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THE OSCILLATION PROPERTIES FOR THE SOLUTIONS OF HALF–LINEAR SECOND ORDER AND HIGHER ORDER DIFFERENTIAL EQUATIONS

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To my family

Abstract

This Thesis consists of five chapters.

The Thesis is dedicated to the investigation of the oscillation properties of half–linear second order and higher order differential equations.

Chapter 1 includes a short history of the problem, the statement of the problems and the main results. In this chapter, we present some well-known auxiliary facts and necessary notation on half–linear second order differential equations and weighted Hardy inequalities.

In Chapter 2, we consider weighted Hardy inequalities on the set of smooth functions with compact support. We obtain new results, which generalize the known results concerning this theme.

In Chapter 3, we investigate the problems of disfocality and disconjugacy of half–linear second order differential equations. We obtain new sufficient conditions and necessary conditions of disfocality and disconjugacy on a given interval. Also, we consider the behavior of some of the solutions on a given interval.

In Chapter 4, we apply the results of Chapter 3 in order to obtain sufficient conditions and necessary conditions of nonoscillation of half–linear second order differential equations with nonnegative coefficients. We also obtain new oscillation and nonoscillation conditions of half–linear differential equations by applying the known results of the theory of weighted Hardy type inequalities. Also, Chapter 4 includes the proofs of general statements establishing the link between results of weighted Hardy type inequalities and results ensuring oscillation and nonoscillation of the solutions of half–linear second order differential equation with nonnegative coefficients. We also obtain new results on conjugacy and oscillation in the case, in which the second coefficient changes sing,

by applying the variational method.

In Chapter 5, we investigate the problem of oscillation and nonoscillation of the solutions of two term linear and half–linear equations of higher order with nonnegative coefficients. It seems that our results concerning general conditions on the coefficients for nonoscillatory solutions of half– linear equations of higher order and for oscillatory and nonoscillatory solutions for linear equations of higher order are new. We establish necessary and sufficient conditions of strong oscillation and strong nonoscillation of the solutions for linear equations.

Riassunto

Questa Tesi consiste di cinque capitoli.

La Tesi è dedicata allo studio delle proprietà di oscillazione di equazioni differenziali semi–lineari del secondo ordine e di ordine superiore.

Il Capitolo 1 contiene una breve storia del problema, la formulazione dei problemi e i risultati principali. In questo capitolo, presentiamo alcuni risultati ausiliari noti e alcune notazioni necessarie relative a equazioni differenziali semi–lineari del secondo ordine e disuguaglianze di Hardy con peso.

Nel Capitolo 2, consideriamo disuguaglianze di Hardy con peso nell'insieme delle funzioni lisce a supporto compatto. Otteniamo risultati nuovi, che generalizzano risultati noti riguardanti questo tema.

Nel Capitolo 3, studiamo i problemi di disfocalità e disconiugazione per equazioni differenziali semi–lineari del secondo ordine. Otteniamo nuove condizioni necessarie e sufficienti di disfocalità e disconiugazione su un intervallo assegnato. Inoltre, consideriamo il comportamento di alcune soluzioni su un intervallo assegnato.

Nel Capitolo 4, applichiamo i risultati del Capitolo 3 al fine di ottenere condizioni necessarie e sufficienti di non–oscillazione di equazioni differenziali semi–lineari del secondo ordine con coefficienti non–negativi. Otteniamo anche nuove condizioni di oscillazione e non–oscillazione per equazioni differenziali semi–lineari applicando risultati noti di teoria delle disuguaglianze di tipo Hardy con peso. Inoltre, il Capitolo 4 include le dimostrazioni di affermazioni generali che stabiliscono il collegamento tra disuguaglianze di tipo Hardy con peso e risultati che assicurano l'oscillazione e la non–oscillazione per le soluzioni di equazioni semi–lineari del secondo ordine con coefficienti non–negativi. Otteniamo anche risultati nuovi sulla coniugazione e l'oscillazione nel caso in cui

il secondo coefficiente cambi di segno, applicando il metodo variazionale.

Nel Capitolo 5, studiamo il problema di oscillazione e non–oscillazione per soluzioni di equazioni lineari e semi–lineari a due termini di ordine superiore con coefficienti non–negativi. Pensiamo che i nostri risultati riguardanti condizioni generali sui coefficienti per soluzioni non–oscillatorie di equazioni semi–lineari del secondo ordine e per soluzioni oscillatorie e non–oscillatorie di equazioni lineari di ordine superiore siano nuovi. Stabiliamo condizioni necessarie e sufficienti per l'oscillazione forte e per la non–oscillazione forte per soluzioni di equazioni lineari.

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Contents

Chapter 1

Introduction

1.1 Preliminaries.

This Thesis is dedicated to the investigation of oscillation properties of the following half–linear second order differential equation

$$\big(\rho(t)|y'(t)|^{p-2}y'(t)\big)' + v(t)|y(t)|^{p-2}y(t) = 0,\ p > 1. \qquad (HL)$$

When $p = 2$ the equation (HL) becomes the linear Sturm – Liouville equation

$$(\rho(t)y'(t))' + v(t)y(t) = 0. \qquad (SL)$$

The investigation of the oscillation properties of the equation (SL) started in the work by Sturm [50]. The investigation has been developing till the present time. The oscillation properties of nonlinear differential equations were investigated at the beginning of the last century. In particular, we refer to the papers [22], [24], [52] and to the book [48]. Many results in this area have been achieved in the first half of the last century. Since the results have been applied in various directions of the qualitative theory of second order differential equations the interest into the subject has increased. For example, in the spectral theory of differential operators [44].

The investigation of the properties of equation (HL) started with the work by Bihari [3] and [4], Elbert [20] and Mirzov [37]. They are generally considered "pioneers" of the qualitative theory (HL).

In the last thirty years, the qualitative theory (HL) has attracted many investigators. At the end of the last century, the similarity of the properties of the solutions of the equations (HL) and (SL) have been established. In particular, Sturm's Comparison and Separation Theorems for the equation (HL) hold. But there exist many differences between the properties of the solutions of the equations (HL) and (SL). Indeed, the set of the solutions of the equation (HL) is not a linear space.

The results of investigations and methods for the equation (HL) up to year 2005 are exposed in the book by Došlý and Řehák [11]. Došlý is one of the leading experts in the oscillation theory of the equation (HL).

The interest into the theory is two fold. On the one hand, the theory can be considered as a generalization of the linear theory. On the other hand, it can be considered as a one–dimensional version of a partial differential equation with the p–Laplacian. The p–Laplacian has many applications in physics, biology and in chemistry and also in non–Newtonian fluid theory and in some models in glaceology [10].

Among the many methods to analyze the oscillation properties of the equation (HL), we mention two basic methods [16], [17], [11]. The former is the "Riccati technique", starting with the theory of the linear equation (SL). And the latter is the variational method.

In the "Riccati technique" the problem of nonoscillation or oscillation of the equation (HL) (for example at $t = \infty$) becomes the problem of existence or non–existence of the solutions of the Riccati type equation

$$w' + v(t) + (p-1)\rho^{1-p'}(t)|w|^{p'-1} = 0, \quad p' = \frac{p}{p-1}, \ p > 1 \qquad (R)$$

in some neighborhood of point $t = \infty$. In this case one uses the different methods of investigating the existence of the solutions of nonlinear differential equations and in particular different approaches in asymptotic analysis. The main results in the qualitative theory of the equation (HL) have been obtained by the application of the "Riccati technique".

In the variational method, the oscillation properties of the equation (HL) are related to the validity or not validity of the following inequality

$$\int_a^b [\rho(t)|f'(t)|^p - v(t)|f(t)|^p]dt > 0, \qquad (F)$$

when f in the space $\overset{\circ}{W}{}^1_p(a,b)$.

The variational method is used relatively little compared to the "Riccati technique", although in the oscillation theory of the linear equation (SL) it plays an important role (see [25]). By applying the variational method, one employ different inequalities, such as the Wirtinger inequality (see [11]).

In the Thesis, we consider the following problems:

– focality and disfocality of the equation (HL) on a given interval;

– conjugacy and disconjugacy of the equation (HL) on a given closed or open interval;

– oscillation and nonoscillation of the equation (HL) on a given interval with a singularity at the endpoint;

– oscillation and nonosillation of two term linear and half–linear equations of higher order.

Our aim is to develop the variational method to study the equation (HL) by applying results of the theory of weighted Hardy type inequalities in terms of the coefficients of the equation (HL). We have obtained the following new results.

1. Firstly, we have obtained new results in the theory of weighted Hardy type inequalities on the set of smooth functions with compact support (Chapter 2). Such results improve the results given in the book of Kufner and Opic [43].

2. Sufficient conditions and necessary conditions of disfocality of the equation (HL) with nonnegative coefficients on the given semi–interval (Chapter 3, Section 3.1).

These results have been obtained by the application of results in the theory of weighted Hardy type inequalities. However, the author is not aware of results of such type in the theory of equations (HL) and (SL).

3. Sufficient conditions and necessary conditions of disfocality of the equation (HL) with nonnegative coefficients on the given interval both with regular and singular points at the endpoint. Also if the equation (HL) is disfocal on the *given interval, we show how to find a nontrivial solution (Chapter 3, Section 3.2).*

These results have been obtained by the application of weighted Hardy type inequalities, which we show in Chapter 2.

4. Sufficient conditions and necessary conditions of nonoscillation for the equation (HL) with nonnegative coefficients (Chapter 4, Section 4.1–4.2).

Here (in Section 4.1), sufficient conditions and necessary conditions of nonoscillation of the equation (HL) have been obtained by the applications of the results of Chapter 3. Here, we also show some properties of the solutions. These sufficient conditions have been known before by another method. In Section 4.2, we obtain sufficient conditions of oscillation and nonoscillation for the solutions of the equation (HL) on the basis of the known results in the weighted Hardy type inequalities. In the same Section, we indicate a general approach of applying the results on weighted Hardy type inequalities in the analysis of oscillation and nonoscillation of the solutions of the equation (HL).

5. The criteria of disconjugacy on the given interval of the equation (HL) without sing condition on the the coefficient v (Chapter 4, Section 4.3).

These results are directly obtained by variational methods. On the basis of such results we have the following new results.

6. Sufficient conditions of oscillation for the equation (HL) without sing condition on the coefficient v (Chapter 4, Section 4.3).

Such sufficient conditions are more general than the previously known results in such direction. They are new for the case when the coefficient $v \geq 0$.

7. Sufficient conditions of nonoscillation of two term half–linear differential equation of higher order (Chapter 5, Section 5.2).

Half–linear differential equations of higher order have already been investigated [11, p. 464]. And to the best of our knowledge, our thesis is one of the first results on nonoscillation of two term half–linear differential equations of higher order.

8. Sufficient conditions of oscillation and nonoscillation of two term half–linear differential equations of higher order with nonnegative conditions (Chapter 5, Section 5.2 and Section 5.3).

These results generalize the known results, which are given in the monograph [25] by Glazman. Here, we have obtained such results and more general conditions. In particular, we generalize the known results by considering one of the coefficients as a power function.

9. Necessary and sufficient conditions of strong oscillation and strong nonoscillation of two term half–linear differential equations of higher order with nonnegative conditions, which find application in the spectral theory of linear differential operators (Chapter 5, Section 5.4).

These results generalize the known related results, when the coefficient of the derivative is constant [25] or when one of the coefficients is a spectral function (see [29], [23] and [15]).

Part of the obtained results have been published in the papers [30], [41] and [42].

We notice that the results of 7–8 have been obtained by the variational method and by applying the latest results in the theory of weighted Hardy inequalities [31] and Hardy type inequalities [49].

We notice that other methods of analysis of some spectral properties of the second order differential operators have been derived again with help of the Hardy inequalities as stated in the paper of Drabek and Kufner [18] and Otelbaev [44], and that the method of investigating differential operators of higher order have been started in the paper of Apyshev and Otelbaev [1].

1.2 Half–linear second order differential equations. Main notions and propositions.

In this section we present basic properties of the solutions of half–linear second order differential equation. We will keep the terminology of the book of O. Došlý and P. Řehák [11].

On the interval $I = (a, b)$, $-\infty \leq a < b \leq +\infty$ we consider the following second order differential equation:

$$\big(\rho(t)|y'(t)|^{p-2}y'(t)\big)' + v(t)|y(t)|^{p-2}y(t) = 0,\ t \in I, \tag{1.1}$$

where $1 < p < \infty$, ρ and v are continuous functions on I. Moreover, $\rho(t) > 0$ for any $t \in I$.

We recall that the terminology half–linear differential equation reflects the fact that the solution space of (1.1) is homogeneous, but not additive.

When $p = 2$ the equation (1.1) becomes the linear Sturm – Liouville equation

$$(\rho(t)y'(t))' + v(t)y(t) = 0. \tag{1.2}$$

Definition 1.1. *A function $y : I \to R$ is said to be a solution of (1.1), if $y(t)$ is continuously differentiable together with $\rho(t)|y'(t)|^{p-2}y'(t)$ and satisfies the equation (1.1) on I.*

We note that the function $y(t) \equiv 0$ is a solution of the equation (1.1). We say that a solution $y(\cdot)$ of (1.1) is nontrivial if there exists at least one point $t_0 \in I$ such that $y(t_0) \neq 0$. If so, the continuity of $y(t)$ ensures that $(t_0 - \delta,\ t_0 + \delta) \subset I$ and $y(t) \neq 0$, for all $t \in (t_0 - \delta,\ t_0 + \delta)$.

Definition 1.2. *A point $t_0 \in I$ is called zero of the solution $y(t)$, if $y(t_0) = 0$.*

Definition 1.3. *A nontrivial solution of the equation (1.1) is called oscillatory at $t = b$ or at $t = a$, if it has an infinite number of zeros converging to b or to a, respectively. Otherwise y is called nonoscillatory, i.e. if there exists $c \in I$ and the solution has not zeros on the interval (c, b) or (a, c), respectively.*

Definition 1.4. *The equation (1.1) is called oscillatory or nonoscillatory at $t = b$ or at $t = a$, if all of its nontrivial solutions are oscillatory or nonoscillatory at $t = b$ or at $t = a$, respectively.*

Let I_0 be an interval contained in I. To investigate the oscillation properties of (1.1), it is proper to use notions such as conjugacy and disconjugacy of the equation (1.1) on the interval I_0.

Definition 1.5. *The points t_1, $t_2 \in I$ are called conjugate points with respect to the equation (1.1), if there exists a nontrivial solution $y(t)$ of the equation (1.1) such that $y(t_1) = y(t_2) = 0$.*

Definition 1.6. *The equation (1.1) is called an equation of disconjugate points on the interval I_0, if all of it any nontrivial solutions have no more than one zero on I_0. Otherwise it is called an equation of conjugate points on the interval I_0.*

Further, by the terminology of [11], we will understand, that the equation (1.1) is disconjugate on the interval I_0, if it is an equation of disconjugate points on the interval I_0. The equation (1.1) is conjugate on the interval I_0, if it is an equation of conjugate points on the interval I_0.

One of the fundamental result in the qualitative theory of half–linear equations in the form (1.1) is a theorem called "Roundabout theorem". The terminology Roundabout theorem (or Reid type Roundabout theorem) is due to Reid [47] (it concerns the linear case), and it is motivated by the fact the proof of this theorem consists of the "roundabout" proof of several equivalent statements.

Theorem 1.7. *[11, Theorem 1.2.2] Let $[\alpha, \beta] \subset I$. The following statements are equivalent:*
(i) The equation (1.1) is disconjugate on the interval $[\alpha, \beta]$.
(ii) There exists a solution of the equation (1.1) having no zero in the interval $[\alpha, \beta]$.
(iii) There exists a solution w of the generalized Riccati equation

$$w' + v(t) + (p-1)\rho^{1-p'}(t)|w(t)|^{p'} = 0$$

defined on the whole interval $[\alpha, \beta]$, where $p' = \frac{p}{p-1}$.
(iv) The functional

$$F(f) = \int_{\alpha}^{\beta} (\rho(t)|f'|^p - v(t)|f|^p)\, dt > 0$$

for any nontrivial $0 \neq f \in \overset{\circ}{W}{}_p^1(\alpha,\beta)$, *where* $\overset{\circ}{W}{}_p^1(\alpha,\beta)$ *is the Sobolev space.*

Recall that the Sobolev space $\overset{\circ}{W}{}_p^1(\alpha,\beta)$ consists of absolutely continuous functions f on the interval $[\alpha,\beta]$ such that $f' \in L_p(\alpha,\beta)$ and $f(\alpha) = f(\beta) = 0$, with the norm $||f||_{W_p^1} = \left(\int\limits_\alpha^\beta [|f'|^p + |f|^p]\, dt\right)^{\frac{1}{p}}$ or with the equivalent norm $||f||_{W_p^1} = \left(\int\limits_\alpha^\beta |f'|^p\, dt\right)^{\frac{1}{p}}$. $L_p(\alpha,\beta)$ is the space of measurable and finite almost everywhere functions g on (α,β) for which the following norm $||g||_p = \left(\int\limits_\alpha^\beta |g|^p\, dt\right)^{\frac{1}{p}}$ is finite.

The qualitative theory of the half - linear oscillation equations as (1.1) is similar to that of the oscillation theory of Sturm–Liouville linear equations as (1.2), and the Sturmian theory extends verbatim to (1.1). The interlacing property of zeros of linearly independent solutions of linear equations is one of the most characteristic properties, which among others justifies the definition of oscillation/ nonoscillation of the equation. The next Sturm type separation theorem claims that this property extends to (1.1) (see [11, Theorem 1.2.3 and Lemma 1.2.2]).

Theorem 1.8. *Two nontrivial solutions of (1.1) which are not proportional cannot have a common zero. If* $t_1 < t_2$ *are two consecutive zeros of a nontrivial solution* y *of (1.1), then any other solution of this equation which is not proportional to* y *has exactly one zero on* $(t_1,\ t_2)$.

The next statement is an extension of well–known Sturm comparison theorem to (1.1).

Theorem 1.9. *Let* R *and* V *be functions of* (a,b) *to* $\mathbb{R}$ *satisfying the same assumptions of the functions* ρ *and* v, *respectively. Let* $t_1 < t_2$ *be consecutive zeros of a nontrivial solution* y *of (1.1) and suppose that*

$$V(t) \geq v(t), \quad \rho(t) \geq R(t) > 0 \tag{1.3}$$

for $t \in [t_1, t_2]$. *Then any solution of the equation*

$$\big(R(t)|x'(t)|^{p-2}x'(t)\big)' + V(t)|x(t)|^{p-2}x(t) = 0 \tag{1.4}$$

has a zero in (t_1,t_2) *or it is a multiple of the solution* y. *The last possibility is excluded if one of the inequalities in (1.3) is strict on a set of positive measure.*

By Theorem 1.7 and Theorem 1.9, we obtain the following theorem (see [11]).

Theorem 1.10. *The equation (1.1) is oscillatory or nonoscillatory at $t = b$ if and only if there exists at least a nontrivial oscillatory or nonoscillatory solution of (1.1) at $t = b$, respectively. The same statement hold by replaying b with a.*

By Theorem 1.7, Theorem 1.9 and Theorem 1.10, we obtain a relation between the notions of oscillatory equation and conjugate equation, nonoscillatory equation and disconjugate equation.

Theorem 1.11. *(i) The equation (1.1) is oscillatory at $t = b$ or at $t = a$ if and only if for any $T \in I$ the equation (1.1) is conjugate on $[T, b)$ or on $(a, T]$, respectively.*
(ii) The equation (1.1) is nonoscillatory at $t = b$ or at $t = a$ if and only if there exists $T \in I$ such that the equation (1.1) is disconjugate on $[T, b)$ or on $(a, T]$, respectively.

A crucial role in the analysis of the oscillation properties of (1.1) is played by the Roundabout theorem (Theorem 1.7), which provides two important methods: the "Riccati technique" and a variational principle. A method is based on the equivalence of (i) and (iii) is called the "Riccati technique". The main results in the qualitative theory of the equation (1.1) are obtained by the application of the "Riccati technique". As an immediate consequence of the equivalence of (i) and (iv) of the Roundabout theorem, we have the following statement which may be widely used in the proofs of oscillation and nonoscillation criteria. When using such method, we say that we are employing a variational method. We present the main statements following from the equivalence of (i) and (iv) in Section 2.2 of Chapter 2, where we consider a class of admissible functions alternative to $\overset{\circ}{W}{}^1_p$ in the Roundabout theorem.

Let $v(t) > 0$ for all $t \in I$ in the equation (1.1). Then the equation

$$(v^{1-p'}(t)|u'(t)|^{p'-2}u'(t))' + \rho^{1-p'}(t)|u(t)|^{p'-2}u(t) = 0 \qquad (1.5)$$

is called the reciprocal equation relative to (1.1), where $p' = \frac{p}{p-1}$. The terminology "reciprocal equation" is motivated by the linear equation (1.2). The reciprocal equation to (1.5) is again the equation (1.1).

It is easy to prove the following theorem (see [11, p. 22]).

Theorem 1.12. *Let $1 < p < \infty$. Let $\rho(t) > 0$ and $v(t) > 0$ for all $t \in I$. Then the equation (1.1) is oscillatory or nonoscillatory if and only if the reciprocal equation is oscillatory or nonoscillatory, respectively.*

Such as result will be referred to as the reciprocity principle.

Now we introduce the following notions (see [11, p. 193]) relative to the equation (1.1), which we consider bellow.

Definition 1.13. *A point β is said to be the first right focal point of a point $\alpha \in I$, with $\alpha < \beta$ with respect to thequation (1.1), if there exists a nontrivial solution y of the equation (1.1) such that $y'(\alpha) = 0 = y(\beta)$ and $y(t) \neq 0$ for all $t \in [\alpha, \beta)$.*

Definition 1.14. *A point α is said to be the first left focal point of a point $\beta \in I$, with $\alpha < \beta$ with respect to the equation (1.1), if there exists a nontrivial solution y of the equation (1.1) such that $y'(\beta) = 0 = y(\alpha)$ and $y(t) \neq 0$ for all $t \in (\alpha, \beta]$.*

Definition 1.15. *The equation (1.1) is said to be the right disfocal on the interval $[\alpha, \beta) \subset I$, if there does not exists a right focal point of the point α with respect to the equation (1.1) in the interval $[\alpha, \beta)$.*

The equation (1.1) is said to be the left disfocal on the interval $(\alpha, \beta] \subset I$, if there does not exists a left focal point of the point α with respect to the equation (1.1) in the interval $(\alpha, \beta]$.

Let $(\alpha, \beta) \subset I$. Suppose that

$$W^1_{p,r}(\alpha, \beta) = \{f \in W^1_p(\alpha, \beta) : \quad f(\beta) = 0\},$$

$$W^1_{p,l}(\alpha, \beta) = \{f \in W^1_p(\alpha, \beta) : \quad f(\alpha) = 0\}.$$

By Theorem 5.8.3, Theorem 5.8.5 and Remark 5.8.1 of [11], we obtain the following variational method for the right/left disfocal points on the interval (α, β).

Theorem 1.16. *Let $a < \alpha < \beta < b$. The equation (1.1) is the right disfocal on the interval $[\alpha, \beta)$ if and only if*

$$F(f) = \int_{\alpha}^{\beta} [\rho(t)|f'(t)|^p - v(t)|f(t)|^p]\, dt \geq 0 \tag{1.6}$$

for all $f \in W^1_{p,r}(\alpha, \beta)$.

Theorem 1.17. *Let $a < \alpha < \beta < b$. The equation (1.1) is the left disfocal on the interval $(\alpha, \beta]$ if and only if*

$$F(f) = \int_{\alpha}^{\beta} [\rho(t)|f'(t)|^p - v(t)|f(t)|^p]\, dt \geq 0 \tag{1.7}$$

for all $f \in W^1_{p,l}(\alpha, \beta)$.

1.3 Hardy's inequality.

In the literature many authors including G.H. Hardy, J.E. Littlewood and G. Pólya [27] consider the continuous Hardy inequality: if $p > 1$ and if f is a nonnegative p integrable function on $(0, \infty)$, then f is integrable over the interval $(0, x)$ for all $x > 0$ and

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p dx < \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)\, dx. \tag{1.8}$$

The constant $\left(\frac{p}{p-1}\right)^p$ in the inequality (1.8) is sharp in the sense that it can not be replaced by any smaller number.

The inequality (1.8) has been generalized and applied in analysis and in the theory of differential equations. Some of these developments, generalizations and applications have been described and discussed in the books [27], [43], [32], [19] and [31].

Below we have just selected a few facts from this development.

In 1928 G.H. Hardy [28] proved the estimate for some integral operators, from which the first "weighted" modification of the inequality (1.8) appeared, namely the inequality

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p x^\varepsilon\, dx < \left(\frac{p}{p-\varepsilon-1} \right)^p \int_0^\infty f^p(x) x^\varepsilon\, dx, \tag{1.9}$$

valid with $p > 1$ and $\varepsilon < p - 1$, for all measurable nonnegative functions f (see [27], Theorem 330), where the constant $\left(\frac{p}{p-\varepsilon-1}\right)^p$ is the best possible. We also mention the following dual inequality, which can be derived from (1.9):

$$\int_0^\infty \left(\frac{1}{x} \int_x^\infty f(t)\, dt \right)^p x^\varepsilon\, dx < \left(\frac{p}{\varepsilon+1-p} \right)^p \int_0^\infty f^p(x) x^\varepsilon\, dx. \tag{1.10}$$

It holds with $p > 1$ and $\varepsilon > p-1$, for all measurable non–negative functions f and the constant $\left(\frac{p}{\varepsilon+1-p}\right)^p$ is the best possible (see again [27], Theorem 330).

During the last decades the inequality (1.9) has been developed to the form

$$\left(\int_a^b \left(\int_a^x f(t)\, dt \right)^q u(x)\, dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b f^p(x) v(x)\, dx \right)^{\frac{1}{p}} \tag{1.11}$$

with

– a, b real number satisfying $-\infty \leq a < b \leq \infty$,

– u, v everywhere positive measurable weight functions in (a, b),

– p, q real parameters, satisfying $0 < q \leq \infty$ and $1 \leq p \leq \infty$.

This is sometimes called the modern form of the continuous Hardy inequality.

A systematic investigation of Hardy inequality, which started in the late fifties and early sixties of the last century, was connected with the name of P.R. Beesack. Beesack, in his paper [2] from 1961, connected the validity of the corresponding inequality (1.8) (with $a = 0$, $b = \infty$, $p = q$) with the existence of a (positive) solution y of the nonlinear ordinary differential equation

$$\frac{d}{dx}\left(v(x)\left(\frac{dy}{dx}(x)\right)^{p-1}\right) + u(x)y^{p-1}(x) = 0, \tag{1.12}$$

which is in fact the Euler–Lagrange equation for the functional

$$J(y) = \int_0^\infty [(y'(x)))^p v(x) - y^p(x)u(x)]\,dx,$$

although Beesack's approach was not the variational one. His approach was extended to a class of the inequalities containing the Hardy inequality as a special case (see [43]).

The problem of finding necessary and sufficient conditions for (1.11) to hold, again for $(a, b) = (0, \infty)$ and $p = q$, was investigated in 1969 by Talenti [51] and 1969 by Tomaselli [53], who also followed Beesack's approach via differential equations and have shown that the solvability of the equation (1.12) is not only sufficient but in a certain sense even necessary for (1.11) to hold. In our opinion the Tomaselli paper [53] plays a fundamental role in the development of the Hardy inequality, since it combines almost all relevant information for the case $p = q$.

In 1969 Talenti [51] and Tomaselli [53] obtained that a necessary and sufficient condition for the estimate

$$\int_0^b \left(\int_0^x f(t)\,dt\right)^p u(x)\,dx \leq C\int_0^b f^p(x)v(x)\,dx \tag{1.13}$$

with $f \geq 0$, $0 < b \leq \infty$ and $1 < p < \infty$ reads as follows

$$B = \sup_{r \in (0,b)} \left(\int_r^b u(x)\, dx \right) \left(\int_0^r v^{1-p'}(x)\, dx \right)^{p-1} < \infty. \tag{1.14}$$

Moreover, $B \leq C \leq \frac{p^p}{(p-1)^{p-1}} B$.

One can we write condition (1.14) with $b = \infty$ in the following modified form

$$A = \sup_{r>0} \left(\int_r^\infty u(x)\, dx \right)^{\frac{1}{p}} \left(\int_0^r v^{1-p'}(x)\, dx \right)^{\frac{1}{p'}} < \infty, \tag{1.15}$$

which is frequently called the Muckenhoupt condition after Muckenhoupt [38] gave a nice and easy direct proof of this result and extended it even to the case with the more general inequality

$$\left(\int_0^\infty \left(\int_0^x f(t) dt \right)^p d\mu \right)^{\frac{1}{p}} \leq C \left(\int_0^\infty f^p(x)\, d\nu \right)^{\frac{1}{p}}, \tag{1.16}$$

$f \geq 0$, with some Borel measures μ and ν in 1972. In this case, a necessary and sufficient condition for (1.16) to hold with a constant C independent of f reads as follows

$$A = \sup_{r>0} [\mu(r,\infty)]^{\frac{1}{p}} \left(\int_0^r \left(\frac{d\widetilde{\nu}}{dx} \right)^{1-p'} dx \right)^{\frac{1}{p'}} < \infty, \tag{1.17}$$

where $\widetilde{\nu}$ denotes the absolutely continuous part of ν. For the special measures $d\mu(t) = u(t)\, dt$, $d\nu(t) = v(t)\, dt$ the condition (1.17) reduces to (1.15). Moreover, the best (=least) constant C in (1.16) satisfies

$$A \leq C \leq p^{\frac{1}{p}} (p')^{\frac{1}{p'}} A$$

for $1 < p < \infty$ and $C = A$ for $p = 1$ (and $p' = \infty$).

He also proved that the best constant C in (1.13) satisfies

$$B^* \leq C \leq \left(\frac{p}{p-1} \right)^p B^*.$$

The main content of the research described above can be formulated as follows.

Theorem 1.18. ***(Talenti–Tomaselli–Muckenhoupt)***. *Let* $1 < p < \infty$. *The inequality (1.11) in case* $p = q$:

$$\int_a^b \left(\int_a^x f(t)\,dt \right)^p u(x)\,dx \le C \int_a^b f^p(x)v(x)\,dx \tag{1.18}$$

holds for all measurable functions $f(x) \ge 0$ *on* (a,b) *if and only if*

$$A_{TTM} = \sup_{z\in(a,b)} \int_z^b u(x)\,dx \left(\int_a^z v^{1-p'}(x)\,dx \right)^{p-1} < \infty.$$

Moreover, the best constant C *in (1.18) satisfies*

$$A_{TTM} \le C \le p \left(\frac{p}{p-1} \right)^{p-1} A_{TTM}.$$

There is also a corresponding result for the dual inequality.

Theorem 1.19. *Let* $1 < p < \infty$. *The inequality*

$$\int_a^b \left(\int_x^b f(t)\,dt \right)^p u(x)\,dx \le C^* \int_a^b f^p(x)v(x)\,dx \tag{1.19}$$

holds for all measurable functions $f(x) \ge 0$ *on* (a,b) *if and only if*

$$A^*_{TTM} = \sup_{z\in(a,b)} \left(\int_z^b v^{1-p'}(x)\,dx \right)^{p-1} \int_a^z u(x)\,dx < \infty.$$

Moreover, the best constant C^* *in (1.19) satisfies*

$$A^*_{TTM} \le C^* \le p \left(\frac{p}{p-1} \right)^{p-1} A^*_{TTM}.$$

Besides (1.14), Tomaselli derived in [53] also some other (equivalent) conditions for the validity of the Hardy inequality (1.13). This result for the inequality (1.18) formulated as follows.

Theorem 1.20. ***(Tomasseli)***. *Let* $1 < p < \infty$. *The inequality (1.18) holds for all measurable functions* $f(x) \ge 0$ *on* (a,b) *if and only if*

$$B_T = \sup_{z\in(a,b)} \left(\int_a^z v^{1-p'}(x)\,dx \right)^{-1} \int_a^z u(x) \left(\int_a^x v^{1-p'}(t)\,dt \right)^p dx < \infty.$$

Moreover, the best constant C *in (1.18) satisfies*

$$B_T \le C \le \left(\frac{p}{p-1} \right)^p B_T.$$

As usual, the results for the dual inequality (1.19) can be obtained by the results for the inequality (1.18), taking into account that the inequality (1.19) is equivalent to the inequality

$$\int_a^b v^{1-p'}(x)\left(\int_a^x g(t)\,dt\right)^{p'} dx \le C\int_a^b u^{1-p'}(x)g^{p'}(x)\,dx, \quad g\ge 0,$$

where the best constant is equivalent to the best constant in (1.19).

Theorem 1.21. *Let $1<p<\infty$. The inequality (1.19) holds for all measurable functions $f(x)\ge 0$ on (a,b) if and only if*

$$B_T^* = \sup_{z\in(a,b)}\left(\int_a^z u(x)\,dx\right)^{-1}\int_a^z v^{1-p'}(t)\left(\int_a^t u(x)\,dx\right)^{p'} dt<\infty.$$

Moreover, the best constant C^* in (1.19) satisfies

$$B_T^*\le C^*\le p^{p'}B_T^*.$$

Next, we will prove the following theorem which we apply later.

Theorem 1.22. *Let $1<p<\infty$. The inequality (1.19) holds for all measurable functions $f(x)\ge 0$ on (a,b) if and only if*

$$B^* = \sup_{z\in(a,b)}\left(\int_z^b v^{1-p'}(x)\,dx\right)^{-1}\int_z^b u(t)\left(\int_t^b v^{1-p'}(x)\,dx\right)^{p} dt<\infty.$$

Moreover, the best constant C^* in (1.19) satisfies

$$B^*\le C^*\le\left(\frac{p}{p-1}\right)^p B^*. \tag{1.20}$$

Proof of Theorem 1.22. In (1.19) we substitute the variables $\tau=e^{-t}$, $s=e^{-x}$ and denote by $\alpha=e^{-b}$, $\beta=e^{-a}$. Then we obtain

$$\int_\alpha^\beta\left(\int_\alpha^s \widetilde{f}(\tau)\,d\tau\right)^p \widetilde{u}(s)\,ds\le C^*\int_\alpha^\beta \widetilde{f}^p(s)\widetilde{v}(s)\,ds, \tag{1.21}$$

where $\widetilde{f}(\tau)=\frac{f(-\ln\tau)}{\tau}$, $\widetilde{u}(s)=\frac{u(-\ln s)}{s}$ and $\widetilde{v}(s)=s^{p-1}v(-\ln s)$.

