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CICLO XXI

THE IMPACT OF DISORDER IN THE CRITICAL DYNAMICS OF MEAN-FIELD MODELS

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To Marco

Sommario. Consideriamo un sistema di particelle interagenti a campo-medio immerso in un ambiente aleatorio i.i.d. e sito-dipendente. Il sistema viene fatto evolvere come una catena di Markov a tempo continuo sullo spazio degli stati. La dinamica dipende da pochi parametri e può essere completamente descritta attraverso quella del parametro d'ordine del modello. Ricaviamo la dinamica di quest'ultimo nel limite di volume infinito e quindi ne studiamo il comportamento per tempi lunghi. Tale dinamica limite risulta essere deterministica e, al variare dei parametri, presenta una transizione di fase. Il nostro interesse principale è lo studio delle fluttuazioni critiche, cioè le fluttuazioni del parametro d'ordine attorno alla dinamica limite quando i parametri assumono i valori tali per cui si verifica la transizione di fase. Lo scopo è l'analisi degli effetti causati dal disordine su di esse, confrontandole con le analoghe fluttuazioni per il caso omogeneo. Trattiamo sistemi di spin e di diffusioni, ma non in totale generalità. Ci concentriamo su dei modelli specifici: il modello di Curie-Weiss con aggiunta di campo aleatorio; un sistema di spin non-reversibile motivato dalla Finanza e il modello di Kuramoto omogeneo e non.

Abstract. We consider a mean-field interacting particle system embedded in a site-dependent and i.i.d. random environment. We make it evolve as a continuous time Markov chain on its state space. The dynamics are given depending on few parameters and they are completely described by that of the order parameter of the model. We derive the dynamics of this last quantity, in the infinite volume limit, and then their long time behavior is studied. The limiting dynamics of the order parameter are deterministic and, depending on the values of the parameters, exhibit a phase transition. Our main interest is the study of the critical fluctuations, that are the fluctuations of the order parameter around its limiting dynamics when the parameters take the values for which the phase transition occurs. We aim at analyzing the effect of the disorder in the dynamics of them, as compared with the homogeneous case. We deal with spin-flip and interacting diffusion systems, but we do not treat the subject in total generality, we focus on specific models: the random Curie-Weiss model; a non-reversible spin-flip system motivated by Finance and the homogeneous and inhomogeneous Kuramoto models.

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Contents

Introduction

Mean-field interacting particle systems are characterized by the complete absence of geometry in the space of configurations, in the sense that each particle interacts with all the others in the same way. The advantage of dealing with this kind of models is that they usually are analytically tractable and it is rather simple derive their macroscopic equations. Even if the hypothesis "all-to-all" may seem too simplistic to describe physical systems, where geometry and short-range interaction are involved, mean-field models have been recently applied to social sciences and finance, as in [DPRST09], [FB08] , [LL07], [BD01], [DPT09] and [DGGL08].
We briefly introduce the general framework we are working in and some of its peculiar features and then we will explain how our work enters within this setting. We consider a mean-field interacting N-particle system evolving as a continuous time Markov chain on its state space, for times belonging to a time interval $[0, T]$. The dynamics are given depending on few parameters and they are completely described by the dynamics of the order parameter of the model. By order parameter we mean a finite or infinite dimensional stochastic process, defined as an Empirical Average of the original process, whose dynamics are Markovian too. By Empirical Average we mean an integral of a function of the state variable with respect to the Empirical Measure

$$\rho_N := \frac{1}{N} \sum_{j=1}^{N} \delta_{j-\text{state variable}} \,.$$

We derive the dynamics of the order parameter, in the limit as N grows to infinity, in a fixed time interval $[0, T]$, via a Large Deviation approach. Then, the long time behavior of the limiting dynamics is studied. Note that this is not necessarily equivalent to the study of the large N behavior of the stationary measure.

Roughly speaking, this would consist in letting $t \longrightarrow +\infty$ for N fixed, and then letting $N \longrightarrow +\infty$. In our approach we reverse the order of the limits. In certain regimes, the two approaches stand be equivalent, as some results in [For09] show. Our dynamic approach has the advantage that allows to deal with non-reversible models, as we will see in Chapter 2 and Chapter 4.
In the infinite volume limit, the limiting dynamics of the order parameter are deterministic (driven by a system of ordinary differential equations) and, depending on the values of the parameters of the model, exhibit a phase transition, which is the appearance of multiple stable equilibria. When the parameters take the values such that the system has exactly one stationary solution, we say the system to be in a subcritical regime; while, when more than one equilibrium appears, we are in a supercritical regime.
Our main interest is the study of the fluctuations of the order parameter around its limiting dynamics. We can capture different features of these fluctuations depending on whether or not the time is rescaled with N. If time is not rescaled and we consider the evolution in a time interval $[0, T]$, with T fixed, a Central Limit Theorem holds true for the order parameter for all regimes; in other words, the fluctuations of the order parameter converge to a Gaussian process, which is the unique solution of a linear diffusion equation. Whenever time is rescaled in such a way T goes to infinity as N does, we may observe different behaviors.

- In a supercritical regime, we expect to find a metastability-type phenomenon; in other words, the system may spend a very long time in a neighborhood of a set of configurations that correspond to a stable equilibrium of the limiting dynamics, and then switches to a neighborhood of another equilibrium. The waiting time for this switch is exponentially large in N.

- In a subcritical regime, the Central Limit Theorem holds *uniformly in time*, as shown in [For09]. Thus, by rescaling time we simply obtain a Central Limit Theorem for the stationary measure.

- In the critical regime, when the parameters of the system are in the boundary between the subcritical and the supercritical regimes, that fluctuations are expected to exhibit a peculiar space-time scaling (*critical fluctuations*), and their limit distribution may be non Gaussian. More generally, critical fluctuations may occur whenever a stationary solution for the limiting dynamics becomes linearly neutrally stable.

The main subject of this thesis is the analysis of the dynamics of the critical fluctuations in disordered mean-field models.
We consider a mean-field model and we add a site-dependent, i.i.d. random environment, acting as an inhomogeneity in the structure of the system; we aim at analyzing the effect of the disorder in the dynamics of critical fluctuations, as compared with the homogeneous case. We deal with spin-flip and interacting diffusion systems, but we do not treat the subject in total generality, we focus on specific models: the random Curie-Weiss model (Chapter 1 and Chapter 5); a non-reversible spin-flip system motivated by Finance (Chapter 2) and the homogeneous and inhomogeneous Kuramoto models (Chapter 3 and Chapter 4, respectively). We are not aware of similar results concerning non-equilibrium critical fluctuations. Static fluctuations for the random Curie-Weiss model have been studied in [AdMFP91].

We now give the basic ideas of how the dynamics of critical fluctuations are determined. The deterministic limiting dynamics of the order parameter is described by a non-linear evolution operator $\mathcal{L}$. The linearization of this equation around a stationary solution gives rise to the so called linearized operator $\mathfrak{L}$. This operator is also related to the normal fluctuation of the process. At the critical point this operator has an eigenvalue with zero real part, while all other elements of the spectrum have negative real part. The eigenspace of the eigenvalue with zero real part is a subspace of the space where the order parameter lives. This subspace will be called *critical direction*, and usually happens to have low dimension: critical phenomena concern the empirical averages corresponding to this subspace. Thus, our analysis follow the following points.

- Locating the critical direction.
- Deciding the correct space-time scaling for the critical fluctuations. This require an approximation of the time evolution of the order parameter that goes beyond the normal approximation.
- Proving that the rescaled fluctuation vanish along non-critical directions. This will be done using the method of “collapsing processes” : it was developed by Comets and Eisele in [CE88] for a geometric long-range interacting

spin system and was previously applied to a homogeneous mean-field spin-flip system in [Sar07]; we extend this method to diffusion systems as well (the details are in Chapter 3). This step requires some control on the whole spectrum of $\mathfrak{L}$.

- Determining the limiting dynamics in the critical direction. It will be done using arguments of perturbation theory for Markov processes, which has been treated in [PSV77], and of tightness, applied to a suitable martingale problem.

In the case the order parameter is infinite dimensional, the spectral analysis of $\mathfrak{L}$ may be hard, in particular when $\mathfrak{L}$ is not self-adjoint in some Hilbert space. For this reason, in the first two chapters of thesis we deal with models whose order parameter is low dimensional. These models are disordered spin systems, where both the spin and the environment are described by $\{\pm 1\}$-valued variables. The analysis of both critical and non-critical fluctuations is obtained by first diagonalizing a low-dimensional matrix, and then proving weak convergence of finite dimensional processes. For these models we proceed directly to the analysis of the disordered systems; the homogeneous case is obtained along the same lines, and it is actually simpler.
In Chapters 3 and 4 we study a model of coupled oscillators, the Kuramoto model. In this case the order parameter is infinite dimensional even in the homogeneous case. Both the problem of determining the critical direction and that of describing the critical fluctuations are considerably harder. For this reason we first discuss the homogeneous case (Chapter 3), and then the inhomogeneous one, under some assumptions on the distribution of the disorder. The main tool here is perturbation theory for Markov processes.
Finally, in Chapter 5, we revisit the Curie-Weiss model. The methods of perturbation theory presented in Chapter 3 apply to this model too, and allow to weaken considerably the assumptions made in Chapter 1 on the distribution of the disorder.
From a qualitatively point of view, our results indicate that when disorder is added, spin systems and rotators belong to two different classes of universality, which is not the case for homogeneous systems. Roughly speaking, in spin systems the fluctuations of the disorder always prevail in the critical regime: these

fluctuations evolve in a time scale of order $N^{1/4}$, while the critical slowing down for homogeneous systems is $N^{1/2}$. For rotators, the disorder does not modify the $N^{1/2}$ slowing down, and the essential features of the non-Gaussian distribution of critical fluctuations are as in the homogeneous case. The effect of the disorder is a sort of "deformation": the critical direction is modified, and becomes disorder-dependent.

In some more details, the structure of the thesis is organized in the following way:

Chapter 1. We consider the random field Curie-Weiss model. Given a sequence of i.i.d., symmetric, Bernoulli random variables $\underline{\eta}$, $\underline{\sigma} = (\sigma_j)_{j=1}^N$ is a N-spin system evolving as a Markov process on its state space $\{-1, +1\}^N$. The dynamics are specified by the requirement that the rates of transition are of the form

$$\sigma_j \longrightarrow -\sigma_j \qquad \text{at rate} \qquad e^{-\beta\sigma_j(m_N^{\sigma} + h\eta_j)} \qquad \beta, h > 0\,.$$

We reduce this system to be finite dimensional. A three dimensional order parameter is necessary to describe the system. Being based on a Large Deviation Principle, we compute the differential equations which drive its evolution in the infinite particle limit (McKean-Vlasov equations) and we derive a Law of Large Number it obeys. Depending on the parameters, we can see there exists phase transition. Our main results consist in the infinite particle limits of the non-critical and critical fluctuation processes. For the non-critical fluctuation process we can provide a Central Limit Theorem and, hence, we show it converges (in the sense of weak convergence of stochastic processes) to a Gaussian process. With regard to the critical fluctuation process, the fluctuations are one-dimensional at the critical point and, in the limit as $N \longrightarrow +\infty$, converge (in the sense of weak convergence of stochastic processes) to a process with constant (but random) drift given by a Gaussian variable with parameters depending on the disorder. Differently, the homogeneous critical fluctuations exist on a longer time-scale and they are non-Gaussian, since they converge (in the sense of weak convergence of stochastic processes) to the unique solution of a non-linear diffusion equation.

