Università
degli Studi
di Padova

Sede amministrativa: Università degli Studi di Padova

Dipartimento di Matematica “Tullio Levi-Civita”

CORSO DI DOTTORATO DI RICERCA IN SCIENZE MATEMATICHE

CURRICOLO MATEMATICA COMPUTAZIONALE

CICLO XXIX

McKean-Vlasov limits, propagation of chaos and long-time behavior of some mean field interacting particle systems

Direttore della Scuola: Ch.mo Prof. Martino Bardi

Supervisore: Ch.mo Prof. Paolo Dai Pra

Dottoranda: Luisa Andreis

A Maurizio e a mia mamma

Riassunto

L'argomento di questa tesi sono i sistemi di particelle con interazione a campo medio e i processi nonlineari ottenuti come limiti di essi. Il lavoro è suddiviso in tre parti, in cui vengono analizzati modelli caratterizzati da tre diversi meccanismi di interazione. Nella prima parte ci occupiamo di un'interazione tramite *salti simultanei*, che prende spunto da alcuni modelli apparsi recentemente in neuroscienze, dove gli autori trattano sistemi di neuroni in comunicazione l'uno con l'altro. Con l'obiettivo di generalizzare questo tipo di modelli consideriamo un sistema di diffusioni con salti che interagiscono tra loro attraverso la componente discontinua: ogni processo compie un *salto principale* con una certa frequenza e, contemporaneamente, forza tutte le altre particelle a compiere anch'esse un salto che però è detto *salto collaterale*, in quanto viene riscalato rispetto alla taglia del sistema. Considerando diverse ipotesi sui coefficienti, ci concentriamo sulla propagazione del caos traiettoriale e sulla dimostrazione di esistenza e unicità delle soluzioni per la corrispondente SDE nonlineare. Nella seconda parte della tesi ci occupiamo di un'interazione di tipo *asimmetrico*. Definiamo un sistema dove ogni particella si muove secondo una passeggiata aleatoria sui naturali, riflessa in zero e con un eventuale drift verso destra. In aggiunte c'è un'interazione asimmetrica, nel senso che ogni particella viene spinta a compiere movimenti verso sinistra sotto l'influenza solo delle particelle che si trovano alla sua sinistra. Ci chiediamo come questo sistema, che in assenza di interazione è transiente, possa diventare ergodico a seconda della forza dell'interazione e studiamo i parametri critici sia nel sistema ad N particelle che nel suo limite termodinamico. In particolare sfruttiamo risultati esistenti su diffusioni che interagiscono attraverso la funzione cumulativa empirica per evidenziare le differenze date dalla dinamica discreta. Nella terza parte ci concentriamo su una dinamica di Langevin per il modello di Curie-Weiss generalizzato alla quale applichiamo un termine di dissipazione. Questo approccio è stato precedentemente usato per rompere la reversibilità nel modello di Curie-Weiss classico ed è stato dimostrato che, in quel caso, il sistema limite ammette una soluzione periodica. Il nostro lavoro conferma l'emergenza di comportamenti periodici anche nel caso del Curie-Weiss generalizzato. In particolare, possiamo dimostrare che un'accurata scelta della funzione di interazione nel modello di partenza è tale da dare luogo ad un sistema limite in cui coesistono molteplici soluzioni periodiche stabili.

Abstract

In this thesis we study mean field interacting particle systems and their *McKean-Vlasov* limiting processes, in particular we focus on three different interaction mechanisms, mainly emerging from biological modelling. The first type of interaction is given by the so called *simultaneous jumps*. We consider a system of interacting jump-diffusion processes that interact by means of the discontinuous component: each particle performs a *main jump* and it simultaneously induces in all the other particles a *simultaneous jump* whose amplitude is rescaled with the size of the system. This peculiar interaction is motivated by recent neuroscience models and here we depict a general framework for this type of processes. We focus on the well-posedness of the *McKean-Vlasov* limits of these particle systems under different assumptions on the coefficients and we prove a pathwise propagation of chaos result. The second interaction we consider is an *asymmetric* one. We describe a system of biased random walks on the positive integers, reflected at zero, where each particle may perform a leftward jump with a rate proportional to the fraction of particles which are strictly at its left. We study the critical interaction strength able to ensure ergodicity to this system, that would be transient in absence of interaction. We compare this model with existing models of diffusions interacting through their CDF and we highlight their differences, mainly caused by the presence of clusters of particles in the discrete model. The third interaction we account for is based on a dynamical version of the *generalized Curie-Weiss* model. We modify a Langevin dynamics for this model with a dissipative evolution of the interaction component, breaking the reversibility of the system. We prove that, in the mean field limit, this gives rise to stable limit cycles, explaining self-sustained periodic behaviors. In particular, we build a flexible model in which a suitable change in the interaction function can result in a system which, in certain regimes of parameters, displays coexistence of stable periodic orbits.

Introduction

Mean-field interacting particles were firstly introduced by Kac with the aim of microscopically justifying the spatially-homogeneous Boltzmann equation [56]. Since then, they have been extensively studied due to their flexibility and their connections with nonlinear PDE, starting from the seminal work of McKean [65] and in a great number of successive works [48, 81, 82, 84]. It is known that the complete graph of interactions among particles and the symmetry of the evolution are not innocent assumptions and these models give an extremely simplified description of the physical phenomena they were introduced for. However, they have recently received more attention because they can be used to describe complex systems coming from biology, social science and finance, where the mean-field assumption seems to be a reasonable one. This type of models consists in a microscopic and a macroscopic description of a phenomenon, in a way that the nonlinearity observed in the macroscopic behavior is explained by an interaction term at the microscopic level. If we consider a fixed number N of particles in this microscopic description, we say that this interaction is of mean field type because its intensity is of order $\mathsf{O}\left(\frac{1}{\mathsf{N}}\right)$. Under suitable assumptions, it is possible to prove that systems of this type have the *propagation of chaos* property, see [83], i.e. when the particles start from i.i.d. initial conditions they maintain an asymptotic stochastic independence, despite the interaction. Indeed, when the size of the system N goes to infinity, particles tend to behave independently and distributed as the correspondent *nonlinear process* characterizing the macroscopic description, which is a particular type of time-inhomogeneous stochastic process, whose dynamics depends on the law of the process itself. Nonlinear processes arising as thermodynamic limits of mean-field interacting particle systems, also called *McKean-Vlasov processes*, are non-trivial processes and the study of their features involves different techniques, usually not needed for classical Markov processes. For instance, stopping times and compactness method are not useful in the proof of well-posedness of the correspondent nonlinear SDE and different approaches are needed, [48, 49, 64]. Moreover, nonlinear processes display a much richer long-time behavior than their correspondent particle systems, they may show stable oscillatory laws [47, 79, 78] or multiple stationary measures, even a continuum of them [54]. This thesis is divided into three parts, in which we focus on models that are characterized by a specific type of interaction, each of them is of mean field type. These interactions arise mainly from biological questions, but their peculiarities make them interesting on their own by a mathematical point of view.

In the first part the key interaction is given by the so-called *simultaneous jumps*. We consider a N-particle system of jump-diffusions in $\mathbb{R}^{\mathsf{d}}$, for $\mathsf{d} \geqslant 1$, that can interact with each other by means of classical mean field interactions. We endow this system with an additional interacting mechanism, inspired by neuroscience problems [29, 43, 75]: each particle performs a jump, that we call *main jump* with a certain rate and it simultaneously induces in all the other particles a *collateral jump*, whose amplitude is of the order $\mathsf{O}\left(\frac{1}{\mathsf{N}}\right)$. There is a dissimilarity in the treatment of the jump terms, since we expect that, in the limit for $\mathsf{N} \to \infty$, the *main jump* component is preserved while the *collateral jump* one, although simultaneous, collapses into an additional nonlinear drift term. Moreover, pathwise propagation of chaos for interacting diffusions with jumps is less widespread in literature than the continuous case, probably because of the discontinuities in the paths and the impossibility to use a compactness approach as in the proof of well-posedness for classical SDE with jumps. Therefore in Chapter 1 and 2 we formally describe a general framework for particle systems with simultaneous jumps, this is the model presented in [3, 4]. We focus on the issues of well-posedness of the correspondent nonlinear limit process and on the proof of pathwise propagation of chaos by means of a coupling method. Being built as a useful tool for modelling purposes, our model is very flexible and it can be adapted to a wide class of processes, enclosing in the same framework nonlinear processes with unbounded jump rates and with diffusive terms, that rarely appear in the mean field literature.

The second part of the thesis is focused on an interaction which is *asymmetric*. We consider a system of N one-dimensional random walks reflected at zero and with a positive bias. We add to this system an interaction that, for each particle, depends on the fraction of particles strictly below the particle itself and it forces the particle to move downward. The reason for this type of interaction comes from population dynamics. We interpret the position of each particle on the line as the *fitness* level of an individual w.r.t. the environment. We suppose that each individual has an intrinsic tendency to improve, given by the biased random walk, but the influence of the individuals worse tham him may decrease its fitness. For this model, presented in [2], the focus is on long-time behavior, rather than on well-posedness of the nonlinear limit. Indeed, after having defined the mean field limit of this system, we aim to understand if this asymmetric interaction can ensure ergodicity to a system that would otherwise be transient. With respect to the first part, we are considering here a *pure jump* process and this plays a crucial role in the analysis of the critical parameters. Indeed, in Chapter 3 we present a slight modification of the system studied in [54, 55, 74], that can be viewed as a continuous analogue of the simplest among the random walks with asymmetric interaction we aim to study in Chapter 4. However, most of the results in Chapter 3 strictly depend on the continuity of the space and the dynamics and they cannot be extended to the discrete model. We highlight in Chapter 4 the differences given by the discontinuous dynamics and how these reflect in the critical parameters of the model.

The third part of the thesis concerns an interaction coming from a *generalized Curie-*

Weiss model [35, 37]. We build a particle system on $\mathbb{R}^N$ that evolves in time according to a Langevin dynamics, i.e. particles move continuously with the aim of minimizing the energy coming from the Hamiltonian of the generalized Curie-Weiss model. We modify this dynamics by providing the interaction term with a dissipative evolution. This is one of the ways in which the reversibility of the model may be broken and it has been proven in [24, 26] that this approach gives rise to self-sustained periodic behaviors in the nonlinear limit. The interest in models of interacting components able to capture collective periodic behaviors is central in several fields, for instance neuroscience, ecology or social science. Indeed, macroscopic oscillatory behaviors are commonly observed in nature even if microscopically there is no tendency to behave periodically. With this in mind, we restrict the class of models with dissipation we defined and we obtain a Gaussian process, which we are able to study completely. We prove that, by suitably modifying the interaction function of the generalized Curie-Weiss model, we may recreate an interacting particle system that shows as many stable limit cycles as we want. This confirms that the add of a dissipation term in the time-evolution of the interaction favors the presence of self-sustained periodic behavior for particle systems without any tendency to behave periodically. Moreover, with a suitable choice of the interaction function we have a model which is extremely flexible and it is able to adapt to multiple situations.

In the following, let us describe precisely the structure of the thesis and the different models we deal with.

Part I: models with simultaneous jumps

In Chapter 1 we define a general mean field model that is characterized by the feature of simultaneous jumps, explaining the motivation coming from neuroscience modelling. We aim to understand if the peculiarity of the simultaneous jumps can create problems in the proof of propagation of chaos in situations different from the ones presented in the neuroscience literature [29, 43, 75], for example in presence of a Brownian component. In this setting, every particle, besides its diffusive dynamics, can perform what we call a *main jump*, that is a jump of a certain amplitude with a certain rate. Every time that a particle performs this jump, it induces a jump in all the other particles' trajectories, but the amplitude of these *collateral jumps* is rescaled according to the size of the system. We consider the McKean-Vlasov limit of this system and in Chapter 2 we prove pathwise propagation of chaos via a coupling technique, under various sets of assumptions. This give a rate of convergence for the W_1 Wasserstein distance between the empirical measures of the two systems on the space of trajectories $\mathbf{D}([0,T],\mathbb{R}^d)$.

The microscopic and macroscopic dynamics

Fix $N \geqslant 2$ and let $X^N = (X_1^N, \ldots, X_N^N) \in \mathbb{R}^{d\times N}$ be the spatial positions of N different particles moving in $\mathbb{R}^d$. We introduce the corresponding *empirical measure*

$$\mu_X^N \doteq \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i^N}.$$

We use the empirical measure to express classical mean field interactions, indeed we describe the evolution of the vector of particles positions $X^N(t)$ as a jump diffusion process whose coefficients depend on it. Moreover, we depict separately a general framework for the peculiar interaction of mean field type represented by the *simultaneous jumps*. Therefore, the following coefficients characterize the i-th particle.

- The **drift coefficient** depends on the spatial position of the particle and on the other particles through the empirical measure, i.e. it is of the form

  $$F(X_i^N(t), \mu_X^N(t))$$

  for some function $F : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^d$ common to all particles.

- The **diffusion coefficient**, equivalently, is written as

  $$\sigma(X_i^N(t), \mu_X^N(t))$$

  for $\sigma : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^{d\times d_1}$, again the same for all particles.

- The **main jump rate**: particle i performs a main jump with rate

  $$\lambda(X_i^N(t), \mu_X^N(t)),$$

  for a positive function $\lambda : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to [0, \infty)$. With this rate, the i-th particle performs a main jump and simultaneously it induces in all the other particles a collateral jump.

- The **main jump amplitude**: particle i perform a main jump that is a random variable

  $$\psi(X_i^N(t), \mu_X^N(t), h_i^N)\ ,$$

  for a function $\psi : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \times [0,1] \to \mathbb{R}^d$. Here h^N is a random variable with values in $[0,1]^N$ and its distribution is given by a symmetric measure ν_N.

- The **collateral jump amplitude**: the i-th particle is induced to jump by main jumps of every other particle. The amplitude of these collateral jumps is given by the function $\Theta : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \times [0,1]^2 \to \mathbb{R}^d$. When the j-th particle jumps (this occurs with rate $\lambda(X_j^N(t), \mu_X^N(t))$, of course) the i-th particle performs a jump of amplitude

  $$\frac{\Theta(X_j^N(t), X_i^N(t), \mu_X^N(t), h_j^N, h_i^N)}{N}\ ,$$

  where h_i^N and h_j^N are components of the random vector h^N, with distribution ν_N.

It is known that a process as X^N is in correspondence with a McKean-Vlasov process, i.e. the process X whose law is the law of the solution of the nonlinear SDE:

$$\begin{aligned} dX(t) =& \left(F(X(t), \mu_t) + \left\langle \mu_t, \lambda(\cdot, \mu_t) \int_{[0,1]^2} \Theta(\cdot, X(t^-), \mu_t, h_1, h_2) \nu_2(dh_1, dh_2) \right\rangle \right) dt \\ &+ \sigma(X(t), \mu_t) dB_t + \int_{[0,\infty) \times [0,1]^{\mathbb{N}}} \psi(X(t^-), \mu_s, h_1) \mathbb{1}_{(0, \lambda(X(t^-), \mu_s)]}(u) \mathcal{N}(dt, du, dh). \end{aligned}$$

Here, B is a d_1-dimensional Brownian motion and $\mathcal{N}$ an independent Poisson random measure with characteristic measure $dtdu\nu(dh)$ on $[0,\infty)^2 \times [0,1]^{\mathbb{N}}$. ν is a symmetric measure on $[0,1]^{\mathbb{N}}$ such that each projection on N coordinates corresponds to ν_N. By $\langle \cdot, \cdot \rangle$ we indicate the integral of a function on its domain with respect to a certain measure; thus, $\langle \mu, \phi \rangle = \int_{\mathbb{R}^d} \phi(y) \mu(dy)$. The equation above is not a standard SDE since the law μ_t of the solution appears as an argument of the coefficients. Processes of this type may be indicated as *nonlinear processes* and the nonlinearity stands in the fact that the coefficients of the SDE depend on the law of the process itself. Informally, we say that these nonlinear terms arise from the mean field interaction in the N particle system; in particular, notice that the simultaneous jumps give rise to a nonlinear drift term. The collateral jumps, due to the rescaling via the size of the system, appear in the limit as being absorbed by an additional drift term, depending on the characteristic measure of the Poisson random measure $\mathcal{N}$, that however is still present in the limit, due to the main jump component.

Well-posedness and propagation of chaos

Because of their peculiarity, well-posedness of nonlinear processes is a delicate issue, in particular in presence of diffusion term and jump component and in literature we find a few examples of this type of processes [48, 49, 50, 67]. However, since classical diffusion processes with jumps are extremely used in various applications, it is natural to look for a flexible approach for the study of their nonlinear analogue in view of the use of particle systems in different frameworks. For this reason, we dedicate Chapter 2 to the study of the nonlinear process that we presented under several sets of assumptions, always allowing for unbounded jump rates.

In Section 2.1 we choose the most classical globally Lipschitz assumptions on all the coefficients, both in the spatial and in the measure variables, w.r.t. the Euclidean and the W_1 Wasserstein distance. These conditions appear in [48], where well-posedness of the nonlinear process is proved. Therefore, we concentrate in the role of the simultaneous jumps and we study their role in the *propagation of chaos*, that is the connection between the microscopic description and the macroscopic one. Let P^N be the law of the particle system X^N on $\mathbf{D}([0,T], \mathbb{R}^d)^N$ and let μ the law of the nonlinear process X on $\mathbf{D}([0,T], \mathbb{R}^d)$. Intuitively, we say that there is propagation of chaos if, whenever the initial conditions of the particles $X_i^N(0)$ are independent and distributed as μ_0, then P^N *is* μ-*chaotic*, i.e. for any $k \geqslant 1$ and any $\phi_1, \dots, \phi_k \in C_b(\mathbf{D}([0,T], \mathbb{R}^d))$

$$\lim_{N\to\infty} \langle P^N, \phi_1 \otimes \cdots \otimes \phi_k \otimes 1 \otimes \dots \rangle = \prod_{i=1}^{k} \langle \mu, \phi_i \rangle.$$

This property states the *asymptotic independence* of the particles despite the interaction and it is often associate to a sort of *Law of Large numbers*. Indeed, it is equivalent to

$$\mu_X^N \xrightarrow{\text{in law}} \mu$$

and when we say that we want to prove *pathwise propagation of chaos* we aim to give a rate of convergence to zero of the distance between μ_X^N and μ w.r.t. some distance between measures, in this case a W_1 Wasserstein distance. Identifying the rate of propagation of chaos for a particular interaction is useful also in view of approximation techniques for the nonlinear process. Indeed, because of their nonlinearity, it is usually hard to simulate numerically the evolution of a McKean-Vlasov process, but the propagation of chaos let us simulate its trajectories by means of the particle system [14, 15]. Of course the propagation of chaos with a rate is a starting point to measure the accuracy of this approximation. In Section 1.2.4 we introduce an *intermediate process* that does not display the collateral jumps, instead it has an additional drift term depending on the empirical measure. This process helps in underlining the role of simultaneous jumps in the pathwise propagation of chaos. Indeed, we couple the two particle systems and in Proposition 2.1.1 we show that the simultaneous jumps give a rate of convergence in W_1 Wasserstein distance of the order $O\left(\frac{1}{\sqrt{N}}\right)$. After that, we couple the intermediate process with N independent copies of the nonlinear process and in Proposition 2.1.2 we prove the property of propagation of chaos along the lines of [48]. From these results it follows the Corollary 2.1.1, in which propagation of chaos for the particle system X^N is proved.

In Section 2.2 we aim to extend the previous results to a more general set of assumptions. Therefore, it is natural to consider a class of systems with a superlinear drift term. In this framework we can incorporate several existing mean field models with continuous paths and extend them to a discontinuous setting [7, 28, 45]. The condition on the drift we are going to consider is the following:

(U) the drift coefficient $F\colon \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^d$ is of the form

$$F(x, \alpha) = -\nabla U(x) + b(x, \alpha),$$

for all $x \in \mathbb{R}^d$ and all $\alpha \in \mathcal{M}(\mathbb{R}^d)$, where U is convex and $\mathcal{C}^1$, while the function b is assumed to be globally Lipschitz in both variables.

All the other coefficients satisfy globally Lipschitz conditions on all variables. Nonlinear processes of this type with an unbounded jump rate seem to be new and we need to verify the well-posedness of the correspondent SDE. We prove it in Theorem 2.2.1, by means of a contraction argument and of a Picard iteration. After that, with Proposition 2.2.1, Proposition 2.2.2 and Corollary 2.2.1, we confirm the results of Section 2.1 on pathwise propagation of chaos under the assumption (U) on the drift coefficient. Notice that we need to perform all the proofs in a L^1 framework, instead of the classical L^2 approach for stochastic calculus. Indeed, we want to have at least globally Lipschitz conditions on the rate function λ and the total jump amplitude, call it Δ^N, and, when dealing with the well-posedness of the nonlinear process, we will need to bound expectations of the supremum over a time interval of an integral w.r.t. the Poisson random measure $\mathcal{N}$. In an L^2 framework, this involves the corresponding compensated martingale $\tilde{\mathcal{N}}$ and it needs bounds of the following type, for X, $Y \in \mathbb{R}^d$,

$$\int_0^\infty \int_{[0,1]^{\mathbb{N}}} \|\Delta^N(X,h)\mathbb{1}_{(0,\lambda(X)]}(u) - \Delta^N(Y,h)\mathbb{1}_{(0,\lambda(Y)]}(u)\|^p du\nu(dh) \leqslant C\|X-Y\|^p,$$

for $p = 2$. However, sometimes this may hold for $p = 1$, but not for $p = 2$, which justifies the choice of getting the L^1 framework, where we do not need to compensate the process $\mathcal{N}$. For instance, if Δ^N is constant and λ is globally Lipschitz, the above inequality holds for $p = 1$ and not $p = 2$.

In Section 2.3 we focus on one of the neuroscience models that inspired the analysis of simultaneous jumps [75] and we slightly generalized it to a d-dimensional framework. Therefore we drop off the diffusive component and we consider the *piecewise deterministic nonlinear Markov process* that solves the following:

$$\begin{aligned} dX(t) =& \mathbf{E}\left[\lambda(X(t))\right] \mathbf{E}\left[V\right] dt - X(t)dt \\ &- \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \left(X(t) - U(h_1)\right) \mathbb{1}_{[0,\lambda(X(t)))}(u)\mathcal{N}(dt, du, dh), \end{aligned}$$

with $\mathcal{N}$ Poisson random measure with characteristic measure $l \times \nu \times l$. We see that the contribution of the collateral jumps creates the additional drift term

$$\mathbf{E}\left[\lambda(X(t))\right] \mathbf{E}\left[V\right] dt.$$

While V and U are two bounded jump functions with values in $\mathbb{R}^d$ (they represents two random variables with values in some bounded subsets of $\mathbb{R}^d$, with abuse of notation we will indicate as expectations their integrals w.r.t. the measure ν), we allow for a superlinear jump rate, of the form prescribed in [75].

(JR) The jump rate of each particle is a non-negative C^1 function of its position, $\lambda : \mathbb{R}^d \to \mathbb{R}_+$, that is written as the sum of two functions:

$$\lambda(\cdot) \doteq b(\|\cdot\|) + h(\cdot).$$

- b is a C^1, positive, non-decreasing function such that

$$b'(r) \leqslant \gamma b(r) + c$$

for some $c > 0$ and $\gamma < \dfrac{1}{5\,\mathbf{E}[\|V\|]}$;

- $h : \mathbb{R}^d \to \mathbb{R}$ is a C^1 bounded function, i.e. there exists $H > 0$ such that $\forall\, x \in \mathbb{R}^d$, $\|h(x)\| \leqslant H$;

To control the jumps of the system when the jump rate is superlinear is particularly hard, especially in the nonlinear case, where we cannot use any compactness method. Notice that, in [43], the authors succeed in proving well-posedness and propagation of chaos with an explicit rate (the expected $\frac{1}{\sqrt{N}}$) for a similar model and for weak moments conditions on the initial values, by defining an ad-hoc distance based on the rate function λ itself. In our study, we choose not to extend this powerful approach to our d-dimensional model and to maintain the same structure of proofs of the previous sections. However, we believe that the computations of [43] would work here and they would give results without the restrictive hypothesis on the bounded support of initial condition that we require in Theorem 2.3.1, where we prove well-posedness of the nonlinear limit for bounded support initial conditions. In the following, by means of a priori bounds on the involved quantities, we end the study with Theorem 2.3.2, Theorem 2.3.3 and Corollary 2.3.1 in which we get pathwise propagation of chaos with the expected $\frac{1}{\sqrt{N}}$ rate.

Part II: models with asymmetric interactions

In this part of the thesis we consider a particle system where the interaction is asymmetric and, if strong enough, it generates ergodicity in a system otherwise transient. Mainly inspired by population models, in Chapter 4 we define and study a class of systems of interacting random walks on the positive integers, reflected in zero to which we add interactions that push each particle towards the origin. Previously, in Chapter 3, we describe a continuous model which is a slight modification of the one in [54] and it represents the continuous analogue of one of the models of Chapter 4. Because of the continuity of the dynamics this model is completely solvable and we use it as a reference for the study of the discrete one.

Interacting random walks with asymmetric interaction

In Chapter 4 we consider a system of N particles on the non-negative integers $\mathbb{N}$, which without interaction evolve as independent random walks, with a drift towards infinity. The interaction induces jumps towards zero, whose size depends on the specific model we consider, and whose rate is proportional to the fraction of particles that are in a lower position than the jumping particle. Let us describe the simplest model we consider. There is a fixed number N of particles on $\mathbb{N}$, where each particle X_i^N, for $i = 1, \dots, N$, makes jumps of size 1. If $X_i^N > 0$, then it goes to

$$\begin{aligned} &X_i^N + 1 \quad \text{with rate } 1 + \delta, \\ &X_i^N - 1 \quad \text{with rate } 1 + \lambda \tfrac{1}{N} \textstyle\sum_{k=1}^N \mathbb{1}(X_k^N < X_i^N). \end{aligned}$$

If $X_i^N = 0$, then the only allowed jump is rightward. Here $\delta \geqslant 0$ indicates a bias rightward, while $\lambda \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}(X_k^N < X_i^N)$ is a bias leftward. We call this model the *small jump model*, while in general we consider a larger class of models where the leftward jump induced by the interaction term may have amplitude wider than 1. One interpretation of these models is as follows. N individuals, each associated with an integer valued *fitness*, have an intrinsic tendency to improve their fitness in time. However, each individual mimicking only the *worse than him* may worsen his fitness. Since the interaction is of mean field type, we associate to the particle system a nonlinear Markov process $\{X(t)\}_{t \geqslant 0}$ whose possible transitions at time $t \geqslant 0$ are as follows:

$$\begin{aligned} &X(t)+1 \quad \text{with rate } 1+\delta, \\ &X(t)-1 \quad \text{with rate } 1+\lambda\mu_t[0, X(t)), \end{aligned}$$

where μ_t is the law of $X(t)$ and, as above, when $X(t)$=0, only the rightward jump is allowed. In Section 4.1 we define a larger class of models, roughly speaking such that the leftward jump induced in the particle X_i^N may have amplitude between 1 and X_i^N itself. We prove well-posedness of the nonlinear process and the property of propagation of chaos, notice that in some special cases this is a particular case of the one described in Chapter 1 and 2.

Then, the question is whether a strong interaction can prevent some individuals from improving forever, i.e. escape towards infinity. At the outset, we make two remarks which we illustrate in the small jump model at the level of the N particle system.

(i) The asymmetry in the drift produces an inhomogeneous system: the rightmost particle, when alone on its site, has a net drift of about $\delta - \lambda$, whereas the leftmost particle has a positive drift δ.

(ii) Particles piled up at the same site do not interact, and this produces a tendency for piles to spread rightward.

It is clear that, when $\lambda = 0$, for any N each particle system has no stationary measure. Indeed, it consists of random walks with a nonnegative drift $\delta \geqslant 0$ and reflection at zero. Our aim is to estimate the *critical interaction strength* above which the system has a stationary measure, we indicate it as

$$\lambda_N^*(\delta) \text{ and } \lambda_\infty^*(\delta)$$

for the N particle system and the nonlinear process, respectively. We focus on the simple model described above since it dominates all others in the class defined in Section 4.1 in stochastic ordering. In particular, ergodicity of the small jump model implies ergodicity of all others. Moreover, in Chapter 3 we describe a model of interacting diffusions that shares the same properties of the small jump model. This is an adaptation of the system of particles interacting through their cumulative density function (CDF) defined in [54]. In this continuous case the critical interaction strength can be explicitly obtained for the N

particle system as well as for the nonlinear process. In Theorem 3.2.2 we prove the critical value for the N particle system is

$$\lambda^*_{N,cont}(\delta) = 2\delta \frac{N}{N-1},$$

while in Theorem 3.2.3 it is proved that the nonlinear process has a critical interaction strength that is

$$\lambda^*_{\infty,cont}(\delta) = 2\delta.$$

Unfortunately, the proofs of these results strictly depend on the continuity of the trajectories and we mainly use them to underline the differences with the discrete dynamics. Indeed, despite the same interacting mechanism, the continuous and the discrete model display a peculiar difference. In the discrete model the particles can form large clusters on a single site. When particles are on the same site, according to our description, they cannot interact and this interferes with ergodicity. On the other hand, the interaction prevents the particles from escaping to infinity and it favors the creations of clusters.

We dedicate Section 4.2 to the study of the long-time behavior and of the critical interaction strength for the N particle system. By means of a Lyapunov function, we prove that, for all $\delta \geqslant 0$, there exists a critical value

$$\lambda^*_{up}(\delta) \doteq 8\delta^2 + 12\delta$$

such that for all $N \geqslant 2$, for all $\lambda > \lambda^*_{up}(\delta)$ the process $X^N = (X^N_1, \dots, X^N_N)$ described in small jump model is exponentially ergodic and there exists a probability measure $\pi^N_{(SJ)}$ on $\mathbb{N}^N$ such that, for any initial condition $X^N(0)$,

$$\|P^N_x((X^N_1(t), \dots, X^N_N(t)) \in \cdot) - \pi^N_{(SJ)}\|_{TV} \leqslant C_N(x)(\rho_N)^t, \ \forall x \in \mathbb{N}^N, \forall t \geqslant 0,$$

where $C_N(x)$ is bounded, $\rho_N < 1$ and $\| \cdot \|_{TV}$ is the total variation norm. $\pi^N_{(SJ)}$ is the unique stationary measure for the process $(X^N_1, \dots, X^N_N)$. These are the results stated in Theorem 4.2.1, in which we prove exponential ergodicity of the particle system under some assumptions and we give an upper bound on $\lambda^*_N(\delta)$ which is uniform in N. On the other hand, it is clear that for $\lambda \leqslant \delta$ the particle system is transient. By means of a linear Lyapunov function, in Theorem 4.2.2 we establish a lower bound on $\lambda^*_N(\delta)$. Indeed, there exists

$$\lambda^*_{N,lower}(\delta) \doteq \big(1 + \rho(\epsilon, N)\big)2\delta, \quad \text{with} \quad \rho(\epsilon, N) \doteq \frac{N^2(\delta+2)}{N(N-1)(\delta+2) - 2\delta} - 1 \longrightarrow 0,$$

such that, for all $\lambda < \lambda^*_{N,lower}(\delta)$, the process $X^N = (X^N_1, \dots, X^N_N)$ is transient. This lower bound in Theorem 4.2.2 is strictly greater than the critical value of the continuum model, highlighting the different role played by the occurrence of piles in our case. We believe that this difference is substantial and it gives rise to a non-trivial expression for $\lambda^*_\infty(\delta)$, unexpected by the analysis of the continuous model.

In Section 4.3 we study the stationary measures of the nonlinear process. In the continuous analogue this is done by directly solving the stationary Fokker-Planck equation and finding that it has a unique solution. This is clearly harder in the discrete case, we could not find a way to prove uniqueness of the stationary measure and we define the critical interaction strength $\lambda^*_\infty(\delta)$ as the value above which the nonlinear process has at least one stationary measure. In Theorem 4.3.1 we prove the existence of at least one stationary distribution by means of a transformation Γ in the space $\mathcal{M}(\mathbb{N})$, for which every stationary distribution of the nonlinear process is a fixed point. This is an approach widely exploited in the study of quasi-stationary distributions (QSD) in countable spaces, see [5, 40, 41]. This gives an upper bound on the critical value that is

$$\lambda^*_{up}(\delta) \doteq 4\delta.$$

In Theorem 4.3.2 we give a simple lower bound on this value, saying that for $\lambda \leqslant 2\delta$ there is no stationary distribution at all.

In Section 4.4, we exploit a link with *Jackson's Networks* [52] to give sharper estimates on the critical values. With a change of variables we study the dynamics of the *gaps* between successive particles and we compare it with a particular queueing system of Jackson's type. This let us derive the exact form of

$$\lambda^*_2(\delta) = 2\delta^2 + 4\delta$$

in Theorem 4.4.2. For $N > 2$ the applicability of this method is still an open problem; however in Section 4.4.3 we define, for each $N \geqslant 3$ a Jackson's Network associated to our particle system. This suggests heuristic computation leading to conjecture the critical interaction strength for all values of N as follows. Fix $N \geqslant 3$, the process X^N is ergodic if and only if

$$(1+\delta)^N < \prod_{k=1}^{N-1}(1+\lambda\frac{k}{N}).$$

Taking the limit as N goes to infinity, a natural conjecture is the critical interaction strength for the nonlinear process. Fix $\delta \geqslant 0$, then for all λ such that

$$(1+\frac{1}{\lambda})\ln(1+\lambda) - 1 > \ln(1+\delta),$$

the nonlinear process X has at least one stationary measure.

Part III: generalized Curie-Weiss model

In Chapter 5 we analyze a particular type of *dissipated interaction* in a dynamical version of the generalized Curie-Weiss model [35, 37], with the aim of proving the existence of *self-sustained periodic behavior* in its nonlinear limit. This interaction has already been proved to originate periodic behavior in a dynamical Curie-Weiss model [26] and in a diffusive model of cooperative behavior [24].

A dissipative dynamics for the generalized Curie-Weiss model

In Section 5.2 we define the dynamical process we are interested in. We recall that the Curie-Weiss model is defined as the sequence of probability measures on $\mathbb{R}^N$, for $N = 1,2,\dots$, given by

$$\mathbf{P}_{N,\beta}(dx_1,\dots,dx_N) = \frac{1}{\mathbf{Z}_N(\beta)}\exp\left(N\beta g\left(\sum_{i=1}^N \frac{x_i}{N}\right)\right)\prod_{i=1}^N \rho(dx_i),$$

where ρ is the symmetric probability measure on $\mathbb{R}$ representing the single-site distribution of a spin, g is the interaction function, β is the inverse absolute temperature of the model and $\mathbf{Z}_N(\beta)$ is the normalizing constant. For each N fixed, a Langevin dynamics associated to the generalized Curie-Weiss model is a diffusion process X^N with values in $\mathbb{R}^N$ such that $\mathbf{P}_{N,\beta}$ is its unique invariant measure. X^N is solution to the following systems of SDE:

$$dX_i^N(t) = \frac{\beta}{2}g'\left(\frac{\sum_{j=1}^N X_j^N(t)}{N}\right)dt - \frac{\rho'(X_i^N(t))}{2\rho(X_i^N(t))}dt + dB_t^i,$$

where $\{B^i\}_{i=1,\dots,N}$ is a family of independent 1-dimensional Brownian motions. This dynamics represents an interacting particle system where each particle follows its own dynamics (given by the last two terms on the right-hand side) and it experiences a mean field interaction, which depends on the empirical mean of the system $m^N(t) \doteq \frac{\sum_{j=1}^N X_j^N(t)}{N}$. Following the approach in [24, 26], we suppose that the motion of each particle depends on a "perceived magnetization" instead of the empirical mean $m^N(t)$. To this aim, we introduce the variables λ_i^N, for $i = 1,\dots,N$, representing the interaction felt by the spin X_i^N. This results in a stochastic process (X^N,λ^N) with values in $\mathbb{R}^{2N}$ where, at every time $t \geqslant 0$, $X_t^N = \left(X_t^{N,1},\dots,X_t^{N,N}\right)$ is the vector of the spins of the N particles and $\lambda_t^N = \left(\lambda_t^{N,1},\dots,\lambda_t^{N,N}\right)$ is the vector of their "perceived magnetizations". (X_t^N,λ_t^N) solves the following system of SDE:

$$\begin{cases} dX_t^{N,i} = \frac{\beta}{2}g'(\lambda_t^{N,i})dt - \frac{\rho'(X_t^{N,i})}{2\rho(X_t^{N,i})}dt + dB_t^{1,i} \\ d\lambda_t^{N,i} = -\alpha\lambda_t^{N,i}dt + \frac{1}{N}\sum_{j=1}^N\left(\frac{\beta}{2}g'(\lambda_t^{N,j}) - \frac{\rho'(X_t^{N,j})}{2\rho(X_t^{N,j})}\right)dt + D\,dB_t^{2,i}, \end{cases}$$

$i = 1,\dots,N$, for $\{(B^{1,i},B^{2,i}\}_{i=1,\dots,N}$ a family of independent 2-dimensional Brownian motions. The constants $\alpha, D \geqslant 0$ are the dissipative and diffusive constants characterizing the evolution of the "perceived magnetization". The interactions are of mean field type; as usual, we define the correspondent nonlinear Markov process (X,λ) on $\mathbb{R}^2$ as the solution of the following nonlinear SDE:

$$\begin{cases} dX_t = \frac{\beta}{2}g'(\lambda_t)dt - \frac{\rho'(X_t)}{2\rho(X_t)}dt + dB_t^1 \\ d\lambda_t = -\alpha\lambda_t dt + \langle \mu_t(x,l), \frac{\beta}{2}g'(l) - \frac{\rho'(x)}{2\rho(x)}\rangle dt + D\,dB_t^2 \\ \mu_t = Law(X_t,\lambda_t), \end{cases}$$

where $B = (B^1, B^2)$ is a two dimensional Brownian motion. In Theorem 5.2.3 we prove well-posedness of this McKean-Vlasov process under some reasonable assumptions and in Theorem 5.2.4 we prove the correspondent property of propagation of chaos.

The Gaussian dynamics

In Section 5.3 we focus on a completely solvable model belonging to the class of models described in Section 5.2. This has no diffusive component in the evolution of λ, i.e. $D = 0$, and the single-site distribution is normally distributed, i.e. $\rho \sim \mathcal{N}(0, \sigma^2)$. This simplification leads to the nonlinear process $(X_t, \lambda_t)_{t \geqslant 0}$ solution of the following nonlinear SDE:

$$\begin{cases} dX_t = \frac{\beta}{2} g'(\lambda_t) dt - \frac{X_t}{2\sigma^2} dt + dB_t, \\ \frac{d\lambda_t}{dt} = -\alpha\lambda_t + \frac{\beta}{2} g'(\lambda_t) - \frac{m_t}{2\sigma^2}, \\ \mu_t = Law(X_t, \lambda_t) \text{ and } m_t = \langle \mu_t(dx, dl), x \rangle, \end{cases}$$

for $\{B_t\}$ Brownian motion. If λ_0 is deterministic, the evolution of the "perceived magnetization" follows a deterministic dynamics, i.e. for all $t > 0$ the law of the process is such that

$$\mu_t(dx, d\lambda) = \nu_t(dx) \times \delta_{\lambda_t}(d\lambda).$$

Moreover, the resulting process is a Gaussian process, specifically it is completely described by the initial condition μ_0 and the quantities $\{(m_t, V_t, \lambda_t)\}_{t \geqslant 0}$, where $V_t = \mathbf{Var}[X_t]$. In Section 5.3.1 we analyze the dynamics without dissipation, i.e. the nonlinear limit of the Langevin dynamics. In Proposition 5.3.1 we study the ODE that rules the evolution of the mean m_t and we derive the set of the *critical* β, while in Theorem 5.3.1 we completely characterize the sets of stationary measures and the long-time behavior of the limiting Langevin dynamics.

In Section 5.3.2 we study the dynamics with dissipation, we reduce the problem to the study of the following system of ODE:

$$\begin{cases} \dot{m}_t = \frac{\beta}{2} g'(\lambda_t) - \frac{m_t}{2\sigma_2}, \\ \dot{\lambda}_t = -\alpha\lambda_t + \frac{\beta}{2} g'(\lambda_t) - \frac{m_t}{2\sigma^2}, \end{cases}$$

because the independence of the evolution of V_t let us consider a two-dimensional instead of a three-dimensional system. With the simple change of variable $y = \frac{1}{2\sigma^2}(\lambda - m)$, we get the system

$$\begin{cases} \dot{y}_t = -\frac{\alpha}{2\sigma^2} \lambda_t, \\ \dot{\lambda}_t = y_t - \left(\alpha + \frac{1}{2\sigma^2}\right) \lambda_t + \frac{\beta}{2} g'(\lambda_t), \end{cases}$$

which is a Liénard system. Among planar differential equations, the systems of Liénard type have been extensively studied, in particular in relation to their limit cycles, [19, 22, 46, 61, 71, 76]. A detailed and complete study of all Liénard systems, with necessary and sufficient conditions for the existence of exactly $k \geqslant 0$ limit cycles, is still an open problem. However, in literature we can find sufficient conditions for the existence of *at least*(or *exactly*) $k \geqslant 0$ limit cycles, [22, 71]. In Theorem 5.3.2 we depict three possible phases of the evolution of (y_t, λ_t) and we give sufficient conditions on the interaction function g and on the value of parameters for them to occur. In general, for an admissible interaction function g we observe the following situations.

i) We can always find a regime of the parameters in which the origin is a global attractor and no limit cycles are present.

ii) Under a simple condition on the derivative of the interaction function, we may find a critical value in which the origin looses its local stability and a stable limit cycle bifurcates from it.

iii) If the previous situation occurs and the interaction function is sufficiently regular at infinity, we can find a regime in which there exists a unique limit cycles, which is attractive.

Then, Theorem 5.3.3 describes the stationary measure of the process (X_t, λ_t) and the invariant sets of measures that characterize periodic solutions.

In Section 5.3.3 we highlight the flexibility of this model, since by a suitable choice of the interaction function g we can observe several interesting phases in the Liénard system and, consequently, in the evolution of the nonlinear process (X_t, λ_t). In particular, we prove that it is possible to find an interaction function that allows, in certain regimes of parameters, *coexistence* of periodic orbits. Indeed, the Liénard system may display the following features.

a) More than one periodic orbit may coexist and they all revolve around the origin. In this case the outer one should be stable, the second should be unstable and then they should alternate.

b) Some periodic orbits may appear even when the origin is still locally stable. These orbits appear through global bifurcations (the Hopf bifurcation is a local one) and they usually appear in pairs, the outer periodic orbit is stable, while the inner one is unstable.

In Proposition 5.3.2 we give sufficient conditions on the interaction function g such that the model admits a regime of parameters in which N limit cycles coexist. In Section 5.3.4 we give an explicit example of two interaction functions that let us observe, in different regimes of the parameters, the two particular situations above. For any $\alpha > 0$, we could find the explicit critical value of β at which the Hopf bifurcation occurs, while we could only estimates the critical values at which the other phase transitions occur.

Contents

Part I

Models with simultaneous jumps

Chapter 1

From Neuroscience to a general framework for simultaneous jumps

In this chapter we study an interacting particle system that displays a particular feature, that we indicate as the *simultaneous jumps*. This characteristic has recently appeared in toy models for interacting neurons, [29, 43, 75]. These models represent the spike of a neuron as a discontinuity in the evolution of its membrane potential. At the same time each spike induces *collateral discontinuities* in the membrane potential of all the other neurons. Those ones are rescaled by the factor $\frac{1}{\mathsf{N}}$, where N is the size of the system, as customary in mean field models. In the limit, these *collateral jumps* collapse into an additional non-linear drift term while the spike component is preserved. This seems to be a new framework in mean field modelling, therefore we aim to depict a general description of this class of models, giving to specialists a general and flexible class of models with simultaneous jumps. In this chapter, we summarize the neuroscience models that have inspired the study and we present at an informal level our general model.

1.1 Interacting particle systems in Neuroscience

Neurons are supposed to spread information by means of electrical impulses, called action potentials or *spikes*. A single neuron has its own membrane potential that varies due to external stimuli, to interactions with other neurons and to its own dynamics. When a neuron spikes its membrane potential is rapidly reset to a resting state and, at the same time, other neurons in the network receive an excitatory or inhibitory influence. Recently, models describing networks of spiking neurons by means of the mean field approach, typical of statistical mechanics, have become widespread in neuroscience. Due to peculiarities of the brain modelling, sometimes these models are raising questions that have their own interest outside the direct brain modelling. In particular, some recent works on piecewise-deterministic Markov processes for the evolution of neurons membrane potential have displayed the interesting feature of simultaneous jumps that we are going to study in the following sections.

1.1.1 Mean field models in Neuroscience

The mean field approach in neuroscience consists in describing large populations of neurons of the same type by means of the behavior of a so-called "typical neuron". The large number of neurons and of connections between them make indeed reasonable to describe the brain, a finite-size network, as the infinite-size limit of a system of particles in mean field interactions, i.e. where the graph of interactions is complete. This approach origins in statistical mechanics, from the seminal work of Kac [56], in which the author builds a microscopic system of interacting Markov processes, representing the molecules of a rarefied gas, to justify the macroscopic description through the spatially homogeneous Boltzmann equation. The link between microscopic and macroscopic level is given by the *propagation of chaos*, see the well-known reference from Sznitmann [83]. Propagation of chaos basically says that, when the size of the system grows to infinity, the particles tends to de-correlate, despite their interaction. As observed by Galves and Löcherbach in [44], it is hard to find a systematical overview on the biological justification and experimental confirmation of propagation of chaos in the brain behavior, although the goodness of this approach seems to be validated in Baladron et al. [7]. There the authors cite experimental results in [34], where de-correlation of neuronal firing in visual cortex is observed. Mean field models account for spikes with different approaches and we do not aim to be complete in describing the extensive literature in this field. However, in the following we summarize some of these approaches.

- The conductance-based models describe in details the role of ions channels in the evolution of membrane potential of each neuron in the network. For instance, Hodgkin-Huxley and FitzHugh-Nagumo models associate to each neuron, respectively, a 4 and a 2 dimensional process, that takes into account the membrane potential, but also other variables, see [7] for analysis of networks of this type. These models consider the evolution of their quantities as continuous path processes, where the *spikes* are rapid changes in the value of the membrane potential and the randomness is expressed by means of a Gaussian process. Usually this approach leads to extremely complicated expressions, however the continuity of paths helps in tackling the problem of propagation of chaos.

- Leaky integrate and fire models are widely studied in the neuroscience community and they represent spikes as discontinuities in the evolution of the membrane potential. A single neuron's membrane potential evolves according to an Ornstein-Uhlenbeck process starting from zero (chosen as the neuron's resting state) and it spikes when it reaches a certain fixed threshold. Then its potential is reset to zero (here is the discontinuity) and the process starts again. In networks of leaky integrate and fire models the interaction is given by the fact that, when a neuron spikes, all the others receive an additional drift (as a positive "kick") of the order $\frac{1}{\mathsf{N}}$, if N is the size of the network. The study of mean field limits for this type of networks requires non-standard techniques, because of the discontinuities given by the threshold and the particular dependence of the nonlinear term on the law of the process itself, see [30] for a probabilistic study and [18, 20] for a PDE approach.

- Models with Poisson spikes account for the intrinsic randomness of spikes describing them by means of inhomogeneous Poisson processes with a rate depending on the membrane potential. In this framework, the membrane potential is modelled by a piecewise deterministic Markov process and the interaction occurs through simultaneous jumps. When a neuron spikes, randomly according to its rate, it is reset to zero as in leaky integrate and fire models, but instead of interacting with other neurons increasing their drifts, it makes their membrane potentials increase of a small quantity, depending on the synaptic weight between them. In this way, the jumps in the network are simultaneous and, even if some of them are of the order $\frac{1}{N}$, they may cause problems when letting the size N of the network going to infinity. The literature on mean field models with jumps is less rich then the one on continuous models, nevertheless in some recent papers the authors prove propagation of chaos for models in this class, see [29, 43, 44, 75].

1.1.2 Neuroscience models with simultaneous jumps

Let us focus in the recent Poisson mean field models, displaying simultaneous jumps, [29, 43, 75]. These models describe the membrane potentials of neurons as quantities on the positive real line. Let $N \geqslant 1$ be a fixed finite number of neurons in an homogeneous network (i.e. where the neurons are all of the same type), we associate to each neuron an index $i = 1, \dots, N$ and we describe the membrane potential of the network with the stochastic process $U^N(t) = \left(U_1^N(t), \dots, U_N^N(t)\right) \in \mathbb{R}_+^N$ for every $t \geqslant 0$, where $U_i^N(t)$ is the membrane potential of the i-th neuron at time t. First of all, the membrane potential of a neuron exponentially decays towards the resting state (here it is 0) due to the *leak current*, a continuous flow of potential. Therefore neuron i has a drift proportional to

$$-U_i^N(t).$$

Then the neurons interact by means of *electrical synapses* and, through the *gap-junction channels*, they constantly communicate. This pushes the system towards the average potential value, that means that the i-th neuron has also a drift proportional to

$$\sum_{j=1}^{N} \frac{U_j^N(t)}{N} - U_i^N(t).$$

Finally, *chemical synapses* cause fast-events, the spikes. A neuron spikes randomly according to a state dependent rate

$$\lambda(U_i^N(t)) \geqslant 0.$$

If $\lambda(0) = 0$, then it is supposed that there is no external stimuli, while a positive value in 0 means that the neuron can spike even when it is at resting state, due to some external input. When neuron i spikes, its membrane potential is reset at 0 by a jump of amplitude $-U_i^N(t^-)$. Simultaneously, the non-spiking neurons receive an additional discrete influence, they increase their potential of a quantity depending on a stochastic *synaptic efficacy.* That results in a jump of amplitude

$$\frac{W_{i,j}}{N}$$

of the membrane potential $U_j^N(t^-)$ when the i-th neuron spikes and this happens simultaneously for all $j \neq i$. The above description corresponds to a piecewise-deterministic Markov evolution for the process U^N, that is solution of the following system of SDEs. For all $i = 1, \dots, N$

$$\begin{aligned} dU_i^N(t) = & -\alpha U_i^N(t)dt - \beta \left(U_i^N(t) - \sum_{j=1}^N \frac{U_j^N(t)}{N} \right) dt - U_i^N(t^-) \int_0^\infty \mathbb{1}_{[0,\lambda(U_i^N(t^-))]}(u) \mathcal{N}^i(du, dt) \\ & + \sum_{j \neq i} \frac{W_{i,j}}{N} \int_0^\infty \mathbb{1}_{[0,\lambda(U_j^N(t^-))]}(u) \mathcal{N}^j(du, dt), \end{aligned} \tag{1.1.1}$$

where $\{\mathcal{N}^i\}_{i=1,\dots,N}$ is a family of independent Poisson random measures with characteristic measure $l \times l$, for l the Lebesgue measure. In the papers [29, 43], the authors study the case with $\alpha = 0$ and $W_{i,j} \equiv 1$ for all $i, j = 1, \dots, N$; while in [75] the authors study the case of $\beta = 0$ and synaptic weights $W_{i,j} = V$ i.i.d. positive bounded random variables. It is clear that the interactions here are all of mean field type, but while the one due to electrical synapses is classical, the one given by chemical synapses is rather peculiar. Indeed these simultaneous jumps, one of which will remain in the limit, while the others collapse in a continuous term because of the rescaling of the order $\frac{1}{N}$, seem to be new in the mean field models framework. In the aforementioned papers, the authors succeed to prove propagation of chaos under super-linear hypothesis on the rate function λ.

1.2 Interacting particle systems with simultaneous jumps

In this section we describe a mean field model that can embed the feature of simultaneous jumps in a more general framework. The idea comes from the desire to understand if the peculiarity of the simultaneous jumps can create problems in the proof of propagation of chaos in situations different from the one described above, for example in presence of a Brownian component. In this setting, every particle, besides its diffusive dynamics, can perform what we call a *main jump*, that is a jump of a certain amplitude with a certain rate. Every time that a particle performs this jump, it induces a jump in all the other particles' trajectories, but the amplitude of these *collateral jumps* is rescaled according to the size of the system. We consider the McKean-Vlasov limit of this system and we want to prove pathwise propagation of chaos via a coupling technique that involves an intermediate process. This would give a rate of convergence for the W_1 Wasserstein distance between the empirical measures of the two systems on the space of trajectories $\mathbf{D}([0, T], \mathbb{R}^d)$. We start at an informal level, introducing both the microscopic and the macroscopic dynamics and illustrating the phenomenon of propagation of chaos. Well-posedness and convergence will be shown under various assumptions in Chapter 2.

1.2.1 The microscopic dynamics

Fix $N \geqslant 2$ and let $X^N = (X_1^N, \dots, X_N^N) \in \mathbb{R}^{d\times N}$ be the spatial positions of N different particles moving in $\mathbb{R}^d$. We introduce the corresponding *empirical measure*

$$\mu_X^N \doteq \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N}.$$

When the time variable appears explicitly in $X^N(t)$, we write $\mu_X^N(t)$ to indicate the time dependence of the empirical measure. Note that $\mu_X^N(t)$ is an element of $\mathcal{M}(\mathbb{R}^d)$, the set of probability measures on the Borel subsets of $\mathbb{R}^d$.

The vector of particles positions $X^N(t)$ evolves as a jump diffusion process with the following specifications for the i-th particle.

- The **drift coefficient** depends on the spatial position of the particle and on the other particles through the empirical measure, i.e. it is of the form

  $$F(X_i^N(t), \mu_X^N(t))$$

  for some function $F : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^d$ common to all particles.

- The **diffusion coefficient**, equivalently, is written as

  $$\sigma(X_i^N(t), \mu_X^N(t))$$

  for $\sigma : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^{d\times d_1}$, again the same for all particles.

- The **main jump rate**: particle i performs a *main jump* with rate

  $$\lambda(X_i^N(t), \mu_X^N(t)),$$

  for a positive function $\lambda : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to [0, \infty)$. With this rate, the i-th particle performs a main jump and simultaneously it induces in all the other particles a *collateral jump.*

- The **main jump amplitude**: particle i performs a main jump that is a random variable

  $$\psi(X_i^N(t), \mu_X^N(t), h_i^N)\ ,$$

  for a function $\psi : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \times [0,1] \to \mathbb{R}^d$. Here h^N is a random variable with values in $[0,1]^N$ and its distribution is given by a symmetric measure ν_N.

- The **collateral jump amplitude**: the i-th particle is induced to jump by main jumps of every other particle. The amplitude of these *collateral jumps* is given by the function $\Theta : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \times [0,1]^2 \to \mathbb{R}^d$. When the j-th particle jumps (this occurs with rate $\lambda(X_j^N(t), \mu_X^N(t))$, of course) the i-th particle performs a jump of amplitude
$$\frac{\Theta(X_j^N(t), X_i^N(t), \mu_X^N(t), h_j^N, h_i^N)}{N},$$
where h_i^N and h_j^N are components of the random vector h^N, with distribution ν_N.

In this description, the classical mean field interactions are already encoded in the dependence of all the coefficients on the empirical measure. Moreover, we highlight the peculiar interaction of mean field type represented by the simultaneous jumps.

In more analytic terms, we are considering a Markov process $X^N = \{X^N(t)\}_{t\in[0,T]}$ with values in $\mathbb{R}^{d\times N}$ whose infinitesimal generator takes the following form on a suitable family of test functions f:

$$\mathcal{L}^N f(\boldsymbol{x}) = \sum_{i=1}^N \Bigg[F(x_i, \mu_{\boldsymbol{x}}^N) \cdot \partial_i f(\boldsymbol{x}) + \frac{1}{2}\sum_{j,k=1}^d a(x_i, \mu_{\boldsymbol{x}}^N)_{jk} \cdot \partial_i^2 f(\boldsymbol{x})_{jk} \\ + \lambda(x_i, \mu_{\boldsymbol{x}}^N) \int_{[0,1]^{\mathbb{N}}} \left(f\left(\boldsymbol{x} + \Delta_i^N(x, \mu_{\boldsymbol{x}}^N, h^N)\right) - f(\boldsymbol{x})\right) \nu_N(dh^N) \Bigg],$$

where $\partial_i f(\boldsymbol{x})$ indicates the vector of first order derivatives w.r.t. x_i, $\partial_i^2 f(\boldsymbol{x})$ indicates the Hessian matrix of the second order derivatives w.r.t. x_i, $a(x_i, \mu_{\boldsymbol{x}}^N) \doteq \sigma(x_i, \mu_{\boldsymbol{x}}^N)\sigma(x_i, \mu_{\boldsymbol{x}}^N)^*$ and

$$\Delta_i^N(x, \mu_{\boldsymbol{x}}^N, h^N)_j \doteq \begin{cases} \frac{\Theta(x_i, x_j, \mu_{\boldsymbol{x}}^N, h_i^N, h_j^N)}{N} & \text{for } j \neq i, \\ \\ \psi(x_i, \mu_{\boldsymbol{x}}^N, h_i^N) & \text{for } j = i. \end{cases}$$

Towards a rigorous construction, allowing the limit as $N \to +\infty$, let us consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbf{P})$ satisfying the usual hypotheses, rich enough to carry an independent family $(B_i, \mathcal{N}^i)_{i\in\mathbb{N}}$ of d-dimensional Brownian motions B_i and Poisson random measures $\mathcal{N}^i$ with characteristic measure $l \times l \times \nu$. Here l is the Lebesgue measure restricted to $[0,\infty)$ and ν is a symmetric probability measure on $[0,1]^{\mathbb{N}}$ such that, for every $N \geqslant 1$, ν_N coincides with the projection of ν on the first N coordinates. We will construct X^N as the solution of the following SDE

$$\begin{aligned} dX_i^N(t) &= F(X_i^N(t), \mu_X^N(t))dt + \sigma(X_i^N(t), \mu_X^N(t))dB_t^i \\ &+ \frac{1}{N}\sum_{j\neq i} \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \Theta(X_j^N(t^-), X_i^N(t^-), \mu_X^N(t^-), h_j, h_i) \mathbb{1}_{(0,\lambda(X_j^N(t^-),\mu_X^N(t^-))]}(u) \mathcal{N}^j(dt, du, dh) \\ &+ \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \psi(X_i^N(t^-), \mu_X^N(t^-), h_i) \mathbb{1}_{(0,\lambda(X_i^N(t^-),\mu_X^N(t^-))]}(u) \mathcal{N}^i(dt, du, dh), \end{aligned} \tag{1.2.1}$$

$i = 1, \ldots, N$. The existence and uniqueness of a solution starting from a vector of initial conditions $\left(X_1^N(0), \ldots, X_N^N(0)\right)$ depends obviously on the assumptions on the coefficients, and we will specify sufficient conditions in the following chapter.

In the latter SDE description, we made the choice of considering separately the jump's rate and amplitude. This is motivated by the fact that the jumps are our main interest and we want to state a clear framework, that we believe could be useful for possible applications. The non-compensated jump component is often represented by a measure that does not directly describe the behavior of the system. Here, we want to highlight the role of the jumps, therefore we describe a diffusion process that at each position has a certain jump rate and a set of possible jumps, represented by the functions λ and Δ^N, respectively. The aim of our study is to give results without uniform boundedness assumptions on the jump rate. In the next sections, we will see that the first natural assumption is to have globally Lipschitz conditions on the functions λ and Δ^N. This is the reason why we need to perform all our proofs in a L^1 framework, instead of the classical L^2 approach for stochastic calculus. Indeed, when dealing with the well-posedness of the nonlinear Markov process, we will need to bound expectations of the supremum over a time interval of an integral w.r.t. the Poisson random measure $\mathcal{N}$. In a L^2 framework, this involves the corresponding compensated martingale $\tilde{\mathcal{N}}$ and it needs bounds of the type, for $X, Y \in \mathbb{R}^d$,

$$\int_0^\infty \int_{[0,1]^{\mathbb{N}}} \|\Delta^N(X,h)\mathbb{1}_{(0,\lambda(X))]}(u) - \Delta^N(Y,h)\mathbb{1}_{(0,\lambda(Y))]}(u)\|^p du\nu(dh) \leqslant C\|X - Y\|^p, \tag{1.2.2}$$

for $p = 2$. However, sometimes (1.2.2) may hold for $p = 1$, but not for $p = 2$, which justifies the choice of getting the L^1 framework, where we do not need to compensate the process $\mathcal{N}$. For instance, if Δ^N is constant and λ is globally Lipschitz, (1.2.2) holds for $p = 1$ and not $p = 2$.

1.2.2 The macroscopic process

We introduce in this section a process that describes macroscopically the above dynamics. Heuristically, suppose the solution X^N of (1.2.1) exists and that its initial condition has a permutation invariant distribution. Fix an arbitrary component i and assume the process X_i^N has a limit in distribution; by symmetry, the law of the limit does not depend on i, so we denote by $X = \{X(t)\}$ the limit process. We make the further assumption that a law of large numbers holds, i.e. for all $t \geqslant 0$

$$\mu_X^N(t) \overset{N\to\infty}{\longrightarrow} \mu_t \doteq Law(X(t)).$$

Then, we define the process X as the one with the law of the solution of the *McKean-Vlasov* SDE:

$$dX(t) = \left(F(X(t), \mu_t) + \left\langle \mu_t, \lambda(\cdot, \mu_t) \int_{[0,1]^2} \Theta(\cdot, X(t^-), \mu_t, h_1, h_2) \nu_2(dh_1, dh_2) \right\rangle \right) dt \qquad (1.2.3)$$
$$+ \sigma(X(t), \mu_t) dB_t + \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \psi(X(t^-), \mu_s, h_1) \mathbb{1}_{(0,\lambda(X(t^-),\mu_s)]}(u) \mathcal{N}(dt, du, dh).$$

Here, B is a d_1-dimensional Brownian motion and $\mathcal{N}$ an independent Poisson random measure with characteristic measure $dtdu\nu(dh)$ on $[0,\infty)^2 \times [0,1]^{\mathbb{N}}$ as above. By $\langle \cdot, \cdot \rangle$ we indicate the integral of a function on its domain with respect to a certain measure; thus, $\langle \mu, \phi \rangle = \int_{\mathbb{R}^d} \phi(y) \mu(dy)$.