Then by Theorem 1.20 and inequality (1.21), the inequality (1.19) holds if and only if

$$\widetilde{B}_T = \sup_{\alpha<y<\beta} \left(\int_\alpha^y \widetilde{v}^{1-p'}(\tau)\,d\tau \right)^{-1} \int_\alpha^y \widetilde{u}(s) \left(\int_\alpha^s \widetilde{v}^{1-p'}(\tau)\,d\tau \right)^p ds < \infty. \quad (1.22)$$

For the best constant we have the following estimate

$$\widetilde{B}_T \le C^* \le \left(\frac{p}{p-1}\right)^p \widetilde{B}_T. \quad (1.23)$$

Now, by substituting $-\ln\tau = x$, $-\ln s = t$ in the integrals of (1.22), and by taking $\ln y = z$, we obtain $\widetilde{B}_T = B$. Then (1.20) follows by (1.23).

Thus the proof of Theorem 1.22 is complete. □

Just recently, several different (but equivalent) scales of the conditions have been derived, mainly for the case $1 < p \le q < \infty$. They can be found in the papers [26], [33]. Let us mention here only one of these results, which is due to A. Wedesting (see her PhD Thesis [54] or [33, Theorem 1]).

Theorem 1.23. *Let $1 < s < p < \infty$. The inequality (1.18) holds for all measurable functions $f(x) \ge 0$ on (a,b) if and only if*

$$A_W(s) = \sup_{z\in(a,b)} \left(\int_a^z v^{1-p'}(t)\,dt \right)^{s-1} \int_z^b u(x) \left(\int_\alpha^x v^{1-p'}(t)\,dt \right)^{p-s} dx < \infty.$$

Moreover, if C is the best possible constant in (1.18), then

$$\sup_{1<s<p} \frac{p^p(s-1)}{p^p(s-1)+(p-s)^p} A_W(s) \le C \le \inf_{1<s<p} \left(\frac{p-1}{p-s}\right)^{p-1} A_W(s).$$

By Theorem 329 of [27] we obtain the following theorem.

Theorem 1.24. *If $p > 1$ and $r > 0$. Then*

$$\int_0^\infty \frac{1}{x^{pr}} \left(\frac{1}{\Gamma(r)} \int_0^x (x-t)^{r-1} f(t)\,dt \right)^p dx < \left\{ \frac{\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(r+1-\frac{1}{p}\right)} \right\}^p \int_0^\infty f^p(x)\,dx \quad (1.24)$$

for all measurable functions $f(x) \ge 0$ $(f \ne 0)$ on $(0,\infty)$.

Moreover, the constant $\left\{ \frac{\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(r+1-\frac{1}{p}\right)} \right\}^p$ is the best constant in (1.24).

Theorem 1.25. *If $p > 1$ and $r > 0$. Then*

$$\int_0^\infty \left(\frac{1}{\Gamma(r)} \int_x^\infty (t-x)^{r-1} f(t)\, dt \right)^p dx < \left\{ \frac{\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(r+\frac{1}{p}\right)} \right\}^p \int_0^\infty (x^r f(x))^p\, dx \quad (1.25)$$

for all measurable functions $f(x) \geq 0$ $(f \neq 0)$ *on* $(0, \infty)$.

Moreover, the constant $\left\{ \frac{\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(r+\frac{1}{p}\right)} \right\}^p$ *is the best constant in (1.25).*

Here $\Gamma(t) = \int_0^\infty s^{t-1} e^{-s} ds$ is the gamma function.

By the results of [49] we have the following theorem.

Theorem 1.26. *Let* $1 < p < \infty$ *and* $n \geq 1$. *Then the inequality*

$$\int_a^b \left(\int_a^x (x-t)^{n-1} f(t)\, dt \right)^p u(x)\, dx \leq C \int_a^b v(x) f^p(x)\, dx \quad (1.26)$$

holds for all measurable functions $f(x) \geq 0$ *on* (a, b) *if and only if*

$$B_n = \max\{B_{1,n},\ B_{2,n}\} < \infty.$$

Moreover, the best constant C *in (1.26) satisfies*

$$B_n \leq C \leq \beta B_n$$

with a constant $\beta \geq 1$ *depending only of* p *and* $n \geq 1$, *where*

$$B_{1,n} = \sup_{z \in (a,b)} \int_z^b u(t)\, dt \left(\int_a^z (z-s)^{p'(n-1)} v^{1-p'}(s)\, ds \right)^{p-1},$$

$$B_{2,n} = \sup_{z \in (a,b)} \int_z^b (t-z)^{p(n-1)} u(t)\, dt \left(\int_a^z v^{1-p'}(s)\, ds \right)^{p-1}.$$

Theorem 1.27. *Let* $1 < p < \infty$ *and* $n \geq 1$. *Then the inequality*

$$\int_a^b \left(\int_x^b (t-x)^{n-1} f(t)\, dt \right)^p u(x)\, dx \leq C^* \int_a^b v(x) f^p(x)\, dx \quad (1.27)$$

holds for all measurable functions $f(x) \geq 0$ *on* (a, b) *if and only if*

$$B_n^* = \max\{B_{1,n}^*,\ B_{2,n}^*\} < \infty.$$

Moreover, the best constant C^* *in (1.27) satisfies*

$$B_n^* \leq C^* \leq \beta^* B_n^*$$

with a constant $\beta^* \geq 1$ *dependent only of* p *and* $n \geq 1$, *where*

$$B^*_{1,n} = \sup_{z\in(a,b)} \int_a^z u(t)\,dt \left(\int_z^b (s-z)^{p'(n-1)} v^{1-p'}(s)\,ds \right)^{p-1},$$

$$B^*_{2,n} = \sup_{z\in(a,b)} \int_a^z (z-t)^{p(n-1)} u(t)\,dt \left(\int_z^b v^{1-p'}(s)\,ds \right)^{p-1}.$$

Chapter 2

Hardy differential inequality for a set of smooth functions with compact support

2.1 The closure of function set with compact support in the weighted Sobolev space.

Let $I = (a, b)$, $-\infty \leq a < b \leq \infty$. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let ρ be a positive and continuous function on I.

Let $L_{p,\rho} \equiv L_{p,\rho}(I)$ be the space of almost everywhere finite measurable functions f on I, for which the following norm

$$||f||_{p,\rho} = \left(\int_a^b \rho(t)|f(t)|^p dt \right)^{\frac{1}{p}}$$

is finite.

We denote by $W^1_{p,\rho} \equiv W^1_p(\rho, I)$ the set of locally absolutely continuous functions f on I with finite norm

$$||f||_{W^1_{p,\rho}} = ||f'||_{p,\rho} + |f(t_0)|, \tag{2.1}$$

where $t_0 \in I$ is a some fixed point.

Let $\overset{\circ}{AC}_p\,(I) = \{f \in W^1_{p,\rho}:\ \text{such } f \text{ compact, } suppf \subset I\}$. Let $AC_{p,l}(I)$ and $AC_{p,r}(I)$ be the set of functions from $W^1_{p,\rho}$ which vanish in at least a left neighborhood of a or a right neighborhood of b, respectively.

We denote by $\overset{\circ}{W}{}^1_p(\rho, I)$, $W^1_{p,l}(\rho, I)$ and $W^1_{p,r}(\rho, I)$ the closures of $\overset{\circ}{AC}_p\,(I)$, $AC_{p,l}(I)$ and $AC_{p,r}(I)$ in the space $W^1_{p,\rho}$, respectively.

Further we need to understand the relation between the spaces $\overset{\circ}{W}{}^1_p(\rho, I)$, $W^1_{p,l}(\rho, I)$, $W^1_{p,r}(\rho, I)$ and $W^1_p(\rho, I)$ depending on the integral behavior of the function $\rho^{1-p'}$ at the end points of the interval I. Such relations can be found in the literature (see e.g. [39], [40]). However, for the sake of competency, we present such results in a form which is suitable for our purpose.

Theorem 2.1. *Let $1 < p < \infty$. Then*

(i) if $\rho^{1-p'} \in L_1(I)$, then for any function $f \in W^1_p(\rho, I)$ the limits $\lim\limits_{t\to a+} f(t) \equiv f(a)$, $\lim\limits_{t\to b-} f(t) \equiv f(b)$ exist and

$$\overset{\circ}{W}{}^1_p(\rho, I) = \{f \in W^1_p(\rho, I):\ \ f(a) = f(b) = 0\},$$

$$W^1_{p,l}(\rho, I) = \{f \in W^1_p(\rho, I):\ \ f(a) = 0\},$$

$$W^1_{p,r}(\rho, I) = \{f \in W^1_p(\rho, I):\ \ f(b) = 0\};$$

(ii) if $\rho^{1-p'} \in L_1(a, c)$ and $\rho^{1-p'} \notin L_1(c, b)$, $c \in I$, then for any function $f \in W^1_p(\rho, I)$ the limit $\lim\limits_{t\to a+} f(t) \equiv f(a)$ exists and

$$\overset{\circ}{W}{}^1_p(\rho, I) = W^1_{p,l}(\rho, I) = \{f \in W^1_p(\rho, I):\ \ f(a) = 0\},$$

$$W^1_{p,r}(\rho, I) = W^1_p(\rho, I);$$

(iii) if $\rho^{1-p'} \notin L_1(a, c)$ and $\rho^{1-p'} \in L_1(c, b)$, $c \in I$, then for any function $f \in W^1_p(\rho, I)$ the limit $\lim\limits_{t\to b-} f(t) \equiv f(b)$ exists and

$$\overset{\circ}{W}{}^1_p(\rho, I) = W^1_{p,r}(\rho, I) = \{f \in W^1_p(\rho, I):\ \ f(b) = 0\},$$

$$W^1_{p,l}(\rho, I) = W^1_p(\rho, I);$$

(iv) if $\rho^{1-p'} \notin L_1(a, c)$ and $\rho^{1-p'} \notin L_1(c, b)$, $c \in I$, then

$$\overset{\circ}{W}{}^1_p(\rho, I) = W^1_{p,l}(\rho, I) = W^1_{p,r}(\rho, I) = W^1_p(\rho, I).$$

Proof of Theorem 2.1. Part (i). Let $\rho^{1-p'} \in L_1(I)$. Then $\rho^{1-p'} \in L_1(a, t_0)$. By the Hölder's inequality we obtain

$$\int_a^{t_0} |f'(t)|\,dt \le \left(\int_a^{t_0} \rho^{1-p'}\right)^{\frac{1}{p'}} \left(\int_a^{b} \rho|f'(t)|^p\,dt\right)^{\frac{1}{p}} < \infty,$$

for all $f \in W_p^1(\rho, I)$.

Therefore the integral $\int_a^{t_0} f'(t)\,dt$ exists. By the Newton-Liebnitz formula for absolutely continuous functions the limit

$$f(a) \equiv \lim_{t\to a+} f(t) = f(t_0) - \lim_{t\to a+} \int_t^{t_0} f'(s)ds = f(c) - \int_a^{t_0} f'(s)ds$$

exists.

The proof of existence of the limit $\lim_{t\to b-} f(t) \equiv f(b)$ is similar.

Let $f \in W_{p,l}^1(\rho, I)$. Then there exists a sequence $\{f_n\} \subset AC_{p,l}(I)$ such that $||f - f_n||_{W_{p,\rho}^1} \to 0$ if $n \to \infty$. If $a < t < t_0 < b$, then

$$|f(t) - f_n(t)| \le \int_t^{t_0} |f'(s) - f_n'(s)|\,ds + |f(t_0) - f_n(t_0)|.$$

Then applying Hölder's inequality we obtain:

$$\sup_{a<t<t_0} |f(t) - f_n(t)| \le \left(\int_a^{t_0} \rho^{1-p'}(s)\,ds\right)^{\frac{1}{p'}} \left(\int_t^{t_0} \rho(s)|f'(s) - f_n'(s)|^p\,ds\right)^{\frac{1}{p}} +$$
$$|f(t_0) - f_n(t_0)| \le \max\left\{1, \left(\int_a^{t_0} \rho^{1-p'}(s)ds\right)^{\frac{1}{p'}}\right\} ||f - f_n||_{W_{p,\rho}^1}.$$

Hence, $\lim_{n\to\infty} \sup_{a<t<t_0} |f(t) - f_n(t)| = 0$. Therefore $\lim_{t\to a+} f(t) \equiv f(a) = 0$. Indeed, $\lim_{t\to a+} f_n(t) = 0$.

The proof of equality $f(b) = 0$ for any function $f \in W_{p,r}^1(\rho, I)$ is similar.

Since $\overset{\circ}{W}{}_p^1(\rho, I) \subset W_{p,l}^1(\rho, I) \bigcap W_{p,r}^1(\rho, I)$, then for any $f \in \overset{\circ}{W}{}_p^1(\rho, I)$ we have $f(a) = f(b) = 0$.

Therefore,

$$W_{p,l}^1(\rho, I) \subset \{f \in W_p^1(\rho, I) : \quad f(a) = 0\},$$

$$W_{p,r}^1(\rho, I) \subset \{f \in W_p^1(\rho, I) : \quad f(b) = 0\},$$

$$\overset{\circ}{W}{}_p^1(\rho, I) \subset \{f \in W_p^1(\rho, I) : \quad f(a) = f(b) = 0\}.$$

We now prove the inverse inclusion. Let $f \in W_p^1(\rho, I)$, $f(a) = 0$. Let $a < \alpha < t_0$. Then by Hölder's inequality we obtain

$$|f(\alpha)| \leq \int_a^\alpha |f'(s)|\,ds \leq \left(\int_a^\alpha \rho^{1-p'}\,ds\right)^{\frac{1}{p'}} \left(\int_a^\alpha \rho|f'(s)|^p\,ds\right)^{\frac{1}{p}}.$$

Hence,

$$|f(\alpha)| \left(\int_a^\alpha \rho^{1-p'}\,ds\right)^{-\frac{1}{p'}} \leq \left(\int_a^\alpha \rho|f'(s)|^p\,ds\right)^{\frac{1}{p}}. \tag{2.2}$$

Let the point $\alpha^* = \alpha^*(\alpha, a) \in (a, \alpha)$ be such that

$$\int_{\alpha^*}^\alpha \rho^{1-p'}\,ds = \int_a^{\alpha^*} \rho^{1-p'}\,ds. \tag{2.3}$$

Next we introduce the function

$$f_\alpha(t) = \begin{cases} 0, & a < t < \alpha^* \\ f(\alpha)\left(\int_{\alpha^*}^t \rho^{1-p'}\,ds\right)\left(\int_{\alpha^*}^\alpha \rho^{1-p'}\,ds\right)^{-1}, & \alpha^* \leq t \leq \alpha \\ f(t), & \alpha < t < b. \end{cases} \tag{2.4}$$

Clearly, that $f_\alpha \in AC_{p,l}(I)$. By (2.2), we have

$$||f - f_\alpha||_{W^1_{p,\rho}} =$$

$$\left(\int_a^b \rho|f' - f'_\alpha|^p\,ds\right)^{\frac{1}{p}} + |f(t_0) - f_\alpha(t_0)| =$$

$$\left(\int_a^\alpha \rho|f' - f'_\alpha|^p\,ds\right)^{\frac{1}{p}} \leq$$

$$\left(\int_a^\alpha \rho|f'|^p\,ds\right)^{\frac{1}{p}} + \left(\int_a^\alpha \rho|f'_\alpha|^p\,ds\right)^{\frac{1}{p}} =$$

$$\left(\int_a^\alpha \rho|f'|^p\,ds\right)^{\frac{1}{p}} + |f(\alpha)|\left(\int_{\alpha^*}^\alpha \rho^{1-p'}\,ds\right)^{-1}\left(\int_{\alpha^*}^\alpha \rho\rho^{p(1-p')}\,ds\right)^{\frac{1}{p}} =$$

$$\left(\int_a^\alpha \rho|f'|^p\,ds\right)^{\frac{1}{p}} + |f(\alpha)|\left(\int_{\alpha^*}^\alpha \rho^{1-p'}\,ds\right)^{-\frac{1}{p'}} =$$

$$\left(\int_a^\alpha \rho|f'|^p\,ds\right)^{\frac{1}{p}} + 2^{\frac{1}{p'}}|f(\alpha)|\left(\int_a^\alpha \rho^{1-p'}\,ds\right)^{-\frac{1}{p'}} \leq$$

$$\left(1+2^{\frac{1}{p'}}\right)\left(\int_a^\alpha \rho|f'|^p ds\right)^{\frac{1}{p}}.$$

Hence, it follows that $||f - f_\alpha||_{W^1_{p,\rho}} \to 0$ if $\alpha \to 0$. Consequently, $f \in W^1_{p,l}(\rho, I)$ and thus $W^1_{p,l}(\rho, I) = \{f \in W^1_p(\rho, I) : \ f(a) = 0\}$.

If $\beta \in (t_0, b)$, we introduce the following function $f_\beta \in AC_{p,r}(I)$

$$f_\beta(t) = \begin{cases} f(t), & a < t < \beta \\ f(\beta)\left(\int\limits_t^{\beta^*} \rho^{1-p'}\,ds\right)\left(\int\limits_\beta^{\beta^*} \rho^{1-p'}\,ds\right)^{-1}, & \beta \leq t \leq \beta^* \\ 0, & \beta^* < t < b, \end{cases} \tag{2.5}$$

where $\beta^* = \beta^*(\beta, b) \in (\beta, b)$ is such that

$$\int_\beta^{\beta^*} \rho^{1-p'}\,ds = \int_{\beta^*}^b \rho^{1-p'}\,ds. \tag{2.6}$$

As indicated above we check that $||f - f_\beta||_{W^1_{p,\rho}} \to 0$ if $\beta \to b$, which in turn implies that $f \in W^1_{p,r}(\rho, I)$. Hence, $W^1_{p,r}(\rho, I) = \{f \in W^1_p(\rho, I) : \ \ f(b) = 0\}$.

Now we introduce the function $f_{\alpha,\beta} \in \overset{\circ}{AC}_p(I)$ by setting

$$f_{\alpha,\beta}(t) = \begin{cases} 0, & a < t < \alpha^* \\ f(\alpha)\left(\int\limits_{\alpha^*}^{t} \rho^{1-p'}\,ds\right)\left(\int\limits_{\alpha^*}^{\alpha} \rho^{1-p'}\,ds\right)^{-1}, & \alpha^* \leq t \leq \alpha \\ f(t), & \alpha < t < \beta \\ f(\beta)\left(\int\limits_t^{\beta^*} \rho^{1-p'}\,ds\right)\left(\int\limits_\beta^{\beta^*} \rho^{1-p'}\,ds\right)^{-1}, & \beta \leq t \leq \beta^* \\ 0, & \beta^* < t < b, \end{cases} \tag{2.7}$$

where α, α^*, β and β^* are chosen as above. Simple computations show that

$$||f - f_{\alpha,\beta}||_{W^1_{p,\rho}} \leq \left(1+2^{\frac{1}{p'}}\right)\left(\int_a^\alpha \rho|f'|^p\,dt\right)^{\frac{1}{p}} + \left(1+2^{\frac{1}{p'}}\right)\left(\int_\beta^b \rho|f'|^p dt\right)^{\frac{1}{p}}. \tag{2.8}$$

Hence, $||f - f_{\alpha,\beta}||_{W^1_{p,\rho}} \to 0$, if $\alpha \to a$ and $\beta \to b$. In other words $f \in \overset{\circ}{W}{}^1_p(\rho, I)$ and $\overset{\circ}{W}{}^1_p(\rho, I) = \{f \in W^1_p(\rho, I) : \ f(a) = f(b) = 0\}$.

The proof of Part (i) is complete.

Part (ii). Let $\rho^{1-p'} \in L_1(a, t_0)$ and $\rho^{1-p'} \notin L_1(t_0, b)$. As in the proof of part (i), the condition $\rho^{1-p'} \in L_1(a, t_0)$ implies that for any $f \in W^1_p(\rho, I)$ there exists $f(a)$ and $f(a) = 0$ for any $f \in W^1_{p,l}(\rho, I)$, and that $W^1_{p,l}(\rho, I) = \{f \in W^1_p(\rho, I) : \ f(a) = 0\}$.

Since $\overset{\circ}{W}{}^1_p(\rho, I) \subset W^1_{p,l}(\rho, I)$, then $\overset{\circ}{W}{}^1_p(\rho, I) \subset \{f \in W^1_p(\rho, I) : \ f(a) = 0\}$. We now prove the converse inclusion. Let $f \in W^1_p(\rho, I)$ and $f(a) = 0$. We denote by $f_{\alpha,\beta} \in \overset{\circ}{AC}_p\ (I)$ the function defined in (2.7), where $a < \alpha < t_0$, $t_0 < \beta < b$ and where $\alpha^* = \alpha^*(a, \alpha) \in (a, \alpha)$ has been defined in (2.3). Here we define the point β^* following way. By the condition $\rho^{1-p'} \notin L_1(t_0, b)$ it follows that $\int\limits_\beta^b \rho^{1-p'}\, ds = \infty$. Therefore for each $\beta \in (t_0, b)$ there exists a point $\beta^* = \beta^*(\beta, b) \in (\beta, b)$ such that

$$|f(\beta)| \left(\int\limits_\beta^{\beta^*} \rho^{1-p'}\, ds \right)^{-\frac{1}{p'}} \leq \left(\int\limits_\beta^b \rho(t)|f'(t)|^p\, dt \right)^{\frac{1}{p}}. \tag{2.9}$$

Then by estimating as above we obtain (2.8). Hence, it follows that $||f - f_{\alpha,\beta}||_{W^1_{p,\rho}} \to 0$ if $\alpha \to a$ and $\beta \to b$. Consequently, $f \in \overset{\circ}{W}{}^1_p(\rho, I)$ and $\overset{\circ}{W}{}^1_p(\rho, I) = W^1_{p,l}(\rho, I) = \{f \in W^1_p(\rho, I) : \ f(a) = 0\}$.

We now prove the equality $W^1_{p,r}(\rho, I) = W^1_p(\rho, I)$. It is suffices to show that $W^1_{p,r}(\rho, I) \supset W^1_p(\rho, I)$. Let $f \in W^1_p(\rho, I)$. Let $\beta \in (t_0, b)$. Let the point $\beta^* \in (\beta, b)$ be chosen as in the condition (2.9). We denote by f_β the function defined by the relation (2.5).

Obviously, $f_\beta \in AC_{p,r}(I)$. Then by taking into account that $f(t_0) = f_\beta(t_0)$ and (2.9) we have

$$||f - f_\beta||_{W^1_{p,\rho}} =$$

$$\left(\int\limits_a^b \rho(t)|f'(t) - f'_\beta(t)|^p dt \right)^{\frac{1}{p}} + |f(t_0) - f_\beta(t_0)| =$$

$$\left(\int_a^b \rho(t)|f'(t)-f'_\beta(t)|^p dt\right)^{\frac{1}{p}} =$$

$$\left(\int_\beta^b \rho(t)|f'(t)-f'_\beta(t)|^p dt\right)^{\frac{1}{p}} \leq$$

$$\left(\int_\beta^b \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}} + |f(\beta)| \left(\int_\beta^{\beta^*} \rho^{1-p'} dt\right)^{-1} \left(\int_\beta^{\beta^*} \rho\rho^{p(1-p')} dt\right)^{\frac{1}{p}} =$$

$$\left(\int_\beta^b \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}} + |f(\beta)| \left(\int_\beta^{\beta^*} \rho^{1-p'} dt\right)^{-\frac{1}{p}} =$$

$$2\left(\int_\beta^b \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}}.$$

Therefore, $||f - f_\beta||_{W^1_{p,\rho}} \to 0$, if $\beta \to b$. Hence, $f \in W^1_p(\rho, I)$ and $W^1_{p,r}(\rho, I) = W^1_p(\rho, I)$.

The proof of Part (ii) is complete.

Part (iii). Let $\rho^{1-p'} \notin L_1(a, t_0)$ and $\rho^{1-p'} \in L_1(t_0, b)$. The proof of part (i) implies that for any $f \in W^1_p(\rho, I)$ the limit $f(\beta)$ exists and thus $W^1_{p,r}(\rho, I) = \{f \in W^1_p(\rho, I) : \ f(b) = 0\}$.

Therefore, $\overset{\circ}{W}{}^1_p(\rho, I) \subset \{f \in W^1_p(\rho, I) : \ f(b) = 0\}$. To prove the converse inclusion, we consider the function $f_{\alpha,\beta} \in \overset{\circ}{AC}_p (I)$, which is defined by (2.7), where β^* is defined as in (2.6) and the point $\alpha^* \in (a, \alpha)$ is defined as follows. Since $\rho^{1-p'} \notin L_1(a, t_0)$, then $\int_a^\alpha \rho^{1-p'}\, ds = \infty$ for any $\alpha \in (a, t_0)$.

We choose the point $\alpha^* = \alpha^*(a, \alpha) \in (a, \alpha)$ such that

$$|f(\alpha)| \left(\int_{\alpha^*}^\alpha \rho^{1-p'}\, ds\right)^{-\frac{1}{p'}} \leq \left(\int_a^\alpha \rho(t)|f'(t)|^p\, dt\right)^{\frac{1}{p}}. \tag{2.10}$$

Then according to (2.2), (2.6), (2.10) and $f(t_0) = f_{\alpha,\beta}(t_0)$, we obtain

$$||f - f_{\alpha,\beta}||_{W^1_{p,\rho}} = \left(\int_a^b \rho(t)|f'(t) - f'_{\alpha,\beta}(t)|^p\, dt\right)^{\frac{1}{p}} + |f(t_0) - f_{\alpha,\beta}(t_0)| =$$

$$\left(\int_a^\alpha \rho(t)|f'(t) - f'_{\alpha,\beta}(t)|^p\, dt + \int_\alpha^\beta \rho(t)|f'(t) - f'_{\alpha,\beta}(t)|^p\, dt+\right.$$

$$\left.\int_\beta^b \rho(t)|f'(t)-f'_{\alpha,\beta}(t)|^p\,dt\right)^{\frac{1}{p}} \le$$

$$\left(\int_a^\alpha \rho(t)|f'(t)-f'_{\alpha,\beta}(t)|^p\,dt\right)^{\frac{1}{p}} + \left(\int_\beta^b \rho(t)|f'(t)-f'_{\alpha,\beta}(t)|^p\,dt\right)^{\frac{1}{p}} \le$$

$$\left(\int_a^\alpha \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_a^\alpha \rho(t)|f'_{\alpha,\beta}(t)|^p dt\right)^{\frac{1}{p}} +$$

$$\left(\int_\beta^b \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_\beta^b \rho(t)|f'_{\alpha,\beta}(t)|^p dt\right)^{\frac{1}{p}} =$$

$$\left(\int_a^\alpha \rho(t)|f'(t)|^p\,dt\right)^{\frac{1}{p}} + |f(\alpha)|\left(\int_{\alpha^*}^\alpha \rho^{1-p'}\,dt\right)^{-1}\left(\int_{\alpha^*}^\alpha \rho\rho^{p(1-p')}\,dt\right)^{\frac{1}{p}} +$$

$$\left(\int_\beta^b \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}} + |f(\beta)|\left(\int_\beta^{\beta^*} \rho^{1-p'}dt\right)^{-1}\left(\int_\beta^{\beta^*} \rho\rho^{p(1-p')}dt\right)^{\frac{1}{p}} =$$

$$\left(\int_a^\alpha \rho(t)|f'(t)|^p\,dt\right)^{\frac{1}{p}} + |f(\alpha)|\left(\int_{\alpha^*}^\alpha \rho^{1-p'}\,dt\right)^{-\frac{1}{p'}} +$$

$$\left(\int_\beta^b \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}} + |f(\beta)|\left(\int_\beta^{\beta^*} \rho^{1-p'}dt\right)^{-\frac{1}{p'}} =$$

$$\left(\int_a^\alpha \rho(t)|f'(t)|^p\,dt\right)^{\frac{1}{p}} + |f(\alpha)|\left(\int_{\alpha^*}^\alpha \rho^{1-p'}\,dt\right)^{-\frac{1}{p'}} +$$

$$\left(\int_\beta^b \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}} + 2^{\frac{1}{p'}}|f(\beta)|\left(\int_\beta^b \rho^{1-p'}dt\right)^{-\frac{1}{p'}} \le$$

$$2\left(\int_a^\alpha \rho(t)|f'(t)|^p\,dt\right)^{\frac{1}{p}} + \left(1+2^{\frac{1}{p'}}\right)\left(\int_\beta^b \rho(t)|f'(t)|^p\,dt\right)^{\frac{1}{p}}. \qquad (2.11)$$

Therefore, $||f-f_{\alpha,\beta}||_{W^1_{p,\rho}} \to 0$, if $\alpha \to a$, $\beta \to b$. Hence, $f \in \overset{\circ}{W}{}^1_p(\rho, I)$. Hence, $\overset{\circ}{W}{}^1_p(\rho, I) = \{f \in W^1_p(\rho, I): \ \ f(b) = 0\} = W^1_{p,r}(\rho, I)$.

We now prove equality $W^1_{p,l}(\rho, I) = W^1_p(\rho, I)$. It is suffices to show the inclusion $W^1_{p,l}(\rho, I) \supset W^1_p(\rho, I)$. In other words, to show that from $f \in W^1_p(\rho, I)$ it follows that $f \in W^1_{p,l}(\rho, I)$. Let $f \in W^1_p(\rho, I)$. We consider the function $f_\alpha \in AC_{p,r}(I)$ defined by formula (2.4), where the point $\alpha^* \in (a, \alpha)$ is defined by the condition (2.10). Then

$$||f - f_\alpha||_{W^1_{p,\rho}} =$$

$$\left(\int_a^b \rho(t)|f'(t) - f'_\alpha(t)|^p dt\right)^{\frac{1}{p}} + |f(t_0) - f_\alpha(t_0)| =$$

$$\left(\int_a^\alpha \rho(t)|f'(t) - f'_\alpha(t)|^p dt + \int_\alpha^b \rho(t)|f'(t) - f'_\alpha(t)|^p dt\right)^{\frac{1}{p}} =$$

$$\left(\int_a^\alpha \rho(t)|f'(t) - f'_\alpha(t)|^p dt\right)^{\frac{1}{p}} \leq$$

$$\left(\int_a^\alpha \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_a^\alpha \rho(t)|f'_\alpha(t)|^p dt\right)^{\frac{1}{p}} =$$

$$\left(\int_a^\alpha \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}} + |f(\alpha)|\left(\int_{\alpha^*}^\alpha \rho^{1-p'} dt\right)^{-1}\left(\int_{\alpha^*}^\alpha \rho\rho^{p(1-p')} dt\right)^{\frac{1}{p}} =$$

$$\left(\int_a^\alpha \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}} + |f(\alpha)|\left(\int_{\alpha^*}^\alpha \rho^{1-p'} dt\right)^{-\frac{1}{p'}} \leq$$

$$2\left(\int_a^\alpha \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}}.$$

Therefore, $||f - f_\alpha||_{W^1_{p,\rho}} \to 0$, if $\alpha \to a$. Consequently, $f \in W^1_{p,l}(\rho, I)$.

The proof of Part (iii) is complete.

Part (iv). Let $\rho^{1-p'} \notin L_1(a, t_0)$ and $\rho^{1-p'} \notin L_1(t_0, b)$. Since $\overset{\circ}{W}{}^1_p(\rho, I) \subset W^1_{p,l}(\rho, I)$ and $\overset{\circ}{W}{}^1_p(\rho, I) \subset W^1_{p,r}(\rho, I)$. It is suffices to show that $\overset{\circ}{W}{}^1_p(\rho, I) = W^1_p(\rho, I)$, that is $\overset{\circ}{W}{}^1_p(\rho, I) \supset W^1_p(\rho, I)$. Let $f \in W^1_p(\rho, I)$.

We consider the function $f_{\alpha,\beta} \in \overset{\circ}{AC}_p(I)$ defined by (2.7), where β^* and α^* are defined by the conditions (2.9) and (2.10), respectively.

Then on the basis of (2.9), (2.10) and (2.11) we obtain

$$||f - f_{\alpha,\beta}||_{W^1_{p,\rho}} \leq$$

$$\left(\int_a^\alpha \rho(t)|f'(t)|^p\, dt\right)^{\frac{1}{p}} + |f(\alpha)|\left(\int_{\alpha^*}^\alpha \rho^{1-p'}\, dt\right)^{-\frac{1}{p'}} +$$

$$\left(\int_a^\alpha \rho(t)|f'(t)|^p dt\right)^{\frac{1}{p}} + |f(\beta)| \left(\int_\beta^{\beta^*} \rho^{1-p'} dt\right)^{-\frac{1}{p'}} \leq$$

$$2\left(\int_a^\alpha \rho(t)|f'(t)|^p\, dt\right)^{\frac{1}{p}} + 2\left(\int_\beta^b \rho(t)|f'(t)|^p\, dt\right)^{\frac{1}{p}}.$$

Therefore, $||f - f_{\alpha,\beta}||_{W^1_{p,\rho}} \to 0$, if $\alpha \to a$, $\beta \to b$. Consequently, $\overset{\circ}{W}{}^1_p(\rho, I) \supset W^1_p(\rho, I)$.

Thus the proof of Theorem 2.1 is complete. □

We now introduce some examples.

