{\pm}(K_p)^\vee \oplus \mathbb{H}_{k_0}^{\pm}(K_p)^\vee & \longrightarrow & D^{FM} \otimes K_p \\ \downarrow & & \downarrow \\ \mathbb{H}_{k_0}^{\pm}(K_p)^\vee & \longrightarrow & \frac{D^{FM}\otimes K_p}{F^{k_0/2}(D^{FM}\otimes K_p)} \end{array}$$

where the vertical arrow is $(x, y) \mapsto x + \mathcal{L}_{FM}y$.

Proof. See Proposition 4.6 and Theorem 4.7 of [33]. □

Remark 4. In Theorem 2 and Theorem 3 of [37], Marco Seveso shows that the $\mathcal{L}$-invariant of Remark 2 coincides with $\mathcal{L}_{FM}$.

By composing with the isomorphism of Theorem 7, we can consider the cohomological Abel Jacobi map as

$$\log \mathrm{AJ} : H_1(\widetilde{\Gamma}, \Delta(P_{k_0-2})) \to \mathbb{H}^{\pm}_{k_0}(K_p)^{\vee} \xrightarrow{\sim} \frac{D^{\mathrm{FM}} \otimes K_p}{F^{k_0/2}(D^{\mathrm{FM}} \otimes K_p)}$$

We will also simultaneously refer to the above map as the p-adic Abel-Jacobi map.

3.5 Rationality of Darmon cycles - I

3.5.1 Bloch-Kato exponential

Recall the *semi-stable* p-adic local Galois representation V_p^{Np} associated to $S_k(\Gamma_0(Np))^{p\text{-new}}$. Let L be a number field in which p is unramified. For every place v of L, we define

$$H^1_{st}(L_v, V_p^{Np}) := \ker\left(H^1(L_v, V_p^{Np}) \to \begin{cases} H^1(L_v^{unr}, V_p^{Np}) & \text{if } v \nmid p \\ H^1(L_v, B_{st} \otimes_{\mathbb{Q}_p} V_p^{Np}) & \text{if } v \mid p \end{cases} \right)$$

The semi-stable *Selmer* group associated to V_p^{Np} is then

$$\mathrm{Sel}_{st}(L, V^{Np}) := \ker\left(H^1(L, V^{Np}) \xrightarrow{\prod res_v} \prod_v \frac{H^1(L_v, V_p^{Np})}{H^1_{st}(L_v, V_p^{Np})} \right)$$

Since the local Galois representation V_p^{Np} is semi-stable, the Bloch-Kato isomorphism (See Section 3 of [6]) induces

$$\exp_{BK} : \frac{D^{FM} \otimes_{\mathbb{Q}_p} K_p}{F^{k_0/2}(D^{FM} \otimes_{\mathbb{Q}_p} K_p)} \xrightarrow{\cong} H^1_{st}(K_p, V_p^{Np}(k_0/2))$$

Composing the p-adic Abel-Jacobi map with the above Bloch-Kato exponential along with the isomorphism of Theorem 7 we have :

$$\mathrm{AJ} : H_1(\widetilde{\Gamma}, \Delta(P_{k_0-2})) \to H^1_{st}(K_p, V_p^{Np}(k_0/2))$$

where $\mathrm{AJ} := \log \mathrm{AJ} \circ \exp_{BK}$.

We consider the image of Darmon cycles under of this map as cohomology classes

$$s_\psi \in H^1_{st}(K_p, V_p^{Np}(k_0/2)), \qquad s_\chi \in H^1_{st}(K_p(\chi), V_p^{Np}(k_0/2))$$

Since p is inert in K, it splits completely in the narrow Hilbert class field H_K^+. Hence the embedding σ_p induces an inclusion $\sigma_p : H_K^+ \hookrightarrow K_p$. Therefore we have

$$\mathrm{res_p} : \mathrm{Sel}_{st}(H_K^+, V_p^{Np}(k_0/2)) \to H^1_{st}(K_p, V_p^{Np}(k_0/2))$$

Conjecture 1 (Conjecture 5.7, [33]). *(1) For $[\psi] \in \widetilde{\Gamma}/Emb(\mathcal{O}, \mathcal{R})$, there exists a global cycle $S_\psi \in Sel_{st}(H_K^+, V_p^{Np}(k_0/2))$ such that*

$$res_p(S_\psi) = AJ(D_{[\psi],k_0})$$

(2) For $\chi : \mathrm{Gal}(H_K^+/K) \to \mathbb{C}^\times$ a character, there exists $S_\chi \in Sel_{st}(H_\chi, V_p^{Np}(k_0/2))^\chi$, where $H_\chi \subseteq H_K^+$ is the extension of K cut out by χ, such that

$$res_p(S_\chi) = AJ(D_{k_0}^\chi)$$

The conjecture is known to be true when χ is a *genus character* of K. This will be discussed in the following chapters.

Chapter 4

p-adic L-functions

Denote by $\mathbb{T}_N$ the Hecke algebra over $\mathbb{Q}$ generated by T_ℓ for $\ell \nmid N$ and U_ℓ for $\ell \mid N$. Then by Theorem 3.51 of [38], $\dim_{\mathbb{Q}} \mathbb{T}_N = \dim_{\mathbb{C}} S_k(\Gamma_0(N))$ where $S_k(\Gamma_0(N))$ is the space of cusp forms of weight k on $\Gamma_0(N)$. For $f \in S_k(\Gamma_0(N))$, let K_f be the number field generated by its Fourier coefficients.

4.1 Modular symbols

Let $P_{k-2}(E)$ denote the space of homogeneous polynomials of degree $k-2$ in two variables over a field E. The group $SL_2(\mathbb{Z})$ acts on the right on $P_{k-2}(E)$ by

$$(P \mid \gamma)(x, y) := P((x, y).\gamma^{-1}) = P(dx - cy, -bx + ay)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. The dual space $V_{k-2}(E) := \mathrm{Hom}_E(P_{k-2}(E), E))$ is endowed with the natural dual left action.

Let $\Delta := \mathrm{Div}(\mathbb{P}^1(\mathbb{Q})$ (resp. $\Delta^0 := \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ denote the space of divisors (resp. divisors of degree zero) over $\mathbb{P}^1(\mathbb{Q})$. Δ and Δ^0 are endowed with a natural left action of $SL_2(\mathbb{Z})$ acting via fractional linear transformations. The space $\mathrm{Hom}(\Delta^0, V_{k-2})$ has an induced right action of $SL_2(\mathbb{Z})$ given by

$$\phi \mid \gamma(D) := \phi(\gamma.D) \mid \gamma$$

where $\gamma \in GL_2(\mathbb{Q})$ and $\phi : \Delta^0 \to V_{k-2}$. For Γ a congruence subgroup of $SL_2(\mathbb{Z})$ (usually $\Gamma_0(N)$ or $\Gamma_1(N)$), let $\mathrm{Symb}_\Gamma(V_{k-2}) \subset \mathrm{Hom}(\Delta^0, V_{k-2})$ be the sub-module invariant under the action of Γ. We call $\mathrm{Symb}_\Gamma(V_{k-2})$ to be the space of *modular symbols* on Γ. The matrix $W_\infty := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ acts as an involution on $V_{k-2}(E)$ (assuming $2 \nmid \mathrm{char}(E)$). We will denote by $V_{k-2}^{w_\infty}$ to be the direct summand of $V_{k-2} = V_{k-2}^{+} \oplus V_{k-2}^{-}$ on which W_∞ acts by $w_\infty \in \{\pm 1\}$.

Consider the $GL_2^+(\mathbb{Q})$-equivariant map

$$\widetilde{\phi} : S_k(\Gamma_0(N), \mathbb{C}) \to \mathrm{Symb}_{\Gamma_0(N)}(V_{k-2}(\mathbb{C}))$$

$$\widetilde{\phi_f}\{x - y\}(P) := 2\pi i \int_x^y f(z)P(z,1)dz \in \mathbb{C}$$

for $P(x,y) \in P_{k-2}(\mathbb{C})$. Since Δ^0 is generated by divisors of the form $\{x - y\}$ for $x, y \in \mathbb{P}^1(\mathbb{Q})$, we extend the map ϕ by linearity to all of Δ^0.

Remark 5. *Note that the map ϕ is similar to the Shintani periods we defined in Chapter 2.*

By Proposition 4.2 of [1], we know that

$$\mathrm{Symb}_{\Gamma_0(N)}(V_{k-2}) \cong H_c^1(\Gamma_0(N), V_{k-2})$$

where H_c^1 denotes the subspace of compactly supported cohomology. See Chapter 8 of [38] for more details about cohomology with compact support H_c^1.

We have a natural map $H_c^1(\Gamma_0(N), V_{k-2}) \to H^1(\Gamma_0(N), V_{k-2})$ which sends a modular symbol ϕ to the 1-cocycle $[\gamma \mapsto \phi(\gamma.x_0 - x_0)]$. By Proposition 2, Appendix of [18], the parabolic cohomology group $H^1_{\text{par}}(\Gamma_0(N), V_{k-2})$ is the image of $H_c^1(\Gamma_0(N), V_{k-2})$ in $H^1(\Gamma_0(N), V_{k-2})$.

Theorem 8 (Eichler-Shimura)**.** *The map ϕ induces an isomorphism*

$$\widetilde{\phi} : \overline{S_k(\Gamma_0(N), \mathbb{C})} \oplus S_k(\Gamma_0(N), \mathbb{C}) \cong H^1_{\text{par}}(\Gamma_0(N), V_{k-2}(\mathbb{C}))$$

Proof. See Theorem 8.4 of [38]. □

We can write $\widetilde{\phi_f} = \widetilde{\phi_f^+} + \widetilde{\phi_f^-}$ where $\widetilde{\phi_f}^{\pm} \in \text{Symb}_\Gamma(V_{k-2}(\mathbb{C})^\pm)$.

