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Singular Perturbations and Ergodic Problems for degenerate parabolic Bellman PDEs in $\mathbb{R}^m$ with Unbounded Data

Direttore della Scuola: Ch.mo Prof. Pierpaolo Soravia

Coordinatore d'indirizzo: Ch.mo Prof. Franco Cardin

Supervisore: Ch.mo Prof. Martino Bardi

Dottorando: João Henrique Branco Meireles

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Resume

In this thesis we treat the first singular perturbation problem of a stochastic model with unbounded and controlled fast variables with success. Our methods are based on the theory of viscosity solutions, homogenisation of fully nonlinear PDEs and a careful analysis of the associated ergodic stochastic control problem in the whole space $\mathbb{R}^m$. The text is divided in two parts.

In the first chapter, we investigate the existence and uniqueness as well as a suitable stability of the solution to the associated ergodic problem that are crucial to characterize the effective Hamiltonian of the limit (effective) Cauchy problem in Chapter II of this thesis. The main achievement obtained in this part is a purely analytical proof for the uniqueness of solution to such ergodic problem. Since the state space of the problem is not compact, in general there are infinitely many solutions to the ergodic problem. However, if one restrict the class of solutions to the set of bounded-below functions, then it is known that uniqueness holds up to an additive constant. The existing proof relies on some probabilistic techniques employing the invariant probability measure for the associated stochastic process. Here we give a new proof, purely analytic, based on the strong maximum principle. We believe that our results can be interesting and useful for researchers in the PDE community.

In the second chapter, we introduce our singular perturbation model of a stochastic control problem and we prove our main result: the convergence of the value function V^ϵ associated to the problem to the solution of the limiting equation. More precisely, we prove that the functions

$$\underline{V}(t,x) := \liminf_{(\epsilon,t',x')\to(0,t,x)} \inf_{y\in\mathbb{R}^m} V^\epsilon(t',x',y)$$

and

$$\bar{V}(t,x) := (\sup_R \bar{V}_R)^*(t,x) \text{ (upper semi-continuous envelope of } \sup_R \bar{V}_R \text{)}$$

where $\bar{V}_R(t,x) := \limsup_{(\epsilon,t',x')\to(0,t,x)} \sup_{y\in B_R(0)} V^\epsilon(t',x',y)$, are, respectively, a super and a subsolution of the effective Cauchy problem. As a corollary of this result, V^ϵ converges to the unique solution V of the effective equation provided the equation admits the comparison principle for discontinuous viscosity solutions. The justification of this convergence is not trivial at all. It especially involves some regularity issues and a careful treatment of viscosity techniques and stochastic analysis. This result has never been obtained

before.

Key words: Singular perturbations, viscosity solutions, optimal stochastic control problems, ergodic control problems in the whole space $\mathbb{R}^m$, PDEs

Riassunto

In questa tesi viene trattato con successo il primo problema di perturbazione singolare di un modello stocastico con variabili veloci controllate e non limitate. I metodi si basano sulla teoria delle soluzioni di viscosità, omogeinizzazione dei PDE completamente non lineari, e su un'attenta analisi del problema stocastico ergodico associato, valido nell'intero spazio $\mathbb{R}^m$. Il testo è diviso in due parti.

Nel primo capitolo, saranno studiate l'esistenza, l'unicità e alcune proprietà di stabilità della soluzione del problema ergodico, riferito sopra, che sono essenziali per caratterizzare il Hamiltoniano effetivo che appare in un Problema di Cauchy "limite", che sarà descritto nel capitolo II di questa tesi. Il principale contributo, presentato in questa parte, è una prova puramente analitica dell'unicità della soluzione di questo problema ergodico. Siccome lo stato dello spazio del problema non è compatto, in generale ci sono un numero infinito di soluzioni a questo problema. Tuttavia, se uno limitasse la classe di soluzioni all'insieme di funzioni limitate inferiormente, allora è noto che l'unicità sarà mantenuta a meno di una costante. La prova esistente si basa su alcune tecniche probabilistiche che impiegano la misura di probabiltà inviariante per l'associato processo stocastico. Qua verrà data una nuova prova, puramente analitica, basata sul principio del massimo. Si ritiene che il risultato potrà essere interessante ed utile per i ricercatori che lavorano all'interno della comunità di ricerca delle Equazioni Differenziali alle derivate Parziali (PDE).

Nel secondo capitolo, sarà introdotto un modello di perturbazione singolare di un problema di controllo stocastico, e provato il risultato principale: la convergenza della funzione valore V^ϵ, associata al nostro problema, per soluzione dell'equazione limite . Più precisamente, sarà provato che le funzioni:

$$\underline{V}(t,x) := \liminf_{(\epsilon,t',x')\to(0,t,x)} \inf_{y\in\mathbb{R}^m} V^\epsilon(t',x',y)$$

e

$$\bar{V}(t,x) := (\sup_R \bar{V}_R)^*(t,x) \text{ (upper semi-continuous envelope of } \sup_R \bar{V}_R \text{)}$$

dove $\bar{V}_R(t,x) := \limsup_{(\epsilon,t',x')\to(0,t,x)} \sup_{y\in B_R(0)} V^\epsilon(t',x',y)$, sono, rispettivamente, una super soluzione e una sottosoluzione del problema effettivo di Cauchy. Come corollario di questo risultato, V^ϵ converge all'unica soluzione V della equazione effettiva se l'equazione limite permette il principio di comparazione per le soluzioni di viscosità discontinue. La motivazione di questa convergenza non è ovvia del tutto. Coinvolge specialmente alcuni problemi di regolarità e un trattamento attento delle tecniche di viscosità e di analisi stocastica. Questo risultato è nuovo e non è mai stato ottenuto, prima d'ora, in la letteratura Matematica.

Parole chiave: Perturbazioni singolari, soluzioni di viscosità, problemi di controllo stocastico ottimale, problemi di controllo ergodico nello spazio $\mathbb{R}^m$, equazioni differenziali alle derivate parziali.

Contents

Introduction

In this thesis we study singular perturbations of a class of optimal stochastic control problems with finite time horizon and with unbounded and controlled fast variables. The problem we treat is for $t \in [0,T]$ and given $\theta^* > 1$ and $\epsilon > 0$

$$\text{minimize in } u \text{ and } \xi: \quad \mathbb{E}^{x,y}[\int_t^T (l(X_s, Y_s, u_s) + \tfrac{1}{\theta^*}|\xi_s|^{\theta^*})ds + g(X_T)]$$

subject to

$$\begin{cases} dX_s = F(X_s, Y_s, u_s)ds + \sqrt{2}\sigma(X_s, Y_s, u_s)dW_s, & X_{s_0} = x \\ dY_s = -\frac{1}{\epsilon}\xi_s ds + \sqrt{\frac{1}{\epsilon}}\tau(Y_s)dW_s, & Y_{s_0} = y \end{cases} \tag{1}$$

where l is a running cost function, g represents a terminal cost, $X_s \in \mathbb{R}^n$, $Y_s \in \mathbb{R}^m$, u_s is a control taking values in a given compact set U, $\xi = (\xi_s)_{0 \leq s \leq T}$ denotes a control process taking its values in $\mathbb{R}^m$, and W_s is a multi-dimensional Brownian motion on some probability space.

Basic assumptions on the drift F and on the diffusion coefficient σ of the slow variables X_s are that they are Lipschitz continuos functions in (x, y) uniformly in u and satisfy the following growth condition at infinity

$$|F| + ||\sigma|| \leq C(1 + |x|).$$

This implies, in particular, the existence and uniqueness of strong solutions to (1).

On the fast process Y_s we will assume that the matrix $\tau\tau^T = \mathbb{I}$ so it is positive definite. No non-degeneracy assumption on the matrix σ will be imposed.

In this thesis, we will deal with continuous running costs l satisfying the following coercivity condition

$$-l_0 + l_0^{-1}|y|^\alpha \leq l(x, y, u) \leq l_0(1 + |y|^\alpha) \text{ for some } l_0 > 0$$

where $\alpha > 1$. As for the terminal costs, we will always assume g continuous and bounded, that is,

$$\exists C_g > 0 \text{ s.t. } |g(x)| \leq C_g.$$

Calling $V^\epsilon(t,x,y)$ the value function of this optimal control problem, i.e.

$$V^\epsilon(t,x,y) = \inf_{u,\xi} \mathbb{E}^{x,y}[\int_t^T (l(X_s,Y_s,u_s) + \frac{1}{\theta^*}|\xi_s|^{\theta^*})ds + g(X_T)],$$

we are interested in the limit V as $\epsilon \to 0$ of V^ϵ and in particular in understanding the PDE satisfied by V. This is a singular perturbation problem for the system above and for the HJB equation associated to it. We treat it by methods of the theory of viscosity solutions to such equations, homogenisation of fully nonlinear PDEs and a careful analysis of the associated ergodic stochastic control problem in the whole space $\mathbb{R}^m$.

In fact our main result is Theorem 10.1 where we prove that if $V^\epsilon(t,x,y)$ is a viscosity solution of the HJB equation then the relaxed semilimits

$$\underline{V}(t,x) = \liminf_{(\epsilon,t',x')\to(0,t,x)} \inf_{y\in\mathbb{R}^m} V^\epsilon(t',x',y) \tag{2}$$

and

$$\bar{V} = (\sup_R \bar{V}_R)^* \tag{3}$$

(the upper semicontinuous envelope of $\sup_R \bar{V}_R$) where $\bar{V}_R$ is defined as

$$\bar{V}_R(t,x) = \limsup_{(\epsilon,t',x')\to(0,t,x)} \sup_{y\in B_R(0)} V^\epsilon(t',x',y) \tag{4}$$

are, respectively, a supersolution and a subsolution of the effective Cauchy Problem

$$\begin{cases} -V_t + \bar{H}(x, D_xV, D^2_{xx}V) = 0 & \text{in } (0,\infty)\times\mathbb{R}^n \\ V(T,x) = g(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{5}$$

This procedure allow us to prove also that in some cases $V^\epsilon(t,x,y)$ converges locally uniformly, as $\epsilon \to 0$, to the only solution $V(t,x)$ of (5). Moreover, the effective Hamiltonian $\bar{H}(x,p,M)$ is the unique constant λ such that the following ergodic PDE

$$(EP) \qquad \lambda - \frac{1}{2}\Delta\phi(y) + \frac{1}{\theta}|D\phi(y)|^\theta = f(y) \quad \text{in } \mathbb{R}^m,$$

has a solution ϕ bounded from below, $\frac{1}{\theta}+\frac{1}{\theta^*} = 1$ and $f(y) = -H(x,y,p,M,0)$, where H is the Bellman Hamiltonian associated to the slow variables of (1) and its last entry is for the mixed derivatives D_{xy}. Such type of equations appear in utility maximisation problems in mathematical finance and were first studied by Naoyuki Ichihara in [25] using probabilistic and analytical arguments.

The main difficulty of (EP) lies in the unbounded nature of our problem: first, the control region of ξ is not compact (all $\mathbb{R}^m$), and consequently the running cost $|\xi|^{\theta^*}$ is unbounded; second, f inherits the coercive growth of l and tends to infinity at infinity; and, third, the superlinear nonlinearity in the gradient implies that $|D\phi(y)| \to +\infty$ as $|y| \to \infty$, in contrast with [37] or [21] where the gradient remains bounded on the whole space. Of course, this unboundedness for the gradient complicates even more the arguments and to prove the solvability of (EP) one needs to know more about the problem or the equation itself. In fact, in our case, due to the form of equation (EP), and the properties on f, we can use the results on the existence of solutions and a priori bounds for second order quasilinear equations to guarantee the existence of classical solutions of (EP) (see Section 1). But if for the existence part the methods used in [25] are analytical, for the study of the uniqueness Ichihara's methods are probabilistic. First, it is showed in [24] that there exists a continuum of λ such that (EP) has a solution. In fact, Ichihara showed that there exists a critical value λ^* such that for any $\lambda < \lambda^*$ (EP) has a solution (see also our Proposition 1.5). And so no uniqueness is expected for a general ϕ but at most for ϕ restricted to some classes. Second, given any solution (λ, ϕ) of (EP) we can define the diffusion process driven by

$$dY(s) = -D_q h(D\phi(Y(s)))dt + dW(s), \quad s \geq 0, \tag{6}$$

where $D_q h(q)$ denotes the gradient of $h(q) := \frac{1}{\theta}|q|^\theta$. Notice that such solution corresponds to the diffusion process obtained from (1) by taking $\epsilon = 1$, and $\tau = \mathbb{I}$ (that we will call for simplicity fast subsystem) and considering $\xi = D_q h(D\phi(Y(s)))$ the optimal control feedback (see Proposition 4.10 of [25]). It can be shown that equation (EP) and diffusion (6) are closely related to a stochastic control problem with ergodic type criterion (which justifies the name of ergodic stochastic control problem for the study of (EP)). Indeed, let $\xi = (\xi(t))_{t\geq 0}$ be an $\mathbb{R}^m$-valued control process belonging to some appropriate admissible class, say $\mathcal{A}$, and let $Y^\xi = (Y^\xi(t))_{t\geq 0}$ denote the associated controlled process driven by

$$Y^\xi(s) = y - \int_0^s \xi(\tau)d\tau + \hat{W}(s), \quad s \geq 0.$$

We consider the stochastic control problem of minimising the long-run average cost

$$J(u,\xi) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}^{x,y}[\int_0^T (l(X_s^{u,\xi}, Y_s^\xi, u_s) + \frac{1}{\theta^*}|\xi_s|^{\theta^*})ds + g(X_T^{u,\xi})].$$

Then, under some suitable conditions, one can expect that

$$\lambda^* = \inf_{u\in\mathcal{U},\xi\in\mathcal{A}} J(u,\xi),$$

where λ^* is the critical constant for the ergodic PDE (1), and that the feedback control $\xi^*(t) := D_q h(D\phi(Y(t)))$ with $Y(t)$ defined by (6) gives the optimal control. That is, only λ^* allows us to define an optimal control ξ^* for which Y_s has good long-time behaviour (ergodicity). Thus studying the ergodicity of diffusion (6) plays a crucial role in solving rigorously this minimization problem. It is proved in [25] that when Y^{ξ^*} is ergodic there exists an invariant probability measure μ for Y^{ξ^*} and one can then prove that (EP) has a unique solution pair (λ, ϕ) such that ϕ is bounded from below and necessarily $\lambda = \lambda^*$.

In this thesis we prove again the uniqueness of (EP) using new and purely analytical proofs. Indeed, if $\theta \geq 2$ we use the transformation $z = -e^{-\phi}$ as the key ingredient to prove uniqueness for solutions of (EP) that are bounded from below (see section 4). In fact, such transformation allows us to show a comparison result for solutions of (EP) in the complement of a large ball and satisfying an apropriate Dirichlet condition. Then we argue by means of the strong maximum principle inside the ball. We mention that with such transformation z we don't need any type of knowledge about the behaviour of the solutions of (EP) (other than the gradient bound that is necessary for the existence). A different phenomenon occurs when $\theta < 2$. In this case the useful transformation is $z = \phi^q$ plus an estimate of the behaviour at infinity of solutions of (EP) bounded from below: they necessarily grow with some specific power that will appear many times in this thesis. All this and more is explained and showed in Section 4.

Properly equipped with these new tools we are able to build up all the results on the ergodic Bellman equation that we need and we can also prove some procedures that are useful for Chapter 2. In this sense, this thesis is self contained. With our new proofs it is easy to see that, if (λ, ϕ) is a pair solution of (EP) such that ϕ is bounded from below, then necessarily we have $\lambda = \lambda^*$ (Section 5). This is the result that we mentioned earlier for the ergodic diffusion Y^{ξ^*} but now we don't need anymore to argue with the ergodic measure. Also, with the new proofs, we can show new non-standard approximation results for (EP) that are extremely useful in the proof of the convergence theorem of Section 10 when we introduce the perturbed test function method. In Section 6, we first treat approximations of (EP) by perturbing f with truncations of f. And then we consider approximations of (EP) by looking at the ergodic PDE defined on the ball $B_R(0)$ and complemented with a boundary condition that reads differently according to the value of θ. In fact, we look at

$$\begin{cases} \lambda_R - \frac{1}{2}\Delta\phi_R(y) + \frac{1}{\theta}|D\phi_R(y)|^\theta = f(y) & \text{in } B_R(0) \\ \phi_R(y) \to +\infty \text{ as } y \to \partial B_R(0) \end{cases}$$

for $1 < \theta \leq 2$

and

$$\begin{cases} \lambda_R - \frac{1}{2}\Delta\phi_R(y) + \frac{1}{\theta}|D\phi_R(y)|^\theta = f(y) & \text{in } B_R(0) \\ \lambda_R - \frac{1}{2}\Delta\phi_R(y) + \frac{1}{\theta}|D\phi_R(y)|^\theta \geq f(y) & \text{on } \partial B_R(0) \end{cases}$$

for $\theta > 2$.

The key features of such approximations are that, first, considering perturbations of f, we can control the growth of the corrector in the perturbed test function method as we wish and, second, considering approximations of (EP) by balls, we consider only correctors that "explode" on the boundary of $B_R(0)$. This is important because it forces the maximum point to be achieved only inside $B_R(0)$, as it will be explored in Section 6, and this is a crucial element for Section 10. For more details we refer the reader to Section 6. Here, we wish to stress that our main motivation in Section 6 is indeed the proof of Theorem 10.1.

We prove a convergence theorem for the singular perturbation problem of our stochastic model with controlled and unbounded fast variables by using non-standard relaxed semilimits and we give some examples that test our result. We are now in conditions to discuss the Convergence Theorem, Theorem 10.1, the main result of this thesis. First we notice that λ^* is the appropriate constant to define the effective Hamiltonian $\bar{H}$ appearing in the Cauchy effective problem. Second, due to the unbounded nature of our problem and the fact that we wish to define the relaxed semilimits in a way that gets rid of y, the standard relaxed semilimits

$$\bar{V}(t,x,y) = \liminf_{(\epsilon,t',x',y')\to(0,t,x,y)} V^\epsilon(t',x',y')$$

and

$$\bar{V}(t',x',y') = \limsup_{(\epsilon,t',x',y')\to(0,t,x,y)} V^\epsilon(t',x',y')$$

cannot be applied successfully. This is a difficulty that we overcome by introducing new relaxed semilimits $\underline{V}$ and $\bar{V}$ as in (2) and (3). The lower relaxed semilimit is similar to the one used in periodic singular perturbations, but something different must be done for $\bar{V}$. The main difference is that we can control how V^ϵ grows from below but not completely from above. The upper relaxed semilimit $\bar{V}$ gives us more troubles but it is also the most interesting to treat. With all the machinery introduced in the earlier sections we can prove that (2) and (3) are respectively a supersolution and a subsolution of the Cauchy effective problem. In the corollaries and examples we test our convergence theorem.

We finish this Introduction with some historical remarks on the theme. Singular perturbations of diffusion processes, with and without controls, have been studied by many authors. For results based on probabilistic methods we refer to the books [27, 29], papers [34, 13], and the references therein. For an approach based on PDE-viscosity methods for the HJB equations we refer the worked developed by Alvarez and Bardi in [1, 2, 3], also [4] for problems with an arbitrary number of scales. It allows to identify the appropriate limit PDE governed by the effective Hamiltonian and gives general convergence theorems of the value function of the singularly perturbed system to the solution of the effective PDE, under assumptions that include deterministic control as well as differential games, in deterministic and stochastic cases. However, this theory originating in periodic homogenisation problems (in papers [33, 19]) was developed mostly for fast variables restricted to a compact set, almost all in the case of the m-dimensional torus. Nonetheless in many financial models the a priori knowledge of the boundedness of the fast variables does not appear to be natural according to the empirical data.

In the papers [7] and [6] the authors present an extension of the methods based on viscosity solutions showed in [1, 2, 3] to singular perturbation problems that have unbounded but uncontrolled fast variables.

This thesis is divided in two chapters. In Chapter I we study (EP) and in Chapter II we study our singular perturbation problem.

Main results of this thesis:

- Chapter I: Study of (EP) by purely PDE methods, introduction of non-standard approximations for (EP) that will be useful in Chapter II.

- Chapter II: The convergence theorem, our main result. Our relaxed semilimits are not typical and the proof of this convergence uses many of the procedures introduced and developed in Chapter I. We present some cases where V^ϵ really converges locally uniformly to V, solution of the effective Cauchy problem, as $\epsilon \to 0$. It is, as far as we know, the first singular perturbation problem of a stochastic model with unbounded and controlled fast variables that is treated with success. We treat everything analytically. We believe that some of our ideas can be applied with success to the study of other singular perturbation models.

Chapter I
The Ergodic Problem

This Chapter is organised as follows. In Section 1 we study the solvability of (EP) and we prove that it has infinitely many solutions and a critical value. Section 2 is devoted to the construction of solution bounded from below of (EP). We prove that there exists a solution $\phi \in C^2(\mathbb{R}^m)$ such that there exists $R > 0$ such that $\phi(y) \geq C|y|^\beta$ for all $|y| \geq R$. These first two sections are connected to Ichihara work, namely, the papers [24] and [25].

Section 3 considers (EP) defined in the ball and complemented with a state constraint boundary condition which is different according to the value of θ. We study such ergodic problems and we give some properties of the ergodic constant. This section is a preparation for the most general approximation results considered in Section 6. The main references in this section are [31], [10], and [38].

Next sections are an original part.

Section 4 deals with the problem of the uniqueness of the ergodic constant and of the corrector (up to additive constants) of (EP). This problem was also studied by Naoyuki Ichihara in [25] but our methods differ very much from his probabilistic arguments. Here we present a full PDE proof.

Section 5 is concerned with some properties of the ergodic constant λ for solutions of (EP) that are bounded from below. We also give some estimates that will be useful in Section 10.

Sections 6 shows new convergence results for (EP). There we consider approximations of (EP) by perturbing f or by specific ergodic problems set on balls. This section is mainly motivated by the problem studied in Chapter 2 though it has interest in itself.

Appendices are devoted to review some well established results or to present some technical computation or estimate needed in the text.

Part I
Existence

1 Solvability of (EP)

This part is concerned with the existence of solutions of (EP). Our problem is to find a pair $(\lambda, \phi) \in \mathbb{R} \times C^2(\mathbb{R}^m)$ such that for given $\theta > 1$

$$(EP) \qquad \lambda - \frac{1}{2}\Delta\phi(y) + \frac{1}{\theta}|D\phi(y)|^\theta = f(y) \quad \text{in } \mathbb{R}^m.$$

Here $D\phi$ and $\Delta\phi$ denotes respectively the gradient and the Laplacian of ϕ.

Definition (**Solution, Subsolution and Supersolution for** (EP)) We will call a pair (λ, ϕ) a solution (resp. subsolution, supersolution) of (EP) if $\lambda \in \mathbb{R}$, $\phi : \mathbb{R}^m \longrightarrow \mathbb{R}$ belongs to $C^2(\mathbb{R}^m)$, and

$$\lambda - \frac{1}{2}\Delta\phi(y) + \frac{1}{\theta}|D\phi(y)|^\theta = f(y) \text{ (resp. } \leq f(y), \geq f(y))$$

for all $y \in \mathbb{R}^m$.

Our assumption on f is

(H1) $f \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^m)$ and there exists $f_0 > 0$ and $\alpha > 0$ such that

$$-f_0 + f_0^{-1}|y|^\alpha \leq f(y) \leq f_0|y|^\alpha + f_0$$

and

$$|Df(y)| \leq f_0(1 + |y|^{\alpha-1})$$

for all $y \in \mathbb{R}^m$.

Notice that this implies that $\forall \lambda, M > 0$ we can find $R > 0$ such that $f(y) - \lambda \geq M$ for all $|y| \geq R$. We will exploit condition $(H1)$ in several occasions.

For simplicity of notation, let G be the operator

$$G[\phi](y) := -\frac{1}{2}\Delta\phi(y) + \frac{1}{\theta}|D\phi(y)|^\theta - f(y).$$

Then (EP) is equivalent to

$$G[\phi](y) = \mu \text{ in } \mathbb{R}^m \tag{7}$$

with $\mu = -\lambda$.

Important remark: Since $\phi \mapsto \frac{1}{\theta}|D\psi|^\theta$ is strictly convex and Laplacian is a linear operator, it is easy to see that G is a stricly convex operator. This convexity will play a crucial role in the arguments.

Definition (**Hölder Spaces**) For given $k \in \mathbb{N} \cup \{0\}$, $\iota \in (0,1]$ and an open set $O \subset \mathbb{R}^m$, we define the Hölder space $C^{k,\iota}(\bar{O})$ by

$$C^{k,\iota}(\bar{O}) := \{v \in C^k(\bar{O}) | \sup_{x,y \in O, x \neq y} \frac{|D^a v(x) - D^a v(y)|}{|x-y|^\iota} < \infty, |a| = k\},$$

where a stands for a multi-index of Differential operator D. We denote by $C^{k,\iota}(\mathbb{R}^m)$ the set of all functions $v \in C^k(\mathbb{R}^m)$ such that $v \in C^{k,\iota}(\bar{O})$ for any bounded O.

Let I be the operator

$$I(y,q) := \frac{1}{\theta}|q|^\theta - f(y) \tag{8}$$

hence we can write

$$G[\phi](y) = -\frac{1}{2}\Delta\phi(y) + I(y, D\phi(y)).$$

Remark: By virtue of $(H1)$ we have that for all $|y| \leq R$ there exists a constant $C_R > 0$ such that $|I(y,q)| \leq C_R(1 + |q|^\theta)$.

1.1 Functions ϕ_β

We start by giving a very important class of functions.

Lemma 1.1 *There are constants* c_0, ν_0 *and* $\rho_0 \in (0,1)$ *such that for any* $\rho \in (-\rho_0, \rho_0)$, $\phi_\beta(y) := \rho(1 + |y|^2)^{\frac{\beta}{2}}$, *satisfies*

$$G[\phi_\beta](y) \leq -c_0|y|^\alpha + \nu_0, \quad y \in \mathbb{R}^m \tag{9}$$

where $\beta \in [0, \frac{\alpha}{\theta} + 1]$.

Proof This proof can be found in [25]. Let $\rho \in (-1, 1)$. We have

$$D\phi_\beta(y) = \beta\rho(1 + |y|^2)^{\frac{\beta-2}{2}} y,$$

and

$$\begin{aligned}\Delta\phi_\beta &= \beta\rho[m(1 + |y|^2)^{\frac{\beta-2}{2}} + (\beta - 2)|y|^2(1 + |y|^2)^{\frac{\beta-4}{2}}] \\ &= \beta\rho[(\beta - 2)|y|^2 + m(1 + |y|^2)](1 + |y|^2)^{\frac{\beta-4}{2}}.\end{aligned}$$

Since $\beta \leq \frac{\alpha}{\theta} + 1 \leq \alpha + 2$ implies $\theta(\beta - 1) \leq \alpha$ and $\beta - 2 \leq \alpha$, we see, in view of our assumption on f, $(H1)$, and $|\rho| \leq 1 \Rightarrow |\rho|^\theta \leq |\rho|$ $(\theta > 1)$, that

$$\begin{aligned} G[\phi_\beta](y) &= -\frac{1}{2}\Delta\phi_\beta(y) + \frac{1}{\theta}|D\phi_\beta(y)|^\theta - f(y) \\ &\leq C_1(1 + |\rho||y|^{\beta-2} + |\rho|^\theta |y|^{\theta(\beta-1)}) + f_0 - f_0^{-1}|y|^\alpha \\ &\leq C_2(1 + |\rho||y|^\alpha + |\rho||y|^\alpha) + f_0 - f_0^{-1}|y|^\alpha \\ &\leq (2|\rho|C_2 - f_0^{-1})|y|^\alpha + f_0 + C_2 \end{aligned}$$

for some constant $C_2 > 0$ independent of ρ and that we can make it independent of β too (by taking larger C_2 if necessary). Now choosing $\rho_0 \in (0, 1)$ so small such that $\rho_0 < (2C_2)^{-1} f_0^{-1}$ and defining $c_0 := f_0^{-1} - 2\rho_0 C_2 > 0$ and $\nu_0 := f_0 + C_2$ we can conclude (9). ∎

Observe that ϕ_β given by Lemma 1.1 satisfies

$$\lim_{|y|\to\infty} G[\phi_\beta](y) = -\infty. \tag{10}$$

Fix any $\phi_\beta := \rho_0(1 + |y|^2)^{\frac{\beta}{2}}$ in the conditions of Lemma 1.1. As we will see next, such function ϕ_β is enough to guarantee the existence of a solution for (EP).

1.2 Existence of solutions for (EP)

The goal of this subsection is to show Theorem 1.2. We follow [24]. The proof proceeds essentially in the same lines of [28, Section 2].

Let $C_c^\infty(\mathbb{R}^m)$ be the set of infinitely differentiable functions on $\mathbb{R}^m$ with compact support. For $k \in \mathbb{N} \cup \{0\}$, $p \in [1, \infty]$ and an open set $O \subset \mathbb{R}^m$, we define the Sobolev space $W^{k,p}(O)$ by the collection of all locally summable functions v on O such that for each multi-index a with $|a| \leq k$, $D^a v$ exists in the weak sense and belongs to $L^p(O)$. We denote by $W^{k,p}_{\text{loc}}(\mathbb{R}^m)$ the set of locally summable functions v on $\mathbb{R}^m$ such that $v\zeta \in W^{k,p}(\mathbb{R}^m)$ for all $\zeta \in C_c^\infty(\mathbb{R}^m)$.

Before starting the proof of Theorem 1.2, we first remark that any weak solution of (7) (and therefore (EP)) in the distribution sense belonging to $W^{1,\infty}_{\text{loc}}(\mathbb{R}^m)$ is indeed a classical solution. This is a direct consequence of the classical regularity theory for quasilinear elliptic equations. Furthermore, in view of the Schauder theory, any classical solution of (7) ((EP)) is a C^3-solution (see [23]).

Theorem 1.2 *There exists a constant μ_0 such that* (7) *has a solution with* $\mu = \mu_0$.

Proof This proof can be found in [24, Proposition 3.2]. Since we have (10) then we know that there exists $\mu_0 \in \mathbb{R}$ such that $G[\phi_\beta](y) < \mu_0 \ \forall y$. This together with Theorem B.1(b) implies that for any $R > 1$, there exists a solution $\phi_R \in C^{2,\iota}(\bar{B}_R)$ of

$$G[\phi_R] = \mu_0 \text{ in } B_R, \quad \phi_R = \phi_\beta \text{ on } \partial B_R.$$

By Corollary B.2, we have that $|D\phi_R|$ is uniformly bounded by a constant not depending on R. Then, using the classical regularity theory for quasilinear elliptic equations (see [23]), we have that the Hölder norm $|D\phi_R|_{\Gamma,B_R}$ for some $\Gamma \in (0,1)$ is bounded by a constant not depending on R. By Schauder's theory for linear elliptic equations, we also have that the Hölder norm $|\phi_R|_{\Gamma,B_R}$ is bounded by a constant not depending on R. In particular, the family $\{\phi_R\}_{R>1}$ is relatively compact. By Arzela-Ascoli Theorem, we can conclude that there exist a sequence $\{R_n\}_n$ diverging to infinity as $n \to \infty$ such that $\{\phi_n\}_n := \{\phi_{R_n}\}_n$ converges to a function $\phi \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^m) = W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m)$ locally uniformly in $\mathbb{R}^m$. Next, we prove that ϕ is is a weak solution of $G[\phi] = \mu_0$ in $\mathbb{R}^m$ in the distribution sense.