Let $a = 0$, $b = +\infty$ and thus $I = R_+$. Let $\rho(t) = t^\gamma$, $\gamma \in R$. If $\gamma \leq p - 1$, then $\gamma(1 - p') + 1 \geq 0$ and thus $\rho^{1-p'}(t) \equiv t^{\gamma(1-p')} \in L_1(0, 1)$ and $\rho^{1-p'}(t) \equiv t^{\gamma(1-p')} \notin L_1(1, \infty)$. In other words the assumptions of part (ii) of Theorem 2.1 hold. Hence,

$$\overset{\circ}{W}{}^1_p(t^\gamma, R_+) = W^1_{p,l}(t^\gamma, R_+) = \{f \in W^1_p(t^\gamma, R_+) : \lim_{t\to 0+} f(t) \equiv f(0) = 0\}, \tag{2.12}$$

$$W^1_{p,r}(t^\gamma, R_+) = W^1_p(t^\gamma, R_+). \tag{2.13}$$

If $\gamma > p - 1$, then $\gamma(1 - p') + 1 < 0$ and thus $\rho^{1-p'}(t) \equiv t^{\gamma(1-p')} \notin L_1(0, 1)$ and $\rho^{1-p'}(t) \equiv t^{\gamma(1-p')} \in L_1(1, \infty)$. It means that the assumptions of part (iii) of Theorem 2.1 hold. Hence,

$$\overset{\circ}{W}{}^1_p(t^\gamma, R_+) = W^1_{p,r}(t^\gamma, R_+) = \{f \in W^1_p(t^\gamma, R_+) : \lim_{t\to\infty} f(t) \equiv f(\infty) = 0\}, \tag{2.14}$$

$$W^1_{p,l}(t^\gamma, R_+) = W^1_p(t^\gamma, R_+). \tag{2.15}$$

Therefore, the following propositions hold.

Proposition 2.2. *Let $1 < p < \infty$. If $\gamma \leq p - 1$, then (2.12) and (2.13) hold. If $\gamma < p - 1$, then (2.14) and (2.15) hold.*

Assume that

$$\rho_0(t) = \begin{cases} t^\gamma, & 0 < t \leq 1 \\ t^\mu, & 1 \leq t < \infty, \end{cases}$$

where $\gamma \in R$, $\mu \in R$.

By the above calculation it follows that if $\gamma \le p-1$ and $\mu > p-1$, then $\rho_0^{1-p'} \in L_1(R_+)$. If $\gamma \le p-1$ and $\mu \le p-1$, then $\rho_0^{1-p'} \in L_1(0,1)$ and $\rho_0^{1-p'} \notin L_1(1,\infty)$. If $\gamma > p-1$ and $\mu > p-1$, then $\rho_0^{1-p'} \notin L_1(0,1)$ and $\rho_0^{1-p'} \in L_1(1,\infty)$. If $\gamma > p-1$ and $\mu \le p-1$, then $\rho_0^{1-p'} \notin L_1(0,1)$ and $\rho_0^{1-p'} \notin L_1(1,\infty)$.

Then Theorem 2.1 implies that

Proposition 2.3. *Let $1 < p < \infty$. Then*

(i) if $\gamma \le p-1$ and $\mu > p-1$, then for any $f \in W_p^1(\rho_0, R_+)$ the limits $f(0)$ and $f(\infty)$ exist and

$$\overset{\circ}{W}{}_p^1(\rho_0, R_+) = \{f \in W_p^1(\rho_0, R_+): \ \ f(0) = f(\infty) = 0\},$$

$$W_{p,l}^1(\rho_0, R_+) = \{f \in W_p^1(\rho_0, R_+): \ \ f(0) = 0\},$$

$$W_{p,r}^1(\rho_0, R_+) = \{f \in W_p^1(\rho_0, R_+): \ \ f(\infty) = 0\};$$

(ii) if $\gamma \le p-1$ and $\mu \le p-1$, then for any $f \in W_p^1(\rho_0, R_+)$ the limit $f(0)$ exists and

$$\overset{\circ}{W}{}_p^1(\rho_0, R_+) = W_{p,l}^1(\rho_0, R_+) = \{f \in W_p^1(\rho_0, R_+): \ \ f(0) = 0\},$$

$$W_{p,r}^1(\rho_0, R_+) = W_p^1(\rho_0, R_+);$$

(iii) if $\gamma > p-1$ and $\mu > p-1$, then for any $f \in W_p^1(\rho_0, R_+)$ the limit $f(\infty)$ exists and

$$\overset{\circ}{W}{}_p^1(\rho_0, R_+) = W_{p,r}^1(\rho_0, R_+) = \{f \in W_p^1(\rho_0, R_+): \ \ f(\infty) = 0\},$$

$$W_{p,l}^1(\rho_0, R_+) = W_p^1(\rho_0, R_+);$$

(iv) if $\gamma > p-1$ and $\mu \le p-1$, then

$$\overset{\circ}{W}{}_p^1(\rho_0, R_+) = W_{p,l}^1(\rho_0, R_+) = W_{p,r}^1(\rho_0, R_+) = W_p^1(\rho_0, R_+).$$

Remark 2.4. *The statement of Theorem 2.1. remains valid if it is assumed that the nonnegative function ρ is measurable and almost everywhere finite on I. The function $\rho^{1-p'}$ is locally summable on I.*

2.2 The main results

In the set $\overset{\circ}{W}{}_p^1(\rho, I)$ we consider the following inequality

$$\int_a^b v(t)|f(t)|^p\,dt \le C\int_a^b \rho(t)|f'(t)|^p\,dt, \tag{2.16}$$

where $1 < p < \infty$, v is a nonnegative function on I, and ρ is a positive continuous function on I. Moreover, $v \neq 0$ on I. The inequality (2.16) is a Hardy inequality in differential form.

Remark 2.5. *The assumption of the continuity of the functions v and ρ is connected with this fact that the results of the inequality (2.16) will apply further when these functions are continuous. However, the results we obtain below hold when the functions v and ρ are nonnegative and measurable on I and the functions v^p, ρ^p and $\rho^{-p'}$ are locally summable on I.*

The inequality (2.16) in the set $\overset{\circ}{W}{}_p^1(\rho, I)$ has been considered in the work of [27, § 8], [43, § 1.2, § 4.6]. Only in [27, § 8] the two-sided estimates of the best constant $C > 0$ in (2.16) has been obtained.

Here, we obtain the more general results that include the results of the papers indicated above, and we give a more precise two-sided estimate of the best constant $C > 0$ in (2.16), than in [27, § 8].

Assume that

$$A_1(a,b,x) = \left(\int_a^x \rho^{1-p'}\,ds\right)^{p-1}\int_x^b v\,dt,$$

$$A_1^*(a,b,x) = \left(\int_x^b \rho^{1-p'}\,ds\right)^{p-1}\int_a^x v\,dt,$$

$$A_2(a,b,x) = \left(\int_a^x \rho^{1-p'}\,ds\right)^{-1}\int_a^x v(t)\left(\int_a^t \rho^{1-p'}\,ds\right)^p dt,$$

$$A_2^*(a,b,x) = \left(\int_x^b \rho^{1-p'}\,ds\right)^{-1}\int_x^b v(t)\left(\int_t^b \rho^{1-p'}ds\right)^p dt,$$

$$A_i(a,b) = \sup_{a<x<b} A_i(a,b,x),$$

$$A_i^*(a,b) = \sup_{a<x<b} A_i^*(a,b,x), \quad i = 1,\ 2.$$

We consider the inequality (2.16) depending on the integral behavior of the function $\rho^{1-p'}$ at the end points of I.

Later, we find convenient to denote by $J_0(a,b)$ the best constant $C > 0$ in (2.16).

Let

$$\int\limits_a^b \rho^{1-p'}(s)\, ds < \infty. \tag{2.17}$$

Definition 2.6. *The point $c_i \in I$, $i = 1, 2$ is called a middle point for (A_i, A_i^*) if $A_i(a, c_i) = A_i^*(c_i, b) \equiv T_{c_i}(a,b) < \infty$, $i = 1, 2$.*

Theorem 2.7. *Let $1 < p < \infty$. Let (2.17) hold. Then inequality (2.16) holds on the set $\overset{\circ}{W}{}_p^1(\rho, I)$ if and only if exists a middle point $c_i \in I$ for (A_i, A_i^*) for at least one $i = 1, 2$. At the same time for the best constant $J_0(a,b)$ in (2.16) the following estimate holds*

$$\max\{T_{c_1}(a,b),\ T_{c_2}(a,b)\} \le J_0(a,b) \le \\ \min\left\{p\left(\frac{p}{p-1}\right)^{p-1} T_{c_1}(a,b),\ \left(\frac{p}{p-1}\right)^{p} T_{c_2}(a,b)\right\}. \tag{2.18}$$

Theorem 2.7 extends the different estimates of $J_0(a,b)$ given in [27, § 8].

For example, in the assumption $A_1(a,a) = A_1^*(b,b) = 0$ of Theorem 8.8 [27] the following inequality has been proved

$$\frac{1}{2} A \le J_0(a,b) \le p\left(\frac{p}{p-1}\right)^{p-1} A,$$

where $A = \inf\limits_{a<c<b} \max\{A_1(a,c),\ A_1^*(c,b)\}$. In these assumptions, it is easy to prove that $A = T_{c_1}(a,b)$.

We consider the following case

$$\int\limits_a^c \rho^{1-p'}(s)ds < \infty, \quad \int\limits_c^b \rho^{1-p'}(s)ds = \infty, \quad c \in I. \tag{2.19}$$

Theorem 2.8. *Let $1 < p < \infty$. Let (2.19) hold. Then the inequality (2.16) holds on the set $\overset{\circ}{W}{}_p^1(\rho, I)$ if and only if $A_i(a,b) < \infty$ for at least one $i = 1, 2$.*

At the same time for the best constant $J_0(a,b)$ in (2.16) the following estimate holds

$$\max\{A_1(a,b),\ A_2(a,b)\} \le J_0(a,b) \le \min\left\{p\left(\frac{p}{p-1}\right)^{p-1} A_1(a,b),\ \left(\frac{p}{p-1}\right)^{p} A_2(a,b)\right\}. \quad (2.20)$$

Now let

$$\int_a^c \rho^{1-p'}(s)ds = \infty, \quad \int_c^b \rho^{1-p'}(s)ds < \infty, \quad c \in I. \qquad (2.21)$$

Theorem 2.9. *Let $1 < p < \infty$. Let (2.21) hold. Then inequality (2.16) holds on the set $\overset{\circ}{W}{}^1_p(\rho, I)$ if and only if $A_i^*(a,b) < \infty$ for at least one $i = 1, 2$. At the same time the following estimate*

$$\max\{A_1^*(a,b),\ A_2^*(a,b)\} \le J_0(a,b) \le \min\left\{p\left(\frac{p}{p-1}\right)^{p-1} A_1^*(a,b),\ \left(\frac{p}{p-1}\right)^{p} A_2^*(a,b)\right\} \quad (2.22)$$

holds for the best constant $J_0(a,b)$ in (2.16).

Finally, let

$$\int_a^c \rho^{1-p'}(s)ds = \infty, \quad \int_c^b \rho^{1-p'}(s)ds = \infty, \quad c \in I. \qquad (2.23)$$

Theorem 2.10. *Let $1 < p < \infty$. Let (2.23) hold. Then inequality (2.16) does not hold on the set $\overset{\circ}{W}{}^1_p(\rho, I)$. Namely $J_0(a,b) = \infty$.*

2.3 Proof of the main results

In the proof of Theorem 2.7, we employ the following statement.

Lemma 2.11. *Let* $1 < p < \infty$. *Let (2.17) hold. Then a middle point for* (A_i, A_i^*), $i = 1, 2$ *exists if and only if*

$$\limsup_{x\to a} A_i(a,c,x) < \infty, \quad \limsup_{x\to b} A_i^*(c,b,x) < \infty, \;\; i = 1,2, \tag{2.24}$$

for all $c \in I$.

Proof of Lemma 2.11. Let a middle point $c_i \in I$ exists for (A_i, A_i^*), $i = 1, 2$. Then by the definition of middle point c_i, we have

$$A_i(a, c_i) = A_i^*(c_i, b) < \infty, \quad i = 1, 2.$$

If $c \geq c_1$ then by the condition (2.17), we obtain

$$\limsup_{x\to a} A_1(a,c,x) =$$
$$\lim_{t\to a}\sup_{a<x<t}\left(\int_a^x \rho^{1-p'}\,ds\right)^{p-1}\int_x^c v\,dt \leq$$
$$\sup_{a<x<c_1}\left(\int_a^x \rho^{1-p'}\,ds\right)^{p-1}\int_x^{c_1} v\,dt + \lim_{t\to a}\sup_{a<x<t}\left(\int_a^x \rho^{1-p'}\,ds\right)^{p-1}\int_{c_1}^c v\,dt =$$
$$A_1(a,c_1) + \lim_{t\to a}\left(\int_a^t \rho^{1-p'}\,ds\right)^{p-1}\int_{c_1}^c v\,dt =$$
$$A_1(a,c_1) < \infty,$$

$$\limsup_{x\to b} A_1^*(c,b,x) =$$
$$\lim_{t\to b}\sup_{t<x<b}\left(\int_x^b \rho^{1-p'}\,ds\right)^{p-1}\int_c^x v\,dt \leq$$
$$\sup_{c_1<x<b}\left(\int_x^b \rho^{1-p'}\,ds\right)^{p-1}\int_{c_1}^x v\,dt$$
$$= A_1^*(c_1, b) < \infty.$$

Similarly, we obtain

$$\limsup_{x\to a} A_1(a,c,x) \le A_1(a,c_1) < \infty,$$

$$\limsup_{x\to b} A_1^*(c,b,x) = \lim_{t\to b}\sup_{t<x<b}\left(\int_x^b \rho^{1-p'}\,ds\right)^{p-1}\int_c^x v\,dt \le$$
$$A_1^*(c_1,b) + \lim_{t\to a}\left(\int_t^b \rho^{1-p'}\,ds\right)^{p-1}\int_c^{c_1} v\,dt =$$
$$A_1^*(c_1,b) < \infty$$

in case $c < c_1$.

In the cases A_2 and A_2^*, we have

$$\limsup_{x\to a} A_2(a,c,x) =$$
$$\lim_{t\to a}\sup_{a<x<t}\left(\int_a^x \rho^{1-p'}\,ds\right)^{-1}\int_a^x v(t)\left(\int_a^t \rho^{1-p'}\,ds\right)^{p} dt \le$$
$$\sup_{a<x<c_1}\left(\int_a^x \rho^{1-p'}\,ds\right)^{-1}\int_a^x v(t)\left(\int_a^t \rho^{1-p'}\,ds\right)^{p} dt =$$
$$A_2(a,c_2) < \infty$$

for all $c \in I$.

Similarly,

$$\limsup_{x\to b} A_2(c,b,x) \le A_2(c_2,b) < \infty.$$

Conversely, let (2.24) hold. Then there exists $t_i \in (a,c)$ and $t_i^* \in (c,b)$, $i = 1,2$ such that

$$\sup_{a<x<t_i} A_i(a,c,x) < \infty, \quad \sup_{t_i^*<x<b} A_i^*(c,b,x) < \infty, \quad i = 1,2. \tag{2.25}$$

Then

$$A_i(a,c) = \sup_{a<x<c} A_i(a,c,x) \le \sup_{a<x<t_i} A_i(a,c,x) + \sup_{t_i<x<c} A_i(a,c,x),$$

$$A_i^*(c,b) = \sup_{c<x<b} A_i^*(c,b,x) \le \sup_{t_i^*<x<b} A_i^*(c,b,x) + \sup_{c<x<t_i^*} A_i^*(c,b,x).$$

Hence, by (2.25) we have $A_i(a,c) < \infty$ and $A_i^*(c,b) < \infty$ for any $c \in I$. By the condition (2.17) and by continuity of the function v on I, we have

$$\sup_{t_1<x<c} A_1(a,c,x) \le \left(\int_a^c \rho^{1-p'}\,ds\right)^{p-1} \left(\int_{t_1}^c v\,dt\right) < \infty,$$

$$\sup_{t_2<x<c} A_2(a,c,x) \le \left(\int_a^{t_2} \rho^{1-p'}\,ds\right)^{-1} \int_a^c v(t) \left(\int_a^t \rho^{1-p'}\,ds\right)^p dt < \infty.$$

Similarly, we have $\sup\limits_{c<x<t_i^*} A_i^*(c,b,x) < \infty$, $i=1,2$.

Now we show that

$$\lim_{t\to b} A_i(a,t) > \lim_{t\to b} A_i^*(t,b), \quad i=1,2. \tag{2.26}$$

Indeed, if

$$\lim_{t\to b} A_i(a,t) \le \lim_{t\to b} A_i^*(t,b) < \infty, \tag{2.27}$$

then

$$\lim_{t\to b} A_1(a,t) = \lim_{t\to b}\sup_{a<x<t} \left(\int_a^x \rho^{1-p'}\,ds\right)^{p-1} \int_x^t v\,d\tau = \sup_{a<x<b} \left(\int_a^x \rho^{1-p'}\,ds\right)^{p-1} \int_x^b v\,d\tau < \infty,$$

$$\infty > \lim_{t\to b} A_2(a,t) = \lim_{t\to b}\sup_{a<x<t} \left(\int_a^x \rho^{1-p'}\,ds\right)^{-1} \int_a^x v(\tau) \left(\int_a^\tau \rho^{1-p'}\,ds\right)^p d\tau = \sup_{a<x<b} \left(\int_a^x \rho^{1-p'}\,ds\right)^{-1} \int_a^x v(\tau) \left(\int_a^\tau \rho^{1-p'}\,ds\right)^p d\tau \ge \sup_{c<x<b} \left(\int_a^x \rho^{1-p'}\,ds\right)^{-1} \int_c^x v(\tau) \left(\int_a^\tau \rho^{1-p'}\,ds\right)^p d\tau \ge \left(\int_a^c \rho^{1-p'}\,ds\right)^p \left(\int_a^b \rho^{1-p'}\,ds\right)^{-1} \int_c^b v(\tau)\,d\tau, \quad c \in I,$$

and thus

$$\int_c^b v(\tau)\,d\tau < \infty, \quad \text{for all } c \in I.$$

Then by (2.17), we have

$$\lim_{t\to b} A_1^*(t,b) = \lim_{t\to b} \sup_{t<x<b} \left(\int_x^b \rho^{1-p'}\,ds\right)^{p-1} \int_t^x v\,d\tau \le \lim_{t\to b} \left(\int_t^b \rho^{1-p'}\,ds\right)^{p-1} \int_t^b v\,d\tau = 0,$$

$$\lim_{t\to b} A_2^*(t,b) = \lim_{t\to b} \sup_{t<x<b} \left(\int_x^b \rho^{1-p'}\,ds\right)^{-1} \int_x^b v(\tau) \left(\int_\tau^b \rho^{1-p'}\,ds\right)^{p} d\tau \le \lim_{t\to b} \left(\int_t^b \rho^{1-p'}\,ds\right)^{p-1} \int_t^b v(\tau)\,d\tau = 0.$$

Hence, (2.27) implies that $\lim_{t\to b} A_i(a,t) = 0$.

The functions $A_i(a,t)$, $i = 1,2$ are nonnegative, nondecreasing and continuous by $t \in I$. Therefore,

$$\lim_{t\to b} A_i(a,t) = A_i(a,b) = 0, \quad i = 1,2.$$

Then

$$\left(\int_a^x \rho^{1-p'}\,ds\right)^{p-1} \int_x^b v\,dt = 0, \quad \forall\, x \in I, \tag{2.28}$$

$$\left(\int_a^x \rho^{1-p'}\,ds\right)^{-1} \int_a^x v(t) \left(\int_a^t \rho^{1-p'}\,ds\right)^{p} dt = 0, \quad \forall\, x \in I. \tag{2.29}$$

Since the function ρ is a positive and continuous on I, the equalities (2.28) and (2.29) hold if and only if $v(t) \equiv 0$ on I, which is a contradiction to the condition imposed on the function v. Such a contradiction shows that (2.26) holds.

Now we show that

$$\lim_{t\to a} A_i^*(t,b) > \lim_{t\to a} A_i(a,t), \quad i=1,2. \tag{2.30}$$

As indicated above, we assume that

$$\lim_{t\to a} A_i^*(t,b) \le \lim_{t\to a} A_i(a,t) < \infty, \quad i=1,2. \tag{2.31}$$

Then

$$\infty > \lim_{t\to a} A_i^*(t,b) =$$
$$\lim_{t\to a} \sup_{t<x<b} \left(\int_x^b \rho^{1-p'}\,ds\right)^{p-1} \int_t^x v\,d\tau =$$
$$\sup_{a<x<b} \left(\int_x^b \rho^{1-p'}\,ds\right)^{p-1} \int_a^x v\,d\tau,$$

$$\infty > \lim_{t\to a} A_2^*(t,b) =$$
$$\lim_{t\to a} \sup_{t<x<b} \left(\int_x^b \rho^{1-p'}\,ds\right)^{-1} \int_x^b v(\tau)\left(\int_\tau^b \rho^{1-p'}\,ds\right)^{p} d\tau =$$
$$\sup_{a<x<b} \left(\int_x^b \rho^{1-p'}\,ds\right)^{-1} \int_x^b v(\tau)\left(\int_\tau^b \rho^{1-p'}\,ds\right)^{p} d\tau \ge$$
$$\sup_{a<x<c} \left(\int_x^b \rho^{1-p'}\,ds\right)^{-1} \int_x^c v(\tau)\left(\int_\tau^b \rho^{1-p'}\,ds\right)^{p} d\tau \ge$$
$$\left(\int_c^b \rho^{1-p'}\,ds\right)^{p} \left(\int_a^b \rho^{1-p'}\,ds\right)^{-1} \int_a^c v(\tau)\,d\tau, \quad c\in I.$$

Hence,

$$\int_a^c v(\tau)\,d\tau < \infty, \quad \text{for all } c\in I,$$

then by applying (2.17), we obtain

$$\lim_{t\to a} A_1(a,t) =$$
$$\lim_{t\to a} \sup_{a<x<t} \left(\int_a^x \rho^{1-p'}\,ds\right)^{p-1} \int_x^t v\,d\tau \le$$

$$\lim_{t\to a}\left(\int_a^t \rho^{1-p'}\,ds\right)^{p-1}\int_a^t v\,d\tau = 0,$$

$$\lim_{t\to a} A_2(a,t) =$$

$$\lim_{t\to a}\sup_{a<x<t}\left(\int_a^x \rho^{1-p'}\,ds\right)^{-1}\int_a^x v(\tau)\left(\int_a^\tau \rho^{1-p'}\,ds\right)^{p} d\tau \le$$

$$\lim_{t\to a}\left(\int_a^t \rho^{1-p'}\,ds\right)^{p-1}\int_a^t v(\tau)\,d\tau = 0.$$

Then by (2.31) it follows that

$$\lim_{t\to a} A_i^*(t,b) = A_i^*(a,b) = 0, \quad i = 1,2.$$

Consequently,

$$\left(\int_x^b \rho^{1-p'}\,ds\right)^{p-1}\int_a^x v\,d\tau = 0, \quad \text{for all } x \in I,$$

$$\left(\int_x^b \rho^{1-p'}\,ds\right)^{-1}\int_x^b v(\tau)\left(\int_\tau^b \rho^{1-p'}\,ds\right)^{p} d\tau = 0, \quad \text{for all } x \in I,$$

which may be by continuity and positivity of the function ρ on I, when $v(\tau) \equiv 0$ on I. But, this is a contradiction with our assumption on v. Thus, (2.30) follows.

By continuity and monotonicity of $A_i(a,t)$, $A_i^*(t,b)$ in the variable $t \in I$, the inequalities (2.26) and (2.30) imply the existence of the points $c_i \in I$ such that $A_i(a,c_i) = A_i^*(c_i,b)$, $i = 1,2$.

Thus the proof of Lemma 2.11 is complete. □

Proof of Theorem 2.7. Let the inequality (2.16) hold for the best constant $C = J_0(a,b)$. Let $a < c^- < c^+ < b$.

We assume that

$$f_0(t) = \begin{cases} \left(\int_a^{c^-} \rho^{1-p'}\,ds\right)^{-1}\int_a^t \rho^{1-p'}\,ds, & a < t < c^- \\ 1, & c^- \le t \le c^+ \\ \left(\int_{c^+}^{b} \rho^{1-p'}\,ds\right)^{-1}\int_t^b \rho^{1-p'}\,ds, & c^+ < t < b. \end{cases} \tag{2.32}$$

The function f_0 is locally absolutely continuous on I and

$$\int_a^b \rho(s)|f_0'(s)|^p ds =$$

$$\int_a^{c^-} \rho(s)|f_0'(s)|^p ds + \int_{c^-}^{c^+} \rho(s)|f_0'(s)|^p ds + \int_{c^+}^{b} \rho(s)|f_0'(s)|^p ds =$$

$$\left(\int_a^{c^-} \rho^{1-p'}\, ds\right)^{-p} \int_a^{c^-} \rho\rho^{p(1-p')}\, ds + \left(\int_{c^+}^{b} \rho^{1-p'}\, ds\right)^{-p} \int_{c^+}^{b} \rho\rho^{p(1-p')}\, ds =$$

$$\left(\int_a^{c^-} \rho^{1-p'}\, ds\right)^{1-p} + \left(\int_{c^+}^{b} \rho^{1-p'}\, ds\right)^{1-p} < \infty. \tag{2.33}$$

Therefore, $f_0 \in W_p^1(\rho, I)$ and $\lim\limits_{t\to a+} f_0(t) \equiv f_0(a) = 0$, $\lim\limits_{t\to b-} f_0(t) \equiv f_0(b) = 0$ by construction. Hence, by the condition (2.17) based on Theorem 2.1 $f_0 \in \overset{\circ}{W}{}_p^1(\rho, I)$. By substituting f_0 in (2.16), we have

$$J_0(a,b) \geq \frac{\int_a^b v(t)|f_0(t)|^p dt}{\int_a^b \rho(t)|f_0'(t)|^p dt}. \tag{2.34}$$

By simple computations, we have

$$\int_a^b v(t)|f_0(t)|^p dt =$$

$$\int_a^{c^-} v(t)|f_0(t)|^p dt + \int_{c^-}^{c^+} v(t)|f_0(t)|^p dt + \int_{c^+}^{b} v(t)|f_0(t)|^p dt =$$

$$\left(\int_a^{c^-} \rho^{1-p'}\, ds\right)^{-p} \int_a^{c^-} v(t) \left(\int_a^{t} \rho^{1-p'}\, ds\right)^{p} dt + \int_{c^-}^{c^+} v(t) dt +$$

$$\left(\int_{c^+}^{b} \rho^{1-p'}\, ds\right)^{-p} \int_{c^+}^{b} v(t) \left(\int_t^{b} \rho^{1-p'}\, ds\right)^{p} dt. \tag{2.35}$$

By applying (2.33), (2.34) and (2.35), we obtain the following two inequalities

$$J_0(a,b) \geq$$

$$\frac{\left(\int_a^{c^-} \rho^{1-p'}\,ds\right)^{-p} \int_a^{c^-} v(t)\left(\int_a^{t} \rho^{1-p'}\,ds\right)^{p} dt}{\left(\int_a^{c^-} \rho^{1-p'}\,ds\right)^{1-p} + \left(\int_{c^+}^{b} \rho^{1-p'}\,ds\right)^{1-p}} + \frac{\left(\int_{c^+}^{b} \rho^{1-p'}\,ds\right)^{-p} \int_{c^+}^{b} v(t)\left(\int_t^{b} \rho^{1-p'}\,ds\right)^{p} dt}{\left(\int_a^{c^-} \rho^{1-p'}\,ds\right)^{1-p} + \left(\int_{c^+}^{b} \rho^{1-p'}\,ds\right)^{1-p}}, \qquad (2.36)$$

$$J_0(a,b) \geq \frac{\int_{c^-}^{c} v(t)dt + \int_{c}^{c^+} v(t)dt}{\left(\int_a^{c^-} \rho^{1-p'}\,ds\right)^{1-p} + \left(\int_{c^+}^{b} \rho^{1-p'}\,ds\right)^{1-p}}, \quad c \in (c^-, c^+). \qquad (2.37)$$

By multiplying terms of the fraction in the right hand sides of (2.36) and (2.37) by $\left(\int_a^{c^-} \rho^{1-p'}\,ds\right)^{p-1}$, we have

$$J_0(a,b) \geq \geq \frac{\left(\int_a^{c^-} \rho^{1-p'}\,ds\right)^{-1} \int_a^{c^-} v(t)\left(\int_a^{t} \rho^{1-p'}\,ds\right)^{p} dt}{1 + \left(\int_a^{c^-} \rho^{1-p'}\,ds\right)^{p-1} \left(\int_{c^+}^{b} \rho^{1-p'}\,ds\right)^{1-p}} + \frac{\left(\int_a^{c^-} \rho^{1-p'}\,ds\right)^{p-1} \left(\int_{c^+}^{b} \rho^{1-p'}\,ds\right)^{-p} \int_{c^+}^{b} v(t)\left(\int_t^{b} \rho^{1-p'}\,ds\right)^{p} dt}{1 + \left(\int_a^{c^-} \rho^{1-p'}\,ds\right)^{p-1} \left(\int_{c^+}^{b} \rho^{1-p'}\,ds\right)^{1-p}}, \qquad (2.38)$$

$$J_0(a,b) \geq \frac{\left(\int_a^{c^-} \rho^{1-p'}\,ds\right)^{p-1} \int_{c^-}^{c} v(t)dt + \left(\int_a^{c^-} \rho^{1-p'}\,ds\right)^{p-1} \int_{c}^{c^+} v(t)dt}{1 + \left(\int_a^{c^-} \rho^{1-p'}\,ds\right)^{p-1} \left(\int_{c^+}^{b} \rho^{1-p'}\,ds\right)^{1-p}}, \qquad (2.39)$$

Since, the left hand sides of (2.38) and (2.39) do not depend on $c^- \in (a, c)$, then we can take the limit as $c^- \to a$ and obtain

$$J_0(a,b) \geq$$

$$\frac{\limsup\limits_{x\to a}\left(\int\limits_a^x \rho^{1-p'}\,ds\right)^{-1}\int\limits_a^x v(t)\left(\int\limits_a^t \rho^{1-p'}\,ds\right)^p dt}{1+\lim\limits_{c^-\to a}\left(\int\limits_a^{c^-}\rho^{1-p'}\,ds\right)^{p-1}\left(\int\limits_{c^+}^b \rho^{1-p'}\,ds\right)^{1-p}} +$$

$$\frac{\lim\limits_{c^-\to a}\left(\int\limits_a^{c^-}\rho^{1-p'}\,ds\right)^{p-1}\left(\int\limits_{c^+}^b \rho^{1-p'}\,ds\right)^{-p}\int\limits_{c^+}^b v(t)\left(\int\limits_t^b \rho^{1-p'}\,ds\right)^p dt}{1+\lim\limits_{c^-\to a}\left(\int\limits_a^{c^-}\rho^{1-p'}\,ds\right)^{p-1}\left(\int\limits_{c^+}^b \rho^{1-p'}\,ds\right)^{1-p}} =$$

$$\limsup_{x\to a}\left(\int\limits_a^x \rho^{1-p'}\,ds\right)^{-1}\int\limits_a^x v(t)\left(\int\limits_a^t \rho^{1-p'}\,ds\right)^p dt =$$

$$\limsup_{x\to a} A_2(a,c,x), \tag{2.40}$$

$$J_0(a,b) \geq$$

$$\frac{\limsup\limits_{x\to a}\left(\int\limits_a^x \rho^{1-p'}\,ds\right)^{p-1}\int\limits_x^c v(t)dt+\lim\limits_{c^-\to a}\left(\int\limits_a^{c^-}\rho^{1-p'}\,ds\right)^{p-1}\int\limits_c^{c^+} v(t)dt}{1+\lim\limits_{c^-\to a}\left(\int\limits_a^{c^-}\rho^{1-p'}\,ds\right)^{p-1}\left(\int\limits_{c^+}^b \rho^{1-p'}\,ds\right)^{1-p}} =$$

$$\limsup_{x\to a}\left(\int\limits_a^x \rho^{1-p'}\,ds\right)^{p-1}\int\limits_x^c v(t)\,dt =$$

$$\limsup_{x\to a} A_1(a,c,x). \tag{2.41}$$

Now, by multiplying both terms of the fractions in the right hand sides of (2.36) and (2.37) by $\left(\int\limits_{c^+}^b \rho^{1-p'}\,ds\right)^{p-1}$ and by taking the limit as $c^+\to b$, we obtain

$$J_0(a,b) \geq$$

$$\frac{\lim\limits_{c^+\to b}\left(\int\limits_{c^+}^b \rho^{1-p'}\,ds\right)^{p-1}\left(\int\limits_a^{c^-}\rho^{1-p'}\,ds\right)^{-p}\int\limits_a^{c^-} v(t)\left(\int\limits_a^t \rho^{1-p'}\,ds\right)^p dt}{\lim\limits_{c^+\to b}\left(\int\limits_{c^+}^b \rho^{1-p'}\,ds\right)^{p-1}\left(\int\limits_a^{c^-}\rho^{1-p'}\,ds\right)^{1-p}+1} +$$

$$\frac{\limsup\limits_{x\to b}\left(\int\limits_x^b \rho^{1-p'}\,ds\right)^{-1}\int\limits_x^b v(t)\left(\int\limits_t^b \rho^{1-p'}\,ds\right)^p dt}{\lim\limits_{c^+\to b}\left(\int\limits_{c^+}^b \rho^{1-p'}\,ds\right)^{p-1}\left(\int\limits_a^{c^-} \rho^{1-p'}\,ds\right)^{1-p}+1} =$$

$$\limsup_{x\to b}\left(\int\limits_x^b \rho^{1-p'}\,ds\right)^{-1}\int\limits_x^b v(t)\left(\int\limits_t^b \rho^{1-p'}\,ds\right)^p dt =$$

$$\limsup_{x\to b} A_2^*(c,b,x), \tag{2.42}$$

$J_0(a,b) \geq$

$$\frac{\lim\limits_{c^+\to b}\left(\int\limits_{c^+}^b \rho^{1-p'}\,ds\right)^{p-1}\int\limits_{c^-}^c v(t)dt + \limsup\limits_{x\to b}\left(\int\limits_x^b \rho^{1-p'}\,ds\right)^{p-1}\int\limits_c^x v(t)dt}{\lim\limits_{c^+\to b}\left(\int\limits_{c^+}^b \rho^{1-p'}\,ds\right)^{p-1}\left(\int\limits_a^{c^-} \rho^{1-p'}\,ds\right)^{1-p}+1} =$$

$$\limsup_{x\to b}\left(\int\limits_x^b \rho^{1-p'}\,ds\right)^{p-1}\int\limits_c^x v(t)\,dt =$$

$$\limsup_{x\to b} A_1^*(c,b,x). \tag{2.43}$$

By applying (2.40), (2.41), (2.42) and (2.43), it follows that (2.24) holds. Then by Lemma 2.11, there exist middle points $c_i \in I$ for (A_i, A_i^*), $i = 1,2$. Consequently, by Definition 2.6 we have $A_i(a, c_i) = A_i^*(c_i, b) \equiv T_{c_i}(a,b) < \infty$, $i = 1,2$.