Chapter 2. We consider a non-reversible spin-flip system, which is a slight generalization of the homogeneous one in [DPRST09]. The interesting aspect of

the latter is its financial application. It models the contagion in a network of N firms active on a market facing credit risk.
Given a sequence of i.i.d., symmetric, Bernoulli random variables $\underline{\eta}$, $(\underline{\sigma}, \underline{\omega}) = (\sigma_j, \omega_j)_{j=1}^N$ is a $2N$-spin system evolving as a non-reversible Markov process on its state space $\{-1,+1\}^{2N}$. The dynamics are specified by the requirement that the rates of transition are of the form

$$\begin{array}{llll} \sigma_j \longrightarrow -\sigma_j & \text{at rate} & e^{-\beta\sigma_j\omega_j} & \beta > 0\,, \\ \omega_k \longrightarrow -\omega_k & \text{at rate} & e^{-\gamma\omega_k(m_N^\sigma + h\eta_k)} & \gamma, h > 0\,. \end{array}$$

We reduce this system to be finite dimensional. A seven dimensional order parameter is necessary to describe this system. Using Large Deviation techniques, we compute the differential equations which drive its evolution in the infinite volume limit (McKean-Vlasov equations) and we derive a Law of Large Number it obeys. Depending on the parameters, we can see there exists phase transition. We then consider the fluctuation processes. We can provide a Central Limit Theorem for the non-critical seven-dimensional fluctuation process, but we skip the proof of this fact since it is completely analogous to the case of the random Curie-Weiss Model discussed in Chapter 1 and we focus on the infinite volume limit of the critical fluctuation process, which represents our main result. As in the previous case, the fluctuations are one-dimensional at the critical point and converge (in the sense of weak convergence of stochastic processes) to a process with constant (but random) drift given by a Gaussian variable with parameters depending on the disorder. Differently, the homogeneous critical fluctuations exist on a longer time-scale and they are non-Gaussian, since they converge (in the sense of weak convergence of stochastic processes) to the unique solution of a non-linear diffusion equation, as proved in [Sar07].

Chapter 3. We consider the homogeneous Kuramoto model. It is a system of mean-field nonlinearly coupled rotators which can be used to describe synchronization phenomena (we refer to [ABPV+05] for a review of the argument).
$\underline{x} = (x_j)_{j=1}^N$ is a N-diffusion system evolving as a Markov process with generator L_N, acting on functions $f : [0, 2\pi]^N \longrightarrow \mathbb{R}$ as follows:

$$L_N f(\underline{x}) = \frac{1}{2} \sum_{j=1}^N \frac{\partial^2 f}{\partial x_j^2}(\underline{x}) + \sum_{j=1}^N \left[\frac{\theta}{N} \sum_{k=1}^N \sin(x_k - x_j) \right] \frac{\partial f}{\partial x_j}(\underline{x})\,.$$

It may not be reduced to a finite dimensional problem. In this case, the order parameter is the Empirical Measure, which is an infinite dimensional Markov process. Being based on a Large Deviation Principle, we compute the differential equations which drive its evolution in the infinite volume limit (McKean-Vlasov equations) and we derive a Law of Large Number it obeys. Depending on the parameters, we can see there exists phase transition. We state these results for completeness; they are already known in literature. They can be deduced from the analogous ones for the inhomogeneous system, studied in [DPdH95] and [dH00]. We use the model as an example by which we explain the methodology for studying the critical dynamics in diffusion systems. The procedure is based on the idea of "collapsing processes", introduced in [CE88] and applied to point processes so far, and on the perturbation theory for Markov processes developed in [PSV77]. We prove that the critical fluctuations are two-dimensional at the critical point and converge (in the sense of weak convergence of stochastic processes) to a non-Gaussian process, which is the unique solution of a cubic stochastic differential equation.

Chapter 4. We consider the random Kuramoto model. Given a sequence of i.i.d., symmetric, Bernoulli random variables $\underline{\omega}$, $\underline{x} = (x_j)_{j=1}^N$ is a N-diffusion system evolving as a non-reversible Markov process with generator L_N, acting on functions $f : [0, 2\pi]^N \longrightarrow \mathbb{R}$ as follows:

$$L_N f(\underline{x}) = \frac{1}{2}\sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_j^2}(\underline{x}) + \sum_{j=1}^{N}\left[\omega_j + \frac{\theta}{N}\sum_{k=1}^{N}\sin(x_k - x_j)\right]\frac{\partial f}{\partial x_j}(\underline{x}) .$$

Even in this case, we may not reduce the system to a finite dimensional one and, then, the order parameter is the Empirical Measure. The results about the McKean-Vlasov limit of the dynamics and the existence of a phase transition are got by [DPdH95] and [dH00]. With regard to the critical fluctuations, applying the method described in Chapter 3, we see that, they are a two-dimensional process at the critical point and converges (in the sense of weak convergence of stochastic processes) to a non-Gaussian process, solution of a cubic stochastic differential equation. This equation looks like the one satisfied by the critical process of the homogeneous model.

Chapter 5. We resume the random field Curie-Weiss model. We generalize the environment, precisely we choose it distributed according to an even distribution, and we approach the study of the critical dynamics using the method developed in Chapter 3. To guarantee the problem can be analytically dealt with, we may consider at most discrete, finite random fields. In this way the spectrum of the linearized operator $\mathfrak{L}$ is only discrete. Our main result is that the random drift appears even in this general case, in which the disorder is not dichotomic, but may assume a finite number of different values. The critical fluctuations continue to be one-dimensional at the critical point and converge (in the sense of weak convergence of stochastic processes) to a deterministic process with constant (but random) drift given by a Gaussian variable with parameters depending on the environment.

We decided to devote an entire chapter to the study of the Kuramoto model without disorder (Chapter 3), since the technique of "collapsing processes" had not been applied to the analysis of the critical dynamics of diffusion systems before. So, we first applied the methodology to the simplest case, which is easily tractable since the operator $\mathfrak{L}$ is self-adjoint, and then we have dealt with the inhomogeneous system.

Part I

Spin-Flip Systems

Chapter 1

The Random Curie-Weiss Model

In this chapter we consider the Curie-Weiss model with the addition of a random site-dependent magnetic field, which acts as random environment. We want to study the dynamical laws of the model, in the infinite volume limit.
We consider N sites and we associate with each of them a spin and a magnetic field value, that we choose to be a dichotomic random variable. We start with a Glauber-type dynamics for the N-particle system, where the spins flip from time to time to another value with a jump intensity depending on the gradient of the Hamiltonian felt by the particle. It is an interacting spin-flip system with a *mean-field* Hamiltonian that depends on the random medium we introduced. In this model there is no spatial geometry in the space of the configurations, since it is subject to a mean-field interaction, meaning that each particle interacts with all the others in the same way.
Three order parameters (magnetization field) are necessary to describe the system. Being based on a Large Deviation Principle, we compute the differential equations which drive their evolution in the infinite particle limit (McKean-Vlasov equations) and we derive a Law of Large Number they obey. Depending on the parameters, we can see there exists phase transition to ferromagnetic states with constant magnetizations.
Our main results are the infinite particle limits of the non-critical and critical fluctuation processes. For the non-critical fluctuation process we can provide a Central Limit Theorem and, hence, we show it converges (in the sense of weak

convergence of stochastic processes) to a Gaussian process. With regard to the critical fluctuation process – besides an appropriate scaling of the space – it require a rescaling of the time in order to keep track of long time fluctuations of the critical direction (critical slowing down). As a result, only the critical structure survives the new scaling, and in the limit, the critical fluctuation process is a lower dimensional process compared with the non-critical one. The fluctuations are one-dimensional at the critical point, while they are three-dimensional for non-critical values. In fact, we prove that, when the size of the system grows towards infinity, two order parameters collapse, while the other converges (in the sense of weak convergence of stochastic processes) to a process with constant (but random) drift given by a Gaussian variable with parameters depending on the magnetic field.

1.1 Description of the Model

Let $\mathscr{S} = \{-1, +1\}$ and $\underline{\eta} = (\eta_j)_{j=1}^N \in \mathscr{S}^N$ be a sequence of independent, identically distributed, symmetric, Bernoulli random variables defined on some probability space $(\Omega, \mathcal{F}, P)$. That is, $P(\eta_j = -1) = P(\eta_j = +1) = \frac{1}{2}$, for any j. We indicate with μ their common law. Given a configuration $\underline{\sigma} = (\sigma_j)_{j=1}^N \in \mathscr{S}^N$ and a realization of the magnetic field $h\underline{\eta}$, $h > 0$, we can define the Hamiltonian $H_N(\underline{\sigma}, \underline{\eta}) : \mathscr{S}^{2N} \longrightarrow \mathbb{R}$ as

$$H_N(\underline{\sigma}, \underline{\eta}) = -\frac{\beta}{2N} \sum_{j,k=1}^{N} \sigma_j \sigma_k - \beta h \sum_{j=1}^{N} \eta_j \sigma_j \,, \tag{1.1}$$

where σ_j is the spin value at site j and η_j the direction of the local magnetic field (of intensity h) associated with the same site. Let β, positive parameter, be the inverse of the temperature. For a fixed realization of $\underline{\eta}$, think of $\underline{\sigma} \longrightarrow H_N(\underline{\sigma}, \underline{\eta})$ as a Hamiltonian in the components σ_j with an inhomogeneous mean-field interaction parametrized by the components η_j. With the expression "mean-field" we mean the sites interact all each other in the same way.

Let us define the dynamics we consider. For given $\underline{\eta}$, $\underline{\sigma}(t) = (\sigma_j(t))_{j=1}^N$, with t belonging to a generic time interval $[0, T]$, where T is fixed, describes a N-spin system evolving as a continuous time Markov chain on $\mathscr{S}^N$, with infinitesimal generator L_N acting on functions $f : \mathscr{S}^N \longrightarrow \mathbb{R}$ as follows:

$$L_N f(\underline{\sigma}) = \sum_{j=1}^{N} e^{-\beta\sigma_j(m_N^{\underline{\sigma}} + h\eta_j)} \nabla_j^{\underline{\sigma}} f(\underline{\sigma}), \tag{1.2}$$

where $\nabla_j^{\underline{\sigma}} f(\underline{\sigma}) = f(\underline{\sigma}^j) - f(\underline{\sigma})$ and the k-th component of $\underline{\sigma}^j$, which has the meaning of a spin flip at site j, is

$$\sigma_k^j = \begin{cases} \sigma_k \text{ for } k \neq j \\ -\sigma_k \text{ for } k = j \end{cases}.$$

The quantity $c_N^{\underline{\eta}}(j, \underline{\sigma}) = e^{-\beta\sigma_j(m_N^{\underline{\sigma}} + h\eta_j)}$ represents the jump rate of the spins; the rate at which the transition $\sigma_j \longrightarrow -\sigma_j$ occurs for some j. The mean-field assumption allows us to suppose that the interaction depends on the value of the magnetization

$$m_N^{\underline{\sigma}}(t) = \frac{1}{N} \sum_{j=1}^{N} \sigma_j(t).$$

The expressions (1.1) and (1.2) describe a system of mean-field ferromagnetically coupled spins, each with its own random magnetic field and subject to Glauber dynamics. The two terms in the Hamiltonian have different effects: the first one tends to align the spins, while the second one tends to point each of them in the direction of its local field.

Remark 1.1.1. For every value of $\underline{\eta}$, (1.2) has a reversible stationary distribution proportional to $\exp[-H_N(\underline{\sigma}, \underline{\eta})]$.

For simplicity, the initial condition $\underline{\sigma}(0)$ is assumed to have product distribution $\lambda^{\otimes N}$, with λ probability measure on $\mathscr{S}$, although weaker conditions could be dealt with. The quantity $\sigma_j(t)$ represents the time evolution on $[0, T]$ of j-th spin value; it is the trajectory of the single j-th spin in time. The space of all these paths is $\mathcal{D}[0, T]$, which is the space of the right-continuous, piecewise-constant functions from $[0, T]$ to $\mathscr{S}$. We endow $\mathcal{D}[0, T]$ with the Skorohod topology, which provides a metric and a Borel σ-field (as we can see in [EK86]).