Indeed, fix any $\zeta \in C^\infty_c(\mathbb{R}^m)$ and some $R > 1$ such that supp $\zeta \subset B_R$. Observe that for any n, we have by integration,

$$\frac{1}{2}\int_{B_R} D\phi_n(y) \cdot D\zeta(y)dy + \int_{B_R} I(y, D\phi_n(y))\zeta(y)dy = \mu_0 \int_{B_R} \zeta(y)dy. \quad (11)$$

where I was introduced in (8). Since $\sup_n |D\phi_n|_{L^\infty(B_R)} < \infty$, we see that $\phi_n - \phi \to 0$ weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^m)$. Moreover, we can verify that $\phi_n - \phi \to 0$ strongly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^m)$ as $n \to \infty$. Indeed, replacing ζ in (11) by the function $(\phi_n - \phi)\zeta$, we first see that

$$\frac{1}{2}\int_{B_R} D\phi_n \cdot (D\phi_n - D\phi)\zeta dy = -\frac{1}{2}\int_{B_R} D\phi_n \cdot (\phi_n - \phi)D\zeta dy$$
$$- \int_{B_R} I(y, D\phi_n)(\phi_n - \phi)\zeta dy + \mu_0 \int_{B_R} (\phi_n - \phi)\zeta dy$$

and then we take into account the equality $D\phi_n(y) = (D\phi_n(y) - D\phi(y)) +$

$D\phi(y)$ and the bound $|I(y,q)| \leq C_R(1+|q|^\theta)$ for all $|y| \leq R$ to conclude that

$$\begin{aligned}
&\int_{B_R} |D\phi_n(y) - D\phi(y)|^2 \zeta(y) dy \\
&\leq - \int_{B_R} (D\phi_n(y) - D\phi(y))\zeta(y) D\phi(y) dy \\
&+ \int_{B_R} |D\phi_n(y)||\phi_n(y) - \phi(y)||D\zeta(y)| dy \\
&+ 2C_R \int_{B_R} (1 + |D\phi_n(y)|^\theta)|\phi_n(y) - \phi(y)||\zeta(y)| dy \\
&+ 2\mu_0 \int_{B_R} (\phi_n(y) - \phi(y))\zeta(y) dy.
\end{aligned}$$

Since the right-hand side converges to zero as $n \to \infty$, we obtain the strong convergence $\phi_n - \phi \to 0$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^m)$. Thus, letting $n \to \infty$ in (11), we get

$$\frac{1}{2} \int_{\mathbb{R}^m} D\phi(y) \cdot D\zeta(y) dy + \int_{\mathbb{R}^m} I(y, D\phi(y))\zeta(y) dy = \mu_0 \int_{\mathbb{R}^m} \zeta(y) dy$$

for all $\zeta \in C_c^\infty(\mathbb{R}^m)$. Hence, ϕ is a weak solution of $G[\phi] = \mu_0$ in $\mathbb{R}^m$ in the distribution sense. By the standard regularity arguments for quasilinear elliptic equations, we conclude that ϕ is indeed a C^2-solution (in fact, C^3-solution). ∎

Corollary 1.3 (EP) *has a solution.*

Important Remark: For the solvability of (EP) the only information that is used is the existence of a function ϕ_β satisfying (10). Therefore the proof works for any other solution satisfying the same condition. We can state

Proposition 1.4 *If there exist $\phi \in C^3(\mathbb{R}^m)$ such that $\lim_{|y|\to\infty} G[\phi](y) = -\infty$ then (EP) has a solution.*

1.3 Infinite number of solutions

Proposition 1.5 *If (λ_1, ϕ) is a sub solution of (EP), then there exist a solution of (EP) for any $\lambda_2 < \lambda_1$.*

Proof Let (λ_1, ϕ) be a sub solution of (EP). Then, $G[\phi] \leq -\lambda_1 < -\lambda_2$. By Theorem B.1(b) (Appendix B), we know that for any $R > 1$ there exists a solution $\phi_R \in C^{2,\iota}(\mathbb{R}^m)$ of

$$G[\phi_R] = -\lambda_2 \text{ in } B_R, \quad \phi_R = \psi \text{ on } \partial B_R.$$

Consider the family $\{\hat{\phi}_R\}_{R>1}$. The conclusion follows by applying the same argument as in in Theorem 1.2. ∎

Remark: Theorem 1.2 and Proposition 1.5 shows that there are infinitely many solutions of (EP).

1.4 Existence of a critical value

Proposition 1.6 *Set $\lambda^* := \sup\{\lambda \in \mathbb{R}|\ (EP)$ has a subsolution$\}$. Then λ^* is finite and there exists a solution associated to it.*

Proof Theorem 1.2 implies $\lambda^* \neq -\infty$ because $\lambda^* \geq -\mu_0$ with $\mu_0 \in \mathbb{R}$. To prove that $\lambda^* < +\infty$, suppose, by contradiction, that $\lambda^* = +\infty$. Then, there exists a sequence of subsolutions $\{(\lambda_k, \phi_k)\}_k$ of (EP) such that $\lim_{k\to\infty} \lambda_k = +\infty$. Fix any $\zeta \in C_c^\infty(\mathbb{R}^m)$. We have, by integration by parts,

$$\frac{1}{2}\int_{\mathbb{R}^m} D\phi_k(y)D\zeta(y)dy + \int_{\mathbb{R}^m} \big(\frac{1}{\theta}|D\phi(y)|^\theta - f(y)\big)\zeta(y)dy \leq -\lambda_k \int_{\mathbb{R}^m} \zeta(y)dy$$

for all k. By Corollary B.3 (gradient bound), we know that the left-hand side is bounded uniformly in k. Therefore, taking $k \to \infty$ yields a contradiction because the right-hand side of the above equality goes to $-\infty$ as $k \to \infty$. Therefore λ^* is finite.

We can now choose any $\{(\lambda_k, \phi_k)\}_k$ such that $G[\phi_k] = -\lambda_k := \mu_k$ in $\mathbb{R}^m$ for each k and $\lim_{k\to\infty} \lambda_k = \lambda^*$. Then, it is an obvious consequence of the gradient bound and the argument in Theorem 1.2, that ϕ_k converges to a function ϕ that is indeed a C^2-solution of (EP) with $\lambda = \lambda^*$. ∎

Notation: Sometimes we will refer to λ^* as $\lambda^*(f)$ to stress the dependence of (EP) on f.

Corollary 1.7 *There exists a constant $\lambda^* \in \mathbb{R}$ such that (EP) admits a classical solution $\phi \in C^2(\mathbb{R}^m)$ if and only if $\lambda \leq \lambda^*$.*

By combining Propositions 1.5 and 1.6 we get the following conclusion.

Proposition 1.8 *(Monotonicity of λ^* with respect to f) Suppose that f_1, f_2 satisfy $(H1)$. If $f_1 \leq f_2$, then $\lambda^*(f_1) \leq \lambda^*(f_2)$.*

Proof By Proposition 1.6 we know that there exists a solution ϕ_1 associated to $\lambda^*(f_1)$ and a solution ϕ_2 associated to $\lambda^*(f_2)$. If $f_1 \leq f_2$ then $(\lambda^*(f_1), \phi_1)$ is a subsolution of (EP) with $f = f_2$. By definition of $\lambda^*(f_2)$, we have that $\lambda^*(f_1) \leq \lambda^*(f_2)$. ∎

Remark: Observe that this result is true for any f_1, f_2 satisfying $(H1)$ possibly with different constants and exponents.

Proposition 1.9 *Assume that f satisfies assumption $(H1)$. Then*

$$\lambda^*(f+c) = \lambda^*(f) + c \text{ where } c \in \mathbb{R}.$$

Proof Let $(\lambda^*(f), \phi_1)$ be a solution of (EP) given by Proposition 1.6 and let $(\lambda^*(f+c), \phi_2)$ be a solution of (EP) with $f = f + c$ given by Proposition 1.6.

We have

$$(\lambda^*(f) + c) - \frac{1}{2}\Delta\phi_1 + \frac{1}{\theta}|D\phi_1|^\theta = f + c$$

Consequently, by definition of $\lambda^*(f+c)$,

$$\lambda^*(f) + c \leq \lambda^*(f+c). \tag{12}$$

On the other hand,

$$(\lambda^*(f+c) - c) - \frac{1}{2}\Delta\phi_2 + \frac{1}{\theta}|D\phi_2|^\theta = (f+c) - c = f$$

and we can see that

$$\lambda^*(f+c) \leq \lambda^*(f) + c. \tag{13}$$

Therefore using (12) and (13), we have that $\lambda^*(f+c) = \lambda^*(f) + c$ as we wished to show. ∎

Proposition 1.10

$$\lambda^*(c|y|^\alpha) = c^{\frac{\theta^*}{\theta^*+\alpha}}\lambda^*(|y|^\alpha).$$

Proof Let $(\lambda^*(|y|^\alpha), \phi_1)$ be a solution of (EP) with $f(y) = |y|^\alpha$ given by Corollary 1.7. We will now construct a solution of (EP) with $f(y) = c|y|^\alpha$ by considering $\phi_2(y) = \beta^{\frac{2-\theta}{\theta-1}}\phi_1(\beta y)$ and the right choice of β.

We have,

$$\begin{aligned}
-\frac{1}{2}\Delta\phi_2(y) + \frac{1}{\theta}|D\phi_2(y)|^\theta &= -\frac{1}{2}\Delta(\beta^{\frac{2-\theta}{\theta-1}}\phi_1(\beta y)) + \frac{1}{\theta}|D(\beta^{\frac{2-\theta}{\theta-1}}\phi_1(\beta y))|^\theta \\
&= -\frac{1}{2}\beta^{\frac{\theta}{\theta-1}}\Delta(\phi_1(\beta y)) + \frac{1}{\theta}\beta^{\frac{\theta}{\theta-1}}|D\phi_1(\beta y)|^\theta \\
&= \beta^{\frac{\theta}{\theta-1}}\big(-\frac{1}{2}\Delta(\phi_1(\beta y)) + \frac{1}{\theta}|D\phi_1(\beta y)|^\theta\big) \\
&= \beta^{\theta^*}\big(\beta^\alpha|y|^\alpha - \lambda^*(|y^\alpha|)\big)
\end{aligned}$$

using the chain rule for the second equality and the fact that $(\lambda^*(|y|^\alpha), \phi_1)$ is a solution of (EP) with $f(y) = |y|^\alpha$ and $\frac{\theta}{\theta-1} = \theta^*$ for the last.

Therefore

$$\beta^{\theta^*}\lambda^*(|y^\alpha|) - \frac{1}{2}\Delta\phi_2(y) + \frac{1}{\theta}|D\phi_2(y)|^\theta = \beta^{\theta^*+\alpha}|y|^\alpha$$

and choosing $\beta^{\theta^*+\alpha} = c$, i.e., $\beta = c^{\frac{1}{\theta^*+\alpha}}$ we arrive at

$$c^{\frac{\theta^*}{\theta^*+\alpha}}\lambda^*(|y^\alpha|) - \frac{1}{2}\Delta\phi_2(y) + \frac{1}{\theta}|D\phi_2(y)|^\theta = c|y|^\alpha$$

By definition of $\lambda^*(c|y|^\alpha)$, we obtain

$$c^{\frac{\theta^*}{\theta^*+\alpha}}\lambda^*(|y^\alpha|) \leq \lambda^*(c|y|^\alpha).$$

The reverse inequality is obtained in an equivalent manner by looking at the solution $(\lambda^*(c|y|^\alpha), \psi_1)$ of (EP) with $f(y) = c|y|^\alpha$ given by Corollary 1.7 and then construction a solution of (EP) with $f(y) = |y|^\alpha$ by considering $\psi_2(y) = \beta^{\frac{2-\theta}{\theta-1}}\psi_1(\beta y)$ and $\beta = (\frac{1}{c})^{\frac{1}{\theta^*+\alpha}}$.

Conclusion: $\lambda^*(c|y|^\alpha) = c^{\frac{\theta^*}{\theta^*+\alpha}}\lambda^*(|y^\alpha|)$. ∎

2 Bounded from below solutions of (EP)

Theorem 1.2 shows that (EP) has a solution for any $\phi_\beta = \rho_0(1+|y|^2)^{\frac{\beta}{2}}$ in the conditions of Lemma 1.1. In this section, our goal is to construct a suitable solution by an analytical approximation procedure (approximation by Dirichlet problems) satisfying a certain growth from below. In fact, we will build a solution $(\lambda, \phi) \in \mathbb{R} \times C^2(\mathbb{R}^m)$ of (EP) such that $\inf_{\mathbb{R}^m}(\phi - \phi_\beta)$ is finite.

2.1 Estimate for solutions of (EP)

Proposition 2.1 *Let (λ, ϕ) be a solution of (EP). Then there exists a $K > 0$ such that*

$$|D\phi(y)| \leq K(1+|y|^{\gamma-1}), \quad |\phi(y)| \leq K(1+|y|^\gamma), \quad y \in \mathbb{R}^m,$$

where $\gamma = \frac{\alpha}{\theta} + 1$.

Proof From Corollary B.3 in Appendix B, we have for all $r > 0$ that there exists a constant $C > 0$ such that

$$\sup_{B_r} |D\phi(y)| \leq C(1 + \sup_{B_{r+1}} |f(y) - \lambda|^{\frac{1}{\theta}} + \sup_{B_{r+1}} |Df(y)|^{\frac{1}{2\theta-1}}).$$

Using now hypothesis $(H1)$, we see that

$$\sup_{B_r} |D\phi| \leq C(1 + (r+1)^{\frac{\alpha}{\theta}} + (r+1)^{\frac{\alpha-1}{2\theta-1}}).$$

for another $C > 0$. Since $\frac{\alpha}{\theta} = \gamma - 1$ and $\frac{\alpha-1}{2\theta-1} < \gamma - 1$, we can conclude that

$$\sup_{B_r} |D\phi| \leq \hat{C} + \hat{C}(r+1)^{\gamma-1}$$

with $\hat{C} > 0$. From this inequality we can deduce the first estimate of this proposition. The second one is deduced by integration from the first one. Hence, we have completed the proof. ∎

An heuristic justification: In fact, from the type of growth assumed on f, we would expect $\phi(y)$ to have a polynomial growth on y. Now assume for simplicity that $\phi(y) = |y|^\beta$ for some power $\beta > 0$ and that we are "away" from $y = 0$. Then $D\phi$ is of order $|y|^{\beta-1}$ while $\Delta\phi$ is of order $|y|^{\beta-2}$. Since $f(y)$

growths like $|y|^\alpha$ at infinity (see $(H1)$) we have, plugging into the equation $\lambda - \frac{1}{2}\Delta\phi(y) + \frac{1}{\theta}|D\phi(y)|^\theta = f(y)$, that

$$O(|y|^{\beta-2} + |y|^{\theta(\beta-1)}) \leq O(|y|^\alpha)$$

(which translates the fact that the left-hand side cannot exceed in growth the right-hand side). But $\beta - 2 < \beta - 1 < \theta(\beta - 1)$ (because $\theta > 1$) hence $O(|y|^{\beta-2} + |y|^{\theta(\beta-1)}) = O(|y|^{\theta(\beta-1)})$ (here we use the fact that y is not close to 0!), and obviously $\theta(\beta - 1) \leq \alpha \Longrightarrow \beta \leq \frac{\alpha}{\theta} + 1 := \gamma$. Hence the maximum growth expected is of order γ!

Notice that, if y is "close to 0", the quantities $|y|^{\beta-2}$, $|y|^{\theta(\beta-1)}$ may explode according to the values of β. The reason why we considered ϕ_0 above, is because ϕ_0 behaves like a polynomial of order β at infinity but is twice differentiable in all $\mathbb{R}^m$.

The exponent γ: From now on, γ will denote the value $\boxed{\gamma := \frac{\alpha}{\theta} + 1}$.

2.2 The class Φ_β

We will denote by $C_p(\mathbb{R}^m)$ the set of continuous functions on $\mathbb{R}^m$ that have polynomial growth, that is,

$$C_p(\mathbb{R}^m) = \{v \in C(\mathbb{R}^m) : \exists q, C > 0 \text{ s.t. } |v(y)| \leq C(1 + |y|^q)\}.$$

As Proposition 2.1 shows if (λ, ϕ) is a solution of (EP) then $\phi \in C_p(\mathbb{R}^m)$.

For given $\beta \in [0, \gamma]$, consider

$$\Phi_\beta := \{v \in C^2(\mathbb{R}^m) \cap C_p(\mathbb{R}^m) | \liminf_{|y|\to\infty} \frac{v(y)}{|y|^\beta} > 0\}.$$

Our aim is to construct a solution of (EP) belonging to Φ_β.

2.3 Existence of a solution of (EP) in Φ_β

For any $\epsilon \in (0, 1)$ and $\beta \in [0, \gamma]$, let us consider the elliptic equation

$$G[\phi] + \epsilon\phi = \epsilon\phi_\beta \text{ in } \mathbb{R}^m \tag{14}$$

for any $\phi_\beta := \rho_0(1 + |y|^2)^{\frac{\beta}{2}}$ satisfying Lemma 1.1 for some $\rho_0 \in (0, 1)$.

Lemma 2.2 *(Similar to Lemma 1.1) There is a constant $\rho_1 > 1$ such that $\psi_\gamma(y) := \rho_1(1+|y|^2)^{\frac{\gamma}{2}}$, satisfies*

$$G[\psi_\gamma](y) \geq -K \ \textit{in}\ \mathbb{R}^m$$

for some $K > 0$.

Proof The proof is similar to Lemma 1.1's proof. We have, in view of $(H1)$ and the computations presented in Lemma 1.1 with $\beta = \gamma$, that

$$G[\psi_\gamma] \geq -\rho_1 C_1(1+|y|^{\gamma-2}) + C_2|\rho_1|^\theta |y|^{\theta(\gamma-1)} - f_0(1+|y|^\alpha)$$

for some constants C_1 and C_2 positive that do not depend on the value of ρ_1 and γ. Since $\theta(\gamma-1) = \alpha$ and $\gamma - 2 \leq \alpha$, we can see that

$$G[\psi_\gamma] \geq (-\rho_1 C_1 + C_2|\rho_1|^\theta - f_0)|y|^\alpha - f_0 - \rho_1 C_1$$

If we choose ρ_1 large enough such that $-\rho_1 C_1 + C_2|\rho_1|^\theta - f_0 \geq 0$ and take $K := f_0 + \rho_1 C_1$ the conclusion follows. ∎

Proposition 2.3 *For any sufficiently small $\epsilon > 0$, there exists a solution $v_\epsilon \in C^2(\mathbb{R}^m)$ of (14) such that $\inf_{\mathbb{R}^m}(v_\epsilon - \phi_\beta)$ is finite. Moreover, we have that $\epsilon v_\epsilon(0)$ is bounded by a constant that does not depend on ϵ.*

Proof The proof is divided into three parts. Fix an arbitrary $\epsilon > 0$ and consider the ψ_γ of Lemma 2.2. Because $0 < \rho_0 < 1 < \rho_1$ and $\beta \leq \gamma$, we have that $\phi_\beta = \rho_0(1+|y|^2)^{\frac{\beta}{2}} \leq \rho_1(1+|y|^2)^{\frac{\gamma}{2}} = \psi_\gamma$ in the whole space $\mathbb{R}^m$.

Let ν_0 be the constant in Lemma 1.1.

Step 1. *$\phi_\beta - \frac{\nu_0}{\epsilon}$ and $\psi_\gamma + \frac{K}{\epsilon}$ are respectively a subsolution and a supersolution of* (14)

We have,

$$G\big[\phi_\beta - \frac{\nu_0}{\epsilon}\big] + \epsilon\big(\phi_\beta - \frac{\nu_0}{\epsilon}\big) = G[\phi_\beta] + \epsilon\phi_\beta - \nu_0 \leq -c_0|y|^\alpha + \epsilon\phi_\beta \leq \epsilon\phi_\beta.$$

Where we use (9) in the first inequality. Therefore $\phi_\beta - \frac{\nu_0}{\epsilon}$ is a subsolution of (14).

Analogously, one can see that

$$G\big[\psi_\gamma + \frac{K}{\epsilon}\big] + \epsilon\big(\psi_\gamma + \frac{K}{\epsilon}\big) = G[\psi_\gamma] + \epsilon\psi_\gamma + K \geq \epsilon\psi_\gamma \geq \epsilon\phi_\beta,$$

and conclude that $\psi_\gamma + \frac{K}{\epsilon}$ is a supersolution of (14). Observe that here we use Lemma 3.2 in the first inequality and $\phi_\beta \leq \psi_\gamma$ in $\mathbb{R}^m$ in the last one.

Step 2. *There exist a C^2-solution v_ϵ of* (8) *such that*

$$\phi_\beta(y) - \frac{\nu_0}{\epsilon} \leq v_\epsilon(y) \leq \psi_\gamma(y) + \frac{K}{\epsilon} \text{ in } \mathbb{R}^m.$$

Fix any $R > 0$ and consider the Dirichlet problem

$$G[v_{\epsilon,R}] + \epsilon v_{\epsilon,R} = \epsilon\phi_\beta \text{ in } B_R, \quad v_{\epsilon,R} = \phi_\beta \text{ on } \partial B_R.$$

By virtue of Theorem B.1(a) in Appendix B, for any $R > 1$ there exists a solution in the class $v_{\epsilon,R} \in C^{2,\iota}(\bar{B}_R)$. Moreover, since $\phi_\beta - \frac{\nu_0}{\epsilon}$ and $\psi_\gamma + \frac{K}{\epsilon}$ are, respectively, sub- and supersolutions of (14) that satisfy $\phi_\beta - \frac{\nu_0}{\epsilon} \leq v_{\epsilon,R} = \phi_\beta \leq \psi_\gamma + \frac{K}{\epsilon}$ on ∂B_R, a standard comparison principle implies that $\phi_\beta - \frac{\nu_0}{\epsilon} \leq v_{\epsilon,R} \leq \psi_\gamma + \frac{K}{\epsilon}$ in $\bar{B}_R$. Furthermore, by Theorem B.2, we have that $\sup_{B_r} |Dv_{\epsilon,R}| \leq C$ where the constant C does not depend on R and ϵ. Thus, by the Ascoli-Arzela theorem and the same argument as in the proof of Theorem 1.2, there exists a weak solution $v_\epsilon \in W^{1,\infty}_{loc}(\mathbb{R}^m)$ of (14) in the distribution sense, which is indeed of C^2-class by the standard regularity arguments.

Step 3. $\inf_{\mathbb{R}^m}(v_\epsilon - \phi_\beta)$ *is finite and* $\epsilon v_\epsilon(0)$ *is bounded by a constant that does not depend on* ϵ.

Since v_ϵ satisfies

$$\phi_\beta(y) - \frac{\nu_0}{\epsilon} \leq v_\epsilon(y) \leq \psi_\gamma(y) + \frac{K}{\epsilon} \text{ in } \mathbb{R}^m,$$

we have that $\inf_{\mathbb{R}^m}(v_\epsilon - \phi_\beta) \geq -\frac{\nu_0}{\epsilon}$. Multiplying now the above inequality by ϵ, we get

$$\epsilon\phi_\beta(0) - \nu_0 \leq \epsilon v_\epsilon(0) \leq \epsilon\psi_\gamma(0) + K \text{ in } \mathbb{R}^m.$$

Hence, $\epsilon|v_\epsilon(0)| \leq C$ for all $\epsilon \in (0, \epsilon_0)$ for some ϵ_0. Thus $\epsilon v_\epsilon(0)$ is bounded by a constant that does not depend on ϵ. ∎

Theorem 2.4 *There exists a solution (λ, ϕ) of (EP) such that* $\inf_{\mathbb{R}^m}(\phi - \phi_\beta)$ *is finite.*

Proof Let v_ϵ be the solution given by Proposition 2.3 and define $w_\epsilon(y) := v_\epsilon(y) - v_\epsilon(0)$ and $\lambda_\epsilon := \epsilon v_\epsilon(0)$. It is obvious that $(\lambda_\epsilon, w_\epsilon)$ is a solution of

$$\lambda_\epsilon + G[w_\epsilon] + \epsilon w_\epsilon = \epsilon\phi_\beta \text{ in } \mathbb{R}^m, \quad w_\epsilon(0) = 0.$$

By Theorem B.2 in Appendix B, we observe that, for any $R>0$, $\sup_{B_R}|Dw_\epsilon|$ is bounded by a constant not depending on ϵ. In particular, repeating the argument of Theorem 1.2, we can prove that the family $\{w_\epsilon\}_{\epsilon>0}$ is relatively compact in $C(\mathbb{R}^m)$. By the Ascoli-Arzela theorem, there exist a sequence $\{\epsilon_k\}_k \to 0$ as $k\to\infty$, a function $\phi\in W^{1,\infty}_{\rm loc}(\mathbb{R}^m)$ and a constant λ such that $\epsilon_k v_{\epsilon_k}(0)\to\lambda$ and $w_{\epsilon_k}\to\phi$ in $C(\mathbb{R}^m)$ as $k\to\infty$. As in the proof of Theorem 1.2, ϕ is a solution of $G[\phi]=-\lambda:=\mu$ in $\mathbb{R}^m$ in the distribution sense and therefore, by the standard regularity arguments, ϕ is a C^2-solution of $G[\phi]=\mu$ in $\mathbb{R}^m$, that is, (λ,ϕ) is a solution of (EP).

We show next that $\inf_{\mathbb{R}^m}(\phi-\phi_\beta)$ is finite. For that, notice first that, by convexity of G, Lemma 1.1 and $(H1)$, we have for any $\delta\in(1/2,1)$ and $y\in\mathbb{R}^m$

$$\begin{aligned}
G[\delta\phi_\beta](y) &= G[\delta\phi_\beta+(1-\delta)0](y)\\
&\le \delta G[\phi_\beta](y)+(1-\delta)G[0](y)\\
&\le -\delta c_0|y|^\alpha+\delta\nu_0+(1-\delta)(-f(y))\\
&\le -\frac{1}{2}c_0|y|^\alpha+\nu_0+(1-\delta)f_0^{-1}-(1-\delta)f_0|y|^\alpha\\
&\le -\frac{1}{2}c_0|y|^\alpha+\nu_0+\frac{1}{2}{f_0}^{-1}.
\end{aligned}$$

Taking into account this last estimate and Proposition 2.3, we can choose $R>0$ so big such that

$$G[\delta\phi_\beta](y)\le -C_1\le \epsilon v_\epsilon(0) \text{ for all } |y|\ge R, \epsilon\in(0,\epsilon_0) \text{ and } \delta\in(1/2,1) \quad (15)$$

and then find an $M_R>0$ such that $\sup_{0<\epsilon<\epsilon_0}\sup_{y\in B_R}(|\phi_\beta|+|w_\epsilon|)\le M_R$. Observe that M_R is finite because $\sup_{B_R}|w_\epsilon|$ is bounded by a constant not depending on ϵ.

We will now prove that w_{ϵ_k} satisfies $w_{\epsilon_k}\ge\delta\phi_\beta-M_R$ in $\mathbb{R}^m$ for all $\delta\in(1/2,1)$. To see this, we will argue in different regions of space. First, we have that

$$w_{\epsilon_k}(y)-\delta\phi_\beta(y)\ge -\sup_{B_R}(|\phi_\beta|+|w_{\epsilon_k}|))=-M_R \text{ for all } |y|\le R. \quad (16)$$

Hence, the claim is true in B_R. From another point of view, since

$$w_{\epsilon_k}(y)-\delta\phi_\beta(y)+M_R=(w_{\epsilon_k}-\phi_\beta)(y)+(1-\delta)\phi_\beta(y)+M_R\to+\infty \quad (17)$$

as $|y|\to+\infty$ (recall that $\inf_{\mathbb{R}^m}(w_{\epsilon_k}-\phi_\beta)$ is finite by Proposition 3.3), the claim also holds for a $R_{\epsilon_k,\delta}>R$ such that

$$w_{\epsilon_k}(y)\ge\delta\phi_\beta(y)-M_R \text{ for all } |y|\ge R_{\epsilon_k,\delta}.$$

Hence, the claim is also true in $B^c_{R_{\epsilon_k,\delta}}$.

It remains to show that it is also verified in the ring $D := \{y \in \mathbb{R}^m | R < |y| < R_{\epsilon_k,\delta}\}$. For that, we will show that $\delta\phi_\beta - M_R$ and w_{ϵ_k} are respectively a sub and a supersolution of

$$G[v] + \epsilon_k v = \epsilon_k \phi_\beta - C_1 \text{ in } D,$$

(C_1 is the constant in (15)) and then conclude by means of the comparison theorem that $\delta\phi_\beta - M_R \leq w_{\epsilon_k}$ in $\bar{D}$.

For any $y \in D$, we have, using (15) and the fact that $M_R > 0$,

$$\begin{aligned} G[\delta\phi_\beta - M_R](y) + \epsilon_k(\delta\phi_\beta(y) - M_R) &= G[\delta\phi_\beta](y) + \epsilon_k(\delta\phi_\beta(y) - M_R) \\ &\leq -C_1 + \epsilon_k(\phi_\beta(y) - M_R) \\ &\leq \epsilon_k\phi_\beta(y) - C_1 \end{aligned}$$

and

$$\begin{aligned} G[w_{\epsilon_k}](y) + \epsilon_k w_{\epsilon_k}(y) &\geq \epsilon_k\phi_\beta(y) - \lambda_{\epsilon_k} \\ &= \epsilon_k\phi_\beta(y) - \epsilon_k v_{\epsilon_k}(0) \\ &\geq \epsilon_k\phi_\beta(y) - C_1. \end{aligned}$$

Therefore, $\delta\phi_\beta - M_R$ and w_{ϵ_k} are respectively a subsolution and a supersolution of

$$G[v] + \epsilon_k v = \epsilon_k \phi_\beta - C_1 \text{ in } D,$$

that satisfy $\delta\phi_\beta - M_R \leq w_{\epsilon_k}$ on ∂D (look at (16) when $|y| = R$ and at (17) when $|y| = R_{\epsilon_k}$) . Applying a standard comparison principle, we obtain $\delta\phi_\beta - M_R \leq w_{\epsilon_k}$ in $\bar{D}$. Therefore, the claim also holds in D.

Consequently, $\delta\phi_\beta - M_R \leq w_{\epsilon_k}$ in $\mathbb{R}^m$ for all $\delta \in (1/2, 1)$. Letting $\delta \to 1$, we conclude that $\phi_\beta - M_R \leq w_{\epsilon_k}$ in $\mathbb{R}^m$. Taking $k \to \infty$ we get $\inf_{\mathbb{R}^m}(\phi - \phi_\beta) \geq -M_R$ as we would like to prove.

∎

Remark: Observe that the convexity of G plays a crucial role in this proof.

It is obvious that if $\inf_{\mathbb{R}^m}(\phi - \phi_\beta)$ is finite then $\liminf_{|y|\to\infty} \frac{\phi(y)}{|y|^\beta} > 0$. Therefore

Corollary 2.5 *There exists a solution (λ, ϕ) of (EP) such that ϕ belongs to*

$$\Phi_\beta := \{v \in C^2(\mathbb{R}^m) \cap C_p(\mathbb{R}^m) | \liminf_{|y|\to\infty} \frac{v(y)}{|y|^\beta} > 0\}.$$

In particular, if we take $\beta = \gamma := \frac{\alpha}{\theta} + 1$, we can conclude that

Corollary 2.6 *There exists a solution* (λ, ϕ) *of* (EP) *such that* ϕ *belongs to*

$$\Phi_\gamma := \{v \in C^2(\mathbb{R}^m) \cap C_p(\mathbb{R}^m) | \liminf_{|y|\to\infty} \frac{v(y)}{|y|^\gamma} > 0\}.$$

We ask the reader to keep in mind this class Φ_γ.

Observation: $\Phi_\gamma \subseteq \Phi_\beta \subseteq \{\text{bounded from below}\}$.

3 Approximations

Let $\theta > 1$ and consider the ergodic problem in $B_R(0)$ ($R > 0$),

$$\lambda_R - \frac{1}{2}\Delta\phi_R(y) + \frac{1}{\theta}|D\phi_R(y)|^\theta = f_R(y) \text{ in } B_R(0) \tag{18}$$

complemented with a state constraint boundary condition which is different in the sub and superquadratic case.

We start with some local gradient bounds for solutions of (18) plus the boundary condition.

Theorem 3.1 *(Local Gradient Bound) Assume that for any fixed λ_R there exist a solution $\phi_R \in W^{2,p}_{loc}(B_R(0))$ ($p < +\infty$) of* (18) *satisfying the boundary condition. Then, if $f_R \in W^{1,\infty}(B_R(0))$, we have for all $0 < R' < R$*

$$|D\phi_R(y)| \leq C_{R'} \text{ if } y \in B_{R'}(0)$$

where $C_{R'}$ depends only on bound on Df_R, upper bounds on $f_R - \lambda_R$ and θ.

Proof This proof can be found in [31, in Appendix]. ∎

3.1 Subquadratic and quadratic cases

For any $R > 0$ consider

$$\begin{cases} \lambda_R - \frac{1}{2}\Delta\phi_R(y) + \frac{1}{\theta}|D\phi_R(y)|^\theta = f_R(y) & \text{in } B_R(0) \\ \phi_R(y) \to +\infty \text{ as } y \to \partial B_R(0). \end{cases} \tag{19}$$

Theorem 3.2 *(Existence and uniqueness of solutions for* (19)*) Assume $1 < \theta \leq 2$ and that $f_R \in C(\bar{B}_R)$. Then, there exists a unique constant $\lambda_R \in \mathbb{R}$ such that the problem* (19) *has a solution $\phi_R \in W^{2,p}_{loc}(B_R(0))$ for every $p > 1$. This solution is unique up to an additive constant.*

Proof See [31, Theorems I.1 and VI.1]. ∎

Lemma 3.3 *Assume $1 < \theta \leq 2$ and let $f_R \in W^{1,\infty}(B_R(0))$ and ϕ_R be a solution of* (19) *given by Theorem 3.2. Then, $\phi_R \in C^{2,\Gamma}_{loc}(B_R(0))$ for all $\Gamma \in (0,1)$.*

Proof This result comes from the additional assumption on f_R and a standard bootstrap argument. In fact, by Theorem 3.2, ϕ_R is in $W^{2,p}_{loc}(B_R(0))$ for all $p > 1$ hence in $C^{1,\Gamma}_{loc}(B_R(0))$ for all $\Gamma \in (0,1)$, and therefore a standard regularity result implies that $\phi_R \in C^{2,\Gamma}_{loc}(B_R(0))$ for all $\Gamma \in (0,1)$ because $|D\phi_R|^\theta$ and f_R are in $C^{0,\Gamma}_{loc}(B_R(0))$. ∎

Remark: In particular, $\phi_R \in C^2_{\text{loc}}(B_R(0))$.

Proposition 3.4 *(Monotonicity property of λ_R with respect to the domain $B_R(0)$) Suppose that for all R $f_R \in W^{1,\infty}(B_R(0))$ and that $f_{R'} \le f_R$ for all $R' > R$. Then, if λ_R and $\lambda_{R'}$ are the ergodic constants associated to* (19) *in B_R and $B_{R'}$ respectively, we have $\lambda_{R'} \le \lambda_R$.*

Examples: $f_R(y) = f(y)$ (f_R does not depend on R) or $f_R(y) = \max_{y \in \bar{B}_{\frac{1}{R}}(0)} f(y)$ (this case will be considered in Section 10) satisfy trivially $f_{R'} \le f_R$ for all $R' > R$.

Proof This is a minor adaptation of Proposition 2.1 of [10]. We include the proof for completeness of the text.