Existence and uniqueness of solutions to (1.2.3) starting from a given initial condition $X(0)$ will be discussed in the following sections. Note that (1.2.3) is not a standard SDE since the law μ_t of the solution appears as an argument of its coefficients. Processes of this type may be indicated as *nonlinear processes* and the nonlinearity stands in the fact that the coefficients of the SDE depend on the law of the process itself. Informally, we say that these nonlinear terms arise from the dependence, in the N particle system, on the empirical measure; this is easy to see in most of the coefficients of (1.2.3). However, also the simultaneous jumps give rise to a nonlinear term: indeed, the collateral jumps, due to the rescaling via the size of the system, appear in the limit as being absorbed by an additional drift term, depending on the characteristic measure of the Poisson random measures $\{\mathcal{N}^i\}_{i\in\mathbb{N}}$.

A SDE of the type of (1.2.3) is often referred to as McKean-Vlasov SDE, as it is customary to call McKean-Vlasov equation the partial differential equation solved, in the weak form, by its law μ_t, that is

$$\langle \mu_t, \phi \rangle - \langle \mu_0, \phi \rangle = \int_0^t \langle \mu_s, \mathcal{L}(\mu_s)\phi \rangle ds,$$

where

$$\mathcal{L}(\mu_t)\phi(x) \doteq F(x, \mu_t)\partial\phi(x) + \frac{1}{2} \sum_{j,k=1}^{d} a(x, \mu_t)_{jk} \partial^2 \phi(x)_{jk}$$
$$+ \left\langle \mu_t, \lambda(\cdot, \mu_t) \int_{[0,1]^2} \Theta(\cdot, x, \mu_t, h_1, h_2) \nu_2(dh_1, dh_2) \right\rangle \partial\phi(x)$$
$$+ \lambda(x, \mu_t) \int_{[0,1]} \left(\phi(x + \psi(x, \mu_t, h_1)) - \phi(x) \right) \nu_1(dh_1).$$

Let us highlight that the Poisson random measures appearing in Equations (1.2.1) and (1.2.3), respectively, have characteristic measure defined on $[0,\infty)^2 \times [0,1]^{\mathbb{N}}$. The equations could equivalently be stated in terms of Poisson random measures with characteristic measures defined on $[0,\infty)^2 \times [0,1]^N$ (namely, $l \times l \times \nu_N$) and on $[0,\infty)^2 \times [0,1]$ (namely, $l \times l \times \nu_1$).

The reason for our seemingly unnatural choice is that it prepares for the coupling argument we will use below to establish propagation of chaos. We will need, for each N, a coupling of the N-particle system with N independent copies of the limit system.

1.2.3 Propagation of chaos

The connection between the microscopic description (1.2.1) and the macroscopic one (1.2.3) is given by *propagation of chaos*, which is an idea introduced by Kac in 1954 in the work "Foundations of kinetic theory" [56]. The author introduced a Markovian model of gas dynamics, to explain, by a microscopic point of view, the spatially homogeneous Boltzmann equation for a rarefied gas with binary collisions. Let us briefly give the idea of what this means, with a particular focus on our model.

We call *chaotic* a configuration of independent particles, i.e. an initial condition $X^N(0)$ such that

$$Law(X^N(0)) = \nu_0^N(dx_1, \dots, dx_N) = \nu_0(dx_1)\dots\nu_0(dx_N),$$

for a certain law ν_0. Of course, the evolution of the microscopic system (1.2.1), since it involves the interactions, destroys the independence of the components. Nevertheless, we will prove that, if we consider only a finite number, say k, of components, when the size of the system N grows to infinity, they tend to behave independently and distributed as k copies of the macroscopic process (1.2.3) with initial condition ν_0. In this sense we say that the system *propagates chaos*, i.e. it preserves asymptotic independence of components. Propagation of chaos depends on the type of interaction (that needs to be of mean field type) and on the exchangeability of the particles in the system, indeed the evolution in (1.2.1) is invariant under all the possible permutations of indexes. A classical reference for the description of propagation of chaos and some particular examples are the lecture notes from A.S. Sznitman [83], from which we take the following definitions.

We rigorously define propagation of chaos by means of the following definition of ν-chaotic sequence of measures, for a certain measure ν.

Definition 1.2.1. Let E be a separable metric space and, for all $N \geqslant 1$, let ν^N be a sequence of symmetric probability measures on E^N. We say that ν^N *is ν-chaotic*, for a measure ν on E if for any $k \geqslant 1$ and any $\phi_1, \dots, \phi_k \in C_b(E)$

$$\lim_{N\to\infty} \langle \nu^N, \phi_1 \otimes \cdots \otimes \phi_k \otimes 1 \otimes \dots \rangle = \prod_{i=1}^{k} \langle \nu, \phi_i \rangle.$$

Chaoticity is often considered as a sort of *law of large numbers*, the reason is explained in the following proposition, proved in Sznitman [83].

Proposition 1.2.1. ν^N *is* ν*-chaotic is equivalent to*

$$\mu_X^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i} \overset{\text{in law}}{\longrightarrow} \delta_\nu,$$

where μ_X^N *is a random variable with values in* $\mathcal{M}(E)$ *(the space of probability measures on* E*),* X_i *indicates the canonical coordinates on* E^N *and* δ_ν *indicates the constant random variable* ν *in* $\mathcal{M}(E)$*.*

In our case, we fix an arbitrary time horizon $T > 0$, and, for every $N \geqslant 1$, we denote by $X^N[0,T] = (X^N(t))_{t\in[0,T]}$ the random path of the solution to (1.2.1), up to time T. $X^N[0,T]$ has law P_T^N on $\mathbf{D}([0,T],\mathbb{R}^d)^N$, i.e. the product of N times the Skorokhod space of *càdlàg* functions. At the same time we denote $X[0,T] = (X(t))_{t\in[0,T]}$ the random path of the solution to (1.2.3), then $X[0,T]$ has a law Q_T on $\mathbf{D}([0,T],\mathbb{R}^d)$.

Definition 1.2.2. For every $N \geqslant 1$, let P^N be the law of the solution of a particle system on $\mathbf{D}(\mathbb{R}^+,\mathbb{R}^d)^N$. We say that *propagation of chaos* holds if, whenever the sequence of initial conditions P_0^N is Q_0-chaotic, for a certain measure Q_0 on $\mathbb{R}^d$, then for all $T \geqslant 0$ the sequence of laws P_T^N is Q_T-chaotic, where Q_T is a law on $\mathbf{D}([0,T],\mathbb{R}^d)^N$ with initial condition Q_0.

The approaches to prove propagation of chaos are essentially two, described in the following.

i) There is a *three-steps* approach. First, the *tightness* of the sequence of empirical measures μ_X^N is proved. The second step consists in proving *consistency* of their limit points, i.e. the limit point of every convergent subsequence belongs to the set of measures that solves the nonlinear limit. Lastly, the *uniqueness* of the measure solving the nonlinear limit is proved, ensuring that the limit of the sequence μ_X^N is deterministic. This method is extremely flexible, it can be used under weak hypothesis on the coefficient but it does not provide any rate of convergence.

ii) An alternative approach consists in proving *pathwise propagation of chaos*, by means of a coupling between the particle system and N independent copies of the limit process. This approach gives a (usually optimal) rate of convergence, but it is less flexible than the previous one. It is mainly used when coefficients satisfy Lipschitz conditions, but it works also under some particular non-Lipschitz conditions.

In Chapter 2 we are interested in getting results on the model with simultaneous jumps with the second approach. However, in the rest of the thesis, we will see the use of both the approaches.

1.2.4 The intermediate process

As we mentioned, we are interested in proving pathwise propagation of chaos, with the aim of getting the rate of convergence due to the *simultaneous jumps*. The general strategy of proof involves the introduction of an *intermediate process* $Y^N = (Y^N(t))_{t\in[0,T]}$ with values in $\mathbb{R}^{d\times N}$. This Markov process Y^N can be given as the solution of the SDE

$$\begin{aligned} dY_i^N(t) =& F(Y_i^N(t), \mu_Y^N(t))dt + \sigma(Y_i^N(t), \mu_i^N(t))dB_t^i \\ &+ \frac{1}{N}\sum_{j=1}^N \lambda(Y_j^N(t^-), \mu_Y^N(t^-)) \int_{[0,1]^2} \Theta(Y_j^N(t^-), Y_i^N(t^-), \mu_Y^N(t^-), h_1, h_2)\nu_2(dh_1, dh_2)dt \\ &+ \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \psi(Y_i^N(t^-), \mu_Y^N(t^-), h)\mathbb{1}_{(0,\lambda(Y_i^N(t^-),\mu_i^N(t^-))]}(u)\mathcal{N}^i(dt, du, dh), \end{aligned} \tag{1.2.4}$$

$i = 1, \dots, N$, where again B^i are independent d-dimensional Brownian motions and $\mathcal{N}^i$ are independent Poisson random measures with characteristic measure $l\times l\times\nu$. It is immediate to see that the process Y^N differs from the original process X^N in the jump terms; indeed, it does not have the collateral jumps anymore. Every particle still performs the main jump with rate given by the function λ, but this does not induce jumps in the other components. As in the macroscopic dynamics (1.2.3), the process Y^N has an additional drift term, that depends on the characteristic measure of $\{\mathcal{N}^i\}_{i\in\mathbb{N}}$ and on the empirical measure μ_Y^N, because, of course, the term in the second line may be rewritten as

$$\langle \mu_Y^N(t), \lambda(\cdot, \mu_Y^N(t^-)) \int_{[0,1]^2} \Theta(\cdot, Y_i^N(t^-), \mu_Y^N(t^-), h_1, h_2)\nu_2(dh_1, dh_2)\rangle.$$

Therefore, the intermediate process Y^N displays only classical mean field interaction terms and the proof of propagation of chaos is easier than for X^N. Furthermore, proving that the laws of the two processes X^N and Y^N get closer as N goes to infinity will help to quantify the role of the simultaneous jumps and the rate at which they tend to collapse into the drift term.

Let us briefly explain the coupling procedure that we will use in the following. We call it *basic coupling* and it is such that it maximizes the chance of two coupled particles to jump together. We use the *same* Brownian motions and the *same* Poisson random measures in (1.2.1) and in (1.2.4), such that the processes X^N and Y^N are *coupled*, i.e. they are realized on the same probability space: it will not be hard to give conditions for the L^1-convergence to zero of $X_1^N[0,T] - Y_1^N[0,T]$. Thus, the fact that the law of X^N is Q-chaotic will follow if one shows that the law of Y^N is Q-chaotic. Since Y^N has no simultaneous jumps, this can be obtained along the lines of the classical approaches. As we said above, the intermediate process has the nice feature of highlighting the role of simultaneous jumps in the rate of convergence in W_1 Wasserstein distance of the empirical measure. Indeed by comparing the empirical measures of X^N and Y^N, we obtain that, under our assumptions, the rate of convergence due to the simultaneous jumps is of the order $\frac{1}{\sqrt{N}}$, while the final rate obviously depends on the moments of initial conditions and of the process itself, see [42].

Chapter 2

Pathwise propagation of chaos for simultaneous jumps

In Section 1.2 of the previous chapter, we present at a heuristic level a general framework for mean-field interacting particle systems with simultaneous jumps. In particular we describe what we mean when we say that a particle system has *simultaneous jumps* and we highlight the role of the microscopic, the intermediate and the macroscopic process. In this chapter, we formally prove pathwise propagation of chaos under various sets of assumptions.

2.1 Globally Lipschitz conditions on all coefficients

We start with the most natural among all the assumptions, i.e. classical Lipschitz conditions on all the coefficients. To state these conditions and the corresponding theorems, let us introduce a suitable metric on spaces of probability measures.

Definition 2.1.1 (W_p Wasserstein distance). For $p \geqslant 1$, let (M, d) be a metric space, we call $\mathcal{M}^p(M)$ be the space of probability on M with finite p^{th} moment:

$$\mathcal{M}^p(M) = \{\mu \in \mathcal{M}(M) : \int d(x, x_0)^p \mu(dx) < +\infty \text{ for some } x_0 \in M\}.$$

We equip this space with the W_p Wasserstein metric defined as follows: for all $\mu, \nu \in \mathcal{M}^p(M)$

$$W_p(\mu, \nu) \doteq \left[\inf\left\{\int_{M\times M} d(x, y)^p \pi(dx, dy);\ \pi \text{ has marginals } \mu \text{ and } \nu\right\}\right]^{1/p}.$$

Therefore, in our case, let $\mathcal{M}^1(\mathbb{R}^d)$ be the space of probability on $\mathbb{R}^d$ with finite first moment:

$$\mathcal{M}^1(\mathbb{R}^d) = \{\mu \in \mathcal{M}(\mathbb{R}^d) : \int \|x\|^1 \mu(dx) < +\infty\}.$$

This space is equipped with the W_1 Wasserstein metric, that by abuse of notation we indicate as follows:

$$\begin{aligned}\rho(\mu,\nu) &\doteq \inf\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\|x-y\|\pi(dx,dy);\ \pi \text{ has marginals } \mu \text{ and } \nu\right\}\\ &= \sup\left\{\langle g,\mu\rangle-\langle g,\nu\rangle : g\colon \mathbb{R}^d\to\mathbb{R},\ \|g(x)-g(y)\|\leqslant\|x-y\|\right\},\end{aligned}$$

where the equality in the latter row is called *Kantorovich-Rubinstein duality* and it characterized the W_p Wasserstein distance when $p=1$, see [87]. We also consider a subset of $\mathcal{M}\left(\mathbf{D}\left([0,T],\mathbb{R}^d\right)\right)$, the set of the probability measures on $\mathbf{D}\left([0,T],\mathbb{R}^d\right)$:

$$\mathcal{M}^1\left(\mathbf{D}\left([0,T],\mathbb{R}^d\right)\right)\doteq\left\{\alpha\in\mathcal{M}\left(\mathbf{D}\left([0,T],\mathbb{R}^d\right)\right):\int_{\mathbf{D}}\sup_{t\in[0,T]}\|x(t)\|\alpha(dx)<+\infty\right\},$$

and provide it with the W_1 Wasserstein metric

$$\rho_T(\alpha,\beta)\doteq\inf\left\{\int_{\mathbf{D}\times\mathbf{D}}\sup_{t\in[0,T]}\|x(t)-y(t)\|P(dx,dy);\ \text{ where } P \text{ has marginals } \alpha \text{ and } \beta\right\}.$$

In what follows, we shall adopt a notion of chaoticity which is stronger than the one of Chapter 1.

Definition 2.1.2. Let $X^N=(X_1^N,X_2^N,\dots,X_N^N)$ be a sequence of random vectors with components $X_i^N\in\mathbb{R}^d$ (resp. $X_i^N\in\mathbf{D}\left([0,T],\mathbb{R}^d\right)$). For $\mu\in\mathcal{M}^1(\mathbb{R}^d)$ (resp. $\mu\in\mathcal{M}^1\left(\mathbf{D}\left([0,T],\mathbb{R}^d\right)\right)$), we say that X^N is μ-*chaotic in* W_1 if its distribution is permutation invariant and, for each $k\in\mathbb{N}$, the law of the vector $(X_1^N,X_2^N,\dots,X_k^N)$ converges to $\mu^{\otimes k}$ with respect to the metric ρ (resp. ρ_T).

Notice that in Definition 1.2.2 we consider weak convergence of the joint law of k components, while in Definition 2.1.2 we consider convergence w.r.t. the metric ρ (or, respectively ρ_T), that gives weak convergence together with the convergence of the first moment.

2.1.1 Assumptions and well-posedness of the SDEs

In the following we list the assumptions on the functions that we informally introduced in Section 1.2. In the L^1-framework that we chose, these are very natural Lipschitz type assumptions and they have the advantage that the proof of well-posedness of the involved SDEs ((1.2.1),(1.2.3) and (1.2.4)) comes as a straightforward consequence of already well-known results on nonlinear diffusion with jumps, [48].

Assumption 2.1.1. *(Li1) The classical global Lipschitz assumption on* F *and* σ*:* $\exists\, L_F, L_\sigma>0$ *such that, for all* $x,y\in\mathbb{R}^d$*, all* $\alpha,\gamma\in\mathcal{M}^1(\mathbb{R}^d)$*,*

$$\begin{aligned}\|F(x,\alpha)-F(y,\gamma)\|&\leqslant L_F\left(\|x-y\|+\rho(\alpha,\gamma)\right),\\ \|\sigma(x,\alpha)-\sigma(y,\gamma)\|&\leqslant L_\sigma\left(\|x-y\|+\rho(\alpha,\gamma)\right).\end{aligned}$$

(Li2) The L^1*-Lipschitz assumption on the jump coefficients:* $\exists\, L_\psi, L_\Theta > 0$ *such that, for all* $x, y \in \mathbb{R}^d$*, all* $\alpha, \gamma \in \mathcal{M}^1(\mathbb{R}^d)$*,*

$$\int_{[0,\infty)\times[0,1]} \|\psi(x,\alpha,h)\mathbb{1}_{(0,\lambda(x,\alpha)]}(u) - \psi(y,\gamma,h)\mathbb{1}_{(0,\lambda(y,\gamma)]}(u)\| du\nu_1(dh) \leqslant L_\psi\left(\|x-y\| + \rho(\alpha,\gamma)\right)$$

and

$$\begin{aligned}\|\langle\alpha, \lambda(\cdot,\alpha)\int_{[0,1]^2}\Theta(\cdot,x,\alpha,h_1,h_2)\nu_2(dh_1,dh_2)\rangle - \langle\gamma,\lambda(\cdot,\gamma)\int_{[0,1]^2}\Theta(\cdot,y,\gamma,h_1,h_2)\nu_2(dh_1,dh_2)\rangle\| \\ \leqslant L_\Theta\left(\|x-y\| + \rho(\alpha,\gamma)\right).\end{aligned}$$

(I1) The integrability condition: for all $N \in \mathbb{N}$*, for all* $\mathbf{x} \in \mathbb{R}^{d\times N}$ *and all* $\alpha \in \mathcal{M}^1(\mathbb{R}^d)$

$$\int_{[0,1]^N} \left\|\Delta_i^N(\mathbf{x},\alpha,h^N)\right\| \nu_N(dh^N) < \infty.$$

(I2) The square-integrability condition on the collateral jumps: for all $x, y \in \mathbb{R}^d$ *and all* $\alpha \in \mathcal{M}^1(\mathbb{R}^d)$

$$\int_{[0,\infty)\times[0,1]^N} \|\Theta(x,y,\alpha,h_1,h_2)\mathbb{1}_{(0,\lambda(x,\alpha)]}(u)\|^2 du\nu_2(dh) < \infty.$$

In the following, we set $L \doteq L_F \vee L_\sigma \vee L_\psi \vee L_\Theta$.

Existence and uniqueness of a square-integrable strong solution of (1.2.1) and (1.2.4) starting from a vector of square-integrable initial conditions $\left(X_1^N(0), \ldots, X_N^N(0)\right)$, independent of the family $(B_i, \mathcal{N}^i)_{i\in\mathbb{N}}$, are ensured by Assumption 2.1.1; see [48, Theorem 1.2]. The same assumptions also guarantee existence and uniqueness of a strong solution of (1.2.3) starting from any square-integrable initial condition $X(0)$; see [48, Theorem 2.1]. We want to highlight that condition *(I2)* on the collateral jumps is necessary only for the proof of propagation of chaos, while it is not needed for well-posedness purposes.

2.1.2 Propagation of chaos

The first step consists in proving the closeness between the original particle system X^N and the intermediate process Y^N. We couple them by means of the basic coupling described in Section 1.2.4. Then the following proposition hold.

Proposition 2.1.1. *Grant Assumptions 2.1.1. Let* X^N *and* Y^N *be the solutions of* (1.2.1) *and* (1.2.4)*, respectively. We assume the two processes are driven by the same Brownian motions and Poisson random measures, and start from the same square-integrable and* *permutation invariant initial condition. Then there exists a constant* $C_T > 0$ *such that, for each fixed* $i \in \mathbb{N}$*, for all* $N \geqslant 1$

$$\mathbf{E}\left[\sup_{t\in[0,T]} \|X_i^N(t) - Y_i^N(t)\|\right] \leqslant \frac{C_T}{\sqrt{N}}. \tag{2.1.1}$$

Proof. To simplify notation, we adopt the following abbreviations:

$$\begin{aligned}
\Theta_{i,j}(X^N(s^-), h) &\doteq \Theta(X_i^N(s^-), X_j^N(s^-), \mu_X^N(s^-), h_i, h_j),\\
\lambda_i(X^N(s^-)) &\doteq \lambda(X_i^N(s^-), \mu_X^N(s^-)),\\
\psi_i(X^N(s^-), h) &\doteq \psi(X_i^N(s^-), \mu_X^N(s^-), h_i),\\
U &\doteq [0,\infty) \times [0,1]^{\mathbb{N}}.
\end{aligned}$$

By permutation invariance of both the initial condition and the dynamics, we have, for every $t \in [0,T]$,

$$\mathbf{E}\left[\sup_{s\in[0,t]} \|X_i^N(s) - Y_i^N(s)\|\right] = \frac{1}{N}\sum_{j=1}^N \mathbf{E}\left[\sup_{s\in[0,t]} \|X_j^N(s) - Y_j^N(s)\|\right].$$

By the same reason we also have a coupling bound for the W_1 distance of the empirical measures of the two systems, i.e. for any $t \geqslant 0$

$$\mathbf{E}\left[\rho\left(\mu_X^N(t), \mu_Y^N(t)\right)\right] \leqslant \frac{1}{N}\sum_{j=1}^N \mathbf{E}\left[\|X_j^N(t) - Y_j^N(t)\|\right].$$

Fix $t \in [0,T]$, and set

$$\begin{aligned}
F_i &\doteq \mathbf{E}\left[\int_0^t \|F(X_i^N(s), \mu_X^N(s)) - F(Y_i^N(s), \mu_Y^N(s))\|ds\right],\\
\sigma_i &\doteq \mathbf{E}\left[\sup_{r\in[0,t]} \left\|\int_0^r \left(\sigma(X_i^N(s), \mu_X^N(s)) - \sigma(Y_i^N(s), \bar{\mu}_Y^N(s))\right) dB_s^i\right\|\right],\\
\Theta_i &\doteq \mathbf{E}\left[\sup_{r\in[0,t]} \left\|\frac{1}{N}\sum_{j\neq i}\int_{[0,r]\times U} \Theta_{j,i}(X^N(s^-), h)\mathbb{1}_{(0,\lambda_j(X^N(s^-)))]}(u)\mathcal{N}^j(ds, du, dh)\right.\right.\\
&\qquad \left.\left. -\frac{1}{N}\sum_{j=0}^N \int_{[0,t]\times U} \Theta_{j,i}(Y^N(s^-), h)\mathbb{1}_{(0,\lambda_j(Y^N(s^-)))]}(u)ds\, du\nu(dh)\right\|\right],\\
\psi_i &\doteq \mathbf{E}\left[\sup_{r\in[0,t]} \left\|\int_{[0,r]\times U} \psi_i(X^N(s^-), h)\mathbb{1}_{(0,\lambda_i(X^N(s^-)))]}(u)\mathcal{N}^i(ds, du, dh)\right.\right.\\
&\qquad \left.\left. -\int_{[0,r]\times U} \psi_i(Y^N(s^-), h)\mathbb{1}_{(0,\lambda_i(Y^N(s^-)))]}(u)\mathcal{N}^i(ds, du, dh)\right\|\right].
\end{aligned}$$

Note that all these quantities do not depend on i, that is therefore omitted in what follows. Then

$$\mathbf{E}\left[\sup_{s\in[0,t]}\|X_i^N(s)-Y_i^N(s)\|\right]\leqslant F+\sigma+\Theta+\psi. \tag{2.1.2}$$

The term F can be easily bounded thanks to the Lipschitz condition (L1) and the coupling bound for the W_1 Wasserstein metric, and we obtain

$$F\leqslant L\int_0^t\mathbf{E}\left[\|X_i^N(s)-Y_i^N(s)\|\right]ds+\frac{L}{N}\sum_{j=1}^N\int_0^t\mathbf{E}\left[\|X_j^N(s)-Y_j^N(s)\|\right]ds.$$

The bound on σ, besides (Li1), involves the Burkholder-Davis-Gundy inequality, and we get, for some constant M not depending on N nor t,

$$\begin{aligned}\sigma&\leqslant M\,\mathbf{E}\left[\left(\int_0^t\left(\|X_i^N(s)-Y_i^N(s)\|+\frac{1}{N}\sum_{j=1}^N\|X_j^N(s)-Y_j^N(s)\|\right)^2ds\right)^{1/2}\right]\\&\leqslant M\sqrt{t}\,\mathbf{E}\left[\sup_{s\in[0,t]}\|X_i^N(s)-Y_i^N(s)\|+\frac{1}{N}\sum_{j=1}^N\sup_{s\in[0,t]}\|X_j^N(s)-Y_j^N(s)\|\right].\end{aligned}$$

The term Θ needs to be treated again with the Burkholder-Davis-Gundy inequality. In what follows, we denote by $\tilde{\mathcal{N}}^i$ the compensated Poisson measure associated to $\mathcal{N}^i$ and it is crucial the fact that $\{\tilde{\mathcal{N}}^i\}_{i=1,\dots,N}$ is a family of orthogonal martingales. First we compensate the Poisson measures and we get

$$\begin{aligned}\Theta&\leqslant\mathbf{E}\left[\sup_{r\in[0,t]}\left\|\frac{1}{N}\sum_{j\neq i}\int_{[0,r]\times U}\Theta_{j,i}(X^N(s^-),h)\mathbb{1}_{[0,\lambda_j(X^N(s^-)))}\tilde{\mathcal{N}}^j(ds,du,dh)\right\|\right]\\&+\mathbf{E}\left[\sup_{r\in[0,t]}\left\|\frac{1}{N}\sum_{j=1}^N\int_{[0,r]\times U}\Theta_{j,i}(X^N(s^-),h)\mathbb{1}_{(0,\lambda_j(X^N(s^-))]}-\Theta_{j,i}(Y^N(s^-),h)\mathbb{1}_{(0,\lambda_j(Y^N(s^-))]}ds\,du\nu(dh)\right\|\right]\\&+\frac{1}{N}\mathbf{E}\left[\sup_{r\in[0,t]}\left\|\int_{[0,r]\times U}\Theta_{i,i}(X^N(s^-),h)\mathbb{1}_{(0,\lambda_i(X^N(s^-))]}ds\,du\,\nu(dh)\right\|\right].\end{aligned}$$

The first term involves a sum of integrals w.r.t. orthogonal martingales and it is treated with Burkholder-Davis-Gundy inequality. Therefore, for a certain constant $K>0$, the constant $L>0$ coming from condition (Li2) and a constant $C>0$ not depending on N nor

t, we have

$$
\begin{aligned}
\Theta \leqslant & \frac{K}{N}\mathbf{E}\left[\left(\sum_{j\neq i}\int_0^t\int_U \left\|\Theta_{j,i}(X^N(s^-),h)\mathbb{1}_{(0,\lambda_j(X^N(s^-))]}(u)\right\|^2 ds\,du\nu(dh)\right)^{1/2}\right]\\
&+\int_0^t \mathbf{E}\left[\left\|\left\langle \mu_s^N, \int_U \Theta_{\cdot,i}(X^N(s^-),h)\mathbb{1}_{[0,\lambda_\cdot(X^N(s^-)))}(u)\,du\nu(dh)\right\rangle\right.\right.\\
&\qquad\left.\left. -\left\langle \bar{\mu}_s^N, \int_U \Theta_{\cdot,i}(Y^N(s^-),h)\mathbb{1}_{[0,\lambda_\cdot(Y^N(s^-)))}(u)\,du\nu(dh)\right\rangle\right\|\right] ds\\
&+\frac{1}{N}\mathbf{E}\left[\int_0^t\int_U \left\|\Theta_{i,i}(X^N(s^-),h)\mathbb{1}_{[0,\lambda_i(X^N(s^-)))}(u)\right\| du\nu(dh)ds\right]\\
\leqslant & \frac{C}{\sqrt{N}} + L\int_0^t \mathbf{E}\left[\|X_i^N(s)-Y_i^N(s)\|\right] ds + \frac{L}{N}\sum_{j=1}^N\int_0^t \mathbf{E}\left[\|X_j^N(s)-Y_j^N(s)\|\right] ds + \frac{C}{N}.
\end{aligned}
$$

The term ψ concerns the main jumps of the particle system and it is bounded by the positivity property of Poisson processes and the Lipschitz condition (Li2):

$$
\begin{aligned}
\psi &\leqslant \mathbf{E}\left[\int_{[0,t]\times U}\left\|\psi_i(X^N(s^-),h)\mathbb{1}_{(0,\lambda_i(X^N(s^-))]}(u)-\psi_i(Y^N(s^-),h)\mathbb{1}_{(0,\lambda_i(Y^N(s^-))]}(u)\right\|\mathcal{N}^i(ds,du,dh)\right]\\
&= \mathbf{E}\left[\int_{[0,t]\times U}\left\|\psi_i(X^N(s^-),h)\mathbb{1}_{(0,\lambda_i(X^N(s^-))]}(u)-\psi_i(Y^N(s^-),h)\mathbb{1}_{(0,\lambda_i(Y^N(s^-))]}(u)\right\| ds\,du\nu(dh)\right]\\
&\leqslant L\int_0^t \mathbf{E}\left[\|X_i^N(s)-Y_i^N(s)\|\right] ds + \frac{L}{N}\sum_{j=1}^N\int_0^t \mathbf{E}\left[\|X_j^N(s)-Y_j^N(s)\|\right] ds.
\end{aligned}
$$

Therefore, recalling (2.1.2), we find that, for every $t\in[0,T]$,

$$
\begin{aligned}
&\mathbf{E}\left[\sup_{s\in[0,t]}\|X_i^N(s)-Y_i^N(s)\|\right]\\
&\quad\leqslant M\sqrt{t}\,\mathbf{E}\left[\sup_{s\in[0,t]}\|X_i^N(s)-Y_i^N(s)\|\right] + M\sqrt{t}\,\mathbf{E}\left[\frac{1}{N}\sum_{j=1}^N\sup_{s\in[0,t]}\|X_j^N(s)-Y_j^N(s)\|\right]\\
&\qquad + 3L\int_0^t \mathbf{E}\left[\|X_i^N(s)-Y_i^N(s)\|\right] ds + \frac{3L}{N}\sum_{j=1}^N\int_0^t \mathbf{E}\left[\|X_j^N(s)-Y_j^N(s)\|\right] ds + \frac{C}{N} + \frac{C}{\sqrt{N}}.
\end{aligned}
$$

Choose $T_0>0$ small enough that $(1-2M\sqrt{T_0})>0$. By summing over the index i in the above inequality and dividing both sides by N, we can move the first two terms on the right-hand side to the left, obtaining, for every $t\in[0,T_0]$,

$$
\begin{aligned}
\frac{1}{N}\sum_{i=1}^N\mathbf{E}\left[\sup_{s\in[0,t]}\|X_i^N(s)-Y_i^N(s)\|\right] \leqslant & \frac{6L}{1-2M\sqrt{t}}\int_0^t\frac{1}{N}\sum_{i=1}^N\mathbf{E}\left[\sup_{s\in[0,r]}\|X_i^N(s)-Y_i^N(s)\|\right] dr\\
&+\frac{C}{N(1-2M\sqrt{t})}+\frac{C}{\sqrt{N}(1-2M\sqrt{t})}.
\end{aligned}
$$

An application of Gronwall's lemma yields

$$\frac{1}{N}\sum_{i=1}^{N}\mathbf{E}\left[\sup_{t\in[0,T_0]}\|X_i^N(t)-Y_i^N(t)\|\right]\leqslant\frac{C_{T_0}}{\sqrt{N}} \tag{2.1.3}$$

for some finite constant C_{T_0} not depending on N. Recall that (2.1.3) holds on a time interval $[0,T_0]$ for T_0 sufficiently small. If T_0 is smaller than T, then we can repeat the procedure of estimates on the interval $[T_0,(2T_0)\wedge T]$. In this case, we find that, for every $t\in[T_0,(2T_0)\wedge T]$,

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}\mathbf{E}\left[\sup_{s\in[T_0,t]}\|X_i^N(s)-Y_i^N(s)\|\right]\leqslant{}&\frac{1}{1-2M\sqrt{t-T_0}}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{E}\left[\sup_{s\in[0,T_0]}\|X_i^N(s)-Y_i^N(s)\|\right]\right)\\&+\frac{6L}{1-2M\sqrt{t-T_0}}\int_{T_0}^{t}\frac{1}{N}\sum_{i=1}^{N}\mathbf{E}\left[\sup_{s\in[T_0,r]}\|X_i^N(s)-Y_i^N(s)\|\right]dr\\&+\frac{C}{N(1-2M\sqrt{t-T_0})}+\frac{C}{\sqrt{N}(1-2M\sqrt{t-T_0})},\end{aligned}$$

where the first term comes from a bound on the initial condition $\frac{1}{N}\sum_{i=1}^{N}\mathbf{E}\left[\|X_i^N(T_0)-Y_i^N(T_0)\|\right]$. Hence, again thanks to Gronwall's lemma, for some constant C_{2,T_0},

$$\frac{1}{N}\sum_{i=1}^{N}\mathbf{E}\left[\sup_{s\in[0,(2T_0)\wedge T]}\|X_i^N(s)-Y_i^N(s)\|\right]\leqslant\frac{C_{2,T_0}}{\sqrt{N}}.$$

We proceed by induction until we cover, after finitely many steps, the entire interval $[0,T]$. By exchangeability of the laws of both the initial and the intermediate process, this yields, for $i=1,\dots,N$

$$\mathbf{E}\left[\sup_{s\in[0,T]}\|X_i^N(s)-Y_i^N(s)\|\right]\leqslant\frac{C_T}{\sqrt{N}}$$

and (2.1.1) holds. □

In the next, we use a similar coupling technique and we show propagation of chaos for Y^N. In this case, for all N, we couple the process Y^N with N independent copies of the process X, solution of (1.2.3).

Proposition 2.1.2. *Grant Assumptions 2.1.1. Let μ_0 be a probability measure on $\mathbb{R}^d$ such that $\int\|x\|^2\mu_0(dx)<+\infty$. For $N\in\mathbb{N}$, let Y^N be a solution of Eq.* (1.2.4) *in $[0,T]$. Assume that $Y^N(0)=(Y_1^N(0),\dots,Y_N^N(0))$, $N\in\mathbb{N}$, form a sequence of square integrable random vectors that is μ_0-chaotic in W_1. Let μ be the law of the solution of Eq.* (1.2.3) *in $[0,T]$ with initial law $\mathbf{P}\circ X(0)^{-1}=\mu_0$. Then Y^N is μ-chaotic in W_1.*

Proof. In order to get the thesis, we set a coupling procedure. Let the processes Y_i^N, $N\in\mathbb{N}$, $i\in\{1,\dots,N\}$ be all defined on the filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geqslant0},\mathbf{P})$ with respect to the family $(B_i,\mathcal{N}^i)_{i\in\mathbb{N}}$ of Brownian motions and Poisson random measures. Since

$(Y^N(0))$ is μ_0-chaotic in W_1 by hypothesis, we assume, as we may, that our stochastic basis carries a triangular array $(\bar{X}_i^N(0))_{i\in\{1,\dots,N\},N\in\mathbb{N}}$ of identically distributed $\mathbb{R}^d$-valued random variables with common distribution μ_0 such that $(\bar{X}_i^N(0))_{i\in\{1,\dots,N\},N\in\mathbb{N}}$ and $(B_i, \mathcal{N}^i)_{i\in\mathbb{N}}$ are independent, the sequence $(\bar{X}_i^N(0))_{i\in\{1,\dots,N\}}$ is independent for each N, and

$$\phi^N \doteq \mathbf{E}\left[\|\bar{X}_i^N(0) - Y_i^N(0)\|\right]$$

tends to zero as $N \to \infty$. For $N \in \mathbb{N}$, $i \in \{1,\dots,N\}$, let $\bar{X}_i^N$ be the unique strong solution of Eq. (1.2.3) in $[0,T]$ with initial condition $X_i^N(0)$, driving Brownian motion B_i and Poisson random measure $\mathcal{N}^i$. Notice that the processes $X_1^N,\dots,X_N^N$ are independent and identically distributed for each N.

Because of the exchangeability of the system (1.2.4), the μ-chaoticity in W_1 of the sequence Y^N is equivalent to

$$\lim_{N\to\infty} \mathbb{E}\left[\rho_T(\mu_Y^N, \mu)\right] = 0.$$

Moreover, by definition of the metric ρ_T, this follows from

$$\lim_{N\to\infty} \mathbf{E}\left[\sup_{t\in[0,T]} \|\bar{X}_i^N(t) - Y_i^N(t)\|\right] = 0, \tag{2.1.4}$$

for every fixed $i \in \mathbb{N}$. However, the limit is the same by exchangeability of components. The term in (2.1.4) is bounded by

$$\mathbf{E}\left[\sup_{t\in[0,T]} \|Y_i^N(t) - \bar{X}_i^N(t)\|\right] \leqslant \phi^N + \bar{F} + \bar{\sigma} + \bar{\Theta} + \bar{\psi}, \tag{2.1.5}$$

where

$$\begin{aligned}
\bar{F} &\doteq \mathbf{E}\left[\int_0^T \|F(Y_i^N(s), \mu_Y^N(s)) - F(\bar{X}_i^N(s), \mu_s)\| ds\right],\\
\bar{\sigma} &\doteq \mathbf{E}\left[\sup_{t\in[0,T]} \left\|\int_0^t \sigma(Y_i^N(s), \mu_Y^N(s)) - \sigma(\bar{X}_i^N(s), \mu_s) dB_s^i\right\|\right],\\
\bar{\Theta} &\doteq \mathbf{E}\left[\sup_{t\in[0,T]} \left\|\int_0^t \left\langle \mu_Y^N(s), \int_U \Theta(\cdot, Y_i^N(s), \mu_Y^N(s), h)\mathbb{1}_{(0,\lambda_j(\cdot,\mu_Y^N(s))]}(u)\, du\nu_2(dh)\right\rangle ds\right.\right.\\
&\qquad \left.\left. - \int_0^t \int_U \langle \mu_s, \Theta(\cdot, \bar{X}_i^N(s), \mu_s, h)\mathbb{1}_{(0,\lambda_j(\cdot,\mu_s)]}(u)\, du\nu(dh)\rangle\, ds\right\|\right],\\
\bar{\psi} &\doteq \mathbf{E}\left[\sup_{t\in[0,T]} \left\|\int_{[0,t]\times U} \psi(Y_i^N(s^-), \mu_Y^N(s), h)\mathbb{1}_{(0,\lambda(Y_i^N(s^-),\mu_Y^N(s^-))]}(u)\right.\right.\\
&\qquad \left.\left. -\psi(\bar{X}_i^N(s^-), \mu_{s^-}, h)\mathbb{1}_{(0,\lambda_i(\bar{X}_i^N(s^-),\mu_{s^-})]}(u)\mathcal{N}^i(dt, du, dh)\right\|\right].
\end{aligned}$$

The terms $\bar{F}$, $\bar{\sigma}$, and $\bar{\psi}$ are treated exactly as in Proposition 2.1.1, whereas the term $\bar{\Theta}$ only requires the application of the Lipschitz condition (Li2). By mimicking the steps in

Proposition 2.1.1, there exists a $T_0 > 0$ small enough and a constant $C_{T_0} \geqslant 0$, independent of N, such that we can apply Gronwall's Lemma and obtain

$$\mathbf{E}\left[\sup_{t\in[0,T_0]} \|Y_i^N(t) - \bar{X}_i^N(t)\|\right] \leqslant C_{T_0}\left(\int_0^{T_0} \mathbf{E}\left[\rho(\mu_Y^N(t), \mu_t)\right] dt + \sqrt{\int_0^{T_0} \mathbf{E}[\rho(\mu_Y^N(t), \mu_t)^2] dt + \varphi^N}\right).$$

By triangular inequality, for every fixed $t \in [0, T_0]$,

$$\begin{aligned}\mathbf{E}\left[\rho(\mu_Y^N(t), \mu_t)\right] &\leqslant \mathbf{E}\left[\rho(\mu_Y^N(t), \mu_{\bar{X}}^N(t))\right] + \mathbf{E}\left[\rho(\mu_{\bar{X}}^N(t), \mu_t)\right] \\ &\leqslant \mathbf{E}\left[\sup_{t\in[0,T_0]} \|Y_i^N(t) - \bar{X}_i^N(t)\|\right] + \mathbf{E}\left[\rho(\mu_{\bar{X}}^N(t), \mu_t)\right].\end{aligned}$$

Then, for a T_0 sufficiently small, using again Gronwall Lemma, there exists a positive constant, depending on T_0, that by abuse of notation we will indicate it again with $C_{T_0} > 0$, such that

$$\mathbf{E}\left[\sup_{t\in[0,T_0]} \|Y_i^N(t) - \bar{X}_i^N(t)\|\right] \leqslant C_{T_0}\left(\int_0^{T_0} \mathbf{E}\left[\rho(\mu_{\bar{X}}^N(t), \mu_t)\right] dt + \sqrt{\int_0^{T_0} \mathbf{E}\left[\rho(\mu_{\bar{X}}^N(t), \mu_t)^2\right] dt + \varphi^N}\right).$$

We see that the bound on (2.1.5) depends on the initial conditions and on $\mathbf{E}\left[\rho(\mu_{\bar{X}}^N(t), \mu_t)\right]$, that is the distance, at every fixed time $t \in [0, T]$, between the empirical measure of N i.i.d. copies of the solution of the process with law μ and the law μ_t itself. The rate of convergence of empirical measures in Wasserstein distance depends on the moments of $\bar{X}(t)$ and on the dimension d, see [42, Theorem 1]. Since

$$\sup_{t\in[0,T]} \mathbf{E}\left[\bar{X}_i^2(t)\right] < +\infty,$$

it follows from [42] that, setting

$$\beta^N \doteq \sup_{t\in[0,T]} \mathbf{E}[\rho(\mu_{\bar{X}}^N(t), \mu_t)],$$

we have

$$\lim_{N\to\infty} \beta^N = 0.$$

Therefore, we know that there exists a constant $C_{T_0} > 0$ such that, for N going to infinity, we have

$$\mathbf{E}\left[\sup_{t\in[0,T_0]} \|Y_i^N(t) - \bar{X}_i^N(t)\|\right] \leqslant C_{T_0}\left(\beta^N + \varphi^N\right).$$

Iterating this procedure as in Proposition 2.1.1, we extend the above result to $[0, T]$, i.e.

$$\mathbf{E}\left[\sup_{t\in[0,T]} \|Y_i^N(t) - \bar{X}_i^N(t)\|\right] \leqslant C_T\left(\beta^N + \varphi^N\right).$$

for a suitable constant C_T. This establishes μ-chaoticity of Y^N in W_1.

□

The property of propagation of chaos for the process X^N comes as a corollary of the two propositions above.

Corollary 2.1.1. *Grant Assumptions 2.1.1. Let μ_0 be a probability measure on $\mathbb{R}^d$ such that $\int \|x\|^2 \mu_0(dx) < +\infty$. For $N \in \mathbb{N}$, let X^N be a solution of Eq.* (1.2.1) *in* $[0, T]$. *Assume that $X^N(0) = (X_1^N(0), \dots, X_N^N(0))$, $N \in \mathbb{N}$, form a sequence of square integrable random vectors that is μ_0-chaotic in W_1. Let μ be the law of the solution of Eq.* (1.2.3) *in* $[0, T]$ *with initial law $\mathbf{P} \circ X(0)^{-1} = \mu_0$. Then X^N is μ-chaotic in W_1.*

Proof. As we said, the propagation of chaos is equivalent to the proof of

$$\lim_{N \to \infty} \mathbb{E}[\rho_T(\mu_X^N, \mu)] = 0.$$

By triangular inequality, it is clear that, with the notations of the previous proofs,

$$\begin{aligned}
\mathbb{E}[\rho_T(\mu_X^N, \mu)] &\leqslant \mathbb{E}[\rho_T(\mu_X^N, \mu_Y^N)] + \mathbb{E}[\rho_T(\mu_Y^N, \mu)] \\
&\leqslant C_T \left(\frac{1}{\sqrt{N}} + \beta^N + \phi^N \right).
\end{aligned}$$

The claim follows by Proposition 2.1.1 and 2.1.2. □

The proof of Proposition 2.1.1 by means of the coupling procedure let us identify the rate at which

$$\mathbb{E}\left[\rho_T(\mu_Y^N, \mu)\right]$$

goes to zero, as N goes to infinity. Indeed, by the results in [13, 42], we know the rate of convergence in Wasserstein distance of the empirical measure mainly depends on the moments of the measure and on the dimension. In general, we can say that the best possible rate we can get is

$$\beta^N = O\left(\frac{1}{\sqrt{N}}\right).$$

Depending on the dimension, in [42] we see that $\beta^N = O\left(\frac{1}{N^{-1/d}}\right)$ except possibly for dimensions $d = 1,2$, where some logarithmic corrections may appear. The same happens for ϕ^N, but note that if the components of the initial condition are i.i.d., then $\phi^N = 0$. In Proposition 2.1.1 we highlight that, in any situation, the simultaneous jumps in the form presented here, do not worsen the rate of convergence due to any other mean-field interaction, since they add a term of order $\frac{1}{\sqrt{N}}$.

2.2 Non-globally Lipschitz drift

In Section 2.1, we develop the coupling procedure and the computations under the most classical assumptions on coefficients. Now we aim to extend this approach to a wider class of processes and to prove pathwise propagation of chaos as well. Unfortunately, results on nonlinear diffusions with jumps are not so common in literature, in particular with unbounded jump rates. Therefore, in the two following sections, we need to prove, besides the equivalent of Proposition 2.1.1, 2.1.2 and Corollary 2.1.1, also the well-posedness of the nonlinear process (1.2.3).

Let us start by relaxing the Lipschitz assumption on the drift, allowing as drift terms the gradients of general convex potentials. This includes relevant examples as those appeared in [28] and [45], extending them to the case with jumps.

2.2.1 Assumptions and well-posedness of the particle systems

The structure of this section recalls the one of Section 2.1.1, in the sense that we list the assumptions on the coefficients, that differ only in the condition of the drift function F.

Assumption 2.2.1. *(U) The drift coefficient* $F\colon \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^d$ *is of the form*

$$F(x, \alpha) = -\nabla U(x) + b(x, \alpha),$$

for all $x \in \mathbb{R}^d$ *and all* $\alpha \in \mathcal{M}(\mathbb{R}^d)$*, where* U *is convex and* $\mathcal{C}^1$*. The function* b *is assumed to be globally Lipschitz in both variables:* $\exists\ L_b > 0$ *such that, for all* $x, y \in \mathbb{R}^d$*, all* $\alpha, \gamma \in \mathcal{M}^1(\mathbb{R}^d)$*,*

$$\|b(x, \alpha) - b(y, \gamma)\| \leqslant L_b \left(\|x - y\| + \rho(\alpha, \gamma)\right).$$

(Li1) The classical global Lipschitz assumption on σ*:* $\exists\ L_\sigma > 0$ *such that, for all* $x, y \in \mathbb{R}^d$*, all* $\alpha, \gamma \in \mathcal{M}^1(\mathbb{R}^d)$*,*

$$\|\sigma(x, \alpha) - \sigma(y, \gamma)\| \leqslant L_\sigma \left(\|x - y\| + \rho(\alpha, \gamma)\right).$$

(Li2) The L^1*-Lipschitz assumption on the jump coefficients:* $\exists\ L_\psi, L_\Theta > 0$ *such that, for all* $x, y \in \mathbb{R}^d$*, all* $\alpha, \gamma \in \mathcal{M}^1(\mathbb{R}^d)$*,*

$$\int_{[0,\infty)\times[0,1]} \|\psi(x, \alpha, h)\mathbb{1}_{(0,\lambda(x,\alpha)]}(u) - \psi(y, \gamma, h)\mathbb{1}_{(0,\lambda(y,\gamma)]}(u)\| du\nu_1(dh) \leqslant L_\psi \left(\|x - y\| + \rho(\alpha, \gamma)\right)$$

and

$$\begin{aligned}\|\langle \alpha, \lambda(\cdot, \alpha)\int_{[0,1]^2} \Theta(\cdot, x, \alpha, h_1, h_2)\nu_2(dh_1, dh_2)\rangle - \langle \gamma, \lambda(\cdot, \gamma)\int_{[0,1]^2} \Theta(\cdot, y, \gamma, h_1, h_2)\nu_2(dh_1, dh_2)\rangle\| \\ \leqslant L_\Theta \left(\|x - y\| + \rho(\alpha, \gamma)\right).\end{aligned}$$

(I1) The integrability condition: for all $N \in \mathbb{N}$*, for all* $\mathbf{x} \in \mathbb{R}^{d\times N}$ *and all* $\alpha \in \mathcal{M}^1(\mathbb{R}^d)$

$$\int_{[0,1]^N} \left\|\Delta_i^N(\mathbf{x}, \alpha, h^N)\right\| \nu_N(dh^N) < \infty.$$

(I2) The square-integrability condition on the collateral jumps*: for all* $x, y \in \mathbb{R}^d$ *and all* $\alpha \in \mathcal{M}^1(\mathbb{R}^d)$

$$\int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \|\Theta(x, y, \alpha, h_1, h_2)\mathbb{1}_{(0,\lambda(x,\alpha))]}(u)\|^2 du\nu_2(dh) < \infty.$$

In the following, we set $L \doteq L_b \vee L_\sigma \vee L_\psi \vee L_\Theta$.

Condition (U) is a natural choice when one wants to relax globally Lipschitz conditions on coefficients. It induces a process whose trajectories are strongly constrained by the convex potential. This attracting drift, even when combined with an unbounded jump rate, should prevent the process from exploding in finite time. We will see that this is what happens provided the jump rate is in some way "controllable", as it is under the Lipschitz assumption $(\mathsf{Li2})$.

We could not find a general result on SDE with unbounded jump's rate and a non globally Lipschitz condition on the drift coefficient, that could ensure existence and uniqueness of solutions to (1.2.1) and (1.2.4) under Assumption 2.2.1. Therefore, we prove some techinical lemmas, that are gathered in Section 2.2.4.

2.2.2 Well-posedness of the McKean-Vlasov SDE

We mentioned that it is not easy to find results on SDE where the rate of jump is unbounded and it is certainly much harder to find such results in the framework of nonlinear SDE. Indeed, the stopping time procedure that we use in the proof of Lemma 2.2.1 is not suitable, since the coefficients depend on the law of the process itself. Thus, in this section, we give a specific proof of well-posedness for nonlinear SDE belonging to this class.

Consider the stochastic differential equation

$$\begin{aligned} dX(t) =& F(X(t), \mu_t)dt + \sigma(X(t), \mu_t)dB_t \\ &+ \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \psi(X(t^-), \mu_{t^-}, h_1)\mathbb{1}_{(0,\lambda(X(t^-),\mu_{t^-})]}(u)\mathcal{N}(dt, du, dh), \end{aligned} \tag{2.2.1}$$

where $\mu_t = \mathrm{Law}(X(t))$, B is a d_1-dimensional Brownian motion and $\mathcal{N}$ is a stationary Poisson random measure with characteristic measure $l \times l \times \nu$.

Theorem 2.2.1. *Let the coefficients of the nonlinear SDE* (2.2.1) *satisfy Assumption 2.2.1. Then for all square integrable initial conditions* $X(0) \in \mathbb{R}^d$*, Eq.* (2.2.1) *admits a unique strong solution.*

Proof. Let P^1 and P^2 two laws on $\mathbf{D}([0,T],\mathbb{R}^d)$ and suppose that X^1 and X^2 are two solutions of the following SDE, for $k = 1,2$:

$$\begin{aligned} dX^k(t) =& F(X^k(t), P_t^k)dt + \sigma(X^k(t), P_t^k)dB_t \\ &+ \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \psi(X^k(t^-), P_{t^-}^k, h_1)\mathbb{1}_{(0,\lambda(X^k(t^-),P_{t^-}^k)]}(u)\mathcal{N}(dt,du,dh), \end{aligned} \tag{2.2.2}$$

defined on the same probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ with the same $\mathcal{F}_t$-Brownian motion B, the same Poisson random measure $\mathcal{N}$ and with initial condition $X^1(0) = X^2(0) = \xi$, $\mathbf{P}$-almost surely. The well-posedness of Eq. (2.2.2) is ensured by Lemma 2.2.3. Let Q^1 and Q^2 be the laws of the solutions on $\mathbf{D}([0,T],\mathbb{R}^d)$ and let Γ be the map that associates Q^k to P^k. We are interested in proving that the map Γ is a contraction for the W_1 Wasserstein norm. Hence, we want to bound the distance

$$\rho_T(Q^1, Q^2) \leqslant \mathbf{E}\left[\sup_{t\in[0,T]} \|X^1(t) - X^2(t)\|\right]. \tag{2.2.3}$$

The idea here, in order to exploit the convexity of U, is to apply Ito's rule. A classical approach consists in applying Ito's rule to a quantity of type $(X_t^1 - X_t^2)^2$; this L^2 approach is not convenient when we have jump terms. For this reason we rather use a L^1 approach. To this aim, for all $\epsilon > 0$ we define the following smooth approximation of the norm

$$f^\epsilon(x) \doteq \|x\|\mathbb{1}(\|x\| > \epsilon) + \left(\frac{\|x\|^2}{2\epsilon} + \frac{\epsilon}{2}\right)\mathbb{1}(\|x\| \leqslant \epsilon). \tag{2.2.4}$$

Then, by Ito's rule and Fatou's Lemma, we have

$$\begin{aligned} \mathbf{E}\left[\sup_{t\in[t_0,t_1]} \|X^1(t) - X^2(t)\|\right] &\leqslant \liminf_{\epsilon\downarrow 0} \mathbf{E}\left[\sup_{t\in[t_0,t_1]} f^\epsilon\left(X^1(t) - X^2(t)\right)\right] \\ &\leqslant \liminf_{\epsilon\downarrow 0}\left(i_\epsilon[t_0,t_1] + u_\epsilon[t_0,t_1] + b_\epsilon[t_0,t_1] + \sigma_\epsilon[t_0,t_1] + \Sigma_\epsilon[t_0,t_1] + \Lambda_\epsilon[t_0,t_1]\right), \end{aligned}$$

where, for $t_1 \in [t_0, T]$, we set

$$\begin{aligned} i_\epsilon[t_0,t_1] &\doteq \mathbf{E}\left[f^\epsilon\left(X^1(t_0) - X^2(t_0)\right)\right], \\ u_\epsilon[t_0,t_1] &\doteq \mathbf{E}\left[\sup_{t\in[t_0,t_1]} -\int_{t_0}^t \nabla f^\epsilon\left(X^1(s) - X^2(s)\right)\cdot\nabla\left(U(X^1(s)) - U(X^2(s))\right)ds\right], \\ b_\epsilon[t_0,t_1] &\doteq \mathbf{E}\left[\sup_{t\in[t_0,t_1]} \int_{t_0}^t \nabla f^\epsilon\left(X^1(s) - X^2(s)\right)\cdot\left(b(X^1(s),P_s^1) - b(X^2(s),P_s^2)\right)ds\right], \\ \sigma_\epsilon[t_0,t_1] &\doteq \frac{1}{2}\mathbf{E}\left[\sup_{t\in[t_0,t_1]} \int_{t_0}^t \mathrm{Tr}\left(\sigma(X^1(s),P_s^1) - \sigma(X^2(s),P_s^2)\right)^T H_{f^\epsilon(X^1(s)-X^2(s))}\left(\sigma(X^1(s),P_s^1) - \sigma(X^2(s),P_s^2)\right)ds\right], \\ \Sigma_\epsilon[t_0,t_1] &\doteq \mathbf{E}\left[\sup_{t\in[t_0,t_1]} \int_{t_0}^t \nabla f^\epsilon\left(X^1(s) - X^2(s)\right)\cdot\left(\sigma(X^1(s),P_s^1) - \sigma(X^2(s),P_s^2)\right)dB_s\right], \\ \Lambda_\epsilon[t_0,t_1] &\doteq \mathbf{E}\left[\sup_{t\in[t_0,t_1]} \int_{t_0}^t \int_{[0,1]}\int_0^\infty f^\epsilon\left(X^1(s) + \psi(X^1(s),P_s^1,h)\mathbb{1}_{u\leqslant\lambda(X^1(s),P_s^1)} - X^2(s)\right.\right. \\ &\qquad \left.\left. -\psi(X^2(s),P_s^2,h)\mathbb{1}_{u\leqslant\lambda(X^2(s),P_s^2)}\right) - f^\epsilon\left(X^1(s) - X^1(s)\right)ds\,du\,\nu_1(dh)\right]. \end{aligned}$$

Notice that, by the assumption of convexity of U, for all $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^d$, it holds

$$\begin{aligned}\nabla f^{\epsilon}(\mathbf{x}-\mathbf{y}) \cdot \nabla\left(U(\mathbf{x})-U(\mathbf{y})\right) =& \frac{\mathbb{1}(\|\mathbf{x}-\mathbf{y}\|>\epsilon)}{\|\mathbf{x}-\mathbf{y}\|}(\mathbf{x}-\mathbf{y}) \cdot \nabla\left(U(\mathbf{x})-U(\mathbf{y})\right) \\ &+ \frac{\mathbb{1}(\|\mathbf{x}-\mathbf{y}\| \leqslant \epsilon)}{\epsilon}(\mathbf{x}-\mathbf{y}) \cdot \nabla\left(U(\mathbf{x})-U(\mathbf{y})\right) \geqslant 0.\end{aligned}$$

Therefore, the term $u_\epsilon[t_0,t_1]$ is easily bounded, since it is always non-positive, i.e.

$$\liminf_{\epsilon \downarrow 0} u_\epsilon[t_0,t_1] \leqslant 0.$$

For the term $b_\epsilon[t_0,t_1]$, we use the global Lipschitz condition on the function b, together with the properties of W_1 Wasserstein distance and inequality (2.2.3):

$$\begin{aligned}b_\epsilon[t_0,t_1] &\leqslant \mathbf{E}\left[\int_{t_0}^{t_1} \left\|b(X^1(s),P^1_s)-b(X^2(s),P^2_s)\right\| ds\right] \\ &\leqslant L\left(\int_{t_0}^{t_1} \mathbf{E}\left[\sup_{s\in[0,t]} \|X^1(s)-X^2(s)\|\right] dt + (t_1-t_0)\rho_{[t_0,t_1]}(P^1,P^2)\right).\end{aligned}$$

To estimate the term $\sigma_\epsilon[t_0,t_1]$, we observe that the Hessian matrix of f^ϵ has the following form:

$$H_{f^\epsilon(x)} = \mathbb{1}(\|x\|>\epsilon)\left(-\frac{1}{\|x\|^3}A+\frac{1}{\|x\|}I\right)+\mathbb{1}(\|x\|\leqslant\epsilon)\frac{1}{\epsilon}I,$$

where A is $d\times d$ matrix such that, for all i,j, $A_{i,j}=x_ix_j$ and I is the identity $d\times d$ matrix. Therefore,

$$\begin{aligned}\sigma_\epsilon[t_0,t_1] \leqslant& \frac{1}{2}\int_{t_0}^{t_0} \mathbf{E}\left[\frac{\mathbb{1}(\|X^1(s)-X^2(s)\|>\epsilon)}{\|X^1(s)-X^2(s)\|}\mathrm{Tr}\left(\sigma(X^1(s),P^1_s)-\sigma(X^2(s),P^2_s)\right)^T\left(\sigma(X^1(s),P^1_s)-\sigma(X^2(s),P^2_s)\right)\right] ds \\ &+\frac{1}{2}\int_{t_0}^{t_0} \mathbf{E}\left[\frac{\mathbb{1}(\|X^1(s)-X^2(s)\|\leqslant\epsilon)}{\epsilon}\mathrm{Tr}\left(\sigma(X^1(s),P^1_s)-\sigma(X^2(s),P^2_s)\right)^T\left(\sigma(X^1(s),P^1_s)-\sigma(X^2(s),P^2_s)\right)\right] ds \\ &+\frac{1}{2}\int_{t_0}^{t_0} \mathbf{E}\left[\frac{\mathbb{1}(\|X^1(s)-X^2(s)\|>\epsilon)}{\|X^1(s)-X^2(s)\|^3}\mathrm{Tr}\left(\sigma(X^1(s),P^1_s)-\sigma(X^2(s),P^2_s)\right)^T\right. \\ &\left.\left((X^1(s)-X^2(s))_i(X^1(s)-X^2(s))_j\right)\left(\sigma(X^1(s),P^1_s)-\sigma(X^2(s),P^2_s)\right)\right] ds.\end{aligned}$$

This term, due to the Lipschitz property of the diffusion coefficient σ gives rise to a new term linear in $\mathbf{E}[\sup_{t\in[t_0,t_1]}\|X^1(t)-X^2(t)\|]$. Indeed, we have, for a certain $K\geqslant 0$,

$$\sigma_\epsilon[t_0,t_1] \leqslant KL\int_{t_0}^{t_1} \mathbf{E}[\sup_{s\in[t_01,t]}\|X^1(s)-X^2(s)\|]dt.$$

The treatment of the term $\Sigma_\epsilon[t_0,t_1]$ involves, in addition to the previous arguments, the

Burkholder-Davis-Gundy inequalities and the global Lipschitz condition (Li1):

$$
\begin{aligned}
\Sigma_\epsilon[t_0,t_1] \leqslant C_1\, \mathbf{E}\Bigg[\Bigg(\int_{t_0}^{t_1} \Big\| \big(\sigma(X^1(s),P^1_s) - \sigma(X^2(s),P^2_s)\big)^{T} \\
\big(\sigma(X^1(s),P^1_s) - \sigma(X^2(s),P^2_s)\big)\Big\|\, ds\Bigg)^{1/2}\Bigg] \\
\leqslant C_1 L\, \mathbf{E}\left[\left(\int_{t_0}^{t_1} \sup_{s\in[t_0,t_1]} \|X^1(s)-X^2(s)\|^2 dt + (t_1-t_0)\rho_{[t_0,t_1]}(P^1,P^2)^2\right)^{1/2}\right] \\
\leqslant C_1 L\sqrt{(t_1-t_0)}\left(\mathbf{E}\left[\sup_{t\in[t_0,t_1]}\|X^1(t)-X^2(t)\|\right] + \rho_{[t_0,t_1]}(P^1,P^2)\right),
\end{aligned}
$$

for some constant C_1 not depending on t_0, t_1. To bound the term $\Lambda_{[t_0,t_1]}$, we make use of the properties of the process $\{\Lambda(t)\}_{t\in[0,T]}$, of the W_1 Wasserstein distance, as well as condition (Li2) and monotone convergence theorem.

$$
\begin{aligned}
\liminf_{\epsilon\downarrow 0} \Lambda_\epsilon[t_0,t_1] = \mathbf{E}\Bigg[\sup_{t\in[t_0,t_1]}\int_{t_0}^{t}\int_{[0,1]}\int_0^\infty \Big\| X^1(s) + \psi(X^1(s),P^1_s,h)\mathbb{1}_{u\leqslant\lambda(X^1(s),P^1_s)} - X^2(s) \\
-\psi(X^2(s),P^2_s,h)\mathbb{1}_{u\leqslant\lambda(X^2(s),P^2_s)}\Big\| - \|X^1(s)-X^1(s)\|\, ds\, du\, \nu_1(dh)\Bigg] \\
\leqslant \mathbf{E}\Bigg[\int_{t_0}^{t_1}\int_{[0,\infty)\times[0,1]} \|\psi(X^1(s^-),P^1_{s^-},h)\mathbb{1}_{(0,\lambda(X^1(s^-),P^1_{s^-})]} \\
-\psi(X^2(s^-),P^2_{s^-},h)\mathbb{1}_{(0,\lambda(X^2(s^-),P^2_{s^-})]}\|\, ds\, du\, \nu(dh)\Bigg] \\
\leqslant L\left(\int_{t_0}^{t_1}\mathbf{E}\left[\sup_{s\in[0,t]}\|X^1(s)-X^2(s)\|\right]dt + (t_1-t_0)\rho_{[t_0,t_1]}(P^1,P^2)\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbf{E}\left[\sup_{t\in[t_0,t_1]}\|X^1(t)-X^2(t)\|\right] \leqslant \mathbf{E}\left[\|X^1(t_0)-X^2(t_0)\|\right] \\
+ L\left((K+1)(t_1-t_0) + C_1\sqrt{t_1-t_0}\right)\rho_{[t_0,t_1]}(P^1,P^2) \\
+ C_1 L\sqrt{(t_1-t_0)}\,\mathbf{E}\left[\sup_{t\in[t_0,t_1]}\|X^1(t)-X^2(t)\|\right] \\
+ L(1+K)(t_1-t_0)\int_{t_0}^{t_1}\mathbf{E}\left[\sup_{s\in[0,t]}\|X^1(s)-X^2(s)\|\right]dt.
\end{aligned}
$$

By hypothesis, $\mathbf{E}[\|X(0)-Y(0)\|] = 0$, then choose $T_0 > 0$ such that $1 - C_1 L\sqrt{T_0} > 0$. Therefore we have

$$
\begin{aligned}
\mathbf{E}\left[\sup_{t\in[0,T_0\wedge T]}\|X^1(t)-X^2(t)\|\right] \leqslant \frac{L(1+K)T_0}{1-C_1L\sqrt{T_0}}\int_0^{T_0\wedge T}\mathbf{E}\left[\sup_{s\in[0,t]}\|X^1(s)-X^2(s)\|\right]dt \\
+ \frac{L\left((1+K)T_0 + C_1\sqrt{T_0}\right)}{1-C_1L\sqrt{T_0}}\rho_{T_0}(P^1,P^2). \qquad (2.2.5)
\end{aligned}
$$

Applying Gronwall's Lemma to (2.2.5), there exists a $T_0 > 0$ sufficiently small such that

$$\rho_{T_0}(Q^1, Q^2) \leqslant \mathbf{E}\left[\sup_{t\in[0,T_0\wedge T]} \|X^1(t) - X^2(t)\|\right] < C_{T_0}\rho_{T_0}(P^1, P^2),$$

for a constant $C_{T_0} < 1$. Therefore, when $P^k \doteq Q^k$, this shows uniqueness of the McKean-Vlasov measure in $\mathcal{M}^1\left(\mathbf{D}([0,T_0],\mathbb{R}^d)\right)$. However, since C_{T_0} depends only on the amplitude of the interval, the same procedure iterated over a finite number of intervals of the type $[T_0\wedge T, 2T_0\wedge T]$, $[2T_0\wedge T, 3T_0\wedge T]$, etc., yields uniqueness of the measure in $\mathcal{M}^1\left(\mathbf{D}([0,T],\mathbb{R}^d)\right)$.