Theorem 9 (Shimura)**.** *If f is a newform on $\Gamma_0(N)$, then there exists complex periods $\Omega^\pm \in \mathbb{C}$ such that*

$$\phi_f^\pm := \frac{\widetilde{\phi_f}^\pm}{\Omega^\pm} \in Symb_{\Gamma_0(N)}(V_{k-2}(K_f)^\pm)$$

The periods $\Omega^\pm$ can be chosen to satisfy

$$\Omega^+\Omega^- = < f, f >$$

Proof. This is Theorem 1 (ii) of [40]. □

Usually we will fix a choice of $w_\infty \in \{\pm 1\}$ and consider $\phi_f := \phi_f^{w_\infty}$.

4.1.1 Overconvergent modular symbols

Let $\mathcal{X}$ denote the rigid analytic p-adic weight space over $\mathbb{Q}_p$. For a finite extension $E/\mathbb{Q}_p$, the rational points are given by $\mathcal{X}(E) = \text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, E^\times)$. We have a natural inclusion, $\mathbb{Z} \subset \mathcal{X}$, given by $k \to [t \mapsto t^{k-2}]$. We can write every $t \in \mathbb{Z}_p^\times$ in the form $t = [t] < t >$ where $[t] \in (\mathbb{Z}/p)^\times$ and $< t > \in 1 + p\mathbb{Z}_p$.

Let $U \subset \mathcal{X}$ be an open affinoid defined over E. Every $\kappa \in U(E)$ can be written uniquely in the form

$$\kappa(t) = \epsilon(t)\chi(t) < t >^s$$

for $\epsilon, \chi : \mathbb{Z}_p^\times \to E^\times$ characters of order $p-1$ and p respectively and $s \in \mathcal{O}_E$. An integer k corresponds to the character $k(t) = [t]^{k-2} < t >^{k-2}$. We will restrict to a neighbourhood U of k_0 such that $\epsilon(t) = [t]^{k_0-2}$ and $\chi = 1$ for all $\kappa \in U(K)$. Note that for all $k \in U$, $[t]^{k-2} = [t]^{k_0-2} \Longrightarrow k \equiv k_0 \bmod (p-1)$.

Denote the set of non-zero vectors in $\mathbb{Q}_p^2$ by W and consider the natural projection to $\mathbb{P}^1(\mathbb{Q}_p)$ which is continuous for the p-adic topology

$$\pi : W = \mathbb{Q}_p^2 - (0,0) \to \mathbb{P}^1(\mathbb{Q}_p)$$

$$\pi(x,y) := \frac{x}{y}$$

Let $L \subset \mathbb{Q}_p^2$ be a $\mathbb{Z}_p$-lattice. Denote by $L' := L - pL$ - its set of primitive (not divisible by p) vectors and by $|L| := p^{\mathrm{ord}_p(\det\gamma)}$ where γ is any $\mathbb{Z}_p$-basis of L written as a 2 x 2 matrix. As in Chapter 3, let $L_* := \mathbb{Z}_p \oplus \mathbb{Z}_p$ and $L_\infty := \mathbb{Z}_p \oplus p\mathbb{Z}_p$ be its neighbour in the Bruhat-Tits tree $\mathcal{T}$. Recall from Definition 5, Chapter 3 that to each edge $e \in \mathcal{E}(\mathcal{T})$, we can associate open compact subsets in W and $\mathbb{P}^1(\mathbb{Q}_p)$ by

$$W_e := L'_{s(e)} \cap L'_{t(e)} \ \ \& \ \ U_e := \pi(W_e)$$

Let e_∞ be the edge between $v_* = [L_*]$ and $v_\infty = [L_\infty]$. Further denote by W_∞ to be the set W_{e_∞}.

Let Y be an open compact subset of either W or $\mathbb{P}^1(\mathbb{Q}_p)$ and denote by $\mathcal{A}(Y)$ to be the space of $\mathbb{Q}_p$-valued locally analytic functions on Y (See Definiton

7, Chapter 3). Let $\mathcal{D}(Y)$ be the continuous $\mathbb{Q}_p$-dual of $\mathcal{A}(Y)$ which will be called the space of locally analytic distributions on Y. For $\mu \in \mathcal{D}(Y)$ and $F \in \mathcal{A}(Y)$, we denote $\mu(F)$ by the measure theoretic notation - $\int\limits_Y F d\mu$. Further for any $X \subset Y$ compact open, write $\int\limits_X F d\mu$ to denote $\mu(F.\chi_X)$ where χ_X is the characteristic function on X.

Consider the left action of $GL_2(\mathbb{Q})$ on $\mathbb{Q}_p^2$ (This is given by matrix multiplication by viewing elements of $\mathbb{Q}_p^2$ as column vectors). This induces an action of $GL_2(\mathbb{Z}_p)$ on L' for any lattice L. Further let $\mathbb{Z}_p^*$ act on the left on L' by multiplication $(t.(x,y) := (tx, ty))$. $\mathcal{D}(\mathbb{Z}_p^*)$ acts on $\mathcal{D}(Y)$ as follows

$$\mathcal{D}(\mathbb{Z}_p^*) \times \mathcal{D}(Y) \to \mathcal{D}(Y) \quad (r, \mu) \mapsto r\mu$$

where $r\mu$ is defined as the distribution

$$\int\limits_{L'_*} F(x,y) d(r\mu)(x,y) := \int\limits_{\mathbb{Z}_p^\times} \Big(\int\limits_{L'_*} F(tx, ty) d\mu(x,y) \Big) dr(t)$$

Let $k \in \mathbb{Z}^{\geq 0}$ and let $U_k \subset \mathcal{X}$ be an affinoid neighborhood of k. The associated affinoid algebra $A(U_k)$ has a natural $\mathcal{D}(\mathbb{Z}_p^\times)$-algebra as follows

$$\mu \mapsto \left[\kappa \mapsto \int\limits_{\mathbb{Z}_p^\times} \kappa(t) d\mu(t) \right]$$

Hence we can consider the completed tensor product over $\mathcal{D}(\mathbb{Z}_p^\times)$,

$$\mathcal{D}_{U_k} := A(U_k) \widehat{\otimes}_{\mathcal{D}(\mathbb{Z}_p^\times)} \mathcal{D}(L'_*)$$

Coleman families. Recall that N is an odd square-free integer such that $p \nmid N$. A *Coleman family* of modular forms on $U \subset \mathcal{X}$ of tamle level $\Gamma_0(N)$ is given by a formal q-expansion

$$\mathbf{f}(q) = \sum_{n \geq 1} \mathbf{a}_n(\kappa) q^n \in \mathcal{O}(U)[[q]]$$

such that for all (even integral) classical weights

$$k \in U^{cl} := \{n \in 2\mathbb{Z}^{\geq 0}\} \cap U$$

the weight k-specialization, $f_k(q) := \sum_{n \geq 1} a_n(k) q^n \in S_k(\Gamma_0(Np))$. We will make an assumption that $f_k(q)$ is N-new for all $k \in U^{cl}$. By Atkin-Lehner-Li-Miyake theory and by the constancy of the *slope of* $\boldsymbol{f}(q)$ - $\mathrm{ord}_p(\mathbf{a}_p(k))$, there is exactly one classical point k_0 for which f_{k_0} is p-new. This happens when $a_p(k_0) = \pm p^{\frac{k_0-2}{2}}$. For every $k \neq k_0 \in U^{cl}$, there exists an eigenform $f^{\#}_{k_0} \in S_k(\Gamma_0(N))^{\mathrm{new}}$ such that

$$f_k(q) = f_k^{\#}(q) - \frac{p^{k-1}}{a_p(k)} f_k^{\#}(q^p) \quad (4.1)$$

See Section 1.3 of [4] for a summary about *Coleman families.*

To each $f_k^{\#}$ (resp. f_k) ($k \neq k_0$) we can attach a modular symbol, $\widetilde{\phi}_k^{\#} \in \mathrm{Symb}_{\Gamma_0(N)}(\mathbb{C})$ (resp. $\widetilde{\phi}_k \in \mathrm{Symb}_{\Gamma_0(Np)}(\mathbb{C})$). Upon fixing $w_\infty \in \{\pm 1\}$ and a period $\Omega^{\#, w_\infty}$, we define

$$\phi_k^{\#} = \phi_k^{\#, w_\infty} = \frac{\widetilde{\phi_k}^{\#, w_\infty}}{\Omega^{\#, w_\infty}} \in \mathrm{Symb}^{w_\infty}_{\Gamma_0(N)}(K_{f_k})$$

Under the Eichler-Shimura isomorphism, (4.1) translates as

$$\phi_k\{r \to s\}(P) = \phi_k^{\#}\{r \to s\}(P) - \frac{p^{k/2-1}}{a_p(k)} \phi_k^{\#}\{r/p \to s/p\}(P(x, y/p)) \quad (4.2)$$

Finally, let $f^{\#}_{k_0} = f_{k_0}$.

Definition 19. *The space of overconvergent modular symbols for* Γ *is defined as the space of modular symbols with coefficients in* $\mathcal{D}_U = A(U)\widehat{\otimes}_{\mathcal{D}(\mathbb{Z}_p^\times)}\mathcal{D}(L'_*)$. *We will denote it by* $Symb_\Gamma(\mathcal{D}_U)$.