Let (λ_R, ϕ_R) and $(\lambda_{R'}, \phi_{R'})$ be the pair of solutions of the ergodic problem (19) in $B_R(0)$ and $B_{R'}(0)$ respectively. From Theorem 3.2, the constants λ_R and $\lambda_{R'}$ are unique whereas the functions ϕ_R, $\phi_{R'}$ are unique up to additive constant. We look at function $\phi_{R'} - \phi_R$.

Pick any $y_0 \in \partial B_R(0)$. Since $\phi_{R'}$ is bounded in $\bar{B}_R(0)$ and $\phi_R(y) \to +\infty$ as $y \to \partial B_R(0)$,

$$\lim_{y \to y_0, y \in \bar{B}_R(0)} (\phi_{R'} - \phi_R)(y) = \phi_{R'}(y_0) - \lim_{y \to y_0, y \in \bar{B}_R(0)} \phi_R(y) = -\infty.$$

Therefore $\phi_{R'} - \phi_R$ has a maximum point $y^* \in B_R(0)$. Going back to the equations solved by $\phi_{R'}$ and ϕ_R, we obtain

$$-\frac{1}{2}\Delta\phi_{R'}(y^*) + \frac{1}{\theta}|D\phi_{R'}(y^*)|^\theta = f_{R'}(y^*) - \lambda_{R'}$$

and

$$-\frac{1}{2}\Delta\phi_R(y^*) + \frac{1}{\theta}|D\phi_R(y^*)|^\theta = f_R(y^*) - \lambda_R.$$

Subtracting and using the properties $D(\phi_{R'} - \phi_R)(y^*) = 0$, $\Delta(\phi_{R'} - \phi_R)(y^*) \le 0$ and $f_{R'} \le f_R$ one gets

$$0 \le -\frac{1}{2}\Delta(\phi_{R'} - \phi_R)(y^*) \le -\lambda_{R'} + \lambda_R,$$

i.e,

$$\lambda_{R'} \le \lambda_R.$$

∎

Proposition 3.5 *(Characterisation of the constant λ_R) Let λ_R be the ergodic constant associated to* (19) *and let us denote by $\mathcal{S}$ the set of all $a \in \mathbb{R}$ such that there exist a subsolution $\psi \in C(\bar{B}_R(0))$ of*

$$a - \frac{1}{2}\Delta\psi + \frac{1}{\theta}|D\psi|^\theta \leq f_R \text{ in } B_R(0). \tag{20}$$

Then

$$\lambda_R = \sup\{a \mid a \in \mathcal{S}\}.$$

Proof See [10, Proposition 2.2] ∎

Remark: Notice that $\mathcal{S} \neq \emptyset$. Indeed, it is easy to see that $-\|f_R\|_\infty \in \mathcal{S}$ because $\psi \equiv 0$ is a (classical) subsolution of

$$-\frac{1}{2}\Delta\psi + \frac{1}{\theta}|D\psi|^\theta \leq f_R + \|f_R\|_\infty \text{ in } B_R(0).$$

3.2 Superquadratic case

Let $R > 0$ and consider the ergodic problem given by

$$\begin{cases} \lambda_R - \frac{1}{2}\Delta\phi_R(y) + \frac{1}{\theta}|D\phi_R(y)|^\theta = f_R(y) & \text{in } B_R(0) \\ \lambda_R - \frac{1}{2}\Delta\phi_R(y) + \frac{1}{\theta}|D\phi_R(y)|^\theta \geq f_R(y) & \text{on } \partial B_R(0). \end{cases} \tag{21}$$

Theorem 3.6 *(Existence and uniqueness of solutions for* (21)*) Assume that $f_R \in W^{1,\infty}(B_R(0))$ and $\theta > 2$. Then there exists $\lambda_R \in \mathbb{R}$ and a function $\phi_R \in C^{0,\frac{\theta-2}{\theta-1}}(\bar{B}_R(0)) \cap C^2_{loc}(B_R(0))$ such that ϕ_R is a viscosity solution of* (21)*. Moreover if the pair (ν, ψ) is a solution of* (21) *such that ψ is a viscosity solution of* (21) *belonging to $C^{0,\frac{\theta-2}{\theta-1}}(\bar{B}_R(0)) \cap C^2_{loc}(B_R(0))$ then $\lambda_R = \nu$ and $\phi_R = \psi + K$ for some constant $K > 0$.*

Proof This is a consequence of Lemma 3.6 of [38] and the standard regularity theory for quasilinear elliptic equations and Schauder's theory. ∎

Proposition 3.7 *(Monotonicity property of λ_R with respect to the domain $B_R(0)$) Suppose that for any R $f_R \in W^{1,\infty}(B_R(0))$ and that $f_{R'} \leq f_R$ for all $R' > R$. Then, if λ_R and $\lambda_{R'}$ are the ergodic constants associated to* (21) *in B_R and $B_{R'}$ respectively, we have $\lambda_{R'} \leq \lambda_R$.*

Proof We know that $(\phi_{R'} - \phi_R)|_{\bar{B}_R(0)}$ has a maximum point at $y^* \in \bar{B}_R(0)$.

Suppose y^* is inside $B_R(0)$. In this case, the proof proceeds exactly as in Proposition 3.4 and one gets $\lambda_{R'} \leq \lambda_R$.

If $y^* \in \partial B_R(0)$, $(\phi_R - \phi_{R'})|_{\bar{B}_R(0)}$ has a minimum at y^*. Then, using the information that ϕ_R is a supersolution of (21), we obtain

$$-\frac{1}{2}\Delta(\phi_{R'})(y^*) + \frac{1}{\theta}|D(\phi_{R'})(y^*)|^\theta \geq f_R(y^*) - \lambda_R$$

with $(\phi_{R'})|_{\bar{B}_R(0)}$ as a test function. But

$$-\frac{1}{2}\Delta(\phi_{R'})(y^*) + \frac{1}{\theta}|D(\phi_{R'})(y^*)|^\theta \leq f_{R'}(y^*) - \lambda_{R'}.$$

Therefore, $f_{R'}(y^*) - \lambda_{R'} \geq f_R(y^*) - \lambda_R$. Since $f_{R'} \leq f_R$, $f_R(y^*) - \lambda_{R'} \geq f_R(y^*) - \lambda_R$ and we arrive at $\lambda_{R'} \leq \lambda_R$. ∎

Proposition 3.8 *(Characterisation of the constant λ_R) The ergodic constant introduced in Theorem 3.6 is characterised as follows:*

$$\lambda_R = \sup\{a \in \mathbb{R} \mid \exists \psi \in C(\bar{B}_R(0)) \text{ with } a - \frac{1}{2}\Delta\psi + \frac{1}{\theta}|D\psi|^\theta \leq f_R \text{ in } B_R(0)\}.$$

Part II
Uniqueness

4 Uniqueness

Theorem 1.2 and Proposition 1.5 show that there are infinitely many solutions (λ, ϕ) of (EP). Hence no uniqueness is expected to be achieved in the general case. Still, is there any hope to prove uniqueness for ϕ for a certain class of functions? We will reformulate this question.

Proposition 2.1 gives us an upper bound for ϕ. It says that ϕ cannot exceed a polynomial growth in y with maximum power γ, that is, $\phi(y) \leq C(1+|y|^\gamma)$ for some $C > 0$ with $\gamma = \frac{\alpha}{\theta}+1$. From another point of view, there are solutions of (EP) bounded from below - Theorem 2.4. Thus, since the growth of ϕ from above is fixed by the gradient bound, a "natural" approach to prove uniqueness for solutions of (EP) is to start looking for conditions restricting the range of ϕ from below.

Indeed our main result is the following

Theorem 4.1 *Assume that f satisfies hypothesis $(H1)$. Let (λ_1, ϕ) and (λ_2, ψ) be two solutions of (EP) bounded from below. Then $\lambda_1 = \lambda_2$ and $\phi = \psi + C$.*

Next subsections are devoted to prove this result.

4.1 Superquadratic and quadratic cases

4.1.1 Transformation $z = -e^{-\phi}$

The key ingredient in the superquadratic case is the transformation $z = -e^{-\phi}$.

Lemma 4.2 *Let (λ, ϕ) be a solution (resp. subsolution, supersolution) of (EP) then $z(y) = -e^{-\phi(y)}$ is a solution (resp. subsolution, supersolution) of*

$$-\frac{1}{2}\Delta z + N(y, z, Dz) = 0 \tag{22}$$

where $N(y, z, Dz) := z\big(\frac{1}{2}\big|\frac{Dz}{z}\big|^2 - \frac{1}{\theta}\big|\frac{Dz}{z}\big|^\theta + f - \lambda\big)$.

Proof By setting $z(y) = -e^{-\phi(y)}$, we have

$$e^{-\phi} = -z \Rightarrow \phi = -\log(-z).$$

Thus

$$D\phi = -\frac{-Dz}{-z} = -\frac{Dz}{z}$$

and

$$\begin{aligned}\Delta\phi &= -\frac{\Delta z}{z} - Dz\cdot(-z^{-2}Dz)\\ &= -\frac{\Delta z}{z} + \frac{|Dz|^2}{z^2}.\end{aligned}$$

Substituting into (EP), we get

$$\lambda - \frac{1}{2}\Big[-\frac{\Delta z}{z} + \frac{|Dz|^2}{z^2}\Big] + \frac{1}{\theta}\Big|\frac{Dz}{z}\Big|^\theta = f \quad (\text{resp. } \le f, \ge f)$$

and multiplying it by $-z$ (observe that $-z \ge 0$) we have

$$-\lambda z - \frac{1}{2}\Delta z + z\Big(\frac{1}{2}\Big|\frac{Dz}{z}\Big|^2 - \frac{1}{\theta}\Big|\frac{Dz}{z}\Big|^\theta + f\Big) = 0 \quad (\text{resp. } \le 0, \ge 0)$$

Set $N(y,z,Dz) := z\big(\frac{1}{2}\big|\frac{Dz}{z}\big|^2 - \frac{1}{\theta}\big|\frac{Dz}{z}\big|^\theta + f - \lambda\big)$. We arrived at equation

$$-\frac{1}{2}\Delta z + N(y,z,Dz) = 0 \quad (\text{resp. } \le 0, \ge 0).$$

∎

Given two arbitrary solutions of (EP), generally not much is known about their behaviour at infinity (other than Proposition 2.1).

In this subsection, we will start considering the particular case when $\phi, \psi \to +\infty$ at infinity (which includes the case $\phi, \psi \in \Phi_\beta$) and then we will show how to adapt the proof of the more general case when ϕ and ψ are bounded from below.

<u>Case</u>: $\phi, \psi \to +\infty$ at infinity

If $\phi \to +\infty$ at infinity, then, $z(y) = -e^{-\phi(y)} \to 0$ as $|y| \to \infty$. Moreover, because $\phi \in C^2(\mathbb{R}^m)$, z is always a non positive regular function. All this properties for z will be crucial.

In the following we will use $(H1)$. Let $R > 0$ be such that

$$f(y) - \lambda > -k \quad \forall |y| \ge R \tag{23}$$

where $k := \min_{Q\in\mathbb{R}^m}\big(-\frac{1}{2}|Q|^2 + (1-\frac{1}{\theta})|Q|^\theta\big)$. Observe that such minimum exists because $\theta \ge 2$ and $1 - \frac{1}{\theta} > 0$.

Proposition 4.3 *Suppose that $\theta \geq 2$ and ϕ, ψ are, respectively, a sub and a supersolution of (EP) such that $\phi, \psi \to +\infty$ at infinity. Let $R > 0$ be as in (23). Then, if $z_1 = -e^{-\phi}$ and $z_2 = -e^{-\psi}$ are such that $z_1 \leq z_2$ on $\partial B_R(0)$ we have*

$$z_1(y) \leq z_2(y) \text{ for all } y \in B_R^c(0).$$

Proof Suppose that the conclusion fails, i.e., there exists a point $y' \in B_R^c$ such that $z_1(y') > z_2(y')$. Then $\sup_{y \in B_R^c}\{z_1(y) - z_2(y)\} \geq \epsilon > 0$. Because $z_i(y) \to 0$ as $|y| \to \infty$ $(i = 1, 2)$, we see that the supremum is actually achieved at some point y^* (it is a maximum point). Moreover, we know that z_1 and z_2 are regular functions and so differentiable at the point y^*. Using now our Dirichlet condition, we see that y^* cannot be in $\partial B_R(0)$ and so it is in $B_R^c(0)$ (that is, it is an interior point). Hence, we know that $Dz_1(y^*) = Dz_2(y^*) =: p$ and $\Delta(z_1 - z_2)(y^*) \leq 0$.

By Lemma 4.2, we have that $z_1(y^*)$ is a subsolution of (22) while $z_2(y^*)$ is a supersolution of (22),

$$-\frac{1}{2}\Delta z_1 + N(y, z_1, p_1) \leq 0 \text{ where } p_1 := D_y z_1,$$

$$-\frac{1}{2}\Delta z_2 + N(y, z_2, p_2) \geq 0 \text{ where } p_2 := D_y z_2.$$

Subtracting the second from the first inequality, we arrive at

$$-\frac{1}{2}\Delta(z_1 - z_2)(y^*) + N(y^*, z_1(y^*), p) - N(y^*, z_2(y^*), p) \leq 0. \tag{24}$$

Denoting by N_z the derivative of N with respect to z, we have

$$\begin{aligned} N_z(y, z, p) &= \frac{1}{2}\Big|\frac{p}{z}\Big|^2 - \frac{1}{\theta}\Big|\frac{p}{z}\Big|^\theta + \big(f(y) - \lambda\big) + z\Big(-\frac{|p|^2}{z^3} - \frac{|p|^\theta}{|z|^{\theta+1}}\Big) \\ &= \frac{1}{2}\Big|\frac{p}{z}\Big|^2 - \frac{1}{\theta}\Big|\frac{p}{z}\Big|^\theta + \big(f(y) - \lambda\big) + z\Big(-\frac{|p|^2}{z^3} - \frac{|p|^\theta}{|z|^\theta(-z)}\Big) \\ &= -\frac{1}{2}\Big|\frac{p}{z}\Big|^2 + (1 - \frac{1}{\theta})\Big|\frac{p}{z}\Big|^\theta + \big(f(y) - \lambda\big) \\ &\geq k + \big(f(y) - \lambda\big). \end{aligned}$$

Therefore

$$N(y^*, z_1(y^*), p) - N(y^*, z_2(y^*), p) \geq [k + \big(f(y^*) - \lambda\big)][z_1(y^*) - z_2(y^*)]. \tag{25}$$

Taking into account (24), (25) and the properties on the laplacian term, we obtain

$$[k + \big(f(y^*) - \lambda\big)][z_1(y^*) - z_2(y^*)] \leq 0.$$

But $[k + \big(f(y^*) - \lambda\big)][z_1(y^*) - z_2(y^*)] > 0$ since, in virtue of our choice of R and $y^* \in B^c_R(0)$, we have $f(y^*) - \lambda > -k$ and because $z_1(y^*) - z_2(y^*) = \sup_{y \in B^c_R}\{z_1(y) - z_2(y)\} > 0$ by our hypothesis. Therefore, we reach a contradiction. Hence,

$$z_1(y) \leq z_2(y) \text{ for all } y \in B^c_R(0).$$

∎

Remark: Observe that hypothesis $\theta \geq 2$ is essential in this proof. Otherwise we cannot guarantee the existence of $\inf_{Q \in \mathbb{R}^m}(-\frac{1}{2}|Q|^2 + (1 - \frac{1}{\theta})|Q|^\theta)$.

Corollary 4.4 *Let $\theta \geq 2$ and take $R > 0$ as in* (23)*. If ϕ and ψ are, respectively, a subsolution and a supersolution of (EP) such that $\phi, \psi \to +\infty$ at infinity and $\phi \leq \psi$ on $\partial B_R(0)$, then*

$$\phi(y) \leq \psi(y) \textit{ for all } y \in B^c_R.$$

Proof The proof follows easily from Proposition 4.3 and Lemma 4.2 by taking $z_1 = -e^{-\phi}$ and $z_2 = -e^{-\psi}$ and observing that $z_1(y) \leq z_2(y) \Leftrightarrow \phi(y) \leq \psi(y)$. ∎

Remark: Notice that our results are true for all solutions of (EP) going to $+\infty$ at infinity. This include a variety of cases such as $\phi(y) \geq \log(|y|)$ or, of course, $\phi \in \Phi_\beta$. Next, we will see how to adapt this proof to the case when ϕ and ψ are bounded from below.

<u>General Case</u>: ϕ, ψ bounded from below

If $\phi(y) \geq -C$ then $0 \geq z(y) = -e^{-\phi(y)} \geq -e^C$ and so $z_1 = -e^{-\phi}$ and $z_2 = -e^{-\psi}$ satisfy $e^C \geq z_1 - z_2 \geq -e^C$. That is, $z_1 - z_2$ is a bounded function in $\mathbb{R}^m$.

In the following, we will consider $R > 1$ such that

$$f(y) - \lambda > \max\big(1, \max_{Q \in \mathbb{R}^m}\{\frac{1}{2}|Q|^2 + (\frac{1}{3} - \frac{1}{\theta^*})|Q|^\theta + \frac{1}{3}\}\big) \qquad (26)$$

for all $|y| \geq R$ where θ^* is the conjugate of θ ($\frac{1}{\theta} + \frac{1}{\theta^*} = 1$).

Remarks:

1. If $\theta > 2$, then $\frac{1}{\theta^*} \geq \frac{1}{2}$ and so $(\frac{1}{3} - \frac{1}{\theta^*}) < 0$. Therefore it is easy to see that there exists a maximum value for $\frac{1}{2}|Q|^2 + (\frac{1}{3} - \frac{1}{\theta^*})|Q|^\theta + \frac{1}{3}$ in $\mathbb{R}^m$.

2. Notice that, due to our choice of R, we always have $f(y) - \lambda > 1$ for all $|y| \geq R$.

Proposition 4.5 *Suppose that $\theta \geq 2$ and ϕ, ψ are respectively a subsolution and a supersolution of (EP) bounded from below. Let $R > 0$ be as in (26). Then, if $z_1 = -e^{-\phi}$ and $z_2 = -e^{-\psi}$ are such that $z_1 \leq z_2$ on $\partial B_R(0)$ we have*

$$z_1(y) \leq z_2(y) \text{ for all } y \in B_R^c(0).$$

Proof The proof follows the same lines of Proposition 4.3's proof but now the main new difficulty is that we cannot guarantee the existence of a maximum point for the function $z_1 - z_2$ in $B_R^c(0)$. Because of that, we will look at $z_1 - z_2 - \delta|y|^2$ for $\delta > 0$ small enough. Our goal being to achieve a contradiction.

Suppose that there exists a point $y' \in B_R^c(0)$ such that $z_1(y') - z_2(y') \geq \epsilon > 0$. Since $z_1 - z_2$ is a bounded function,

$$(z_1 - z_2)(y) - \delta|y|^2 \to -\infty \text{ as } |y| \to \infty.$$

Therefore $z_1 - z_2 - \delta|y|^2$ has a maximum point y_δ^* in $\bar{B}_R^c(0)$. Suppose that $y_\delta^* \in \partial B_R(0)$. Then, by our hypothesis (Dirichlet condition), $z_1(y_\delta^*) - z_2(y_\delta^*) \leq 0$ and we would get that

$$0 \geq z_1(y_\delta^*) - z_2(y_\delta^*) - \delta|y_\delta^*|^2 \geq z_1(y') - z_2(y') - \delta|y'|^2 \geq \epsilon - \delta|y'|^2.$$

A contradiction letting $\delta \to 0$. Then $y_\delta^* \in B_R^c(0)$ and $M_\delta := \max_{B_R^c(0)}\{z_1 - z_2 - \delta|y|^2\} > 0$ for any δ small enough. We can also conclude that $M_\delta \to \sup_{B_R^c(0)}(z_1 - z_2)$ (> 0) as $\delta \to 0$.

Considering such a small δ, we have that

$$D(z_1 - z_2)(y_\delta^*) = 2\delta y_\delta^*$$

and

$$\Delta(z_1 - z_2)(y_\delta^*) \leq 2\delta m.$$

Arguing now as in Proposition 4.3, we arrive at

$$-\frac{1}{2}\Delta(z_1 - z_2)(y_\delta^*) + N(y_\delta^*, z_1(y_\delta^*), Dz_1(y_\delta^*)) - N(y_\delta^*, z_2(y_\delta^*), Dz_2(y_\delta^*)) \leq 0,$$

i.e.,

$$N(y_\delta^*, z_1(y_\delta^*), Dz_2(y_\delta^*) + 2\delta y_\delta^*) - N(y_\delta^*, z_2(y_\delta^*), Dz_2(y_\delta^*)) \leq \delta m. \tag{27}$$

Let $t \in [0, 1]$ and define $X(t) := tz_1(y_\delta^*) + (1-t)z_2(y_\delta^*)$, $Y(t) := Dz_2(y_\delta^*) + 2t\delta y_\delta^*$ and $h(t) := N(y_\delta^*, X(t), Y(t))$. Then,

$$\begin{aligned}
&N(y_\delta^*, z_1(y_\delta^*), Dz_2(y_\delta^*) + 2\delta y_\delta^*) - N(y_\delta^*, z_2(y_\delta^*), Dz_2(y_\delta^*)) \\
&= N(y_\delta^*, X(1), Y(1)) - N(y_\delta^*, X(0), Y(0)) \\
&= h(1) - h(0) \\
&= \int_0^1 h'(t)dt
\end{aligned}$$

and

$$h'(t) = \frac{\partial N}{\partial X}(y_\delta^*, X(t), Y(t))(M_\delta + \delta|y_\delta^*|^2) + \frac{\partial N}{\partial Y}(y_\delta^*, X(t), Y(t)) \cdot (2\delta y_\delta^*).$$

Thus inequality (27) can be re-written as

$$\int_0^1 \big[\frac{\partial N}{\partial X}(y_\delta^*, X(t), Y(t))(M_\delta + \delta|y_\delta^*|^2) + \frac{\partial N}{\partial Y}(y_\delta^*, X(t), Y(t)) \cdot (2\delta y_\delta^*)\big] \leq \delta m \tag{28}$$

Set $Q := \frac{Y}{X}$. We did the computation for $\frac{\partial N}{\partial X}$ in Proposition 4.3,

$$\frac{\partial N}{\partial X}(y, X, Y) = -\frac{1}{2}\big|Q\big|^2 + (1 - \frac{1}{\theta})\big|Q\big|^\theta + \big(f(y) - \lambda\big).$$

We now compute $\frac{\partial N}{\partial Y}$,

$$\begin{aligned}\frac{\partial N}{\partial Y}(y, X, Y) &= X\big(\frac{Y}{X^2} - \frac{|Y|^{\theta-2}}{|X|^\theta}Y\big) = Q - \frac{X|Y|^{\theta-2}}{|X|^\theta}Y \\ &= Q + \frac{|Y|^{\theta-2}}{|X|^{\theta-1}}Y\end{aligned}$$

where we used $|X| = -X$.

- Case $\theta = 2$

In this case, $\frac{\partial N}{\partial X}(y, X, Y) = f(y) - \lambda$ and $\frac{\partial N}{\partial Y}(y, X, Y) = 0$. Then, (28) is reduced to

$$(f(y_\delta^*) - \lambda)(M_\delta + \delta|y_\delta^*|^2) \leq \delta m$$

a contradiction because $f(y_\delta^*) - \lambda > 1$ and $M_\delta > 0$ when letting $\delta \to 0$.

- Case $\theta > 2$

First we notice, by Cauchy-Schwarz inequality, that

$$\frac{\partial N}{\partial Y} \cdot (2\delta y_\delta^*) \geq -2\delta\big|\frac{\partial N}{\partial Y}\big||y_\delta^*|.$$

Therefore (28) implies

$$\int_0^1 \big[\frac{\partial N}{\partial X}(y_\delta^*, X(t), Y(t))(M_\delta + \delta|y_\delta^*|^2) - 2\delta\big|\frac{\partial N}{\partial Y}(y_\delta^*, X(t), Y(t))\big|\big|y_\delta^*\big|\big] \leq \delta m. \tag{29}$$

We now come back to our choice of R ((26)). It is easy to see that

$$\begin{aligned}
&-\frac{1}{2}|Q|^2+(1-\frac{1}{\theta})|Q|^\theta+f(y)-\lambda\\
&=-\frac{1}{2}|Q|^2+\frac{1}{\theta^*}|Q|^\theta+f(y)-\lambda\\
&\geq -\frac{1}{2}|Q|^2+\frac{1}{\theta^*}|Q|^\theta+\frac{1}{2}|Q|^2+(\frac{1}{3}-\frac{1}{\theta^*})|Q|^\theta+\frac{1}{3}\\
&=\frac{1}{3}(1+|Q|^\theta)
\end{aligned}$$

for all $|y|\geq R$. Therefore

$$\frac{\partial N}{\partial X}(y_\delta^*,X,Y)\geq\frac{1}{3}\big(1+\big|Q\big|^\theta\big)$$

and we can also see that $\frac{\partial N}{\partial X}(y_\delta^*,X,Y)\geq\frac{1}{3}$ for all $\delta>0$. From another point of view, we have

$$\big|\frac{\partial N}{\partial Y}(y_\delta^*,X,Y)\big|\leq\big|Q\big|+\big|Q\big|^{\theta-1}\leq 2\big(1+\big|Q\big|^\theta\big).$$

Indeed, the first inequality follows immediately from the computation of $\frac{\partial N}{\partial Y}(y,X,Y)$ while the second comes by noticing that if $\big|Q\big|\leq 1$ then $\big|Q\big|+\big|Q\big|^{\theta-1}\leq 2$ and if $\big|Q\big|>1$ then $\big|Q\big|<\big|Q\big|^{\theta-1}<\big|Q\big|^\theta$ because $\theta>2$.

Hence,

$$\big|\frac{\partial N}{\partial Y}(y_\delta^*,X,Y)\big|\leq 6\frac{\partial N}{\partial X}(y_\delta^*,X,Y).$$

Then (29) implies

$$\int_0^1\big[\frac{\partial N}{\partial X}(y_\delta^*,X(t),Y(t))(M_\delta+\delta|y_\delta^*|^2)-12\delta\frac{\partial N}{\partial X}(y_\delta^*,X(t),Y(t))\big|y_\delta^*\big|\big]\leq\delta m$$

$$\Leftrightarrow\int_0^1\frac{\partial N}{\partial X}(y_\delta^*,X(t),Y(t))(M_\delta+\delta|y_\delta^*|^2-12\delta|y_\delta^*|)\leq\delta m. \qquad (30)$$

Given that $M_\delta\to\sup_{B_R^c(0)}(z_1-z_2)$ (see the beginning of this proof), $\delta|y_\delta^*|^2\to 0$ and $\delta|y_\delta^*|\to 0$ as $\delta\to 0$, we can see that

$$M_\delta+\delta|y_\delta^*|^2-12\delta\big|y_\delta^*\big|\to\sup_{B_R^c(0)}(z_1-z_2)>0 \text{ when } \delta\to 0.$$

Then, we have

$$\int_0^1 \frac{\partial N}{\partial X}(y^*_{\delta_0}, X(t), Y(t))(M_{\delta_0} + \delta_0 |y^*_{\delta_0}|^2 - 12\delta_0 |y^*_{\delta_0}|) > 0$$

for all $\delta_0 \leq \delta$ small enough since $\frac{\partial N}{\partial X}(y^*_{\delta_0}, X(t), Y(t)) > 0$. This is a contradiction with (30) by taking $\delta_0 \to 0$.

Conclusion,

$$z_1(y) \leq z_2(y) \text{ for all } y \in B^c_R(0).$$

$\blacksquare$

Corollary 4.6 *Let $\theta \geq 2$ and take $R > 0$ as in Proposition 4.4. If ϕ and ψ are, respectively, a subsolution and a supersolution of (EP) bounded from below and such that $\phi \leq \psi$ on $\partial B_R(0)$, then*

$$\phi(y) \leq \psi(y) \textit{ for all } y \in B^c_R.$$

Proof Obvious from the previous proposition. $\blacksquare$

4.1.2 The Strong Maximum Principle

We refer the reader to [18] and [23] for more about the strong maximum principle for smooth solutions of linear parabolic and elliptic equations and to [9] and [16] for viscosity solutions of fully nonlinear degenerate elliptic and parabolic operator. The result we are concerned with is the following

Lemma 4.7 *Let $C > 0$ and let $\mathcal{O}$ be an open set. Any upper semicontinuous viscosity subsolution of*

$$-\Delta w - C|Dw| = 0 \quad \textit{in} \quad \mathcal{O}$$

that attains its maximum at some point of $\mathcal{O}$ is a constant in $\mathcal{O}$. In particular,

$$\max_{\bar{\mathcal{O}}} w = \max_{\partial \mathcal{O}} w.$$

For the proof, see [9].

4.1.3 Uniqueness of (EP)

Theorem 4.8 *Let $\theta \geq 2$. If (λ_1, ϕ) and (λ_2, ψ) are two solutions of (EP) such that ϕ and ψ are bounded from below, then $\lambda_1 = \lambda_2$ and $\phi = \psi + C$.*

Proof Let $R > 1$ be as in (26) and suppose that $\lambda_1 \geq \lambda_2$ (otherwise we change the roles of ϕ and ψ in the argument). Observe that this implies that (λ_1, ϕ) is a subsolution of (EP) with $\lambda = \lambda_2$. We will first prove that $\phi = \psi$ in $\mathbb{R}^m$ and then conclude that $\lambda_1 = \lambda_2$.

Step 1: *Adding constants to ϕ and ψ we may assume that* $\sup_{\partial B_R(0)}(\phi^* - \psi^*) = 0$.

Indeed, consider the compact set $\partial B_R(0)$. Since ϕ and ψ are continuous functions, we know that there exists $\sup_{\partial B_R(0)}(\phi - \psi) = S$. Set $\phi^* := \phi$ and $\psi^* := \psi + S$. It is obvious then, that $\sup_{\partial B_R(0)}(\phi^* - \psi^*) = \sup_{\partial B_R(0)}(\phi - \psi - S) = S - S = 0$. Hence, $\phi^* = \psi^*$ at some point of $\partial B_R(0)$.

Step 2: *In $B_R(0)$*

We have

$$-\frac{1}{2}\Delta\phi^* + \frac{1}{\theta}|D\phi^*|^\theta = f - \lambda_1 \leq f - \lambda_2$$

and

$$-\frac{1}{2}\Delta\psi^* + \frac{1}{\theta}|D\psi^*|^\theta = f - \lambda_2.$$

Therefore $\phi^* - \psi^*$ satisfies

$$-\frac{1}{2}\Delta(\phi^* - \psi^*) + \frac{1}{\theta}\big(|D\phi^*|^\theta - |D\psi^*|^\theta\big) \leq 0.$$

Let $t \in [0,1]$ and define $h(t)$ by $h(t) := |tD\phi^* + (1-t)D\psi^*|^\theta$. Then $\frac{1}{\theta}\big(|D\phi^*|^\theta - |D\psi^*|^\theta\big) = \frac{1}{\theta}\big(h(1) - h(0)\big) = \frac{1}{\theta}\int_0^1 h'(t)dt$, that is,

$$\frac{1}{\theta}\big(|D\phi^*|^\theta - |D\psi^*|^\theta\big) = \frac{1}{\theta}\int_0^1 \theta|tD\phi^* + (1-t)D\psi^*|^{\theta-2}(tD\phi^* + (1-t)D\psi^*)\cdot D(\phi^* - \psi^*)dt.$$

Thus we see that $\phi^* - \psi^*$ is a weak subsolution of

$$-\frac{1}{2}\Delta w + \mathcal{I}(y)\cdot Dw = 0 \tag{31}$$

where $\mathcal{I}(y) = \int_0^1 |tD\phi^*(y) + (1-t)D\psi^*(y)|^{\theta-2}(tD\phi^*(y) + (1-t)D\psi^*(y))dt$. We notice that $\mathcal{I}(y) \in L^\infty(B_R(0))$, more precisely

$$\begin{aligned}|\mathcal{I}(y)| &\leq \int_0^1 |tD\phi^*(y) + (1-t)D\psi^*(y)|^{\theta-1}dt \\ &\leq \int_0^1 K(1+|y|^{(\theta-1)(\gamma-1)})dt \\ &\leq K(1+R^{(\theta-1)(\gamma-1)})\end{aligned}$$

where we used Proposition 2.1 in the second inequality and that $y \in B_R(0) \Rightarrow |y| \leq R$ in the third. Then we can apply the classical maximum principle to (31) in $B_R(0)$ (see [23]) to obtain

$$\sup_{B_R(0)} (\phi^* - \psi^*) = \sup_{\partial B_R(0)} (\phi^* - \psi^*). \tag{32}$$

Remark: Observe that this gives a comparison in $B_R(0)$. In fact, since $\sup_{\partial B_R(0)}(\phi^* - \psi^*) = 0$, (32) implies $\phi^* \leq \psi^*$ in $\bar{B}_R(0)$.

Step 3: *In* $B_R^c(0)$

Let R' be any real number such that $R' > R$. We want to prove that the maximum of $\phi^* - \psi^*$ in $B_{R'}(0)$ is attained on the boundary of $B_R(0)$.

