Università degli Studi di Padova
Dipartimento di Matematica
Scuola di Dottorato di Ricerca in
Scienze Matematiche
Indirizzo: Matematica
XXVII Ciclo

Université de Bordeaux
École Doctorale de
Mathématiques et Informatique
Spécialité: Mathématiques
Pures

Finite morphisms of p-adic curves

Direttore della Scuola

Prof. Pierpaolo Soravia

Coordinatore dell'indirizzo

Prof. Franco Cardin

Supervisori

Prof. Francesco Baldassarri

Prof. Denis Benois

Dottorando

Velibor Bojković

Abstract

In this thesis we study finite morphisms $\varphi : Y \to X$ of quasi-smooth k-analytic curves which admit finite semistable triangulations, and where k is algebraically closed field, complete with respect to a non-trivial, nonarchimedean valuation and of mixed characteristic. We introduce the notion of (strictly) φ-compatible (strictly) semistable triangulations of Y and X, respectively, and prove their existence as well as various consequences in terms of "compatible partitions" of Y and X, and at the same time provide a new proof of existence of strictly semistable formal models of Y and X, respectively, to which φ extends as a finite morphism.

We introduce and study the main properties of the pro-category $\mathbb{W}$ whose objects are built from systems of wide open curves and inclusions. It is a full subcategory of the pro-category of k-analytic curves. We introduce a Grothendieck topology on $\mathbb{W}$ making it a site, and exploit the "pro" structure of the objects which makes them behave particularly nice in coverings, to study (hyper)cohomology groups of complexes of coherent sheaves on k-analytic curves, and in particular we provide a new point of view on dagger curves and their de Rham cohomology.

Finally, we state and prove the Riemann-Hurwitz formula for finite morphisms of pro-wide open curves, which in particular gives Riemann-Hurwitz formula for compact, connected, quasi-smooth k-analytic curves.

Riassunto

In questa tesi studiamo morfismi finiti $\varphi : Y \to X$ di curve quasi-lisce k-analitiche, che ammettono triangolazioni finite semistabili, dove k è un campo algebricamente chiuso, completo rispetto ad una valutazione non-archimedea, non-triviale, in caratteristica mista. Introduciamo la nozione di triangolazioni (strettamente) semistabili (strettamente) φ-compatibili di Y ed X, rispettivamente, e dimostriamo la loro esistenza, così come varie conseguenze in termini di "partizioni compatibili" di Y ed X, ed allo stesso tempo otteniamo una nuova dimostrazione dell'esistenza dei modelli formali strettamente semistabili di Y ed X, rispettivamente, ai quali φ si estende come morfismo finito.

Introduciamo e studiamo le proprietà principali della pro-categoria $\mathbb{W}$, i cui oggetti sono ottenuti da sistemi di curve largamente aperte e inclusioni. È una sottocategoria piena della categoria di curve k-analitiche. Introduciamo una topologia di Grothendieck su $\mathbb{W}$, trasformandola in un sito, e utilizziamo la "pro" struttura degli oggetti, che li fa comportare particolarmente bene rispetto ai rivestimenti, per studiare i gruppi di (iper)coomologia dei complessi di fasci coerenti su curve k-analitiche ed in particolare otteniamo un nuovo punto di vista per le curve dagger e la loro coomologia di De Rham.

Infine, enunciamo e dimostriamo la formula di Riemann-Hurwitz per i morfismi finiti di curve pro-largamente aperte, che in particolare fornisce la formula di Riemann-Hurwitz per curve k-analitiche, quasi-lisce, connesse e compatte.

Resumé

Dans cette thèse, nous étudions les morphismes finis $\phi : Y \to X$ entre des courbes k-analytiques quasi-lisses admettant une triangulation semistable finie, pour k un corps algébriquement clos et complet par rapport à une valuation non archimédienne (non triviale), de caractéristique mixte. On introduit la notion de triangulations (strictement) semistables (strictement) phi-compatibles de Y et X, respectivement, et on prouve leur existence ainsi que quelque conséquences en terme de "partitions compatibles" de ces courbes, tout en donnant une nouvelle preuve de l'existence de mod̀les formels semistables de Y et X sur lesquels ϕ se relève en un morphisme fini.

On présente et étudie les propriétés principales de la pro-catégorie $\mathbb{W}$ dont les objets sont construits à partir de systèmes de courbes largement ouvertes munis d'inclusions. C'est une sous catégorie pleine de la pro-catégorie des courbes k-analytiques.On définit une topologie de Grothendieck sur $\mathbb{W}$, ce qui en fait un site, et les objets se comportent particulièrement bien vis à vis des recouvrements grâce à leur structure de pro objet. Nous étudions ainsi les groupes de (hyper)cohomologie de complexes de faisceaux cohérents sur les courbes k-analytiques, en donnant un nouveau point de vue sur les courbes dagues et leur cohomologie de De Rham.

Enfin, on énonce et prouve la formule de Riemann-Hurwitz pour les morphismes finis entre courbes pro-largement ouvertes, ce qui donne en particulier la formule pour les courbes k-analytiques compactes, connexes et quasi-lisses.

Acknowledgments

I would like to express my deepest and utmost gratitude to my supervisor Francesco Baldassarri for suggesting me the topic of my research and for his patience and guidance throughout the whole period of my PhD.

I am very thankful to my supervisor Denis Benois for his unconditional help and support and hospitality during my stay in Bordeaux, and especially for unveiling to me new points of view on the things that I was studying.

I would also like to thank to professors Bruno Chiarellotto, Marco Garuti, Qing Liu, Michel Matignon and Michael Temkin for answering my questions and in particular to Jérôme Poineau and Antoine Ducros for going through this thesis and for their most valuable comments on how to improve the text (both mathematically and aesthetically). I thank Federico Bambozzi and Vinsent Mineo for discussing with me many topics of rigid geometry.

Working on math problems was much more enjoyable because of my friends. I thank them all from the bottom of my heart.

Finally, I dedicate this thesis to my family whose love and support mean everything to me.

To Dušan, Ljubinka and Vladimir

Contents

0 Introduction **3**

1 Compact, connected, quasi-smooth k-analytic curves **15**

1.1 Structure of compact, connected, quasi-smooth k-analytic curves . . . 15

1.2 Skeleton . . . 24

1.3 Strong topology on k-analytic curves . . . 29

1.3.1 (Big) metric on k-analytic curves . . . 29

1.3.2 Strong topology on k-analytic curves . . . 34

2 Finite morphism between k-analytic curves **37**

2.1 Compatible models and triangulations . . . 37

2.1.1 Compatible triangulations of discs . . . 46

2.1.2 Compatible triangulations of annuli and discs . . . 50

2.1.3 Compatible triangulations of k-analytic curves . . . 55

2.1.4 Compatible partitions of curves . . . 61

2.2 Finite étale morphisms of open annuli . . . 64

2.3 Factoring finite morphisms . . . 68

2.4 Ramification locus . . . 70

2.4.1 Finite extensions of complete residue fields and discriminant function 70

2.4.2 Inseparable ramification locus for the liftings of relative Frobenius morphisms 77

3 Pro-wide open curves **81**

3.1 Category $\mathbb{W}$ 81

3.1.1 Cohomology on pro-wide open curves 92

4 Riemann-Hurwitz formula **99**

4.1 Generalized Riemann-Hurwitz formula 99

4.1.1 Finite morphisms of pro-wide open curves 99

4.1.2 Riemann-Hurwitz formula for compact k-analytic curves 102

4.1.3 Riemann-Hurwitz formula for pro-wide open curves 106

4.2 Riemann-Hurwitz formula for curves in characteristic $p > 0$ 107

Chapter 0

Introduction

The motivational and starting point of this thesis is to study the Riemann-Hurwitz formula for a finite morphisms

$$\varphi : Y \to X \tag{0.0.0.1}$$

where Y and X are compact, connected, quasi-smooth, strict Berkovich curves over an algebraically closed field k which is complete with respect to a nontrivial, nonarchimedean valuation, and of mixed characteristic $(0, p)$ [1]. As such curves are either affinoid, or projective (i.e. come from analytification of smooth, connected, projective k-algebraic curves), we are set in two cases: where the curves involved are affinoid, or projective.

One may note that in the later case the Riemann-Hurwitz formula is the classical (algebraic) one. However, if one stops here, the analytic and topological structure of the Berkovich curves remains unexploited. An example to see this is perhaps the first treatment of the Riemann-Hurwitz formula in the context of Berkovich *projective* curves which appeared in a paper by John Welliaveetil [39], where the author studied Riemann-Hurwitz formula having in mind the the *inner* structure of the curves, namely constructing

[1] For linearity of exposition, we choose to threat in this thesis only the case of a field with a nontrivial valuation. However, the general case will be considered in the following papers

compatible skeleta in Y and X to which the curves retract. This showed that the rich structure of the Berkovich curves provides a fertile ground for studying finite morphisms from new points of view.

Recently, some instances of Riemann-Hurwitz formula for affinoid curves started appearing and finding application in the theory of p-adic differential equations (see for example [3, 34]). Still, a separate treatment of the case where Y and X are affinoid spaces or the systematic treatment of both cases seems to be missing from the literature [2]. We hope to start filling in this gap with the present thesis.

Let us go back in time for a moment and start by (oversimplifying) a proof of classical Riemann-Hurwitz formula, which will serve us to regularly compare the notions and results that we introduce in the p-adic counterpart. So, for the moment let (0.0.0.1) denote a finite morphism between compact Riemann surfaces. We divide the proof in the following major steps:

(0) One proves that the ramification locus of φ (*i.e.* the set of ramified points *i.e.* the points in Y in whose neighborhoods φ fails to be 1-1) is finite;

(1) Let R be the ramification locus locus and $Q = \varphi(R)$ the branching locus. One takes a triangulation T of X such that the set of vertices $V(T)$ of triangles in T contains the points in Q and such that $\varphi^{-1}(T)$ is a triangulation of Y. Note that $R \subset V(\varphi^{-1}(T))$.

(2) One proves that over open edges and interiors of elements in T, the map φ is an n-fold covering, where n is the degree of φ. Around a ramified point $P \in Y$, the map φ is of degree $e_p \geq 2$. Around nonramified points, the map φ is of degree one, and for each point $x \in X$, one has $\sum_{y \in Y, \varphi(y)=x} e_y = n$

(3) Finally, one compares $\chi(Y) = v_{\varphi^{-1}(T)} - e_{\varphi^{-1}(T)} + f_{\varphi^{-1}(T)}$ with $\chi(X) = v_T - e_T + f_T$, where v_*, e_* and f_* stand for the number of vertices, edges and faces, respectively, in the set "$*$". It follows from (2) that $e_{\varphi^{-1}(T)} = n \cdot e_T$, $f_{\varphi^{-1}(T)} = n \cdot f_T$, while, because of

[2] This is not quite true, very recently a paper by Amina-Temkin-Trushin appeared where the authors proved using different methods the Riemann-Hurwitz formula discussed in this thesis. In the future we intend to provide a detailed comparison of two methods

ramification and (2) we have $\#V(\varphi^{-1}(T)) = n\#V(T) - \sum_{P\in V(\varphi^{-1}(T))}(e_P - 1)$. All in all, we conclude that

$$\chi(Y) - n\chi(X) = -\sum_{P\in V(\varphi^{-1}(T))}(e_P - 1) = -\sum_{P\in Y}(e_P - 1).$$

As we said in the beginning we take as an analogue of the compact Riemann surface in the p-adic world to be a strict, compact, connected, quasi-smooth k-analytic curve which by the result of Fresnel-Matignon is either projective *i.e.* an analytification of a smooth, connected projective k-algebraic curve or a k-affinoid space. In this introduction we will refer to them simply as *our curves*.

Let us move to the point (2) of the "proof" above where important notion of triangulations enter into the picture. To make the parallelism possible between the classical and p-adic settings, we permit ourselves to loosen the rigor and to dwell further into abstraction to notice that triangulation is rather a partition of a Riemann surface S into finitely many simple pieces/topological triangles which together with the gluing data along their edges capture the main characteristics of the surface. From another point of view, the vertices and edges of triangles in a triangulation form a *finite* graph structure Γ in S, such that $S \setminus \Gamma$ is a disjoint union of *finitely* many pieces all of which are isomorphic to open discs. What can we say about potential triangulations of our p-adic curves? Well, having in mind their *tree-like*[3] topological structure, asking for topological triangles is perhaps too much, but following the alternative point of view it turns out that the parallelism between the two worlds, classical and p-adic, continues. Namely, we can find a finite graph-like structure Γ in our p-adic curve X such that $X \setminus \Gamma$ is a disjoint union of open analytic domains isomorphic to open discs, but (unfortunately), their number is infinite.

We stop for a moment with comparison and say what a (strictly) semistable triangulation of our curve X is: a finite set $\mathcal{T} \subset X$ consisting of type two points such that $X \setminus \mathcal{T}$ decomposes as a disjoint union of finitely many connected components isomorphic to open

[3]When we say " $*$ "-*like*, we actually mean resembling " $*$ " in its main mathematical properties

strict annuli and infinitely many connected components all of which are isomorphic to open unit discs. Our triangulation will have the role which have the vertices of topological triangles in a triangulation of S, while the role of edges goes to the presence of open annuli in the connected components in $X \setminus \mathcal{T}$ (in fact, the complement of all open discs in an open annulus is a segment/edge!). The points in triangulation together with the edges in open annuli will constitute a graph that we talked about before. As in the classical case, the existence of (strictly) semistable triangulations of our curves is a highly nontrivial fact and moreover in the p-adic world it is closely related to (strictly) semistable formal models.[4]

A (resp. strictly) semistable formal model of our curve X is a k°-formal scheme $\mathfrak{X}$ such that $\mathfrak{X}_\eta = X$, where $\mathfrak{X}_\eta$ is the generic fiber of $\mathfrak{X}$, and such that the special fiber $\mathfrak{X}_s$, which is a $\tilde{k}$-algebraic curve (where $\tilde{k}$ is the residue field of k), has only regular singular points (resp. and irreducible components don't self-intersect). There is also a specialization map $\mathrm{sp}_{\mathfrak{X}} : X \to \mathfrak{X}_s$ which is anticontinuous[5] and surjective. From a (strictly) semistable formal model we obtain the corresponding (strictly) semistable triangulation by taking its points to be the inverse images of the generic points of the irreducible components of the formal model by the map $\mathrm{sp}_{\mathfrak{X}}$.

Having in mind (0.0.0.1) and the fact that our p-adic curve has (many) semistable and strictly semistable formal models, the question arises: Do there exist semistable or even strictly semistable formal models $\mathfrak{Y}$ and $\mathfrak{X}$ of Y and X, respectively, and a finite morphism $\Phi : \mathfrak{Y} \to \mathfrak{X}$, such that $\Phi_\eta = \varphi$? This question has been studied in the literature and was answered positively if one looks for semistable $\mathfrak{Y}$ and $\mathfrak{X}$ and where Y and X are projective, by Robert Coleman in his paper [15]. Let us also mention that Coleman's proof (almost algebraic in the nature) proceeds by taking the Galois closure of φ and using the results concerning the reduction of affinoids constructs a semistable covering on Y which via φ induces a semistable covering of X (a semistable covering of a curve X-another

[4] which justifies the terminology used: (strictly) semistable triangulation

[5] Poetically speaking, a formal model lives between the two worlds of $\tilde{k}$-algebraic and k-analytic curves; its underlying topological space belongs to the former, while the structure sheaf resembles the structure sheaf of the later

notion equivalent to that of a semistable formal model- is a finite covering of X by strict[6] wide open curves (*i.e.* open analytic subdomains isomorphic to projective curves minus finitely many connected components isomorphic to closed unit disc) and which have as intersections open annuli, see Remark 1.1.7 for details).

In the present article we provide another proof of Coleman's theorem, which pretend to be almost analytic in nature. Namely, we consider an equivalent problem of finding compatible (strictly) semistable triangulations $\mathcal{S}$ and $\mathcal{T}$ of Y and X, respectively, and where compatible means that $\mathcal{S} = \varphi^{-1}(\mathcal{T})$. An observant reader will immediately notice the similarity with the conditions on triangulations in the end of step (1) of the proof of the classical Riemann-Hurwitz formula. Since we are working now with semistable triangulations, a notion which makes sense for other type of curves and not only for ones having semistable formal models, we are able to prove a similar result on existence of compatible triangulations for finite morphisms of wide open curves as well (which are a slight generalization of wide open curves introduced by Coleman).

As a consequence, we prove a partition result (Corollary 2.1.32). Loosely speaking, we proved that for a morphism (0.0.0.1) (or a finite morphism between wide open curves), we can partition our curve Y (resp. X) as a disjoint union of elements of two *finite* sets $\mathcal{C}_Y$ and $\mathcal{A}_Y$ (resp. $\mathcal{C}_X$ and $\mathcal{A}_X$) such that the elements of $\mathcal{C}_Y$ (resp. $\mathcal{C}_X$) are affinoid domains with good canonical reduction and the elements of $\mathcal{A}_Y$ (resp. $\mathcal{A}_X$) are open annuli, such that: The elements in $\mathcal{C}_X$ (resp. $\mathcal{A}_X$) are the images of elements in $\mathcal{C}_Y$ (resp. $\mathcal{A}_Y$) by φ, the inverse images of elements in $\mathcal{C}_X$ (resp. $\mathcal{A}_X$) by φ are disjoint unions of elements in $\mathcal{C}_Y$ (resp. $\mathcal{A}_Y$) and finally, φ restricted to any of the elements in $\mathcal{C}_Y$ (resp. $\mathcal{A}_Y$) is a finite (resp. finite étale) morphism. This result proves to be useful since we can reduce questions considering finite morphisms between our curves to finite morphisms between open annuli or between affinoids with good canonical reduction (we come back to this point later).

While preparing this article, another proof of Coleman's theorem appeared (see [1]).

[6] We denote by *strict* wide open curve for what R. Coleman used wide open curve

Although, in our opinion, the underlying idea of the proof presented here and in *loc.cit.* are similar, we tend to think that the respective realizations are different, and as such the present proof may (or may not) shed some new light on the problem. We return to the comparison of the two methods again in Remark 2.1.9.

Let us go back to the analysis of the proof of the classical Riemann-Hurwitz formula, in particular to steps (0) and (1) where one proves that the ramification locus is *finite* and uses it to construct compatible triangulations. If we accept the classical attitude to say that a ramified point is a point in our curve such that in any of its open neighborhoods the morphism fails to be 1-1, we face a new phenomena: that the ramification locus in general is not finite, but rather an infinite closed[7] subset of Y (see [20] for the case of a finite morphism between Berkovich projective lines). This is due to the fact that there are new types of points present in our curves, rather than just k-rational ones (which are analogues of points in a Riemann surface). In fact, it can even happen that φ is a finite étale morphism, but still to have a nonempty ramification locus, as we will see in the examples later on. Although,we note that when we are on connected components which don't meet the branching locus (*i.e.* the image of the ramification locus), our morphism φ will be an n-fold covering over them, so there is similarity with the classical situation. But, the main question is, is it possible to read off a *finite* set of numerical data from such a rich ramification locus which is enough to express the difference between Euler-Poincaré characteristics $\chi(Y) - \deg(\varphi)\chi(X)$? (We will say later what we mean by $\chi(Y)$ and $\chi(X)$.) A natural answer would be to take just the ramification indices of ramified k-rational points (and indeed, there are finitely many of those), and in fact, the answer would be correct if we assume that Y and X are projective, in which case the Riemann-Hurwitz formula is the classical (algebraic) one. How about when Y and X are affinoid curves?

As one may suspect, in this case the Riemann-Hurwitz formula needs to be modified

[7] in strong topology

and the following basic example will demonstrate why.

Suppose we are given two finite morphisms $f_1' : \mathbb{P}^1_k \to \mathbb{P}^1_k$, $x \mapsto y = x^p$ for all $x \in \mathbb{P}^1_k$, and $f_2' : \mathbb{P}^1_k \to \mathbb{P}^1_k$, $x \mapsto y = x^p - x$. Both maps induce finite morphisms $f_1, f_2 : D(0,1^+) \to D(0,1^+)$ of the closed unit disc to itself.

The first map, when restricted to $D(0,1^+)$ is classically ramified[8] only at $x = 0$ with the ramification index $e_0 = p$ (although the ramification locus for example contains all the points connecting $x = 0$ with the Gauss point $\eta_{0,1}$, and even more than this), while the second one is classically unramified. Assuming for the moment without goint into details that the Euler-Poincaré characteristic (see Definition 3.1.34) of the closed unit disc (equipped with the overconvergent structure sheaf) is equal to 1. So in this case, the classical Riemann-Hurwitz formula would give us $1 = p \cdot 1 - \sum_{P \in D(0,1^+)(k)} (e_P - 1)$ which is true only for the first map.

To see what goes wrong with the second map, we could proceed as follows: Deduce the Riemann-Hurwitz formula for the map f_2 from the classical one by considering the full map f_2' and see how the relevant invariants change when removing the open disc at infinity.

The classical ramification locus for the map f_2' consists of the points $x_i = \zeta^i p^{-1/(p-1)}$ together with the point $x_\infty = \infty$, where ζ is any primitive $(p-1)$th root of 1 (and again, the full ramification locus is much richer). Each x_i has the ramification index $e_{x_i} = 2$, while $e_{x_\infty} = p$ so the classical Riemann-Hurwitz formula yields (recall $\chi(\mathbb{P}^1_k) = 2$): $2 = p \cdot 2 - \sum_{i=1}^{p-1} (2-1) - (p-1)$ but for the moment let us write $\chi(\mathbb{P}^1_k) = \deg(f_2') \cdot \chi(\mathbb{P}^1) - \sum_{P \in D(\infty,1^-)(k)} (e_p - 1)$, where $D(\infty, 1^-)$ is the open unit disc with the center at ∞, i.e. $\mathbb{P}^1_k \setminus D(0,1^+)$. Using the fact that $\chi(D(0,1^+)) = \chi(\mathbb{P}^1_k) - 1$, we obtain $\chi(D(0,1^+)) = \deg(f_2) \cdot \chi(D(0,1^+)) + (\deg(f_2) - 1) - \sum_{P \in D(\infty,1^-)(k)} (e_P - 1)$, so we are led to think that the term $((\deg(f_2) - 1) - \sum_{P \in D(\infty,1^-)(k)} (e_P - 1))$ should count the defect. A natural question is, can we read off the defect out of the properties of the map f_2, without using

[8] Classical ramification, *i.e.* the ramification with support in rational points; classically ramified points are also called critical points, as in [20]

its extension f_2'?

It turns out that the answer is positive under the condition that the map f_2' is *overconvergent*, meaning that it extends to some extent over the boundary of the unit disc, and for this we need to stray in the area of finite étale morphisms of open annuli. Let us put here $A(0;r,1)$ to denote the open annulus $D(0,1^-)\setminus D(0,r^+)$ (note that every strict open annulus can be put in this form by a suitable isomorphism). Let $\varphi : A(0;r,1)\to A(0;r^n,1)$ be a finite étale morphism of degree n. After introducing suitable coordinates at 0, the derivative of $dS/dT = d\varphi_{\#}(T)/dT$ is an invertible function on $A(0;r,1)$ and let us denote its order[9] by σ. Suppose that our morphism φ extends to a finite morphism $\psi : D(0,1^+)\to D(0,1^+)$, classically ramified at rational points x_i, $i=1,\dots,l$. Then, one can prove that $\sigma=\sum_{i=1}^{l}(e_{x_i}-1)$.

Returning to our morphism f_2', introducing coordinates at ∞ and considering the restriction of f_2' to some annulus $A(0;r,1)$ (corresponding to the annulus $A(0;1,r')$ before introducing the coordinates at ∞, hence one can see the overconvergence of f_2 in the fact that it extends also to some open annulus $A(0;1,r')$, for some $r'>1$), it follows that the aforementioned defect in fact is equal to the value $\nu:=\sigma-\deg(f_{2,\infty})+1$, where $f_{2,\infty}$ is the degree of f_2 restricted to the open annulus $A(0;r,1)$.

Let us consider now a more general situation, a finite morphism $f:Y\to X$ of quasi-smooth, connected 1-dimensional affinoids. By a result of Van Der Put [36, Theorem 1.1] we know that that Y (resp X) can be embedded in a smooth, projective curve Y' (resp. X'), s.t. $Y'\backslash Y=\uplus_{i=1}^{m}D_i^Y$ (resp. $X'\backslash X=\uplus_{j=1}^{n}D_j^X$), where D_i^Y (resp. D_j^X) are isomorphic to open unit discs. Suppose that f extends to a finite morphism $f':Y'\to X'$. Let f_i be the restriction of f' to the disc D_i^Y, and let σ_i be its derivative when we restrict f_i to a small open annulus living at the boundary of D_i^Y. Then, with similar arguments as before

[9]Since $\frac{dS}{dT}$ is invertible, we can put it in the form $\frac{dS}{dT}=\epsilon T^{\sigma}(1+h(T))$, where for each $\rho\in(r,1)$, $|h(T)|_{\eta_{0,\rho}}<1$. We say that σ is the order of $\frac{dS}{dT}$

and with a bit more effort one can prove that

$$\chi(Y) = \deg(f) \cdot \chi(X) - \sum_{i=1}^{m} \nu_i - \sum_{P \in Y(k)} (e_P - 1)$$

where $\nu_i = \sigma_i - \deg(f_i) + 1$. This is a special case of our Riemann-Hurwitz formula 4.1.10.1.

In the previous paragraph we made an essential assumption in order for the arguments to work, that the morphism φ extends to a finite morphisms of curves Y' and X' which are made by adding *open discs* to the affinoids Y and X, respectively. Is it always possible to do so, *i.e.* to *compactify* our morphism of affinoids? If we allow ourselves to add to our affinoids more than just open discs but rather wide open curves of higher complexity (*i.e.* genus), than the answer is yes by a result of Garuti [22, Proposition 2.4]). Unfortunately, the above arguments work if we just add open discs to our affinoids, so we needed to find a way around this obstacle.

Finally, we turn our attention to the definition of the Euler-Poincaré characteristic of our curves. Although genus of our curves can be defined combinatorially/topologically via the Euler characteristic of the graph coming from the semistable triangulations (one has to take into account also the genus of the points themselves) to compute the Euler-Poincaré characteristic one also has to count the number of "missing" discs *i.e.* the number of discs needed to "projectify" our affinoid curve in the sense discussed before. For the right choice of cohomology on our curve, this is achieved with the Euler-Poincaré characteristic defined as the alternating sum of the dimensions/ranks of the corresponding cohomology groups. Here one faces a choice which cohomology theory to use, *i.e.* for which cohomology we will get the "right" Euler-Poincaré numbers which will appear in Riemann-Hurwitz formula. One choice could be the l-adic étale cohomology ([18]) Fand indeed, one would obtain the right numbers. However, in this thesis we opted for the de Rham cohomology, as we had in mind future applications in the area of p-adic differential equations. It is well known that considering structure sheaf to compute de Rham cohomology yields, in general, in k-vector spaces of infinite dimension and a way to remedy this is to use the *overconvergent* structure sheaf which implies the finite dimensionality. However, overconvergent rings of analytic functions are naturally assigned to pro-objects, and this is precisely what motivated us to introduce and study the category of pro-wide open curves.

The way to pass from pro-wide open curves to our affinoid or projective curves is by using the "heart"-functor which sends the pro-wide open curve to its "heart" which is the intersection of all wide open curves making the given pro-wide open curve. We classify all the curves that can appear as hearts of the pro-wide open curves and we show that a pro-wide open curve is essentially determined by its heart [10]. This allow us to translate many of the notions used for our curves into the setting of pro-wide open curves and it is the category of pro-wide open curves that is the right setting when one considers the open-closed coverings of our curves with affinoid domains and open subsets, and because of their pro-structure they behave nicely when one considers questions related to de Rham cohomology.

We now describe the content of the thesis.

In the first chapter we recall some basic notions concerning Berkovich p-adic curves focusing our attention to quasi-smooth, compact curves and to wide open curves, as these are the principal objects of our study. A relation between (strictly) semistable formal models and (strictly) semistable triangulations is explained as well as the structure of our curves using the reduction with respect to some model. We introduce what is classically known as the skeleton of the curve in the form of the *skeleton function* (acting on the subsets of our curve) and study its main properties. Finally, we introduce the strong topology which, as far as we know, was mainly treated for the projective line. The strong topology on the curves is particularly convenient when one studies the properties of the ramification locus for the finite morphisms.

In the second chapter we prove the Coleman's theorem *i.e.* the existence of strictly φ-compatible triangulations (see Definition 2.1.2) for the curves Y and X and a morphism

[10] Let us say here that there is a pro-wide open curve which is empty-hearted and which corresponds to what is classically known as the Robba ring

φ like in (0.0.0.1) and for a finite morphisms of wide open curves as well. We also show the equivalence of the problem of extension of φ to some semistable models and the problem of finding compatible triangulations. The general path that we follow consists in studying the problem for the case of a finite morphism between more simple curves and then gradually increasing the generality. It is here where we also explain how to construct strictly compatible triangulations for a finite morphism between an annulus and a disc, which allow us to pass from semistable models/triangulations to strictly semistable ones. We further apply the results to find compatible partitions of curves Y and X (whether they are compact or wide open curves); we introduce some invariants for finite étale morphisms of open annuli which will play an important role in Riemann-Hurwitz formula. We also deal with an interesting problem of factoring finite morphisms into a morphism which is residually separable (whose reduction is separable) and a morphism which is residually purely inseparable[11]. Finally, we turn our attention to ramification locus, which we study using the discriminant function, which was already studied by Lütkebohmert [31, 32], but also by other authors in [37, 13]. We end by describing the ramification locus for a finite morphism of affinoids with good canonical reduction and which is a lifting of Frobenius morphism[12].

In the third chapter we introduce and study the basic properties of the category $\mathbb{W}$ of pro-wide open curves which are pro-objects built out of wide open curves. We focus on their connection with our curves, and use their structure to study de Rham cohomology of our curves[13]. Because of the overconvergent nature of the functions on pro-wide open curves, the cohomology groups are finite dimensional. But moreover, their pro structure makes them behave nicely in the coverings: for example, we explain how to recover the Euler-Poincaré characteristic of our curves if we know Euler-Poincaré characteristic of its

[11] Our contribution here is merely to revive a beautiful result of R. Coleman which concerns lifting of morphisms from characteristic p to characteristic 0

[12] A case we are particularly interested in because of the applications in studying p-adic differential equations

[13] In this aspect pro-wide open curves are very similar to dagger affinoid spaces; In fact, we used a lot results on dagger affinoid spaces to study pro-wide open curves

(nice enough) subsets, which is a sort of a Mayer-Vietoris theorem. In particular, $\mathbb{W}$ provides a fertile context to study Robba rings, which correspond to a nonempty objects in W, which we call Robba proannulus (Definition 3.1.18).

Finally, in the last chapter we turn back to the motivational point of this thesis: The Riemann-Hurwitz formula for a morphism (0.0.0.1). We formulate and study the formula in the context of pro-wide open curves, putting in use the results of the previous chapters. We provide different proof for residually separable morphisms (using the Riemann-Hurwitz formula in the reduction), but also a general proof of Riemann-Hurwitz formula. Finally, we provide a different point of view for a Riemann-Hurwitz formula of curves in characteristic $p > 0$, regardless whether the morphism is separable or not.

Chapter 1

Compact, connected, quasi-smooth k-analytic curves

From now on, let k be an algebraically closed field which is complete with respect to a nonarchimedean valuation and of mixed characteristic $(0, p)$. [1] The norm on k will be denoted by $|\cdot|$. We denote by k° the set of integers of k, i.e. the set $\{a \in k, |a| \leq 1\}$, and by $k^{\circ\circ}$ the maximal ideal of k°, i.e. the set $\{a \in k, |a| < 1\}$. Residual field $\widetilde{k}$ is by definition $k^\circ/k^{\circ\circ}$ and it is an algebraically closed field of characteristic p.

1.0.1. Unless otherwise stated, let X be a strict, compact, connected, quasi-smooth k-analytic curve. Note that X is not empty.

1.1 Structure of compact, connected, quasi-smooth k-analytic curves

1.1.1. *Basic pieces* Recall that the Berkovich projective line $\mathbb{P}^1_k$ is the one point compactification of the Berkovich affine line $\mathbb{A}^1_k$. The points of the affine line correspond to the multiplicative k-seminorms on the polynomial algebra $k[T]$. For $a \in k$ and $r \in \mathbb{R}_{\geq 0}$

[1] As we already noted in the Introduction, for the sake of a straight exposition, we choose to threat only this case, but the case of characteristic $(0, 0)$ will be considered on equal basis in the following papers

we denote by $\eta_{a,r}$ the point in the affine line $\mathbb{A}^1_k$ corresponding to the multiplicative seminorm which is given by: for $f(t) \in k[T]$, $|f(\eta_{a,r})| = \max_{i\geq 0} |\frac{f^{(i)}(a)}{i!}|r^i$. For the seminorm corresponding to the point $\eta_{a,\rho}$ we will write $|\cdot|_{a,\rho}$ or $|\cdot(\eta_{a,\rho})|$. If $a = 0$ we also write $|\cdot|_\rho$ instead of $|\cdot|_{0,\rho}$. We identify points in k with rational points in $\mathbb{P}^1_k$ via $a \in k \longleftrightarrow \eta_{a,0} \in \mathbb{P}^1_k$.

For $a \in k$ and $r \in \mathbb{R}_{\geq 0}$ we denote by $D(a, r)$ (resp. $D(a, r^-)$) the Berkovich closed (resp. open) disc centered at a point a and of radius r. A point $\eta_{b,\rho}$ is in $D(a, r)$ (resp. $D(a, r^-)$) iff $b \in D(a, r)$ and $\rho \leq r$ (resp. $\rho < r$). Note that according to the previous, a rational point is a closed disc of radius 0. Similarly, we denote by $A[a; r_1, r_2]$ (resp. $A(a; r_1, r_2)$) a closed (resp. open) Berkovich annulus with radii r_1 and r_2, where $r_2 \geq r_1 \in \mathbb{R}_{>0}$ (resp. $r_2 \geq r_1 \in \mathbb{R}_{\geq 0}$). Note that $A(a; r_1, 0)$ is a punctured open disc *i.e.* the open disc $D(a; r_2^-)$ punctured in the point $\eta_{a,0}$. For the later purposes, we will agree to consider a complement in a projective line of a closed disc to be an open disc. Note that under this assumptions, affine line $\mathbb{A}^1_k$ is an open disc (for which we agree to say to be of infinite radius).

In general, let X be a connected, quasi-smooth k-analytic curve. A subset $D \subset X$ is called a standard open disc (or just an open disc) if there is an isomorphism $T : D \xrightarrow{\sim} D(0, r^-)$. A subset $A \in X$ is called a standard open annulus (or just an open annulus) if there is an isomorphism $T : A \xrightarrow{\sim} A(0; r_1, r_2)$. A standard open disc D is strict if there exists an isomorphism $D \xrightarrow{\sim} D(0, 1^-)$, and similarly, a standard open annulus is strict if there exists an isomorphism $A \xrightarrow{\sim} A(0; , r_1, r_2)$, and both $r_1, \ldots, r_2 \in |k|$.

Open discs and open annuli are special cases of *wide open curves* (or wide open spaces). We slightly generalize the definition given in [15].

Definition 1.1.2. *Let X be as in 1.0.1 and assume that it is projective. An open analytic subset U of X is called a wide open (resp. strict wide open) in X if $X \setminus U$ is a finite disjoint union (possibly empty) of closed discs (resp. strict closed discs) in X. In general, an analytic curve (*i.e. *a 1-dimensional analytic spaces) is called a (strict) wide open curve if it is isomorphic to a wide open subset in some curve X as before.*

An open analytic subset U of X, where X is an affinoid curve like in 1.0.1 such that

$X \setminus U$ is a finite nonempty *disjoint union of open discs (resp. strict open discs) is called a generalized wide open curve (resp. a generalized strict wide open curve) in X.*

If $U \subseteq X$ is a wide open curve embedded in a curve X, the points in the set $\bar{U} \setminus U$, where $\bar{U}$ is the closure of U in X, are called endpoints of U in X.

For example, smooth projective k-analytic curves (*i.e.* analytifications of smooth projective k-algebraic curves) are wide open curves, but so are the analytifications of smooth affine k-algebraic curves.

Definition 1.1.3. *Let U be an analytic domain in X where X is like in 1.0.1. Then, any isomorphism (resp. étale morphism) $T : U \xrightarrow{\sim} V$ where V is an analytic domain in $\mathbb{P}^1_k$ is called a* coordinate *(resp. an* étale coordinate*) .*

For example, for a standard strict open annulus in some curve, there is a coordinate $T : A \xrightarrow{\sim} A(0; r_1, 1)$ where $r \in |k^\times|$ and two standard strict open annuli $T_1 : A_1 \xrightarrow{\sim} A(0; r_1, 1)$ and $T_2 : A_2 \xrightarrow{\sim} A(0; r_2, 1)$ are isomorphic if and only if $r_1 = r_2$.

1.1.4. *Semistable triangulations* Let X be as in 1.0.1. Then, X is either projective, i.e. analytification of a smooth k-algebraic curve, or it is an affinoid [21, Théorème 2] (recall that we assume our affinoids to be strict, unless otherwise stated). Structure of such curves as well as various constructions concerning their (strictly) semistable reduction is described in details in [2]. The topic of more general k-analytic curves is thoroughly studied in the coming book [17], which also contains the (analytic) proof of the existence of semistable triangulations. For the purpose of this paper, we explain the relation between strictly semistable triangulations of X and strictly semistable models of X.