Since $A_i(a, c_i, x)$, $A_i^*(c_i, b, x)$ are continuous in x on $(a, c_i]$ and $[c_i, b)$, respectively, and $A_i(a,c_i) \geq \limsup\limits_{x\to a} A_i(a,c_i,x)$, $A_i^*(c_i,b) \geq \limsup\limits_{x\to b} A_i^*(c_i,b,x)$, then there exist points c_i^-, c_i^+: $a < c_i^- \leq c_i$, $c_i \leq c_i^+ < b$ that $A_i(a,c_i) = A_i(a,c_i,c_i^-)$, $A_i^*(c_i,b) = A_i^*(c_i,b,c_i^+)$ and that $c_1^- \neq c_1$, $c_i^+ \neq c_1$.

Let $c^- = c_2^-$, $c^+ = c_2^+$ in (2.36), and $c = c_1$, $c^- = c_1^-$, $c^+ = c_2^+$ in (2.37).

Then, the inequalities (2.36) and (2.37) imply that:

$J_0(a,b) \geq$

$$\frac{\left(\int\limits_a^{c_2^-} \rho^{1-p'}\,ds\right)^{-p}\int\limits_a^{c_2^-} v(t)\left(\int\limits_a^t \rho^{1-p'}\,ds\right)^p dt}{\left(\int\limits_a^{c_2^-} \rho^{1-p'}\,ds\right)^{1-p}+\left(\int\limits_{c_2^+}^b \rho^{1-p'}\,ds\right)^{1-p}} +$$

$$\frac{\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\,ds\right)^{-p} \int\limits_{c_2^+}^{b} v(t) \left(\int\limits_{t}^{b} \rho^{1-p'}\,ds\right)^{p} dt}{\left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\,ds\right)^{1-p} + \left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\,ds\right)^{1-p}} =$$

$$\frac{\left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\,ds\right)^{-1} \int\limits_{a}^{c_2^-} v(t) \left(\int\limits_{a}^{t} \rho^{1-p'}\,ds\right)^{p} dt \left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\,ds\right)^{p-1}}{\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\,ds\right)^{p-1}} +$$

$$\frac{\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\,ds\right)^{-1} \int\limits_{c_2^+}^{b} v(t) \left(\int\limits_{t}^{b} \rho^{1-p'}\,ds\right)^{p} dt \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\,ds\right)^{p-1}}{\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\,ds\right)^{p-1}} =$$

$$\frac{A_2(a, c_2, c_2^-) \left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + A_2^*(c_2, b, c_2^+) \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\,ds\right)^{p-1}}{\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\,ds\right)^{p-1}} =$$

$$\frac{A_2(a, c_2) \left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + A_2^*(c_2, b) \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\,ds\right)^{p-1}}{\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\,ds\right)^{p-1}} =$$

$$T_{c_2}(a, b) \frac{\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\,ds\right)^{p-1}}{\left(\int\limits_{c_2^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + \left(\int\limits_{a}^{c_2^-} \rho^{1-p'}\,ds\right)^{p-1}} = T_{c_2}(a, b) \qquad (2.44)$$

and that

$J_0(a, b) \geq$

$$\frac{\int\limits_{c_1^-}^{c_1} v(t)dt + \int\limits_{c_1}^{c_1^+} v(t)dt}{\left(\int\limits_{a}^{c_1^-} \rho^{1-p'}\,ds\right)^{1-p} + \left(\int\limits_{c_1^+}^{b} \rho^{1-p'}\,ds\right)^{1-p}} =$$

$$\frac{\left(\int_a^{c_1^-} \rho^{1-p'}\,ds\right)^{p-1} \int_{c_1^-}^{c_1} v(t)dt \left(\int_{c_1^+}^{b} \rho^{1-p'}\,ds\right)^{p-1}}{\left(\int_{c_1^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + \left(\int_a^{c_1^-} \rho^{1-p'}\,ds\right)^{p-1}} +$$

$$\frac{\left(\int_{c_1^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} \int_{c_1}^{c_1^+} v(t)dt \left(\int_a^{c_1^-} \rho^{1-p'}\,ds\right)^{p-1}}{\left(\int_{c_1^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + \left(\int_a^{c_1^-} \rho^{1-p'}\,ds\right)^{p-1}} =$$

$$\frac{A_1(a, c_1, c_1^-)\left(\int_{c_1^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + A_1^*(c_1, b, c_1^+)\left(\int_a^{c_1^-} \rho^{1-p'}\,ds\right)^{p-1}}{\left(\int_{c_1^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + \left(\int_a^{c_1^-} \rho^{1-p'}\,ds\right)^{p-1}} =$$

$$\frac{A_1(a, c_1)\left(\int_{c_1^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + A_1^*(c_1, b)\left(\int_a^{c_1^-} \rho^{1-p'}\,ds\right)^{p-1}}{\left(\int_{c_1^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + \left(\int_a^{c_1^-} \rho^{1-p'}\,ds\right)^{p-1}} =$$

$$T_{c_1}(a,b)\frac{\left(\int_{c_1^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + \left(\int_a^{c_1^-} \rho^{1-p'}\,ds\right)^{p-1}}{\left(\int_{c_1^+}^{b} \rho^{1-p'}\,ds\right)^{p-1} + \left(\int_a^{c_1^-} \rho^{1-p'}\,ds\right)^{p-1}} = T_{c_1}(a,b), \quad (2.45)$$

respectively.

The inequalities (2.44) and (2.45) imply the left hand side of the estimate (2.18).

Sufficiency. Assume that there exists middle point $c_i \in I$ for (A_i, A_i^*), $i = 1, 2$. Then $A_i(a, c_i) = A_i^*(c_i, b) = T_{c_i}(a, b) < \infty$, $i = 1, 2$. By the condition (2.17) we have $f(a) = f(b) = 0$ for $f \in \overset{\circ}{W}{}_p^1(\rho, I)$, and thus the function $f \in \overset{\circ}{W}{}_p^1(\rho, I)$ can be written in the form

$$f(x) = \int_a^x f'(s)\,ds, \quad x \in (a, c_i), \quad f(x) = -\int_x^b f'(s)\,ds, \quad x \in (c_i, b).$$

in the intervals (a, c_i) and (c_i, b), respectively.

Then

$$\int_a^b v(t)|f(t)|^p dt =$$

$$\int_a^{c_i} v(t)|f(t)|^p dt + \int_{c_i}^b v(t)|f(t)|^p dt =$$

$$\int_a^{c_i} v(t)\left|\int_a^t f'(s)ds\right|^p dt + \int_{c_i}^b v(t)\left|\int_t^b f'(s)ds\right|^p dt. \qquad (2.46)$$

According to Theorems 1.18–1.20 and Theorem 1.22, we obtain

$$\int_a^{c_i} v(t)\left|\int_a^t f'(s)ds\right|^p dt \le \gamma_i A_i(a, c_i)\int_a^{c_i} \rho(s)|f'(s)|^p\, ds, \quad i = 1, 2,$$

$$\int_{c_i}^b v(t)\left|\int_t^b f'(s)ds\right|^p dt \le \gamma_i A_i^*(c_i, b)\int_{c_i}^b \rho(s)|f'(s)|^p\, ds, \quad i = 1, 2,$$

where $\gamma_1 = p\left(\frac{p}{p-1}\right)^{p-1}$, $\gamma_2 = \left(\frac{p}{p-1}\right)^p$. By applying (2.46), we obtain

$$\int_a^b v(t)|f(t)|^p dt \le$$

$$\gamma_i A_i(a, c_i)\int_a^{c_i} \rho(s)|f'(s)|^p\, ds + \gamma_i A_i^*(a, c_i)\int_{c_i}^b \rho(s)|f'(s)|^p\, ds =$$

$$\gamma_i T_{c_i}(a, b)\left(\int_a^{c_i} \rho(s)|f'(s)|^p\, ds + \int_{c_i}^b \rho(s)|f'(s)|^p\, ds\right) =$$

$$\gamma_i T_{c_i}(a, b)\int_a^b \rho(s)|f'(s)|^p\, ds.$$

Namely, the inequality (2.16) holds. Then by (2.18) the following estimate

$$J_0(a, b) \le \min\{\gamma_1 T_{c_1}(a, b),\ \gamma_2 T_{c_2}(a, b)\},$$

holds for the best constant $C = J_0(a, b)$ in (2.16).

Thus the proof of Theorem 2.7 is complete. □

Proof of the Theorem 2.8. The condition (2.19) holds, by the assumptions of Theorem 2.8. Then by the part (ii) of Theorem 2.1 we have

$\overset{\circ}{W}{}^1_p(\rho, I) = \{f \in W^1_p(\rho, I) : \ f(a) = 0\}$. Consequently, the equalities $f' = g,\ f(x) = \int\limits_a^x g(s)ds$ define one-to-one mapping between spaces $\overset{\circ}{W}{}^1_p(\rho, I)$ and $L_{p,\rho}(I)$. Then the inequality (2.16) on the set $\overset{\circ}{W}{}^1_p(\rho, I)$ is equivalent to the following inequality

$$\int\limits_a^b v(t) \left| \int\limits_a^t g(s)ds \right|^p dt \le C \int\limits_a^b \rho(s) \left| g(s) \right|^p ds, \quad g \in L_{p,\rho}(I). \tag{2.47}$$

In addition, the best constant in (2.16) and in (2.47) coincide. By Theorems 1.18–1.20 the inequality (2.47) holds if and only if $A_i(a, b) < \infty$ for at least one $i = 1, 2$ and the estimate (2.20) holds for the best constant $C = J_0(a, b)$.

Thus the proof of Theorem 2.8 is complete. □

Proof of Theorem 2.9. The condition (2.21) holds, by the assumptions of Theorem 2.9. Then by the part (iii) of Theorem 2.1 we have $\overset{\circ}{W}{}^1_p(\rho, I) = \{f \in W^1_p(\rho, I) : \ f(b) = 0\}$. Consequently, the equality $f' = g,\ f(x) = -\int\limits_x^b g(s)ds$ define one-to-one mapping between the spaces $\overset{\circ}{W}{}^1_p(\rho, I)$ and $L_{p,\rho}(I)$. Then the inequality (2.16) on the set $\overset{\circ}{W}{}^1_p(\rho, I)$ is equivalent to the following inequality

$$\int\limits_a^b v(t) \left| \int\limits_t^b g(s)ds \right|^p dt \le C \int\limits_a^b \rho(s) \left| g(s) \right|^p ds, \quad g \in L_{p,\rho}(I), \tag{2.48}$$

with the best constant $C = J_0(a, b)$. Then by Theorem 1.19 and Theorem 1.22 the inequality (2.48) holds if and only if $A_i^*(a, b) < \infty$ for at least one $i = 1, 2$ and the estimate (2.22) holds for the best constant $J_0(a, b)$.

Thus the proof of Theorem 2.9 is complete. □

Proof of Theorem 2.10 The condition (2.23) holds, by the assumptions of Theorem 2.10. Then by the part (iv) of Theorem 2.1. we have $\overset{\circ}{W}{}^1_p(\rho, I) = W^1_p(\rho, I)$. For the function $f(x) \equiv 1 \in W^1_p(\rho, I)$ the inequality (2.16) does not hold. Hence, $J_0(a, b) = \infty$.

Thus the proof of Theorem 2.10 is complete. □

Chapter 3

Disfocal and disconjugate half–linear second order differential equations on a given interval

Notions such as disfocal and disconjugate equations play a very important role in the qualitative theory of differential equations (see e.g., [45], [21], [5], [7], [36], [8], [9], [46]).

In theory of linear and half–linear second order differential equations disfocality and disconjugacy properties on a given interval with regular and singular endpoints have been investigated comparatively less than nonoscillatory properties of these equations. The Riccati technique and the Lyapunov, La Vallee–Poussin and Opial equations are often used to find disfocality and disconjugacy properties on a given interval (see e.g., [11, Chapter 5]).

In this Section, by using the results on the weighted Hardy inequalities we get necessary and sufficient conditions for half–linear second order differential equation to be disfocal and disconjugate on a given interval. In Section 3.1 we investigate disfocality properties. The main results on disconjugacy properties are in Section 3.2. The corresponding proofs are in Section 3.3.

3.1 Disfocality.

Let $I = (a, b)$, $-\infty \le a < b \le +\infty$. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $v \ge 0$ and $\rho > 0$ be continuous functions on I. Moreover, $v \not\equiv 0$ on I.

On I, we consider the half–linear second order differential equation

$$\big(\rho(t)|y'(t)|^{p-2}y'(t)\big)' + v(t)|y(t)|^{p-2}y(t) = 0. \tag{3.1}$$

Let $a \le \alpha < \beta < b$. According to Definitions 1.13 and 1.14, we consider the problem of the existence of a left focal point of the point β on the interval (α, β) with respect to the equation (3.1) and the left disfocality property on the interval (α, β).

If $a < \alpha < \beta \le b$, we consider the existence of a right focal point of the point α on the interval (α, β) with respect to the equation (3.1) and the right disfocality property on the interval (α, β).

Theorem 3.1. *Let $1 < p < \infty$. Let $a \le \alpha < \beta < b$. Then the validity of the condition*

$$\max\{A_1(\alpha, \beta),\ A_2(\alpha, \beta)\} \le 1 \tag{3.2}$$

is necessary and the validity of one of the conditions

$$A_1(\alpha, \beta) \le \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}, \quad A_2(\alpha, \beta) \le \left(\frac{p-1}{p}\right)^{p} \tag{3.3}$$

is sufficient for the equation (3.1) to be left disfocal on the interval $(\alpha, \beta]$.

Corollary 3.2. *Let the assumptions of Theorem 3.1 hold. If the condition*

$$\max\{A_1(\alpha, \beta),\ A_2(\alpha, \beta)\} > 1 \tag{3.4}$$

holds, then there exists a left focal point of the point β with respect to the equation (3.1) on the interval (α, β).

Theorem 3.3. *Let $1 < p < \infty$. Let $a < \alpha < \beta \le b$. Then the validity of the condition*

$$\max\{A_1^*(\alpha, \beta),\ A_2^*(\alpha, \beta)\} \le 1 \tag{3.5}$$

is necessary and the validity of one of the conditions

$$A_1^*(\alpha, \beta) \le \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}, \quad A_2^*(\alpha, \beta) \le \left(\frac{p-1}{p}\right)^{p} \tag{3.6}$$

is sufficient for the equation (3.1) to be left disfocal on the interval $[\alpha, \beta)$.

Corollary 3.4. *Let the assumptions of Theorem 3.3 hold. If the condition*

$$\max\{A_1^*(\alpha,\beta),\ A_2^*(\alpha,\beta)\} > 1 \tag{3.7}$$

holds, then there exists a right focal point of the point α with respect to the equation (3.1) on the interval (α,β).

The values $A_i(\alpha,\beta)$ and $A_i^*(\alpha,\beta)$, $i = 1,2$, are defined in Section 2.2 of Chapter 2.

For the proof of Theorem 3.1 we need the following statement.

Lemma 3.5. *Let the assumptions of Theorem 3.1 hold. Then the equation (3.1) is left disfocal on the interval $(\alpha,\beta]$ if and only if*

$$\int_\alpha^\beta v(x)|f(x)|^p\,dx \le \int_\alpha^\beta \rho(x)|f'(x)|^p\,dx, \quad \textit{for all } f \in W^1_{p,l}(\rho,(\alpha,\beta)). \tag{3.8}$$

Proof of Lemma 3.5. First, we start with the case $a < \alpha < \beta < b$. In this case the norm of the space $W^1_{p,l}(\alpha,\beta)$ is equivalent to the norm

$$||f||_{W^1_p} = \left(\int_\alpha^\beta |f'(t)|^p\,dt\right)^{\frac{1}{p}}.$$

The norm of the space $W^1_{p,l}(\rho,(\alpha,\beta))$ is equivalent to the norm

$$||f||_{W^1_p(\rho,I)} = \left(\int_\alpha^\beta \rho(t)|f'(t)|^p\,dt\right)^{\frac{1}{p}}.$$

Since $\rho > 0$ and it is continuous on I, then

$$0 < m = \min_{\alpha<t<\beta} \rho(t) \le \max_{\alpha<t<\beta} \rho(t) = M < \infty.$$

Therefore,

$$m\int_\alpha^\beta |f'(x)|^p\,dx \le \int_\alpha^\beta \rho(x)|f'(x)|^p\,dx \le M\int_\alpha^\beta |f'(x)|^p\,dx.$$

Hence, the spaces $W^1_{p,l}(\alpha,\beta)$ and $W^1_{p,l}(\rho,(\alpha,\beta))$ coincide and have equivalent norms. Then on the basis of Theorem 2.8 the equation (3.1) is left disfocal on the interval $(\alpha,\beta]$ if and only if

$$\int_\alpha^\beta \left(\rho(t)|f'(t)|^p - v(t)|f(t)|^p\right)dt \ge 0, \quad \text{for all } f \in W^1_{p,r}(\rho,(\alpha,\beta)),$$

a condition which is equivalent to (3.8). Indeed, the integral $\int\limits_{\alpha}^{\beta} \rho(t)|f'(t)|^p\,dt$ is finite for all $f \in W^1_{p,r}(\rho,(\alpha,\beta))$.

Now, let $a = \alpha < \beta < b$. Assume that the inequality (3.8) holds if $\alpha = a$, but the equation (3.1) is not left disfocal on (a,β), or in other words that there exists a left focal point $\gamma \in (a,\beta)$ of the point β. Then by Theorem 2.8 there exists $\alpha_1 \in (a,\gamma)$ and $\widetilde{f} \in W^1_{p,l}(\alpha_1,\beta)$, such that

$$\int\limits_{\alpha_1}^{\beta} v(t)|\widetilde{f}(t)|^p\,dt > \int\limits_{\alpha_1}^{\beta} \rho(t)|\widetilde{f}'(t)|^p\,dt.$$

Since $\widetilde{f}(\alpha_1) = 0$, then we can extend the function $\widetilde{f}$ take zero on the semi–interval $(a,\alpha_1]$ and obtain an extension $\widetilde{f}_0$ on (α,β), such that $\widetilde{f}_0 \in W^1_{p,l}(\rho,(a,\beta))$ and

$$\int\limits_{a}^{\beta} v(t)|\widetilde{f}_0(t)|^p\,dt > \int\limits_{a}^{\beta} \rho(t)|\widetilde{f}_0'(t)|^p\,dt,$$

in contradiction with the validity of (3.8). Hence, the equation (3.1) is left disfocal on $(a,\beta]$.

Let the equation (3.1) be left disfocal on $(a,\beta]$. Assume that the inequality (3.8) does not hold for $\alpha = a$. Then there exists a function $\widehat{f} \in W^1_{p,l}(\rho,(a,\beta))$ such that

$$\int\limits_{a}^{\beta} v(t)|\widehat{f}(t)|^p\,dt > \int\limits_{a}^{\beta} \rho(t)|\widehat{f}'(t)|^p\,dt. \tag{3.9}$$

Since $\widehat{f} \in W^1_{p,l}(\rho,(a,\beta))$ there exists a sequence $\{\widehat{f}_n\} \in AC_{p,l}(a,\beta)$ such that $||\widehat{f} - \widehat{f}_n||_{W^1_p(\rho,(a,\beta))} \to 0$ for $n \to \infty$. Since each function $\widehat{f}_n \in AC_{p,l}(a,\beta)$ equals zero in a suitable right neighborhood of the point a, then (3.8) holds for $\widehat{f}_n$. In other words we have

$$\int\limits_{a}^{\beta} \rho(t)|\widehat{f}_n'(t)|^p\,dt \geq \int\limits_{a}^{\beta} v(t)|\widehat{f}_n(t)|^p\,dt. \tag{3.10}$$

Hence, the sequence $\{\widehat{f}_n\}$ is fundamental in the complete space $L_{p,v}(a,\beta)$. Therefore, by taking the limit as $n \to \infty$ in (3.10) we have

$$\int\limits_{a}^{\beta} \rho(t)|\widehat{f}'(t)|^p\,dt \geq \int\limits_{a}^{\beta} v(t)|\widehat{f}(t)|^p\,dt,$$

in contradiction with (3.9), and thus (3.8) holds.

Thus the proof of Lemma 3.5 is complete. □

Let

$$J_l(\alpha,\beta) = \sup_{0\neq g\in L_{p,\rho}(\alpha,\beta)} \frac{\int\limits_\alpha^\beta v(t)\left|\int\limits_\alpha^t g(s)\,ds\right|^p\,dt}{\int\limits_\alpha^\beta \rho(t)\,|g(t)|^p\,dt}. \tag{3.11}$$

Lemma 3.6. *Let the assumptions of Theorem 3.1 hold. Then the equation (3.1) is left disfocal on the interval $(\alpha,\beta]$ if and only if $J_l(\alpha,\beta)\le 1$.*

Proof of Lemma 3.6. Let the equation (3.1) be left disfocal on $(\alpha,\beta]$. Then by Lemma 3.5 the inequality (3.8) holds. Then $W^1_{p,l}(\rho,(\alpha,\beta))\neq W^1_p(\rho,(\alpha,\beta))$. By the proof of Theorem 1.21 it follows that the inequality (3.8) does not hold on the set $W^1_p(\rho,(\alpha,\beta))$. Hence, by Theorem 2.1 we have

$$W^1_{p,l}(\rho,(\alpha,\beta)) = \{f\in W^1_p(\rho,(\alpha,\beta)):\ \ f(\alpha)=0\}. \tag{3.12}$$

The map of $L_{p,\rho}(\alpha,\beta)$ to $W^1_{p,l}(\rho,(\alpha,\beta))$ which takes $g\in L_{p,\rho}(\alpha,\beta)$ to the function f define by $f(t)=\int\limits_\alpha^t g(s)ds$ for all $t\in(\alpha,\beta)$ is a one–to–one correspondence.

Therefore, the inequality (3.8) is equivalent to

$$\int\limits_\alpha^\beta v(t)\left|\int\limits_\alpha^t g(s)\,ds\right|^p\,dt \le \int\limits_\alpha^\beta \rho(t)\,|g(t)|^p\,dt,\quad g\in L_{p,\rho}(\alpha,\beta). \tag{3.13}$$

Consequently, by (3.11) and (3.13), we have $J_l(\alpha,\beta)\le 1$.

Now, let $J_l(\alpha,\beta)\le 1$. Since $J_l(\alpha,\beta)$ is the best constant in the inequality of the type (3.13), then by Theorem 1.18 we obtain $A_1(\alpha,\beta)\le 1$. Hence, $\int\limits_\alpha^\beta \rho^{1-p'}(s)\,ds<\infty$. Then by Theorem 2.1 we have (3.12). Moreover, by $J_l(\alpha,\beta)\le 1$ and by (3.11) and by the equivalence of (3.12) and (3.8) it follows that (3.13) holds. Hence, by Lemma 3.5 the equation (3.1) is left disfocal on $(\alpha,\beta]$.

Thus the proof of Lemma 3.6 is complete. □

Proof of Theorem 3.1. Let the equation (3.1) be left disfocal on $(\alpha,\beta]$. Then by Lemma 3.6, we have $J_l(\alpha,\beta)\le 1$. By definition, $J_l(\alpha,\beta)$ is the best constant in the inequality

$$\int\limits_\alpha^\beta v(t)\left|\int\limits_\alpha^t g(s)\,ds\right|^p\,dt \le J_l(\alpha,\beta)\int\limits_\alpha^\beta \rho(t)\,|g(t)|^p\,dt,\quad g\in L_{p,\rho}(\alpha,\beta). \tag{3.14}$$

Then by Theorem 1.18 and Theorem 1.20 we obtain

$$A_1(\alpha,\beta) \le J_l(\alpha,\beta) \le p\left(\frac{p}{p-1}\right)^{p-1} A_1(\alpha,\beta), \tag{3.15}$$

$$A_2(\alpha,\beta) \le J_l(\alpha,\beta) \le \left(\frac{p}{p-1}\right)^{p} A_2(\alpha,\beta). \tag{3.16}$$

Since $J_l(\alpha,\beta) \le 1$, then we obtain (3.2).

Let one of the conditions (3.3) holds. Then by (3.15) and (3.16) it follows that $J_l(\alpha,\beta) \le 1$. Therefore, by Lemma 3.6 the equation (3.1) is left disfocal on $(\alpha,\beta]$.

Thus the proof of Theorem 3.1 is complete. □

Proof of Corollary 3.2. Let (3.4) hold. Then by (3.15) and by (3.16) it follows that $J_l(\alpha,\beta) > 1$.

The following two cases $J_l(\alpha,\beta) = \infty$ and $1 < J_l(\alpha,\beta) < \infty$ are possible. In the first case from the definition of $J_l(\alpha,\beta)$, we have that for any $N > 0$ there exists $g_N \in L_{p,\rho}(\alpha,\beta)$ such that

$$\int_\alpha^\beta v(t)\left|\int_\alpha^t g_N(s)\,ds\right|^p dt > N\int_\alpha^\beta \rho(t)\,|g_N(t)|^p\,dt.$$

In particular, for $N = 1$ there exists $g_1 \in L_{p,\rho}(\alpha,\beta)$ such that

$$\int_\alpha^\beta v(t)\left|\int_\alpha^t g_1(s)\,ds\right|^p dt > \int_\alpha^\beta \rho(t)\,|g_1(t)|^p\,dt. \tag{3.17}$$

Assume that $f_1(t) = \int_\alpha^t g_1(s)\,ds$. Then $f_1(\alpha) = 0$ and $f_1'(t) = g_1(t) \in L_{p,\rho}(\alpha,\beta)$. Hence, $f_1 \in W^1_{p,l}(\rho,(\alpha,\beta))$ and

$$\int_\alpha^\beta v(t)|f_1(t)|^p\,dt > \int_\alpha^\beta \rho(t)|f_1'(t)|^p\,dt, \tag{3.18}$$

i.e., (3.8) does not holds. Then by Lemma 3.5 there exists a left focal point of the point β with respect to the equation (3.1).

Since $J_l(\alpha,\beta)$ is the best constant in the inequality (3.14), in the second case, there exists $g_1 \in L_{p,\rho}(\alpha,\beta)$ such that (3.17) holds. Consequently, (3.18) holds. This gives the existence of the left focal point of the point β with respect to (3.1).

Thus the proof of Corollary 3.2 is complete. □

In the same way, on the basis of Lemma 3.7 and Lemma 3.8 we can prove Theorem 3.3 and Corollary 3.4. The proofs of Lemma 3.7 and Lemma 3.8 are similar to the proofs of Lemma 3.5 and Lemma 3.6, respectively.

Lemma 3.7. *Let the assumptions of Theorem 3.3 hold. Then the equation (3.1) is right disfocal on the interval* $[\alpha, \beta)$ *if and only if*

$$\int_\alpha^\beta v(x)|f(x)|^p\,dx \le \int_\alpha^\beta \rho(x)|f'(x)|^p\,dx, \quad \textit{for all } f \in W^1_{p,r}(\rho,(\alpha,\beta)).$$

Lemma 3.8. *Let the assumptions of Theorem 3.3 hold. Then the equation (3.1) is right disfocal on the interval* $[\alpha, \beta)$ *if and only if* $J_r(\alpha,\beta) \le 1$*, where*

$$J_r(\alpha,\beta) = \sup_{0 \ne g \in L_{p,\rho}(\alpha,\beta)} \frac{\int_\alpha^\beta v(t)\left|\int_t^\beta g(s)\,ds\right|^p\,dt}{\int_\alpha^\beta \rho(t)\,|g(t)|^p\,dt}.$$

3.2 Disconjugacy. Main results.

In this Section, we investigate the problem of the disconjugacy of the half–linear second order differential equation (3.1) on a given interval $(\alpha, \beta) \subseteq I$. By assuming the same conditions on the coefficients of the equation (3.1) as in Section 3.1, we write the following remark.

Remark 3.9. *If the continuous function v changes sing, then we set that $v^+(t) = \max\{0, v(t)\}$ and we consider the equation*

$$\big(\rho(t)|y'(t)|^{p-2}y'(t)\big)' + v^+(t)|y(t)|^{p-2}y(t) = 0 \tag{3.19}$$

instead of the equation (3.1). Then on the basis of the Sturm comparison Theorem (Theorem 1.9), the disconjugacy on (α, β) of the equation (3.19) follows by the diconjugacy on (α, β) of the equation (3.1). Therefore, without loss of generality we will assume that $v \geq 0$ and $\rho > 0$ are continuous functions on I and that $v \neq 0$ on I as in Section 3.1.

Let $a \leq \alpha < \beta \leq b$. We consider the problem of the disconjugacy on (α, β) depending on the integral behavior of the function $\rho^{1-p'}$ on (α, β).

Let

$$\int_\alpha^\beta \rho^{1-p'}(t)\, dt < \infty. \tag{3.20}$$

Theorem 3.10. *Assume that $1 < p < \infty$ and that (3.20) hold. If the equation (3.1) is disconjugate on (α, β), then there exists a middle point $c_i \in (\alpha, \beta)$ for $(A_i(\alpha, \beta), A_i^*(\alpha, \beta))$ and $A_i(\alpha, c_i) = A_i^*(c_i, \beta) \equiv T_{c_i}(\alpha, \beta) \leq 1$, $i = 1, 2$. Conversely, if there exists a middle point $c_i \in (\alpha, \beta)$ for $(A_i(\alpha, \beta), A_i^*(\alpha, \beta))$ and $T_{c_i}(\alpha, \beta) < \gamma_i$ for at least one $i = 1, 2$, where $\gamma_1 = \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}$, $\gamma_2 = \left(\frac{p-1}{p}\right)^p$, then the equation (3.1) is disconjugate on (α, β).*

Corollary 3.11. *Let the assumptions of Theorem 3.10 hold. Let there exist a middle point $c_{i_0} \in (\alpha, \beta)$ for $(A_{i_0}(\alpha, \beta), A_{i_0}^*(\alpha, \beta))$, $1 \leq i_0 \leq 2$. If $T_{c_{i_0}}(\alpha, \beta) < \gamma_{i_0}$, then the equation (3.1) is left disfocal on $(\alpha, c_{i_0}]$ and right disfocal on $[c_{i_0}, \beta)$. Moreover, any solution satisfying the condition $y'(c_{i_0}) = 0$, $y(c_{i_0}) \neq 0$ is not identically equal to zero on (α, β).*

By the Roundabout theorem (Theorem 1.7) the disconjugacy on (α, β) of the equation (3.1) is equivalent to the existence of a solution of (3.1) which is not identically zero on (α, β). The last statement in Corollary 3.11 shows how to find such a solution.

Corollary 3.12. *Let the asumptions of Corollary 3.11 hold. If* $T_{c_{i_0}} > 1$, *then all solutions of the equation (3.1) satisfying the condition* $y'(c_{i_0}) = 0$, $y(c_{i_0}) \neq 0$ *has at least one zero on each of the intervals* (α, c_{i_0}) *and* (c_{i_0}, β). *Therefore, the equation (3.1) is conjugate on* (α, β).

The assumptions of Corollary 3.12 imply the existence of a solution that has at least two zeros on a given interval.

Corollary 3.13. *Let the assumptions of Theorem 3.10 hold. If at least one of the limits* $\limsup_{x\to a} A_i(\alpha, c, x)$ *and* $\lim_{x\to\beta} A_i^*(c, \beta, x)$, $c \in (\alpha, \beta)$, *is infinite, then the equation (3.1) is conjugate on* (α, β).

Now, we consider the case

$$\int_{\alpha}^{c} \rho^{1-p'}(s)\, ds < \infty, \quad \int_{c}^{\beta} \rho^{1-p'}(s)\, ds = \infty, \quad c \in (\alpha, \beta). \tag{3.21}$$

We note that the second condition in (3.21) holds if and only if $\beta = b$.

Theorem 3.14. *Let* $1 < p < \infty$. *Let (3.21) hold. Then the condition*

$$\max\{A_1(\alpha, \beta),\ A_2(\alpha, \beta)\} \leq 1 \tag{3.22}$$

is necessary and the validity of one of the conditions

$$A_1(\alpha, \beta) < \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}, \quad A_2(\alpha, \beta) < \left(\frac{p-1}{p}\right)^{p} \tag{3.23}$$

is sufficient for the equation (3.1) to be disconjugate on (α, β).

We now introduce the following definition.

Definition 3.15. *The equation (3.1) is said to be left or right disfocal on all the interval* (α, β), *if for any* $c \in (\alpha, \beta)$ *it is left or right disfocal on* $(\alpha, c]$ *or on* $[c, \beta)$, *respectively.*

The next theorem defines more exactly the assumption of Theorem 3.14.

Theorem 3.16. *Let the assumptions of Theorem 3.14 hold. Then the condition (3.22) is necessary and the validity of one of the conditions of (3.23) is sufficient for the equation (3.1) to be left disfocal on all the interval* (α, β).

By Theorem 3.16 we can deduce the following corollary.

Corollary 3.17. *Let (3.21) hold. Then for any* $c \in (\alpha, \beta)$ *the condition (3.22) is necessary and the validity of at least one of the conditions of (3.23) is sufficient for the solutions of the equation (3.1) with the initial condition* $y(c) = 0$, $y'(c) \neq 0$ *to be strictly monotone on* $[c, \beta)$.

Let

$$\int\limits_{\alpha}^{c} \rho^{1-p'}(s)\,ds = \infty, \quad \int\limits_{c}^{\beta} \rho^{1-p'}(s)\,ds < \infty, \quad c \in (\alpha, \beta). \tag{3.24}$$

We note that the first condition in (3.24) holds if and only if $\alpha = a$.