Arguing as above, we can conclude that

$$\sup_{B_{R'}(0)} (\phi^* - \psi^*) = \sup_{\partial B_{R'}(0)} (\phi^* - \psi^*).$$

We now observe that, since (λ_1, ϕ^*) and (λ_2, ψ^*) are, respectively, a subsolution and a solution of (EP) with $\lambda = \lambda_2$, by Corollary 4.6, we have that $\phi^* \leq \psi^*$ in $B_R^c(0)$. In particular, $\phi^* \leq \psi^*$ in $\partial B_{R'}(0)$. Hence,

$$\sup_{B_{R'}(0)} (\phi^* - \psi^*) = \sup_{\partial B_{R'}(0)} (\phi^* - \psi^*) \leq 0 = \sup_{\partial B_R(0)} (\phi^* - \psi^*), \quad \forall R' > R.$$

This implies,

$$\sup_{\bar{B}_{R'}(0)} (\phi^* - \psi^*) = \sup_{\partial B_R(0)} (\phi^* - \psi^*), \quad \forall R' > R. \tag{33}$$

Combining results (32) and (33), we arrive to

$$\sup_{\bar{B}_R(0)} (\phi^* - \psi^*) = \sup_{\partial B_R(0)} (\phi^* - \psi^*) = \sup_{\bar{B}_{R'}(0)} (\phi^* - \psi^*), \quad \forall R' > R.$$

Therefore the global maximum of $\phi^* - \psi^*$ in $B_{R'}(0)$ is achieved on $\partial B_R(0)$ at some point $y_0 \in \partial B_R(0)$.

Step 4: *$\phi^* - \psi^*$ is a constant in $\mathbb{R}^m$*

Arguing as above, we can see that $\phi^* - \psi^*$ satisfies

$$-\frac{1}{2}\Delta_y w + \mathcal{I}(y) \cdot D_y w \leq 0.$$

Since $-\frac{1}{2}\Delta_y w - |\mathcal{I}(y)||D_y w| \leq -\frac{1}{2}\Delta_y w + \mathcal{I}(y) \cdot D_y w$ and $\mathcal{I}(y) \in L^\infty(B_R(0))$, we can conclude that there exists a constant $C > 0$ (for example, $C := \|\mathcal{I}(y)\|_{L^\infty(B_R(0))}$) such that $\phi^* - \psi^*$ is an upper semi continuous viscosity subsolution of

$$-\frac{1}{2}\Delta_y w - C|D_y w| = 0 \quad \text{in } B_{R'}(0)$$

that we showed that attains its maximum at some point $y_0 \in \partial B_R(0) \subset B_{R'}(0)$. Applying Lemma 4.7, we can deduce that $\phi^* - \psi^*$ is a constant in $B_{R'}$, for all $R' > R$. It follows that $\phi^* - \psi^*$ is a constant in $\mathbb{R}^m$ by continuity.

Step 5: $\lambda_1 = \lambda_2$

Since $\phi^* - \psi^*$ is a constant in $\mathbb{R}^m$, we have $\phi = \psi + C$ and we can deduce immediately from (EP) that $\lambda_1 = \lambda_2$. ∎

4.2 Sub quadratic case

As pointed out in the previous subsection, when $\theta < 2$ there isn't a minimum value for the function $-\frac{1}{2}|Q|^2 + (1 - \frac{1}{\theta})|Q|^\theta$ with $Q \in \mathbb{R}^m$. This implies that technically the proof needs changes.

4.2.1 Behaviour at infinity of the solutions of the ergodic problem

Proposition 4.9 *Let $\theta > 1$. If ϕ is a solution of (EP) bounded from below then there exists $c > 0$ such that $\phi(y) \geq c|y|^\gamma - c^{-1}$ where $\gamma = \frac{\alpha}{\theta} + 1$.*

Proof Adding constants to ϕ if necessary we may assume that $\phi \geq 0$. We already know that ϕ satisfies:

$$|D\phi(y)| \leq K(1 + |y|^{\gamma-1}) \quad \text{(Proposition 2.1)}$$

and

$$|\phi(y)| \le K(1+|y|^\gamma) \quad \text{(consequence of the previous estimate)}$$

for some constant K.

We argue by contradiction assuming that there exists a sequence $|y_\epsilon| \to +\infty$ such that $\frac{\phi(y_\epsilon)}{|y_\epsilon|^\gamma} \to 0$. We set $\Gamma_\epsilon = \frac{|y_\epsilon|}{2}$ and we introduce

$$v_\epsilon(y) = \frac{\phi(y_\epsilon + \Gamma_\epsilon y)}{{\Gamma_\epsilon}^\gamma} \text{ for } |y| \le 1.$$

Because of the above estimates on ϕ, we have $|v_\epsilon|$, $|Dv_\epsilon|$ uniformly bounded and v_ϵ satisfies

$$-\frac{1}{2}\Gamma_\epsilon^{\gamma-2-\alpha}\Delta v_\epsilon + \frac{1}{\theta}|Dv_\epsilon|^\theta = \Gamma_\epsilon^{-\alpha}\big(f(y_\epsilon+\Gamma_\epsilon y) - \lambda\big) \text{ in } B_1(0).$$

Then we notice

$$\gamma - 2 - \alpha = \frac{\alpha}{\theta} - 1 - \alpha = \alpha(\frac{1}{\theta} - 1) - 1 < 0$$

and therefore $\Gamma_\epsilon^{\gamma-2-\alpha} \to 0$ as $\epsilon \to 0$.

$$f(y_\epsilon + \Gamma_\epsilon y) \ge f_0^{-1}|\Gamma_\epsilon|^\alpha - f_0$$

since $|y_\epsilon + \Gamma_\epsilon y| \ge |y_\epsilon| - \Gamma_\epsilon \ge \Gamma_\epsilon$.

Since (v_ϵ) is precompact in $C(\bar{B}_1(0))$, we can apply Ascoli's Theorem and pass to the limit in the viscosity sense: if $v_\epsilon \to v$ then

$$\frac{1}{\theta}|Dv|^\theta \ge f_0^{-1} \text{ in } B_1(0)$$

and

$$v \ge 0 \text{ on } \partial B_1(0) \text{ since } \phi \ge 0$$

therefore v is a supersolution of the equation $\frac{1}{\theta}|Du|^\theta = f_0^{-1}$ with null boundary condition for which the unique solution is $(\theta f_0^{-1})^{1/\theta} d(y, \partial B(0,1))$. By comparison principle for the eikonal equation the supersolution is above the solution. Then, $v(y) \ge (\theta f_0^{-1})^{\frac{1}{\theta}} d(y, \partial B_1(0))$ and $v(0) \ge (\theta f_0^{-1})^{\frac{1}{\theta}}$.

But this is a contradiction since $v_\epsilon(0) = 2^\gamma \frac{\phi(y_\epsilon)}{|y_\epsilon|^\gamma} \to 0$ by our hypothesis. ∎

Corollary 4.10 *If ϕ is a bounded from below solution of (EP) then $\phi \in \Phi_\gamma$.*

4.2.2 Uniqueness of (EP)

We will show that the analog of the transformation $z = -e^{-\phi}$ when $\theta < 2$ is the transformation ψ^q where $q > 1$ is very close to 1.

Lemma 4.11 *Let (λ, ϕ) be a solution of (EP) such that $\phi \in \Phi_\gamma$ and $\phi \geq 1$. Then, there exists $R > 1$ and $q_0 > 1$ such that for all $q \in (1, q_0)$, (λ, ϕ^q) is a strict supersolution of (EP) in $B_R^c(0)$.*

Proof We wish to prove that, there exists $R > 1$ and $q_0 > 1$ such that for all $q \in (1, q_0)$

$$Q(y) > 0 \quad \text{for all } y \in B_R^c(0)$$

where $Q(y) := -\frac{1}{2}\Delta\phi^q(y) + \frac{1}{\theta}|D\phi^q(y)|^\theta - (f(y) - \lambda)$. We have,

$$D\phi^q = q\phi^{q-1}D\phi$$

and

$$\Delta\phi^q = q(q-1)\phi^{q-2}|D\phi|^2 + q\phi^{q-1}\Delta\phi.$$

Then Q becomes

$$Q = -\frac{1}{2}\big(q(q-1)\phi^{q-2}|D\phi|^2 + q\phi^{q-1}\Delta\phi\big) + \frac{1}{\theta}\big|q\phi^{q-1}D\phi\big|^\theta - (f - \lambda).$$

By adding and subtracting $\frac{1}{\theta}q\phi^{q-1}|D\phi|^\theta$, using the equation (EP) and noticing that $(1 - q\phi^{q-1}) = [(1 - \phi^{q-1}) - (q-1)\phi^{q-1}]$, we arrive at

$$\begin{aligned}
Q &= -\frac{1}{2}q(q-1)\phi^{q-2}|D\phi|^2 + \frac{1}{\theta}\big(q^\theta\phi^{\theta(q-1)} - q\phi^{q-1}\big)|D\phi|^\theta \\
&\quad - [(1 - \phi^{q-1}) - (q-1)\phi^{q-1}](f - \lambda) \\
&= -\frac{1}{2}q(q-1)\phi^{q-2}|D\phi|^2 + \frac{1}{\theta}\big(q^\theta\phi^{\theta(q-1)} - q\phi^{q-1}\big)|D\phi|^\theta - (1 - \phi^{q-1})(f - \lambda) \\
&\quad + (q-1)\phi^{q-1}(f - \lambda).
\end{aligned}$$

But

$$\frac{1}{\theta}\big(q^\theta\phi^{\theta(q-1)} - q\phi^{q-1}\big)|D\phi|^\theta \geq 0 \text{ and } -(1 - \phi^{q-1})(f - \lambda) \geq 0$$

because $q > 1$, $\phi \geq 1$ and for R large enough we have $f - \lambda > 0$ in $B_R^c(0)$. Therefore, if we prove that there exist large $R > 1$ such that

$$Q_1 > 0 \text{ for all } y \in B_R^c(0)$$

where $Q_1 := -\frac{1}{2}q(q-1)\phi^{q-2}|D\phi|^2 + (q-1)\phi^{q-1}(f - \lambda)$, we would have $Q > 0$ for all $y \in B_R^c(0)$.

We have,

$$Q_1 > 0 \Leftrightarrow \frac{1}{2}q\phi^{q-2}|D\phi|^2 < \phi^{q-1}(f-\lambda)$$
$$\Leftrightarrow \frac{1}{2}q\frac{|D\phi|^2}{\phi} < f-\lambda \quad (\text{because } \phi \geq 1).$$

By Proposition 2.1, assumption $(H1)$ and the fact that $\phi \in \Phi_\gamma$ we can see that there are constants $K, M > 0$ such that

$$\frac{1}{2}q\frac{|D\phi(y)|^2}{\phi(y)} \leq K|y|^{\gamma-2}.$$

and

$$f(y) - \lambda \geq M|y|^\alpha$$

for all y in the complementary of a (possible large) ball B_R. Therefore, to have inequality $Q_1 > 0$ (at least for R large) it is enough to have $\gamma - 2 < \alpha$. But

$$\alpha - (\gamma - 2) = \alpha - \frac{\alpha}{\theta} + 1 = \frac{\alpha}{\theta^*} + 1 > 0.$$

Consequently, there exist a $R > 1$ such that $Q_1 > 0$ for all $y \in B_R^c(0)$. It is worth remarking that such R is independent of $q \in (1, q_0)$. ∎

Proposition 4.12 *Suppose that ϕ and ψ are respectively a subsolution and a supersolution of (EP) such that $\phi, \psi \in \Phi_\gamma$, $\psi \geq 1$ and $\phi \leq \psi$ on $\partial B_R(0)$. If $R > 1$ is as in Lemma 4.11, then $\phi \leq \psi^q$ in $B_R^c(0)$.*

Proof By Lemma 4.11, we know that (λ, ψ^q) is a strict supersolution of (EP) in $B_R^c(0)$. We wish to prove that $\phi \leq \psi^q$ in $B_R^c(0)$.

Suppose that $\exists y' \in B_R^c(0)$ such that $\phi(y') > \psi^q(y')$. Then $\sup_{y \in B_R^c(0)}\{\phi(y) - \psi^q(y)\} \geq \epsilon > 0$. Consider the function $(\phi - \psi^q)(y)$. Since $\psi \in \Phi_\gamma$ implies $\psi^q \in \Phi_{q\gamma}$ and $q\gamma > \gamma$, we can conclude that ψ grows more than $|y|^\gamma$ at infinity (at least like $|y|^{q\gamma}$). From another point of view, Proposition 2.1 showed that $\phi(y) \leq K(1 + |y|^\gamma)$. Therefore, we can conclude

$$(\phi - \psi^q)(y) \to -\infty \text{ as } |y| \to \infty.$$

Hence there exists a maximum point $y^* \in \bar{B}_R^c(0)$ of $\phi - \psi^q$. If $y^* \in \partial B_R(0)$, $\phi(y^*) \leq \psi(y^*) \leq \psi^q(y^*)$ $(\psi \geq 1)$ (here we used the Dirichlet condition) and then we would have that $\phi(y^*) - \psi^q(y^*) \leq 0$ a contradiction because y^* is the maximum of $\phi - \psi^q$ in $B_R^c(0)$ and $\phi > \psi^q$ at y'. Therefore at y^*, we know that

$$D\phi = D\psi^q$$

and

$$\Delta(\phi - \psi^q) \leq 0.$$

We then arrive at

$$f(y^*)-\lambda \geq -\frac{1}{2}\Delta\phi(y^*)+\frac{1}{\theta}|D\phi(y^*)|^\theta \geq -\frac{1}{2}\Delta\psi^q(y^*)+\frac{1}{\theta}|D\psi^q(y^*)|^\theta > f(y^*)-\lambda \text{ in } B_R^c(0)$$

a contradiction. Therefore, $\phi \leq \psi^q$ in $B_R^c(0)$. ∎

Corollary 4.13 *Suppose that ϕ and ψ are, respectively, a subsolution and a supersolution of (EP) such that $\phi, \psi \in \Phi_\gamma$, $\psi \geq 1$ and $\phi \leq \psi$ on $\partial B_R(0)$. If $R > 1$ is as in Lemma 4.11, then $\phi \leq \psi$ in $B_R^c(0)$.*

Proof Since R in Lemma 4.11 is independent of q, the conclusion follows by letting $q \to 1$ in Proposition 4.12. ∎

Theorem 4.14 *Let $\theta < 2$ and suppose that (λ_1, ϕ) and (λ_2, ψ) are two solutions of (EP) such that $\phi, \psi \in \Phi_\gamma$. Then, $\phi = \psi + C$ and $\lambda_1 = \lambda_2$.*

Proof Suppose that $\lambda_1 \geq \lambda_2$. Otherwise we exchange the roles of ϕ and ψ in the argument. Notice that if $\lambda_1 \geq \lambda_2$, then (λ_1, ϕ) is a subsolution of (EP) with $\lambda = \lambda_2$.

We saw in Step 1 of Theorem 4.8 that for a fixed $R > 1$ we can always add constants to ϕ and ψ and ask that $\sup_{\partial B_R(0)}(\phi^* - \psi^*) = 0$ and $\psi^* \geq 1$. We now look at $R > 1$ given by Lemma 4.11 for which we know that $(\lambda_2, (\psi^*)^q)$ for $q > 1$ is a strict supersolution of (EP) with $\lambda = \lambda_2$. Corollary 4.13 give us now $\phi^* \leq \psi^*$ in $B_R^c(0)$. From another point of view, $\sup_{\partial B_R(0)}(\phi^* - \psi^*) = 0$ implies a comparison in the ball, $\phi^* \leq \psi^*$ in $\bar{B}_R(0)$. Therefore, $\phi^* \leq \psi^*$ in $\mathbb{R}^m$ and we can repeat the argument of Theorem 4.8 to conclude that $\phi - \psi = C$ is a constant in $\mathbb{R}^m$. Consequently, from the equation of (EP), $\lambda_1 = \lambda_2$. ∎

Since solutions of (EP) that are bounded from below belong to Φ_γ (Proposition 4.9), we have

Theorem 4.15 *(Uniqueness result) Let $\theta < 2$ and suppose that (λ_1, ϕ) and (λ_2, ψ) are two solutions of (EP) bounded from below. Then, $\phi = \psi + C$ and $\lambda_1 = \lambda_2$.*

Important comment: It is worth pointing out that all the arguments of this subsection, particularly, both Theorems 4.8 and 4.15 are valid for any $\theta > 1$. Subsection 4.1 gives another analytical proof in the superquadratic case $\theta \geq 2$.

4.3 Conclusion

Combining Theorems 4.8 and 4.15 we can conclude Theorem 4.1.

Part III
Consequences of the Uniqueness

5 Remark on the properties of λ

Corollary 1.7 shows the existence of a critical value

$$\lambda^* := \sup\{\lambda \in \mathbb{R} |\ (EP) \text{ has a subsolution}\}$$

such that (EP) admits a classical subsolution $\phi \in C^2(\mathbb{R}^m)$ if and only if $\lambda \leq \lambda^*$.

Proposition 5.1 *Suppose $1 < \theta < 2$ and let (λ, ϕ) be a solution of (EP) bounded from below. Then $\lambda = \lambda^*$.*

Proof Without loss of generality we may assume $\phi \geq 1$. Let ψ be a solution of (EP) with $\lambda = \lambda^*$ (see Proposition 1.6). We have that $\lambda \leq \lambda^*$. It remains to check that $\lambda^* \leq \lambda$. Suppose that $\lambda^* > \lambda$.

If $\lambda^* > \lambda$, then (λ^*, ϕ) is a supersolution of (EP). By adding a constant to ϕ we may assume that $\psi \leq \phi$ in $\partial B_R(0)$. In particular, we saw that this implies that $\psi \leq \phi$ in $\bar{B}_R(0)$. From another point of view, since $\phi \in \Phi_\gamma$ (Corollary 4.10) and $\phi \geq 1$, by Lemma 4.11 there exists $R > 1$ and $q > 1$ such that (λ^*, ϕ^q) is a strict supersolution of (EP) with $\lambda = \lambda^*$ in $B_R^c(0)$. We can now repeat Proposition 4.12 to conclude that $\psi \leq \phi^q$ in $B_R^c(0)$ and consequently, by letting $q \to 1$, $\psi \leq \phi$ in $B_R^c(0)$ (Corollary 4.13).

Since $\psi \leq \phi$ in $\bar{B}_R(0)$ and $\psi \leq \phi$ in $B_R^c(0)$, we have $\psi \leq \phi$ in $\mathbb{R}^m$. Therefore, we can repeat the argument shown in Theorem 4.8 to conclude that $\phi - \psi$ is a constant in $\mathbb{R}^m$. Consequently, from the equation, $\lambda = \lambda^*$ which is a contradiction with our hypothesis.

Therefore, $\lambda = \lambda^*$ ∎

Proposition 5.2 *Suppose $\theta \geq 2$ and let (λ, ϕ) be a solution of (EP) such that ϕ is bounded from below. Then $\lambda = \lambda^*$.*

Proof Let ψ be a solution of (EP) with $\lambda = \lambda^*$ (look at Proposition 1.6). If ψ is bounded from below, then it is a direct consequence of Theorem 4.8 (uniqueness result) that $\lambda = \lambda^*$. Admit now that ψ is not bounded from below. We already know that $\lambda \leq \lambda^*$. Therefore all we have to check is that $\lambda^* \leq \lambda$.

Suppose $\lambda^* > \lambda$. Then (λ^*, ψ) is a subsolution of (EP) and, by Lemma 4.2, $z_1 = e^{-\psi(y)}$ and $z_2 = e^{-\phi(y)}$ are respectively a subsolution and a solution of (22). Translating, if necessary, ϕ and ψ in the compact set $\bar{B}_R(0)$ we can always ask that $\psi \leq \phi$ in $\bar{B}_R(0)$. In particular $\psi \leq \phi$ on $\partial B_R(0)$. We now use the hypotheses that ϕ is bounded from below in $\mathbb{R}^m$ but ψ is not, to see that $e^{-\phi}$ is a bounded function but $-e^{-\psi}$ (≤ 0) is not bounded from below. Therefore we have

$$-e^{-\psi(y)} + e^{-\phi(y)} - \delta|y|^2 \to -\infty$$

at infinity and we can repeat Proposition 4.5 to conclude that $z_1 \leq z_2$ in $B_R^c(0)$. That is, to conclude that $\psi \leq \phi$ in $B_R^c(0)$. Hence $\psi \leq \phi$ in $\mathbb{R}^m$ and repeating the argument of Theorem 4.8 we arrive at $\psi = \phi + C$ in $\mathbb{R}^m$. Thus, $\lambda = \lambda^*$ a contradiction because we assumed $\lambda^* > \lambda$. Therefore, $\lambda^* \leq \lambda$, as we wished to prove.

Conclusion, $\lambda = \lambda^*$. ∎

Remark: From the proof of these proposition, one can also conclude that (EP) with $\lambda = \lambda^*$ does not have a subsolution which is strict at some point in $\mathbb{R}^m$.

In the next Proposition f and g are two functions satisfying hypothesis $(H1)$ with the same exponent α.

Proposition 5.3 *Let $0 < \mu \leq 1$ and suppose that $(\lambda^*(f), \phi_1)$, $(\lambda^*(g), \phi_2)$ are, respectively, the unique solution pair of (EP) such that ϕ_1 is bounded from below and the unique solution pair of (EP) with $f = g$ such that ϕ_2 is bounded from below. Then,*

$$\mu\lambda^*(f) - \lambda^*(g) \leq \sup_{y \in \mathbb{R}^m} (\mu f(y) - g(y))^+$$

where $r^+ = r \vee 0 := \max(r, 0)$.

Proof Adding constants to ϕ_2 we can always assume that $\phi_2 \geq 1$. Let $q > 1$. By Lemma 4.11 we know that there exists $R > 1$, independent of q, such that $(\lambda^*(g), \phi_2^q)$, is a (strict) supersolution of (EP) with $f = g$ in $B_R^c(0)$. That is,

$$\lambda^*(g) - \frac{1}{2}\Delta\phi_2^q(y) + \frac{1}{\theta}|D\phi_2^q(y)|^\theta \geq g(y) \text{ in } B_R^c(0).$$

For this fixed R we add constants to ϕ_1 and ϕ_2 and ask that $\phi_1 \leq \phi_2$ in $\bar{B}_R(0)$.

Let $\mu \in (0, 1)$. We study the maximum of $\mu\phi_1 - \phi_2^q$. We have,

$$(\mu\phi_1 - \phi_2^q)(y) \to -\infty \text{ as } |y| \to \infty.$$

Therefore there exists a maximum point y^* of $(\mu\phi_1 - \phi_2^q)(y)$ in $B_R^c(0)$ and $y^* \notin \partial B_R(0)$ since we asked $\phi_1 \leq \phi_2$ on $\partial B_R(0)$. Thus, y^* is inside $B_R^c(0)$ and we have $\Delta(\mu\phi_1 - \phi_2^q)(y^*) \leq 0$, $D(\mu\phi_1 - \phi_2^q)(y^*) = 0$.

Since $\mu \geq \mu^\theta$,

$$\begin{aligned}\mu f(y) &= \mu\lambda^*(f) - \frac{1}{2}\Delta(\mu\phi_1(y)) + \frac{1}{\theta}\mu|D(\phi_1(y))|^\theta \\ &\geq \mu\lambda^*(f) - \frac{1}{2}\Delta(\mu\phi_1(y)) + \frac{1}{\theta}|D(\mu\phi_1(y))|^\theta\end{aligned}$$

for all $y \in \mathbb{R}^m$. Consequently we have computed at y^*

$$\begin{aligned}\mu f &\geq \mu\lambda^*(f) - \frac{1}{2}\Delta(\mu\phi_1) + \frac{1}{\theta}|D(\mu\phi_1)|^\theta \\ &\geq \mu\lambda^*(f) - \frac{1}{2}\Delta(\phi_2^q) + \frac{1}{\theta}|D(\phi_2^q)|^\theta \\ &\geq \mu\lambda^*(f) + (g - \lambda^*(g)).\end{aligned}$$

That is, we have

$$\mu\lambda^*(f) - \lambda^*(g) \leq (\mu f - g)(y^*) \leq (\mu f - g)^+$$

as we wished to prove. ∎

Comment: In many cases, the right-hand side of this estimate is $+\infty$ but when it is finite this result is very useful. Indeed, one of our main examples in subsection 10.2 uses this formula and the right-hand side is finite.

Important observation: If $\mu = 1$ in the previous proposition we can see that

$$\lambda^*(f) - \lambda^*(g) \leq \sup_{y\in\mathbb{R}^m} |f(y) - g(y)|$$

and changing the roles of $(\lambda^*(f), \phi_1)$ and $(\lambda^*(g), \phi_2)$ in the argument, we get

$$|\lambda^*(f) - \lambda^*(g)| \leq \sup_{y\in\mathbb{R}^m} |f(y) - g(y)|. \tag{34}$$

Corollary 5.4 *Suppose that, in addition to the assumptions of Proposition 5.3, we know that* $\sup_{y\in\mathbb{R}^m}\left(\left|\frac{|(f-g)(y)|}{1+|y|^\alpha}\right|\right) \leq \epsilon$. *Then*

$$|\lambda^*(f) - \lambda^*(g)| \leq K\epsilon \max(\lambda^*(f), \lambda^*(g))$$

for some constant $K > 0$.

Proof Translating if necessary f to $f + C$ and g to $g + C$ and moving accordingly $\lambda^*(f)$ and $\lambda^*(g)$, we may assume that

$$f(y) \geq a(1 + |y|^\alpha) \tag{35}$$

for some $a > 0$.

If $\sup_{y\in\mathbb{R}^m}\left(\left|\frac{|(f-g)(y)|}{1+|y|^\alpha}\right|\right) \leq \epsilon$, then choosing $\mu - 1 = -\frac{\epsilon}{a}$ we have

$$\begin{aligned}\mu f - g &= (f-g) + (\mu - 1)f \\ &= (f-g) - \frac{\epsilon}{a}f \\ &\leq (f-g) - \frac{\epsilon}{a}a(1+|y|^\alpha) \quad \text{(using (35))} \\ &\leq 0 \quad \text{(because of our hypothesis).}\end{aligned}$$

Therefore, by the previous proposition,

$$\mu\lambda^*(f) - \lambda^*(g) \leq 0.$$

That is,

$$\lambda^*(f) - \lambda^*(g) \leq \frac{\epsilon}{a}\lambda^*(f).$$

Exchanging the roles of f and g in the argument, we obtain

$$\lambda^*(g) - \lambda^*(f) \leq \frac{\epsilon}{a}\lambda^*(g).$$

Hence,

$$|\lambda^*(g) - \lambda^*(f)| \leq \frac{\epsilon}{a}\max(|\lambda^*(f), \lambda^*(g)|).$$

∎

6 Approximations of (EP)

In this section we present new convergence results for (EP). This section is mainly motivated by the singular perturbation problem that will be studied in Chapter 2.

We recall (EP)

$$\lambda - \frac{1}{2}\Delta\phi(y) + \frac{1}{\theta}|D\phi(y)|^\theta = f(y) \quad \text{in } \mathbb{R}^m$$

with f satisfying

(H1) $f \in W^{1,\infty}_{loc}(\mathbb{R}^m)$ and there exists $f_0 > 0$ and $\alpha > 0$ such that

$$-f_0 + f_0^{-1}|y|^\alpha \leq f(y) \leq f_0|y|^\alpha + f_0$$

and

$$|Df(y)| \leq f_0(1 + |y|^{\alpha-1})$$

for all $y \in \mathbb{R}^m$.

6.1 Convergence of approximations to (EP) by perturbations of f

For $R > 0$ we will consider the following ergodic problem

$$\lambda_R - \frac{1}{2}\Delta\phi_R(y) + \frac{1}{\theta}|D\phi_R(y)|^\theta = \mathcal{F}_R(y) \quad \text{in } \mathbb{R}^m \tag{36}$$

with $\mathcal{F}_R$ satisfying the following set of assumptions:

(i) $\mathcal{F}_R$ satisfies $(H1)$ with coercive growth $\alpha_1 > 0$,

(ii) $\mathcal{F}_R \uparrow f$ (f above) as $R \to +\infty$ on any compact subset of $\mathbb{R}^m$.

Remark: Observe that (ii) implies $\mathcal{F}_R \leq \mathcal{F}_{R'}$ for all $R' > R$ and $\mathcal{F}_R \leq f$ for all R. And this last one also implies $\alpha_1 \leq \alpha$.

Examples: The example that we have in mind for $\mathcal{F}_R$ is to consider $\mathcal{F}_R(y) = f(y) \wedge (f_0^{-1}|y|^{\alpha_1} + R)$ where we are using the notation $a \wedge b := \min(a, b)$.

Proposition 6.1 *Assume that $\mathcal{F}_R$ satisfies (i) above. Then, there exist a unique bounded from below solution pair (λ_R, ϕ_R) of* (36). *Moreover, we know that $\phi_R \in \Phi_{\gamma_{\alpha_1}}$, with $\gamma_{\alpha_1} = \frac{\alpha_1}{\theta} + 1$, and $\lambda_R = \lambda^*_R$.*

Proof The existence and uniqueness comes from Corollary 2.6 and Theorem 4.1 by considering (EP) given by (36). $\phi_R \in \Phi_{\gamma_{\alpha_1}}$ comes from Corollary 4.10 and $\lambda_R = \lambda_R^*$ from Section 5. ∎

Theorem 6.2 *Assume that $\mathcal{F}_R$ satisfies (i)-(ii) above. Then, there exists a sequence $R_j \to +\infty$ as $j \to +\infty$ such that the solution pair $(\lambda_{R_j}^*, \phi_{R_j})$ of (36) with $R = R_j$ and $\phi_{R_j} \in \Phi_{\gamma_{\alpha_1}}$ converges to $(\lambda^*, \phi + C)$ where (λ^*, ϕ) is the unique solution of (EP) such that ϕ is bounded from below.*

Proof In view of Theorem B.2 in Appendix B, we know that $\forall 0 < R' < R$ there exists a constant $C > 0$ depending only on R', θ and m such that

$$\sup_{B_{R'}} |D\phi_R| \leq C(1 + \sup_{B_{R'+1}} |\mathcal{F}_R(y) - \lambda_R|^{\frac{1}{\theta}} + \sup_{B_{R'+1}} |D\mathcal{F}_R(y)|^{\frac{1}{2\theta-1}}).$$

If $\alpha_1 \geq 1$ we can use directly the estimates in (i) and the fact that $\frac{\alpha_1}{\theta} = \gamma_{\alpha_1} - 1$ and $\frac{\alpha_1 - 1}{2\theta - 1} < \gamma_{\alpha_1} - 1$ to conclude

$$\sup_{B_{R'}} |D\phi_R| \leq C(1 + (R' + 1)^{\gamma_{\alpha_1} - 1})$$

for a bigger $C > 0$. Otherwise, for the second estimate we use the information that $\mathcal{F}_R$ is locally Lipchitz on any compact subset of $\mathbb{R}^m$ and so $\sup_{B_{R'+1}} |D\mathcal{F}_R|$ is finite.

In particular, we observe that $\forall R > R' > 0$, $\sup_{B_{R'}} |D\phi_R|$ is bounded by a constant not depending on R. These facts together with the classical theory for quasilinear elliptic equations, imply that the Hölder norm $|D\phi_R|_{\Gamma, B_{R'}}$ for some $\Gamma \in (0,1)$ is bounded by a constant not depending on $R > R'$. Applying Schauder's theory for quasi linear elliptic equations, we also see that the Hölder norm $|\phi_R|_{2+\Gamma, B_{R'}}$ is bounded by a constant not depending on $R > R'$. In particular, $\{\phi_R\}_{R > R'}$ is relatively compact in $C^2(\mathbb{R}^m)$, namely $\exists R_j \to +\infty$ as $j \to +\infty$ and a function $v \in C^2(\mathbb{R}^m)$ such that

$$\phi_{R_j}, D\phi_{R_j}, D^2_{R_j} \text{ converge respectively to}$$

$$v, Dv, D^2 v \text{ uniformly on any compact subset of } \mathbb{R}^m \text{ as } j \to +\infty.$$

Since $\mathcal{F}_{R_j} \leq f$ and $\mathcal{F}_{R_j} \leq \mathcal{F}_{R_{j+1}}$ for all j, Proposition 1.8 gives that

- $\lambda_{R_j}^* \leq \lambda^* \ \forall j$,
- $\lambda_{R_j}^* \leq \lambda_{R_{j+1}}^*, \ \forall j$.

Hence $\{\lambda^*_{R_j}\}_j$ is monotone and bounded and so it converges to a constant: $\lambda^*_{R_j} \to c \in \mathbb{R}$.

We have that

$$\lambda^*_{R_j} - \frac{1}{2}\Delta\phi_{R_j}(y) + \frac{1}{\theta}|D\phi_{R_j}(y)|^\theta = \mathcal{F}_{R_j}(y)$$

converges as $j \to +\infty$ to

$$c - \frac{1}{2}\Delta v(y) + \frac{1}{\theta}|Dv(y)|^\theta = f(y)$$

where we use (ii).

Since $\phi_{R_j} \in \Phi_{\gamma_{\alpha_1}}$ for all j we can see that $v := \lim_{j\to+\infty}\phi_{R_j}$ belongs to Φ_{γ_1}. In particular, v is a bounded from below solution of (EP) and therefore, by corollary 4.10, it has to belong to Φ_γ. Hence, v is in the right class for which we have uniqueness for (EP) and we can conclude $v = \phi + C$ and $c = \lambda^*$ (Theorem 4.1). ∎

6.2 Convergence of approximations to (EP) by restrictions to balls

We recall (19)

$$\begin{cases} \lambda_R - \frac{1}{2}\Delta\phi_R(y) + \frac{1}{\theta}|D\phi_R(y)|^\theta = f_R(y) & \text{in } B_R(0) \\ \phi_R(y) \to +\infty \text{ as } y \to \partial B_R(0) \end{cases} \tag{37}$$

for $1 < \theta \le 2$

and (21)

$$\begin{cases} \lambda_R - \frac{1}{2}\Delta\phi_R(y) + \frac{1}{\theta}|D\phi_R(y)|^\theta = f_R(y) & \text{in } B_R(0) \\ \lambda_R - \frac{1}{2}\Delta\phi_R(y) + \frac{1}{\theta}|D\phi_R(y)|^\theta \ge f_R(y) & \text{on } \partial B_R(0) \end{cases} \tag{38}$$

for $\theta > 2$.