The proof of existence is obtained via a Picard iteration argument, starting from (2.2.2). Let $P^k \doteq Q^{k-1}$, then (2.2.2) gives a sequence of laws $\{Q^k\}_{k\in\mathbb{N}}$, that is a Cauchy sequence for the metric ρ_{T_0} on $\mathcal{M}^1\left(\mathbf{D}([0,T_0],\mathbb{R}^d)\right)$. Consequently, it is a Cauchy sequence also for a weaker Wasserstein metric based on a complete Skorohod metric, that yields existence of a solution of (2.2.2) on $[0, T_0 \wedge T]$. Again, iterating the procedure over a finite number of intervals gives the existence of a weak solution on the time interval $[0,T]$.

According to Yamada-Watanabe theorem, the two previous steps ensure existence and uniqueness of strong solutions. □

2.2.3 Propagation of chaos

This section is an adaptation of Section 2.1.2 to the framework of Assumption 2.2.1, this is the reason why the proofs here are rather short and mostly remind to Section 2.1.2. We recall that we use the same approach, i.e. we make use again of the sequence of *intermediate processes* $\{Y^N\}_{N\in\mathbb{N}}$, where each process $Y^N = \{Y^N(t)\}_{t\in[0,T]}$ is defined as the solution of the system (1.2.4).

Proposition 2.2.1. *Grant Assumption 2.2.1. Let* X^N *and* Y^N *be the solution of* (1.2.1) *and* (1.2.4)*, respectively. We assume the two processes are driven by the same Brownian motions and Poisson random measures, and start from the same square-integrable and permutation invariant initial condition. Then, for each fixed* $i \in \mathbb{N}$*,*

$$\lim_{N\to+\infty} \mathbf{E}\left[\sup_{t\in[0,T]} \|X_i^N(t) - Y_i^N(t)\|\right] = 0.$$

Proof. By the permutation invariance of the systems we write

$$\begin{aligned}\mathbf{E}\left[\sup_{t\in[0,T]} \|X_i^N(t) - Y_i^N(t)\|\right] &= \frac{1}{N}\sum_{i=1}^N \mathbf{E}\left[\sup_{t\in[0,T]} \|X_i^N(t) - Y_i^N(t)\|\right] \\ &= \liminf_{\epsilon\downarrow 0} \frac{1}{N}\sum_{i=1}^N \mathbf{E}\left[\sup_{t\in[0,T]} f^\epsilon(X_i^N(t) - Y_i^N(t))\right],\end{aligned}$$

where f^ϵ is the smooth approximation of the norm, defined in Theorem 2.2.1. Then, we use the techniques of Theorem 2.2.1, as the use of Ito's rule with the function f^ϵ. This, together with the computations in Proposition 2.1.1 and the usual application of Gronwall Lemma iteratively over a finite number of intervals of the type $[0, T_0 \wedge T]$, $[T_0, 2T_0 \wedge T]$, etc. yields, for some constant $C_T > 0$,

$$\frac{1}{N}\sum_{i=1}^{N} \mathbf{E}\left[\sup_{t\in[0,T]} \|X_i^N(t) - Y_i^N(t)\|\right] \leqslant \frac{C_T}{\sqrt{N}},$$

that gives the thesis. □

Proposition 2.2.2. *Grant Assumption 2.2.1. Let μ_0 be a probability measure on $\mathbb{R}^d$ such that $\int \|x\|^2\mu_0(dx) < +\infty$. For $N \in \mathbb{N}$, let Y^N be a solution of Eq.* (1.2.4) *in* $[0, T]$*. Assume that $Y^N(0) = (Y_1^N(0), \dots, Y_N^N(0))$, $N \in \mathbb{N}$, form a sequence of square-integrable random vectors that is μ_0-chaotic in W_1. Let Q be the law of the solution of Eq.* (1.2.3) *in* $[0, T]$ *with initial law $\mathbf{P} \circ X(0)^{-1} = \mu_0$. Then Y^N is Q chaotic in W_1.*

Proof. We follow the steps of Proposition 2.1.2 to define the coupling procedure. We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbf{P})$ with respect to the family $(B_i, \mathcal{N}^i)_{i\in\mathbb{N}}$ of independent Brownian motions and Poisson random measures. For each $N \in \mathbb{N}$, we couple the process Y^N with the process $\bar{X}^N = \left\{\bar{X}_i^N(t),\, i = 1, \dots, N\right\}_{t\in[0,T]}$ defined thanks to Theorem 2.2.1, where the initial condition is $Law(\bar{X}^N(0)) = \otimes^N \mu_0$ and each component $\bar{X}_i^N$ is a solution of SDE (2.2.1). Successively, we use the techniques of the previous theorems, we iterate the computations over a finite number of time intervals to cover all $[0, T]$ and we obtain

$$\frac{1}{N}\sum_{i=1}^{N} \mathbf{E}\left[\sup_{t\in[0,T]} \|Y_i^N(t) - \bar{X}_i^N(t)\|\right] \overset{N\to\infty}{\to} 0,$$

that implies Q chaoticity of the law of Y^N. □

Corollary 2.2.1. *Grant Assumption 2.2.1. Let μ_0 be a probability measure on $\mathbb{R}^d$ such that $\int \|x\|^2\mu_0(dx) < +\infty$. For $N \in \mathbb{N}$, let X^N be a solution of Eq.* (1.2.1) *in* $[0, T]$*. Assume that $X^N(0) = (X_1^N(0), \dots, X_N^N(0))$, $N \in \mathbb{N}$, form a sequence of square-integrable random vectors that is μ_0-chaotic in W_1. Let Q be the law of the solution of Eq.* (1.2.3) *in* $[0, T]$ *with initial law $\mathbf{P} \circ X(0)^{-1} = \mu_0$. Then X^N is Q chaotic in W_1.*

Proof. This follows, of course, from the same procedure of the proof of Corollary 2.1.1 and it is based on the results of Proposition 2.2.1 and 2.2.2. □

2.2.4 Some technical lemmas

We gather in this section the lemmas necessary to prove well-posedness of the particle systems (1.2.1) and (1.2.4) and the nonlinear stochastic differential equation (1.2.3), under Assumption 2.2.1. These lemmas are simply an application of classical approach, see for example Ikeda Watanabe [51], together with the trick used in the proof of Theorem 2.2.1. Well-posedness of equations (1.2.1) and (1.2.4) is clearly a consequence of Lemma 2.2.3, where there is no parametrizing measure, the state space of the process is $\mathbb{R}^{N\times d}$ and we have a finite number (precisely N) of Poisson integrals, instead of the one described in the statement of the lemma.

Lemma 2.2.1. *Consider the SDE parametrized by two measures α and $\beta \in \mathcal{M}(\mathbf{D}([0,T],\mathbb{R}^d))$*

$$\begin{aligned} dX(t) =& F(X(t),\alpha_t)dt + \sigma(X(t^-),\alpha_t)dB_t \\ &+ \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \psi(Y(t^-),\alpha_{t^-},h_1)\mathbb{1}_{(0,\lambda(Y(t^-),\alpha_{t^-})]}(u)\mathcal{N}(dt,du,dh), \end{aligned} \tag{2.2.6}$$

with $\mathsf{Law}(Y) = \beta$. *If the coefficients satisfy Assumption 2.2.1, then for every α and $\beta \in \mathcal{M}^1(\mathbf{D}([0,T],\mathbb{R}^d))$, every square-integrable initial condition, there exists a unique strong solution to Eq.* (2.2.6).

Moreover, let $\mu \doteq \mathsf{Law}((X(t))_{t\in[0,T]})$ *be the law of the solution of* (2.2.6) *starting from the square-integrable initial condition* $X(0)$ μ_0*-distributed, then* $\mu \in \mathcal{M}^1(\mathbf{D}([0,T],\mathbb{R}^d))$.

Proof. Let B be an $(\mathcal{F}_t)$-brownian motion, p be a $(\mathcal{F}_t)$-stationary Poisson point process with characteristic measure $l \times \nu$ and ξ be a $\mathcal{F}_0$-measurable square-integrable r.v.. Let $D \doteq \{s \in D_p \text{ s.t. } p(s) \in \bar{U}_s = (0,\lambda(Y(s^-),\alpha_{s^-})]\times[0,1]\times[0,1]\times\dots\}$. Let us call $\sigma_1 < \sigma_2 < \dots$ the elements of D. Each σ_n is an $\mathcal{F}_t$ -stopping time and $\lim_{n\to\infty}\sigma_n = \infty$ a.s.. Indeed, for every $T > 0$ and for a fixed $n \in \mathbb{N}^*$,

$$\mathbf{P}(\sigma_n \leqslant T) = \mathbf{P}\left(\int_0^T\int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \mathbb{1}_{(0,\lambda(Y(t^-),\alpha_{t^-})]}(u)\mathcal{N}(du,dh,dt) \geqslant n\right) \leqslant \frac{\mathbf{E}\left[\lambda(Y(T),\alpha_T)\right]}{n} \leqslant \frac{C_T}{n},$$

for a certain constant C_T. By Lemma 2.2.2, we get the claim. Then we start by showing $\exists!$ of a solution for (2.2.6) on $[0,\sigma_1]$. Consider the equation

$$Z(t) = X(0) + \int_0^t F(Z(s),\alpha_s)ds + \int_0^t \sigma(Z(s^-),\alpha_s)dB_s. \tag{2.2.7}$$

Existence and uniqueness of a strong solution for (2.2.7) are ensured by the classical Hasminskii's test for non-explosion (see e.g. [66] with the Lyapunov function $V(z) = \|z\|^2$). The test's conditions are guaranteed by the inequality

$$\sup_{\alpha\in\mathcal{M}^1(\mathbb{R}^d)} z\cdot F(z,\alpha) + \mathrm{tr}(\sigma(z,\alpha)\sigma^T(z,\alpha)) \leqslant C(1+\|z\|^2), \tag{2.2.8}$$

for some $C > 0$, for all $z \in \mathbb{R}^d$. Indeed, fix an $\alpha \in \mathcal{M}^1(\mathbb{R}^d)$. Then, under (U) from Assumption 2.2.1, we have

$$z\cdot F(z,\alpha) = -(z-\underline{0})\cdot(\nabla U(z) - \nabla U(\underline{0})) + z\cdot\nabla U(\underline{0}) + z\cdot b(z,\alpha) \leqslant C\left(\|z\|^2+1\right),$$

due to the convexity of U and the linear growth of b. A similar bound is obtained for the second summand in the l.h.s of (2.2.8), which has uniform quadratic growth in the z

variable. Then, for every integrable initial condition, there exists a unique strong solution to (2.2.7). Let π_1 be the projection defined as

$$\begin{array}{rccc} \pi_1 : & [0,1]^{\mathbb{N}} \times [0,\infty) & \mapsto & [0,1] \\ & (h,u) & \to & h_1, \end{array}$$

we define

$$X_1(t) = \begin{cases} Z^1(t) & t \in [0,\sigma_1), \\ Z^1(\sigma_1^-) + \psi(Z^1(\sigma_1^-), \alpha(\sigma_1^-), \pi_1 \circ p(\sigma_1)) & t = \sigma_1, \end{cases} \tag{2.2.9}$$

where $\{Z^1(t)\}_{t\geqslant 0}$ is solution of (2.2.7) with initial condition $Z^1(0) = \xi$ a.s.. We see that $X^1(t)$ is solution of (2.2.6) for $t \in [0,\sigma_1]$. We iterate the procedure by setting $\bar{\xi} \doteq X_1(\sigma_1)$, $\bar{B} \doteq (B(t+\sigma_1) - B(\sigma_1))_{t\geqslant 0}$ and $\bar{p} \doteq (p(t+\sigma_1))_{t\geqslant 0}$. We define $\bar{X}_1(t)$ for $t \in [0,\bar{\sigma}_1]$ as we did for $X_1(t)$ in (2.2.9), where $\bar{\sigma}_1$ is the smallest time such that $\bar{p}_s$ belongs to $\bar{U}_{\sigma_1+s}$ and coincides with $\sigma_2 - \sigma_1$. We define

$$X_2(t) = \begin{cases} X_1(t) & t \in [0,\sigma_1], \\ \bar{X}_1(t-\sigma_1) & t \in [\sigma_1,\sigma_2]. \end{cases}$$

Clearly X_2 is solution of (2.2.6) for $t \in [0,\sigma_2]$. Since $\lim_{n\to\infty} \sigma_n = \infty$ a.s., we can iterate this procedure to cover the entire time interval $[0,T]$.

To prove that the law μ of a solution of (2.2.6) belongs to $\mathcal{M}^1(\mathbf{D}([0,T],\mathbb{R}^d))$, we will show that there exists a filtered probability space $(\Omega, \mathbf{P}, (\mathcal{F}_t), \mathcal{F})$, with a $\mathcal{F}_t$-Brownian motion B, an adapted $\mathcal{F}_t$ Poisson random measure $\mathcal{N}$ with characteristic measure $l \times l \times \nu$ and a $\mathcal{F}_0$-measurable initial condition $X(0)$ μ_0-distributed such that $\mathbf{E}\left[\sup_{t\in[0,T]} \|X(t)\|\right] < \infty$. We consider the process $X(t)$, for all $t \geqslant 0$, solution of (2.2.6). Now, we use the trick of applying Ito's rule to the smooth approximation f^ϵ of $\|\cdot\|$ and taking the limit for $\epsilon \downarrow 0$, to exploit the properties of the potential function U. For the details of the approach, see the proof of Theorem 2.2.1. Then, for the properties of coefficients and quantities involved, there exist three positive constants D_1, D_2 and D_3 s.t.

$$\mathbf{E}\left[\sup_{t\in[0,T]} \|X(t)\|\right] \leqslant \mathbf{E}\left[\|X(0)\|\right] + D_1 T + D_2 T\, \mathbf{E}\left[\sup_{t\in[0,T]} \|Y(t)\|\right] + D_1 \int_0^T \mathbf{E}\left[\sup_{s\in[0,t]} \|X(s)\|\right] dt.$$

We apply Gronwall Lemma and we get the desired bound. □

Lemma 2.2.2. *Let* $\{\sigma_n\}_{n\in\mathbb{N}^*}$ *be a sequence of strictly increasing stopping times. If, for all* $T > 0$*, there exists a constant* $C_T \geqslant 0$ *such that*

$$\mathbf{P}(\sigma_n \leqslant n) \leqslant \frac{C_T}{n},$$

then $\lim_{n\to\infty} \sigma_n = \infty$ *a.s..*

Proof. We start by proving that, for all $T > 0$, there exists a measurable set Λ_T with probability one, such that for all $\omega \in \Lambda_T$, there exists $n_0(\omega, T)$ and for all $n \geqslant n_0(\omega, T)$ it holds $\sigma_n(w) > T$.

Let $A_n \doteq \{\sigma_{n^2} \leqslant T\}$ and $A \doteq \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$, therefore we have

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) \leqslant \sum_{n=1}^{\infty} \frac{C_T}{n^2} < \infty$$

and for Borel Cantelli $\mathbf{P}(A) = 0$. Let $\Lambda_T \doteq A^C$, then it has probability one and for all $\omega \in \Lambda_T$ there exists $\bar{n}_0(\omega, T)$ such that for all $n \geqslant \bar{n}_0(\omega, T)$ we have $\sigma_{n^2} > T$. Since the σ_n are increasing, we have the claim that there exists $n_0(\omega, T)$ such that for all $n \geqslant n_0(\omega, T)$, $\sigma_n > T$.

Now, let $\tilde{\Lambda} \doteq \bigcap_{T \in \mathbb{N}} \Lambda_T$, then $\mathbf{P}(\tilde{\Lambda}) = 1$ and for all $\omega \in \tilde{\Lambda}$ for all $T \geqslant 0$ there exists $n_0(\omega, T)$ s.t. for all $n \geqslant n_0(\omega, T)$ then $\sigma_n(\omega) > T$. This implies $\sigma_n \nearrow \infty$ a.s.. □

We are now ready to prove the most important lemma of this section, the one that allows us to prove well-posedness of SDE with jumps if the drift satisfies condition (U) and that it is crucial in the proof of Theorem 2.2.1.

Lemma 2.2.3. *Consider the SDE parametrized by a measure $\alpha \in \mathcal{M}^1(\mathbf{D}([0,T], \mathbb{R}^d))$*

$$\begin{aligned} dX(t) =& F(X(t), \alpha_t)dt + \sigma(X(t), \alpha_t)dB_t \\ &+ \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \psi(X(t^-), \alpha_{t^-}, h_1)\mathbb{1}_{(0,\lambda(X(t^-),\alpha_{t^-})]}(u)\mathcal{N}(dt, du, dh). \end{aligned} \tag{2.2.10}$$

If the coefficients satisfy Assumption 2.2.1, then for every $\alpha \in \mathcal{M}^1(\mathbf{D}([0,T], \mathbb{R}^d))$ and every square-integrable initial condition, there exists a unique strong solution to Eq. (2.2.10)*.*

Proof. First let X^1 and X^2 be two integrable stochastic processes on $[0, T]$ with values in $\mathbb{R}^d$. We define the map that associates the law of X^k to the law of the solution of

$$\begin{aligned} dY^k(t) =& F(Y^k(t), \alpha_t)dt + \sigma(Y^k(t), \alpha_t)dB_t \\ &+ \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \psi(X^k(t^-), \alpha_{t^-}, h_1)\mathbb{1}_{(0,\lambda(X^k(t^-),\alpha_{t^-})]}(u)\mathcal{N}(dt, du, dh), \end{aligned} \tag{2.2.11}$$

that is well-defined for Lemma 2.2.1. With the same computation of the proof of Theorem 2.2.1, we get that, for a small enough $T_0 > 0$, there exists a constant $C_{T_0} < 1$ such that

$$\mathbf{E}\left[\sup_{t\in[0,T_0]} \|Y^1(t) - Y^2(t)\|\right] \leqslant C_{T_0}\, \mathbf{E}\left[\sup_{t\in[0,T_0]} \|X^1(t) - X^2(t)\|\right].$$

This shows *pathwise uniqueness* for solution of (2.2.10). By means of (2.2.11), we define a Picard iteration argument that gives a sequence of laws $\{Q^n\}_{n\in\mathbb{N}}$ on $\mathbf{D}([0,T], \mathbb{R}^d)$. Again, there exists a $T_0 > 0$ small enough such that $\{Q^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence for ρ_{T_0} and hence for a weaker but complete Wasserstein metric on $\mathcal{M}^1(\mathbf{D}([0,T_0],\mathbb{R}^d))$. Iterating the procedure over a finite number of time intervals, to cover $[0,T]$, yields tweak existence of a solution. The integrability property is proved as in the proof of Lemma 2.2.1. Then Yamada-Watanabe theorem concludes the proof. □

Let us highlight that, in the proof of Lemma 2.2.3, we need to define the map by means of (2.2.11) and we could not straightly substitute X^k in the whole right-hand side of (2.2.10). In fact, we need to control the jumps by means of a known process, but at the same time, we need to have the same variable as argument of the drift coefficient to exploit the convexity of the potential function U.

2.3 Non-globally Lipschitz jump rate

In this section we want to extend our study to a class of systems in which the jump rate is super-linear. This is mainly motivated by the neuroscience models we introduced in Section 1.1.2, from which comes the inspiration of this study. In [29, 43, 75] the authors present two piece-wise deterministic Markov processes (PDMPs) of interacting neurons with the feature of simultaneous jumps, that we summarized (at the microscopic level) in (1.1.1). In this section we extend the model to a d-dimensional framework and we slightly generalize the jumps' amplitude and rate functions, but we neglect the term modelling *electrical synapses*, by choosing $\beta = 0$. This results in a d-dimensional extension of the model in [75].

2.3.1 Assumptions and well-posedness of the particle system

We consider, as the initial particle system, the PDMP X^N solution of the following SDE:

$$\begin{aligned} dX_i^N(t) =& -X_i^N(t)dt + \frac{1}{N}\sum_{j\neq i}\int_{[0,\infty]\times[0,1]^{\mathbb{N}}} V(h_j,h_i)\mathbb{1}_{[0,\lambda(X_j^N(t)))}(u)\mathcal{N}^j(dt,du,dh) \\ & -\int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \left(X_i^N(t) - U(h_i)\right)\mathbb{1}_{[0,\lambda(X_i^N(t)))}(u)\mathcal{N}^i(dt,du,dh) \end{aligned} \tag{2.3.1}$$

for all $i = 1,\dots,N$. As before, $(\mathcal{N}^i)_{i\in\mathbb{N}}$ is an independent family of Poisson random measures $\mathcal{N}^i$, each of them with characteristic measure $l\times l\times\nu$. Remember that ν is a symmetric probability measure on $[0,1]^{\mathbb{N}}$ such that it exists a consistent family of symmetric probability measures $(\nu_N)_{N\in\mathbb{N}}$, each of them defined respectively on $[0,1]^N$ and coinciding with the projections of ν on N coordinates.

Assumption 2.3.1. *The coefficients of the system* (2.3.1) *obey the following properties:*

(JR) the jump rate of each particle is a non-negative C^1 *function of its position,* $\lambda:\mathbb{R}^d\to\mathbb{R}_+$*, that is written as the sum of two functions:*

$$\lambda(\cdot) \doteq b(\|\cdot\|) + h(\cdot).$$

- b *is a* C^1*, positive, non-decreasing function such that*

$$b'(r) \leqslant \gamma b(r) + c \tag{2.3.2}$$

for some $c > 0$ *and* $\gamma < \dfrac{1}{5\,\mathbf{E}[\|V\|]}$*;*

- $h : \mathbb{R}^d \to \mathbb{R}$ *is a* C^1 *bounded function, i.e. there exists* $H > 0$ *such that* $\forall\, x \in \mathbb{R}^d$*,* $\|h(x)\| \leqslant H$*;*

(JA) the jump amplitudes, V *and* U*, are two bounded functions from respectively* $[0,1]^2$ *and* $[0,1]$ *to* $\mathbb{R}^d$ *(since they represents two random variables with values in some bounded subsets of* $\mathbb{R}^d$*, with abuse of notation we will indicate as expectations their integrals w.r.t. the measure* ν *).*

Notice that the form of the function b is exactly the one suggested by [75]. The assumption

$$\gamma < \frac{1}{5\,\mathbf{E}[\|V\|]}$$

allows to obtain apriori bounds on the moments of $\lambda(X(t))$, where $X(t)$ is the solution of the corresponding McKean-Vlasov equation, see (2.3.3), and it is used in the proofs of next Lemmas 2.3.4 and 2.3.5. It is interesting to notice that Assumption 2.3.1 allows to consider non-globally Lipschitz functions; in particular, this covers all the cases where $b(r)$ is of the form r^α, for $\alpha \geqslant 1$. We also remark that the condition on b here is a little stronger than in [75], due to the coupling method (vs. the martingale approach) in the proof, which in particular allows to identify the rate of convergence, which is of the order $O\left(\frac{1}{\sqrt{N}}\right)$. This requires $\gamma < \dfrac{1}{K\,\mathbf{E}[\|V\|]}$ with $K = 5$ rather than $K = 3$, as in [75].

We will deal with initial conditions with bounded support and, if the function b is convex, we could adapt our computations to include a drift towards the barycenter of the system, that would be an extention to the model in [43]. However, in [43], the authors succeed in proving propagation of chaos with an explicit rate (namely, the expected $\frac{1}{\sqrt{N}}$) even for weaker conditions on the initial values, by defining an ad-hoc distance based on the rate function λ itself. In our study, we choose not to extend this powerful approach to our d-dimensional model and to maintain the same structure of proofs of the previous sections. However, we believe that the computations of [43] would work here and they would give results without the restrictive hypothesis on the bounded support of initial condition that we require from Section 2.3.2 to the end of the chapter.

Let us start by proving well-posedness of (2.3.1), this relies on a truncation argument on the function λ.

Lemma 2.3.1. *Under Assumption 2.3.1, for every integrable initial condition* $X^N(0) \in \mathbb{R}^{d\times N}$*, the SDE* (2.3.1) *admits a unique solution.*

Proof. The main issue is represented by the fact that the function λ is not bounded, neither globally Lipschitz continuous. For λ bounded or globally Lipschitz continuous, existence and uniqueness of solutions for (2.3.1) are consequences of standard results, see [51].

Let us consider the truncate function $\lambda^K \doteq \lambda \wedge K$, for $K \in \mathbb{N}$, and the solution $X^{N,K}(t)$ of (2.3.1) with the function λ^K instead of λ. This solution exists and it is unique for all $t \in [0,T]$. By pathwise uniqueness, it holds $X^{N,K}(t) = X^{N,K+1}(t)$ for all $t \in \tau^K$, where

$$\tau^K \doteq \inf\left\{t \,/ \|X^{N,K}(t)\| \geqslant K\right\}.$$

Therefore $\tau^K \leqslant \tau^{K+1}$ a.s. and there exists a pathwise unique solution $X(t)$ to (2.3.1), defined for all $t \in [0,\tau)$, where $\tau \doteq \sup_{K\in\mathbb{N}} \tau^K$. We are left to prove that $\mathbf{P}(\tau > T) = 1$.

Let us fix $i \in \{1,\dots,N\}$ and $\epsilon > 0$. We compute, by means of Ito's formula, $f^\epsilon(X_i^N(t))$, where f^ϵ is the function defined in (2.2.4). We get

$$\begin{aligned}
f^\epsilon(X_i^N(t)) &\leqslant f^\epsilon(X_i^N(0)) \\
&\quad + \sum_{j\neq i} \int_0^t \int_{[0,1]^{\mathbb{N}}} \int_0^\infty \left(f^\epsilon\left(X_i^N(s) + \frac{V(h_j,h_i)}{N}\right) - f^\epsilon\left(X_i^N(s)\right)\right) \mathbb{1}_{(0,\lambda(X_j^N(s))]}(u)\mathcal{N}^j(ds,du,dh) \\
&\quad + \int_0^t \int_{[0,1]^{\mathbb{N}}} \int_0^\infty \left(f^\epsilon\left(U(h_i)\right) - f^\epsilon\left(X_i^N(s)\right)\right) \mathbb{1}_{(0,\lambda(X_i^N(s))]}(u)\mathcal{N}^i(ds,du,dh),
\end{aligned}$$

that, of course, is bounded by the following expression

$$\begin{aligned}
f^\epsilon(X_i^N(t)) \leqslant f^\epsilon(X_i^N(0)) &+ \frac{1}{N}\sum_{j=1}^N \int_0^t \int_{[0,1]^{\mathbb{N}}} \int_0^\infty f^\epsilon\left(V(h_j,h_i)\right) \mathbb{1}_{(0,\lambda(X_j^N(s))]}(u)\mathcal{N}^j(ds,du,dh) \\
&+ \int_0^t \int_{[0,1]^{\mathbb{N}}} \int_0^\infty \left(f^\epsilon\left(U(h_i)\right) - f^\epsilon\left(X_i^N(s)\right)\right) \mathbb{1}_{(0,\lambda(X_i^N(s))]}(u)\mathcal{N}^i(ds,du,dh).
\end{aligned}$$

Summing on all $i = 1,\dots,N$ and taking expectation, by the application of Fatou's Lemma, we get:

$$\begin{aligned}
\mathbf{E}\left[\frac{1}{N}\sum_{i=1}^N \|X_i^N(t)\|\right] \leqslant \liminf_{\epsilon\downarrow 0} \Bigg(&\mathbf{E}\left[\frac{1}{N}\sum_{i=1}^N f^\epsilon(X_i^N(0))\right] - \int_0^t \mathbf{E}\left[\frac{1}{N}\sum_{i=1}^N f^\epsilon(X_i^N(s))\lambda(X_i^N(s))\right] ds \\
&+ \int_0^t \left(\mathbf{E}[f^\epsilon(V)] + \mathbf{E}[f^\epsilon(U)]\right) \mathbf{E}\left[\frac{1}{N}\sum_{i=1}^N \lambda(X_i^N(s))\right] ds\Bigg)
\end{aligned}$$

Then, by monotone convergence, we have

$$\begin{aligned}
\mathbf{E}\left[\frac{1}{N}\sum_{i=1}^N \|X_i^N(t)\|\right] \leqslant &\mathbf{E}\left[\frac{1}{N}\sum_{i=1}^N \|X_i^N(0)\|\right] - \int_0^t \mathbf{E}\left[\frac{1}{N}\sum_{i=1}^N \|X_i^N(s)\|\lambda(X_i^N(s))\right] ds \\
&+ \int_0^t \left(\mathbf{E}[\|V\|] + \mathbf{E}[\|U\|]\right) \mathbf{E}\left[\frac{1}{N}\sum_{i=1}^N \lambda(X_i^N(s))\right] ds
\end{aligned}$$

Since b is increasing and h is bounded, there exists a positive constant C, depending on $\mathbf{E}\left[\frac{1}{N}\sum_{i=1}^{N}\|X_i^N(0)\|\right]$, such that

$$\sup_{t\geqslant 0}\mathbf{E}\left[\frac{1}{N}\sum_{i=1}^{N}\|X_i^N(t)\|\right]\leqslant C,$$

implying $\mathbf{P}(\tau > T) = 1$. □

2.3.2 Well-posedness of the McKean-Vlasov SDE

This section is devoted to analyze the McKean-Vlasov equation whose law is the limit of the sequence of empirical measures corresponding to system (2.3.1), that is

$$\begin{aligned} dX(t) = &\mathbf{E}[\lambda(X(t))]\,\mathbf{E}[V]\,dt - X(t)dt \\ &- \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} (X(t) - U(h_1))\,\mathbb{1}_{[0,\lambda(X(t)))}(u)\mathcal{N}(dt,du,dh), \end{aligned} \tag{2.3.3}$$

with $\mathcal{N}$ Poisson random measure with characteristic measure $l\times\nu\times l$. We see that the contribution of the *collateral jumps* creates the additional drift term

$$\mathbf{E}[\lambda(X(t))]\,\mathbf{E}[V]\,dt.$$

As we said, the model that we treat is basically an extension in d-dimension of the model presented in [75]. Therefore, techniques for proving existence and uniqueness of solutions for the nonlinear Markov process (2.3.3) are adaptations of the techniques presented in that paper. The procedure relies on a priori bounds on moments of the solution and of the expectation of $\lambda(X(t))$, we will present the main steps here, while we gather the technical details in Section 2.3.4.

We start by proving well-posedness of a time-inhomogeneous PDMP, associated to (2.3.3).

Lemma 2.3.2. *Let $f:\mathbb{R}_+\to\mathbb{R}^d$ be a locally bounded Borel function, then there exists a unique solution $(Z_f(t))$ to the SDE*

$$dZ_f(t) = -Z_f(t)dt + f(t)dt - \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} (Z_f(t) - U(h_1))\,\mathbb{1}_{[0,\lambda(Z_f(t)))}(u)\mathcal{N}(dt,du,dh) \tag{2.3.4}$$

with initial condition x and coefficients satisfying Assumption 2.3.1. Moreover, for every pair of locally bounded Borel functions f and g, for every $T>0$ there exists a constant $C_T>0$ such that

$$\mathbf{E}\left[\sup_{t\in[0,T]}\|Z_f(t) - Z_g(t)\|\right]\leqslant C_T\int_0^T \sup_{s\in[0,t]}\|f(s)-g(s)\|\,dt. \tag{2.3.5}$$

It is clear that, if we choose the function f in a suitable way, i.e.

$$f(t) \doteq \mathbf{E}\left[\lambda(X(t))\right]\mathbf{E}\left[V\right],$$

the solution Z_f to (2.3.4) coincides with the solution X to (2.3.3). Then we derive a priori bounds for any solution of (2.3.3), that are necessary to perform the iteration that yields to the existence and uniqueness of the nonlinear process itself. The following lemma provides the required bounds.

Lemma 2.3.3. *Suppose Assumption 2.3.1 is satisfied. Let* X *be a solution of* (2.3.3) *with integrable initial condition* $X(0)$*; then we have that* $\sup_{t\geqslant 0}\mathbf{E}\left[\|X(t)\|\right] < \infty$*. Moreover, for* $p = 1,2,3,4$*, if* $\mathbf{E}\left[\lambda^p(X(0))\right] < \infty$ *then* $\sup_{t\geqslant 0}\mathbf{E}\left[\lambda^p(X(t))\right] \leqslant C < \infty$*, where* C *only depends on* $\mathbf{E}\left[\lambda^p(X(0))\right]$ *and on the parameters of equations* (2.3.3)*.*

Last, we prove well-posedness of the nonlinear PDMP (2.3.3).

Theorem 2.3.1 (Solution of the McKean-Vlasov equation)**.** *Under Assumption 2.3.1, for any initial condition* $X(0)$ *with bounded support and independent of* $\mathcal{N}$*, there exists a unique strong solution* $\{X(t)\}_{t\in[0,T]}$ *for* (2.3.3)*.*

Let us sketch the idea of the proof. We prove well-posedness of a nonlinear process, in which we truncate the nonlinear drift term above a certain threshold. Since Lemma 2.3.3 gives a priori bounds on the drift term itself, we can identify (2.3.3) with processes belonging to this class and we get the thesis.

Proof of Theorem 2.3.1. Fix a constant $C > 0$, and consider the following Picard iteration: $Z_0^C(t) \equiv X(0)$ and

$$\begin{cases} dZ_n^C(t) = -Z_n^C(t)dt + \left(\mathbf{E}\left[\lambda(Z_{n-1}^C(t))\right] \wedge C\right)\mathbf{E}\left[V\right]dt \\ \qquad\qquad - \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \left(Z_n^C(t) - U(h_1)\right)\mathbb{1}_{[0,\lambda(Z_n^C(t)))}(u)\mathcal{N}(dt,du,dh), \\ Z_n^C(0) = X(0) \end{cases}$$

The following almost sure apriori bound is essentially obvious: for any $n \geqslant 1$

$$\|Z_n^C(t)\| \leqslant K + tC\,\mathbf{E}[\|V\|],$$

for a suitable $K > 0$ depending on the support of $X(0)$ and the range of $U(h)$. Indeed, when $\|Z^C(t)\|$ is large, the linear term $-Z_n^C(t)dt$ as well as the jumps can only decrease the norm. From Lemma 2.3.2 we now that there exists a constant C_T such that

$$\mathbf{E}\left[\sup_{t\in[0,T]}\|Z_{n+1}^C(t) - Z_n^C(t)\|\right] \leqslant C_T\,\mathbf{E}\left[\|V\|\right]\int_0^T \left\|\mathbf{E}\left[\lambda(Z_n^C(s))\right] - \mathbf{E}\left[\lambda(Z_{n-1}^C(s))\right]\right\| ds.$$

Thanks to the a.s. bounds on $\|Z_n^C(t)\|$, we can exploit the local Lipschitzianity of λ and get, for a certain constant $K_T > 0$,

$$\begin{aligned} \mathbf{E}\left[\sup_{t\in[0,T]}\|Z_{n+1}^C(t) - Z_n^C(t)\|\right] &\leqslant C_T\,\mathbf{E}\left[\|V\|\right]K_T\int_0^T \mathbf{E}\left[\sup_{s\in[0,t]}\|Z_n^C(s) - Z_{n-1}^C(s)\|\right]dt \\ &\leqslant \cdots \leqslant \frac{(K_T C_T\,\mathbf{E}\left[\|V\|\right]T)^n}{n!}\mathbf{E}\left[\sup_{s\in[0,t]}\|Z_1^C(s) - Z_0^C(s)\|\right]. \end{aligned}$$

Therefore the sequence $\{Z^C_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and its limit Z^C is a solution of the SDE

$$
\begin{aligned}
dZ^C(t) = &- Z^C(t)dt + \left(\mathbf{E}\left[\lambda(Z^C(t))\right] \wedge C\right) \mathbf{E}[V]\, dt \\
&- \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \left(Z^C(t^-) - U(h_1)\right) \mathbb{1}_{[0,\lambda(Z^C(t^-)))}(u)\mathcal{N}(dt, du, dh).
\end{aligned}
$$

By Lemma 2.3.3, we can choose C so that $\mathbf{E}\left[\lambda(Z^C(t))\right] \leqslant C$ for all t, so that Z^C is indeed a solution of (2.3.3). Uniqueness is given by considering two solutions Z_1 and Z_2. Using the above apriori bound, (2.3.5) and the Gronwall Lemma their equality follows from standard arguments. □

2.3.3 Propagation of Chaos

As in the previous sections, we use the intermediate process $\{Y^N(t)\}_{t\in[0,T]}$ that, in this case, is the solution of the system: for all $i = 1, \dots, N$

$$
\begin{aligned}
dY^N_i(t) =&-Y^N_i(t)dt + \frac{1}{N}\sum_{j=1}^{N} \mathbf{E}[V]\, \lambda(Y^N_j(t))dt \\
&- \int_{[0,\infty)\times[0,1]^{\mathbb{N}}} \left(Y^N_i(t) - U(h_i)\right) \mathbb{1}_{[0,\lambda(Y^N_i(t)))}(u)\mathcal{N}^i(dt, du, dh).
\end{aligned}
\tag{2.3.6}
$$

Well-posedness of (2.3.6) follows from Lemma 2.3.1. In order to use a coupling procedure to prove propagation of chaos, we need to set some a priori bounds on the involved quantities.

Lemma 2.3.4. *For $N > 0$, under Assumption 2.3.1, let X^N and Y^N be solutions, respectively, of (2.3.1) and (2.3.6), starting from initial conditions s.t. $\mathbf{E}\left[\langle\mu^N_X(0), \lambda^4(\cdot)\rangle\right] < \infty$ and $\mathbf{E}\left[\langle\mu^N_Y(0), \lambda^4(\cdot)\rangle\right] < \infty$. Then there exists a certain $N_0 > 0$ such that it holds*

$$
\begin{aligned}
&\sup_{N\geqslant N_0}\sup_{t\geqslant 0} \mathbf{E}\left[\langle\mu^N_X(t), \lambda^4(\cdot)\rangle\right] < \infty, \\
&\sup_{N\geqslant N_0}\sup_{t\geqslant 0} \mathbf{E}\left[\langle\mu^N_Y(t), \lambda^4(\cdot)\rangle\right] < \infty.
\end{aligned}
$$

Lemma 2.3.4 is crucial for proving that the number of jumps of the system in a compact time interval is proportional to N with probability increasing with N. This bound is stated in the following lemma.

Lemma 2.3.5 (Bound on the number of jumps)**.** *Assume that Assumption 2.3.1 is satisfied and that, for any $N > 0$, X^N and Y^N are solutions, respectively, of (2.3.1) and (2.3.6), starting from initial conditions that are μ_0-chaotic. Here μ_0 is a probability measure on $\mathbb{R}^d$ s.t. $\mathbf{E}_{\mu_0}\left[\lambda^3(X)\right] < \infty$. Then, for any $T > 0$, there exists a positive constant H_T and a natural number $N_0 > 0$ such that, for certain positive constants K_T and $\tilde{K}_T$*

$$
\begin{aligned}
&\mathbf{P}\left(\frac{C_N(T)}{N} \geqslant H_T\right) \leqslant \frac{K_T}{N} \\
&\mathbf{P}\left(\int_0^T \langle\mu^N_Y(s), \lambda\rangle ds \geqslant H_T\right) \leqslant \frac{\tilde{K}_T}{N},
\end{aligned}
$$

for all $N > N_0$*. Here* $C_N(T)$ *is the number of jumps performed by system* (2.3.1) *up to time* T*.*

The bounds on the number of *collateral* jumps and of the corresponding drift in a compact time interval plays a role in the proof of propagation of chaos, since they let us exploit the local Lipschitzianity of the function λ when we start from initial conditions with bounded support. The proofs of these lemmas involve the form of the function λ and they are in Section 2.3.4. In the following we state and prove the result on propagation of chaos and also in this case, the simultaneous jumps result in a rate of the order $\frac{1}{\sqrt{N}}$. As in the previous sections, we start with the comparison between the particle system X^N and the intermediate system Y^N.

Theorem 2.3.2. *Let Assumptions 2.3.1 be satisfied and let* X^N *and* Y^N *be the solution, respectively, of* (2.3.1) *and* (2.3.6) *with permutation invariant initial condition with compact support* $X^N(0) = Y^N(0)$ *a.s. that are* μ_0*-chaotic, with* μ_0 *probability measure on* $\mathbb{R}^d$ *with compact support. We assume the two processes are driven by the same Poisson random measures. Then, for each fixed* $i \in \mathbb{N}$,

$$\lim_{N\to+\infty} \mathbf{E}\left[\sup_{t\in[0,T]} \|X_i^N(t) - Y_i^N(t)\|\right] = 0.$$

Proof. As in previous sections, by permutation invariance of the initial conditions and of the dynamics, we have

$$\mathbf{E}\left[\sup_{t\in[0,T]} \|X_i^N(t) - Y_i^N(t)\|\right] = \frac{1}{N}\sum_{i=1}^N \mathbf{E}\left[\sup_{t\in[0,T]} \|X_i^N(t) - Y_i^N(t)\|\right].$$

Let us start with

$$\mathbf{E}\left[\sup_{t\in[0,T]} \|X_i^N(t) - Y_i^N(t)\|\right] \leqslant \mathbf{E}\left[\int_0^T \|X_i^N(t) - Y_i^N(t)\|\,dt\right] + V_{X_i^N,Y_i^N}(T) + U_{X_i^N,Y_i^N}(T),$$

where, for simplicity, we have set:

$$\begin{aligned} V_{X_i^N,Y_i^N}(T) \doteq \mathbf{E}\Bigg[\sup_{t\in[0,T]} \Bigg\| &\frac{\mathbf{E}[V]}{N}\sum_{j=1}^N \int_0^t \lambda(X_j^N(s)) - \lambda(Y_j^N(s))ds - \frac{\mathbf{E}[V]}{N}\int_0^t \lambda(X_i^N(s))ds \\ &+ \frac{1}{N}\sum_{j\neq i}\int_0^t\int_{[0,1]^{\mathbb{N}}}\int_0^\infty V(h_i,h_j)\mathbb{1}_{[0,\lambda(X_j^N(s))}(u)\tilde{\mathcal{N}}^j(ds,du,dh)\Bigg\|\Bigg]; \end{aligned}$$

$$\begin{aligned} U_{X_i^N,Y_i^N}(T) \doteq \mathbf{E}\Bigg[\sup_{t\in[0,T]} \Bigg\| &-\int_0^t\int_{[0,1]^{\mathbb{N}}}\int_0^\infty (X_i^N(s) - U(h_i))\mathbb{1}_{[0,\lambda(X_i^N(s)))}(u) \\ &-(Y_i^N(s) - U(h_i))\mathbb{1}_{[0,\lambda(Y_i^N(s)))}(u)\mathcal{N}^i(ds,du,dh)\Bigg\|\Bigg]. \end{aligned}$$

With the notation of Lemma 2.3.5, we consider the positive constant H_T and the event

$$E_N \doteq \left\{ \frac{C_N(T)}{N} \leqslant H_T \right\} \cap \left\{ \int_0^T \langle \mu_Y^N(s), \lambda \rangle ds \leqslant H_T \right\},$$

such that $\mathbf{P}(E_N^c) \to 0$ for $N \to \infty$. Obviously, under the event E_N, for all $i = 1, \ldots, N$, the quantities $\sup_{t\in[0,T]} \lambda(X_i^N(t))$ and $\sup_{t\in[0,T]} \lambda(Y_i^N(t))$ are uniformly bounded and we can exploit local Lipschitzianity of λ (we will indicate its Lipschitz constant as L_{H_T}). Thus, we bound the first terms in $V_{X_i^N, Y_i^N}(T)$ in the following way:

$$\mathbf{E}\left[\sup_{t\in[0,T]} \left\| \frac{\mathbf{E}[V]}{N} \sum_{j=1}^N \int_0^t \lambda(X_j^N(s)) - \lambda(Y_j^N(s)) ds \right\| \right]$$
$$\leqslant \frac{\mathbf{E}[\|V\|]}{N} \sum_{j=1}^N \mathbf{E}\left[\left(\int_0^T L_{H_T} \|X_j^N(s) - Y_j^N(s)\| ds \right) \mathbb{1}_{E_N} \right] + \mathbf{E}[\|V\|]\, \mathbf{E}\left[\left(\int_0^T \frac{1}{N} \sum_{j=1}^N |\lambda(X_j^N(s))| + |\lambda(Y_j^N(s))| ds \right) \mathbb{1}_{E_N^C} \right]$$

$$\leqslant L_{H_T}\, \mathbf{E}[\|V\|] \int_0^T \frac{1}{N} \sum_{j=1}^N \mathbf{E}\left[\sup_{s\in[0,t]} \|X_j^N(s) - Y_j^N(s)\| \right] dt + \int_0^T \frac{\mathbf{E}[\|V\|]}{N} \sum_{j=1}^N \sqrt{\mathbf{P}(E_N^C)} \sqrt{\mathbf{E}\left[|\lambda(X_j^N(s))|^2 \right]} ds$$
$$+ \int_0^T \frac{\mathbf{E}[\|V\|]}{N} \sum_{j=1}^N \sqrt{\mathbf{P}(E_N^C)} \sqrt{\mathbf{E}\left[|\lambda(Y_j^N(s))|^2 \right]} ds$$
$$\leqslant L_{H_T}\, \mathbf{E}[\|V\|] \int_0^T \frac{1}{N} \sum_{j=1}^N \mathbf{E}\left[\sup_{s\in[0,t]} \|X_j^N(s) - Y_j^N(s)\| \right] dt + \int_0^T \mathbf{E}[\|V\|] \sqrt{\mathbf{P}(E_N^C)} \sqrt{\mathbf{E}\left[\langle \mu_X^N(s), |\lambda(\cdot)|^2 \rangle \right]} ds$$
$$+ \int_0^T \mathbf{E}[\|V\|] \sqrt{\mathbf{P}(E_N^C)} \sqrt{\mathbf{E}\left[\langle \mu_Y^N(s), |\lambda(\cdot)|^2 \rangle \right]} ds.$$

By Lemma 2.3.4 there exists $N_0 > 0$ such that for all $N > N_0$ $\sup_{t\geqslant 0} \mathbf{E}\left[\langle \mu_X^N(s), |\lambda(\cdot)|^2 \rangle \right]$ and $\sup_{t\geqslant 0} \mathbf{E}\left[\langle \mu_Y^N(s), |\lambda(\cdot)|^2 \rangle \right]$ are bounded. By Lemma 2.3.5, there exists a constant $K_T \geqslant 0$ such that $\mathbf{P}(E_N^C) \leqslant \frac{K_T}{N}$. The second term in $V^N_{X_i^N, Y_i^N}(T)$ is bounded by exchangeability of the X_i^N and by Lemma 2.3.4. Indeed, we have

$$\mathbf{E}\left[\sup_{t\in[0,T]} \left\| \frac{\mathbf{E}[V]}{N} \int_0^t \lambda(X_i^N(s)) ds \right\| \right] \leqslant \frac{\mathbf{E}[V]}{N} \int_0^T \sup_{t\in[0,T]} \mathbf{E}\left[\langle \mu_X^N(t), |\lambda(\cdot)| \rangle \right] dt.$$

To bound the third term we use Burkholder-Davis-Gundy inequality, the orthogonality of the martingales $\{\tilde{\mathcal{N}}^j\}_{j\in\mathbb{N}}$ and Lemma 2.3.4.

$$\mathbf{E}\left[\sup_{t\in[0,T]} \left\| \frac{1}{N} \sum_{j\neq 1}^N \int_0^t \int_0^t \int_{[0,1]^{\mathbb{N}}} \int_0^\infty V(h_i, h_j) \mathbb{1}_{(0, \lambda(X_j^N(s))](u)} \tilde{\mathcal{N}}^j(ds, du, dh) \right\| \right]$$
$$\leqslant \frac{M}{N} \mathbf{E}\left[\left(\sum_{j\neq i}^N \int_0^T \mathbf{E}[\|V\|^2] \lambda(X_j^N(s)) ds \right)^{1/2} \right] \leqslant \sqrt{\frac{\mathbf{E}[\|V\|^2]}{N}}\, \mathbf{E}\left[\left(\int_0^T \langle \mu_X^N(t), \lambda(\cdot) \rangle dt \right)^{1/2} \right].$$

Therefore we get that there exists three constants C_T,K_T and M_T such that, for all $N > N_0$,

$$V^N_{X_i^N,Y_i^N}(T) \leqslant C_T \int_0^T \mathbf{E}\left[\sup_{s\in[0,t]} \|X^N(s) - Y^N(s)\|^2\right] dt + \frac{K_T}{\sqrt{N}} + \frac{M_T}{N}.$$

With a similar argument, we get a bound of the same type for $U_{X_i^N,Y_i^N}(T)$.

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^N U_{X_i^N,Y_i^N}(T) \leqslant{}& C_T \int_0^T \frac{1}{N}\sum_{i=1}^N \|X_i^N(t) - Y_i^N(t)\| dt + \mathbf{E}\left[\mathbb{1}_{E_N^C}\int_0^T \frac{1}{N}\sum_{i=1}^N \|X_i^N(t)\|\lambda(X_i^N(t))dt\right]\\ &+ \mathbf{E}\left[\mathbb{1}_{E_N^C}\int_0^T \frac{1}{N}\sum_{i=1}^N \|Y_i^N(t)\|\lambda(Y_i^N(t))dt\right] + \mathbf{E}[\|U\|]\,\mathbf{E}\left[\mathbb{1}_{E_N^C}\int_0^T \frac{1}{N}\sum_{i=1}^N \lambda(X_i^N(t))dt\right]\\ &+ \mathbf{E}[\|U\|]\,\mathbf{E}\left[\mathbb{1}_{E_N^C}\int_0^T \frac{1}{N}\sum_{i=1}^N \lambda(Y_i^N(t))dt\right].\end{aligned}$$

As before, we wish to get a bound of the order $O\left(\frac{1}{\sqrt{N}}\right)$ for the last terms. We do that by means of Cauchy-Schwartz inequality, Lemma 2.3.4 and Lemma 2.3.5. We also exploit that, by definition of λ , it holds $\|x\| \leqslant B\lambda(x) + c$ for a positive constant B and a constant c. Take, for instance, the second term of the right-hand side, it holds

$$\begin{aligned}\mathbf{E}\left[\mathbb{1}_{E_N^C}\int_0^T \frac{1}{N}\sum_{i=1}^N \|X_i^N(t)\|\lambda(X_i^N(t))dt\right] &\leqslant \int_0^T \sqrt{\mathbf{P}(E_N^C)}\sqrt{\mathbf{E}\left[\left(\frac{1}{N}\sum_{i=1}^N \|X_i^N(s)\|\lambda(X_i^N(s))\right)^2\right]}ds\\ &\leqslant T\sqrt{\mathbf{P}(E_N^C)}\sqrt{\mathbf{E}\left[\sup_{t\in[0,T]} \langle \mu_X^N(t), \|\cdot\|^2\rangle\langle \mu_X^N(t), \lambda(\cdot)^2\rangle\right]}\\ &\leqslant T\sqrt{\mathbf{P}(E_N^C)}\sqrt{\mathbf{E}\left[\sup_{t\in[0,T]} \langle \mu_X^N(t), \lambda(\cdot)^4\rangle\right]}.\end{aligned}$$

The same holds for the remaining right-hand side terms. Thus, there exists two constants $\tilde{C}_T$ and $\tilde{K}_T$ and a $N_0 > 0$, such that for all $N > N_0$ it holds

$$\frac{1}{N}\sum_{i=1}^N U_{X_i^N,Y_i^N}(T) \leqslant \tilde{C}_T \int_0^T \mathbf{E}\left[\sup_{s\in[0,t]} \|X^N(s) - Y^N(s)\|\right] dt + \frac{\tilde{K}_T}{\sqrt{N}}.$$

Thus, there exist three constants, that with abuse of notation we will indicate as C_T,K_T and M_T, depending only on T, and $N_0 >$ such that, for all $N > N_0$ it holds

$$\mathbf{E}\left[\sup_{t\in[0,T]} \|X^N(t) - Y^N(t)\|\right] \leqslant C_T \int_0^T \mathbf{E}\left[\sup_{s\in[0,t]} \|X^N(s) - Y^N(s)\|\right] dt + \frac{K_T}{\sqrt{N}} + \frac{M_T}{N}.$$

By applying Gronwall lemma we get the thesis. $\square$

Theorem 2.3.3 (Propagation of Chaos for Y^N). *Grant Assumptions 2.3.1. Let* μ_0 *be a probability measure on* $\mathbb{R}^d$ *with compact support. For* $N \in \mathbb{N}$*, let* Y^N *be a solution of Eq.* (2.3.6) *in* $[0,T]$*. Assume that* $Y^N(0) = (Y_1^N(0), \dots, Y_N^N(0))$*,* $N \in \mathbb{N}$*, form a sequence of compact support random vectors that is* μ_0*-chaotic in* W_1*. Let* Q *be the law of the solution of Eq.* (2.3.3) *in* $[0,T]$ *with initial law* $\mathbf{P} \circ X(0)^{-1} = \mu_0$*. Then* Y^N *is* Q *chaotic in* W_1*.*

The proof of this theorem is a combination of the computations done for proving Theorem 2.3.2 and the coupling techniques for propagation of chaos used in the previous sections.

Proof of Theorem 2.3.3. We follow the steps of Proposition 2.1.2 and 2.2.2 to define the coupling procedure. We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbf{P})$ with respect to the family $(B_i, \mathcal{N}^i)_{i \in \mathbb{N}}$ of independent Brownian motions and Poisson random measures. For each $N \in \mathbb{N}$, we couple the process Y^N with the process $\bar{X}^N = \left\{\bar{X}_i^N(t), i = 1, \dots, N\right\}_{t \in [0,T]}$ defined thanks to Theorem 2.2.1, where the initial condition is $Law(\bar{X}^N(0)) = \otimes^N \mu_0$ and each component $\bar{X}_i^N$ is a solution of SDE (2.2.1). We start with

$$\mathbf{E}\left[\sup_{t \in [0,T]} \|Y_i^N(t) - \bar{X}_i^N(t)\|\right] \leqslant \mathbf{E}\left[\int_0^T \left\|Y_i^N(t) - \bar{X}_i^N(t)\right\| dt\right] + U_{Y_i^N, \bar{X}_i^N}(T)$$
$$+ \mathbf{E}\left[\sup_{t \in [0,T]} \left\| \frac{\mathbf{E}[V]}{N} \sum_{j=1}^N \int_0^t \lambda(Y_j^N(s)) - \mathbf{E}[\lambda(\bar{X}_i^N(s))] ds \right\| \right],$$

where $U_{Y_i^N, \bar{X}_i^N}(T)$ is defined as in the proof of Theorem 2.3.2. We use Lemma 2.3.3, 2.3.4 and 2.3.5, together with local Lipschitzianity of λ and μ_0-choaticity of the initial conditions to get

$$\frac{1}{N} \sum_{i=1}^N \mathbf{E}\left[\sup_{t \in [0,T]} \|Y_i^N(t) - \bar{X}_i^N(t)\|\right] \overset{N \to \infty}{\to} 0,$$

that implies Q chaoticity of the law of Y^N. □

As in the previous sections, these results imply propagation of chaos for X^N, as the following corollary states.

Corollary 2.3.1 (Propagation of Chaos for X^N). *Grant Assumptions 2.3.1. Let* μ_0 *be a probability measure on* $\mathbb{R}^d$ *with compact support. For* $N \in \mathbb{N}$*, let* X^N *be a solution of Eq.* (2.3.1) *in* $[0,T]$*. Assume that* $X^N(0) = (X_1^N(0), \dots, X_N^N(0))$*,* $N \in \mathbb{N}$*, form a sequence of compact support random vectors that is* μ_0*-chaotic in* W_1*. Let* Q *be the law of the solution of Eq.* (2.3.3) *in* $[0,T]$ *with initial law* $\mathbf{P} \circ X(0)^{-1} = \mu_0$*. Then* X^N *is* Q *chaotic in* W_1*.*

2.3.4 Additional lemmas and proofs

We collect here the proofs of the various lemmas stated in Section 2.3 and some other technical result necessary for these proofs. First, we prove Lemma 2.3.2 and, thanks to two technical lemmas, we give the proof of Lemma 2.3.3, crucial for the existence and uniqueness of solution of the nonlinear process (2.3.3). Then, we give the proofs of Lemma 2.3.4 and Lemma 2.3.5, that we use in the propagation of chaos section. Notice that, the key ingredient here is represented by the fact that all the main jumps of the processes are such that they make the process go back inside a compact set (the support of U). To exploit that, we need to apply Ito's rule for a process with jumps (notice that here we do not have a diffusion term). Since all the functions of interest ($\|\cdot\|$ and $\lambda(\cdot)$) have singularities in the origin, we use the smooth approximation of the norm $\|\cdot\|$ defined in the proof of Theorem 2.2.1, for all $\epsilon > 0$, we define

$$f^{\epsilon}(x) \doteq \|x\|\mathbb{1}(\|x\| > \epsilon) + \left(\frac{\|x\|^2}{2\epsilon} + \frac{\epsilon}{2}\right)\mathbb{1}(\|x\| \leqslant \epsilon).$$

We start with the proof of existence and uniqueness of solutions of (2.3.3) for compact support initial condition. This proof relies on a straightforward adaptation of the arguments of [75] to our framework, therefore we write the proof of Lemma 2.3.2 only for completeness.