Theorem 3.18. *Let* $1 < p < \infty$. *Let (3.24) hold. Then the condition*

$$\max\{A_1^*(\alpha, \beta),\ A_2^*(\alpha, \beta)\} \leq 1 \tag{3.25}$$

is necessary and the validity of one of the conditions

$$A_1^*(\alpha, \beta) < \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}, \quad A_2^*(\alpha, \beta) < \left(\frac{p-1}{p}\right)^{p} \tag{3.26}$$

is sufficient for the equation (3.1) to be disconjugate on (α, β).

Theorem 3.19. *Let the assumptions of Theorem 3.18 hold. Then the condition (3.25) is necessary and the validity of one of the conditions of (3.26) is sufficient for the equation (3.1) to be right disfocal on all the interval* (α, β).

By Theorem 3.19 we can deduce the following corollary.

Corollary 3.20. *Let (3.24) hold. Then for any* $c \in (\alpha, \beta)$ *the condition (3.25) is necessary and the validity of at least one of the conditions of (3.26) is sufficient for the solutions of the equation (3.1) with the initial condition* $y(c) = 0$, $y'(c) \neq 0$ *to be strictly monotone on* $(\alpha, c]$.

At last we consider the case when

$$\int\limits_{\alpha}^{c} \rho^{1-p'}(s)\,ds = \infty, \quad \int\limits_{c}^{\beta} \rho^{1-p'}(s)\,ds = \infty, \quad c \in (\alpha, \beta). \tag{3.27}$$

In this case we have $\alpha = a$, $\beta = b$.

Theorem 3.21. *Let $1 < p < \infty$. Let (3.27) hold. Then the equation (3.1) is conjugate on $(\alpha, \beta) = (a, b)$. Moreover, for any $c \in (a, b)$ the solution of the equation (3.1) with the initial condition $y'(c) = 0$, $y(c) \neq 0$ has at least one zero on each of the intervals (a, c) and (c, b).*

Consider the equation (3.1) with the parameter $\lambda \in \mathbb{R}$:

$$(\rho(t)|y'(t)|^{p-2}y'(t))' + \lambda v(t)|y(t)|^{p-2}y(t) = 0. \tag{3.28}$$

The set of values $\lambda \in \mathbb{R}$ for which the equation (3.28) is disconjugate on $I = (a, b)$ is called the disconjugacy domain of the equation (3.28). In [11, Lemma 5.3.1] it has been shown that if the equation (3.28) is disconjugate for all $\lambda \in \mathbb{R}$, then $v(t) \equiv 0$. But if $v(t) \not\equiv 0$, then there exists $\lambda_0 \in \mathbb{R}$ such that for $\lambda < \lambda_0$ the equation (3.28) is diconjugate on I. If so, λ_0 is said to be the disconjugacy constant. For $\lambda > \lambda_0$ the equation (3.28) has conjugacy points on I.

By Theorem 3.10, Theorem 3.14, Theorem 3.16 and Theorem 3.18, we have the following theorems, respectively.

Theorem 3.22. *Let $1 < p < \infty$. Let (3.20) hold. Then the equation (3.28) is disconjugate on I for*

$$\lambda < \left[\min\{\gamma_1^{-1} T_{c_1}(a, b),\ \gamma_2^{-1} T_{c_2}(a, b)\}\right]^{-1}$$

and the equation (3.28) has conjugacy points on I for

$$\lambda > \left[\max\{T_{c_1}(a, b),\ T_{c_2}(a, b)\}\right]^{-1}.$$

Recall that $\gamma_1 = \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}$, $\gamma_2 = \left(\frac{p-1}{p}\right)^{p}$ and that c_i is a middle point of $(A_i(\alpha, \beta), A_i^*(\alpha, \beta))$, $i = 1, 2$.

Theorem 3.23. *Let $1 < p < \infty$. Let (3.21) hold. Then the equation (3.28) is disconjugate on I for*

$$\lambda < \left[\min\{\gamma_1^{-1} A_1(a, b),\ \gamma_2^{-1} A_2(a, b)\}\right]^{-1}$$

and the equation (3.28) has conjugacy points on I for

$$\lambda > \left[\max\{A_1(a, b),\ A_2(a, b)\}\right]^{-1}.$$

Theorem 3.24. *Let $1 < p < \infty$. Let (3.24) hold. Then the equation (3.28) is disconjugate on I for*

$$\lambda < \left[\min\{\gamma_1^{-1}A_1^*(a,b),\ \gamma_2^{-1}A_2^*(a,b)\}\right]^{-1}$$

and the equation (3.28) has conjugacy points on I for

$$\lambda > [\max\{A_1^*(a,b),\ A_2^*(a,b)\}]^{-1}.$$

Theorem 3.25. *Let $1 < p < \infty$. Let (3.27) hold. Then the equation (3.28) has conjugacy points on I for all $\lambda > 0$.*

In these theorems there is a gap between the domains of conjugacy and disconjugacy that means that the disconjugacy constants of the equation (3.28) are not defined. Since ρ and v are arbitrary functions, the problem to define the disconjugacy constants of an equation of the type of (3.28) is difficult even in our case in which $\rho > 0$ and $v \geq 0$. However, when ρ and v are some power functions the disconjugacy constants can be defined.

Let $a = 0$ and $b = \infty$, i.e., $I = (0, \infty)$. On I, we consider the Euler type half–linear equation

$$(t^{\mu}|y'(t)|^{p-2}y'(t))' + \lambda t^{\gamma}|y(t)|^{p-2}y(t) = 0. \tag{3.29}$$

We denote by (μ, γ) the points of the plane R^2 and we assume that $\Omega = \{(\mu,\gamma) : \mu \in R \setminus \{p-1\},\ \gamma = \mu - p\}$.

Theorem 3.26. *Let $1 < p < \infty$. If $(\mu, \gamma) \in \Omega$ and $\mu < p-1$ or $\mu > p-1$, then the constant*

$$\lambda_0 = \left(\frac{|p-\mu-1|}{p}\right)^p$$

is the disconjugacy constant of the equation (3.29) on $I = (0, \infty)$. Moreover, for $\lambda = \lambda_0$ the equation (3.29) is disconjugate on I.

If $(\mu, \gamma) \notin \Omega$, then the equation (3.29) is disconjugate on I for all $\lambda \leq 0$ and the equation (3.29) has conjugacy points on I, i.e., $\lambda_0 = 0$ is the disconjugacy constant of the eqiuation (3.29) on I for all $\lambda > 0$.

Corollary 3.27. *Let $1 < p < \infty$. Let $\gamma = \mu - p$ and $\mu \neq p - 1$. Then the equation (3.29) is nonoscillatory if and only if*

$$\lambda \leq \lambda_0 = \left(\frac{|p-\mu-1|}{p}\right)^p.$$

We note that our Corollary 3.27 is Theorem 1.4.4 of [11].

3.3 Disconjugacy. Proofs of the main results.

Lemma 3.28. *Let $1 < p < \infty$. Let $a \le \alpha < \beta \le b$. The equation (3.1) is disconjugate on (α, β) if and only if*

$$\int_\alpha^\beta v(t)|f(t)|^p\,dt < \int_\alpha^\beta \rho(s)|f'(s)|^p\,ds, \quad \text{for all } f \in \overset{\circ}{AC}_p(\alpha,\beta). \tag{3.30}$$

Proof of Lemma 3.28. For $a < \alpha < \beta < b$ the validity of Lemma 3.28 follows by the Roundabout theorem (Theorem 1.7). Indeed, in this case $\overset{\circ}{W}{}^1_p(\alpha,\beta) = \overset{\circ}{W}{}^1_p(\rho,(\alpha,\beta))$ and $\overset{\circ}{AC}_p(\alpha,\beta)$ is dense in $\overset{\circ}{W}{}^1_p(\rho,(\alpha,\beta))$. Hence, it suffices to consider ones of the following cases $a = \alpha < \beta < b$, $a < \alpha < \beta = b$ and $a = \alpha < \beta = b$. We prove the last case when $a = \alpha < \beta = b$. The other cases can be proved similarly.

Thus, we assume that

$$\int_a^b v(t)|f(t)|^p\,dt < \int_a^b \rho(s)|f'(s)|^p\,ds, \quad \text{for all } f \in \overset{\circ}{AC}_p(\alpha,\beta). \tag{3.31}$$

If the equation (3.1) is conjugate on I, i.e., if it has conjugacy points $t_1 \in I$, $t_2 \in I$, $t_1 < t_2$, then there exists $\alpha \in (a, t_1)$, $\beta \in (t_2, b)$ such that the equation (3.1) is conjugate on (α,β). Then there exists $\widetilde{f} \in \overset{\circ}{AC}_p(I)$ such that

$$\int_\alpha^\beta v(t)|\widetilde{f}(t)|^p\,dt \ge \int_\alpha^\beta \rho(s)|\widetilde{f}'(s)|^p\,ds, \tag{3.32}$$

in contradiction with (3.30). Hence, we have the disconjugacy of the equation (3.1) on I.

Conversely, we assume that the equation (3.1) is disconjugate on I, but that (3.31) does not hold. That means that there exists $\widehat{f} \in \overset{\circ}{AC}_p(I)$ such that

$$\int_a^b v(t)|\widehat{f}(t)|^p\,dt \ge \int_a^b \rho(s)|\widehat{f}'(s)|^p\,ds. \tag{3.33}$$

By the definition of $\overset{\circ}{AC}_p(I)$, the membership of $\widehat{f}$ in $\overset{\circ}{AC}_p(I)$ implies that $supp\widehat{f} \subset I$. In other words there exist $\alpha,\ \beta \in I$, $\alpha < \beta$ with $supp f \subseteq [\alpha,\beta]$. Then the inequality (3.33) has the form (3.32) in contradiction with (3.30). Hence, (3.30) follows.

Thus the proof of Lemma 3.28 is complete. □

Let $a \le \alpha < \beta \le b$. Assume that

$$J_0(\alpha,\beta) = \sup_{0\ne f\in \overset{\circ}{W}{}^1_p(\rho,(\alpha,\beta))} \frac{\int_\alpha^\beta v(t)|f(t)|^p\,dt}{\int_\alpha^\beta \rho(t)|f'(t)|^p\,dt}.$$

Lemma 3.29. *Let $1 < p < \infty$. Let $a \le \alpha < \beta \le b$. The condition $J_0(\alpha,\beta) \le 1$ is necessary and the condition $J_0(\alpha,\beta) < 1$ is sufficient for the equation (3.1) to be disconjugate on (α,β).*

Proof of Lemma 3.29. Let the equation (3.1) be disconjugate on (α,β). Then by Lemma 3.28 it follows that (3.30) holds. In view of the density of $\overset{\circ}{AC}_p\,(I)$ in $\overset{\circ}{W}{}^1_p(\rho,(\alpha,\beta))$, implies that

$$1 \ge \sup_{0\ne f\in \overset{\circ}{AC}_p\,(I)} \frac{\int_\alpha^\beta v(t)|f(t)|^p\,dt}{\int_\alpha^\beta \rho(t)|f'(t)|^p\,dt} =$$

$$\sup_{0\ne f\in \overset{\circ}{W}{}^1_p(\rho,(\alpha,\beta))} \frac{\int_\alpha^\beta v(t)|f(t)|^p\,dt}{\int_\alpha^\beta \rho(t)|f'(t)|^p\,dt} = J_0(\alpha,\beta)$$

Conversely, let $J_0(\alpha,\beta) < 1$. Then by the definition $J_0(\alpha,\beta)$ we have

$$\int_\alpha^\beta v(t)|f(t)|^p\,dt \le J_0(\alpha,\beta)\int_\alpha^\beta \rho(t)|f'(t)|^p\,dt, \quad \forall\ f\in \overset{\circ}{W}{}^1_p(\rho,(\alpha,\beta)),$$

and thus (3.30) holds. Hence, by Lemma 3.28 the equation (3.1) is conjugate on (α,β).

Thus the proof of Lemma 3.29 is complete. □

Remark 3.30. *If the functional F does not have an extremal function f_0 in $\overset{\circ}{AC}_p\,(\alpha,\beta)$ such that*

$$\int_\alpha^\beta v(t)|f_0(t)|^p\,dt = J_0(\alpha,\beta)\int_\alpha^\beta \rho(t)|f_0'(t)|^p\,dt,$$

then the condition $J_0(\alpha,\beta) \le 1$ is necessary and sufficient for the equation (3.1) to be disconjugate on (α,β).

Proof of Theorem 3.10. Let the equation (3.1) be disconjugate on (α, β). Then by Lemma 3.29 we have $J_0(\alpha, \beta) \leq 1$. The finite value $J_0(\alpha, \beta)$ is by definition the best constant in the inequality

$$\int_\alpha^\beta v(t)|f(t)|^p \, dt \leq J_0(\alpha, \beta) \int_\alpha^\beta \rho(t)|f'(t)|^p \, dt, \quad \text{for all } f \in \overset{\circ}{W}{}_p^1(\rho, (\alpha, \beta)).$$

Then by Theorem 1.18 there exists a middle point $c_i \in (\alpha, \beta)$ for $(A_i(\alpha, \beta), A_i^*(\alpha, \beta))$, $i = 1, 2$, and the following estimate

$$\max\{T_{c_1}(\alpha, \beta),\ T_{c_2}(\alpha, \beta)\} \leq J_0(\alpha, \beta) \leq \min\left\{p\left(\frac{p}{p-1}\right)^{p-1} T_{c_1}(\alpha, \beta),\ \left(\frac{p}{p-1}\right)^p T_{c_2}(\alpha, \beta)\right\} \quad (3.34)$$

holds, where $T_{c_i}(\alpha, \beta) = A_i(\alpha, c_i) = A_i^*(c_i, \beta)$.

Then by the left estimate in (3.34) and by $J_0(\alpha, \beta) \leq 1$, we get $T_{c_i}(\alpha, \beta) \leq 1$, $i = 1, 2$.

Conversely, if at least one of the conditions

$$T_{c_1}(\alpha, \beta) < \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}, \quad T_{c_2}(\alpha, \beta) < \left(\frac{p-1}{p}\right)^p,$$

holds, then

$$\min\left\{p\left(\frac{p}{p-1}\right)^{p-1} T_{c_1}(\alpha, \beta),\ \left(\frac{p}{p-1}\right)^p T_{c_2}(\alpha, \beta)\right\} < 1.$$

Therefore, by the right estimate in (3.34), we have $J_0(\alpha, \beta) < 1$ and thus Lemma 3.29 implies the disconjugacy of the equation (3.1) on (α, β).

Thus the proof of Theorem 3.10 is complete. □

Proof of Corollary 3.11. By the condition $T_{c_0}(\alpha, \beta) < \gamma_0$ we have $A_{i_0}(\alpha, c_0) < \gamma_{i_0}$ and $A_{i_0}^*(c_0, \beta) < \gamma_{i_0}$. Then by Theorem 3.1 and Theorem 3.2 the equation (3.1) is left disfocal on $(\alpha, c_0]$ and right disfocal on $[c_0, \beta)$, respectively. Then a solution of the equation (3.1) with the initial condition $y'(c_0) = 0$, $y(c_0) \neq 0$ does not have zeros in $(\alpha, c_0]$ and $[c_0, \beta)$, i.e., it does not turn to zero in (α, β).

Thus the proof of Corollary 3.11 is complete. □

Proof of Corollary 3.12. If $T_{c_{i_0}}(\alpha, \beta) > 1$, $1 \leq i_0 \leq 2$, then $A_{i_0}(\alpha, c_{i_0}) > 1$ and $A_{i_0}^*(c_{i_0}, \beta) > 1$. Then

$$\max\{A_1(\alpha, c_{i_0}),\ A_2(\alpha, c_{i_0})\} > 1, \quad \max\{A_1^*(c_{i_0}, \beta),\ A_2^*(c_{i_0}, \beta)\} > 1.$$

Moreover, according to Corollary 3.2 and Corollary 3.4 a solution of the equation (3.1) with the initial condition $y'(c_{i_0}) = 0$, $y(c_{i_0}) \neq 0$ has at least one zero in each of the intervals (α, c_{i_0}), (c_{i_0}, β), respectively. Hence, the equation (3.1) is conjugate on (α, β).

Thus the proof of Corollary 3.12 is complete. □

Proof of Corollary 3.13. If one of the limits from the condition of Corollary 3.13 is infinite, then by Lemma 2.11 there does not exist a middle point for $(A_i(\alpha,\beta), A_i^*(\alpha,\beta))$. Then by Theorem 2.7 there can not exist a constant $C > 0$ such that the inequality

$$\int_\alpha^\beta v(t)|f(t)|^p\,dt \leq C \int_\alpha^\beta \rho(t)|f'(t)|^p\,dt,$$

holds for all f in $\overset{\circ}{W}{}_p^1(\rho, I)$ or in $\overset{\circ}{AC}_p\ (\alpha,\beta)$. Then for $C = 1$ there exists $f_1 \in \overset{\circ}{AC}_p\ (\alpha,\beta)$ such that

$$\int_\alpha^\beta v(t)|f(t)|^p\,dt > \int_\alpha^\beta \rho(t)|f'(t)|^p\,dt.$$

Then Lemma 3.28 implies the conjugacy of the equation (3.1) on (α,β).

Thus the proof of Corollary 3.13 is complete. □

Proof of Theorem 3.14. Let the equation (3.1) be disconjugate on (α,β). Then by Lemma 3.29, we have $J_0(\alpha,\beta) \leq 1$. Since by the assumptions of Theorem 3.14 it follows that (3.21) holds, then we obtain

$$\max\{A_1(\alpha,\beta),\ A_2(\alpha,\beta)\} \leq J_0(\alpha,\beta) \leq \min\left\{p\left(\frac{p}{p-1}\right)^{p-1} A_1(\alpha,\beta),\ \left(\frac{p}{p-1}\right)^p A_2(\alpha,\beta)\right\}, \tag{3.35}$$

by Theorem 2.8. By the left estimate (3.35) and by $J_0(\alpha,\beta) \leq 1$, we have (3.22).

Conversely, if one of the conditions of (3.23) hold, then

$$\min\left\{p\left(\frac{p}{p-1}\right)^{p-1} A_1(\alpha,\beta),\ \left(\frac{p}{p-1}\right)^p A_2(\alpha,\beta)\right\} < 1.$$

Hence, by the right estimate in (3.35) we have $J_0(\alpha,\beta) < 1$. Therefore, by Lemma 3.29 the equation (3.1) is disconjugate on (α,β).

Thus the proof of Theorem 3.14 is complete. □

Proof of Theorem 3.16. Let the equation (3.1) be left disfocal on all the interval (α, β). Then according to Definition 3.15, for any $c \in (\alpha, \beta)$ the equation (3.1) is left disfocal on $(\alpha, c]$. Therefore, by Theorem 3.1, we obtain $A_i(\alpha, c) \leq 1$, $i = 1, 2$. Since the expression $A_i(\alpha, c)$ does not decrease in $c \in (\alpha, \beta)$, then we can take the limit as c tends to β and obtain $\lim_{c \to \beta} A_i(\alpha, c) = A_i(\alpha, \beta) \leq 1$, $i = 1, 2$. Such a limiting relation is equivalent to the condition (3.22).

Let at least one of the conditions of (3.23) hold. Since $A_i(\alpha, c) \leq A_i(\alpha, \beta)$ for any $c \in (\alpha, \beta)$, then it follows at least one of the conditions

$$A_1(\alpha, c) < \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}, \quad A_2(\alpha, c) < \left(\frac{p-1}{p}\right)^{p}, \quad \text{for all } c \in (\alpha, \beta),$$

respectively. Then by Theorem 3.1, the equation (3.1) is left disfocal on $(\alpha, c]$ for all $c \in (\alpha, \beta)$. Thus by Definition 3.15 the equation (3.1) is left disfocal on all the interval (α, β).

Thus the proof of Theorem 3.16 is complete. □

Proof of Corollary 3.17. Let (3.21) hold. Assume that $c \in (\alpha, \beta)$ a solution of the equation (3.1) with initial condition $y(c) = 0$, $y'(c) \neq 0$ is strictly monotone on $[c, \beta)$. Then, for even solution of (3.1) we have $y'(t) \neq 0$ for all $t \in [c, \beta)$. Then a solution $y_1(t)$ of the equation (3.1) with initial condition $c_1 \in (\alpha, \beta)$, $y_1'(c_1) = 0$, $y_1(c_1) \neq 0$ does not have zeros in (α, c_1). Indeed, if a zero $c \in (\alpha, c_1)$ of y_1 exists, then the nontriviality of the solution $y_1(t)$ $(y_1(c_1) \neq 0)$ would imply that $y_1(c) = 0$, $y_1'(c) \neq 0$. However, by the assumed strictly monotonicity of y_1 on $[c, \beta)$ for such solution we would have $y'(c_1) \neq 0$. The obtained contradiction gives that the equation (3.1) is left disfocal on all the interval (α, β). Then by Theorem 3.16 we have that (3.22) holds.

Conversely, let (3.23) hold. Then by Theorem 3.21 the equation (3.1) is left disfocal on all the interval (α, β). Then for any $c_1 \in (\alpha, \beta)$ the solution $y_1(t)$ of the equation (3.1) with initial condition $y_1'(c_1) = 0$, $y_1(c_1) \neq 0$ does not have zeros in the interval (α, c_1). Hence, for any $c \in (\alpha, \beta)$ the solution $y(t)$ of the equation (3.1) with initial condition $y(c) = 0$, $y'(c) \neq 0$ satisfies the condition $y'(t) \neq 0$ for all $t \in [c, \beta)$.

Thus the proof of Corollary 3.17 is complete □

Theorem 3.18, Theorem 3.19 and Corollary 3.20 can be proved in the same way as Theorem 3.14, Theorem 3.16 and Corollary 3.17, respectively.

Proof of Theorem 3.21. Let (3.27) hold. Then on the basis of Theorem 2.10 the inequality (2.16) does not hold on the set $\overset{\circ}{W}{}^1_p(\rho, I)$. That means that $J_0(a,b) = \infty$. Therefore, there exists a nontrivial function $\widetilde{f} \in \overset{\circ}{AC}_p\,(I)$ such that

$$\int_a^b v(t)|\widetilde{f}(t)|^p\,dt > \int_a^b \rho(t)|\widetilde{f}'(t)|^p\,dt.$$

To prove the existence of such a nontrivial function we take $a < \alpha < \alpha + h < \beta - h < \beta < b$, $h > 0$ and then we consider the function $\widetilde{f}_0$ of (2.32) when we replace a by α, c^- by $\alpha + h$, c^+ by $\beta - h$, b by β. Suppose that $\widetilde{f}_{\alpha,\beta}(t) = \widetilde{f}_0(t)$ for all $\alpha \le t \le \beta$ and $\widetilde{f}_{\alpha,\beta}(t) = 0$ for all $t \in I \setminus (\alpha, \beta)$. Then $\widetilde{f}_{\alpha,\beta} \in \overset{\circ}{AC}_p\,(I)$ and in view of (2.32), (2.35), (3.27) if $\alpha \to a$ and $\beta \to b$, then we have $\int_a^b \rho(t)|\widetilde{f}'_{\alpha,\beta}(t)|^p\,dt \to 0$. Moreover, by the inequality $|\widetilde{f}_{\alpha,\beta}| \ge |\widetilde{f}_0|$ in I, we deduce that $\int_a^b v(t)|\widetilde{f}_{\alpha,\beta}(t)|^p\,dt \ge \int_a^b v(t)|\widetilde{f}_0(t)|^p\,dt > 0$. Hence, when α and β are close enough to a and b, respectively, we obtain that

$$\int_a^b v(t)|\widetilde{f}_{\alpha,\beta}(t)|^p\,dt > \int_a^b \rho(t)|\widetilde{f}'_{\alpha,\beta}(t)|^p\,dt.$$

Therefore, by Lemma 3.28 it follows that the equation (3.1) is conjugate on I. By the condition (3.27) we have $A_1(a,c) = \infty$ and $A_1^*(c,b) = \infty$ for any $c \in I$. Then by Theorem 3.1 and Theorem 3.3 the equation (3.1) has a left focal point of the point c on $(a,c]$ and a right focal point of the point c on $[c,b)$. Thus, a solution of the equation (3.1) with the initial condition $y'(c) = 0$, $y(c) \ne 0$ has at least one zero on each of the intervals (a,c) and (c,b).

Thus the proof of Theorem 3.21 is complete. □

Theorem 3.22, Theorem 3.23, Theorem 3.24 and Theorem 3.25 directly follow from Theorem 3.10, Theorem 3.14, Theorem 3.18 and Theorem 3.21, respectively. For example, we can prove Theorem 3.24. By Theorem 3.18 the equation (3.28) is disconjugate on I, if at least one of the two conditions

$$\lambda A_1^*(a,b) < \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}, \quad \lambda A_2^*(a,b) < \left(\frac{p-1}{p}\right)^{p}, \tag{3.36}$$

holds. If

$$\lambda < \max\left\{\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}(A_1^*(a,b))^{-1},\ \left(\frac{p-1}{p}\right)^{p}(A_2^*(a,b))^{-1}\right\},$$

then at least one of the conditions in (3.36) holds. Such inequalities are equivalent to the conditions of disconjugacy in Theorem 3.24.

If $\lambda A_i^*(a,b) > 1$, $i = 1,2$, then Theorem 3.18 implies that the necessary condition of disconjugacy for (3.28) on I does not hold. Hence, the equation (3.28) is conjugate on I, i.e., for all values

$$\lambda > [\max\{A_1^*(a,b),\ A_2^*(a,b)\}]^{-1}$$

the equation (3.28) has conjugacy points on I.

Thus the proof of Theorem 3.24 is complete. □

Proof of Theorem 3.26. Let $(\mu,\gamma) \in \Omega$ and $\mu < p-1$. In this case for $c \in (0,\infty)$ we have $\int\limits_c^\infty t^{\mu(1-p')}\,dt = \infty$, $\int\limits_0^c t^{\mu(1-p')}\,dt < \infty$. In other words the condition (3.21) holds at $\alpha = 0$, $\beta = \infty$ for the function $\rho(t) = t^\mu$. Then Theorem 2.1 implies that $\overset{\circ}{W}{}_p^1(t^\mu, R_+) = W_{p,l}^1(t^\mu, R_+)$. Then by (1.9), we obtain

$$\begin{aligned} J_0(R_+) = \sup_{0\neq f\in \overset{\circ}{W}{}_p^1(t^\mu,R_+)} \frac{\lambda\int\limits_0^\infty t^\gamma |f(t)|^p\,dt}{\int\limits_0^\infty t^\mu |f'(t)|^p\,dt} &= \\ \lambda \sup_{0\neq f\in W_{p,l}^1(t^\mu,R_+)} \frac{\int\limits_0^\infty t^\gamma |f(t)|^p\,dt}{\int\limits_0^\infty t^\mu |f'(t)|^p\,dt} &= \\ \lambda \sup_{0\neq g\in L_p(t^\mu,R_+)} \frac{\int\limits_0^\infty t^\gamma \left|\int\limits_0^t g(s)\,ds\right|^p dt}{\int\limits_0^\infty t^\mu |g(t)|^p\,dt} &= \\ \lambda \sup_{0\neq g\in L_p(t^\mu,R_+)} \frac{\int\limits_0^\infty \left(\frac{1}{t}\int\limits_0^t g(s)\,ds\right)^p t^\mu\,dt}{\int\limits_0^\infty t^\mu g^p(t)\,dt} &= \\ \lambda\left(\frac{p}{p-\mu-1}\right)^p. \end{aligned} \tag{3.37}$$

Moreover, (1.9) for $\lambda > 0$ implies that

$$\lambda\int_0^\infty t^\gamma |f(t)|^p\,dt < \lambda\left(\frac{p}{p-\mu-1}\right)^p \int_0^\infty t^\mu |f'(t)|^p\,dt,\quad f\in \overset{\circ}{W}{}_p^1(t^\mu,R_+) \tag{3.38}$$

Then on the basis of Lemma 3.29 and Remark 3.30 the equation (3.29) is disconjugate on R_+ if and only if

$$\lambda \leq \lambda_0 = \left(\frac{p-\mu-1}{p}\right)^p.$$

If $\mu > p-1$, then the condition (3.24) holds for the function $\rho(t) = t^\mu$ and Theorem 2.1 implies that $\overset{\circ}{W}{}_p^1(t^\mu, R_+) = W_{p,r}^1(t^\mu, R_+)$. Now, by exploiting (1.10), we obtain

$$J_0(R_+) = \lambda \left(\frac{p}{\mu+1-p}\right)^p$$

and

$$\lambda \int_0^\infty t^\gamma |f(t)|^p\,dt < \lambda \left(\frac{p}{\mu+1-p}\right)^p \int_0^\infty t^\mu |f'(t)|^p\,dt, \quad f \in \overset{\circ}{W}{}_p^1(t^\mu, R_+) \qquad (3.39)$$

for all $\lambda > 0$.

Consequently, according to Lemma 3.29 and Remark 3.30 the equation (3.29) is disconjugate on R_+ if and only if

$$\lambda \leq \lambda_0 = \left(\frac{\mu+1-p}{p}\right)^p.$$

Now, let $(\mu,\gamma) \notin \Omega$. We consider the three cases

(1) $\mu = p-1$;

(2) $\mu < p-1$, $\gamma \neq \mu - p$;

(3) $\mu > p-1$, $\gamma \neq \mu - p$.

In the case $\mu = p-1$ the condition (3.27) holds for the function $\rho(t) = t^\mu$ and Theorem 3.25 implies that the equation (3.29) has conjugacy points in R_+ for all $\lambda > 0$.

In the case $\mu < p-1$, $\gamma \neq \mu - p$ the condition (3.21) holds for the function $\rho(t) = t^\mu$ and simple computation imply that $A_1(0,\infty) = \infty$. Then by Theorem 3.23 the equation (3.29) has conjugacy points in I for all $\lambda > 0$.

In the last case $\mu > p-1$, $\gamma \neq \mu - p$ the condition (3.24) holds for the function $\rho(t) = t^\mu$ and simple computations imply that $A_1^*(0,\infty) = \infty$. Then by Theorem 3.24 the equation (3.29) has conjugacy points in R_+ for all $\lambda > 0$.

For $\lambda \leq 0$, we have

$$F(f) = \int_0^\infty (t^\mu |f'(t)|^p - \lambda t^\gamma |f(t)|^p\,dt > 0, \quad \text{for all } f \in \overset{\circ}{AC}_p (R_+).$$

Therefore, on the basis of the Roundabout theorem (Theorem 1.7) the equation (3.29) is disconjugate on any closed interval in R_+. Hence, it is disconjugate in R_+.

Thus the proof of Theorem 3.26 is complete. □

To prove Corollary 3.27 we note that if $\lambda \leq \lambda_0$, then by Theorem 3.26 the equation (3.29) is disconjugate in R_+, and if the equation (3.29) is nonoscillatory for $t = \infty$, then there exists $t_0 > 0$ such that the equation (3.29) is disconjugate on $[t_0, \infty)$. However, the inequalities (3.37), (3.38) hold for all functions $f \in \overset{\circ}{W}{}_p^1(t^\mu, [t_0, \infty))$ equal to zero on $(0, t_0]$. Thus, by Lemma 3.29 we have $\lambda \leq \lambda_0$.

Thus the proof of Theorem 3.26 is complete. □

Chapter 4

The oscillation properties of half–linear second order differential equations

In this Chapter, we investigate the oscillation properties of the equation (3.1). This Chapter consists of three Sections. In the first and the second Sections we assume that the coefficients of the equation (3.1) do not change sing. In the first Section we establish sufficient conditions and necessary conditions of nonoscillation for the equation (3.1) by exploiting the results of Chapter 3. In the same Section we show existence of solutions which do not vanish in a given interval.

Note that the obtained necessary conditions of nonoscillation of the first Section have obtained before by other methods.

The results of Chapter 3, which we employed in the first Section of the present Chapter have been obtained by the variational method as by using the results on weighted Hardy inequalities. Therefore in the second Section we introduce a general method to use the results on Hardy inequalities in the theory of oscillation and nonoscillation for the equation (3.1). On the basis of one of the results on Hardy inequalities, we establish sufficient conditions of oscillation and nonoscillation in new terms. It is our opinion that one can not obtain such results as a consequence of other known results.

In the third Section we assume that ρ is a positive function and that v is a function with arbitrary sing. Primarily, we establish our results by using the variational principle and obtain a conjugacy result for the equation (3.1) on a given interval. Accordingly, we obtain different conditions for oscillation of the equation (3.1), which generalize known results in such a case.

4.1 The criteria of oscillation and nonoscillation for half–linear second order differential equations with nonnegative coefficients

We consider the equation (3.1):

$$(\rho(t)|y'(t)|^{p-2}y'(t))' + v(t)|y(t)|^{p-2}y(t) = 0 \tag{4.1}$$

on $I = (a, b)$, $-\infty \le a < b \le \infty$, $1 < p < \infty$.

We suppose that ρ is a positive continuous function and that v is a nonnegative continuous function on I (Cf. Remark 3.9). Moreover, we assume that v is not identically 0 in I.

Let

$$\int_c^b \rho^{1-p'}(s)ds < \infty, \quad c \in I. \tag{4.2}$$

Let $\alpha \in I$. Then by continuity of the functions $\rho^{1-p'}$ and v on $[\alpha, c]$, we have $\lim\limits_{x\to\alpha} A_i(\alpha, c, x) = 0$, $i = 1, 2$. Therefore, if $\limsup\limits_{x\to b} A^*(c, b, x) < \infty$, $i = 1, 2$, then Lemma 2.11 implies that there exists a middle point c_i for $(A_i(\alpha, b),\ A_i^*(\alpha, b))$, i.e. $A_i(\alpha, c_i) = A_i^*(c_i, b)$, $i = 1, 2$.