For every $k \in U^{cl}$, we can define a weight k-specialization map

$$\rho_k : \mathrm{Symb}_{\Gamma_0(N)}(\mathcal{D}_U) \to \mathrm{Symb}_{\Gamma_0(Np)}(V_{k-2}(\overline{\mathbb{Q}_p}))$$

$$\rho_k(I)\{r \to s\}(P) := \int_{W_\infty} P(x,y)dI\{r \to s\}(x,y)$$

Theorem 10 (G. Stevens)**.** *There exists* $\Phi_* \in Symb_{\Gamma_0(N)}(\mathcal{D}_U(W_\infty))$ *such that*

- *for any* $k \in U^{cl}$, *the weight* k*-specialization,* $\rho_k(\Phi_*) = \lambda(k)\phi_k$ *for some constant* $\lambda(k) \in \overline{\mathbb{Q}_p}^\times$.
- $\rho_{k_0}(\Phi_*) = \phi_{k_0}$

Proof. See Theorem 6.4.1 of [2]. □

We can define a family of distributions $\{\Phi_L\} \in \mathrm{Symb}_{\Gamma_0(N)}(\mathcal{D}_U)$, indexed by lattices $L \subset \mathbb{Q}_p^2$, as follows : for all $F \in \mathcal{A}(L')$

$$\Phi_{L_*} := \Phi_*; \qquad \Phi_L\{r \to s\}(F) := \Phi_{L_*}\{\gamma r \to \gamma s\}(F \mid \gamma^{-1})$$

where $\gamma.L = L_*$. We can now describe (4.2) in terms of Φ_*.

Corollary 2. *For every* $k \in U^{cl}$ *and* $P \in P_{k-2}$

$$\rho_k(\Phi_*)\{r \to s\}(P) = \lambda(k)\left(1 - \frac{p^{k-2}}{a_p(k)^2}\right)\phi_k^{\#}\{r \to s\}(P).$$

Proof. This is Corollary 4.6 in [36]. □

4.2 The Stevens - Mazur - Kitagawa p-adic L-function

In this section, we recall the construction and interpolation property of the two variable Stevens - Mazur - Kitagawa p-adic L-function. The original construction of Mazur and Kitagawa ([21]) was only for the ordinary case (Hida families) and was extended by G. Stevens to the finite slope case using overconvergent modular symbols in [43].

Let $g \in S_k(\Gamma_0(N))$ be a cusp form and let $\phi_g \in \mathrm{Symb}^{w_\infty}_{\Gamma_0(N)}(K_g)$ be the modular symbol attached to g. For $n \in \mathbb{Z}^{>0}$, define

$$\phi_{g,n} : P_{k-2}(K_g) \times \mathbb{Z}/n\mathbb{Z} \to K_g$$

$$\phi_{g,n}(P, a) := \phi_g\{\infty \to a/n\}(P)$$

Since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & n \end{pmatrix} = \begin{pmatrix} 1 & -a+n \\ 0 & n \end{pmatrix}$$

and that ϕ_g is invariant under the action of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N), \phi_{g,n}$ depends only on the class of $a \in \mathbb{Z}/n$.

Let

$$\tau(\chi) := \sum_{a \in \mathbb{Z}/n\mathbb{Z}} \chi(a) e^{2\pi i a/n}$$

be the Gauss sum associated to χ - a primitive Dirichlet character mod n. For

$$\tilde{L}(g, \chi, j) := \frac{(j-1)!\tau(\chi)}{(-2\pi i)^{j-1}\Omega_g} L(g, \chi, j)$$

the 'algebraic part' of $L(g,\chi,j)$ and $P_{j,a} := \left(x - \frac{a}{n}y\right)^{j-1} y^{k-j-1} \in P_{k-2}(K_g)$, we have the following results :

Proposition 6. *For every integer $1 \leq j \leq k-1$ such that $\chi(-1) = (-1)^{j-1}w_\infty$, we have*

$$\sum_{a \in \mathbb{Z}/n\mathbb{Z}} \chi(a)\phi_{g,n}(P_{j,a}, a) = \tilde{L}(g,\chi,j)$$

Proof. This is a straight forward calculation relating L-values and modular symbols. This has been shown in Sect. 7 of [28]. The relevant calculation for the twisted L-values is in Sect. 8 of [28]. □

Remark 6. *The proposition shows that $\tilde{L}(g,\chi,j)$ belongs to $K_g(\chi)$. In particular these quantities are algebraic and hence can be seen as p-adic numbers thus making it possible to interpolate them p-adically. In the thesis, we deal with square free level and quadratic twists which implies that K_g is a totally real field and that $\tilde{L}(g,\chi,j) \in \mathbb{R}$.*

Let $(x,y) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ and let $p \nmid n$. Then we have

$$x - \frac{pa}{n}y \in \mathbb{Z}_p^\times + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$$

Hence, for $\kappa \in U$, the *locally analytic function*

$$F_{s,pa} := \left(x - \frac{pa}{n}y\right)^{s-1} y^{\kappa-s-1} \in \mathcal{A}_U(L'_*)$$

Definition 20. *Let $\mathbf{f}(\mathbf{q})$ be the Coleman family of tame level N and let $\chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ be a Dirichlet character of conductor n such that $p \nmid n$. We define the Stevens - Mazur- Kitagawa p-adic L-function as follows :*

$$\mathcal{L}_p^{SMK}(\boldsymbol{f},\chi,\kappa,s) : U \times \mathbb{Z}_p \to \mathbb{C}_p$$

$$(\kappa, s) \mapsto \mathcal{L}_p^{SMK}(\boldsymbol{f}, \chi, \kappa, s)$$

$$\mathcal{L}_p^{SMK}(\boldsymbol{f}, \chi, \kappa, s) := \sum_{a \in \mathbb{Z}/n\mathbb{Z}} \chi(ap) \int\limits_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} F_{s,pa} d\Phi_* \{\infty \to \frac{pa}{n}\}$$

where Φ_* is the big modular symbol from Theorem 10.

Interpolation of special values. We now recall an important result about the interpolation of classical L-values by the Stevens - Mazur- Kitagawa p-adic L-function $\mathcal{L}_p^{\mathrm{SMK}}$.

Theorem 11. *Let $k \in U^{cl}$ be a classical weight and χ a primitive character. For all integers $1 \leq j \leq k-1$ such that $\chi(-1) = (-1)^{j-1} w_\infty$ we have*

$$\mathcal{L}_p^{SMK}(\mathbf{f}, \chi, k, j) = \lambda(k) \left(1 - \chi(p) \frac{p^{j-1}}{a_p(k)} \right) \tilde{L}(f_k, \chi, j)$$

Proof. The interpolation property of the two variable p-adic L-function follows from Proposition 3.23 in [4] along with Theorem 10 , due to G. Stevens, stated above. □

We will be particularly interested in the specialization of $\mathcal{L}_p^{\mathrm{SMK}}$ to the central line $j = k/2$. Since we are working with the newforms $f_k^\#$, the following relation between the L-values of f_k and $f_k^\#$ will be useful

$$\tilde{L}(f_k, \chi, j) = \left(1 - \chi(p) \frac{p^{k-j-1}}{a_p(k)} \right) \tilde{L}(f_k^\#, \chi, j) \tag{4.3}$$

Corollary 3. *For $k \neq k_0 \in U^{cl}$, suppose $\chi(-1) = (-1)^{\frac{k-2}{2}} w_\infty$. Then we have*

$$\mathcal{L}_p^{SMK}(\mathbf{f}, \chi, k, k/2) = \lambda(k) \left(1 - \chi(p) \frac{p^{\frac{k-2}{2}}}{a_p(k)} \right)^2 \tilde{L}(f_k^\#, \chi, k/2)$$

For $k = k_0$,

$$\mathcal{L}_p^{SMK}(\mathbf{f}, \chi, k_0, k_0/2) = \left(1 - \chi(p)\frac{p^{\frac{k_0-2}{2}}}{a_p(k)}\right)\tilde{L}(f_{k_0}^{\#}, \chi, k_0/2)$$

where $f_{k_0}^{\#} = f_{k_0}$.

Proof. This follows from Theorem 11 and equation (4.3). □

4.3 p-adic L-functions attached to real quadratic fields

4.3.1 p is inert in K.

In this section we will recall the construction and interpolation properties of a p-adic L-function attached to real quadratic fields due to M. Seveso in [36]. Recall that $K/\mathbb{Q}$ is a real quadratic field such that

• All the primes dividing N split in K while

• p is inert in K.

Recall the set $\mathrm{Emb}_+(\mathcal{O}, \mathcal{R})$ introduced in Chapter 3. Let $\Psi \in \mathrm{Emb}_+(\mathcal{O}, \mathcal{R})$ have conductor prime to D_K and Np (i.e. the order $\mathcal{O}$ has conductor prime to D_K and Np). Consider the triple $(\tau_\Psi, P_\Psi, \gamma_\Psi)$ associated to Ψ and a lattice $L_\Psi \subset \mathbb{Q}_p^2$ such that $v_\Psi = [L_\Psi]$.