And in both cases we will assume that $f_R = f|_{B_R(0)}$ with f as in the beginning of this section.

Because we are considering such f_R we can use all the results of section 3.

Proposition 6.3 *Let (λ, ϕ) be any solution of (EP) and let (λ_R, ϕ_R) be the solution of* (37) *(respectively of* (38)*) given by Theorem 3.2 (respectively Theorem 3.6). Then, $\lambda \le \lambda_R$.*

Proof The proof is identical to the proof of Proposition 3.4 ($1 < \theta \leq 2$) or Proposition 3.7 ($\theta > 2$) with $\phi_{R'}$ replaced by ϕ. ∎

Theorem 6.4 *There exists a sequence $R_j \to +\infty$ as $j \to +\infty$ such that the solution pair $(\lambda_{R_j}, \phi_{R_j})$ of* (37) *(respectively* (38)*) given by Theorem 3.2 (respectively Theorem 3.6) converges to $(\lambda^*, \phi+C)$ where (λ^*, ϕ) is the unique solution of (EP) such that ϕ is bounded from below.*

Remember: By Propositions 3.5 and 3.8,

$$\lambda_R = \sup\{a \in \mathbb{R} \mid \exists \psi \in C(\bar{B}_R(0)) \text{ with } a - \frac{1}{2}\Delta\psi + \frac{1}{\theta}|D\psi|^\theta \leq f_R \text{ in } B_R(0)\}.$$

Proof By Theorem 3.1, we know that $\forall 0 < R' < R$ there exists a constant C depending only on bounds on f_R, upper bounds on $f_R - \lambda_R$ and θ such that

$$\sup_{B_{R'}} |D\phi_R| \leq C.$$

In particular, $\forall R > R' > 0$ $\sup_{B_{R'}} |D\phi_R|$ is bounded by a constant not depending on R and thus $|\phi_R|_{2+\Gamma, B_{R'}}$ by the standard regularity arguments and Schauder's estimates. Since $\{\phi_R\}_{R>R'}$ is pre compact in $C^2(B_R(0))$, we can apply Ascoli-Arzela's Theorem to conclude that $\exists R_j \to +\infty$ as $j \to +\infty$ and a function $v \in C^2(\mathbb{R}^m)$ such that ϕ_{R_j}, $D\phi_{R_j}$, $D^2\phi_{R_j}$ converge respectively to v, Dv, D^2v uniformly on any compact subset of $\mathbb{R}^m$ as $j \to \infty$.

By Proposition 3.4 (respectively 3.7) we have that $\lambda_{R_{j+1}} \leq \lambda_{R_j}$ for all j and by Proposition 6.3 $\lambda^* \leq \lambda_{R_j}$ for all j. Therefore, it exists ${R'}_j$ a subsequence of R_j such that $\lambda_{{R'}_j} \to c \in \mathbb{R}$ as $j \to +\infty$ and we have that

$$\lambda_{{R'}_j} - \frac{1}{2}\Delta\phi_{{R'}_j}(y) + \frac{1}{\theta}|D\phi_{{R'}_j}(y)|^\theta = f_{R_j}(y)$$

converges, as ${R'}_j \to +\infty$ ($j \to +\infty$), to

$$c - \frac{1}{2}\Delta v + \frac{1}{\theta}|Dv|^\theta = f(y).$$

We now wish to prove that $(c, v) = (\lambda^*, \phi + C)$ and for that we need to know that $v \in C^2(\mathbb{R}^m)$ is bounded from below.

By Lemma 3.3 (respectively Theorem 3.6) we know that $\phi_{R'_j} \in C^2_{\text{loc}}(B_{R_j}(0))$. Since $\phi_{R'_j} \in C^2_{\text{loc}}(B_{R_j}(0))$ and, when $1 < \theta \leq 2$, the solution blows up on the boundary, we can conclude that the minimum point of $\phi_{R'_j}$, call it $y^*_{R'_j}$, is achieved inside $B_{R'_j}(0)$. (When $\theta > 2$, the minimum cannot be on the boundary due to the boundary condition in the viscosity solution's sense- see [31]). Therefore, we have

$$D\phi_R(y^*_{R'_j}) = 0 \text{ and } \Delta\phi_R(y^*_{R'_j}) \geq 0$$

and the equation implies $f_{R'_j}(y^*_{R'_j}) - \lambda_{R'_j} \leq 0 \Leftrightarrow f_{R'_j}(y^*_{R'_j}) \leq \lambda_{R'_j}$. But $\lambda_{R'_j}$ are bounded (decreasing sequence converging to c) and f tends to $+\infty$ at infinity, therefore $y^*_{R'_j}$ remains in a bounded region of $\mathbb{R}^m$ as $j \to +\infty$. By compactness of $\phi_{R'_j}$ (j large) in bounded regions, there exist a subsequence of R'_j, that we will still denote by R'_j, such that $\phi_{R'_j}(y^*_{R'_j}) \to w(y^*)$ as $j \to +\infty$. Therefore, $v(y) := \lim_{j\to+\infty} \phi_{R'j}(y) \geq \lim_{j\to+\infty} \phi_{R'j}(y^*_{R'_j}) = w(y^*)$ is bounded from below.

Since $v \in C^2(\mathbb{R}^m)$ and v is bounded from below, we can apply Theorem 4.1. Hence, $v = \phi + C$ and $c = \lambda^*$ as we wished to prove. ∎

Chapter II
The Singular Perturbation Problem

This chapter is organised as follows. In section 7 we present our singular perturbation problem for a class of optimal stochastic control problems and the HJB equation associated to it. Section 8 studies the initial value problem satisfied by V^ϵ. Section 9 is devoted to the effective Hamiltonian and its properties. In section 10 we prove our main result, Theorem 10.1, and we give some examples on the convergence of V^ϵ to the solution of the effective Cauchy problem.

7 The stochastic control system

7.1 The two-scale system

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a complete filtered probability space on which is defined an $(\mathcal{F}_t)$-adapted standard Brownian motion $(W_t)_{t\geq 0}$ in $\mathbb{R}^d$, $d \geq m$. We consider stochastic control systems with small parameter $\epsilon > 0$ of the form:

$$\begin{cases} dX_s = F(X_s, Y_s, u_s)ds + \sqrt{2}\sigma(X_s, Y_s, u_s)dW_s, & X_{s_0} = x \in \mathbb{R}^n \\ dY_s = -\frac{1}{\epsilon}\xi_s ds + \sqrt{\frac{1}{\epsilon}}\tau(Y_s)dW_s, & Y_{s_0} = y \in \mathbb{R}^m. \end{cases} \tag{39}$$

It is a model of systems where some state variables, Y_s here, evolve at a much faster time scale than the other variables, X_s. Passing to the limit as $\epsilon \to 0^+$ is a classical singular perturbation problem. Here u and ξ are two different control processes. u is a control process affecting only the slow variables X_s and taking values in a given compact set U, subset of a separable complete normed space (for example $\mathbb{R}^k$), while ξ is a control process taking its values in $\mathbb{R}^m$ and driving only the fast variables Y_s.

The first equation is a general *stochastic differential equation* (SDE):

$$dX_s = F(X_s, Y_s, u_s)ds + \sqrt{2}\sigma(X_s, Y_s, u_s)dW_s, \quad X_{s_0} = x \in \mathbb{R}^n. \tag{40}$$

This is a shorthand way of writing

$$X_s = x + \int_{s_0}^{s} F(X_\tau, Y_\tau, u_\tau)d\tau + \sqrt{2}\int_{s_0}^{s} \sigma(X_\tau, Y_\tau, u_\tau)dW_\tau. \tag{41}$$

Here the drift F and diffusion coefficient σ are measurable functions. The first integral in (41) stands for a Lebesgue-Stieltjes Integral while the second one stands for a Stochastic Integral.

The second equation in (39),

$$dY_s = -\frac{1}{\epsilon}\xi_s ds + \sqrt{\frac{1}{\epsilon}}\tau(Y_s)dW_s, \quad Y_{s_0} = y \in \mathbb{R}^m, \tag{42}$$

is a stochastic differential equation that translates the situation of a moving particle starting in y at instant s_0 for which we can control the drift ξ without constraints but that it is affected by a "noise" perturbing its trajectory. Such type of equations appear naturally in many models of Physics as well as Mathematical Finance. Minimisation problems for functionals associated to such type of equations fall in the category of optimal stochastic control problems or stochastic calculus of variations. In this chapter we will deal with such problems.

On the fast process Y_s we will assume that the matrix $\tau = \left[\begin{array}{cc} \mathbb{I}_m & O_{m\times(d-m)} \end{array} \right]$ (recall that $d \geq m$). Hence $\tau\tau^T = \mathbb{I}_m$ and the matrix $\tau\tau^T$ is positive definite. No non-degeneracy assumption on the matrix σ will be imposed.

We now define the set of admissible control processes for u and ξ. We remind that $t \in [0, T]$.

Definition (admissible control processes ξ) ξ progressively measurable process taking values in $\mathbb{R}^m$ is admissible if for all compact $K \in \mathbb{R}^m$, there exists $C_K > 0$ such that

$$\begin{aligned} &\mathbb{E}\big[(|\xi|^{\theta^*} + |Y^\xi|^{\theta(\alpha-1)\vee\alpha})ds|Y^\xi_t = y\big] \\ &=: \mathbb{E}^y\big[(|\xi|^{\theta^*} + |Y^\xi|^{\theta(\alpha-1)\vee\alpha})ds\big] \leq C_K \end{aligned}$$

for all $y \in \mathbb{R}^m$ and where Y^ξ is governed by (42) with $s_0 = t$ and we are using the notation $a \vee b = \max(a, b)$.

We denote by $\mathcal{A}$ the totality of admissible control processes ξ.

Definition (admissible control processes u) We say that u is an admissible control process if it is a progressively measurable process taking values in U.

We denote by $\mathcal{U}$ the totality of admissible control processes u taking values in U.

Remarks:

1. The first definition is not standard. It comes from [25] and is motivated by the unbounded nature of the problem.

2. $\mathcal{A}$ is not empty.

   Indeed, consider $\xi = 0$. Then it is an easy consequence of the known result that any Brownian motion has finite moments of all orders and $\|\tau\|$ is bounded (since τ is a constant matrix) that $\xi = 0 \in \mathcal{A}$.

   Changing slightly the argument we can also see that ξ bounded implies $\xi \in \mathcal{A}$.

7.2 The optimal control problem

For a given $\theta^* > 1$, we consider Payoff Functionals for $t \in [0,T]$ of the form

$$J^\epsilon(t,x,y,u,\xi) = \mathbb{E}[\int_t^T (l(X_s,Y_s,u_s) + \frac{1}{\theta^*}|\xi_s|^{\theta^*})ds + g(X_T,Y_T)|X_t = x, Y_t = y] \tag{43}$$

$$\Big(=: \mathbb{E}^{x,y}[\int_t^T (l(X_s,Y_s,u_s) + \frac{1}{\theta^*}|\xi_s|^{\theta^*})ds + g(X_T,Y_T)] \Big).$$

Our optimal control problem consists in determining the optimal trajectories (X,Y) which minimize $J^\epsilon(t,x,y,u,\xi)$ among admissible controls u and ξ and all solutions (X,Y) of (39) satisfying the initial condition $X_t = x$ and $Y_t = y$.

Therefore the Value Function of the terminal value optimal control problem is:

$$V^\epsilon(t,x,y) = \inf_{u,\xi} J^\epsilon(t,x,y,u,\xi), \quad 0 \le t \le T \tag{44}$$

In our definition of Payoff Functional, l and $\frac{1}{\theta^*}|\xi_s|^{\theta^*}$ represent running costs associated respectively to the controls u and ξ. They mean the amount of expend to perform some desire task. Our running costs will be continuous functions having specific growth conditions (see Section 8).

7.3 The HJB equation

In this subsection let L be $L(x,y,u,\xi) := l(x,y,u) + \frac{1}{\theta^*}|\xi|^{\theta^*}$.

The HJB equation associated via Dynamic Programming to the value function V^ϵ is

$$-V_t^\epsilon + \mathcal{H}\big(x,y,D_xV^\epsilon, \frac{D_yV^\epsilon}{\epsilon}, D_{xx}^2V^\epsilon, \frac{D_{yy}^2V^\epsilon}{\epsilon}, \frac{D_{xy}^2V^\epsilon}{\sqrt{\epsilon}}\big) = 0 \text{ in } (0,T)\times\mathbb{R}^n\times\mathbb{R}^m$$

complemented with the obvious terminal condition

$$V^\epsilon(T,x,y) = g(x,y).$$

This is a fully nonlinear degenerate parabolic equation.

The Hamiltonian $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^n \times \mathbb{S}^m \times \mathbb{M}^{n,m} \to \mathbb{R}$ is defined as

$$\mathcal{H}(x,y,p,q,M,N,Z) := H(x,y,p,M,Z) - \frac{1}{2}\text{trace}(N) + h(q) \tag{45}$$

where

$$H(x,y,p,M,Z) := \sup_{u\in U}\{-\text{trace}(\sigma\sigma^T M) - F\cdot p - \sqrt{2}\text{trace}(\sigma\tau^T Z^T) - l\} \quad (46)$$

and

$$h(q) := \sup_{\xi}\{\xi\cdot q - \frac{1}{\theta^*}|\xi|^{\theta^*}\} = \frac{1}{\theta}|q|^{\theta}, \quad \frac{1}{\theta^*}+\frac{1}{\theta} = 1, \quad (47)$$

is the Legendre transform associated to the cost $\frac{1}{\theta^*}|\xi|^{\theta^*}$.

Justification:

Observation: We would like to point out that the justification that we present here is not rigorous. We don't know for now if the value function V^ϵ has enough regularity to apply the argument of the Dynamic Programming Principle. This will be done in the next section.

For a fixed control $u \in U$ the generator of the diffusion process (39) is

$$\text{trace}(\sigma\sigma^T D^2_{xx}) + \sqrt{\frac{2}{\epsilon}}\text{trace}(\sigma\tau^T\big(D^2_{xy}\big)^T) + F\cdot D_x + L$$
$$+ \frac{1}{2\epsilon}\text{trace}\big(\tau\tau^T D^2_{yy}\big) + \frac{1}{\epsilon}(-\xi)\cdot D_y$$

$$= \text{trace}(\sigma\sigma^T D^2_{xx}) + \sqrt{2}\text{trace}(\sigma\tau^T\big(\frac{D^2_{xy}}{\sqrt{\epsilon}}\big)^T) + F\cdot D_x +$$
$$l + \frac{1}{\theta^*}|\xi|^{\theta^*} + \frac{1}{2\epsilon}\Delta_y + (-\xi)\cdot\frac{D_y}{\epsilon}$$

since $\tau\tau^T = \mathbb{I}_m$.

To derive the HJB equation associated via Dynamic Programming to the value function in this case we first maximise in u

$$-V^\epsilon_t + \sup_{u\in U}\{-\text{trace}(\sigma\sigma^T D^2_{xx}V^\epsilon) - F\cdot D_x V^\epsilon - \sqrt{2}\text{trace}(\sigma\tau^T\big(\frac{D^2_{xy}V^\epsilon}{\sqrt{\epsilon}}\big)^T) - l - \frac{1}{\theta^*}|\xi|^{\theta^*}\}$$
$$-[-\xi\cdot\frac{D_y V^\epsilon}{\epsilon} + \frac{1}{2\epsilon}\Delta_y V^\epsilon] = 0 \Leftrightarrow$$
$$-V^\epsilon_t + \sup_{u\in U}\{-\text{trace}(\sigma\sigma^T D^2_{xx}V^\epsilon) - F\cdot D_x V^\epsilon - \sqrt{2}\text{trace}(\sigma\tau^T\big(\frac{D^2_{xy}V^\epsilon}{\sqrt{\epsilon}}\big)^T) - l\} - \frac{1}{\theta^*}|\xi|^{\theta^*}$$
$$+\xi\cdot\frac{D_y V^\epsilon}{\epsilon} - \frac{1}{2\epsilon}\Delta_y V^\epsilon = 0, \qquad \text{(because } -\frac{1}{\theta^*}|\xi|^{\theta^*} \text{ doesn't depend on } u\text{)}$$

Next we maximise in ξ and the HJB equation associated via Dynamic Programming to the value function of our control problem becomes

$$\begin{cases} -V_t^\epsilon + H(x,y,D_xV^\epsilon, D_{xx}^2V^\epsilon, \frac{D_{xy}^2V^\epsilon}{\sqrt{\epsilon}}) - \frac{1}{2\epsilon}\Delta_y V^\epsilon + \sup_\xi\{\xi\cdot\frac{D_yV^\epsilon}{\epsilon} - \frac{1}{\theta^*}|\xi|^{\theta^*}\} = 0 \\ V^\epsilon(T,x,y) = g(x,y) \end{cases}$$

Claim: Now we wish to prove that $h\big(\frac{D_yV^\epsilon}{\epsilon}\big) := \sup_\xi\{\xi\cdot\frac{D_yV^\epsilon}{\epsilon} - \frac{1}{\theta^*}|\xi|^{\theta^*}\}$ is also equal to $\frac{1}{\theta}\big|\frac{D_yV^\epsilon}{\epsilon}\big|^\theta$ where $\frac{1}{\theta^*}+\frac{1}{\theta}=1$.

For this, define $\varphi(\xi) := \xi\cdot\frac{D_yV^\epsilon}{\epsilon} - \frac{1}{\theta^*}|\xi|^{\theta^*}$. We have,

$$D_\xi\varphi(\xi) = 0 \Leftrightarrow \frac{D_yV^\epsilon}{\epsilon} - \frac{1}{\theta^*}\theta^*|\xi|^{\theta^*-1}\frac{\xi}{|\xi|} = 0 \Leftrightarrow \frac{D_yV^\epsilon}{\epsilon} = \frac{|\xi|^{\theta^*-1}}{|\xi|}\xi \tag{48}$$

then

$$(48) \Rightarrow \big|\frac{D_yV^\epsilon}{\epsilon}\big| = |\xi|^{\theta^*-1} \Rightarrow \big|\frac{D_yV^\epsilon}{\epsilon}\big|^\theta = |\xi|^{\theta(\theta^*-1)} = |\xi|^\theta \quad \text{because} \quad \frac{1}{\theta^*}+\frac{1}{\theta} = 1.$$

Substituting the implicit relation (48) in φ, we get

$\xi\cdot\frac{|\xi|^{\theta^*-1}}{|\xi|}\xi - \frac{1}{\theta^*}|\xi|^{\theta^*} = |\xi|^{\theta^*} - -\frac{1}{\theta^*}|\xi|^{\theta^*} = (1-\frac{1}{\theta^*})|\xi|^{\theta^*} = \frac{1}{\theta}\big|\frac{D_yV^\epsilon}{\epsilon}\big|^\theta.$

Therefore

$$h\big(\frac{D_yV^\epsilon}{\epsilon}\big) = \frac{1}{\theta}\big|\frac{D_yV^\epsilon}{\epsilon}\big|^\theta.$$

Conclusion:

$$\begin{cases} -V_t^\epsilon + H(x,y,D_xV^\epsilon, D_{xx}^2V^\epsilon, \frac{D_{xy}^2V^\epsilon}{\sqrt{\epsilon}}) - \frac{1}{2\epsilon}\Delta_y V^\epsilon + \frac{1}{\theta}|\frac{D_yV^\epsilon}{\epsilon}|^\theta = 0 \text{ in } (0,T)\times\mathbb{R}^n\times\mathbb{R}^m \\ V^\epsilon(T,x,y) = g(x,y). \end{cases}$$

We close this section recalling the duality between $\frac{1}{\theta^*}|\xi|^{\theta^*}$ and h defined in (47).

Proposition 7.1 *(Convex duality)* $\frac{1}{\theta^*}|\xi|^{\theta^*} + h(q) \geq \xi\cdot q$ *for all* $\xi, q \in \mathbb{R}^m$ *where* h *is defined in* (47)*. Moreover, the equality hold if and only if* $\xi = D_qh(q)$*.*

Proof See, for example, [15, Theorem A.2.5]. ∎

8 The Cauchy Problem for the HJB equation

We would like to characterise the value function V^ϵ as the unique continuous viscosity solution under some growth to the parabolic problem with terminal data

$$\begin{cases} -V_t^\epsilon + H(x,y,D_xV^\epsilon, D_{xx}^2V^\epsilon, \frac{D_{xy}^2V^\epsilon}{\sqrt{\epsilon}}) - \frac{1}{2\epsilon}\Delta_y V^\epsilon + \frac{1}{\theta}|\frac{D_yV^\epsilon}{\epsilon}|^\theta = 0 \text{ in } (0,T)\times\mathbb{R}^n\times\mathbb{R}^m \\ V^\epsilon(T,x,y) = g(x,y). \end{cases} \tag{49}$$

For that reason we will present a variant of a result shown in [7, 2010]. The main difficulty now is that the cost $\frac{1}{\theta^*}|\xi|^{\theta^*}$ has super linear growth ($\theta^* > 1$). The idea then is to adapt their proof using a result of Da-Lio and Ley for convex Hamilton-Jacobi equations under super linear growth conditions on data (see [17, 2011]).

8.1 Assumptions

We start by discussing the conditions on the data.

In the following we will assume:

(A1) U is a compact subset of a separable complete normed space;

(A2) $F : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{M}^{n,d}$ are Lipschitz functions in (x,y) uniformly w.r.t. u;

(A3) For some $C > 0$, $|F(x,y,u)| + \|\sigma(x,y,u)\| \le C(1+|x|)$;

(A4) $l : \mathbb{R}^n \times \mathbb{R}^m \times U \to \mathbb{R}$ is continuous and there exist $l_0 > 0$ and $\alpha > 1$ such that for all $(x,y,u) \in \mathbb{R}^n \times \mathbb{R}^m \times U$, we have

$$-l_0 + l_0^{-1}|y|^\alpha \le l(x,y,u) \le l_0(1+|y|^\alpha)$$

and for every $R > 0$, there exists a modulus of continuity m_R such that for all $x, x' \in B_R(0)$ and $u \in U$

$$|l(x,y,u) - l(x',y,u)| \le (1+|y|^\alpha)m_R(|x-x'|);$$

(A5) $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous and there exists $C_g > 0$ such that $|g(x,y)| \le C_g$ for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$ (i.e, g is bounded).

Remarks:

- Under the current hypotheses, the Hamiltonian H is finite (which might not happen if the diffusion coefficient σ depend on the control u, look at Example 2.1 of [17, 2011]). Note that H is convex with respect to the third variable.
- Observe that (A2) implies that F and σ are continuous functions in (x, y) uniformly w.r.t. u.
- l is bounded on x and u.

8.2 A subsolution and a supersolution for V^ϵ

Proposition 8.1 *(A subsolution for V^ϵ) For any $\beta_1 \in [0, \alpha \wedge \gamma]$ there exists $\rho_1 \in (0, 1)$ such that*

$$v^\epsilon(t, x, y) = (T - t)(\epsilon\rho_1(1 + |y|^2)^{\frac{\beta_1}{2}} - 2f_0) - C_g \tag{50}$$

is a classical subsolution of (49) *where C_g is the constant appearing in* $(A5)$ *and $\gamma = \frac{\alpha}{\theta} + 1$. Moreover, we have $-C < v^\epsilon \leq V^\epsilon$ with V^ϵ defined in* (44).

Proof We first notice that we can write (49) as

$$\begin{cases} -V^\epsilon_t - \frac{1}{2\epsilon}\Delta_y V^\epsilon + \frac{1}{\theta}|\frac{D_y V^\epsilon}{\epsilon}|^\theta = f(x, y, D_x V^\epsilon, D^2_{xx}V^\epsilon, \frac{D^2_{xy}V^\epsilon}{\sqrt{\epsilon}}) \quad \text{in} \quad Q := (0, T) \times \mathbb{R}^n \times \mathbb{R}^m \\ V^\epsilon(T, x, y) = g(x, y) \end{cases}$$

with $f(x, y, D_x V^\epsilon, D^2_{xx}V^\epsilon, \frac{D^2_{xy}V^\epsilon}{\sqrt{\epsilon}}) := -H(x, y, D_x V^\epsilon, D^2_{xx}V^\epsilon, \frac{D^2_{xy}V^\epsilon}{\sqrt{\epsilon}})$.

We wish to prove that

$$v^\epsilon(T, x, y) \leq g(x, y)$$

and

$$-v^\epsilon_t - \frac{1}{2\epsilon}\Delta_y v^\epsilon + \frac{1}{\theta}|\frac{D_y v^\epsilon}{\epsilon}|^\theta \leq f(x, y, D_x v^\epsilon, D^2_{xx}v^\epsilon, \frac{D^2_{xy}v^\epsilon}{\sqrt{\epsilon}}).$$

For that, notice that, by the definition of f and the fact that v^ϵ does not depend on x, that f inherit the coercive growth of l in $(A4)$. That is, we have

$$f_0^{-1}|y|^\alpha - f_0 \leq f(x, y, D_x v^\epsilon, D^2_{xx}v^\epsilon, \frac{D^2_{xy}v^\epsilon}{\sqrt{\epsilon}}) = f(x, y, 0, 0, 0) \leq f_0(1 + |y|^\alpha)$$

for some $f_0 > 0$ and for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$.

If $\beta_1 = 0$ there is not so much to show. Indeed, in this case, $v^\epsilon(t,x,y) = (T-t)(\epsilon\rho_1 - 2f_0) - C_g$ and then $v^\epsilon(T,x,y) = -C_g \le g(x,y)$ and $-v_t^\epsilon - \frac{1}{2\epsilon}\Delta_y v^\epsilon + \frac{1}{\theta}|\frac{D_y v^\epsilon}{\epsilon}|^\theta = (\epsilon\rho_1 - 2f_0)$ which is less or equal to $-f_0 + f_0^{-1}|y|^\alpha$ by taking any $\rho_1 \in (0,1)$ small enough such that $\epsilon\rho_1 - f_0 \le 0$.

If $\beta_1 > 0$, by (50) v^ϵ satisfies

$$v^\epsilon(T,x,y) = -C_g,$$

$$-v_t^\epsilon = \epsilon\rho_1(1+|y|^2)^{\frac{\beta_1}{2}} - 2f_0,$$
$$D_y v^\epsilon = (T-t)\epsilon\beta_1\rho_1(1+|y|^2)^{\frac{\beta_1-2}{2}}y$$

and

$$\Delta_y v^\epsilon = (T-t)\epsilon\beta_1\rho_1[(\beta_1-2)|y|^2 + m(1+|y|^2)](1+|y|^2)^{\frac{\beta_1-4}{2}}.$$

Hence,

$$\begin{aligned}
&-v_t^\epsilon - \frac{1}{2\epsilon}\Delta_y v^\epsilon + \frac{1}{\theta}|\frac{D_y v^\epsilon}{\epsilon}|^\theta \\
&= \epsilon\rho_1(1+|y|^2)^{\frac{\beta_1}{2}} - 2f_0 - \frac{1}{2\epsilon}(T-t)\epsilon\beta_1\rho_1[(\beta_1-2)|y|^2 + m(1+|y|^2)](1+|y|^2)^{\frac{\beta_1-4}{2}} \\
&+ \frac{1}{\theta}|\frac{(T-t)(\epsilon\beta_1\rho_1(1+|y|^2)^{\frac{\beta_1-2}{2}}y}{\epsilon}|^\theta \\
&\le C(\rho_1)(1+|y|^{\beta_1} + |y|^{\beta_1-2} + |y|^{\theta(\beta_1-1)}) - 2f_0
\end{aligned}$$

for some constant $C(\rho_1)$ that depends linearly on ρ_1 as $\rho_1 \to 0$. Since $\beta_1 \le \alpha \wedge \gamma$ then $\beta_1 - 2 \le \gamma - 2 \le \alpha$ and $\beta_1 \le \alpha$ and we arrive at

$$-v_t^\epsilon - \frac{1}{2\epsilon}\Delta_y v^\epsilon + \frac{1}{\theta}|\frac{D_y v^\epsilon}{\epsilon}|^\theta \le C(\rho_1)(1+3|y|^\alpha) - 2f_0 = \big(C(\rho_1) - 2f_0\big) + 3C(\rho_1)|y|^\alpha.$$

Since $C(\rho_1) \to 0$ as $\rho_1 \to 0$, we can choose $\rho_{10} \in (0,1)$ small enough such that $C(\rho_{10}) \le \min(\frac{f_0^{-1}}{3}, f_0)$ and have

$$\big(C(\rho_{10}) - 2f_0\big) + 3C(\rho_{10})|y|^\alpha \le -f_0 + f_0^{-1}|y|^\alpha \le f(x,y,0,0,0).$$

That is,

$$-v_t^\epsilon - \frac{1}{2\epsilon}\Delta_y v^\epsilon + \frac{1}{\theta}|\frac{D_y v^\epsilon}{\epsilon}|^\theta \le f(x,y,D_x v^\epsilon, D_{xx}^2 v^\epsilon, \frac{D_{xy}^2 v^\epsilon}{\sqrt{\epsilon}}). \tag{51}$$

Therefore, using (51) and $v^\epsilon(T,x,y) \le g(x,y)$ for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, we can conclude that v^ϵ is a subsolution of (49) for every $\rho_1 \le \rho_{1_0}$. Since v^ϵ is bounded from below, we have $-C < v^\epsilon$ for some $C \in \mathbb{R}$.

We will now prove that any of such v^ϵ's satisfy $v^\epsilon \le V^\epsilon$ with V^ϵ defined in (44). For that, fix any $\xi \in \mathcal{A}$ and $u \in \mathcal{U}$. By Proposition 7.1,

$$\frac{1}{\theta^*}|\xi|^{\theta^*} + \frac{1}{\theta}\Big|\frac{D_y v^\epsilon}{\epsilon}\Big|^\theta \ge \xi \cdot \frac{D_y v^\epsilon}{\epsilon}. \tag{52}$$

Applying Itô's formula (Appendix A) to $v^\epsilon(s, Y_s^\xi) := v^\epsilon(s,x,Y_s^\xi)$ (since it is independ on x), we have

$$\begin{aligned} v^\epsilon(T\wedge\tau_R, Y^\xi_{T\wedge\tau_R}) = v^\epsilon(t,y)& \\ + \int_t^{T\wedge\tau_R} &\big(v_s d_s + D_y v \cdot dY_s^\xi\big) \\ + \frac{1}{2}\int_t^{T\wedge\tau_R} &\mathrm{trace}(D^2_{yy}v^\epsilon) d < Y^\xi >_s \end{aligned}$$

where $< Y^\xi >_s$ denotes the quadratic variation of Y_s^ξ, see Appendix A for the definition. Therefore,

$$\begin{aligned} v^\epsilon(T\wedge\tau_R,&x, Y^\xi_{T\wedge\tau_R}) + \int_t^{T\wedge\tau_R}\big(\frac{1}{\theta^*}|\xi_s|^{\theta^*} + f(X_s^{u,\xi}, Y_s^\xi)\big)ds \\ &= v^\epsilon(t,x,y) + \int_t^{T\wedge\tau_R} v^\epsilon_s ds + \int_t^{T\wedge\tau_R} D_y v^\epsilon \cdot (-\frac{1}{\epsilon}\xi_s ds + \sqrt{\frac{1}{\epsilon}}\tau(Y_s)dW_s) \\ &+ \frac{1}{2\epsilon}\int_t^{T\wedge\tau_R}\mathrm{trace}(D^2_{yy}v^\epsilon)\tau(Y_s^\xi)\tau^T(Y_s^\xi)ds + \int_t^{T\wedge\tau_R}\big(\frac{1}{\theta^*}|\xi_s|^{\theta^*} + f(X_s^{u,\xi}, Y_s^\xi)\big)ds \\ &= v^\epsilon(t,x,y) + \int_t^{T\wedge\tau_R} v^\epsilon_s ds + \int_t^{T\wedge\tau_R} -\frac{1}{\epsilon}\xi_s \cdot D_y v^\epsilon ds + \sqrt{\frac{1}{\epsilon}}\int_t^{T\wedge\tau_R} D_y v^\epsilon \tau(Y_s)dW_s \\ &+ \frac{1}{2\epsilon}\int_t^{T\wedge\tau_R}\Delta_y v^\epsilon ds + \int_t^{T\wedge\tau_R}\big(\frac{1}{\theta^*}|\xi_s|^{\theta^*} + f(X_s^{u,\xi}, Y_s^\xi)\big)ds \end{aligned}$$

since $\tau\tau^T = \mathbb{I}$ and where we are using the notation $f(x,y)$ to say $f(x,y,0,0,0) = f(x,y,D_x v^\epsilon, D^2_{xx}v^\epsilon, \frac{D^2_{xy}v^\epsilon}{\sqrt{\epsilon}})$. Then, noting the subsolution property for v^ϵ ,

$$\frac{1}{2\epsilon}\Delta_y v^\epsilon \ge \frac{1}{\theta}\Big|\frac{D_y v^\epsilon}{\epsilon}\Big|^\theta - f(x,y) - v^\epsilon_t,$$

and (52), we can see that

$$v^\epsilon(T\wedge\tau_R,x,Y^\xi_{T\wedge\tau_R})+\int_t^{T\wedge\tau_R}\big(\frac{1}{\theta^*}|\xi_s|^{\theta^*}+f(X^{u,\xi}_s,Y^\xi_s)\big)ds$$
$$\geq v^\epsilon(t,x,y)+\sqrt{\frac{1}{\epsilon}}\int_t^{T\wedge\tau_R}D_yv^\epsilon\tau(Y_s)dW_s.$$

Taking the expectation and observing that $M_t:=\int_t^{T\wedge\tau_R}D_yv^\epsilon\tau(Y_s)dW_s$ is a martingale, we get

$$\mathbb{E}^{x,y}\big[v^\epsilon(T\wedge\tau_R,x,Y^\xi_{T\wedge\tau_R})+\int_t^{T\wedge\tau_R}\big(\frac{1}{\theta^*}|\xi_s|^{\theta^*}+f(X^{u,\xi}_s,Y^\xi_s)\big)ds\big]\geq v^\epsilon(t,x,y)$$

for all admissible controls $\xi\in\mathcal{A}$ and $u\in\mathcal{U}$.