Definition 1.1.5. *Let X be as in 1.0.1 or a wide open curve. A semistable triangulation of X is a finite set $\mathcal{T}$ of type two points of X such that $X \setminus \mathcal{T}$ decomposes into a disjoint union of open discs and finitely many open annuli.*

Assume that the semistable triangulation $\mathcal{T}$ consists of at least two points or that $\mathcal{T}$ has at least one point and X is not isomorphic to a projective curve. Let $\xi \in \mathcal{T}$. We *denote*

(i) *C_ξ to be a maximal subaffinoid in X with respect to $\mathcal{T}$, with good canonical reduction and maximal point ξ. With respect to $\mathcal{T}$ means that for residual classes of C_ξ we only take maximal open disks in $X \setminus \mathcal{T}$ which are attached to ξ. By $\mathcal{C}_\mathcal{T}$, we denote the set $\{C_\xi, \xi \in \mathcal{T}\}$.*

(ii) *$\mathcal{A}_T$ to be the set of connected components in $X \setminus \mathcal{T}$ which are open annuli. If X is compact then this is the set of all open annuli in X which have both end-points in $\mathcal{T}$ and which don't contain any point of $\mathcal{T}$. For $\xi \in \mathcal{T}$, let $\mathcal{A}_\xi = \{A \in \mathcal{A}_\mathcal{T}, A$ has an endpoint in $\xi\}$.*

(iii) *For $\xi \in \mathcal{T}$, W_ξ to be the union of C_ξ and all annuli in $\mathcal{A}_\xi$. W_ξ is an open connected subset of X which is a wide open curve. Let $W_\mathcal{T} = \{W_\xi, \xi \in \mathcal{T}\}$.*

Remark 1.1.6. If, for example $X = \mathbb{P}^1_k$ then any type two point in $x \in X$ is a triangulation of X. This means that the the set of all open discs in X attached to x, together with x, is not an affinoid domain but rather the whole projective curve. That is why in the previous definition we assume that $\mathcal{T}$ has at least two points or simply that X is not isomorphic to a projective curve.

Remark 1.1.7. We note here that in the terminology of [15], elements of the set $W_\mathcal{T}$ define a semistable covering of X. Recall that the later is a finite collection $\mathcal{C}$ of wide open curves in X satisfying the following conditions: (i) for any two different elements $\mathcal{U}, \mathcal{V} \in \mathcal{C}$, the intersection $\mathcal{U} \cap \mathcal{V}$ is a finite union of open annuli in X, (ii) for any three distinct elements $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathcal{C}$, $\mathcal{U} \cap \mathcal{V} \cap \mathcal{W} = \emptyset$, and finally (iii) for any $\mathcal{U} \in \mathcal{C}$, the *underlying affinoid domain* U^u which is by definition the affinoid domain $\mathcal{U}^u := \mathcal{U} \setminus \cup_{\mathcal{V} \in \mathcal{C}, \mathcal{V} \neq \mathcal{U}} \mathcal{V}$ has a canonical reduction which has only regular singular points. As we have seen, to each semistable triangulation $\mathcal{T}$ of X, we can assign a semistable covering $\mathcal{W}_\mathcal{T}$. On the other side, it can be seen that semistable covering $\mathcal{C}$ of X comes from a triangulation if and only if for each element $\mathcal{U} \in \mathcal{C}$, the underlying affinoid $\mathcal{U}^u$ has good canonical reduction.

Definition 1.1.8. *For a semistable triangulation $\mathcal{T}$ we say that it is strictly semistable if each open annulus in $\mathcal{A}_{\mathcal{T}}$ has distinct endpoints.*

Remark 1.1.9. Given a semistable triangulation $\mathcal{T}$, we can always construct a strictly semistable one by picking the open annuli in $A \in \mathcal{A}_{\mathcal{T}}$ whose endpoints coincide, and then add to $\mathcal{T}$ a type 2 point from the skeleton of A (see Definition 1.2.1 for the definition of the skeleton).

We extend the Definition 1.1.2 by introducing special classes of wide open curves.

Definition 1.1.10. *A wide open curve U is called simple if U has a strictly semistable triangulation consisting of one point (not necessarily unique).*

Similarly, we say that a generalized wide open curve is simple if it has a strictly semistable triangulation consisting of one point.

1.1.11. *Semistable formal models and triangulations* Assume X to be as in 1.0.1. To any semistable triangulation of X we can assign a formal semistable model of X, which we denote by $\mathfrak{X}_{\mathcal{T}}$ in the following way. We start by constructing formal schemes Spf $\mathcal{O}^\circ(W_\xi)$, for $\xi \in \mathcal{T}$ and glue them along Spf $\mathcal{O}^\circ(W_{\xi_1} \cap W_{\xi_2})$ where $W_{\xi_1} \cap W_{\xi_2} \neq \emptyset$. The special fiber of $\mathfrak{X}_{\mathcal{T}}$, denoted by $\mathfrak{X}_{\mathcal{T},s}$ is a reduced $\widetilde{k}$-scheme, with only regular singular points. There is a well defined specialization map spe : $\mathfrak{X}_{\mathcal{T}} \to \mathfrak{X}_{\mathcal{T},s}$, whose main properties are recalled in the following theorem ([10, Proposition 2.2 and 2.3] and [5, Proposition 2.4.4]).

Theorem 1.1.12. (Bosch-Lütkebohmert-Berkovich) *Let X be as in 1.0.1, $\mathcal{T}$ a semistable triangulation of X, $\mathfrak{X}_{\mathcal{T}}$ the corresponding semistable formal model and spe : $\mathfrak{X}_{\mathcal{T}} \to \mathfrak{X}_{\mathcal{T},s}$ the specialization map. Let $x \in \mathfrak{X}_{\mathcal{T},s}$. Then*

(i) The mapping spe induces a 1-1 correspondence between the irreducible components of $\mathfrak{X}_{\mathcal{T},s}$ (or generic points of irreducible components of $\mathfrak{X}_{\mathcal{T},s}$) and of points in $\mathcal{T}$;

(ii) If x is a smooth point in $\mathfrak{X}_{\mathcal{T},s}$ belonging to the irreducible component with generic point $\tilde{x}$, then spe$^{-1}(x)$ is an open disc with the endpoint spe$^{-1}(\tilde{x})$;

(iii) *If x is a regular singular point in $\mathfrak{X}_{\mathcal{T},s}$ belonging to the irreducible components with generic points $\tilde{x}_1$ and $\tilde{x_1}$ ($\tilde{x}_1$ and $\tilde{x}_2$ may coincide), then spe$^{-1}(x)$ is an open annulus with endpoints spe$^{-1}(\tilde{x}_1)$ and spe$^{-1}(\tilde{x}_2)$.*

Using this theorem, given a strictly semistable model $\mathfrak{X}$ of X, we obtain a corresponding strictly semistable triangulation $\mathcal{T}_{\mathfrak{X}}$ in the following way. Let $\{\mathfrak{c}_1, \dots, \mathfrak{c}_r\}$ be the irreducible components of $\mathfrak{X}_s$, let $\mathrm{sm}(\mathfrak{c})$ denote the smooth part of the component $\mathfrak{c}$, and let $\mathrm{Sing}(\mathfrak{X}_s)$ denote the singular locus of $\mathfrak{X}_s$. If $\mathrm{gen}(\mathfrak{c}_i)$ denotes the generic point of the component $\mathfrak{c}_i$, then $\mathcal{T}_{\mathfrak{X}} = \{\mathrm{spe}^{-1}(\mathrm{gen}(\mathfrak{c}_i)), i = 1, \dots, r\}$. If $r > 1$ we furthermore have

$$C_{\mathcal{T}_{\mathfrak{X}}} = \{\mathrm{spe}^{-1}(\mathrm{sm}(\mathfrak{c}_i)), i = 1, \dots, r\}$$
$$A_{\mathcal{T}_{\mathfrak{X}}} = \{\mathrm{spe}^{-1}(x), x \in \mathrm{Sing}(\mathfrak{X}_s)\}$$
$$W_{\mathcal{T}_{\mathfrak{X}}} = \{\mathrm{spe}^{-1}(\mathfrak{c}_i), i = 1, \dots, r\}.$$

1.1.13. *Minimal semistable triangulations.* Given a curve X like in 1.0.1, one can ask whether exists a minimal semistable triangulation $\mathcal{T}$ of X, *i.e.* a triangulation such that every other semistable triangulation of X necessarily contains it. The following theorem gives us necessary and sufficient conditions and is a translation of a corresponding result for semistable formal models given in [2, Theorem 1.2.9].

Theorem 1.1.14. *Let X be a smooth, connected, projective k-analytic curve. Then, X has a minimal semistable triangulation if and only if $g(X) \geq 2$ or $g(X) = 1$ and X has a smooth formal model, where $g(X)$ is the genus of X.*

As one can nottice, the cases where X doesn't have a minimal semistable triangulation is when $X = \mathbb{P}^1_k$ or when X is a *Tate curve i.e.* an analytification of a smooth, projective k-algebraic curve of genus 1, which doesn't have a smooth projective model over the ring of integers k°. For example, every type two point of $\mathbb{P}^1_k$ is a semistable triangulation of $\mathbb{P}^1_k$.

Assumption 1.1.15. *From now on we will assume, unless otherwise stated, that all the* *triangulations and formal models are strictly semistable, and therefore we will sometimes omit saying* strictly semistable*.*

1.1.16. *Tangent space at a point.* Let $x \in X$ be a point. The inductive limit over the open connected neighborhoods U of x in X of connected components of $U \setminus \{x\}$ is called the tangent space at x and is denoted by T_xX. We will sometimes call the elements of the set T_xX *the tangential points* and use "$\vec{t}$" notation for them. If x is of type 1 or 4, then the set T_xX consists of one point [3, Section 4.2], this is because in these cases a fundamental system of neighborhoods consists of a decreasing sequence of discs D_i in X where $D_i \setminus \{x\}$ has only one connected component. If x is of type three, then there exists an open neighborhood A of x in X which is an open annulus with different endpoints (this follows from the existence of semistable triangulations of X). In this case, T_xX consists of two elements, corresponding to two connected components of $A \setminus \{x\}$ (or to the two endpoints of A in X).

If x is a type 2 point, the space X can be described in another way. Namely, by choosing a triangulation $\mathcal{T}$ of X such that $x \in \mathcal{T}$, then from 1.1.12 it follows that x corresponds to an irreducible component $\mathfrak{c}_x$ of $\mathfrak{X}_{\mathcal{T},s}$. Then, T_xX can be identified with the rational points of the curve $\mathfrak{c}_x(\widetilde{k})$ via the correspondence $\mathfrak{c}_x(\widetilde{k}) \to T_xX$ where the point is sent to the class of its preimage under the specialization map using the 1.1.12. It is clear that we can identify the tangential points in T_xX with the residual classes which have an endpoint in x.

Suppose now that X is an affinoid. It is well known (see [36]) that X can be embedded in the analytification X' of a smooth, projective k-algebraic curve $\mathscr{X}'$, s.t. $X' \setminus X$ is a finite union of disjoint open discs. Moreover, the number of discs is independent of the choice of X' (see Definition 3.1.44 and the remark that follows it). Let us fix one such X', and let $X' \setminus X = \uplus_{i=1}^{n} B_i$, where B_i are disjoint open discs. For $i = 1, \ldots, n$, let $\xi_i \in X$ be the (necessarily) type two endpoint of the disc B_i, and let $\vec{t_i} \in T_{\xi_i}X'$ be the tangent vector corresponding to B_i.

Definition 1.1.17. *We define TX to be the set of tangent vectors $\{\vec{t_1}, \ldots, \vec{t_n}\}$ constructed in the previous paragraph. If X is projective, we agree for TX to denote the empty set.*

Remark 1.1.18. Let $X' \setminus X = \uplus_{i=1}^{n} B_i$ be as above. Then, the discs B_i correspond to the points at infinity of the smooth compactification of the canonical reduction of X.

1.1.19. *Change of a base field.* Because of the different natures of the various complete residue fields of points in a curve, i.e. different types of points in X, sometimes the notions defined over points of one type may not be directly defined over points of other types. We will see an example of this when we introduce discriminant at a point with respect to the given finite morphism where we *have to* work over points of type 2. Another glimpse of this can be seen in the theory of p-adic differential equations when one introduces the radius of convergence at a point of an integrable connection (where we one needs the point to be rational). For this aspect, see for example [2].

A way to deal with these problems is to choose a suitable extension of the base field K/k and to consider the curve X_K. In this paragraph we collect some of the main properties of this operation, detailed in [3, Section 3.1].

We start with a compact, connected, quasi-smooth k-analytic curve X. Let $x \in X$ be a type 2 or 3 point in X (this will be the particular cases that we are interested in) and let $\mathscr{H}(x)$ denote it's usual complete residue field. Let K/k be a complete valued extension of k, and let us put $X_K := X\widehat{\otimes}_k K$ (see [5, 2.1] for the construction of $X\widehat{\otimes}_k K$). We may proceed locally and assume that X is a k-affinoid space corresponding to the affinoid algebra $\mathcal{A}$, in which case X_K is a K-affinoid space corresponding to the K-affinoid algebra $A\widehat{\otimes}_k K$. We have a natural projection map $\psi_K : X_K \to X$ corresponding to the isometric embedding $\mathcal{A} \hookrightarrow \mathcal{A}_K$. For a point $x \in X$, it can be proved that the topological spaces $\psi_K^{-1}(x)$ and $\mathcal{M}(\mathscr{H}(x)\widehat{\otimes}_k K)$ are homeomorphic ([3, Section 3.1]).

Suppose now that there is an isometric embedding $\iota : \mathscr{H}(x) \hookrightarrow K$. Then we can introduce the following definition (*loc.cit.*)

Definition 1.1.20. *The canonical rational point $x_{\iota,K}$ above x in X_K is the unique rational point in X_K corresponding to the character $\chi_{x_{\iota,K}} = \iota \circ \chi_x : \mathcal{A}\widehat{\otimes}_k K \to K$, where $\chi : A \to \mathscr{H}(x)$ is the character corresponding to the point x. In particular, when $K = \mathscr{H}(x)$ and $\iota = \mathrm{id}_{\mathscr{H}_x}$, we write $x_{\mathscr{H}(x)}$ for $x_{\mathrm{id},\mathscr{H}(x)}$.*

Remark 1.1.21. (i) Moreover, the set of K-rational points in the fiber $\psi_K^{-1}(x)$ is in one-to-one correspondence with the set of different isometric embeddings $\mathscr{H}(x) \hookrightarrow K$ and any k-rational point $x \in X$ lifts to a unique rational point $x_K \in X_K$ (*loc.cit.* Proposition 3.3 and Corollary 3.5).

(ii) The previous definition and construction are meaningful even if the base field k is not algebraically closed but rather just a complete valued field.

However, when k is algebraically closed, each point $x \in X$ is *universal i.e.* the complete residue field $\mathscr{H}(x)$ is *universally multiplicative* which means that for each complete valued extension $K/\mathscr{H}(x)$, the tensor norm on $\mathscr{H}(x)\widehat{\otimes}K$ is multiplicative ([33, Definition 3.2]). In particular, $\mathcal{M}(\mathcal{H}(x)\widehat{\otimes}_k K)$ contains a point which corresponds to the (multiplicative) norm on $\mathscr{H}(x)\widehat{\otimes}_k K$, denoted by $\sigma_K(x)$. The map of topological spaces $\sigma_K : X \to X_K$ is continuous (*loc.cit.* Corollaire 3.7).

Definition 1.1.22. *We call the point $\sigma_K(x)$ the K*-generic point above x in X_K.

Remark 1.1.23. It is shown in [3, Section 3.1] that $\sigma_K(x)$ corresponds to the unique point in the Shilov boundary of the multiplicative spectrum $\mathcal{M}(\mathscr{H}(x)\widehat{\otimes}_k K)$ of K-Banach algebra $\mathscr{H}(x)\widehat{\otimes}_k K$. In particular, if $|\mathscr{H}(x)| \subset |K|$ and x is a type two or three, then the K-generic point $\sigma_K(x)$ is of type two. This will be of importance for us in the Section 2.4.1 and definitions 2.4.8 and 2.4.9.

Example 1.1.24. To get a feeling of the introduced notions, we start with an example in the form of a closed annulus $X = \mathcal{M}(k\{T, r^{-1}T^{-1}\})$, $r \in (0,1)$. Let $x \in X$ be a point corresponding to the point $\eta_{0,\rho}$, $\rho \in (r,1)$ on the skeleton of the annulus $\mathcal{M}(k\{T, r^{-1}T^{-1}\})$ and let K be a valued extension of k. Then, $X_K = \mathcal{M}(K\{T, r^{-1}T^{-1}\})$ and we have a canonical inclusion $k\{T, r^{-1}T^{-1}\} \hookrightarrow K\{T, r^{-1}T^{-1}\}$ which induces a projective map $\psi_K : \mathcal{M}(k\{T, r^{-1}T^{-1}\}) \to \mathcal{M}(K\{T, r^{-1}T^{-1}\})$.

Let us take $K : \mathscr{H}(x)$ and let t be the image of the variable T under the canonical inclusion $k\{T, r^{-1}T^{-1}\} \hookrightarrow K$. Then t is precisely a canonical rational point above x in X_K.

If we take K to be the completion of the algebraic closure of $\mathscr{H}(x)$, then $\sigma_K(x)$ is the point $\zeta_{0,\rho}^K$ i.e. a point corresponding to the seminorm defined by $|(\sum a_i T^i)(\zeta_{0,\rho}^K)| = \max_i |a_i|\rho^i$ where $\sum_i a_i T^i \in K\{T, r^{-1}T^{-1}\}$.

1.2 Skeleton

We assume X to be as in 1.0.1 or a wide open curve.

Definition 1.2.1. The skeleton function Γ^X *is the function* $\Gamma^X : P(X) \to P(X)$, *where* $P(X)$ *is the power set of* X, *that sends a subset* $S \subset X$, *to a subset* $\Gamma^X(S) \subset X$ *which is the complement of the union of all open discs in* X *having an empty intersection with* S. *If* X *is clear from the context, we also write* $\Gamma(S)$ *or* Γ_S *instead of* $\Gamma^X(S)$.

An arbitrary subset of $P(X)$ *which is of the form* Γ_S^X *for some subset* S *is called the skeleton of* X *or* (X, S)*-skeleton if we want to emphasize dependence on* S.

Example 1.2.2.

1. $\Gamma_\emptyset^X = \emptyset$ if and only if $X = \mathbb{P}_k^1$ or X an open disc (recall that we also consider the affine line to be an open disc of infinite radius).

2. Suppose that X is an open annulus and $S = \emptyset$. Then $\Gamma_\emptyset$ is the usual skeleton of the annulus. More precisely, if $T : X \xrightarrow{\sim} A(0; r, 1)$ is a coordinate on X, $\Gamma_\emptyset$ is the set of points $\{\eta_{0,\rho}, \quad \rho \in (r, 1)\}$.

3. Let S be a semistable triangulation of X. Then Γ_S^X is a connected closed subset of X consisting of the points in S, and skeleta of the open annuli in $\mathcal{A}_S$. Clearly, Γ_S^X is a subset of the later set, and the equality follows from the Lemma 1.2.3 and Remark 1.2.4. If $\mathfrak{X}$ is a semistable model of X, we will write $\Gamma_{\mathfrak{X}}^X$ for the skeleton $\Gamma_{\mathcal{T}_{\mathfrak{X}}}^X$.

4. Let X be as in 1.0.1 and assume that the X is not a projective line. Then $\Gamma_{\emptyset}^{X}$ is classically called *the skeleton* of the curve X. We will come back to this later.

5. For any dense subset S of X, $\Gamma^X(S) = X$.

Lemma 1.2.3. *Let X be as in 1.0.1 or a wide open curve. Then, for each $S \subset X$, $\Gamma^X(S)$ is a connected closed subset of X.*

Proof. The set $\Gamma^X(S)$ is closed by its very definition. Assume for the moment that X is compact. Let $s_1, s_2 \in S$ be distinct points in S and let $\mathcal{T}$ be any strictly semistable triangulation of X such that s_1, s_2 don't belong to the same open disc in $X \setminus \Gamma_{\mathcal{T}}^X$. Let $\xi_1, \xi_2 \in \mathcal{T}$ such that $s_i \in C_{\xi_i}$. Then, there is a path P from s_1 to s_2 consisting of the canonical paths from s_i to ξ_i and of the path from ξ_1 to ξ_2 along the set $\Gamma_{\mathcal{T}}^X$. Moreover, we can assume that every point of the path is simple.

Suppose that every point of the path $P \setminus \{s_1, s_2\}$ belongs to an open disc which has an empty intersection with S and let $p \in P \setminus \{s_1, s_2\}$ be a type 2 point belonging to some open disc D' where $D' \cap S = \emptyset$. Then each except possibly one residual class at the point p is an open disc and more over, the set of open discs in X with endnpoint in p, together with p is a closed disc in X and in D' which we denote by D_p. The set $\mathcal{D}$ of all open discs containing D_p and with an empty intersection with S is nonempty $D' \in \mathcal{D}$ and is ordered by inclusion (if D_1 and D_2 are two open discs with a nonempty intersection, then one of the two is a subset of the other), their union D is an open disc in X not intersecting S. Let us denote by ξ_D the endpoint of D in X. By the construction we can conclude that $s_1, s_2 \notin D$ and that $\xi_D \in \Gamma^X(S)$. Moreover, path P contains the point ξ_D because it contains the point p and since we assumed that none of the points on P are double, we obtain a contradiction. $\square$

Remark 1.2.4. We proved even more, namely, for every two distinct points $s_1, s_2 \in S$, any simple path between them is contained in the set $\Gamma^X(S)$.

Definition 1.2.5. *Let Γ^X be the skeleton function and S a subset of X. An endpoint of the skeleton $\Gamma^X(S)$ is a point $x \in \Gamma^X(S)$ such that $\Gamma^X(S)$ intersects only one residual class attached to x. A branching point of the skeleton $\Gamma^X(S)$ is a point $x \in \Gamma^X(S)$ such that $\Gamma^X(S)$ intersects at least three residual classes attached to x. We say that the skeleton Γ^X_S is finite if it has finitely many endpoints.*

In the proof of next theorem, we will use the following results.

Lemma 1.2.6. *Let X be as in 1.0.1 or a wide open curve and suppose Γ is a finite skeleton in X with type two endpoints. Then, every open disc/connected components in $X \setminus \Gamma$ is strict* i.e. *has a type two endpoint in X.*

Proof. Let D be an open disc/connected component in $X \setminus \Gamma$. The endpoint η_D of D necessarily belongs to Γ. If η_D is a type three point, then there are only two tangent directions emanting from it, one of which corresponds to D. The other one necessarily has a nonempty intersection with Γ, which by definition means that η_D is an endpoint, which is a contradiction. □

Lemma 1.2.7. *Let X be as in the previous lemma and let $\mathcal{T}$ be a nonempty semistable triangulation of X. Let D be an open disc/connected component in $X \setminus \Gamma_{\mathcal{T}}$, Let η_D be the endpoint of D in X (necessarily belonging to $\Gamma_{\mathcal{T}}$) and suppose that $D \cap \mathcal{T} \neq \emptyset$. Then, $\mathcal{T}' : \{\eta_D\} \cup (\mathcal{T} \setminus (\mathcal{T} \cap D))$ is a semistable triangulation of X.*

Proof. First of all we nottice that the set $\mathcal{T}'' := \{\eta_D\} \cup \mathcal{T}$ is a strictly semistable triangulation of X. The set of connected components in $X \setminus \mathcal{T}''$ can be sorted in two families: namely, connected components which are subsets of the set $X \setminus D$ and the connected components which are contained in D. More precisely, let $I_{D,1}$ be the set of open discs/connected components of $X \setminus \mathcal{T}''$ which are contained in $X \setminus D$, $I_{A,1}$ be the open annuli/connected components of $X \setminus \mathcal{T}''$ which are contained in $X \setminus D$, $I_{D,2}$ be the set of open discs/connected components of $X \setminus \mathcal{T}''$ which are contained in D, and $I_{A,2}$ be the open annuli/connected components of $X \setminus \mathcal{T}''$ which are contained in D. Note that $I_{A,i}$, $i = 1, 2$ are finite sets. We can write

$$X \setminus \mathcal{T}'' = (\biguplus_{B \in I_{D,1}} B) \biguplus (\biguplus_{A \in I_{A_1},} A) \biguplus (\biguplus_{B \in I_{D,2}} B) \biguplus (\biguplus_{A \in I_{A,2}} A).$$

On the other hand, we simply have

$$X \setminus \mathcal{T}' = (\biguplus_{B \in I_{D,1}} B) \biguplus (\biguplus_{A \in I_{A_1},} A) \biguplus D,$$

so it follows that $\mathcal{T}'$ is a semistable triangulation of X. $\square$

We are ready for the main result of this section, namely, finite graphs with type two endpoints are precisely the ones coming from the semistable triangulations as in Example 1.2.2(3).

Theorem 1.2.8. *Let X be as in 1.0.1 and let Γ be a (nonempty) finite skeleton only with type two endpoints. Then there exists a (strictly) semistable triangulation $\mathcal{T}$ of X such that $\Gamma = \Gamma_{\mathcal{T}}$.*

Proof. Let E be the set of endpoints of Γ (possibly empty). Let us prove that there exists a strictly semistable triangulation $\mathcal{S}$ of X which contains E, has a nonempty intersection with Γ (the first condition will be used later in the proof while the second one is there to simplify the argument in the case that $E = \emptyset$) and such that $\mathcal{S} \subset \Gamma$. Let $\mathcal{S}'$ be any strictly semistable triangulation of Γ which contains E, has a nonempty intersection with Γ and let $\mathcal{S}'$ be such that the number $n = |\mathcal{S}' \setminus \Gamma|$ is the minimal possible. If $n > 0$ let $\xi \in \mathcal{S}' \setminus \Gamma$. By construction, $\xi \in D$, where D is an open disc/connected component in $X \setminus \Gamma$. Note that the endpoint η_D of disc D is a type two point by Lemma 1.2.6, and it belongs to Γ but also to $\Gamma_{\mathcal{S}'}$ because $\Gamma_{\mathcal{S}'}$ is connected, has a nonempty intersection with Γ and contains a point in D. It follows that $\mathcal{S}'' := \mathcal{S}' \cup \eta_D$ is also a strictly semistable triangulation of X. But then, $\mathcal{S}'' \setminus (\mathcal{S}' \cap D)$ is a strictly semistable triangulation of X by Lemma 1.2.7 and we have $|(\mathcal{S}'' \setminus (\mathcal{S}' \cap D)) \setminus \Gamma| < n$ which is a contradiction. We conclude $n = 0$.

As a consequence, there exists a strictly semistable triangulation $\mathcal{S}$ of X such that $E \subset \mathcal{S}$ and $S \subset \Gamma$. Next we prove that $\Gamma_{\mathcal{S}} = \Gamma$. As $\mathcal{S} \subset \Gamma$ it follows $\Gamma_{\mathcal{S}} \subset \Gamma_{\Gamma}^{X} = \Gamma$. Suppose that $\Gamma_{\mathcal{S}} \subsetneq \Gamma$ *i.e.* there exists an open disc/connected component D in $X \setminus \Gamma_{\mathcal{S}}$ such that $D \cap \Gamma \neq \emptyset$. If we prove that D contains an endpoint of Γ then we obtain the contradiction as by the constraction $E \subset \mathcal{S}$. So, the proof of the Theorem will follow from the following:

Lemma 1.2.9. *Let Γ be a finite skeleton in X and D an open disc in X having a nonempty intersection with Γ. Then, D contains an endpoint of Γ.*

Proof. We identify D with an open disc $D(c, t^-)$, where $t \in \mathbb{R}_{>0}$. Since D can be represented as a union of increasing sequence of closed discs contained in D, there is a closed disc $D_0 = D(a_0, \rho_0)$ contained in D such that $D_0 \cap \Gamma \neq \emptyset$. As Γ is connected, the endpoint η_{a_0,ρ_0} of D_0 belongs to Γ. Let (ϵ_n) be a decreasing sequence of positive real numbers converging to 0. Let $r_0 := \inf\{\rho,$ s.t. $\eta_{a,\rho} \in D_0 \cap \Gamma$ for some $a \in D_0(k)\}$ and let $\eta_{a_1,\rho_1} \in D_0 \cap \Gamma$ such that $\rho_1 < r_0 + \epsilon_0$. More generally, given η_{a_n,ρ_n} we put $r_n = \inf\{\rho,$ s.t. $\eta_{a,\rho} \in D(a_n, \rho_n) \cap \Gamma$ for some $a \in D(a_n, \rho_n)(k)\}$, and take $\eta_{a_{n+1},\rho_{n+1}} \in D(a_n, \rho_n) \cap \Gamma$ with ρ_{n+1} satisfying $\rho_{n+1} < r_n + \epsilon_n$.

In this way, we obtained a sequence of points (η_{a_n,ρ_n}) such that the sequence of closed discs $D(a_n, \rho_n)$ is nested (meaning that for all n, $D(a_{n+1}, \rho_{n+1}) \subseteq D(a_n, \rho_n)$). It is known then that the intersection $\cap_n D(a_n, \rho_n)$ is either a closed disc $D(a, r)$ or a type 4 point.

Let $\eta_{a,\rho}$ be the maximal point of the $\cap_n D(a_n, \rho_n)$ in case the intersection is a closed disc $D(a, \rho)$, or just the intersection itself, in the case the later is a type 4 point. The point $\eta_{a,\rho}$ must belong to Γ, because otherwise there exists an open disc in $D \setminus \Gamma$ which contains it and this implies that the intersection $\cap_n D(a_n, \rho_n)$ is strictly bigger than $D(a, \rho)$. Finally, let us prove that $\eta_{a,\rho}$ is an endpoint of Γ. If $\eta_{a,\rho}$ is a type 4 point, then it is automatically an endpoint, so let us assume that $\eta_{a,\rho}$ is not of type 4. Suppose that $\eta_{a,\rho}$ is not an endpoint of Γ and let D' be an open disc in $D(a, \rho)$ attached to $\eta_{a,\rho}$ such that $D' \cap \Gamma \neq \emptyset$ and let

$\eta_{b,r} \in D' \cap \Gamma$ for some $b \in D'(k)$. Necessarily $r < \rho$ and let $i \in \mathbb{N}$ be such that $\epsilon_i < \rho - r$. Then, by the construction of the sequence (η_{a_n,ρ_n}) we have $\rho_{i+1} < r_i + \epsilon_i \leq r + \epsilon_i < \rho$ which is a contradiction. Hence, $\eta_{a,\rho}$ is an endpoint of Γ. $\square$

$\square$

1.2.10. *Skeleton and minimal triangulations.* For this paragraph, we assume that X is a smooth, projective, connected k-analytic curve.

Theorem 1.2.11. *Suppose that X has a minimal semistable triangulation and let $\mathcal{T}$ be the minimal semistable triangulation of X. Then, Γ^X contains $\mathcal{T}$.*

Proof. Suppose that there is a point $t \in \mathcal{T}$ such that $t \notin \Gamma^X$. Let D be the connected component/open disc in $X \setminus \Gamma^X$ which contains t and let η_D be the endpoint of D.

Lemma 1.2.12. *The point η_D belongs to $\Gamma^X_{\mathcal{T}}$.*

Proof. If not, then, $\Gamma^X_{\mathcal{T}}$ being connected and closed, and containing the point t, is completely contained in disc D. Then it follows that $X = \mathbb{P}^1_k$, which is a contradiction since X has a minimal triangulation. $\square$

We can apply then Lemma 1.2.7 to conclude that $\{\eta_D\} \cup (\mathcal{T} \setminus (\mathcal{T} \cap D))$, which doesn't contain $\mathcal{T}$, hence $\mathcal{T}$ is not the minimal semistable triangulation of X, which is a contradiction. $\square$

1.3 Strong topology on k-analytic curves

1.3.1 (Big) metric on k-analytic curves

1.3.1. Let $D(0,1^-)$ be the (Berkovich) open unit disc. Let us introduce the function $d_{D(0,1^-)} : D(0,1^-) \times D(0,1^-) \to \mathbb{R}_{\geq 0}$ in the following way: For two points $\eta_{a,\rho_1}, \eta_{a,\rho_2} \in D(0,1^-)$ we put $d_{D(0,1^-)}(\eta_{a,\rho_1}, \eta_{a,\rho_2}) = d_{D(0,1^-)}(\eta_{a,\rho_2}, \eta_{a,\rho_1}) := |\rho_2 - \rho_1|_\infty$ where the $|\cdot|_\infty$ as usual denotes the ordinary archimedean absolute value on $\mathbb{R}$. In general, for two points $\eta_{a_1,\rho_1},\eta_{a_2,\rho_2} \in D(0,1^-)$, let $\eta_{a,\rho} = \eta_{a_1,\rho} = \eta_{a_2,\rho}$ be the maximal point of the minimal closed disc in $D(0,1)$ containing both points (if $\eta_{a,\rho}$ doesn't coincide with one of the points η_{a_1,ρ_1} and η_{a_2,ρ_2}, then it can be seen that $\eta_{a,\rho}$ is the *unique* branching point of the skeleton $\Gamma^{D(0,1^-)}_{\{\eta_{a_1,\rho_1},\eta_{a_2,\rho_2}\}}$). In this case, we put $d_{D(0,1^-)}(\eta_{a_1,\rho_1},\eta_{a_2,\rho_2}) := d_{D(0,1^-)}(\eta_{a_1,\rho_1},\eta_{a_1,\rho}) + d_{D(0,1^-)}(\eta_{a_2,\rho_2},\eta_{a_2,\rho})$.

Lemma 1.3.2. *The function $d_{D(0,1^-)}$ is a metric on $D(0,1^-)$.*

Proof. The proof is straightforward but we present it for the sake of the reader. We put for the moment $d := d_{D(0,1^-)}$. As the other properties for a metric are clear, we just prove the triangle inequality: For three points $\eta_1,\eta_2,\eta_3 \in D(0,1^-)$, $d(\eta_1,\eta_2) \le d(\eta_1,\eta_3) + d(\eta_2,\eta_3)$. As the other cases are trivial, we assume that $\eta_1 \neq \eta_2 \neq \eta_3 \neq \eta_1$.

Let $\eta = \eta_{a,\rho}$ be the maximal point of the minimal closed disc which contains both of the points η_1 and η_2. We distinguish the following cases.

The point η_3 doesn't belong to the closed disc $D(a,\rho)$ and let η_4 be the maximal point of the minimal closed disc containing η_2,η_2 (hence η) and η_3. We have

$$\begin{aligned} d(\eta_1,\eta_3) + d(\eta_2,\eta_3) &= d(\eta_1,\eta_4) + d(\eta_4,\eta_3) + d(\eta_2,\eta_4) + d(\eta_4,\eta_3) \\ &= d(\eta_1,\eta) + d(\eta,\eta_4) + +d(\eta_4,\eta_3) + d(\eta_2,\eta) + d(\eta,\eta_4) + d(\eta_4,\eta_3) \\ &\ge d(\eta_1,\eta) + d(\eta_2,\eta) = d(\eta_1,\eta_2) \end{aligned}$$

so the triangle inequality is satisfied.

Suppose that $\eta_3 \in D(a,\rho)$. If $\eta_3 = \eta$ the triangle inequality is trivial so assume that $\eta_3 \neq \eta$. We distinguish two cases (a) and (b) below.

(a) The open disc attached to η which contains η_3 also contains one of the two points η_1 or η_2. Without loss of generality, we assume that η_1 and η_3 belong to the same residual class at η and let η_4 be the maximal point of the minimal closed disc containing both points η_1 and η_3. Note also that the maximal point of the minimal closed disc containing η_4 and η_2 (as well as η_3 and η_2) is η. Then,

$$\begin{aligned}
d(\eta_1,\eta_3)+d(\eta_2,\eta_3) &= d(\eta_1,\eta_4)+d(\eta_4,\eta_3)+d(\eta_3,\eta)+d(\eta,\eta_2)\\
&= d(\eta_1,\eta_4)+d(\eta_4,\eta_3)+d(\eta_3,\eta_4)+d(\eta_4,\eta)+d(\eta,\eta_2)\\
&= d(\eta_1,\eta_4)+d(\eta_4,\eta)+d(\eta,\eta_2)+2d(\eta_3,\eta_4)\\
&\geq d(\eta_1,\eta)+d(\eta_2,\eta)=d(\eta_1,\eta_2).
\end{aligned}$$

(b) All the three points η_1,η_2,η_3 belong to different open discs attached to η. We have

$$\begin{aligned}
d(\eta_1,\eta_3)+d(\eta_2,\eta_3) &= d(\eta_1,\eta)+d(\eta,\eta_3)+d(\eta_2,\eta)+d(\eta,\eta_3)\\
&\geq d(\eta_1,\eta)+d(\eta_2,\eta)=d(\eta_1,\eta_2).
\end{aligned}$$

□

In general, for a strict open disc D and a coordinate on it $T: D \xrightarrow{\sim} D(0,1^-)$, we can equip D with a metric $d_{D,T}$ by setting $d_{D,T} := T^* d_{D(0,1^-)}$ *i.e.* $d_{D,T}$ is the pullback via T of the metric $d_{D(0,1^-)}$. For the points $x,y\in D$ we have $d_D(x,y)=d_{D(0,1^-)}(T(x),T(y))$. In fact, $d_{D,T}$ doesn't depend on T, because automorphisms of the open unit disc preserve the radius of the points, therefore for such a metric on D we will simply write d_D.

1.3.3. Let $A(0;r,1)$ be the open annulus with inner radius r and the outer radius 1. For two points $x,y\in A(0;r,1)$ we define the function $d_{A(0;r,1)}: A(0;r,1)\times A(0;r,1)\to\mathbb{R}_{\geq 0}$ as $d_{A(0;r,1)}(x,y) := d_{D(0,1^-)}(x,y)$, where $d_{D(0,1^-)}$ is like in the previous paragraph. In other words $d_{A(0;r,1)} = (d_{D(0,1^-)})_{|A(0;r,1)}$. It follows that $d_{A(0;r,1)}$ is a metric on $A(0;r,1)$.