Theorem 4.1. *Let (4.2) hold.*

(i) The equation (4.1) is nonoscillatory at $t = b$, then

$$\lim_{c\to b} A_i^*(c, b) \le 1, \quad i = 1, 2. \tag{4.3}$$

If at least one of the following conditions

$$\lim_{c\to b} A_i^*(c, b) < \gamma_i, \quad i = 1, 2$$

holds, i.e. if

$$\lim_{c\to b} \sup_{c<x<b} \left(\int_x^b \rho^{1-p'}(s)ds\right)^{p-1} \int_c^x v(t)dt < \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}, \tag{4.4}$$

$$\lim_{c\to b} \sup_{c<x<b} \left(\int_x^b \rho^{1-p'}(s)ds\right)^{-1} \int_x^b v(t)\left(\int_t^b \rho^{1-p'}(s)ds\right)^p dt < \left(\frac{p-1}{p}\right)^p, \tag{4.5}$$

then the equation (4.1) is nonoscillatory at $t = b$.

(ii) If

$$\lim_{c\to b} A_1^*(c, b) < \gamma_1,$$

then for any point $z \in (c_0, b)$ *there exist* $\alpha_z \in (\alpha, z)$ *such that a solution of the equation (4.1) with initial condition* $y'(z) = 0$, $y(z) \neq 0$ *is not identically zero in the interval* (α_z, b), *where* $c_0 = \max\{c_1,\ c^*\}$, $c^* = \inf\{c \in I,\ A_1^*(c, b) < \gamma_1\}$.

(iii) If at least one of the two conditions

$$\lim_{c\to b} A_1^*(c, b) > 1, \quad \lim_{c\to b} A_2^*(c, b) > 1 \tag{4.6}$$

holds, then the equation (4.1) is oscillatory at $t = b$.

Proof of Theorem 4.1. Part (i). Let the equation (4.1) be nonoscillatory at $t = b$. Then by Theorem 1.11 (4.1) is disconjugate on (T, b) at some $T \in I$. Hence, by Theorem 3.10 there exists a middle point $T_i \in (T, b)$ for $(A_i(T, b),\ A_i^*(T, b))$ and $A_i^*(T_i, b) \le 1$. Since $A_i(c, b)$ is nonincreasing in the variable $c \in I$, then (4.3) holds.

Let one of the conditions (4.4) or (4.5) hold. Then there exists $d_i \in I$ such that $A_i^*(c, b) < \gamma_i$ at $c > d_i$. Now if $\alpha \in (a, d_i)$ then we can take that a middle point c_i for $(A_i(\alpha, \beta),\ A_i^*(\alpha, \beta))$ and we have $c_i > d_i$. Then by Theorem 3.10 the equation (4.1) is disconjugate on (α, b). Hence, (4.1) is nonoscillatory at $t = b$.

Part (ii). If (4.4) exists, then for every $z \ge c_0$ by the continuity and monotonicity of $A_1(\alpha, z)$ in the variable α, there exists $\alpha_z \in (\alpha, z)$ such that $A_1(\alpha_z, z) = A_1^*(z, b) < \gamma_1$. Then by Corollary 3.11 a solution of the equation (4.1) with initial condition $y'(z) = 0$, $y(z) \neq 0$ is not identically zero on the interval (α_z, b).

Part (iii). Let one of the conditions (4.6) hold. Suppose that $\lim\limits_{c\to b} A_{i_0}^*(c, b) > 1$, $1 \le i_0 \le 2$ holds. If $\limsup\limits_{x\to b} A_{i_0}^*(c, b, x) = \infty$, then by Corollary 3.13 the equation (4.1) is conjugate on the interval (α, b) for all $\alpha \in I$. Therefore, by Theorem 1.11 the equation (4.1) is oscillatory.

If $\limsup\limits_{x\to b} A_{i_0}^*(c, b, x) < \infty$, then by Lemma 2.11, for any $\alpha \in I$ there exists a middle point $c_{i_0} \in (\alpha, b)$ for $(A_{i_0}(\alpha, b),\ A_{i_0}^*(\alpha, b))$ and $A_{i_0}(\alpha, c_{i_0}) = A_{i_0}^*(c_{i_0}, b) \equiv T_{c_0}(\alpha, b)$. Since $A_{i_0}(c, b)$ does not increase in $c \in I$, then (4.6) implies that $A_{i_0}(c_{i_0}, b) \equiv T_{c_0}(\alpha, b) > 1$. Thus, by Corollary 3.12 the equation (4.1) is conjugate on (α, b) for all $\alpha \in I$. Hence, by Theorem 1.11 the equation (4.1) is oscillatory.

Thus the proof of Theorem 4.1 is complete. □

Now we consider the following case

$$\int_c^b \rho^{1-p'}(s)ds = \infty, \quad c \in I. \tag{4.7}$$

Theorem 4.2. *Let (4.7) hold.*

(i) The equation (4.1) is nonoscillatory at $t = b$, then

$$\lim_{c \to b} A_i(c, b) \le 1, \quad i = 1, 2. \tag{4.8}$$

If at least one of the two following conditions

$$\lim_{c \to b} A_1(c, b) < \gamma_1, \quad \lim_{c \to b} A_2(c, b) < \gamma_2,$$

holds, i.e. if

$$\lim_{c \to b} \sup_{c<x<b} \left(\int_c^x \rho^{1-p'}(s)ds \right)^{p-1} \int_x^b v(t)dt < \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}, \tag{4.9}$$

$$\lim_{c \to b} \sup_{c<x<b} \left(\int_c^x \rho^{1-p'}(s)ds \right)^{-1} \int_c^x v(t) \left(\int_c^t \rho^{1-p'}(s)ds \right)^p dt < \left(\frac{p-1}{p}\right)^p, \tag{4.10}$$

then the equation (4.1) is nonoscillatory at $t = b$.

(ii) If at least one of the conditions (4.9) or (4.10) holds, i. e. if

$$\lim_{c \to b} A_{i_0}(c, b) < \gamma_{i_0}, \quad 1 \le i_0 \le 2,$$

then for any $z \in (c_0, b)$ a solution of the equation (4.1) with initial condition $y(z) = 0$, $y'(z) \ne 0$ is strictly monotone on $[z, b)$, where $c_0 = \inf\{c \in I,\ A_{i_0}(c, b) < \gamma_{i_0}\}$.

(iii) If at least one of the following conditions

$$\lim_{c \to b} A_i(c, b) > 1, \quad i = 1, 2 \tag{4.11}$$

holds, then the equation (4.1) is oscillatory at $t = b$.

Proof of Theorem 4.2. Part (i). Let the equation (4.1) be nonoscillatory at $t = b$. Then by Theorem 1.11 the equation (4.1) is disconjugate on the interval (T, b) at some $T \in I$. Hence, Theorem 3.14 implies that $A_i(T, b) \leq 1$, $i = 1, 2$ and thus we obtain (4.8). Let at least one of the conditions (4.9) or (4.10) hold, i. e. $\lim\limits_{c \to b} A_{i_0}(c, b) < \gamma_{i_0}$, $1 \leq i_0 \leq 2$, then $A_{i_0}(c, b) < \gamma_{i_0}$ for any $c \in (c_0, b)$. Hence, Theorem 3.14 implies that the equation (4.1) is disconjugate on (c, b), which means that (4.1) is nonoscillatory at $t = b$.

Part (ii). If at least one of the conditions (4.9) or (4.10) hold, then the equation (4.1) is disconjugate on (c, b) for any $c \in (c_0, b)$. Hence, Corollary 3.17 implies that for all $z \in (c_0, b)$ a solution of the equation (4.1) with initial condition $y(z) = 0$, $y'(z) \neq 0$ is strictly monotone on $[z, b)$.

Part (iii). If at least one of the conditions (4.11) hold, i. e. if $\lim\limits_{c \to b} A_{i_0}(c, b) > 1$, $1 \leq i_0 \leq 2$, then there exists $c \in (T, b)$ with $A_{i_0}(c, b) > 1$. Hence, Theorem 3.14 implies that the equation (4.1) is conjugate on the interval (c, b) for all $c \in (T, b)$. Therefore, Theorem 1.11 implies that the equation (4.11) is oscillotory at $t = b$.

Thus the proof of Theorem 4.2 is complete. □

The following two theorems can be proved similarly. Such theorems establish oscillation and nonoscillation of the equation (4.1) at $t = a$.

Theorem 4.3. *Let*

$$\int_a^c \rho^{1-p'}(s)ds < \infty, \quad c \in I.$$

(i) The equation (4.1) is nonoscillatory at $t = a$, then

$$\lim_{c \to a} A_i(a, c) \leq 1, \quad i = 1, 2.$$

If at least one of the following conditions

$$\lim_{c \to a} A_i(a, c) < \gamma_i, \quad i = 1, 2$$

holds, then the equation (4.1) is nonoscillatory at $t = a$.

(ii) If

$$\lim_{c \to a} A_1(a, c) < \gamma_1,$$

then for any $z \in (a, c_0)$ there exists $\beta_z \in (z, b)$ such that a solution of the equation (4.1) with initial conditions $y'(z) = 0$, $y(z) \neq 0$ is not identically zero *in the interval* (a, β_z), *where* $c_0 = \min\{c_1,\ c^*\}$, $c^* = \sup\{c \in I,\ A_1(a, c) < \gamma_1\}$ *and* c_1 *is a middle point for* $(A_1(a, \beta),\ A_1^*(a, \beta))$, $\beta \in I$.

(iii) If at least one of the following conditions

$$\lim_{c\to a} A_i(a, c) > 1, \quad i = 1, 2,$$

holds, then the equation (4.1) is oscillatory at $t = a$.

Theorem 4.4. *Let*

$$\int_a^c \rho^{1-p'}(s)ds = \infty, \quad c \in I.$$

(i) The equation (4.1) is nonoscillatory at $t = a$, *then*

$$\lim_{c\to a} A_i^*(a, c) \le 1, \quad i = 1, 2.$$

If at least one of the two conditions

$$\lim_{c\to a} A_i^*(a, c) < \gamma_i, \quad i = 1, 2$$

holds, then the equation (4.1) is nonoscillatory at $t = a$.

(ii) If

$$\lim_{c\to a} A_{i_0}^*(a, c) < \gamma_{i_0}, \quad 1 \le i_0 \le 2,$$

then for any $z \in (a, c_0)$ *a solution of the equation (4.1) with initial condition* $y(z) = 0$, $y'(z) \ne 0$ *is strictly monotone on* $(a, z]$, *where* $c_0 = \sup\{c \in I,\ A_{i_0}(a, c) < \gamma_{i_0}\}$.

(iii) If at least one of the following conditions

$$\lim_{c\to a} A_i^*(a, c) > 1, \quad i = 1, 2,$$

holds, then the equation (4.1) is oscillatory at $t = a$.

Remark 4.5. *The criteria of nonoscillation (4.4) and (4.5) under the assumption (4.2) and the criteria of oscillation (4.6), and also the criteria of nonoscillation (4.9) or (4.10) under the assumption (4.7), and the criteria of oscillation (4.11) are known. They have been obtained earlier by different methods. The corresponding analysis can be found in [11, Chapter 3]. In Theorem 4.1 and Theorem 4.2 we establish necessary conditions of nonoscillation for the equation (4.1) and the corresponding behavior of the solutions by* *another method. Such conditions are more general than sufficient conditions for the equation (4.1) and have been obtained in Chapter 3, by the variational principle and by using the results on weighted Hardy inequalities.*

4.2 Applications of weighted Hardy inequalities to oscillation theory of half–linear differential equations

In this section we obtain new sufficient conditions of oscillation and nonoscillation for the equation (4.1) by exploiting Theorem 1.23 on weighted Hardy inequalities. We assume that the coefficients of (4.1) satisfy the conditions of Section 4.1.

Let $a \le \alpha < \beta \le b$. By Theorem 1.23, we have

$$\sup_{1<s<p} \frac{p^p(s-1)}{p^p(s-1)+(p-s)^p} A_W(s,\alpha,\beta) \le J_l(\alpha,\beta) \le \inf_{1<s<p} \left(\frac{p-1}{p-s}\right)^{p-1} A_W(s,\alpha,\beta), \tag{4.12}$$

where

$$A_W(s,\alpha,\beta) = \sup_{\alpha<x<\beta} \left(\int_\alpha^x \rho^{1-p'}(t)dt\right)^{s-1} \int_x^b v(\tau) \left(\int_\alpha^\tau \rho^{1-p'}(t)dt\right)^{p-s} d\tau.$$

Let

$$\int_c^b \rho^{1-p'}(t)dt = \infty, \quad c \in I. \tag{4.13}$$

By (4.12), we have

Theorem 4.6. *Assume that the condition (4.13) hold. If there exists $s \in (1,p)$ such that*

$$\lim_{c\to b} \sup_{c<x<b} \left(\int_c^x \rho^{1-p'}(t)dt\right)^{s-1} \int_x^b v(\tau) \left(\int_c^\tau \rho^{1-p'}(t)dt\right)^{p-s} d\tau < \left(\frac{p-s}{p-1}\right)^{p-1}, \tag{4.14}$$

then the equation (4.1) is nonoscillatory at $t = b$.

However, if there exists $s \in (1,p)$ such that

$$\lim_{c\to b} \sup_{c<x<b} \left(\int_c^x \rho^{1-p'}(t)dt\right)^{s-1} \int_x^b v(\tau) \left(\int_c^\tau \rho^{1-p'}(t)dt\right)^{p-s} d\tau > 1 + \frac{\left(1-\frac{s}{p}\right)^p}{s-1}, \tag{4.15}$$

then the equation (4.1) is oscillatory at $t = b$.

Proof of Theorem 4.6. Indeed, if for some $\tau \in (1,p)$ the inequality (4.14) holds, then there exists $c \in (a,b)$ such that $A_W(\tau,c,b) < \left(\frac{p-\tau}{p-1}\right)^{p-1}$. Hence, $\inf\limits_{1<s<p} \left(\frac{p-1}{p-s}\right)^{p-1} A_W(s,c,b) < 1$ and $J_l(c,b) < 1$ by (4.12). Since $J_l(c,b) = J_0(c,b)$ and we obtain that $J_0(c,b) < 1$ by (4.12). Then by Lemma 3.29 the equation (4.1) is disconjugate on (c,b), which means that the equation (4.1) is nonoscillatory at $t = b$.

If there exists $\tau \in (1,p)$ such that the condition (4.15) holds. Since $A_W(\tau,c,b)$ is nonincreasing with respect to $c \in (a,b)$, we obtain that $A_W(\tau,c,b) > 1 + \frac{\left(1-\frac{\tau}{p}\right)^p}{\tau-1}$ for all $c \in (a,b)$, and it follows that

$$\sup_{1<s<p} \frac{p^p(s-1)}{p^p(s-1)+(p-s)^p} A_W(s,c,b) > 1$$

for all $c \in (a,b)$. Hence, (4.12) and (4.13) imply that $J_0(c,b) > 1$ for all $c \in (a,b)$. Then Lemma 3.29 implies that the equation (4.1) is conjugate on (c,b) for all $c \in (a,b)$. Thus, Theorem 1.11 implies that the equation (4.1) is oscillatory at $t = b$.

Thus the proof of Theorem 4.6 is complete. □

When (4.13) holds, Theorem 2.1 implies that $W^1_{p,L}(\rho, I_c) = \overset{\circ}{W}{}^1_p(\rho, I_c)$, where $I_c = (c,b)$ and thus we have $J_0(I_c) = J_l(I_c)$.

By the efforts of many mathematicians, numerous estimates for $J_l(I_c)$ of the following form

$$k_p A_p(\rho, v, c, b) \leq J_l(I_c) \leq K_p A_p(\rho, v, c, b), \tag{4.16}$$

where the positive constants k_p and K_p depend only on p, and where $A_p(\rho, v, c, b)$ does not increase with respect to $c \in I$ have been obtained.

Theorem 4.7. *Assume that the condition (4.13) hold. If the inequality (4.16) holds for $J_l(I_c)$, then the condition*

$$\lim_{c\to b^-} A_p(\rho, v, c, b) \leq \frac{1}{k_p}$$

is necessary and the condition

$$\lim_{c\to b^-} A_p(\rho, v, c, b) < \frac{1}{K_p}$$

is sufficient for the equation (4.1) to be nonoscillatory at $t = b$.

Proof of Theorem 4.7. If the equation (4.1) is nonoscillatory at $t = b$, then Theorem 1.11 implies that there exists $c \in I$ such that the equation (4.1) is conjugate on I_c. Then Lemma 3.29 implies that $J_0(I_c) = J_0(c, b) \leq 1$. Since $J_0(I_c) = J_l(I_c)$, (4.13) and (4.16) imply that $k_p A_p(\rho, v, c, b) \leq 1$. Hence, $\lim\limits_{c \to b^-} A_p(\rho, v, c, b) \leq \frac{1}{k_p}$.

Conversely, let $\lim\limits_{c \to b^-} A_p(\rho, v, c, b) < \frac{1}{K_p}$. Then there exists $c \in I$ such that $A_p(\rho, v, c, b) < \frac{1}{K_p}$, i.e., $K_p A_p(\rho, v, c, b) < 1$. Therefore, (4.16) implies that $J_0(I_c)$ satisfies $J_0(I_c) < 1$ by (4.13) and the equation (4.1) is disconjugate on I_c by Lemma 3.29, which means that the equation (4.1) is nonoscillatory at $t = b$.

Thus the proof of Theorem 4.7 is complete. □

Theorem 4.8. *Assume that the condition (4.13) hold. If the inequality (4.16) holds for $J_l(I_c)$, then the condition $\lim\limits_{c \to b^-} A_p(\rho, v, c, b) \geq \frac{1}{K_p}$ is necessary and the condition $\lim\limits_{c \to b^-} A_p(\rho, v, c, b) > \frac{1}{k_p}$ is sufficient for the equation (4.1) to be oscillatory at $t = b$.*

Proof of Theorem 4.8. Let the equation (4.1) be oscillatory at $t = b$. By Theorem 1.11, for any $c \in I$ the equation (4.1) is conjugate on I_c. Then by Lemma 3.29 we have $J_0(I_c) \geq 1$. Since $J_0(I_c) = J_l(I_c)$, the condition (4.13) and Theorem 2.1 imply that $J_l(I_c) \geq 1$. Hence, we have that $K_p A_p(\rho, v, c, b) \geq 1$ for all $c \in I$ by (4.16). Therefore, $\lim\limits_{c \to b^-} A_p(\rho, v, c, b) \geq \frac{1}{K_p}$.

Conversely, let $\lim\limits_{c \to b^-} A_p(\rho, v, c, b) > \frac{1}{k_p}$. Since $A_p(\rho, v, c, b)$ is nonincreasing with respect to $c \in I$ we have that $A_p(\rho, v, c, b) > \frac{1}{k_p}$ for all $c \in I$. Hence, (4.16) implies that $J_l(I_c) > 1$ for all $c \in I$. By (4.13) and Theorem 2.1 we obtain $J_0(I_c) > 1$ for all $c \in I$. Then by Lemma 3.29 the equation (4.1) is conjugate on (c, b) for all $c \in I$. Therefore, by Theorem 1.11 the equation (4.1) is oscillatory at $t = b$.

Thus the proof of Theorem 4.7 is complete. □

We now consider the case

$$\int_c^b v(t)dt = \infty \quad \text{and} \quad \int_c^b \rho^{1-p'}(t)dt < \infty.$$

We assume that $\rho > 0$ and $v > 0$ and we analyze the equation

$$(v^{1-p'}(t)|y'(t)|^{p'-2}y'(t))' + \rho^{1-p'}(t)|y(t)|^{p'-2}y(t) = 0$$

of Theorem 4.6–4.8 by exploiting the reciprocity principle (see Chapter 1). We obtain conditions of oscillation and nonoscillation for the equation (4.1) at $t = b$.

In the previous theorems, there are sufficiently large gaps between the conditions for oscillation and nonoscillation. To obtain sharper conditions, we need to know the precise values of coefficients k_p and K_p. The above mentioned gap would be zero, if $k_p = K_p$. If $A_p(\rho, v, c, b)$ does not depend on $c \in I$, then nonoscillation (oscillation) of the equation (4.1) at $t = b$ is equivalent to its disconjugacy (conjugacy) on some interal (c, b). For example, for the equation

$$(t^{\alpha}|y'(t)|^{p-2}y'(t))' + \gamma t^{-(p-\alpha)}|y(t)|^{p-2}y(t) = 0, \quad \gamma > 0, \quad t > 0$$

at $\alpha \neq p-1$ the condition $\gamma \leq \left(\frac{|p-\alpha-1|}{p}\right)^p$ is necessary and sufficient for disconjigacy on some interval (c, ∞), with $c > 0$ and the equation (4.1) is oscillatory at $t = \infty$. The condition $\gamma > \left(\frac{|p-\alpha-1|}{p}\right)^p$ is necessary and sufficient for conjigacy on some interval (c, ∞) and the equation (4.1) is oscillatory at $t = \infty$ (see Chapter 3).

Finally we would like to note that some spectral properties of the differential operators related to the equation (4.1) have been derived, again with the help of Hardy–type inequalities in the paper of Drabek and Kufner [18].

4.3 Conjugacy and oscillation of half–linear second order differential equation with sing–variable potential

We consider the equation (4.1)

$$(\rho(t)|y'(t)|^{p-2}y'(t))' + v(t)|y(t)|^{p-2}y(t) = 0 \tag{4.17}$$

on $I = (a, b)$, $-\infty \le a < b \le \infty$, $1 < p < \infty$.

Here, in contrast to our previous work we do not make any assumption on the sing of v. Hence, we just assume that ρ and v are continuous functions. Moreover, we require that ρ is a positive function on I.

We now introduce the following notation. Let $a \le \alpha < \beta \le b$.

$$\Phi^-(\alpha, c) = \inf_{\alpha<z<c} \left[\left(\int_z^c \rho^{1-p'} ds \right)^{1-p} - \left(\int_z^c \rho^{1-p'} ds \right)^{-p} \int_z^c v(t) \left(\int_z^t \rho^{1-p'} ds \right)^p dt \right],$$

$$\Phi^+(d, \beta) = \inf_{d<z<\beta} \left[\left(\int_d^z \rho^{1-p'} ds \right)^{1-p} - \left(\int_d^z \rho^{1-p'} ds \right)^{-p} \int_d^z v(t) \left(\int_t^z \rho^{1-p'} ds \right)^p dt \right],$$

$$\varphi^-(\alpha, c) = \inf_{\alpha<z<c} \left[\left(\int_z^c \rho^{1-p'} ds \right)^{1-p} + \int_z^c v_-(t) dt \right],$$

$$\varphi^+(d, \beta) = \inf_{d<z<\beta} \left[\left(\int_d^z \rho^{1-p'} ds \right)^{1-p} + \int_d^z v_+(t) dt \right],$$

where $\alpha < c < d < \beta$, $v_\pm(t) = \max\{0, \pm v(t)\}$.

Theorem 4.9. *Let $a \le \alpha < \beta \le b$. If there exist points c, d such that $\alpha < c < d < \beta$ and that*

$$\int_c^d v(t) dt > \Phi^-(\alpha, c) + \Phi^+(d, \beta), \tag{4.18}$$

then the equation (4.17) is conjugate on (α, β).

Corollary 4.10. *Let $a \le \alpha < \beta \le b$. If there exist points c, d such that $\alpha < c < d < \beta$ and that*

$$\int_c^d v(t) dt > \varphi^-(\alpha, c) + \varphi^+(d, \beta), \tag{4.19}$$

then the equation (4.17) has conjugate points on (α, β).

Proof of Theorem 4.9. Let $\alpha < c < d < \beta$. Let (4.18) hold. Then by definition of infimum there exist a point $z^-(\alpha, c) = z^-$ and a point $z^+(d, \beta) = z^+$ such that the following inequality holds

$$\int_c^d v(t)dt \geq$$

$$\left(\int_{z^-}^{c} \rho^{1-p'} ds\right)^{1-p} -$$

$$\left(\int_{z^-}^{c} \rho^{1-p'} ds\right)^{-p} \int_{z^-}^{c} v(t) \left(\int_{z^-}^{t} \rho^{1-p'} ds\right)^{p} dt +$$

$$\left(\int_{d}^{z^+} \rho^{1-p'} ds\right)^{1-p} -$$

$$\left(\int_{d}^{z^+} \rho^{1-p'} ds\right)^{-p} \int_{d}^{z^+} v(t) \left(\int_{t}^{z^+} \rho^{1-p'} ds\right)^{p} dt$$

or

$$\left(\int_{z^-}^{c} \rho^{1-p'} ds\right)^{-p} \int_{z^-}^{c} v(t) \left(\int_{z^-}^{t} \rho^{1-p'} ds\right)^{p} dt +$$

$$\int_c^d v(t)dt + \left(\int_{d}^{z^+} \rho^{1-p'} ds\right)^{-p} \int_{d}^{z^+} v(t) \left(\int_{t}^{z^+} \rho^{1-p'} ds\right)^{p} dt \geq$$

$$\left(\int_{z^-}^{c} \rho^{1-p'} ds\right)^{1-p} + \left(\int_{d}^{z^+} \rho^{1-p'} ds\right)^{1-p}. \tag{4.20}$$

We denote by f_0 the function defined by

$$f_0(t) = \begin{cases} 0, & a < t < z^- \\ \left(\int_{z^-}^{c} \rho^{1-p'}\, ds\right)^{-1} \int_{z^-}^{t} \rho^{1-p'}\, ds, & z^- \leq t \leq c \\ 1, & c < t < d \\ \left(\int_{d}^{z^+} \rho^{1-p'}\, ds\right)^{-1} \int_{t}^{z^+} \rho^{1-p'}\, ds, & d \leq t \leq z^+ \\ 0, & z^+ < t < \beta, \end{cases}$$

Obviously, $f_0 \in \overset{\circ}{AC}_p(\alpha,\beta)$. We now compute the integrals $\int\limits_\alpha^\beta \rho(t)|f_0'(t)|^p dt$ and $\int\limits_\alpha^\beta v(t)|f_0(t)|^p dt$.

$$\int\limits_\alpha^\beta \rho(t)|f_0'(t)|^p dt =$$

$$\int\limits_{z^-}^{c} \rho(t)|f_0'(t)|^p dt + \int\limits_{c}^{d} \rho(t)|f_0'(t)|^p dt + \int\limits_{d}^{z^+} \rho(t)|f_0'(t)|^p dt =$$

$$\left(\int\limits_{z^-}^{c} \rho^{1-p'} ds\right)^{-p} \int\limits_{z^-}^{c} \rho(t)\rho^{p(1-p')}(t)dt +$$

$$\left(\int\limits_{d}^{z^+} \rho^{1-p'} ds\right)^{-p} \int\limits_{d}^{z^+} \rho(t)\rho^{p(1-p')}(t)dt =$$

$$\left(\int\limits_{z^-}^{c} \rho^{1-p'} ds\right)^{1-p} + \left(\int\limits_{d}^{z^+} \rho^{1-p'} ds\right)^{1-p}, \tag{4.21}$$

$$\int\limits_\alpha^\beta v(t)|f_0(t)|^p dt =$$

$$\int\limits_{z^-}^{c} v(t)|f_0(t)|^p dt + \int\limits_{c}^{d} v(t)|f_0(t)|^p dt + \int\limits_{d}^{z^+} v(t)|f_0(t)|^p dt =$$

$$\left(\int\limits_{z^-}^{c} \rho^{1-p'} ds\right)^{-p} \int\limits_{z^-}^{c} v(t)\left(\int\limits_{z^-}^{t} \rho^{1-p'} ds\right)^{p} dt + \int\limits_{c}^{d} v(t)dt +$$

$$\left(\int\limits_{d}^{z^+} \rho^{1-p'} ds\right)^{-p} \int\limits_{d}^{z^+} v(t)\left(\int\limits_{t}^{z^+} \rho^{1-p'} ds\right)^{p} dt. \tag{4.22}$$

The inequality (4.20) and the equalities (4.21) and (4.22) imply that

$$\int\limits_\alpha^\beta v(t)|f_0(t)|^p dt \geq \int\limits_\alpha^\beta \rho(t)|f_0'(t)|^p dt.$$

Thus by Lemma 3.28 the equation (4.17) has conjugate points on (α,β).

Thus the proof of Theorem 4.9 is complete. □

Proof of Corollary 4.10. For any $z \in (\alpha, c)$, we have

$$\left(\int_z^c \rho^{1-p'} ds\right)^{1-p} + \int_z^c v_-(t)dt \geq$$

$$\left(\int_z^c \rho^{1-p'} ds\right)^{1-p} + \left(\int_z^c \rho^{1-p'} ds\right)^{-p} \int_z^c v_-(t) \left(\int_z^t \rho^{1-p'} ds\right)^p dt \geq$$

$$\left(\int_z^c \rho^{1-p'} ds\right)^{1-p} - \left(\int_z^c \rho^{1-p'} ds\right)^{-p} \int_z^c v_+(t) \left(\int_z^t \rho^{1-p'} ds\right)^p dt +$$

$$\left(\int_z^c \rho^{1-p'} ds\right)^{-p} \int_z^c v_-(t) \left(\int_z^t \rho^{1-p'} ds\right)^p dt =$$

$$\left(\int_z^c \rho^{1-p'} ds\right)^{1-p} -$$

$$\left(\int_z^c \rho^{1-p'} ds\right)^{-p} \int_z^c (v_+(t) - v_-(t)) \left(\int_t^c \rho^{1-p'} ds\right)^p dt =$$

$$\left(\int_z^c \rho^{1-p'} ds\right)^{1-p} - \left(\int_z^c \rho^{1-p'} ds\right)^{-p} \int_z^c v(t) \left(\int_t^c \rho^{1-p'} ds\right)^p dt.$$

Hence, $\varphi^-(\alpha, c) \geq \Phi^-(\alpha, c)$. Similarly, $\varphi^+(\alpha, c) \geq \Phi^+(\alpha, c)$. Then by (4.19) (4.18) follows. By Theorem 4.9 the equation (4.17) is conjugate on (α, β).

Thus the proof of Corollary 4.10 is complete. □

Theorem 4.11. *Let* $\int_d^b \rho^{1-p'}(s)ds < \infty$. *Let the limiting relation*

$$\lim_{y \to b} \int_d^y v(t) \left(\int_t^y \rho^{1-p'}(s)ds\right)^p dt = \int_d^b v(t) \left(\int_t^b \rho^{1-p'}(s)ds\right)^p dt. \quad (4.23)$$

hold. If for any $c \in I$

$$\limsup_{d \to b} \left[\left(\int_d^b \rho^{1-p'} ds\right)^{p-1} \int_c^d v(t)dt + \left(\int_d^b \rho^{1-p'} ds\right)^{-1} \int_d^b v(t) \left(\int_t^b \rho^{1-p'} ds\right)^p dt\right] > 1, \quad (4.24)$$

then the equation (4.17) is oscillatory at $t = b$.

Proof of Theorem 4.11. Let $c \in I$. By definition lim sup there exists a sequence $\{d_k\}_{k=1}^{\infty} \subset (c, b)$ such that $k \geq k_0$

$$F(d_k, b) \equiv \left(\int_{d_k}^{b} \rho^{1-p'} ds\right)^{p-1} \int_{c}^{d_k} v(t)dt+$$
$$\left(\int_{d_k}^{b} \rho^{1-p'} ds\right)^{-1} \int_{d_k}^{b} v(t) \left(\int_{t}^{b} \rho^{1-p'} ds\right)^{p} dt > 1. \tag{4.25}$$

We take k_0 such that for $\alpha \in (a, c)$ the inequality

$$\left(\int_{d_k}^{b} \rho^{1-p'} ds\right)^{p-1} \int_{c}^{d_k} v(t)dt+$$
$$\left(\int_{d_k}^{b} \rho^{1-p'} ds\right)^{-1} \int_{d_k}^{b} v(t) \left(\int_{t}^{b} \rho^{1-p'} ds\right)^{p} dt >$$
$$1 + \left(\int_{d_k}^{b} \rho^{1-p'} ds\right)^{p-1} \Phi^{-}(\alpha, c)$$

holds for $k \geq k_0$. Indeed, $\lim\limits_{d \to b} \int_{d}^{b} \rho^{1-p'} ds = 0$.

Hence,

$$\int_{c}^{d_k} v(t)dt > \left(\int_{d_k}^{b} \rho^{1-p'} ds\right)^{p-1} -$$
$$\left(\int_{d_k}^{b} \rho^{1-p'} ds\right)^{-p} \int_{d_k}^{b} v(t) \left(\int_{d_k}^{b} \rho^{1-p'} ds\right)^{p} dt +$$
$$\Phi^{-}(\alpha, c). \tag{4.26}$$

By (4.26) and by the condition $\int_{d_k}^{b} \rho^{1-p'}(s)ds < \infty$ and by (4.23), we obtain $\int_{c}^{d_k} v(t)dt > \Phi^{-}(\alpha, c) + \Phi^{+}(d_k, b)$, $k \geq k_0$.

By the arbitrariness of $c \in I$ and $\alpha \in (a, c)$ Theorem 4.9 implies that the equation (4.17) has conjugate points on (α, b) for all $\alpha \in I$. Thus by Theorem 1.11 the equation (4.17) is oscillatory at $t = b$.

Thus the proof of Theorem 4.11 is complete. □

Now we consider the following case

$$\int_{d}^{b} \rho^{1-p'}(s)ds = \infty, \quad d \in I. \tag{4.27}$$

We assume that

$$\lim_{d\to b}\int_c^d v(t)dt = \int_c^b v(t)dt, \quad c \in I. \tag{4.28}$$

Theorem 4.12. *Let (4.27) hold. Let the limiting (4.23) and (4.28) exist finite.*

If

$$\lim_{\alpha\to b}\sup_{b>c>\alpha}\left[\left(\int_\alpha^c \rho^{1-p'}ds\right)^{p-1}\int_c^b v(t)dt + \left(\int_\alpha^c \rho^{1-p'}ds\right)^{-1}\int_\alpha^c v(t)\left(\int_\alpha^t \rho^{1-p'}ds\right)^p dt\right] > 1, \tag{4.29}$$

then the equation (4.17) is oscillatory at $t = b$.