1.2 Limiting Dynamics

We now derive the dynamics of the process (1.2), in the limit as $N \longrightarrow +\infty$, in a fixed time interval $[0, T]$, via a Large Deviation approach. Later, the large time behavior of the limiting dynamics will be studied. Note that this is not necessarily equivalent to the study of the large N behavior of the stationary measure $\exp[-H_N(\underline{\sigma}, \underline{\eta})]$. Roughly speaking, this would consist in letting $t \longrightarrow +\infty$ for N fixed, and then letting $N \longrightarrow +\infty$. In our approach we reverse the order of the limits. In certain regimes, the two approaches are equivalent, as some results in [For09] show. Our dynamic approach has the advantage that allows to deal with non-reversible model, as we will see in Chapter 2.

So, let $(\sigma_j[0,T])_{j=1}^N \in (\mathcal{D}[0,T])^N$ denote a path of the system in the time interval $[0,T]$, with T positive and fixed. If $f(\sigma_j[0,T])$ is a function of the trajectory of a single spin, we are interested in the asymptotic behavior of *empirical averages* of the form

$$\frac{1}{N}\sum_{j=1}^{N} f(\sigma_j[0,T]) =: \int f d\rho_N \,,$$

where $\{\rho_N\}_{N\geq 1}$ is the sequence of *empirical measures*

$$\rho_N := \frac{1}{N}\sum_{j=1}^{N} \delta_{(\sigma_j[0,T],\eta_j)} \,.$$

Remark 1.2.1. The measure ρ_N is a joint measure of the process and the environment.

We may think of ρ_N as a random element of $\mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S})$, the space of probability measures on $\mathcal{D}[0,T] \times \mathscr{S}$ endowed with the weak convergence topology. First, we want to determine the weak limit of ρ_N in $\mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S})$ as N grows to infinity; i.e. for $f \in \mathcal{C}_b$ we look for $\lim_{N\to+\infty} \int f d\rho_N$. It corresponds to a Law of Large Number with the limit being a deterministic measure. Being an element of $\mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S})$, such a limit can be viewed as a stochastic process, which describes the dynamics of the system in the infinite volume limit.

1.2.1 Empirical Measure and Large Deviations

Let $W \in \mathcal{M}_1(\mathcal{D}[0,T])$ denote the law of the $\mathscr{S}$-valued process $(\sigma(t))_{t\in[0,T]}$ such that the initial condition $\sigma(0)$ has distribution λ and the spin signs change with constant rate equal to 1. By $W^{\otimes N}$ we mean the product of N copies of W, which represents the law of the N-spin system whose generator is (1.2) where we have set $c_N^\eta \equiv 1$; in other words, the law of our system in absence of interaction. Moreover, we shall write $P_N^{\underline{\eta}}$ the law of $\underline{\sigma}([0,T]) = (\underline{\sigma}(t))_{t\in[0,T]}$, the process with infinitesimal generator (1.2) and initial distribution $\lambda^{\otimes N}$, for a given $\underline{\eta}$.

Consider $Q \in \mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S})$, if $\Pi_t Q$ indicates the marginal distribution of Q at time t, then we have

$$m^\sigma_{\Pi_t Q} := \int_{\mathscr{S}^2} \sigma \Pi_t Q(d\sigma, d\eta)$$

and for a given path $\sigma([0,T]) \in \mathcal{D}[0,T]$, we define

$$F(Q) := \int \Bigg\{ \int_0^T \left(1 - e^{-\beta\sigma(t)\left(m^\sigma_{\Pi_t Q} + h\eta\right)}\right) dt \\ - \frac{\beta}{2}\left[\sigma(T) m^\sigma_{\Pi_T Q} - \sigma(0) m^\sigma_{\Pi_0 Q} + h\eta\left(\sigma(T) - \sigma(0)\right)\right] \Bigg\} dQ\,. \quad (1.3)$$

Remark 1.2.2. The function $F(Q)$ is continuous and bounded.

We can obtain a representation of $P_N^{\underline{\eta}}$ in terms of ρ_N, as follows:

Lemma 1.2.1. *For a fixed realization $\underline{\eta}$,*

$$\frac{dP_N^{\underline{\eta}}}{dW^{\otimes N}}(\underline{\sigma}([0,T])) = \exp[NF(\rho_N(\underline{\sigma}([0,T]), \underline{\eta})) + O(1)]$$

where, for $Q \in \mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S})$, $F(Q)$ is expressed by (1.3).

Proof. The proof is essentially an application of the analogous of the Girsanov's Formula in the case we work with stochastic integrals with respect to point processes (see [Bré81] or [LS01]). This formula clarifies the fact that an absolutely continuous change of probability measures is described by its Radon-Nikodym derivative in terms of the change of the intensities of the point processes involved.

Let $(\mathcal{N}_t^{\underline{\sigma}}(j))_{j=1}^N$ be the multivariate Poisson process counting the jumps of σ_j, for $j = 1, \ldots, N$. If we read $\sigma_j(t-) = \lim_{s\to t-} \sigma_j(s)$ and $m^{\underline{\sigma}}_{\rho_N(t-)} = \lim_{s\to t-} m^{\underline{\sigma}}_{\rho_N(s)}$, it yields

$$\frac{dP_N^{\underline{\eta}}}{dW^{\otimes N}}(\underline{\sigma}([0,T])) = \exp\Bigg\{ \sum_{j=1}^N \Bigg[\int_0^T \log e^{-\beta\sigma_j(t-)\left(m^{\underline{\sigma}}_{\rho_N(t-)} + h\eta_j\right)} d\mathcal{N}_t^{\underline{\sigma}}(j) \\ - \int_0^T \left(e^{-\beta\sigma_j(t)\left(m^{\underline{\sigma}}_{\rho_N(t)} + h\eta_j\right)} - 1\right) dt \Bigg]\Bigg\}$$

but $\underline{\sigma}$ has no simultaneous jumps $W^{\otimes N}$–almost surely, therefore

$$= \exp\left\{ \sum_{j=1}^{N} \left[\int_0^T \left(1 - e^{-\beta\sigma_j(t)\left(m^{\sigma}_{\rho_N(t)} + h\eta_j\right)}\right) dt \right.\right.$$
$$\left.\left. - \beta \int_0^T (-\sigma_j(t)) \left(\left(m^{\sigma}_{\rho_N(t)} - \frac{2}{N}\sigma_j(t)\right) + h\eta_j\right) d\mathcal{N}^{\sigma}_t(j) \right]\right\}$$

and because $\int \mathcal{N}^{\sigma}_T d\rho_N < +\infty$ almost surely with respect to $W^{\otimes N}$,

$$= \exp\left\{ \sum_{j=1}^{N} \left[\int_0^T \left(1 - e^{-\beta\sigma_j(t)\left(m^{\sigma}_{\rho_N(t)} + h\eta_j\right)}\right) dt \right.\right.$$
$$\left.\left. + \beta \int_0^T \sigma_j(t) \left(m^{\sigma}_{\rho_N(t)} + h\eta_j\right) d\mathcal{N}^{\sigma}_t(j) + O\left(\frac{1}{N}\right) \right]\right\}$$
$$= \exp\left\{ \sum_{j=1}^{N} \left[\int_0^T \left(1 - e^{-\beta\sigma_j(t)\left(m^{\sigma}_{\rho_N(t)} + h\eta_j\right)}\right) dt \right.\right.$$
$$- \frac{\beta}{2}\Big(\sigma_j(T) m^{\sigma}_{\rho_N(T)} - \sigma_j(0) m^{\sigma}_{\rho_N(0)}$$
$$\left.\left. + h\eta_j \left(\sigma_j(T) - \sigma_j(0)\right)\Big) + O\left(\frac{1}{N}\right) \right]\right\}$$

and this leads us to the expression (1.3) for F. The last equality is due to a general result for reversible spin-flip systems we can find in [DPdH96], Lemma 3. ■

⌜

Definition 1.2.1. Let $\mathscr{X}$ be a Polish[1] space with distance $d : \mathscr{X} \times \mathscr{X} \longrightarrow [0, +\infty)$. The function $i : \mathscr{X} \longrightarrow [0, +\infty]$ is called a rate function if $i \neq \infty$ and i is lower semi-continuous with compact level-sets (that is, for every $k > 0$, the set $\{x : i(x) \leq k\}$ is compact in the weak topology).

⌟

⌜

Definition 1.2.2. Let $\mathscr{X}$ be a Polish space with distance $d : \mathscr{X} \times \mathscr{X} \longrightarrow [0, +\infty)$. A sequence of probability measures $\{\mathscr{P}_n\}_{n\geq 1}$ on $\mathscr{X}$ is said to satisfy a *Large Deviation Principle* with rate function $i : \mathscr{X} \longrightarrow [0, +\infty]$ if for $\mathscr{O}, \mathscr{C} \subseteq \mathscr{X}$ respectively open and closed set for the weak topology, it yields

$$\liminf_{n\to+\infty} \frac{1}{n} \log \mathscr{P}_n(\mathscr{O}) \geq - \inf_{x\in\mathscr{O}} i(x) \tag{1.4a}$$

[1]We recall that a Polish space is a complete and separable metric space.

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathscr{P}_n(\mathscr{C}) \leq - \inf_{x\in\mathscr{C}} i(x) \,. \tag{1.4b}$$

⌟

Lemma 1.2.1 allows us to deduce a Large Deviation Principle for ρ_N, from which we can derive its asymptotic behavior as $N \longrightarrow +\infty$.
Define

$$\mathcal{P}_N(\cdot) := \int \mu^{\otimes N}(d\underline{\eta}) P_N^{\underline{\eta}}(\rho_N \in \cdot) \,,$$

which is an element of $\mathcal{M}_1(\mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S}))$ and represents the law of ρ_N under the joint distribution of the process and the environment.
If $Q \in \mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S})$, we denote by

$$H(Q|W\otimes\mu) := \begin{cases} \int dQ \log \frac{dQ}{d(W\otimes\mu)} & \text{if} \quad Q \ll W\otimes\mu \quad \text{and} \quad \log\frac{dQ}{d(W\otimes\mu)} \in L^1(Q) \\ +\infty & \text{otherwise} \end{cases}$$

the relative entropy between Q and $W \otimes \mu$.

Remark 1.2.3. Let us consider $W \otimes \mu$ fixed. Then the relative entropy $H(\cdot|W \otimes \mu)$ is a nonnegative, convex function on $\mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S})$ and $H(Q|W \otimes \mu) = 0$ if and only if $Q = W \otimes \mu$. Besides, it is lower semi-continuous on $\mathcal{D}[0,T] \times \mathscr{S}$ endowed with the weak topology. (We refer to [DZ93], Chapter VI, Section 2, for complete statements and proofs.)

Proposition 1.2.1. *The laws $\{\mathcal{P}_N\}_{N\geq 1}$ of ρ_N (under the joint distribution of the process and the medium) obey a Large Deviation Principle with rate function*

$$I(Q) := H(Q|W\otimes\mu) - F(Q) \,. \tag{1.5}$$

Proof. The key tools are Sanov's Theorem and Varadhan's Lemma (we refer respectively to Theorem 3.2.17 in [DS89] and Theorem 2.2 in [Var84]). We give here the statements for completeness and convenience.

⌜

Theorem 1.2.1 (Sanov). *Let $\mathscr{P}$ be a probability measure on the Polish space $\mathscr{X}$ and let $\widetilde{\mathscr{P}}_n \in \mathcal{M}_1(\mathcal{M}_1(\mathscr{X}))$ be the distribution under $\mathscr{P}_n$ of the random probability*

$$\rho_n(\underline{x}) = \frac{1}{n}\sum_{j=1}^{n}\delta_{x_j}$$

with $\underline{x} = (x_1, \ldots, x_n) \in \mathscr{X}^n$. Moreover consider the relative entropy with respect to $\mathscr{P}$, $H(\cdot|\mathscr{P})$. Then, $H(\cdot|\mathscr{P})$ is a good, convex rate function on $\mathcal{M}_1(\mathscr{X})$ and $\{\widetilde{\mathscr{P}}_n\}_{n\geq 1}$ satisfies a Large Deviation Principle with rate function $H(\cdot|\mathscr{P})$.