For a general strict open annulus A and a coordinate $T: A \xrightarrow{\sim} A(0;r,1)$ we define metric $d_{A,T}$ as the pullback $T^* d_{A(0;r,1)}$ of the metric $d_{A(0;r,1)}$ via coordinate T. Note that contrary to the case of an open disc, $d_{A,T}$ depends on the coordinate T as there are two orientations of the open annulus. More precisely, assuming that A is strict open annulus, if we introduce the coordinate $T_1: A \xrightarrow{\sim} A(r,1)$ which sends the point $x\in A$ to $\frac{\alpha}{T(x)}$, where $\alpha \in k$ such that $|\alpha| = r$, then the radius of the points is changed hence we get a different metric on A. But if the two coordinates have the same orientation, the induced metrics are the same.

1.3.4. Let X be a compact, connected, quasi-smooth k-affinoid with good canonical reduction and let ξ be the maximal point of X *i.e.* the unique point in the Shilov boundary of X. We define the function $d_X : X \times X \to \mathbb{R}_{\geq 0}$ as follows.

Let x_1, x_2 be two points in X contained in the same open disc D which is a residual class attached to ξ. In this case we put $d_X(x_1, x_2) := d_D x_1, x_2$ where d_D is like in the end of the Paragraph 1.3.1.

Let $x \in X$ be a point and let D be the open disc attached to ξ that contains x. We set $d_X(x, \xi) = d_X(\xi, x) := 1 - \rho_D(x)$, where $\rho_D(x)$ is the radius of the point of x taken for (any) coordinate $T : D \xrightarrow{\sim} D(0, 1^-)$.

Finally, for two points $x_1, x_2 \in X$ and not belonging to the same residual class attached to ξ, we put $d_X(x_1, x_2) := d_X(x_1, \xi) + d_X(x_2, \xi)$.

Lemma 1.3.5. *The function d_X is a metric on X.*

Proof. As before, we just prove the triangle inequality. Let $x_1 \neq x_2 \neq x_3 \neq x_1$ be three points in X.

If x_1, x_2 belong to the same residual class D attached to ξ, and $x_3 = \xi$ or x_3 belongs to the residual class at ξ which doesn't contain x_1, x_3, let x_4 be the maximal point of the minimal closed disc in D containing x_1, x_2. Then we can check directly the following equalities

(1.3.5.1)

$$d_X(x_1, x_3) + d_X(x_2, x_3) = d_X(x_1, x_4) + d_X(x_3, x_4) + d_X(x_2, x_4) + d_X(x_4, x_3) \geq d_X(x_1, x_2).$$

If $x_3 = \xi$ and x_1, x_2 belong to the different residual classes at ξ, then the triangle inequality becomes equality.

Similarly, if one of the pairs $\{x_1, x_3\}$, $\{x_2, x_3\}$ is contained in one residual class at ξ (without loss of generality we assume that $\{x_1, x_3\}$ is contained in an open disc D attached to ξ) and if $x_2 = \xi$ or x_2 belongs to an open disc attached to ξ and different from D, we can refer again to (1.3.5.1).

If all the three points belong to different residual classes at ξ, then the triangle inequality becomes equality and is easy to check. Finally, the only remaining case is the case where all the three points belong to the same residual class at ξ, in which case we refer to Paragraph 1.3.1 and Lemma 1.3.2. $\square$

1.3.6. Let W be a simple wide open (see Definition 1.1.10), let $\{\xi\}$ be a strictly semistable triangulation of W, let W_0 be the underlying affinoid domain with the maximal point W_0. Let $W \setminus W_0 = \uplus_{i=1}^{s} A_i$, where A_i are disjoint open annuli. For each $i = 1, \ldots, s$, let $T_i : A \xrightarrow{\sim} A(0; r, 1)$ be a coordinate on A_i such that the sequence of points η_{0,ρ_n}, $\rho_n \to 1-$ is converging to ξ (note that even if the annulus A_i is not strict, such a coordinate does exist as A_i is attached to ξ).

The function $d_{W,\xi} : W \times W \to \mathbb{R}_{\geq 0}$ is defined as follows. Suppose $w_1, w_2 \in W_0$ (resp. to some A_i as above). Then, we put $d_{W,\xi}(w_1, w_2) := d_{W_0}(w_1, w_2)$ (resp. $d_{W,\xi}(w_1, w_2) := d_{A_i,T_i}(w_1, w_2)$). where d_{W_0} (resp. d_{A_i,T_i}) is as in Paragraph 1.3.4 (resp. Paragraph 1.3.3).

If $w \in W$, and w belongs to some maximal open disc D in W attached to ξ (resp. to some open annulus A_i as above), we put $d_{W,\xi}(w, \xi) = d_{W,\xi}(\xi, w) := 1 - \rho_D(w)$ (resp. $d_{W,\xi}(w, \xi) = d_{W,\xi}(\xi, w) := 1 - \rho_{A_i,T_i}(w)$).

Finally, if two points $w_1, w_2 \in W$ don't belong to the same residual class in W attached to ξ, we set $d_{W,\xi}(w_1, w_2) = d_{W,\xi}(w_1, \xi) + d_{W,\xi}(w_2, \xi)$.

Theorem 1.3.7. *The function* $d_{W,\xi} : W \times W \to \mathbb{R}_{\geq}$. $(w_1, w_2) \mapsto d_{W,\xi}(w_1, w_2)$ *is a metric on* W*.*

Proof. The proof follows similar arguments as those of Lemmas 1.3.2 and 1.3.5. $\square$

Remark 1.3.8. Following the previous paragraph, we can introduce the big metric for a bigger class of analytic curves. For example, if U is a k-analytic curve such that there exists a morphism $i : U \to W$, where W is a simple wide open can take the pullback metric via i of the restriction of $d_{W,\xi}$ to $i(U)$, that is we have the metric on U which is $d_{i,W,\xi} := i^*((d_{W,\xi})_{|i(U)})$. Even for a wider class of k-analytic curves a similar metric can be defined, but this is not necessary for our purposes.

We agree that when we write $d_{i,W,\xi}$ for a metric on U that it is precisely the metric described above.

1.3.2 Strong topology on k-analytic curves

1.3.9. Let U be a quasi-smooth k-analytic curve and $i : U \to W$ a morphism such that W is a simple wide open curve and such that i induces an isomorphism $U \xrightarrow{\sim} i(U)$. For a given strictly semistable triangulation $\{\xi\}$, we have a metric $d_{i,W,\xi}$ defined as in Paragraph 1.3.8.

Definition 1.3.10. *We denote the topology on U induced by the metric $d_{i,W,\xi}$ by $\tau_{i,W,\xi}$.*

1.3.11. It follows from the Remark 1.3.8 that for the same curve U, we can introduce different metrics $d_{i,W,\xi}$ depending on the morphism $i : U \to W$ but also on the choice of the strictly semistable triangulation $\{\xi\}$ of W, if it is not unique. For example, if we take the Berkovich projective line $\mathbb{A}^1_k$, for any type two point ξ in $\mathbb{A}^1_k$ we obtain different metric on $\mathbb{A}^1_k$. However, although a priori for each such a metric $d_{i,W,\xi}$ we have a different induced topology $\tau_{i,W,\xi}$, it turns out that the topologies are equivalent.

Theorem 1.3.12. *Let U be a quasi-smooth k-analytic curve and $i_1 : U \to W_1$ and $i_2 : U \to W_2$ two morphisms such that W_1 and W_2 are simple wide open curves and such that $U \xrightarrow{\sim} i_1(U)$ and $U \xrightarrow{\sim} i_2(U)$. Then, the topologies τ_{i_1,W_1} and τ_{i_2,W_2} on U are equivalent.*

Proof. □

1.3.13. Let X be like in 1.0.1 or a wide open curve, and let $\mathcal{T}$ be a nonempty strictly semistable triangulation of X, and assume for simplicity that $\mathcal{T}$ has at least two elements. Recall the sets $\mathcal{W}_{\mathcal{T}}$ from the Definition 1.1.5. For each element $U \in \mathcal{W}_{\mathcal{T}}$ we can introduce a topology τ on it by embedding $i : U \to W$, where W is a simple wide open curve and such that i induces an isomorphism $i : U \to i(U)$. More precisely, U is a simple wide open curve or a generalized simple wide open curve. In the first case we can take $i : U \to W$ to be id $: U \to U$ while in the second case, which only occurs if X is an affinoid space and U corresponds to a point $\xi \in \mathcal{T}$ which is in the Shilov boundary of X, we proceed as follows.

From the Van der Put's theorem ([36]), X can be embedded in the analytification X' of a smooth, connected, projective k-algebraic curve such that $X' \setminus X$ is a finite union of disjoint open discs, some of which are necessarily attached to the point ξ. Adding boundary open annuli of these discs to the the point ξ and to the set U we obtain a simple wide open curve W containing U. We take $i : U \to W$ to be id $: U \to W$.

In this way, each of the elements in $U \in \mathcal{W}_{\mathcal{T}}$ can be equipped with a strong topology τ_U which by Theorem 1.3.12 does not depend on the choices made. In the same way we can equip annuli in $\mathcal{A}_{\mathcal{T}}$ with the strong topology and if $A \in \mathcal{A}_{\mathcal{T}}$ is such that $A \subset U$, for a $U \in \mathcal{W}_{\mathcal{T}}$, then the restriction of the topology τ_U to A coincides with the strong topology on A. Moreover, as the spaces $U \in \mathcal{W}_{\mathcal{T}}$ have as intersections finite unions of elements in $\mathcal{A}$ we can glue topological spaces (U, τ_U), $U \in \mathcal{W}_{\mathcal{T}}$ along (A, τ_A) to obtain a topology $\tau_{X,s}$ on our initial curve X. It doesn't depend on the strictly semistable triangulation $\mathcal{T}$.

Definition 1.3.14. *We call the topology τ_X the strong topology on the curve X.*

Chapter 2

Finite morphism between k-analytic curves

2.0.15. *Setting* Throughout the paper, unless otherwise stated, $\varphi : Y \to X$ will denote a finite morphism between compact, connected, quasi-smooth k-analytic curves.

2.1 Compatible models and triangulations

Definition 2.1.1. *Let $\varphi : Y \to X$ be as in 2.0.15, and let $\mathfrak{Y}$ and $\mathfrak{X}$ be some formal (resp. strictly) semistable models of Y and X, respectively. We say that φ extends to a finite morphism between models $\mathfrak{Y}$ and $\mathfrak{X}$ if there exists a finite morphism $\Phi : \mathfrak{Y} \to \mathfrak{X}$ such that $\varphi = \Phi_\eta$, where Φ_η is the generic fiber of Φ. If φ extends to a finite morphism between models $\mathfrak{Y}$ and $\mathfrak{X}$, we also say that $\mathfrak{Y}$ and $\mathfrak{X}$ are (resp. strictly) φ-compatible. We sometimes just say compatible, if φ is understood from the context.*

We have a similar notions for triangulations.

Definition 2.1.2. *Let $\varphi : Y \to X$ be as in 2.0.15. We say that semistable triangulations $\mathcal{S}$ and $\mathcal{T}$ of Y and X, respectively, are φ-compatible (or just compatible if φ is understood from the context) if $\mathcal{S} = \varphi^{-1}(\mathcal{T})$. We say that they are strictly φ-compatible (or just strictly* *compatible if φ is understood from the context) if no two adjacent vertices of $\mathcal{S}$ are mapped to a single vertex of $\mathcal{T}$. Two vertices of a triangulation are said to be adjacent if they are endpoints of an open annulus which has an empty intersection with the triangulation itself.*

Theorem 2.1.3. *Let $\varphi : Y \to X$ be as in 2.0.15 and let $\mathfrak{Y}$ and $\mathfrak{X}$ be semistable formal models of Y and X, respectively. Then, $\mathfrak{Y}$ and $\mathfrak{X}$ are φ-compatible if and only if the triangulations $\mathcal{T}_{\mathfrak{Y}}$ and $\mathcal{T}_{\mathfrak{X}}$ are φ-compatible.*

Proof. Let $\Phi : \mathfrak{Y} \to \mathfrak{X}$ be the extension of φ. Then it is known that the generic points of the irreducible components of $\mathfrak{Y}_s$ are surjectively mapped to the generic points of the irreducible components of $\mathfrak{X}_s$ which implies that the triangulations $\mathcal{T}_{\mathfrak{Y}}$ and $\mathcal{T}_{\mathfrak{X}}$ are φ-compatible.

We postpone the proof of the converse implication until after the Corollary 2.1.32. □

The main results of this section are the following theorems.

Theorem 2.1.4. *Let $\varphi : Y \to X$ be a finite morphism of compact, connected, quasi-smooth, k-analytic curves. Then, there exists triangulations $\mathcal{S}$ and $\mathcal{T}$ of Y and X, respectively, which are (strictly) φ-compatible.*

Theorem 2.1.5. *Let $\varphi : T \to X$ be as in 2.0.15. Then, there exist (strictly) semistable formal models $\mathfrak{Y}$ and $\mathfrak{X}$ of Y and X, respectively, such that φ extends to a finite morphism $\Phi : \mathfrak{Y} \to \mathfrak{X}$.*

There is a counterpart for the finite morphisms of wide open curves.

Theorem 2.1.6. *Let $\varphi : W \to V$ be a finite morphisms of wide open curves. Then, there exists triangulations $\mathcal{S}$ and $\mathcal{T}$ of W and V, respectively, which are (strictly) φ-compatible.*

Before going to the proofs, we give some remarks.

Remark 2.1.7. Suppose that $\mathcal{S}$ and $\mathcal{T}$ are φ-compatible semistable triangulations of Y and X, respectively, then, at the level of skeletons, $\Gamma^Y_{\mathcal{S}} = \varphi^{-1}(\Gamma^X_{\mathcal{T}})$. As was seen before,

$\Gamma^Y_{\mathcal{S}}$ (resp. $\Gamma^X_{\mathcal{T}}$) is a disjoint union of the skeleta of the annuli in $\mathcal{A}_{\mathcal{S}}$ (resp. $\mathcal{A}_{\mathcal{T}}$) and $\mathcal{S}$ (resp. $\mathcal{T}$). It will be shown later (Corollary 2.1.32) that each $A_2 \in \mathcal{A}_{\mathcal{T}}$ is of the form $\varphi(A_1)$ for some $A_1 \in \mathcal{A}_{\mathcal{S}}$ and for each $A_1 \in \mathcal{A}_{\mathcal{S}}$ there exists an $A_2 \in \mathcal{A}_{\mathcal{T}}$ such that φ restricts to a finite morphism $\varphi : A_1 \to A_2$. Having in mind this, to prove $\Gamma^Y_{\mathcal{S}} = \varphi^{-1}(\Gamma^X_{\mathcal{T}})$ it is enough to prove that if $\varphi : A_1 \to A_2$ is a finite morphism of open annuli, then $\Gamma^{A_1} = \varphi^{-1}(\Gamma^{A_2})$. But this is a consequence of the fact that, for a choice of coordinates on A_1 and A_2, the valuation polygon of the function φ consists of a single segment of nonzero slope without breaks, and essentially (the second and the third paragraph of) the proof of Lemma 2.1.17.

Remark 2.1.8. Let $\varphi : Y \to X$, $\mathcal{S}$ and $\mathcal{T}$ be as in Theorem 2.1.4 and suppose that $\xi \in \Gamma^X_{\mathcal{T}} \setminus \mathcal{T}$ is a type two point. Necessarily, ξ belongs to a skeleton of an open annulus in $\mathcal{A}_{\mathcal{T}}$ and therefore $\mathcal{T}_1 := \mathcal{T} \cup \{\xi\}$ is a strictly semistable triangulation of X. Then $\mathcal{S}_1 = \varphi^{-1}(\mathcal{T}_1) = \mathcal{S} \cup \varphi^{-1}(\xi)$ is a (strictly semistable) triangulation of Y which is φ-compatible with $\mathcal{T}_1$. Again, this is a consequence of Corollary 2.1.32 and the previous remark. Namely, we simply note that every point in $\varphi^{-1}(\xi)$ belongs to a skeleton of some open annulus in $\mathcal{A}_{\mathcal{S}}$.

Remark 2.1.9. As we have seen in the Remark 2.1.7 if we have a finite morphism of quasi-smooth, k-analytic curves $\varphi : Y \to X$ and strictly φ-compatible triangulations $\mathcal{S}$ and $\mathcal{T}$ of Y and X, respectively, then it follows that for skeleta $\Gamma_{\mathcal{S}} = \varphi^{-1}(\Gamma_{\mathcal{T}})$. In the other direction, if we have $\Gamma_1 = \varphi^{-1}(\Gamma_2)$, where Γ_1 and Γ_2 are finite skeleta with type 2 endpoints, then we can find strictly semistable triangulations $\mathcal{S}'$ and $\mathcal{T}'$ of Y and X, respectively, such that $\Gamma_1 = \Gamma^Y_{\mathcal{S}'}$ and $\Gamma_2 = \Gamma^X_{\mathcal{T}'}$ (Lemma 1.2.8). Then, for the reasons similar as in the previous remarks $\mathcal{T} := \varphi(\mathcal{S}') \cup \mathcal{T}'$ is a semistable triangulation of $\mathcal{X}$ and $\mathcal{S} := \varphi^{-1}(\mathcal{T}')$ is a semistable triangulation of Y and $\mathcal{S}$ and $\mathcal{T}$ are *strictly* φ-compatible.

In fact, this gives a way to prove the Theorem 2.1.4, which was adapted in article [1] of which the author learned while writing this note. We explain, in our own words the main idea of [1], and compare it to our method. Namely, in *loc.cit.* Section 4.10, it is proved that, in our terminology, given a semistable skeleton Γ_1 in the curve Y, and a semistable skeleton Γ_2 in the curve X, we can find a semistable skeleton Γ_3 in X containing $\Gamma_2 \cup \varphi(\Gamma_1)$. As semistable skeleton Γ_1 corresponds to a semistable triangulation $\mathcal{T}_{\Gamma_2}$ of X which as a complement has a finite disjoint union of open annuli and a disjoint union of open discs, one is led to prove that for each open annulus $A \in \mathcal{A}_{\mathcal{T}_{\Gamma_2}}$, $\varphi(\Gamma_1) \cap A$ is a finite (semistable) skeleton in A and for each open disc/connected component in $X \setminus \Gamma_2$, $\varphi(\Gamma_1) \cap D$ is a finite (semistable) skeleton in D. Similarly, in the other direction one shows that $\varphi^{-1}(\Gamma_3)$ is a semistable skeleton of Y, again by studying intersections of $\varphi^{-1}(\Gamma_3)$ with the open discs in $Y \setminus \Gamma_1$ and with the open annuli in $\mathcal{A}_{\mathcal{S}_{\Gamma_1}}$.

Let us explain the underlying idea of the proof of Theorem 2.1.4. Let $\mathcal{S}_1$ be any semistable triangulation of Y and let $\mathcal{T}_1$ be any semistable triangulation of X containing the points in the set $\varphi(\mathcal{S}_1)$. Then, the connected components of $Y \setminus \varphi^{-1}(\mathcal{T}_1)$ are open discs and wide open curves isomorphic to wide open curves in a projective line $\mathbb{P}^1_k$. Then, because of the next theorem, our morphism φ restricts to a finite morphism between such wide open curves in Y and open annuli and open discs in X and we are led to find compatible triangulations for these type of objects.

We start by considering simple cases first, compatible triangulations for a finite family of finite morphisms of open discs to an open disc. We describe in details how to find compatible triangulations for a finite morphism from an open annulus to an open disc, and the main tool here is the valuation polygon for our morphism/function φ as we have the advantage of coordinates in this situation. We end up with the more involved case of finding up compatible triangulations for a finite family of finite morphisms between wide open curves in a projective line and an open annulus (as it turns out, we don't have to consider more general case than this).

For the proof of the Theorem 2.1.4, we use the following well known results. As the proof are short, we choose to present them for the convenience of the reader.

Theorem 2.1.10. *Let $\varphi : Y \to X$ be a finite morphism of k-analytic spaces. Let $V \subset X$ be an analytic domain in X and let $U \subset Y$ be a connected component of $\varphi^{-1}(V)$. Then,*

φ restricts to a finite morphism $\varphi_U : U \to V$.

Remark 2.1.11. For the definition and basic properties of analytic domains see [4, Section 1.3].

Proof. We will need the following (well known) result, proved in [5].

Lemma 2.1.12. *Let X be a k-affinoid space. Then, X has finitely many connected components and arbitrary union of connected components is a k-affinoid domain in X.*

Proof. That X has only finitely many connected components follows from the fact that X is compact. To prove the second part of the assertion, we show that connected components are closely related to the canonical maps $O_X(X) \to O_{X,x}$, where $x \in X$. More precisely:

X is connected if and only if for all points $x \in X$ the canonical map $O_X(X) \to O_{X,x}$ is injective.

For the "only if" part, let $f \in O_X(X), f \neq 0$, and suppose that for some point $x \in X$, the image f_x of f in $O_{X,x}$ is 0. This means that f is identically zero on some open neighborhood of x and by the identity theorem, it vanishes on the whole connected component containing x. Since $f \neq 0$, it follows that X is not connected. On the other side, suppose that X is not connected and let $U \subset X$ be a connected component and let $x \in U$. Choose a function $f^U \in O_X(X)$ s.t. $f^U_{|U} = 0$ and $f^U_{|X\setminus U} = 1$. Then $f^U \neq 0$ and $f^U_x = 0$.

Furthermore, if $U_1, \dots U_n$ are some connected components of the affinoid X, then $U = U_1 \cup \dots \cup U_n$ can be described as the set $\{x \in X, |f^U(x)| \leq 1/2\}$ for the function f^U constructed above, so U is an affinoid domain. $\square$

Let us prove that U is an analytic domain in Y. Let $y \in U$. Since $\varphi^{-1}(V)$ is an analytic domain and $y \in \varphi^{-1}(V)$, there exists affinoid domains $V_1, \dots, V_n \subset \varphi^{-1}(V)$ s.t. $y \in V_1 \cap \dots \cap V_n$ and $V_1 \cup \dots \cup V_n$ is a neighborhood of y. The intersection $W_i = V_i \cap U$ $i = 1, \dots, n$ is an affinoid domain because of the previous lemma and contains y. Also, $W_1 \cup \dots \cup W_n = U \cap (V_1 \cup \dots \cup V_n)$ which implies that $W_1 \cup \dots \cup W_n$ is a neighborhood of y, hence U is an analytic domain.

Next we show that the inclusion $j : U \hookrightarrow \varphi^{-1}(V)$ is a closed embedding (thus a finite morphism). We use the characterisation [4, Lemma 1.3.7] (we note here that in *loc.cit.* there is a condition missing in the statement of this Lemma, namely: in *a*) it should also be stated that $V_1 \cup \cdots \cup V_n$ is a neighborhood of the point x). Let $W \subset X$ be an affinoid domain in X. Then, $j^{-1}(W) = W \cap U$. It follows from the previous lemma that $W \cap U$ is an affinoid domain. Then the inclusion $W \cap U \to W$ is a closed embedding given by admissible epimorphism $O_X(W) \to O_X(W \cap U)$.

Finally, the restriction $\varphi_U : U \to V$ factors through the inclusion $U \hookrightarrow X$ followed by φ, which is a composition of finite maps, which itself then is finite. □

Corollary 2.1.13. *Let $\varphi : Y \to X$ be a finite morphism where Y and X are compact, connected, quasi-smooth, k-analytic curves or wide open curves. Let A_1 be an open annulus in Y with endpoints η_1 and η_2 in Y, $\eta_1 \neq \eta_2$ and let $\xi_i = \varphi(\eta_i)$, $i = 1, 2$. Suppose that $\varphi^{-1}(\xi_1) \cap A_1 = \varphi^{-1}(\xi_2) \cap A_1 = \emptyset$.*

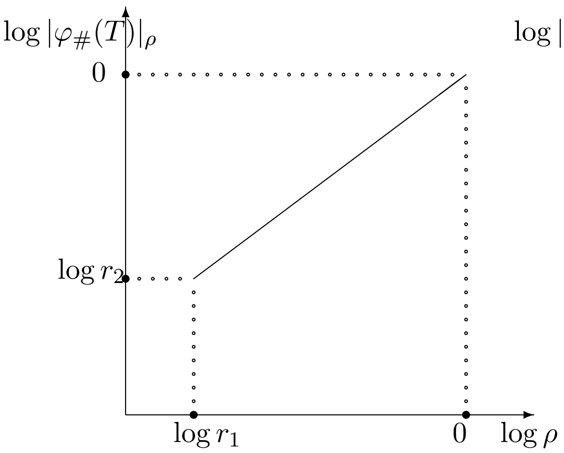
(i) Suppose that $\xi_1 \neq \xi_2$ and that ξ_1 and ξ_2 are endpoints of an open annulus A_2 in X, such that there exists a point $\eta \in A_1$ with $\varphi(\eta) \in A_2$. Then, $A_2 = \varphi(A_1)$ and φ restricts to a finite morphism $\varphi_{A_1} : A_1 \to A_2$.

(ii) Suppose that $\xi_1 = \xi_2$ and that D is an open disc in X attached to the point ξ_1 such that there exists a point $\eta \in A_1$ with $\varphi(\eta) \in D$. Then, $D = \varphi(A_1)$ and φ restricts to a finite morphism $\varphi_{A_1} : A_1 \to D$.

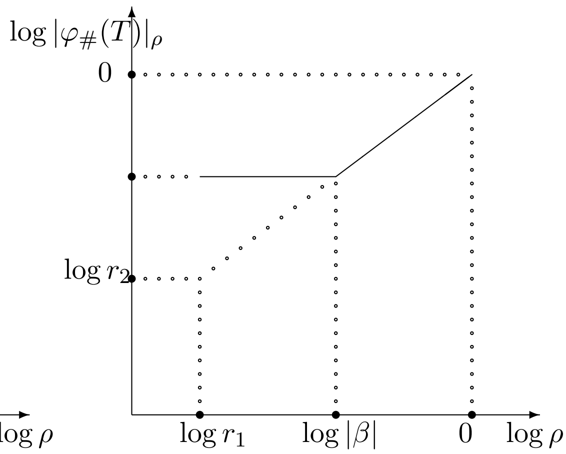
Proof. (*i*) Note that $\varphi(A_1)$ is path connected since φ is continuous and A_1 is path connected. Since $\varphi(A_1)$ doesn't contain neither of the points ξ_1, ξ_2, and since $\varphi(\eta) \in A_2$, it follows that $\varphi(A_1) \subseteq A_2$. Let us prove that equality holds.

Let y_n, $n = 1, 2, \ldots$ be a sequence of points in A_1 converging to the point η_1 (resp. η_2). Then, $\varphi(y_n)$, $n = 1, 2, \ldots$ converges to the point ξ_1 (resp. ξ_2), which together with the previous remark implies that the image of the skeleton of A_1 contains the skeleton of A_2 (any path contained in A_2 connecting the endpoints of A_2 contains the skeleton of

A_2). To prove that $\varphi(A_1) = A_2$, it is enough to show surjectivity on rational points. For this, let T (resp. S) be a coordinate on A_1 (resp. A_2), which identifies A_1 (resp. A_2) with $A(0; r_1, 1)$ (resp. $A(0; r_2, 1)$). The morphsim φ can be expressed in coordinates S and T as $S = \varphi_{\#}(T) = \sum_{i\in\mathbb{Z}} a_i T^i$. Since $\varphi_{\#}(T)$ doesn't have zeros, it follows that its valuation polygon consists of a line segment without breaks (with necessarily nonzero slope equal to the degree of φ!), as shown in the Picture 1. Let $\beta \in A(0; r_2, 1)(k)$. Then, the valuation polygon of the function $\varphi_{\#}(T) - \beta$ defined on $A(0; r_1, 1)$ and shown in Picture 2, has a break at the level $\log|\beta|$, hence a zero in $A(0; r_1, 1)$. As noted before, this implies that $\varphi(A_1) = A_2$.



Pic. 1

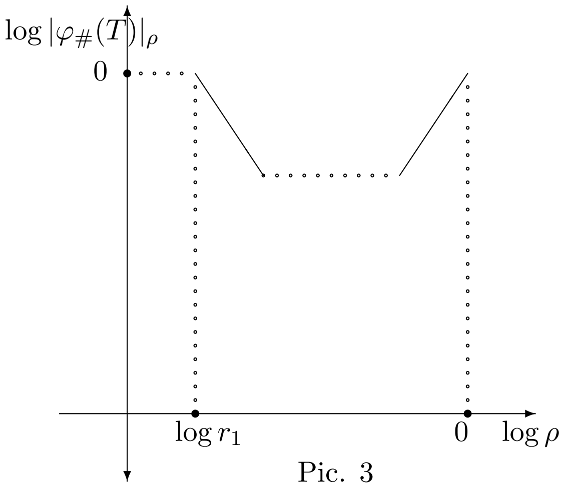


Pic. 2

On the other side, by the conditions of lemma, one of the connected components of the set $\varphi^{-1}(A_2)$ is A_1, hence by Lemma 2.1.10, φ restricts to a finite morphism $\varphi_{A_1} : A_1 \to A_2$.

(ii) As before, we can conclude that $\varphi(A_1)$ is path connected and that $\varphi(A_1) \subseteq D$. We first prove the equality $\varphi(A_1) = D$. It is enough, as before, to prove surjectivity on rational points.

Let $x \in D(k)$ be a rational point, and let us fix a coordinate S on A_1, which identifies A_1 with $A(0; r_1, 1)$ and let us choose a coordinate $T : D \xrightarrow{\sim} D(0, 1^-)$, with $T(x) = 0$. We have a coordinate representation $S = \varphi_{\#}(T) = \sum_{i\in\mathbb{Z}} a_i T^i$. If we take a sequence of points y_i, $i = 1, 2, \ldots$ converging to the point η_1 (resp. η_2), then $\varphi(y_i)$, $i = 1, 2, \ldots$ converges to the point ξ_1. This implies that the valuation polygon, which is a convex function of the variable $\log\rho$, $\log\rho \in (\log r_1, 0)$, near the boundary points for the function $\varphi_{\#}(T)$ looks like in the Picture 3 (note that it cannot be a line segment) *i.e.* it has nonzero slopes of the opposite signs.



Pic. 3

It follows that the function $\log|\varphi_{\#}(T)|_\rho$ has a break, which means there exists a rational point $\alpha \in A(0; r_1, 1)(k)$, with $\varphi(\alpha) = 0$ i.e. there exists $y \in A_1(k)$, s.t. $\varphi(y) = x$, that is φ is surjective on rational points.

Once again, the conditions of the lemma imply that one of the connected components of the set $\varphi^{-1}(D)$ is A_1, which together with Lemma 2.1.10 finishes the proof.

□

With similar arguments one can prove the following

Corollary 2.1.14. *Let $\varphi : Y \to X$ be a finite morphism between quasi-smooth k-analytic curves, and suppose that D_1 is an open disc in Y whose image is contained in an open disc D_2 in X. Then, $\varphi(D_1)$ is an open disc in X and φ restricts to a finite morphism $\varphi : D_1 \to \varphi(D_1)$.*

Proof. Let ξ be the end point of D_1 and let $\eta = \varphi(\xi)$. It is enough to prove that $\varphi^{-1}(\eta) \cap D_1 = \emptyset$. Suppose the contrary, that there exists a point ξ_1 in D_1 which is mapped to

η. Let $y \in D_1(k)$ be a rational point such that the canonical path from y to ξ contains the point ξ_1. Let $T : D_1 \xrightarrow{\sim} D(0, r_1^-)$ be a coordinate on D_1 with $T(y) = 0$ and $S : D_2 \xrightarrow{\sim} D(0, r_2^-)$ a coordinate on D_2 with $S(\varphi(y)) = 0$. Then, the restriction of φ on D_1 has a a coordinate representation which is a power series $S = \varphi_{\#}(T) = \sum_{n \geq 0} a_n T^n$. Since the valuation polygon of $\varphi_{\#}(T)$ is a strictly increasing, convex function it follows that $\varphi(\xi_1) \in D_2$ which is a contradiction. $\square$

Remark 2.1.15. Keep the notation from the Theorem 2.1.4. Let $x \in X$ be a point and let B_x be an open disc which is a connected component in $X \setminus \Gamma_{\mathcal{T}}$ containing x. Let B_y be any open disc which a connected component in $Y \setminus \Gamma_{\mathcal{S}}$ which contains a point $y \in B_y$ with $\varphi(y) = x$. Then φ restricts to a finite morphism $\varphi : B_y \to B_x$. This is almost an immediate consequence of the previous Corollary and the Remark 2.1.7. Namely, the endpoint of B_y is mapped to the endpoint of B_x and since $\Gamma_{\mathcal{S}}^Y = \varphi^{-1}(\Gamma_{\mathcal{T}}^X)$ we are in the situation of the Corollary 2.1.14.

Corollary 2.1.16. *Let $\varphi_i : W_i \to V$, $i = 1, \ldots, n$ be a finite family of finite morphisms of wide open curves, and assume that W_i (hence also V) is not projective. Then, for a sufficiently small boundary (see below) open annulus A_V of V, there is a boundary open annulus $A_i \in W_i$, for each i, such that φ induces a finite morphism $\varphi_{i|A_i} : A_i \to A_V$.*

Let U be a wide open curve in a curve X (see Definition 1.1.2), and let $\{u_1, \ldots, u_n\}$ be the endpoints of U in X. Then, any open annulus in U and not equal to U with an endpoint in a u_i is called a *boundary* open annulus in U. For a general wide open curve U, an open annulus A in U is said to be boundary if $i(A)$ is a boundary open annulus in U', where U' is a wide open curve in X and $i : U \to U'$ is an isomorphism from the Definition 1.1.2. In other words, an open annulus A in a wide open curve U is called boundary if it is not precompact in U.

Proof. Let V_0 be an affinoid domain in V. Then, $W_{i,0} = \varphi_i^{-1}(V_0)$ is an affinoid domain in W_i, hence compact and not containing any boundary open annuli in W_i. As V and W_i are exhausted by affinoid domains, by increasing V_0 we may achieve that $V \setminus V_0$ is a finite union of open boundary annuli and that $W_i \setminus W_{i,0}$ is a finite union of boundary open annuli. Let A_V be a boundary open annulus in V which is a connected component of $V \setminus V_0$ and let A_i be a boundary open annulus which is a connected component of $W_i \setminus W_{i,0}$ such that $\varphi_i(A_i)$ has a nonempty intersection with A_V. Then, $\varphi_i(A_i)$ being connected, is in fact contained in A_V. On the other side, $\varphi_i^{-1}(A_V)$ contains A_i and has an empty intersection with $W_{i,0}$, therefore A_i is a connected component of it which by Theorem 2.1.10 implies that $\varphi_{i|A_i} : A_i \to A_V$ is finite. □

2.1.1 Compatible triangulations of discs

Lemma 2.1.17. *Let $\varphi : D_1 \to D_2$ be a finite morphism between closed unit discs and let $\xi \in D_2$ be a type two point. Then, there exists triangulations $\mathcal{T}_1$ of D_1 and $\mathcal{T}_2$ of D_2 such that $\xi \in \mathcal{T}_2$ and the triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ are strictly compatible.*

Proof. Let $x \in D_2(k)$ be a rational point such that the canonical path from x to the maximal point of D_2 contains the point ξ and let $S = \varphi^{-1}(x) = \{y_2, \dots, y_s\}$. Let us prove that $\Gamma_S = \varphi^{-1}(\Gamma_x)$.