Definition 21. *Let* $s \in \mathbb{P}^1(\mathbb{Q})$ *be an arbitrary base point. For* $\mathbf{f}(\mathbf{q})$ *the Coleman family and* $\Psi \in \mathrm{Emb}_+(\mathcal{O}, \mathcal{R})$*, we define the partial p-adic L-function* $L_p^{Sev}(\mathbf{f}/K, \Psi, -) : U \to \mathbb{C}_p$ *as follows*

$$L_p^{Sev}(\mathbf{f}/K, \Psi, \kappa) := |L_\psi|^{-\frac{k_0-2}{2}} \int_{L'_\Psi} < P_\Psi(x,y) >^{\frac{\kappa-k_0}{2}} P_\Psi^{\frac{k_0-2}{2}}(x,y) d\Phi_{L_\Psi}\{s \to \gamma_\Psi s\}.$$

For $\chi : \mathrm{Gal}(H_{\mathcal{O}}^{+}/K) \to \mathbb{C}^{\times}$ *a character, we define*

$$L_p^{Sev}(\mathbf{f}/K, \chi, \kappa) := \sum_{\sigma \in \mathrm{Gal}(H_{\mathcal{O}}^{+}/K)} \chi^{-1}(\sigma) L_p^{Sev}(\mathbf{f}/K, \sigma\Psi, \kappa).$$

The p-adic L-function $\mathcal{L}_p^{Sev}(\mathbf{f}/K, \chi, -) : U \to \mathbb{C}_p$ *is then defined as*

$$\mathcal{L}_p^{Sev}(\mathbf{f}/K, \chi, \kappa) = L_p^{Sev}(\mathbf{f}/K, \chi, \kappa)^2.$$

Remark 7. *Unlike the Stevens - Mazur- Kitagawa p-adic L-function, the definition of* $\mathcal{L}_p^{Sev}$ *depends on the class of the embedding* $\Psi \in \widetilde{\Gamma}/Emb_+(\mathcal{O}, \mathcal{R})$*. We make a suitable choice for* L_Ψ *as follows : choose* $\gamma \in \widetilde{\Gamma}$ *such that* $\gamma v_\Psi = v_*$*. This is possible since* $\widetilde{\Gamma}$ *acts transitively on* $\mathcal{V}^+$*. Thus* $v_* = v_{\gamma\Psi\gamma^{-1}}$ *and* $L_* = L_{\gamma\Psi\gamma^{-1}}$ *is associated to the embedding* $\gamma\Psi\gamma^{-1} \in [\Psi]$ *and the choice of the modular symbols* Φ_{L_Ψ} *can be taken to be the big modular symbol* Φ_**. This will allow us to compare* L_p^{Sev} *with the Stevens - Mazur- Kitagawa p-adic L function.*

Recall that a *genus character* $\chi : \mathrm{Gal}(H_K^{+}/K) \to \mathbb{C}^{\times}$ corresponds to a biquadratic extension $\mathbb{Q}(\sqrt{D_1}, \sqrt{D_2})$ of $\mathbb{Q}$ where $D_K = D = D_1.D_2$ is a factorization into coprime factors of the discriminant of $K = \mathbb{Q}(\sqrt{D})$ (See Appendix A).

Let χ_{D_i} denote the Dirichlet character associated to $\mathbb{Q}(\sqrt{D_i})$. Then $\chi_D = \chi_{D_1}.\chi_{D_2}$ and since K is real quadratic, we have

$$1 = \chi_D(-1) = \chi_{D_1}(-1)\chi_{D_2}(-1)$$

Since p is inert in K, $D_K \in \mathbb{Z}_p^{\times}$ and $D_K^{\frac{k-2}{2}}$ extends to the analytic function on U, $< D_K >^{\frac{\kappa-2}{2}}$. We now state the result about the factorization of

$\mathcal{L}_p^{\mathrm{Sev}}(f/K,\chi,\kappa)$.

Theorem 12. *Suppose* $\chi(-1)=(-1)^{\frac{k_0-2}{2}}w_\infty$. *Then*

$$\mathcal{L}_p^{Sev}(f/K,\chi,\kappa)=D_K^{\frac{\kappa-2}{2}}\mathcal{L}_p^{SMK}(f,\chi_{D_1},\kappa,\kappa/2)\mathcal{L}_p^{SMK}(f,\chi_{D_2},\kappa,\kappa/2)$$

Proof. This is Theorem 5.9 of [36]. □

4.3.2 p is split in K.

Only in this section (4.3.2) we assume that K is a real quadratic field that satisfies the *Heegner hypothesis*, i.e. :

- All primes dividing Np are split in K.

Let $\Psi\in\mathrm{Emb}_+(\mathcal{O},\mathcal{R})$ be as above and let

$$e_1:=\begin{pmatrix}\tau_\Psi\\1\end{pmatrix}\qquad e_2:=\begin{pmatrix}\overline{\tau}_\Psi\\1\end{pmatrix}$$

Note that (e_1,e_2) is a $\mathbb{Z}_p$-basis for L_Ψ. Let $L''_\Psi:=\mathbb{Z}_p^\times e_1\oplus\mathbb{Z}_p^\times e_2$. We will now recall the construction of a p-adic L-function due to Greenberg - Seveso - Shahabi in this setting (See Section 5.1 of [17]).

Definition 22. *For* $s\in\mathbb{P}^1(\mathbb{Q})$ *an arbitrary base point and* $\Psi\in Emb_+(\mathcal{O},\mathcal{R})$, *we define the partial p-adic L-function* $L_p^{GSS}(\mathbf{f}/K,\Psi,-):U\to\mathbb{C}_p$ *as follows*

$$L_p^{GSS}(\mathbf{f}/K,\Psi,\kappa):=|L_\psi|^{-\frac{k_0-2}{2}}\int_{L''_\Psi}<P_\Psi(x,y)>^{\frac{\kappa-k_0}{2}}P_\Psi^{\frac{k_0-2}{2}}(x,y)d\Phi_{L_\Psi}\{s\to\gamma_\Psi s\}.$$

For $\chi:\mathrm{Gal}(H_{\mathcal{O}}^+/K)\to\mathbb{C}^\times$ *a character, we define*

$$L_p^{GSS}(\mathbf{f}/K,\chi,\kappa):=\sum_{\sigma\in\mathrm{Gal}(H_{\mathcal{O}}^+/K)}\chi^{-1}(\sigma)L_p^{GSS}(\mathbf{f}/K,\sigma\Psi,\kappa).$$

We then define the p-adic L-function $\mathcal{L}_p^{GSS}(\mathbf{f}/K,\chi,-):U\to\mathbb{C}_p$ *to be*

$$\mathcal{L}_p^{GSS}(\mathbf{f}/K,\chi,\kappa)=L_p^{GSS}(\mathbf{f}/K,\chi,\kappa)^2.$$

4.4 Derivative of $L_p^{\mathbf{Sev}}$ and Darmon cycles

We are now back to the case when K is a real quadratic field in which p is split. Recall the *cohomological Abel-Jacobi* map from Chapter 3

$$\log \mathrm{AJ}: H_1(\tilde{\Gamma},\Delta(P_{k-2}))\to\mathbb{H}_k^{w_\infty,\vee}$$

Let $D_{[\Psi],k}$ be the *Darmon cycle* associated to the class of the embedding $[\Psi]$. By the Eichler-Shimura isomorphism, $\log \mathrm{AJ}(D_{[\Psi],k})$ can be considered as element in $\mathrm{Symb}_{\Gamma_0(Np)}(V_{k-2})^\vee$. We can consider the derivative of the partial p-adic L-function along the weight direction, i.e $\frac{d}{d\kappa}[L_p^{\mathrm{Sev}}(\mathbf{f}/K,\Psi,\kappa)]$. We have the following result of M. Seveso:

Theorem 13.

$$\frac{d}{d\kappa}[L_p^{Sev}(\mathbf{f}/K,\Psi,\kappa)]_{\kappa=k_0}=\frac{1}{2}D_K^{\frac{k_0-2}{4}}\Big(\log \mathrm{AJ}(D_{[\Psi],k_0})(\phi_{k_0}) + (-1)^{k_0/2}\log \mathrm{AJ}(D_{[\overline{\Psi}],k_0})(\phi_{k_0})\Big) \quad (4.4)$$

Let $\chi:\mathrm{Gal}(H_{\mathcal{O}}/K)\to\pm1$ be the genus character corresponding to a factorization $D=D_1.D_2$. Since all primes dividing N split in K, we have

$$1=\chi_D(N)=\chi_{D_1}(N)\chi_{D_2}(N)$$

Since $\chi_{D_1}(-1)=\chi_{D_2}(-1)$, we have $\chi_{D_1}(-N)=\chi_{D_2}(-N)$ which we will simply write as $\chi_{D_i}(-N)$.

Corollary 4.

$$\frac{d}{d\kappa}[L_p^{Sev}(\mathbf{f}/K,\chi,\kappa)]_{\kappa=k_0} = \frac{1}{2}D_K^{\frac{k_0-2}{4}}\Big(1+(-1)^{k_0/2}w_N\chi_{D_i}(-N)\Big)\log \mathrm{AJ}(D_{k_0}^{\chi})(\phi_{k_0})$$

Remark 8. *Since p is inert in K, we have that $\chi_D(p) = -1$. This implies that $\chi_{D_1}(-p) = -\chi_{D_2}(-p)$.*

Chapter 5

Main Theorem

5.1 Rationality of Darmon cycles - II

Recall that in the end of Chapter 3, we discussed about the Rationality Conjecture of Darmon cycles. Here we will state some known results towards this conjecture which are relevant to the case we are interested in. Let $\mathcal{M}_{k_0}$ be the motive over $\mathbb{Q}$ associated to the space of cusp forms $S_{k_0}(\Gamma_0(Np), \mathbb{Q})$. See [35] for the construction. For any number field $L/\mathbb{Q}$, let $\mathcal{M}_{k_0} \otimes L$ be the base change to L. Let $\mathrm{CH}_0^{k_0/2}(\mathcal{M}_{k_0} \otimes L)$ be the Chow group of co-dimension $k_0/2$ cycles that are homologous to the zero cycle. By [30], we have a *global p-adic Abel-Jacobi* map

$$\mathrm{cl}_{0,L}^{k_0/2} : \mathrm{CH}_0^{k_0/2}(\mathcal{M}_{k_0} \otimes L) \to \mathrm{Sel}_{st}(L, V_p^{Np}(k_0/2))$$

For a discussion about the motive $\mathcal{M}_{k_0}$ and the Abel-Jacobi map $\mathrm{cl}_{0,L}^{k_0/2}$, see Sections 1 - 4 of [30].