⌟

⌜

Lemma 1.2.2 (Varadhan). *Let $\{\mathscr{P}_n\}_{n\geq 1}$ be a sequence of probability measures in $\mathcal{M}_1(\mathscr{X})$ satisfying a Large Deviation Principle with rate function $i : \mathscr{X} \longrightarrow [0, +\infty]$. Then, for every bounded function $f : \mathscr{X} \longrightarrow \mathbb{R}$ which is continuous on the set $\{x : i(x) < +\infty\}$, it holds*

$$\liminf_{n\to+\infty} \frac{1}{n}\int_{\mathscr{O}} \exp[nf] d\mathscr{P}_n \geq -\inf_{x\in\mathscr{O}}[f(x) - i(x)] \tag{1.6a}$$

$$\limsup_{n\to+\infty} \frac{1}{n}\int_{\mathscr{C}} \exp[nf] d\mathscr{P}_n \leq -\inf_{x\in\mathscr{C}}[f(x) - i(x)]\,. \tag{1.6b}$$

⌟

Denote by R_N the distribution of ρ_N under $W^{\otimes N} \times \mu^{\otimes N}$; in other words, if $A \in \mathcal{B}(\mathcal{D}[0,T] \times \mathscr{S})$ is a Borelian set, then $R_N(A) = (W^{\otimes N} \times \mu^{\otimes N})(\rho_N^{-1}(A))$. Under R_N, the pairs $(\sigma_j[0,T], \eta_j)$ are independent, identically distributed random variables.

Now, because of the result proved in Lemma 1.2.1, we have

$$\begin{aligned}
\mathcal{P}_N(\cdot) &= \int \mu^{\otimes N}(d\underline{\eta}) P_N^{\underline{\eta}}(\rho_N(d\underline{\sigma}[0,T], \underline{\eta}) \in \ \cdot) \\
&= \int \mu^{\otimes N}(d\underline{\eta}) \int W^{\otimes N}(d\underline{\sigma}[0,T]) \frac{dP_N^{\underline{\eta}}}{dW^{\otimes N}}(\underline{\sigma}[0,T]) \mathbf{1}_{\{\rho_N(d\underline{\sigma}[0,T],\underline{\eta}) \in \ \cdot\}} \\
&= \int d(W^{\otimes N} \times \mu^{\otimes N}) \exp[NF(\rho_N)] \mathbf{1}_{\{\rho_N \in \ \cdot\}}
\end{aligned}$$

$$= \int R_N(dQ) \exp[NF(Q)] \mathbf{1}_{\{Q \in \cdot\}}\,, \tag{1.7}$$

with $Q = \rho_N$. The last identity (1.7) means that

$$\frac{d\mathcal{P}_N}{dR_N}(Q) = \exp[NF(Q)]\,. \tag{1.8}$$

Since $\mathcal{D}[0,T] \times \mathscr{S}$ is a Polish space, by Sanov's Theorem (Theorem 1.2.1) we can deduce that $\{R_N\}_{N \geq 1}$ satisfies a Large Deviation Principle with rate function $H(Q|W \otimes \mu)$. Therefore, we can apply Varadhan's Lemma (Lemma 1.2.2) to obtain an upper bound of type (1.4b). In fact, if $C \in \mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S})$ is a closed set,

$$\limsup_{N \to +\infty} \frac{1}{N} \log \mathcal{P}_N(C) = \limsup_{N \to +\infty} \frac{1}{N} \log \int R_N(dQ) \exp[NF(Q)] \mathbf{1}_{\{Q \in C\}}$$

$$\leq \sup_{Q \in C} [F(Q) - H(Q|W \otimes \mu)] = - \inf_{Q \in C} I(Q)\,,$$

where the definition of $I(Q)$ is in (1.5). The lower bound of type (1.4a) is proved similarly.

■

1.2.2 McKean-Vlasov Equation

Given $Q \in \mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S})$ and $\eta \in \mathscr{S}$, we can associate with Q a Markov process on $\mathscr{S}$ with law $P^{\eta,Q}$, initial distribution λ and time-dependent infinitesimal generator

$$\mathcal{L}_t^{\eta,Q} f(\sigma) = e^{-\beta\sigma(m^\sigma_{\Pi_t Q} + h\eta)} \nabla^\sigma f(\sigma)\,,$$

acting on $f : \mathscr{S} \longrightarrow \mathbb{R}$.

Proposition 1.2.2. *For every $Q \in \mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S})$ such that $I(Q) < +\infty$,*

$$I(Q) = H(Q|P^Q)\,, \tag{1.9}$$

where $P^Q \in \mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S})$ is defined by

$$P^Q(d\sigma[0,T], d\eta) = P^{\eta,Q}(d\sigma[0,T])\mu(d\eta).$$

Proof. First we need to verify that the following representation for $F(Q)$ (defined in (1.3)) holds

$$F(Q) = \int Q(d\sigma[0,T], d\eta) \log \frac{dP^{\eta,Q}}{dW}(\sigma[0,T]) \,.$$

We begin by observing that, since by assumption $I(Q) < +\infty$, we have also $H(Q|W \otimes \mu) < +\infty$ and so, by the entropy equality (see (6.2.14) in [DZ93]), it follows that Q belongs to the set $\{Q \in \mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S}) : \int \mathcal{N}_T^\sigma dQ < +\infty$, with $\mathcal{N}_t^\sigma$ the process counting the jumps of $\sigma\}$, which implies the integrals below are well defined. Using again the Girsanov's Formula for Markov Chains, we get

$$\begin{aligned}
\int dQ \log \frac{dP^{\eta,Q}}{dW}(\sigma[0,T]) &= \int \Bigg[\int_0^T \left(1 - e^{-\beta\sigma(t)\left(\int \sigma \Pi_t Q(d\sigma, d\eta) + h\eta\right)}\right) dt \\
&\qquad - \beta \int_0^T \sigma(t-) \left(\int \sigma \Pi_{t-} Q(d\sigma, d\eta) + h\eta \right) d\mathcal{N}_t^\sigma \Bigg] dQ \\
&= \int \Bigg[\int_0^T \left(1 - e^{-\beta\sigma(t)\left(\int \sigma \Pi_t Q(d\sigma, d\eta) + h\eta\right)}\right) dt \\
&\qquad + \beta \int_0^T \sigma(t) \left(\int \sigma \Pi_t Q(d\sigma, d\eta) + h\eta \right) d\mathcal{N}_t^\sigma \Bigg] dQ \\
&= \int \Bigg[\int_0^T \left(1 - e^{-\beta\sigma(t)\left(m^\sigma_{\Pi_t Q} + h\eta\right)}\right) dt \\
&\qquad + \beta \int_0^T \sigma(t) \left(m^\sigma_{\Pi_t Q} + h\eta \right) d\mathcal{N}_t^\sigma \Bigg] dQ = F(Q)
\end{aligned}$$

where the last equality holds thanks to the reversibility of the system. We refer again to [DPdH96], Lemma 3. By combining what we obtained, we can compute

$$\begin{aligned}
I(Q) = H(Q|W \otimes \mu) - F(Q) &= \int dQ \log \frac{dQ}{d(W \otimes \mu)} - \int dQ \log \frac{dP^{\eta,Q}}{dW} \\
&= \int dQ \log \frac{dQ}{d(P^{\eta,Q} \otimes \mu)} = \int dQ \log \frac{dQ}{dP^Q} = H(Q|P^Q) \,.
\end{aligned}$$

■

Theorem 1.2.2. *Suppose that the initial distribution of the Markov process* $(\underline{\sigma}(t))_{t\geq 0}$ *with generator* (1.2) *is such that the random variables* $(\sigma_j(0))_{j=1}^N$ *are independent and identically distributed with law* λ*. Then the equation* $I(Q) = 0$ *admits a unique solution* $Q_* \in \mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S})$*, such that its marginals* $q_t^\eta = \Pi_t Q_*^\eta \in \mathcal{M}_1(\mathscr{S})$ *are weak solutions of the nonlinear McKean-Vlasov equation*

$$\begin{cases} \frac{\partial q_t^\eta}{\partial t} = \mathcal{L}^\eta q_t^\eta & (t \in [0,T],\ \eta \in \mathscr{S}) \\ q_0^\eta \ = \ \lambda \end{cases} \tag{1.10}$$

where, for all the pairs $(\sigma, \eta) \in \mathscr{S}^2$*, the operator* $\mathcal{L}^\eta$ *acts*

$$\mathcal{L}^\eta q_t^\eta(\sigma) = \nabla^\sigma \left[e^{-\beta\sigma\left(m_{q_t}^\sigma + h\eta\right)} q_t^\eta(\sigma) \right] \tag{1.11}$$

and q_t *is defined by*

$$q_t(\sigma) = \int_{\mathscr{S}} q_t^\eta(\sigma)\mu(d\eta).$$

Moreover, with respect to a metric $d(\cdot,\cdot)$ *inducing the weak topology,* $\rho_N \longrightarrow Q_*$ *in probability with exponential rate, i.e.* $\mathcal{P}_N\{d(\rho_N, Q_*) > \varepsilon\}$ *is exponentially small in* N*, for each* $\varepsilon > 0$.

Proof. We know that if the relative entropy between two measures is zero then the two measures must be equal (see Remark 1.2.3). By this property, from (1.9) we have $I(Q) = 0$ translates into $Q = P^Q$. Let us suppose Q_* is a solution of this last equation. Then, in particular, for a given η, $q_t^\eta := \Pi_t Q_*^\eta = \Pi_t P^{Q_*^\eta}$. The marginals of a Markov process are solutions of the corresponding forward equation that, in this case, leads to the fact that q_t^η is a solution of (1.10). This differential equation, being an equation in finite dimension with locally Lipschitz coefficients, has at most one solution in $[0,T]$. Since $P^{Q_*^\eta}$ is totally determined by the flow q_t^η, it follows that equation $Q = P^Q$ has at most one solution. The existence of a solution derives from the fact that $I(Q)$ is the rate function of a Large Deviation Principle and therefore it has to have at least one zero: indeed, by the bound of type (1.4a) with $O = \mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{S})$, we get $\inf_{Q\in O} I(Q) = 0$. Since I is lower semi-continuous, it attains this null value and so this infimum is actually a minimum.

It remains to prove the Law of Large Numbers for ρ_N: with respect to a metric $d(\cdot,\cdot)$ inducing the weak topology, $\rho_N \xrightarrow{N\to+\infty} Q_*$ in probability with exponential rate, i.e. $\mathcal{P}_N\{d(\rho_N, Q_*) > \varepsilon\}$ is exponentially small in N, for each $\varepsilon > 0$.

Let Q_* be the unique solution of equation $Q = P^Q$ and let B_{Q_*} be an arbitrary open neighborhood of Q_*. By the Large Deviation upper bound (type (1.4b)), we have

$$\limsup_{N\to+\infty} \frac{1}{N} \log \mathcal{P}_N(\rho_N \notin B_{Q_*}) \leq - \inf_{Q\notin B_{Q_*}} I(Q) < 0\,,$$

where the last inequality comes from the lower semi-continuity of I, the compactness of its level-sets and the fact that $I(Q) > 0$ for $Q \neq Q_*$. Indeed, if $\inf_{Q\notin B_{Q_*}} I(Q) = 0$, then there exists a sequence $(Q_n)_n \notin B_{Q_*}$ such that $I(Q_n) \xrightarrow{n\to+\infty} 0$. By the compactness of level-sets, the sequence $(Q_n)_n$ admits a subsequence $(Q_{n_k})_{n_k}$ converging to $\overline{Q} \notin B_{Q_*}$, when $n_k \longrightarrow +\infty$. Thanks to the lower semi-continuity of I, it follows $I(\overline{Q}) \leq \liminf_{n_k\longrightarrow+\infty} I(Q_{n_k}) = 0$, which contradicts $I(Q) > 0$ for $Q \neq Q_*$. Thus, from the above inequality, we deduce that there exists a positive constant A such that

$$\mathcal{P}_N(\rho_N \notin B_{Q_*}) \leq A e^{-N \inf_{Q\notin B_{Q_*}} I(Q)}.$$

It means that, if we denote with $d(\cdot,\cdot)$ any metric which induces the weak topology on $\mathcal{M}_1$, for every $\varepsilon > 0$, the probability $\mathcal{P}_N(\rho_N \notin B_{Q_*}) = \mathcal{P}_N\{d(\rho_N, Q_*) \geq \varepsilon\}$ converges toward zero exponentially fast with respect to N and this concludes the proof of the Law of Large Numbers. ■

1.2.3 Stationary Solution(s)

The equation (1.10) describes the behavior of the system governed by generator (1.2) in the infinite volume limit. We are interested in the detection of the t-stationary solution(s) of this equation. We recall that to be t-stationary solution for (1.10) means to satisfy the equation $\mathcal{L}^\eta q^\eta = 0$ for every t.