Proof of Lemma 2.3.2. We want to get an almost sure bound for $\|Z_f(t)\|$, in order to use locally Lipschitzianity of λ in the following computations. Intuitively, the jumps have an increasing role only if we are inside the support of the random variable U, otherwise they force the norm to decrease. Therefore, a.s., we can bound the process $\|Z_f(t)\|$ with the deterministic expression

$$K_0 + \int_0^t \|f(s)\|ds,$$

where $K_0 \doteq \max\{\|x\|, \sup_{h\in[0,1]} \|U(h)\|\}$. This almost sure bound for $\|Z_f(t)\|$ and the continuity of the coefficients ensure the existence and uniqueness of a non-explosive solution Z_f on $[0, T]$. Let Z_f and Z_g two solutions of (2.3.4) corresponding to two different locally bounded Borellian functions f and g, we have

$$\begin{aligned}\mathbf{E}\left[\sup_{t\in[0,T]} \|Z_f(t) - Z_g(t)\|\right] &\leqslant \int_0^T \mathbf{E}\left[\sup_{s\in[0,t]} \|Z_f(s) - Z_g(s)\|\right] ds + \int_0^T \sup_{s\in[0,t]} \|f(s) - g(s)\|dt \\ &+ \mathbf{E}\left[\int_0^T \int_{[0,1]\times[0,\infty)} \|(Z_f(s^-) - U(h))\mathbb{1}_{[0,\lambda(Z_f(s^-)))}(u) - (Z_g(s^-) - U(h))\mathbb{1}_{[0,\lambda(Z_g(s^-)))}(u)\| \, ds du \nu_1(dh)\right].\end{aligned}$$

The almost sure bounds on $\|Z_f(t)\|$ and $\|Z_g(t)\|$ let us define two positive constant $b_{f,g}(T)$ and $L_{f,g}(T)$, such that we get

$$\begin{aligned}\mathbf{E}\left[\sup_{t\in[0,T]} \|Z_f(t) - Z_g(t)\|\right] \leqslant & \int_0^T \mathbf{E}\left[\sup_{s\in[0,t]} \|Z_f(s) - Z_g(s)\|\right] ds + \int_0^T \sup_{s\in[0,t]} \|f(s) - g(s)\|dt \\ & + (b_{f,g}(T) + H)\int_0^T \mathbf{E}\left[\sup_{s\in[0,t]} \|Z_f(s) - Z_g(s)\|\right] ds \\ & + L_{f,g}(T)\left(\sup_{t\in[0,T]} \|Z_f(t)\|\right)\int_0^T \mathbf{E}\left[\sup_{s\in[0,t]} \|Z_f(s) - Z_g(s)\|\right] ds.\end{aligned}$$

We apply now Gronwall lemma and we obtain (2.3.5). $\square$

The proof of Lemma 2.3.3 requires two technical lemmas adapted to our case from [75].

Lemma 2.3.6. *Let* $x(t)$ *be a non-negative* C^1 *function on* $\mathbb{R}_+$*. If the following inequality holds for any* $0 \leqslant s \leqslant t$*:*

$$x(t) \leqslant x(s) - \bar{K}\int_s^t x^k(u)du + \int_s^t P_\delta\left(x(u)\right)du$$

where $k, \bar{K} > 0$ *and* $P_\delta(\cdot)$ *is a polynomial of degree* $\delta < k$*, then*

$$\sup_{t \geqslant 0} x(t) \leqslant C_0 < \infty.$$

Proof. Consider that for $x \to \infty$, then

$$-\bar{K}x^k + P_\delta(x) \to -\infty.$$

Therefore it exists a value $\bar{C}_0$ such that, as soon as the trajectory exceeds $\bar{C}_0 \geqslant 0$ its derivative becomes strictly negative and the trajectory is forced toward zero. Thus, defining

$$C_0 := \max\{\bar{C}_0, x(0)\},$$

we get the desired bound. □

Lemma 2.3.7. *If the function* b *satisfies the assumption* (2.3.2)*, then for any* $\epsilon > 0$ *and* $p \in [1, 4+2\epsilon]$*, there exists a constant* $\gamma_1 < (4+2\epsilon)\gamma$*,* $c_1 > 0$ *and a value* $\eta > 0$*, such that, for all* $a \in \mathbb{R}^d$ *with* $\|a\| \leqslant \eta$ *and for all* $x \in \mathbb{R}^d$*, it holds*

$$|b^p(\|x+a\|) - b^p(\|a\|)| \leqslant \|a\|\left(\gamma_1 b^p(\|x\|) + c_1\right).$$

Proof. The proof of this lemma comes directly from Lemma 8 in the appendix of [75]. □

Notice that the constant $\gamma_1 < (4+2\epsilon)\gamma$, together with the condition of Lemma 2.3.6 on the negativity of the coefficient $\bar{K}$, cause the condition on γ w.r.t. $\mathbf{E}[\|V\|]$ in Assumption 2.3.1. This condition plays a crucial role in all the proofs of the boundedness for the moments of $\lambda(X(t))$ and of $\lambda(X_i^N(t))$ for all i. Now that we have stated these two results, we are ready to prove Lemma 2.3.3, that provides a priori uniform bounds on the first moment of the solution to (2.3.3) and on the moments of $\lambda(X(t))$.

Proof of Lemma 2.3.3. Fix $\epsilon > 0$, by means of Ito's rule, we have

$$\begin{aligned}\mathbf{E}\left[f^\epsilon(X(t))\right] \leqslant \mathbf{E}\left[f^\epsilon(X(0))\right] &- \int_0^t \mathbf{E}\left[\|X(s)\|\mathbb{1}(\|X(s)\| > \epsilon)\right]ds \\ &- \int_0^t \mathbf{E}\left[\epsilon\mathbb{1}(\|X(s)\| \leqslant \epsilon)\right]ds + \int_0^t \mathbf{E}\left[(\mathbf{E}\left[\|V\|\right] + \mathbf{E}\left[\|U\|\right] - f^\epsilon(X(s)))h(X(s))\right]ds \\ &+ \int_0^t \mathbf{E}\left[b(\|X(s)\|)\left(\mathbf{E}\left[\|V\|\right] + \mathbf{E}\left[\|U\|\right] - f^\epsilon(X(s))\right)\right]ds.\end{aligned}$$

For the monoticity assumption on b, we know that there exist $\Lambda > 0$ and $\beta \geqslant 0$ such that $b(r)\left(\mathbf{E}[\|V\|] + \mathbf{E}[\|U\|] - r\right) \leqslant -\Lambda r + \beta$. Therefore, by Fatou's lemma and monotone convergence theorem,

$$\mathbf{E}[\|X(t)\|] \leqslant \mathbf{E}[\|X(0)\|] + \int_0^t [H(\mathbf{E}[\|V\|] + \mathbf{E}[\|U\|]) + \beta]\, ds - \Lambda \int_0^t \mathbf{E}[\|X(s)\|]\, ds,$$

that gives the boundedness of $\sup_{t\geqslant 0} \mathbf{E}[\|X(t)\|]$.

Let $p = 1$, clearly, to get a bound for $\mathbf{E}[\lambda(X(t))]$, it is sufficient to bound $\mathbf{E}[b(\|X(t)\|)]$. Thus, again, we use Ito's rule to compute $b(f^\epsilon(X(t)))$ for $\epsilon > 0$.

$$\begin{aligned}
\mathbf{E}[b(f^\epsilon(X(t)))] \leqslant \mathbf{E}[b(f^\epsilon(X(0)))] &- \int_0^t \mathbf{E}\left[b'(f^\epsilon(X(s)))\|X(s)\|\mathbb{1}(\|X(s)\| > \epsilon)\right] ds \\
&- \int_0^t \mathbf{E}\left[b'(f^\epsilon(X(s)))\frac{\|X(s)\|^2}{\epsilon}\mathbb{1}(\|X(s)\| \leqslant \epsilon)\right] ds \\
&+ \int_0^t \mathbf{E}\left[b'(f^\epsilon(X(s)))\,\mathbf{E}[b(\|X(s)\|)]\,\frac{X(s)\cdot \mathbf{E}[V]}{\|X(s)\|}\mathbb{1}(\|X(s)\| > \epsilon)\right] ds \\
&+ H\int_0^t \mathbf{E}\left[b'(f^\epsilon(X(s)))\frac{X(s)\cdot \mathbf{E}[V]}{\|X(s)\|}\mathbb{1}(\|X(s)\| > \epsilon)\right] ds \\
&+ \int_0^t \mathbf{E}\left[b'(f^\epsilon(X(s)))\,\mathbf{E}[b(\|X(s)\|)]\,\frac{X(s)\cdot \mathbf{E}[V]}{\epsilon}\mathbb{1}(\|X(s)\| \leqslant \epsilon)\right] ds \\
&+ H\int_0^t \mathbf{E}\left[b'(f^\epsilon(X(s)))\frac{X(s)\cdot \mathbf{E}[V]}{\epsilon}\mathbb{1}(\|X(s)\| \leqslant \epsilon)\right] ds \\
&+ \int_0^t \mathbf{E}[b(\|X(s)\|)]\,\mathbf{E}[b(f^\epsilon(U))]\, ds + \int_0^t \mathbf{E}[h(X(s))]\,\mathbf{E}[b(f^\epsilon(U))]\, ds \\
&- \int_0^t \mathbf{E}[b(f^\epsilon(X(s)))b(\|X(s)\|)]\, ds - \int_0^t \mathbf{E}[b(f^\epsilon(X(s)))]\,\mathbf{E}[h(X(s))]\, ds.
\end{aligned}$$

Again we use Fatou's lemma and monotone convergence theorem (indeed $b(f^\epsilon(\cdot))$ converges monotonically to $b(\|\cdot\|)$, thanks to the increasing property of b). Since b' is positive, we disregard the two terms with minus sign in the first two rows, we use properties of b' to bound the remaining terms and we get

$$\begin{aligned}
\mathbf{E}[b(\|X(t)\|)] \leqslant{}& \mathbf{E}[b(\|X(0)\|)] + (Hc\,\mathbf{E}[\|V\|] + H\,\mathbf{E}[b(\|U\|)])\,t + (\gamma\,\mathbf{E}[\|V\|] - 1)\int_0^t \mathbf{E}[b(\|X(s)\|)]^2\, ds \\
&(c\,\mathbf{E}[\|V\|] + H\gamma\,\mathbf{E}[\|V\|] + \mathbf{E}[b(\|U\|)] + H)\int_0^t \mathbf{E}[b(\|X(s)\|)]\, ds.
\end{aligned}$$

With Lemma 2.3.6 we conclude the boundedness for $\mathbf{E}[b(\|X(t)\|)]$. The same argument is used to get a uniform bound for $\mathbf{E}[b^p(\|X(t)\|)]$ when $p = 2,3,4$. □

While the uniform bounds for $\mathbf{E}[\|X(t)\|]$ and $\mathbf{E}[b(\|X(t)\|)]$ are needed for the well-posedness of the nonlinear process itself, higher moments of λ are needed only for the proof of propagation of chaos. The same a priori bounds for the moments of λ appear also in the case of the particle system. Their proof is similar to the nonlinear case, relies on Lemma 2.3.6 and Lemma 2.3.7, together with an argument based on orthogonal martingales.

Proof of Lemma 2.3.4. We only prove it for μ_X^N, then for μ_Y^N the steps are basically the same. Fix $\delta > 0$, of course it is sufficient to prove the boundedness of

$$\sup_{N \geqslant N_0} \sup_{t \geqslant 0} \mathbf{E}[\langle \mu_X^N(t), b^4(f^\delta(\cdot))\rangle].$$

Let us define for $K > 0$ the stopping time

$$\tau_K := \inf\left\{t \geqslant 0 : \langle \mu_X^N(t), b^5(f^\delta(\cdot))\rangle \geqslant K\right\}.$$

Obviously the random variables $\langle \mu_X^N(t\wedge\tau_K), b^p(f^\delta(\cdot))\rangle$ for $1 \leqslant p \leqslant 5$ and $\langle \mu_X^N(t\wedge\tau_K), f^\delta(\cdot)\rangle$ are integrable. Recall that, for all $\epsilon > 0$, the process $\{M_\epsilon^N(t)\}_{t\geqslant[0,T]}$ is a martingale; where, for $t \in [0,T]$ we have

$$\begin{aligned}
M_\epsilon^N(t) \doteq\; & \langle \mu_X^N(t), f^\epsilon(\cdot)\rangle - \langle \mu_X^N(0), f^\epsilon(\cdot)\rangle \\
& + \frac{1}{N}\sum_{i=1}^N \left(\int_0^t \frac{X_i^N(s)\cdot X_i^N(s)}{\|X_i^N(s)\|}\mathbb{1}(\|X_i^N(s)\| > \epsilon)ds + \int_0^t \frac{X_i^N(s)\cdot X_i^N(s)}{\epsilon}\mathbb{1}(\|X_i^N(s)\| \leqslant \epsilon)ds\right) \\
& - \frac{1}{N}\sum_{i=1}^N \sum_{j\neq i} \int_0^t \int_{[0,1]^{\mathbb{N}}} \lambda(X_j^N(s)) \left(f^\epsilon\left(X_i^N(s) + \frac{V(h_i,h_j)}{N}\right) - f^\epsilon\left(X_i^N(s)\right)\right)\nu(dh)ds \\
& - \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{[0,1]}^{\mathbb{N}} \lambda(X_i^N(s)) \left(f^\epsilon\left(U(h_i)\right) - f^\epsilon\left(X_i^N(s)\right)\right)\nu(dh)ds.
\end{aligned}$$

Then, for the optional stopping theorem, it holds

$$\begin{aligned}
\mathbf{E}\left[\langle \mu_X^N(t\wedge\tau_K), f^\epsilon(\cdot)\rangle\right] \leqslant\; & \mathbf{E}\left[\mu_X^N(0), f^\epsilon(\cdot)\rangle\right] - \mathbf{E}\left[\int_0^{t\wedge\tau_K} \langle \mu_X^N(s), \|\cdot\|\mathbb{1}(\|\cdot\| > \epsilon)\rangle ds\right] \\
& - \mathbf{E}\left[\int_0^{t\wedge\tau_K} \langle \mu_X^N(s), \frac{\|\cdot\|^2}{\epsilon}\mathbb{1}(\|\cdot\| \leqslant \epsilon)\rangle ds\right] \\
& + N\,\mathbf{E}\left[\int_0^{t\wedge\tau_K} \langle \mu_X^N(s), \lambda(\cdot)\rangle \langle \mu_X^N(s), \int_{[0,1]^2} f^\epsilon\left(\cdot + \frac{V(h_1,h_2)}{N}\right) - f^\epsilon(\cdot)\nu_2(dh)\rangle ds\right] \\
& - \mathbf{E}\left[\int_0^{t\wedge\tau_K} \langle \mu_X^N(s), \lambda(\cdot)\int_{[0,1]} f^\epsilon\left(\cdot + \frac{V(h_1,h_1)}{N}\right) - f^\epsilon(\cdot)\nu_1(dh)\rangle ds\right] \\
& + \mathbf{E}\left[\int_0^{t\wedge\tau_K} \mathbf{E}[f^\epsilon(U)]\langle \mu_X^N(s), \lambda(\cdot)\rangle - \langle \mu_X^N(s), \lambda(\cdot)f^\epsilon(\cdot)\rangle ds\right].
\end{aligned}$$

Again, we use the monotone convergence of $f^\epsilon(x)$ to $\|x\|$, to get

$$\begin{aligned}
\mathbf{E}\left[\mathbb{1}(t \leqslant \tau_K)\langle \mu_X^N(t), \|\cdot\|\rangle\right] &\leqslant \liminf_{\epsilon\downarrow 0} \mathbf{E}\left[\mathbb{1}(t \leqslant \tau_K)\langle \mu_X^N(t), f^\epsilon(\cdot)\rangle\right] \\
&\leqslant \liminf_{\epsilon\downarrow 0} \mathbf{E}\left[\langle \mu_X^N(t\wedge\tau_K), f^\epsilon(\cdot)\rangle\right].
\end{aligned}$$

By arguments close to the one in the proof of Lemma 2.3.3, there exists $\Lambda > 0$ and $\beta > 0$, such that we get the following inequality

$$\begin{aligned}
\mathbf{E}\left[\mathbb{1}(t \leqslant \tau_K)\langle \mu_X^N(t), \|\cdot\|\rangle\right] &\leqslant \mathbf{E}\left[\mu_X^N(0), \|\cdot\|\rangle\right] \\
&+ \int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), \left(\mathbf{E}[\|V\|] + \frac{\mathbf{E}[\|V\|]}{N} + \mathbf{E}[\|U\|] - \|\cdot\|\right)\lambda(\cdot)\rangle\right] ds \\
&\leqslant \mathbf{E}\left[\mu_X^N(0), \|\cdot\|\rangle\right] \\
&+ \left[H\left(\mathbf{E}[\|V\|] + \frac{\mathbf{E}[\|V\|]}{N} + \mathbf{E}[\|U\|]\right) + \beta\right] t - \Lambda \int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), \|\cdot\|\rangle\right] ds.
\end{aligned}$$

This, together with Lemma 2.3.6, gives the boundedness of

$$\sup_{t \geqslant 0} \mathbf{E}\left[\mathbb{1}(t \leqslant \tau_K)\langle \mu_X^N(t), \|\cdot\|\rangle\right].$$

Since this bound does not depend on K, letting K go to infinity gives the bound on

$$\sup_{t \geqslant 0} \mathbf{E}\left[\langle \mu_X^N(t), \|\cdot\|\rangle\right].$$

Now we apply the same argument to the martingale $\{M_{b^4}^N(t)\}_{t \geqslant [0,T]}$. By deleting some of the negative terms, applying Lemma 2.3.7 and repeating the previous steps, we obtain the following bound

$$\begin{aligned}
&\mathbf{E}\left[\mathbb{1}(\tau_K \leqslant t)\langle \mu_X^N(t), b^4(\|\cdot\|)\rangle\right] \leqslant \mathbf{E}\left[\langle \mu_X^N(0), b^4(\|\cdot\|)\rangle\right] \\
&+ \gamma_1 \mathbf{E}[\|V\|] \int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle\langle \mu_X^N(s), b(\|\cdot\|)\rangle\right] ds \\
&+ H\gamma_1 \mathbf{E}[\|V\|] \int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle\right] ds \\
&+ c_1 \mathbf{E}[\|V\|] \int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b(\|\cdot\|)\rangle\right] ds + c_1 H \mathbf{E}[\|V\|] \int_0^t \mathbf{E}[\mathbb{1}(s \leqslant \tau_K)] ds \\
&+ \gamma_1 \frac{\mathbf{E}[\|V\|]}{N} \int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^5(\|\cdot\|)\rangle\right] ds + c_1 \frac{\mathbf{E}[\|V\|]}{N} \int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b(\|\cdot\|)\rangle\right] ds \\
&+ H\gamma_1 \frac{\mathbf{E}[\|V\|]}{N} \int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle\right] ds + c_1 H \frac{\mathbf{E}[\|V\|]}{N} \int_0^t \mathbf{E}[\mathbb{1}(s \leqslant \tau_K)] ds \\
&+ \mathbf{E}\left[b^4(\|U\|)\right] \int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b(\|\cdot\|)\rangle\right] ds + \mathbf{E}\left[b^4(\|U\|)\right] Ht \\
&- \int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^5(\|\cdot\|)\rangle\right] ds + H \int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle\right] ds.
\end{aligned}$$

By Hölder and Jensen inequalities, we get the following expression

$$
\begin{aligned}
&\mathbf{E}\left[\langle \mathbb{1}(\tau_K \leqslant t)\mu_X^N(t), b^4(\|\cdot\|)\rangle\right] \leqslant \mathbf{E}\left[\langle \mu_X^N(0), b^4(\|\cdot\|)\rangle\right] \\
&+\gamma_1 \mathbf{E}\left[\|V\|\right]\int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle\right]^{5/4} ds + H\gamma_1 \mathbf{E}\left[\|V\|\right]\int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle\right] ds \\
&+ c_1 \mathbf{E}\left[\|V\|\right]\int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle\right]^{1/4} ds + c_1 H \mathbf{E}\left[\|V\|\right] t \\
&+ c_1 \frac{\mathbf{E}\left[\|V\|\right]}{N}\int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle\right]^{1/4} ds + H\gamma_1 \frac{\mathbf{E}\left[\|V\|\right]}{N}\int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle\right] ds \\
&+ \left(c_1 H \frac{\mathbf{E}\left[\|V\|\right]}{N} + \mathbf{E}\left[b^4(\|U\|)\right]\right) t + \mathbf{E}\left[b^4(\|U\|)\right]\int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle^{1/4}\right] ds \\
&+ H\int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle\right] ds + \left(\gamma_1 \frac{\mathbf{E}\left[\|V\|\right]}{N} - 1\right)\int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle\right]^{5/4} ds,
\end{aligned}
$$

where we have exploited the fact that $\left(\gamma_1 \frac{\mathbf{E}[\|V\|]}{N} - 1\right) < 0$, for N large enough, and that $\langle \mu_X^N(s), b^5\rangle \geqslant \langle \mu_X^N(s), b^4\rangle^{5/4}$. Reordering, we get

$$
\begin{aligned}
\mathbf{E}\left[\mathbb{1}(\tau_K \leqslant t)\langle \mu_X^N(t), b^4(\|\cdot\|)\rangle\right] \leqslant &\langle \mathbf{E}\left[\mu_X^N(0), b^4(\|\cdot\|)\rangle\right] \\
&+ \left(c_1 \mathbf{E}[\|V\|] + c_1 \frac{\mathbf{E}[\|V\|]}{N} + \mathbf{E}\left[b^4(\|U\|)\right]\right)\int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle\right]^{1/4} ds \\
&+ \left(H\gamma_1 \mathbf{E}[\|V\|] + H\gamma_1 \frac{\mathbf{E}[\|V\|]}{N} + H\right)\int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle\right] ds \\
&+ \left(\gamma_1 \mathbf{E}\left[\|V\|\right] + \gamma_1 \frac{\mathbf{E}\left[\|V\|\right]}{N} - 1\right)\int_0^t \mathbf{E}\left[\mathbb{1}(s \leqslant \tau_K)\langle \mu_X^N(s), b^4(\|\cdot\|)\rangle\right]^{5/4} ds.
\end{aligned}
$$

Since, by hypothesis, there exists N_0 such that, for all $N \geqslant N_0$ it holds

$$
\left(\gamma_1 \mathbf{E}\left[\|V\|\right] + \gamma_1 \frac{\mathbf{E}\left[\|V\|\right]}{N} - 1\right) < 0,
$$

we use Proposition 2.3.6 and this gives a bound on $\mathbf{E}\left[\mathbb{1}(t \leqslant \tau_K)\langle \mu_X^N(t), b^4(\|\cdot\|)\rangle\right]$ independent of N and K; therefore letting K go to infinity proves the thesis. □

As mentioned before, Lemma 2.3.4 plays a crucial role in the proof of Lemma 2.3.5, where we bound the number of jumps of a single particle for the particle system (2.3.1) and the contribution of the collateral drift term for the particle system (2.3.6).

Proof. of Lemma 2.3.5. We develop the computations for the proof just in the case of (2.3.1), since for the system (2.3.6) they are almost the same. Let us start by describing the quantity $C_N(T)$, that is

$$
C_N(T) = \sum_{i=1}^N \int_0^T \int_{[0,1]^{\mathbb{N}}} \int_0^\infty \mathbb{1}_{[0,\lambda(X_i^N(s))}(u)\mathcal{N}^i(ds, du, dh).
$$

We can rewrite this quantity as the sum of orthogonal martingales, that we will indicate as $M^N(t)$, plus a term depending on the empirical measure, as follows:

$$\frac{C_N(T)}{N} = \frac{1}{N}\sum_{i=1}^{N}\int_0^T\int_{[0,1]^{\mathbb{N}}}\int_0^\infty \mathbb{1}_{[0,\lambda(X_i^N)(s)}(u)\tilde{\mathcal{N}}^i(ds,du,dh) + \int_0^T \langle \mu_X^N(s), \lambda(\cdot)\rangle ds$$
$$\doteq M^N(T) + \int_0^T \langle \mu_X^N(s), \lambda(\cdot)\rangle ds.$$

Let us consider a positive constant $H_T > 0$, then

$$\mathbf{P}\left(\frac{C_N(T)}{N} \geqslant H_T\right) \leqslant \mathbf{P}\left(M^N(T) \geqslant H_T\right) + \mathbf{P}\left(\int_0^T \langle \mu_X^N(s), \lambda\rangle ds \geqslant H_T\right).$$

Of course, since $\{M^N(t)\}_{t\in[0,T]}$ is a martingale, we have

$$\mathbf{P}\left(M^N(T) \geqslant H_T\right) \leqslant \frac{\mathbf{E}[M^N(T)]}{H_T} = 0.$$

Therefore, we want to get a bound for the probability $\mathbf{P}\left(\int_0^T \langle \mu_X^N(s), \lambda\rangle ds \geqslant H_T\right)$. Let $\delta > 0$ be fixed, the first step consists in proving that there exists $C_T > 0$ such that

$$\mathbf{E}\left[\sup_{t\in[0,T]} M_{b,\delta}^N(t)^2\right] \leqslant \mathbf{E}\left[\langle M_{b,\delta}^N(T)\rangle\right] \leqslant \frac{C_T}{N},$$

where $\{M_{b,\delta}^N(t)\}_{t\in[0,T]}$ is the martingale arising from the compensated Poisson measure in the computation of $\langle \mu_X^N(t), b(f^\delta(\cdot))\rangle$ with Ito rule, that is

$$M_{b,\delta}^N(t) \doteq \frac{1}{N}\sum_{i=1}^{N}\int_0^t\int_{[0,1]^{\mathbb{N}}}\int_0^\infty \mathbb{1}_{(0,\lambda(X_i^N(s))]}\Big[b(f^\delta(U(h_i))) - b(f^\delta(X_i^N(s)))$$
$$+ \sum_{j\neq i} b\left(f^\delta\left(X_j^N(s) + \frac{V(h_i,h_j)}{N}\right)\right) - b\left(f^\delta\left(X_j^N(s)\right)\right)\Big]\tilde{\mathcal{N}}^i(ds,du,dh)$$

and $\langle M_{b,\delta}^N(t)\rangle$ is its quadratic variation. We use the fact that $\{\tilde{\mathcal{N}}^i\}_{i=1,2\ldots}$ is a family of orthogonal martingales, therefore

$$\langle M_{b,\delta}^N(t)\rangle = \frac{1}{N^2}\sum_{i=1}^{N}\int_0^t\int_{[0,1]^{\mathbb{N}}} \lambda(X_i^N(s))\Big[b(f^\delta(U(h_i))) - b(f^\delta(X_i^N(s)))$$
$$+ \sum_{j\neq i} b\left(f^\delta\left(X_j^N(s) + \frac{V(h_i,h_j)}{N}\right)\right) - b\left(f^\delta\left(X_j^N(s)\right)\right)\Big]^2 \nu(dh)ds.$$

Let us write

$$\langle M_{b,\delta}^N(t)\rangle \doteq \frac{1}{N^2}\sum_{i=1}^{N} M_{b,\delta,i}^N(t).$$

We fix i and we compute $M_{b,\delta,i}(t)$ as follows.

$$\begin{aligned}
M_{b,\delta,i}(t) \leqslant & \, 2\int_0^t \int_{[0,1]^{\mathbb{N}}} b(f^\delta(X_i^N(s)))b^2(f^\delta(U)) + Hb^2(f^\delta(U)) + b^3(f^\delta(X_i^N(s))) + Hb^2(f^\delta(X_i^N(s)))\nu(dh)ds \\
& + \int_0^t \int_{[0,1]^{\mathbb{N}}} b(f^\delta(X_i^N(s)))(N-1)\sum_{j\neq i}\left(\frac{f^\delta(V)}{N}(\gamma_1 b(f^\delta(X_j^N(s))) + c_1)\right)^2 \nu(dh)ds \\
& + \int_0^t \int_{[0,1]^{\mathbb{N}}} H(N-1)\sum_{j\neq i}\left(\frac{f^\delta(V)}{N}(\gamma_1 b(f^\delta(X_j^N(s))) + c_1)\right)^2 \nu(dh)ds \\
& + 2\int_0^t \int_{[0,1]^{\mathbb{N}}} b(f^\delta(X_i^N(s)))(b(f^\delta(U)) - b(f^\delta(X_i^N(s))))\sum_{j\neq i}\left(\frac{f^\delta(V)}{N}(\gamma_1 b(f^\delta(X_j^N(s))) + c_1)\right)\nu(dh)ds \\
& + 2H\int_0^t \int_{[0,1]^{\mathbb{N}}} (b(f^\delta(U)) - b(f^\delta(X_i^N(s))))\sum_{j\neq i}\left(\frac{f^\delta(V)}{N}(\gamma_1 b(f^\delta(X_j^N(s))) + c_1)\right)\nu(dh)ds
\end{aligned}$$

$$\begin{aligned}
M_{b,\delta,i}(t) \leqslant & \left(2H\,\mathbf{E}[b^2(f^\delta(U))] + Hc_1^2\,\mathbf{E}[f^\delta(V)^2]\frac{N-1}{N} + 2c_1\,\mathbf{E}[b(f^\delta(U))]\,\mathbf{E}[f^\delta(U)]H\right)t \\
& + \left(2\,\mathbf{E}[b^2(f^\delta(U))] + c_1^2\,\mathbf{E}[f^\delta(V)^2]\frac{N-1}{N} + 2c_1\,\mathbf{E}[b(f^\delta(U))]\,\mathbf{E}[f^\delta(V)] + 2c_1\,\mathbf{E}[f^\delta(V)]H\right)\int_0^t b(f^\delta(X_i^N(s)))ds \\
& + \left(2H + 2c_1\,\mathbf{E}[f^\delta(V)]\right)\int_0^t b^2(f^\delta(X_i^N(s)))ds + \int_0^t b^3(f^\delta(X_i^N(s)))ds \\
& + \left(2\gamma_1\,\mathbf{E}[b(f^\delta(U))]\,\mathbf{E}[f^\delta(V)] + 2\gamma_1\,\mathbf{E}[f^\delta(V)]H\right)\int_0^t b(f^\delta(X_i^N(s)))\langle\mu_X^N(s), b(f^\delta(\cdot))\rangle ds \\
& + \gamma_1^2\,\mathbf{E}[f^\delta(V)^2]\frac{N-1}{N}\int_0^t b(f^\delta(X_i^N(s)))\langle\mu_X^N(s), b^2(f^\delta(\cdot))\rangle ds \\
& + H\gamma_1\,\mathbf{E}[f^\delta(V)^2]\frac{N-1}{N}\int_0^t \langle\mu_X^N(s), b^2(f^\delta(\cdot))\rangle ds + 2\gamma_1\,\mathbf{E}[b(f^\delta(U))]\,\mathbf{E}[f^\delta(V)]\int_0^t \langle\mu_X^N(s), b(f^\delta(\cdot))\rangle ds \\
& + 2\gamma_1\,\mathbf{E}[f^\delta(V)]\int_0^t b^2(f^\delta(X_i^N(s)))\langle\mu_X^N(s), b(f^\delta(\cdot))\rangle ds
\end{aligned}$$

Summing over all $i = 1, \ldots, N$ and dividing by N^2, we can find four positive constants K_1, K_2, K_3 and K_4 such that $\langle M_b^N(t)\rangle$ is bounded by the expression

$$\frac{K_1}{N}t + \frac{K_2}{N}\int_0^t \langle\mu_X^N(s), b^3(f^\delta(\cdot))\rangle^{1/3}ds + \frac{K_3}{N}\int_0^t \langle\mu_X^N(s), b^3(f^\delta(\cdot))\rangle^{2/3}ds + \frac{K_4}{N}\int_0^t \langle\mu_X^N(s), b^3(f^\delta(\cdot))\rangle ds$$

Using the result of Lemma 2.3.4, we know that there exists a certain N_0, such that the expectation of all the terms involved is bounded uniformly in $N > N_0$. Therefore, for such N we have

$$\mathbf{E}\left[\sup_{t\in[0,T]} M_{b,\delta}^N(t)\right] \leqslant \frac{C_T}{N}.$$

By Chebychev and Doob inequalities this leads to

$$\mathbf{P}\left(\sup_{t\in[0,T]} M_{b,\delta}^N(t) \geqslant 1\right) \leqslant \mathbf{E}\left[\sup_{t\in[0,T]}\left(M_{b,\delta}^N(t)\right)^2\right] \leqslant \mathbf{E}\left[\langle M_{b,\delta}^N(T)\rangle\right] \leqslant \frac{C_T}{N}.$$

Now, we compute $\langle \mu_X^N(t), b(f^\delta(\cdot))\rangle$ with Ito's rule, that gives the following bound:

$$\begin{aligned}\langle \mu_X^N(t), b(f^\delta(\cdot))\rangle \leqslant \langle \mu_X^N(0), b(f^\delta(\cdot))\rangle + M_{b,\delta}^N(t) + \left(\mathbf{E}[f^\delta(V)]\gamma_1\left(1+\frac{1}{N}\right)-1\right)\int_0^t \langle \mu_X^N(t), b^2(f^\delta(\cdot))\rangle ds \\ + \left(H\,\mathbf{E}[f^\delta(V)]\gamma_1\left(1+\frac{1}{N}\right)+H+\mathbf{E}[f^\delta(V)]c_1\left(1+\frac{1}{N}\right)+\mathbf{E}[b^2(f^\delta(U))]\right)\int_0^t \langle \mu_X^N(t), b^2(f^\delta(\cdot))\rangle^{1/2} ds \\ +H\left(\mathbf{E}[f^\delta(V)]c_1+\mathbf{E}[b^2(f^\delta(U))]\right)t.\end{aligned}$$

Since, for hypothesis, $b(f^\delta(\cdot))$ is integrable with respect to the law of $X(0)$, for the law of large number, we know that

$$\mathbf{P}\left(\langle \mu_X^N(0), b(f^\delta(\cdot))\rangle \geqslant 1+\mathbf{E}[b(f^\delta(X(0)))]\right) \leqslant \frac{Var\left(b(f^\delta(X(0)))\right)}{N}.$$

Let us consider the event

$$\left\{\langle \mu_X^N(0), b(f^\delta(\cdot))\rangle < 1+\mathbf{E}[b(f^\delta(X(0)))]\right\} \cup \left\{\sup_{t\in[0,T]} M_{b,\delta}^N(t) < 1\right\},$$

that has a probability greater than $1-2\frac{C}{N}$. Under this event, we apply Lemma 2.3.6 to get a bound for $\langle \mu_X^N(T), b(f^\delta(\cdot))\rangle$. Since, for all $\delta > 0$, $\lambda(\cdot) \leqslant b(f^\delta(\cdot)) + H$ a.s., this is equivalent to a bound for $\sup_{t\in[0,T]}\langle \mu_X^N(t), \lambda(\cdot)\rangle$, that leads to the existence of a positive constant K_T such that

$$\mathbf{P}\left(\int_0^T \langle \mu_X^N(s), \lambda\rangle ds \geqslant H_T\right) \leqslant \frac{K_T}{N},$$

and therefore to the desired bound for $\mathbf{P}\left(\frac{C_N(T)}{N} \geqslant H_T\right)$.

□

Part II

Models with asymmetric interactions

Chapter 3

A system of rank-based interacting diffusions

In this chapter we study a slight modification of the particle system and the nonlinear process presented in [54], that we will use as a comparison for the system of interacting random walks that we study in Chapter 4. The proofs of this Chapter basically comes from the adaptations of known results to the case of a reflecting barrier in zero. We mainly exploit the theory of competing Brownian particles, see for instance [8, 21, 80] and references therein, and the works on particles interacting through their cumulative distribution function (CDF) [53, 54, 55, 74].

3.1 The model and propagation of chaos

We start from the model presented in [54, 53], in order to study the viscous scalar conservation law with flux function $-A$:

$$\begin{cases} \partial_t F_t(x) = \frac{1}{2}\partial_{xx} F_t(x) + \partial_x(A(F_t(x))), \\ F_0(x) = H * m(x), \end{cases} \tag{3.1.1}$$

where m is a probability function on $\mathbb{R}$, $H(x) = \mathbb{1}(x \geqslant 0)$ is the Heaviside function and $*$ indicates the spatial convolution. The authors study the correspondent nonlinear process:

$$\begin{cases} dX_t = -A'(H * P_t(X_t))dt + dB_t, \\ P_t = Law(X_t), \\ m = Law(X_0) \end{cases} \tag{3.1.2}$$

with B real Brownian motion independent from the initial condition X_0. The two equations are linked in the sense that $H * P_t(x)$ is the unique bounded weak solution of (3.1.1). In this chapter we choose a particular form of the function A and we add to (3.1.2) a reflecting barrier in zero, in order to get useful results that will serve as basis of comparison for the model presented in Chapter 4.

3.1.1 The particle system

We consider a system of N particles, each of them moving on the positive half-line $\mathbb{R}^+$ with a reflecting barrier in 0. Each particle evolves according to:

- an intrinsic dynamics given by a Brownian motion with a positive drift $\delta > 0$;
- a rank-dependent interaction, that is an additional drift term depending on the parameter $\lambda > 0$ and on the cumulative distribution function of the empirical measure μ^N.

The infinitesimal generator $\mathcal{L}^N$ of the system acts on suitable C^2 functions $f : D_N \to \mathbb{R}$ in the following way:

$$\mathcal{L}^N f(x) = \sum_{i=1}^{N} \frac{1}{2}\frac{\partial^2}{\partial x_i^2} f(x) + \left(\delta - \frac{\lambda}{N}\sum_{k=1}^{N} \mathbb{1}(x_k \leqslant x_i)\right)\frac{\partial}{\partial x_i} f(x),$$

where

$$D_N \doteq \{x \in \mathbb{R}^N / x_i \geqslant 0 \, \forall \, i = 1, \dots, N\}$$

and the domain of the generator $\mathcal{L}^N$ contains the following set of functions:

$$\mathcal{D}\left(\mathcal{L}^N\right) \supseteq \left\{ f \in C^\infty \colon |n_i(x)^T \nabla f(x)| = 0 \text{ for } x \in \partial_i D_N, \text{ for } i = 1, \dots, N \right\}.$$

We indicate with $\partial_i D_N$ a "face" of the boundary of D_N, i.e.

$$\partial_i D_N \doteq \{x \in D_N \colon x_i = 0\},$$

and $n_i(x)$ is the inward normal vector to $\partial_i D_N$.

Let us indicate as $X^N(t) = (X_1^N(t), \dots, X_N^N(t)) \in \mathbb{N}^N$ the vector of the particles' positions at a fixed time $t \geqslant 0$ and the empirical measures as

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i^N(t)}.$$

Given an initial condition $X^N(0) \in D_N$, we say that the process X^N is the solution to a system on SDEs with reflection of ∂D_N if the following holds:

- $X^N(t) \in D_N$ for all $t \geqslant 0$;
- K^N is a continuous adapted process with values in $\mathbb{R}^N$ and finite variation on bounded intervals, such that $K_0^N = 0$ and for $i = 1, \dots, N$,

$$\begin{cases} X_i^N(t) &= X_i^N(0) + B_t^i + \int_0^t \left(\delta - \frac{\lambda}{N}\sum_{k=1}^{N} \mathbb{1}(X_k^N(s) \leqslant X_i^N(s))\right) ds + K_i^N(t), \\ K_i^N(t) &= \int_0^t \mathbb{1}(X_i^N(s) = 0) dK_i^N(s), \end{cases} \tag{3.1.3}$$

where B is a N-dimensional Brownian motion.

Proposition 3.1.1. *The system* (3.1.3) *of SDEs with reflection on* ∂D_N *has a unique strong solution, for all measurable initial condition* $X^N(0)$.

Proof. Fix a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})t \geqslant 0, \mathbf{P})$, we know by [54, Proposition 1.3] that the non-reflected system (3.1.3) has a unique solution. Therefore, we may build a weak solution to (3.1.3) pathwise with the local times in zero of the particles $\{X_i^N(t)\}_{t\geqslant 0}$ for every $i = 1, \dots, N$.

According to Yamada-Watanabe theorem, if we prove pathwise uniqueness, we obtain existence and uniqueness of strong solutions. Let $(X_1^N(0), \dots, X_N^N(0))$ and $(Y_1^N(0), \dots, Y_N^N(0))$ be two initial conditions and let $\{(X_1^N(t), \dots, X_N^N(t))\}_{t\geqslant 0}$ and $\{(Y_1^N(t), \dots, Y_N^N(t))\}_{t\geqslant 0}$ two solutions of (3.1.3) coupled by means of the same Brownian motion. By Ito-Tanaka formula, we write:

$$\begin{aligned}\sum_{i=1}^N \left(X_i^N(t) - Y_i^N(t)\right)^2 = \sum_{i=1}^N \left(X_i^N(0) - Y_i^N(0)\right)^2 \\ +2\lambda \int_0^t \sum_{i=1}^N \left(X_i^N(s) - Y_i^N(s)\right) \left(\frac{1}{N}\sum_{j=1}^N \mathbb{1}(Y_j^N(s) \leqslant Y_i^N(s)) - \frac{1}{N}\sum_{j=1}^N \mathbb{1}(X_j^N(s) \leqslant X_i^N(s))\right) ds \\ +2\int_0^t \sum_{i=1}^N \left(X_i^N(s) - Y_i^N(s)\right) \left(\mathbb{1}(X_i^N(s) = 0) dK_{i,X}^N(s) - \mathbb{1}(Y_i^N(s) = 0) dK_{i,Y}^N(s)\right).\end{aligned}$$

While the last term is obviously less or equal than zero, by construction, the term in the second row is treated as in [54, Proposition 1.3]. We know that $d\mathbf{P} \otimes ds$ a.e. the positions of the particles are distinct, therefore we find a unique permutation of indexes τ_s^X (resp. τ_s^Y) such that

$$X_{\tau_s^X(1)}^N(s) < X_{\tau_s^X(2)}^N(s) < \dots < X_{\tau_s^X(N)}^N(s) \text{ (resp. } Y_{\tau_s^Y(1)}^N(s) < Y_{\tau_s^Y(2)}^N(s) < \dots < Y_{\tau_s^Y(N)}^N(s) \text{).}$$

Then we rewrite the argument of the second row integral as

$$\sum_{i=1}^N \frac{i}{N} \left(\left(X_{\tau_s^Y(i)}^N(s) - Y_{\tau_s^Y(i)}^N(s) \right) - \left(X_{\tau_s^X(i)}^N(s) - Y_{\tau_s^X(i)}^N(s) \right) \right). \tag{3.1.4}$$

We exploit now a result on non decreasing sequences of real numbers [54, Lemma 1.4].

Lemma 3.1.1. *For any pair of non decreasing sequences of real numbers* $(a(i))_{i=1,\dots,N}$ *and* $(b(i))_{i=1,\dots,N}$ *and any permutation of indexes* $\tau \in \mathcal{S}_N$

$$\sum_{i=1}^N a(i)b(\tau(i)) \leqslant \sum_{i=1}^N a(i)b(i).$$

First, we consider the sequences $(a(i))_{i=1,\dots,N} = \left(\frac{i}{N}\right)_{i=1,\dots,N}$, $(b(i))_{i=1,\dots,N}$ $\left(X_{\tau_s^X(i)}^N(s)\right)_{i=1,\dots,N}$ and the permutation $\tau = (\tau_s^X)^{-1} \circ \tau_s^Y$. Then, we apply the same result to the sequences

$(a(i))_{i=1,\dots,N} = \left(\frac{i}{N}\right)_{i=1,\dots,N}$, $(b(i))_{i=1,\dots,N} = \left(Y^N_{\tau^Y_s(i)}(s)\right)_{i=1,\dots,N}$ and the permutation $\tau = (\tau^Y_s)^{-1} \circ \tau^X_s$. This implies that (3.1.4) is less or equal than zero $\mathbf{dP} \otimes \mathrm{d}s$ a.e.. This implies that

$$\sum_{i=1}^{N} \left(X^N_i(t) - Y^N_i(t)\right)^2 = \sum_{i=1}^{N} \left(X^N_i(0) - Y^N_i(0)\right)^2 \quad \text{a.s.},$$

that implies pathwise uniqueness.

□

Let us highlight the following property of our system (3.1.3), that concerns collisions of particles and it is simply implied by the results in [16].

Proposition 3.1.2. *Fix a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$ *and an initial condition* $X^N(0) = (X^N_1(0), \dots, X^N_N(0))$, *let* $\{(X^N_1(t), \dots, X^N_N(t))\}_{t\geqslant 0}$ *be the solution of* (3.1.3). *Then the following hold:*

i) a.s. there does not exists a $t > 0$ *such that there is a* triple collision *at* t, *i.e.*

$$X^N_i(t) = X^N_j(t) = X^N_h(t) \ \textit{for some} \ i \neq j \neq h;$$

ii) a.s. there does not exists a $t > 0$ *such that there are two or more* simultaneous collisions, *i.e.*

$$X^N_i(t) = X^N_j(t) \ \textit{and} \ X^N_h(t) = X^N_k(t) \ \textit{for some} \ i \neq j \neq h \neq k.$$

This proposition is a simple consequence of [16, Theorem 1.1] and it is crucial in the comparison with the model on Chapter 4. Notice that the absence of triple collisions is also a necessary condition for the strong existence of solutions to (3.1.3), that otherwise would have only weak ones.

3.1.2 Propagation of chaos and the nonlinear process

The nonlinear process associated to the particle system (3.1.3) is the pair $(X_t, K_t)_{t\geqslant 0}$, solution to the following nonlinear SDE:

$$\begin{cases} dX_t = dB_t + (\delta - \lambda H * P_t(X_t))\, dt + dK_t \\ K_t = \int_0^t \mathbb{1}(X_s = 0) dK_s \\ Q_t = Law(X_t, K_t) \ \text{ and } \ P_t = Q \circ X_t^{-1} \end{cases} \tag{3.1.5}$$

where B is a Brownian motion, independent from the initial condition $(X_0, 0)$.

Theorem 3.1.1. *The nonlinear SDE with reflection* (3.1.5) *admits a unique strong solution for all measurable initial conditions* X_0.

We prove existence and uniqueness of solutions to (3.1.5) by means of a propagation of chaos result. Indeed, we build one weak solution to (3.1.5) taking the limit of a converging subsequence of empirical measures $\{\mu_t^{N_k}\}_{t\geqslant 0}$. Therefore, we first want to show that this limit exists (tightness of the sequence of empirical measures) and that it actually has the law of a solution of (3.1.5) (consistency of the limit). This, together with uniqueness of solutions to (3.1.5) proves propagation of chaos according to the first approach described in Section 1.2.3. Notice that the coefficients of (3.1.5) satisfies the globally Lipschitz condition of Assumption 2.1.1, indeed the drift term is such that, for all $x, y \in \mathbb{R}^+$ and $\alpha, \beta \in \mathcal{M}(\mathbb{R}^+)$ it holds

$$\begin{aligned}\lambda\left|\int_0^x d\alpha - \int_0^y d\beta\right| \leqslant & \lambda\left|\int_0^x d\alpha - \int_0^y d\alpha\right| + \lambda\left|\int_0^y d\alpha - \int_0^y d\beta\right| \\ \leqslant & \lambda|x-y| + \lambda \sup_{\|f\|\in 1-Lip}\left|\int_0^{+\infty} f d\alpha - \int_0^{+\infty} f d\beta\right| \\ \leqslant & \lambda(|x-y| + \rho(\alpha,\beta)).\end{aligned}$$

We could simply adapt the results of Chapter 2 for well-posedness and propagation of chaos to the case of a reflecting barrier in zero, which is quite straightforward since we are in dimension one (in higher dimension the problem would of course be harder). However, we choose to highlight the other approach described in Section 1.2.3 and to use the *martingale problem*, which is defined as follows.

Definition 3.1.1. We say that the law Q on $\mathbf{C}(\mathbb{R}^+, \mathbb{R}^+)^2$ of any $(X_t, K_t)_{t\geqslant 0}$ satisfying (3.1.5) is the solution of the *martingale problem* correspondent to (3.1.5) if the following hold:

i) $Q \circ (X_0, K_0)^{-1} = \mu_0 \otimes \delta_0$, for μ_0 measure on $[0, \infty)$;

ii) $\forall\ \phi \in \mathbf{C}(\mathbb{R}^+, \mathbb{R})$

$$\phi\left(X_t - K_t\right) - \phi\left(X_0 - K_0\right) - \int_0^t \frac{1}{2}\phi''\left(X_s - K_s\right) - \left(\delta - \lambda H * P_s(X_s)\right)\phi'\left(X_s - K_s\right) ds$$

is a Q-martingale;

iii) $\forall\ t \geqslant 0$ $\int_0^t dK_s < \infty$ and $K_t = \int_0^t \mathbb{1}(X_s = 0) dK_s$ Q-a.s..

Proposition 3.1.3. *Fix a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$ *and an initial condition* $X^N(0)$ *whose law is* μ_0*-chaotic, for a law* μ_0 *on* $\mathbb{R}^+$*. Let* $(X^N, K^N)_{t\geqslant 0}$ *be the solution to* (3.1.3)*. Then:*

a) the sequence of empirical measures

$$\left\{\mu^N_{(X,K)} = \frac{1}{N}\sum_{i=1}^N \delta_{(X_i^N, K_i^N)}\right\}$$

is tight in $\mathcal{M}\left(\mathbf{C}(\mathbb{R}^+, \mathbb{R}^+)^2\right)$*;*

b) any limit point μ *of* $\mu^N_{(X,K)}$ *solves the martingale problem correspondent to* (3.1.5).

Proof. a) The system is exchangeable, therefore, by a well-known result by Sznitman [83, Proposition 2.2], proving point *i)* is equivalent to proving the following:

$$\text{the sequence } (X_1^N(t), K^N(t))_{t\geqslant 0} \text{ is tight in } \mathbf{C}(\mathbb{R}^+,\mathbb{R}^+)^2.$$

By writing as $\{Y^N(t)\}_{t\geqslant 0}$ the evolution on $\mathbb{R}$ of the particle system without reflection, the sequence $\{Y_1^N(t)\}_{t\geqslant 0}$ is clearly tight. Then, the map associating to $\{Y_1^N(t)\}_{t\geqslant 0}$ the solution $(X_1^N(t), K_1^N(t))_{t\geqslant 0}$ is continuous, see [60], and this gives the desired tightness.

b) Let Q_∞ be a limit point of a converging subsequence $\{\mu^{N_k}_{(X,K)}\}$. We need to prove that it satisfies Definition 3.1.1.
Point *i)* is clearly satified, because the sequence of initial condition is μ_0-chaotic.

For point *ii)* we follow the approach of Sznitman [81] and we define a functional

$$F\colon \mathcal{M}\left(\mathbf{C}(\mathbb{R}^+,\mathbb{R}^+)^2\right) \longrightarrow \mathbb{R}$$

that is zero on the solutions of the martingale problem defined in Definition 3.1.1. Let $f \in \mathbf{C}^\infty(\mathbb{R},\mathbb{R})$, let $g_1,\dots,g_h$ be continuous and bounded functions on $\mathbb{R}^2$ and let $0 \leqslant s_1 \leqslant \dots \leqslant s_h \leqslant s \leqslant t$ be positive real numbers:

$$\begin{aligned}F(Q) = \langle Q, &g_1(x_{s_1},k_{s_1})\dots g_h(x_{s_h},k_{s_h})\times \\ &\left[f(x_t-k_t)-f(x_s-k_s)-\int_s^t \frac{1}{2}f''(x_r-k_r)-(\delta-\lambda H * Q_r(x_r))\,f'(x_r-k_r)dr\right]\rangle.\end{aligned}$$

Let us write

$$\mathbf{E}\left[F(\mu^N_{(X,K)})^2\right] = \mathbf{E}\left[\left(\frac{1}{N}\sum_{i=1}^N g_1(X_i^N(s_1),K_i^N(s_1))\dots g_h(X_i^N(s_h),K_i^N(s_h))(M_f^i(t)-M_f^i(s))\right)^2\right],$$

where, for all $i = 1,\dots,N$

$$\begin{aligned}M_f^i(t) = &f(X_i^N(t)-K_i^N(t)) - f(X_i^N(0)-K_i^N(0)) \\ &-\int_0^t \frac{1}{2}f''(X_i^N(r)-K_i^N(r)) - \left(\delta-\lambda\frac{1}{N}\sum_{j=1}^N \mathbb{1}(X_j^N(r)\leqslant X_i^N(r))\right) f'(X_i^N(r)-K_i^N(r))dr\end{aligned}$$

are orthogonal martingales, i.e. $\langle M_f^i, M_f^j\rangle = 0$ for any $i \neq j$. Then, because of the orthogonality of the martingales, we get, for a certain constant $C \geqslant 0$

$$\lim_{k\to\infty}\mathbf{E}\left[F(\mu^{N_k}_{(X,K)})^2\right] \leqslant \lim_{k\to\infty}\frac{C}{{N_k}^2}\sum_{i=1}^{N_k}\mathbf{E}[\langle M_f^i\rangle_t - \langle M_f^i\rangle_s] \leqslant \lim_{k\to\infty}\frac{C}{N_k} = 0.$$

We are left to prove that

$$\mathbf{E}_{Q_\infty}[F(Q)^2] = \lim_{k\to\infty} \mathbf{E}\left[F(\mu^{N_k}_{(X,K)})^2\right].$$

This is a consequence of the fact that the functional F is continuous. Indeed let $P_\infty = Q_\infty \circ X^{-1}$, then one can prove that P_∞ does not weight points dt-a.e.. This implies that, if we call $P_N = \mu^N_{(X,K)} \circ X^{-1}$, then $H * P_N \to H * P_\infty$ uniformly and therefore F is continuous in Q_∞. This implies that $F = 0$ Q_∞-a.s..

For point *iii)* we find that

$$\mathbf{E}_{Q_\infty}[K_T] \leqslant \limsup_{k\to\infty} \mathbf{E}[\frac{1}{N}\sum_{i=1}^N K_i^N(T)] = \limsup_{k\to\infty} \mathbf{E}[K_1^N(T)] < \infty,$$

this means that Q_∞-a.s. for all $T \geqslant 0$ $K_T < \infty$. Moreover, for any g positive continuous function with compact support in $(0,\infty)$

$$\mathbf{E}_{Q_\infty}[\int_0^T g(X_s)dK_s] \leqslant \limsup_{k\to\infty} \mathbf{E}[\frac{1}{N}\sum_{i=1}^N \int_0^T g(X_i^N(s))dK_i^N(s)] = \limsup_{k\to\infty} \mathbf{E}[\int_0^T g(X_1^N(s))dK_1^N(s)] = 0$$

and this means that Q_∞-a.s. for all $T \geqslant 0$ $K_T = \int_0^T \mathbb{1}(X_s = 0)dK_s$. □

Proof of Theorem 3.1.1. By Proposition 3.1.3, for any initial measure μ_0 we get weak existence of a solution, that is obtained as limit of the sequence of empirical measures $\{\mu^N_{(X,K)}\}$. Uniqueness in law is given by uniqueness of solutions of the associated PDE, obtained in [54, 53]. By Yamada-Watanabe theorem, pathwise uniqueness ensures the thesis.

Fix a probability measure Q on $\mathbf{C}(\mathbb{R}^+,\mathbb{R}^+)^2$ and let $P_t = Q \circ X_t^{-1}$ be the flow of its first coordinate time marginals. Let Γ be the map that associates to Q the solution $(X_t, K_t)_{t\geqslant 0}$ to the SDE

$$\begin{cases} dX_t = dB_t + (\delta - \lambda H * P_t(X_t))\,dt + dK_t \\ K_t = \int_0^t \mathbb{1}(X_s = 0)dK_s. \end{cases}$$

Suppose that $(X_t^1, K_t^1)_{t\geqslant 0}$ and $(X_t^2, K_t^2)_{t\geqslant 0}$ be two solutions, then by Ito-Tanaka formula we have

$$\begin{aligned}\left(X_t^1 - X_t^2\right)^2 = \left(X_0^1 - X_0^2\right)^2 &+ 2\int_0^t \lambda\left(X_s^1 - X_s^2\right)\left(H * P_s(X_s^2) - H * P_s(X_s^1)\right)ds \\ &+ 2\int_0^t \left(X_s^1 - X_s^2\right)\left(\mathbb{1}(X_s^1 = 0)dK_s^1 - \mathbb{1}(X_s^2 = 0)dK_s^2\right).\end{aligned}$$

Since $H * P_t(\cdot)$ is non-decreasing, the second term in the right-hand side is a.s. less or equal than zero and the same is true for the third term. This implies that, for all $t \geqslant 0$, a.s.

$$\left(X_t^1 - X_t^2\right)^2 \leqslant \left(X_0^1 - X_0^2\right)^2.$$

Let Q be the unique law of a solution to (3.1.5), this implies pathwise uniqueness.

□

3.1.3 Pathwise propagation of chaos

We know by Section 2.1, that for the system without reflection, it is possible to prove a pathwise propagation of chaos. The problem of adding a reflecting barrier in zero, in dimension one, can be handled easily and, following also the approach of [54, Theorem 1.5], we do the same in this case. To this aim we set the basic coupling procedure of Chapter 2 between the particle system $(X^N_t, K^N_t)_{t\geqslant 0}$ solving (3.1.3) and N i.i.d. copies of the nonlinear process solving (3.1.5). Notice that, since we do not have jump terms here, we adopt the classical L^2 approach of stochastic calculus, estimating the rate of convergence to zero of the W_2 distance between the empirical measure and its mean field limit.

We fix a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$ and let μ_0 be a probability measure on $\mathbb{R}^+$. For any $N \in \mathbb{N}$ let

$$\{(X^N(t), K^N(t))\}_{t\geqslant 0} = \{(X^N_1(t), K^N_1(t), \dots, X^N_N(t), K^N_N(t))\}_{t\geqslant 0}$$

be solutions to (3.1.3), all defined on the same filtered probability space, with respect to the family $\{B^i\}_{i\in\mathbb{N}}$ of one-dimensional independent Brownian motions with initial condition such that, for all $i = 1, \dots, N$, $(X^N_i(0), K^N_i(0))$ are independent and distributed as $\mu_0 \otimes \delta_0$. Let us define the vector

$$(\bar{X}_t, \bar{K}_t)_{t\geqslant 0} = (\bar{X}^1_t, \bar{K}^1_t, \dots, \bar{X}^N_t, \bar{K}^N_t)_{t\geqslant 0}$$

such that, for any $i = 1, \dots, N$ the pair $(\bar{X}^i_t, \bar{K}^i_t)_{t\geqslant 0}$ is a solution to (3.1.5) with Brownian motion B^i and initial condition $(\bar{X}^i_0, \bar{K}^i_0) = (X^N_i(0), K^N_i(0))$ a.s.. Now, for any $N \in \mathbb{N}$, we have coupled the solution to (3.1.3) with N independent copies of the solution to (3.1.5). The following result states the trajectorial propagation of chaos.

Theorem 3.1.2. *For all* $t \geqslant 0$*, there exists a positive constant* $C_t < \infty$ *such that*

$$\mathbf{E}[\sup_{s\leqslant t}(X^N_1(s) - \bar{X}^1_s)^2] \leqslant \frac{C_t}{N}. \tag{3.1.6}$$

Proof. We recall that, by exchangeability of the particle system, we write:

$$\mathbf{E}[\sup_{s\leqslant t}(X^N_1(s) - \bar{X}^1_s)^2] = \mathbf{E}\left[\sup_{s\leqslant t}\frac{1}{N}\sum_{i=1}^{N}(X^N_i(s) - \bar{X}^i_s)^2\right]. \tag{3.1.7}$$

We make use of the Ito-Tanaka formula to write:

$$\begin{aligned}\sum_{i=1}^{N}\left(X_i^N(t)-\bar{X}_t^i\right)^2 &= \sum_{i=1}^{N}\left(X_i^N(0)-\bar{X}_0^i\right)^2\\ &+2\lambda\int_0^t\sum_{i=1}^{N}\left(X_i^N(s)-\bar{X}_s^i\right)\left(\frac{1}{N}\sum_{j=1}^{N}\mathbb{1}(\bar{X}_s^j\leqslant\bar{X}_s^i)-\frac{1}{N}\sum_{j=1}^{N}\mathbb{1}(X_j^N(s)\leqslant X_i^N(s))\right)ds\\ &+2\lambda\int_0^t\sum_{i=1}^{N}\left(X_i^N(s)-\bar{X}_s^i\right)\left(H*P_s(\bar{X}_s^i)-\frac{1}{N}\sum_{j=1}^{N}\mathbb{1}(\bar{X}_s^j\leqslant\bar{X}_s^i)\right)ds\\ &+2\int_0^t\sum_{i=1}^{N}\left(X_i^N(s)-\bar{X}_s^i\right)\left(\mathbb{1}(X_i^N(s)=0)dK_i^N(s)-\mathbb{1}(\bar{X}_s^i=0)d\bar{K}_s^i\right).\end{aligned}$$

By the computations in the proof of Proposition 3.1.1, we get that the terms in the second and fourth rows are a.s. less or equal than zero. We consider, for a fixed $t\geqslant 0$

$$\mathbf{E}\left[\left(H*P_s(\bar{X}_s^i)-\frac{1}{N}\sum_{j=1}^{N}\mathbb{1}(\bar{X}_s^j\leqslant\bar{X}_s^i)\right)^2\right],$$

where the $\{\bar{X}_s^i\}_{i=1,\dots,N}$ are i.i.d. random variables with common law P_s, $H*P_s(\bar{X}_s^i)$ is uniformly distributed in $[0,1]$ and $\mathbf{E}[\mathbb{1}(\bar{X}_s^j\leqslant\bar{X}_s^i)|\bar{X}_s^i]=H*P_s(\bar{X}_s^i)$. This means that

$$\mathbf{E}\left[\left(H*P_s(\bar{X}_s^i)-\frac{1}{N}\sum_{j=1}^{N}\mathbb{1}(\bar{X}_s^j\leqslant\bar{X}_s^i)\right)^2\right]=\frac{1}{N}\mathbf{E}[H*P_s(\bar{X}_s^i)(1-H*P_s(\bar{X}_s^i))]=\frac{1}{6N}.$$

Then we have, by Cauchy-Schwarz inequality and previous computations,

$$\begin{aligned}\mathbf{E}[\sup_{s\leqslant t}\sum_{i=1}^{N}\left(X_i^N(s)-\bar{X}_s^i\right)^2] &\leqslant \mathbf{E}[\sum_{i=1}^{N}\left(X_i^N(0)-\bar{X}_0^i\right)^2]\\ &+2\lambda\int_0^t\sqrt{\mathbf{E}\left[\sup_{r\leqslant s}\left(\sum_{i=1}^{N}(X_i^N(r)-\bar{X}_r^i)\right)^2\right]\mathbf{E}\left[\sup_{r\leqslant s}\left(H*P_r(\bar{X}_r^i)-\frac{1}{N}\sum_{j=1}^{N}\mathbb{1}(\bar{X}_r^j\leqslant\bar{X}_r^i)\right)^2\right]}ds\\ &\leqslant\frac{2\lambda}{\sqrt{6}}\int_0^t\sqrt{\mathbf{E}\left[\sup_{r\leqslant s}\sum_{i=1}^{N}(X_i^N(r)-\bar{X}_r^i)^2\right]}ds.\end{aligned}$$

This, by comparison with the ODE $\dot{y}=K\sqrt{y}$, implies that there exists a $C_t\geqslant 0$ such that

$$\mathbf{E}[\sup_{s\leqslant t}\sum_{i=1}^{N}\left(X_i^N(s)-\bar{X}_s^i\right)^2]\leqslant C_t.$$

This, together with (3.1.7), implies (4.1.9).