Let $D > 0$ be an odd square free integer. Let H_K^+ denote the narrow ring class field (See Appendix A for the definition) of the real quadratic field

$K = \mathbb{Q}(\sqrt{D})$. Recall from Chapter 3 the *Abel-Jacobi* map

$$\mathrm{AJ} : H^1(\widetilde{\Gamma}, \Delta(P_{k_0-2})) \to H^1_{st}(K_p, V_p^{Np}(k_0/2)).$$

Denote the image of Darmon cycles $\mathrm{AJ}(D_{[\Psi],k_0})$ (resp. $\mathrm{AJ}(D^{\chi}_{k_0})$) by $s_\Psi \in H^1_{st}(K_p, V_p^{Np}(k_0/2))$ (resp. $s_\chi \in H^1_{st}(K_p(\chi), V_p^{Np}(k_0/2))$. Recall that we have the restriction at p, $\mathrm{res}_p : \mathrm{Sel}_{st}(H_K^{+,\chi}, V_p^{Np}(k_0/2)) \to H^1_{st}(K_p(\chi), V_p^{Np}(k_0/2))$.

Let $\chi : \mathrm{Gal}(H_K^+/K) \to \mathbb{C}^\times$ be the genus character of K corresponding to the factorization $D = D_1^*.D_2^*$. Note that we have $\chi_{D_1^*}(-N) = \chi_{D_2^*}(-N)$ while $\chi_{D_1^*}(-p) = -\chi_{D_2^*}(-p)$. We now recall the main results concerning the rationality of Darmon cycles:

Theorem 14. *Assume* $(-1)^{k_0/2} w_N \chi_{D_i^*}(-N) = 1$ *for* $i \in \{1, 2\}$. *Then, there exists a global cycle*

$$d_{k_0}^{\chi_{D_2^*}} \in CH_0^{k_0/2}(\mathcal{M}_{k_0} \otimes \mathbb{Q}(\sqrt{D_2^*}))^{\chi_{D_2^*}} \subset (\mathcal{M}_{k_0} \otimes \mathbb{Q}(\sqrt{D_2^*}, \sqrt{D_1^*}))$$

and a constant $s_f \in K_{f_{k_0}}^\times$ *such that*

$$res_p(cl^{k_0/2}_{0,H_K^+}(d_{k_0}^{\chi_{D_2^*}})) = s_f AJ(D^{\chi}_{k_0})$$

Proof. This is Theorem 6.2 of [36]. The notation used here is different from loc cit. but it is similar to Theorem 6.11 of [17] which is a generalization of the above theorem to the general quaternionic setting. □

Remark 9. *Note that the global algebraic cycle* $d_{k_0}^{\chi_{D_2^*}}$ *depends only on* D_2 *and not on* D_1.

5.2 Normalized Fourier coefficients $\widetilde{c}(D,k)$

Recall the *Coleman family* of cusp forms of tame level $\Gamma_0(N)$

$$\mathbf{f}(q) = \sum_{n\geq 1} \mathbf{a}_n(\kappa) q^n \in \mathcal{O}(U)[[q]]$$

and the classical cusp forms $f_k^{\#}$, for $k \in U^{cl}$. Note that $f_k^{\#} \in S_k(\Gamma_0(N))^{\mathrm{new}}$ for $k \neq k_0$ while $f_{k_0}^{\#} = f_{k_0} \in S_{k_0}(\Gamma_0(Np))^{p-\mathrm{new}}$.

Recall from Theorem 4, Chapter 2 that the map $\Theta_{k,N,m}$ is a Hecke-equivariant isomorphism between $S_k^{new}(\Gamma_0(Np))$ (resp. $S_k^{new}(\Gamma_0(N))$) and $S^{+,new}_{\frac{k+1}{2}}(\Gamma_0(4Np))$ (resp. $S^{+,new}_{\frac{k+1}{2}}(\Gamma_0(4N))$). In this chapter, we denote $\Theta_{k,N,m}$ just by θ for brevity.

For all $k \neq k_0 \in U^{cl}$, let

$$g_k := \sum_{D>0} c(D,k) q^D \in S^{+,new}_{\frac{k+1}{2}}(\Gamma_0(4N))$$

be the Shintani lifting of $f_k^{\#}$ and

$$g_{k_0} := \sum_{D>0} c(D,k_0) q^D \in S^{+,new}_{\frac{k_0+1}{2}}(\Gamma_0(4Np))$$

be the Shintani lifting of $f_{k_0} = f_{k_0}^{\#}$.

In view of Proposition 2 of Chapter 2, the values of the discriminant D for which $c(D,k)$ need not necessarily vanish can be classified into two types - viz. Type I and Type II.

<u>Type I</u> : All $D > 0$ such that $\chi_{D^*}(p) = w_p$.

<u>Type II</u> : All $D > 0$ such that $\chi_{D^*}(p) = -w_p$.

Remark 10. *For D a Type II discriminant, $L(f_{k_0}, D^*, k_0/2) = 0$ since the sign in the function equation $w(f_{k_0}, D^*) = -1$ and hence $c(D, k_0) = 0$. On the other hand, by the non-vanishing results for quadratic twists of Modular L-functions due to Waldspurger* (See Theorem 1.1 of [29])*, there exists infinitely many fundamental discriminants D_1 of Type I such that $L(f, \chi_{D_1^*}, k_0/2) \neq 0$ and consequently $c(D_1, k_0) \neq 0$.*

We fix a Type I discriminant D_1 such that $c(D_1, k_0) \neq 0$.

Lemma 6. *Upto further shrinking of U, $c(D_1, k)$ is non - vanishing for all $k \in U^{cl}$.*

Proof. This is Lemma 3.2 of [25] and has been reproduced here. By Proposition 2, Chapter 2

$$c(D_1, k) \neq 0 \Leftrightarrow L(f_k^{\#}, \chi_{D_1^*}, k/2) \neq 0$$

Recall the algebraic part of the central L-value

$$\tilde{L}(f_k^{\#}, \chi_{D_1^*}, k/2) := \frac{(k/2-1)!\tau(\chi_{D_1^*})}{(-2\pi i)^{k/2-1}\Omega_{f_k^{\#}}} L(f_k^{\#}, \chi, k/2)$$

By the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, we can look at $\tilde{L}(f_k^{\#}, \chi_{D_1^*}, k/2)$ as p-adic numbers. It suffices to show the non-vanishing of $\tilde{L}(f_k^{\#}, \chi_{D_1^*}, k/2)$ in a neighbourhood of k_0. Fix the choice of w_∞ such that $\chi_{D_1^*}(-1) = (-1)^{\frac{k_0-2}{2}} w_\infty$. By the interpolation property of the Stevens - Mazur - Kitagawa p-adic L-function attached to $\mathbf{f}(q)$ (Corollary 2, Chapter 4), we have

$$\mathcal{L}_p^{SMK}(\mathbf{f}, \chi_{D_1^*}, k_0, k_0/2) = \left(1 - \chi_{D_1^*}(p)\frac{p^{\frac{k_0-2}{2}}}{a_p(k_0)}\right)\tilde{L}(f_{k_0}^{\#}, \chi_{D_1^*}, k_0/2)$$

Note that $a_p(k_0) = -w_p p^{\frac{k_0-2}{2}}$. Since D_1 is a Type I discriminant, $\chi_{D_1^*}(p) = w_p$. Hence the Euler like factor $\left(1-\chi_{D_1^*}(p)\frac{p^{\frac{k_0-2}{2}}}{a_p(k_0)}\right)$ is non-zero. This establishes the non-vanishing of $\mathcal{L}_p^{SMK}$ at $(k_0, k_0/2)$. Since the Stevens - Mazur - Kitagawa p-adic L-function is a non-zero p-adic analytic function, upto shrinking U, we have the non-vanishing of $\mathcal{L}_p^{SMK}$. Thus the non-vanishing result for $c(D_1,k)$ follows. □

We would like to interpolate the Fourier coefficients $c(D,k)$, for $k \in U^{cl}$, by a p-adic analytic function over U. We now introduce a normalization of the Fourier coefficients $c(D,k)$ (See Proposition 1.3 of [13] and Proposition 3.3 of [25]). For D^* a fundamental discriminant of either Type and for every $k \in U^{cl}$, define the normalized Fourier coefficient as follows :

$$\widetilde{c}(D,k) := \frac{\left(1-\chi_{D^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\right)c(D,k)}{\left(1-\chi_{D_1^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\right)c(D_1,k)}$$

Proposition 7. *Up to shrinking, the normalized coefficients $\widetilde{c}(D,k)$ extends to a p-adic analytic function in a neighbourhood of k_0.*

Proof. The proof is a higher weight analogue of Proposition 3.3 of [25]. We write

$$\frac{c(D,k)}{c(D_1,k)} = \frac{c(D,k)\overline{c(D_1,k)}}{\mid c(D_1,k)\mid^2}$$

Assuming D is relatively prime to D_1, from Theorem 5 and Proposition 2 of Chapter 2, we can interpret the right hand side as

$$\frac{\pi^{k/2}(-2i)^{k/2}2^{\nu(N)}r_{k,N}(f_k^{\#}, D_1^*, D^*)}{2^{\nu(N)}(k/2-1)!|D_1|^{\frac{k-1}{2}}L(f_k^{\#}, D_1^*, k/2)}$$

which simplifies as

$$\frac{(-2\pi i)^{k/2} r_{k,N}(f_k^{\#}, D_1^*, D^*)}{(k/2-1)!|D_1|^{\frac{k-1}{2}} L(f_k^{\#}, D_1^*, k/2)}$$