We now send $R\to\infty$ and use the fact that $f(x,y,0,0,0)$ is bounded from below, to conclude that

$$\mathbb{E}^{x,y}\big[v^\epsilon(T,x,Y^\xi_T)+\int_t^T\big(\frac{1}{\theta^*}|\xi_s|^{\theta^*}+f(X^{u,\xi}_s,Y^\xi_s)\big)ds\big]\geq v^\epsilon(t,x,y)$$

for all admissible controls $\xi\in\mathcal{A}$ and $u\in\mathcal{U}$.

But $v^\epsilon(T,x,Y^\xi_T)\leq g(X^{u,\xi}_T,Y^\xi_T)$ and $f(X^{u,\xi}_s,Y^\xi_s)\leq l(X^{u,\xi}_s,Y^\xi_s,u_s)$. Thus, by monotonicity,

$$\mathbb{E}^{x,y}\big[g(X^{u,\xi}_T,Y^\xi_T)+\int_t^T\big(\frac{1}{\theta^*}|\xi_s|^{\theta^*}+l(X^{u,\xi}_s,Y^\xi_s,u_s)\big)ds\big]\geq v^\epsilon(t,x,y)$$

for all admissible controls $\xi\in\mathcal{A}$ and $u\in\mathcal{U}$. Taking the $\inf_{u\in\mathcal{U},\xi\in\mathcal{A}}$, we obtain $v^\epsilon\leq V^\epsilon$ as we wished to prove. ∎

Proposition 8.2 *(A supersolution for V^ϵ) For any $\beta_2\in[\gamma,\alpha+2]$, there exists $\rho_2>1$ such that*

$$w^\epsilon(t,x,y)=\epsilon\rho_2(1+|y|^2)^{\frac{\beta_2}{2}}+\rho_2^2(T-t)+C_g \tag{53}$$

is a classical supersolution of (49) *where C_g is the constant appearing in* (A5) *and $\gamma=\frac{\alpha}{\theta}+1$. Moreover, we have $w^\epsilon\geq V^\epsilon$.*

Observation: Notice, again, that γ is always less than $\alpha+2$.

Proof As in the proof of Proposition 8.1, we first notice that we can write (49) as

$$\begin{cases} -V_t^\epsilon - \frac{1}{2\epsilon}\Delta_y V^\epsilon + \frac{1}{\theta}|\frac{D_y V^\epsilon}{\epsilon}|^\theta = f(x,y,D_x V^\epsilon, D^2_{xx}V^\epsilon, \frac{D^2_{xy}V^\epsilon}{\sqrt{\epsilon}}) & \text{in} \quad Q := (0,T)\times\mathbb{R}^n\times\mathbb{R}^m \\ V^\epsilon(T,x,y) = g(x,y) \end{cases}$$

with $f(x,y,D_x V^\epsilon, D^2_{xx}V^\epsilon, \frac{D^2_{xy}V^\epsilon}{\sqrt{\epsilon}}) := -H(x,y,D_x V^\epsilon, D^2_{xx}V^\epsilon, \frac{D^2_{xy}V^\epsilon}{\sqrt{\epsilon}})$.

We wish to prove that

$$w^\epsilon(T,x,y) \geq g(x,y)$$

and

$$-w_t^\epsilon - \frac{1}{2\epsilon}\Delta_y w^\epsilon + \frac{1}{\theta}|\frac{D_y w^\epsilon}{\epsilon}|^\theta \geq f(x,y,D_x w^\epsilon, D^2_{xx}w^\epsilon, \frac{D^2_{xy}w^\epsilon}{\sqrt{\epsilon}}).$$

Similarly to the proof of Proposition 8.1, we observe that, by the definition of f and the fact that w^ϵ does not depend on x, f inherit the coercive growth of l in $(A4)$. That is, we have

$$f_0^{-1}|y|^\alpha - f_0 \leq f(x,y,D_x w^\epsilon, D^2_{xx}w^\epsilon, \frac{D^2_{xy}w^\epsilon}{\sqrt{\epsilon}}) = f(x,y,0,0,0) \leq f_0(1+|y|^\alpha)$$

for some $f_0 > 0$ and for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$.

w^ϵ satisfies

$$w^\epsilon(T,x,y) \geq C_g \geq g(x,y),$$

$$-w_t^\epsilon = \rho_2^2,$$

$$D_y w^\epsilon = \epsilon\beta_2\rho_2(1+|y|^2)^{\frac{\beta_2-2}{2}}y,$$

and

$$\Delta_y w^\epsilon = \epsilon\beta_2\rho_2[(\beta_2-2)|y|^2 + m(1+|y|^2)](1+|y|^2)^{\frac{\beta_2-4}{2}}.$$

Hence,

$$\begin{aligned} &-w_t^\epsilon - \frac{1}{2\epsilon}\Delta_y w^\epsilon + \frac{1}{\theta}|\frac{D_y w^\epsilon}{\epsilon}|^\theta \\ &\geq \rho_2^2 - \rho_2 C_1(1+|y|^{\beta_2-2}) + C_2|\rho_2|^\theta|y|^{\theta(\beta_2-1)} \end{aligned}$$

for some positive constants C_1 and C_2 independent on the value of ρ_2 and β_2.

Suppose that $|y| \leq 1$ ($\Leftrightarrow -|y| \geq -1$), then

$$-w_t^\epsilon - \frac{1}{2\epsilon}\Delta_y w^\epsilon + \frac{1}{\theta}|\frac{D_y w^\epsilon}{\epsilon}|^\theta \geq \rho_2^2 - 2\rho_2 C_1 + C_2|\rho_2|^\theta |y|^{\theta(\beta_2 - 1)}$$

and taking $\rho_2 > 1$ so large such that

$$\rho_2^2 - 2\rho_2 C_1 \geq 2f_0$$

we can see that

$$-w_t^\epsilon - \frac{1}{2\epsilon}\Delta_y w^\epsilon + \frac{1}{\theta}|\frac{D_y w^\epsilon}{\epsilon}|^\theta \geq 2f_0 \geq f_0(1 + |y|^\alpha)$$

for all $|y| \leq 1$. This proves that

$$-w_t^\epsilon - \frac{1}{2\epsilon}\Delta_y w^\epsilon + \frac{1}{\theta}\Big|\frac{D_y w^\epsilon}{\epsilon}\Big|^\theta \geq f(x, y, D_x w^\epsilon, D_{xx}^2 w^\epsilon, \frac{D_{xy}^2 w^\epsilon}{\sqrt{\epsilon}}) = f(x, y, 0, 0, 0)$$

for all $|y| \leq 1$, since

$$f_0(1 + |y|^\alpha) \geq f(x, y, 0, 0, 0).$$

Assume now that $|y| \geq 1$. Since $\beta_2 \in [\gamma, \alpha + 2]$, then $\theta(\beta_2 - 1) \geq \alpha$ and $\beta_2 - 2 \leq \alpha$ and consequently

$$-w_t^\epsilon - \frac{1}{2\epsilon}\Delta_y w^\epsilon + \frac{1}{\theta}\Big|\frac{D_y w^\epsilon}{\epsilon}\Big|^\theta \geq \rho_2^2 + (-\rho_2 C_1 + C_2|\rho_2|^\theta)|y|^\alpha - \rho_2 C_1$$

for all $|y| \geq 1$. Taking $\rho_2 > 1$ enough large such that

$$(-\rho_2 C_1 + C_2|\rho_2|^\theta)|y|^\alpha - \rho_2 C_1 \geq f_0(1 + |y|^\alpha)$$

we can see that

$$-w_t^\epsilon - \frac{1}{2\epsilon}\Delta_y w^\epsilon + \frac{1}{\theta}\Big|\frac{D_y w^\epsilon}{\epsilon}\Big|^\theta \geq f_0(1 + |y|^\alpha)$$

holds and hence

$$-w_t^\epsilon - \frac{1}{2\epsilon}\Delta_y w^\epsilon + \frac{1}{\theta}\Big|\frac{D_y w^\epsilon}{\epsilon}\Big|^\theta \geq f(x, y, D_x w^\epsilon, D_{xx}^2 w^\epsilon, \frac{D_{xy}^2 w^\epsilon}{\sqrt{\epsilon}}) = f(x, y, 0, 0, 0)$$

since

$$f_0(1 + |y|^\alpha) \geq f(x, y, 0, 0, 0).$$

Therefore if we take $\rho_2 > 1$ enough large such that

$$\rho_2^2 - 2\rho_2 C_1 \geq 2f_0$$

and

$$(-\rho_2 C_1 + C_2|\rho_2|^\theta)|y|^\alpha - \rho_2 C_1 \geq f_0(1 + |y|^\alpha),$$

w^ϵ is a classical supersolution of (49).

We will now show that any of such w^ϵ's satisfy $w^\epsilon \geq V^\epsilon$ with V^ϵ the value function defined in (44). We know that

$$\frac{1}{\theta^*}|\xi^*|^{\theta^*} + \frac{1}{\theta}\Big|\frac{D_y w^\epsilon}{\epsilon}\Big|^\theta = \xi^* \cdot \frac{D_y w^\epsilon}{\epsilon} \tag{54}$$

where $\xi^* = D_q h(q)$ and h is defined in (47) (see Proposition 7.1).

Applying Itô's formula to $w^\epsilon(s, Y_s^{\xi^*}) := w^\epsilon(s, x, Y_s^{\xi^*})$, we have

$$\begin{aligned} w^\epsilon(T \wedge \tau_R, Y_{T\wedge\tau_R}^{\xi_s^*}) &= w^\epsilon(t, y) \\ &+ \int_t^{T\wedge\tau_R} w_s^\epsilon ds + \int_t^{T\wedge\tau_R} D_y w^\epsilon \cdot dY_s^{\xi^*} \\ &+ \frac{1}{2}\int_t^{T\wedge\tau_R} \mathrm{trace}(D_{yy}^2 w^\epsilon) d < Y^{\xi^*} >_s . \end{aligned}$$

Therefore,

$$\begin{aligned} w^\epsilon(T \wedge \tau_R,&x, Y_{T\wedge\tau_R}^{\xi^*}) + \int_t^{T\wedge\tau_R} \big(\frac{1}{\theta^*}|\xi_s^*|^{\theta^*} + f(X_s^{u,\xi^*}, Y_s^{\xi^*})\big)ds \\ &= w^\epsilon(t, x, y) + \int_t^{T\wedge\tau_R} w_s^\epsilon ds + \int_t^{T\wedge\tau_R} D_y w^\epsilon \cdot dY_s^{\xi^*} \\ &+ \frac{1}{2\epsilon}\int_t^{T\wedge\tau_R} \mathrm{trace}(D_{yy}^2 w^\epsilon)\tau(Y_s^{\xi^*})\tau^T(Y_s^{\xi^*})ds \\ &+ \int_t^{T\wedge\tau_R} \big(\frac{1}{\theta^*}|\xi_s^*|^{\theta^*} + f(X_s^{u,\xi^*}, Y_s^{\xi^*})\big)ds \\ &= w^\epsilon(t, x, y) + \int_t^{T\wedge\tau_R} -\frac{1}{\epsilon}\xi_s^* \cdot D_y w^\epsilon ds + \sqrt{\frac{1}{\epsilon}}\int_t^{T\wedge\tau_R} D_y w^\epsilon \tau(Y_s^{\xi^*})dW_s \\ &+ \int_t^{T\wedge\tau_R} w_s^\epsilon ds + \frac{1}{2\epsilon}\int_t^{T\wedge\tau_R} \Delta_y w^\epsilon ds + \int_t^{T\wedge\tau_R} \big(\frac{1}{\theta^*}|\xi_s^*|^{\theta^*} + f(X_s^{u,\xi^*}, Y_s^{\xi^*})\big)ds \end{aligned}$$

since $\tau\tau^T = \mathbb{I}$ and where we are using the notation $f(x,y)$ to say $f(x,y,0,0,0) = f(x,y,D_x w^\epsilon, D^2_{xx} w^\epsilon, \frac{D^2_{xy} w^\epsilon}{\sqrt{\epsilon}})$. Then, noting the supersolution property for w^ϵ ,

$$\frac{1}{2\epsilon}\Delta_y w^\epsilon \leq \frac{1}{\theta}\Big|\frac{D_y w^\epsilon}{\epsilon}\Big|^\theta - f(x,y) - w^\epsilon_t,$$

and (52), we can see that

$$w^\epsilon(T\wedge\tau_R, x, Y^{\xi^*}_{T\wedge\tau_R}) + \int_t^{T\wedge\tau_R} \big(\frac{1}{\theta^*}|\xi^*_s|^{\theta^*} + f(X^{u,\xi^*}_s, Y^{\xi^*}_s)\big)ds$$
$$\leq w^\epsilon(t,x,y) + \sqrt{\frac{1}{\epsilon}}\int_t^{T\wedge\tau_R} D_y w^\epsilon \tau(Y^{\xi^*}_s)dW_s.$$

Taking the expectation and observing that $M_t := \int_t^{T\wedge\tau_R} D_y w^\epsilon \tau(Y^{\xi^*}_s)dW_s$ is a martingale, we get

$$\mathbb{E}^{x,y}\big[w^\epsilon(T\wedge\tau_R, x, Y^{\xi*}_{T\wedge\tau_R}) + \int_t^{T\wedge\tau_R} \big(\frac{1}{\theta^*}|\xi^*_s|^{\theta^*} + f(X^{u,\xi^*}_s, Y^{\xi^*}_s)\big)ds\big] \leq w^\epsilon(t,x,y).$$

Since $\frac{1}{\theta^*}|\xi_s|^{\theta^*}$, f and w^ϵ are bounded from below, sending $R\to\infty$ we can conclude that

$$\mathbb{E}^{x,y}\big[w^\epsilon(T,x,Y^{\xi^*}_T) + \int_t^T \big(\frac{1}{\theta^*}|\xi^*_s|^{\theta^*} + f(X^{u,\xi^*}_s, Y^{\xi^*}_s)\big)ds\big] \leq w^\epsilon(t,x,y).$$

But $w^\epsilon(T, X^{u,\xi^*}_T, Y^{\xi^*}_T) \geq g(X^{u,\xi^*}_T, Y^{\xi^*}_T)$ and $f(X^{u,\xi^*}_s, Y^{\xi^*}_s) \geq l(X^{u,\xi^*}_s, Y^{\xi^*}_s, u_s)$ for all control $u \in \mathcal{U}$. Thus, by monotonicity,

$$\mathbb{E}^{x,y}\big[g(X^{u,\xi^*}_T, Y^{\xi^*}_T) + \int_t^T \big(\frac{1}{\theta^*}|\xi^*_s|^{\theta^*} + l(X^{u,\xi^*}_s, Y^{\xi^*}_s, u_s)\big)ds\big] \leq w^\epsilon(t,x,y)$$

for all control $u\in\mathcal{U}$. Since $\xi^*\in\mathcal{A}$ in view of $(A4)$, using the definition of the value function V^ϵ, we obtain $w^\epsilon \geq V^\epsilon$, the required estimate. ∎

We are now in conditions to prove the main result of this section.

8.3 Well posedness

Under the previous assumptions we wish to characterize the value function V^ϵ as the unique continuous viscosity solution with α-growth to the parabolic problem with terminal data (49).

Theorem 8.3 *Suppose $\theta = \alpha^*$ where α^* is the conjugate number of α, that is, $\alpha^* = \frac{\alpha}{\alpha-1}$. For any $\epsilon > 0$, the function V^ϵ defined in* (44) *is the unique continuous viscosity solution to the Cauchy problem* (49) *with at most α-growth in x and y. Moreover the functions V^ϵ are locally equibounded.*

Proof We divided the proof in 4 steps.

Step 1 (bounds on V^ϵ)

By Propositions 8.3 and 8.4, we know that there exists v^ϵ, defined in (50), and there exists w^ϵ, defined in (53), such that

$$v^\epsilon(t,x,y) \le V^\epsilon(t,x,y) \le w^\epsilon(t,x,y),$$

for all $t \in [0,T]$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. So using the previous inequality and the definitions of v^ϵ and w^ϵ, we can conclude that there exist $K > 0$ such that

$$|V^\epsilon(t,x,y)| \le K(1+|y|^{\beta_2}), \tag{55}$$

for all $t \in [0,T]$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $\beta_2 \in [\gamma, \alpha+2]$. This estimate in particular implies that the sequence V^ϵ is locally equibounded.

Step 2 (the upper and lower semicontinuous envelopes are sub and supersolutions respectively)

We define the lower and upper semicontinuous envelopes of V^ϵ as:

$$V^\epsilon_*(t,x,y) = \liminf_{(t',x',y')\to(t,x,y)} V^\epsilon(t',x',y'),$$

$$(V^\epsilon)^*(t,x,y) = \limsup_{(t',x',y')\to(t,x,y)} V^\epsilon(t',x',y')$$

where $(t',x',y') \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^m$.

By definition, we have $V^\epsilon_* \le V^\epsilon \le (V^\epsilon)^*$ and moreover both V^ϵ_* and $(V^\epsilon)^*(t,x,y)$ satisfy the growth condition (55).

A standard argument in viscosity solution theory, based on the dynamic programming (see [14]), gives that V^ϵ_* is a viscosity supersolution and $(V^\epsilon)^*$ is a viscosity subsolution of (49). We remark that in this result, the space of controls U just need to be closed and that it is very important to know a priori that the functions V^ϵ are locally equibounded.

Step 3 (V^ϵ attains continuously the final data)

We show that the value function V^ϵ attains continuously the final data locally uniformly with respect to (x,y). This means that $\lim_{t\to T} V^\epsilon(t,x,y) = g(x,y)$ locally uniformly in $(x,y) \in \mathbb{R}^n\times\mathbb{R}^m$.

By definition of V^ϵ and $V^\epsilon(T,x,y) = g(x,y)$, we get

$\forall \eta, \exists u \in \mathcal{U}, \exists \xi \in \mathcal{A}$ such that

$$\begin{aligned}|V^\epsilon(t,x,y) - V^\epsilon(T,x,y)| \leq & \mathbb{E}^{x,y}\big[\int_t^T \big|l(X_s^{u,\xi}, Y_s^{u,\xi}, u_s) + \frac{1}{\theta^*}|\xi_s|^{\theta^*}\big|ds\big] \\ & + \mathbb{E}^{x,y}[|g(X_T^{u,\xi}, Y_T^{u,\xi}) - g(x,y)|] + \eta.\end{aligned}$$

Exchanging the integral with respect to the time variable with the expectation (Fubini's Theorem), we can see that

$$\mathbb{E}^{x,y}\big[\int_t^T \big|l(X_s^{u,\xi}, Y_s^{u,\xi}, u_s) + \frac{1}{\theta^*}|\xi_s|^{\theta^*}\big|ds\big] = \int_t^T \mathbb{E}^{x,y}\big[\big|l(X_s^{u,\xi}, Y_s^{u,\xi}, u_s) + \frac{1}{\theta^*}|\xi_s|^{\theta^*}\big|\big]ds.$$

Using assumption $(A4)$ and the admissibility of the control process $\xi \in \mathcal{A}$, we have for all compact $K \subset \mathbb{R}^m$ that

$$\begin{aligned}&\mathbb{E}^{x,y}\big[\big|l(X_s^{u,\xi}, Y_s^{u,\xi}, u_s) + \frac{1}{\theta^*}|\xi_s|^{\theta^*}\big|ds\big] \\ &\quad \leq \mathbb{E}^{x,y}\big[l_0(1 + |Y_s^{u,\xi}|^\alpha) + \frac{1}{\theta^*}|\xi_s|^{\theta^*}\big|ds\big] \leq L\end{aligned}$$

for all $x \in \mathbb{R}^n$ and $y \in K$ and where L is a constant depending on C_K. Then, by monotonicity of the integral

$$\int_t^T \mathbb{E}^{x,y}\big[\big|l(X_s^{u,\xi}, Y_s^{u,\xi}, u_s) + \frac{1}{\theta^*}|\xi_s|^{\theta^*}\big|\big]ds \leq \int_t^T L = L(T-t) \to 0 \text{ as } t \to T,$$

for all $x \in \mathbb{R}^n$ and $y \in K$. We just proved that

$$\mathbb{E}^{x,y}\big[\int_t^T \big|l(X_s^{u,\xi}, Y_s^{u,\xi}, u_s) + \frac{1}{\theta^*}|\xi_s|^{\theta^*}\big|ds\big] \to 0 \text{ as } t \to T$$

locally uniformly in $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$.

What about $\mathbb{E}^{x,y}[|g(X_T^{u,\xi}, Y_T^{u,\xi}) - g(x,y)|]$?

For that we write the integral form of (42), take the modulus and apply the triangle inequality

$$|Y_T^{u,\xi} - y| \leq \frac{1}{\epsilon}\int_t^T |\xi_s|ds + \sqrt{\frac{1}{\epsilon}}\big|\int_t^T \tau(Y_s^{u,\xi})dW_s\big|.$$

Next we use Hölder's inequality for the first term and Burkholder-Davis-Gundy inequality for the last. We obtain

$$\mathbb{E}^{x,y}[|Y_T^{u,\xi} - y|] \leq \mathbb{E}^{x,y}\big[\frac{1}{\epsilon}\big(\int_t^T |\xi_s|^{\theta^*} ds\big)^{\frac{1}{\theta^*}}(T-t)^{\frac{1}{\theta}}\big] + \mathbb{E}^{x,y}\big[\sqrt{\frac{1}{\epsilon}}C_1(T-t)^{\frac{1}{2}}\big]$$

for some $C_1 > 0$ and consequently

$$\mathbb{E}^{x,y}[|Y_T^{u,\xi} - y|] \leq \frac{1}{\epsilon}C_2(T-t)^{\frac{1}{\theta}} + \sqrt{\frac{1}{\epsilon}}C_1(T-t)^{\frac{1}{2}} \tag{56}$$

where $C_2 := \mathbb{E}^{x,y}[\big(\int_0^T |\xi_s|^{\theta^*} ds\big)^{\frac{1}{\theta^*}}] < +\infty$ (by the admissibility of the control process ξ).

For every $M > 0$ and $\delta > 0$, and for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ such that $|x|, |y| \leq M$ define

$$A := \{|X_T - x| \geq \delta\} \cup \{|Y_T - y| \geq \delta\}.$$

Because of property (56), relation $\mathbb{P}^{x,y}(\{|Y_T - y| \geq \delta\}) = \mathbb{E}^{x,y}(1_{\{|Y_T - y| \geq \delta\}})$ and the Markov inequality, we can conclude that there exists a constant $C(\xi, \delta, \epsilon)$, depending on the admissibility of the control process ξ, on δ, and on ϵ such that the conditional probability

$$\mathbb{P}^{x,y}[|Y_T - y| \geq \delta] \leq C(\xi, \delta, \epsilon)[(T-t)^{\frac{1}{\theta}} + (T-t)^{\frac{1}{2}}]. \tag{57}$$

We can now apply Theorems 1 and 4 of [22, Ch 2]) or Appendix D of [20] to equation (41) (that depends on Y) to conclude that there exists a constant $C(\xi, M, \delta, \epsilon)$ depending on the Lipschitz constants of F and σ and possible also on $C(\xi, \delta, \epsilon)$ such that

$$\mathbb{P}^{x,y}(|X_T - x| \geq \delta) \leq C(\xi, M, \delta, \epsilon)(T-t)^{\frac{1}{2}}.$$

Therefore

$$\begin{aligned}\mathbb{P}^{x,y}(A) &\leq C(\xi, M, \delta, \epsilon)(T-t)^{\frac{1}{2}} + C(\xi, \delta)[(T-t)^{\frac{1}{\theta}} + (T-t)^{\frac{1}{2}}] \\ &\leq K(\xi, M, \delta, \epsilon)[(T-t)^{\frac{1}{\theta}} + (T-t)^{\frac{1}{2}}].\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}^{x,y}[|g(X_T^{u,\xi}, Y_T^{u,\xi}) - g(x,y)|] \leq &\mathbb{E}^{x,y}[1_{\Omega - A}|g(X_T^{u,\xi}, Y_T^{u,\xi}) - g(x,y)|] \\ &+ \mathbb{E}^{x,y}[1_A|g(X_T^{u,\xi}, Y_T^{u,\xi}) - g(x,y)|]\end{aligned}$$

and we can compute the last term by using the estimate on $\mathbb{P}^{x,y}(A)$ and the information that g is bounded $(A5)$ to say that

$$\mathbb{E}^{x,y}[1_A|g(X_T^{u,\xi}, Y_T^{u,\xi}) - g(x,y)|] \leq K'(\xi, M, \delta, \epsilon)[(T-t)^{\frac{1}{\theta}} + (T-t)^{\frac{1}{2}}] \to 0$$

uniformly in (x,y) as $T \to t$.

The term $\mathbb{E}^{x,y}[1_{\Omega-A}|g(X_T^{u,\xi}, Y_T^{u,\xi}) - g(x,y)|]$ can be estimated as follows

$$\mathbb{E}^{x,y}[1_{\Omega-A}|g(X_T^{u,\xi}, Y_T^{u,\xi}) - g(x,y)|] \leq \mathbb{E}^{x,y}[w_{g,M}(|X_T^{u,\xi} - x|, |Y_T^{u,\xi} - y|)] \to 0$$

uniformly in (x,y) as $T \to t$, where $\delta < M$ and $w_{g,M}$ is the modulus of continuity of g restricted to $\{(x,y)||x| \leq 2M, |y| \leq 2M\}$.

Consequenlty

$$\limsup_{t\to T} |V^\epsilon(t,x,y) - V^\epsilon(T,x,y)| \leq \eta$$

locally uniformly in (x,y) and we conclude by the arbitrariness of η.

Finally, using the definitions, it is easy to prove that

$$V_*^\epsilon(T,x,y) = (V^\epsilon)^*(T,x,y) = g(x,y), \quad \forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^m$$

Step 4 (Comparison Principle and conclusion)

We now use a comparison result of Da-Lio and Ley between sub and supersolutions to parabolic problems satisfying an α-growth condition. We mention that to apply this result we need to know a priori that $\theta \leq \alpha^*$ (see [17, page 314]).

We already observed that the estimate (55) holds also for V_*^ϵ and $(V^\epsilon)^*$, so they both satisfy the appropriate growth condition if $\beta_2 \leq \alpha$. Since $\beta_2 \in [\gamma, \alpha+2]$, $\beta_2 \leq \alpha$ implies $\theta \geq \alpha^*$. Moreover we saw in the previous step that $V_*^\epsilon(T,x,y) = (V^\epsilon)^*(T,x,y) = g(x,y)$ for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$. Therefore we can apply the comparison principle and conclude that $(V^\epsilon)^*(T,x,y) \leq V_*^\epsilon(T,x,y)$ for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$ provided that we have $\theta = \alpha^*$.

Now using the definition of upper and lower envelopes and the comparison result, we get $(V^\epsilon)^*(t,x,y) = V^\epsilon(t,x,y) = V_*^\epsilon(t,x,y)$ for every $(t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m$. Then V^ϵ is the unique continuous viscosity solution to (49) satisfying an α-growth in x and y. ∎

9 The effective Hamiltonian

9.1 The effective Hamiltonian

We are interested in the limit V as $\epsilon \to 0$ of V^ϵ and in particular in understanding the PDE satisfied by V. In this section we deal with the problem of finding the candidate limit Cauchy problem of the singularly perturbed problem as $\epsilon \to 0$.

Ansatz: $V^\epsilon(t,x,y) = V(t,x) + \epsilon\chi(y)$, with $\chi(y) \in C^2(\mathbb{R}^m)$.

We have:

$$V_t^\epsilon = V_t, \quad D_x V^\epsilon = D_x V, \quad D^2_{xx}V^\epsilon = D^2_{xx}V, \quad D_y V^\epsilon = \epsilon D_y \chi, \quad D^2_{yy}V^\epsilon = \epsilon D^2_{yy}\chi,$$
$$D^2_{xy}V^\epsilon = 0, \quad \Delta_y V^\epsilon = \epsilon \Delta_y \chi.$$

Substituting into (49) we get

$$0 = -V_t + H(x,y,D_xV,D^2_{xx}V,0) - \frac{1}{2\epsilon}\epsilon\Delta_y\chi + \frac{1}{\theta}\left|\frac{\epsilon D_y\chi}{\epsilon}\right|^\theta$$

that is,

$$0 = -V_t + H(x,y,D_xV,D^2_{xx}V,0) - \frac{1}{2}\Delta_y\chi + \frac{1}{\theta}|D_y\chi|^\theta$$

with $H(x,y,p,M,0) = \sup_u\{-\text{trace}(\sigma\sigma^T M) - F\cdot p - l\}$.

We wish that

$$\bar{H}(x,D_xV,D^2_{xx}V) = H(x,y,D_xV,D^2_{xx}V,0) - \frac{1}{2}\Delta_y\chi + \frac{1}{\theta}|D_y\chi|^\theta.$$

The idea is to freeze $(\bar{t},\bar{x})$, set $\bar{p} := D_xV(\bar{t},\bar{x})$ and $\bar{M} := D^2_{xx}V(\bar{t},\bar{x})$ and let only y vary. Hence, $\bar{H}(\bar{x},\bar{p},\bar{M})$ is a constant that we will denote by $-\lambda$ thus we call $f(y) := -H(\bar{x},y,\bar{p},\bar{M},0)$ and impose $\chi(0) = 0$, to avoid the ambiguity of additive constants with respect to χ, and we arrive at the following problem:

$$\begin{cases} \lambda - \frac{1}{2}\Delta_y\chi + \frac{1}{\theta}|D_y\chi|^\theta = f(y) & \text{in } \mathbb{R}^m \\ \chi(0) = 0, \end{cases}$$

where unknown is the pair $(\lambda,\chi) \in \mathbb{R}\times C^2(\mathbb{R}^m)$.

Proposition 9.1 *Assume that all the assumptions* $(A1)-(A5)$ *of Section 8 holds. Then* $f(y) := -H(\bar{x},y,\bar{p},\bar{M},0)$ *satisfies*

$$-f_0 + f_0^{-1}|y|^\alpha \le f(y) \le f_0(1+|y|^\alpha)$$

for some $f_0 > 0$.

Proof This is an obvious consequence of the fact that f is a function of y only, the definition of H and the assumptions $(A3)$ and $(A4)$. ∎

We arrived at exactly the ergodic problem studied in Chapter I.

9.2 The standing assumptions

Generally, we don't know the regularity of $f := -H$. Basically, all we know is that the supremum of continuous functions is lower semicontinuous. As a consequence, to use the results of the previous chapter we have to assume the regularity of f.

The standing assumptions: We will assume besides $(A1)-(A5)$ of Section 8 the following hypothesis:

$$(H0) \quad \begin{cases} y \mapsto f(y) \text{ belongs to } W^{1,\infty}_{\text{loc}}(\mathbb{R}^m) \\ |Df(y)| \leq f_0(1+|y|^{\alpha-1}) \text{ for all } y \in \mathbb{R}^m \end{cases}$$

where f_0 is the constant appearing in Proposition 9.1.

Therefore the standing assumptions imply that $f = -H$ satisfy $(H1)$ of Chapter I. As a consequence we can use all the results of Chapter I here.

Proposition 9.2 *Assume all the standing assumptions. Then, for any fixed* $\bar{x}, \bar{p}, \bar{M}$ $f(y) := -H(\bar{x}, y, \bar{p}, \bar{M}, 0)$ *satisfies* $(H1)$ *of Chapter I and we have that there exist a unique solution pair* $(\bar{H}, \phi) \in \mathbb{R}^m \times C^2(\mathbb{R}^m)$ *of*

$$\begin{cases} -\frac{1}{2}\Delta_y \phi + \frac{1}{\theta}|D_y \phi|^\theta + H(\bar{x}, y, \bar{p}, \bar{M}, 0) = \bar{H}(\bar{x}, \bar{p}, \bar{M}) \\ \phi(0) = 0, \end{cases}$$

such that ϕ *is bounded from below. Moreover,* ϕ *belongs to* Φ_γ *and* $|\phi(y)| \leq C(1+|y|^\gamma)$ *for some* $C > 0$ *with* $\gamma = \frac{\alpha}{\theta} + 1$.

Definition (**Effective Hamiltonian**) The constant $\bar{H}(\bar{x}, \bar{p}, \bar{M})$ in the conditions of Proposition 9.2 is called effective Hamiltonian.