First of all, we proceed to reformulate the "original" McKean-Vlasov equation (1.10) in terms of $m^{\sigma\eta}_{q_t}$, $m^{\sigma}_{q_t}$ and $m^{\eta}_{q_t}$ defined as follows:

$$m^\eta_{q_t} := m^\eta_t = \frac{1}{2} \sum_{\sigma\in\mathscr{S}} \sum_{\eta\in\mathscr{S}} \eta\, q^\eta_t(\sigma)\,, \tag{1.12}$$

$$m^\sigma_{q_t} := m^\sigma_t = \frac{1}{2} \sum_{\sigma\in\mathscr{S}} \sum_{\eta\in\mathscr{S}} \sigma\, q^\eta_t(\sigma) = \sum_{\sigma\in\mathscr{S}} \sigma\, q_t(\sigma) \tag{1.13}$$

and

$$m^{\sigma\eta}_{q_t} := m^{\sigma\eta}_t = \frac{1}{2} \sum_{\sigma\in\mathscr{S}} \sum_{\eta\in\mathscr{S}} \sigma\eta\, q^\eta_t(\sigma)\,, \tag{1.14}$$

where q_t and q_t^η have the meaning explained in Theorem 1.2.2. We introduce these expectations because the probability measure q_t on $\mathscr{S}^2$ is completely determined by them.

Lemma 1.2.3. *Equations* (1.10) *can be rewritten in the following form:*

$$\begin{aligned}
\dot m_t^\eta &= 0 \\
\dot m_t^\sigma &= -2\, m_t^\sigma \cosh(\beta h)\cosh(\beta m_t^\sigma) - 2\, m_t^{\sigma\eta}\sinh(\beta h)\sinh(\beta m_t^\sigma) \\
&\qquad +2\cosh(\beta h)\sinh(\beta m_t^\sigma) \\
\dot m_t^{\sigma\eta} &= -2\, m_t^\sigma \sinh(\beta h)\sinh(\beta m_t^\sigma) - 2\, m_t^{\sigma\eta}\cosh(\beta h)\cosh(\beta m_t^\sigma) \\
&\qquad +2\sinh(\beta h)\cosh(\beta m_t^\sigma)\,,
\end{aligned} \tag{1.15}$$

with initial condition $m_0^\eta = m_{(\lambda,\mu)}^\eta = 0$, $m_0^\sigma = m_{(\lambda,\mu)}^\sigma$ *and* $m_0^{\sigma\eta} = m_{(\lambda,\mu)}^{\sigma\eta}$.

Proof. By definition (1.13) and Theorem 1.2.2 we deduce

$$\dot m_t^\sigma = \sum_{\sigma\in\mathscr{S}} \sigma\, \dot q_t(\sigma) = \frac{1}{2}\sum_{\sigma\in\mathscr{S}}\sum_{\eta\in\mathscr{S}} \sigma\, \dot q_t^\eta(\sigma) = \frac{1}{2}\sum_{\sigma\in\mathscr{S}}\sum_{\eta\in\mathscr{S}} \sigma\, \mathcal{L}^\eta q_t^\eta(\sigma)$$

$$=\frac{1}{2}\sum_{\sigma\in\mathscr{S}}\sum_{\eta\in\mathscr{S}} \sigma \nabla^\sigma \left[e^{-\beta\sigma(m_t^\sigma+h\eta)} q_t^\eta(\sigma)\right]$$

$$=\frac{1}{2}\sum_{\sigma\in\mathscr{S}}\sum_{\eta\in\mathscr{S}} \sigma \Big[e^{\beta\sigma(m_t^\sigma+h\eta)}\underbrace{q_t^\eta(-\sigma)}_{=1-q_t^\eta(\sigma)} - e^{-\beta\sigma(m_t^\sigma+h\eta)} q_t^\eta(\sigma)\Big]$$

$$=\frac{1}{2}\sum_{\sigma\in\mathscr{S}}\sum_{\eta\in\mathscr{S}} \sigma e^{\beta\sigma(m_t^\sigma+h\eta)} - \sum_{\sigma\in\mathscr{S}}\sum_{\eta\in\mathscr{S}} \sigma\cosh(\beta\sigma(m_t^\sigma+h\eta)) q_t^\eta(\sigma)$$

$$\begin{aligned}
=&\frac{1}{2}\sum_{\sigma\in\mathscr{S}} \sigma e^{\beta\sigma m_t^\sigma}\sum_{\eta\in\mathscr{S}} e^{\beta h\sigma\eta} - \sum_{\sigma\in\mathscr{S}}\sum_{\eta\in\mathscr{S}} \sigma\cosh(\beta h\sigma\eta)\cosh(\beta\sigma m_t^\sigma) q_t^\eta(\sigma) \\
&\quad - \sum_{\sigma\in\mathscr{S}}\sum_{\eta\in\mathscr{S}} \sigma\sinh(\beta h\sigma\eta)\sinh(\beta\sigma m_t^\sigma) q_t^\eta(\sigma)
\end{aligned}$$

$$\begin{aligned}
=&2\cosh(\beta h)\sinh(\beta m_t^\sigma) - \cosh(\beta h)\cosh(\beta m_t^\sigma)\sum_{\sigma\in\mathscr{S}}\sum_{\eta\in\mathscr{S}} \sigma\, q_t^\eta(\sigma) \\
&\quad - \sinh(\beta h)\sinh(\beta m_t^\sigma)\sum_{\sigma\in\mathscr{S}}\sum_{\eta\in\mathscr{S}} \sigma\eta\, q_t^\eta(\sigma)
\end{aligned}$$

$$
\begin{aligned}
&= 2\cosh(\beta h)\sinh(\beta m_t^\sigma) - 2\, m_t^\sigma \cosh(\beta h)\cosh(\beta m_t^\sigma) \\
&\quad - 2\, m_t^{\sigma\eta} \sinh(\beta h)\sinh(\beta m_t^\sigma)\,,
\end{aligned}
$$

where the last equality holds thanks to (1.13) and (1.14). So the first equation of (1.15) is proved. Similarly, we can obtain the other ones. ∎

Remark 1.2.4. Notice that $m_t^\eta \equiv 0$ for every t; it is a static variable.

Regarding to Remark 1.2.4, any equilibrium solution of the system (1.15) is of the form

$$
m_*^\sigma = \frac{1}{2}\left[\tanh(\beta(m_*^\sigma + h)) + \tanh(\beta(m_*^\sigma - h))\right] \tag{1.16a}
$$

$$
m_*^{\sigma\eta} = \frac{1}{2}\left[\tanh(\beta(m_*^\sigma + h)) - \tanh(\beta(m_*^\sigma - h))\right]\,. \tag{1.16b}
$$

To discover the presence of phase transition(s) (multiple equilibria) and the stability of equilibria, it is sufficient to study equation (1.16a), since $m_*^{\sigma\eta} = m_*^{\sigma\eta}(m_*^\sigma)$ and hence $\lim_{t\to+\infty} m_t^{\sigma\eta} = m_*^{\sigma\eta}$ when $\lim_{t\to+\infty} m_t^\sigma = m_*^\sigma$.

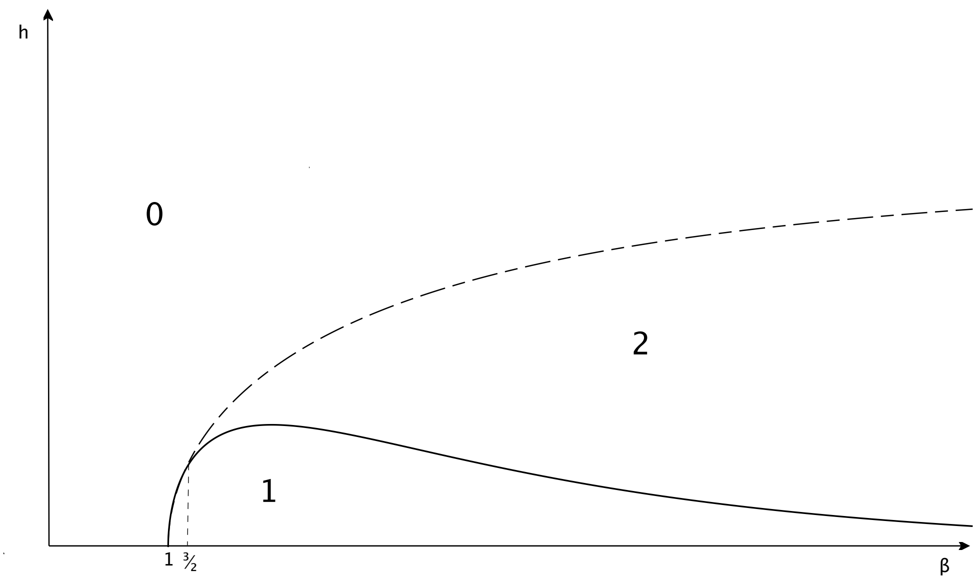

Figure 1.1: Phase diagram

The phase diagram is qualitatively drawn in Figure 1.1. There are three phases, corresponding to 0, 1 and 2 ferromagnetic solutions respectively. The continuous separation curve is

$$h = h(\beta) = \frac{1}{\beta}\operatorname{arccosh}(\sqrt{\beta}) \qquad \beta \in [1, +\infty)\,, \tag{1.17}$$

while the dotted one is obtained numerically and it is due to the fact that the function

$$m_*^\sigma \longmapsto \frac{1}{2}\left[\tanh(\beta(m_*^\sigma + h)) + \tanh(\beta(m_*^\sigma - h))\right]$$

is not always concave. The two curves coincide for $\beta \in [1, 3/2]$ and separate at the "tricritical" point $(3/2, h(3/2))$.

Theorem 1.2.3. *Consider the system of equations* (1.16a) *and* (1.16b).

(a) If (β, h) *belongs to the region* 0 *of Figure 1.1, then the only solution is* $(0, \tanh(\beta h))$.

(b) If (β, h)*, with* $\beta \in [1, +\infty)$*, is below the curve* (1.17)*, then there are three solutions:* $(0, \tanh(\beta h))$*,* $(m_*, m_*^{\sigma\eta}(m_*))$ *and* $(-m_*, m_*^{\sigma\eta}(m_*))$*, where* m_* *is the unique positive solution of* (1.16a).