Let T_i be a coordinate on D_1 such that $T_i(y_i) = 0$ and T a coordinate on D_2 with $T(x) = 0$. With respect to coordinates T and T_i, φ has representation $T = \sum_{n \geq 0} a_{i,n} T_i^n$. Let (ρ_j, ρ_{j+1}) be any segment contained in $(0, 1)$ such that the valuation polygon of φ over $(\log \rho_j, \log \rho_{j+1})$ consists of a segment with constant slope (see Figure 2.1). Then, we can write $T = \varphi_\#(T_i) = a_d T_i^d (1 + h(T_i))$ where d is the slope of $\varphi_\#(T_i)$ over $(\log \rho_j, \rho_{j+1})$ and $h(T_i)$ is an analytic function convergent on the open annulus $A(0; \rho_j, \rho_{j+1})$ and such that $|h|_\rho < 1$ for all $\rho \in (\rho_j, \rho_{j+1})$. Let us prove that $\varphi(\eta_{0,\rho}) = \eta_{0,|a_d|\rho^d}$ for all $\rho \in (\rho_j, \rho_{j+1})$. It is enough to check that for any analytic function $f(T)$ in $k\{T\}$ we have $|f(\varphi(\eta_{0,\rho}))| = |f|_{0,|a_d|\rho}$ and since the ring of polynomials $k[T]$ is everywhere dense in $k\{T\}$ with respect to the norm $|\cdot|_\rho$ it is enough to check the equality $|f(\varphi(\eta_{0,\rho}))| = |f|_{0,|a_d|\rho}$ for a polynomial $f(T) = b_t T^t + \cdots + b_0 \in k[T]$. But the last claim follows from the fact that

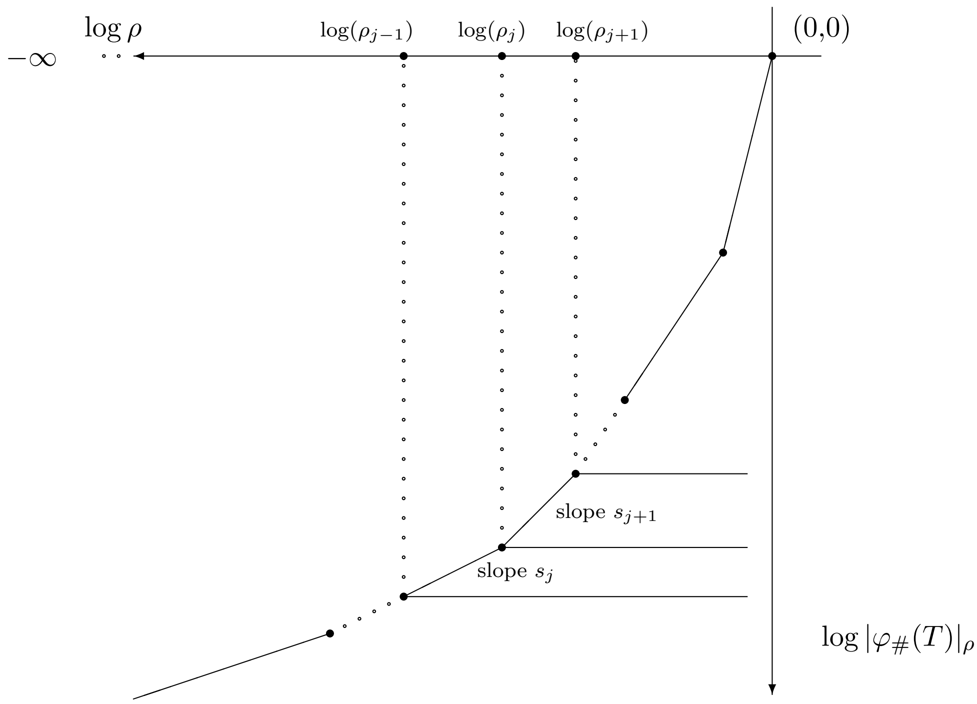


Figure 2.1: The valuation polygon for the function $T = \varphi_{\#}(T_i)$; The number of zeros of the function $\varphi_{\#}(T_i)$ of norm ρ_j is $s_{j+1} - s_j$

$|f(\varphi(\eta_{0,\rho}))| = |f(\varphi_{\#}(T_i))|_\rho = \max_{1\leq l\leq t}|b_l||T_i^{dl}|_\rho = \max_{1\leq l\leq t}|b_l||a_dT_i^{dl}|_\rho = |f|_{0,|a_d|\rho^d}$. By continuity it follows that $\varphi(\Gamma_{y_i}) = \Gamma_x$.

Suppose that $\eta \in D_1$ is a type two point such that $\varphi(\eta) \in \Gamma_x$ and let $y \in D_1(k)$ be such that the canonical path from y to the maximal point of D_1 contains η. If we choose a coordinate T_0 on D_1 with $T_0(y) = 0$, then we may write $\eta = \eta_{0,\rho}$ for some $\rho \in (0,1)$. The valuation polygon of the function $T = \varphi_{\#}(T_0)$ cannot be constant as $\varphi(\eta_{0,\rho}) \in \Gamma_x$ which implies there is a zero in $D_1(k)$ for the equation $\varphi_{\#}(T_0) = 0$ with norm less or equal to ρ. It follows that $\eta \in \Gamma_{\{y,\dots,y_s\}}$. We conclude that $\varphi^{-}(\Gamma_x) = \Gamma_{\{y_1,\dots,y_s\}}$.

At this point we can invoke the Remark 2.1.9 and conclude the proof. □

In a similar way, we can prove as well the following corollary.

Corollary 2.1.18. *Let $\varphi : D_1 \to D_2$ be a finite morphism between open unit discs and let $\xi \in D_2$ be a type two point. Then, there exist triangulations $\mathcal{T}_1$ of D_1 and $\mathcal{T}_2$ of D_2 such that $\xi \in \mathcal{T}_2$ and the triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ are strictly compatible.*

Proof. It follows from Corollary 2.1.16 that we can decompose D_1 (resp. D_2) as a disjoint union of a closed disc D'_1 (resp. D'_2) and a boundary open annulus A_1 (resp. A_2) such that the restrictions $\varphi_{|D'_1} : D'_1 \to D'_2$ and $\varphi_{|A_1} : A_1 \to A_2$ are finite morphisms. Then we can apply the previous lemma. □

Corollary 2.1.19. *Let $\varphi_i : D_i \to D$, $i = 1,\dots,s$ be a finite family of finite morphisms of closed (resp. open) unit discs and let $\xi \in D$ be a type two point. Then, we can find triangulations $\mathcal{T}_i$ of D_i, $i = 1,\dots,s$ and $\mathcal{T}$ of D such that $\mathcal{T}_i$ and $\mathcal{T}$ are strictly φ_i-compatible for each $i = 1,\dots,s$ and $\xi \in \mathcal{T}$.*

Proof. We can reduce the case where the discs involved are open to the case where the discs are closed by using the Corollary 2.1.16, similarly like in the previous corollary, therefore we just consider the case where all discs are closed. It follows from Lemma 2.1.17 that for each $i = 1,\dots,s$ there exist triangulations $\mathcal{T}'_i$ of D_i and $\mathcal{T}^i$ of D which are strictly φ_i-compatible and such that $\xi \in \mathcal{T}^i$. By construction of $\mathcal{T}^i$ and $\mathcal{T}'_i$ in Lemma 2.1.17 we have $\Gamma_\xi = \Gamma_{\mathcal{T}^i}$, so the union $\mathcal{T} := \cup_i \mathcal{T}^i$ is a triangulation of D and moreover $\mathcal{T}_i := \varphi^{-1}(\mathcal{T})$ is a triangulation of D_i which is strictly φ_i-compatible with $\mathcal{T}$. $\square$

Theorem 2.1.20. *Let $\varphi_i : D_i \to D$, $i = 1, \ldots, s$, be a finite family of finite morphisms of closed (resp. open) discs and let $S \subset D$ be a finite set of type two points. Then we can find triangulations $\mathcal{T}_i$ of D_i, $i = 1, \ldots, s$ and $\mathcal{T}$ of D such that $S \subset \mathcal{T}$ and such that $\mathcal{T}_i$ and $\mathcal{T}$ are strictly φ_i-compatible for each $i = 1, \ldots, s$.*

Proof. We proceed by induction on the number of points in S. For S an empty set or S having a one element, we can use Corollary 2.1.19. Suppose that the Theorem 2.1.20 is true for any S with $|S| \leq n-1$. Let S be a subset of D consisting of n type two points, let $\xi \in S$ and put $S' = S \setminus \{\xi\}$. Then there exist triangulations $\mathcal{T}_{S'}$ of D and for each $i = 1, \ldots, s$, $\mathcal{T}_{S',i}$ of D_i such that $S' \subset \mathcal{T}_{S'}$ and $\mathcal{T}_{S'}$ and $\mathcal{T}_{S',i}$ are strictly φ_i-compatible.

There are two possibilities for the point ξ: either $\xi \in \Gamma_{S'}$, or ξ belongs to an open disc attached to $\Gamma_{S'}$. In the first case we can take $\mathcal{T} := \mathcal{T}_{S'} \cup \{\xi\}$ and $\mathcal{T}_i := \mathcal{T}_{S',i} \cup \varphi_i^{-1}(\xi)$ (see Remark 2.1.8) so let us assume that we are in the second case. Let B be a maximal open disc in D attached to $\Gamma_{S'}$ and containing ξ and let for each $i = 1, \ldots, s$, $B_{i,j}$, $j = 1, \ldots, \alpha(i)$ denote maximal open discs in D_i, attached to $\Gamma_{S',i}$ and which have a non empty intersection with $\varphi_i^{-1}(\xi)$. Let us also put η to be the (necessarily type two) point in $\Gamma_{S'}$ to which the disc B is attached. Then, φ_i restricts to a finite morphism $\varphi_j : B_{i,j} \to B$ for each i and j and we can apply the Theorem 2.1.19 to find triangulations $\mathcal{T}_{i,j}$ of $B_{i,j}$ and $\mathcal{T}'$ of B such that $\xi \in \mathcal{T}'$ and $\mathcal{T}_{i,j}$ and $\mathcal{T}'$ are φ_i-compatible. Then we can take $\mathcal{T}_i := \mathcal{T}_{S'} \cup \varphi_i^{-1}(\mathcal{T}' \cup \{\eta\})$ and $\mathcal{T} := \mathcal{T}_{S'} \cup \mathcal{T}' \cup \{\eta\}$. $\square$

We are ready to prove the following lemma which in some forms we have already seen in this section.

Lemma 2.1.21. *Let $\varphi : Y \to X$ be a finite morphism where Y and X are like in 2.0.15* i.e. *compact, connected, quasi-smooth k-analytic curves or Y and X are simultaneously wide open curves. Suppose that $\mathcal{S}$ and $\mathcal{T}$ are strictly semistable triangulations of Y and*

X, respectively, which are (strictly) φ-compatible and let S be a finite set of type two points in X not contained in $\Gamma^X_{\mathcal{T}}$. Then we can find strictly φ-compatible triangulations $\mathcal{S}'$ and $\mathcal{T}'$ of Y and X which refine $\mathcal{S}$ and $\mathcal{T}$, respectively, such that $S \subset \mathcal{T}'$.

More precisely, let D_i, $i = 1, \ldots, n$ be all the open discs in X which have a nonempty intersection with S and with endpoints in $\Gamma_{\Gamma^X_{\mathcal{T}}}$. Then, we can find $\mathcal{T}'$ such that $\Gamma^X_{\mathcal{T}'} = \Gamma^X_{\mathcal{T}} \cup \cup_{i=1}^n \Gamma^{D_i}_{D_i \cap S}$.

Proof. For each $i = 1, \ldots, n$, let $D_{i,j}$, $j = 1, \ldots, \alpha(i)$ be the open discs which are connected components of $\varphi^{-1}(D_i)$. Then φ restricts to finite morphisms $\varphi_{|D_{i,j}} : D_{i,j} \to D$ (Remark 2.1.15). Let us fix i and let $S_i = S \cap D_i = \{s_{i,1}, \ldots, s_{i,\beta(i)}\}$. Then we can apply Theorem 2.1.20 for the family of morphisms $\varphi_{|D_{i,j}} : D_{i,j} \to D_i$, $j = 1, \ldots, \alpha(i)$ and subset $S_i \subset D_i$ to find $\varphi_{|D_{i,j}}$-compatible triangulations $\mathcal{T}_{i,j}$ and $\mathcal{T}_i$ for $D_{i,j}$ and D_i with $S_i \subset \mathcal{T}_i$. But from the proof of the same theorem we can conclude that $\Gamma^{D_i}_{\mathcal{T}_i} = \Gamma^{D_i}_{S_i} = \cup_{l=1}^{\beta(i)} \Gamma^{D_i}_{s_{i,l}}$ and that $\Gamma^{D_{i,j}}_{\mathcal{T}_{i,j}} = \varphi^{-1}_{|D_{i,j}}(\Gamma^{D_i}_{\mathcal{T}_i})$. To conclude the proof, we notice that $\Gamma^Y_{\mathcal{S} \cup \cup_{i,j} \mathcal{T}_{i,j}} = \varphi^{-1}(\Gamma^X_{\mathcal{T} \cup \cup i \mathcal{T}_i})$ and recall the Remark 2.1.9 to prove the existence of strictly φ-compatible triangulations $\mathcal{S}''$ and $\mathcal{T}''$ of Y and X, respectively such that $\Gamma^Y_{\mathcal{S}''} = \Gamma^Y_{\mathcal{S} \cup \cup_{i,j} \mathcal{T}_{i,j}}$ and $\Gamma^X_{\mathcal{T}_i} = \Gamma^{D_i}_{\mathcal{T}_i}$. Then we can take $\mathcal{T}' = \mathcal{T}'' \cup \mathcal{T}$ and $\mathcal{S}' = \varphi^{-1}(\mathcal{T}')$ (note that S belongs to $\mathcal{T}''$ automatically as being the endpoints of the corresponding skeleton). □

2.1.2 Compatible triangulations of annuli and discs

To motivate this section, we again recall the difference between strictly compatible and compatible triangulations $\mathcal{S}$ and $\mathcal{T}$. Namely, in the case of compatible triangulations we may have an open annulus in $\mathcal{A}_{\mathcal{S}}$ which is mapped to an open disc with the boundary point in $\mathcal{T}$. We would like to know how to refine triangulations such that an open annulus in $\mathcal{A}_{\mathcal{S}}$ is mapped to an open annulus in $\mathcal{A}_{\mathcal{T}}$. Therefore, we proceed by studying with some extra details finite morphisms between annuli and discs.

Theorem 2.1.22. *Let $\varphi : A \to D$ be a finite morphism between a closed annulus and a closed disc. Then there exist strictly φ-compatible triangulations $\mathcal{S}$ and $\mathcal{T}$ of A and D,* *respectively.*

Proof. Let us start with the case where both annulus and disc are strict. If we fix coordinates $T : A \xrightarrow{\sim} A[0; r, 1]$ and $S : D \xrightarrow{\sim} D(0, 1)$ where $r \in (0, 1) \cap |k^\times|$, then φ can be expressed as $S = \varphi_\#(T) = \sum_{n\in\mathbb{Z}} a_n T^n$ and since φ can always be decomposed as a morphism $S = \varphi'_\#(T) = \sum_{n\in\mathbb{Z}, n\neq 0} a_n T^n$ followed by translation by a_0, and translation is an isomorphism, we will assume that in coordinate representation of φ, $a_0 = 0$. In this case, valuation polygon of the function $\varphi_\#(T)$ (see Figure 2) doesn't have flat components and achieves its minimum in a unique point $\log(\rho_{\min})$.

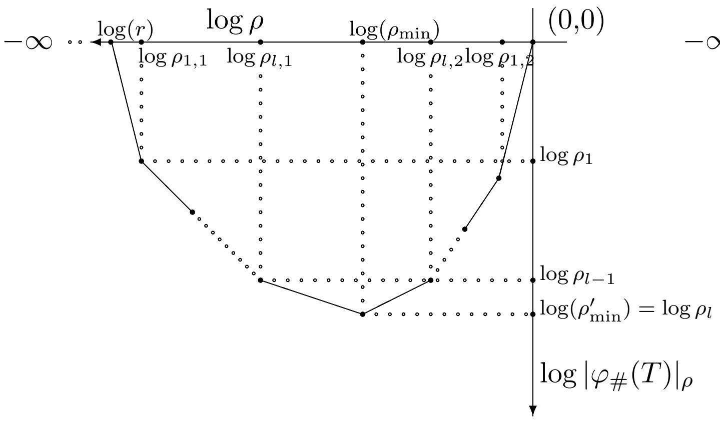


Figure 2.2: Valuation polygon for the function $\varphi_\#(T)$

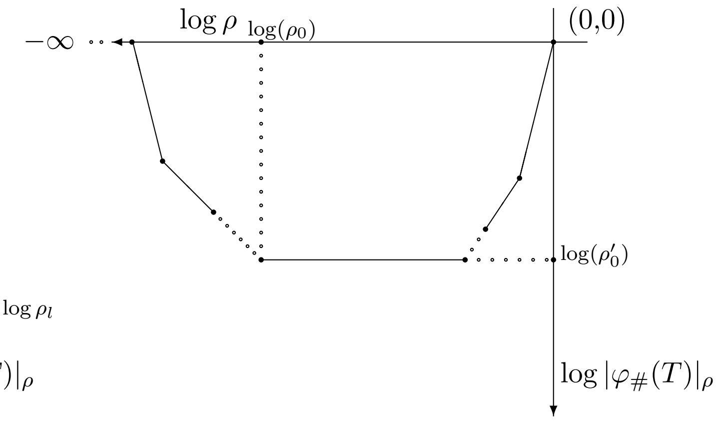


Figure 2.3: Valuation polygon for the function $\varphi_\#(T) - a$, $|a| = \rho'_0$

Let $(\rho_j, \rho_{j+1}) \subseteq [r, 1]$ be a segment such that valuation polygon of the function $\varphi_\#(T)$ over $(\log\rho_j, \log\rho_{j+1})$ consists of a segment with a single slope, say d. Then $\varphi_\#(T)$ restricted to the open annulus $A(0; \rho_j, \rho_{j+1})$ can be written as $\varphi_\#(T) = a_d T^d(1 + h(T))$, where $h(T) = \sum_{n\in\mathbb{Z}} b_n T^n$ is an analytic function on $A(0; \rho_j, \rho_{j+1})$ with $|h(T)|_\rho < 1$ for all $\rho \in (\rho_j, \rho_{j+1})$. It follows as in the proof of Lemma 2.1.17 that $\varphi(\eta_{0,\rho}) = \eta_{0,|a_d|\rho^d}$ and more precisely by continuity we have $\varphi(\Gamma^A) = \Gamma^D_{\eta_{0,\rho'_{\min}}}$, where $\eta_{0,\rho'_{\min}} = \varphi(\eta_{0,\rho_{\min}})$.

Let $\rho_0 \in (r, 1) \cap |k^\times|$. Depending whether the valuation polygon of the function $\varphi_\#(T)$

has a break in the point $\log(\rho_0)$ we have the following two lemmas.

Lemma 2.1.23. *Suppose that the valuation polygon of $\varphi_\#(T)$ doesn't have a break point in $\log(\rho_0)$. Then, φ restricts to a finite morphism $\varphi : A[0;\rho_0,\rho_0] \to A[0;\rho_0',\rho_0']$, where $\eta_{0,\rho_0'} = \varphi(\eta_{0,\rho})$.*

Proof. Because of Theorem 2.1.10 it is enough to prove that $A[0;\rho_0,\rho_0]$ is a connected component of the set $\varphi^{-1}(A[0;\rho_0',\rho_0'])$. First we note that $\varphi(A[0;\rho_0,\rho_0](k)) = A[0;\rho_0',\rho_0'](k)$: this follows from the valuation polygon of the function $\varphi_\#(T) - a$, for $a \in A[0;\rho_0',\rho_0'](k)$ (see Figure 3) and the fact that the valuation polygon of $\varphi_\#(T)$ doesn't have a break in $\log(\rho_0)$. Moreover, for $\varepsilon \in \mathbb{R}_{>0}$ and ε sufficiently small, $\varphi^{-1}(A[0;\rho_0',\rho_0'](k)) \cap A(0;\rho_0 - \varepsilon,\rho_0+\varepsilon) = A[0;\rho_0,\rho_0](k)$ and since rational points are everywhere dense, we conclude that $\varphi^{-1}(A[0;\rho_0',\rho_0']) \cap A(0;\rho_0-\varepsilon,\rho_0+\varepsilon) = A[0;\rho_0,\rho_0]$, hence the conclusion. $\square$

Lemma 2.1.24. *Suppose that the valuation polygon of $\varphi_\#(T)$ has a break point in $\log(\rho_0)$ (we allow ρ_0 to be 1 or r as well). Then*

(i) There exist finitely many open discs $D_{\rho_0,i}$, $i = 1,\dots,\alpha(\rho_0)$ attached to the point η_{0,ρ_0} such that φ restricts to a finite morphism $\varphi : A[0;\rho_0,\rho_0] \setminus \cup_{i=1}^{\alpha(\rho_0)} D_{\rho_0,i} \to A[0;\rho_0',\rho_0']$ where as before $\varphi(\eta_{0,\rho_0}) = \eta_{0,\rho_0'}$;

(ii) For each $i = 1,\dots,\alpha(\rho_0)$, φ restricts to a finite morphism of open discs $\varphi : D_{\rho_0,i} \to D(0,\rho_0'^-)$.

Proof. Let $y_1,\dots,y_s$ be all the zeros of $\varphi_\#(T)$ in $A[0;\rho_0,\rho_0](k)$, and let $D_{\rho_0,1},\dots,D_{\rho_0,\alpha(\rho_0)}$ be all the maximal open discs in $A[0;\rho_0,\rho_0]$ containing at least one of the points $y_1,\dots,y_s$. Since for each $i = 1,\dots,\alpha(\rho_0)$, there is a point $y_j \in D_{\rho_0,i}$, we can choose a coordinate $T_i : D_{\rho_0,i} \xrightarrow{\sim} D(0,\rho_0^-)$, $T_i(y_j) = 0$ and because the valuation polygon for the function $\varphi_\#(T_i)$ is convex, strictly increasing function it follows that $D_{\rho_0,i} \cap \varphi^{-1}(\eta_{0,\rho_0'}) = \emptyset$. Hence, φ restricts to a finite morphism $\varphi : D_{\rho_0,i} \to D(0,\rho_0'^-)$ (Corollary 2.1.14) and this proves (ii).

To prove the claim (ii), let us note that $\varphi^{-1}(D(0,\rho_0'^-)) \cap A[0;\rho_0,\rho_0] = \cup_{i=1}^{\alpha(\rho_0)} D_{\rho_0,i}$. Indeed, any maximal open disc in $A[0;\rho_0,\rho_0]$ which has a nonempty intersection with $\varphi^{-1}(D(0,\rho_0'^-))$ necessarily contains a point whose immage by φ is $0 = \eta_{0,0} \in D(0,\rho_0'^-)$, as follows from the second paragraph of the proof of Lemma 2.1.17, hence contains one of the points y_i. This implies that one of the connected components of $\varphi^{-1}(A[0;\rho_0',\rho_0'])$ is $A[0;\rho_0,\rho_0] \setminus \cup_{i=1}^{\alpha(\rho_0)} D_{\rho_0,i}$, which proves (i). □

We continue the proof of Theorem 2.1.22. Let $\log\rho_0' = 0 > \log\rho_1' > \cdots > \log\rho_l' = \log\rho_{\min}'$ be in $[\log\rho_{\min}', 0]$ such that for all $i = 1,\ldots,l$ there exists $\rho \in [r,1] \cap |k^\times|$ such that $\varphi(\eta_{0,\rho}) = \eta_{0,\rho_i'}$ and the valuation polygon of $\varphi_\#(T)$ has a break in $\log(\rho)$ (see Figure 2). For each $i = 0,\ldots,l-1$ let $\rho_{i,1} < \rho_{i,2}$ be the two points in $[r,1] \cap |k^\times|$ such that $(\log(\rho_{i,j}), \log\rho_i')$, $j = 1,2$, belongs to the valuation polygon of $\varphi_\#(T)$. For each $i = 0,\ldots,l$, let $D_{i,1},\ldots,D_{i,\beta(i)}$ be all the open discs attached to one of the points $\eta_{0,\rho_{i,1}}$ or $\eta_{0,\rho_{i,2}}$ from Lemma 2.1.24.

We construct strictly φ-compatible triangulations $\mathcal{S}$ and $\mathcal{T}$ in the following way. Put $\mathcal{T}_l = \eta_{0,\rho_m'}$ and inductively, for $i = l-1,\ldots,0$, let $\mathcal{T}_i$ and $\mathcal{S}_i$ be the strictly φ-compatible triangulations from Theorem 2.1.20 applied to the family of morphisms $\varphi_{|D_{i,j}} \to D(0,\rho_j'^-)$, $j = 1,\ldots,\beta(i)$ and with set $S = \mathcal{T}_{i+1} \cup \{\eta_{0,\rho_{i+1}}\}$. Then we can take $\mathcal{T} := \mathcal{T}_0 \cup \{\eta_{0,1}\}$ and $\mathcal{S} := \varphi^{-1}(\mathcal{T})$. It follows from Lemmas 2.1.23 2.1.24 and the construction of $\mathcal{T}_i$ that $\mathcal{S}$ is a strict triangulation of A and that $\mathcal{S}$ and $\mathcal{T}$ are strictly φ-compatible.

The remaining case where both A and D are non-strict is done in a quite similar way, again choosing coordinates on A and D and studying properties of the valuation polygon of the respective coordinate representation of our morphism φ. □

Corollary 2.1.25. *Let $\varphi : A \to D$ be a finite morphism between an open annulus and an open disc. Then there exist strictly φ-compatible triangulations of A and D.*

Proof. From Corollary 2.1.16 we may take a sufficiently small open boundary annulus A_D in D such that $\varphi^{-1}(A_D)$ is a disjoint union of two open boundary annuli in A whose complement is a closed annulus A_1 and such that $\varphi_{|A_1} : A_1 \to D \setminus A_D$ is finite. Then we can apply the Theorem 2.1.22 to the morphism $\varphi_{|A_1} : A_1 \to D \setminus A_D$. □

Corollary 2.1.26. *Let $\varphi : A \to D$ be a finite morphism between a closed (resp. open) annulus and a closed (resp. open) disc, let $\mathcal{S}$ be a strictly semistable triangulations of A and $\mathcal{T}$ a strictly semistable triangulation of D such that $\mathcal{S}$ and $\mathcal{T}$ are strictly φ-compatible. Let S be a finite set of type two points in D. Then, there exist strictly φ-compatible triangulations $\mathcal{S}'$ and $\mathcal{T}'$ of A and D, respectively, such that $S \subset \mathcal{T}'$ and $\mathcal{S}'$ refines $\mathcal{S}$ while $\mathcal{T}'$ refines $\mathcal{T}$.*

Proof. The proof now follows from Theorem 2.1.22 and Corollary 2.1.25 combined with Lemma 2.1.21. □

We summarize the results of this section in the following theorem.

Theorem 2.1.27. *Let $\varphi_i : Y_i \to D$, $i = 1, \dots, n$ be a finite family of finite morphisms where each Y_i is a closed (resp. open) disc or annulus and D is a closed (resp. open) disc. Let $S \subset D$ be a finite set of type two points in D. Then, there exist strictly semistable triangulations $\mathcal{S}_i$ of Y_i and $\mathcal{T}$ of D such that $S \subset \mathcal{T}$ and $\mathcal{S}_i$ and $\mathcal{T}$ are strictly φ_i-compatible. Moreover, for any finite subset S of type two points in D, we can choose $\mathcal{T}$ such that $S \in \mathcal{T}$.*

Proof. We only consider the case where all the curves involved are closed, the case when they are open being done in a similar fashion with help of Corollary 2.1.16. For each $i = 1, \dots, n$ let $\mathcal{S}_i'$ and $\mathcal{T}_i'$ be φ_i-compatible triangulations of Y_i and D, respectively, given by Lemma 2.1.17 and Theorem 2.1.22. Let us put $\Gamma = \cup_i \Gamma^D_{\mathcal{T}_i'}$, and let $S_i := \mathcal{T}_i' \setminus \cup_{j \neq i} \mathcal{T}_j'$. Then we can apply Lemma 2.1.21 to find refinements $\mathcal{S}_i''$ and $\mathcal{T}_i''$ of $\mathcal{S}_i'$ and $\mathcal{T}_i'$ with $S_i \subset \mathcal{T}_i''$. Note that we have that $\Gamma_{\mathcal{T}_i''} = \Gamma_{\mathcal{T}_j''}$ for each $i, j = 1, \dots, n$ (for the first time we use here the second part of Lemma 2.1.21). Let $\mathcal{T} := \cup_i \mathcal{T}_i''$ and put $\mathcal{S}_i := \varphi_i^{-1}(\mathcal{T})$. Then $\mathcal{S}_i$, $i = 1, \dots, n$ and $\mathcal{T}$ satisfy the claim of the theorem. □

2.1.3 Compatible triangulations of k-analytic curves

After studying compatible triangulations for a finite family of finite morphisms from closed (resp. open) annuli and discs to closed (resp. open) discs, we are ready to move to a more general situation, that is to study compatible triangulations for finite morphisms of more complicated curves. We do so by starting the

Proof of the Theorem 2.1.4. Let $\mathcal{S}_1$ and $\mathcal{T}_1$ be any (nonempty) strictly semistable triangulations of Y and X, respectively, and let $\mathcal{T}_2$ be a strictly semistable triangulation of X containing $\varphi(\mathcal{S}_1) \cup \mathcal{T}_1$. The connected components of $Y \setminus \varphi^{-1}(\mathcal{T}_2)$ consists of open discs and *finitely many* wide open curves isomorphic to wide open curves in the projective line and each of the connected components is mapped to a wide open annulus or wide open disc in $X \setminus \mathcal{T}_2$. So we are led to study compatible triangulations for the finite family of finite morphisms $\varphi_i : W_i \to V$, where W_i's are some wide open curves (we can even assume that they are isomorphic to wide open curves in a projective line) and where V is an open disc or an open annulus. In lemmas 2.1.28 and 2.1.30 we study these two situations, in the former V is an open disc, while in the later V is an open annulus.

Lemma 2.1.28. *Let $\varphi : W \to D$ be a finite morphism between wide open curves and suppose that D is an open disc. Then*

(i) There exist strictly φ-compatible triangulations of W and D.

(ii) For a given finite subset of type two points S of D, we can find strictly φ-compatible triangulations $\mathcal{S}$ and $\mathcal{T}$ refining any given triangulations $\mathcal{S}_1$ and $\mathcal{T}_1$ of W and D, respectively, such that $S \subset \mathcal{T}$.

(iii) Let $\varphi_i : W_i \to D$, $i = 1, \ldots, n$ be finitely many finite morphisms of wide open curves where D is an open disc. Then there exist strictly semistable triangulations $\mathcal{S}_i$ of W_i, $i = 1, \ldots, n$ and $\mathcal{T}$ of D such that $\mathcal{S}_i$ and $\mathcal{T}$ are strictly φ_i-compatible.

Remark 2.1.29. This lemma is a generalization of lemmas 2.1.26 and 2.1.17.

Proof. (i) We find it easier to work with skeleta in this contexts rather than with triangulations, so in the proof we will have in mind Remark 2.1.9. Let $\mathcal{S}_1$ be a strictly semistable triangulation of W and let $\Gamma^W_{\mathcal{S}_1}$ be the corresponding skeleton. Then

Claim 1. The image $\varphi(\Gamma^W_{\mathcal{S}_1})$ is a finite skeleton with type two endpoints in D.

Proof of the Claim 1. To find the image $\varphi(\Gamma^W_{\mathcal{S}_1})$ it is enough to find the images $\varphi(\Gamma^A)$, where A is an open annulus in $W \setminus \mathcal{S}_1$ and add to it the images of the points in $\mathcal{S}_1$. Let A be an connected component/open annulus in $W \setminus \mathcal{S}_1$ *i.e.* in $\mathcal{A}_{\mathcal{S}_1}$. Then, by using valuation polygon of the function φ restricted to A (after a suitable choice of coordinates on A and D so that the mentioned valuation polygon doesn't have flat parts), similarly like in the proof of Lemma 2.1.17 or Lemma 2.1.22, it follows that $\varphi(\Gamma^A)$ is a skeleton of an annulus in disc D. The endpoints of $\varphi(\Gamma^A)$ are necessarily type two points as they are the images of the endpoints of Γ^A or the images of a point $\eta_{0,\rho}$ such that the valuation polygon of $\varphi_{|A}$ has a break in $\log\rho$ (compare for example with the valuation polygon in Figure 2). As there are only finitely many elements in $\mathcal{A}_{\mathcal{S}_1}$, then the image $\varphi(\Gamma^W_{\mathcal{S}_1})$ is connected subset consisting of finitely many skeleta of open annuli in D and has finitely many endpoints. It remains to prove that $\varphi(\Gamma^W_{\mathcal{S}_1})$ is not contained in any open disc D' which is strictly smaller than D (if this were the case, then $D \setminus \varphi(\Gamma^W_{\mathcal{S}_1})$ would have an open annulus "at the boundary" as a connected component, hence $\varphi(\Gamma^W_{\mathcal{S}_1})$ would not be a skeleton in D). But, if $\varphi(\Gamma^W_{\mathcal{S}_1})$ is contained in such a D', then it would be contained in a proper compact subset of D, and this would imply that $\Gamma^W_{\mathcal{S}_1}$ is contained in a compact subset of W (recall that φ is a proper map), which is a contradiction. □

Claim 2. $\varphi^{-1}(\varphi(\Gamma^W_{\mathcal{S}_1}))$ is a finite skeleton with type two endpoints in W.

Proof of the Claim 2. Similarly to what we did in the previous claim, let A be an open annulus in $\mathcal{A}_{\mathcal{S}_1}$ equipped with a coordinate $T : A \xrightarrow{\sim} A(0;r,1)$. With a right choice of coordinate $S : D \xrightarrow{\sim} D(0,1)$ we can assume that the valuation polygon of the function $S = \varphi_{\#}(T)$ doesn't contain segments with zero slope. Again, from the valuation polygon for the function $\varphi_{\#}(T)$ we conclude, like in Lemmas 2.1.24 and 2.1.23, that there are only finitely many discs in the annulus A whose endpoints belong to Γ^A and whose images have a nonempty intersection with $\varphi(\Gamma^A)$, these will be precisely the maximal open discs in A that contain a zero of the function $\varphi_{\#}(T)$. Also, there are only finitely many open discs with endpoint in Γ^A whose image has a nonempty intersection with $\varphi(\Gamma^W_{\mathcal{S}_1})$. To prove it, let D' be one such disc. Then $\varphi(D')$ is an open disc attached to $\varphi(\Gamma^A)$ (Corollary 2.1.14) and because of the assumption on D' there is a part of $\varphi(\Gamma^W_{\mathcal{S}_1})$ contained in $\varphi(D')$. We can be more precise, either $\varphi(D')$ has a nonempty intersection with $\varphi(\Gamma^A)$ or $\varphi(D')$ is disjoint from $\varphi(\Gamma^A)$ and the endpoint of $\varphi(D')$ belongs to $\varphi(\Gamma^W_{\mathcal{S}_1} \setminus \Gamma^A)$. The first case has already been discussed, so let us assume that we are in the second one. Let the corresponding endpoint be η'. To conclude, there are only finitely many branches of the skeleton $\varphi(\Gamma^W_{S_1} \setminus \Gamma^A)$ emanating from η', hence only finitely many open discs containing these branches and attached to η', and therefore only finitely many open discs attached to Γ^A which are mapped to these disc, because of the finiteness of the morphism. Let us prove that there are only finitely many points η' as above. First of all, point η' belongs to $\varphi(\Gamma^A)$, and as we assumed that $\varphi(\Gamma^W_{\mathcal{S}_1})$ has a nonempty intersection with D', η' must be a branching point. We proved in Claim 1. that $\varphi(\Gamma^W_{\mathcal{S}_1})$ is a finite skeleton in D having type two endpoints, so it follows that there are only finitely many branching points.

The same conclusion goes for the open discs attached to points of $\mathcal{S}_1$, namely, there are only finitely many of them whose image by φ intersects the skeleton $\varphi(\Gamma^W_{\mathcal{S}_1})$. Let us consider a fixed point $s \in \mathcal{S}_1$ and let us show that there are only finitely many open discs with endpoint s whose image has a nonempty intersection with $\varphi(\Gamma^W_{S_1})$. Similar as before, $\varphi(s)$ is a type two point of $\varphi(\Gamma^W_{S_1})$ and there are only finitely many branches of $\varphi(\Gamma^W_{S_1})$ emanating from $\varphi(s)$. Hence only finitely many open discs attached to $\varphi(s)$ which potentially contain one of the branches, and therefore finitely many open discs with endpoint in s whose images have a nonempty intersection with $\varphi(\Gamma^W_{S_1})$.

What we showed is that the there are only finitely many maximal open discs in $W \setminus \Gamma^W_{S_1}$

attached to $\Gamma^W_{\mathcal{S}_1}$ whose images have a nonempty intersection with $\varphi(\Gamma^W_{S_1})$. Let $D_1, \dots, D_s$ be these open discs and let us fix $i \in \{1, \dots, s\}$. The intersection $\varphi(D_i) \cap \varphi(\Gamma^W_{S_1})$ is a finite skeleton with type two endpoints in $\varphi(D_i)$ connected to the endpoint of $\varphi(D_i)$. It follows from the proof of Lemma 2.1.17 (see also Lemma 2.1.21) that the inverse image of $\varphi(D_i) \cap \varphi(\Gamma_{S_1})$ is a finite skeleton with type two endpoints in D_i connected to the endpoint of D_i. The claim follows. □

We end the proof of Lemma 2.1.28(i) by recalling the Remark 2.1.9 together with Claims 1 and 2.

The part (ii) of lemma is a consequence of (i) and Lemma 2.1.21. while (iii) is proved arguing like in Theorem 2.1.27 and having in mind arguments in Remark 2.1.21 and claims (i) and (ii). □

The following is an equivalent statement for finite morphisms of wide open curves to an open annulus.

Lemma 2.1.30. *Let $\varphi : W \to V$ be a finite morphisms of wide open curves and assume that V is an open annulus. Then*

(i) There exist strictly φ-compatible triangulations of W and V.

(ii) For a given finite subset S of type two points in V, we can find strictly φ-compatible triangulations $\mathcal{S}$ and $\mathcal{T}$ refining any given triangulations $\mathcal{S}_1$ and $\mathcal{T}_1$ of W and V, respectively, such that $S \subset \mathcal{T}$.

(iii) Let $\varphi_i : W_i \to V$, $i = 1, \dots, n$ be finitely many finite morphisms of wide open curves where V is an open annulus. Then, there exist strictly semistable triangulations $\mathcal{S}_i$ of W_i, $i = 1, \dots, n$ and $\mathcal{T}$ of V such that $\mathcal{S}_i$ and $\mathcal{T}$ are strictly φ_i-compatible.

Proof. (i) The idea of the proof is to reduce the problem to the case of Lemma 2.1.28, so for this let $\{w_1, \dots, w_\delta\}$ be a strictly semistable triangulation of W (if W is an open annulus or an open disc we may take empty triangulations, so we may assume that W is not an open annulus or an open disc, so that the later set is nonempty). For each $j = 1, \ldots, \delta$, let C_j be an affinoid domain with good canonical reduction in W with maximal point w_j and containing all open discs in W attached to w_j and not containing any of the points w_l, $l \neq j$. Since φ, seen as a function on W, has no zeros or poles on W, the absolute value (we fix once and for all a coordinate on V: $T : V \xrightarrow{\sim} A(0; r, 1)$) of the function φ over C_j is constant, and we denote it by d_j (*cf.* [31, Lemma 2.3]). We also note that the absolute value of φ can vary only over the open annuli connecting C_j.