Proof Checking that f satisfies $(H1)$ is easy due to Proposition 9.1 and hypothesis $(H0)$. The remaining conclusions are all consequence of Chapter I. ∎

9.3 Some results for $\bar{H}$

Proposition 9.3 *$\bar{H}$ is continuous at $(\bar{x}, \bar{p}, \bar{M})$.*

Proof Suppose that $x_n \to \bar{x}$, $p_n \to \bar{p}$ and $M_n \to \bar{M}$. We wish to show that then $\bar{H}(x_n, p_n, M_n) \to \bar{H}(\bar{x}, \bar{p}, \bar{M})$.

For every $n \in \mathbb{N}$, consider the ergodic problem

$$-\frac{1}{2}\Delta\phi_n + \frac{1}{\theta}|D\phi_n(y)|^\theta + H(x_n, y, p_n, M_n, 0) = \bar{H}(x_n, p_n, M_n) := -\lambda_n \quad (58)$$

First we notice that by continuity of H we have $H(x_n, y, p_n, M_n, 0) \to H(\bar{x}, y, \bar{p}, \bar{M}, 0)$ and then, by the results of Chapter I, we know that there exists a unique solution pair (λ_n, ϕ_n) of (58) such that ϕ_n is bounded from below. Moreover, Corollary 4.10 shows that $\phi_n \in \Phi_\gamma$ and Section 5 that $\lambda_n = \lambda^*(f_n)$.

In view of Theorem B.2 (Appendix B), we have that, for any $R > 0$, $\sup_{B_R} |\phi_n|$ and $\sup_{B_R} |D\phi_n|$ are bounded by a constant not depending on n. In particular, we can see that the Hölder norm $|\phi_n|_{2+\Gamma, B_R}$ for some $\Gamma \in (0,1)$ is bounded uniformly in n. Hence, the family $\{\phi_n\}_{n\in\mathbb{N}}$ is relatively compact in $C(\mathbb{R}^m)$. By the Ascoli-Arzela theorem, there exists a sequence $n_j \to +\infty$ as $j \to +\infty$ and a function $w \in C^2(\mathbb{R}^m)$ such that $\phi_{n_j} \to w$, $D\phi_{n_j} \to Dw$, $D^2\phi_{n_j} \to D^2 w$ locally uniformly in $\mathbb{R}^m$.

We now observe that for every $n \in \mathbb{N}$, $f_n(y) := -H(x_n, y, p_n, M_n, 0)$ satisfy (by Proposition 9.1)

$$-f_{n0} + f_{n0}^{-1}|y|^\alpha \le f_n(y) \le f_{n0}(1 + |y|^\alpha)$$

for some $f_{n0} > 0$. Therefore, by Propositions 1.8, 1.9 and 1.10, we have

$$-f_{n0} + (f_{n0}^{-1})^{\frac{\theta^*}{\theta^*+\alpha}} \lambda^*(|y|^\alpha) \le \lambda^*(f_n(y)) \le f_{n0} + (f_{n0})^{\frac{\theta^*}{\theta^*+\alpha}} \lambda^*(|y|^\alpha)).$$

This tell us that λ_n and thus $\bar{H}_n$ is uniformly bounded by a constant.

Hence there exists a subsequence, that we will still denote by n_j, such that

$$\lambda_{n_j} - \frac{1}{2}\Delta\phi_{n_j} + \frac{1}{\theta}|D\phi_{n_j}(y)|^\theta = -H(x_{n_j}, y, p_{n_j}, M_{n_j}, 0)(y) \quad (59)$$

converges to, as $n_j \to +\infty$,

$$c - \frac{1}{2}\Delta w + \frac{1}{\theta}|Dw(y)|^\theta = -H(\bar{x}, y, \bar{p}, \bar{M}, 0) \quad (60)$$

with $c \in \mathbb{R}$.

Since $\phi_{n_j} \in \Phi_\gamma$ for all n_j, $w \in \Phi_\gamma$ and we are in the right class for which there is uniqueness of (EP). Thus $c = -\bar{H}(\bar{x}, \bar{p}, \bar{M})$ and $w = \phi + C$.

In particular, we showed that (x_n, p_n, M_n) has a unique limit point for all converging subsequences, therefore (x_n, p_n, M_n) converges to this limit point and we can conclude that $\bar{H}(x_n, p_n, M_n) \to \bar{H}(\bar{x}, \bar{p}, \bar{M})$ as we wished to show. $\blacksquare$

Proposition 9.4 *$\bar{H}$ is degenerate elliptic, i.e., $\bar{H}(\bar{x}, \bar{p}, \bar{M}) \leq \bar{H}(\bar{x}, \bar{p}, \bar{M}')$ for all $\bar{M} \geq \bar{M}'$ all $\bar{p}$.*

Proof If $\bar{M} \geq \bar{M}'$, the degenerate ellipticity of H implies that $f_{\bar{M}'}(y) = -H(\bar{x}, y, \bar{p}, \bar{M}', 0) \leq -H(\bar{x}, y, \bar{p}, \bar{M}, 0) = f_{\bar{M}}(y)$. By the monotonicity of λ^* (Proposition 1.8), $\lambda^*(f_{\bar{M}'}(y)) \leq \lambda^*(f_{\bar{M}}(y))$, i.e., $\bar{H}(\bar{x}, \bar{p}, \bar{M}) \leq \bar{H}(\bar{x}, \bar{p}, \bar{M}')$ for all $\bar{p}$. $\blacksquare$

Remark: Note that the degeneracy ellipticity of $\bar{H}$ is a requisite for considering viscosity solutions of the effective equation.

10 Convergence theorem

The main result of this chapter is the following convergence result.

10.1 Convergence theorem

Theorem 10.1 *Assume the standing assumptions, condition $\theta = \alpha^*$, and that the terminal data g is independent of y, i.e., $g(x, y) = g(x)$. Then, the semilimits*

$$\underline{V}(t, x) := \liminf_{(\epsilon,t',x')\to(0,t,x)} \inf_{y\in\mathbb{R}^m} V^\epsilon(t', x', y) \tag{61}$$

and

$$\bar{V}(t, x) := (\sup_R \bar{V}_R)^*(t, x) \text{ (upper semi-continuous envelope of } \sup_R \bar{V}_R \text{)} \tag{62}$$

where

$$\bar{V}_R(t, x) := \limsup_{(\epsilon,t',x')\to(0,t,x)} \sup_{y\in B_R(0)} V^\epsilon(t', x', y)$$

and V^ϵ was defined in (44)*, are, respectively, a supersolution and a subsolution of the effective Cauchy problem*

$$\begin{cases} -V_t + \bar{H}(x, D_x V, D^2_{xx} V) = 0 & \text{in } (0, T) \times \mathbb{R}^n \\ V(T, x) = g(x). \end{cases} \tag{63}$$

Proof The proof is divided in several steps.

Step 1: *Relaxed semi limits.*

First recall that by (55) the functions V^ϵ are locally equibounded in $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m$, uniformly in ϵ. Second recall that by Proposition 8.1 V^ϵ is bounded from below and in fact we have

$$V^\epsilon \geq v^\epsilon \geq -C.$$

These facts allow us to define the upper and lower relaxed semilimits in $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ as in (61) and (62).

It is immediate to get from the definitions that $\underline{V}$ is a lower semi-continuous (LSC) function ($\liminf_{(t,x)\to(t_0,x_0)} \underline{V}(t, x) = \underline{V}(t_0, x_0)$) and $\bar{V}$ is an upper semi-continuous (USC) function, they do not depend on y and we have that $\underline{V} \leq V^\epsilon$ and $\underline{V} \leq \bar{V}_R \leq \bar{V}$. It is also obvious by the definitions and estimate (55) that there exists a constant $K > 0$ such that

$$|\bar{V}_R(t, x)|, |\bar{V}(t, x)|, |\underline{V}(t, x)| \leq K \tag{64}$$

for all $t \in [0, T]$.

Step 2: *$\underline{V}$ is a supersolution of the limit PDE in $(0,T) \times \mathbb{R}^n$*

We would like to prove that $\underline{V}$ is a supersolution of the limit PDE (63) in $(0,T) \times \mathbb{R}^n$. For that purpose, we fix an arbitrary $(\bar{t}, \bar{x}) \in (0,T) \times \mathbb{R}^n$ and we show that $\underline{V}$ is a supersolution at that point of (63). This means that if ψ is a smooth function such that $\psi(\bar{t}, \bar{x}) = \underline{V}(\bar{t}, \bar{x})$ and $\underline{V} - \psi$ has a minimum at $(\bar{t}, \bar{x})$ then

$$-\psi_t(\bar{t}, \bar{x}) + \bar{H}(\bar{x}, D_x\psi(\bar{t}, \bar{x}), D^2_{xx}\psi(\bar{t}, \bar{x})) \geq 0.$$

Without loss of generality we may assume that $\underline{V} - \psi$ has a strict minimum at $(\bar{t}, \bar{x})$ in $B_r(\bar{t}, \bar{x}) \cap ([0,T] \times \mathbb{R}^n)$.

We will argue by contradiction.

Assume that there exists $\eta > 0$ such that

$$-\psi_t(\bar{t}, \bar{x}) + \bar{H}(\bar{x}, D_x\psi(\bar{t}, \bar{x}), D^2_{xx}\psi(\bar{t}, \bar{x})) < -2\eta < 0. \tag{65}$$

Set

$$f_{t,x}(y) := -H(x, y, D_x\psi(t,x), D^2_{xx}\psi(t,x), 0). \tag{66}$$

By Proposition 9.1, we know that for all fixed t and x there exist $f_0 > 0$ such that

$$-f_0 + f_0^{-1}|y|^\alpha \leq f_{t,x}(y) \leq f_0(1 + |y|^\alpha).$$

That is, $f_{t,x}$ satisfies $(H1)$.

For $\delta \in (0,1)$ consider the ergodic problem

$$\lambda_\delta - \frac{1}{2}\Delta\phi_\delta(y) + \frac{1}{\theta}|D\phi_\delta(y)|^\theta = \inf_{|x-\bar{x}|,|t-\bar{t}| \leq \delta} f_{t,x}(y) = f_\delta(y). \tag{67}$$

We observe that $f_\delta(y)$ satisfies $(H1)$ with growth α and that we have $f_\delta \leq f_{t,x}$, $\{f_\delta\}_{\delta \in (0,1)}$ increases as δ goes to 0, $f_\delta \to f_{\bar{t},\bar{x}}$ and $\sup_{B_R} |Df_\delta|$ is uniformly bounded ($(H0)$). Furthermore, by the results of Chapter 1, we know that (67) has a unique solution pair $(\lambda_\delta, \phi_\delta)$ such that ϕ_δ is bounded from below.

We now look at (36) with

$$\mathcal{F}_{R,\delta}(y) := f_\delta(y) \wedge (f_0^{-1}|y|^{\beta_1} + R). \tag{68}$$

where $0 < \beta_1 < \min(\theta(\alpha-1), \alpha)$ (this choice of β_1 is motivated by Proposition 8.1 as it will be clarified later).

First we define $\bar{H}_{R,\delta} = -\lambda_{R,\delta}$ and then we notice that by Proposition 6.1 there exist a unique bounded from below solution pair $(\lambda_{R,\delta}, \phi_{R,\delta})$ of (36) with $\mathcal{F}_R$ given by (68) and one also has that $\phi_{R,\delta} \in \Phi_{\gamma_{\beta_1}}$, $\gamma_{\beta_1} = \frac{\beta_1}{\theta} + 1$. In particular, using Proposition 2.1 with $f = \mathcal{F}_R$, we have the upper bound $\phi_{R,\delta}(y) \le M(1 + |y|^{\gamma_{\beta_1}})$ and, since $\phi_{R,\delta}$ is bounded from below, we can then write

$$m \le \phi_{R,\delta}(y) \le M(1 + |y|^{\gamma_{\beta_1}}). \tag{69}$$

All this characteristics for $\phi_{R,\delta}$ will be essential next.

By Theorem 6.2 (applied twice, one for $F_{R,\delta}$ be letting R going to infinity and another for f_δ with $\delta \to 0$), there exist a sequence $R_j \to +\infty$ and a sequence $\delta_j \to 0$ as $j \to +\infty$ such that, for some C

$$(\lambda_{R_j,\delta_j}, \phi_{R_j,\delta_j}) \to (\lambda, \phi + C) \text{ as } j \to +\infty$$

where $-\lambda$ is exactly $\bar{H}(\bar{x}, D_x\psi, D^2_{xx}\psi)|_{(\bar{t},\bar{x})}$ that we will also denote by $\bar{H}$ (see the definition of effective Hamiltonian in Section 9).

We now consider the perturbed test function

$$\psi^\epsilon(t, x, y) = \psi(t, x) + \epsilon\phi_{R,\delta}(y).$$

for $R = R_j$ and $\delta = \delta_j$ with j large enough such that

$$|\bar{H}_{R,\delta} - \bar{H}| < \eta, \tag{70}$$

and satisfying also

$$|t - \bar{t}|, |x - \bar{x}| \le \delta \Longrightarrow |\psi_t(t, x) - \psi_t(\bar{t}, \bar{x})| \le \eta \quad (\psi \text{ is smooth}) \tag{71}$$

and

$$\delta < r \tag{72}$$

($\underline{V} - \psi$ has a strict minimum at $(\bar{t}, \bar{x})$ in $B_r(\bar{t}, \bar{x}) \cap ([0, T] \times \mathbb{R}^n)$ therefore considering δ as in (72) will imply later for us that we will be in a region that is strictly contained in $B_r(\bar{t}, \bar{x}) \cap ([0, T] \times \mathbb{R}^n)$).

We have $\forall t, x, y$

$$
\begin{aligned}
&-\psi_t^\epsilon(t,x,y) + H(x,y,D_x\psi^\epsilon(t,x,y),D_{xx}^2\psi^\epsilon(t,x,y),0) - \frac{1}{2\epsilon}\Delta_y\psi^\epsilon(t,x,y) + \frac{1}{\theta}\Big|\frac{D_y\psi^\epsilon(t,x,y)}{\epsilon}\Big|^\theta \\
&= -\psi_t(t,x) + H(x,y,D_x\psi(t,x),D_{xx}^2\psi(t,x),0) - \frac{1}{2}\Delta_y\phi_{R,\delta}(y) + \frac{1}{\theta}\big|D_y\phi_{R,\delta}(y)\big|^\theta \\
&\leq -\psi_t(t,x) + \sup_{|x-\bar{x}|,|t-\bar{t}|\leq\delta} H(x,y,D_x\psi(t,x),D_{xx}^2\psi(t,x),0) \vee (-f_0^{-1}|y|^\beta - R) \\
&- \frac{1}{2}\Delta_y\phi_{R,\delta}(y) + \frac{1}{\theta}\big|D_y\phi_{R,\delta}(y)\big|^\theta \\
&= -\psi_t(t,x) - \mathcal{F}_{R,\delta} - \frac{1}{2}\Delta\phi_{R,\delta}(y) + \frac{1}{\theta}|D\phi_{R,\delta}(y)|^\theta \\
&= -\psi_t(t,x) + \bar{H}_{R,\delta} \\
&\leq -\psi_t(\bar{t},\bar{x}) + \bar{H} + 2\eta \text{ (by (70) and (71))} \\
&< 0 \quad \text{(by (65))}.
\end{aligned}
$$

Consequently ψ^ϵ satisfies

$$
-\psi_t^\epsilon + H(x,y,D_x\psi^\epsilon,D_{xx}^2\psi^\epsilon,0) - \frac{1}{2\epsilon}\Delta_y\psi^\epsilon + \frac{1}{\theta}\Big|\frac{D_y\psi^\epsilon}{\epsilon}\Big|^\theta < 0 \tag{73}
$$

in

$$
Q_\delta =]\bar{t}-\delta,\bar{t}+\delta[\times B_\delta(\bar{x}) \times \mathbb{R}^m. \tag{74}
$$

From another point of view, by Proposition 8.1 and the definition of ψ^ϵ, we have that

$$
\begin{aligned}
V^\epsilon(t,x,y) - \psi^\epsilon(t,x,y) \geq &(T-t)(\epsilon\rho(1+|y|^2)^{\frac{\min(\alpha,\gamma)}{2}} - 2f_0^{-1}) + \inf_{\mathbb{R}^n} g(x) \\
&- \psi(t,x) - \epsilon\phi_{R,\delta}(y).
\end{aligned}
$$

Taking into account estimate (69), we can conclude that

$$
\begin{aligned}
V^\epsilon(t,x,y) - \psi^\epsilon(t,x,y) \geq &(T-t)(\epsilon\rho(1+|y|^2)^{\frac{\min(\alpha,\gamma)}{2}} - 2f_0^{-1}) + \inf_{\mathbb{R}^n} g(x) \\
&- \psi(t,x) - \epsilon K\big(1+|y|^{\gamma_\beta}\big) > -C_\epsilon
\end{aligned}
$$

since $\min(\alpha,\gamma) > \gamma_{\beta_1}$ due to our choice of β_1 in (68).

Hence there exists

$$
\liminf_{\epsilon\to 0,(t',x')\to(t,x)} \inf_{y\in\mathbb{R}^m} (V^\epsilon - \psi^\epsilon)(t',x',y)
$$

and we have

$$\liminf_{\epsilon\to 0,(t',x')\to(t,x)} \inf_{y\in\mathbb{R}^m} (V^\epsilon - \psi^\epsilon)(t',x',y) \geq (\underline{V} - \psi)(t,x)$$

because ψ^ϵ is bounded from below in y.

We now use the fact that $(\bar{t},\bar{x})$ is a strict minimum point of $\underline{V}-\psi$ in $B_r(\bar{t},\bar{x})\cap([0,T]\times\mathbb{R}^n)$ such that $\underline{V}(\bar{t},\bar{x}) = \psi(\bar{t},\bar{x})$ and condition $\delta < r$ to conclude that $(\underline{V} - \psi) > 0$ on ∂Q_δ and thus $\liminf_{\epsilon\to 0,(t',x')\to(t,x)} \inf_{y\in\mathbb{R}^m}(V^\epsilon - \psi^\epsilon) > 0$. Therefore, we can find $\zeta > 0$ such that

$$V^\epsilon - \zeta \geq \psi^\epsilon \text{ on } \partial Q_\delta \text{ for } \epsilon \text{ small.} \tag{75}$$

But ψ^ϵ grows at maximum like $|y|^{\gamma_{\beta_1}}$ with $\gamma_{\beta_1} < \alpha$, for this reason it belongs to the class $\mathcal{C}^\alpha$:

$\exists C > 0$ s.t. $|u(t,x,y)| \leq C(1+|x|^\alpha+|y|^\alpha)$, for all $(t,x,y) \in \mathbb{R}^n\times\mathbb{R}^m\times[0,T]$.

Therefore to apply Da Lio-Ley comparison principle we need to guarantee that V^ϵ also belongs to $\mathcal{C}^\alpha$. Since V^ϵ satisfy estimate (55) (that comes from Proposition 8.2) with $\beta_2 \in [\gamma, \alpha+2]$, it belongs to $\mathcal{C}^\alpha$ if $\beta_2 \leq \alpha$. But $\beta_2 \in [\gamma,\alpha+2]$ and $\beta_2 \leq \alpha$ implies $\gamma \leq \alpha$ and as a result $\alpha^* \leq \theta$. Once we are assuming condition $\theta = \alpha^*$, $\gamma = \alpha = \beta_2$ and so V^ϵ belongs to $\mathcal{C}^\alpha$. By the comparison result proved in [17], we have that

$$V^\epsilon - \zeta \geq \psi^\epsilon \text{ in } Q_\delta.$$

Since $V^\epsilon - \zeta \geq \psi^\epsilon$ in Q_δ we can pass to the lim inf inf and obtain

$$\underline{V}(t,x) - \zeta \geq \psi(t,x) \text{ for all } (t,x) \in\ ]\bar{t}-\delta,\bar{t}+\delta[\times B_\delta(\bar{x}).$$

In particular at $(\bar{t},\bar{x})$ we have $\underline{V}(\bar{t},\bar{x}) - \zeta \geq \psi(\bar{t},\bar{x})$ a contradiction since $\psi(\bar{t},\bar{x}) = \underline{V}(\bar{t},\bar{x})$ and $\zeta > 0$.

<u>Conclusion</u>: $\underline{V}$ is a super solution of the limit PDE in $(0,T)\times\mathbb{R}^n$.

Step 3 ($\bar{V}_R$ **is a subsolution of**

$$-V_t + \bar{H}_R(x, D_xV, D^2_{xx}V) = 0 \quad \textbf{in } (0,T)\times\mathbb{R}^n \tag{76}$$

where $\bar{H}_R := \bar{H}(x,p,M)$ **is the ergodic constant obtained by looking at** (37) **or** (38) **with** $f_R = f_{\bar{t},\bar{x}}$**,** $-\lambda_R = \bar{H}_R$.

$\bar{V}_R$ is a subsolution of (76) in $(0,T)\times\mathbb{R}^n$ if for any $(\bar{t},\bar{x})\in(0,T)\times\mathbb{R}^n$ and for any test function ψ such that $\psi(\bar{t},\bar{x})=\bar{V}_R(\bar{t},\bar{x})$ and $\bar{V}_R-\psi$ has a strict maximum at $(\bar{t},\bar{x})$ in $B_r(\bar{t},\bar{x})\cap([0,T]\times\mathbb{R}^n)$ then

$$-\psi_t(\bar{t},\bar{x})+\bar{H}_R(\bar{x},D_x\psi(\bar{t},\bar{x}),D^2_{xx}\psi(\bar{t},\bar{x}))\le 0.$$

By contradiction, assume that there exists $\eta>0$ such that

$$-\psi_t(\bar{t},\bar{x})+\bar{H}_R(\bar{x},D_x\psi(\bar{t},\bar{x}),D^2_{xx}\psi(\bar{t},\bar{x}))>2\eta>0. \tag{77}$$

Recall $f_{t,x}$ defined in (66). For $(t,x,y)\in(0,T)\times\mathbb{R}^n\times\mathbb{R}^m$ we define

$$f^\delta(y):=\sup_{|x-\bar{x}|,|t-\bar{t}|\le\delta} f_{t,x}(y) \tag{78}$$

and we look at the ergodic problem

$$\lambda_R^\delta-\frac{1}{2}\Delta_y\phi_R^\delta(y)+\frac{1}{\theta}\left|D_y\phi_R^\delta(y)\right|^\theta=f_\delta(y)\quad\text{in } B_R(0) \tag{79}$$

complemented with one of the boundary conditions introduced in the section about approximations, namely,

$$\phi_R^\delta(y)\to+\infty \text{ as } y\to\partial B_R(0)\quad\text{if } 1<\theta\le 2$$

or

$$\lambda_R^\delta-\frac{1}{2}\Delta_y\phi_R^\delta(y)+\frac{1}{\theta}\left|D_y\phi_R^\delta(y)\right|^\theta\ge f_\delta(y)\text{ in }\bar{B}_R(0)\quad\text{if }\theta>2.$$

By Theorem 3.2 ($1<\theta\le 2$) or 3.6 ($\theta>2$), we know that there exists a unique solution pair $(\lambda_R^\delta,\phi_R^\delta)$ of (79) such that ϕ_R^δ satisfies the associated boundary condition. We denote $-\lambda_R^\delta$ by $\bar{H}_\delta^\delta$ as in Section 9.

Since f_δ forms a decreasing sequence as δ decreases and $f_\delta\to f_{\bar{t},\bar{x}}$ as $\delta\to 0$, we can repeat the arguments of Section 6 and conclude by uniqueness of solutions (Theorem 3.2 or Theorem 3.6) that there exists a sequence $\delta_j\to 0$ as $j\to+\infty$ such that, for some $c\in\mathbb{R}$,

$$(\lambda_R^{\delta_j},\phi_R^{\delta_j})\to(\lambda_R,\phi_R+c)\text{ as } j\to+\infty$$

where (λ_R,ϕ_R) is the solution given by Theorem 3.2 or 3.6 for (79) with the right hand side equal to $f_{\bar{t},\bar{x}}(y)$. $-\lambda_R=\bar{H}_R$ with the notation of Section 9.

We now consider the perturbed test function

$$\psi^\epsilon(t,x,y) = \psi(t,x) + \epsilon\phi_R^\delta(y)$$

for $\delta = \delta_j$ small enough such that

$$|\bar{H}_R^\delta - \bar{H}_R| \leq \eta, \tag{80}$$

$$|t-\bar{t}|, |x-\bar{x}| \leq \delta \Longrightarrow |\psi_t(t,x) - \psi_t(\bar{t},\bar{x})| \leq \eta \tag{81}$$

and

$$\delta < r. \tag{82}$$

Then, for all $(t,x,y) \in (0,T) \times \mathbb{R}^n \times B_R(0)$

$$\begin{aligned}
&-\psi_t^\epsilon(t,x,y) + H(x,y,D_x\psi^\epsilon(t,x,y), D_{xx}^2\psi^\epsilon(t,x,y),0) - \frac{1}{2\epsilon}\Delta_y\psi^\epsilon(t,x,y) + \frac{1}{\theta}\Big|\frac{D_y\psi^\epsilon(t,x,y)}{\epsilon}\Big|^\theta \\
&= -\psi_t(t,x) + H(x,y,D_x\psi(t,x), D_{xx}^2\psi(t,x),0) - \frac{1}{2}\Delta_y\phi_R^\delta(y) + \frac{1}{\theta}\big|D_y\phi_R^\delta(y)\big|^\theta \\
&\geq -\psi_t(t,x) + \inf_{|x-\bar{x}|,|t-\bar{t}|\leq\delta}\big(-f_{t,x}(y)\big) - \frac{1}{2}\Delta_y\phi_R^\delta(y) + \frac{1}{\theta}\big|D_y\phi_R^\delta(y)\big|^\theta \\
&= -\psi_t(t,x) - f_\delta(y) - \frac{1}{2}\Delta_y\phi_R^\delta(y) + \frac{1}{\theta}\big|D_y\phi_R^\delta\big|^\theta \\
&= -\psi_t(t,x) + \bar{H}_R^\delta \\
&\geq -\psi_t(\bar{t},\bar{x}) - \eta + \bar{H}_R - \eta \text{ (by (81) and (80))} \\
&> 0 \text{ (by (77))}.
\end{aligned}$$

Consequently,

$$-\psi_t^\epsilon + H(x,y,D_x\psi^\epsilon, D_{xx}^2\psi^\epsilon, D_{xy}^2\psi^\epsilon) - \frac{1}{2\epsilon}\Delta_y\psi^\epsilon + \frac{1}{\theta}\Big|\frac{D_y\psi^\epsilon}{\epsilon}\Big|^\theta > 0 \tag{83}$$

in

$$Q_{R,\delta} :=]\bar{t}-\delta, \bar{t}+\delta[\times B_\delta(\bar{x}) \times B_R(0).$$

On the other hand,

$$\limsup_{\epsilon\to 0,(t',x')\to(t,x)} \sup_{y\in B_R(0)} (V^\epsilon - \psi^\epsilon)(t',x',y) \leq (\bar{V}_R - \psi)(t,x).$$

But $(\bar{t},\bar{x})$ is a strict maximum point of $\bar{V}_R - \psi$ in $B_r(\bar{t},\bar{x}) \cap ([0,T]\times\mathbb{R}^n)$ such that $\bar{V}_R(\bar{t},\bar{x}) = \psi(\bar{t},\bar{x})$ and δ satisfy condition $\delta < r$ (82), therefore $Q_{R,\delta}$ is strictly contained in $(B_r(\bar{t},\bar{x}) \cap ([0,T]\times\mathbb{R}^n)) \times B_R(0)$ and we have that $(\bar{V}_R -$

$\psi) < 0$ on $\partial Q_{R,\delta}$. Consequently $\limsup_{(\epsilon,t',x')\to(0,t,x)} \sup_{y\in B_R(0)}(V^\epsilon - \psi^\epsilon) < 0$ on $\partial Q_{R,\delta}$ and we can find $\zeta > 0$ such that

$$V^\epsilon + \zeta \le \psi^\epsilon \text{ on } \partial Q_{R,\delta} \text{ for } \epsilon \text{ small.} \tag{84}$$

Since V^ϵ is a viscosity solution of (49) in $Q_{R,\delta}$ and ψ^ϵ is a classical supersolution of (49) in $Q_{R,\delta}$ that satisfies one of the boundary condition (the correspondent for each θ), we can conclude by the comparison result proved in [9] (when $\theta \le 2$) or in Theorem 2.1 of [38] (when $\theta > 2$), that

$$V^\epsilon + \zeta \le \psi^\epsilon \text{ in } Q_{R,\delta}.$$

In particular by taking $\sup_{y\in B_R(0)}$ and $\limsup_{(\epsilon,t',x')\to(0,t,x)}$ we reach a contradiction with $\bar{V}_R(\bar{t},\bar{x}) = \psi(\bar{t},\bar{x})$ and $\zeta > 0$.

<u>Conclusion</u>: $\bar{V}_R$ is a subsolution of (76).

Step 4 (**Behaviour of $\bar{V}_R$ and $\underline{V}$ at time T**)

The arguments in this step are based on analogous results given in [1, Theorem 14] and [2, Theorem 3] in the periodic setting, with some corrections due to the unboundedness of our domain.

1. $\underline{V}(T,x) \ge g(x)$ for all $x \in \mathbb{R}^n$

By Proposition 8.1 we know that for all $\epsilon > 0$, $(t,x,y) \in (0,T) \times \mathbb{R}^n \times \mathbb{R}^m$ there exists a constant $M \in \mathbb{R}$ such that

$$V^\epsilon(t,x,y) \ge v^\epsilon(t,x,y) \ge M. \tag{85}$$

We now fix $\bar{x} \in \mathbb{R}^n$ and $r > 0$. Moreover we consider a smooth nonpositive function ψ such that $\psi(\bar{x}) = 0$ and $\psi(x) + \inf_{|z-\bar{x}|\le r} g(z) \le M$ for every $x \in \partial B_r(\bar{x})$. Next, we consider a positive constant C such that

$$H(x,y,D_x\psi,D^2_{xx}\psi,0) < C \quad \text{for all } x \in \bar{B}_r(\bar{x}) \text{ and } y \in \mathbb{R}^m \tag{86}$$

where H is defined in (46) and we are using assumptions $(A3)$ and $(A4)$.

We define the function

$$\Psi(t,x) = \inf_{|z-\bar{x}|\le r} g(z) + \psi(x) - C(T-t) \tag{87}$$

and we check that it is a subsolution to the parabolic problem

$$\begin{cases} -V_t + \mathcal{H}(x, y, D_x V, \frac{D_y V}{\epsilon}, D^2_{xx} V, \frac{D^2_{yy} V}{\epsilon}, \frac{D^2_{xy} V}{\sqrt{\epsilon}}) = 0 & \text{in } (0, T) \times B_r(\bar{x}) \times \mathbb{R}^m \\ V(t, x, y) = M & \text{in } (0, T) \times \partial B_r(\bar{x}) \times \mathbb{R}^m \\ V(T, x, y) = g(x) & \text{in } B_r(\bar{x}) \times \mathbb{R}^m \end{cases} \tag{88}$$

where $\mathcal{H}$ is given by (45). Indeed Ψ is smooth and

$$\begin{aligned} & -\Psi_t + \mathcal{H}(x, y, D_x\Psi, \frac{D_y\Psi}{\epsilon}, D^2_{xx}\Psi, \frac{D^2_{yy}\Psi}{\epsilon}, \frac{D^2_{xy}\Psi}{\sqrt{\epsilon}}) \\ & = -C + H(x, y, D_x\psi, D^2_{xx}\psi, 0) \leq 0 \end{aligned}$$

by (86) and for all $t \in (0, T)$, $x \in B_r(\bar{x})$ and $y \in \mathbb{R}^m$. Moreover, for all $x \in B_r(\bar{x})$ and $y \in \mathbb{R}^m$

$$\Psi(T, x) = \inf_{|z-\bar{x}| \leq r} g(z) + \psi(x) \leq g(x)$$

because ψ is ≤ 0. Finally, for every $t \in (0, T)$, $x \in \partial B_r(\bar{x})$ and $y \in \mathbb{R}^m$

$$\Psi(t, x) = \inf_{|z-\bar{x}| \leq r} g(z) + \psi(x) - C(T - t) \leq M$$

because of the definition of ψ and $-C(T - t) \leq 0$.

Therefore Ψ is a viscosity subsolution of (88). Since V^ϵ is a viscosity solution to (49) that satisfies (85) then it is a viscosity supersolution to (88). Then, by Da Lio-Ley comparison principle we can conclude that $\Psi(t', x') \leq V^\epsilon(t', x', y)$ for all $\epsilon > 0$, $0 < t' \leq T$ and $x' \in \bar{B}_r(\bar{x})$, $y \in \mathbb{R}^m$. Taking first $\inf_{y \in \mathbb{R}^m}$ on both sides of the above expression and then the $\liminf_{(\epsilon, t', x') \to (0, t, x)}$, we deduce that $\inf_{|z-\bar{x}| \leq r} g(z) + \psi(x) - C(T - t) \leq \underline{V}(t, x)$. Letting $r \to 0$, $(t, x) \to (T, \bar{x})$ and recalling that $\psi(\bar{x}) = 0$ we obtain

$$g(\bar{x}) \leq \underline{V}(T, \bar{x})$$

This proves that $\underline{V}$ is a supersolution at the terminal boundary.

2. $\bar{V}_R(T, x) \leq g(x)$ for all R and $x \in \mathbb{R}^n$

The proof is similar. The main differences being the choice of M and the domain of y which is $B_R(0)$ now.