(c) If we choose the parameters above the curve (1.17) *and* h *is small enough, in other words* (β, h) *belongs to the region* 2 *of Figure 1.1, then two further solutions arise.*

Proof. We refer to [DPdH95] for the proof concerning the phase diagram of the system and the stability of its equilibria. ■

We are going to focus on the critical regime corresponding to the critical values for the parameters $\beta = \cosh^2(\beta h)$, meaning that we are on the curve (1.17). For these values of the parameters, the equilibrium $(0, \tanh(\beta h))$ is neutrally stable for the linearized system. In fact, denoting by

$$\begin{aligned} V : [-1,+1] \times [-1,+1] &\longrightarrow \mathbb{R}^2 \\ (x_1, x_2) &\longmapsto (V_1(x_1,x_2), V_2(x_1,x_2))\,, \end{aligned}$$

with

$$\begin{aligned} V_1(x_1,x_2) := &-2x_1\cosh(\beta h)\cosh(\beta x_1) - 2x_2\sinh(\beta h)\sinh(\beta x_1) \\ &+ 2\cosh(\beta h)\sinh(\beta x_1) \end{aligned}$$

$$V_2(x_1, x_2) := -2x_1 \sinh(\beta h) \sinh(\beta x_1) - 2x_2 \cosh(\beta h) \cosh(\beta x_1) + 2 \sinh(\beta h) \cosh(\beta x_1) ,$$

the vector field of the system (1.15), we obtain the linearized matrix evaluated in the stationary solution is

$$DV(0, \tanh(\beta h)) = 2 \begin{bmatrix} \frac{\beta - \cosh^2(\beta h)}{\cosh(\beta h)} & 0 \\ 0 & -\cosh(\beta h) \end{bmatrix} .$$

Its eigenvalues are $\lambda_1 = 2\frac{\beta - \cosh^2(\beta h)}{\cosh(\beta h)}$ and $\lambda_2 = -2\cosh(\beta h)$; $\lambda_2 < 0$ for every value of β, h and instead it is easy to see that

- if $\beta < \cosh^2(\beta h)$, then $\lambda_1 < 0$ and thus $(0, \tanh(\beta h))$ is linearly stable;
- if $\beta = \cosh^2(\beta h)$, then $\lambda_1 = 0$ and thus $DV(0, \tanh(\beta h))$ has a neutral direction;
- if $\beta > \cosh^2(\beta h)$, then $\lambda_1 > 0$ and thus $(0, \tanh(\beta h))$ is a saddle point for the linearized system.

1.3 Normal Fluctuations and Central Limit Theorem

Thanks to Theorem 1.2.2 we established a Law of Large Numbers for the empirical measure ρ_N: $\rho_N \longrightarrow Q_*$. We are going to analyze the Normal fluctuations around the limit Q_*. We are also interested in the N-asymptotic distribution of $\rho_N - Q_*$.

We use a weak convergence-type approach based on uniform convergence of the infinitesimal generators. It is deeply explained in [EK86] and the result we need can be summarized in the following theorem, whose proof can be found in [EK86], Chapter 4, Corollary 8.7 .

⌜

Theorem 1.3.1. *Let $X_n(t)$ be a sequence of Markov processes with values in $\mathscr{X}_n$ and denote by $\mathscr{L}_n$ the corresponding infinitesimal generators, defined on $\mathcal{D}(\mathscr{L}_n)$. Moreover, let $\mathscr{L}$, defined on $\mathcal{D}(\mathscr{L})$, be the infinitesimal generator of another Markov process $X(t)$ with values on $\mathscr{X}$, and let $\mathscr{C}$ be a core for $\mathscr{L}$.*
Assume that for every n $\mathscr{X}_n \subset \mathscr{X}$ and each function in $\mathscr{C}$ is an element of $\mathcal{D}(\mathscr{L}_n)$, when restricted to $\mathscr{X}_n$. If the condition

$$\lim_{n\to+\infty} \sup_{x\in\mathscr{X}_n} |\mathscr{L}_n(f(x)) - \mathscr{L}(f(x))| = 0 \tag{$\star$}$$

holds for every $f \in \mathscr{C}$ and $X_n(0)$ converges to $X(0)$ in distribution, then the sequence of processes $X_n(t)$ converges to the process $X(t)$ in distribution.

⌟

Let $f : \mathscr{S} \longrightarrow \mathbb{R}$ be a function and define $\rho_N(t)$, the marginal distribution of ρ_N at time t, by

$$\int f(\sigma)\, d\rho_N(t) = \frac{1}{N}\sum_{j=1}^{N} f(\sigma_j(t)).$$

We have $m_N^\sigma(t) = m^\sigma_{\rho_N(t)}$. For each fixed t, $\rho_{N(t)}$ is a probability on $\mathscr{S}$ and so, by the considerations which led as to introduce the expectations (1.13), (1.14) and (1.12), we can proceed similarly saying $\rho_{N(t)}$ is completely determined by the triple $(m^\eta_{\rho_N(t)}, m^\sigma_{\rho_N(t)}, m^{\sigma\eta}_{\rho_N(t)})$ and seeing it as a three-dimensional object. Thus $(\rho_N(t))_{t\in[0,T]}$ is a three-dimensional flow. A simple consequence of Theorem 1.2.2 is the following convergence of flows:

$$(\rho_N(t))_{t\in[0,T]} \longrightarrow (q_t)_{t\in[0,T]}\,, \tag{1.18}$$

where the convergence is meant in probability, with respect to the weak topology for measure-valued processes. Since the flow of marginals contains less information than the full measure of paths, the Law of Large Numbers in (1.18) is weaker than the one in Theorem 1.2.2. However, the corresponding fluctuation flow

$$(N^{1/2}(\rho_N(t) - q_t))_{t\in[0,T]}$$

is also a finite-dimensional flow, whose limiting distribution can be explicitly characterized.

Lemma 1.3.1. *Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain on a finite state space S, admitting an infinitesimal generator L. Let $g : S \longrightarrow S'$ be a given function, where S' is a finite set. Assume that for every $f : S \longrightarrow \mathbb{R}$, $L(f \circ g)$ is a function of $g(x)$, i.e. $L(f \circ g) = (Kf) \circ g$. Then this last identity defines a linear operator K; moreover, $g(X_t)$ is a Markov process with infinitesimal generator K.*

Proof. Obviously K is linear. Observing that

$$e^{tL}(f \circ g) = (e^{tK} f) \circ g\,, \tag{1.19}$$

we can conclude. In fact, X_t is a Markov process with generator L, then we have

$$\begin{aligned} E[(f \circ g)(X_t)|X_0 = x] &= e^{tL}(f \circ g)(x) \\ &\overset{(1.19)}{=} e^{tK} f(g(x)) \\ &= E[f(g(X_t))|g(X_0) = g(x)] \end{aligned}$$

and the last inequality holds since e^{tK} is a Markov semigroup and $L(f \circ g) = (Kf) \circ g$. Hence, $g(X_t)$ is a Markov process with infinitesimal generator K. ∎

Lemma 1.3.2. *The stochastic process $(m^{\eta}_{\rho_N(t)}, m^{\sigma}_{\rho_N(t)}, m^{\sigma\eta}_{\rho_N(t)})$ is an* order parameter *for the model; it means its evolution is Markovian.*

Proof. To prove that $(m^{\eta}_{\rho_N(t)}, m^{\sigma}_{\rho_N(t)}, m^{\sigma\eta}_{\rho_N(t)})$ is a Markov process, we determine the expression of the infinitesimal generator $\mathcal{K}_N$ driving its dynamics. We apply Lemma 1.3.1.
The process $\{\underline{\sigma}(t)\}_{t\geq 0}$ is a continuous time Markov chain on the finite state space $\mathscr{S}^N$, with infinitesimal generator L_N, defined by (1.2). Consider the function

$$\begin{aligned} \zeta : \mathscr{S}^N &\longrightarrow [-1, +1]^3 \\ \underline{\sigma} &\longmapsto (m^{\eta}_{\rho_N}, m^{\sigma}_{\rho_N}, m^{\sigma\eta}_{\rho_N})\,, \end{aligned}$$

it plays the role of g in Lemma 1.3.1; then, for every $\phi : \mathscr{S}^N \longrightarrow \mathbb{R}$, we have

$$L_N(\phi \circ \zeta) = (\mathcal{K}_N \phi) \circ \zeta$$

and $\zeta(\underline{\sigma})$ is a Markov process with generator $\mathcal{K}_N$ given by

$$\mathcal{K}_N \phi(m^{\eta}_{\rho_N}, m^{\sigma}_{\rho_N}, m^{\sigma\eta}_{\rho_N}) = \sum_{j,k \in \mathscr{S}} |A_{\rho_N}(j,k)|\, e^{-\beta j (m^{\sigma}_{\rho_N} + kh)} \cdot \\ \cdot \left[\phi\left(m^{\eta}_{\rho_N}, m^{\sigma}_{\rho_N} - j\frac{2}{N}, m^{\sigma\eta}_{\rho_N} - jk\frac{2}{N}\right) - \phi(m^{\eta}_{\rho_N}, m^{\sigma}_{\rho_N}, m^{\sigma\eta}_{\rho_N})\right], \quad (1.20)$$

where $A_{\rho_N}(j,k)$ is the set of all pairs (σ_i, η_i), $i \in \{1, \ldots, N\}$, such that $\sigma_i = j$, $\eta_i = k$, with $j, k \in \mathscr{S}$; hence

$$|A_{\rho_N}(j,k)| = \frac{N}{4}\left[1 + k m^{\eta}_{\rho_N} + j m^{\sigma}_{\rho_N} + jk m^{\sigma\eta}_{\rho_N}\right]. \quad (1.21)$$

■

Theorem 1.3.2. *In the limit as $N \longrightarrow +\infty$, the three-dimensional fluctuation process $(r_N(t), x_N(t), y_N(t))$, defined by*

$$\begin{aligned} r_N(t) &:= N^{1/2} m^{\eta}_{\rho_N(t)} \\ x_N(t) &:= N^{1/2}\left(m^{\sigma}_{\rho_N(t)} - m^{\sigma}_t\right) \\ y_N(t) &:= N^{1/2}\left(m^{\sigma\eta}_{\rho_N(t)} - m^{\sigma\eta}_t\right), \end{aligned}$$

converges (in the sense of weak convergence of stochastic processes) to a limiting three-dimensional Gaussian process $(r(t), x(t), y(t))$, which is the unique solution of the linear stochastic differential equation

$$\begin{aligned} dr(t) &= 0 \\ \begin{bmatrix} dx(t) \\ dy(t) \end{bmatrix} &= 2\,\mathscr{H}\, A_1(t) dt + 2A_2(t) \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} dt + D(t) \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}, \end{aligned} \quad (1.22)$$

where B_1, B_2 are independent Standard Brownian motions, $\mathscr{H}$ is a Standard Gaussian random variable, $A_1(t)$, $A_2(t)$ and $\frac{D(t)D'(t)}{2}$ are respectively

$$\begin{bmatrix} \sinh(\beta h)\cosh(\beta m^{\sigma}_t) \\ \\ \cosh(\beta h)\sinh(\beta m^{\sigma}_t) \end{bmatrix},$$

$$\begin{bmatrix} (\beta-1)\cosh(\beta h)\cosh(\beta m^{\sigma}_t) - \beta m^{\sigma}_t \cosh(\beta h)\sinh(\beta m^{\sigma}_t) - \beta m^{\sigma\eta}_t \sinh(\beta h)\cosh(\beta m^{\sigma}_t) & -\sinh(\beta h)\sinh(\beta m^{\sigma}_t) \\ \\ (\beta-1)\sinh(\beta h)\sinh(\beta m^{\sigma}_t) - \beta m^{\sigma}_t \sinh(\beta h)\cosh(\beta m^{\sigma}_t) - \beta m^{\sigma\eta}_t \cosh(\beta h)\sinh(\beta m^{\sigma}_t) & -\cosh(\beta h)\cosh(\beta m^{\sigma}_t) \end{bmatrix},$$

$$\begin{bmatrix} -m_t^{\sigma}\mathrm{ch}(\beta h)\mathrm{sh}(\beta m_t^{\sigma}) - m_t^{\sigma\eta}\mathrm{sh}(\beta h)\mathrm{ch}(\beta m_t^{\sigma}) + \mathrm{ch}(\beta h)\mathrm{ch}(\beta m_t^{\sigma}) & -m_t^{\sigma}\mathrm{sh}(\beta h)\mathrm{ch}(\beta m_t^{\sigma}) - m_t^{\sigma\eta}\mathrm{ch}(\beta h)\mathrm{sh}(\beta m_t^{\sigma}) + \mathrm{sh}(\beta h)\mathrm{sh}(\beta m_t^{\sigma}) \\ -m_t^{\sigma}\mathrm{sh}(\beta h)\mathrm{ch}(\beta m_t^{\sigma}) - m_t^{\sigma\eta}\mathrm{ch}(\beta h)\mathrm{sh}(\beta m_t^{\sigma}) + \mathrm{sh}(\beta h)\mathrm{sh}(\beta m_t^{\sigma}) & -m_t^{\sigma}\mathrm{ch}(\beta h)\mathrm{sh}(\beta m_t^{\sigma}) - m_t^{\sigma\eta}\mathrm{sh}(\beta h)\mathrm{ch}(\beta m_t^{\sigma}) + \mathrm{ch}(\beta h)\mathrm{ch}(\beta m_t^{\sigma}) \end{bmatrix}$$

and $(r(0), x(0), y(0))$ has a centered Gaussian distribution with covariance matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - (m^{\sigma}_{(\lambda,\mu)})^2 & -m^{\sigma}_{(\lambda,\mu)} m^{\sigma\eta}_{(\lambda,\mu)} \\ 0 & -m^{\sigma}_{(\lambda,\mu)} m^{\sigma\eta}_{(\lambda,\mu)} & 1 - (m^{\sigma\eta}_{(\lambda,\mu)})^2 \end{bmatrix}. \tag{1.23}$$