Let $c_1, \ldots, c_\alpha$ be numbers d_j put in an increasing order and without repetition. We chose a triangulation on $V = A(0; r, 1)$ given by the points η_{0,c_j}, $j = 1, \ldots, \alpha$. Then we have the following finite morphisms induced by restriction of φ:

$$\varphi_{|\Delta^j} : \Delta^j := W \times_V A(0; c_{j-1}, c_j) \to A(0; c_{j-1}, c_j) \quad j = 2, \ldots, \alpha$$

and

$$\varphi^j := \varphi_{|\Theta^j} : \Theta^j = W \times_V A[0; c_j, c_j] \to A[0; c_j, c_j], \quad j = 1, \ldots, \alpha.$$

We note here that Δ^j is a disjoint union of open annuli, while Θ^j is an affinoid domain in W. At this point we would like to find compatible triangulations for a family of morphisms $\varphi^j_l := \varphi^j_{|\Theta^j_l} : \Theta^j_l \to A[0; c_j, c_j]$, $l = 1, \ldots, \beta$, and where Θ^j_l, $l = 1, \ldots, \beta$ are connected components of the affinoid Θ^j. Let $\mathrm{Sh}(\Theta^j_l)$ be the Shilov boundary of Θ^j_l. Then $(\varphi^j_l)^{-1}(\eta_{0,c_j}) = \mathrm{Sh}(\Theta^j_l)$. For each $\xi \in \mathrm{Sh}(\Theta^j_l)$, let C_ξ be the maximal affinoid in Θ^j_l with good canonical reduction and with the maximal point ξ. Then φ maps connected components of $\Theta^j_l \setminus \cup_{\xi \in \mathrm{Sh}(\Theta^j_l)} C_\xi$ (these are all wide open curves and more precisely the connected components of $\Theta^j_l \setminus \mathrm{Sh}(\Theta^j_l)$ which are *not* open discs) to some open discs in $A[0; c_j, c_j]$ attached to η_{0,c_j}. More precisely, if U is a connected component of $\Theta^j_l \setminus \cup_{\xi \in \mathrm{Sh}(\Theta^j_l)} C_\xi$, there is a maximal open disc D' in $A[0; c_j, c_j]$ containing $\varphi^j_l(U)$, since U is connected and since $(\varphi^j_l)^{-1}(\eta_{0,c_j}) \cap U = \emptyset$. But then U is a connected component of the set $(\varphi^j_l)^{-1}(D')$, from which we conclude with help of Theorem 2.1.10 that φ restricts to a finite morphism

$\varphi_U : U \to \varphi(U)$, where $\varphi(U)$ is an open disc in $A[0; c_j, c_j]$ with the endpoint η_{0,c_j}. Now let $\mathcal{D}$ be the set of all the maximal open discs D in $A[0; c_j, c_j]$ such that one of the connected components of the set $(\varphi^j)^{-1}(D)$ is a wide open component in $\Theta_l^j \setminus \cup_{\xi \in \mathrm{Sh}(\Theta_l^j)} C_\xi$ for some $l = 1, \dots, \beta$ and not equal to an open disc (note that at this point we are considering φ^j instead of φ_l^j since we want to construct a simultaneous compatible strictly semistable triangulation of D and of connected components in $(\varphi^j)^{-1}(D)$, as follows; also note that we ask that at least one component in $(\varphi^j)^{-1}(D)$ is not equal to an open disc to ensure that $\mathcal{D}$ is finite). Then for each $D \in \mathcal{D}$, and a family of finite morphisms $\varphi_{|U} : U \to D$, U a connected component in $(\varphi^j)^{-1}(D)$, we can find strictly $\varphi_{|U}$-compatible triangulations $\mathcal{S}_U$ and $\mathcal{T}_D$ (Lemma 2.1.28 (iii)). Then we construct strictly φ_l^j-compatible triangulations $\mathcal{S}_l^j$ and $\mathcal{T}_l^j$ of Θ_l^j and $A[0; c_j, c_j]$, respectively, by taking $\mathcal{S}_l^j = \mathrm{Sh}(\Theta_l^j) \cup \cup_U \mathcal{S}_U$, where U goes through the connected components of $(\varphi_l^j)^{-1}(D)$ and D goes through $\mathcal{D}$, while $\mathcal{T}_l^j = \{\eta_{0,c_j}\} \cup_D \mathcal{T}_D$, where D goes through $\mathcal{D}$. Moreover, from the previous constructions it follows that we can find compatible triangulations $\mathcal{S}^j$ and $\mathcal{T}^j$ of Θ^j and $A[0; c_j, c_j]$, respectively, by simply taking $\mathcal{S}^j = \cup_l \mathcal{S}_l^j$, $l = 1, \dots, \beta$ and $\mathcal{T}^j = \cup_l \mathcal{T}_l^j$, $l = 1, \dots, \beta$. Finally, since for $j_1 \neq j_2$, $\varphi(\Theta^{j_1})$ is disjoint from $\varphi(\Theta^{j_2})$, we conclude that $\mathcal{S}_{W_i} = \cup_j \mathcal{S}_{i,j}$ and $\mathcal{T}_V = \cup_j \mathcal{T}_{i,j}$ are strictly φ_i-compatible triangulations of W_i and V, respectively. The part (ii) is done with help of Lemma 2.1.21 while part (iii) is proved like Theorem 2.1.27 and with help of Lemma 2.1.21, and claims (i) and (ii). □

Continuation of the Proof of Theorem 2.1.4. We continue using the notation from the beginning of the proof. Let $\mathcal{W}'$ be a finite set of connected components in $Y \setminus \varphi^{-1}(\mathcal{T}_2)$ which are not isomorphic to open discs (if there is no such a component, then $\varphi^{-1}(\mathcal{T}_2)$ is a strictly semistable triangulation of Y and $\mathcal{T}_2$ and $\varphi^{-1}(\mathcal{T}_2)$ are strictly φ-compatible). From Theorem 2.1.10 we have for each $W \in \mathcal{W}'$, φ restricts to a finite morphisms $\varphi_{|W} : W \to \varphi(W)$ and $\varphi(W)$ is a connected component in $X \setminus \mathcal{T}_2$ *i.e.* an open disc or an open annulus. Let $\mathcal{V} := \{\varphi(W), W \in \mathcal{W}'\}$ and let $\mathcal{W}$ be the set of connected components of $\varphi^{-1}(\varphi(W))$, while $W \in \mathcal{W}'$. Let $V \in \mathcal{V}$ (V is necessarily an open disc or an open annulus) and let $\mathcal{W}_V$

be the finite set of connected components of $\varphi^{-1}(V)$. Then for each $V \in \mathcal{V}$ we have finitely many finite morphisms $\varphi_{|W} : W \to V$, $W \in \mathcal{W}_V$ (Theorem 2.1.10) and since we are in one of the situations discussed in Lemmas 2.1.28 or 2.1.30 we have strictly $\varphi_{|W}$-compatible strictly semistable triangulations $\mathcal{S}_W$ of W, for $W \in \mathcal{W}_V$ and $\mathcal{T}_V$ of V. To finish the proof we can take $\mathcal{T} := \mathcal{T}_2 \cup \cup_{V\in\mathcal{V}}\mathcal{T}_V$ and for $\mathcal{S} := \varphi^{-1}(\mathcal{T}) = \mathcal{S}_2 \cup_{V\in\mathcal{V}} \cup_{W\in\mathcal{W}_V}\mathcal{S}_W$. □

Proof of the Theorem 2.1.6. From Corollary 2.1.16 we can find an affinoid domain W_0 in W and an affinoid domain V_0 in V such that the restriction of φ to W_0, $\varphi_{|W_0} : W_0 \to V_0$ is a finite morphism and such that each connected component A of $W\backslash W_0$ (resp. A' of $V\backslash V_0$) is an open annulus in W (resp. V) and while A' goes through the set of connected components of $V \setminus V_0$, connected components of $\varphi^{-1}(A')$ go through the connected components of $W \setminus W_0$. It follows that for the strictly φ-compatible strictly semistable triangulations $\mathcal{S}$ and $\mathcal{T}$ of W and V, respectively, we can take the triangulations from the Theorem 2.1.4 applied to the finite morphism of affinoid domains $\varphi_{|W_0} : W_0 \to V_0$. □

Remark 2.1.31. For the practical purposes we emphasize here once again that we can find compatible triangulations in Theorem 2.1.4 (resp. 2.1.6) containing any given finite set of type two points in Y and X (resp. W and V).

2.1.4 Compatible partitions of curves

Given a finite morphism $\varphi : Y \to X$ like in 2.0.15 or where Y and X are wide open curves, we use Theorem 2.1.4 to partition (understood here as division into finitely many analytic subdomains) our curves into pieces on which φ induces finite morphisms and which often have a simpler structure compared to those of Y and X.

Corollary 2.1.32. *Let $\varphi : Y \to X$ be a finite morphism between compact, connected, quasi-smooth k-analytic curves or a finite morphism between wide open curves. Then, there exist partitions of Y and X, $\mathcal{P}_Y = \{\mathcal{A}_Y, \mathcal{B}_Y, \mathcal{C}_Y\}$ and $\mathcal{P}_X = \{\mathcal{A}_X, \mathcal{B}_X, \mathcal{C}_X\}$, respectively, where $\mathcal{A}_Y$ (resp. $\mathcal{A}_X$) is a finite set of disjoint open annuli, $\mathcal{B}_Y$ (resp. $\mathcal{B}_X$) is a finite* *set of disjoint open discs and $\mathcal{C}_Y$ (resp. $\mathcal{C}_X$) is a finite set of disjoint affinoids with good canonical reduction such that*

(i) For all $A \in \mathcal{A}_Y$, $B \in \mathcal{B}_Y$ and $C \in \mathcal{C}_Y$ (resp. For all $A \in \mathcal{A}_X$, and all $B \in \mathcal{B}_X$ and $C \in \mathcal{C}_X$), $A \cap B = \emptyset$, $A \cap C = \emptyset$ and $B \cap C = \emptyset$,

(ii) $\mathcal{A}_X = \{\varphi(A), A \in \mathcal{A}_Y\}$, $\mathcal{B}_X = \{\varphi(B), B \in \mathcal{B}_Y\}$ and $\mathcal{C}_X = \{\varphi(C), C \in \mathcal{C}_Y\}$ and for all $A \in \mathcal{A}_X$ (resp. $B \in \mathcal{B}_X$, resp. $C \in \mathcal{C}_X$) $\varphi^{-1}(A)$ is a disjoint union of elements in $\mathcal{A}_Y$ (resp. $\mathcal{B}_Y$, resp. $\mathcal{C}_Y$),

(iii) For each $C \in \mathcal{C}_Y$, φ restricts to a finite, étale morphism $\varphi : C \to \varphi(C)$ of affinoid spaces,

(iv) For each $A \in \mathcal{A}_Y$, φ restricts to a finite, étale morphism $\varphi : A \to \varphi(A)$ of open annuli,

(v) For each $B \in \mathcal{B}_Y$, φ restricts to a finite morphism $\varphi : B \to \varphi(B)$ of open discs.

Proof. Let $\{y_1, \dots, y_s\} \subset Y(k)$ be the saturation of the set of rational points of Y which are ramified, and let $\mathcal{S}$ and $\mathcal{T}$ be any strictly φ-compatible semistable triangulations of Y and X, respectively such that no two different points in $\{y_1, \dots, y_s\}$ belong to the same open disc attached to $\Gamma_{\mathcal{S}}^Y$ (note that we can always achieve this by choosing sufficiently small disjoint closed discs containing points $\{y_1, \dots, y_s\}$ and adding their maximal points to $\mathcal{S}$ as in the Remark 2.1.31).

Let $\mathcal{B}_Y$ be the finite family of open discs in Y attached to the points in $\mathcal{S}$ and having nonempty intersection with $\{y_1, \dots, y_s\}$. Then, the set $\mathcal{B}_X := \{\varphi(B), B \in \mathcal{B}_Y\}$ is a finite family of open discs attached to points in $\mathcal{T}$ and containing the rational branching locus of φ in X (see Remark 2.1.15). Moreover, for each $B \in \mathcal{B}_Y$, φ restricts to a finite morphism $\varphi : B \to \varphi(B)$ and by the choice of the set $\{y_1, \dots, y_s\}$, we have $\varphi^{-1}(B)$, for $B \in \mathcal{B}_X$ is a disjoint union of elements in $\mathcal{B}_Y$. This proves (v).

Let us put $\mathcal{A}_Y := \mathcal{A}_{\mathcal{S}}$ and $\mathcal{A}_X := \mathcal{A}_{\mathcal{T}}$. Since $\mathcal{S}$ and $\mathcal{T}$ are strictly φ-compatible, for each $A \in \mathcal{A}_Y$, φ restricts to a finite morphism $\varphi : A \to \varphi(A)$, where $\varphi(A) \in \mathcal{A}_X$, and for each $A \in \mathcal{A}_X$, $\varphi^{-1}(A)$ is a finite union of elements in $\mathcal{A}_Y$. As there are no ramified rational points belonging to annuli in $\mathcal{A}_Y$ the (iv) follows.

Finally, for each point $y \in \mathcal{S}$ let C_y be the affinoid domain in Y with maximal point y and containing all the open discs attached to y and having an empty intersection with $\mathcal{S}$ and with discs in $\mathcal{B}_Y$ and let $\mathcal{C}_Y$ be the set of all such C_y. Then for each $C \in \mathcal{C}_Y$, φ restricts to a finite étale morphism $\varphi : C \to \varphi(C)$, where C is an affinoid domain in X with good canonical reduction and with the maximal point $\varphi(y) \in \mathcal{T}$ and having an empty intersection with the set $\mathcal{T} \setminus \{\varphi(y)\}$ and with the open discs in $\mathcal{B}_X$. We put $\mathcal{C}_X := \{\varphi(C), C \in \mathcal{C}_Y\}$. Note that for each element $C \in \mathcal{C}_X$, $\varphi^{-1}(C)$ is a disjoint union of elements in $\mathcal{C}_Y$ and moreover the elements of the sets $\mathcal{C}_Y$, $\mathcal{A}_Y$, $\mathcal{B}_Y$ (resp. $\mathcal{C}_X$, $\mathcal{A}_X$, $\mathcal{B}_X$) cover Y (resp. X), which together with the previous finishes the proof of the corollary. $\square$

Remark 2.1.33. From the proof of Corollary 2.1.32, we see that the partitions $\mathcal{P}_Y$ and $\mathcal{P}_X$ depend on the chosen triangulations $\mathcal{S}$ and $\mathcal{T}$, respectively. If we want to emphasize this fact, we will write in the subscript $\mathcal{P}_{\mathcal{S}}$ and $\mathcal{P}_{\mathcal{T}}$. If the triangulations $\mathcal{S}$ and $\mathcal{T}$ come from semistable models $\mathfrak{Y}$ and $\mathfrak{X}$, respectively, we will write $\mathcal{P}_{\mathfrak{Y}}$ and $\mathcal{P}_{\mathfrak{X}}$ for the corresponding compatible partitions. Similarly, we will write $\mathcal{A}_{\mathcal{S}}$, $\mathcal{A}_{\mathcal{T}}$ ($\mathcal{A}_{\mathfrak{Y}}$, $\mathcal{A}_{\mathfrak{X}}$) and so on.

Proof of Theorem 2.1.3. Let us suppose that the triangulations $\mathcal{T}_{\mathfrak{Y}}$ and $\mathcal{T}_{\mathfrak{X}}$, which are nonempty and have at least two points, are strictly φ-compatible, and let $\mathcal{W}_{\mathcal{T}_{\mathfrak{Y}}}$ and $\mathcal{W}_{\mathcal{T}_{\mathfrak{X}}}$ be as in the Definition 1.1.5. From the proof of the previous Corollary it follows that the triangulations $\mathcal{T}_{\mathfrak{Y}}$ and $\mathcal{T}_{\mathfrak{X}}$ give us compatible partitions of Y and X, respectively. Moreover, for each $W \in \mathcal{W}_{\mathcal{T}_{\mathfrak{Y}}}$, φ induces a finite morphism $\varphi : W \to \varphi(W)$, where $\varphi(W) \in \mathcal{W}_{\mathcal{T}_{\mathfrak{X}}}$ and for each element $W \in \mathcal{W}_{\mathcal{T}_{\mathfrak{X}}}$, $\varphi^{-1}(W)$ is a disjoint union of elements in $\mathcal{W}_{\mathcal{T}_{\mathfrak{Y}}}$ (more precisely, an element $W \in \mathcal{W}_{\mathcal{T}_{\mathfrak{X}}}$ consists of the affinoid in $\mathcal{C}$). But then morphisms $\varphi : W \to \varphi(W)$ induce morphisms $\varphi : \mathcal{O}(\varphi(W)) \to \mathcal{O}(W)$ which again induce morphism $\varphi : \mathcal{O}^{\circ}(\varphi(W)) \to \mathcal{O}^{\circ}(W)$ of elements of spectral norm smaller or equal than 1, which we can glue along the induced morphisms (for $W_1, W_2 \in \mathcal{W}_{\mathcal{T}_{\mathfrak{Y}}}$, $W_1 \neq W_2$) $\varphi : \mathcal{O}^{\circ}(W_1 \cap W_2) \to \varphi(\mathcal{O}^{\circ}(W_1 \cap W_2))$ which are all open annuli in $\mathcal{A}_{\mathcal{T}_{\mathfrak{Y}}}$ and $\mathcal{A}_{\mathcal{T}_{\mathfrak{X}}}$. This precisely means that we have a finite morphism of formal models $\Phi : \mathfrak{Y} \to \mathfrak{X}$. $\square$

Proof of Theorem 2.1.5. now follows from Theorems 2.1.4 and 2.1.3. Namely, it is enough to find compatible triangulations of Y and X, respectively, which have more than two points (see Remark 1.1.7). $\square$

Finally, we give the following corollary which can be proved with a little effort with the help of Corollary 2.1.32.

Corollary 2.1.34. *Let $\varphi : Y \to X$ be a finite morphism like in 2.0.15 or a finite morphism between wide open curves, and let $\mathcal{S}$ and $\mathcal{T}$ be nonempty strictly φ-compatible semistable triangulations and suppose that they have at least two points each. Let $y \in \Gamma_{\mathcal{S}}$ be a type 2 point, let $x = \varphi(y)$ and let $C_{\mathcal{S},y}$ (resp. $C_{\mathcal{T},x}$) be an affinoid domain in Y (resp. X) with maximal point y (resp. x) and containing all the open discs in Y (resp. X) attached to y (resp. x) and having an empty intersection with $\Gamma_{\mathcal{S}}$ (resp. $\Gamma_{\mathcal{T}}$). Then, φ induces a finite morphism $\varphi : C_{\mathcal{S},y} \to C_{\mathcal{T},x}$.*

Proof. We note that $\mathcal{T}' := \mathcal{T} \cup \{x\}$ is a strictly semistable triangulation of X and similarly $\mathcal{S}' := \mathcal{S} \cup \varphi^{-1}(x)$ is a strictly semistable triangulation of Y and $\mathcal{S}'$ and $\mathcal{T}'$ are strictly φ-compatible. Moreover, $\Gamma^X_{\mathcal{T}'} = \Gamma^X_{\mathcal{T}}$ and $\Gamma^Y_{\mathcal{S}'} = \Gamma^Y_{\mathcal{S}}$ (this all follows from the Remark 2.1.8). It follows that affinoid domains $C_{\mathcal{S},y}$ and $C_{\mathcal{T},x}$ are either elements in the sets $\mathcal{C}_{\mathcal{S}'}$ and $\mathcal{C}_{\mathcal{T}'}$, respectively, or $C_{\mathcal{S},y}$ and $C_{\mathcal{T},x}$ are qa disjoint union of an element in $\mathcal{C}_{\mathcal{S}'}$ and $\mathcal{C}_{\mathcal{T}'}$ and some elements in $\mathcal{B}_{\mathcal{S}'}$ and $\mathcal{B}_{\mathcal{T}'}$, respectively. In either case, it follows from Corollary 2.1.32 that $C_{\mathcal{S},y}$ is a connected component of the set $\varphi^{-1}(C_{\mathcal{T},x})$, which ends the proof. $\square$

2.2 Finite étale morphisms of open annuli

Let $\varphi : A_1 \to A_2$ be a finite étale morphism of open annuli of degree d. Let $S : A_2 \xrightarrow{\sim} A(0;\rho^d,1)$ (resp. $T : A_1 \xrightarrow{\sim} A(0;\rho,1)$) be a coordinate on A_1 (resp. A_2). Then φ can be represented as $S = \varphi_{\#}(T) = a_d T^d u(T)$, where $|u(T)-1|_{\rho_0} < 1$ for every $\rho_0 \in (\rho,1)$ and with $|a_d| = 1$. By choosing a different coordinate T, we may achieve that $a_d = 1$ so we will assume, unless otherwise stated that $a_d = 1$. Since φ is étale, derivative of φ is an invertible function on A_1, hence has the following coordinate representation $\frac{dS}{dT} = \varphi'_{\#}(T) = \epsilon T^{\sigma} v(T)$, where again $|v(T) - 1|_{(\rho,1)} < 1$ and $\epsilon \in k^{\circ\circ}$. We put

$$\nu = \sigma - d + 1 \tag{2.2.0.1}$$

If we want to emphasize the dependence on the morphism φ, we will write $\sigma(\varphi)$, $d(\varphi)$, $\nu(\varphi)$, $\epsilon(\varphi)$, and so on. We will see later that sometimes our annulus will correspond to a tangent vector $\vec{t}$ belonging to the tangent space of a type two point in a curve. If this is the case, to emphasize also the dependence on $\vec{t}$, we may write $\sigma(\varphi,\vec{t})$, $d(\varphi,\vec{t})$, $\nu(\varphi,\vec{t})$, $\epsilon(\varphi,\vec{t})$.

We collect some properties of finite étale maps of open annuli in the following lemma.

Lemma 2.2.1. *Let $\varphi : A(0;\rho,1) \to A(0;\rho^{d(\varphi)},1)$ and $\psi : A(0;\rho^{d(\varphi)},1) \to A(0;\rho^{d(\varphi)d(\psi)},1)$ be finite étale morphisms of open annuli of degree d_φ and d_ψ, respectively. Then*

(i) $|\epsilon(\varphi)| \geq |d(\varphi)|$ (here $|\cdot|$ is the norm of the field k).

(ii) Let $S_1 = r^d/S$ and $T_1 = r/T$, where $r \in k$, $|r| = \rho$ be "inverted" coordinates on $A(0;\rho,1)$ and $A(0;\rho^d,1)$, respectively. Then,

$$\sigma_1(\varphi) = -\sigma(\varphi) + 2d(\varphi) - 2,$$
$$\epsilon_1(\varphi) = r^{\nu(\varphi)} \cdot \epsilon(\varphi)$$
$$\nu_1(\varphi) = -\nu(\varphi).$$

(iii) We have $\nu(\psi \circ \varphi) = d(\psi)\nu(\varphi) + \nu(\psi)$ and $\sigma(\psi \circ \varphi) = d(\varphi)\sigma(\psi) + \sigma(\varphi)$.

Proof. (*i*) We can write $\varphi_{\#}(T) = \sum_{l\in\mathbb{Z}} a_l T^l$, so $\frac{d}{dT}\varphi_{\#}(T) = \sum_{l\in\mathbb{Z}} l a_l T^{l-1}$, which implies that $\epsilon(\vec{t}) = a_{\sigma_l+1}(\sigma_l+1)$. Then $|\epsilon(\vec{t})| = |a_{\sigma_l+1}(\sigma_l+1)| \geq |da_d| = |d|$. For (ii), we have

$$S_1 = \frac{r^d}{\varphi(\frac{r}{T_1})} = T_1^d(u(\frac{r}{T_1}))^{-1} = T_1^d u_1(T_1)$$

and

$$\begin{aligned}
\frac{dS_1}{dT_1} &= r^d \cdot \frac{1}{\big(\varphi(\frac{r}{T_1})\big)^2} \cdot \varphi'_{\#}(\frac{r}{T_1}) \cdot \frac{r}{T_1^2} \\
&= r^d \cdot \frac{T_1^{2d}}{r^{2d}} \cdot (u_1(T))^2 \cdot \epsilon\frac{r^\sigma}{T_1^\sigma} \cdot v(\frac{r}{T_1}) \cdot \frac{r}{T_1^2} \\
&= r^\nu \cdot \epsilon \cdot T_1^{-\sigma+2d-2} \cdot v_1(T_1) \\
&= \epsilon_1 \cdot T_1^\sigma \cdot v_1(T_1).
\end{aligned}$$

We just note here that $u_1(T_1) = u(r/T_1)$ and $v_1(T_1) = \big(u_1(T_1)\big)^2 v(r/T_1)$ are units so formulae follow.

(iii) If we introduce coordinates U, S and T on $A(\rho^{d_\varphi d_\psi}, 1)$, $A(\rho^{d_\varphi}, 1)$ and $A(\rho, 1)$, respectively, we may write $U = \psi_\#(S) = S^{d(\psi)}h_1(S)$, and $S = \varphi_\#(T) = T^{d(\varphi)}h_2(T)$, where h_1 and h_2 are units in their respective rings, and $|h_2|_{(\rho,1)} = 1$ and $|h_1|_{(\rho^d,1)} = 1$. Let us write $\frac{dU}{dS} = \epsilon(\psi)S^{\sigma(\psi)}g_1(S)$ and $\frac{dS}{dT} = \epsilon(\varphi)T^{\sigma(\varphi)}g_2(T)$ with usual assumptions on functions g_1 and g_2. Then it is a straightforward computation using the chain rule:

$$\begin{aligned}
\frac{dU}{dT} &= \frac{d}{dT}(\psi_\#(\varphi_\#(T))) = \frac{d\psi_\#}{dS}(\varphi_\#(T))\frac{d\varphi_\#}{dT}(T) \\
&= \epsilon(\psi)(\varphi_\#(T))^{\sigma(\psi)}g_1(\varphi_\#(T))\epsilon(\varphi)T^{\sigma(\varphi)}g_2(T) \\
&= \epsilon(\varphi)\epsilon(\psi)T^{d(\varphi)\sigma(\psi)+\sigma(\varphi)}g_1(\varphi_\#(T))g_2(T).
\end{aligned}$$

which implies $\sigma(\psi\circ\varphi) = d(\varphi)\sigma(\psi) + \sigma(\varphi)$. Then, $\nu(\psi\circ\varphi) = \sigma(\psi\circ\varphi) - d(\varphi)d(\psi) + 1 = d(\varphi)(\sigma(\psi) - d(\psi) + 1) + \sigma(\varphi) - d(\varphi) + 1 = d(\psi)\nu(\varphi) + \nu(\psi)$. □

Remark 2.2.2. A priori, for a given morphism of an open annulus as above, the terms σ and consequently ν depend on the choice of coordinates. But, taking a different choice of coordinates doesn't affect the values because the change of coordinate(s) just represents an isomorphism of the corresponding open annulus. Then, we can use the previous Lemma to conclude because for such an isomorphism, the corresponding terms are $d = 1$, $\sigma = 0$, $\nu = 0$ and $\epsilon = 1$.

Lemma 2.2.3. *Suppose that $\varphi : A_1 \to A_2$ extends to a finite map of the whole open disc $B(0, 1^-)$ to itself with ramified points $x_1, \ldots, x_s$ and ramification indexes $e_{x_1}, \ldots, e_{x_s}$, respectively. Then $\sigma = \sum_{1\leq i\leq s}(e_{x_i} - 1)$.*

Proof. The map φ has a coordinate representation as a power series $S = \varphi_{\#}(T) = \sum_{i\geq 0} a_i T^i$, and by a choice of coordinates we can assume that zero is not a ramified point of φ. The derivative of φ is again a power series and we can factor it as $\frac{dS}{dT} = P(T)g(T)$, where $P(T) = (T - x_1)^{e_{x_1}-1} \cdots (T - x_s)^{e_{x_s}-1}$, while $g(T)$ is invertible on $B(0, 1^-)$ [12, Proposition 2.24]. As $\rho \in (0, 1)$ approaches 1, the theory of valuation polygons tells us that the the logarithmic derivative $\mathrm{dlog}^-|\frac{dS}{dT}|_\rho(1)$ is, on one side equal to the number of zeros in $B(0, 1^-)$ of $\frac{dS}{dT}$ counted with multiplicities, so exactly $\sum_{1\leq i\leq s}(e_{x_i} - 1)$, but on the other side it is equal to σ. □

Corollary 2.2.4. *Suppose that φ is ramified at only one point in $B(0, 1^-)$. Then $\sigma = d - 1$.*

2.2.5. *The norm of the operator $\frac{d}{dT}$.* Let $T : A \xrightarrow{\sim} A(0; \rho, 1)$ be a coordinate on an open annulus A. Then, every function f on A can be seen via T as a function $f(T) = \sum_{i\in\mathbb{Z}} a_i T^i$, where coefficients $a_i \in k$ satisfy the condition: for each $\rho_0 \in (\rho, 1)$, $\lim_{|i|_\infty\to\infty} |a_i|\rho_0^i = 0$, where $|\cdot|_\infty$ is the usual archimedean absolute value.

On the other side, the derivative $\frac{d}{dT} f(T) = \sum_{i\in\mathbb{Z}} a_i i T^{i-1}$ can also be seen as a function on $A(0; \rho, 1)$ because for all $i \in \mathbb{Z}$, $|i| \leq 1$ and therefore for each $\rho_0 \in (\rho, 1)$ we have $\lim_{|i|_\infty\to\infty} |a_i||i|\rho_0^i \to \infty = 0$. In this way we can see $\frac{d}{dT}$ as an operator acting on the space of fuctions on $A(0; \rho, 1)$.

Lemma 2.2.6. *Let $\rho_0 \in (\rho, 1)$ and let $|\frac{d}{dT}|_{\rho_0}$ be the operator norm of the operator $\frac{d}{dT}$ seen as acting on the space of functions on $A(0; \rho, 1)$ equipped with the norm $|\cdot|_\rho$. Then,*

$|\frac{d}{dT}|_{\rho_0} = \rho_0^{-1}$.

Proof. The proof is straightforward: if $f(T)$ is an analytic function on $A(0;\rho,1)$, $f(T) = \sum_{i\in\mathbb{Z}} a_i T^i$, then $\frac{d}{dT}f(T) = \sum_{i\in\mathbb{Z}} a_i i T^{i-1}$. Furthermore,

$$|T\frac{d}{dT}f(T)|_{\rho_0} = \max_{i\in\mathbb{Z}} |a_i||i|\rho_0^i \leq \max_{i\in\mathbb{Z}} |a_i|\rho_0^i = |f(T)|_{\rho_0}$$

which implies $|\frac{d}{dT}f(T)|_{\rho_0} \leq \rho_0^{-1}|f(T)|_{\rho_0}$. Finally, since $|\frac{d}{dT}(T)|_{\rho_0} = \rho_0^{-1}|T|_{\rho_0}$, the proof follows. □

2.3 Factoring finite morphisms

2.3.1. *Absolute and relative Frobenius.* (We recall that $\mathrm{char}(\widetilde{k}) = p > 0$). Recall that the absolute Frobenius morphism of a field $\widetilde{k}$ is the automorphism $F_{abs} : \widetilde{k} \to \widetilde{k}$ given by $F_{abs}(a) = a^p$. We will also denote the induced morphism $\mathrm{Spec}\,\widetilde{k} \to \mathrm{Spec}\,\widetilde{k}$ by F_{abs}. More generally, for a smooth, connected curve X over the field $\widetilde{k}$, absolute Frobenius is the morphism $F_{abs} : X \to X$ which at the level of sections acts as raising to the p-th power, and which (consequently) is the identity at the topological level acting on the structure sheaf as rising functions to the p-th power. Then it can be shown that absolute Frobenius acts as the identity map on the points of X. It is easy to see that F_{abs} is not $\widetilde{k}$-linear.

Let $X^{(p)}$ denote the fiber product $X \times_{\mathrm{Spec}\,(\widetilde{k})} \mathrm{Spec}\,(\widetilde{k})$ with respect to the structure morphism $X \to \mathrm{Spec}\,\widetilde{k}$ and the absolute Frobenius $F_{abs} : \mathrm{Spec}\,(\widetilde{k}) \to \mathrm{Spec}\,(\widetilde{k})$. The relative Frobenius morphism (also called $\widetilde{k}$-linear Frobenius morphism) F_{rel} is a $\widetilde{k}$-morphisms $F_{rel} : X \to X^{(p)}$, given as a $F_{rel} = (F_{abs}, \mathrm{Id})$. In more concrete terms, if we put $X = \mathrm{Spec}\,(\widetilde{k}[T_1,\ldots,T_n]/(f_1,\ldots,f_m))$ where T_i are indeterminates and $f_i \in \widetilde{k}[T_1,\ldots,T_n]$, then $X^{(p)} = \mathrm{Spec}\,(\widetilde{k}[T_1,\ldots,T_n]/(f_1^p,\ldots,f_m^p))$. The relative Frobenius morphisms is then induced by a $\widetilde{k}$-morphism $\widetilde{k}[T_1,\ldots,T_n]/(f_1^p,\ldots,f_m^p) \to \widetilde{k}[T_1,\ldots,T_n]/(f_1,\ldots,f_m)$ given by $\sum a_I T^I \mapsto a_I T^{pI}$. More generally, we define F_{rel}^r to be r-uple composition of F_{rel} with itself, that is $F_{rel}^1 := F_{rel}$ and for $r \geq 1$, $F_{rel}^r := F_{rel} \circ F_{rel}^{r-1}$. For $r \geq 1$, we denote $Y^{(p^r)} :=$

$Y^{(p^{r-1})} \times_{\mathrm{Spec}\, \widetilde{k}} \mathrm{Spec}\, \widetilde{k}$, where again the fiber product is taken with respect to the structure morphism $Y^{(p^{r-1})} \to \mathrm{Spec}\,(\widetilde{k})$ and the absolute Frobenius $F_{abs} : \mathrm{Spec}\,(\widetilde{k}) \to \mathrm{Spec}\,(\widetilde{k})$.

2.3.2. Recall that if $f : Y \to X$ is a finite morphism of smooth, connected $\widetilde{k}$-curves, then f factors as $f = f_{sep} \circ f_{insep}$ where $f_{insep} = Y \to Z$ is purely inseparable morphism, while $f_{sep} : Z \to X$ is a generically étale morphism. This corresponds to the inclusion of the fields $K(X) \hookrightarrow K(Z) \hookrightarrow K(Y)$, where $K(X)$ and $K(Y)$ are function fields of X and Y respectively, while $K(Z)$ is the separable closure of $K(X)$ in $K(Y)$. More precisely, $Z \simeq Y^{(p^r)}$ where p^r is the degree of the morphism f_{insep} and $f_{insep} = F^r_{rel}$ (see [27, Chapter IV, Proposition 2.5]).

2.3.3. *Composite Hensel's lemma.* Since we study affinoids over a field of characteristic 0, we cannot speak about purely inseparable morphisms between them. Nevertheless, since the residue field $\widetilde{k}$ is of positive characteristic, we introduce the following definition.

Definition 2.3.4. *Let $\varphi : Y \to X$ be a finite morphism of quasismooth, connected, 1-dimensional k-affinoids with good canonical reduction. We say that φ is residually separable (resp. purely inseparable) if the reduction morphism $\widetilde{\varphi} : \widetilde{Y} \to \widetilde{X}$ is separable (resp. purely inseparable). In the case where φ is residually purely inseparable of degree p^r, we also say that φ is a lifting of the relative Frobenius morphism F^r_{rel}.*

We are particularly interested in the case when we can lift the decomposition of a finite morphism of the previous paragraph to the decomposition of a morphism between suitable affinoids. We tend to think of the next result as a sort of a *composite Hensel's lemma.*

Theorem 2.3.5. *Let $\varphi : Y \to X$ be a finite étale morphism of one-dimensional, quasi-smooth connected k-affinoid spaces which have good canonical reduction. Then, after possibly removing finitely many residual classes from Y and X, φ factors through $\varphi = \varphi_1 \circ \varphi_2$ where φ_1 is residually separable and φ_2 is residually purely inseparable morphism.*

Proof. Consider the reduction $\widetilde{\varphi} : \widetilde{Y} \to \widetilde{X}$ of $\varphi : Y \to X$. We know that $\widetilde{\varphi}$ factors as $\widetilde{\varphi_2} : \widetilde{Y} \to \widetilde{Y}^{(p^r)}$ which is purely inseparable (say of degree p^r i.e. the r-fold relative

Frobenius map) and $\widetilde{\varphi_1} : \widetilde{Y}^{(p^r)} \to \widetilde{X}$ which is generically étale ([30, p. 291]). Moreover, after removing some points from $\widetilde{Y}$ and $\widetilde{X}$, we may assume that $\widetilde{\varphi}_1$ is smooth.