Proof of Theorem 4.12. The inequality (4.29) implies that there exists $\alpha_0 \in I$ and $c \in (\alpha, b)$ for each $\alpha > \alpha_0$ such that

$$\left(\int_\alpha^c \rho^{1-p'}ds\right)^{p-1}\int_c^b v(t)dt + \left(\int_\alpha^c \rho^{1-p'}ds\right)^{-1}\int_\alpha^c v(t)\left(\int_\alpha^t \rho^{1-p'}ds\right)^p dt > 1.$$

Hence,

$$\int_c^b v(t)dt > \left(\int_\alpha^c \rho^{1-p'}ds\right)^{1-p} - \left(\int_\alpha^c \rho^{1-p'}ds\right)^{-p}\int_\alpha^c v(t)\left(\int_\alpha^t \rho^{1-p'}ds\right)^p dt \tag{4.30}$$

The existence of the limiting (4.28) implies that there exists $d \in (c, b)$ such that

$$\int_c^d v(t)dt > \left(\int_\alpha^c \rho^{1-p'}ds\right)^{1-p} - \left(\int_\alpha^c \rho^{1-p'}ds\right)^{-p}\int_\alpha^c v(t)\left(\int_\alpha^t \rho^{1-p'}ds\right)^p dt \geq \Phi^-(\alpha, c). \tag{4.31}$$

By (4.27) and by the finiteness of the limiting (4.23), we obtain

$$\lim_{y\to b}\left[\left(\int_d^y \rho^{1-p'}ds\right)^{1-p} - \left(\int_d^y \rho^{1-p'}ds\right)^{-p}\int_d^y v(t)\left(\int_t^y \rho^{1-p'}ds\right)^p dt\right] = 0.$$

Hence, $\Phi^+(d, b) \leq 0$. Then the inequality (4.31) implies that there exist $\alpha < c < d < b$ such that $\int_c^d v(t)dt > \Phi^-(\alpha, c) + \Phi^+(d, b)$ for any $\alpha \geq \alpha_0$. By Theorem 4.9 thus means that the equation (4.17) is conjugate on (α, b) for any $\alpha \in (\alpha_0, b)$. Hence, Theorem 1.11 implies that the equation (4.17) is oscillatory at $t = b$.

Thus the proof of Theorem 4.12 is complete. □

Similarly, we have the following two theorems. Such theorems establish the oscillation of the equation (4.17) at $t = a$.

Theorem 4.13. *Assume that there exists $c \in I$ such that*

$$\int_a^c \rho^{1-p'}(s)ds < \infty$$

and that there

$$\lim_{y\to a}\int_y^c v(t)\left(\int_y^t \rho^{1-p'}ds\right)^p dt = \int_a^c v(t)\left(\int_a^t \rho^{1-p'}ds\right)^p dt < \infty. \tag{4.32}$$

If

$$\limsup_{c\to a}\left[\left(\int_a^c \rho^{1-p'}ds\right)\int_c^d v(t)dt + \left(\int_a^c \rho^{1-p'}ds\right)^{-1}\int_a^c v(t)\left(\int_a^t \rho^{1-p'}ds\right)^p dt\right] > 1,$$

for all $d \in I$, then the equation (4.17) is oscillatory at $t = a$.

Theorem 4.14. *Assume that there exists $c \in I$ such that*

$$\int_a^c \rho^{1-p'}(s)ds = \infty$$

and that there (4.32) holds and that

$$\lim_{c\to a}\int_c^d v(t)dt = \int_a^d v(t)dt < \infty.$$

If

$$\lim_{\beta\to a}\sup_{a<d<\beta}\left[\left(\int_d^\beta \rho^{1-p'}ds\right)^{p-1}\int_a^d v(t)dt + \left(\int_d^\beta \rho^{1-p'}ds\right)^{-1}\int_d^\beta v(t)\left(\int_t^\beta \rho^{1-p'}ds\right)^p dt\right] > 1,$$

then the equation (4.17) is oscillatory at $t = a$.

By Theorem 4.11 and Theorem 4.12, we obtain the following corollary, which includes the results of both Theorem 4.1 and Theorem 4.2 concerning the oscillation of the equation (4.7).

Corollary 4.15. *Let*

$$\int_d^b \rho^{1-p'}(s)ds < \infty.$$

If $v(t) \geq 0$ for t in a left neighborhood of b and if the condition (4.24) holds, then the equation (4.17) is oscillatory $t = b$.

Corollary 4.16. *Let*

$$\int_d^b \rho^{1-p'}(s)ds = \infty.$$

If $v(t) \geq 0$ for t in a left neighborhood of b and if the condition (4.29) holds, then the equation (4.17) is oscillatory $t = b$.

Remark 4.17. *In the monograph [11], Theorem 3.1.4 and Theorem 3.1.6 have been proved under the assumptions of Corollary 4.15. Indeed, in Theorem 3.1.4 and Theorem 3.1.6, the oscillation of the equation (4.17) at $t = b = \infty$ has been proved under the assumptions*

$$\limsup_{x\to\infty} \left(\int_x^\infty \rho^{1-p'} ds \right)^{p-1} \int_c^x v(t)dt > 1, \tag{4.33}$$

and

$$\limsup_{x\to\infty} \left(\int_x^\infty \rho^{1-p'} ds \right)^{-1} \int_x^\infty v(t) \left(\int_t^\infty \rho^{1-p'} ds \right)^p dt > 1, \tag{4.34}$$

respectively.

Obviously, the assumptions (4.33) and (4.34) with $v(t) \geq 0$ for $t > 0$ large enough imply the validity of (4.24) at $b = \infty$. Otherwise, we can not expect that (4.24) holds at $b = \infty$.

Therefore, the statements of Corollary 4.15 includes the statements of Theorem 3.1.4 and of Theorem 3.1.6 of [11].

Remark 4.18. *In the monograph [11], Theorem 3.1.2 and Theorem 3.1.7 have been proved under the assumptions of Theorem 4.12 at $b = \infty$. Indeed, in Theorem 3.1.2 and Theorem 3.1.7 the oscillation of the equation (4.17) at $t = b = \infty$ has been proved under the assumptions*

$$\limsup_{x\to\infty} \left(\int_c^x \rho^{1-p'} ds \right)^{p-1} \int_x^\infty v(t)dt > 1,$$

$$\left(\int_c^x \rho^{1-p'} ds\right)^{-1} \int_c^x v(t) \left(\int_c^t \rho^{1-p'} ds\right)^p dt \geq 0 \tag{4.35}$$

for $x > 0$ large enough and

$$\limsup_{x\to\infty} \left(\int_c^x \rho^{1-p'} ds\right)^{-1} \int_c^x v(t) \left(\int_c^t \rho^{1-p'} ds\right)^p dt > 1,$$

$$\left(\int_c^x \rho^{1-p'} ds\right)^{p-1} \int_x^\infty v(t) dt \geq 0 \tag{4.36}$$

for $t > 0$ large enough, respectively.

Obviously, the conditions (4.35) and (4.36) imply the validity of the condition (4.29) of Theorem 4.12. Thus the statement of Theorem 4.12 generalizes the statement of Theorem 3.1.2 and Theorem 3.1.7 of [11].

Chapter 5

Oscillation and nonoscillation of two term linear and half - linear equations of higher order

5.1 Introduction

Let $I = [0, \infty)$, $n > 1$. Let $1 < p < \infty$. We consider the following higher order differential equation

$$(-1)^n(\rho(t)|y^{(n)}(t)|^{p-2}y^{(n)}(t))^{(n)} - v(t)|y(t)|^{p-2}y(t) = 0 \qquad (5.1)$$

on I, where v is a nonnegative ($v \neq 0$) continuous function and ρ is a positive continuous function on I. When the principle of reciprocity is used for the linear equation ($p = 2$), we assume that the functions v and ρ are positive and continuous on I.

A function $y : I \to R$ is said to be a solution of the equation (5.1), if $y(t)$ and $\rho(t)|y^{(n)}(t)|^{p-2}y^{(n)}(t)$ are n-times continuously differentiable and $y(t)$ satisfies the equation (5.1) on I.

Following the linear case [25], the equation (5.1) is called oscillatory at infinity if for any $T \geq 0$ there exist points $t_1 > t_2 > T$ and a nonzero solution $y(\cdot)$ of the equation (5.1) such that $y^{(i)}(t_k) = 0$, $i = 0, 1, ..., n-1$, $k = 1, 2$. Otherwise the equation (5.1) is called nonoscillatory.

If $p = 2$, then the equation (5.1) becomes a higher order linear equation

$$(-1)^{(n)}(\rho(t)y^{(n)}(t))^{(n)} - v(t)y(t) = 0. \tag{5.2}$$

The variational method to investigate the oscillatory properties of higher order linear equations and their relations to spectral characteristics of the corresponding differential operators have been well presented in the monograph [25]. Another method is the transition from a higher order linear equation to a Hamilton in system of equations [7]. However, to obtain the conditions of oscillation or nonoscillation of a higher order linear equation by this method, we need to find the principal solutions of a Hamilton in system (see e.g [13], [12]), which is not an easy task.

However, the general method of the investigation of the oscillatory properties for the equation (5.1) has not been developed yet. In the monograph [11, p. 464] by O. Došlý, one of the leading experts in the oscillation theory of half–linear differential equations, and his colleagues, the oscillation theory of half–linear equations of higher order is compared with "terra incognita".

The main aim of this Chapter is to establish the conditions of oscillation and nonoscillation of the equations (5.1) and (5.2) in terms of their coefficients by applying the latest results in the theory of weighted Hardy type inequalities.

Let $I_T = [T, \infty)$, $T \geq 0$. Let $1 < p < \infty$. Suppose that $L_{p,\rho} \equiv L_p(\rho, I_T)$ is the space of measurable and finite almost everywhere functions f, for which the following norm

$$\|f\|_{p,\rho} = \left(\int_T^\infty \rho(t)|f(t)|^p dt \right)^{\frac{1}{p}}$$

is finite.

We shall consider the weighted Hardy inequalities (see, Chapter 1)

$$\left(\int_T^\infty v(t) \left| \int_T^t f(s)ds \right|^p dt \right)^{\frac{1}{p}} \leq C \left(\int_T^\infty \rho(t)|f(t)|^p dt \right)^{\frac{1}{p}}, \quad f \in L_{p,\rho}, \tag{5.3}$$

and

$$\left(\int_T^\infty v(t) \left| \int_t^\infty f(s)ds \right|^p dt \right)^{\frac{1}{p}} \leq C \left(\int_T^\infty \rho(t)|f(t)|^p dt \right)^{\frac{1}{p}}, \quad f \in L_{p,\rho}, \tag{5.4}$$

where $C > 0$ does not dependent of f.

Let

$$J_l(T) \equiv J_l(\rho, v; T) = \sup_{0 \neq f \in L_{p,\rho}} \frac{\int_T^\infty v(t) \left| \int_T^t f(s)ds \right|^p dt}{\int_T^\infty \rho(t)|f(t)|^p dt},$$

$$J_r(T) \equiv J_r(\rho, v; T) = \sup_{0 \neq f \in L_{p,\rho}} \frac{\int_T^\infty v(t) \left| \int_t^\infty f(s)ds \right|^p dt}{\int_T^\infty \rho(t)|f(t)|^p dt}.$$

The criteria for $J_l(T)$ and $J_r(T)$ to be finite, which is equivalent to the validity of the inequalities (5.3) and (5.4), is given in Theorem 5.1 and Theorem 5.2 (see Theorems 1.18–1.22).

Theorem 5.1. *Let $1 < p < \infty$. Then $J_l(T) \equiv J_l(\rho, v; T) < \infty$ if and only if $A_1(T) < \infty$ or $A_2(T) < \infty$, where*

$$A_1(T) \equiv A_1(\rho, v; T) = \sup_{x > T} \int_x^\infty v(t)dt \left(\int_T^x \rho^{1-p'}(s)ds \right)^{p-1},$$

$$A_2(T) \equiv A_2(\rho, v; T) = \sup_{x > T} \left(\int_T^x \rho^{1-p'}(s)ds \right)^{-1} \int_T^x v(t) \left(\int_T^t \rho^{1-p'}(s)ds \right)^p dt.$$

Moreover, $J_l(T)$ can be estimated from above and from below, i.e.,

$$A_1(T) \leq J_l(T) \leq p \left(\frac{p}{p-1} \right)^{p-1} A_1(T), \tag{5.5}$$

$$A_2(T) \leq J_l(T) \leq \left(\frac{p}{p-1} \right)^p A_2(T). \tag{5.6}$$

Theorem 5.2. *Let $1 < p < \infty$. Then $J_r(T) \equiv J_r(\rho, v; T) < \infty$ if and only if $A_1^*(T) < \infty$ or $A_2^*(T) < \infty$, where*

$$A_1(T)^* \equiv A_1^*(\rho, v; T) = \sup_{x > T} \int_T^x v(t)dt \left(\int_x^\infty \rho^{1-p'}(s)ds \right)^{p-1},$$

$$A_2^*(T) \equiv A_2^*(\rho, v; T) = \sup_{x > T} \left(\int_x^\infty \rho^{1-p'}(s)ds \right)^{-1} \int_x^\infty v(t) \left(\int_t^\infty \rho^{1-p'}(s)ds \right)^p dt.$$

Moreover, $J_r(T)$ can be estimated from above and from below, i.e.,

$$A_1^*(T) \leq J_r(T) \leq p \left(\frac{p}{p-1} \right)^{p-1} A_1^*(T), \tag{5.7}$$

$$A_2^*(T) \le J_r(T) \le \left(\frac{p}{p-1}\right)^p A_2^*(T). \tag{5.8}$$

In [35] it is shown that the constant $p\left(\frac{p}{p-1}\right)^{p-1}$ in (5.5) is the best possible. Next, we consider the following expression

$$J_{p,l}^n(T) \equiv J_{p,l}^n(\rho, v; T) = \sup_{0\ne f\in L_{p,\rho}} \frac{\int_T^\infty \left|\int_T^t (t-s)^{n-1} f(s)ds\right|^p dt}{\int_T^\infty \rho(t)|f(t)|^p dt}.$$

$$J_{p,r}^n(T) \equiv J_{p,r}^n(\rho, v; T) = \sup_{0\ne f\in L_{p,\rho}} \frac{\int_T^\infty \left|\int_t^\infty (s-t)^{n-1} f(s)ds\right|^p dt}{\int_T^\infty \rho(t)|f(t)|^p dt}.$$

By Theorem 1.26 and Theorem 1.27 we have:

Theorem 5.3. *Let $1 < p < \infty$. Then*

$$J_{p,l}^n(T) \equiv J_{p,l}^n(\rho, v; T) < \infty$$

if and only if $B_{p,1}^n(T) < \infty$ and $B_{p,2}^n(T) < \infty$, where

$$B_{p,1}^n(T) \equiv B_{p,1}^n(\rho, v; T) = \sup_{x>T} \int_x^\infty v(t)dt \left(\int_T^x (x-s)^{p(n-1)} \rho^{-1}(s)ds\right)^{p-1},$$

$$B_{p,2}^n(T) \equiv B_{p,2}^n(\rho, v; T) = \sup_{x>T} \int_x^\infty v(t)(t-x)^{p(n-1)}dt \left(\int_T^x \rho^{-1}(s)ds\right)^{p-1}.$$

Moreover, there exists a constant $\beta \ge 1$ independent of ρ, v and T such that

$$B_p^n(T) \le J_n(T) \le \beta B_p^n(T), \tag{5.9}$$

where $B_p^n(T) = \max\{B_{p,1}^n(T), B_{p,2}^n(T)\}$.

Theorem 5.4. *Let $1 < p < \infty$. Then*

$$J_{p,r}^n(T) \equiv J_{p,r}^n(\rho, v; T) < \infty$$

if and only if $B_{p,1}^{*,n}(T) < \infty$ and $B_{p,2}^{*,n}(T) < \infty$, where

$$B_{p,1}^{*,n}(T) \equiv B_{p,1}^{*,n}(\rho, v; T) = \sup_{x>T} \int_T^x v(t)dt \left(\int_x^\infty (x-s)^{p(n-1)} \rho^{-1}(s)ds\right)^{p-1},$$

$$B_{p,2}^{*,n}(T) \equiv B_{p,2}^{*,n}(\rho, v; T) = \sup_{x>T} \int_T^x v(t)(t-x)^{p(n-1)}dt \left(\int_x^\infty \rho^{-1}(s)ds \right)^{p-1}.$$

Moreover, there exists a constant $\beta^* \geq 1$ independent of ρ, v and T such that

$$B_p^{*,n}(T) \leq J_n(T) \leq \beta^* B_p^{*,n}(T), \tag{5.10}$$

where $B_p^{*,n}(T) = \max\{B_{p,1}^{*,n}(T), B_{p,2}^{*,n}(T)\}$.

Assume that $AC_p^{n-1}(\rho, I_T)$ is the set of all functions f that have absolutely continuous $n-1$ order derivatives on $[T, N]$ for any $N > T$ and $f^{(n)} \in L_p$. Let $AC_{p,L}^{n-1}(\rho, I_T) = \{f \in AC_p^{n-1}(\rho, I_T) : \ f^{(i)}(T) = 0, \ i = 0, 1, ..., n-1\}$.

Suppose that $A^0C_p^{n-1}(\rho, I_T)$ is the set of all functions from $AC_{p,L}^{n-1}(\rho, I_T)$ that are equal to zero in a neighborhood of infinity. The function f from $AC_{p,L}^{n-1}(\rho, I_T)$ is called nontrivial if $\|f^{(n)}\|_p \neq 0$. If so, we write that $f \neq 0$.

By applying the variational method for higher order linear equations [25] we have:

Theorem 5.5. *The equation (5.1)*

(i) is nonoscillatory if and only if there exists $T \geq 0$ such that

$$\int_T^\infty (\rho(t)|f^{(n)}(t)|^2 - v(t)|f(t)|^2)dt > 0 \tag{5.11}$$

for every nontrivial $f \in A^0C_2^{n-1}(\rho, I_T)$;

(ii) is oscillatory if and only if for every $T \geq 0$ there exists a nontrivial function $\tilde{f} \in A^0C_2^{n-1}(\rho, I_T)$ such that

$$\int_T^\infty (\rho(t)|\tilde{f}^{(n)}(t)|^2 - v(t)|\tilde{f}(t)|^2)dt \leq 0. \tag{5.12}$$

Theorem 9.4.4 of [11] implies the validity of the following statement.

Theorem 5.6. *Let $1 < p < \infty$. If there exists $T \geq 0$ such that*

$$\int_T^\infty (\rho(t)|f^{(n)}(t)|^p - v(t)|f(t)|^p)dt > 0 \tag{5.13}$$

for all nontrivial $f \in A^0C_p^{n-1}(\rho, I_T)$, then the equation (5.1) is nonoscillatory.

Suppose that $W_p^n \equiv W_p^n(\rho, I_T)$ is the set of functions f that have n order generalized derivatives on I_T and for which the norm

$$\|f\|_{W_p^n} = \|f^{(n)}\|_p + \sum_{i=0}^{n-1} |f^{(i)}(T)| \tag{5.14}$$

is finite.

It is obvious that $A^0C_p^{n-1}(\rho, I_T) \subset AC_{p,L}^{n-1}(\rho, I_T) \subset W_p^n(\rho, I_T)$. We denote by $\overset{\circ}{W}{}_p^n \equiv \overset{\circ}{W}{}_p^n(\rho, I_T)$ and $W_{p,L}^n \equiv W_{p,L}^n(\rho, I_T)$ the closures of the sets $A^0C_p^{n-1}(\rho, I_T)$ and $AC_{p,L}^{n-1}(\rho, I_T)$ with respect to the norm (5.14), respectively. Since $\rho(t) > 0$ for $t \geq 0$, we have that

$$f^{(i)}(T) = 0, \quad i = 0, 1, ..., n-1 \tag{5.15}$$

for any $f \in W_{p,L}^n(\rho, I_T)$.

5.2 Nonoscillation of two term linear and half - linear equations of higher order

In this section we consider nonoscillation of the equations (5.1) and (5.2). Assume that there exists $T > 0$ such that

$$\int_x^\infty v(t)(t-T)^{p(n-1)}dt < \infty. \tag{5.16}$$

Theorem 5.7. *Let $1 < p < \infty$. Assume that v is a nonnegative continuous function and that ρ is a positive continuous function on I and that (5.16) hold. If one of the following conditions*

$$\lim_{T\to\infty}\sup_{x>T}\left(\int_T^x \rho^{1-p'}(s)ds\right)^{p-1}\int_x^\infty v(t)(t-T)^{p(n-1)}dt < \frac{1}{p-1}\left[\frac{(n-1)!(p-1)}{p}\right]^p \tag{5.17}$$

or

$$\lim_{T\to\infty}\sup_{x>T}\left(\int_T^x \rho^{1-p'}(s)ds\right)^{-1}\int_T^x v(t)(t-T)^{p(n-1)}\times \left(\int_T^t \rho^{1-p'}(s)ds\right)^p dt < \left[\frac{(n-1)!(p-1)}{p}\right]^p \tag{5.18}$$

holds, then the equation (5.1) is nonoscillatory.

In the case $p = 2$ Theorem 5.7 implies nonoscillation of the equation (5.2).

Theorem 5.8. *Assume that v is a non-negative and ρ is a positive continuous functions on I and that (5.16) holds with $p = 2$. If one of the following conditions*

$$\lim_{T\to\infty}\sup_{x>T}\int_T^x \rho^{-1}(s)ds\int_x^\infty v(t)(t-T)^{2(n-1)}dt < \left[\frac{(n-1)!}{2}\right]^2$$

or

$$\lim_{T\to\infty}\sup_{x>T}\left(\int_T^x \rho^{-1}(s)ds\right)^{-1}\int_T^x v(t)(t-T)^{2(n-1)}\left(\int_T^t \rho^{-1}(s)ds\right)^2 dt < \left[\frac{(n-1)!}{2}\right]^2$$

holds, then the equation (5.2) is nonoscillatory.

Proof of Theorem 5.7. If we show that one of conditions (5.17) or (5.18) hold, it follows that there exists $T \geq 0$ such that

$$F_{p,0}(T) \equiv F_{p,0}(\rho, v; T) = \sup_{0 \neq f \in A^0 C_p^{n-1}(\rho, I_T)} \frac{\int_T^\infty v(t)|f(t)|^p dt}{\int_T^\infty \rho(t)|f^{(n)}(t)|^p dt} = \sup_{0 \neq f \in \overset{\circ}{W}{}_p^n} \frac{\int_T^\infty v(t)|f(t)|^p dt}{\int_T^\infty \rho(t)|f^{(n)}(t)|^p dt} < 1, \tag{5.19}$$

then Theorem 5.6 implies that the equation (5.1) is nonoscillatory.

We define

$$F_{p,L}(T) \equiv F_{p,L}(\rho, v; T) = \sup_{0 \neq f \in W_{p,L}^n} \frac{\int_T^\infty v(t)|f(t)|^p dt}{\int_T^\infty \rho(t)|f^{(n)}(t)|^p dt}. \tag{5.20}$$

Since $\overset{\circ}{W}{}_p^n \subset W_{p,L}^n$, then

$$F_{p,0}(T) \leq F_{p,L}(T). \tag{5.21}$$

By (5.15) the mapping

$$f^{(n)} = g, \quad f(t) = \frac{1}{(n-1)!} \int_T^t (t-s)^{n-1} g(s) ds \tag{5.22}$$

gives one-to-one correspondence of $W_{p,L}^n$ and L_p. Therefore, by replacing $f \in W_{p,L}^n$ with $g \in L_p$, we have

$$F_{p,L}(T) = \frac{1}{[(n-1)!]^p} \sup_{0 \neq g \in L_p} \frac{\int_T^\infty v(t) \left| \int_T^t (t-s)^{n-1} g(s) ds \right|^p dt}{\int_T^\infty \rho(t)|g(t)|^p dt} \leq \frac{1}{[(n-1)!]^p} \sup_{0 \neq g \in L_p} \frac{\int_T^\infty v(t)(t-T)^{p(n-1)} \left| \int_T^t g(s) ds \right|^p dt}{\int_T^\infty \rho(t)|g(t)|^p dt} = \frac{J(\rho, \tilde{v}; T)}{[(n-1)!]^p}, \tag{5.23}$$

where $\tilde{v} = v(t)(t-T)^{p(n-1)}$.

Thus, by the estimates (5.5) and (5.6) of Theorem 5.1, we obtain

$$\frac{J(\rho,\tilde{v};T)}{[(n-1)!]^p} \le (p-1)\left[\frac{(n-1)!(p-1)}{p}\right]^{-p} \times \sup_{x>T}\int_x^\infty v(t)(t-T)^{p(n-1)}dt\left(\int_T^x \rho^{1-p'}(s)ds\right)^{p-1} \tag{5.24}$$

and

$$\frac{J(\rho,\tilde{v};T)}{[(n-1)!]^p} \le \left[\frac{(n-1)!(p-1)}{p}\right]^{-p} \times \sup_{x>T}\left(\int_T^x \rho^{1-p'}(s)ds\right)^{-1} \times \int_T^x v(t)(t-T)^{p(n-1)}\left(\int_T^t \rho^{1-p'}(s)ds\right)^p dt. \tag{5.25}$$

If either (5.17) or (5.18) is satisfied, then there exists $T \ge 0$ such that the left–hand side of (5.24) or (5.25) becomes less than one, respectively. By the assumptions of Theorem 5.7 there exists $T \ge 0$ such that

$$\frac{J(\rho,\tilde{v};T)}{[(n-1)!]^p} < 1.$$

Then (5.19) follows from (5.23) and (5.21).

Thus the proof of Theorem 5.7 is complete. □

We consider the equation

$$(-1)^n(|y^{(n)}|^{p-2}y^{(n)})^{(n)} - \frac{\gamma}{t^{np}}|y|^{p-2}y = 0, \tag{5.26}$$

where $\gamma \in R$.

By the proof of Theorem 5.7 follows that if

$$\gamma F_{p,L}(0) = \frac{\gamma}{[(n-1)!]^p}\sup_{0\ne g\in L_p}\frac{\int_0^\infty\left|\frac{1}{t^n}\int_0^t (t-s)^{n-1}g(s)ds\right|^p dt}{\int_0^\infty |g(t)|^p dt} < 1,$$

then the equation (5.26) is nonoscillatory.

By Theorem 1.24 we have

$$\gamma F_{p,L}(0) = \gamma\left[\frac{\Gamma(1-\frac{1}{p})}{\Gamma\left(n+1-\frac{1}{p}\right)}\right]^p < 1. \tag{5.27}$$

Here $\Gamma(t) = \int\limits_0^\infty s^{t-1}e^{-s}ds$ is a gamma–function. By using the reduction formula $\Gamma(q+1) = q\Gamma(q)$, $q > 0$, we have

$$\Gamma\left(n+1-\frac{1}{p}\right) = \prod_{k=1}^{n}\left(k-\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right).$$

Taking into account (5.27), we obtain that the equation (5.26) is nonoscillatory if

$$\gamma < \prod_{k=1}^{n}\left(k-\frac{1}{p}\right)^{p} = p^{-np}\prod_{k=1}^{n}(kp-1)^{p}. \tag{5.28}$$

We notice that the condition (5.28) has been obtained in Theorem 9.4.5 of [11] by another way.

Assume that there exists $T > 0$ such that

$$\int\limits_T^\infty \rho^{1-p'}(s)(s-T)^{p'(n-1)}ds < \infty. \tag{5.29}$$

Theorem 5.9. *Let* $1 < p < \infty$. *Let (5.29) hold. Assume that* v *is a non-negative continuous function and* ρ *is a positive continuous function on* I. *If one of the following conditions*

$$\limsup_{T\to\infty}\sup_{x>T}\left(\int\limits_x^\infty \rho^{1-p'}(s)(s-T)^{p'(n-1)}ds\right)^{p-1}\int\limits_T^x v(t)dt < \frac{1}{p-1}\left[\frac{(n-1)!(p-1)}{p}\right]^{p} \tag{5.30}$$

or

$$\limsup\left(\int\limits_x^\infty \rho^{1-p'}(s)(s-T)^{p'(n-1)}ds\right)^{-1}\times \int\limits_x^\infty v(t)\left(\int\limits_t^\infty \rho^{1-p'}(s)(s-T)^{p'}ds\right)^{p}dt < \left[\frac{(n-1)!(p-1)}{p}\right]^{p} \tag{5.31}$$

holds, then the equation (5.1) is nonoscillatory at $t=\infty$.

In the case $p=2$ Theorem 5.9 implies nonoscillation of the equation (5.2).

Theorem 5.10. *Let (5.29) hold with* $p=2$. *Assume that* v *is a non-negative continuous function and that* ρ *is a positive continuous function on* I. *Then,* *if one of the following conditions*

$$\lim_{T\to\infty}\sup_{x>T}\left(\int_x^\infty \rho^{-1}(s)(s-T)^{2(n-1)}ds\right)\int_T^x v(t)dt < \frac{[(n-1)!]^2}{4}$$

or

$$\limsup\left(\int_x^\infty \rho^{-1}(s)(s-T)^{2(n-1)}ds\right)^{-1}\int_x^\infty v(t)\times \left(\int_t^\infty \rho^{-1}(s)(s-T)^{p'}ds\right)^2 dt < \frac{[(n-1)!]^2}{4}$$

holds, then the equation (5.2) is nonoscillatory at $t=\infty$.

Proof of Theorem 5.9. We define

$$F_{p,R}(T)\equiv F_{p,R}(\rho,v;T)=\sup_{0\neq f\in W^n_{p,R}(\rho,I_T)}\frac{\int_T^\infty v(t)|f(t)|^p dt}{\int_T^\infty \rho(t)|f^{(n)}(t)|^p dt}. \tag{5.32}$$

Since $\overset{\circ}{W}{}^n_p(\rho,I_T)\subset W^n_{p,R}(\rho,I_T)$, then

$$F_{p,0}(T)\leq F_{p,R}(T). \tag{5.33}$$

Here, instead of (5.22), the mapping

$$f^{(n)}=g,\quad f(t)=\frac{(-1)^n}{(n-1)!}\int_t^\infty (s-t)^{n-1}g(s)ds$$

gives one-to-one correspondence of $W^n_{p,R}(\rho,I_T)$ and $L_p(\rho,I_T)$. Therefore, by replacing $f\in W^n_{p,R}(\rho,I_T)$ with $g\in L_p(\rho,I_T)$, we have

$$F_{p,R}(T)=$$
$$\frac{1}{[(n-1)!]^p}\sup_{0\neq g\in L_{p,\rho}}\frac{\int_T^\infty v(t)\left|\int_t^\infty (s-t)^{n-1}g(s)ds\right|^p dt}{\int_T^\infty \rho(t)|g(t)|^p dt}\leq$$
$$\frac{1}{[(n-1)!]^p}\sup_{0\neq g\in L_{p,\rho}}\frac{\int_T^\infty v(t)\left|\int_t^\infty (s-T)^{(n-1)}g(s)ds\right|^p dt}{\int_T^\infty \rho(t)|g(t)|^p dt}=$$

$$\frac{1}{[(n-1)!]^p} \sup_{0\neq\varphi\in L_{p,\widetilde{\rho}}} \frac{\int_T^\infty v(t) \left| \int_t^\infty \varphi(s)ds \right|^p dt}{\int_T^\infty \widetilde{\rho}(t)|\varphi(t)|^p dt} =$$

$$\frac{J_R(\widetilde{\rho}, v; T)}{[(n-1)!]^p}, \tag{5.34}$$

where $\varphi(s) = (s-T)^{n-1}g(s)$ and where $\widetilde{\rho} = \rho(t)(t-T)^{-p(n-1)}$.

Thus, by the estimates (5.7) and (5.8) of Theorem 5.2, we have

$$\frac{J_R(\widetilde{\rho}, v; T)}{[(n-1)!]^p} \leq$$

$$(p-1)\left[\frac{(n-1)!(p-1)}{p}\right]^{-p} \times$$

$$\sup_{x>T} \int_T^x v(t)dt \left(\int_x^\infty \rho^{1-p'}(s)(s-T)^{p'(n-1)}ds \right)^{p-1} \tag{5.35}$$

and

$$\frac{J_R(\widetilde{\rho}, v; T)}{[(n-1)!]^p} \leq$$

$$\left[\frac{(n-1)!(p-1)}{p}\right]^{-p} \times$$

$$\sup_{x>T} \left(\int_x^\infty \rho^{1-p'}(s)(s-T)^{p'(n-1)}ds \right)^{-1} \int_x^\infty v(t) \times$$

$$\left(\int_t^\infty \rho^{1-p'}(s)(s-T)^{p'(n-1)ds} \right)^p dt. \tag{5.36}$$

If either (5.30) or (5.31) is satisfied, then there exists $T \geq 0$ such that the left–hand side of (5.35) or (5.36) becomes less than one, respectively. Therefore in any case there exists $T \geq 0$ such that

$$\frac{J_R(\widetilde{\rho}, v; T)}{[(n-1)!]^p} < 1.$$

The inequalities (5.33) and (5.34) imply the validity of the inequality (5.19).

Thus the proof of Theorem 5.9 is complete. □

Next we consider the equation

$$(-1)^n(t^{pn}|y^{(n)}|^{p-2}y^{(n)})^{(n)} - \gamma|y|^{p-2}y = 0, \tag{5.37}$$

where the function $\rho(t) = t^{pn}$ satisfies the condition (5.29).

By the proof of Theorem 5.9, it follows that, if

$$\gamma F_{p,R}(0) = \frac{\gamma}{[(n-1)!]^p} \sup_{0\neq g\in L_p} \frac{\int\limits_0^\infty \left| \int\limits_0^t (t-s)^{n-1} g(s)ds \right|^p dt}{\int\limits_0^\infty |t^n g(t)|^p dt} < 1,$$

then the equation (5.37) is nonoscillatory.

By Theorem 1.25 we have

$$\gamma F_{p,R}(0) = \lambda \left\{ \frac{\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(n+\frac{1}{p}\right)} \right\}^p < 1. \tag{5.38}$$

Since $\Gamma\left(n+\frac{1}{p}\right) = \prod\limits_{k=0}^{n-1}\left(k+\frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right)$, then (5.38) implies that the equation (5.37) is nonoscillatory, if $\gamma < \prod\limits_{k=0}^{n-1}\left(k+\frac{1}{p}\right) = p^{-np}\prod\limits_{k=0}^{n-1}(pk+1)^p$.

5.3 Oscillation of linear equations of higher order

Now, we consider the problem of oscillation for the equation (5.2).