Proof. We know that by the previous Lemma the infinitesimal generator $\mathcal{K}_N$ drives the evolution of $(m^{\eta}_{\rho_N}, m^{\sigma}_{\rho_N}, m^{\sigma\eta}_{\rho_N})$ as in (1.20). To use this information, we operate the following change of variables

$$\begin{aligned} m^{\eta}_{\rho_N(t)} &\longrightarrow r_N(t) = N^{1/2} m^{\eta}_{\rho_N(t)} \\ m^{\sigma}_{\rho_N(t)} &\longrightarrow x_N(t) = N^{1/2}\left(m^{\sigma}_{\rho_N(t)} - m^{\sigma}_t\right) \\ m^{\sigma\eta}_{\rho_N(t)} &\longrightarrow y_N(t) = N^{1/2}\left(m^{\sigma\eta}_{\rho_N(t)} - m^{\sigma\eta}_t\right). \end{aligned}$$

Notice that $(r_N(t), x_N(t), y_N(t))$ is obtained from $(m^{\eta}_{\rho_N(t)}, m^{\sigma}_{\rho_N(t)}, m^{\sigma\eta}_{\rho_N(t)})$ through a time-dependent, linear and invertible transformation; so, let us consider its inverse $\varphi(r_N(t), x_N(t), y_N(t))$. Then we have

$$\begin{aligned} \mathcal{M}^t_{N,\varphi} = \varphi(r_N(t), x_N(t), y_N(t)) - \int_0^t [\mathcal{K}_N(\varphi(r_N(s), x_N(s), y_N(s))) \\ + \partial_s \varphi(r_N(s), x_N(s), y_N(s))]\, ds\,, \end{aligned}$$

where $\mathcal{M}^t_{N,\varphi}$ is a martingale and $\varphi(r_N(t), x_N(t), y_N(t))$ has to be seen as a function of $m^{\eta}_{\rho_N(t)}$, $m^{\sigma}_{\rho_N(t)}$ and $m^{\sigma\eta}_{\rho_N(t)}$. Now, if we consider $\varphi(r_N(t), x_N(t), y_N(t))$ as a function of $r_N(t)$, $x_N(t)$ and $y_N(t)$, then $(r_N(t), x_N(t), y_N(t))$ is itself a time inhomogeneous Markov process, whose generator $\mathcal{H}_{N,t}$ acting on functions $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$,

$\varphi \in \mathcal{C}_b^3$, is given by

$$\mathcal{H}_{N,t}\varphi(r,x,y) = \mathcal{K}_N\varphi(r,x,y) + \partial_t\varphi(r,x,y) =$$

$$\begin{aligned}
&= \sum_{j,k\in\mathscr{S}} |A_N(j,k)| e^{-\beta j\left(\frac{x}{N^{1/2}}+m_t^\sigma+kh\right)}\bigg[\varphi\bigg(r, x-j\frac{2}{N^{1/2}}, y-jk\frac{2}{N^{1/2}}\bigg) \\
&\quad - \varphi(r,x,y)\bigg] - N^{1/2}\,\dot{m}_t^\eta\,\varphi_r(r,x,y) - N^{1/2}\,\dot{m}_t^\sigma\,\varphi_x(r,x,y) \\
&\quad - N^{1/2}\,\dot{m}_t^{\sigma\eta}\,\varphi_y(r,x,y)\,,
\end{aligned}$$

where

$$|A_N(j,k)| = \frac{N}{4}\left[1 + k\frac{r}{N^{1/2}} + j\frac{x}{N^{1/2}} + jk\frac{y}{N^{1/2}} + jm_t^\sigma + jkm_t^{\sigma\eta}\right];$$

we develop φ around (r,x,y) by a Taylor expansion stopped at second order and the exponential function around 0 at first order:

$$\begin{aligned}
\mathcal{H}_{N,t}\varphi(r,x,y) &= \sum_{j,k\in\mathscr{S}} |A_N(j,k)|\, e^{-\beta j(m_t^\sigma+kh)}\bigg(1 - j\frac{\beta x}{N^{1/2}} + o\bigg(\frac{1}{N^{1/2}}\bigg)\bigg)\cdot \\
&\quad \cdot \bigg[-j\frac{2}{N^{1/2}}\varphi_x - jk\frac{2}{N^{1/2}}\varphi_y + \frac{2}{N}\varphi_{xx} + \frac{2}{N}\varphi_{yy} + k\frac{4}{N}\varphi_{xy} + o\bigg(\frac{1}{N}\bigg)\bigg] \\
&\quad - N^{1/2}\,\dot{m}_t^\sigma\,\varphi_x(x,y) - N^{1/2}\,\dot{m}_t^{\sigma\eta}\,\varphi_y(x,y)
\end{aligned}$$

$$\begin{aligned}
&= \varphi_x\,\frac{N^{1/2}}{2}\bigg[e^{-\beta m_t^\sigma}\bigg(1 - \frac{\beta x}{N^{1/2}} + o\bigg(\frac{1}{N^{1/2}}\bigg)\bigg)\bigg(-2\cosh(\beta h) \\
&\quad + \frac{2r}{N^{1/2}}\sinh(\beta h) - \frac{2x}{N^{1/2}}\cosh(\beta h) + \frac{2y}{N^{1/2}}\sinh(\beta h) \\
&\quad - 2m_t^\sigma\cosh(\beta h) + 2m_t^{\sigma\eta}\sinh(\beta h)\bigg) \\
&\qquad + e^{\beta m_t^\sigma}\bigg(1 + \frac{\beta x}{N^{1/2}} + o\bigg(\frac{1}{N^{1/2}}\bigg)\bigg)\bigg(2\cosh(\beta h) \\
&\quad + \frac{2r}{N^{1/2}}\sinh(\beta h) - \frac{2x}{N^{1/2}}\cosh(\beta h) - \frac{2y}{N^{1/2}}\sinh(\beta h) \\
&\quad - 2m_t^\sigma\cosh(\beta h) - 2m_t^{\sigma\eta}\sinh(\beta h)\bigg) - 2N^{1/2}\,\dot{m}_t^\sigma\bigg]
\end{aligned}$$

$$+ \varphi_y\,\frac{N^{1/2}}{2}\bigg[e^{-\beta m_t^\sigma}\bigg(1 - \frac{\beta x}{N^{1/2}} + o\bigg(\frac{1}{N^{1/2}}\bigg)\bigg)\bigg(2\sinh(\beta h)$$

$$
\begin{aligned}
& - \frac{2r}{N^{1/2}} \cosh(\beta h) + \frac{2x}{N^{1/2}} \sinh(\beta h) - \frac{2y}{N^{1/2}} \cosh(\beta h) \\
& + 2m_t^{\sigma} \sinh(\beta h) - 2m_t^{\sigma\eta} \cosh(\beta h)\Big) \\
& \qquad + e^{\beta m_t^{\sigma}} \left(1 + \frac{\beta x}{N^{1/2}} + o\left(\frac{1}{N^{1/2}}\right)\right)\Big(2\sinh(\beta h) \\
& + \frac{2r}{N^{1/2}} \cosh(\beta h) - \frac{2x}{N^{1/2}} \sinh(\beta h) - \frac{2y}{N^{1/2}} \cosh(\beta h) \\
& - 2m_t^{\sigma} \sinh(\beta h) - 2m_t^{\sigma\eta} \cosh(\beta h)\Big) - 2N^{1/2}\, \dot{m}_t^{\sigma\eta}\Big]
\end{aligned}
$$

$$
\begin{aligned}
+ \varphi_{xx} \frac{1}{2} \Big[& e^{-\beta m_t^{\sigma}} \left(1 - \frac{\beta x}{N^{1/2}} + o\left(\frac{1}{N^{1/2}}\right)\right)\Big(2\cosh(\beta h) \\
& - \frac{2r}{N^{1/2}} \sinh(\beta h) + \frac{2x}{N^{1/2}} \cosh(\beta h) - \frac{2y}{N^{1/2}} \sinh(\beta h) \\
& + 2m_t^{\sigma} \cosh(\beta h) - 2m_t^{\sigma\eta} \sinh(\beta h)\Big) \\
& \qquad + e^{\beta m_t^{\sigma}} \left(1 + \frac{\beta x}{N^{1/2}} + o\left(\frac{1}{N^{1/2}}\right)\right)\Big(2\cosh(\beta h) \\
& + \frac{2r}{N^{1/2}} \sinh(\beta h) - \frac{2x}{N^{1/2}} \cosh(\beta h) - \frac{2y}{N^{1/2}} \sinh(\beta h) \\
& - 2m_t^{\sigma} \cosh(\beta h) - 2m_t^{\sigma\eta} \sinh(\beta h)\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
+ \varphi_{yy} \frac{1}{2} \Big[& e^{-\beta m_t^{\sigma}} \left(1 - \frac{\beta x}{N^{1/2}} + o\left(\frac{1}{N^{1/2}}\right)\right)\Big(2\cosh(\beta h) \\
& - \frac{2r}{N^{1/2}} \sinh(\beta h) + \frac{2x}{N^{1/2}} \cosh(\beta h) - \frac{2y}{N^{1/2}} \sinh(\beta h) \\
& + 2m_t^{\sigma} \cosh(\beta h) - 2m_t^{\sigma\eta} \sinh(\beta h)\Big) \\
& \qquad + e^{\beta m_t^{\sigma}} \left(1 + \frac{\beta x}{N^{1/2}} + o\left(\frac{1}{N^{1/2}}\right)\right)\Big(2\cosh(\beta h) \\
& + \frac{2r}{N^{1/2}} \sinh(\beta h) - \frac{2x}{N^{1/2}} \cosh(\beta h) - \frac{2y}{N^{1/2}} \sinh(\beta h) \\
& - 2m_t^{\sigma} \cosh(\beta h) - 2m_t^{\sigma\eta} \sinh(\beta h)\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \varphi_{xy} \Bigg[e^{-\beta m_t^\sigma} \left(1 - \frac{\beta x}{N^{1/2}} + o\left(\frac{1}{N^{1/2}} \right) \right) \Bigg(- 2\sinh(\beta h) \\
&\quad + \frac{2r}{N^{1/2}} \cosh(\beta h) - \frac{2x}{N^{1/2}} \sinh(\beta h) + \frac{2y}{N^{1/2}} \cosh(\beta h) \\
&\quad - 2m_t^\sigma \sinh(\beta h) - 2m_t^{\sigma\eta} \cosh(\beta h) \Bigg) \\
&\qquad + e^{\beta m_t^\sigma} \left(1 + \frac{\beta x}{N^{1/2}} + o\left(\frac{1}{N^{1/2}} \right) \right) \Bigg(2\sinh(\beta h) \\
&\quad + \frac{2r}{N^{1/2}} \cosh(\beta h) - \frac{2x}{N^{1/2}} \sinh(\beta h) - \frac{2y}{N^{1/2}} \cosh(\beta h) \\
&\quad - 2m_t^\sigma \sinh(\beta h) - 2m_t^{\sigma\eta} \cosh(\beta h) \Bigg) \Bigg] \\
&+ o(1)
\end{aligned}
$$