We have the following diagrams (the diagram on the left is commutative while on that one on the right we look for a lifting "?" which will make the diagram commute):

$$\begin{tikzcd} & \widetilde{Y}^{(p^r)} \arrow[d, "\widetilde{\varphi}_1"] \\ \widetilde{Y} \arrow[ur, dotted, "\widetilde{\varphi}_2"] \arrow[r, "\widetilde{\varphi}"] & \widetilde{X} \end{tikzcd} \qquad \begin{tikzcd} & Y^{(p^r)} \arrow[d, "\varphi_1"] \\ Y \arrow[ur, dotted, "?"] \arrow[r, "\varphi"] & X \end{tikzcd}$$

where in the diagram on the right, $\varphi_1 : Y^{(p^r)} \to X$ is any lifting of the morphism $\widetilde{\varphi_1} : \widetilde{Y}^{(p^r)} \to \widetilde{X}$ (a lifting exists as $\widetilde{X}$ is smooth). Since we assumed that $\widetilde{Y}^{(p^r)}$ is smooth over $\widetilde{X}$ we can apply Coleman's theorem about lifting morphisms [16, Theorem 1.1, p. 115], to obtain a morphism $\varphi_2 : Y \to Y^{(p^r)}$ which lifts the morphism $\widetilde{\varphi_2} : \widetilde{Y} \to \widetilde{Y}^{(p^r)}$ in such a way that the diagram on the right is commutative. □

2.3.6. Factoring étale morphisms of open annuli

2.4 Ramification locus

2.4.1 Finite extensions of complete residue fields and discriminant function

2.4.1. *Stable fields* Let $(K, |\cdot|_K)$ be a normed field of *characteristic 0*, where $|\cdot|_K$ is nontrivial and nonarchimedean. As usual, we denote by $K^\circ = \{x \in K, |x|_K \leq 1\}$ the ring of integers of K, by $K^{\circ\circ}$ its maximal ideal and by $\widetilde{K}$ the residual field $K^\circ/K^{\circ\circ}$. Recall that a finite extension of normed fields $(L, |\cdot|_L)$ of $(K, |\cdot|_K)$ (we assume that $|\cdot|_L$ extends $|\cdot|_K$) is called K-cartesian (or for brevity just cartesian) if there exists a basis $x_1, \ldots, x_d$ of L over K such that for each $m = m_1x_1 + \cdots + m_dx_d \in L$, $m_1, \ldots, m_d \in K$, we have $|m|_L = \max_i |m_i|_K|x_i|_L$. Such a basis is called orthogonal or K-cartesian and if its elemets in addition have norm 1, we call the basis orthonormal. We say that the field K is stable if every finite extension of it is cartesian.

For some criteria for stability of complete normed fields, we refer to [9, Section 3.6.2, Propositions 1 to 6]. In particular, a complete field K is stable if it is discretely valued or algebraically closed. If K is complete, then it is stable if and only if for each finite extension L/K of degree n we have $n = e \cdot f$, where $f = f(\widetilde{L}/\widetilde{K})$ is the degree of the residual extension $\widetilde{L}/\widetilde{K}$ and $e = e(\widetilde{L}/\widetilde{K})$ is the ramification index of the valuation groups, that is, $e = \big||L^*| : |K^*|\big|$.

In what follows we assume that K is stable and that the valuation group $|K^*|$ is discrete or divisible. Let L/K be a finite extension of valued fields. Then, by [32, Lemma 1.6] this implies that the integral closure of K° in L (i.e. L°) is finite and free module over K°. In particular, following Lütkebohmert ([31, 32]) we give

Definition 2.4.2. *The fractional ideal $\mathcal{C}_{L/K} := \{y \in L, \mathrm{tr}_{L/K}(yL^\circ \subset K^\circ)\}$, where* tr *is the trace pairing for the extension L/K, is called codifferent of the extension L/K. It is a fractional principal ideal of L°. A base is given by the dual base of L° over K° with respect to trace pairing. Its inverse $\mathcal{D}_{L/K} := (\mathcal{C}_{L/K})^{-1}$ is called different of the extension L/K (see [32, Definition 1.8, Remark 1.9.1]).*

Definition 2.4.3. *Let $y_1, \ldots, y_n$ be a basis of L° over K°. Then the principal ideal $\mathfrak{d} \subset K^\circ$ generated by the element $(\det(\mathrm{tr}(y_i y_j))_{i,j=1,\ldots,n}$ is called the discriminant ideal of the extension L/K. ([32, Definition 1.8]). Its ideal norm is denoted by $\delta_{L/K}$ and is called discriminant.*

We collect some of the main properties of the objects just introduced in the following lemma.

Lemma 2.4.4.

1. *If $y_1, \ldots, y_n$ is a K°-basis of L°, then the dual basis $y_1^*, \ldots, y_n^*$ with respect to the trace pairing $\mathrm{tr}_{\mathrm{L/K}}$ is a K°-basis of the codifferent $\mathcal{C}_{L/K}$.*
2. *$\mathfrak{d}_{L/K} = N_{L/K}(\mathcal{D}_{L/K})K^\circ$, where $N_{L/K}$ is the ideal norm function.*
3. *If $L = K(a)$ is a separable extension and if $L^\circ = K^\circ[a]$, then we have $\mathcal{D}_{L/K} =$*

$f'_a(a)L^\circ$, where f'_a is the usual derivative of the minimal polynomial f_a of a over K. Consequently, $\delta_{L/K} = |f'_a(a)|_L^n$.

4. *If M/L and L/K are finite separable extensions, then $\mathcal{D}_{M/K} = \mathcal{D}_{M/L}\mathcal{D}_{L/K}$ and $\mathfrak{d}_{M/K} = N_{L/K}(\mathfrak{d}_{M/L})\mathfrak{d}_{L/K}^{[M:L]}$.*

5. *Let l/k be the residue field extension of the finite extension L/K and suppose $[l : k] = [L : K]$. Then*

   (a) *Any lifting of a k-basis of l to L° is an orthonormal K°-basis of L° (a basis of L° over K° is orthonormal if it is orthogonal, in a similar sense as above, and its elements have norm 1).*

   (b) *If both extensions L/K and l/k are separable, then $\delta_{L/K} = |\mathfrak{d}_{L/K}|_K = 1$.*

Proof. [32, Remark 1.9, Lemma 1.13]. $\square$

2.4.5. *Stability of completed residue field $\mathscr{H}(y)$.* The following theorem is well known, but nevertheless we give the proof.

Theorem 2.4.6. *Let Y be a compact, connected, quasi-smooth k-analytic curve and let y be a type two point. Then the residue field $\mathscr{H}(y)$ is stable.*

Proof. First suppose that $\mathscr{H}(y)$ has genus zero. This means that the point y can be embedded into analytic projective line, in particular $\mathscr{H}(y)$ is isomorphic to the completion of the fraction field of the affinoid algebra $k\{T\}$ with respect to the Gauss norm. The later field is stable by the theorem of Grauert-Remmert and Gruson [9, Theorem 5.3.2/1].

If $\mathscr{H}(y)$ has a positive genus, then we can take an affinoid neighborhood $V \subset Y$ of y and a finite morphism $f : V \to W$, where W is an affinoid domain in $\mathbb{P}^1_k$. Put $x = f(y)$. Then, we have an induced finite extension $\mathscr{H}(y)/\mathscr{H}(x)$ and since $\mathscr{H}(x)$ is complete and stable by the previous case, we conclude that $\mathscr{H}(y)$ is stable as well by [9, 3.6. Corollary 7]. $\square$

2.4.7. *Function δ.* In this paragraph we return to our initial setting and a finite morphism $\varphi : Y \to X$ like in 2.0.15 or a finite morphisms between wide open curves. We agree in this section to denote the set of type 2 (resp. type 3, resp. type 2 or 3) points in a curve Z by Z^{II} (resp. Z^{III}, resp. $Z^{H'}$). Let $y \in Y$ be a type two point and put $x = \varphi(y)$. Then, our morphism φ induces a finite field extension $\mathscr{H}(y)/\mathscr{H}(x)$ which is of degree $\nu_\varphi(y)$ (recall that $\nu_{\varphi(y)}$ is the geometric ramification index of φ at y, defined in [4, 6.3, Remark 6.3.1]. For the basic properties of the function $\nu_\varphi(y)$ we refer to the *loc.cit.*). As we have just seen, $\mathscr{H}(x)$ is stable and we know that the group $|\mathscr{H}(x)^\times|$ is divisible because $|k^*|$ itself is divisible, hence we have a well defined discriminant $\delta_{\mathscr{H}(y)/\mathscr{H}(x)}$.

Definition 2.4.8. *We define function $\delta : Y^{II} \to (0, 1]$ to be defined as $\delta(y) = \delta_{\mathscr{H}(y)/\mathscr{H}(x)}$ and call it the discriminant function.*

Contrary to the case when $x \in X^{II}$, if $x \in X^{III}$ the integral closure of $\mathscr{H}(x)^\circ$ in a finite extension of $\mathscr{H}(x)$ is not finite over $\mathscr{H}(x)^\circ$, except if the extension is trivial. This could be seen from [32, Lemma 1.6] or [11, VI, Section 8, Theorem 2]. Therefore, we cannot *a priori* extend the function δ to the points of type 3. However, there is a way to overcome this problem.

Definition 2.4.9. *(Continuing 2.4.8) Let $y \in Y^{III}$. We define $\delta(y)$ to be $\delta_{\sigma_K(y)/\sigma_K(x)}$, where $K = \widehat{\mathscr{H}(x)}$ (see Paragraph 1.1.19 for the definition of K and for an explanation why $\sigma_K(x)$ is a type 2 point).*

Combining the previous two definitions, we have a function (still denoted by) δ defined on the subset $Y^{H'}$ of Y consisting of type 2 and type 3 points.

Lemma 2.4.10. *Let $\varphi : Y \to X$ be a finite étale morphism of degree n of strict open annuli. Then, the discriminant function $\delta : Y^{H'} \to (0, 1]$ is continuous along the skeleton of Y. Moreover, $\delta(\eta_\rho) = |\epsilon|^n |\rho|^{n\nu}$ and, in particular, when $\nu = 0$, $\delta(\eta_\rho) = |n|^n$.*

Proof. Let $T : Y \xrightarrow{\sim} A(r, 1)$ and $S : X \xrightarrow{\sim} A(r^n, 1)$ be coordinates on Y and X, where

n is the degree of the map φ, so that we have $S = \varphi_{\#}(T) = T^n(1 + h(T))$ where $h(T)$ is convergent on $A(r, 1)$ and $|h(T)|_\rho < 1$ for each $\rho \in (r, 1)$.

Suppose $\rho \in |k^*|$. Then, φ restricts to a finite morphism $\varphi_\rho : A[\rho, \rho] \to A[\rho^n, \rho^n]$ and we have the corresponding morphism of affinoid algebras $\varphi_\rho^{\#} : k\{\rho^n S, \rho^{-n} S^{-1}\} \to k\{\rho T, \rho^{-1} T^{-1}\}$ is given by $S \mapsto \varphi_{\#}(T)$. Note that $\varphi_\rho^{\#}$ extends to a finite morphism of fields $\varphi_\rho^{\#} : \mathscr{H}(\eta_{\rho^n}) \to \mathscr{H}(\eta_\rho)$. Let s and t be the images of S and T, respectively under the canonical embeddings $k\{\rho^n S, \rho^{-n} S^{-1}\} \hookrightarrow \mathscr{H}(\eta_{\rho^n})$ and $k\{\rho T, \rho^{-1} T^{-1}\} \hookrightarrow \mathscr{H}(\eta_\rho)$, respectively, and let us put for the moment $\mathscr{H}_{\rho^n, s} := \mathscr{H}(\eta_{\rho^n})$ and $\mathscr{H}_{\rho, t} := \mathscr{H}(\eta_\rho)$. We have the following commutative diagram

$$\begin{tikzcd} k\{\rho^n S, \rho^{-n} S^{-1}\} \arrow[r, "\varphi^{\#}"] \arrow[d, hook] & k\{\rho T, \rho^{-1} T^{-1}\} \arrow[d, hook] \\ \mathscr{H}_{\rho^n, s} \arrow[r, "\varphi_\rho^{\#}"] & \mathscr{H}_{\rho, t} \end{tikzcd}$$

where the vertical maps are canonical embeddings.

After introducing *integral* coordinates $S_1 = \alpha^{-n} S$ and $T_1 = \alpha^{-1} T$, where $\alpha \in k$ and $|\alpha| = \rho$ so that $S_1 = \phi(T_1) = \alpha^{-n} \varphi(\alpha T_1)$, the previous diagram is transformed into a commutative diagram

$$\begin{tikzcd} k\{S_1, S_1^{-1}\} \arrow[r, "\phi^{\#}"] \arrow[d, hook] & k\{T_1, T_1^{-1}\} \arrow[d, hook] \\ \mathscr{H}_{1, s_1} \arrow[r, "\phi_1^{\#}"] & \mathscr{H}_{1, t_1} \end{tikzcd}$$

where now $s_1 = \alpha^{-n} s$ and $t_1 = \alpha^{-1} t$ with $|s_1| = |t_1| = 1$. The next thing to note is that the elements $1, t_1, \dots, t_1^{n-1}$ form an orthonormal $\mathscr{H}_{1,s_1}^\circ$-basis for $\mathscr{H}_{1,t_1}$. This follows from Lemma 2.4.4 *5.(a)*. Let $P_{t_1}(X) \in \mathscr{H}_{1,s_1}[X]$ be the minimal polynomial of t_1 over $\mathscr{H}_{1,s_1}$. Then from Lemma 2.4.4 we have $\delta(\eta_\rho) = |P'_{t_1}(t_1)|^n$. On the other side, t_1 is a zero of the series equation given by $\phi(X) - s_1 = 0$. Let us prove that $|\phi'(t_1)| = |P'_{t_1}(t_1)|$.

We proceed by proving an equivalent (after returning the former coordinates) equation $|\varphi'_{\#}(t)| = |P'_t(t)|$. From the valuation polygon of the function $\varphi(X) - s$ (seen as a function with coefficients in $\mathscr{H}_{\rho^n, s}$) we see that it has a break at the point $(\frac{1}{n} \log |s|, \log |s|)$ so that $\varphi(X)-s$ has n zeros (counted with multiplicity and t being one of them) of the norm $|t|$. Moreover, $\varphi(X)-s$ factorizes as the product $Q(X)(1+q(X))$, where $Q(X)\in\mathscr{H}_{\rho^n,s}[X]$ is of degree n and has the same zeros (with multiplicities) as $\varphi(X)-s$, while $|1+q(t)|=1$. We conclude that $P_t(X)$ divides $Q(X)$ and again from the valuation polygons we see that the norm of the leading coefficient of $Q(X)$ is 1. Hence $P_t(X)$ and $Q(X)$ are equal after the possible multiplication by a unit in $\mathscr{H}_{\rho^n,s}$. Finally, we have $|\varphi'_{\#}(t)|=|Q'(t)(1+q(t))+Q(t)q'(t)|=|Q'(t)|=|P'_t(t)|$ which implies $|P'_{t_1}(t_1)|=|\phi'(t_1)|$.

Furthermore, note that $|\phi'(t_1)|=|\alpha^{\sigma-n+1}\epsilon t_1^{\sigma}(1+h(t_1))|=|\epsilon|\rho^{\nu}$ which implies $\delta(\eta_\rho)=|\epsilon|^n\rho^{n\nu}$.

The same calculations apply for a $\rho\in(r,1)\setminus|k^*|$ after extension from a base field k to $K=\widehat{\mathscr{H}(\eta_\rho)}$. The Lemma follows. $\square$

2.4.11. *Function đ.* In general, function $\delta: Y^{H'}\to(0,1]$ is not continuous. However, a cousin function đ that we introduce below is.

Definition 2.4.12. *Let $\varphi: Y\to X$ be as in 2.0.15 and let $x\in X^{H'}$. We define the function $\text{đ}: X^{H'}\to[0,1]$ as $\text{đ}(x)=\prod_{y\in Y,\varphi(y)=x}\delta(y)$.*

Remark 2.4.13. In [31] Lütkebohmert introduced and studied another function closely related with our function đ. In fact in the papers [31] and [32] Lütkebohmert's function is denoted by δ. To show the relation between the two functions, we choose a symbol δ' for the Lütkebohmert's one. We recall the definition of δ' in a slightly more general situation than those considered in [31, 32]. Let $\varphi: Y\to X$ be as in 2.0.15. For a type two point $x\in X$ let C_x be an affinoid domain in X with good canonical reduction and maximal point x. We define $\delta'(x)$ (we also write $\delta'(x,\varphi)$ if we want to emphasize the dependence on φ) to be the discriminant of the extension $\mathcal{M}(\varphi^{-1}(C_x))/\mathcal{M}(C_x)$, where $\mathcal{M}(\varphi^{-1}(C_x))$ and $\mathcal{M}(C_x)$ are the total fraction rings of meromorphic functions on affinoid domains $\varphi^{-1}(C_x)$ and C_x, respectively, and where the norms involved are coming from the corresponding maximal points. Another way to see this is, if $y_1,\dots,y_r\in Y$ such that $\varphi(y_i)=x$, if C_x

is small enough and if C_{y_i} is an affinoid domain in Y with good canonical reduction and with maximal point y_i and which is a connected component of $\varphi^{-1}(C_x)$, for $i = 1, \ldots, r$, then $\delta'(x) = \prod_{i=1}^{r} \delta'(x, \varphi_{|C_{y_i}})$.

We obtain Lütkebohmert definition if we take X to be a unit disc $D(0,1)$ and $C_x = D(0,r)$, where $x = \eta_{0,r}$, and in fact, the function δ' is the function on the radius $r \in (0,1) \cap |k^\times|$ rather than the function of the point $\eta_{0,r}$.

Lemma 2.4.14. *Let $\varphi : Y \to X$ as in 2.0.15 and let $x \in X$ be a type two point. Then, $đ(x) = \delta'(x)$.*

Proof. Having in mind the previous remark it is enough to prove the lemma in the case where Y and X are with good canonical reduction and where x is the maximal point of X. Let $y \in Y$ be the maximal point of Y so that $\varphi(y) = x$. Note that in this case $đ(x) = \delta(y)$ as $\varphi^{-1}(x) = y$. Suppose that $t_1, \ldots, t_n \in \mathcal{M}(Y)^\circ$ is an orthonormal $\mathcal{M}(X)^\circ$-basis, and let $i : \mathcal{M}(Y) \hookrightarrow \mathscr{H}(y)$ be the canonical inclusion. Then $(i(t_1), \ldots, i(t_n))$ is an orthonormal $\mathscr{H}(x)^\circ$-basis because of the Lemma 2.4.4 5.(a). Moreover we have $(\det(\mathrm{tr}(t_i t_j))_{i,j=1,\ldots,n} = (\det(\mathrm{tr}(i(t_i)t(t_j))_{i,j=1,\ldots,n}$ which implies $\delta'(x) = \delta(y) = đ(x)$. $\square$

Remark 2.4.15. The previous lemma also shows that δ' is well defined, *i.e.* it doesn't depend on the choice of the affinoid domain C_x in Remark 2.4.13.

Theorem 2.4.16. *For $\varphi : Y \to X$ as before, $đ : X^{H'} \to [0,1]$ is continuous in the strong topology of $X^{H'}$.*

Proof. By taking a finite morphism from X to an affinoid domain in a projective line $\mathbb{P}^1_k$ and having in mind 2.4.4 4, we can assume that X is a unit affinoid disc. In this case, by a choice of coordinate on X we may write $X = D(0,1)$. Let $\rho \in (0,1)$ and let $X(\rho) := Y \times_X D(0,\rho)$. If we (ambiguously) put $\zeta : (0,1) \to \zeta_\rho = \zeta_{0,\rho}$, then we know that: (1) The function $\rho \mapsto \delta'(\zeta_\rho)$ is continuous on the set $(0,1] \cap |k^\times|$ which follows from [32, Proposition 3.6], and (2) $\delta'(\zeta_\rho) = đ(\zeta_\rho)$ which follows from Lemma 2.4.14. The two assertions imply that the $đ \circ \zeta$ is continuous on $(0,1] \cap |k^\times|$. If $\rho \notin (0,1] \cap |k^\times|$, the type three point ζ_ρ has a neighborhood which is an open annulus A, such that $\varphi^{-1}(A)$ is a disjoint union of open annuli A_i, $i = 1, \dots, s$ and $\varphi_{A_i} : A_i \to A$ is a finite étale morphism. Recall that for each $i = 1, \dots, s$ $\varphi^{-1}(\Gamma^A) = \Gamma^{A_i}$. Then, the continuity of $\text{đ} \circ \zeta$ follows from Lemma 2.4.10 and definition of the function đ.

In conclusion, $\text{đ} \circ \zeta$ is continuous on $(0, 1]$ or that đ is continuous along the canonical path (excluding 0) from the point 0 to point $\zeta_{0,1}$. On the other side, if we choose another coordinate $T : X \to D(0, 1)$, $T(x) = 0$ for an arbitrary rational point $x \in X$, we obtain that đ is continuous along the canonical path (excluding x) from x to the maximal point of X. This precisely means that đ is continuous on $X^{H'}$ in the strong topology. □

2.4.2 Inseparable ramification locus for the liftings of relative Frobenius morphisms

We apply the previous results to study the ramification locus for liftings of relative Frobenius morphisms (Definition 2.3.4). More precisely, let $\varphi : Y \to X$ be a finite morphism where Y and X are like in 2.0.15 or wide open curves and let $y \in Y$ be a type two point and let $x = \varphi(y)$. Than we can find affinoid domain C_y in Y and C_x in X, with good canonical reductions and with maximal points y and x, respectively, such that $\varphi_y := \varphi_{|C_y} : C_y \to C_x$ is a finite morphism. We have canonical reduction $\widetilde{\varphi}_y$ of the morphism φ_y. Recalling the Definition 2.3.4, we introduce the following one (compare with [20, Section 5]).

Definition 2.4.17. *We say φ is residually separable (resp. inseparable, resp. purely inseparable) at a type two point y if the morphism $\widetilde{\varphi}_y$ is separable (resp. inseparable, resp. purely inseparable).*

If y is a type three point in Y, we say that φ is residually separable (resp. inseparable, resp. purely inseparable) at y if the morphism $\varphi\hat{\otimes}K : Y\hat{\otimes}K \to X\hat{\otimes}K$ is residually separable (resp. inseparable, resp. purely inseparable) at $\sigma_K(y)$, where $K = \widehat{\mathscr{H}(y)}$.

We denote by $\mathcal{R}_{\text{sep}}(\varphi)$ (resp. $\mathcal{R}_{\text{ins}}(\varphi)$) the set of points $y \in Y$ at which φ is residually separable (resp. residually inseparable).

Remark 2.4.18. (i) The previous definition does not depend on the choice of the affinoid C_y.

(ii) It follows from Lemma 2.4.4 that φ is inseparable at y if and only if $\delta(y) = \delta_\varphi(y) < 1$.

We are ready to state the main theorem.

Theorem 2.4.19. *Let $\varphi : Y \to X$ be a finite étale morphism of degree $p > 0$, where p is a prime number, of quasi-smooth k-affinoid curves with good canonical reduction and let η and ξ be the maximal points of Y and X, respectively. Suppose that φ is residually purely inseparable (*i.e. *a relative Frobenius morphism). Then, $\mathcal{R}_{\text{ins}}(\varphi) = \{y \in Y, d_Y(\eta, y) < \gamma\}$, where d_Y is metric on Y defined in Paragraph 1.3.4 and where $\gamma = \delta(\eta)^{\frac{1}{p(p-1)}}$.*

Proof. We start with a simple observation:

Lemma 2.4.20. *Let $y \in Y^{H'}$. Then, $y \notin \mathcal{R}_{\text{ins}}(\varphi)$ implies $\text{đ}(\varphi(y)) = 1 (= \delta(y))$.*

Proof. It is enough to show that φ is residually separable at all the points $z \in \varphi^{-1}(\varphi(y))$. But, this is true since $\deg(\varphi) = \sum_{z \in \varphi^{-1}(\varphi(y))} \nu_\varphi(z) = p$ and where $\nu_\varphi(z)$ is the geometric ramification index of φ at z ([4, 6.3.1.(iii)]), so for each $z \in \varphi^{-1}(\varphi(y))$, $\nu_\varphi(z) < p$, therefore φ cannot be residually inseparable at z. □

It follows that $\mathcal{R}_{\text{ins}}(\varphi) = \varphi^{-1}(\{x \in X, \text{đ}(x) < 1\})$. On the other hand, if for a point $x \in X$, $\text{đ}(x) < 1$, it follows that there is only one point in the set $\varphi^{-1}(x)$ and $\text{đ}(x) = \delta(\varphi^{-1}(x))$. This implies that the function $\delta : Y^{H'} \to (0, 1]$ is continuous on $Y^{H'}$ for the strong topology, because the function $\text{đ} : X^{H'} \to (0, 1]$ is.

Let D be a residual class attached to the point η. Then, $\varphi(D)$ is a residual class attached to ξ and φ restricts to a finite morphism $\varphi_D : D \to \varphi(D)$ which is necessarily of degree p as φ is purely inseparable at η. It follows that for any residual class D_X attached to ξ, $\varphi^{-1}(D_X)$ is a single residual class attached to η. We are now going to study more precisely the set $\mathcal{R}_{\text{ins}}(\varphi_D) = \mathcal{R}_{\text{ins}}(\varphi) \cap D$.

For this, let $T : D \to D(0,1)$ and $S : \varphi(D) \to D(0,1)$ be coordinates on D and $\varphi(D)$. Let $A = A(0;r,1)$ be an open annulus living at the boundary of the disc D such that $\varphi_A : A(0;r,1) \to \varphi(A) = A(0;r^p,1)$ is a finite étale morphism of open annuli of degree p. As φ is étale morphism it follows that $\sigma(\varphi_D) = 0$ which implies that $\nu_D := \nu(\varphi_A) = -p+1$. We conclude that for a point $\eta_\rho = \eta_{0,\rho} \in A(0;r,1)$, $\rho \in (0,1)$, the discriminant $\delta(\eta_\rho) = |\epsilon(\varphi_A)|^p \rho^{-p(p-1)}$ (Lemma 2.4.10). Because of the continuity of δ along the skeleton it follows that $\delta(\eta_r) = |\epsilon(\varphi_A)|^p r^{-p(p-1)}$. Also, the continuity of δ established above implies that the minimal $r \in (0,1)$ such that φ restricts to a finite étale morphism $\varphi_A : A = A(0;r,1) \to A(0;r^p,1)$ is such an r for which $\delta(\eta_r) = 1 = |\epsilon(\varphi_A)|^p r^{-p(p-1)}$, *i.e.* $r = |\epsilon(\varphi_A)|^{\frac{1}{p-1}}$. We also note here that $|\epsilon(\varphi_A)|^p = \delta(\eta)$, for the reasons of continuity as before. In a nutshell, a point $\eta_{0,\rho}$ on the canonical path from 0 to η in the disc $D(0,1)$ is in $\mathcal{R}_{\text{ins}}(\varphi_D)$ if and only if $\rho \in (|\epsilon(\varphi_A)|^{\frac{1}{p-1}}, 1)$ if and only if $\rho \in (\delta(\eta)^{\frac{1}{p(p-1)}}, 1)$. The computations are independent on the choice of coordinates T and S, so we conclude that $\mathcal{R}_{\text{ins}}(\varphi_D) = \{y \in D, d_Y(\eta, y) < \delta(\eta)^{\frac{1}{p(p-1)}}\}$.

Finally, $\mathcal{R}_{\text{ins}}(\varphi) = \{\eta\} \cup \cup_D \mathcal{R}_{\text{ins}}(\varphi_D) = \{y \in Y, d_Y(\eta, y) < \delta(\eta)^{\frac{1}{p(p-1)}}\}$. $\square$

Chapter 3

Pro-wide open curves

3.1 Category $\mathbb{W}$

In order to find a nice setting for the objects that we will have to work with latter on (*cf.* definition 3.1.18 Robba proannuli, Robba rings), and inspired by Lemma 3.1.43, we introduce a pro-category of wide open curves.

3.1.1. Recall that a wide open curve (Definition 1.1.2) is an open analytic subset isomorphic to a complement of a disjoint union (possibly empty) of finitely many closed discs in a smooth, connected, projective k-analytic curve.

By $\mathbf{W}$ we denote a full subcategory of category of k-analytic curves whose objects are wide open curves. We can thus form a category pro-$\mathbf{W}$, of pro objects in the category $\mathbf{W}$. Let us recall what the later category is.

The elements of $\mathrm{Ob}(\mathbf{pro}{-}\mathbf{W})$ are functors $I \to \mathbf{W}$, where I is a small cofiltered category. We also denote them by $"\varprojlim"_{t\in I} X_t$, or simply $"\varprojlim_I"X_t$ or $"\varprojlim"X_t$ if the indexing set is understood to be known. For two objects $X = I \to \mathbf{W} = "\varprojlim"_t X_t$ and $Y = J \to \mathbf{W} = "\varprojlim"_s Y_s$, the set of morphisms between them is given by $\mathrm{Hom}_{pro-\mathbf{W}}(X,Y) := \varprojlim_t \varinjlim_s \mathrm{Hom}_{\mathbf{W}}(X_t, Y_s)$.

3.1.2. However, for our purposes we will restrict our attention to a subcategory of pro-$\mathbf{W}$, denoted by $\mathbb{W}$ and which we call category of pro-wide open curves.

Definition 3.1.3. *Objects of the category of pro-wide open curves $\mathbb{W}$, consists of pro-objects* $"\varprojlim_I"U_i$*, where the system* $(U_i)_i \in I$ *is subject to the following conditions:*

(i) Each U_i*,* $i \in I$ *is an open subset in* X *isomorphic to a wide open curve, where* X *is a smooth, connected, projective* k*-analytic curve* X *is the same for all* U_i*,*

(ii) For all $i, j \in I$*,* $j \to i$ *we have an inclusion* $U_j \hookrightarrow U_i$*. Furthermore, we ask that* $U_i \setminus U_j$ *is a (finite) disjoint union of* n *open annuli. The number* n *depends only on* U *and not on* i, j*.*

Morphisms are morphisms of the category pro-$\mathbf{W}$*. We will call the above curve* X*,* the curve of definition for $"\varprojlim_I"U_i$*, and we will assume without necessarily emphasizing it that* X *is the part of the datum for a pro-wide open curve* $"\varprojlim_I"U_i$*.*

Remark 3.1.4. We can simplify objects and morphisms in category $\mathbb{W}$ in the following way. By definition of a pro-category, an element $f \in \mathrm{Hom}("\varprojlim"_t U_t, "\varprojlim"_s V_s)$ is an inverse system of morphisms $\{f_s : "\varprojlim"_t U_t \to V_s\}_{s \in J}$, and each f_s is an element of $\varinjlim_t \mathrm{Hom}(U_t, V_s)$. By [19, Proposition 2.1.4], we can reindex the elements $"\varprojlim"_t U_t$ and $"\varprojlim"_s V_s$ by the same index category I such that f is represented by a family of compatible morphisms $f_i : U_i \to V_i$, $i \in I$. Furthermore, this can be done uniformly for any finite diagram without loops in the category $\mathbb{W}$ [19, Proposition 2.1.5].

As for the objects $"\varprojlim"_t U_t \in \mathbb{W}$, we note here that we can assume the index category to be cofinite and strongly directed [19, Theorem, 2.1.6], and in fact, we may use the set $\mathbb{N}$ for the indexing category.

We refer to [19] for all the details and notions.

Remark 3.1.5. Studying pro-objects in the setting of k-analytic spaces is nothing new. For example, the category of pro-k-analytic spaces was studied by Berkovich in [6] as an approach to dagger spaces, and we introduce the category $\mathbb{W}$ just to restrict our attention to particular pro-objects which are well behaved for the topic of our study.

To further simplify objects, we introduce the following definition.

Definition 3.1.6. *Let $U = \text{"}\varprojlim\text{"}_t U_t \in \mathbb{W}$ with the curve of definition X. If U_t is a wide open curve in X, then we call such U a* standard *pro-wide open curve.*

Remark 3.1.7. Every pro-wide open curve $U = \text{"}\varprojlim\text{"}_{t\in I} U_t$ is isomorphic to a standard pro-wide open curve in the following way. Pick a $t \in I$ and the curve U_t. As U_t is isomorphic to a wide open curve, there exists a smooth projective k-analytic curve Y, such that U_t can be seen as a wide open curve in Y. Then, for every $s \in I$ and $s \to t$, the curve U_s is a wide open curve in Y. Finally, we note that pro-objects $\text{"}\varprojlim\text{"}_{s\in I, s\to t} U_s$ and $\text{"}\varprojlim\text{"}_{t\in I} U_t$ are isomorphic.

3.1.8. *The heart of a pro-wide open curve.* Let $U = \text{"}\varprojlim\text{"} U_n$ be a pro-wide open curve, and let us denote by $\heartsuit(U)$ the intersection $\cap_n U_n$ and call it the *heart of* U. (Note that taking intersection makes sense, as all U_n are subsets of the curve of definition of U.) The following two theorems show a strong bond between an object in $\mathbb{W}$ and its heart.

Theorem 3.1.9. *Let $U = \text{"}\varprojlim_{n\in\mathbb{N}}\text{"} U_n \in \mathbb{W}$. Then there are the following possibilities for $\heartsuit(U)$:*

1. *$\heartsuit(U) = X$ where X is the curve of definition of U;*

2. *$\heartsuit(U) = \emptyset$;*

3. *$\heartsuit(U) = x$, where x is a type 1, 3 or 4 point in X;*

4. *$\heartsuit(U)$ is a wide open set;*

5. *$\heartsuit(U)$ is an affinoid curve;*

6. *$\heartsuit(U)$ is a semi-open affinoid curve.*

We say that X is a semi-open affinoid curve if it is isomorphic to a complement of finitely many open, and finitely many closed discs in a smooth, projective, connected k-analytic curve.

Proof. It is easy to show that each of the 6 possibilities above appear. By taking each U_i to be equal to the curve of definition X, we are in situation 1.. Then, taking $X = \mathbb{P}^1_k$ and U_i to be a family of open annuli $A(0, 1-1/n, 1)$ (with respect to the standard projective coordinate), we obtain a pro-object in the situation 2.. Similarly, taking any type 3 point (resp. type 1, resp. type 4) in a projective curve and a suitable system of open neighborhoods consisting of strict open annuli (resp. strict open discs, resp. strict open discs), we obtain a pro-object from 3.. The cases 4, 5, and 6. are dealt with similarly. For example, if we start with a smooth, connected, projective k-analytic curve X, and V an affinoid domain in X such that $X \setminus V$ is a disjoint union of open discs, then, a suitable system of open neighborhoods of V in X consisting of wide open curves will give us a pro-wide open curve from the situation 4.

To show that situations $1-6$ are the only ones that can occur, we may assume that U is a standard pro-wide open curve. Let X be the curve of definition of U and let us write $\heartsuit(U) = X \setminus (\uplus_{i=1}^m \cup_n D_{i,n})$, where $D_{i,n}$ are closed discs and $D_{i,n} \subset D_{i,l}$ for $l < n$. Then we distinguish the following cases:

1) The curve X has a semistable triangulation. Then, every complement of finitely many disjoint open discs in X contains Γ^X which is nonempty as it contains the minimal semistable triangulation of X (Theorem 1.2.11). In particular, for each $i = 1, \ldots, n$, every closed disc $D_{i,n}$ has either an empty intersection with Γ^X, or its Shilov point. In the former case, the whole family $(D_{i,n})_n$ is contained in an open disc in $X \setminus \Gamma^X$ and $\cup_n D_{i,n}$ is an open or a closed disc in X. In the later case, the family $(D_{i,n})_n$ is stationary (as two closed disc having a nonempty intersection and a common maximal point coincide) and again the union $\cup_n D_{i,n}$ is a closed disc. In this case, the Theorem follows.

2) The curve X is a Tate curve. Since Γ^X is again nonempty, we argue just like in the

previous case.

3) The curve X is a projective line $\mathbb{P}^1_k$. In this case, the computations are direct and we omit them. $\square$

Remark 3.1.10. It follows directly from the previous theorem that for any *standard* pro-wide open curve U with the curve of definition X, such that $\heartsuit(U)$ is a point (of necessarily type 1, 3 or 4) or $\heartsuit(U) = \emptyset$, the X is $\mathbb{P}^1_k$, as these cases only occur if $\Gamma^X = \emptyset$ *i.e.* $X = \mathbb{P}^1_k$.

Theorem 3.1.11. *Let $U_1, U_2 \in \mathbb{W}$ with hearts which are nonempty and not points. Then $\heartsuit(U_1) \simeq \heartsuit(U_2)$ (in the category of k-analytic curves) implies $U_1 \simeq U_2$. There are three types of pro-wide open curves with empty heart.*

Proof. We may assume that the pro-wide open curves are standard. We will consider the case where the hearts are strict analytic curves, the non-strict case being done in a similar way.

Let us start with the case where $\heartsuit(U_1) \simeq \heartsuit(U_2)$ is an affinoid curve and let us put $U_1 = X_1^\dagger$ and $U_2 = X_2^\dagger$, with $\heartsuit(X_1^\dagger) = X_1$ and $\heartsuit(X_2^\dagger) = X_2$. Let X_1' and X_2' be the curves of definition for $X_1^\dagger$ and $X_2^\dagger$, respectively. Then, we may write $X_1 = X_1' \setminus (D_{1,1} \cup \cdots \cup D_{1,n})$ (resp. $X_2 = X_2' \setminus D_{2,1} \cup \cdots \cup D_{2,m}$), where $D_{1,1}, \ldots, D_{1,n}$ (resp. $D_{2,1}, \ldots, D_{2,m}$) is a finite number of disjoint open discs in X_1' (resp. X_2') each of which is isomorphic to the open unit disc (in fact, we will have $n = m$, but *a priori* we don't know that, but also at the moment we don't need it). Note that it is enough to prove that some open neighborhoods of X_1 and X_2 in X_1' and X_2' are isomorphic.