By Theorem 5.8, it is easy to prove that if both the integrals

$$\int_T^\infty \rho^{-1}(s)ds$$

and

$$\int_T^\infty v(t)(t-T)^{2(n-1)}dt$$

are finite, then the equation (5.2) is nonoscillatory.

Therefore, we are interested into the case when at least one of these integrals is infinite.

We start with the case

$$\int_T^\infty \rho^{-1}(s)ds = \infty. \tag{5.39}$$

Theorem 5.11. *Let (5.39) hold. If either one of the inequalities*

$$\lim_{T\to\infty} \sup_{x>T} \int_T^x \rho^{-1}(s)ds \int_x^\infty v(t)(t-x)^{2(n-1)}dt > [(n-1)!]^2$$

or

$$\lim_{T\to\infty} \sup_{x>T} \int_T^x \rho^{-1}(s)(x-s)^{2(n-1)}ds \int_x^\infty v(t)dt > [(n-1)!]^2$$

holds, then the equation (5.2) is oscillatory.

Proof of Theorem 5.11. If we show that

$$F_{2,0}(T) > 1 \tag{5.40}$$

for any $T \geq 0$, then the equation (5.2) is oscillatory.

Indeed, (5.40) implies that for every $T \geq 0$ there exists a nontrivial function $\tilde{f} \in A^0C_p^{n-1}(\rho, I_T)$ such that the inequality (5.12) holds. Consequently, by Theorem 5.5 the equation (5.2) is oscillatory.

According to the results of [34] the condition (5.39) implies that $\overset{\circ}{W}{}_2^n = W_{2,L}^n$. Then

$$F_{2,0}(T) = F_{2,L}(T) \tag{5.41}$$

and (5.22) implies that

$$F_{2,0}(T) = \sup_{0\neq f\in W_{2,L}^n} \frac{\int_T^\infty v(t)|f(t)|^2 dt}{\int_T^\infty \rho(t)|f^{(n)}(t)|^2 dt} = \frac{1}{[(n-1)!]^2} \sup_{0\neq g\in L_2} \frac{\int_T^\infty v(t)\left|\int_T^t (t-s)^{n-1} g(s)ds\right|^2 dt}{\int_T^\infty \rho(t)|g(t)|^2 dt} = \frac{J_n(T)}{[(n-1)!]^2}. \tag{5.42}$$

The estimate (5.9) of Theorem 5.3 implies that

$$\frac{B(T)}{[(n-1)!]^2} \leq F_{2,0}(T) \leq \beta \frac{B(T)}{[(n-1)!]^2}. \tag{5.43}$$

The left–hand side of the inequality (5.43) and the assumptions of Theorem 5.11 imply that the inequality (5.40) holds. Thus, the equation (5.2) is oscillatory.

Thus the proof of Theorem 5.11 is complete. □

Now, we apply Theorem 5.11 to the equation

$$(-1)^n \left(t^{-\alpha} y^{(n)(t)}\right)^{(n)} - v(t)y(t) = 0. \tag{5.44}$$

Let $k = \lim\limits_{T\to\infty} \sup\limits_{x>T} \int_T^x s^\alpha (x-s)^{2(n-1)} ds \int_x^\infty v(t)dt$ and $\gamma > 1$.

Then

$$\sup_{x>T} \int_T^x s^\alpha (x-s)^{2(n-1)} ds \int_x^\infty v(t)dt \geq \int_T^{\gamma T} s^\alpha (\gamma T - s)^{2(n-1)} ds \int_{\gamma T}^\infty v(t)dt = \frac{1}{\gamma^{2n-1+\alpha}} \int_1^\gamma s^\alpha (\gamma - s)^{2(n-1)} ds (\gamma T)^{2n-1+\alpha} \int_{\gamma T}^\infty v(t)dt.$$

If

$$\sup_{\gamma>1}\frac{1}{\gamma^{2n-1+\alpha}}\int_1^{\gamma}s^{\alpha}(\gamma-s)^{2(n-1)}ds\times$$

$$\lim_{x\to\infty}x^{2n-1+\alpha}\int_x^{\infty}v(t)dt>[(n-1)!]^2, \tag{5.45}$$

then $k>[(n-1)!]^2$ and by Theorem 5.11 the equation (5.44) is oscillatory.

In [14] the exact values of the oscillation constants of the equation (5.44) are obtained for the different values $\alpha\in R$. We collect the main oscillation conditions which had been proved in Proposition 2.2 of [14]. If we compare the conditions (5.45) and the conditions of Proposition 2.2 for $\alpha\geq 0$, that we can see that the conditions (5.45) are better than the conditions of Proposition 2.2. For example, when $n=2$ and $\alpha=0$, we have that

$$\sup_{\gamma>1}\frac{1}{\gamma^3}\int_1^{\gamma}(\gamma-s)^2ds=\frac{1}{3}\sup_{\gamma>1}\left(1-\frac{1}{\gamma}\right)^3=\frac{1}{3}.$$

Therefore, (5.45) implies that the equation $y^{(IV)}(t)=v(t)y(t)$ is oscillatory if $\lim\limits_{x\to\infty}x^3\int_x^{\infty}v(t)dt>3$. The analogous condition of Proposition 2.2 has the form $\lim\limits_{x\to\infty}x^3\int_x^{\infty}v(t)dt>12$.

We assume that the functions v and ρ are positive and continuous on I. Then by the principle of reciprocity [13] the equation (5.2) and the reciprocal equation

$$(-1)^n(v^{-1}(t)y^{(n)})^{(n)}-\rho^{-1}(t)y=0 \tag{5.46}$$

are simultaneously oscillatory or nonoscillatory. Applying the principle of reciprocity in the case

$$\int_T^{\infty}v(t)dt=\infty, \tag{5.47}$$

we obtain the following theorem.

Theorem 5.12. *Let (5.47) hold. Assume that v and ρ are positive continuous functions on I. Then, if one of the following inequalities*

$$\lim_{T\to\infty}\sup_{x>T}\int_T^{x}v(t)dt\int_x^{\infty}\rho^{-1}(s)(s-x)^{2(n-1)}ds>[(n-1)!]^2$$

or

$$\lim_{T\to\infty}\sup_{x>T}\int_T^x v(t)(x-t)^{2(n-1)}dt\int_x^\infty \rho^{-1}(s)ds > [(n-1)!]^2$$

holds, then the equation (5.2) is oscillatory.

Indeed, if the assumptions of Theorem 5.12 is satisfied, then by Theorem 5.11 the equation (5.46) is oscillatory. Therefore, the equation (5.2) is also oscillatory.

Now we apply the statement of Theorem 5.12 to the equation (5.44) when the condition (5.47) holds and $\alpha > 2n-1$. Then

$$\int_x^\infty \rho^{-1}(s)(s-x)^{2(n-1)}ds = \int_x^\infty s^{-\alpha}(s-x)^{2(n-1)}ds = x^{2n-\alpha-1}\int_1^\infty t^{-\alpha}(t-1)^{2(n-1)}dt,$$

where $s = xt$.

Integrating by the parts we have

$$\int_1^\infty t^{-\alpha}(t-1)^{2(n-1)}dt = \frac{(2n-1)!}{\prod\limits_{k=1}^{2n-1}(\alpha-k)}.$$

Therefore

$$\int_x^\infty \rho^{-1}(s)(s-x)^{2(n-1)}ds = x^{2n-\alpha-1}\frac{(2n-1)!}{\prod\limits_{k=1}^{2n-1}(\alpha-k)}$$

and by Theorem 5.12 the equation (5.44) is oscillatory if one of the following conditions

$$\lim_{T\to\infty}\sup_{x>T} x^{2n-\alpha-1}\int_T^x v(t)dt > \frac{[(n-1)!]^2\prod\limits_{k=1}^{2n-1}(\alpha-k)}{(2n-1)!} \tag{5.48}$$

or

$$\lim_{T\to\infty}\sup_{x>T} x^{1-\alpha}\int_T^x v(t)(x-t)^{2(n-1)}dt > (\alpha-1)[(n-1)!]^2$$

holds.

We now show that if the inequality

$$\lim_{x\to\infty} x^{2n-\alpha-1}\int_1^x v(t)dt > \frac{[(n-1)!]^2\prod_{k=1}^{2n-1}(\alpha-k)}{(2n-1)!} \tag{5.49}$$

holds, then also the inequality (5.48) holds.

Let $d = \lim_{T\to\infty}\sup_{x>T} x^{2n-\alpha-1}\int_T^x v(t)dt$. If $d=\infty$, then for any $N>0$ there exists $T_N>0$ such that

$$x^{2n-\alpha-1}\int_{T_N}^x v(t)dt > N$$

for all $x > T_N$.

Then by $\alpha > 2n-1$, we have $\lim_{x\to\infty} x^{2n-\alpha-1}\int_{T_N}^x v(t)dt = \lim_{x\to\infty} x^{2n-\alpha-1}\int_1^x v(t)dt > N$. By arbitrariness of $N>0$, we obtain $\lim_{x\to\infty} x^{2n-\alpha-1}\int_1^x v(t)dt = \infty$, i.e. the limiting relation (5.49) implies (5.48).

Let now $d<\infty$. Then for any $\varepsilon>0$ there exists $T_\varepsilon>0$ such that

$$x^{2n-\alpha-1}\int_{T_\varepsilon}^x v(t)dt \le d+\varepsilon$$

for all $x > T_\varepsilon$.

Then

$$\lim_{x\to\infty} x^{2n-\alpha-1}\int_1^x v(t)dt \le d.$$

Hence, the limiting relation (5.49) implies (5.48). Thus, the equation (5.44) is oscillatory with the condition (5.49). Now we compare the obtained results with results known in the literature for the equation (5.44). Let $\alpha = 2n$. Then the condition (5.49) has the following form

$$\lim_{x\to\infty}\frac{1}{x}\int_1^x v(t)dt > [(n-1)!]^2.$$

By Proposition 2.2 and Theorem 3.1 of [14] at $m=n-1$, $\alpha=2n$, we obtain

$$\lim_{x\to\infty}\frac{1}{x}\int_1^x v(t)dt > 4[(n-1)!]^2$$

and

$$\lim_{x\to\infty} \frac{1}{x}\int_1^x v(t)dt > \frac{1}{12}[(n-1)!]^2,$$

respectively.

Hence, Theorem 5.12 is proved under general assumptions. The application of Theorem 5.12 to an equation of the form (5.44) gives better results if we take ρ in the form $\rho(t) = t^{\alpha}$. Moreover, in Theorem 3.1 of [14] we also need to consider the conditions at $m = 0, 1, ..., n-1$.

5.4 Strong oscillation and spectral properties of linear equations of higher order

We now turn to the equation (5.2) with parameter $\lambda > 0$ in the following form.

$$(-1)^n(\rho(t)y^{(n)})^{(n)} - \lambda v(t)y = 0. \tag{5.50}$$

If the equation (5.50) is oscillatory or nonoscillatory for any $\lambda > 0$, then the equation (5.50) is called strongly oscillatory or strongly nonoscillatory, respectively.

Theorem 5.13. *Assume that v and ρ are positive and continuous functions on I. If the condition (5.39) is satisfied, then the equation (5.50)*

(i) is strongly nonoscillatory if and only if

$$\lim_{x\to\infty} \int_0^x \rho^{-1}(s)ds \int_x^\infty v(t)(t-x)^{2(n-1)}dt = 0 \tag{5.51}$$

and

$$\lim_{x\to\infty} \int_0^x \rho^{-1}(s)(x-s)^{2(n-1)}ds \int_x^\infty v(t)dt = 0; \tag{5.52}$$

(ii) is strongly oscillatory if and only if at least one of the following conditions

$$\limsup_{x\to\infty} \int_0^x \rho^{-1}(s)ds \int_x^\infty v(t)(t-x)^{2(n-1)}dt = \infty \tag{5.53}$$

or

$$\limsup_{x\to\infty} \int_0^x \rho^{-1}(s)(x-s)^{2(n-1)}ds \int_x^\infty v(t)dt = \infty. \tag{5.54}$$

holds.

Proof of Theorem 5.13. Let the equation (5.50) be nonoscillatory for any $\lambda > 0$. Then by the criterion of nonoscillation (5.11) of Theorem 5.5 for every $\lambda > 0$ there exists $T_\lambda \geq 0$ such that $\lambda F_{2,0}(T_\lambda) \leq 1$. Then $\lim\limits_{\lambda\to\infty} F_{2,0}(T_\lambda) = 0$. However, if the equation (5.50) is nonoscillatory for $\lambda = \lambda_0 > 0$, then by (5.11) it is nonoscillatory for any $0 < \lambda \leq \lambda_0$. Therefore, T_λ does not decrease. Hence,

$$\lim_{T\to\infty} F_{2,0}(T) = 0. \tag{5.55}$$

Thus, by the left–hand side of the inequality (5.43) and by (5.55) it follows that $\lim\limits_{T\to\infty} B(T) = 0$, where $B(T) = \max\{B_1(T), B_2(T)\}$ and

$$B_1(T) = \sup_{x>T} \int_x^\infty v(t)dt \int_T^x (x-s)^{2(n-1)}\rho^{-1}(s)ds,$$

$$B_2(T) = \sup_{x>T} \int_x^\infty v(t)(t-x)^{2(n-1)}dt \int_T^x \rho^{-1}(s)ds.$$

Then for any $\varepsilon > 0$ there exists $T_\varepsilon^1 > 0$ such that for every $x \geq T_\varepsilon^1$ we have

$$\int_{T_\varepsilon^1}^x \rho^{-1}(s)ds \int_x^\infty v(t)(t-x)^{2(n-1)}dt \leq \frac{\varepsilon}{2}$$

and there exists $T_\varepsilon \geq T_\varepsilon^1$ such that for every $x \geq T_\varepsilon$ we have

$$\int_0^{T_\varepsilon^1} \rho^{-1}(s)ds \int_x^\infty v(t)(t-x)^{2(n-1)}dt \leq \frac{\varepsilon}{2}.$$

Indeed, $\lim\limits_{x\to\infty} \int_x^\infty v(t)(t-x)^{2(n-1)}dt = 0$.

Therefore, for every $x \geq T_\varepsilon$ we have

$$\int_0^x \rho^{-1}(s)ds \int_x^\infty v(t)(t-x)^{2(n-1)}dt \leq \varepsilon,$$

which means that the equality (5.51) is satisfied. The equality (5.52) can be proved similarly.

Now, we shall prove that if the equalities (5.51) and (5.52) hold, then the equation (5.50) is strongly nonoscillatory.

Since the equalities (5.51) and (5.52) hold, then $\lim\limits_{T\to\infty} B(T) = 0$. Therefore, the right–hand side of the inequality (5.43) follows by the equality (5.55). Hence, for every $\lambda > 0$ there exists $T_\lambda \geq 0$ such that $\lambda F_{2,0}(T_\lambda) < 1$. Then the equation (5.50) is strongly nonoscillatory. Thus, statement (i) is proved.

We now prove statement (ii). Let the equation (5.50) be strongly oscillatory. By Theorem 5.5 we have that $\lambda F_{2,0}(T) \geq 1$ for every $\lambda > 0$ and for every $T \geq 0$. Therefore, $F_{2,0}(T) \geq \sup\limits_{\lambda>0} \frac{1}{\lambda} = \infty$ for every $T \geq 0$.

Thus, from the right–hand side of the inequality (5.43), it follows that $B(T) = \infty$ for every $T \geq 0$, so at least $B_1(T) = \infty$ or $B_2(T) = \infty$. This means that the equality (5.53) or (5.54) holds.

Suppose that for every $T \geq 0$ one of the conditions (5.53) or (5.54) holds. Then either $B_1(T) = \infty$ or $B_2(T) = \infty$. Therefore, $B(T) = \infty$ for any $T \geq 0$. Then, the left–hand side of the inequality (5.43) implies that $F_{2,0}(T) = \infty$ for any $T \geq 0$. Consequently, $\lambda F_{2,0}(T) > 1$ for any $\lambda > 0$ and $T \geq 0$, which by (5.12) means the oscillation of the equation (5.50) for $\lambda > 0$.

Thus the proof of Theorem 5.13 is complete. □

Corollary 5.14. *Let $T \geq 0$. If the conditions (5.39) and*

$$\int_T^\infty v(t)(t-T)^{2(n-1)}dt = \infty$$

are satisfied, then the equation (5.2) is strongly oscillatory.

As an example let us consider the equation

$$(-1)^n \left(t^\alpha y^{(n)}(t)\right)^{(n)} - \lambda v(t)y(t) = 0,\ n > 1, \tag{5.56}$$

where $\alpha \in R$ and v is a non–negative continuous function on I. For $\alpha < 1$ the conditions (5.39) for the equation (5.56) is valid.

Since

$$\int_0^x s^{-\alpha}(x-s)^{2(n-1)}ds = x^{2n-1-\alpha}\int_0^1 s^{-\alpha}(1-s)^{2(n-1)}ds,$$

then the conditions (5.52) and (5.54) for the equation (5.56) are equivalent to the conditions

$$\lim_{x\to\infty} x^{2n-1-\alpha}\int_x^\infty v(t)dt = 0 \tag{5.57}$$

and

$$\limsup_{x\to\infty} x^{2n-1-\alpha}\int_x^\infty v(t)dt = \infty, \tag{5.58}$$

respectively.

By applying the L'Hospital rule $2(n-1)$ times, it is easy to see that from (5.57) follows the condition (5.51)

$$\lim_{x\to\infty} x^{1-\alpha}\int_x^\infty v(t)(t-x)^{2(n-1)}dt = 0$$

for the equation (5.56).

Thus, by Theorem 5.13 the equation (5.56) is strongly nonoscillatory if and only if (5.57) holds. Moreover, it is strongly oscillatory if and only if (5.58) holds. This yields the validity of Theorems 15 and 16 of the monograph [25] for $\alpha = 0$.

The proof of the following theorem is based on the principle of reciprocity.

Theorem 5.15. *Assume that v and ρ are positive and continuous functions on I. Let (5.47) hold. Then the equation (5.50)*

(i) is strongly nonoscillatory if and only if

$$\lim_{x\to\infty} \int_0^x v(t)dt \int_x^\infty \rho^{-1}(s)(s-x)^{2(n-1)}ds = 0 \tag{5.59}$$

and

$$\lim_{x\to\infty} \int_0^x v(t)(x-t)^{2(n-1)}dt \int_x^\infty \rho^{-1}(s)ds = 0; \tag{5.60}$$

(ii) is strongly oscillatory if and only if one of the following conditions

$$\limsup_{x\to\infty} \int_0^x v(t)dt \int_x^\infty \rho^{-1}(s)(s-x)^{2(n-1)}ds = \infty \tag{5.61}$$

or

$$\limsup_{x\to\infty} \int_0^x v(t)(x-t)^{2(n-1)}dt \int_x^\infty \rho^{-1}(s)ds = \infty \tag{5.62}$$

holds.

Corollary 5.16. *Let $T \geq 0$. If the conditions (5.47) and*

$$\int_T^\infty \rho^{-1}(t)(t-T)^{2(n-1)}dt = \infty$$

are satisfied, then the equation (5.2) is strongly oscillatory.

Now we apply Theorem 5.15 to the equation (5.56), when (5.47) holds and $\alpha > 2n-1$, as in Section 5.3. By (5.48), we have

$$\int_x^\infty \rho^{-1}(s)(s-x)^{2(n-1)}ds = \int_x^\infty s^{-\alpha}(s-x)^{2(n-1)} = x^{2n+\alpha-1}\frac{(2n-1)!}{\prod_{k=1}^{2n-1}(\alpha-k)}.$$

Therefore the conditions (5.59) and (5.61) of Theorem 5.15 have the following form

$$\lim_{x\to\infty} x^{2n-\alpha-1} \int_1^x v(t)dt = 0 \tag{5.63}$$

and

$$\limsup_{x\to\infty} x^{2n-\alpha-1} \int_1^x v(t)dt = \infty, \tag{5.64}$$

respectively.

By applying the L'Hospital rule $2(n-1)$ times, it is easy to see that the condition (5.60) holds if and only if the condition (5.63) holds. Thus, the equation (5.56) is strongly nonoscillatory with the condition (5.47) and $\alpha > 2n-1$ if and only if the condition (5.63) holds; and the equation (5.56) is strongly oscillatory if and only if the condition (5.64) holds.

Now we consider the following equation

$$(-1)^n(\rho(t)y^{(n)}(t))^{(n)} - \lambda t^{-\beta}y(t) = 0. \tag{5.65}$$

Let at first $\beta > 2n-1$. Then the conditions (5.51) and (5.53) of Theorem 5.13 have the following form

$$\lim_{x\to\infty} x^{2n-\beta-1} \int_1^x \rho^{-1}(s)ds = 0 \tag{5.66}$$

and

$$\limsup_{x\to\infty} x^{2n-\beta-1} \int_1^x \rho^{-1}(s)ds = \infty, \tag{5.67}$$

respectively.

Moreover, (5.66) implies that (5.52) holds for $v(t) = t^{\beta}$. Thus, the equation (5.65) is strongly nonoscillatory with $\beta > 2n-1$ if and only if the condition (5.66) holds; and the equation (5.65) is strongly oscillatory if and only if the condition (5.67) holds.

Now let $\beta < 1$. Then we apply Theorem 5.15 to the equation (5.65) and we obtain that the equation (5.65) is strongly nonoscillatory with $\beta < 1$ if and only if the condition

$$\lim_{x\to\infty} x^{2n-\beta-1} \int_x^\infty \rho^{-1}(t)dt = 0, \tag{5.68}$$

holds. Thus the equation (5.65) is strongly oscillatory if and only if the condition

$$\lim_{x\to\infty} \sup x^{2n-\beta-1} \int_x^\infty \rho^{-1}(t)dt = \infty, \tag{5.69}$$

holds.

Next we deal with spectral properties of the differential expression

$$l(y) = (-1)^n \frac{1}{v(t)} (\rho(t) y^{(n)})^{(n)},$$

where v and ρ are continuous functions and $\rho(t) > 0$, $v(t) > 0$ for $t \in [T, \infty)$.

The maximal differential operator L_{max} generated by the differential expression l (i.e. $L_{max}(y) = l(y)$) is the operator with domain

$$D(L_{max}) = \{y : [T, \infty) \to R, \quad y^i \in AC[T, \infty),\ i = 0, 1, ...n-1, \text{ and } \ ly \in L_2(T, \infty)\}.$$

The minimal operator L_{min} is defined as the adjoint operator to the maximal operator, i.e. $L_{min} \equiv (L_{max})^*$. The domain of every self–adjoint extension L of the minimal operator L_{min} satisfies the inclusions

$$D(L_{min}) \subset D(L) \subset D(L_{max}).$$

It is known that all self–adjoint extensions of the minimal operator have the same essential spectrum, see [25], [39].

Next we focus our attention to the following spectral properties of the operator L.

Definition 5.17. *The operator L is said to have the BD property if every self–adjoint extension of L_{min} has bounded below and discrete spectrum.*

The link between oscillatory and spectral properties of the operator L is shown in the following statement.

Proposition 5.18. *The operator L has the BD property if and only if the equation $L(y) = \lambda y$ is strongly nonoscillatory.*

By Theorem 5.13 and Theorem 5.15, we have the following theorem.

Theorem 5.19. *Assume that v and ρ are positive and continuous functions on I.*

(i) If the condition (5.39) holds, then the operator L has the BD property if and only if both (5.51) and (5.52) hold.

(ii) If the condition (5.47) holds, then the operator L has the BD property if and only if both (5.59) and (5.60) hold.

We now consider the differential operators

$$\widetilde{L}(y) = (-1)^n \frac{1}{v(t)} (t^\alpha y^{(n)})^{(n)},$$

$$\widehat{L}(y) = (-1)^n t^\beta (\rho(t) y^{(n)})^{(n)}.$$

By applying the previous oscillation and nonoscillation criteria at t_0 to the equations (5.56) and (5.65), we obtain the following necessary and sufficient condition for the BD property of $\widetilde{L}$ and $\widehat{L}$ to hold, respectively.

Proposition 5.20. *(i) If $\alpha < 1$ and the condition (5.39) holds, then the operator $\widetilde{L}$ has the BD property if and only if (5.57) holds.*

(ii) If $\alpha > 2n - 1$ and the condition (5.47) holds, then the operator $\widetilde{L}$ has the BD property if and only if (5.63) holds.

Proposition 5.21. *(i) If $\beta < 1$ and the condition (5.47) holds, then the operator $\widehat{L}$ has the BD property if and only if (5.68) holds.*

(ii) If $\beta > 2n - 1$ and the condition (5.39) holds, then the operator $\widehat{L}$ has the BD property if and only if (5.66) holds.

We note, that the statement (ii) of Proposition 5.21 is new with respect to the results of Theorem 3 in [15]. Indeed, the authors of [15] assume that $\int_T^\infty \rho^{-1}(s) ds < \infty$.

Bibliography

ApOt43 [1] O.D. Apyshev and M. Otelbayev, On the spectrum of one class of differential operators and some embedding theorems, Izv. Akad. Nauk SSSR, Ser. Mat., 43 (1979), pp. 739-764 (in Russian).

Bees11 [2] P. R. Beesack, Hardy's inequality and its extensions, Pacific J. Math. 11(1969), pp. 39–61.

Bih2 [3] I. Bihari, Ausdehung der turmschen Oszillations and vergleichungen zweiter Ordnung, Publ. Math. Ins. Hunger. Acad. Sci.,2 (1957), pp. 159–173.

Bih8 [4] I. Bihari, An oscillation theorem concerning the half–linear differential equation of the second order, Publ. Math. Ins. Hunger. Acad. Sci. A 8 (1964), pp. 275–280.

Brown [5] R. C. Brown, Separation and disconjugacy, J. of Inequalities in pure and applied mathematics, 4(3), 56 (2003).

ChLU [6] T. Chantladze, A. Lomtatidze and D. Ugulava, Conjugacy and disconjugacy for second order linear ordinary differential equations, Archivum mathematicum, BRNO, 36 (2000), pp. 313–323.

Cop220 [7] W.A. Coppel, Disconjugacy, Lectures Notes in Math., Springer Verlag, Berlin-Heidelberg, 220 (1971).

Derr08 [8] V. Ya Derr, The theory of disconjugacy for a second order linear differential equation, arXiv: 0811.4636v1 [math. CA] 2008.

Derr10 [9] V. Ya Derr, Discongugacy of second order linear differential equation and periodic solutions, arXiv: 1005.5646v1 [math. CA] 2010.

Di85 [10] J. I. Diaz, Nonlinear partial differential equations and free boundaries, Vol I. Elleptic Equations, Pitman, London, 1985.

DoRe202 [11] O. Došlý and P. Řehák, Half–linear differential equations, Math. Studies, North-Holland, 202 (2005).

Dos182 [12] O. Došlý, Nehari-type oscillation criteria for self-adjoint linear differential equations, S. Math. anal and appl., 182 (1994).

Dos30 [13] O. Došlý, Oscillation and spectral properties of self-adjoint differential operators, Nonlinear Anal. Theory, Methods and Appl., (3)30 (1997).

Dos34 [14] O. Došlý, Constants in oscillation theory of higher order Sturm–Liouville differential equations, Electron. J. Differ. Equ., 34 (2002).

Dos7 [15] O. Došlý, Oscillation and spectral proporties of self–adjoint even order differential operators with middle terms, E. J. Qualitative Theory of Diff. Equ., Proc 7th. Coll. QTDE, 7 (2004), pp. 1–21.

Dos50 [16] O. Došlý, Methods of oscillation theory of half–liner second order differential equations, Czechoslovak Math. J. 50 (125) (2000), pp. 657–671.

Dos127 [17] O. Došlý, Qualitative theory of half–linear second order differential equation, Math. Bohem 127 (2002), pp. 181–185.

DrKu18 [18] P. Drabek and A. Kufner, Hardy inequality and properties of the quasilinear Sturm-Liouville problem, Rend. Lincei Mat. Appl. 18 (2007), pp. 125–138.

EdEv04 [19] D. E. Edmunds and W.D. Evans, Hardy Operators, Function Spaces and Embeddings, Springer–Verlag, Berlin, 2004.

Elb30 [20] A. Elbert, A half–linear second order differential equations, Colloq. Math. Soc., Janos Bolyai, 30(1979), pp. 158–180.

El81 [21] U. Elias, Nessesary conditions and sufficient conditions for disfocality and disconjugacy of a differential equation, Pacific J. of mathematics, 81(2), 1979.

EmGas [22] Emden R. Gaskugeln, Anwendungen der mechanischen Warmen theorie auf Kosmologie and metheorologische Probleme, Leipzig, 1907.

Fieg [23] F. Fiegler, Oscillation criteria for a class of 2n–order ordinary differential equations, Math. Nachr. 131 (1987), pp. 205–218.

Fow91 [24] R. H. Fowler, The solutions of Emden's and similar differential solutions, Mothly Notices Roy. Astronom. Soc. 91 (1930), pp. 63–91.

Glaz [25] I. M. Glazman, Direct methods of qualitative analysis of singular differtial operators, Jerusalim, 1965.

GKPW29 [26] A. Gogatishvili, A. Kufner, L.–E. Persson and A. Wedestig, An equivalece theorem for integral conditions related to Hardy's inequality, Real Anal. Exchage 29(2003/04), pp. 867–880.

HaLP2 [27] G. H. Hardy, J.E. Littlewood and G. Pólya, Inequalities, 2–nd ed., Cambrige University Press, 1952.

Har57 [28] G. H. Hardy, Note on some points in the integral calculus. LXIV. Further inequalities between integrals, Messenger of Math. 57(1927), pp.12–16.

HiLe [29] D. B. Hinton and R. T. Lewis, Singular differential operators with spectra discrete and bounded below, Proc. Roy. Soc., Edinburgh, 84A(1979) pp. 117–134.

KuOi2 [30] S.Y. Kudabayeva and R. Oinarov, The criteria of disconjugate of half–linear second order differential equations, Math. J., Kazakhstan, T.10, No. 2 (36), 2010, pp. 56-67. (in Russian).

KuMP07 [31] A. Kufner, L. Malegranda and L.–E. Persson, The Hardy inequality. About its history and some related results, Pilsen, 2007.

KuPe03 [32] A. Kufner and L.–E. Person, Weighted Inequalities of Hardy type, Word Scientific Publishing Co., Singapore/ New Jersey/ London/ Hong Kong, 2003.

KuPW64 [33] A. Kufner, L.–E. Persson and A. Wedestig, A study of some constans characterizing the weighted Hardy inequality, Banach Center Publ. 64, Orlicz Centenary Volume, Polish Acad. Sci., Warsaw 2004, pp. 135–146.

Liz239 [34] P. I. Lizorkin, On closer of the set of infinitely differentiable function with compact support in the weighted space $W^l_{p,\phi}$, DAN USSR, 239(4) (1978) (in Russian).

Man24 [35] V. M. Manakov, On the best constant in weighted inequalities for Riemann - Liouville integrals, Bull. London Math. Soc., 24 (1992).

MiHa88 [36] A. B. Mingarelli and S. G. Halvorsen, Non–oscillation domains of differential equations with two parameters, Lecture Notes in Mathematics, Spinger–Verlag Berlin Heidelberg, 1988.

Mir53 [37] J. D. Mirzov, On some analogs of Sturm's and Kneser's theorems for nonlinear systems, J. Math. Anal. Appl. 53(1976), pp. 418–425.

Muck44 [38] B. Muckenhoupt, Hardy's inequalities with weights, Studia Math. 44(1972), pp. 31–38.

MyOt88 [39] K. T. Mynbaev and M. Otelbayev, Weghted function spaces and the spectrum of differential operators, Nauka, Moscow, 1988 (in Russian).

Oin303 [40] R. Oinarov, About density of the functions of compact support on weighted spaces and weighted inequalities, Dokl. Akad. Nauk SSSR 303 (1988), pp. 559–563; English transl. in Sovet Math. Dokl. 38 (1989), pp. 575–579.

OiRa42 [41] R. Oinarov and S.Y. Rakhimova, Weighted Hardy inequalities and their application to oscillation theory of half-linear differential equation, Eurasian Math. J., Kazakhstan, V.1, No. 42 (2010), pp. 110–124.

OiRa49 [42] R. Oinarov and S.Y. Rakhimova, Oscillation and nonoscillation of two terms linear and half-linear equations of higher order, E. J. Qualitative Theory of Diff. Equ., Hungary, No. 49 (2010), pp. 1-15.

OpKu219 [43] B. Opic and A. Kufner, Hardy–type Inequalities. Pitman Research Notes in Mathimatics 219, Longman Scientific and Technical, Harlow, 1990.

Otel [44] M. Otelbayev, Estimates of the spectrum of the Sturm –Liouville operator, Gylym, Alma-Ata, 1990 (in Russian).

Pan23 [45] S. Panigrahi, Criteria for disfocality and disconjugacy for third order differential equations, E. J. Qualitative Theory of Diff. Equ., 23 (2009), pp. 1–17.

Pen35 [46] S. Peña, Conjugacy criteria for half–linear differential equations, Arch. Math. (Brno) 35 (1999), pp. 1–11.

Reid [47] W. T. Reid, Ordinary differential equations, Wiley, New York, 1971.

Sans [48] G. Sansone, Equazioni diferenziali nel campo reale I, II, Zanichelli, Bologna, 1949.

Step187 [49] V. D. Stepanov, On one weighted Hardy type inequality for derivatives higher order, Trudy MIAN USSR, 187(1989) (in Russian).

Stm1 [50] C.Sturm, Sur les equations differentielles du second ordre, J. Math. Pures. Appl., 1 (1836), pp. 106–186.

Tal39 [51] G. Tallenti, Osservazioni sopra una class di disuguaglianze, Rend. Sem. Mat. Fis. Milano 39(1969), pp. 171–185.

Thom23 [52] L. H. Thomos, The calculation of atomic fields, Proc. Cambridge Phil. Soc., 23 (1927), pp. 542–548.

Tom2 [53] G. Tomaselli, A class of inequalities, Boll. Un. Mat. Ital. 2(1969), pp. 622–631.

Wed17 [54] A. Wedestig, Weighted inequalities of Hardy–type and their limiting inequalities, PhD Thesis, Luliá University of Technology, 17(2003), pp. 1–106.