Expressing the central L-value in terms of its 'algebraic part'

$$L(f_k^{\#}, \chi_{D_1^*}, k/2) = \frac{(-2\pi i)^{\frac{k-2}{2}} \Omega_{f_k^{\#}}}{(k/2-1)!\tau(\chi_{D_1})} \tilde{L}(f_k^{\#}, \chi_{D_1^*}, k/2)$$

we have

$$\frac{c(D,k)}{c(D_1,k)} = \frac{-\tau(\chi_{D_1^*})(2\pi i) r_{k,N}(f_k^{\#}, D^*, D_1^*)}{|D_1|^{\frac{k-1}{2}} \tilde{L}(f_k^{\#}, D_1^*, k/2) \Omega_{f_k^{\#}}}$$

Using the interpolation formula for the Stevens - Mazur - Kitagawa p-adic L-function and that $|\tau(\chi_{D_1^*})| = D_1^{1/2}$, we have

$$\widetilde{c}(D,k) = \frac{\lambda(k)(2\pi i)\big(1 - \chi_{D^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\big)\big(1 - \chi_{D_1^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\big) r_{k,N}(f_k^{\#}, D^*, D_1^*)}{|D_1|^{\frac{k-2}{2}} \mathcal{L}_p^{SMK}(\mathbf{f}, \chi_{D_1^*}, k, k/2) \Omega_{f_k^{\#}}}$$

Now, suppose D is of Type II.

Then, $\chi_{D^*}(p) = -\chi_{D_1^*}(p)$. Hence

$$\Big(1 - \chi_{D^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\Big).\big(1 - \chi_{D_1^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\Big) = 1 - \frac{p^{k-2}}{a_p(k)^2}.$$

Since the primes dividing N split in the real quadratic field $\mathbb{Q}(\sqrt{D^* D_1^*})$ while p is inert, we can write

$$\frac{\lambda(k).\big(1 - \frac{p^{k-2}}{a_p(k)^2}\big).(2\pi i).r_{k,N}(f_k^{\#}, D^*, D_1^*)}{\Omega_{f_k^{\#}}} = L_p^{\mathrm{Sev}}(\mathbf{f}/\mathbb{Q}(\sqrt{D^* D_1^*}), \chi_{D^* D_1^*}, k/2).$$

This calculation is shown in detail in the paragraph after Lemma 7. Plugging it back, we have

$$\widetilde{c}(D,k) = \frac{L_p^{\mathrm{Sev}}(\mathbf{f}/\mathbb{Q}(\sqrt{D^*D_1^*}))}{|D_1|^{\frac{k-2}{2}}\mathcal{L}_p^{SMK}(\mathbf{f},\chi_{D_1^*},k,k/2)}.$$

up to some constant. Since $\widetilde{c}(D,k)$ is the ratio of p-adic analytic functions on some neighbourhood of k_0, we conclude the same about the normalized coefficients.

Now suppose D is also of Type I and $(D,D_1)=1$. Note that in this case, all the primes dividing Np split in the real quadratic field $\mathbb{Q}(\sqrt{D^*D_1^*})$. This is the *Heegner hypothesis*. Also

$$\Big(1-\chi_{D^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\Big).\Big(1-\chi_{D_1^*}(p)\frac{p^{\frac{k-2}{2}}}{a_p(k)}\Big) = \Big(1-\chi_{D_1^*}(p)\frac{p^{k-2}}{a_p(k)^2}\Big)^2.$$

In this case,

$$\frac{\lambda(k).\big(1-\chi_{D_1^*}(p)\frac{p^{k-2}}{a_p(k)^2}\big)^2.(2\pi i)r_{k,N}(f_k^{\#},D^*,D_1^*)}{\Omega_{f_k^{\#}}} = L_p^{\mathrm{GSS}}(\mathbf{f}/\mathbb{Q}(\sqrt{D^*D_1^*}),\chi_{D^*D_1^*},k/2). \quad (5.1)$$

When $(D,D_1)\neq 1$, choose a Type I discriminant D_1', prime to both D_1 and D, such that $c(D_1',k_0)\neq 0$. Then we can write

$$\frac{\widetilde{c}(D,k)}{\widetilde{c}(D_1,k)} = \Big(\frac{\widetilde{c}(D,k)}{\widetilde{c}(D_1',k)}\Big).\Big(\frac{\widetilde{c}(D_1',k)}{\widetilde{c}(D_1,k)}\Big)$$

and we repeat the same as above for each individual factor in the product. □

5.3 Proof of the Main Theorem

Let D_2 be a Type II discriminant relatively prime to D_1. Let $D := D_1^*.D_2^*$ and $K = \mathbb{Q}(\sqrt{D})$ be the real quadratic field of discriminant $D_K = D$.

Note that we have a bijection between $\mathcal{F}_D/\Gamma_0(N)$, G_D and $\mathrm{Gal}(H_K^+/K)$ where $\mathcal{F}_D$ is the set of Heegner forms of level N of discriminant D, G_D is the $SL_2(\mathbb{Z})$-equivalence class of primitive integral binary quadratic forms of discriminant D and H_K^+ is the narrow Hilbert class field of K (See Appendix A). The last isomorphism comes from class field theory.

Since D_2 is of Type II and relatively prime to D_1, by Proposition 8 we have

$$\widetilde{c}(D_2, k) = \frac{\lambda(k)(2\pi i)\Big(1 - \frac{p^{\frac{k-2}{2}}}{a_p(k)^2}\Big) r_{k,N}(f_k^{\#}, D_2^*, D_1^*)}{|D_1|^{\frac{k-2}{2}} \mathcal{L}_p^{SMK}(\mathbf{f}, \chi_{D_1^*}, k, k/2)\Omega_{f_k^{\#}}} \tag{5.2}$$

Let $\Psi_* \in \mathrm{Emb}_+(\mathcal{O}_K, \mathcal{R})$ be the optimal embedding such that $v_{\Psi_*} = v_* = [\mathbb{Z}_p^2]$. Note that

$$r_{k,N}(f_k^{\#}, D_1^*, D_2^*) = \sum_{Q \in \mathcal{F}_D/\Gamma_0(N)} \omega_{D_1^*, D_2^*}(Q) r(f_k^{\#}, Q)$$

where $\omega_{D_1^*, D_2^*}$ is the genus character corresponding to the bi-quadratic extension $\mathbb{Q}(\sqrt{D_1^*}, \sqrt{D_2^*})$ of K. By Proposition 9 of Appendix A, we will consider the character $\omega_{D_1^*, D_2^*}$ as $\chi_{D_1^*, D_2^*} : \mathrm{Gal}(H_K^+/K) \to \{\pm 1\}$. Hence we re-write

$$r_{k,N}(f_k^{\#}, D_1^*, D_2^*) = \sum_{\sigma \in \mathrm{Gal}(H_K^+/K)} \chi_{D_1^*, D_2^*}(\sigma) r(f_k^{\#}, P_{\sigma.[\Psi_*]}(x, y))$$

Since D is not a perfect square, for all $\Psi \in \mathrm{Emb}_+(\mathcal{O}_K, \mathcal{R})$, we have :

$$(2\pi i) r(f_k^{\#}, P_\Psi(x, y)) = \widetilde{\phi}_k^{\#}\{r \to \gamma_\Psi.r\}(P_\Psi^{k/2-1}(x, y))$$

Hence we re-write equation (5.2) as

$$|D_1|^{\frac{k-2}{2}}\mathcal{L}_p^{SMK}(\mathbf{f},\chi_{D_1^*},k,k/2)\widetilde{c}(D_2,k)=(1-a_p(k)^{-2}p^{k-2})$$
$$\sum_{\sigma\in\mathrm{Gal}(H_K^+/K)}\chi_{D_1^*,D_2^*}(\sigma)\phi_{f_k^\#}(P_{\sigma.[\Psi_*]}^{k/2-1}(x,y))\{r\to\gamma_{\Psi_*}.r\} \quad (5.3)$$

where $\widetilde{\phi}_k^\#/\Omega_{f_k^\#}=\phi_k^\#$.

Lemma 7. *For all $k\in U^{cl}$ and $P(x,y)\in P_{k-2}$,*

$$\int_{(\mathbb{Z}_p^2)'}P(x,y)d\Phi_*\{r\to s\}=\lambda(k)(1-a_p(k)^{-2}p^{k-2})\phi_{f_k^\#}\{r\to s\}(P(x,y))$$

Proof. See Corollary 4.6 of [36] and Proposition 2.4 of [5]. □

Now since $P_{\Psi_*}(x,y)\in\mathbb{Z}_p^\times$ for all $(x,y)\in(\mathbb{Z}_p^2)'$, we have by Lemma 4.1 of [36]

$$P_{\Psi_*}^{\frac{k-2}{2}}(x,y)=P_{\Psi_*}^{\frac{k-k_0}{2}}(x,y).P_{\Psi_*}^{\frac{k_0-2}{2}}(x,y)=<P_{\Psi_*}(x,y)>^{\frac{k-k_0}{2}}P_{\Psi_*}^{\frac{k_0-2}{2}}$$

By (5.3) and Lemma 7, we have

$$|D_1|^{\frac{k-2}{2}}\mathcal{L}_p^{SMK}(\mathbf{f},\chi_{D_1^*},k,k/2)\widetilde{c}(D_2,k)=\sum_{\sigma\in\mathrm{Gal}(H_K^+/K)}\chi_{D_1^*,D_2^*}(\sigma)$$
$$\int_{(\mathbb{Z}_p')^2}<P_{\Psi_*}(x,y)>^{\frac{k-k_0}{2}}P_{\Psi_*}^{\frac{k_0-2}{2}}d\Phi_*\{r\to\gamma_{\Psi_*}.r\} \quad (5.4)$$

Since $\Phi_{L_{\Psi_*}}=\Phi_*$, the integral above is the value of the partial p-adic L-function L_p^{Sev} at $k\in U^{cl}$. Hence we have

$$\widetilde{c}(D_2,k)=\frac{L_p^{Sev}(\mathbf{f}/K,\chi_{D_1^*,D_2^*},k)}{|D_1|^{\frac{k-2}{2}}\mathcal{L}_p^{SMK}(\mathbf{f},\chi_{D_1^*},k,k/2)} \quad (5.5)$$

We will now compute the derivative of the analytic function $\widetilde{c}(D_2, \kappa)$ along the weight direction around a neighbourhood of k_0.