By a theorem of Fresnel-Matignon, [21, Theorem 1], there exists a rational function f_1 (resp. f_2) on X_1' (resp. X_2') s.t. $X_1 = \{x \in X_1', \quad |f_1(x)| \leq 1\}$ (resp. $X_2 = \{x \in X_2', \quad |f_2(x)| \leq 1\}$). Let us put, for an $\epsilon > 1$, $X_{i,\epsilon} := \{x \in X_i', |f_i(x)| \leq \epsilon\}$, for $i = 1, 2$. Then, for $i = 1, 2$, $X_{i,\epsilon}$ is an affinoid domain in X_i' and $X_i = X_{i,1}$ is contained in the interior of $X_{i,\epsilon}$ for $\epsilon > 1$.

Furthermore, let us fix some $\epsilon_0 > 1$ and introduce the temporary notation and put A_ϵ to denote the corresponding affinoid algebra of the affinoid domain $X_{1,\epsilon}$ and let B

be the affinoid algebra corresponding to the affinoid domain X_{2,ϵ_0}. Then, we are exactly in the situation of [8] (with the notation as used in *loc.cit.*) and we can conclude that the isomorphism that we started with $i : X_1 \to X_2$ can be deformed to an isomorphism $i_1 : X_{1,\epsilon} \to i_1(X_{1,\epsilon})$, for some $\epsilon > 1$ and such that $i_1(X_1 = X_2)$. This implies that there are open neighborhoods of X_1 and X_2 in X_1' and X_2', respectively, which are isomorphic and the theorem follows.

We continue with the case where $\heartsuit(U_1) \cong \heartsuit(U_2)$ is a semi-open affinoid. Let X_1' and X_2' be the curves of definition for U_1 and U_2, respectively. An isomorphism $\heartsuit(U_1) \xrightarrow{\sim} \heartsuit(U_2)$ induces an isomorphism $X_1 \xrightarrow{\sim} X_2$ where X_1 (resp. X_2) is an affinoid subdomain in $\heartsuit(U_1)$ (resp. $\heartsuit(U_2)$) such that $\heartsuit(U_1) \setminus X_1$ (resp. $\heartsuit(U_2) \setminus X_2$) is a finite disjoint union of open annuli. Then, X_1 (resp. X_2) is a heart of a pro-object $X_1^\dagger$ (resp. $X_2^\dagger$) with the curve of definition X_1' (resp. X_2'). We argue as in the previous case to obtain an isomorphism of some open neighborhoods of X_1 and X_2 in X_1' and X_2', respectively, which restricts to an isomorphism between X_1 and X_2. By approximating $\heartsuit(U_1)$ by affinoids X_1 (which in turn induces an approximation of $\heartsuit(U_2)$ by affinoids X_2), arguing as before, and using that U_1 and U_2 are standard, we obtain an isomorphism of open neighborhoods of $\heartsuit(U_1)$ and $\heartsuit(U_2)$, and hence of the pro-objects U_1 and U_2.

At last, we discuss the case where $\heartsuit(U_1) = \emptyset$. In this case, the curve of definition is necessarily $\mathbb{P}^1_k$, see Remark 3.1.10. It is easy to show, using the projective coordinate on $\mathbb{P}^1_k$ that an arbitrary union of a nondecreasing family of closed discs is either an open or closed disc of finite radius, or $\mathbb{P}^1_k$ minus a type 1 or type 4 point. As before, we can write $\heartsuit(U_1) = \mathbb{P}^1_k \setminus \cup_{i=1}^n \cup_t B_t^i$, where B_t^i are closed discs in $\mathbb{P}^1_k$, and for $t > s$, $B_s^i \subset B_t^i$. Suppose first that, (after fixing a projective coordinate on $\mathbb{P}^1_k$) for some $i = 1, \ldots, n$, say $i = 1$, the radii of the family of discs B_t^1 go to infinity, their union is then $\mathbb{P}^1_k \setminus \{x\}$, where x is a type 1 or a type 4 point. This is because the complements form a family of open discs whose radii go to zero, hence their intersection is a type 1 or a type 4 point. Becuase the heart is empty, the complement, *i.e.* the point x, must be equal to the union $\cup_{i=2}^n \cup_t B_t^i$. This is possible only if $n = 2$, x is of type 1 and for every t, $B_t^2 = \{x\}$. After introducing a projective coordinate with $T(x) = 0$, we obtain that U is of the form $" \varprojlim "_\epsilon A(0; 0, \epsilon)$.

Now suppose that we don't have a family $(B_t^i)_t$ whose radii go to infinity in any projective coordinate. Let $B_i = \cup_t B_t^i$. We fix a suitable coordinate and we identify B_1 with $D(0, r)$ or $D(0, r^-)$, depending whether it is an open or closed disc. Assume for the moment that B_1 is an open disc. It follows that $\cup_{i=2}^n \cup_t B_t^i := \uplus_{i=2}^n B_i$ covers the complement which is a closed disc in $\mathbb{P}_k^1$. It is easy to show that a closed or an open disc cannot be nontrivially covered with a finite disjoint union of closed or open discs, hence we conclude that again $n = 2$ and $B_2 = \mathbb{P}_k^1 \setminus B_1$. With a bit more effort and the right choice of coordinates, one can show that in this case the pro-wide open curve U can be put in the form $U = " \varprojlim "_\epsilon A(0; r - \epsilon, r)$. Furthermore, if $r \in |k^\times|$, U is isomorphic to $" \varprojlim "_\epsilon A(0; 1 - \epsilon, 1)$.

We end the theorem with the following lemma.

Lemma 3.1.12. *The pro-objects* $" \varprojlim "_\epsilon A(0; 1 - \epsilon, 1)$, $" \varprojlim "_\epsilon A(0; r - \epsilon, r)$, *where* $r \notin |k|$ *and* $" \varprojlim "_\epsilon A(0; r - \epsilon, r)$ *are not isomorphic pairwise.*

Proof. □

□

Remark 3.1.13. The category $\mathbb{W}$ is very similar to the category of germs of analytic spaces, introduced and studied by Vladimir Berkovich in [4, 7]. However, a germ is defined as a pair (S, X), where X is a k-analytic space and S is a subset of the underlying topological space (similarly as an object in $\mathbb{W}$ with a non-empty heart is defined by its heart and the curve of definition), and as such, it is not suitable for studying pro-objects which have empty hearts.

3.1.14. For a subset $U \subset X$ of a quasi-smooth, connected, projective k-analytic curve X, which is a heart of some pro-wide open, we will write $U^\dagger$ for the corresponding pro-wide open. By a type 1,2,3 or 4 point of a pro-wide open, we mean a type 1,2,3 or 4 point of its heart. The set of rational points in a pro-wide open U is denoted by $U(k)$.

We give special names to the objects that we will mostly work with.

Definition 3.1.15. *Let $U \in \mathbb{W}$, and suppose that $\heartsuit(U)$ is an (strict) affinoid curve (resp. (strict) semiopen affinoid). Then we call such an U a* (strict) dagger affinoid *(resp.* semiopen (strict) dagger affinoid*). If $\heartsuit(U)$ is a curve of definition of U, then we call U a projective curve.*

Remark 3.1.16. A semiopen affinoid curve X is called strict if it can be embedded in a smooth-projective curve X', such that $X' \setminus X$ is a finite union of disjoint strict open and closed discs.

We say that a dagger affinoid $X^{\dagger}$ has a good canonical reduction, if X has a good canonical reduction. In this case when we refer to the maximal *i.e.* Shilov point of $X^{\dagger}$, we mean the Shilov point of X.

Remark 3.1.17. We do not intend to change the classical terminology of dagger affinoids but rather to express how one might want to think about some particular pro-wide open curves: namely, as their name suggests.

Definition 3.1.18. *Suppose that $U \in \mathbb{W}$ and $\heartsuit(U) = \emptyset$. Then, as follows from the previous theorem, U is isomorphic to a pro-wide open of the form $\text{"}\varprojlim\text{"}_{\epsilon} A(0; 1-\epsilon, 1)$ or $\text{"}\varprojlim\text{"}_{\epsilon} A(0; 0, \epsilon)$ or $\text{"}\varprojlim\text{"}_{\epsilon} A(0; r-\epsilon, r)$ for some real number r such that $r \notin |k^{\times}|$. In the first case we call such an U* Robba proannulus of the first type *or simply Robba proannulus, in the second case we say that U is a* Robba proannulus of the second type *while in the third case we say that U is a* Robba proannulus of the third type*.*

Remark 3.1.19. If $U = \text{"}\varprojlim\text{"} U_i \in \mathbb{W}$ and $\heartsuit(U)$ is a wide open, then U is isomorphic to a stable pro-object (meaning that starting from some i on, all U_i are equal), so in fact isomorphic to the constant pro-object $\text{"}\varprojlim\text{"}\heartsuit(U)$. In this case, risking the ambiguity, we will sometimes call U a wide open curve as well.

3.1.20. *Intersections and fiber products.* Let $U = \text{"}\varprojlim\text{"}_t U_t$ and $V = \text{"}\varprojlim\text{"}_t V_t$ be pro-wide open curves, both with the same curve of definition X. Then, $U_t \cap V_t$ is a finite union of disjoint *wide open curves.* Increasing t stabilizes the number of connected components of $U_t \cap V_t$ and we may write $U_t \cap V_t = \uplus_{i=1}^n W_{i,t}$, where $W_{i,t}$ are wide open curves. Then, we define $U \times_X V$ (and also write $U \cap_X V$, or simply $U \cap V$ if X is clear from the context) to be a collection of pro-wide open curves $W_i = \text{"}\varprojlim\text{"}_t W_{i,t}$, $i = 1, \ldots, n$ where X is the curve of definition for all of them. We will still *symbolically* write $U \cap V = \uplus_{i=1}^n W_i$. If $U \cap V = V$, then we say that V *is a sub-pro-wide open curve of* U and write $V \subset U$. For a sub-pro-wide open $V \subset U$, we define $U \setminus V$ as a collection of pro-wide open curves V_i and write $U \setminus V := \uplus V_i$, where V_i are pro-wide open curves whose hearts are in one-to-one correspondence with the connected components of $\heartsuit(U) \setminus \heartsuit(V)$.

More generally, suppose that both U and V can be embedded in some smooth, connected, projective k-analytic curve X. This means that for some t, there exists open embeddings $U_t \hookrightarrow X$ and $V_t \hookrightarrow X$ (X doesn't have to be a curve of definition of U and V). Let $i_U : U \to X$ and $i_V : V \to X$ be the open embeddings. Then, we define $U \times_X V$ to be $i_U(U) \cap i_V(V)$ and call it the fiber product of $i_U : U \to X$ and $i_V : V \to X$ or just the fiber product of U and V in X if i_U and i_V are understood from the context.

Definition 3.1.21. *A finite collection of open embeddings of pro-wide open curves $\mathcal{U} = \{i_i : U_i \to U, i = 1, \ldots, n\}$ is said to be a* finite admissible covering of a pro-wide open U *if $\{\heartsuit(i_i(U_i)), i = 1, \ldots, n\}$ is a topological covering of $\heartsuit(U)$. We say that a finite admissible covering $\mathcal{U}$ of U is* basic, *if the intersections $i_i(U_i) \cap i_j(U_j)$, $i \neq j$ are either empty or Robba proannuli. Basic coverings will be studied in more details later on.*

Theorem 3.1.22. *Category $\mathbb{W}$ together with the assignment to each object $U \in \mathbb{W}$ a set of finite admissible coverings is a site.*

3.1.23. *Presheaf on $\mathbb{W}$.*

We endow $\mathbb{W}$ with a presheaf in the following way. Let $U = \text{"}\varprojlim\text{"} U_i \in \mathbb{W}$ be a pro-wide open curve. Then we assign to U a ring of functions $O(U) := \varinjlim O(U_i)$, where $O(U_i)$

is the ring of analytic functions assigned to a wide open curve in the curve of definition.

For example, suppose that U is a Robba proannulus of type one (resp. two). Then, $O(U)$ is isomorphic to the ring of power series $f(T) = \sum_{i\in\mathbb{Z}} a_i T^i$ s.t. f converges on some open annulus $A(0; 1-\varepsilon, 1)$ (resp. $A(0;\varepsilon, 0)$). Note that the later is equivalent to $\lim_{|i|\to\infty} |a_i| r^i \to 0$, for all $r \in (1-\varepsilon, 1)$ (resp $r \in (0, \varepsilon)$).

If $U \in \mathbb{W}$ is a wide open, then it follows that $O(U) = O(\heartsuit(U))$, the ring of analytic functions on the open set (in the curve of definition) $\heartsuit(U)$.

3.1.24. *Triangulations and formal models of pro-wide open curves.* We may introduce many notions connected to k-analytic curves by simply doing so in a naive way, by taking the corresponding notion for the heart of the pro-wide open curve (when this makes sense). In particular we introduce semistable triangulations, reductions and formal models.

Let $U \in \mathbb{W}$ be a pro-wide open such that $\heartsuit(U)$ is a wide open curve, a semiopen affinoid or an affinoid. We say that a finite set of type two points $\mathcal{T} \subset \heartsuit(U)$ is a (strictly) semistable triangulation of U if $\mathcal{T}$ is a (strictly) semistable triangulation of $\heartsuit(U)$.

Remark 3.1.25. A semistable triangulation for a semiopen affinoid X is introduced analogously as for k-analytic curves. The same goes for X an affinoid which is not strict, except we don't ask that the connected complements of the triangulation to be strict open discs or annuli.

Note that for a strictly semistable triangulation $\mathcal{T}$ of a pro-wide open U, the elements of sets $\mathcal{C}_\mathcal{T}, \mathcal{A}_\mathcal{T}$ and $\mathcal{W}_\mathcal{T}$ (with respect to $\heartsuit(U)$) are naturally hearts of pro-wide open curves with the same curve of definition as for U. We write $\mathcal{C}^\dagger_\mathcal{T} = \{C^\dagger_\xi, \xi \in \mathcal{T}\}$, and similarly $\mathcal{A}^\dagger_\mathcal{T} = \{A^\dagger, A \in \mathcal{A}_\mathcal{T}\}$, $\mathcal{W}^\dagger_\mathcal{T} = \{W^\dagger, W \in \mathcal{W}_\mathcal{T}\}$. However, there are some more interestings sets here whose elements are Robba proannuli coming from the intersection of the elements of $\mathcal{C}^\dagger_\mathcal{T}$ and $\mathcal{A}^\dagger_\mathcal{T}$. They will be of importance for us when later when we will deal with cohomology issues on pro-wide open curves, so we set $\mathcal{R}^\dagger_\xi$ to denote the finite set of Robba proannuli coming from the intersections of $C^\dagger_\xi$ with the elements of $\mathcal{A}^\dagger_\xi$, and we put $\mathcal{R}^\dagger_\mathcal{T} = \{R, R \in \mathcal{R}^\dagger_\xi, \xi \in \mathcal{T}\}$. Finally, we set for $A \in \mathcal{A}_\mathcal{T}$, $\mathcal{R}^\dagger_A$ to be the set of two

Robba proannuli embedded at the "ends" of the proannulus $A^\dagger$ i.e. elements of $\mathcal{R}_A^\dagger$ are intersections of A with the elements of $\mathcal{C}_\mathcal{T}^\dagger$.

Next, we define reduction with respect to a given strictly semistable triangulation $\mathcal{T}$ of a strict dagger affinoid $X^\dagger$ or of a pro-wide open projective curve, as the reduction of its heart with respect to $\mathcal{T}$.

For dagger affinoids and projective curves, we can do more and study their formal models. By formal model of a dagger affinoid we simply mean the formal model of its heart. Then hopefully, the reader will not find bigger difficulties in translating the notation from the Subsection 1.1 by just putting a superfix † for the corresponding elements and sets. Once again, for a formal model $\mathfrak{X}$ of $X^\dagger$, we introduce set $\mathcal{R}_\mathfrak{X}^\dagger$ which is a set of Robba proannuli made by intersecting the elements of $\mathcal{C}_\mathfrak{X}^\dagger$ and $\mathcal{A}_\mathfrak{X}$.

3.1.26. *Tangent space.* For a pro-wide open curve U with a nonempty heart, and a point $x \in U$, we set $T_xU := T_x\heartsuit(U)$. For a dagger affinoid U, we can define as well TU as $T\heartsuit(U)$, where the last set is always taken with respect to the curve of definition of U. Recall that TU is then a (finite) set $\{\vec{t_1}, \dots, \vec{t_n}\}$ of "outer" tangent vectors living at the boundary of the affinoid $\heartsuit(U)$, i.e. they correspond to the connected components of the set $U_t \setminus \heartsuit(U) = \uplus_{i=1}^n A_{i,t}$. Each $A_{i,t}$ is an open annulus, and the collection $\{A_{i,t}\}$ forms a projective system, hence a pro-object $R_{\vec{t_i}} = R_i :=$ "$\varprojlim$"$A_{i,t}$ which is a Robba proannulus. A similar procedure goes for a pro-wide open curve with the heart which is a type 1 or type 4 point, or trivially when U is projective.

However, for a pro-wide open curve U, with the heart which is a wide open or a semiopen affinoid, the situation is a bit different. Namely, let X be the curve of definition of U. Then $X \setminus \heartsuit(U) = \biguplus_{i=1}^n B_i \uplus \biguplus_{j=1}^m D_j$, where B_i are closed and D_j are open disjoint discs. For each $i = 1, \dots, n$, let η_{B_i} be the maximal point of B_i, and let $\vec{t}_{B_i}$ be the unique tangent vector in TB_i. We denote the by $T_{\text{in}}U$ the set of tangent vectors t_{B_i}, $i = 1, \dots, n$ and call it the set of *inner tangent vector*. Note that for each inner tangent vector there is a natural assignment of a Robba proannuli corresponding to it. More precisely, for each

$i = 1, \dots, n$ we put $R_{\vec{t}_{B_i}}$ to denote the Robba proannuli corresponding to the intersection of pro-wide open curves $U \cap B_i^{\dagger}$.

On the other hand, for each $j = 1, \dots, n$, we can apply the previous construction for the open disc D_j to obtain set $T_{\text{in}} D_j$ consisting of only one tangent vector which we denote $\vec{t}_{D_j}$. We put $T_{\text{out}} U := \{\vec{t}_{D_j}, j = 1, \dots, m\}$ and call it the set of *outer tangent vectors* of U. Note as well that to each $\vec{t}_{D_j}$ we can naturally assign a Robba proannulus $R_{\vec{t}_{D_j}}$ corresponding to the intersection of the pro-wide curves U and D_j.

In the end we put $TU := T_{\text{in}} U \cup T_{\text{out}} U$. We note as well that for U a dagger affinoid $T_{\text{in}} U = \emptyset$, and the previous construction coincides with the standard one.

3.1.1 Cohomology on pro-wide open curves

3.1.27. Recall that a quasi-Stein space is an increasing countable union $\cup_n X_n$ of affinoid subdomains which correspond to morphisms of affinoid algebras $i_n : A_{n+1} \to A_n$ such that $i_n(A_{n+1})$ is dense in A_n with respect to the spectral seminorm on A_n. Wide open curves are quasi-Stein spaces, they can indeed be presented as an increasing union of affinoid subdomains, and the condition about morphism of affinoid algebras can be seen for example from the Runge theorem in rigid geometry[38, Corollaire 3.5.1].

In particular, for a coherent $\mathcal{O}_U$-module $\mathcal{F}$ on a wide open curve U, F is acyclic for the Čech resolution and higher cohomology groups $H^i(U, \mathcal{F})$, $i > 0$ vanish (see [29, Satz 2.4]).

Now, let U be a wide open curve and let $\mathcal{F}^*$ be a complex of sheaves of coherent O_U-modules on U (we will always assume that $\mathcal{F}^*$ is zero in negative degrees). Let $\mathbb{H}^i(U, \mathcal{F}^*)$ denote the ith hypercohomology group of the complex $\mathcal{F}^*$. For an inclusion of wide open curves $j : V \hookrightarrow U$, we have a natural morphism of sheaves on U, $\mathcal{F}^* \to j_* \mathcal{F}^*_{|V}$ which induces a natural morphism of the groups $\mathbb{H}^i(U, \mathcal{F}^*) \to \mathbb{H}^i(U, j_* \mathcal{F}^*_{|V})$. The later group can be identified (because both U and V are quasi-Stein) with the group $\mathbb{H}^i(V, \mathcal{F}^*_{|V})$ and we will (ambiguously) also write $\mathbb{H}^i(V, \mathcal{F}^*)$. This allows us to introduce the following definition.

Definition 3.1.28. *Let $U = {}''\varprojlim{}''_t U_t \in \mathbb{W}$. A complex of sheaves $\mathcal{F}^*$ on U is any complex of sheaves of coherent $\mathcal{O}_{U_t}$-modules on some wide open U_t. Furthermore, we define $\mathbb{H}^i(U, \mathcal{F}^*) := \varinjlim_t \mathbb{H}^i(U_t, \mathcal{F}^*)$ and call it i*th hypercohomology of the complex $\mathcal{F}^*$ on U.

To study hypercohomology on pro-wide open curves, we begin with the following simple, but useful observation in the form of Mayer-Vietoris sequence on wide open curves which then easily transfers to the Mayer-Vietoris sequence of pro-wide open curves.

Lemma 3.1.29. *Let U be a wide open and let $\mathcal{V} = \{V_i\}_{i \in I}$ be a finite covering of U by wide open curves s.t. $V_i \cap V_j \cap V_k = \emptyset$ whenever $i \neq j \neq k \neq i$. For a subset $J \subseteq I$, let us denote by $V_J = \cap_{j \in J} V_j$ and by $|J|$ the cardinality of J. Then we have a long exact sequence*

(3.1.29.1)

$$0 \to \mathbb{H}^0(U, \mathcal{F}^*) \to \bigoplus_{J \subset I, |J|=1} \mathbb{H}^0(V_J, \mathcal{F}^*_{|V_J}) \to \bigoplus_{J \subset I, |J|=2} \mathbb{H}^0(V_J, \mathcal{F}^*_{|V_J}) \to \mathbb{H}^1(U, \mathcal{F}^*) \to \cdots .$$

Proof. For, $J \subset I$, let $j_J : V_J \to U$ be the inclusion of V_J into U and let $\mathcal{F}^*_J = j_{J*}(\mathcal{F}^*_{|V_J})$. First we observe that the sequence of complexes $0 \to \mathcal{F}^* \to \oplus_{|J|=1} \mathcal{F}^*_J \to \oplus_{|J|=2} \mathcal{F}^*_J \to 0$ is exact, as for each $i \geq 0$, $0 \to \mathcal{F}^i \to \oplus_{|J|=1} \mathcal{F}^i_J \oplus_{|J|=2} \mathcal{F}^i_J \to 0$ is a Čech resolution of $\mathcal{F}^i$ with respect to covering $\mathcal{V}$. Then, such an exact sequence of complexes induces a long exact sequence of the form

$$0 \to \mathbb{H}^0(U, \mathcal{F}^*) \to \bigoplus_{J \subset I, |J|=1} \mathbb{H}^0(U, \mathcal{F}^*_J) \to \bigoplus_{J \subset I, |J|=2} \mathbb{H}^0(U, \mathcal{F}^*_J) \to \mathbb{H}^1(U, \mathcal{F}^*) \to \cdots .$$

We end the proof by noticing that $\mathbb{H}^l(U, \mathcal{F}^*_J) = \mathbb{H}^l(V_J, \mathcal{F}^*_{|V_J})$ because of the reasons we mentioned above. $\square$

The following theorem is inspired and based on [26].

Theorem 3.1.30. *Let $\mathcal{V} = \{V_i\}_{i\in I}$ be a finite admissible covering of a pro-wide open U given by sub-pro-wide open curves such that any three distinct elements of $\mathcal{U}$ have an empty intersection and let us put $\mathcal{V}' := \{V_i \cap V_j,\ i \neq j, \quad i,j \in I\}$. Let $\mathcal{F}^*$ be a complex of sheaves on U. Then, we have a long exact sequence*

$$0 \to \mathbb{H}^0(U,\mathcal{F}^*) \to \bigoplus_{V\in\mathcal{V}} \mathbb{H}^0(V,\mathcal{F}^*_{|V}) \to \bigoplus_{V\in\mathcal{V}'} \mathbb{H}^0(V,\mathcal{F}^*_{|V}) \to \mathbb{H}^1(U,\mathcal{F}^*) \to \cdots . \tag{3.1.30.1}$$

Proof. Let us write $\mathcal{V}' = \{W_l\}_{l\in L}$. Since we have a finite amount of data, to simplify the notation we can use the same indexing set T for all the pro-wide open curves involved (see Remark 3.1.4) and by reindexing if needed we can assume that for $t \in T$ and a wide open U_t ($U = \text{"}\varprojlim\text{"}_t U_t$), we can find for each $i \in I$ a wide open $V_{i,t}$ ($V_i = \text{"}\varprojlim\text{"}_t V_{i,t}$) s.t. $\mathcal{V}_T = \{V_{i,t}, i \in I\}$ is a covering of U_t satisfying the conditions of Lemma 3.1.29 and s.t. for each element $W_l \in \mathcal{V}'$ ($W_l = \text{"}\varprojlim\text{"}_t W_{l,t}$), $W_{l,t} = V_{i,t} \cap V_{j,t}$ for some $i,j \in I$ $i \neq j$. Lemma 3.1.29 then implies

$$0 \to \mathbb{H}^0(U_t,\mathcal{F}^*) \to \bigoplus_{i\in I} \mathbb{H}^0(V_{i,t},\mathcal{F}^*_{|V_{i,t}}) \to \bigoplus_{l\in L} \mathbb{H}^0(W_{l,t},\mathcal{F}^*_{|W_{l,t}}) \to \mathbb{H}^1(U_t,\mathcal{F}^*) \to \cdots .$$

Since inductive limit is an exact functor in the category of modules, by taking $\varinjlim_t$ we obtain the required result. $\square$

Definition 3.1.31. *Suppose U is a pro-wide open, and let $\mathcal{F}^*$ be a complex of sheaves of coherent O_U-modules on U. If there are finitely many $s \geq 0$ s.t. $\mathbb{H}^s(U,\mathcal{F}^*) \neq 0$ and if for all such s, $\mathbb{H}^s(U,\mathcal{F}^*)$ is a finite dimensional k-vector space, then the quantity $\chi(U,\mathcal{F}^*) = \sum_{s\geq 0}(-1)^s \dim_k \mathbb{H}^s(U,\mathcal{F}^*)$ is well defined and will be called* Euler-Poincaré characteristic of U with coefficients in $\mathcal{F}^*$.

Corollary 3.1.32. *Keep the notation from the Theorem 3.1.30. Suppose that the long exact sequence* (3.1.30.1) *is finite and all the groups in it are finite dimensional vector* *spaces over k. Then,*

$$\chi(U,\mathcal{F}^*) = \sum_{i\in I}\chi(V_i,\mathcal{F}^*_{V_i}) - \sum_{l\in L}\chi(W_l,\mathcal{F}^*_{|W_l}). \tag{3.1.32.1}$$

Proof. Under the assumptions of the corollary, we can take the alternating sum of the dimensions of the groups in (3.1.30.1) which is 0. The corollary follows. □

3.1.33. *De Rham cohomology.*

Definition 3.1.34. *For a pro-wide open $U = \text{"}\varprojlim\text{"}U_t$ we define $\mathrm{H}^i_{dR}(U)$ as $\mathbb{H}^i(U,\Omega^*_U)$, where Ω^*_U is the de Rham complex on a wide open curve U (recall that this means that Ω^*_U is in fact the de Rham complex on some wide open U_t). Furthermore, if all k-vector spaces $\mathrm{H}^i_{dR}(U), i \geq 0$ are finite dimensional (as it turns out in Theorem 3.1.39, they are), we set $\chi(U) = \sum_{i\geq 0}(-1)^i \dim_k \mathrm{H}^i_{dR}(U)$ and call $\chi(U)$ the Euler-Poincare characteristic of U.*

Remark 3.1.35. A priori, de Rham cohomology groups depend on the chosen U_t from the system defining $U \in \mathbb{W}$. However, it follows from the very definition that this is not the case.

Remark 3.1.36. Note that when $U \in \mathbb{W}$ is a projective curve, the $\mathrm{H}^i_{dR}(U)$ coincides with the algebraic de Rham cohomology of the curve U (by GAGA). Furthermore, if U is a proper wide open, then $\mathrm{H}^1_{dR}(U)$ is isomorphic to the group $\Omega^1(U)/dO(U)$. This is because in this case U is a quasi-Stein space, hence all the higher sheaf cohomology groups for a coherent sheaf vanish, which implies the degeneration of the spectral sequence (used to compute hypercohomology) at the first sheet.

Remark 3.1.37. In [35], a notion of overconvergent presentation is introduced. For an affinoid X with corresponding affinoid algebra A, the overconvergent presentation (φ, A) (*loc.cit.* , Definition on page 224.) coincides (up to an isomorphism) with $O(X^\dagger)$. Then, following *loc.cit.* , we can introduce the module $\Omega^1(X^\dagger)$ of continuous differentials of $O(X^\dagger)$, which coincides with $\Omega^1(\varphi, A)^\dagger$ in *loc.cit.* . Accordingly, we can define de Rham complex which depends on the curve of definition, but however the corresponding de Rham cohomology groups don't and they are isomorphic to the groups in 3.1.34.

Remark 3.1.38. When U is a dagger affinoid, our de Rham cohomology groups are isomorphic to the de Rham cohomology groups for dagger spaces introduced by Grosse-Klönne ([25]).

Theorem 3.1.39. *Let $U = "\varprojlim"_t U_t \in \mathbb{W}$ and let X be a curve of definition of U. Assume that $X \setminus \heartsuit(U) \neq \emptyset$. Then $X \setminus U_t = \uplus_{i=1}^n D_i$ is a disjoint union of closed discs. For any disc D_i, let us choose a rational point $x_i \in D_i(k)$ and put $S = \{x_1, \dots, x_n\}$. Let $\mathcal{H}^1_{dr}(X \setminus S)$ be algebraic de Rham cohomology of the curve $X \setminus S$. Then,*

(i) (Compare with [14, Theorem 4.2], and [35, Corollary 2.7]) $\mathrm{H}^1_{dR}(U)$ *is finite dimensional over k and the natural restriction map $\mathcal{H}^1_{dR}(X \setminus S) \to \mathrm{H}^1_{dR}(U)$ is an isomorphism;*

(ii) (Compare with [35, Theorem 2.1]) *The de Rham cohomology groups of U are:* $\mathrm{H}^0_{dR}(U) \cong k$, $\mathrm{H}^1_{dR}(U) \cong k^{2g-1+n}$ *and* $\mathrm{H}^i_{dR}(U) = 0$, $i > 1$, *where g is the genus of X.*

Proof. (i) Due to Grothendieck, we have a natural isomorphism of groups $\mathcal{H}^i_{dR}(X \setminus S)$ and $\mathrm{H}^i_{dR}(X \setminus S)$ (where in the later group, $X \setminus S$ is seen as an analytic curve, or a pro-wide open curve, as it doesn't change the group itself), so we we will prove that the natural map from $\mathrm{H}^i_{dR}(X \setminus S)$ to $\mathrm{H}^i_{dR}(U)$ is an isomorphism. Consider the finite admissible covering of $(X \setminus S)^\dagger$ given by U and $\{U_i\}_{i=1}^n$, where U_i's are pro-wide open curves whose hearts are connected components of $(X \setminus S) \setminus \heartsuit(U)$. Two distinct elements U_i and U_j have an empty intersection, and intersection $U_i \cap U$ is a Robba proannulus denoted by W_i. Since each of the pro-wide open curves $(X \setminus U)^\dagger, U, U_i, W_i$, $i = 1, \dots, n$ has $\mathrm{H}^0_{dR} = k$ because the kernel of the derivation on any wide open curve is 0, the long exact sequence (3.1.30.1) splits for dimension reasons into short exact sequences $0 \to \mathrm{H}^0_{dR}(X \setminus S) \to \mathrm{H}^0_{dR}(U) \oplus_{i=1}^n \mathrm{H}^0_{dR}(W_i) \to \oplus_{i=1}^n \mathrm{H}^0_{dR}(W_i) \to 0$ and $0 \to \mathrm{H}^1_{dR}(X \setminus S) \to \mathrm{H}^1_{dR}(U) \oplus_{i=1}^n \mathrm{H}^1_{dR}(W_i) \to \oplus_{i=1}^n \mathrm{H}^1_{dR}(W_i) \to 0$. From the later sequence we conclude that $\mathrm{H}^1_{dR}(U)$ has a finite dimension over k since

$\mathrm{H}^1_{dR}(W_i) = k$, and that the natural morphism $\mathrm{H}^1_{dR}(X\backslash S) \to \mathrm{H}^1_{dR}(U)$ is injective. But, from Corollary 3.1.32.1 we have $\chi(X \setminus S) = \chi(U)$, which implies that the injective morphism $\mathrm{H}^1_{dR}(X \setminus S) \to \mathrm{H}^1_{dR}(U)$ is in fact an isomorphism.

(ii) We could use (i) but we give a slightly different argument which at the same time proves finite dimensionality of $\mathrm{H}^i_{dR}(X \setminus S)$. Take a finite admissible covering of X given by U and pro-wide open curves U_i, $i = 1, \dots, n$ whose hearts correspond to connected components of $X \setminus \heartsuit(U)$ and set $W_i = U \cap U_i$ (note that $\heartsuit(U_i)$ is either a point, open disc or a closed disc and that W_i is a Robba proannulus). Then $\chi(U_i) = 1$ and the $\chi(W_i) = 0$ for every i, so (3.1.30.1) gives us $\chi(U) = \chi(X) - n$. Once again, using that $\mathrm{H}^0_{dR}(U) = \mathrm{H}^0_{dR}(X) = k$, $\mathrm{H}^1_{dR}(X) = k^{2g}$ and $\mathrm{H}^2_{dR}(X) = k$ yields the result. □

Corollary 3.1.40. *Let $U \hookrightarrow V$ be an open embedding of pro-wide open curves s.t. $\heartsuit(V) \setminus \heartsuit(U)$ is a finite disjoint union of semi-open, open or closed annuli and Robba proannuli. Then, the natural restriction $H^1_{dR}(V) \to H^1_{dR}(U)$ is an isomorphism.*

Remark 3.1.41. The previous corollary in fact tells us that "chopping out" finitely many Robba proannuli or annuli such that the remaining part is still connected, doesn't affect the cohomology and in particular Euler-Poincaré characteristic of a pro-wide open curve.

3.1.42. *Basic coverings and genus*

Putting $\mathcal{F}^* = \Omega^*_U$ as in 3.1.30 and using that $\chi(W) = 0$ for a Robba proannulus W, we have the following:

Corollary 3.1.43. *Let U be a pro-wide open and $\mathcal{U}$ be a basic covering of U. Then*

$$\chi(U) = \sum_{V \in \mathcal{U}} \chi(V).$$

When U is a dagger affinoid or a projective curve, basic coverings can naturally be assigned to a given strictly semistable triangulation $\mathcal{T}$. Namely, elements of the sets $\mathcal{C}^\dagger_{\mathcal{T}}$ and $\mathcal{A}^\dagger_{\mathcal{T}}$ form a basic covering of U.

Definition 3.1.44. *Let U be a* standard *pro-wide open curve with the curve of definition X. We define* the genus of U, *$g(U)$ to be the genus of the curve X. We also set $\delta(U)$ to denote the number of connected components of $X \setminus \heartsuit(U)$ and call it* defect*.*

It follows from the Theorem 3.1.39 that the defect is in fact an invariant for a pro-wide open, i.e. isomorphic pro-wide open curves have the same defect.

As a direct application of the Corollary 3.1.43 (using $\chi(U) = 2 - 2g(U) - \delta(U)$, for a pro-wide open U, as is computed directly from Theorem 3.1.39), we deduce the genus formula.

Theorem 3.1.45. (The genus formula) *Let U be a dagger affinoid or a projective curve, and let $\mathcal{U} = \{U_i\}_{i=1}^n$ be a basic covering of U. Then,*

$$g(U) = \sum_{i=1}^{n} g(U_i) + \frac{1}{2}(\sum_{i=1}^{n} \delta(U_i) - \delta(U)) - (n-1). \tag{3.1.45.1}$$

We end this section noting that in the case when U is projective, i.e. when $\delta(U) = 0$, then the sum $\frac{1}{2}(\sum_{i=1}^{n} \delta(U_i) - \delta(U)) + (n-1)$ is the Betti number $\beta(\Gamma_{\mathcal{U}})$ of the graph $\Gamma_{\mathcal{U}}$ associated to the covering $\mathcal{U}$ (the graph $\Gamma_{\mathcal{U}}$ is formed by taking its vertices to be in one-one correspondence with elements of $\mathcal{U}$, and edges correspond to the intersections of the elements in $\mathcal{U}$).