□

3.2 Long-time behavior of the model

In this section we study the stability properties of the model in Section 3.1. Of course, when $\lambda = 0$, the particle system (3.1.3) and its nonlinear limit (3.1.5) have no chance of having a stationary measure, since they are just Brownian motions with a positive drift, reflected in zero. We are interested in finding under which conditions on λ the Markov processes have a stationary measure and whether they converge towards it as the time goes to infinity.

3.2.1 Background: stability of Markov processes

Let us introduce some mathematical background for the study of stability and long-time behavior of Markov processes. We start with the definition of a distance between probability measures, different from the Wasserstein distance defined in Chapter 2.

Definition 3.2.1 (Total variation distance)**.** Let μ, ν be two probability measures on the same Polish metric space (M, d), then the *total variation distance* between μ, ν is defined as follows:

$$\begin{aligned}\|\mu - \nu\|_{TV} &= \frac{1}{2} \sup \left\{ \int f d\mu - \int f d\nu \colon \|f\|_\infty \leqslant 1 \right\} \\ &= \inf \left\{ \mathbf{P}(X \neq Y) \colon \text{ for a coupling } (X, Y) \text{ s.t. } Law(X) = \mu;\, Law(Y) = \nu \right\}.\end{aligned}$$

Clearly the *total variation distance* is equivalent to the W_1 Wasserstein distance when $d(x, y) = \mathbb{1}(x \neq y)$ on M.

The stability of Markov processes has been extensively studied in literature, among the others, let us cite the three well-known papers from Meyn and Tweedie [68, 69, 70]. In particular in [70] we find a criterion for exponential ergodicity of continuous-time Markov processes.

Definition 3.2.2 (Exponential ergodicity [70])**.** Let $X = \{X_t\}_{t \geqslant 0}$ be a Markov process with values in a measurable space $(E, \mathcal{E})$ and stationary measure π. We say that X is exponentially ergodic if there exist a positive constant $\beta < 1$ and a finite valued function B such that

$$\|\mathbf{P}\left(X_t \in \cdot\right) - \pi\|_{TV} \leqslant B(x)\beta^t \quad \text{for all } \; t \geqslant 0,\, x \in E,$$

when $X_0 = x$ a.s..

In [70] the authors give a Foster-Lyapunov criterion for exponential ergodicity, in the sense that they use the infinitesimal generator $\mathcal{L}$ of the Markov process and a Lyapunov function defined as follows.

Assumption 3.2.1 (Foster-Lyapunov condition)**.** *Let* $V \colon E \to \mathbb{R}_+$ *a positive, measurable function in the domain* $\mathcal{D}(\mathcal{L})$ *of the generator of the Markov process* X*, that is* norm-like, *i.e. the level sets* $\{x \in E \colon V(x) \leqslant K\}$ *are precompact for each* $K > 0$*. The function* V *is such that there exist two positive constants* $\gamma > 0$ *and* $0 < H < \infty$ *such that*

$$\mathcal{L}V(x) \leqslant -\gamma V(x) + H,$$

for all $x \in E$*.*

Any function V satisfying Assumption 3.2.1 is called Lyapunov function for the Markov process X, a terminology that comes from the theory of dynamical systems.

Theorem 3.2.1 (Theorem 6.1 in [70])**.** *Let* $X = \{X_t\}_{t \geqslant 0}$ *be an irreducible and positive recurrent Markov process with stationary measure* π*. Suppose that it exists a function* V *satisfying Assumption 3.2.1, then the process is exponentially ergodic, i.e. there exists two positive constant* $\beta < 1$ *and* $C > 0$ *such that*

$$\|\mathbf{P}\left(X_t \in \cdot\right) - \pi\|_{TV} \leqslant C(V(x)+1)\beta^t \quad \textit{for all} \ \ t \geqslant 0,\ x \in E,$$

when $X_0 = x$ *a.s.. Moreover, we have the following estimate on* π*:*

$$\int_E V(x)\pi(dx) \leqslant \frac{H}{\gamma}. \tag{3.2.1}$$

Theorem 3.2.1 gives an estimate of rate of convergence of the law of the process towards its stationary measure and it gives also an estimate on the stationary measure itself, with the bound (3.2.1).

None of the results above with Lyapunov functions can be applied in the case of nonlinear Markov processes, see [32]. Indeed, the infinitesimal generator of a McKean-Vlasov process does not even exist. However, if X is a McKean-Vlasov process, we can intuitively define an operator $\mathcal{L}_\mu$ that depends on a measure μ and such that, for all time $t \geqslant 0$

$$\frac{d}{dt}\mathbf{E}[f(X_t)] = \mathbf{E}[\mathcal{L}_{\mu_t} f(X_t)],$$

where $\mu_t = Law(X_t)$. The stability study for nonlinear Markov processes is highly nontrivial and it usually relies on some *ad hoc* procedures. The stationary measures are obtained by solving the stationary Fokker-Planck equation, which, by definition is actually nonlinear. We find in literature different situations. In the case of [9, 63], the authors prove the existence and uniqueness of a stationary measure and they prove it is attractive. The approach of [63] relies on the proof of a uniform (in time) propagation of chaos and a functional inequality (in this case a log-Sobolev inequality) for the N-particle system stationary measure, with a constant that does not depend on N. The nonlinear model in [43] has exactly two stationary measures and, under some reasonable conditions, the authors prove that only one of these two is attractive. In [54], the authors prove that, under reasonable conditions, the process (3.1.2) has a continuum of stationary measures. In [55], the authors prove that each of these stationary measures attract solutions starting from its *basin of attraction*, the subset of probability measures with the same first moment.

We know that a mean field interacting particle system is characterized by the fact that, for all $t \geqslant 0$, the empirical measure of the particle system is such that

$$\mu^N(t) \to \mu_t,$$

where μ_t is the law of a nonlinear Markov process, for which we may wonder whether it converges to some stationary measure when t goes to infinity. At the same time, we know that, if the N particle system is ergodic, the empirical measure converges to the *stationary emprirical measure* when the time goes to infinity, i.e.

$$\mu^N(t) \to \mu^N_\infty.$$

Then, we study if the measure μ^N_∞ has a weak limit when N grows to infinity, i.e. if the sequence of stationary measures is chaotic. This means that, in some situations, it is possible to find a measure μ_∞ such that the following diagram commutes:

$$\begin{array}{ccccc} & \mu^N(t) & \xrightarrow{t\to\infty} & \mu^N_\infty & \\ N\to\infty & \Big\downarrow & & \Big\downarrow & N\to\infty \\ & \mu_t & \xrightarrow[t\to\infty]{} & \mu_\infty & \end{array} \tag{3.2.2}$$

We will prove in the following that the model presented in Section 3.1 belongs to the class of models for which this study may be performed.

3.2.2 Exponential ergodicity of the particle systems

In this Section we fix $N > 0$ and we study the long time behavior of the system with N particles. We study the particle system along the lines of the work [72] on one-dimensional Brownian particles with rank dependent drift. Indeed, we highlight that, by considering the increasing reordering of the vector X^N as $Y^N = (Y^N_1, \dots, Y^N_N)$, we can express the evolution of our particle system as the process

$$dX^N_i(t) = dB^1_t + \sum_{k=1}^N (\delta - \lambda\frac{k-1}{N})\mathbb{1}(X^N_i(t) = Y^N_k(t))dt + dK^N_i(t). \tag{3.2.3}$$

By classical results, this is equivalent to

$$\begin{cases} Y^N_i(t) = Y^N_i(0) + (\delta - \lambda\frac{i-1}{N})t + \beta^i_t + V^{i,i-1}_t - V^{i,i+1}_t, \\ V^{i,i-1}_t = \int_0^t \mathbb{1}(Y^N_i(s) = Y^N_{i-1}(s))dV^{i,i-1}_s, \text{ for } i = 2, \dots, N, \\ V^{1,0}_t = \int_0^t \mathbb{1}(Y^N_1(s) = 0)dV^{1,0}_s, \\ V^{i,i+1}_t = \int_0^t \mathbb{1}(Y^N_i(s) = Y^N_{i+1}(s))dV^{i,i+1}_s, \text{ for } i = 1, \dots, N-1, \end{cases} \tag{3.2.4}$$

where $\beta_t^i = \sum_{k=1}^N \int_0^t \mathbb{1}(X_k^N(s) = Y_i^N(s)) dB_s^k$ is a Brownian motion. Weak existence and uniqueness of solutions of a system like (3.2.4) is given by the theory of reflecting Brownian motions in polyhedra, see [72, 88]. Here we do not have any problem of well-posedness because (3.2.4) is simply a reordering of (3.2.3), for which Proposition 3.1.1 and Proposition 3.1.2 ensure existence and uniqueness of strong solutions. Let us state a Lemma from [72, Lemma 9], deduced by general results in [88], that is crucial for the study of stationary distribution of (3.2.3).

Lemma 3.2.1. *Let $Z = \{Z_t\}_{t \geqslant 0}$ be a N-dimensional Brownian motion in a domain*

$$\Lambda_N = \left\{ x \in \mathbb{R}^N : b_i(x) \geqslant 0 \text{ for } i = 1, \dots, N \right\},$$

where $\{b_i\}_{i=1,\dots,N}$ are N linearly independent functionals. Let Z have identity covariance matrix, normal reflection at the boundaries and constant drift vector D such that

$$\sum_{i=1}^N D_i x_i = -\sum_{i=1}^N a_i b_i(x) \quad \text{for all } x \in \mathbb{R}^N. \tag{3.2.5}$$

The process Z has a stationary probability distribution if and only if $a_i > 0$ for all $i = 1, \dots, N$. Moreover, in the stationary state (when it exists), the $b_i(Z)$ are independent exponential random variables with parameter $2a_i$ and the process in its stationary state is reversible.

The proof of Lemma 3.2.1 is a particular case of [88, Theorem 1.2]. It is based on the observation that Z is in duality with itself w.r.t. the measure ρ on the domain whose density is $\exp(2D \cdot x)$. With the change of variable $y_i = b_i(x)$ and the relation (3.2.5), we see that the distribution of the $b_i(x)$ under ρ has joint density $\prod_{i=1}^N \exp(-2a_i y_i)$, which has finite mass if and only if $a_i > 0$ for all $i = 1, \dots, N$.

Let us apply Lemma 3.2.1 to our case.

Theorem 3.2.2. *The process X^N, solution to (3.1.3), has a unique stationary distribution π^N if and only if*

$$\lambda > 2\delta \frac{N}{N-1}. \tag{3.2.6}$$

Moreover, this stationary distribution is such that the gaps

$$(Y_1^N, Y_2^N - Y_1^N, Y_3^N - Y_2^N, \dots, Y_N^N - Y_{N-1}^N)$$

are independent exponential random variables, respectively with parameter $2a_i$, where

$$a_i = \frac{\lambda}{2N} \left[(N + 1 - i) \left(i - \frac{\lambda(2-N) + 2\delta N}{\lambda} \right) \right]. \tag{3.2.7}$$

Proof. This proof comes from the direct application of Lemma 3.2.1. We consider the process Y^N defined in (3.2.4), that belongs to the domain

$$\Lambda_N = \left\{ y \in \mathbb{R}^N : y_1 \geqslant 0, y_2 - y_1 \geqslant 0, y_3 - y_2 \geqslant 0, \dots, y_N - y_{N-1} \geqslant 0 \right\},$$

i.e. the functionals are defined as $b_1(x) = x_1$ and $b_i(x) = x_i - x_{i-1}$ for all $i = 2, \dots, N$. The drift term D of the process Y^N has the following components:

$$D_i = \delta - \lambda \frac{i-1}{N},$$

for all $i = 1, \dots, N$. By solving (3.2.5), we clearly get (3.2.7). Therefore, the necessary and sufficient condition for ergodicity ($a_i > 0$ for all $i = 1, \dots, N$) in our case it is reduced simply to condition (3.2.6). $\square$

The exponential convergence of the process X^N to its stationary distribution π^N from any initial condition is ensured by the recent results in [77], in which the author proves, via Lyapunov function argument, exponential ergodicity for Brownian motions reflected in a convex polyhedral cone. Therefore, we define the **gap process** $G^N \doteq \{G^N(t)\}_{t \geqslant 0}$ such that, for all $t \geqslant 0$

$$G^N(t) \doteq \left(Y_1^N(t), Y_2^N(t) - Y_1^N(t), \dots, Y_N^N(t) - Y_{N-1}^N(t) \right).$$

In the terminology of [77], G^N is a reflected Brownian motion in $\mathbb{R}_+^N$ whose reflection matrix on the boundary is

$$R = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & \dots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \dots & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & \dots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & -\frac{1}{2} & 1 \end{pmatrix},$$

the covariance matrix A and the drift vector μ are

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 1 & -1 & \dots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix}, \quad \mu = \begin{pmatrix} \delta \\ -\lambda \frac{1}{N} \\ -\lambda \frac{2}{N} \\ \vdots \\ -\lambda \frac{N-1}{N} \end{pmatrix}.$$

Under the conditions for the existence of a unique stationary measure for G^N, following [77], we define the function

$$V(g) \doteq e^{a\varphi(g^T R^{-1} g)},$$

for any $g \in \mathbb{R}_+^N$ and φ a positive C^∞ function such that, for some $0 < s_1 < s_2 < \infty$

$$\varphi(s) \doteq \begin{cases} 0 & \text{for } s \leqslant s_1, \\ s & \text{for } s \geqslant s_2. \end{cases}$$

There exists a sufficiently small and two constants $0 < s_1 < s_2 < +\infty$ such that function V satisfies Assumption 3.2.1 and the process G^N (consequently also the process X^N) is exponentially ergodic.

3.2.3 Stationary distribution for the nonlinear process

The study of the invariant measures for the *nonlinear process* relies on the analysis of the correspondent stationary Fokker-Planck equation. Indeed, the nonlinear process (3.1.5) has a density function $p_t(x)$ that is the solution of the following PDE with boundary conditions:

$$\begin{cases} \partial_t p_t(x) = \frac{1}{2}\partial_x^2 p_t(x) + \partial_x(\lambda H * p_t(x) - \delta)p_t(x)) & \forall x > 0; \\ (\delta - \lambda H * P_t(0))p_t(0) = \frac{1}{2}\partial_x p_t(x)|_{x=0} & \forall t \geqslant 0. \end{cases}$$

Theorem 3.2.3. *The process* X*, solution to* (3.1.5)*, has a unique stationary measure* π *if and only if*

$$\lambda > 2\delta.$$

The proof of this theorem follows the line of [54, Lemma 2.1], we sketch here the main steps.

Proof. Consider the stationary Fokker-Planck equation with boundary conditions

$$\begin{cases} \frac{1}{2}\partial_x^2 \pi + \partial_x(\lambda H * \pi - \delta)\pi) = 0 & \forall x > 0; \\ (\delta - \lambda H * \pi(0))\pi(0) = \frac{1}{2}\partial_x \pi(x)|_{x=0}. \end{cases} \tag{3.2.8}$$

Then any probability measure π solving (3.2.8) in the weak sense is a stationary measure for X. Let us consider a solution π to (3.2.8), that must be absolutely continuous w.r.t. the Lebesgue measure, thanks to the regularizing effect of the second derivative. Therefore, we write (3.2.8) as

$$\partial^2_{xx}(\pi - 2\delta H * \pi + \lambda(H * \pi)^2) = 0.$$

Let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be the density function of a solution of (3.2.8) and let $F(x) = \int_0^x f(y)dy$ be its cumulative distribution function, then F must be the solution of the following Cauchy problem

$$\begin{cases} \frac{dF}{dx} = 2\delta F - \lambda F^2 + \beta, \\ F(0) = 0, \end{cases} \tag{3.2.9}$$

for certain constants β to be determined. The solution to (3.2.9) is of the following form:

$$F(x) = \frac{\delta + \sqrt{\delta^2 + \beta\lambda}}{\lambda} \left[\frac{1 - e^{-2x\sqrt{\delta^2+\beta\lambda}}}{1 - \frac{\delta+\sqrt{\delta^2+\beta\lambda}}{\delta-\sqrt{\delta^2+\beta\lambda}} e^{-2x\sqrt{\delta^2+\beta\lambda}}} \right]. \tag{3.2.10}$$

Therefore, we must find β such that $\lim_{x\to\infty} F(x) = 1$, that is $\beta = \lambda - 2\delta$. For all $\delta, \lambda > 0$ there exists a unique solution F_∞ to (3.2.9) with $\beta = \lambda - 2\delta$

$$F(x) = \frac{1 - e^{-2x(\lambda-\delta)}}{1 + \frac{\lambda}{\lambda-2\delta} e^{-2x(\lambda-\delta)}},$$

this F is the CDF of a probability measure if and only if $\frac{dF(x)}{dx} > 0$ for all $x > 0$, that means if and only if $\lambda > 2\delta$.

Therefore, if and only if $\lambda > 2\delta$, there exists a unique stationary measure π on $[0, \infty)$, which is absolutely continuous w.r.t. the lebesgue measure and it has CDF F.

$\square$

Adapting the approach in [55] to our framework, we can get a result on the long-time behavior of any solution of (3.1.5), in the sense that we have the convergence to the stationary measure in Wasserstein distance, without any rate of convergence. Indeed, starting from any initial condition μ_0 with finite first moment and $W_2(\mu_0, \pi) < +\infty$, we consider P_t the law of the solution to (3.1.5). Then, for all $p \geqslant 2$ such that $W_p(\mu_0, \pi) < +\infty$, we have

$$\lim_{t\to\infty} W_q(P_t, \pi) = 0, \quad \text{for all } 1 \leqslant q < p.$$

As the authors observe in [55, Section 3.3], in our framework we may as well obtain an exponential rate of convergence in W_2 Wassertein distance when the process (3.1.5) starts from an initial condition sufficiently close to the stationary measure.

3.2.4 Propagation of chaos for the stationary measures

At the end of Section 3.2.1 we mentioned that, in some cases, it is possible to prove the chaoticity of the sequence of the N particle system's stationary measures. In the previous sections, we see that for

$$\lambda > 4\delta$$

there is exponential ergodicity of the particle system (3.1.3) for every $N \geqslant 2$, there exists a unique and attracting stationary measure for the nonlinear Markov process (3.1.5) and, of course, there is propagation of chaos for every fixed $t \geqslant 0$. We wonder if we can close a diagram like (3.2.2), since so far we proved the following

$$\begin{array}{cccc} & \mu^N(t) & \xrightarrow{t\to\infty} & \mu^N_\infty \\ N\to\infty & \Big\downarrow & & ? \\ & P_t & \xrightarrow[t\to\infty]{} & \pi \end{array}$$

The missing step is therefore to prove is the sequence of stationary measures $\{\pi^N\}$ is π-chaotic.

In the paper [74] the author proves, for $p \geqslant 1$, the W_p-chaoticity of the sequence of stationary measures by means of the convergence of the Laplace transforms, together with the uniform boundedness of the sequence $|X_1^N|^p$ under the N particle system's stationary measure. We believe that, by mimicking the same computations, we could get W_p-chaoticity of the sequence $\{\pi^N\}$.

Let us underline, that proving that the sequence of stationary measures π^N is π-chaotic is equivalent to prove weak convergence of the sequence of stationary empirical measures to π. Fix $\delta > 0$ and $\lambda > 4\delta$. The stationary measure π has density function:

$$f(x) = \frac{4(\lambda-\delta)^2(\lambda-2\delta)e^{-2(\lambda-\delta)x}}{(\lambda-2\delta+\lambda e^{-2(\lambda-\delta)x})^2}. \tag{3.2.11}$$

If we call m^* the median of the probability measure π, i.e. such that $F(m^*) = 1/2$, it is clear from (3.2.11) that

$$m^* = \frac{1}{2(\lambda-\delta)}\log\left[\frac{3\lambda-4\delta}{\lambda-2\delta}\right].$$

At the level of N particle system, we know as well the median of the stationary empirical measure μ_∞^N, that is the position $Y^N_{\lceil\frac{N}{2}\rceil}$ of the $\lceil\frac{N}{2}\rceil^{th}$ ranked particle, where Y^N is the reordered particle system (3.2.4) under π^N. We know that, in the stationary regimes, $Y^N_{\lceil\frac{N}{2}\rceil}$ is the sum of $\lceil\frac{N}{2}\rceil$ independent exponential random variables, with known parameters. For instance, we can compute its mean and variance.

$$\begin{aligned}
\mathbf{E}_{\pi^N}[Y^N_{\lceil\frac{N}{2}\rceil}] &= \sum_{i=1}^{\lceil\frac{N}{2}\rceil}\frac{1}{2a_i} = \sum_{i=1}^{\lceil\frac{N}{2}\rceil}\frac{N}{\lambda(N+1-i)\left(i-\frac{\lambda(2-N)+2\delta N}{\lambda}\right)}\\
&\leqslant \frac{N}{\lambda(2N-1)-2\delta N}\left[\ln\left(\frac{N\left(\frac{3}{2}\lambda-2\delta\right)-2\lambda}{N(\lambda-2\delta)-2\lambda}\right)+\ln(2)+O\left(\frac{1}{N}\right)\right]\\
&\overset{N\to\infty}{\longrightarrow}\frac{1}{2(\lambda-\delta)}\ln\left(\frac{3\lambda-4\delta}{\lambda-2\delta}\right) = m^*
\end{aligned}$$

$$\mathbf{Var}_{\pi^N}[Y^N_{\lceil\frac{N}{2}\rceil}] = \sum_{i=1}^{\lceil\frac{N}{2}\rceil}\frac{1}{4a_i^2} = \sum_{i=1}^{\lceil\frac{N}{2}\rceil}\frac{N^2}{\lambda^2(N+1-i)^2\left(i-\frac{\lambda(2-N)+2\delta N}{\lambda}\right)^2}\overset{N\to\infty}{\longrightarrow} 0$$

We can make the same computations for other quantiles of the measure π, compared with the quantiles of the stationary empirical measure μ_∞^N, for which we know the explicit distribution in terms of the independent gaps between successive particles. Therefore, for any $p \in (0,1)$, the sequence of p-quantiles of the stationary empirical measure $\{x_p^N\}$, which is a sequence of random variables, converges a.s. to the deterministic value x_p, that is the p-quantile of π. The convergence of any p-quantile such that the pseudo inverse of the limit CDF $F^{-1}(p)$ is continuous in p is equivalent to weak convergence. Since π admits a density, we have that a.s. the sequence of stationary empirical measures converges weakly to π.

Chapter 4

A system of random walks with asymmetric interaction

In this chapter we study a system of mean field interacting random walks on the positive integers, reflected at zero, presented in [2]. Each particle has a drift $\delta \geqslant 0$ towards infinity and a parameter $\lambda > 0$ that tunes an interaction. This interaction is asymmetric in the sense that it pushes each particle towards the origin, but it depends on the number of particles at the left of the affected one. We are interested in studying the mean field limit of this model and its stability properties.

4.1 The model

In this section we describe all the details of the model we are interested in. As in the previous chapters, we start with the description of the N particle system, then we heuristically describe its mean field limit and we prove well-posedness and propagation of chaos.

4.1.1 The particle system

We fix $N \geqslant 2$ and we consider N particles, each of them moving on the nonnegative integers. Let $X^N = (X_1^N, \ldots, X_N^N) \in \mathbb{N}^N$ be the vector of the particles' positions. Each particle has an intrinsic dynamics and it experiences an interaction.

- The **intrinsic dynamics** is given by a simple biased random walk, with jump amplitude one, independent of the other particles and reflected at zero. This is described by 2 independent Poisson clocks for each particle, one with rate 1, governing the downward jump and the other with rate $1 + \delta$, $\delta \geqslant 0$, governing the upward jump.

- The **interaction dynamics** is tuned by a parameter $\lambda > 0$. Every pair of particles, for example

$$(X_i^N, X_j^N)$$

is activated with a space-dependent rate

$$\frac{\lambda}{N}\phi(X_i^N, X_j^N).$$

Here $\phi\colon \mathbb{N}^2 \to [0,1]$ is a bounded interaction function, symmetric in its argument. If the two particles are in the same position, i.e. $X_i^N = X_j^N$, then nothing happens. Otherwise, if they are in different sites, for example $X_i^N < X_j^N$, then the one in the highest position (in the example X_j^N) is encouraged to move down. This means that its position makes a backward jump of amplitude

$$\psi(X_j^N, X_i^N),$$

where $\psi\colon \mathbb{N}^2 \to \mathbb{N}$ is a symmetric function such that $1 \leqslant \psi(x,y) \leqslant x \vee y$ for all $(x,y) \in \mathbb{N}^2$.

Let $X^N(t) = (X_1^N(t), \dots, X_N^N(t))$ be Markov process with the above dynamics at each time $t \geqslant 0$, then for all $i = 1, \dots, N$ the particle $X_i^N(t)$ does the following moves:

$$X_i^N(t) - \psi(X_i^N(t), X_j^N(t)) \text{ with rate } \frac{\lambda}{N}\phi(X_i^N(t), X_j^N(t)),$$

for all $j = 1, \dots, N$ and only if $X_i^N(t) > X_j^N(t)$. The infinitesimal generator $\mathcal{L}^N$ of this Markov process acts on bounded measurable function $f : \mathbb{N}^N \to \mathbb{R}$ in the following way:

$$\begin{aligned} \mathcal{L}^N f(z) =& \sum_{i=1}^N \left(\mathbb{1}(z_i > 0)(f(z - \mathbf{e}_i) - f(z)) + (1+\delta)(f(z+\mathbf{e}_i) - f(z))\right) \\ &+ \frac{\lambda}{N}\sum_{i=1}^N \sum_{k=1}^N \mathbb{1}(z_k < z_i)\phi(z_k, z_i)\left(f(z - \mathbf{e}_i\psi(z_k, z_i)) - f(z)\right), \end{aligned} \tag{4.1.1}$$

where $\mathbf{e}_i$ is the vector $(0,\dots,0,1,0,\dots,0)$ with the i-th coordinate equal to 1 and zero otherwise. Since the jump rates are bounded, the process is well defined and admit a solution for every initial condition in $\mathbb{N}^N$. It will be useful to notice that, for all filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$, X^N is solution of the following system of SDEs: for $i = 1, \dots, N$,

$$\begin{aligned} dX_i^N(t) =& -\mathbb{1}(X_i^N(t^-) > 0)\int_0^\infty \mathbb{1}_{[0,1]}(u)\mathcal{N}^i_{(-)}(du, dt) + \int_0^\infty \mathbb{1}_{[0,1+\delta]}(u)\mathcal{N}^i_{(+)}(du, dt) \\ &- \int_{[0,1]}\int_0^\infty \sum_{k=0}^{X_i^N(t^-)-1} \psi(k, X_i^N(t^-))\mathbb{1}_{I_k(X_i^N(t^-), \mu^N_{t^-})}(h)\mathbb{1}_{[0,\lambda]}(u)\mathcal{N}^i(du, dh, dt), \end{aligned} \tag{4.1.2}$$

where $\{\mathcal{N}^i_{(-)}, \mathcal{N}^i_{(+)}, \mathcal{N}^i\}_{i=1,\dots,N}$ are independent stationary Poisson processes with characteristic measures, respectively, $du dt$, $du dt$ and $du dh dt$, and

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N(t)}$$

indicates the empirical measures, as usual. The intervals are such that, for all $k > 0$ we have

$$I_k(x,\mu) \doteq \left(\sum_{y=0}^{k-1} \phi(y,x)\mu(y), \sum_{y=0}^{k} \phi(y,x)\mu(y)\right]$$

and $I_0 \doteq (0, \phi(0,x)\mu(0)]$.

4.1.2 The nonlinear processes

In this section we introduce, at a heuristic level, the *nonlinear process*, that stands for the macroscopic description of the model presented in Section 4.1.1. As we said in the previous chapters, heuristically the mean field limit is obtained under the assumption that there exists a law μ_t that is the weak limit of the sequence of empirical measures, i.e.

$$\{\mu_t^N\}_{t\geqslant 0} \stackrel{N\to\infty}{\longrightarrow} \{\mu_t\}_{t\geqslant 0}.$$

In this framework, let us consider the nonlinear SDE defined as follows

$$\begin{aligned} dX(t) = &- \mathbb{1}(X(t^-) > 0)\int_0^\infty \mathbb{1}_{[0,1]}(u)\mathcal{N}_{(-)}(du,dt) + \int_0^\infty \mathbb{1}_{[0,1+\delta]}(u)\mathcal{N}_{(+)}(du,dt) \\ &- \int_{[0,1]}\int_0^\infty \sum_{k=0}^{X(t^-)-1} \psi(k,X(t^-))\mathbb{1}_{I_k(X(t^-),\mu_{t^-})}(h)\mathbb{1}_{[0,\lambda]}(u)\mathcal{N}(du,dh,dt), \end{aligned} \tag{4.1.3}$$

where $\mu_t = \text{Law}(X(t))$, $\{\mathcal{N}_{(-)}, \mathcal{N}_{(+)}, \mathcal{N}\}$ are independent stationary Poisson processes with characteristic measures, respectively, $dudt$, $dudt$ and $dudhdt$. The intervals $I_k(x,\mu)$ are defined as in (4.1.2). The well-posedness of (4.1.3) for every initial condition supported on $\mathbb{N}$ will be proved together with the mean field limit, as in Chapter 3. Indeed, the existence of a process that solves (4.1.3) is ensured by the tightness of the sequence of empirical measures.

In order to ensure uniqueness of the nonlinear system, we require the backward jumps to satisfy the following condition.

Assumption 4.1.1. *There exists* $C < \infty$ *such that for all* $x, y \in \mathbb{N}$ *and* $\alpha, \beta \in \mathcal{M}(\mathbb{N})$ *probability measure on* $\mathbb{N}$

$$\left|\sum_{k=0}^{x\vee y-1} \psi(k, x\vee y)\,|I_k(x,\alpha)\Delta I_k(y,\alpha)|\right| \leqslant C|x-y|,$$

where for A, B *two intervals of the real line* $A\Delta B \doteq A\backslash B \cup B\backslash A$ *and*

$$\left|\sum_{(x,y,z)\in\mathcal{A}} \alpha(y)\alpha(z) - \beta(y)\beta(z)\right| \leqslant C\sum_{x\in\mathbb{N}} |\alpha(x)-\beta(x)|,$$

where $\mathcal{A} \doteq \{(x,y,z)\in\mathbb{N}^3 \colon z > x,\ z > y,\ z - \psi(y,z) = x\}$.

Assumption 4.1.1 is rather technical; it resembles a Lipschitz-type condition on the coefficients and it is sufficient for the proof of uniqueness via Gronwall inequalities, as the following result shows. However, it is more general than any condition on jumps we considered in Chapter 2. Indeed, we can prove pathwise propagation of chaos only for a small subclass of models among the ones described here. This is the reason why we adopt the approach via the solution of the *martingale problem*, which is more flexible than the coupling procedure. In Section 4.1.4, we present a particular model of this class that satisfies Assumption 2.1.1 and for which pathwise propagation of chaos holds, with the expected rate of $\frac{1}{\sqrt{N}}$.

Proposition 4.1.1. *Grant Assumption 4.1.1, then pathwise uniqueness holds for the non-linear SDE* (4.1.3) *in the class of processes with initial conditions supported on* $\mathbb{N}$.

Proof. Step 1: uniqueness in law. Let μ be the law of X and μ_t its time-marginal. We consider the following equation, for all $x \in \mathbb{N}$

$$\begin{aligned}\frac{d}{dt}\mu_t(x) =& \mu_t(x+1) - (1+\delta)\mu_t(x) + \mathbb{1}(x>0)((1+\delta)\mu_t(x-1) - \mu_t(x)) \\ &+ \lambda\left(\sum_{(h,k)\in\mathcal{A}_x} \mu_t(h)\mu_t(k)\phi(h,k) - \mu_t(x)\sum_{k=0}^{x-1}\phi(k,x)\mu_t(k)\right),\end{aligned} \tag{4.1.4}$$

where $\mathcal{A}_x \doteq \{(h,k) \in \mathbb{N}^2\colon\ h<k,\ k>x,\ k-\psi(h,k)=x\}$. Since we are looking for processes with initial condition supported on $\mathbb{N}$, μ_0 is a measure on $\mathbb{N}$, thus the same is true for μ_t for all $t \geqslant 0$ and (4.1.4) is actually the equation for the time evolution of the law μ_t. Set $x_k(t) \doteq \mu_t(k)$ for all $k \geqslant 0$, then (4.1.4) is equivalent to the following infinite dimensional system of ODEs:

$$\begin{cases} \dot{x}_0 = x_1 - (1+\delta)x_0 + \lambda \displaystyle\sum_{(h,k)\in\mathcal{A}_0} x_h x_k \phi(h,k) \\ \dot{x}_n = x_{n+1} - x_n + (1+\delta)(x_{n-1}-x_n) + \lambda\left(\displaystyle\sum_{(h,k)\in\mathcal{A}_n} x_h x_k\phi(h,k) - x_n\sum_{k=0}^{n-1} x_k\phi(k,x)\right) \\ n = 1,2,\dots \end{cases} \tag{4.1.5}$$

Therefore, we are looking for the uniqueness of the solution of (4.1.5) in the subspace

$$M^1 = \left\{x \in l^1 \text{ s.t. } \|x\|_1 = 1 \text{ and } x_i \in [0,1]\,\forall i\right\}.$$

Let $x(t)$ and $y(t)$ be two solutions of (4.1.5) with the same initial condition. Fix $T>0$, we want to prove that $\|x(t)-y(t)\|_1 = 0$ for all $t \in [0,T]$. By a simple integration of (4.1.5) and some bound, we get

$$
\begin{aligned}
\|x(t)-y(t)\|_1 =& \sum_{n=0}^{\infty} |x_n(t)-y_n(t)| \\
\leqslant& \|x(0)-y(0)\|_1 + \int_0^t (4+2\delta) \sum_{n=0}^{\infty} |x_n(s)-y_n(s)| ds \\
&+\lambda \int_0^t \sum_{n=0}^{\infty} \Bigg| \Bigg(\sum_{(h,k)\in\mathcal{A}_n} x_h(s)x_k(s)\phi(h,k) - \sum_{k=0}^{n-1} x_n(s)x_k(s)\phi(k,n) \Bigg) \\
&- \Bigg(\sum_{(h,k)\in\mathcal{A}_n} y_h(s)y_k(s)\phi(h,k) - \sum_{k=0}^{n-1} y_n(s)y_k(s)\phi(k,n) \Bigg) \Bigg| ds.
\end{aligned}
$$

Now the role of Assumption 4.1.1 is clear, since we have

$$
\begin{aligned}
\|x(t)-y(t)\|_1 \leqslant& \|x(0)-y(0)\|_1 + 2(2+\delta) \int_0^t \|x(s)-y(s)\|_1 ds \\
&+ \lambda \int_0^t \sum_{n=0}^{\infty} |x_n(s)-y_n(s)| \left| \sum_{k=n+1}^{\infty} x_k(s) - \sum_{k=0}^{n-1} x_k(s) \right| ds \\
&+ \lambda \int_0^t \sum_{n=0}^{\infty} |y_n(s)| \left| \sum_{k=n+1}^{\infty} (x_k(s)-y_k(s)) - \sum_{k=0}^{n-1} (x_k(s)-y_k(s)) \right| ds \\
\leqslant& \|x(0)-y(0)\|_1 + 2(2+\delta+(C+1)\lambda) \int_0^t \|x(s)-y(s)\|_1 ds.
\end{aligned}
$$

By applying Gronwall Lemma, since $\|x(0)-y(0)\|_1 = 0$, we get $\|x(t)-y(t)\|_1 = 0$ for all $t \in [0,T]$. By the arbitrariness of $T>0$, we get uniqueness for μ.

Step 2: pathwise uniqueness. We fix a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ and we consider two solutions $X^1 = (X^1(t))_{t\geqslant 0}$ and $X^2 = (X^2(t))_{t\geqslant 0}$ driven by the same Poisson processes and such that a.s. $X^1(0) = X^2(0) \in \mathbb{N}$. We know, by point *i)*, that these two solutions coincide in law, i.e. $\mu^1 = \mu^2$.

Fix $T>0$, then we want to prove that $\mathbf{E}\left[\sup_{t\in[0,T]} |X^1(t)-X^2(t)|\right] = 0$.

$$
\begin{aligned}
\mathbf{E}\left[\sup_{t\in[0,T]} |X^1(t)-X^2(t)|\right] \leqslant& \mathbf{E}\Bigg[\int_0^T \left|\mathbb{1}(X^1(t)>0) - \mathbb{1}(X^2(t)>0)\right| dt \\
&+\lambda \int_0^T \int_{[0,1]} \left| \sum_{k=0}^{X^1(t)-1} \psi(k,X^1(t)) \mathbb{1}_{I_k(X^1(t),\mu^1(t))}(h) - \sum_{k=0}^{X^2(t)-1} \psi(k,X^2(t)) \mathbb{1}_{I_k(X^2(t),\mu^2(t))}(h) \right| dh ds\Bigg].
\end{aligned}
$$

First, we know by the previous step that there is weak uniqueness, i.e. $\mu^1(t) = \mu^2(t)$. Moreover, by hypothesis we have that $|\phi(\cdot,\cdot)| \leqslant 1$. Therefore we perform the following bound:

$$
\begin{aligned}
\mathbf{E}\left[\sup_{t\in[0,T]}|X^1(t)-X^2(t)|\right] \leqslant & \int_0^T \mathbf{E}\left[\sup_{s\in[0,t]}|X^1(t)-X^2(t)|\right]dt \\
&+\lambda\int_0^T \mathbf{E}\left[|X^1(t)-X^2(t)|\sum_{k=0}^{X^1(t)\wedge X^2(t)-1}\mu^1(t)(\{k\})\right]dt \\
&+\lambda\int_0^T \mathbf{E}\left[\left|\sum_{k=0}^{X^1(t)\wedge X^2(t)-1}\psi(k,X^1(t))|I_k(X^1(t),\mu^1_t)\Delta I_k(X^2(t),\mu^1_t)|\right|\right]dt \\
&+\lambda\int_0^T \mathbf{E}\left[\sum_{k=X^1(t)\wedge X^2(t)}^{X^1(t)\vee X^2(t)-1}|X^1(t)\vee X^2(t)-k|\mu^1(t)(\{k\})\right]dt.
\end{aligned}
$$

We bound the third term in the right-hand side by means of Assumption 4.1.1. Then, we consider the fourth term and, obviously, we have that, for all $t \geqslant 0$,

$$
\sum_{k=X^1(t)\wedge X^2(t)}^{X^1(t)\vee X^2(t)-1}|X^1(t)\vee X^2(t)-k|\mu^1(t)(\{k\}) \leqslant |X^1(t)-X^2(t)|\sum_{k=X^1(t)\wedge X^2(t)}^{X^1(t)\vee X^2(t)-1}\mu^1(t)(\{k\}).
$$

Moreover, since $\mu^1(t)$ is a probability measure for all $t \geqslant 0$, we recall that, obviously,

$$
\sum_{k=0}^{X^1(t)\vee X^2(t)-1}\mu^1(t)(\{k\}) \leqslant 1.
$$

Therefore, we obtain

$$
\mathbf{E}\left[\sup_{t\in[0,T]}|X^1(t)-X^2(t)|\right] \leqslant (1+(C+1)\lambda)\int_0^T \mathbf{E}\left[\sup_{s\in[0,t]}|X^1(s)-X^2(s)|\right]dt.
$$

Now, we apply Gronwall Lemma and we get the thesis.

□

As we mentioned, weak existence of a solution to (4.1.3) is a consequence of propagation of chaos and it will be shown in the following section. Therefore, we end this section by stating the result on well-posedness of (4.1.3), but we postpone its proof to the end of Section 4.1.3.

Theorem 4.1.1. *Grant Assumption 4.1.1, then for every $\mathcal{F}_0$-measurable initial condition X_0 with values in $\mathbb{N}$, there exists a unique strong solution to* (4.1.3).

4.1.3 Propagation of chaos

In the proof of propagation of chaos for the particle system (4.1.2), we progress step by step with the same approach of Section 3.1.2. This means that we first prove tightness and consistency of the sequence of empirical measures. To this aim, let us define the martingale problem associated to (4.1.3).

Definition 4.1.1. We say that the law Q on $\mathbf{D}(\mathbb{R}^+, \mathbb{R}^+)$ of any $(X_t)_{t\geqslant 0}$ satisfying (4.1.3) is the solution of the *martingale problem* correspondent to (4.1.3) if the following properties hold.

i) $Q \circ X_0^{-1} = \mu_0$, for μ_0 measure on $\mathbb{N}$.

ii) Let $\mathcal{L}_\mu$ be the generator defined on every bounded function f by

$$\mathcal{L}_\mu f(x) = \mathbb{1}(x>0)(f(x-1)-f(x))+(1+\delta)(f(x+1)-f(x))+\lambda\sum_{k=0}^{x-1}\phi(k,x)\mu(k)(f(x-\psi(k,x))-f(x)).$$

Then, for all $f \in C_b$, for all $t \geqslant 0$

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{L}_{\mu_s} f(X_s)ds$$

is a Q-martingale, where $\mu_s = \mu \circ X_s^{-1}$.

Proposition 4.1.2. *Fix a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$ *and an initial condition* $X^N(0)$ *whose law is* μ_0*-chaotic, for a law* μ_0 *on* $\mathbb{N}$*. Let* ${X^N}_{t\geqslant 0}$ *be the solution to* (4.1.2)*. Then:*

a) the sequence of empirical measures $\left\{\mu^N\right\}$ *is tight in* $\mathcal{M}\left(\mathbf{D}(\mathbb{R}^+)^2\right)$*;*

b) any limit point μ *of the sequence* μ^N *solves the martingale problem correspondent to* (4.1.3)*.*

Proof. a) We know that, under our conditions of exchangeability of the components, proving *point a)* is equivalent to prove that the sequence of processes $\{(X_1^N(t))_{t\geqslant 0}, N \geqslant 2\}$ is tight in $\mathbf{D}(\mathbb{R}^+, \mathbb{R}^+)$, see [83]. To this aim, we use Aldous's criterion, see [12], and we want prove the two following statements.

i) For all $T > 0$,

$$\lim_{K\uparrow\infty}\sup_{N\geqslant 1}\mathbf{P}\left(\sup_{t\in[0,T]} X_1^N(t) > K\right) = 0.$$

ii) For all $T > 0$, $\epsilon > 0$

$$\lim_{\rho\downarrow 0}\limsup_{N\to\infty}\sup_{(S,S')\in A_T^\rho}\mathbf{P}\left(|X_1^N(S) - X_1^N(S')| > \epsilon\right) = 0,$$

where $A_T^\rho \doteq \{(\tau, \tau')$ stopping times s.t. $0 \leqslant \tau \leqslant \tau' \leqslant \tau + \rho \leqslant T$ a.s.$\}$.

- Condition ***i)*** follows immediately from the fact that, if we fix $T > 0$, $\forall\, N \geqslant 1$ we have

$$\mathbf{P}\left(\sup_{t\in[0,T]} X_1^N(t) > K\right) \leqslant \frac{\mathbf{E}\left[\sup_{t\in[0,T]}|X_1^N(t)|\right]}{K}$$

(we take into account only the jumps on the right)

$$\leqslant \frac{1}{K}\mathbf{E}\left[\sup_{t\in[0,T]}\int_0^t\int_0^\infty \mathbb{1}_{[0,1+\delta]}(u)\mathcal{N}^i_{(+)}(du,ds)\right] \leqslant \frac{(1+\delta)T}{K}.$$

- To prove condition ***ii)***, we know that

$$\mathbf{P}\left(|X_1^N(S)-X_1^N(S')|>\epsilon\right) \leqslant \mathbf{P}\left(|X_1^N(S)-X_1^N(S')|>0\right).$$

We write $|X_1^N(S)-X_1^N(S')|$ by means of (4.1.2):

$$\begin{aligned}|X_1^N(S)-X_1^N(S')| \leqslant &\left|-\int_S^{S'}\mathbb{1}(X_1^N(t^-)>0)\int_0^\infty \mathbb{1}_{[0,1]}(u)\mathcal{N}^1_{(-)}(du,dt)\right| \\ &+\left|\int_S^{S'}\int_{[0,1]}\int_0^\infty \sum_{k=0}^{X_1^N(t^-)-1}\psi(k,X_1^N(t^-))\mathbb{1}_{I_k(X_1^N(t^-),\mu^N_{t-})}(h)\mathbb{1}_{[0,\lambda]}(u)\mathcal{N}^1(du,dh,dt)\right| \\ &+\left|\int_S^{S'}\int_0^\infty \mathbb{1}_{[0,1+\delta]}(u)\mathcal{N}^1_{(+)}(du,dt)\right| \doteq A^-_{(S,S')}+B_{(S,S')}+A^+_{(S,S')}.\end{aligned}$$

Since we have 3 terms involving only integrals with respect to Poisson random measures, the probability of those terms being strictly greater than 0 is equal to the probability that there is at least one jump in the time interval $[S,S']$, therefore

$$\begin{aligned}&\mathbf{P}(A^-_{(S,S')}+B_{(S,S')}+A^+_{(S,S')}>0) \leqslant \mathbf{P}\left(A^-_{(S,S')}>0\right)+\mathbf{P}\left(B_{(S,S')}>0\right)+\mathbf{P}\left(A^+_{(S,S')}>0\right) \\ &\leqslant \mathbf{P}(A^-_{(S,S')}\geqslant 1)+\mathbf{P}\left(\int_S^{S'}\int_0^1\int_0^\infty \mathbb{1}_{[0,\lambda)}(u)\mathcal{N}^1(du,dh,dt)\geqslant 1\right)+\mathbf{P}(A^+_{(S,S')}\geqslant 1) \\ &\leqslant \mathbf{E}[A^-_{(S,S')}]+\mathbf{E}\left[\int_S^{S'}\int_0^1\int_0^\infty \mathbb{1}_{[0,\lambda)}(u)\mathcal{N}^1(du,dh,dt)\right]+\mathbf{E}[A^+_{(S,S')}] \leqslant \rho(2+\lambda+\delta).\end{aligned}$$

By taking the limit for $\rho \downarrow 0$ we get the thesis.

b) A probability measure $Q \in \mathcal{M}(\mathbf{D}(\mathbb{R}^+,\mathbb{R}^+))$ that is solution to the martingale problem defined in Definition 4.1.1 must satisfy the two conditions *i)* and *ii)*.

i) Clearly $\mu \circ X_0^{-1} = \mu_0$, since $\mu \circ X_0^{-1}$ is the limit of the sequence $\frac{1}{N}\sum_{i=1}^N \delta_{X_i^N(0)}$ that, by μ_0-chaoticity clearly converges weakly to μ_0.

ii) **Step 1)** As in Section 3.1.2, we follow the approach of Sznitman, [81], and we define a functional

$$F\colon \mathcal{M}(\mathbf{D}(\mathbb{R}^+,\mathbb{R}^+)) \to \mathbb{R}$$

that is zero on the measures satisfying this martingale problem defined in Definition 4.1.1. Fix $\varphi \in C_b$, $0 \leqslant s_1 \leqslant \cdots \leqslant s_q \leqslant s \leqslant t$, $g_1, \ldots, g_q \in C_b$ and define

$$F(Q) \doteq \langle Q, \left(\phi(X(t)) - \phi(X(s)) - \int_s^t \mathcal{L}_{Q_u}\phi(X(u))du\right) g_1(X(s_1))\ldots g_q(X(s_q))\rangle.$$

We want to prove that for all Q limit point of μ^N, $F(Q) = 0$ a.s..

Step 2) We firstly prove that $\lim_{N\to\infty} \mathbf{E}\left[F(\mu^N)^2\right] = 0$. We consider

$$\begin{aligned}\mathbf{E}\left[F(\mu^N)^2\right] = \mathbf{E}&\left[\frac{1}{N^2}\sum_{i=1}^N (M_t^{\varphi_i} - M_s^{\varphi_i})^2 g_1^2(X_i^N(s_1))\ldots g_q^2(X_i^N(s_q))\right]\\ +\mathbf{E}&\left[\frac{1}{N^2}\sum_{i\neq j}^N (M_t^{\varphi_i} - M_s^{\varphi_i})(M_t^{\varphi_j} - M_s^{\varphi_j}) g_1(X_i^N(s_1)) g_1(X_j^N(s_1))\ldots\right],\end{aligned}$$

where $M_t^{\varphi_i} = \varphi(X_i^N(t)) - \varphi(X_i^N(0)) - \int_0^t \mathcal{L}_{\mu^N(u)}\varphi(X_i^N(u))du$ for $i = 1, \ldots, N$ are orthogonal martingales, i.e.

$$\langle M^{\varphi_i}, M^{\varphi_j}\rangle = 0 \text{ for all } i \neq j.$$

Indeed, by applying Ito's rule for jump processes to $\varphi(X_i^N(t)) - \varphi(X_i^N(0))$, we see that $M_t^{\varphi_i} - M_s^{\varphi_i}$ is simply the sum of integrals w.r.t. the three martingales $\tilde{\mathcal{N}}^i_{(-)}$, $\tilde{\mathcal{N}}^i_{(+)}$ and $\tilde{\mathcal{N}}^i$. Therefore, by hypothesis, it is orthogonal to $M_t^{\varphi_j} - M_s^{\varphi_j}$ for all $j \neq i$. We also know how to rewrite the quadratic variation:

$$\begin{aligned}\langle M_t^{\varphi_i} - M_s^{\varphi_i}\rangle &= \int_s^t \int_0^\infty \left(\varphi(X_i^N(r) + \mathbb{1}_{[0,1]}(u)) - \varphi(X_i^N(r))\right)^2 dudr\\ &+ \int_s^t \int_0^\infty \left(\varphi(X_i^N(r) - \mathbb{1}_{[0,1]}(u)) - \varphi(X_i^N(r))\right)^2 \mathbb{1}(X_i^N(r) > 0)dudr\\ &+ \int_s^t\int_0^\infty\int_{[0,1]} \left(\varphi\left(X_i^N(r) - \sum_{k=0}^{X_i^N(r)-1} \psi(k, X_i^N(r))\mathbb{1}_{I_k(X_i^N(r),\mu_r^N)}(h)\,\mathbb{1}_{[0,\lambda)}(u)\right) - \varphi(X_i^N(r))\right)^2 dhdudr.\end{aligned}$$

Then, for a constant $K \geqslant 0$, depending on $\{g_i\}_{i=1,\ldots,q}$, and $C \geqslant 0$, depending on the function φ, we have

$$\begin{aligned}\mathbf{E}\left[F(\mu^N)^2\right] &\leqslant \frac{K}{N^2}\sum_{i=1}^N \mathbf{E}\left[\langle M_t^{\varphi_i} - M_s^{\varphi_i}\rangle\right] + \frac{K}{N^2}\sum_{i\neq j}^N \mathbf{E}\left[\langle M_t^{\varphi_i} - M_s^{\varphi_i}, M_t^{\varphi_j} - M_s^{\varphi_j}\rangle\right]\\ &\leqslant \frac{K}{N^2}\sum_{i=1}^N C(2+\lambda)(t-s) \leqslant \frac{KC(2+\lambda)(t-s)}{N},\end{aligned}$$

therefore $\lim_{N\to\infty} \mathbf{E}\left[F(\mu^N)^2\right] = 0$.

Step 3) We are left to prove that $\mathbf{E}\left[F(Q)^2\right] = \lim_{N\to\infty} \mathbf{E}\left[F(\mu^N)^2\right]$, from Sznitman [83] we know that it is sufficient to verify

$$Q\left(\{\Delta X(t) \neq 0\} \cup \{\Delta X(s) \neq 0\} \cup \{\Delta X(s_1) \neq 0\} \cup \cdots \cup \{\Delta X(s_q) \neq 0\}\right) = 0 \text{ a.s.},$$

where $\Delta X(t) = X(t) - X(t^-)$.

By contradiction, assume that there exists a $\bar{t} \in \{s_1, \ldots, s_q, s, t\}$ such that $Q(\Delta X(\bar{t}) \neq 0) > 0$ with positive probability. That is, there exists a constant $b > 0$ such that the event

$$E \doteq \{Q(\Delta X(\bar{t}) > 0) > b\}$$

has positive probability. For every $\epsilon > 0$ we can define the open set of $\mathbf{D}(\mathbb{R}^+, \mathbb{R}^+)$

$$D^\epsilon \doteq \{x \in \mathbf{D}(\mathbb{R}^+, \mathbb{R}^+) \text{ s.t. } \sup_{s\in(\bar{t}-\epsilon, \bar{t}+\epsilon)} |\Delta x_s| > 0\}$$

and the open set of $\mathcal{M}(\mathbf{D}(\mathbb{R}^+, \mathbb{R}^+))$

$$P^d_\epsilon \doteq \{\mu \text{ s.t. } \mu(D^\epsilon) > d\}.$$

We see that $E \subset \{Q(D^\epsilon_a) > d\}$ and, by Portmanteau theorem,

$$\liminf_{N\to\infty} \mathbf{P}(\mu^N \in P^d_\epsilon) \geqslant \mathbf{P}(Q \in P^d_\epsilon) \geqslant \mathbf{P}(E) > 0.$$

We bound the term on the left-hand by means of

$$\{\mu_N \in P^d_\epsilon\} \subset \left\{\frac{1}{N}\sum_{i=1}^N \mathbb{1}(X^N_i \text{ performs at least one jump in } (t-\epsilon, t+\epsilon)) \geqslant d\right\}.$$

Since the particle are exchangeable and their jump rates are constants, we have that

$$\mathbf{P}(\mu^N \in P^d_\epsilon) \leqslant \frac{2\epsilon(2+\lambda+\delta)}{d}.$$

This leads to the contradiction

$$0 = \liminf_{\epsilon\downarrow 0} \liminf_{N\to\infty} \mathbf{P}(\mu_N \in P^d_\epsilon) \geqslant \mathbf{P}(E) > 0$$

and it proves continuity of F.

□

Weak uniqueness of solutions of the martingale problem in Definition 4.1.1 is proved in Proposition 4.1.1, this implies that the limit of any convergent subsequence of $\{\mu^N\}$ is the same deterministic element of the space $\mathcal{M}(\mathbf{D}(\mathbb{R}^+, \mathbb{R}^+))$. As we anticipated, the previous results lead to the proof of the well-posedness of the nonlinear SDE (4.1.3).

Proof of Theorem 4.1.1. Existence of a weak solution to (4.1.3) is ensured by Proposition 4.1.2. Indeed, for all initial condition $\mu_0 \in \mathcal{M}(\mathbb{N})$, we can construct a sequence of processes $X^N = (X^N(t))_{t \geqslant 0}$. For all $N \geqslant 2$, each process has as initial condition $(X_1^N(0), \ldots, X_N^N(0))$, where $X_i^N(0)$ are i.i.d. random variables μ_0-distributed and X^N solves (4.1.2). Then the limit of the sequence of empirical measures $\{\mu^N\}_{N \geqslant 2}$ for $N \to \infty$ is a solution to (4.1.3). Pathwise uniqueness of this solution is given by Proposition 4.1.1. By Yamada-Watanabe theorem, pathwise uniqueness together with weak existence gives existence and uniqueness of strong solutions, see [57]. $\square$

Finally we can state and prove a complete result of propagation of chaos, that is simply a consequence of the previous results.

Theorem 4.1.2 (Propagation of chaos)**.** *For every* μ_0 *probability measure on* $\mathbb{N}$*, let* $P^N \in \mathcal{M}(\mathbf{D}(\mathbb{R}^+, \mathbb{R})^N)$ *be the law of the solution of system* (4.1.2) *with initial condition* $P_0^N = P^N \circ (X_0^N)^{-1}$ *that is* μ_0*-chaotic. Then the sequence* P^N *is* μ*-chaotic, where* $\mu \in \mathcal{M}(\mathbf{D}(\mathbb{R}^+, \mathbb{R}))$ *is the law of the unique solution of* (4.1.3) *with initial condition* μ_0*.*

Proof. We prove propagation of chaos with the tightness/consistency/uniqueness approach, see [81, 83].

- From Proposition 4.1.2, *point i)* we have tightness of the sequence of empirical measures $\{\mu^N\}_{N \in \mathbb{N}}$ in $\mathcal{M}(\mathbf{D}(\mathbb{R}^+, \mathbb{R}^+))$.

- From Proposition 4.1.2, *point ii)* we have that any limit point of a converging subsequence $\{\mu^{N_k}\}_{k \in \mathbb{N}}$ is a solution of (4.1.3).

- In Proposition 4.1.1 we proved uniqueness of solution of (4.1.3), this let us conclude that the limit of the sequence of empirical measures is deterministic.

The three steps above imply the property of propagation of chaos for the particle system (4.1.2). $\square$

4.1.4 Motivation and examples

The class of models that we introduce in this chapter is motivated by genetics, indeed it can be used as a description of the evolution of genetics traits. Our N particle system may be interpreted as a population of N individuals. Each individual is characterized by its fitness level, that is an integer number greater or equal than 0, that is the worst possible fitness value. Each time that a particle moves, we imagine that the corresponding individual dies and gives birth to a child whose fitness level is greater or smaller than its own. The individuals have an intrinsic tendency to improve (given by the biased random walk).

However, by mimicking the worst individual of the population (the one with lower fitness level), they may give birth to a child that is much worse than themselves; this of course corresponds to the leftward jumps due to the asymmetric interaction.

Let us give some explicit examples of models belonging to this class, by specifying the involved functions.

The *small jumps* model

The simplest model in our class is such that the size of the jumps induced by the asymmetric interaction is the minimal, i.e. they are of size 1. We refer to this model as the model with *small jumps*. In this case both the rate and the jump function are constantly equal to 1: for all $x \neq y \in \mathbb{N}$

$$\begin{aligned} \phi(x,y) &= 1, \\ \psi(x,y) &= 1. \end{aligned}$$

Let us describe in details this model. For a fixed number N of particles on $\mathbb{Z}^+$, each particle X_i^N, for $i = 1, \dots, N$, makes the following moves: if $X_i^N > 0$, then it goes to

$$\begin{array}{ll} X_i^N + 1 & \text{with rate } 1+\delta, \\ X_i^N - 1 & \text{with rate } 1+\lambda \frac{1}{N}\sum_{k=1}^N \mathbb{1}(X_k^N < X_i^N), \end{array} \tag{4.1.6}$$

while when $X_i^N = 0$, the only allowed jump is the one upward. It is clear that here $\delta \geqslant 0$ indicates a bias rightward, while $\lambda \frac{1}{N}\sum_{k=1}^N \mathbb{1}(X_k^N < X_i^N)$ is a bias leftward. The infinitesimal generator of (4.1.6) is given by

$$\mathcal{L}^N_{(SJ)} f(z) = \sum_{i=1}^N \left(\mathbb{1}(z_i > 0)\nabla_i^- f(z) + (1+\delta)\nabla_i^+ f(z)\right) + \frac{\lambda}{N}\sum_{i=1}^N \nabla_i^- f(z) \sum_{k=1}^N \mathbb{1}(z_k < z_i). \tag{4.1.7}$$

Here $\nabla_i^- f(z) = f(z - \mathbf{e}_i) - f(z)$ and $\nabla_i^+ f(z) = f(z + \mathbf{e}_i) - f(z)$. We associate to (4.1.6) its correspondent nonlinear Markov process, that is a Markov process $\{X_t\}_{t \geqslant 0}$ whose possibile transitions at time $t \geqslant 0$ are the following:

$$\begin{array}{ll} X_t + 1 & \text{with rate } 1+\delta, \\ X_t - 1 & \text{with rate } 1+\lambda\mu_t[0, X_t), \end{array} \tag{4.1.8}$$

where $\mu_t = \text{Law}(X_t)$ and, as in (4.1.6), when X_t=0, only the upward jump is allowed.

Let us underline that this *small jumps* model has a direct link with the diffusion model described in Chapter 3, that may be seen as its continuous analogue. Each particle performs a random walk with a bias that depends on its rank with respect to all the others, with the same form of the drift in Chapter 3. Indeed, the rightmost particle, when alone on its site, has a net drift of $\delta - \lambda \frac{N-1}{N}$, whereas the leftmost particle has a positive drift

δ. For this reason, we will use the continuous model as a basis for comparisons in the analysis of the *small jumps* model. Despite their similarities, the two models display peculiar differences that emerge in the study of the long-time behavior. In the discrete model particles can form big clusters on a single site. By our rule, particles in the same site *do not* interact: thus the formation of clusters tends to prevent the stabilization of the process.