Theorem 15. *There exists a global cycle*

$$d_{k_0}^{\chi_{D_2^*}} \in CH_0^{k_0/2}(\mathcal{M}_{k_0} \otimes \mathbb{Q}(\sqrt{D_2^*}))^{\chi_{D_2^*}} \subset (\mathcal{M}_{k_0} \otimes \mathbb{Q}(\sqrt{D_2^*}, \sqrt{D_1^*}))$$

and a constant $s_f \in K_{f_{k_0}}^{\times}$ such that

$$\frac{d}{dk}[\widetilde{c}(D_2, k)]_{k=k_0} = \frac{|D_1|^{\frac{k_0-2}{4}}}{|D_2|^{\frac{k_0-2}{4}}}.s_f.\exp_{\mathrm{BK}}{}^{-1}(\mathrm{res}_p(\mathrm{cl}_{0,H_K^+}^{k_0/2}(d_{k_0}^{\chi_{D_2^*}})))(\phi_{k_0})$$

Proof. Taking the derivative w.r.t κ on both sides of (5.5), we have

$$\begin{aligned}\frac{d}{d\kappa}\widetilde{c}(D_2, \kappa) = &\frac{|D_1|^{\kappa/2-1}\mathcal{L}_p^{SMK}(\mathbf{f}, \chi_{D_1^*}, \kappa, \kappa/2)\frac{d}{d\kappa}[L_p^{Sev}(\mathbf{f}/K, \chi_{D_1^*, D_2^*}, \kappa)]}{|D_1|^{\kappa-2}\mathcal{L}_p^{SMK}(\mathbf{f}, \chi_{D_1^*}, \kappa, \kappa/2)^2} \\ &+ \frac{L_p^{Sev}(\mathbf{f}/K, \chi_{D_1^*, D_2^*}, \kappa)\frac{d}{d\kappa}[|D_1|^{\kappa/2-1}\mathcal{L}_p^{SMK}(\mathbf{f}, \chi_{D_1^*}, \kappa, \kappa/2)]}{|D_1|^{\kappa-2}\mathcal{L}_p^{SMK}(\mathbf{f}, \chi_{D_1^*}, \kappa, \kappa/2)^2} \qquad (5.6)\end{aligned}$$

At $\kappa = k_0$, we know that $L_p^{Sev}(\mathbf{f}/K, \chi_{D_1^*, D_2^*}, k_0) = 0$ (See Proposition 5.7 of [36]). Hence (5.6) can be simplified as

$$\frac{d}{d\kappa}[\widetilde{c}(D_2, \kappa)]_{\kappa=k_0} = \frac{\frac{d}{d\kappa}[L_p^{Sev}(\mathbf{f}/K, \chi_{D_1^*, D_2^*}, \kappa)]_{\kappa=k_0}}{|D_1|^{\frac{k_0-2}{2}}\mathcal{L}_p^{SMK}(\mathbf{f}, \chi_{D_1^*}, k_0, k_0/2)} \qquad (5.7)$$

By Corollary 3 (See Chapter 4, p. 64), we write (5.7) as

$$\frac{d}{d\kappa}[\widetilde{c}(D_2, \kappa)]_{\kappa=k_0} = \frac{D^{\frac{k_0-2}{4}}\log \mathrm{AJ}(D_{k_0}^{\chi})(\phi_{k_0})}{|D_1|^{\frac{k_0-2}{2}}2\widetilde{L}(f_{k_0}, \chi_{D_1^*}, k_0, k_0/2)}$$

Since $D = |D_1|.|D_2|$ and that $2\widetilde{L}(f_{k_0}, \chi_{D_1^*}, k_0/2) \in K_{f_{k_0}}^{\times}$, the theorem follows from Theorem 14 on the rationality of Darmon cycles. □

Remark 11. *The additional factor of $\frac{|D_2|}{|D_1|}^{\frac{k_0-2}{4}}$ is 1 in [13] since they consider* *the $k_0 = 2$ case.*

5.4 Future directions

5.4.1 Kohnen-Shintani correspondence for Shimura curves

A natural extension of our result would be to study the case of eigenforms on a Shimura curve attached to a quaternion algebra $\mathcal{B}/\mathbb{Q}$. Shimura in [41] proved the existence of a Shimura-Shintani correspondence in this case. We would like to construct a p-adic lifting in this case. We hope to use the Jacquet-Langlands correspondence (which is compatible with the formation of finite slope p-adic families) to construct the Kohnen-Shimura-Shintani lifting for p-adic families of modular forms over $\mathcal{B}$. We remark that a Λ-adic lifting in the quaternionic setting is given in [26].

5.4.2 Darmon cycles over Shimura curves

We hope to prove an analogue of Kohnen's theorem. In the quaternion case, rationality results about Darmon cycles have been established in [17] and we hope to use these results to prove an analogue of Theorem 15 in this scenario.

Appendix A

Quadratic forms and Quadratic fields

In this appendix, we briefly recall the relation between quadratic forms and quadratic fields. Let

$$Q(x, y) := ax^2 + bxy + cy^2$$

be an integral (i.e. $a, b, c \in \mathbb{Z}$) primitive ($(a, b, c) = 1$) quadratic form. Assume that $D := b^2 - 4ac$, the discriminant of $Q(x, y)$, is positive and square-free. The set of quadratic forms of discriminant D are related to the (fractional ideals of) real quadratic field $K = \mathbb{Q}(\sqrt{D})$.

Since D is a square free discriminant, we have

$$\mathcal{O}_K \simeq \mathbb{Z} \oplus \mathbb{Z}[\frac{1+\sqrt{D}}{2}]$$

Definition 23. *A fractional ideal, $\mathfrak{b}$, of K is a finitely generated $\mathcal{O}_K$-submodule of K.*

Remark 12. *It follows that every fractional ideal $\mathfrak{a}$ is generated by at most* *two elements of K.*

The set of fractional ideals $\mathcal{J}_K$ are a monoid under multiplication. Let $\mathcal{P}_K^+$ be the set of principal fractional ideals generated by totally positive elements, i.e.

$$\mathcal{P}_K^+ := \{(\gamma) : \gamma = r/s;\ r, s \in \mathcal{O}_K\ \gamma, \gamma^\sigma > 0\}$$

where σ is the non-trivial element in $\mathrm{Gal}(K/\mathbb{Q})$.

The quotient $C\ell^+(K) := \mathcal{J}_K/\mathcal{P}_K^+$ is a group under multiplication, called the narrow ideal class group.

Definition 24. *A genus character of K is a quadratic character of $C\ell^+(K)$.*

Definition 25. *The maximal abelian extension of K unramified at all finite places of K is called the **narrow** Hilbert class field, denoted by H_K^+.*

Remark 13. *By Class field theory, we have the following isomorphism* (See Theorem 4.2 of [31]) *:*

$$rec^+ : C\ell^+(K) \simeq \mathrm{Gal}(H_K^+/K)$$

Hence we can alternatively consider a genus character as a quadratic character of $\mathrm{Gal}(H_K^+/K)$*.*

Corresponding to each factorisation $D = D_1^*.D_2^*$, we define a genus character $\chi_{D_1^*,D_2^*} : C\ell_K^+ \to \mathbb{C}^\times$ by defining on prime ideals:

$$\chi_{D_1^*,D_2^*}(\mathfrak{p}) := \begin{cases} \left(\frac{D_1^*}{(N(\mathfrak{p})}\right) & \text{if } \mathfrak{p} \nmid D_1 \\ \left(\frac{D_2^*}{N(\mathfrak{p})}\right) & \text{if } \mathfrak{p} \nmid D_2 \end{cases}$$

and extending multiplicatively to $C\ell_K^+$. We exclude the trivial factorisation $D = D.1$. The genus character $\chi_{D_1^*,D_2^*}$ cuts out the bi-quadratic field

$\mathbb{Q}(\sqrt{D_1^*}, \sqrt{D_2^*})$ (See Section 2 of [13]).

We say that a fractional ideal $\mathfrak{a} = (a_1, a_2)$ has a normalized basis if $a_1 a_2^\sigma - a_2 a_1^\sigma > 0$. Note that either (a_1, a_2) or (a_2, a_1) is a normalized basis. To each fractional ideal with a normalized basis $\mathfrak{a} = (a_1, a_2)$, we attach a quadratic form by

$$Q_{\mathfrak{a}}(x, y) := N(\mathfrak{a})^{-1} N_{K/\mathbb{Q}}(a_1 x + a_2 y)$$

where N denotes norm.