Chapter 4

Riemann-Hurwitz formula

4.1 Generalized Riemann-Hurwitz formula

4.1.1 Finite morphisms of pro-wide open curves

Definition 4.1.1. *We say that a morphism of pro-wide open curves $\varphi : U \to V$ is finite, if there exists indexing of U and V, $U = " \varprojlim "_t U_t$ and $V = " \varprojlim "_t V_t$, s.t. φ can be represented by $\varphi_t : U_t \to V_t$, and s.t. φ_t is is a finite morphism of wide open curves. We say that φ is finite étale if φ_t is finite and étale. As a finite morphism of smooth curves is flat, it has a well defined degree as long as the target is non-empty and connected. So we can define the degree of φ, $\deg(\varphi)$, to be the degree of some (and hence all) φ_t. A point $x \in U$ is said to be ramified if it is a ramification point of some φ_t. Similarly we define branching points.*

4.1.2. *Finite morphisms of dagger affinoids and pro-wide projective curves*

Lemma 4.1.3. *Let $\varphi : Y^\dagger \to X^\dagger$ be a finite morphism of dagger affinoids. Then, φ induces a finite morphism $\varphi_\heartsuit : Y \to X$.*

Proof. Let $\varphi_t : Y_t \to X_t$ be a level t representation of φ, and let us fix some t_0 from the index set. Then, for all $t > t_0$, $\varphi_t = \varphi_{t_0 | Y_t}$, so we will write φ_{t_0} instead of φ_t and assume

$t > t_0$. Since $\varphi_{t_0}^{-1}(X_t) = Y_t$, it follows that $\varphi_{t_0}^{-1}(X) = Y$ and since φ_{t_0} is finite, we are done. □

Definition 4.1.4. *The ramification index of a point $x \in U$ is the ramification index of the point x with respect to the induced morphism $\varphi_{\heartsuit}$.*

Remark 4.1.5. In fact, a stronger statement is true. Namely, every morphism $\varphi : Y \to X$ compactifies (see [22, Proposition 2.4]) in the sense that it extends to a finite morphism $\varphi' : Y' \to X'$, where Y' and X' are smooth, connected, projective curves containing Y and X, respectively. Compactification implies that we have a finite morphism of some open neighborhoods of Y and X, isomorphic to wide open curves and this in turn implies that we also have a finite morphism between our pro-wide open curves $Y^\dagger$ and $X^\dagger$.

The previous lemma and remark give us the proof of the following result.

Theorem 4.1.6. *The category of quasi-smooth, compact, connected, k-analytic curves with finite morphisms is equivalent to the category of dagger k-affinoids and pro-wide open projective curves with finite morphisms. The equivalence functor is given by $X \mapsto \heartsuit(X)$ and $\varphi : Y \to X \mapsto \varphi_{\heartsuit} : \heartsuit(Y) \to \heartsuit(X)$.*

Now we can translate Theorem 2.1.5 and Corollary 2.1.32 in the setting of dagger affinoids. For the convenience, we do so only for the Corollary 2.1.32.

Corollary 4.1.7. *Let $\varphi : Y \to X$ be a finite morphism between dagger affinoids or pro-wide projective curves. Then, there exist coverings of Y and X, by the elements of the sets $\mathscr{A}_Y^\dagger, \mathscr{B}_Y^\dagger, \mathscr{C}_Y^\dagger$ and $\mathscr{A}_X^\dagger, \mathscr{B}_X^\dagger, \mathscr{C}_X^\dagger$, respectively, where $\mathscr{A}_Y^\dagger$ (resp. $\mathscr{A}_X^\dagger$) is a finite set of disjoint pro-wide open annuli, $\mathscr{B}_Y^\dagger$ (resp. $\mathscr{B}_X^\dagger$) is a finite set of disjoint pro-wide open discs and $\mathscr{C}_Y^\dagger$ (resp. $\mathscr{C}_X^\dagger$) is a finite set of disjoint dagger affinoids with good canonical reduction s.t.*

(i) Both coverings are basic

(ii) $\mathscr{A}_X^\dagger = \{\varphi(A), A \in \mathscr{A}_Y^\dagger\}$, $\mathscr{B}_X^\dagger = \{\varphi(B), B \in \mathscr{B}_Y^\dagger\}$ and $\mathscr{C}_X^\dagger = \{\varphi(C), C \in \mathscr{C}_Y^\dagger\}$ and for all $A \in \mathscr{A}_X^\dagger$ (resp. $B \in \mathscr{B}_X^\dagger$, resp. $C \in \mathscr{C}_X^\dagger$) $\varphi^{-1}(A)$ is a disjoint union of elements

in $\mathscr{A}_Y^\dagger$ (resp. $\mathscr{B}_Y^\dagger$, resp. $\mathscr{C}_Y^\dagger$) (for a pro-wide open U, $\varphi^{-1}(U)$ is defined as the union of pro-wide open curves whose hearts correspond to the connected components of $\varphi_\heartsuit^{-1}(\heartsuit(U))$, where $\varphi_\heartsuit$ is the induced morphism from Lemma 4.1.3)

(iii) For each $C^\dagger \in \mathscr{C}_Y^\dagger$, φ restricts to a finite, étale morphism $\varphi : C^\dagger \to \varphi(C^\dagger)$ of dagger affinoids

(iv) For each $A \in \mathscr{A}_Y^\dagger$, φ restricts to a finite, étale morphism $\varphi : A \to \varphi(A)$ of pro-wide open annuli

(v) For each $B \in \mathscr{B}_Y^\dagger$, φ restricts to a finite morphism $\varphi : B \to \varphi(B)$ of pro-wide open discs.

(vi) Let $\mathscr{R}_Y^\dagger$ (resp. $\mathscr{R}_X^\dagger$) be the set of Robba proannuli coming from the basic covering of Y (resp. X). Then for each $R \in \mathscr{R}_Y^\dagger$, $\varphi(R) \in \mathscr{R}_X^\dagger$, φ restricts to a finite étale morphism $\varphi : R \to \varphi(R)$, and for each $R \in \mathscr{R}_X^\dagger$, $\varphi^{-1}(R)$ is a disjoint union of elements in $\mathscr{R}_Y^\dagger$.

Proof. The only new part is the claim (vi). Let $f : A_1 := A(0; r, 1) \to A_2 := A(0; r^n, 1)$ be a general finite étale map of open annuli expressed in corresponding coordinates as $S = f(T) = T^n(1 + h(T))$, with $|h(T)|_{(r,1)} < 1$. Then, as T^n is the dominating term, it is easy to show with the help of Newton polygons that $f^{-1}(A(0, \rho, 1)) = A(0; \rho^{1/n}, 1)$ for all $\rho \in (r^n, 1)$, which means that f induces a finite morphism $f : A(0; r_1, 1) \to A(0; r_1^n, 1)$, for all $r_1 \in (r, 1)$, which in turn induces a finite morphism of Robba proannuli "$\varprojlim_\epsilon$"$A(0, 1-\epsilon, 1) \to$ "$\varprojlim_\epsilon$"$A(0, 1-\epsilon, 1)$. From this we deduce all the claims in (vi). □

4.1.8. *Finite morphisms of Robba proannuli* Let $\varphi :$ "$\varprojlim_t$"$A_t \to$ "$\varprojlim_t$"A_t, where $A_t = A(0; 1-t, 1)$, be a finite étale morphism of Robba proannuli. Then, it induces a finite étale morphism at some sheet $\varphi_t : A_t \to A_t$, so we have a coordinate representation as well as the terms $d(\varphi_t), \sigma(\varphi_t), \nu(\varphi_t)$ and $\epsilon(\varphi_t)$. As they don't depend on t, we will write $d(\varphi), \sigma(\varphi), \nu(\varphi)$ and $\epsilon(\varphi)$.

If $f : A_1 = A(0; r, 1) \to A_2 = A(0; r^n, 1)$ is a finite étale morphism of open annuli, then we have naturally two induced isomorphisms of Robba proannuli living at the "ends" of A_1 and A_2. The relation among the parameters σ, ν, ϵ will be the same as in Lemma 2.2.1, so the Lemma naturally translates to the setting of finite étale morphism of Robba proannuli. We choose not to repeat the lemma here, hoping that the its formulation in the context of Robba proannuli is clear.

4.1.2 Riemann-Hurwitz formula for compact k-analytic curves

We will use the following reformulation of the classical Riemann-Hurwitz formula.

Theorem 4.1.9. *Let $\varphi : Y \to X$ be a finite morphism of projective, quasi-smooth, connected, (pro-wide open) k-analytic curves. Let $\mathcal{S}$ and $\mathcal{T}$ be strictly φ-compatible triangulations of Y and X, respectively, such that the elements of $\mathcal{A}_S$ don't contain rational ramified points. Then*

$$\chi(Y) = \deg(\varphi)\chi(X) - \sum_{y \in \mathcal{S}} \sum_{\vec{t} \in T_y Y} \sigma(\vec{t}).$$

Proof. We start with the classical Riemann-Hurwitz formula for φ which gives us $\chi(Y) = \deg(\varphi)\chi(X) - \sum_{P \in Y(k)}(e_p - 1)$, where e_p is the ramification index at the rational point $P \in Y(k)$. We note also that under the assumptions in the theorem, all the ramified points are contained in the open discs attached to points $y \in \mathcal{S}$ and having a nonempty intersection with $\Gamma_{\mathcal{S}}^Y$. Let D be one such a disc. By Remark 2.1.15, the restriction $\varphi_{|D} : D \to \varphi(D)$ is a finite morphism of open discs where $\varphi(D)$ is attached to $\Gamma_{\mathcal{T}}^X$. We finally invoke Lemma 2.2.3 to conclude the proof. □

Theorem 4.1.10. *Let $\varphi : Y^\dagger \to X^\dagger$ be a finite morphism of compact, connected, quasi-smooth pro-wide open k-analytic curves. Then*

$$\chi(Y^\dagger) = \deg(\varphi) \cdot \chi(X^\dagger) - \sum_{\vec{t} \in TY^\dagger} \nu_{\vec{t}} - \sum_{P \in Y^\dagger(k)} (e_P - 1) \tag{4.1.10.1}$$

Before proving the theorem, we consider a special case.

Lemma 4.1.11. *Let $\varphi : Y^\dagger \to X^\dagger$ be as in theorem 4.1.10 and assume that $Y^\dagger$ (hence also $X^\dagger$) has good canonical reduction. Then, formula* (4.1.10.1) *is true.*

Proof. First we show that if (4.1.10.1) is valid for finite étale morphisms, then it is valid for finite morphisms. Indeed, let $D_j^{X^\dagger}, j = 1, \dots, n$ be a finite number of disjoint maximal open discs in $X^\dagger$ containing the rational branching locus of φ and let $\uplus_{i=1}^m D^{Y_i^\dagger} = \varphi^{-1}(\cup_{j=1}^n D^{X_j^\dagger})$, where $D^{Y_i^\dagger}, i = 1, \dots, m$ are disjoint maximal open discs in $Y^\dagger$. Then φ restricts to a finite étale morphism

$$\varphi : Y^\dagger \setminus \uplus_{i=1}^m D^{Y_i^\dagger} \to X^\dagger \setminus \cup_{j=1}^n D^{X_j^\dagger}, \tag{4.1.11.1}$$

and the following are equivalent

$$\begin{aligned}
&\chi\big(Y^\dagger \setminus \uplus_{i=1}^m D^{Y_i^\dagger}\big) = \deg(\varphi)\cdot\chi\big(X^\dagger \setminus \cup_{j=1}^n D^{X_j^\dagger}\big) - \sum_{\vec{t}\in T\left(Y^\dagger\setminus \uplus_{i=1}^m D_i^Y\right)} \nu(\vec{t})\\
&\iff \chi(Y^\dagger) = \deg(\varphi)\cdot\chi(X^\dagger) + (m - \deg(\varphi)n) - \sum_{i=1}^m \nu(\vec{t}_i) - \sum_{\vec{t}\in TY^\dagger}\nu(\vec{t})\\
&\iff \chi(Y^\dagger) = \deg(\varphi)\cdot\chi(X^\dagger) - \sum_{i=1}^m (d(\vec{t}_i) - 1) - \sum_{i=1}^m \nu(\vec{t}_i) - \sum_{\vec{t}\in TY^\dagger}\nu(\vec{t})\\
&\iff \chi(Y^\dagger) = \deg(\varphi)\cdot\chi(X^\dagger) - \sum_{i=1}^m \sigma(\vec{t}_i) - \sum_{\vec{t}\in TY^\dagger}\nu(\vec{t})\\
&\iff \chi(Y^\dagger) = \deg(\varphi)\cdot\chi(X^\dagger) - \sum_{P\in Y^\dagger(k)} (e_P - 1) - \sum_{\vec{t}\in TY^\dagger}\nu(\vec{t}),
\end{aligned} \tag{4.1.11.2}$$

where in the last implication we used Lemma 2.2.3.

Next we show that we can assume that $X^\dagger$ is a dagger affinoid isomorphic to $Z^\dagger = \mathbb{P}_k^1 \setminus \uplus_{i=1}^n D(c_i, 1^-)$, where $D(c_i, 1^-)$, $i = 1, \dots, n$ are disjoint open unit discs in the wide-open projective line $\mathbb{P}_k^1$, one of which contains ∞. For this we first note that there exists a finite morphism $\psi : X^\dagger \to D^\dagger(0, 1^+)$ (= dagger closed unit disc) because there exists a finite morphism $X \to D(0, 1^+)$. Next, we take away from $Y^\dagger, X^\dagger$ and $D(0, 1^+)$ finitely many maximal open discs such that ψ becomes étale morphism between the remaining spaces. So we have morphisms $\varphi : Y^\dagger \to X^\dagger$ and $\psi : X^\dagger \to Z^\dagger$ where φ and ψ are finite étale. Lemma 2.2.1(b) implies that if (4.1.10.1) holds for any two of the morphisms $\psi \circ \varphi$, φ, ψ, then it must hold for the third one. Therefore, we may and will assume that $\varphi : Y^\dagger \to X^\dagger$ is a finite étale morphism where $X^\dagger$ is isomorphic to $\mathbb{P}^1_k \setminus \uplus_{i=1}^n D(c_i, 1^-)$.

Let Y' be a smooth projective curve s.t. $Y' \setminus \heartsuit(Y^\dagger) = \uplus_{i=1}^l D_i^Y$, where D_i^Y, $i = 1, \ldots, l$ are disjoint (unit) open discs, like in Theorem 1.1 in [36]. For each $i = 1, \ldots, l$, chose a point $y_i \in D_i^Y(k)$. By Runge theorem in rigid geometry (Corollaire 3.5.2. [38]) the set of rational functions on Y' with poles at most in the points y_i is dense in the ring of analytic functions on Y (w.r.t spectral norm, which in our case coincide with the norm $|\cdot|_\eta$) and hence in the ring of functions in $O(Y^\dagger)$. Suppose that φ can be approximated with a rational function f (recall that φ comes from a function φ_t defined on some open wide Y_t containing $\heartsuit(Y^\dagger)$) s.t. for each $\vec{t} \in T_\eta Y^\dagger$ we have $\sigma(\varphi, \vec{t}) = \sigma(f, \vec{t})$, where η is the maximal point of $Y^\dagger$. Then, f defines a finite covering of the projective line $\mathbb{P}^1_k$, which is possibly ramified only in the points of D_i^Y, $i = 1, \ldots, l$, so we have Riemann-Hurwitz formula 4.1.9 (note that $\deg(f) = \deg(\varphi)$ as long as $|f - \varphi|_\eta < 1$)

$$\chi(Y') = \deg(\varphi)\chi(\mathbb{P}^1_k) - \sum_{\vec{t} \in TY^\dagger} (\sigma(\varphi, \vec{t})).$$

from which we can deduce (4.1.10.1) arguing like in (4.1.11.2). We will now exhibit a sufficient approximation of φ by a rational function f.

For a point $\vec{t} \in T_\eta Y^\dagger$ and the corresponding residual class $R_{\vec{t}}$ (recall that $R_{\vec{t}}$ is either an open disc or a pro-annulus depending on whether $\vec{t} \in TY^\dagger$ or not) we choose a coordinate $T_{\vec{t}}$ and similarly we choose a coordinate $S_{\varphi(\vec{t})} = S_{\vec{v}}$ for the residual class $R_{\varphi(\vec{t})} = \varphi(R_{\vec{t}})$ attached to the maximal point ξ of $X^\dagger$ (in fact ξ is the Gauss point of $\mathbb{P}^1_k$). We then may write $S_{\vec{v}} = \varphi_\#(T_{\vec{t}}) = T_{\vec{t}}^{d_{\vec{t}}} \cdot h(T_{\vec{t}})$ and $\frac{dS_{\vec{v}}}{dT_{\vec{t}}} = \epsilon_{\vec{t}} \cdot T_{\vec{t}}^{\sigma_{\vec{t}}} \cdot g(T_{\vec{t}})$. Let us suppose now that $R_{\vec{t}}$ is an open disc. In this case, since φ is étale, $\sigma(\vec{t}) = 0$, hence $\frac{dS_{\vec{v}}}{dT_{\vec{t}}}$ has constant norm $|\epsilon(\vec{t})|$ all over $R_{\vec{t}}$. In Lemma 2.2.1(i), we proved that $|\epsilon(\vec{t})| \geq |d(\varphi, \vec{t})|$, and since $d(\varphi, \vec{t}) \in \{1, \ldots, d\}$, there exists a global constant $c_1 \in \mathbb{R}_{>0}$ s.t. for all $\vec{t} \in T_\eta Y^\dagger$, $|\epsilon(\vec{t})| > c_1$. On the other hand, for $\vec{t} \in TY^\dagger$, we have the equality $|\frac{dS_{\vec{v}}}{dT_{\vec{t}}}|_{\vec{t},\rho} = |\epsilon|\rho^{\sigma(\vec{t})}$, so as long as we fix some $\rho_0 \in (0,1)$ and big enough and only allow $\rho \in (\rho_0, 1)$, we can bound $|\frac{dS_{\vec{v}}}{dT_{\vec{t}}}|_{\vec{t},\rho}$ from below with some constant $c_2 \in \mathbb{R}_{>0}$ not depending on $\vec{t} \in TY^\dagger$. In final conclusion, there exists a positive constant $C \in \mathbb{R}_{>0}$ s.t. for all $\vec{t} \in T_\eta Y^\dagger$ and ρ close to 1, $|\frac{dS_{\vec{v}}}{dT_{\vec{t}}}|_{\vec{t},\rho} > C$. Finally we prove that a rational function f, s.t. $|f - \varphi|_\eta < \rho_0 C$ satisfies our needs from the previous paragraph.

Indeed, for a vector $\vec{t} \in T_\eta Y^\dagger$, we have (in the second inequality we use Lemma 2.2.6) $|\frac{d}{dT_{\vec{t}}} f - \frac{d}{dT_{\vec{t}}}\varphi|_{\vec{t},\rho} \leq |\frac{d}{dT_{\vec{t}}}|_\rho |f-\varphi|_{\vec{t},\rho} \leq \rho^{-1}\rho_0 C \leq C < |\frac{d}{dT_{\vec{t}}}\varphi|_{\vec{t},\rho}$, hence $|\frac{d}{dT_{\vec{t}}} f|_{\vec{t},\rho} = |\frac{d}{dT_{\vec{t}}}\varphi|_{\vec{t},\rho}$, which implies $\sigma(f,\vec{t}) = \sigma(\varphi,\vec{t})$, which finishes the proof of lemma.

□

Proof of theorem 4.1.10. If $Y^\dagger$ is projective, then so is $X^\dagger$, and (4.1.10.1) is then classic. So let us assume that $Y^\dagger$ is a dagger affinoid.

Let $\mathcal{S}$ and $\mathcal{T}$ be compatible triangulations of $Y^\dagger$ and $X^\dagger$, respectively. Corollary 3.1.43 implies that

$$\chi(Y^\dagger) = \sum_{C \in \mathcal{C}^\dagger_{\mathcal{S}}} \chi(C) \quad \text{and} \quad \chi(X^\dagger) = \sum_{C \in \mathcal{C}^\dagger_{\mathcal{T}}} \chi(C) = \sum_{C \in \mathcal{C}^\dagger_{\mathcal{S}}} \chi(\varphi(C)).$$

Previous lemma implies that for all $C \in \mathcal{C}^\dagger_{\mathcal{S}}$,

$$\chi(C) = \deg(\varphi_C)\chi(\varphi(C)) - \sum_{P \in C(k)} (e(P) - 1) - \sum_{\vec{t} \in \partial C^\dagger} \nu_{\vec{t}}. \tag{4.1.11.3}$$

Moreover, for a dagger affinoid $C \in \mathcal{C}^\dagger_{\mathcal{T}}$, if we denote $\varphi^{-1}(C) = \{C_1, \ldots, C_l\} \subset \mathcal{C}^\dagger_{\mathcal{S}}$, we have $\sum_{i=1}^l \deg(\varphi_{C_i}) = \deg(\varphi)$. Finally we note that whenever $\vec{t} \in \partial C^\dagger$ doesn't belong to $TY^\dagger$, there exists a dagger affinoid $C' \in \mathcal{C}^\dagger_{\mathcal{S}}$, $C \neq C'$ and a tangential vector $\vec{v} \in \partial C'^\dagger$, $\vec{v} \notin TY^\dagger$ s.t. the corresponding annuli $A_{\vec{t}}, A_{\vec{v}} \in \mathcal{A}^\dagger_{\mathcal{S}}$ coincide. Then lemma 2.2.1(*ii*) implies that $\nu(\varphi_C, \vec{t}) = -\nu(\varphi_{C'}, \vec{v})$. Having in mind all of this, and summing (4.1.11.3) over $C \in \mathcal{C}^\dagger_{\mathcal{S}}$ yields the proof. □

4.1.3 Riemann-Hurwitz formula for pro-wide open curves

Theorem 4.1.12. *Let $\varphi : U \to V$ be a finite morphism of pro-wide open curves with nonempty hearts. Then,*

$$\chi(U) = \deg\varphi\chi(V) - \sum_{P\in U(k)}(e_P - 1) - \sum_{\vec{t}\in T_{\mathrm{out}}U}\nu_{\vec{t}} + \sum_{\vec{t}\in T_{\mathrm{in}}U}\nu_{\vec{t}}.$$

Proof. Let V_0 be a dagger affinoid domain in V such that $V \setminus V_0$ is a disjoint union of open annuli and Robba proannuli. Then, $U_0 := \varphi^{-1}(V_0)$ is a dagger affinoid domain in U and by increasing V_0 we may assume that $U \setminus U_0$ is a disjoint union of open annuli and Robba proannuli. Let $\mathcal{R}_{U_0}$ (resp. $\mathcal{R}_{V_0}$) be the disjoint union of Robba proannuli in $U \setminus U_0$ (resp in $V \setminus V_0$) and let $\mathcal{A}_{U_0}$ (resp. $\mathcal{A}_{V_0}$) be the disjoint union of open annuli in $U \setminus U_0$ (resp. $V \setminus V_0$). Note that for each element $W \in \mathcal{R}_{U_0}$ (resp. $W \in \mathcal{A}_{U_0}$), φ restricts to a finite étale morphism $\varphi_{|W} : W \to \varphi(W)$ and $\varphi(W) \in \mathcal{R}_{V_0}$ (resp. $\varphi(W) \in \mathcal{A}_{V_0}$), and furthermore for each element $W \in \mathcal{R}_{V_0}$ (resp. $W \in \mathcal{A}_{V_0}$), $\varphi^{-1}(W)$ is a disjoint union of elements in $\mathcal{R}_{U_0}$ (resp. in $\mathcal{A}_{U_0}$).

From Theorem 4.1.10 we obtain

$$\chi(U_0) = \deg(\varphi)\chi(V_0) - \sum_{P\in U_0(k)}(e_P - 1) - \sum_{\vec{t}\in TU_0}\nu_{\vec{t}}. \tag{4.1.12.1}$$

First, we notice that $\chi(U_0) = \chi(U)$ and $\chi(V_0) = \chi(V)$. Then, the set TU_0 decomposes as a disjoint union of set $T_{\mathrm{out}}U$ and a set of tangential points corresponding to the intersection of U_0 and open annuli in $\mathcal{A}_{U_0}$. For each $A \in \mathcal{A}_{U_0}$, there are two inner tangential vectors $\vec{t}_{A,1}$ and $\vec{t}_{A,2}$, one of which (say $\vec{t}_{A,1}$) corresponds to the intersection of U_0 and A. In other words, the Robba proannulus attached to the vector $\vec{t}_A \in TU_0$ which "points" towards A coincides with the Robba proannulus corresponding to the vector $\vec{t}_{A,1}$. From Lemma

2.2.1(ii) it follows that

$$\nu(\vec{t}_A, \varphi_{|R_{\vec{t}_A}}) = \nu(\vec{t}_{A,1}, \varphi_{|R_{\vec{t}_{A,1}}}) = -\nu(\vec{t}_{A,2}, \varphi_{|R_{\vec{t}_{A,2}}}),$$

therefore the formula (4.1.12.1) can be rewritten as

$$\chi(U_0) = \deg(\varphi)\chi(V_0) - \sum_{P \in U(k)} (e_P - 1) - \sum_{\vec{t} \in T_{\text{out}} U} \nu_{\vec{t}} + \sum_{A \in \mathcal{A}_{U_0}} \nu_{\vec{t}_{A,2}}.$$

To conclude the proof, we note that the set of tangential points $\{\vec{t}_{A,2}, A \in \mathcal{A}_{U_0}\}$ is equal to the set of inner tangential vectors $T_{\text{in}} U$. □

Remark 4.1.13. The previous theorem is a generalization of [28, Proposition 5.7].

4.2 Riemann-Hurwitz formula for curves in characteristic $p > 0$

Let $\widetilde{\varphi} : \widetilde{Y'} \to \widetilde{X'}$ be a finite morphism of smooth, projective $\tilde{k}$-algebraic curves. When one classically looks for a Riemann-Hurwitz formula for a morphism φ *i.e.* for a formula of the form

$$\chi(\widetilde{Y'}) = \deg(\widetilde{\varphi})\chi(\widetilde{X'}) - R,$$

where R should be counted for the degree of the *ramification divisor*, one faces some problems. First of all, the *true* ramification divisor may be too big, as for example in the case where $\widetilde{\varphi}$ is purely inseparable the ramification locus has support on the whole curve $\widetilde{Y'}$. One is led to decompose the morphism $\widetilde{\varphi}$ into purely inseparable and separable parts, and one only considers the Riemann-Hurwitz formula for the separable part (the purely inseparable part doesn't change the Euler-Poincare characteristic of $\widetilde{Y'}$). However, in this way we break the similarity with the classical Riemann-Hurwitz formula for curves in characteristic 0. In this section, we propose another approach to study Riemann-Hurwitz formula for the morphism $\widetilde{\varphi}$.

We start by asking the following questions:

1. How can one choose a divisor on $\widetilde{Y'}$ whose degree will be equal to R?

2. Can we choose D in a canonical way?

To answer the question 1, we proceed as follows: Let $\widetilde{x} \in \widetilde{X'}(\widetilde{k})$ be a rational point, and let $\widetilde{\varphi}^{-1}(\widetilde{x}) = \{\widetilde{y_1}, \dots, \widetilde{y_s}\}$, and let us put $\widetilde{X} := \widetilde{X'} \setminus \{\widetilde{x}\}$ and $\widetilde{Y} := \widetilde{Y'} \setminus \{\widetilde{y_1}, \dots, \widetilde{y_s}\}$. Then, $\widetilde{\varphi}$ restricts to a finite morphism (which we still denote by $\widetilde{\varphi}$) $\widetilde{\varphi} : \widetilde{Y} \to \widetilde{X}$ of smooth *affine* curves.

Definition 4.2.1. *We say that $\widetilde{\varphi} : \widetilde{Y} \to \widetilde{X}$ lifts to characteristic 0 (or simply lifts) if there exist k-affinoid curves Y and X, and a finite morphism $\varphi : Y \to X$ such that the canonical reduction of $\varphi : Y \to X$ is $\widetilde{\varphi} : \widetilde{Y} \to \widetilde{X}$. We also say that $\varphi : Y \to X$, or just φ, is a lift of $\widetilde{\varphi} : \widetilde{Y} \to \widetilde{X}$.*

We know that in our case the morphism $\widetilde{\varphi} : \widetilde{Y} \to \widetilde{X}$ lifts to a finite morphism $\varphi : Y \to X$ of quasi-smooth dagger affinoids with good canonical reduction, because $\widetilde{X}$ is affine and $\widetilde{Y}$ is smooth. We put η to be the maximal point of Y and $\xi = \varphi(\eta)$ the maximal point of X. To each rational point in $\widetilde{Y'}$ which is different from $\widetilde{y_1}, \dots, \widetilde{y_s}$ corresponds a vector/point in a tangent space $T_\eta Y$, which again corresponds to a residual class in Y which is an open disc. On the other side, to each of the points $\widetilde{y_1}, \dots, \widetilde{y_s}$ corresponds a vector/point in $T_\eta Y$ which corresponds to a Robba proannulus attached to η. In conclusion, let the correspondence be given by $\widetilde{y} \leftrightarrow \vec{t}_{\widetilde{y}}$ and let $R_{\vec{t}}$ be a residual class of Y corresponding to the tangent vector $\vec{t}$.

We also introduce smooth projective curves Y' (resp. X'), containing Y (resp. X), such that $Y' \setminus Y$ (resp. $X' \setminus X$) is a disjoint union of open disc attached to η (resp. ξ). Recall that $\chi(Y') = \chi(\widetilde{Y'})$ (see for example [23, Theorem p.139]) and $\chi(X') = \chi(\widetilde{X'})$.

To each vector $\vec{t} \in T_\eta Y$ we can assign the degree/order $\sigma(\vec{t}) = \sigma(\vec{t}, \varphi)$ of the derivative of the φ restricted to the corresponding residual class $R_{\vec{t}}$ (more precisely to the small enough annulus at the boundary of the class $R_{\vec{t}}$). We have the Riemannn-Hurwitz formula

$$\chi(Y) = \deg(\varphi)\chi(X) - \sum_{y \in Y(k)} (e_y - 1) - \sum_{i=1}^{s} (\sigma(\vec{t}_{\widetilde{y}_i}) - d(\vec{t}_{\widetilde{y}_i}) + 1),$$

which is equivalent to

$$\chi(Y') = \deg(\varphi)\chi(X') - \sum_{\widetilde{y} \in \widetilde{Y'}(\widetilde{k})} \sigma(\vec{t}_{\widetilde{y}})$$

which, by using the standard isomorphism for Euler-Poincare characteristics (reference needed?), is equivalent to

$$\chi(\widetilde{Y'}) = \deg(\widetilde{\varphi})\chi(\widetilde{X'}) - \sum_{\widetilde{y} \in \widetilde{Y'}(\widetilde{k})} \sigma(\vec{t}_{\widetilde{y}}).$$

Note that for almost all $\widetilde{y} \in \widetilde{Y'}(\widetilde{k})$, $\sigma(\vec{t}_{\widetilde{y}})$ is 0.

Finally, we can proceed as follows. We assign a divisor

$$D(\varphi) := \sum_{\widetilde{y} \in \widetilde{Y'}(\widetilde{k})} \sigma(\widetilde{y}, \varphi)\widetilde{y}$$

where $\sigma(\widetilde{(y)}, \varphi) := \sigma(\vec{t}_{\widetilde{y}}, \varphi)$ (which has a support in finitely many points) and the Riemann-Hurwitz formula can be expressed in the following form:

$$\chi(\widetilde{Y'}) = \deg(\widetilde{\varphi})\chi(\widetilde{X'}) - \deg(D(\varphi)).$$

This gives a possible answer to the question 1 above. For the question 2, the following remark suggests that we shouldn't raise our hopes.

Remark 4.2.2. For two lifts φ_1 and φ_2 of $\widetilde{\varphi} : \widetilde{Y'} \to \widetilde{X'}$ and for a $\widetilde{y} \in \widetilde{Y'}(\widetilde{k})$, in general we have $\sigma(\widetilde{y}, \varphi_1) \neq \sigma(\widetilde{y}, \varphi_2)$. For example, take two mappings $\varphi_1, \varphi_2 : D(0,1) \to D(0,1)$, where $D(0,1)$ is a dagger affinoid unit disc and where $S = \varphi_1(T) = T^p$ and $S = \varphi_2(T) = (T-1)^p + 1$. Then, both φ_1 and φ_2 are lifts of a Frobenius covering of the affine line by affine line over the field $\widetilde{k}$, while we have $\sigma(\widetilde{0}, \varphi_1) = p - 1 \neq 0 = \sigma(\widetilde{0}, \varphi_2)$. This implies that, using the approach above, the answer to the question 2 is negative.

To reconcile, we propose the following theorem.

Theorem 4.2.3. *For two lifts φ_1 and φ_2 of $\widetilde{\varphi} : \widetilde{Y'} \to \widetilde{X'}$, we have divisors $D(\varphi_1)$ and $D(\varphi_2)$ are linearly equivalent.*

Proof. $\square$

Bibliography

[1] Omid Amini, Matthew Baker, Erwan Brugallé, Joseph Rabinoff. Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta Preprint: *arXiv:1303.4812v3*

[2] Francesco Baldassarri. Continuity of the radius of convergence of differential equations on p-adic analytic curves *Invent. Math.* 182(3): 513–584, 2010.

[3] Francesco Baldassarri, Kiran Kedlaya. Harmonic functions attached to meromorphic connections on non-archimedean curves *in preparation*

[4] Vladimir Berkovich. Étale cohomology for non-archimedean analytic spaces *Publications mathématiques de l'I.H.É.S.* 78: 5–161, 1993.

[5] Vladimir Berkovich. Spectral theory and analytic geometry over non-archimedean fields *vol. 33 Mathematical surveys and monographs.* American Mathematical Society, Providence RI, 1990.

[6] Vladimir Berkovich. Vanishing cycles for non-Archimedean analytic spaces *Journal of Amererican Mathematical Society* 9 (4), 1187–1209 1996.

[7] Vladimir Berkovich Integration of one-forms on p-adic analytic spaces *Annals of Mathematics Studies* Princeton University Press, Princeton, NJ vol 162, 2007.

[8] Siegfried Bosch. A rigid analytic version of M. Artin's theorem on analytic equations *Mathematische Annalen* 255: 395–404, 1981

[9] Siegfried Bosch, Ulrich Güntzer, Reinhold Remmert. Non-Archimedean analysis *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin vol 261, 1986.

[10] Siegfried Bosch, Werner Lütkebohmert. Stable reduction and uniformization of abelian varieties I *Mathematischen Annalen.* 270(3): 349–379, 1985.

[11] Nicolas Bourbaki. Commutative algebra *Elements of mathematics.* Hermann, Paris 1972.

[12] Gilles Christol. Le theorem Turrittin p-adique *http://www.math.jussieu.fr/ christol/courspdf.pdf*

[13] Adina Cohen, Michael Temkin, Dmitri Trushin Morphisms of Berkovich curves and the different function Preprint *arXiv:1408.2949v2*

[14] Robert F. Coleman. Reciprocity laws on curves *Compositio Mathematica* 72(2): 205–235, 1989.

[15] Robert F. Coleman. Stable maps of curves *Documenta Math. Extra Volume Kato* 217–225, 2003.

[16] Robert Coleman Torsion points on curves and p-adic Abelian integrals *Annals of mathematics* Vol. 121(1): 111–168, 1995

[17] Antoine Ducros. La structure des courbes analytiques *http://webusers.imj-prg.fr/ antoine.ducros/livre.html*

[18] Antoine Ducros. Triangulations et cohomologie étale sur une courbe analytique p-adique *Journal of Algebraic Geometry* 17(3): 503–575, 2008

[19] David A. Edwards, Harold M. Hastings. Čech and Steenrod homotopy theories with application to geometric topology *Lecture notes in mathematics* 542, 1986

[20] Xander Faber. Topology and geometry of the Berkovich ramification locus for rational functions I Preprint: *arXiv:1102.1432v4*

[21] Jean Fresnel, Michel Matignon. Sur les espaces analytique quasi-compacts de dimension 1 sur un corps valué, complet, ultramétrique *Ann. Mat. Pura. Appl.* CXLV(IV): 159–210, 1986

[22] Marco A. Garuti. Prolongement de revêtements galoisiens en géométrie rigide *Compositio Mathematica.* 104(3): 305–331, 1996

[23] Lothar Gerritzen, Marius Van der Put. Schottky groups and Mumford curves *Springer Vol. 817.* 1980

[24] Elmar Grosse-Klönne. Rigid analytic spaces with overconvergent structure sheaf *J. reine angew. Math* 519: 73–95, 2000

[25] Elmar Grosse-Klönne. De Rham cohomology of rigid spaces *Mathematische Zeitschrift* 247(2): 223–240, 2004

[26] Elmar Grosse-Klönne. Remark on the Čech cohomology of a coherent sheaf on dagger spaces *A letter to Francesco Baldassarri* 2013

[27] Robin Hartshorne. Algebraic geometry *Graduate texts in mathematics 52* Springer-Verlag, 1977.

[28] Kazuya Kato. Vanishing cycles, ramification of valuations and class field theory *Duke Mathematical Journal* 55(3):629–659, 1987

[29] Reindhard Kiehl. Theorem A und Theorem B in der nicharchimedischen Funktionentheorie *Inventines math.* 2: 256–273, 1967

[30] Qing Liu. Algebraic geometry and arithmetic curves *Oxford Graduate Texts in Mathematics 6* Oxford University Press, 2002

[31] Werner Lütkebohmert. Riemann's existence problem for a p-adic field *Inventiones Mathematicae* 111: 309–330, 1993

[32] Werner Lütkebohmert, Thorsten Schmechta. Étale covers of a punctured p-adic discs *Quaestiones Mathematicae* 28:2 195–242, 2005

[33] Jérôme Poineau. Les espaces de Berkovich sont angelique *Bulletin de la Société Mathématique de France* 141(2): 267–297, 2013

[34] Jérôme Poineau, Andrea Pulita. The convergence Newton polygon of p-adic differential equation IV: Local and global index theorem Preprint *arXiv:1309.3940v1*

[35] Marius van der Put. De Rham cohomology of affinoid spaces *Compositio Mathematica* 73(2): 223–239, 1990

[36] Marius van der Put. The class group of a one-dimensional affinoid space *Ann. Inst. Fourier* 30(4): 155–164, 1980.

[37] Lorenzo Ramero. Local monodromy in non-archimedean analytic geometry *Publications Mathé matiques de l'IHÉS* 102: 167–280, 2005.

[38] Michel Raynaud. Revetêments de la droite affine en caractéristique $p > 0$ et conjecture de Abhyankar

[39] John Welliaveetil. A Riemann-Hurwitz formula for skeleta in non-archimedean geometry Preprint: *arXiv:1303.0164*