Before the study of the long-time behavior, motivated by Theorem 3.1.2, we look for a result of trajectorial propagation of chaos. To this aim, we define a coupling procedure between a Markov process defined by the generator (4.1.7) and N copies of the nonlinear Markov process defined in (4.1.8). We fix a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, \mathbf{P})$ and let μ_0 be a probability measure on $\mathbb{N}$. For any $N \in \mathbb{N}$ let

$$\{(X^N(t), \bar{X}_t)\}_{t \geqslant 0} = \{(X_1^N(t), \bar{X}_t^1, \dots, X_N^N(t), \bar{X}_t^N)\}_{t \geqslant 0}$$

be a Markov process with initial conditions such that $X_i^N(0) = \bar{X}_0^i$ a.s., independent for all $i = 1, \dots, N$ and μ_0-distributed. Then, for all $i = 1, \dots, N$, the pair $(X_i^N(t), \bar{X}_t^i)$ jumps in the following positions:

$$\begin{array}{lll}
(X_i^N(t) + 1, \bar{X}_t^i + 1) & \text{with rate} & 1 + \delta, \\
(X_i^N(t) - 1, \bar{X}_t^i - 1) & \text{"} & \mathbb{1}(X_i^N(t) > 0)\mathbb{1}(\bar{X}_t^i > 0) + \lambda \left(\mu_t^N[0, X_i^N(t)) \wedge \mu_t[0, \bar{X}_t^i)\right), \\
(X_i^N(t) - 1, \bar{X}_t^i) & \text{"} & \mathbb{1}(X_i^N(t) > 0)\mathbb{1}(\bar{X}_t^i = 0) + \lambda \left(\mu_t^N[0, X_i^N(t)) - \mu_t[0, \bar{X}_t^i)\right)_+, \\
(X_i^N(t), \bar{X}_t^i - 1) & \text{"} & \mathbb{1}(X_i^N(t) = 0)\mathbb{1}(\bar{X}_t^i > 0) + \lambda \left(\mu_t[0, \bar{X}_t^i) - \mu_t^N[0, X_i^N(t))\right)_+.
\end{array}$$

This is equivalent to the so-called *basic coupling* that we introduced in Chapter 1. Indeed, we assign to every pair of particles $(X_i^N(t), \bar{X}_t^i)_{t \geqslant 0}$ the same Poisson clocks (the Poisson random measures) and it maximizes the chances of the two particles to jump together. Of course, this means that for any continuous and bounded function $f \colon \mathbb{N}^{2N} \to \mathbb{R}$

$$f(X^N(t), \bar{X}_t) - f(X^N(0), \bar{X}_0) - \int_0^t \mathbf{L}_{\mu_s} f(X^N(s), \bar{X}_s)$$

is a martingale, where $\mu_t = \mathrm{Law}(\bar{X}_t^i)$ for any $i = 1, \dots, N$ and

$$\begin{aligned}
\mathbf{L}_\mu f(x, y) = \sum_{i=1}^N & (1 + \delta)(f(x + \delta_i, y + \delta_i) - f(x, y)) \\
& + \left[\mathbb{1}(x_i > 0)\mathbb{1}(y_i > 0) + \lambda \left(\frac{1}{N} \sum_{k=1}^N \mathbb{1}(x_k < x_i) \wedge \mu[0, y_i)\right)\right] (f(x - \delta_i, y - \delta_i) - f(x, y)) \\
& + \left[\mathbb{1}(x_i > 0)\mathbb{1}(y_i = 0) + \lambda \left(\frac{1}{N} \sum_{k=1}^N \mathbb{1}(x_k < x_i) - \mu[0, y_i)\right)_+\right] (f(x - \delta_i, y) - f(x, y)) \\
& + \left[\mathbb{1}(x_i = 0)\mathbb{1}(y_i > 0) + \lambda \left(\mu[0, y_i) - \frac{1}{N} \sum_{k=1}^N \mathbb{1}(x_k < x_i)\right)_+\right] (f(x, y - \delta_i) - f(x, y)).
\end{aligned}$$

The following result states the trajectorial propagation of chaos, which is a consequence of Proposition 2.1.2.

Theorem 4.1.3. *For all* $t \geqslant 0$*, there exists a positive constant* $C_t < \infty$ *such that*

$$\mathbf{E}[\sup_{s \leqslant t} |X_1^N(s) - \bar{X}_s^1|] \leqslant \frac{C_t}{\sqrt{N}}. \tag{4.1.9}$$

Proof. Notice that the jump coefficients here satisfy Assumption 2.1.1, indeed, for all $x, y \in \mathbb{N}$ and α, β in $\mathcal{M}(\mathbb{N})$ it holds:

$$\begin{aligned}|-\mathbb{1}(x>0)\lambda\alpha[0,x) + \mathbb{1}(y>0)\lambda\beta[0,y)| \leqslant& \lambda|\alpha[0,x) - \beta[0,y)| + 2\lambda|\mathbb{1}(x=0) - \mathbb{1}(y=0)| \\ \leqslant& 3\lambda|x-y| + \lambda\rho(\alpha,\beta).\end{aligned}$$

Then, we use Proposition 2.1.2, and we get that, for all $t \geqslant 0$

$$\mathbf{E}[\sup_{s \leqslant t} |X_1^N(s) - \bar{X}_s^1|] \leqslant \beta_N,$$

where $\beta_N \doteq \sup_{s\in[0,t]} \mathbf{E}[\rho(\mu^N_{\bar{X}_s}, \mu_s)$ is the W_1 Wasserstein distance between the empirical measure of the N copies of the nonlinear process (4.3.1) and its law. We know that there exists a constant $C_t > 0$ such that $\beta_N \leqslant \frac{C_t}{\sqrt{N}}$, which proves (4.1.9).

□

A branching and selection mechanism

Let us cite another interesting model belonging to our class of interacting random walks; we can relate this model to a branching-and-selection mechanism on the positive integers. Branching and selection particle systems are popular models in population dynamics, starting from the work of Brunet and Derrida [17] and followed by many others, for instance [10, 33, 62]. This models are studied in relation to the Fisher-Kolmogorov Petrovsky Piscounov equation (F-KPP):

$$\frac{\partial h}{\partial x} = \Delta h + h(1-h),$$

for $h = h(x,t)$, $x \in \mathbb{R}$, $t \geqslant 0$.

In our case, we imagine that any individual reproduces with a rate λ and he gives birth to a child. The fitness of this child is uniformly chosen among the ones of the other individuals. If the fitness of the newborn is strictly smaller than his parent's one, the child kills him, on the other case the child does not survive. In the terminology we introduced, this means that, for all $x \neq y \in \mathbb{N}$

$$\begin{aligned}\phi(x,y) &= 1, \\ \psi(x,y) &= |x-y|.\end{aligned}$$

It is natural, in this case, to imagine that the individuals are characterized by their *unfitness* rather than their *fitness*, such that lower values are related to stronger genetic traits. This model is clearly very interesting and we may develop its continuous-space analogue as we did for the *small jumps* model. Of course, the continuous-space dynamics would not have continuous paths in this case, but each particle would behave according a diffusion with jumps. Following the approach of Chapter 3, we define the infinitesimal generator $\mathcal{L}^N_{(BS)}$ of the system on suitable C^2 functions $f : D_N \to \mathbb{R}$ in the following way:

$$\mathcal{L}^N_{(BS)} f(x) = \sum_{i=1}^{N} \frac{1}{2} \frac{\partial^2}{\partial x_i^2} f(x) + \delta \frac{\partial}{\partial x_i} f(x) + \lambda \int_0^{x_i} \left(f(x - \mathbf{e}_i(x_i - y)) - f(x)\right) \mu^N(dy),$$

the domain of the generator $\mathcal{L}^N_{(BS)}$ coincides with $\mathcal{D}\left(\mathcal{L}^N\right)$ from Chapter 3. Its mean field limit is a process on $\mathbb{R}_+$ such that its law has a density μ_t that solves the following Fokker-Planck equation with boundary conditions

$$\begin{cases} \partial_t \mu_t(x) = \frac{1}{2}\partial_x^2 \mu_t(x) - \delta \partial_x \mu_t(x) + \lambda \mu_t(x) \int_x^\infty \mu_t(y) dy - \lambda \mu_t(x) \int_0^x \mu_t(y) dy \quad \forall x > 0; \\ \delta \partial_x \mu_t(x)|_{x=0} - \lambda \mu_t(0) = \frac{1}{2} \partial_x^2 \mu_t(x)|_{x=0}. \end{cases} \tag{4.1.10}$$

Of course, integrating (4.1.10) from 0 to x, for all $x \geqslant 0$ and setting $F_t(x) = \int_0^x \mu_t(y) dy$ its CDF, we get the equivalence with the following:

$$\partial_t F_t(x) = \frac{1}{2} \partial_x^2 F_t(x) - \delta \partial_x F_t(x) + \lambda F_t(x)(1 - F_t(x)),$$

for all $x > 0$ and $F_t(x) = 0$ for $x \leqslant 0$, for all $t \geqslant 0$, that links this model with the F-KPP equation.

4.2 Exponential ergodicity of the particle system

In this Section we study the long time behavior of the system with N particles. The main question we address is whether the asymmetric interaction can balance the drift to infinity, i.e. we want to understand if the interaction can ensure ergodicity in the N particle system. Clearly, when $\lambda = 0$, the model has no chance of having a stationary measure, because each particle perform a simple random walk with a nonnegative drift $\delta \geqslant 0$ and reflection in zero. We aim to determine (or to give bounds to) the *critical* interaction strength

$$\lambda_N^*(\delta)$$

above which any system described in Section 4.1.1 has a stationary measure.

We restrict the analysis to the specific model with *small jumps*, defined by (4.1.6), since it stochastically dominates all the other model in the class we presented in Section 4.1.1, when the interaction function $\phi(x, y) \equiv 1$. To this aim, following the approach in [58], we define the stochastic ordering $\preceq$ between probability measures. Let $\mathcal{X}$ be a compact metric space, in which we can define a partial order $\leqslant$. Let $\mathcal{M}$ define the set of continuous functions on $\mathcal{X}$, which are monotone, i.e.

$$\mathcal{M} \doteq \{f \colon f(x) \leqslant f(y) \text{ for all } x \leqslant y\}.$$

Definition 4.2.1 (Stochastic ordering)**.** Let μ, ν be two probability measures on $\mathcal{X}$, we say that

$$\mu \preceq \nu$$

if and only if

$$\int_{\mathcal{X}} f d\mu \leqslant \int_{\mathcal{X}} f d\nu$$

for all $f \in \mathcal{M}$.

We say that a stochastic process $\{X_t\}_{t \geqslant 0}$ dominates another stochastic process $\{Y_t\}_{t \geqslant 0}$ if, whenever $\mathcal{L}aw(Y_0) \preceq \mathcal{L}aw(X_0)$ then $\mathcal{L}aw(Y_t) \preceq \mathcal{L}aw(X_t)$, for all $t \geqslant 0$. In this sense, we have the following result on the model with *small jumps* w.r.t. all the other models presented in Section 4.1.

Proposition 4.2.1. *Let P^N and P^N_S be the law on $\mathbf{D}(\mathbb{R}^+, \mathbb{N})$ of the trajectories of the Markov processes described, respectively, by the generator* (4.1.1) *with $\phi(x, y) \equiv 1$ and* (4.1.7)*. For any measure μ_0 on $\mathbb{N}$, if $P^N(0) \preceq P^N_S(0)$, then for all $t \geqslant 0$ $P^N(t) \preceq P^N_S(t)$.*

A way to prove stochastic ordering between two measures $\mu \preceq \nu$ consists in finding a coupling (X, Y) such that $\mathcal{L}aw(X) = \mu$, $\mathcal{L}aw(Y) = \nu$ and

$$\mathbf{P}(X \leqslant Y) = 1,$$

see Theorem 2.4 in [58]. We will use this equivalence in the proof, by finding a coupling that preserves the order at any time $t \geqslant 0$.

Proof of Proposition 4.2.1. Let us fix one particular model among the ones defined in Section 4.1. We define the basic coupling procedure between this model and the model with *small jumps*, the coupling that maximizes the chances of two coupled particles to jump together. We fix a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, \mathbf{P})$ and let μ_0 be a probability measure on $\mathbb{N}$. For any $N \in \mathbb{N}$ let

$$\{(X^N(t), Y^N(t)\}_{t \geqslant 0} = \{(X^N_1(t), Y^N_1(t), \dots, X^N_N(t), Y^N_N(t))\}_{t \geqslant 0}$$

be a Markov process with initial conditions such that $X^N_i(0) \leqslant Y^N_i(0)$ a.s., for all $i = 1, \dots, N$. Then, for all $i = 1, \dots, N$, the pair $(X^N_i(t), Y^N_i(t))$ jumps in the following positions:

$$
\begin{array}{lll}
(X^N_i + 1, Y^N_i + 1) & \text{with rate} & 1 + \delta, \\
(X^N_i - 1, Y^N_i - 1) & \text{''} & \mathbb{1}(X^N_i > 0)\mathbb{1}(Y^N_i > 0), \\
(X^N_i - 1, Y^N_i) & \text{''} & \mathbb{1}(X^N_i > 0)\mathbb{1}(Y^N_i = 0), \\
(X^N_i, Y^N_i - 1) & \text{''} & \mathbb{1}(X^N_i = 0)\mathbb{1}(Y^N_i > 0), \\
\text{and for all } j \neq i & & \\
(X^N_i - \psi(X^N_i, X^N_j), Y^N_i - 1) & \text{with rate} & \frac{\lambda}{N}\left(\mathbb{1}(X^N_j \leqslant X^N_i) \wedge \mathbb{1}(Y^N_j \leqslant Y^N_i)\right), \\
(X^N_i, Y^N_i - 1) & \text{''} & \frac{\lambda}{N}\left(\mathbb{1}(Y^N_j \leqslant Y^N_i) - \mathbb{1}(X^N_j \leqslant X^N_i)\right)_+, \\
(X^N_i - \psi(X^N_i, X^N_j), Y^N_i) & \text{''} & \frac{\lambda}{N}\left(\mathbb{1}(X^N_j \leqslant X^N_i) - \mathbb{1}(Y^N_j \leqslant Y^N_i)\right)_+,
\end{array}
$$

where we omit the time index for simplicity. This coupling is characterized by its own generator $\mathbf{L}^N_{(X^N,Y^N)}$. We aim to prove that, since $\mathbf{P}\left(X_i^N(0) \leqslant Y_i^N(0),\ \text{for } i = 1,\dots,N\right) = 1$, then for all $t > 0$

$$\mathbf{P}\left(X_i^N(t) \leqslant Y_i^N(t),\ \text{for } i = 1,\dots,N\right) = 1.$$

Therefore, we consider the generator on the function $\mathbb{1}(x_1 \leqslant y_1,\dots,x_N \leqslant y_N)$, that summarize in the following few terms

$$\begin{aligned}
\mathbf{L}^N_{(X^N,Y^N)}\mathbb{1}\left(x_1 \leqslant y_1,\dots,x_N \leqslant y_N\right) &= \sum_{i=1}^N \mathbb{1}\left(x_i = 1, y_i = 0, x_k \leqslant y_k\, k \neq i\right) \\
&+ \sum_{i=1}^N \sum_{j\neq i} \mathbb{1}\left(x_i - y_i > \psi(x_i,x_j), x_k \leqslant y_k\right) \frac{\lambda}{N} \left(\mathbb{1}(x_j \leqslant x_i) \wedge \mathbb{1}(y_j \leqslant y_i)\right) \\
&- \sum_{i=1}^N \sum_{j\neq i} \mathbb{1}\left(x_i = y_i, x_k \leqslant y_k\right) \frac{\lambda}{N} \left(\mathbb{1}(y_j \leqslant y_i) - \mathbb{1}(x_j \leqslant x_i)\right)_+ \\
&+ \sum_{i=1}^N \sum_{j\neq i} \mathbb{1}\left(x_i - y_i > \psi(x_i,x_j), x_k \leqslant y_k\right) \frac{\lambda}{N} \left(\mathbb{1}(x_j \leqslant x_i) - \mathbb{1}(y_j \leqslant y_i)\right)_+.
\end{aligned}$$

Let us focus on the third term in the r.h.s., the one with a minus sign. Fix $i = 1,\dots,N$ and $j \neq i$, the first indicator function says that we are in the case $x_j \leqslant y_j \leqslant y_i = x_i$, but then two following indicators are both equal to 1, meaning that this term is zero. Therefore we proved that

$$\mathbf{L}^N_{(X^N,Y^N)}\mathbb{1}\left(x_1 \leqslant y_1,\dots,x_N \leqslant y_N\right) \geqslant 0,$$

i.e. for all $t \geqslant 0$

$$\frac{d}{dt}\mathbf{P}\left(X_i^N(t) \geqslant Y_i^N(t),\ i = 1,\dots,N\right) = \mathbf{E}\left[\mathbf{L}^N_{(X^N,Y^N)}\mathbb{1}\left(X_i^N(t) \leqslant Y_i^N(t), i = 1,\dots,N\right)\right] \geqslant 0.$$

By hypothesis, this implies that, for all $t \geqslant 0$,

$$\mathbf{P}\left(X_i^N(t) \geqslant Y_i^N(t),\ i = 1,\dots,N\right) = \mathbf{P}\left(X_i^N(0) \geqslant Y_i^N(0),\ i = 1,\dots,N\right) = 1.$$

$\square$

Proposition 4.2.1 implies that, for fixed $N \geqslant 2$, $\delta \geqslant 0$ and $\lambda > 0$ such that the model with *small jumps* has a unique stationary measure, say $\pi^N_{(SJ)}$, then every other model described in Section 4.1 has a stationary measure π^N as well and $\pi^N \preceq \pi^N_{(SJ)}$. Then, we look for the critical interaction strength $\lambda_N^*(\delta)$ for the *small jumps* model. Unfortunately, in contrast with the continuous analogue of Chapter 3, we could not obtain it in an explicit form and we give an upper and a lower bound for it. However, the lower bound has the interesting feature of highlighting the difference between the continuous and the discrete model, indeed we prove that

$$\lambda_N^*(\delta) > 2\delta\frac{N}{N-1},$$

where in the right-hand side of the inequality we have the exact critical value of the continuous model, obtained in Theorem 3.2.2.

4.2.1 Upper bound for the critical interaction strength in the particle system

In this section, by means of a Lyapunov function, we give an upper bound on $\lambda_N^*(\delta)$. This upper bound is uniform in $N \geqslant 2$. We determine sufficient conditions for the assumptions of Theorem 3.2.1 to hold and, therefore, this gives exponential ergodicity of the process X^N.

Theorem 4.2.1. *For all $\delta \geqslant 0$, there exists a critical value*

$$\lambda_{up}^*(\delta) \doteq 8\delta^2 + 12\delta$$

such that for all $N \geqslant 2$, for all $\lambda > \lambda_{up}^*(\delta)$ the process $X^N = (X_1^N, \dots, X_N^N)$ described in (4.1.6) is exponentially ergodic. There exists a probability measure $\pi_{(SJ)}^N$ on $\mathbb{N}^N$ such that, for any initial condition $X^N(0)$,

$$\|P_x^N((X_1^N(t), \dots, X_N^N(t)) \in \cdot) - \pi_{(SJ)}^N\|_{TV} \leqslant C_N(x)(\rho_N)^t, \ \forall x \in \mathbb{N}^N, \forall t \geqslant 0,$$

where $C_N(x)$ is bounded, $\rho_N < 1$ and $\|\cdot\|_{TV}$ is the total variation norm. $\pi_{(SJ)}^N$ is the unique stationary measure for the process $(X_1^N, \dots, X_N^N)$.

Our purpose is to prove Theorem 4.2.1 by means of a Lyapunov function. We choose a function that is the product of two exponential functions, encoding two characteristics of the particle system: the center of mass and the height of the highest "pile" of particles. By pile of particles we mean the number of particles in the same spatial position. A configuration $\mathbf{x} = (x_1, \dots, x_N)$ of $\mathbb{N}^N$ shows piles as soon as there exists $i \neq j$ such that $x_i = x_j$. In our dynamics the piles play a crucial role, since particles belonging to the same pile do not interact. When particles are widespread in the space, the asymmetric interaction favors the moves that push particles one towards the other. However, when particles are gathered in the same position they do not feel the interaction and they tend to spread rightward. This is a peculiarity of the discrete space model, since in the continuous one we know from Proposition 3.1.2 that multiple collisions do not occur a.s.. This means that the highest possible pile is of height 2 and moreover it instantaneously disappears, while in the discrete dynamics piles last for a certain amount of time.

The candidate Lyapunov function depends on two positive parameters α and β that we tune in order to satisfy the criterion. Let us define, for all $x \in \mathbb{N}^N$

$$\psi(x) = \frac{1}{N}\sum_{i=1}^{N} e^{\alpha x_i},$$
$$\phi(x) = e^{+\frac{\beta}{N}\bar{\eta}},$$

where $\bar{\eta} \doteq \max_{\nu\in\mathbb{N}} \sum_{i=1}^{N} \mathbb{1}(x_i = \nu)$, that is the high of the highest pile of the configuration x. Then, let

$$V_{\alpha,\beta}^N(x) \doteq \psi(x)\phi(x)$$

be our candidate Lyapunov function. We briefly describe the idea of the proof. We exploit the multiplicative form of $V^N_{\alpha,\beta}(x)$ and the fact that we can write

$$\mathcal{L}^N_{(SJ)}\psi\phi = \psi\mathcal{L}^N_{(SJ)}\phi + \phi\mathcal{L}^N_{(SJ)}\psi + 2\Gamma^N_{(SJ)}(\phi,\psi),$$

where $\Gamma^N_{(SJ)}$ is the operator carré du champ associated to $\mathcal{L}^N_{(SJ)}$, defined for every pair of functions f, g

$$\Gamma^N_{(SJ)}(f,g) = \frac{1}{2}\left[\mathcal{L}^N_{(SJ)}fg - f\mathcal{L}^N_{(SJ)}g - g\mathcal{L}^N_{(SJ)}f\right].$$

By the form of the jumps and of the involved functions, $\Gamma^N_{(SJ)}(\phi,\psi)$ can be bounded with a term proportional to

$$(e^\beta - 1)(e^\alpha - 1)V^N_{\alpha,\beta}(x).$$

For α sufficiently small and $\beta = C\alpha$ such that the constant $C > 0$ is admissible (here the admissibility of C depends on the values of δ and λ), we find $\gamma > 0$ and a constant $H \geqslant 0$ for which

$$\mathcal{L}^N_{(SJ)}V^N_{\alpha,\beta}(x) \leqslant -\gamma V^N_{\alpha,\beta}(x) + H,$$

i.e. the functio $V_{\alpha,\beta}$ satisfies Assumption 3.2.1. This prove the exponential ergodicity criterion of Meyn and Tweedie, see Theorem 3.2.1.

We treat separately the terms $\mathcal{L}^N_{(SJ)}\psi_\alpha(x)$, $\mathcal{L}^N_{(SJ)}\phi_\beta(x)$ and $\Gamma^N_{(SJ)}(\psi_\beta,\phi_\alpha)(x)$. We divide the space $\mathbb{N}^N$ into two unbounded subsets, such that we bound the values of $\mathcal{L}^N_{(SJ)}V^N_{\alpha,\beta}(x)$ with two different approaches. One subset of $\mathbb{N}^N$ is the region of space such that where there is **one single tall pile of particles** (by **tall pile** we intend that it contains more then the half of particles), i.e. the region

$$\Lambda_N \doteq \{x \in \mathbb{N}^N : \bar{\eta}(x) > \frac{N}{2}\}.$$

The other region is its complementary Λ^C_N, where **the particles are widespread in different positions**, there may be a single pile taller than the others, but it does not contain more than half of the particles.

Proof of Theorem 4.2.1. Fix $\delta \geqslant 0$ and $N \geqslant 2$. It is sufficient to prove that the exponential ergodicity criterion from Meyn and Tweedie, [70] holds for all values of λ greater than $\lambda^*_{up}(\delta)$.

Let α, β be two positive parameters, as we mentioned, we aim to bound the following function

$$\mathcal{L}^N_{(SJ)}V^N_{\alpha,\beta}(x) = \phi(x)\mathcal{L}^N_{(SJ)}\psi(x) + \psi(x)\mathcal{L}^N_{(SJ)}\phi(x) + \Gamma^N_{(SJ)}(\phi,\psi)(x).$$

The bound on $\mathcal{L}^N_{(SJ)}\psi_\alpha(x)$ relies basically on the following observation. It is possible to give a lower bound on the quantity $K_N \doteq \frac{1}{N}\sum_{i=1}^N \mu^N[0,x_i)$ in terms of $\bar{\eta}(x)$. Indeed, this term can be rewritten as the number of the unordered pairs of particles in distinct positions,

$$K_N = \frac{1}{N}\sum_{i=1}^{N} \mu^N[0,x_i) = \frac{1}{2N^2}\sum_{i,j=1}^{N} \mathbb{1}(x_j \neq x_i) \geqslant \frac{1-\frac{\bar{\eta}(x)}{N}}{2}.$$

We will use this bound in Λ_N, while we will keep the exact expression of K_N in Λ_N^C to compensate the term coming from $\mathcal{L}^N_{(SJ)}\phi(x)$. We start with the bound on $\mathcal{L}^N_{(SJ)}\psi(x)$:

$$\begin{aligned}
\mathcal{L}^N_{(SJ)}\psi(x) =& \sum_{i=1}^{N}(1+\delta)\nabla_i^+\psi(x) + \nabla_i^-\psi(x) - \sum_{i=1}^{N}\mathbb{1}(x_i=0)\nabla_i^-\psi(x) + \lambda\sum_{i=1}^{N}\mu^N[0,x_i)\nabla_i^-\psi(x)\\
=&(e^\alpha+e^{-\alpha}-2)\psi(x) + \delta(e^\alpha-1)\psi(x) + (1-e^{-\alpha})\frac{1}{N}\sum_{i=1}^{N}\mathbb{1}(x_i=0)\\
&-\lambda(1-e^{-\alpha})\frac{1}{N}\sum_{i=1}^{N} e^{\alpha x_i}\mu^N[0,x_i)
\end{aligned}$$

We highlight that, since the functions $e^{\alpha x}$ and $\mu^N[0,x)$ are non-decreasing, for KFG inequality, we have:

$$\frac{1}{N}\sum_{i=1}^{N} e^{\alpha x_i}\mu^N[0,x_i) \geqslant \psi(x)\frac{1}{N}\sum_{i=1}^{N}\mu^N[0,x_i).$$

Then we have

$$\begin{aligned}
\mathcal{L}^N_{(SJ)}\psi(x) \leqslant& (e^\alpha+e^{-\alpha}-2)\psi(x) + \delta(e^\alpha-1)\psi(x) + (1-e^{-\alpha})\frac{1}{N}\sum_{i=1}^{N}\mathbb{1}(x_i=0)\\
&-\lambda(1-e^{-\alpha})\psi(x)K_N\mathbb{1}(\Lambda_N^C) - \lambda(1-e^{-\alpha})\psi(x)\frac{1-\frac{\bar{\eta}(x)}{N}}{2}\mathbb{1}(\Lambda_N).
\end{aligned}$$

The bound on $\mathcal{L}^N_{(SJ)}\phi(x)$, instead, is performed as follows.

i) For all $x \in \Lambda_N$ we know that there exists **one single tall pile**, i.e. a unique

$$\nu^*(x) = \arg\max_{\nu\in\mathbb{N}} \sum_{i=1}^{N}\mathbb{1}(x_i=\nu).$$

Therefore, the function $\phi(x)$ changes values under the effect of $\mathcal{L}^N_{(SJ)}$ only because of the moves of the particles in three positions: $\nu^*(x)-1$, $\nu^*(x)$ and $\nu^*(x)+1$. This means that we can write the action of the generator $\mathcal{L}^N_{(SJ)}$ as follows:

$$\begin{aligned}
\mathcal{L}^N_{(SJ)}\phi(x) =& \sum_{i=1}^{N}(1+\delta)\nabla_i^+\phi(x) + \nabla_i^-\phi(x) - \sum_{i=1}^{N}\mathbb{1}(x_i=0)\nabla_i^-\phi(x) + \lambda\sum_{i=1}^{N}\mu^N[0,x_i)\nabla_i^-\phi(x)\\
=&-\bar{\eta}(x)(1-e^{-\beta/N})\left[1+\delta+\mathbb{1}(\nu^*(x)>0)+\lambda\mu_N[0,\nu^*(x))\right]\phi(x)\\
&+(e^{\beta/N}-1)\left[\eta(\nu^*(x)-1)(1+\delta)+\eta(\nu^*(x)+1)(1+\lambda\mu^N[0,\nu^*(x)+1))\right]\phi(x)
\end{aligned}$$

Then, disregarding the non-positive term $-\bar{\eta}(x)(1-e^{-\beta/N})\mathbb{1}(\nu^*(x)>0)\phi(x)$ and bounding the number of particles $\eta(k)$ in position k with $(N-\bar{\eta}(x))$ for any $k \neq \nu^*(x)$, we have

$$\begin{aligned}\mathcal{L}^N_{(SJ)}\phi(x) \leqslant \Big[&-\frac{\bar{\eta}(x)}{N}N(1-e^{-\beta/N})(1+\delta) - \frac{\bar{\eta}(x)}{N}N(1-e^{-\beta/N})\lambda\mu^N[0,\nu^*(x))\\ &+(N-\bar{\eta}(x))\lambda\mu^N[0,\nu^*(x))(e^{\beta/N}-1) + (1+\delta)(N-\bar{\eta}(x))(e^{\beta/N}-1)\\ &+(N-\bar{\eta}(x))(e^{\beta/N}-1)\lambda\frac{\bar{\eta}(x)}{N}\Big]\phi(x).\end{aligned}$$

ii) In the region Λ_N^C, we bound $\mathcal{L}^N_{(SJ)}\phi(x)$ with the pessimistic assumption that every jump increases $\phi(x)$ of the quantity $(e^{\beta/N}-1)\phi(x)$, this means that we bound with

$$\mathcal{L}^N_{(SJ)}\phi(x) \leqslant \left(N(2+\delta) + \lambda\sum_{i=1}^N \mu^N[0,x_i)\right)(e^{\beta/N}-1)\phi(x).$$

In the right-hand side the term $\sum_{i=1}^N \mu^N[0,x_i) = NK_N$ appears and it will compensate the same term coming from $\mathcal{L}^N_{(SJ)}\psi(x)$.

The carré du champ term $\Gamma^N_{(SJ)}(\phi,\psi)(x)$, because of the fixed jump amplitude of the process, is bounded as follows:

$$|\Gamma^N_{(SJ)}(\psi,\phi)(x)| \leqslant N(2+\lambda+\delta)(e^\alpha-1)(e^{\beta/N}-1)V^N_{\alpha,\beta}(x).$$

Given these bounds, we want to identify if there exist λ, α, β positive such that

$$\mathcal{L}^N_{(SJ)}V^N_{\alpha,\beta}(x) \leqslant -\gamma_N V^N_{\alpha,\beta}(x) + H,$$

for some constants $\gamma_N > 0$ and $H \geqslant 0$, that is the condition for the ergodicity criterion to hold. In the two complementary regions we have the following bounds, up to terms bounded by $H = (1-e^{-\alpha})e^\beta$:

A) for $x \in \Lambda_N$:

$$\begin{aligned}\mathcal{L}^N_{(SJ)}V^N_{\alpha,\beta}(x) \leqslant \Bigg[&(e^\alpha+e^{-\alpha}-2) + \delta(e^\alpha-1) - \lambda(1-e^{-\alpha})\frac{1-\frac{\bar{\eta}(x)}{N}}{2}\\ &-\frac{\bar{\eta}(x)}{N}N(1-e^{-\beta/N})\lambda\mu^N[0,\nu^*(x)) + (N-\bar{\eta}(x))\lambda\mu^N[0,\nu^*(x))(e^{\beta/N}-1)\\ &-\frac{\bar{\eta}(x)}{N}N(1-e^{-\beta/N})(1+\delta) + (1+\delta)(N-\bar{\eta}(x))(e^{\beta/N}-1)\\ &+(N-\bar{\eta}(x))(e^{\beta/N}-1)\lambda\frac{\bar{\eta}(x)}{N} + N(2+\delta+\lambda)(e^{\beta/N}-1)(e^\alpha-1)\Bigg]V^N_{\alpha,\beta}(x);\end{aligned}$$

B) for $x \in \Lambda_N^C$:

$$\begin{aligned}\mathcal{L}^N_{(SJ)}V^N_{\alpha,\beta}(x) \leqslant \Big[&(e^\alpha+e^{-\alpha}-2) + \delta(e^\alpha-1) - \lambda(1-e^{-\alpha})K_N\\ &+ \left(N(2+\delta)+\lambda NK_N\right)(e^{\beta/N}-1) + N(2+\delta+\lambda)(e^{\beta/N}-1)(e^\alpha-1)\Big]V^N_{\alpha,\beta}(x).\end{aligned}$$

We want to make the two above parenthesis negative; we start by choosing $\beta = C\alpha$, for a certain $C > 0$ and to make α sufficiently small.

Let us look at point **A)**. We can not say anything about the terms

$$-\frac{\bar{\eta}(x)}{N}N(1-e^{-\beta/N})\lambda\mu^N[0,\nu^*(x)) + (N-\bar{\eta}(x))\lambda\mu^N[0,\nu^*(x))(e^{\beta/N}-1), \tag{4.2.1}$$

but we know that, for β sufficiently small,

$$(1-e^{-\beta/N}) \simeq (e^{\beta/N}-1) \simeq \frac{\beta}{N}.$$

In this case, since $\frac{\bar{\eta}(x)}{N} > \frac{1}{2}$, the expression (4.2.1) is negative and we can neglect it. In the same way, we disregard also the terms

$$\begin{aligned} N(2+\delta+\lambda)(e^{\beta/N}-1)(e^{\alpha}-1) &= o(\alpha) \\ (e^{\alpha}+e^{-\alpha}-2) &= o(\alpha). \end{aligned}$$

We are left to find λ and C such that the expression

$$\delta(e^{\alpha}-1) - \lambda(1-e^{-\alpha})\frac{1-\xi}{2} - \xi N(1-e^{-\beta/N})(1+\delta) + (1+\delta)(1-\xi)N(e^{\beta/N}-1) + (1-\xi)N(e^{\beta/N}-1)\lambda\xi$$

is negative for all $\xi \in (\frac{1}{2},1]$. Then, for α sufficiently small, this condition becomes

$$\left[\delta - (1-\xi)\left(\frac{\lambda}{2} - C(1+\lambda\xi+\delta)\right) - C\xi(1+\delta)\right]\alpha + o(\alpha) < 0,$$

for all $\xi \in (1/2,1]$, that gives the conditions on C:

$$\begin{cases} C \leqslant \frac{\lambda-4\delta}{\lambda} \\ C \geqslant \frac{\delta}{1+\delta}. \end{cases}$$

Now we look at point **B)**. Again, we neglect the terms

$$N(2+\delta+\lambda)(e^{\beta/N}-1)(e^{\alpha}-1) + (e^{\alpha}+e^{-\alpha}-2).$$

We look for conditions under which

$$\delta(e^{\alpha}-1) - \lambda(1-e^{-\alpha})K_N + (N(2+\delta)+\lambda N K_N)(e^{\beta/N}-1)$$

is negative for all values assumed by K_N when $x \in \Lambda_N^C$. This means, for α small,

$$[\delta - \lambda K_N + C(2+\delta+\lambda K_N)]\,\alpha + o(\alpha) \leqslant 0,$$

that gives an additional conditions on C:

$$C \leqslant \frac{\lambda k - \delta}{2 + \delta + \lambda k},$$

for every $k \in [1/4, 1]$.

Now, we already see that the conditions are independent of N and they are are admissible only if

$$\lambda > 12\delta + 8\delta^2 = \lambda^*_{up}(\delta).$$

$\square$

Let us fix $N \geqslant 2$, $\delta > 0$ and $\lambda > \lambda^*_{up}(\delta)$, then for α sufficiently small and $\beta = C\alpha$, we have that the constants for which $V_{\alpha,\beta}$ satisfies Assumption 3.2.1 have the following form:

$$\begin{aligned}
\gamma_N = -\max\Bigg\{ & \sup_{\xi \in (\frac{1}{2},1]} \Big[(e^{\alpha} + e^{-\alpha} - 2) + \delta(e^{\alpha} - 1) - \lambda(1 - e^{-\alpha})\frac{1-\xi}{2} - \frac{\bar{\eta}}{N} N(1 - e^{-\beta/N})\lambda\mu^N[0, \nu^*) \\
& \quad + (N - \bar{\eta})\lambda\mu^N[0, \nu^*)(e^{\beta/N} - 1) + (1-\xi)N(e^{\beta/N} - 1)\lambda\xi + N(2 + \delta + \lambda)(e^{\beta/N} - 1)(e^{\alpha} - 1) \Big], \\
& \sup_{k \in [\frac{1}{4},1]} \Big[(e^{\alpha} + e^{-\alpha} - 2) + \delta(e^{\alpha} - 1) - \lambda(1 - e^{-\alpha})k + (N(2+\delta) + \lambda N k)(e^{\beta/N} - 1) \\
& \quad + N(2 + \delta + \lambda)(e^{\beta/N} - 1)(e^{\alpha} - 1) \Big] \Bigg\};
\end{aligned}$$

$$H = (1 - e^{-\alpha})e^{\beta}.$$

It would be desirable to understand the dependence of the quantities $C_N(x)$ and ρ_N on the size of the system N. If they could be chosen independent of N, this would be a crucial step in the proof of chaoticity of the sequence of the stationary measures. Indeed, this, together with a uniform in time trajectorial propagation of chaos, would give the desired result. However, a uniform in time propagation of chaos for the system (4.1.6), as in the continuous model, seems to be very hard to get.

On the other hand, Theorem 3.2.1 implies that

$$\mathbf{E}_{\pi^N_{(SJ)}} \left[V^N_{\alpha,\beta}(X^N) \right] < \frac{H}{\gamma_N},$$

which is clearly bounded for any $N \in \mathbb{N}$. Since we may write $V_{\alpha,\beta}$ as a continuous and unbounded function of the empirical measure as follows:

$$V^N_{\alpha,\beta}(X^N) = \langle \mu^N, e^{\alpha \cdot} \rangle e^{\beta \sup_{x \in \mathbb{N}} \mu^N(x)}.$$

This implies the tightness of the sequence

$$\{Law(\mu^N_{\pi^N})\}_{N \in \mathbb{N}}$$

of the stationary empirical measures. Then, one may try to adopt the classical approach for the proof of propagation of chaos, verifying that any limit point of a convergent subsequence of $\mu^N_{\pi^N}$ is stationary for the nonlinear process (4.3.1) and that the stationary measure of (4.3.1) is unique.

4.2.2 Lower bound for the critical interaction strength in the particle system

The aim of this section is to highlight, by means of the lower bound for the critical interaction strength $\lambda^*_N(\delta)$, the difference between the continuous model presented in Chapter 3 and the *small jumps* model presented in Section 4.1.4.

As in Section 3.2.2, we consider the increasing reordering of the vector $X^N(t)$ that we now call

$$(X^N_{(1)}(t), \dots, X^N_{(N)}(t)),$$

such that $X^N_{(1)}(t) \leqslant X^N_{(2)}(t) \leqslant \cdots \leqslant X^N_{(N)}(t)$ for all $t \geqslant 0$. According to the dynamics (4.1.6), the element $X^N_{(1)}(t)$ perform an upward jump of amplitude 1 with rate $1+\delta$, a backward jump of amplitude 1 with rate 1 and it is reflected when $X^N_{(1)}(t) = 0$ and when $X^N_{(1)}(t) = X^N_{(2)}(t)$. The same happens for $X^N_{(2)}(t)$, with the difference that the rate of backward jump is $1 + \lambda\frac{1}{N}$ and the reflection is upward when when $X^N_{(2)}(t) = X^N_{(1)}(t)$ and it is backward when $X^N_{(2)}(t) = X^N_{(3)}(t)$. This is a random walk in a wedge, i.e. in the region $\mathcal{W}_N \subset \mathbb{N}^N$ defined as

$$\mathcal{W}_N \doteq \left\{ x \in \mathbb{N}^N \text{ s.t. } 0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \right\}.$$

The dynamics in the interior of $\mathcal{W}_N$ is the following, for all $i = 1, \dots, N$

$$\begin{aligned} x &\to x + \mathbf{e}_i \quad \text{with rate } 1+\delta, \\ x &\to x - \mathbf{e}_i \quad \text{with rate } 1 + \lambda\frac{i-1}{N}, \end{aligned}$$

The dynamics at the boundaries of $\mathcal{W}_N$ depends on which "face" of the wedge the point is in. For instance, fix an index $i < N$, an $N-1$-dimensional "face" of the wedge is the subset

$$\mathcal{B}_{i,i+1} \doteq \{x \in \mathcal{W}_N : x_i = x_{i+1} \text{ and } x_j < x_{j+1}\, \forall j \neq i\}.$$

The dynamics on $\mathcal{B}_{i,i+1}$ is the same as the interior one for all jumps $\pm\mathbf{e}_j$, $j \neq i, i+1$, while the difference is the following

$$\begin{aligned} x &\to x + \mathbf{e}_{i+1} \quad \text{with rate } 2(1+\delta), \\ x &\to x - \mathbf{e}_i \quad \text{with rate } 2(1 + \lambda\frac{i-1}{N}). \end{aligned}$$

In the same way, we define the dynamics on all the other lower dimensional “faces” of the wedge, according to the dynamics (4.1.6) and the number of particles in the same position.

It is clear that these “faces” corresponds to configurations of particles that show some piles of particles. For instance the $N-1$-dimensional face $\mathcal{B}_{i,i+1}$ corresponds to the set of configurations that display one single pile of particles of height 2 involving the i^{th} and $(i+1)^{th}$ ranked particles. Therefore, we may identify the whole boundary of $\mathcal{W}_N$ with the set of configurations with at least one pile of particles. We already mentioned that are exactly these “faces” that creates the main issue in understanding the stationary measure. Indeed, we defined $(Y_1^N(t),\dots,Y_N^N(t))$ the process that evolves according to (3.2.4) in the region

$$\mathbf{W}_N \doteq \{y \in D_N \text{ s.t. } 0 \leqslant y_1 \leqslant y_2 \leqslant \cdots \leqslant y_N\}.$$

We highlighted in Proposition 3.1.2 that, in this case, there is a.s. no triple collision. This means that the non-smooth parts of the boundary of the wedge are of no importance and that it is sufficient to consider reflection conditions on the hyperplanes of dimension $N-1$ that bound the wedge and not on their edges. In the discrete case the piles of particles actually matter and this is confirmed by the following result. We found, for all $N \geqslant 2$, a lower bound for the critical value $\lambda_N^*(\delta)$ that is strictly greater than the critical value for the continuous space model, proved in Theorem 3.2.2. Therefore, in the random walk, the strength of the interaction needed to get ergodicity is higher than in the diffusive case.

Theorem 4.2.2. *For all $N \geqslant 2$ and for all $\delta \geqslant 0$, there exists*

$$\lambda^*_{N,lower}(\delta) \doteq \big(1+\rho(\epsilon,N)\big)2\delta, \quad \textit{with} \quad \rho(\epsilon,N) \doteq \frac{N^2(\delta+2)}{N(N-1)(\delta+2)-2\delta} - 1 \longrightarrow 0, \quad (4.2.2)$$

such that, for all $\lambda < \lambda^*_{N,lower}(\delta)$, the process $X^N = (X_1^N,\dots,X_N^N)$ generated by (4.1.6) is transient.

It is clear that, for all fixed $N \geqslant 2$, this lower bound satisfies

$$\lambda^*_{N,lower}(\delta) > 2\delta\frac{N}{N-1},$$

i.e. it is strictly greater than the critical interaction strength of the continuous model found in Theorem 3.2.2.

The proof of this lower bound is made by means of a Lyapunov function. We exploit the following result on the transience of Markov chains, that is a simplified version of Theorem 2.2.7 in [39].

Theorem 4.2.3 (Theorem 2.2.7 in [39]). *Let $\{X_t\}_{t\geqslant 0}$ an irreducible Markov process on a countable space $\mathcal{X}$ with infinitesimal generator $\mathcal{L}$ and bounded jumps. Suppose there exists two positive constants $\epsilon, C > 0$ and a positive function f such that $\mathcal{A}_C \doteq \{x \in \mathcal{X} \colon f(x) > C\} \neq \emptyset$ and*

$$\mathcal{L}f(x) > \epsilon \quad \textit{for all } x \in \mathcal{A}_C.$$

Then the process $\{X_t\}_{t\geqslant 0}$ is transient.

Therefore, we want to define a linear function $f_{[N,\lambda,\delta]}\colon \mathcal{W}_N \to \mathbb{R}$ such that for all $\lambda > \lambda^*_{N,lower}(\delta)$

$$\mathcal{L}^N_{ord} f_{[N,\lambda,\delta]}(x) > 0, \tag{4.2.3}$$

for all $x \in \mathcal{W}_N$, where $\mathcal{L}^N_{ord}$ is the generator of the reordered process $(X^N_{(1)}(t), \dots, X^N_{(N)}(t))$. The idea for the construction of this Lyapunov function is the following. Firstly, we consider the barycenter of the particle system

$$B(x) = \sum_{k=1}^{N} x_k$$

as a candidate Lyapunov function. We see that, if $\lambda < 2\delta\frac{N}{N-1}$,

$$\mathcal{L}^N_{ord} B(x) > 0$$

for all $x \in \mathcal{W}_N$. If the parameters satisfy this condition, the mean drift of the lowest $\lfloor \frac{N}{2} \rfloor$ particles is positive, i.e. for $k = 1, \dots, \lfloor \frac{N}{2} \rfloor$ it holds

$$\mathcal{L}^N_{ord} x_k > 0$$

for all x in the interior of $\mathcal{W}_N$. Then, we may consider, instead of the classic barycenter, a modification of it that gives more "weight" to the first particles. Therefore, we consider the vector $v_\epsilon = (1+\epsilon, 1, 1, 1, \dots)$ and define the Laypunov function

$$f_\epsilon(x) = \langle v_\epsilon, x \rangle = (1+\epsilon)x_1 + \sum_{k=2}^{N} x_k$$

for some $\epsilon > 0$. We look for the maximal $\epsilon > 0$ that improves the condition on λ, i.e. we want to find if there exists $\rho(\epsilon, N) > 0$ such that $\mathcal{L}^N_{ord} f_\epsilon(x) > 0$ for all $x \in \mathcal{W}_N$, when

$$\lambda < 2\delta\frac{N}{N-1} + \rho(\epsilon, N).$$

Proof of Theorem 4.2.2. We fix $N \geqslant 2$ and $\delta \geqslant 0$, then we consider the N dimensional vector $v_\epsilon = (1+\epsilon, 1, 1, \dots, 1)$ and the function $f_\epsilon(x) = \langle v_\epsilon, x \rangle$, defined on $\mathcal{W}_N$. For x in the interior of $\mathcal{W}_N$ the condition for (4.2.3) to hold is

$$N\delta + \epsilon - \lambda\frac{N-1}{2} > 0, \tag{4.2.4}$$

our aim is to find an admissible $\epsilon > 0$, that increases the maximal λ satisfying (4.2.3). We need to check the admissibility of ϵ in those regions of $\mathcal{W}_N$ where the mean drift of the first particle is negative, that are the regions where the first particle belongs to a pile. Let first consider the subsets

$$\mathcal{W}_{(N,k)} := \{x \in \mathcal{W}_N : x_1 = \cdots = x_k < x_{k+1}\},$$

where the first particle belongs to a pile of exactly height k, for $k = 2, \dots, N$. Then we say that, for all $k = 2, \dots, N$ and for all $x \in \mathcal{W}_{(N,k)}$, it holds

$$\mathcal{L}^N_{ord} f_\epsilon(x) \geqslant N\delta - k\epsilon - \lambda\frac{N-1}{2} + \lambda\frac{k(k-1)}{2N}.$$

Therefore we want to find the maximal $\epsilon > 0$ such that

$$\min_{k=2,\dots,N} N\delta - k\epsilon - \lambda\frac{N-1}{2} + \lambda\frac{k(k-1)}{2N} \geqslant 0,$$

that is

$$\epsilon_{max} = \frac{\lambda}{N(\delta+2)}.$$

Substituting ϵ_{max}, we get that (4.2.4) holds for all

$$\lambda < \frac{2N^2(\delta+2)\delta}{N(N-1)(\delta+2) - 2\delta} = \lambda^*_{N,lower}(\delta).$$

□

4.3 Stationary measures for the nonlinear process

Following the approach of Section 4.2, we focus on the stationary measures for the nonlinear process with *small jumps*, i.e. the process described in (4.1.8), that corresponds to the solution $\{X(t)\}_{t\geqslant 0}$ of the following nonlinear SDE

$$\begin{aligned} dX(t) = -\mathbb{1}(X(t^-) > 0)\int_0^\infty \mathbb{1}_{[0,1]}(u)\mathcal{N}_{(-)}(du,dt) + \int_0^\infty \mathbb{1}_{[0,1+\delta]}(u)\mathcal{N}_{(+)}(du,dt) \\ - \int_{[0,1]}\int_0^\infty \mathbb{1}_{[0,\mu_{t^-}([0,X(t^-)))}(h)\mathbb{1}_{[0,\lambda]}(u)\mathcal{N}(du,dh,dt), \quad (4.3.1) \end{aligned}$$

where $\mu_t = \text{Law}(X(t))$ and $\{\mathcal{N}_{(-)}, \mathcal{N}_{(+)}, \mathcal{N}\}$ are independent stationary Poisson processes with characteristic measures, respectively, $dudt$, $dudt$ and $dudhdt$.

We know that the study of stationary measures for nonlinear processes is much more difficult than the one for classical Markov processes. Motivated by the stability study of its continuous analogue in Theorem 3.2.3, we conjecture that a stationary measure for (4.3.1), when it exists, it should be unique. However, proving uniqueness of the stationary measure for nonlinear processes is a delicate issue. For this reason, in this section we look for the exact critical value

$$\lambda^*(\delta)$$

above which the nonlinear process (4.3.1) has at least one stationary measure. As in Section 4.2, we could not find the exact value $\lambda^*(\delta)$, so we give an upper and a lower bound for it.

4.3.1 Upper bound for the critical interaction strength in the nonlinear process

In this section we give a sufficient condition for the existence of at least one stationary measure for the nonlinear process, solution to (4.3.1). This provides an upper bound for the interaction strength $\lambda^*(\delta)$ and it is stated in the next theorem.

Theorem 4.3.1. *For all $\delta \geqslant 0$, there exists a value*

$$\lambda^*_{up}(\delta) \doteq 4\delta$$

such that, for all $\lambda > \lambda^*_{up}(\delta)$, the nonlinear process(4.3.1) *has at least one stationary distribution.*

We prove the existence of at least one stationary distribution by means of a transformation Γ in the space $\mathcal{M}(\mathbb{N})$, for which every stationary distribution of (4.3.1) is a fixed point. This is an approach widely exploited in the study of quasi-stationary distributions (QSD) in countable spaces, see [5, 40, 41].

We define a continuous time Markov chain on $\mathbb{N}$, parametrized by a measure. Fix $\mu \in \mathcal{M}(\mathbb{N})$, then let $\{X^\mu(t)\}_{t\geqslant 0}$ be the process with infinitesimal generator defined as follows. For $f \in C_b$, and $x \in \mathbb{N}$

$$\mathcal{L}^\mu f(x) = (1+\delta)(f(x+1) - f(x)) + \mathbb{1}(x > 0)(1 + \lambda\mu[0,x))(f(x-1) - f(x)). \tag{4.3.2}$$

It is clear that $\{X^\mu(t)\}_{t\geqslant 0}$ is a *birth and death* process. Assuming $\lambda > \delta$, for every measure μ, the process $\{X^\mu(t)\}_{t\geqslant 0}$ is ergodic, and π^μ denotes its unique stationary distribution. Define the map

$$\begin{array}{rccc} \Gamma\colon & \mathcal{M}(\mathbb{N}) & \to & \mathcal{M}(\mathbb{N}) \\ & \mu & \mapsto & \pi^\mu, \end{array}$$

Notice that, by definition, μ^* is a stationary distribution for (4.3.1) if and only if it is a fixed point of Γ.

Proof of Theorem 4.3.1. The proof of the upper bound consists of three steps. First we define an auxiliary map that stochastically dominates the map Γ, then we prove that this map preserves a certain subset of $\mathcal{M}(\mathbb{N})$, finally we prove that Γ admits at least one fixed point in that subset.

Step 1. Given $m \in \mathbb{R}^+$, consider the birth and death process with infinitesimal generator

$$\mathcal{L}_m f(x) = (1+\delta)(f(x+1) - f(x)) + (\mathbb{1}(x > 0) + \frac{\lambda}{2}\mathbb{1}(x > m))(f(x-1) - f(x)),$$

Since we are assuming $\lambda > 4\delta$ (here $\lambda > 2\delta$ would suffice), this process is ergodic, and we denote by π_m its stationary distribution. We claim that for all $\mu \in \mathcal{M}(\mathbb{N})$, we have $\pi^\mu \preceq \pi_{med(\mu)}$, where $med(\mu)$ denotes the median of μ and $\preceq$ is defined in Definition 4.2.1.

This is proved by using the *basic coupling* between $\mathcal{L}^{\mu}$ and $\mathcal{L}_{med(\mu)}$, i.e. we consider the Markov process (X_t, Y_t) on $\mathbb{N}^2$ that, at every time $t \geqslant 0$, jumps in the following positions:

$$\begin{array}{lll}
(X_t+1, Y_t+1) & \text{with rate} & 1+\delta, \\
(X_t-1, Y_t-1) & \text{"} & \mathbb{1}(X_t>0)\mathbb{1}(Y_t>0)+\lambda\left(\mu[0,X_t)\wedge \frac{\mathbb{1}(Y_t>med(\mu))}{2}\right), \\
(X_t-1, Y_t) & \text{"} & \mathbb{1}(X_t>0)\mathbb{1}(Y_t=0)+\lambda\left(\mu[0,X_t)-\frac{\mathbb{1}(Y_t>med(\mu))}{2}\right)_+, \\
(X_t, Y_t-1) & \text{"} & \mathbb{1}(X_t=0)\mathbb{1}(Y_t>0)+\lambda\left(\frac{\mathbb{1}(Y_t>med(\mu))}{2}-\mu[0,X_t)\right)_+.
\end{array}$$

We start by proving that, if $Law(X_0) \preceq Law(Y_0)$, then $Law(X_t) \preceq Law(Y_t)$ for all $t \geqslant 0$, i.e.

$$\mathbf{P}(X_t \leqslant Y_t) = \mathbf{P}(X_0 \leqslant Y_0) = 1.$$

We apply the infinitesimal generator of the process (X_t, Y_t) to the function $\mathbb{1}(x \leqslant y)$:

$$\begin{aligned}
\mathcal{L}\mathbb{1}(x \leqslant y) =& \mathbb{1}(y=0)\mathbb{1}(x=1)+\lambda\left(\mu[0,x)-\frac{\mathbb{1}(y>med(\mu))}{2}\right)_+\mathbb{1}(x-1=y) \\
&-\lambda\left(\frac{\mathbb{1}(y>med(\mu))}{2}-\mu[0,x)\right)_+\mathbb{1}(x=y),
\end{aligned}$$

but the last term is always 0, then

$$\mathcal{L}\mathbb{1}(x \leqslant y) = \mathbb{1}(y=0)\mathbb{1}(x=1)+\lambda\left(\mu[0,x)-\frac{\mathbb{1}(y>med(\mu))}{2}\right)_+\mathbb{1}(x-1=y) \geqslant 0.$$

Thus

$$\frac{d}{dt}\mathbf{P}(X_t \leqslant Y_t) = \mathbf{E}[\mathcal{L}\mathbb{1}(X_t \leqslant Y_t)] \geqslant 0$$

and, if $\mathbf{P}(X_0 \leqslant Y_0) = 1$ then $\mathbf{P}(X_t \leqslant Y_t) = 1$ for all $t > 0$.

Since X evolves according to $\mathcal{L}^{\mu}$ and Y to $\mathcal{L}_{med(\mu)}$, which are both ergodic, the order is preserved in equilibrium, i.e.

$$\pi^{\mu} \preceq \pi_{med(\mu)}$$

as desired. We also observe that, by a similar (simpler) coupling argument, $\pi_m \preceq \pi_{m'}$ for $m \leqslant m'$.

Step 2. We now show that if m^* is large enough and $\mu \preceq \pi_{m^*}$, then $\pi^{\mu} \preceq \pi_{m^*}$. By Step 1, this follows if we show that $\pi_{med(\mu)} \preceq \pi_{m^*}$, which amounts to $med(\mu) \leqslant m^*$; since $\mu \preceq \pi_{m^*}$.Thus, it is enough to show that for some m^*, $med(\pi_{m^*}) \leqslant m^*$. To see this, we use the explicit formula for the stationary measure of a birth and death process, obtained by the detailed balance equation: for Z^* normalizing constant,

$$\begin{cases}
\pi_{m^*}(x) = \frac{1}{Z^*}(1+\delta)^x & \text{for } x \leqslant m^*; \\
\pi_{m^*}(x) = \frac{1}{Z^*}(1+\delta)^{m^*}\left(\frac{1+\delta}{1+\lambda/2}\right)^{x-m^*} & \text{for } x > m^*.
\end{cases}$$

The desired inequality $\mathtt{med}(\pi_{m^*}) \leqslant m^*$ follows if we show that

$$\pi_{m^*}[0, m^*] > \pi_{m^*}(m^*, \infty).$$

Indeed, this is equivalent to

$$\frac{(1+\delta)^{\lfloor m^* \rfloor + 1} - 1}{\delta} > (1+\delta)^{\lfloor m^* \rfloor} \frac{1+\delta}{\lambda/2 - \delta}$$

and, by simplifying,

$$\frac{\lambda/2 - 2\delta}{\lambda/2 - \delta} > \frac{1}{(1+\delta)^{\lfloor m^* \rfloor + 1}},$$

which holds for m^* sufficiently large.

Step 3. Define the set

$$\mathcal{M}_{m^*}(\mathbb{N}) \colon = \{\mu \in \mathcal{M}(\mathbb{N}) \colon \mu \preceq \pi_{m^*}\},$$

where m^* has been determined in step 2. We have seen that the function Γ maps $\mathcal{M}_{m^*}$ into itself. Moreover, $\mathcal{M}_{m^*}$ is clearly convex, and it is compact for the weak topology, being closed and tight. The existence of a fixed point follows from Schauder-Tychonov fixed point theorem if we show that Γ is continuous. Let $\mu_n \to \mu$ in $\mathcal{M}_{m^*}$. By the formula for the stationary distribution of a birth and death process we have

$$\pi^{\mu_n}(x) = \frac{1}{Z_n^*} \frac{(1+\delta)^k}{\prod_{h=0}^{k-1}(1 + \lambda\mu_n[0,h))},$$

with

$$Z_n^* \colon = \sum_{k=0}^{\infty} \frac{(1+\delta)^k}{\prod_{h=0}^{k-1}(1 + \lambda\mu_n[0,h))}.$$

Since

$$\frac{(1+\delta)^k}{\prod_{h=0}^{k-1}(1 + \lambda\mu_n[0,h))} \leqslant \frac{(1+\delta)^k}{\prod_{h=0}^{k-1}(1 + \lambda\pi_{m^*}[0,h))},$$

by the Dominated Convergence Theorem

$$Z_n^* \to Z^* := \sum_{k=0}^{\infty} \frac{(1+\delta)^k}{\prod_{h=0}^{k-1}(1 + \lambda\mu[0,h))},$$

and $\pi^{\mu_n} \to \pi^{\mu}$, which establishes continuity. □

Let us underline the importance of this approach with the fixed point argument. It gives an upper bound for the critical value $\lambda_\infty^*(\delta)$ which is linear in δ. Indeed, based on some basic numerical computation, we have the feeling that the condition on λ is not quadratic in δ, as the one emerging from Theorem 4.2.1. Clearly the one found in Theorem 4.3.1 is not optimal and in the following sections we propose conjectures for the critical interaction strength in both the particle system and the nonlinear limit equation.

4.3.2 Lower bound for the critical interaction strength in the nonlinear process

In this section we give a simple lower bound for the critical value $\lambda^*(\delta)$. Although we believe that $\lambda^*(\delta)$ should be strictly greater than its continuous analogue, we could not find a way to prove it and the lower bound here coincides exactly with the continuous critical interaction strength of Theorem 3.2.3.

Theorem 4.3.2. *For all* $\delta \geqslant 0$*, there exists a*

$$\lambda^*_{lower}(\delta) \doteq 2\delta$$

such that, for all $\lambda < \lambda^*_{lower}(\delta)$*, there is no stationary distribution for the nonlinear process* (4.3.1)*.*

Proof. We show that, for $\lambda \leqslant 2\delta$, the nonlinear system has no stationary distributions. Let us remark, to begin with, that for $\lambda \leqslant \delta$ the conclusion is essentially obvious: indeed, the nonlinear Markov process can be coupled, monotonically from below, with a reflected random walk with forward rate $1+\delta$ and backward rate $1+\lambda$, whose distribution at time t tends to concentrate in $+\infty$ as $t \uparrow +\infty$, for any initial distribution.

So assume $\lambda > \delta$, and suppose there exists a stationary distribution μ. The Markov process generated by $\mathcal{L}^\mu$ in (4.3.2) has a strictly negative drift for sufficiently large positions; this implies that its stationary distribution, that is μ by assumption (because μ must be a fixed point of the map Γ), has tails not larger than exponentials. In particular, denoting by $(X_t)_{t\geqslant 0}$ the associated stationary process,

$$E(X_t) < +\infty.$$

Moreover, denoting by id the identity map on $\mathbb{N}$,

$$0 = \frac{d}{dt}E(X_t) = E\left[\mathcal{L}^\mu \mathrm{id}(X_t)\right] = \delta - \lambda \sum_{x\geqslant 0} \mu[0,x)\mu(x). \tag{4.3.3}$$

But

$$\begin{aligned}
\sum_{x\geqslant 0} \mu[0,x)\mu(x) &= \sum_{x\geqslant 0} \mu[0,x-1]\left(\mu[0,x]-\mu[0,x-1]\right) \\
&= \sum_{x\geqslant 0}\left(\mu^2[0,x]-\mu^2[0,x-1]\right) - \sum_{x\geqslant 0}\mu[0,x]\left(\mu[0,x]-\mu[0,x-1]\right) \\
&= 1 - \sum_{x\geqslant 0}\mu[0,x-1]\left(\mu[0,x]-\mu[0,x-1]\right) - \sum_{x\geqslant 0}\mu^2(x)
\end{aligned}$$

which implies

$$\sum_{x\geqslant 0}\mu[0,x)\mu(x) < \frac{1}{2}.$$

Inserting this in (4.3.3), we get $\lambda > 2\delta$, which completes the proof. $\square$

4.4 The exact critical interaction strength

In this section we tackle the problem of getting the exact critical interaction strength by looking at the dynamics of the gaps between successive particles. With a simple linear transformation of the process $(X^N_{(1)}, \ldots, X^N_{(N)})$, we define the **gap process**

$$G^N = \{(G^N_1(t), \ldots, G^N_N(t))\}_{t \geqslant 0},$$

where $G^N_1 \doteq X^N_{(1)}$ and $G^N_i \doteq X^N_{(i)} - X^N_{(i-1)}$ for $i = 2, \ldots, N$, that is a reflected random walk in $\mathbb{N}^N$. In the continuous analogue, this process is a diffusion reflected in R^N_+ and we know, from Theorem 3.2.2, its stationary measure for each fixed N. In the stationary regime the gaps are independent and exponentially distributed with different parameters. The admissibility of these parameters determines the critical interaction strength in the continuous model. We do not expect independence of the gaps for all $N \geqslant 2$ in this discrete setting, because of the importance of triple (or more) collisions of particles. In the following we give a complete treatment in the case $N = 2$ and we conjecture the critical value $\lambda^*_N(\delta)$ for $N > 2$. To this aim we make use of the theory of Jackson networks, that we briefly introduce in the following.