We fix $\bar{x} \in \mathbb{R}^n$, $r > 0$ and a constant M such that $V^\epsilon(t, x, y) \leq M$ for every $\epsilon > 0$, $x \in \bar{B}_r(\bar{x})$ and $y \in \bar{B}_R(0)$. Observe that this is possible by estimates (55). Moreover, we consider a smooth nonnegative function $\psi_1(x)$ such that $\psi_1(\bar{x}) = 0$ and $\psi_1(x) + \sup_{|x-\bar{x}|\le r} g(x) \ge M$ for every $x \in \partial B_r(\bar{x})$, and the solution $\phi_2 \ge 0$ of

$$\inf_{|x-\bar{x}|\le r} H(x, y, D_x\psi_1, D^2_{xx}\psi_1, 0) - \frac{1}{2}\Delta\phi_2(y) + \frac{1}{\theta}|D\phi_2(y)|^\theta = -\lambda_2 \quad \text{in } B_R(0)$$

and complemented with one of the boundary conditions introduced in the section about approximations, namely,

$$\phi_2(y) \to +\infty \text{ as } y \to \partial B_R(0) \quad \text{if } 1 < \theta \le 2$$

or

$$\inf_{|x-\bar{x}|\le r} H(x, y, D_x\psi_1, D^2_{xx}\psi_1, 0) - \frac{1}{2}\Delta\phi_2(y) + \frac{1}{\theta}|D\phi_2(y)|^\theta \ge -\lambda_2 \text{ in } \bar{B}_R(0) \quad \text{if } \theta > 2.$$

Observe that by Theorem 3.2 ($\theta \le 2$) or 3.6 ($\theta > 2$) there exits a unique solution pair (λ_2, ϕ_2) of the above problem.

Next, we consider a constant $C \ge \lambda_2$ and we define Ψ as follows

$$\Psi(t, x, y) = \sup_{|z-\bar{x}|\le r} g(z) + \psi_1(x) + \epsilon\phi_2(y) + C(T - t)$$

and we claim that Ψ is a supersolution to

$$\begin{cases} -V_t + \mathcal{H}(x, y, D_xV, \frac{D_yV}{\epsilon}, D^2_{xx}V, \frac{D^2_{yy}V}{\epsilon}, \frac{D^2_{xy}V}{\sqrt{\epsilon}}) = 0 & \text{in } (0, T) \times B_r(\bar{x}) \times B_R(0) \\ V(t, x, y) = M & \text{in } (0, T) \times \partial B_r(\bar{x}) \times B_R(0) \\ V(t, x, y) = +\infty & \text{in } (0, T) \times B_r(\bar{x}) \times \partial B_R(0) \\ V(T, x, y) = g(x) & \text{in } B_r(\bar{x}) \times B_R(0) \end{cases} \tag{89}$$

if $\theta \le 2$, and a supersolution to

$$\begin{cases} -V_t + \mathcal{H}(x, y, D_xV, \frac{D_yV}{\epsilon}, D^2_{xx}V, \frac{D^2_{yy}V}{\epsilon}, \frac{D^2_{xy}V}{\sqrt{\epsilon}}) = 0 & \text{in } (0, T) \times B_r(\bar{x}) \times B_R(0) \\ V(t, x, y) = M & \text{in } (0, T) \times \partial B_r(\bar{x}) \times B_R(0) \\ -V_t + \mathcal{H}(x, y, D_xV, \frac{D_yV}{\epsilon}, D^2_{xx}V, \frac{D^2_{yy}V}{\epsilon}, \frac{D^2_{xy}V}{\sqrt{\epsilon}}) = 0 & \text{in } (0, T) \times B_r(\bar{x}) \times \partial B_R(0) \\ V(T, x, y) = g(x) & \text{in } B_r(\bar{x}) \times B_R(0) \end{cases} \tag{90}$$

if $\theta > 2$.

In fact,

$$
\begin{aligned}
& -\Psi_t + \mathcal{H}(x, y, D_x\Psi, \frac{D_y\Psi}{\epsilon}, D^2_{xx}\Psi, \frac{D^2_{yy}\Psi}{\epsilon}, \frac{D^2_{xy}\Psi}{\sqrt{\epsilon}}) \\
&= C + H(x, y, D_x\psi_1, D^2_{xx}\psi_1, 0) - \frac{1}{2}\Delta\phi_2 + \frac{1}{\theta}|D\phi_2|^\theta \\
&\geq C + \inf_{|x-\bar{x}|\leq r} H(x, y, D_x\psi_1, D^2_{xx}\psi_1, 0) - \frac{1}{2}\phi_2 + \frac{1}{\theta}|D\phi_2|^\theta \\
&\geq C - \lambda_2 \geq 0
\end{aligned}
$$

because C was chosen to be $C \geq \lambda_2$ and this for all $t \in (0, T)$, $x \in B_r(\bar{x})$ and $y \in B_R(0)$. Furthermore, for all $x \in B_r(\bar{x})$ and $y \in B_R(0)$

$$
\Psi(T, x, y) = \sup_{|z-\bar{x}|\leq r} g(z) + \psi_1(x) + \epsilon\phi_2(y) \geq g(x)
$$

because ψ_1 and ϕ_2 are nonnegative. Also,

$$
\Psi(t, x, y) = \sup_{|z-\bar{x}|\leq r} g(z) + \psi_1(x) + \epsilon\phi_2(y) + C(T-t) \geq M
$$

for every $t \in (0, T)$, $x \in \partial B_r(\bar{x})$ and $y \in B_R(0)$ because of our definition of ψ_1 and the fact that ϕ_2 is ≥ 0, and $C(T-t) \geq 0$. Finally, for every $t \in (0, T)$, $x \in B_r(\bar{x})$ and $y \in \partial B_R(0)$

$$
\Psi(t, x, y) = +\infty \geq M \text{ if } \theta \leq 2 \text{ since } \phi_2 \text{ blows up at the boundary}
$$

and

$$
-\Psi_t + \mathcal{H}(x, y, D_x\Psi, \frac{D_y\Psi}{\epsilon}, D^2_{xx}\Psi, \frac{D^2_{yy}\Psi}{\epsilon}, \frac{D^2_{xy}\Psi}{\sqrt{\epsilon}}) \geq C - \lambda_2 \geq 0 \text{ if } \theta > 2
$$

as done above for $(0, T) \times B_r(\bar{x}) \times B_R(0)$.

From our choice of M, we get that V^ϵ is a subsolution to (89) ($\theta \leq 2$) or (90) ($\theta > 2$). Since Ψ is a supersolution of (89) ($\theta \leq 2$) or (90) ($\theta > 2$) in $(0, T) \times B_r(\bar{x}) \times B_R(0)$ that satisfies the boundary condition correspondent for each θ, we can conclude by the comparison result proved in [9] ($\theta \leq 2$) or or in Theorem 2.1 of [38] (when $\theta > 2$), that

$$
V^\epsilon(t, x, y) \leq \Psi(t, x, y) = \sup_{|z-\bar{x}|\leq r} g(z) + \psi_1(x) + \epsilon\phi_2(y) + C(T-t)
$$

for every $\epsilon > 0$ and $(t, x, y) \in [0, T] \times \bar{B}_r(\bar{x}) \times \bar{B}_R(0)$. In particular, we have

$$
V^\epsilon(t', x', y') \leq \Psi(t', x', y') = \sup_{|z-\bar{x}|\leq r} g(z) + \psi_1(x') + \epsilon\phi_2(y') + C(T-t')
$$

for every $\epsilon > 0$ and $(t', x', y') \in (0,T) \times B_r(\bar{x}) \times B_R(0)$. We compute the upper relaxed semilimit on both sides of the above expression as $(\epsilon, t', x', y') \to (0, t, x, y)$ for $t \in (0,T)$, $x \in B_r(\bar{x})$, $y \in B_R(0)$ and get

$$\bar{V}_R(t,x) \leq \sup_{|z-\bar{x}| \leq r} g(z) + \psi_1(x) + C(T-t).$$

Remark: Observe that ϕ_2 is regular in $B_R(0)$, a bounded domain, and we are not including the behaviour of ϕ_2 at the boundary, therefore $\epsilon\phi_2(y) \to 0$ as $\epsilon \to 0$ and $\underline{y \in B_R(0)}$.

Taking the limit for $r \to 0$ and $(t,x) \to (T, \bar{x})$ and recalling that $\psi_1(\bar{x}) = 0$ we obtain

$$\bar{V}_R(T, \bar{x}) \leq g(\bar{x})$$

as we wished to show. This proves that $\bar{V}_R$ is a subsolution at the terminal boundary.

<u>Conclusion</u>: Using the previous Steps, we can now conclude that $\underline{V}$ is a viscosity super solution of the limit PDE (63) whereas $\bar{V}_R$ is a viscosity subsolution of

$$\begin{cases} -V_t + \bar{H}_R(x, D_x V, D^2_{xx} V) = 0 & \text{in } (0,T) \times \mathbb{R}^n \\ V(T,x) = g(x). \end{cases} \tag{91}$$

Step 5 ($\bar{V}$ **is a subsolution of the limit PDE** (63))

We just showed that for all fixed R $\bar{V}_R$ is a viscosity subsolution to (91). Since $\bar{H}_R \geq \bar{H}$ (this is Proposition 6.3), $\bar{V}_R$ being a subsolution with $\bar{H}_R$ implies that it is a subsolution with $\bar{H}$ for the same R. Since as R tends to infinity, $\bar{V}_R$ form an increasing sequence and we have estimate (64) for all R, $\sup_R \bar{V}_R$ is finite. We want now conclude that $\sup_R \bar{V}_R$ is a viscosity subsolution of the limit PDE (63) but in general it is not upper semi-continuous. Therefore we consider the upper semi-continuous envelope $\bar{V} = (\sup_R \bar{V}_R)^*$ which coincides with $\limsup^*_R \bar{V}_R$ by Lemma 2.18 of Chapter V in [5]. Then, we have

$$-\bar{V}_t + \bar{H}(x, D_x \bar{V}, D^2_{xx} \bar{V}) \leq 0 \text{ if } 0 < t < T \tag{92}$$

and

$$\text{either } \bar{V}(T,x) \leq g(x) \text{ or } -\bar{V}_t + \bar{H}(x, D_x \bar{V}, D^2_{xx} \bar{V}) \leq 0 \text{ at } t = T \tag{93}$$

where we also use that $\bar{H}$ is degenerate elliptic (Proposition 9.4).

We will now show that $\bar{V}$ satisfies the terminal data in the classical sense, i.e., $\bar{V}(T,x) \leq g(x)$. For all fixed $x \in \mathbb{R}^n$, $\delta > 0$ and $C > 0$ large, we define

$$\chi(t,z) = \bar{V}(t,z) - \frac{|z-x|^2}{\delta} + C(T-t). \tag{94}$$

Since $\bar{V}(t,z)$ is locally bounded (we have estimate (64)) and for all $t \in (0,T)$ we have that $-\frac{|z-x|^2}{\delta} + C(T-t) \to -\infty$ as $|z| \to \infty$, then χ has a local maximum point (t^*, z^*) in a neighborhood of (T,x).

Suppose $t^* < T$. Then using χ as a test function, condition (92) and the properties of the maximum point, we can conclude that

$$C + \bar{H}\big(z^*, \frac{2(z^*-x)}{\delta}, \frac{2}{\delta}\mathbb{I}\big) \leq 0. \tag{95}$$

But if we fix $\delta > 0$ small and we take C large enough, then (95) cannot occur. Therefore $t^* = T$ and we necessarily have that $\bar{V}(T,z^*) \leq g(z^*)$ because of condition (93) and the argument just showed.

Since (T,z^*) is a local maximum point of χ in a neighborhood of (T,x), then $\chi(T,z^*) \geq \chi(T,x)$, i.e., $\bar{V}(T,z^*) \geq \bar{V}(T,x) + \frac{|z^*-x|^2}{\delta} \geq \bar{V}(T,x)$. Consequently

$$\bar{V}(T,x) \leq g(z^*)$$

since we know that $\bar{V}(T,z^*) \leq g(z^*)$. Letting now $\delta \to 0$, keeping C large enough, we see that $z^* \to x$ and so

$$\bar{V}(T,x) \leq g(x)$$

in the limit.

Hence $\bar{V} = (\sup_R \bar{V}_R)^*$ is an upper semi-continuous viscosity subsolution of

$$\begin{cases} -V_t + \bar{H}(x, D_x V, D^2_{xx} V) = 0 & \text{in } (0,T) \times \mathbb{R}^n \\ V(T,x) = g(x) \end{cases} \tag{96}$$

as we wished to show. ∎

Corollary 10.2 *Suppose that, in addition to the assumptions of Theorem 10.1, V^ϵ converges uniformly on the compact subsets of $[0,T] \times \mathbb{R}^n \times \mathbb{R}^m$ to some function $V(t,x)$ such that $V = g$ at $t = T$. Then V is a viscosity solution of* (63).

Proof First, we see that $\underline{V} = \bar{V} = V$ in $[0,T) \times \mathbb{R}^n$, by using the convergence of V^ϵ for $t < T$ and $V = g$ at $t = T$. This implies that V is continuous and it is a viscosity solution of the limit PDE (63). ∎

The second corollary is most useful because it proves the local uniform convergence of V^ϵ. It supposes that the comparison principle holds for the limit equation (63) in the sense that every upper-semicontinuous viscosity subsolution must be smaller than every lower-semicontinuous viscosity supersolution.

Definition (**Comparison principle**) Let $\mathbb{H}$ be a Hamilton-Jacobi-Bellman operator defined in $(0,T)\times\mathbb{R}^n$ and consider the evolutive equation

$$u_t + \mathbb{H}(x, D_x u, D^2_{xx}u) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n. \tag{97}$$

We say that $\mathbb{H}$ satisfy the comparison principle if for every upper-semicontinuous viscosity subsolution of (97) and for every lower semi-continuous viscosity supersolution of (97) such that $u_1(T,x)\le u_2(T,x)$ for all $x\in\mathbb{R}^n$, then $u_1(t,x)\le u_2(t,x)$ for all $[0,T]\times\mathbb{R}^n$.

Corollary 10.3 *Suppose that, in addition to the assumptions of Theorem 10.1,* $\bar H$ *satisfy the comparison principle for* (63)*. Then* V^ϵ *converges uniformly on the compact subsets of* $[0,T)\times\mathbb{R}^n\times\mathbb{R}^m$ *to the unique viscosity solution of* (63)*.*

Proof Observe that by definition $\underline{V}\le\bar V$. Moreover, the Hamiltonian $\bar H$ satisfies the usual regularity assumptions so as to get comparison result between upper-semicontinuous viscosity subsolution and lower-semicontinuous viscosity supersolution of (63). Then we deduce $\underline{V}\ge\bar V$. Therefore $\underline{V}=\bar V := V$. In particular V is continuous (is a LSC and an USC function). We will now show that V^ϵ converges locally uniformly to V as $\epsilon\to 0^+$ in $[0,T)\times\mathbb{R}^n\times\mathbb{R}^m$.

Assume by contradiction that there exist a compact set $K\subset[0,T)\times\mathbb{R}^n\times\mathbb{R}^m$, $\delta>0$, $\epsilon_n\to 0^+$, and $(t_n,x_n,y_n)\in K$ such that either

$$V^{\epsilon_n}(t_n,x_n,y_n) - V(t_n,x_n) > \delta$$

or

$$V^{\epsilon_n}(t_n,x_n,y_n) - V(t_n,x_n) < -\delta.$$

Since K is a compact set, we may assume that $(t_n,x_n,y_n)\to(t,x,y)$. Therefore passing to the relaxed semilimits as $(\epsilon_n,t_n,x_n,y_n)\to(0,t,x,y)$ and using the continuity of V we arrive at either

$$\bar V(t,x) - V(t,x) > \delta$$

or

$$\underline{V}(t,x) - V(t,x) < -\delta$$

which are both in contradiction with $\underline{V}=\bar V := V$.

In particular, we proved that V^ϵ converges uniformly on the compact subsets of $[0,T)\times\mathbb{R}^n\times\mathbb{R}^m$ to V. Corollary 10.2 ensures that V is a viscosity solution of (63). It is the unique solution by comparison. ∎

10.2 Examples

In this subsection we present some examples in which our convergence theorem can be applied.

We start by recalling the definition of $H(x, y, p, M, 0)$,

$$H(x,y,p,M,0) = \sup_{u\in U}\{-\text{trace}(\sigma(x,y,u)\sigma(x,y,u)^T M) - F(x,y,u)\cdot p - l(x,y,u)\}.$$

Example 1 (F and σ independent of y, bounded in x, and l does not depend on u)

In this first example we are assuming that besides the standing assumptions F and σ do not depend on y and are bounded in x for all u fixed and l is independent of u and we also have that

$$\exists C_1, C_2 > 0 \text{ such that } C_1(1+|y|^\alpha) \leq l(x,y) \leq C_2(1+|y|^\alpha).$$

In this case, we can write H as

$$H(x,y,p,M,0) = H'(x,p,M) + l(x,y)$$

where

$$H'(x,p,M) = \sup_{u\in U}\{-\text{trace}(\sigma\sigma^T(x,u)M) - F(x,u)\cdot p\}$$

and it is easy to check, using our assumptions on F and σ, that

$$|H'(x,p,M) - H'(x',p',M')| \leq L|x-x'|(1+|M|+|p|) + C(|M-M'|+|p-p'|)$$

for some constants $L, C > 0$.

By Proposition 1.9 we know that

$$\bar{H}(x,p,M) = H'(x,p,M) + \bar{l}(x)$$

because H' is a constant in y.

On the other hand, we have for all $x, x' \in B_R(0)$ that

$$\sup_{y\in\mathbb{R}^m}\left(\left|\frac{|l(x,y)-l(x',y)|}{1+|y|^\alpha}\right|\right) \leq m_R(|x-x'|)$$

since $|l(x,y) - l(x',y)| \leq (1+|y|^\alpha)m_R(|x-x'|)$ for all $x, x' \in B_R(0)$ and $y \in \mathbb{R}^m$ by $(A4)$.

Therefore we can use Corollary 5.4 and we have for all $x, x' \in B_R(0)$

$$|\bar{l}(x) - \bar{l}(x')| \le K m_R(|x - x'|) \max(|\bar{l}(x)|, |\bar{l}(x')|).$$

We now remind that

$$C_1(1 + |y|^\alpha) \le l(x, y) \le C_2(1 + |y|^\alpha).$$

By monotonicity of $\lambda^*(l) = -\bar{l}(x)$ (Proposition 1.8) we have

$$\lambda^*\big(C_1(1 + |y|^\alpha)\big) \le \lambda^*(l) \le \lambda^*\big(C_2(1 + |y|^\alpha)\big).$$

Then, by Propositions 1.9 and 1.10, we obtain

$$C_1 + C_1^{\frac{\theta^*}{\theta^*+\alpha}} \lambda^*(|y|^\alpha) \le \lambda^*(l) \le C_2 + C_2^{\frac{\theta^*}{\theta^*+\alpha}} \lambda^*(|y|^\alpha).$$

This tell us that $\lambda^*(l)$ (and thus $\bar{l}$) is uniformly bounded by a constant which depends only on C_1 and C_2. The dependence in x appears only through C_1 and C_2 and $\lambda^*(|y|^\alpha)$ depends only on α which is fixed.

Then, for all $x, x' \in B_R(0)$ we have

$$|\bar{l}(x) - \bar{l}(x')| \le K m_R(|x - x'|)$$

for some bigger constant $K > 0$.

Hence

$$\begin{aligned}
|\bar{H}(x, p, M) - \bar{H}(x', p', M')| &= |(H'(x, p, M) + \bar{l}(x)) - (H'(x', p', M') + \bar{l}(x'))| \\
&\le |H'(x, p, M) - H'(x', p', M')| + |\bar{l}(x) - \bar{l}(x')| \\
&\le |H'(x, p, M) - H'(x', p', M')| + |\bar{l}(x) - \bar{l}(x')| \\
&\le L|x - x'|(1 + |M| + |p|) + C(|M - M'| + |p - p'|) \\
&+ K m_R(|x - x'|)
\end{aligned}$$

for all $x, x' \in B_R(0)$.

By Da Lio-Ley comparison principle, we can conclude that $\bar{H}$ satisfies the comparison.

A special case of this example would be decorrelating the variables x_1 and x_2:

$$H(x_1, x_2, y, p, M) = H'(x_1, p, M) + l(x_2, y).$$

Example 2 (using formula (34))

In this example we suppose, besides the standing assumptions, that $l(x, y, u) = l_1(x, u) + l_2(y)$ where l_2 and l_1 are continuous functions that satisfy

$$\exists l_0 > 0 \text{ such that } -l_0 + l_0^{-1}|y|^\alpha \leq l_2(y) \leq l_0(1 + |y|^\alpha)$$

and

$$|l_1(x, u) - l_2(x', u)| \leq m_R(|x - x'|)$$

for all $x, x' \in B_R(0)$ and all $u \in U$. We will also assume that F and σ are bounded in all variables.

Then H can be written as

$$H(x, y, p, M, 0) = H'(x, y, p, M) + l_2(y)$$

where

$$H'(x, y, p, M) = \sup_{u \in U}\{-\text{trace}(\sigma(x, y, u)\sigma(x, y, u)^T M) - F(x, y, u) \cdot p - l_1(x, u)\}.$$

and it is easy to see that it satisfies for all $x, x' \in B_R(0)$

$$|H'(x, y, p, M) - H'(x', y, p', M')| \leq L|x - x'|(1 + |M| + |p|) + C(|M - M'| + |p - p'|) + m_R(|x - x'|).$$

Using formula (34) we can conclude that

$$|\bar{H}(x, p, M) - \bar{H}(x', p', M')| \leq \sup_{y \in \mathbb{R}^m} |H'(x, y, p, M) - H'(x', y, p', M')|$$

because the l_2 in both H's cancel out.

Then

$$|\bar{H}(x, p, M) - \bar{H}(x', p', M')| \leq L|x - x'|(1 + |M| + |p|) + C(|M - M'| + |p - p'|) + m_R(|x - x'|)$$

Again, we can apply the comparison result of Da Lio-Ley.

This example is also interesting because we can prove in another way the continuity of $\bar{H}$ from the above expression. Indeed,

$$|\bar{H}(x, p, M) - \bar{H}(x, p', M')| \leq C(|p - p'| + |M - M'|)$$

that is, $\bar{H}$ is Lipschitz in the variables p and M, and

$$|\bar{H}(x,p,M) - \bar{H}(x',p,M)| \le L|x-x'|(1+|M|+|p|) + m_R(|x-x'|)$$

which shows the continuity in x.

Corollary 10.4 *Let $\theta = \alpha^*$ and suppose that we are in one of the Examples 1 or 2. Then, V^ϵ converges uniformly on the compact subsets of $[0,T)\times\mathbb{R}^n\times\mathbb{R}^m$ to the unique viscosity solution of* (63).

Proof By Proposition 8.3 V^ϵ defined in (44) is the unique continuous viscosity solution to the Cauchy problem (49) with at most α-growth in x and y. Since in all these cases $\bar{H}$ satisfy the comparison of Da Lio-Ley and both $\underline{V}$ and $\bar{V}$ satisfy the growth condition (64), we deduce that $\underline{V} \ge \bar{V}$. Therefore $\underline{V} = \bar{V} := V$ and we conclude by Corollary 10.3. ∎

Appendix A - Some Stochastic Results

We start by reviewing some definitions and give some well known results.

A collection of σ-fields F_t such that $F_s \subseteq F_t$ if $s \leq t$ is called a *filtration*. A σ-field $\mathcal{G}$ is complete if $N \in \mathcal{G}$ whenever $\mathbb{P}(N) = 0$. A filtration is right continuous if $F_{t^+} = F_t$, where $F_{t^+} = \cap_{\epsilon>0} F_{t+\epsilon}$. We say that the filtration satisfies the usual conditions if each F_t is complete and the filtration is right continuous. In this text we consider only filtrations that satisfy the usual conditions.

We say that a stochastic process X is *adapted* to a filtration $\{F_t\}$ if X_t is F_t measurable for each t. Often one starts with a stochastic process X and defines F_t to be the smallest σ-field with respect to which $\{X_s : s \leq t\}$ is measurable. In such a case X is automatically adapted.

Definition (Brownian motion or Wiener process) A one-dimensional *Brownian motion* or *Wiener process* adapted to a filtration $\mathcal{F}_t$ is a process W_t started at 0 if

- $W_0 = 0$ a.s.
- $W_t - W_s$ is a normal random variable with mean 0 and variance $t - s$ whenever $s < t$.
- $W_t - W_s$ is independent of the σ-field generated by $\{W_r : r \leq s\}$.
- With probability 1 the map $t \to W_t(\omega)$ is continuous as a function of t.

If instead of $W_0 = 0$ we have $W_0 = x$, we say we have a Brownian motion started at x. A d-dimensional Brownian motion is a d-dimensional process whose components are independent one-dimensional Brownian motions. An example of brownian motion is tea diffusing in water, the particles swirl in random directions.

First hitting time and first exit time of a Borel subset: If X_t is a stochastic process and A a Borel subset of $\mathbb{R}^d$, we write

$$T_A = T(A) = \inf\{t \geq 0 : X_t \in A\}$$

and

$$\tau_A = \tau(A) = \inf\{t > 0 : X_t \notin A\}$$

for the first hitting time and first exit time of A, respectively.

Definition (Stopping times) A random variable $T : \Omega \mapsto [0, \infty)$ is called a *stopping time* with respect to the filtrarion $\mathcal{F}_t$ provided

$$\{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0.$$

This says that the set of all $\omega \in \Omega$ such that $T(\omega) \leq t$ is an $\mathcal{F}_t$-measurable set. Note that T is allowed to take on the value ∞, and also any constant is a stopping time.

Definition (Martingale) A process X_t is a *martingale* if for each t and $s < t$ the random variable X_t is integrable and adapted to $\mathcal{F}_t$ and $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ almost surely (a.s.).

Example: The concept of the martingale is a generalisation of the sequence of partial sums arising from a sequence $\{X_n\}$ of independent and identically distributed random variables with mean 0. Let $S_n = X_1 + ... + X_n$. Then the sequence $\{S_n\}$ is a martingale.

Definition (Local martingale and semimartingale) A process X_t is a *local martingale* if there exist stopping times $T_n \uparrow \infty$ such that $X_{t \wedge T_n}$ is a martingale for each n and it is a *semimartingale* if it is the sum of a local martingale and a process that is locally of finite bounded variation (i.e., finite bounded variation on every interval $[0, t]$).

Remark: Semimartingales are "good integrators", forming the largest class of processes with respect to which the stochastic integral can be defined (see below). The class of semimartingales is quite large (including, for example, all continuously differentiable processes and Brownian motion).

We will be dealing exclusively with continuous processes, so all of our processes will have continuous paths.

Definition (Quadratic variation) We say that a local martingale M_t has quadratic variation $< M >_t$ (sometimes written $< M, M >_t$) if $< M >_t$ is the unique increasing continuous process such that $M_t^2 - < M >_t$ is a local martingale. In case that X_t is a semimartingale, i.e. $X_t = M_t + A_t$ where M_t is a local martingale and A_t has paths of locally finite bounded variation, we define $< X_t >$ to be $< M_t >$.

If X_t and Y_t are two semimartingales, we define $< X, Y >_t$ by polarization: $< X, Y >_t = \frac{1}{2}(< X + Y >_t - < X >_t - < Y >_t)$.

Definition (Stochastic Integral) Let M_t be a local martingale and let H_t be a process adapted to the filtration $\{F_t\}$ satisfying the usual conditions. If $\int_0^t H_s^2 d < M >_s < \infty$ for all t, we define the *stochastic integral* $N_t = \int_0^t H_s dM_s$ to be the local martingale such that $< N, L >_t = \int_0^t H_s d < M, L >_s$ for all martingales L_t adapted to $\{\mathcal{F}_t\}$.

For $X_t = M_t + A_t$ a semimartingale, the stochastic integral $\int_0^t H_s dX_s$ is defined by

$$\int_0^t H_s dX_s = \int_0^t H_s dM_s + \int_0^t H_s dA_s,$$

where the first integral on the right is a stochastic integral and the second integral on the right is a Lebesgue-Stieltjes integral.

Observation: We observe that we can extend this construction to more general processes H_s by linearity and taking limits in L^2. We refer the reader to [12] to see this.

The most important theorem of stochastic integration is Itô's formula. This is also known as the change of variables formula.

Itô's formula: Let X_t be a semimartingale and $f \in C^2(\mathbb{R})$. Itô's formula is the equation

$$f(X_t) - f(x_0) = \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d < X >_s .$$

There is a multivariate version of Itô's formula,

$$f(X_t) - f(x_0) = \int_0^t \sum_{i=1}^d \partial_i f(X_s) dX_s^i + \frac{1}{2} \int_0^t \sum_{i,j=1}^d \partial_{ij} f(X_s) d < X^i, X^j >_s .$$

Here $X_t = (X_t^1, ..., X_t^d)$ is a d-dimensional semimartingale, that is, a process in $\mathbb{R}^d$, each of whose components is a semimartingale and $f \in C^2(\mathbb{R}^d)$.

We next list some important results which will be used in the following text.

Lemma A.1 *(Markov's inequality) If X is any nonnegative integrable random variable and $a > 0$, then*

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$

Lemma A.2 *(Gronwall's lemma) Suppose $g : [0, \infty) \to \mathbb{R}$ is bounded on each finite interval, is measurable, and there exist A and B such that for all t $g(t) \leq A + B \int_0^t g(s) ds$. Then $g(t) \leq A \exp(Bt)$ for all t.*

Lemma A.3 *(Burkholder-Davis-Gundy inequality) For $p > 0$ there exist a constant $c_1(p)$ such that if M_t is a continuous martingale and T a stopping time, then* $\mathbb{E}\big[\sup_{s\leq T}|M_s|^p\big] \leq c_1(p)\mathbb{E}\big[< M >_T^{p/2} \big]$

All these proofs can be found in [11] or [12].

Appendix B - Gradient Estimate

In this Appendix, we present some results and estimates needed for this thesis.

We recall that

$$G[\phi](y) = -\frac{1}{2}\Delta\phi(y) + \frac{1}{\theta}|D\phi(y)|^{\theta} - f(y)$$

and that (EP) can be written equivalently as

$$G[\phi](y) = \mu \text{ in } \mathbb{R}^m$$

with $\mu = -\lambda$.

Let I be the operator

$$I[\phi](y) := \frac{1}{\theta}|D\phi|^{\theta} - f(y).$$

We begin with the following result

Theorem B.1 *Let $R > 1$, $f_1 \in C^1(B_R)$ and $g_1 \in C^{2,\iota}(\partial B_R)$ where $\iota \in (0, 1)$. Then,*

(a) For any $\epsilon > 0$, the Dirichlet problem

$$G[\phi] + \epsilon\phi = f_1 \text{ in } B_R, \quad \phi = g_1 \text{ on } \partial B_R,$$

has a solution in the class $C^{2,\iota}(\bar{B}_R)$.

(b) The Dirichlet problem

$$G[\phi] = f_1 \text{ in } B_R, \quad \phi = g_1 \text{ on } \partial B_R,$$

has a solution in the class $C^{2,\iota}(\bar{B}_R)$ provided that there exists a function $u \in C^2(B_R) \cap C(\bar{B}_R)$ such that

$$G[u] < f_1 \text{ in } B_R, \quad u = g_1 \text{ on } \partial B_R.$$

Proof Claim (a) is a particular case of [30]. Claim (b) can be found in Theorem A.1 of [24] and it uses the convexity of the operator I and Theorem 6.14 of [23]. ∎

Theorem B.2 *Let Ω and Ω' be two bounded open sets in $\mathbb{R}^m$ such that $\bar{\Omega}' \subset \Omega$. For given $\epsilon \in [0,1)$ and $f_1 \in W^{1,\infty}_{loc}(\mathbb{R}^m)$, let $\phi \in C^2(\mathbb{R}^m)$ be a solution of the elliptic equation*

$$-\frac{1}{2}\Delta\phi + \frac{1}{\theta}|D\phi|^\theta + \epsilon\phi = f_1 \text{ in } \Omega.$$

Then, there exists a constant $K > 0$ depending only on m, θ and dist$(\Omega', \partial\Omega)$ such that

$$\sup_{\Omega'} |D\phi| \leq K(1 + \sup_{\Omega} |\epsilon\phi|^{\frac{1}{\theta}} + \sup_{\Omega} |f_1|^{\frac{1}{\theta}} + \sup_{\Omega} |Df_1|^{\frac{1}{2\theta-1}}).$$

Proof The proof is the same as in Theorem B.1 of [25]. ∎

Comment: To apply the classical Bernstein method, we need $\phi \in C^3$. However, one can see by taking suitable approximations that the same estimate holds for $\phi \in C^2$ provided all the coefficients are regular enough.

Next, we give a priori gradient estimate for C^2-solutions of (EP).

Corollary B.3 *(Gradient estimate for solutions of (EP)) Fix any $r > 0$. There exists $K > 0$ depending only on m and $\theta > 1$ such that*

$$\sup_{B_r} |D\phi| \leq K(1 + \sup_{B_{r+1}} |f - \lambda|^{\frac{1}{\theta}} + \sup_{B_{r+1}} |Df|^{\frac{1}{2\theta-1}}).$$

for any solution $\phi \in C^2(\mathbb{R}^m)$.

Proof Follows easily from the previous theorem by taking $\Omega' = B_r$, $\Omega = B_{r+1}$, $\epsilon = 0$ and $f_1 = f - \lambda$. ∎

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