Lemma 8. *A positive integer Γ is represented by $Q_{\mathfrak{a}}$ if and only if Γ is the norm of an integral ideal $\mathfrak{b}$ in the ideal class of $\mathfrak{a}$.*

Proof. Let $\Gamma = N(\mathfrak{b})$ where $\mathfrak{b} = (\gamma)\mathfrak{a}$ is integral and γ is totally positive. Then $\mathfrak{b}^\sigma = (\gamma^\sigma N(\mathfrak{a})\mathfrak{a}^{-1}$. Hence $\Gamma = N(\mathfrak{b}) = N(\alpha)N(\mathfrak{a}^{-1})$, where $\alpha = \gamma^\sigma N(\mathfrak{a})$ is totally positive. Since $N(\mathfrak{a})$ divides $N_{K/\mathbb{Q}}(\alpha)$, we have $\alpha \in \mathfrak{a}$. Therefore $\Gamma = N(\mathfrak{a})^{-1}.N_{K/\mathbb{Q}}(a_1 x + a_2 y) = Q_{\mathfrak{a}}(x, y)$ for some $x, y \in \mathbb{Z}$. Conversely, let $\Gamma = Q_{\mathfrak{a}}(x, y) = N(\mathfrak{a})^{-1}.N_{K/\mathbb{Q}}(\alpha)$ for some $\alpha \in \mathfrak{a}$. Then $\Gamma = N((\alpha)\mathfrak{a}^{-1}) > 0$ and hence $(\alpha)\mathfrak{a} \in \mathcal{O}_K$ is an integral ideal. □

Theorem 16. *The map $\Theta : \mathfrak{a} \mapsto Q_{\mathfrak{a}}(x, y)$ is a bijection between G_D - the set of $SL_2(\mathbb{Z})$-equivalence classes of binary quadratic forms of discriminant D and $C\ell_K^+$.*

Proof. A nice discussion can be found in Sections 5.1 and 5.2 of [8]. □

Recall that we had defined the character $\omega_{D_1^*, D_2^*}$ on G_D as

$$\omega_{D_1^*, D_2^*}(Q) := \begin{cases} \left(\frac{D_1^*}{Q(m,n)}\right) & \text{when } \gcd(D_2^*, Q(m,n)) = 1 \\ \left(\frac{D_1^*}{Q(m,n)}\right) & \text{when } \gcd(D_2^*, Q(m,n)) = 1 \end{cases}.$$

Proposition 8.

$$\chi_{D_1^*,D_2^*} = \Theta \circ \omega_{D_1^*,D_2^*}$$

Proof. By the Lemma above, there exists an integral ideal $\mathfrak{b}$ in the class of $\mathfrak{a}$ such that $N(\mathfrak{b}) = \Gamma = Q_{\mathfrak{a}}(m,n)$. Note that $Q_{\mathfrak{a}} \sim Q_{\mathfrak{b}}$. Hence we re - write

$$\omega_{D_1^*,D_2^*}(Q_{\mathfrak{b}}) := \begin{cases} \left(\frac{D_2^*}{N(\mathfrak{b})}\right) & \text{when } (D_2^*, N(\mathfrak{b})) = 1 \\ \left(\frac{D_1^*}{N(\mathfrak{b})}\right) & \text{when } \gcd(D_1^*, N(\mathfrak{b})) = 1 \end{cases}.$$

Now the proof follows by the multiplicative property of the Kronecker symbol $\left(\frac{D^*}{N(\mathfrak{p})}\right)$ over the prime ideals $\mathfrak{p}$ dividing $\mathfrak{b}$. □

Bibliography

[1] A. Ash and G. Stevens, *Modular forms in characteristic ℓ and special values of their L-functions.* Duke Math. J. 53 (1986) no. 3, 849-868.

[2] A. Ash and G. Stevens, *p-adic deformations of arithmetic cohomology.* - http://math.bu.edu/people/ghs/preprints/Ash-Stevens-02-08.pdf

[3] A. Atkin and J. Lehner, *Hecke operators on $\Gamma_0(m)$.* Math. Ann. 185 (1970) 134 - 160.

[4] J. Bellaiche, *Critical p-adic L-functions.* Invent. Math. 189 (2012) 1 - 60.

[5] M. Bertolini and H. Darmon, *The rationality of Stark-Heegner points over genus fields of real quadratic fields.* Ann. of Math. (2) 170 (2009) 343 - 370.

[6] S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives.* The Grothendieck Festschrift Vol. I (1990) 333 - 400.

[7] K. S. Brown, *Cohomology of Groups.* Graduate Texts in Mathematics 87 (1982).

[8] H. Cohn, *A course in computational number theory.* Second Corrected Printing (1995).

[9] P. Colmez, Zéros supplémentaires de fonctions L p-adiques de formes modulaires. Algebra and Number Theory (2005), 193 - 210.

[10] P. Colmez and J. M. Fontaine *Construction des représentations p-adiques semi-stables.* Invent. Math. 140 (2000) 1 - 43.

[11] H. Darmon, *Integration on* $\mathcal{H}_p \times \mathcal{H}$ *and arithmetic applications.* Ann. of Math. (2) 154 (2001) 589-639.

[12] H. Darmon, *Rational points on modular elliptic curves.* American Mathematical Society (2004).

[13] H. Darmon and G. Tornaria, *Stark - Heegner points and the Shimura correspondence.* Compos. Math. 144 (2008) no. 5 1155-1175.

[14] S. Dasgupta and J. Teitelbaum, *The p-adic upper half plane.* Notes from 2007 Arizona Winter School (2007).

[15] J. M. Fontaine, *Represéntations p-adiques semi-stables.* Astérisque 223 (1994), 113 - 184.

[16] J. Fresnel and M. van der Put, *Rigid Analytic Geometry and its Applications.* Progress in Mathematics (2003).

[17] M. Greenberg, M. Seveso and S. Shahabi, *Module p-adic L-functions attached to real quadratic fields and arithemtic applications.* To appear in Crelle's Journal.

[18] H. Hida, *Elementary theory of L-functions and Eisenstein series.* London Mathematical Society Student Texts 26 (1993).

[19] A. Iovita and M. Spiess, *Derivatives of p-adic L functions, Heegner cycles and monodromy modules attached to modular forms.* Invent. Math. 154 (2003) 333-384.

[20] H. Iwaniec, *Topics in Classical Automorphic Forms.* Graduate Studies in Mathematics Volume 17. (1997).

[21] K. Kitagawa, *On standard p-adic L-functions of families of elliptic cusp forms.* Contemp. Math. 165 (1994) 81 - 110.

[22] W. Kohnen, *Fourier coefficients of modular forms of half-integral weight.* Ann. of Math. 271 (1985) 237-268.

[23] W. Kohnen, *Newforms of half-integral weight.* Crelle's Journal 333 (1982) 32-72.

[24] W. Kohnen and D. Zagier, *Values of L-series of Modular Forms at the Center of the Critical Strip.* Invent. Math. 64 (1981) 175 - 198.

[25] M. Longo and Z. Mao, *Kohnen's formula and a conjecture of Darmon and Tornaria.* submitted

[26] M. Longo and M. H. Nicole, *The* Λ*-adic Shintani-Shimura-Waldspurger correspondence.* Doc. Math. 18 (2003) 1-21.

[27] B. Mazur, *On monodromy invariants occuring in global arithmetic of Fontaine's theory.* Contemp. Math. 165 (1994) 1-20.

[28] B. Mazur, J. Tate and J. Teitelbaum, *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer.* Invent. Math. 84 (1986) 1-48.

[29] M. R. Murty and V.K. Murty, *Non-vanishing of L-functions and applications*, Birkhauser, 1997.

[30] J. Nekovár, *Kolyvagin's method for Chow groups of Kuga-Sato varieties.* Invent. Math. 107 (1992) 99 - 125.

[31] J. Neukrich, *Class Field Theory.* Grundlehren der mathematischen Wissenchaften 280 (1985).

[32] K. Ono and C. Skinner, *Fourier coefficients of half-integral weight modular forms modulo ℓ.* Ann. of Math. (2) 147 (1998) 453 - 470.

[33] V. Rotger and M. Seveso, *L-invariants and Darmon cycles attached to modular forms.* Journal of the European Mathematical Society 14 Issue 6 (2012), 1955-1999.

[34] P. Schneider and U. Stuhler, *The cohomology of p-adic symmetric spaces.* Invent. Math. 105 (1991) no. 1 47-122.

[35] A. J. Scholl, *Motives for modular forms*, Invent. Math 100 (1990), 419 - 430.

[36] M. Seveso, *p-adic L functions and the rationality of Darmon cycles.* Canadian Journal of Mathematics 64 (5) (2012) 1122-1181.

[37] M. Seveso, *The Teitelbaum conjecture in the indefinite setting.* Amer. J of Math. 135 no. 6 (2013), 1525-1557.

[38] Shimura G, *Introduction to the arithmetic theory of automorphic functions.* Princeton University Press (1971).

[39] G. Shimura, *On modular forms of half-integral weight.* Ann. of Math. 97 (1973) 440-481.

[40] G. Shimura, *On the Periods of Modular Forms.* Math. Ann. 229 (1971) 211 - 221.

[41] G. Shimura, *The periods of certain automorphic forms of arithmetic type.* J. Fac. Sci. Univ. Tokyo. Sect. IA Math 28 (1981) 605 - 632.

[42] T. Shintani, *On construction of holomorphic cusp forms of half-integral weight.* Nagoya Math. J. 58 (1975) 83 - 126.

[43] G. Stevens, *Rigid Analytic Modular Symbols.* - (Preprint) http://math.bu.edu/people/ghs/preprints/OC-Symbs-04-94.pdf.

[44] J-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier*, J. Math. pures Appl. 60 (1981) p. 375-484.

[45] J. Weinstein, *Cusp forms and the Eichler-Shimura relation.* - http://tinyurl.com/ln8mb5k.

[46] T. Zink, *Catiertheorie commutativer formaler Gruppen.* Teubner Texte zur Mathematik 68 Leipzig (1984).