4.4.1 Jackson networks

Jackson networks are queueing models, firstly introduced by Jackson [52], that proved the product form of their stationary distribution. An open Jackson network with N nodes is a Markov process Z^N with values in $\mathbb{N}^N$ such that, at every time $t \geqslant 0$, the vector

$$Z^N(t) \doteq (Z^N_1(t), \ldots, Z^N_N(t))$$

represents the length of N queues. We assume independent Poissonian inputs with parameters λ_i at node i, for $i = 1, \ldots, N$. The servers have exponential service times with parameters μ_i and each customer of node i, after being served, has a probability $p_{i,0}$ of exiting the system and a probability $p_{i,j}$ of being transferred to node j, for $j = 1, \ldots, N$. Therefore, the Markov process Z^N performs a jump of amplitude $\mathbf{j} = (j_1, \ldots, j_N)$ with the following rate:

$$\mathrm{rate}(\mathbf{j}) \doteq \begin{cases} \lambda_i & \text{for } \mathbf{j} = \mathbf{e}_i, \\ \mu_i p_{i,0} & \text{for } \mathbf{j} = -\mathbf{e}_i, \\ \mu_i p_{i,j} & \text{for } \mathbf{j} = -\mathbf{e}_i + \mathbf{e}_j. \end{cases} \tag{4.4.1}$$

The rates do not change according to the current value of the process Z^N, with the only exception that, of course, if the i-th component is equal to zero, i.e. the queue is empty, the jumps that decrease that component are suppressed.

Let us discuss the condition for stationarity in Jackson networks. It involves the so-called *Jackson's system*: for all $i = 1, \ldots, N$

$$\nu_i = \lambda_i + \sum_{j=1}^{N} \nu_j p_{j,i}. \tag{4.4.2}$$

Here, if we suppose to be in a stationary regime, the solution $\nu = (\nu_1, \dots, \nu_N)$ of the system (4.4.2) represents the mean number of customers entering each node in a unit time interval, coming from the outside or from the other nodes. First, the system (4.4.2) admits a solution if the spectral radius of the matrix $\{p_{i,j}\}_{i,j=1,dots,N}$ is strictly less than one, i.e. every customer leaves the network with probability one. Then, we state whether a stationary measure for the process with rates (4.4.1) exists in the following theorem, due to Jackson [52].

Theorem 4.4.1. *The Jackson network* Z^N *with rates* (4.4.1) *is ergodic if and only if*

$$\nu_i < \mu_i$$

for all $i = 1, \dots, N$. *In that case, the stationary measure is given by*

$$\pi_{N,Jack}(z) = \prod_{i=1}^{N} \left(\frac{\nu_i}{\mu_i}\right)^{z_i} (1 - \frac{\nu_i}{\mu_i}),$$

for all $z \in \mathbb{N}^N$.

In [38] the authors also prove exponential ergodicity of Jackson networks under the same conditions of Theorem 4.4.1, by means of a linear Lyapunov function geometrically constructed.

4.4.2 Exact study of gap process for $N = 2$

Let us focus on the *small jumps* model (4.1.6) with $N = 2$. Its gap process $G^2 = \{(G_1^2(t), G_2^2(t))\}_{t \geqslant 0}$ is a reflected random walk in the positive quadrant. It jumps according the following rates:

$$\begin{array}{llll}
\mathbf{g} \text{ s.t. } g_1 > 0,\ g_2 > 0 & \to & \mathbf{g} + (1,-1) & \text{with rate} & 1+\delta \\
 & & \mathbf{g} + (0,-1) & \text{''} & 1+\frac{\lambda}{2} \\
 & & \mathbf{g} + (-1,1) & \text{''} & 1 \\
 & & \mathbf{g} + (0,1) & \text{''} & 1+\delta \\
 & & & & \\
\mathbf{g} \text{ s.t. } g_1 = 0,\ g_2 > 0 & \to & \mathbf{g} + (1,-1) & \text{''} & 1+\delta \\
 & & \mathbf{g} + (0,-1) & \text{''} & 1+\frac{\lambda}{2} \\
 & & \mathbf{g} + (0,1) & \text{''} & 1+\delta \\
 & & & & \\
\mathbf{g} \text{ s.t. } g_1 > 0,\ g_2 = 0 & \to & \mathbf{g} + (-1,1) & \text{''} & 2 \\
 & & \mathbf{g} + (0,1) & \text{''} & 2+2\delta \\
 & & & & \\
(0,0) & \to & (0,1) & \text{''} & 2+2\delta.
\end{array}$$

The following results gives the exact value of the critical interaction strength $\lambda_2^*(\delta)$ and the expression of the stationary measure for the model.

Theorem 4.4.2. *The process* G^2 *is exponentially ergodic if and only if* $\lambda > 2\delta^2 + 4\delta$. *Moreover, when it exists, the unique stationary measure* π_2 *has the following explicit form:*

$$\begin{aligned}
&\pi_2(0,0) = \tfrac{C}{2} && \\
&\pi_2(0,y) = C\left(\tfrac{1+\delta}{1+\frac{\lambda}{2}}\right)^y && y \geqslant 1, \\
&\pi_2(x,0) = \tfrac{C}{2}\left(\tfrac{(1+\delta)^2}{1+\frac{\lambda}{2}}\right)^x && x \geqslant 1, \\
&\pi_2(x,y) = C\left(\tfrac{(1+\delta)^2}{1+\frac{\lambda}{2}}\right)^x\left(\tfrac{1+\delta}{1+\frac{\lambda}{2}}\right)^y && x \geqslant 1, y \geqslant 1,
\end{aligned}$$

for $C \doteq \frac{2(\frac{\lambda}{2}-\delta)(\frac{\lambda}{2}-2\delta-\delta^2)}{(\frac{\lambda}{2}+\delta 2)(\frac{\lambda}{2}+1)}$.

The proof of exponential ergodicity is based on the link between the gap process G^2 and a particular Jackson network. Indeed, because of the nature of the jumps of the gap process G^2, notice that, except that for the "last gap", the increase by one unit of a component causes the decrease by one unit of another. Therefore, we associate to the gap process G^2 a two dimensional Jackson network. Let Z^2 be such that its parameters take the following values:

$$\begin{aligned}
&\lambda_1 = 0, && \mu_1 = 1, && p_{1,0} = 0, && p_{1,2} = 1, \\
&\lambda_2 = 1+\delta, && \mu_2 = 2 + \tfrac{\lambda}{2} + \delta, && p_{2,0} = \tfrac{+\frac{\lambda}{2}}{\mu_2}, && p_{2,1} = \tfrac{1+\delta}{\mu_2}.
\end{aligned} \tag{4.4.3}$$

The process Z^2 defined in this way has the same jumps and the same rates of G^2 in the internal region $\mathbb{N}_* \times \mathbb{N}_*$, while has a slight difference in the rates on the boundaries.

In Figure 4.1 we see the admissible jumps of the gap process G^2 and their rates. In Figure 4.2 we see that the admissible jumps of the Jackson network Z^2 are the same of G^2 but the rates of the jumps performed from the x axis are halved w.r.t. the ones of G^2. Therefore, it is easy to see that, if we consider the embedded Markov chain of each process, the two Markov chains have the same transition matrix. This implies that conditions on stationary measures and on ergodicity for one process are the same for the other and it is the key of the following proof.

Proof of Theorem 4.4.2. Consider the 2 nodes Jackson network Z^2 with same rates of G^2 in the internal region, i.e. the one described by the parameters in (4.4.3). Let (ν_1, ν_2) be the solution of the *Jackson's system* (4.4.2), that here we write as:

$$\begin{cases} \nu_1 = \lambda_1 + \nu_2 p_{2,1}, \\ \nu_2 = \lambda_2 + \nu_1 p_{1,2}. \end{cases}$$

We know from Theorem 4.4.1 that Z^2 is ergodic if and only if $\nu_i < \mu_i$, for i=1,2. In our case this condition becomes

$$\begin{cases} \frac{(1+\delta)^2}{(1+\frac{\lambda}{2})} < 1, \\ \frac{(1+\delta)\mu_2}{(1+\frac{\lambda}{2})} < \mu_2, \end{cases}$$

that gives $\lambda > 2\delta^2 + 4\delta$. As we mentioned, in [39], by the use of a Lyapunov function, the authors prove that this is the necessary and sufficient condition for exponential ergodicity

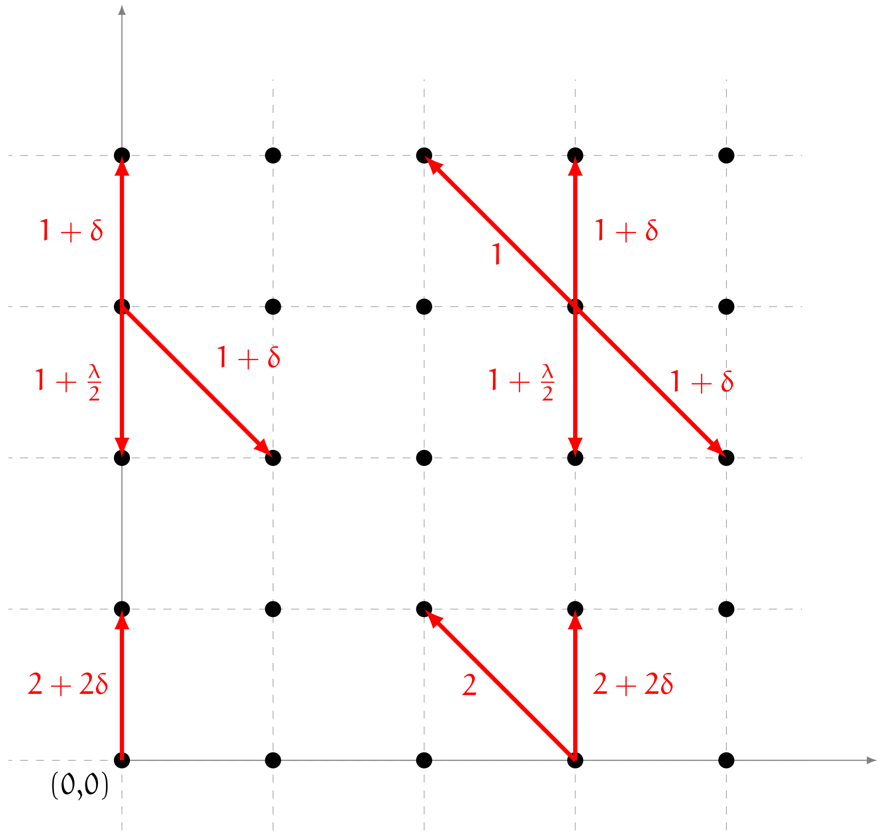

Figure 4.1: Gap process with rates when $N = 2$.

of the process Z^2 and, consequently, for G^2.

The invariant measure of the common embedded Markov chain has the following form:

$$\begin{aligned}
&\eta_2(0,0) = 1+\delta \\
&\eta_2(0,y) = \left(3+2\delta+\tfrac{\lambda}{2}\right)\left(\tfrac{1+\delta}{1+\frac{\lambda}{2}}\right)^y && y \geqslant 1, \\
&\eta_2(x,0) = (2+\delta)\left(\tfrac{(1+\delta)^2}{1+\frac{\lambda}{2}}\right)^x && x \geqslant 1, \\
&\eta_2(x,y) = \left(4+2\delta+\tfrac{\lambda}{2}\right)\left(\tfrac{(1+\delta)^2}{1+\frac{\lambda}{2}}\right)^x\left(\tfrac{1+\delta}{1+\frac{\lambda}{2}}\right)^y && x \geqslant 1, y \geqslant 1.
\end{aligned}$$

Then, for all $(x,y) \in \mathbb{N}^N$ we divide $\eta_2(x,y)$ by the sum of G^2-rates of exiting from (x,y) and we normalize. Of course, the obtained measure coincide with the explicit form of π_2. This is validated by verifying that π_2 solves the stationary equation, i.e. for all bounded measurable functions f it holds:

$$\sum_{(x,y)\in\mathbb{N}^2} \mathbf{L}^2 f(x,y)\pi_2(x,y) = 0,$$

where $\mathbf{L}^2$ is the infinitesimal generator of G^2. $\square$

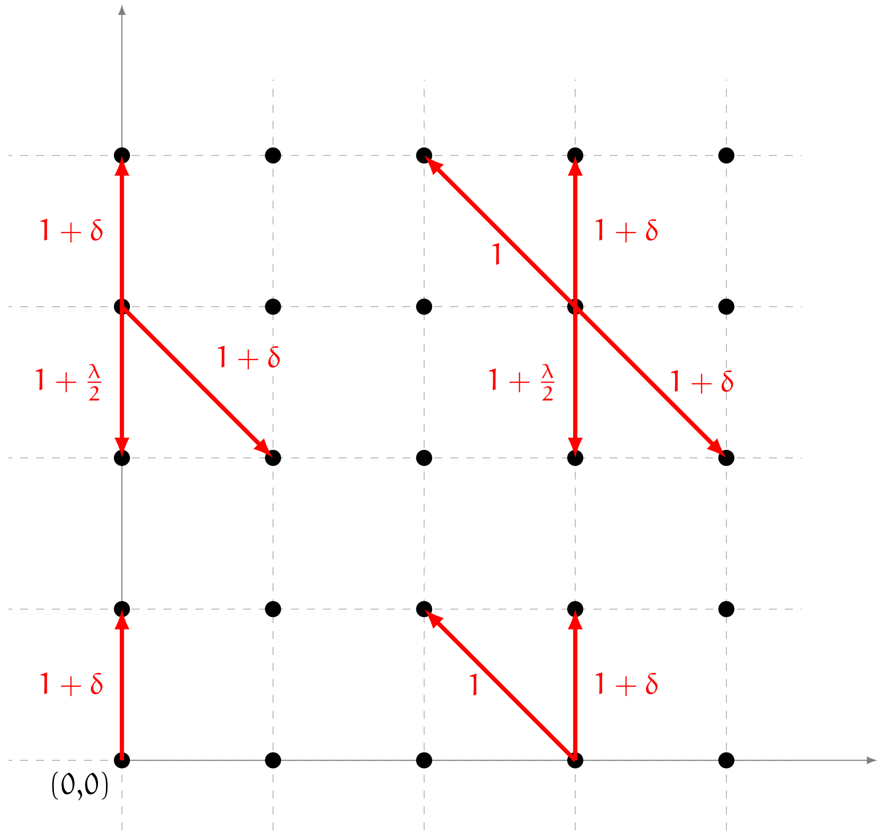

Figure 4.2: Associated Jackson network rates when $N = 2$.

Theorem 4.4.2 gives the exact critical value $\lambda_2^*(\delta)$ for the ergodicity of the system and we see that it is quadratic in δ. Moreover, the explicit expression of π_2 proves that, in the stationary regime, the gaps G_1^2 and G_2^2 are independent. Notice that the lower bound on $\lambda_2^*(\delta)$ obtained in Theorem 4.2.2 is optimal in this case.

4.4.3 Some conjectures

The link between the gap process and a Jackson network for $N = 2$ suggests an association with N nodes Jackson network for every fixed N. Unfortunately, when $N \geqslant 3$ the transition matrix of the embedded Markov chains of G^N and Z^N are not the same. However we can propose a conjecture on the critical value $\lambda_N^*(\delta)$ based on the properties of the associated Jackson network. First of all, let us define the Jackson network Z^N to associate with the gap process G^N, for a fixed $N \geqslant 3$. Z^N must be such that the transition rates in the internal region $\mathbb{N}_*^N$ correspond to the ones of the gap process G^N. For all $i = 1, \dots, N-1$

$$
\begin{array}{lllll}
z & \to & z - \mathbf{e}_i + \mathbf{e}_{i+1} & \text{with rate} & 1 + \lambda\frac{i-1}{N}, \\
z & \to & z + \mathbf{e}_i - \mathbf{e}_{i+1} & \text{"} & 1 + \delta, \\
z & \to & z - \mathbf{e}_N & \text{"} & 1 + \lambda\frac{N-1}{N}, \\
z & \to & z + \mathbf{e}_N & \text{"} & 1 + \delta,
\end{array}
\tag{4.4.4}
$$

where $\mathbf{e}_i$ is the vector $(0,\dots,0,1,0,\dots,0)$ with the i-th coordinate equal to 1.

Proposition 4.4.1. *Fix* $N \geqslant 3$, *the* N *node Jackson network* Z^N *with transition rates* (4.4.4) *is ergodic if, and only if, we have*

$$\frac{(1+\delta)^N}{\prod_{k=1}^{N-1}(1+\lambda\frac{k}{N})} < 1. \tag{4.4.5}$$

Proof. The Jackson network Z^N is such that

$$\begin{aligned}
&\lambda_N = 1+\delta,\\
&\lambda_j = 0 \text{ for all } j = 1,\dots,N-1,\\
&\\
&\mu_1 = 0,\\
&\mu_j = 2+\delta+\lambda\tfrac{j-1}{N}, \text{ for all } j = 2,\dots,N,\\
&\\
&p_{1,2} = 1, p_{1,k} = 0, \text{ for all } k \neq 2,\\
&\\
&p_{j,j+1} = \tfrac{1+\lambda\frac{j-1}{N}}{\mu_j},\ p_{j,j-1} = \tfrac{1+\delta}{\mu_j} \text{ for all } j = 2,\dots,N-1,\\
&p_{j,k} = 0 \text{ for all } j = 2,\dots,N-1, \text{ and all } k \neq j+1, j-1,\\
&\\
&p_{N,0} = \tfrac{1+\lambda\frac{N-1}{N}}{\mu_N},\ p_{N,N-1} = \tfrac{1+\delta}{\mu_j}\\
&p_{N,k} = 0 \text{ for all } k \neq N,0.
\end{aligned}$$

Let us recall the *Jackson system*:

$$\nu_j = \lambda_j + \sum_{i=1}^{N} \nu_i p_{i,j}, \quad \text{for } j = 1,\dots,N.$$

It is easy to verify that the solution $(\nu_1,\dots,\nu_N)$ of this is system has the following form:

$$\nu_j = \mu_j \prod_{k=1}^{N+1-j} \frac{(1+\delta)}{(1+\lambda\frac{N-k}{N})}, \quad \text{for all } i = 1,\dots,N,$$

that by classical result on Jackson networks gives the following condition:

$$\prod_{k=1}^{i} \frac{(1+\delta)}{(1+\lambda\frac{N-k}{N})} < 1, \quad \text{for all } i = 1,\dots,N,$$

that is equivalent to (4.4.5). □

Conjecture 4.4.1. *Fix* $N \geqslant 3$, *the gap process* G^N *is ergodic if, and only if,*

$$\frac{(1+\delta)^N}{\prod_{k=1}^{N-1}(1+\lambda\frac{k}{N})} < 1.$$

This would give an exact critical value $\lambda_N^*(\delta)$, i.e. for each $N \geqslant 3$ and each $\delta \geqslant 0$ would be the solution of

$$\frac{(1+\delta)^N}{\prod_{k=1}^{N-1}(1+\lambda\frac{k}{N})} = 1.$$

In the continuous framework, the sequence of critical values (that by abuse of notation we indicate in the same way) $\lambda_N^*(\delta)$ converges, as N goes to ∞ to the critical value $\lambda_\infty^*(\delta)$ for the nonlinear process. In our case we could not understand if this can be true or not, since we do not even know if there is a value such that there exists a unique stationary measure. However we make a conjecture on the critical value for which there exists at least one stationary measure based on the sequence $\lambda_N^*(\delta)$.

Conjecture 4.4.2. *Fix $\delta \geqslant 0$, then for all λ such that*

$$(1+\frac{1}{\lambda})\ln(1+\lambda) - 1 > \ln(1+\delta),$$

the nonlinear process (4.3.1) *has at least one stationary measure.*

Based on the comparison with the continuous model from Chapter 3, we conjecture also that, when a stationary measure for (4.3.1) exists, then it is unique and that the chaoticity of the stationary measures holds true also in this case. The Lyapunov function of Theorem 4.2.1 seems very promising in this direction, since it ensures the tightness of the sequence of empirical measures, as we observed at the end of Section 4.2.1. Indeed, given $\alpha > 0$ from Theorem 4.2.1, there exists $K \doteq \sup_N \frac{H}{\gamma_N} > 0$ such that, for all N, the stationary measure $\pi_{(SJ)}^N$ is such that

$$\mathbf{E}_{\pi_{(SJ)}^N}\left[\sum_{k=0}^{\infty} e^{\alpha k}\mu_{X^N}^N(k)\right] < K, \tag{4.4.6}$$

where $\mu_{X^N}^N(\cdot) = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N}$ and $Law(X^N) = \pi_{(SJ)}^N$. Then we may wish to apply the approach in [6], where the authors prove chaoticity of the stationary measures of the Fleming-Viot particle system linked to the subcritical Galton-Watson process. Let us adapt this approach to our case. Fix $\delta > 0$ and $\lambda > \lambda_{up}^*(\delta)$, we wish to prove

$$\lim_{N\to\infty} \mathbf{E}_{\pi_{X^N}^N}\left[\left\|\mu_{X^N}^N(\cdot) - \pi(\cdot)\right\|_{TV}\right] = 0, \tag{4.4.7}$$

that implies weak convergence of the stationary empirical measures to the stationary measure π of (4.3.1) identified in Theorem 4.3.1. Notice that (4.4.7) is equivalent to

$$\lim_{N\to\infty} \mathbf{E}_{\pi_{X^N}^N}\left[\left|\mu_{X^N}^N(k) - \pi(k)\right|\right] = 0,$$

for all $k \in \mathbb{N}$. Then fox k, by definition of stationary measure, we know that

$$\mathbf{E}_{\pi_{(SJ)}^N}\left|\mu_{X^N}^N(k) - \pi(k)\right| = \mathbf{E}_{\pi_{(SJ)}^N}\left|\mu_t^N(k) - \pi(k)\right|,$$

where μ_t^N is the empirical measure of the process $X^N(t)$ such that $Law(X^N(0)) = \pi^N_{(SJ)}$. Let us also denote by $\mu_t(\cdot,\nu)$ the law at time $t>0$ of the solution to (4.3.1), such that ν is the law of the initial condition. For any $C>0$, we consider the following subset of $\mathbb{N}^N$

$$S_\alpha(C) \doteq \left\{ x \in \mathbb{N}^N \colon \sum_{k=0}^{\infty} e^{\alpha k}\mu_x^N(k) \leqslant C, \ \text{ for } \mu_x^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} \right\}.$$

Then it holds

$$\begin{aligned}
\mathbf{E}_{\pi^N_{(SJ)}}\left|\mu_t^N(k)-\pi(k)\right| \leqslant& \mathbf{E}_{\pi^N_{(SJ)}}\left[\mathbb{1}\left(S_\alpha(C)^C\right)\right] + \mathbf{E}_{\pi^N_{(SJ)}}\left[\mathbb{1}\left(S_\alpha(C)\right)\left|\mu_t^N(k)-\pi(k)\right|\right] \\
&\leqslant \mathbf{E}_{\pi^N_{(SJ)}}\left[\mathbb{1}\left(S_\alpha(C)^C\right)\right] + \mathbf{E}_{\pi^N_{(SJ)}}\left[\mathbb{1}\left(S_\alpha(C)\right)\left|\mu_t(k,\mu^N_{X^N})-\pi(k)\right|\right] \\
&\quad + \mathbf{E}_{\pi^N_{(SJ)}}\left[\mathbb{1}\left(S_\alpha(C)\right)\left|\mu_t^N(k)-\mu_t(k,\mu^N_{X^N})\right|\right].
\end{aligned}$$

Let $\epsilon>0$, the first term is treated by means of (4.4.6), such that

$$\mathbf{E}_{\pi^N_{(SJ)}}\left[\mathbb{1}\left(\sum_{h=0}^{\infty} e^{\alpha h}\mu^N_{X^N}(h) > C\right)\right] \leqslant \frac{K}{C}$$

and we can choose C sufficiently large such that this term is smaller than $\frac{\epsilon}{3}$. Now, suppose that we could prove that, for any initial condition ν belonging to the following set

$$\mathcal{S}_\alpha(C) \doteq \left\{ \nu \in \mathcal{M}(\mathbb{N}) \colon \sum_{k=0}^{\infty} e^{\alpha k}\nu(k) \leqslant C \right\},$$

the (4.3.1) is ergodic, i.e. for all $\nu \in \mathcal{S}_\alpha(C)$

$$\lim_{t\to\infty} \|\mu_t(\cdot,\nu)-\pi(\cdot)\|_{TV} = 0.$$

Of course, in the event $\{X^N \in S_\alpha(C)\}$ is equivalent to the event $\{\mu^N_{X^N} \in \mathcal{S}_\alpha(C)\}$. This would let us choose $t>0$ sufficiently large such that the second term is smaller than $\frac{\epsilon}{3}$. Then, for fixed $C, t > 0$ the pathwise propagation of chaos proved in Theorem 4.1.3, let us choose N sufficiently large such that the third term is smaller than $\frac{\epsilon}{3}$, as well. This would prove (4.4.7). We conclude by highlighting that the missing step is the proof of ergodicity of (4.3.1), which seems to us a reasonable result, but not a trivial one.

Part III

Generalized Curie-Weiss model

Chapter 5

Periodic behavior in a generalized Curie-Weiss model with dissipation

Among the interesting phenomena observed in complex systems, there is the occurrence of rhythmic behaviors. It is natural to model this systems by means of large families of interacting units and one question is how periodic behaviors emerge in these systems when the single units have no tendency to behave periodically.

5.1 Self-sustained periodic behavior

In a particle system we say that we observe a *self-sustained periodic behavior* if there is a phase in which the evolution of the macroscopic law of the system has a stable limit cycle, without the action of any periodic force. Often this periodicity, even if easily detectable by numerical simulations of the particles for large system's size, it is a peculiarity of the thermodynamic limit and it is quite hard to formally prove it, since it is an infinite dimensional problem. However, some recent works [27, 47] try to investigate the minimal hypothesis needed to create self-sustained periodic behavior in mean field interacting particle systems. In this framework, one key step is to consider interactions that favor cooperation among units of the same type, but, opposed to the classical models originated from statistical mechanics, the reversibility of the dynamics seems to be in contrast with the occurrence of periodic behavior [11, 47]. Therefore, different mechanisms that perturb classical reversible models have been introduced.

- In [24, 26] the authors introduce particle systems where each particle has its own *local field* that undergoes a *diffusive and dissipative* dynamics. The add of a dissipative term seems to be a useful tool to create stable periodic orbit in systems otherwise in equilibrium. The simplest model of this class is the dynamical Curie-Weiss model with a dissipative term studied in [26], which already displays an interesting behavior due to the dissipation and the rise of a periodic orbit via a *Hopf bifurcation*. In [24]we find a general way of introducing dissipation in systems of interacting diffusions, together with the study of the dissipative counterpart of the model by Dawson [28].

- In [47] authors consider active rotators models, one among the others the stochastic Kuramoto model. They propose different dynamics and interactions for which they prove periodicity with a very general method. In particular adding a simple *disorder* in the initial phase of each rotator for the Kuramoto model, see [1, 23], gives origin to oscillating behavior in the stationary solutions.

- The role of *noise* in the dynamics is crucial. It may induce periodicity in systems which cannot exhibit periodic solutions, giving rise to the phenomenon of *noise-induced* periodicity [27, 78, 86]. Moreover, noise has a role in enhancing periodicity in dynamical systems already proved to have limit cycles, a phenomenon known as *excitability by noise* [24, 59].

- The add of a *delay* in the interactions is proved to give self-sustained periodicity in multi-populated models. In [85] a fixed time delay is sufficient to create stable oscillations in a bi-populated spin system with non-cooperative interaction. In [31] a multi-populated system of interacting Hawkes processes with a delay given by Erlang kernels is studied and sufficient conditions for the existence of at least one stable limit cycle are given.

- In [25] the *interaction network* between two populations is enough to generate periodicity, without the need of a delay in the interaction. In particular, it is a *frustrated* interaction that, if strong enough, generates the limit cycles. By frustrated interaction we mean the situation in which one population "wants to copy" the other, but this one has the tendency to behave oppositely to the first one.

The aim of this chapter is to extend the first approach to the so-called generalized Curie-Weiss model. We confirm that the *dissipative* interaction is able to give origin to self-sustained periodic behaviors in a wide class of mean field particle systems.

5.2 The model

In this section we build step-by-step the class of processes we are interested in. We start with a brief description of the classical Curie-Weiss spin model and of the class of its generalized counterpart; then we define a stochastic process related to this class and we describe how we can break its reversibility by means of the dissipative dynamics.

5.2.1 The Curie-Weiss model

The Curie-Weiss model origins as a mean field approximation of the Ising model for ferromagnetism. In this case a magnet is modelled, at microscopic level, by a configuration of N spins; each spin takes values in the set $\{-1, +1\}$. We associate to each configuration $\sigma^{\mathsf{N}} \in \{-1, +1\}^{\mathsf{N}}$ an energy, by the Hamiltonian, which is a quadratic function of the mean spin value:

$$H_N(\sigma^N) \doteq -\frac{1}{2N}\left(\sum_{i=1}^{N}\sigma_i^N\right)^2. \tag{5.2.1}$$

For any inverse temperature $\beta = \frac{1}{T} > 0$ we define the probability measure, called the Gibbs measure, on $\{-1,+1\}^N$

$$\mathbb{P}_{N,\beta}(d\sigma_1^N \dots d\sigma_N^N) \doteq \frac{1}{Z_N(\beta)} e^{-\beta H_N(\sigma^N)},$$

where $Z_N(\beta) \doteq \sum_{\sigma^N \in \{-1,+1\}}^{N} e^{-\beta H_N(\sigma^N)}$ is the normalizing constant. The Gibbs measure gives higher probability to configurations with minimal energy.

The Curie-Weiss model is defined as the sequence of probability measures $\{\mathbb{P}_{N,\beta}\}_{N\in\mathbb{N}^*}$ and it shows a **phase transition**, usually identified as a breakdown in the *Law of Large Numbers*, see for example [36]. The previous means that the following weak limit holds:

$$\frac{\sum_{i=1}^{N}\sigma_i^N}{N} \xrightarrow{N\to\infty} \begin{cases} \delta_0 & \text{for } \beta \leqslant 1, \\ \frac{1}{2}(\delta_{m(\beta)} + \delta_{-m(\beta)}) & \text{for } \beta > 1, \end{cases} \tag{5.2.2}$$

for a certain increasing function $m(\cdot)\colon (1,\infty) \to (0,1)$, called the *spontaneous magnetization*. Therefore, the value $\beta_c \doteq 1$ is a critical value for the Curie-Weiss. When $\beta < \beta_c$ (i.e. the temperature is sufficiently high) we see that the spins behave as they were i.i.d. random variables with mean 0. On the other hand, when $\beta > \beta_c$ (i.e. for low temperature) the limit of the empirical mean is a random variable.

5.2.2 The generalized Curie-Weiss model

A natural extension of the previous model is the so-called *generalized Curie-Weiss model*, see [37, 35]. The quadratic interaction function in (5.2.1) is replaced by a more general even function and the spin's single site distribution in absence of interaction (that in the classical Curie-Weiss is $\frac{1}{2}(\delta_{-1} + \delta_1)$) is some symmetric distribution on $\mathbb{R}$. Therefore, we consider a sequence of probability measures on $\mathbb{R}^N$, for $N = 1,2,\dots$, given by

$$\mathbf{P}_{N,\beta}(dx_1,\dots,dx_N) = \frac{1}{\mathbf{Z}_N(\beta)} \exp\left(N\beta g\left(\sum_{i=1}^{N}\frac{x_i}{N}\right)\right)\prod_{i=1}^{N}\rho(dx_i), \tag{5.2.3}$$

where ρ is the symmetric probability measure on $\mathbb{R}$ representing the single-site distribution of a spin, g is the interaction function, β is again the inverse absolute temperature of the model and $\mathbf{Z}_N(\beta)$ is the normalizing constant. We summarize in the following some assuptions on these quantities, mainly coming from [35], that are sufficient for the existence of such a model.

Assumption 5.2.1. *The function g and the probability measure ρ satisfy the following conditions.*

i) $g\colon \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ *is an even,* $C^2(\mathbb{R})$ *function, strictly increasing on* $[0, \infty)$ *with* $g(0) = 0$. *It is* two-sided real analytic, *i.e.* $\forall\, x \in \mathbb{R}$ *there exists* $\delta > 0$ *and two real analytic functions* g_1 *and* g_2 *on* $(x - \delta, x + \delta)$ *such that*

$$g\colon = \begin{cases} g_1 & \text{on } (x-\delta, x] \\ g_2 & \text{on } [x, x+\delta). \end{cases}$$

ii) ρ *is a symmetric Borel probability measure on* $\mathbb{R}$*, absolutely continuous w.r.t. the Lebesgue measure and, by abuse of notation, we denote its density function with* $\rho(x)$.

iii) *There exists a symmetric, nonconstant, convex function* h *on* $\mathbb{R}$ *such that*

$$g(x) \leqslant h(x) \ \text{ for all } x \in \mathbb{R},$$
$$\int_{\mathbb{R}} e^{a h(x)} \rho(x) dx < \infty \ \text{ for all } a > 0. \tag{5.2.4}$$

The key function in the analysis of the asymptotic behavior of the sequence of measures $\{\mathbf{P}_{N,\beta}\}$ is the *specific Gibbs free energy* $\psi(\beta)$, defined, for all $\beta > 0$, as

$$-\beta\psi(\beta) = \lim_{N\to\infty} \frac{1}{N} \log \mathbf{Z}_N(\beta).$$

This is known to be equivalent to the variational formula:

$$-\beta\psi(\beta) = \sup_{u\in\mathbb{R}} \{\beta g(u) - i(u)\}, \tag{5.2.5}$$

where $i(u)$ is defined as the Legendre-Fenchel transform $i(u) = \sup_{t\in\mathbb{R}}\{tu - c(t)\}$ of the quantity $c(t) = \log \int_{\mathbb{R}} e^{tx} \rho(dx)$, for all $t \in \mathbb{R}$. The role of this formulation appears in the following theorem, that is the main result of [35].

Theorem 5.2.1 (Theorem 1.2 in [35]). *Suppose that Assumption 5.2.1 is satisfied. Then there exists a non-empty set of* critical values

$$\mathcal{P} \doteq \{0 < \beta_1 < \beta_2 < \dots\},$$

whose elements are either finite or countably many and converging to infinity.

i) *There exists a function* $m\colon (0, \infty) \setminus \mathcal{P} \longrightarrow \mathbb{R}_+$ *such that*

$$m(\beta) \begin{cases} = 0 & \text{for } \beta < \beta_1, \\ > 0 & \text{for } \beta \in (\beta_1, \infty) \setminus \mathcal{P}, \end{cases}$$

it is strictly increasing in $(\beta_1, \infty)\setminus\mathcal{P}$ *and real analytic on every connected subset of* $(0, \infty)\setminus\mathcal{P}$*, but it is not the restriction of one real analytic function in any neighborhood of a critical value* β_i.

ii) *The function* m *is such that, for all* $\beta \in (0,\infty)\setminus\mathcal{P}$, *the supremum in the formula* (5.2.5) *is attained at the points* $u = \pm m(\beta)$ *(of course, for* $\beta < \beta_1$ *it is attained at* 0*).*

iii) *For any* $N \geqslant 1$, *let* $(X_1^N, \ldots, X_N^N)$ *be a random variable with values in* $\mathbb{R}^N$ *and distribution* $\mathbf{P}_{N,\beta}$. *Then the following weak limit holds:*

$$\frac{\sum_{i=1}^N X_i^N}{N} \xrightarrow{N\to\infty} \begin{cases} \delta_0 & \text{for } \beta \leqslant \beta_1, \\ \frac{1}{2}(\delta_{m(\beta)} + \delta_{-m(\beta)}) & \text{for } \beta \in (\beta_1,\infty)\setminus\mathcal{P}. \end{cases}$$

Theorem 5.2.1 is true also for the classical Curie-Weiss model, where we have $\mathcal{P} = \{1\}$ and in (5.2.2) the value of the *spontaneous magnetization* is exactly the point at which the supremum of (5.2.5) is attained, when $\rho = \frac{1}{2}(\delta_{-1} + \delta_1)$. In general, the nature of phase transitions of the generalized Curie-Weiss model may be of two type, depending on the continuity of the function m in the critical value.

Definition 5.2.1. We say that there is a **phase transition** for the generalized Curie-Weiss model at the critical value β^* if either one of the two following conditions is satisfied:

i) $\lim_{\beta\uparrow\beta^*} m(\beta) < \lim_{\beta\downarrow\beta^*} m(\beta)$, in this case we have a *first-order phase transition*;

ii) $\lim_{\beta\uparrow\beta^*} m(\beta) = \lim_{\beta\downarrow\beta^*} m(\beta)$, but $\lim_{\beta\downarrow\beta^*} m'(\beta) = +\infty$, in this case we have a *second-order phase transition.*

5.2.3 The Langevin dynamics for the generalized Curie-Weiss model

Let μ be a probability density on $\mathbb{R}^d$, for $d \geqslant 1$, sufficiently regular, then the Langevin dynamics $\{X_t\}_{t\geqslant 0}$ associated to μ is a diffusion process in $\mathbb{R}^d$ such that μ is its unique stationary measure. We define this process as the solution of the following SDE

$$dX_t = \frac{1}{2}\nabla \log \mu\,(X_t)\,dt + dB_t, \tag{5.2.6}$$

where B is a d-dimensional Brownian motion. Let us state a classical result on well-posedness and long-time behavior of Langevin diffusions.

Theorem 5.2.2. *Let* μ *be a probability density function on* $\mathbb{R}^d$, *for* $d \geqslant 1$, *such that* $\log\mu \in C^2(\mathbb{R}^d)$ *and for all* $x \in \mathbb{R}^d$

$$x^T \nabla \log \mu(x) \leqslant C(1 + \|x\|^2),$$

for a certain $C > 0$. *The SDE* (5.2.6) *admits a unique strong solution, for any square-integrable initial condition. Moreover,* $\mu(x)dx$ *is the unique stationary measure of* (5.2.6) *and, for all* $x \in \mathbb{R}^d$ *such that* $X_0 = x$ *a.s. then*

$$\lim_{t\to\infty} \|\mathbf{P}_x(X_t \in \cdot) - \mu(\cdot)\|_{TV} = 0.$$

Therefore, for each N fixed, a Langevin dynamics associated to (5.2.3) is a diffusion process X^N with values in $\mathbb{R}^N$ such that $\mathbf{P}_{N,\beta}$ is its unique invariant measure, i.e. X^N is solution to the following systems of SDE

$$dX_i^N(t) = \frac{\beta}{2} g'\left(\frac{\sum_{j=1}^N X_j^N(t)}{N}\right) dt - \frac{\rho'(X_i^N(t))}{2\rho(X_i^N(t))} dt + dB_t^i, \tag{5.2.7}$$

where $\{B^i\}_{i=1,\dots,N}$ is a family of independent 1-dimensional Brownian motions. The dynamics in (5.2.7) represents an interacting particle system where each particle follows its own dynamics, given by the last two terms on the right-hand side, and it experiences a mean field interaction, which depends on the empirical mean of the system $m^N(t) := \frac{\sum_{j=1}^N X_j^N(t)}{N}$.

Assumption 5.2.2. *The function g and the probability measure ρ satisfy the following conditions.*

i) g' *is uniformly Lipschitz continuous, i.e. there exists a finite constant* $L \geqslant 0$ *such that for all* $x, y \in \mathbb{R}$

$$|g'(x) - g'(y)| \leqslant L|x - y|.$$

ii) We require $\log(\rho(x)) \in C^2$ *and that there exists* $K > 0$ *s.t. for all* $x, y \in \mathbb{R}$

$$(x - y)\left(\frac{\rho'(x)}{\rho(x)} - \frac{\rho'(y)}{\rho(y)}\right) \leqslant K(1 + (x - y)^2). \tag{5.2.8}$$

Assumptions 5.2.1 and 5.2.2 ensures well-posedness of (5.2.7) and, as we show later, of its mean field limit; of course, in this case, Theorem 5.2.2 holds.

5.2.4 The dissipative dynamics

We aim to suitably modify the Langevin dynamics (5.2.7) in order to observe the emergence of self-sustained periodic behavior. Therefore, we choose to break reversibility by following the approach in [24, 26], where the interaction in the particle system undergoes its own stochastic dynamics, characterized by a dissipative term.

We suppose that the motion of each particle depends on a "perceived magnetization" instead of the empirical mean $m^N(t)$. To this aim, we introduce the variables λ_i^N, for $i = 1, \dots, N$, representing the interaction felt by the spin X_i^N. They evolve as the magnetization of the system but they undergo a dissipative and diffusive evolution:

$$d\lambda_i^N(t) = -\alpha\lambda_i^N(t)dt + D dB_t^{2,i} + dm^N(t),$$

where $\{B^{2,i}\}_{i=1,\dots,N}$ are independent Brownian motions. This results in a stochastic process (X^N, λ^N) with values in $\mathbb{R}^{2N}$ where, at every time $t \geqslant 0$, $X^N(t) = \big(X_1^N(t), \dots, X_N^N(t)\big)$ is the vector of the spins of the N particles and $\lambda^N = \left(\lambda_1^N(t), \dots, \lambda_N^N(t)\right)$ is the vector of the "perceived magnetizations". The Markov process $(X^N(t), \lambda^N(t))$ has infinitesimal generator

$$\mathcal{L}^N f(x,\lambda) = \sum_{i=1}^N \left[\frac{1}{2}\left(\beta g'(\lambda_i) - \frac{\rho'(x_i)}{\rho(x_i)}\right)\frac{\partial}{\partial x_i} f(x,\lambda) + \frac{1}{2}\frac{\partial^2}{\partial x_i^2} f(x,\lambda) \right.$$
$$\left. + \left(\frac{1}{2N}\sum_{j=1}^N \left(\beta g'(\lambda_j) - \frac{\rho'(x_j)}{\rho(x_j)}\right) - \alpha\lambda_i\right)\frac{\partial}{\partial \lambda_i} f(x,\lambda) + \frac{D}{2}\frac{\partial^2}{\partial \lambda_i^2} f(x,\lambda)\right],$$

i.e. $(X^N(t), \lambda^N(t))$ solves the following system of SDE:

$$\begin{cases} dX_t^{N,i} = \frac{\beta}{2} g'(\lambda_t^{N,i})dt - \frac{\rho'(X_t^{N,i})}{2\rho(X_t^{N,i})}dt + dB_t^{1,i} \\ d\lambda_t^{N,i} = -\alpha\lambda_t^{N,i}dt + \frac{1}{N}\sum_{j=1}^N \left(\frac{\beta}{2} g'(\lambda_t^{N,j}) - \frac{\rho'(X_t^{N,j})}{2\rho(X_t^{N,j})}\right)dt + D dB_t^{2,i}, \end{cases} \tag{5.2.9}$$

$i = 1, \dots, N$, for $\{(B^{1,i}, B^{2,i}\}_{i=1,\dots,N}$ a family of independent 2-dimensional Brownian motions. Well-posedness of this system, under Assumptions 5.2.1 and 5.2.2 is a simple consequence of previous results.

This approach has been proved to break reversibility of the (otherwise reversible) Langevin dynamics in one particular case, in a way that collective periodic behavior occurs, see [24]. That model is a particular case of the framework we depicted in this section. Indeed, in [24] the authors consider the particle system $\{(Y_t^{N,i}, \lambda_t), i = 1, \dots, N\}_{t \geqslant 0}$ that solves the following, for $i = 1, \dots, N$

$$\begin{cases} dY_t^{N,i} = (-(Y_t^{N,i})^3 + Y_t^{N,i})dt - \lambda_t dt + \sigma dB_t^{N,i} \\ \frac{d}{dt}\lambda_t = -(\alpha - \theta)\lambda_t - \theta\mathbf{E}[-Y_t^3 + Y_t], \end{cases}$$

where $\{B^i\}_{i=1,\dots,N}$ are independent Brownian motions. By the change of variable $X = \frac{Y}{\sigma}$, we see that the previous system correspond to (5.2.9) with following specifications: $\beta = \theta$, $D = 0$ and

$$\begin{aligned} g(x) =& x^2; \\ \rho(x) =& \frac{1}{Z^*}\exp\left(\frac{x^2}{2}\left(1 - \frac{x^2\sigma^2}{2}\right)\right), \quad \text{for } Z^* \text{ normalizing constant.} \end{aligned}$$

The model in [26] follows the same lines we described above, but since the state space is discrete the dynamics is given by a (slight modification of a) Glauber dynamics for the classical Curie-Weiss.

5.2.5 The nonlinear process and propagation of chaos

The interactions in (5.2.11) are of mean field type and we define the correspondent nonlinear Markov process (X, λ) on $\mathbb{R}^2$ as the solution of the following nonlinear SDE:

$$\begin{cases} dX_t = \frac{\beta}{2} g'(\lambda_t)dt - \frac{\rho'(X_t)}{2\rho(X_t)}dt + dB_t^1 \\ d\lambda_t = -\alpha\lambda_t dt + \langle \mu_t(x,l), \frac{\beta}{2} g'(l) - \frac{\rho'(x)}{2\rho(x)}\rangle dt + D dB_t^2 \\ \mu_t = Law(X_t, \lambda_t), \end{cases} \tag{5.2.10}$$

where $B = (B^1, B^2)$ is a two dimensional Brownian motion. Well-posedness of (5.2.10) is stated in the following theorem.

Theorem 5.2.3. *The nonlinear process* (5.2.10) *is well-defined, i.e. there exists a unique strong solution for all square-integrable initial condition* $(X_0, \lambda_0) \in \mathbb{R}^2$.

The proof of Theorem 5.2.3 follows the approach via pathwise estimates we extensively used in Chapter 2. Notice that the drift coefficient of (5.2.10) satisfies Assumption 2.2.1 in its spatial coordinate, but not in the measure one. Indeed, it is not always true that, under Assumptions 5.2.1 and 5.2.2 it holds a Lipschitz condition w.r.t. to the W_1 Wasserstein distance. This is because of the assumption (5.2.8), which implies that the nonlinear term involves expectations of non-globally Lipschitz functions. However, we will follow the same approach of Section 2.2, just adapting the proofs by means of an *ad hoc* treatment of the mean field term. Indeed, the nonlinear term coincides with the time-derivative of $\mathbf{E}[X_t]$, which drastically simplifies the situation. Due to the absence of jumps in the dynamics, we use the usual L^2 approach. We fix a time $T > 0$ and wemake use of the W_2 Wasserstein distance on the set $\mathcal{M}^2(\mathbf{C}([0,T], \mathbb{R}^2))$ of square-integrable measures: for all $\mu, \nu \in \mathcal{M}^2(\mathbf{C}([0,T], \mathbb{R}^2))$

$$W_{2,T}(\mu,\nu)^2 = \inf\Big\{ \int \sup_{t\in[0,T]} \|x(s) - y(s)\|^2 m(dx, dy),$$
$$\text{with } m \in \mathcal{M}^2(\mathbf{C}([0,T],\mathbb{R}^2) \times \mathbf{C}([0,T],\mathbb{R}^2)), \pi_1 \circ m = \mu,\ \pi_2 \circ m = \nu \Big\}.$$

Proof of Theorem 5.2.3. Given any square-integrable law μ_0 on $\mathbb{R}^2$, we define a map Γ that associates to a measure $Q \in \mathcal{M}^2(\mathbf{C}([0,T], \mathbb{R}^2))$ the law of the solution $\{(X_t, \lambda_t)\}_{t\in[0,T]}$ of the SDE

$$\begin{cases} dX_t = \frac{\beta}{2} g'(\lambda_t)dt - \frac{\rho'(X_t)}{2\rho(X_t)}dt + dB^1_t \\ d\lambda_t = -\alpha\lambda_t dt + \langle Q_t(dx, dl), \frac{\beta}{2}g'(l) - \frac{\rho'(x)}{2\rho(x)}\rangle dt + D dB^2_t, \end{cases}$$

that, for μ_0 initial condition, admits a unique strong solution for classical results, see [51]; of course a solution to (5.2.10) is a fixed point of Γ. We use a coupling argument to prove existence (via a Picard iteration) and uniqueness of the fixed point of Γ. Let us start with the proof of uniqueness, if Q^1 and Q^2 are two fixed point of Γ, i.e. two measures in $\mathcal{M}^2(\mathbf{C}([0,T], \mathbb{R}^2))$ such that $Q^1 = \Gamma(Q^1)$ and $Q^2 = \Gamma(Q^2)$. We couple them as follows. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$ be a filtered probability space and $\{B_t\}_{t\in[0,T]}$ a two-dimensional Brownian motion. Then we write

$$\begin{cases} dX^1_t = \frac{\beta}{2} g'(\lambda^1_t)dt - \frac{\rho'(X^1_t)}{2\rho(X^1_t)}dt + dB^1_t \\ d\lambda^1_t = -\alpha\lambda^1_t dt + \langle Q^1_t(dx, dl), \frac{\beta}{2}g'(l) - \frac{\rho'(x)}{2\rho(x)}\rangle dt + D dB^2_t, \end{cases}$$

and

$$\begin{cases} dX^2_t = \frac{\beta}{2} g'(\lambda^2_t)dt - \frac{\rho'(X^2_t)}{2\rho(X^2_t)}dt + dB^1_t \\ d\lambda^2_t = -\alpha\lambda^2_t dt + \langle Q^2_t(dx, dl), \frac{\beta}{2}g'(l) - \frac{\rho'(x)}{2\rho(x)}\rangle dt + D dB^2_t, \end{cases}$$

where the initial conditions are $(X_0^1, \lambda_0^1) = (X_0^2, \lambda_0^2)$ a.s., μ_0 distributed. We estimate the distance between Q^1 and Q^2 by means of the above coupling, i.e.

$$W_{2,T}(Q^1, Q^2) \leqslant \sqrt{\mathbf{E}\left[\sup_{t\in[0,T]} (X_t^1 - X_t^2)^2 + (\lambda_t^1 - \lambda_t^2)^2\right]}.$$

The SDE for λ^1 and λ^2 is linear, then we write explicitly

$$\lambda_t^1 - \lambda_t^2 = \int_0^t e^{\alpha(s-t)} \langle Q_s^1(dx, dl) - Q_s^2(dx, dl), \frac{\beta}{2} g'(l) - \frac{\rho'(x)}{2\rho(x)} \rangle ds.$$

Notice that $\langle Q_t^1(dx, dl) - Q_t^2(dx, dl), \frac{\beta}{2} g'(l) - \frac{\rho'(x)}{2\rho(x)} \rangle = \frac{d}{dt} \mathbf{E}\left[X_t^1 - X_t^2\right]$, that gives

$$\lambda_t^1 - \lambda_t^2 = \mathbf{E}\left[X_t^1 - X_t^2\right] - \alpha \int_0^t \mathbf{E}\left[X_s^1 - X_s^2\right] e^{-\alpha(t-s)} ds.$$

On the other hand, we use Ito's formula to obtain

$$(X_t^1 - X_t^2)^2 = 2\int_0^t (X_s^1 - X_s^2) \left(\frac{\beta}{2} g'(\lambda_s^1) - \frac{\beta}{2} g'(\lambda_s^1) - \frac{\rho'(X_s^1)}{2\rho(X_s^1)} + \frac{\rho'(X_s^2)}{2\rho(X_s^2)} \right) ds.$$

Therefore, there exists C_T such that

$$\mathbf{E}\left[\sup_{t\in[0,T]} (X_t^1 - X_t^2)^2 + (\lambda_t^1 - \lambda_t^2)^2\right] \leqslant C_T \int_0^T \mathbf{E}\left[\sup_{t\in[0,s]} (X_t^1 - X_t^2)^2 + (\lambda_t^1 - \lambda_t^2)^2\right] ds,$$

and by Gronwall Lemma this gives $W_{2,T}(Q^1, Q^2) = 0$. With a Picard iteration of the type $Q^n = \Gamma(Q^{n-1})$ and with the above arguments, we obtain that

$$\begin{aligned}\mathbf{E}\left[\sup_{t\in[0,T]} (X_t^n - X_t^{n-1})^2 + (\lambda_t^n - \lambda_t^{n-1})^2\right] \leqslant & L \int_0^T \mathbf{E}\left[\sup_{t\in[0,s]} (X_t^n - X_t^{n-1})^2 + (\lambda_t^n - \lambda_t^{n-1})^2\right] ds \\ & + \alpha^2 T \int_0^T W_{2,s}(Q^{n-1}, Q^{n-2})^2 ds,\end{aligned}$$

that gives $W_{2,T}(Q^n, Q^{n-1})^2 \leqslant \frac{(e^{LT} T\alpha^2)^n}{n!} \int_0^T W_{2,s}(Q^1, Q^0)^2 ds$, i.e. $\{Q^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence for $W_{2,T}$ and therefore for a weaker, but complete, metric on $\mathcal{M}^2(\mathbf{C}([0,T], \mathbb{R}^2))$.

□

We prove propagation of chaos for the particle system (5.2.9) with the same pathwise approach.

Theorem 5.2.4. *Let $(X^N(t), \lambda^N(t))_{t\geqslant 0}$ be the Markov process solution to* (5.2.9) *starting from i.i.d. initial conditions* $\mathrm{Law}((X_i^N(0), \lambda_i^N(0))) = \mu_0$ *on* $\mathbb{R}^2$, *where* $\int_{\mathbb{R}^2} (x^2 + \lambda^2) \mu_0(dx, d\lambda) < \infty$, *and denote with* P^N *its law on* $\mathbf{C}([0,T], \mathbb{R}^{2N})$. *Let* $(X(t), \lambda(t))_{t\geqslant 0}$ *be the solution to* (5.2.10) *with initial condition* μ_0, *and denote with* μ *its law on* $\mathbf{C}([0,T], \mathbb{R}^2)$. *Then, the sequence* $(P^N)_{N\in\mathbb{N}}$ *is* μ*-chaotic.*

As we said repeatedly in the previous chapters, by the exchangeability of the components and of the dynamics, it is well-known that this claim is implied by $\mathbf{E}\left[W_{2,T}(\mu^N,\mu)\right] \longrightarrow 0$ as $N \to +\infty$, where $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{(X^N,\lambda^N)}$ (see [83]). Therefore, we apply this approach to prove the theorem.

Proof of Theorem 5.2.4. On a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$, for any $N \in \mathbb{N}$, take a $2N$-dimensional Brownian motion $\{B_t\}_{t\in[0,T]}$ and consider the coupled processes given by

$$\begin{cases} dX_t^{N,i} = \frac{\beta}{2} g'(\lambda_t^{N,i})dt - \frac{\rho'(X_t^{N,i})}{2\rho(X_t^{N,i})}dt + dB_t^{1,i} \\ d\lambda_t^{N,i} = -\alpha\lambda_t^{N,i}dt + \frac{1}{N}\sum_{j=1}^N \left(\frac{\beta}{2}g'(\lambda_t^{N,j}) - \frac{\rho'(X_t^{N,j})}{2\rho(X_t^{N,j})}\right)dt + D\,dB_t^{2,i}, \end{cases} \tag{5.2.11}$$

$i = 1, \dots, N$, and

$$\begin{cases} d\bar{X}_t^{N,i} = \frac{\beta}{2} g'(\bar{\lambda}_t^{N,i})dt - \frac{\rho'(\bar{X}_t^{N,i})}{2\rho(\bar{X}_t^{N,i})}dt + dB_t^{1,i} \\ d\bar{\lambda}_t^{N,i} = -\alpha\bar{\lambda}_t^{N,i}dt + \langle \mu_t(dx, dl), \frac{\beta}{2}g'(l) - \frac{\rho'(x)}{2\rho(x)}\rangle dt + D\,dB_t^{2,i}, \end{cases}$$

$i = 1, \dots, N$, where the initial conditions are $(X_0^{N,i}, \lambda_0^{N,i}) = (\bar{X}_0^{N,i}, \bar{\lambda}_0^{N,i})$ a.s., $\mu_0^{\otimes N}$ distributed. Let $\bar{\mu}^N = \frac{1}{N}\sum_{j=1}^N \delta_{(\bar{X}^{N,j},\bar{\lambda}^{N,j})}$, then, similarly to the proof of Theorem 5.2.3, it's easy to see that it holds

$$\begin{aligned} \mathbf{E}\left[W_{2,T}(\mu^N, \bar{\mu}^N)^2\right] \leqslant & \mathbf{E}\left[\sup_{t\in[0,T]} (X_t^{N,1} - \bar{X}_t^{N,1})^2 + (\lambda_t^{N,1} - \bar{\lambda}_t^{N,1})^2\right] \\ \leqslant & L\int_0^T \mathbf{E}\left[\sup_{t\in[0,s]} (X_t^{N,1} - \bar{X}_t^{N,1})^2 + (\lambda_t^{N,1} - \bar{\lambda}_t^{N,1})^2\right] ds \\ & + \alpha^2 T \int_0^T \mathbf{E}\left[W_{2,s}(\mu^N, \mu)^2\right] ds, \end{aligned}$$

which, by an application of Gronwall's Lemma, implies that there exists $C_T > 0$ such that

$$\mathbf{E}\left[W_{2,T}(\mu^N, \bar{\mu}^N)^2\right] \leqslant C_T \int_0^T \mathbf{E}\left[W_{2,s}(\mu^N, \mu)^2\right] ds. \tag{5.2.12}$$

Moreover, it is well known that $\mathbf{E}\left[W_{2,T}(\bar{\mu}^N, \mu)\right] \leqslant \beta(N)$ for some sequence $\beta(N)$ such that $\lim_{N\to\infty} \beta(N) = 0$. Then, using (5.2.12), we have

$$\begin{aligned} \mathbf{E}\left[W_{2,T}(\mu^N, \mu)^2\right] \leqslant & \mathbf{E}\left[W_{2,T}(\mu^N, \bar{\mu}^N)^2\right] + \mathbf{E}\left[W_{2,T}(\bar{\mu}^N, \mu)^2\right] \\ \leqslant & C_T \int_0^T \mathbf{E}\left[W_{2,s}(\mu^N, \mu)^2\right] ds + \beta(N) \leqslant K_T \beta(N) \end{aligned}$$

for some $K_T > 0$, which concludes the proof. $\square$

5.3 Focus on the Gaussian dynamics

The study of the stability and the long-time behavior of (5.2.10) is particularly hard and we aim to focus on some particular cases. We choose as single site distribution of spins the Normal distribution with mean zero and variance σ^2. As a consequence we restrict the interaction function g to the class of functions such that, there exists a symmetric, nonconstant, convex function h on $\mathbb{R}$ with $g(x) \leqslant h(x)$ for all $x \in \mathbb{R}$ and

$$\int_{\mathbb{R}} e^{ah(x)} e^{-x^2} dx < \infty \text{ for all } a > 0.$$

Moreover, let $D = 0$ and let us consider as initial condition measures of the form

$$\mu_0(dx, d\lambda) = \nu_0(dx) \times \delta_{\lambda_0}(d\lambda),$$

where ν_0 is a square-integrable measure on $\mathbb{R}$ and δ_{λ_0} is a Dirac delta centered in $\lambda_0 \in \mathbb{R}$. This drastically simplifies the treatment. The nonlinear process $(X(t), \lambda(t))_{t \geqslant 0}$ solution of the following nonlinear SDE:

$$\begin{cases} dX_t = \frac{\beta}{2} g'(\lambda_t) dt - \frac{X_t}{2\sigma^2} dt + dB_t, \\ \frac{d\lambda_t}{dt} = -\alpha\lambda_t + \frac{\beta}{2} g'(\lambda_t) - \frac{m_t}{2\sigma^2}, \\ \mu_t = Law(X_t, \lambda_t) \text{ and } m_t = \langle \mu_t(dx, dl), x \rangle, \end{cases} \tag{5.3.1}$$

for $\{B_t\}$ Brownian motion. The evolution of the "perceived magnetization" follows a deterministic dynamics, i.e. for all $t > 0$ the law of the process is such that

$$\mu_t(dx, d\lambda) = \nu_t(dx) \times \delta_{\lambda_t}(d\lambda).$$

Moreover, the resulting process is a Gaussian process, specifically it is completely described by the initial condition μ_0 and the quantities $\{(m_t, V_t, \lambda_t)\}_{t \geqslant 0}$, where $V_t = \mathbf{Var}[X_t]$. In the following we study the stability properties of (5.3.1) and to compare the behavior of the process with and without the dissipation in the perceived magnetization.

5.3.1 The case without dissipation, $\alpha = 0$

We start with the stability study of (5.3.1) without the dissipative term. Let us underline that, in this case, the variable λ_t has the same evolution of m_t, then if $\lambda_0 = \mathbf{E}[X_0]$ the nonlinear process (5.3.1) when $\alpha = 0$, coincides with the nonlinear limit of a sequence of particle systems X^N, each of them evolving according to the Langevin dynamics (5.2.7) w.r.t. the Gibbs measure $\mathbf{P}_{N,\beta}$. Therefore, we may consider this process as the dynamical generalized Curie-Weiss model in the Gaussian case. We restrict the study to the following system of ODE:

$$\begin{cases} \dot{m}_t = \frac{\beta}{2} g'(m_t) - \frac{m_t}{2\sigma_2} \\ \dot{V}_t = 1 - \frac{V_t}{\sigma^2}. \end{cases} \tag{5.3.2}$$

By independence of the two variables, we focus the attention on the one-dimensional ODE for the evolution of m_t. We define the function

$$f_\beta(x) \doteq \beta g'(x) - \frac{x}{\sigma^2} \tag{5.3.3}$$

and notice that, for a fixed $\beta > 0$, the equilibrium points of (5.3.2) belong to the set $\{(x, \sigma^2), x \in \Lambda(\beta)\}$, where

$$\Lambda(\beta) \doteq \{x \in \mathbb{R} \text{ s.t. } f_\beta(x) = 0\}.$$

We call **phase transition** any change in the number of equilibrium points, according to the following definition.

Definition 5.3.1. We say that the system (5.3.2) has a **phase transition** in β^* if, for any neighborhood of β^* the cardinality of the set $\Lambda(\beta)$ is not constant.

In the following proposition we study phase transitions and stability of the system 5.3.2, by focusing on its coordinate m, whose evolution and stability points are less trivial than the ones for V.

Proposition 5.3.1. *Consider the dynamics*

$$\dot{m}_t = \frac{\beta}{2} g'(m_t) - \frac{m_t}{2\sigma_2}, \tag{5.3.4}$$

for a fixed $\beta > 0$, the set of equilibrium points of (5.3.4) *is given by $\Lambda(\beta)$. Moreover, there exists a nonempty set $\mathsf{CV} = \{0 < \beta_1 < \beta_2 < \dots < \beta_i < \dots\}$ of critical values for* (5.3.4), *that are either finite in number or countably many and divergent to infinity and such that $\beta \in \mathsf{CV}$ if and only if $\exists\ x\ \mathbb{R}$ for which $f_\beta(x) = 0$ and $f'_\beta(x) = 0$.*

- *If $\beta \notin \mathsf{CV}$, let $m^* \geqslant 0$ be a non-negative equilibrium point. Then the two points $\pm m^*$ are asymptotically stable (resp. unstable) if there exists $\epsilon > 0$ such that $f'_\beta(y) < 0$ (resp. > 0) for all $y \in (m^* - \epsilon, m^* + \epsilon)$.*
- *If $\beta \in \mathsf{CV}$, let $m^* \geqslant 0$ be a non-negative equilibrium point such that $f'_\beta(m^*) = 0$. If there exists $\epsilon > 0$ such that $f_\beta(x) \geqslant 0$ for all $x \in (m^* - \epsilon, m^* + \epsilon)$, then m^* is asymptotically stable from the left and unstable from the right, while the opposite happens for $-m^*$ (obviously everything is inverted if $f_\beta(x) \leqslant 0$ for all $x \in (m^* - \epsilon, m^* + \epsilon)$). The stability of all the other equilibrium points $x^* \in \Lambda(\beta)$ such that $f'_\beta(x^*) \neq 0$ follows the above description.*

Proof. Obviously, since (5.3.4) can be rewritten as

$$\dot{m}_t = \frac{1}{2} f_\beta(m_t),$$

its equilibrium points and their stability depend on the function f_β and on its zeros. It is clear that, since g' is globally Lipschitz continuous, for β sufficiently small the origin is the only equilibrium points and it is a global attractor. Moreover, the set CV contains all the values of β such that the line $y = \frac{x}{\sigma^2\beta}$ is tangent to the graph $y = g'(x)$. By the property (5.2.4) we get that the set CV is not empty, while its cardinality depends on the regularity of the function g. The stability of the equilibrium points of (5.3.4) follows from a standard analysis of the sign of the function f_β. □

By means of Proposition 5.3.1, we plan to study stability and long-time behavior of the solution to (5.3.1) when $\alpha = 0$. To this aim, we state and prove a lemma concerning the long-time behavior of the following time-inhomogeneous SDE:

$$\begin{cases} dY_t = a(t)dt - \frac{Y_t}{2\sigma^2}dt + dB_t \\ Y_0 \in L^2(\Omega) \end{cases} \tag{5.3.5}$$

where $a(t)$ is a deterministic function such that

$$\lim_{t\to+\infty} a(t) = a^* \in \mathbb{R}. \tag{5.3.6}$$

The solution of (5.3.5) will be used as an auxiliary process to prove long-behavior of the solution of (5.3.1). When $\alpha = 0$, let $\{X_t\}_{t\geqslant 0}$ be the first component of a solution to (5.3.1) and $\{Y_t\}_{t\geqslant 0}$ be the solution to (5.3.5). Then, if $\mathsf{Law}(X_0) = \mathsf{Law}(Y_0)$ and

$$\begin{cases} a(t) = \frac{\beta g'(m_t)}{2}, \\ \dot{m}_t = \frac{\beta}{2} g'(m_t) - \frac{m_t}{2\sigma_2}, \end{cases}$$

with $m_0 = \mathbf{E}[X_0]$, then, for all $t > 0$

$$\mathsf{Law}(X_t) = \mathsf{Law}(Y_t).$$

The same argument holds true when $\alpha > 0$, replacing m_t with λ_t.

Lemma 5.3.1. *Let* $\{Y_t\}$ *be the solution of* (5.3.5) *and* $P_t(Y_0,\cdot)$ *be its law. Then,*

$$\lim_{t\to+\infty} \|P_t(Y_0,\cdot) - \nu_{a^*}(\cdot)\|_{TV} = 0$$

where $\nu_{a^*}(dx) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(x-a^*)^2}{2\sigma^2}} dx.$

Proof. First of all, notice that

$$\lim_{t\to+\infty} \left| \int_0^t e^{\frac{s-t}{2\sigma^2}} a(s)ds - a^* \right| = 0. \tag{5.3.7}$$

In fact, fix $\varepsilon > 0$, then by (5.3.6), there exists t^*_ε such that $|a(t) - a^*| < \varepsilon$ for any $t > t^*_\varepsilon$. So,

$$\begin{aligned} \left| \int_0^t e^{\frac{s-t}{2\sigma^2}} a(s)ds - a^* \right| &\leqslant e^{-\frac{t}{2\sigma^2}} \int_0^t e^{\frac{s}{2\sigma^2}} |a(s) - a^*| ds \\ &= e^{-\frac{t}{2\sigma^2}} \int_0^{t^*_\varepsilon} e^{\frac{s}{2\sigma^2}} |a(s) - a^*| ds + e^{-\frac{t}{2\sigma^2}} \int_{t^*_\varepsilon}^t e^{\frac{s}{2\sigma^2}} \varepsilon\sigma^2 ds \\ &\leqslant e^{-\frac{t}{2\sigma^2}} t^*_\varepsilon e^{\frac{t^*_\varepsilon}{2\sigma^2}} \max_{s\in[0,t^*_\varepsilon]} |a(s) - a^*| + \frac{\varepsilon}{2}, \end{aligned}$$

then, taking t^{**}_ε such that

$$e^{-\frac{t^{**}_\varepsilon}{2\sigma^2}} t^*_\varepsilon e^{\frac{t^*_\varepsilon}{2\sigma^2}} \max_{s\in[0,t^*_\varepsilon]} |a(s)-a^*| < \frac{\varepsilon}{2},$$

for any $t > t^{**}_\varepsilon$ it holds that

$$\left| \int_0^t e^{\frac{s-t}{2\sigma^2}} a(s)ds - a^* \right| < \varepsilon$$

and (5.3.7) is proved. By the theory of linear stochastic differential equations it's well-known that

$$(Y_t|Y_0 = y) \sim \mathcal{N}\left(ye^{-\frac{t}{2\sigma^2}} + \int_0^t e^{\frac{s-t}{2\sigma^2}} a(s)ds, \sigma^2\left(1-e^{-\frac{t}{\sigma^2}}\right)\right),$$

then, if $\mu_0(\cdot) = Law(Y_0)$,

$$\begin{aligned}
||P_t(Y_0,\cdot) - \nu_{a^*}(\cdot)||_{TV} &= \int_{\mathbb{R}} ||P_t(y,\cdot) - \nu_{a^*}(\cdot)||_{TV} d\nu_0(y) \\
&= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \left| \frac{\exp\left(\frac{\left(x-ye^{-\frac{t}{2\sigma^2}} - \int_0^t e^{\frac{s-t}{2\sigma^2}} a(s)ds\right)^2}{2\sigma^2(1-e^{-\frac{t}{2\sigma^2}})} \right)}{\sqrt{2\pi\sigma^2(1-e^{-\frac{t}{2\sigma^2}})}} - \frac{e^{\frac{(x-a^*)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \right| dx d\nu_0(y)
\end{aligned}$$

which converges to 0 as $t \to +\infty$ thanks to (5.3.7) and the Dominated Convergence Theorem. □

We are now ready to prove the result on the stability points and long-time behavior of the Markov process (X_t, λ_t), which is solution to (5.3.1) when $\alpha = 0$. Let us define what we mean by **phase transition** for this dynamical generalized Curie-Weiss model.

Definition 5.3.2. Let β^* be a value such that, for any neighborhood of β^*, the number of stationary measures for (5.3.1) when $\alpha = 0$ is not constant. Then, we say that β^* is a **phase transition** for the dynamical generalized Curie-Weiss model in the Gaussian case.

Theorem 5.3.1. *The process (X_t, λ_t) described by (5.3.1) has a* ***phase transition*** *as defined in Definition 5.3.2, for any $\beta \in CV$. Fix $\beta > 0$, then (X_t, λ_t) has exactly $\mathrm{Card}(\Lambda(\beta))$ stationary solution given by the measures*

$$\mu^*_m(dx, dl) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx \times \delta_m(dl)$$

for all $m \in \Lambda(\beta)$. Moreover, for all $\mu_0(dx, d\lambda) = \nu_0(dx) \times \delta_{m_0}(d\lambda)$, square-integrable initial conditions with $m_0 = \langle \mu_0, x\rangle$

$$\lim_{t\to\infty} ||\mu_t(\cdot) - \mu^*_m(\cdot)||_{TV} = 0, \tag{5.3.8}$$

where m is the equilibrium point of (5.3.4) such that m_0 belongs to the domain of attraction of m.

Proof. It is clear that the evolution given by (5.3.1) when $\alpha = 0$ must have a law $\mu_t(dx, dl) = \nu_t(dx) \times \delta_{m_t}(dl)$ where δ_{m_t} is a Dirac delta centered in $m_t = \int_{\mathbb{R}} y\nu_t(dy)$. Then the stationary Fokker-Planck gives

$$0 = \frac{1}{2}\frac{d^2}{dx^2}\nu^*(x) - \frac{d}{dx}\left[\left(\frac{\beta}{2}g'(m^*) - \frac{x}{2\sigma^2}\right)\nu^*(x)\right]$$

with $m^* = \int_{\mathbb{R}} x\nu^*(x)dx$ and $\mu^*(dx, dl) = \nu^*(dx)\delta_{m^*}(dl)$. Then, there exists $K \in \mathbb{R}$ such that

$$\frac{d}{dx}\nu^*(x) = K + \left(\frac{\beta}{2}g'(m^*) - \frac{x}{2\sigma^2}\right)\nu^*(x).$$

Thus $\nu^*(x)$ solves a linear ODE, i.e. there exists $C \in \mathbb{R}$ such that

$$\nu^*(x) = \exp(\beta g'(m)x - \frac{x^2}{2\sigma^2})\left(C + K\int_{\mathbb{R}} \exp(-\beta g'(m)y + \frac{y^2}{2\sigma^2})dy\right).$$

Let us define the values of the constants:

- $K = 0$, indeed, when $K \neq 0$ $\nu^*(x)$ is not integrable;
- $C = \int_{\mathbb{R}} \exp(\beta g'(m)x - \frac{x^2}{2\sigma^2})dx$, such that $\nu^*(x)$ is the density function of a random variable.

The admissible functions ν^* are such that

$$m^* = \int_{\mathbb{R}} x\nu^*(x)dx = \beta\sigma^2 g'(m^*),$$

this identifies them as the ones corresponding to $m^* \in \Lambda(\beta)$.
Now, let us prove the long-time behavior of μ_t for any square-integrable initial condition of the type $\mu_0 = \nu_0 \times \delta_{m_0}$. As we said, this implies that $\mu_t = \nu_t \times \delta_{\lambda_t}$ and $\lambda_t = \mathbf{E}[X_t]$ for all $t > 0$. We introduce an auxiliary process Y_t, solution of

$$dY_t = a(t)dt - \frac{Y_t}{2\sigma^2}dt + dB_t$$

with initial condition $Y_0 \sim \mu_0$ and $a(t) = \frac{\beta g'(\lambda_t)}{2}$ for all $t \geqslant 0$. Denoting $P_t(Y_0, \cdot) = Law(Y_t)$, it is clear that

$$\|P_t(Y_0, \cdot) - \nu_t(\cdot)\|_{TV} = 0,$$

for all $t \geqslant 0$. Then (5.3.8) follows directly from Lemma 5.3.1. □

Let us compare the dynamical Curie-Weiss model we described here with its static counterpart. We recall that we are considering a situation in which the single site distribution of spin is Gaussian $\sim \mathcal{N}(0, \sigma^2)$. This implies that, in this case, (5.2.5) becomes

$$-\beta\psi(\beta) = \sup_{x\in\mathbb{R}}\left\{\beta g(x) - \frac{x^2}{2\sigma^2}\right\}.$$

According to Definition 5.2.1 and Theorem 5.2.1 the phase transitions of the generalized Curie-Weiss model depend on the points in which the supremum of the function

$$F_\beta(x) \doteq \beta g(x) - \frac{x^2}{2\sigma^2}$$

is attained. The dynamical approach does not differ too much in a sense that we are interested in the local maxima and minima of the function $F_\beta(x)$ instead that in its supremum. Indeed, it is clear that the function $f_\beta(x)$ defined in (5.3.3) coincides with the first derivative of $F_\beta(x)$. According to Definition 5.3.2, a critical value a value $\bar{\beta}$ for the dynamical model is such that, for any neighborhood of $\bar{\beta}$, the number of minima and maxima of the function F_β is not constant. It is clear that the two sets of critical values CV and $\mathcal{P}$ may be very different, but it is interesting to keep in mind their link through the function F_β .

5.3.2 The case with dissipation, $\alpha > 0$

Let us now focus on the system in presence of a dissipative behavior for λ_t. We again reduce the problem to the study of a system of ODE, that is the following:

$$\begin{cases} \dot{m}_t = \frac{\beta}{2} g'(\lambda_t) - \frac{m_t}{2\sigma_2}, \\ \dot{\lambda}_t = -\alpha\lambda_t + \frac{\beta}{2} g'(\lambda_t) - \frac{m_t}{2\sigma^2}, \end{cases} \tag{5.3.9}$$

where, as before, the independence of the evolution of V_t let us consider a two-dimensional instead of a three-dimensional system. We consider a simple change of variable $y = \frac{1}{2\sigma^2}(\lambda - m)$, then we get the system

$$\begin{cases} \dot{y}_t = -\frac{\alpha}{2\sigma^2}\lambda_t, \\ \dot{\lambda}_t = y_t - \left(\alpha + \frac{1}{2\sigma^2}\right)\lambda_t + \frac{\beta}{2} g'(\lambda_t), \end{cases} \tag{5.3.10}$$

which is a Liénard system. The link with Liénard systems is important; indeed, among planar differential equations, the systems of this class have been extensively studied, in particular in relation to their limit cycles, [19, 22, 46, 61, 71, 76]. A system of Liénard type has the following form:

$$\begin{cases} \dot{x} = y - A(x), \\ \dot{y} = -b(x), \end{cases}$$

for two suitable functions A, b. The usual hypothesis require that $a = A'$ and b are C^1 functions, $b(0) = 0$ and $b(x)x > 0$ for x small enough. A detailed and complete study of all Liénard systems, with necessary and sufficient conditions for the existence of exactly $k \geqslant 0$ limit cycles, is still an open problem. However, in literature we can find sufficient conditions for the existence of *at least*(or *exactly*) $k \geqslant 0$ limit cycles, [22, 71]. In this setting, by a slight abuse of notation, we define the function

$$f_{\alpha,\beta}(x) := \left(\alpha + \frac{1}{2\sigma^2}\right) x - \frac{\beta}{2} g'(x);$$

of course, this generalizes (5.3.3), indeed $f_{0,\beta} = f_\beta$. For any fixed triplet of parameters $(\alpha, \beta, \sigma^2)$, it is clear that (5.3.10) is a Liénard system with $A(x) = f_{\alpha,\beta}(x)$ and $b(x) = \frac{\alpha}{2\sigma^2}x$.

In this case, by **phase transition** we mean any change in the number or in the stability of equilibrium points and limit cycles of the ODE (5.3.10). In the following theorem we depict three possible phases of the system and we give sufficient conditions for them to occur.

i) We can always find a regime of the parameters in which the origin is a global attractor and there is not any limit cycle.

ii) Under a simple condition on the derivative of the interaction function, we may find a critical value in which the origin looses its local stability and a stable limit cycle bifurcates from it.

iii) If the previous situation occurs and the interaction function is sufficiently regular at infinity, we can find a regime in which there exists a unique limit cycles, which is attractive.

Let us explain in details what are the conditions under which the above situations are possible.

Theorem 5.3.2. *Fix* $\sigma^2 > 0$ *and* $\alpha > 0$ *and consider the dynamical system* (5.3.10) *under Assumptions 5.2.1 and 5.2.2.*

i) There exists $\beta^* > 0$ *such that* $\forall\ \beta \in (0, \beta^*)$ *the origin is a global attractor for* (5.3.10).

ii) If $g''(0) > 0$, *the origin looses stability via a Hopf bifurcation at the critical value* $\beta_H = \frac{2\alpha + \frac{1}{\sigma^2}}{g''(0)}$.

iii) If $g''(0) > 0$ *and there exists* $C > 0$ *such that for all* $x \in (C, \infty)$ *the function* $g'(x)$ *is concave, then there exists a* β_{UC} *such that for all* $\beta > \beta_{UC}$ *there exists a unique limit cycles for* (5.3.10).

Proof. i) The strategy consists in finding a Lyapunov function (in the sense of dynamical systems, in contrast with Section 3.2.1) for the system (5.3.10). Let us consider the function

$$W(y, \lambda) = \frac{\alpha}{4\sigma^2}\lambda^2 + \frac{y^2}{2},$$

it is clear that

$$\frac{d}{dt}W(y(t), \lambda(t)) = -\frac{\alpha}{2\sigma^2}\lambda\left((\alpha + \frac{1}{2\sigma^2})\lambda - \frac{\beta}{2}g'(\lambda)\right) = -\frac{\alpha}{2\sigma^2}\lambda f_{\alpha,\beta}(\lambda). \tag{5.3.11}$$

As in the proof of Proposition 5.3.1, the problem reduces to consider the intersection of the graph of the function

$$y = g'(\lambda)$$

with a line, that in this case is the line

$$y = \frac{2\alpha + \frac{1}{\sigma^2}}{\beta}\lambda.$$

This, indeed, determines the sign of the function $f_{\alpha,\beta}(\lambda)$. We see that there exists a β^* sufficiently small, such that $\forall\ \beta < \beta^*$ the only intersection is the origin, meaning that (5.3.11) is strictly negative except than at $(0,0)$, in which it is zero. Therefore W is a global Lyapunov function for the system (5.3.10), proving global attractivity of the origin.

ii) A Hopf bifurcation occurs when a stable periodic orbit arises from an equilibrium point that loses its (local) stability. Such a bifurcation can be detected looking at the linearized system around this stable equilibrium and finding the values of the parameters for which a pair of complex eigenvalues crosses the imaginary axis [73, Theorem 2, Chapter 4.4]. Therefore, we consider the system (5.3.10) linearized around the point $(0,0)$, that gives the linear system:

$$\begin{pmatrix} \dot{y} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\alpha}{2\sigma^2} \\ 1 & -\left(\alpha + \frac{1}{2\sigma^2}\right) + \frac{\beta}{2} g''(0) \end{pmatrix} \begin{pmatrix} y \\ \lambda \end{pmatrix}$$

with eigenvalues

$$x_\pm = \frac{1}{2}\left(\frac{\beta}{2}g''(0) - \alpha - \frac{1}{2\sigma^2} \pm \sqrt{\left(\frac{\beta}{2}g''(0) - \alpha - \frac{1}{2\sigma^2}\right)^2 - \frac{2\alpha}{\sigma^2}}\right).$$

It is clear that, when $\beta = \beta_H$, we have a Hopf bifurcation.

iii) Under these hypothesis, there exists a β_{UC} sufficiently large such that $\forall\ \beta > \beta_{UC}$, the function $f_{\alpha,\beta}$ has exactly three zeros $-x^* < 0 < x^*$ and satisfies the following: $f_{\alpha,\beta}$ is negative on $(0, x^*)$ and positive and monotonically increasing on (x^*, ∞). In this way, for all $\beta > \beta_{UC}$, the system (5.3.10) satisfies the conditions for the existence and uniqueness of a limit cycle presented in Theorem 1.1 of [19]. The proof follows the usual approach for Liénard systems, used also in [26].
First it is shown that the y axis and the function $y = f_{\alpha,\beta}(\lambda)$ divides the (λ, y)-plane in four regions:

$$\begin{aligned} \mathrm{I} &\doteq \{(\lambda, y) \colon \lambda > 0; y > f_{\alpha,\beta}(\lambda)\}; \\ \mathrm{II} &\doteq \{(\lambda, y) \colon \lambda > 0; y < f_{\alpha,\beta}(\lambda)\}; \\ \mathrm{III} &\doteq \{(\lambda, y) \colon \lambda < 0; y < f_{\alpha,\beta}(\lambda)\}; \\ \mathrm{IV} &\doteq \{(\lambda, y) \colon \lambda < 0; y > f_{\alpha,\beta}(\lambda)\}. \end{aligned}$$

In each of these four regions the vector field pushes the trajectories to cross either the y axis or the graph $y = f_{\alpha,\beta}(\lambda)$. Therefore each trajectory is forced to revolve clockwise around the origin.

Then, for $y_0 > 0$, we consider a trajectory starting from the point $(0, y_0)$ and we call $y_1 > 0$ its first intersection with the y-axis in the negative half-plane. We define the function

$$\Delta W(y_0) = W(0, y_1) - W(0, y_0).$$

Of course, when $\Delta W(\bar{y}) = 0$, the trajectory starting from $(0, \bar{y})$ is a periodic orbit. Let $y_0^* > 0$ be such that the trajectory starting from $(0, y_0^*)$ passes through $(x^*, 0)$, the positive zero of the function $f_{\alpha,\beta}(\lambda)$. It is possible to prove that $\Delta W(y) > 0$ for all $y \leqslant y_0^*$. Then $\Delta W(y)$ decreases monotonically to $-\infty$, when $y \to \infty$, meaning that there exists a unique $\bar{y}$ for which it is zero. □

As in the case without dissipation, the results on the dynamical system (5.3.9) immediately extend to the Markov process (X_t, λ_t) solution to (5.3.1). In this case, a periodic orbit will be a set of measures, which does not contain a single measure and it is invariant under the dynamics. We define as **phase transition** any change in the number of these disjoint invariant sets and in the long-time behavior of the process.

Theorem 5.3.3. *Fix $\alpha, \beta > 0$, then the process (X_t, λ_t) described by (5.3.1) has exactly one stationary solution given by the measures*

$$\mu^*_{(0,0)}(dx, dl) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \times \delta_0(dl).$$

Let γ be a limit cycle of (5.3.9), then the set

$$\Gamma = \left\{ \mu^*_{(m,\lambda)}(dx, dl) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx \times \delta_\lambda(dl), \quad \textit{for all } (m, \lambda) \in \gamma \right\}$$

is an invariant set for the dynamics (5.3.1). Moreover, for all $\mu_0(dx, d\lambda) = \nu_0(dx) \times \delta_{\lambda_0}(d\lambda)$, square-integrable initial conditions with $\lambda_0 \in \mathbb{R}$,

$$\lim_{t\to\infty} \inf_{(m,\lambda)\in\gamma} \|\mu_t(\cdot) - \mu^*_{(m,\lambda)}(\cdot)\|_{TV} = 0,$$

where γ is the attractor of the trajectory starting from $(\langle \mu_0, x \rangle, \lambda_0)$ in the dynamical system (5.3.9); here with γ we mean either a limit cycle or simply the origin.

Proof. The proof follows the same approach of the proof of Theorem 5.3.1, using the stationary Fokker-Planck equation and the auxiliary results on long-time behavior given by Lemma 5.3.1. □

Theorem 5.3.2 together with Theorem 5.3.3 show that the generalized Curie-Weiss model with dissipation (5.3.1), at least in this Gaussian case, may undergo different phases. In particular we find a combination of parameters under which an unstable stationary measure $\mu^*_{(0,0)}$ coexists with an attractive periodic orbit, which is unique. However, the framework depicted by these results is far from being complete and we will show that several other phases may be recreated in a generalized Curie-Weiss model with dissipation, by suitably choosing the interaction function g and the parameters.

5.3.3 Coexistence of limit cycles

When one introduces dissipation in a classical Curie-Weiss model [26], a Hopf bifurcation identifies the transition from disorder to a phase in which a unique globally stable limit cycle is present. The model introduced here generalizes this scheme: as seen in Theorem 5.3.2, a Hopf bifurcation may occur but, according to the form of the interaction function $g(x)$, the limiting dynamics may display a richer behavior. Let us summarize some of the most interesting situation that may occur in this model.

- More than one periodic orbit may coexist and they all revolve around the origin. In this case the outer one should be stable, the second should be unstable and then they should alternate.
- Some periodic orbits may appear even when the origin is still locally stable. These orbits appears through global bifurcations (the Hopf bifurcation is a local one) and they usually appear in pairs, the outer periodic orbit is stable, while the inner one is unstable.

The number of limit cycles and their stability mainly depends on the function $f_{\alpha,\beta}(x)$, which plays a key role in the study of a Liénard system such as (5.3.10). In general, some tools to determine the exact number of limit cycles in a Liénard system are available in literature (see [22, 71] and references therein). However, their application may be cumbersome in a general setting, since several features of the function $f_{\alpha,\beta}(x)$ should be studied, such as the position of its zeroes, its local minima and maxima, their height and so on. Nevertheless, playing with the form of the interaction function g, we can always create a Gaussian Curie-Weiss model with dissipation with a customized number of phase transitions and of coexisting limit cycles.

Let us briefly underline the role of the function g in the occurrence of limit cycles in the dynamics of (5.3.9). To this aim, we rewrite the Liénard system (5.3.10):

$$\begin{cases} \dot{y}_t = -\frac{\alpha}{2\sigma^2}\lambda_t, \\ \dot{\lambda}_t = y_t - f_{\alpha,\beta}(\lambda_t). \end{cases}$$

In the rich literature on Liénard system, we see that the form of the function $f_{\alpha,\beta}$ plays a fundamental role in the number of limit cycles of the system. In particular, from the results in [71], we can state the following.

Proposition 5.3.2. *Fix $\sigma^2, \alpha > 0$ and suppose that there exists a β^* such that the following conditions are satisfied:*

i) the function f_{α,β^*} has N positive zeros $x_0 := 0 < x_1 < \cdots < x_N (< x_{N+1}$ a bound $)$ at which it changes sign;

ii) for every $k = 1, \dots, N$ there is a C^1 mapping $\phi_k : [x_{k-1}, x_k] \to [x_k, x_{k+1}]$ such that

$$\phi_k(x)\phi_k'(x) \geqslant x \quad \textit{and} \quad |f_{\alpha,\beta^*}(\phi_k(x))| \geqslant |f_{\alpha,\beta^*}(x)|;$$

iii) the function f_{α,β^*} *on each interval* $[x_{k-1}, \phi_{k-1}(x_{k-1})]$ *for* $2 \leqslant k \leqslant N+1$ *has an extremum at a unique point* y_k *and its derivative is weakly monotone.*

Then the generalized Curie-Weiss model with dissipation has at least one regime in which it has exactly N *limit cycles. The outer cycle is stable, then the others alternate as unstable and stable, respectively.*

The proof of this result is a simple application of the results in [71]. It is easy to see that the function

$$f_{\alpha,\beta}(x) = \left(\alpha + \frac{1}{2\sigma^2}\right)x - \frac{\beta}{2}g'(x)$$

depends on the choice of the interaction function g. Since Assumptions 5.2.1 and 5.2.2 are not very restrictive, g can be manipulated to obtain a system that admits a regime with the desired number of limit cycles. It is clear that, when the origin is stable, with Proposition 5.3.2 we can create an even number of periodic orbits, half of them stable and half of them unstable. On the other hand, if the origin is unstable, we can create an odd number of periodic orbits, such that the inner and the outer are both stable, while the others alternate.

Let us highlight the links with the model without dissipation. One may think that the existence of periodic orbits in the Liénard system (5.3.10) depends only on the zeros of the function $f_{\alpha,\beta}(x)$. By the form of this function, this would lead to a direct comparison with the phase transitions in the model without dissipation and its critical values CV. Therefore, let $\beta^* \in CV$ be a critical value for (5.3.2) such that two equilibrium points appear (one stable and one unstable). One could immagine that the value

$$\beta^*_\alpha \doteq \beta^*(1 + 2\alpha\sigma^2) \tag{5.3.12}$$

is a critical value for the system (5.3.10) and it is such that two periodic orbit appear, one stable and one unstable. Unfortunately, this is true only when we the origin bifurcates in two stable points (when $\alpha = 0$) or in one stable periodic orbit (the Hopf bifurcation when $\alpha > 0$). In all the other case, the critical values in CV could not be obtained by choosing $\alpha = 0$ in the dissipated case. By numerical evidence, for fixed $\alpha, \sigma^2 > 0$, we see that the emergence of two periodic orbits occurs for a value of $\mathtt{beta}$ slightly greater than the one expected from (5.3.12), while the disappearance of two periodic orbits occurs at smaller values of β than expected. We suppose that this is linked to conditions as the points *ii)* and *iii)* of Proposition 5.3.2.

5.3.4 A suitable interaction function for the coexistence of periodic orbits

By means of an explicit example, we show how we can manipulate the interaction function g in order to observe the coexistence of two stable limit cycles. Let us define the function

$$g(x) = \tanh\left(ax^2 + bx^4 + cx^6\right),$$

with a, b, c suitable constants such that g stays strictly increasing on $[0, \infty)$. Fix $\sigma^2 > 0$, then the pair (g, ρ), with $\rho \sim \mathcal{N}(0, \sigma^2)$ clearly satisfies Assumptions 5.2.1 and 5.2.2 and it defines a generalized Curie-Weiss model. We consider two triplets of constants $(1/2, -1, 2)$ and $(1,1,0)$ in order to observe some particular regimes that do not exist for the classical Curie-Weiss model with dissipation.

Case A: triplet $(\frac{1}{2}, -1, 2)$

We see from Figure 5.1 the changes in the concavity of $g'(x)$. This causes, in the dynamics without dissipation, three critical values of β and the four following regimes:

- for $\beta < \beta_1$ the origin is a global attractor;
- for $\beta \in (\beta_1, \beta_2)$ the origin is locally stable, but there are four other equilibrium points $-x_2 < -x_1 < 0 < x_1 < x_2$, such that $\pm x_2$ are stable and $\pm x_1$ are unstable;
- for $\beta \in (\beta_2, \beta_3)$ the origin becomes unstable and two additional stable equilibrium points appear, $\cdots - x_1 < -x_3 < 0 < x_3 < x_1 \ldots$;
- for $\beta = \beta_3$ the pairs of equilibrium points $\{x_3, x_1\}$ and $\{-x_3, -x_1\}$ collapse and disappear, such that for $\beta > \beta_3$ there are three equilibrium points $-x_2 < 0 < x_2$, the outer two are stable and the origin is unstable.

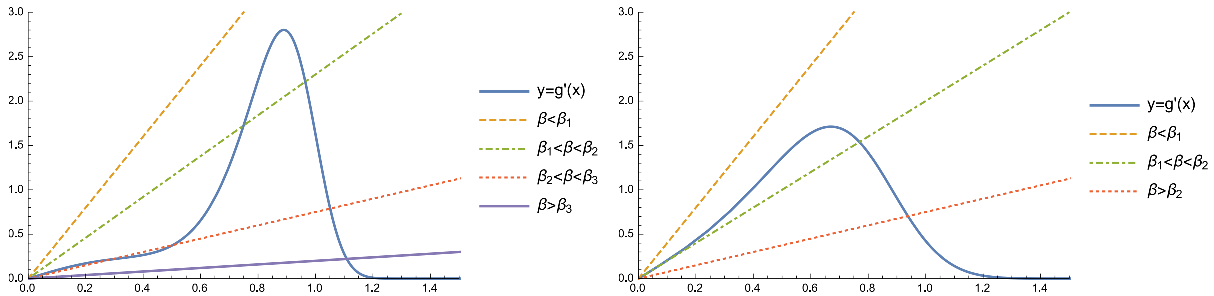

Figure 5.1: The plot of the function g' and of lines $y = \frac{1}{\beta\sigma^2}x$ for different values of β. The number of intersections gives the number of equilibrium points in the positive axes. Left: the case A. Right: the case B.

The exact critical values may be obtained numerically, and the behavior of the dynamical system is clear from Proposition 5.3.1. As we expect, in this case the dissipated dynamics (5.3.9) actually shows four different regimes as well, but the critical values $\hat{\beta}_1(\alpha)$, $\hat{\beta}_2(\alpha)$, $\hat{\beta}_3(\alpha)$ are not straightforwardly obtained with the same procedure of the elements of CV. To be precise, if β_1 corresponds to the smallest value of β in which the line $y = \frac{x}{\sigma^2\beta}$ is tangent to the graph of $y = g'(x)$, the value $\hat{\beta}_1(\alpha)$ is strictly greater than the smallest value of β such that the line $y = \frac{2\alpha+\frac{1}{\sigma^2}}{\beta}x$ is tangent to the graph of $y = g'(x)$. This means that there exists a β^* such that the line $y = \frac{2\alpha+\frac{1}{\sigma^2}}{\beta^*}x$ intersects the graph of $y = g'(x)$

but any limit cycle occurs. Nevertheless, the system displays a regime of *coexistence of stable limit cycles*. Let us better explain the four regimes that we observe in system (5.3.9) (actually the computations and the plots refer to system (5.3.10), since the link with the function $f_{\alpha,\beta}$ is more clear in this case).

- For $\beta < \hat{\beta}_1(\alpha)$ the origin is a global attractor. Notice that, numerically we can see that $\hat{\beta}_1(\alpha)$ is greater that the β^* obtained in Theorem 5.3.2; indeed it is not necessary that the function $f_{\alpha,\beta}(x)$ is strictly greater than zero for all $x > 0$. It is reasonable to believe, see Figure 5.2(a), that for those β such that the negative part of the function is "small enough" do not necessary give rose to a periodic orbit.

- For $\beta \in (\hat{\beta}_1(\alpha), \hat{\beta}_2(\alpha))$ the origin is locally stable and, through a global bifurcation, two periodic orbits have arised, the larger one is stable and the smaller one is unstable. Indeed, we can prove numerically that there exists a value $\beta^* \in (\hat{\beta}_1(\alpha), \hat{\beta}_2(\alpha))$ such that the function f_{α,β^*} satisfies the conditions of Theorem A and B in [71] for the existence of exactly two limit cycles. See Figure 5.2(b) in which we can observe the stable outer cycle and the attractivity of the origin.

- Notice that $\beta_H = \hat{\beta}_2(\alpha)$ from Theorem 5.3.2. Therefore, for $\beta \in (\hat{\beta}_2(\alpha), \hat{\beta}_3(\alpha))$ we observe two attractive limit cycles, a smaller one spreading from the origin (that is now unstable) while the bigger one remains from the previous regime: see Figure 5.2(c). The basin of attraction of the two stable orbits are separated by a third unstable periodic orbit. This is the regime in which we see the coexistence of two stable periodic orbits, one inside the other. The existence of this regime is again a consequence of Theorem A and B in [71], because we can numerically find a β^{**} that satisfies the hypothesis for the existence of exactly three limit cycles, two stable and one unstable.

- For $\beta > \hat{\beta}_3(\alpha)$ we see that only the largest periodic orbit has survived. Indeed, for in $\hat{\beta}_3(\alpha)$ the smallest stable orbit and the unstable one collapse and disappear. Of course, we see that this $\hat{\beta}_3(\alpha) = \beta_{UC}$ defined in Theorem 5.3.2; but from numerical evidence we suppose that for this value of β the function $f_{\alpha,\beta}$ has more than one single zero in the positive half-line, but that the other two zeros are not distant enough to admit the existence of the two inner orbits. Of course when β is such that there exists a unique positive zero for $f_{\alpha,\beta}$, we analitically prove the existence and uniqueness of the limit cycle (see Theorem 5.3.2) while for lower values we can only show it numerically, see Figure 5.2(d).

Case B: triplet $(1,1,0)$

We see in Figure 5.1 that the shape of g' basically allows three different regimes for the case without dissipation. Indeed, in (5.3.2), the set CV has cardinality 2, i.e. we have $\beta_1 < \beta_2 = \frac{2}{\sigma^2}$. The three regimes are the following:

- for $\beta < \beta_1$ the origin is a global attractor;

- for $\beta \in (\beta_1, \beta_2)$ there are five equilibrium points $-x_2 < -x_1 < 0 < x_1 < x_2$, where $\pm x_1$ are unstable, while the others are stable;
- at $\beta = \beta_1$ the two points $\pm x_1$ collapse in the origin that becomes unstable, such that for $\beta > \beta_1$ the origin is unstable and the points $\pm x_2$ are stable.

We treat this example in the dissipated case (5.3.9) (by means of the Liénard system (5.3.10)). We expect three regimes and, in particular, we will observe an atypical behavior at the Hopf bifurcation, where we will not have a small limit cycle bifurcating from the origin, but the already existing stable limit cycle that becomes a global attractor. In Figure 5.3 we compare the regimes immediately below and above the Hopf bifurcation.

- For $\beta < \beta_1(\alpha)$, the origin is a global attractor. As is Case A the value $\beta_1(\alpha)$ is strictly greater than the value in which the line first touches the graph $y = g'(x)$.
- For $\beta \in (\beta_1(\alpha), \beta_H)$ the origin is stable and we have an unstable periodic orbit contained in a stable periodic one. When β increase the inner orbit shrinks and the outer expands.
- For $\beta = \beta_H$ the Hopf bifurcation is such that the origin looses stability, but this happens simultaneously to the collapse of the unstable periodic orbit on it. Therefore, after the bifurcation, we do not see the usual periodic orbit expanding form the origin because the unique orbit is the stable one (from the previous regime) that becomes globally stable.

This case is interesting because the Hopf bifurcation do not originates a small periodic orbit. However, the phenomenon is still a local one, because it is a small unstable orbit that collapses on the origin changing its stability.

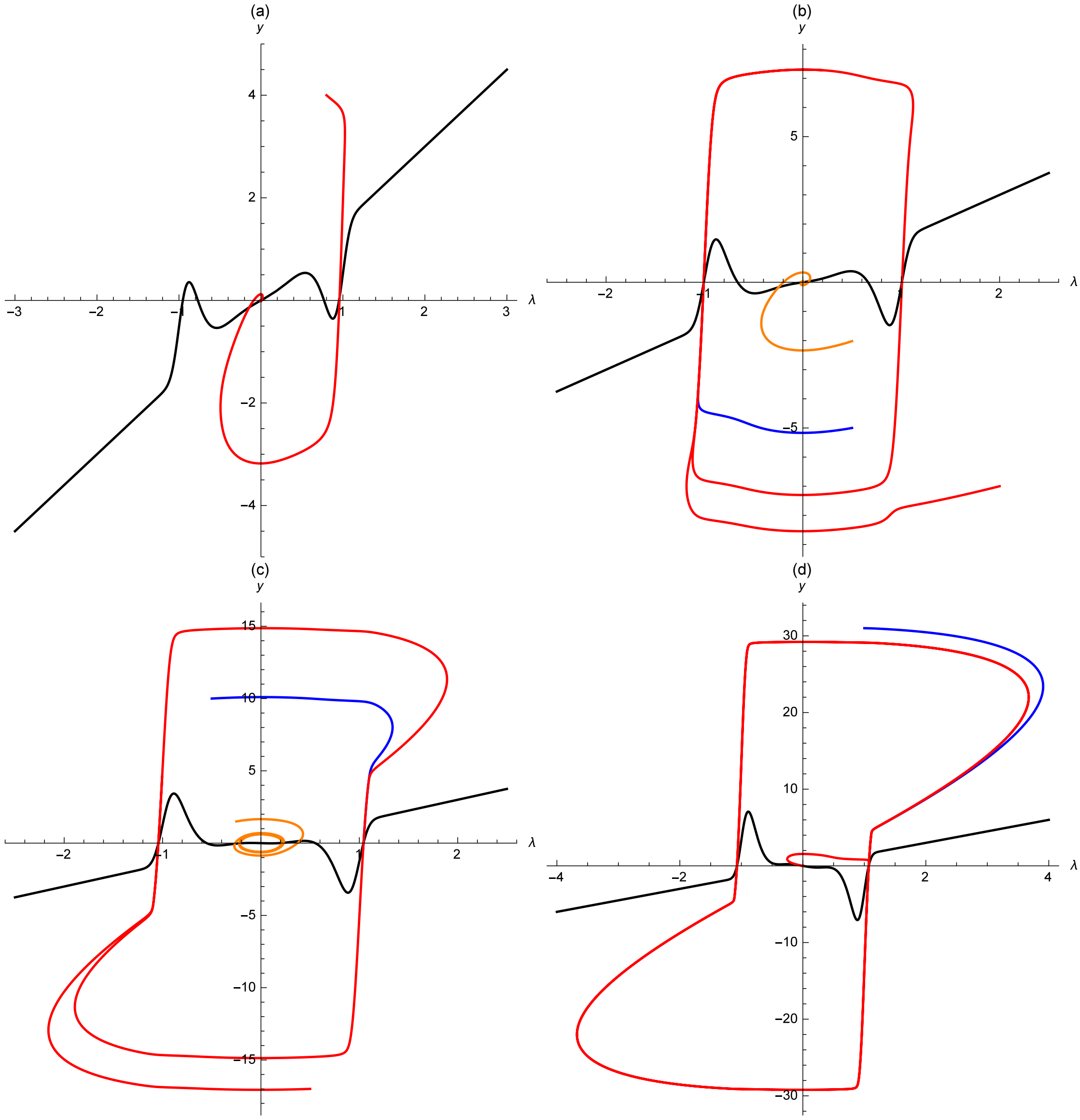

Figure 5.2: Different regimes for case A. In all the pictures, the black line represents the graph of $y = f_{\alpha,\beta}(\lambda)$ and we fixed $\alpha = \sigma^2 = 1$. In (a), the regime $\beta < \hat{\beta}_1(\alpha)$ ($\beta = 1.2$): the red line represents the solution starting from $\lambda(0) = 1$, $y(0) = 4$, which is definitely attracted by the globally stable origin. In (b), the regime $\beta \in (\hat{\beta}_1, \hat{\beta}_2)$ ($\beta = 2$): the red-colored solution, starting from $\lambda(0) = 2$, $y(0) = -7$, and the blue-colored solution, starting from $\lambda(0) = 0.5$, $y(0) = -5$, are attracted by a stable limit cycle. Here, the origin is locally stable (the orange-colored solution with initial condition $\lambda(0) = 0.5$, $y(0) = -2$ is attracted by it) and its basin of attraction is surrounded by an unstable limit cycle. In (c), the regime $\beta \in (\hat{\beta}_2, \hat{\beta}_3)$ ($\beta = 3.4$): the red and blue lines, here representing solution starting from $\lambda(0) = 0.5$, $y(0) = -17$ and $\lambda(0) = -0.5$, $y(0) = 10$ respectively, are again attracted by the outer cycle but now the origin is unstable and another stable cycle is born via the Hopf bifurcation. The orange-colored solution, with initial condition $\lambda(0) = -0.25$, $y(0) = 1.5$, is attracted by the smallest cycle. The basins of attraction of the stable orbits is separated by an unstable cycle. In (d), the regime $\beta > \hat{\beta}_3$ ($\beta = 6$): only the external orbit is survived and it has become globally attractive, as shown by the red and blue lines, with initial conditions $\lambda(0) = 0$, $y(0) = -0.005$ and $\lambda(0) = 1.5$, $y(0) = 31$ respectively.

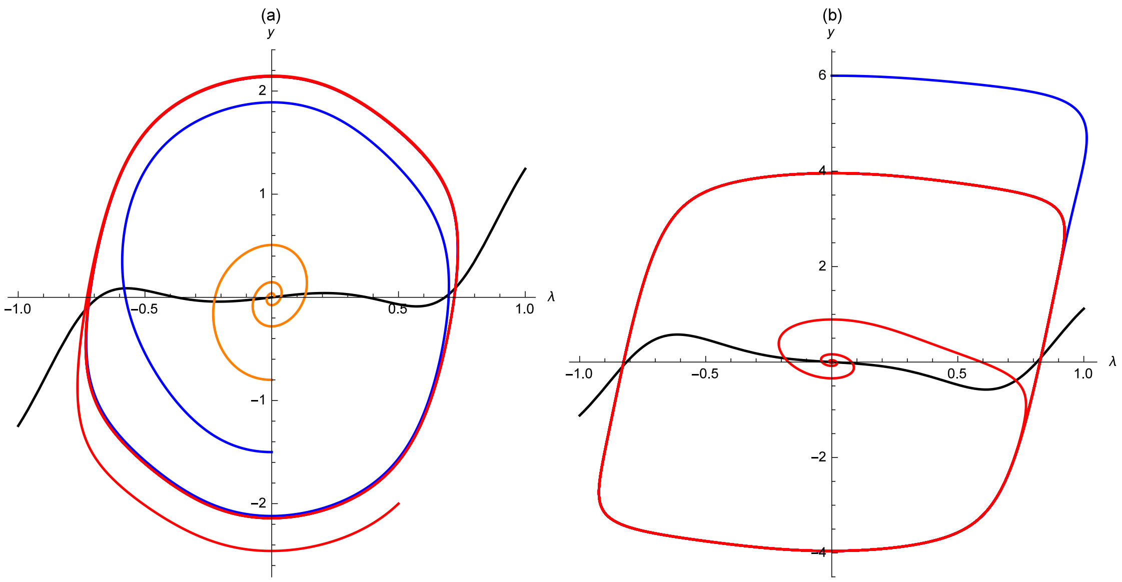

Figure 5.3: Different regimes for case B close to the Hopf bifurcation. In both pictures, the black line represents the graph of $y = f_{\alpha,\beta}(\lambda)$ and we fixed $\alpha = \sigma^2 = 1$. In (a), the regime $\beta \in (\hat{\beta}_1(\alpha), \beta_H)$ ($\beta = 1.2$): the situation is qualitatively the same of Figure 5.2(b). The red, blue and orange lines represent solution starting from $\lambda(0) = 0.5$, $y(0) = -2$, $\lambda(0) = 0$, $y(0) = -1.5$ and $\lambda(0) = 0$, $y(0) = -0.8$ respectively. In (b), the regime $\beta > \beta_H$ ($\beta = 1.8$): the system has undergone through a Hopf bifurcation but the stable limit cycle spreading from the origin is not present here, due to the collapse of the unstable cycle in the origin, leaving the outer orbit to become globally attractive. The red-colored and blue-colored solutions have initial conditions $\lambda(0) = 0$, $y(0) = -0.005$ and $\lambda(0) = 0$, $y(0) = 6$ respectively.

Bibliography

[1] J. A. Acebrón, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler. The Kuramoto model: A simple paradigm for synchronization phenomena. *Reviews of modern physics*, 77(1):137, 2005.

[2] L. Andreis, A. Asselah, and P. Dai Pra. Ergodicity of a system of interacting random walks with asymmetric interaction. *arXiv preprint arXiv:1702.02754*, 2017.

[3] L. Andreis, P. Dai Pra, and M. Fischer. McKean-Vlasov limit for interacting systems with simultaneous jumps. *arXiv preprint arXiv:1704.01052*, 2016.

[4] L. Andreis, P. Dai Pra, and M. Fischer. Simultaneous jumps in interacting particle systems: from neuronal networks to a general framework. *Submitted for publication*, 2017.

[5] A. Asselah and F. Castell. Existence of quasi-stationary measures for asymmetric attractive particle systems on z^{d}. *Annals of Applied Probability*, 13(4):1569–1590, 2003.

[6] A. Asselah, P. A. Ferrari, P. Groisman, and M. Jonckheere. Fleming–Viot selects the minimal quasi-stationary distribution: The Galton–Watson case. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 52(2), pages 647–668. Institut Henri Poincaré, 2016.

[7] J. Baladron, D. Fasoli, O. Faugeras, and J. Touboul. Mean-field description and propagation of chaos in networks of Hodgkin-Huxley and FitzHugh-Nagumo neurons. *The Journal of Mathematical Neuroscience*, 2(1):1, 2012.

[8] A. D. Banner, R. Fernholz, and I. Karatzas. Atlas models of equity markets. *The Annals of Applied Probability*, 15(4):2296–2330, 2005.

[9] S. Benachour, B. Roynette, and P. Vallois. Nonlinear self-stabilizing processes–II: Convergence to invariant probability. *Stochastic processes and their applications*, 75(2):203–224, 1998.

[10] J. Bérard and J.-B. Gouéré. Brunet-Derrida behavior of branching-selection particle systems on the line. *Communications in Mathematical Physics*, 298(2):323–342, 2010.

[11] L. Bertini, G. Giacomin, and K. Pakdaman. Dynamical aspects of mean field plane rotators and the Kuramoto model. *Journal of Statistical Physics*, 138(1-3):270–290, 2010.

[12] P. Billingsley. *Convergence of probability measures.* Wiley, 1999.

[13] S. Bobkov and M. Ledoux. One-dimensional empirical measures, order statistics and kantorovich transport distances. *preprint*, 2014.

[14] M. Bossy and B. Jourdain. Rate of convergeance of a particle method for the solution of a 1d viscous scalar conservation law in a bounded interval. *The Annals of Probability*, 30(4):1797–1832, 2002.

[15] M. Bossy and D. Talay. Convergence rate for the approximation of the limit law of weakly interacting particles: application to the burgers equation. *The Annals of Applied Probability*, 6(3):818–861, 1996.

[16] C. Bruggeman and A. Sarantsev. Multiple collisions in systems of competing Brownian particles. *arXiv preprint arXiv:1309.2621*, 2013.

[17] E. Brunet and B. Derrida. Shift in the velocity of a front due to a cutoff. *Physical Review E*, 56(3):2597, 1997.

[18] M. J. Cáceres, J. A. Carrillo, and B. Perthame. Analysis of nonlinear noisy integrate & fire neuron models: blow-up and steady states. *The Journal of Mathematical Neuroscience*, 1(1):1–33, 2011.

[19] T. Carletti and G. Villari. A note on existence and uniqueness of limit cycles for Liénard systems. *Journal of mathematical analysis and applications*, 307(2):763–773, 2005.

[20] J. A. Carrillo, M. d. M. González, M. P. Gualdani, and M. E. Schonbek. Classical solutions for a nonlinear Fokker-Planck equation arising in computational neuroscience. *Communications in Partial Differential Equations*, 38(3):385–409, 2013.

[21] S. Chatterjee and S. Pal. A phase transition behavior for Brownian motions interacting through their ranks. *Probability theory and related fields*, 147(1-2):123–159, 2010.

[22] X. Chen, J. Llibre, and Z. Zhang. Sufficient conditions for the existence of at least n or exactly n limit cycles for the Liénard differential systems. *Journal of Differential Equations*, 242:11–23, 2007.

[23] F. Collet and P. Dai Pra. The role of disorder in the dynamics of critical fluctuations of mean field models. *Electron. J. Probab*, 17(26):1–40, 2012.

[24] F. Collet, P. Dai Pra, and M. Formentin. Collective periodicity in mean-field models of cooperative behavior. *Nonlinear Differential Equations and Applications NoDEA*, 22(5):1461–1482, 2015.

[25] F. Collet, M. Formentin, and D. Tovazzi. Rhythmic behavior in a two-population mean-field Ising model. *Physical Review E*, 94(4):042139, 2016.

[26] P. Dai Pra, M. Fischer, and D. Regoli. A Curie-Weiss model with dissipation. *Journal of Statistical Physics*, 152(1):37–53, 2013.

[27] P. Dai Pra, G. Giacomin, and D. Regoli. Noise-induced periodicity: some stochastic models for complex biological systems. In *Mathematical Models and Methods for Planet Earth*, pages 25–35. Springer, 2014.

[28] D. A. Dawson. Critical dynamics and fluctuations for a mean-field model of cooperative behavior. *Journal of Statistical Physics*, 31(1):29–85, 1983.

[29] A. De Masi, A. Galves, E. Löcherbach, and E. Presutti. Hydrodynamic limit for interacting neurons. *Journal of Statistical Physics*, 158(4):866–902, 2014.

[30] F. Delarue, J. Inglis, S. Rubenthaler, and E. Tanré. Global solvability of a networked integrate-and-fire model of McKean–Vlasov type. *The Annals of Applied Probability*, 25(4):2096–2133, 2015.

[31] S. Ditlevsen and E. Löcherbach. Multi-class oscillating systems of interacting neurons. *Stochastic Processes and their Applications*, 2016.

[32] P. Dupuis and M. Fischer. On the construction of Lyapunov functions for nonlinear markov processes via relative entropy. *submitted for publication*, 2011.

[33] R. Durrett and D. Remenik. Brunet–Derrida particle systems, free boundary problems and Wiener—Hopf equations. *The Annals of Probability*, 39(6):2043–2078, 2011.

[34] A. S. Ecker, P. Berens, G. A. Keliris, M. Bethge, N. K. Logothetis, and A. S. Tolias. Decorrelated neuronal firing in cortical microcircuits. *Science*, 327(5965):584–587, 2010.

[35] T. Eisele and R. S. Ellis. Multiple phase transitions in the generalized Curie-Weiss model. *Journal of statistical physics*, 52(1-2):161–202, 1988.

[36] R. Ellis. *Entropy, large deviations, and statistical mechanics.* Springer, 2007.

[37] R. S. Ellis and C. M. Newman. Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 44(2):117–139, 1978.

[38] G. Fayolle, V. Malyshev, M. Menshikov, and A. Sidorenko. Lyapounov functions for Jackson networks. *Mathematics of operations research*, 18(4):916–927, 1993.

[39] G. Fayolle, V. A. Malyshev, and M. V. MenÊźshikov. *Topics in the constructive theory of countable Markov chains.* Cambridge university press, 1995.

[40] P. Ferrari, H. Kesten, S. Martinez, and P. Picco. Existence of quasi-stationary distributions. A renewal dynamical approach. *The Annals of Probability*, pages 501–521, 1995.

[41] P. A. Ferrari and N. Maric. Quasi stationary distributions and fleming-viot processes in countable spaces. *Electron. J. Probab*, 12(24):684–702, 2007.

[42] N. Fournier and A. Guillin. On the rate of convergence in Wasserstein distance of the empirical measure. *Probability Theory and Related Fields*, 162(3-4):707–738, 2015.

[43] N. Fournier and E. Löcherbach. On a toy model of interacting neurons. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 52(4):1844–1876, 2016.

[44] A. Galves and E. Löcherbach. Modeling networks of spiking neurons as interacting processes with memory of variable length. *Journal de la Société Française de Statistique*, 157(1):17–32, 2016.

[45] J. Garnier, G. Papanicolaou, and T.-W. Yang. Large deviations for a mean field model of systemic risk. *SIAM Journal on Financial Mathematics*, 4(1):151–184, 2013.

[46] A. Gasull, H. Giacomini, and J. Llibre. New criteria for the existence and non-existence of limit cycles in Liènard differential systems. *Dynamical Systems*, 24(2):171–185, 2009.

[47] G. Giacomin and C. Poquet. Noise, interaction, nonlinear dynamics and the origin of rhythmic behaviors. *Brazilian Journal of Probability and Statistics*, 29(2):460–493, 2015.

[48] C. Graham. McKean-Vlasov Itô-Skorohod equations, and nonlinear diffusions with discrete jump sets. *Stochastic processes and their applications*, 40(1):69–82, 1992.

[49] C. Graham. Nonlinear diffusion with jumps. In *Annales de l'IHP Probabilités et statistiques*, volume 28(3), pages 393–402, 1992.

[50] C. Graham and S. Méléard. Stochastic particle approximations for generalized boltzmann models and convergence estimates. *The Annals of probability*, 25(1):115–132, 1997.

[51] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*. Elsevier, 2014.

[52] J. R. Jackson. Jobshop-like queueing systems. *Management science*, 10(1):131–142, 1963.

[53] B. Jourdain. Diffusion processes associated with nonlinear evolution equations for signed measures. *Methodology and computing in applied probability*, 2(1):69–91, 2000.

[54] B. Jourdain and F. Malrieu. Propagation of chaos and Poincaré inequalities for a system of particles interacting through their CDF. *The Annals of Applied Probability*, 18(5):1706–1736, 2008.

[55] B. Jourdain and J. Reygner. Propagation of chaos for rank-based interacting diffusions and long time behaviour of a scalar quasilinear parabolic equation. *Stochastic partial differential equations: analysis and computations*, 1(3):455–506, 2013.

[56] M. Kac. Foundations of Kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, volume 3, pages 171–197, 1954.

[57] T. G. Kurtz. Weak and strong solutions of general stochastic models. *Electronic Communications in Probability*, 19(58):1–16, 2014.

[58] T. Liggett. *Interacting particle systems*, volume 276. Springer-Verlag Berlin Heidelberg, 2012.

[59] B. Lindner, J. Garcıa-Ojalvo, A. Neiman, and L. Schimansky-Geier. Effects of noise in excitable systems. *Physics reports*, 392(6):321–424, 2004.

[60] P.-L. Lions and A.-S. Sznitman. Stochastic differential equations with reflecting boundary conditions. *Communications on Pure and Applied Mathematics*, 37(4):511–537, 1984.

[61] J. Llibre and X. Zhang. Limit cycles of the classical Liénard differential systems: A survey on the Lins Neto, de Melo and Pugh's conjecture. *Expositiones Mathematicae*, 2016.

[62] P. Maillard. Speed and fluctuations of N-particle branching brownian motion with spatial selection. *Probability Theory and Related Fields*, 166(3-4):1061–1173, 2016.

[63] F. Malrieu. Logarithmic sobolev inequalities for some nonlinear PDE's. *Stochastic processes and their applications*, 95(1):109–132, 2001.

[64] F. Malrieu. Convergence to equilibrium for granular media equations and their euler schemes. *The Annals of Applied Probability*, 13(2):540–560, 2003.

[65] H. P. McKean. A class of Markov processes associated with nonlinear parabolic equations. *Proceedings of the National Academy of Sciences*, 56(6):1907–1911, 1966.

[66] H. P. McKean. *Stochastic integrals*, volume 353. American Mathematical Soc., 1969.

[67] S. Méléard. Asymptotic behaviour of some interacting particle systems; mckean-vlasov and boltzmann models. In *Probabilistic models for nonlinear partial differential equations*, pages 42–95. Springer, 1996.

[68] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes I: Criteria for discrete-time chains. *Advances in Applied Probability*, 24(03):542–574, 1992.

[69] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes II: Continuous-time processes and sampled chains. *Advances in Applied Probability*, 25(3):487–517, 1993.

[70] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes. *Advances in Applied Probability*, 25(3):518–548, 1993.

[71] K. Odani. Existence of Exactly N Periodic Solutions for Liénard Systems. *Funkcialaj Ekvacioj*, 39:217–234, 1996.

[72] S. Pal and J. Pitman. One-dimensional brownian particle systems with rank-dependent drifts. *The Annals of Applied Probability*, pages 2179–2207, 2008.

[73] L. Perko. *Differential equations and dynamical systems*, volume 7. Springer Science & Business Media, 2013.

[74] J. Reygner. Chaoticity of the stationary distribution of rank-based interacting diffusions. *Electronic Communications in Probability*, 20, 2015.

[75] P. Robert and J. Touboul. On the dynamics of random neuronal networks. *Journal of Statistical Physics*, 165(3):545–584, 2016.

[76] M. Sabatini and G. Villari. Limit cycle uniqueness for a class of planar dynamical systems. *Applied mathematics letters*, 19(11):1180–1184, 2006.

[77] A. Sarantsev. Reflected brownian motion in a convex polyhedral cone: Tail estimates for the stationary distribution. *Journal of Theoretical Probability*, pages 1–24, 2016.

[78] M. Scheutzow. Noise can create periodic behavior and stabilize nonlinear diffusions. *Stochastic processes and their applications*, 20(2):323–331, 1985.

[79] M. Scheutzow. Some examples of nonlinear diffusion processes having a time-periodic law. *The Annals of Probability*, pages 379–384, 1985.

[80] M. Shkolnikov. Competing particle systems evolving by interacting Lévy processes. *The Annals of Applied Probability*, 21(5):1911–1932, 2011.

[81] A.-S. Sznitman. Équations de type de Boltzmann, spatialement homogenes. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 66(4):559–592, 1984.

[82] A.-S. Sznitman. Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated. *Journal of functional analysis*, 56(3):311–336, 1984.

[83] A.-S. Sznitman. Topics in propagation of chaos. In *Ecole d'Eté de Probabilités de Saint-Flour XIX—1989*, pages 165–251. Springer, 1991.

[84] H. Tanaka. Limit theorems for certain diffusion processes with interaction. *North-Holland Mathematical Library*, 32:469–488, 1984.

[85] J. Touboul. The hipster effect: When anticonformists all look the same. *arXiv preprint arXiv:1410.8001*, 2014.

[86] J. Touboul, G. Hermann, and O. Faugeras. Noise-induced behaviors in neural mean field dynamics. *SIAM Journal on Applied Dynamical Systems*, 11(1):49–81, 2012.

[87] C. Villani. *Optimal transport: old and new*, volume 338. Springer-Verlag Berlin Heidelberg, 2008.

[88] R. J. Williams. Reflected brownian motion with skew symmetric data in a polyhedral domain. *Probability Theory and Related Fields*, 75(4):459–485, 1987.

Ringraziamenti

I would like to deeply thank my supervisor Paolo Dai Pra for his careful guidance and support. I will always be grateful for his great expertise, passion and patience, that have been milestones in my research experience and in the writing of this thesis.

A sincere thank goes to Amine Asselah for all the enriching discussions and useful suggestions. Moreover he gave us a great hospitality in Marseille, we spent an amazing and very productive time there. Thank you to Markus Fischer, for its knowledge and its careful help, in particular in the first moments of my research experience. A thank to all the Probability and Finance group in Padova, for many useful comments and conversations; in particular thank you to Tiziano Vargiolu for giving me the opportunity to remain here and letting me spend time in finishing this thesis.

A huge thank to my friend Daniele Tovazzi, who happened to be also my collaborator. Working with a close friend is something incredibly special, thank you for all the time spent together and for all your encouragement and support. I would like to thank also all the people that I met during these years and that made this experience unique. When I started the PhD more than three years ago, I would have never imagined to meet so many people that I am now proud to call friends. I thank also my family, in which I clearly include all my lifetime friends from Boves. I am grateful to my mother, for her endless support and love. Finally, I will never be able to express all my gratitude to Maurizio for constantly being by my side.