and, recalling equations (1.15), it yields

$$
\begin{aligned}
&= 2\varphi_x \Bigg[r \sinh(\beta h) \cosh(\beta m_t^\sigma) - x \cosh(\beta h) \cosh(\beta m_t^\sigma) \\
&\quad - y \sinh(\beta h) \sinh(\beta m_t^\sigma) + \beta x \cosh(\beta h) \cosh(\beta m_t^\sigma) \\
&\quad + \frac{\beta r x}{N^{1/2}} \sinh(\beta h) \sinh(\beta m_t^\sigma) - \frac{\beta x^2}{N^{1/2}} \cosh(\beta h) \sinh(\beta m_t^\sigma) \\
&\quad - \frac{\beta x y}{N^{1/2}} \sinh(\beta h) \cosh(\beta m_t^\sigma) - \beta x m_t^\sigma \cosh(\beta h) \sinh(\beta m_t^\sigma) \\
&\quad - \beta x m_t^{\sigma\eta} \sinh(\beta h) \cosh(\beta m_t^\sigma) \Bigg] \\
&+ 2\varphi_y \Bigg[r \cosh(\beta h) \sinh(\beta m_t^\sigma) - x \sinh(\beta h) \sinh(\beta m_t^\sigma) \\
&\quad - y \cosh(\beta h) \cosh(\beta m_t^\sigma) + \beta x \sinh(\beta h) \sinh(\beta m_t^\sigma) \\
&\quad + \frac{\beta r x}{N^{1/2}} \cosh(\beta h) \cosh(\beta m_t^\sigma) - \frac{\beta x^2}{N^{1/2}} \sinh(\beta h) \cosh(\beta m_t^\sigma) \\
&\quad - \frac{\beta x y}{N^{1/2}} \cosh(\beta h) \sinh(\beta m_t^\sigma) - \beta x m_t^\sigma \sinh(\beta h) \cosh(\beta m_t^\sigma) \\
&\quad - \beta x m_t^{\sigma\eta} \cosh(\beta h) \sinh(\beta m_t^\sigma) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\varphi_{xx}\Bigg[\cosh(\beta h)\cosh(\beta m_t^\sigma) + \frac{r}{N^{1/2}}\sinh(\beta h)\sinh(\beta m_t^\sigma) \\
&\qquad - \frac{x}{N^{1/2}}\cosh(\beta h)\sinh(\beta m_t^\sigma) - \frac{y}{N^{1/2}}\sinh(\beta h)\cosh(\beta m_t^\sigma) \\
&\qquad - m_t^\sigma\cosh(\beta h)\sinh(\beta m_t^\sigma) - m_t^{\sigma\eta}\sinh(\beta h)\cosh(\beta m_t^\sigma)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\varphi_{yy}\Bigg[\cosh(\beta h)\cosh(\beta m_t^\sigma) + \frac{r}{N^{1/2}}\sinh(\beta h)\sinh(\beta m_t^\sigma) \\
&\qquad - \frac{x}{N^{1/2}}\cosh(\beta h)\sinh(\beta m_t^\sigma) - \frac{y}{N^{1/2}}\sinh(\beta h)\cosh(\beta m_t^\sigma) \\
&\qquad - m_t^\sigma\cosh(\beta h)\sinh(\beta m_t^\sigma) - m_t^{\sigma\eta}\sinh(\beta h)\cosh(\beta m_t^\sigma)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ 4\varphi_{xy}\Bigg[\sinh(\beta h)\sinh(\beta m_t^\sigma) + \frac{r}{N^{1/2}}\cosh(\beta h)\cosh(\beta m_t^\sigma) \\
&\qquad - \frac{x}{N^{1/2}}\sinh(\beta h)\cosh(\beta m_t^\sigma) - \frac{y}{N^{1/2}}\cosh(\beta h)\sinh(\beta m_t^\sigma) \\
&\qquad - m_t^\sigma\sinh(\beta h)\cosh(\beta m_t^\sigma) - m_t^{\sigma\eta}\cosh(\beta h)\sinh(\beta m_t^\sigma)\Bigg]
\end{aligned}
$$

$$
+ o(1)
$$

Now, as $N \longrightarrow +\infty$, thanks to the Central Limit Theorem for independent identically distributed random variables, the variable r converges to $\mathscr{H}$, which is a Standard Gaussian random variable.
Moreover, $\mathcal{H}_{N,t}f(r,x,y) \xrightarrow{N\to+\infty} \mathcal{H}_t f(r,x,y)$, where:

$$
\begin{aligned}
\mathcal{H}_t\varphi(r,x,y) = 2\varphi_x\Bigg[&x\Bigg((\beta-1)\cosh(\beta h)\cosh(\beta m_t^\sigma) - \beta m_t^\sigma\cosh(\beta h)\sinh(\beta m_t^\sigma) \\
&-\beta m_t^{\sigma\eta}\sinh(\beta h)\cosh(\beta m_t^\sigma)\Bigg) - y\sinh(\beta h)\sinh(\beta m_t^\sigma) \\
&+\mathscr{H}\sinh(\beta h)\cosh(\beta m_t^\sigma)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
+2\varphi_y\Big[&x\Big((\beta-1)\sinh(\beta h)\sinh(\beta m_t^\sigma) - \beta m_t^\sigma\sinh(\beta h)\cosh(\beta m_t^\sigma) \\
&-\beta m_t^{\sigma\eta}\cosh(\beta h)\sinh(\beta m_t^\sigma)\Big) - y\cosh(\beta h)\cosh(\beta m_t^\sigma)
\end{aligned}
$$

$$+\mathscr{H}\cosh(\beta h)\sinh(\beta m_t^\sigma)\Big]$$

$$+2\varphi_{xx}\Big[-m_t^\sigma\cosh(\beta h)\sinh(\beta m_t^\sigma)-m_t^{\sigma\eta}\sinh(\beta h)\cosh(\beta m_t^\sigma)$$
$$+\cosh(\beta h)\cosh(\beta m_t^\sigma)\Big]$$

$$+2\varphi_{yy}\Big[-m_t^\sigma\cosh(\beta h)\sinh(\beta m_t^\sigma)-m_t^{\sigma\eta}\sinh(\beta h)\cosh(\beta m_t^\sigma)$$
$$+\cosh(\beta h)\cosh(\beta m_t^\sigma)\Big]$$

$$+4\varphi_{xy}\Big[-m_t^\sigma\sinh(\beta h)\cosh(\beta m_t^\sigma)-m_t^{\sigma\eta}\cosh(\beta h)\sinh(\beta m_t^\sigma)$$
$$+\sinh(\beta h)\sinh(\beta m_t^\sigma)\Big]$$

The just found generator $\mathcal{H}_t$ is the infinitesimal generator of the linear diffusion process which corresponds to the unique solution of (1.22). In view of Theorem 1.3.1, we complete the proof if we show that $(r_N(0), x_N(0), y_N(0))$ converges in distribution to $(r(0), x(0), y(0))$, when $N \longrightarrow +\infty$, and if we can prove the analogous condition of $(\star)$.
The first statement is implied by the Central Limit Theorem for independent, identically distributed random variables: in fact, by hypothesis, $(\sigma_j(0))_{j=1}^N$ and $(\eta_j)_{j=1}^N$ are independent with common law λ and μ respectively and (1.23) is the covariance matrix under the joint measure (λ, μ) of $(\sigma(0), \eta(0))$.
In regard to $(\star)$, we need to overcome the fact that the generators $\mathcal{H}_{N,t}$ and $\mathcal{H}_t$, we are dealing with, for which we need convergence, are time-dependent. The trick works as follows: we consider time as an additional variable. It means we introduce another process $\tau(t) := t$ (which is deterministic and therefore clearly Markovian) and we substitute the unknown vector (r, x, y) with the new one (r, x, y, τ). In this way, $\mathcal{H}_{N,t}\varphi(r, x, y) = \mathcal{H}_N\varphi(r, x, y, \tau)$ and $\mathcal{H}_t\varphi(r, x, y) = \mathcal{H}\varphi(r, x, y, \tau)$ and the problem is solved.
Moreover, for $\varphi \in \mathcal{C}_b^3$ and $(r, x, y, \tau) \in [-1, +1]^3 \times [0, T]$ (which plays the role of $\mathscr{X}_n$ in Theorem 1.3.1), it is obviously true that

$$\lim_{N\to+\infty}\sup_{(r,x,y,\tau)}|\mathcal{H}_N\varphi(r,x,y,\tau)-\mathcal{H}\varphi(r,x,y,\tau)|=0\,,$$

because the difference between the generators is of order $o(1)$ with respect to N. ■

Remark 1.3.1. The drift terms $\mathscr{H}\, A_1(t)dt$ in (1.22) marks the relevant difference with respect to the homogeneous case. The survival of this term in the critical regime is responsible for the dynamics of critical fluctuations, as shown in next section.

Theorem 1.3.2 guarantees that the distribution of $(x_N(t), y_N(t))$ is asymptotically Gaussian for every $t > 0$ and provides a method to compute the limiting covariance matrix. Indeed, if we denote by Σ_t the covariance matrix of $(x(t), y(t))$, then we can verify

Proposition 1.3.1. *The covariance matrix Σ_t solves the linear Lyapunov equation*

$$\frac{d\Sigma_t}{dt} = 2A_2(t)\Sigma_t + 2\Sigma_t A_2'(t) + D(t)D'(t)\,. \tag{1.24}$$

Proof. We consider the column vector $X(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and its transpose, the row vector $X'(t) := (x(t), y(t))$. Thanks to (1.22), we know these vectors have the following stochastic differentials:

$$\begin{aligned} dX(t) &= 2A_1(t)dt + 2A_2(t)X(t)dt + D(t)dB(t) \\ dX'(t) &= 2A_1'(t)dt + 2X'(t)A_2'(t)dt + dB'(t)D'(t)\,, \end{aligned}$$

where $dB(t)$ is the column vector whose components are the independent, Standard Brownian motions appearing in the statement of Theorem 1.3.2. Using Itô's Formula, we get

$$\begin{aligned} d(X(t)X'(t)) &= X(t)dX'(t) + dX(t)X'(t) + dX(t)dX'(t) \\ &= X(t)[2A_1'(t)dt + 2X'(t)A_2'(t)dt + dB'(t)D'(t)] \\ &\quad + [2A_1(t)dt + 2A_2(t)X(t)dt + D(t)dB(t)]X'(t) + D(t)D'(t)dt \\ &= 2X(t)A_1'(t)dt + 2X(t)X'(t)A_2'(t)dt + X(t)dB'(t)D'(t) \\ &\quad + 2A_1(t)X'(t)dt + 2A_2(t)X(t)X'(t)dt + D(t)dB(t)X'(t) \\ &\quad + D(t)D'(t)dt \end{aligned}$$

and thus, if we take the expectation, we obtain

$$\begin{aligned} d\Sigma_t = dE[X(t)X'(t)] &= 2E[X(t)X'(t)]A_2'(t)dt + 2A_2(t)E[X(t)X'(t)] \\ &\quad + D(t)D'(t)dt \\ &= 2\Sigma_t A_2'(t)dt + 2A_2(t)\Sigma_t dt + D(t)D'(t)dt\,, \end{aligned}$$

since we have $E[X(t)dB'(t)D'(t)] = E[D(t)dB(t)X'(t)] = \mathbf{0}$, for the properties of Brownian motion, and $E[X(t)A_1'(t)] = E[A_1(t)X'(t)] = \mathbf{0}$, because the vectors $X(t)$ and $X'(t)$ are centered Gaussian vectors. ■

In order to solve equation (1.24), it is convenient to interpret Σ_t as a vector in $\mathbb{R}^{2\times 2} = \mathbb{R}^2 \otimes \mathbb{R}^2$, where $\otimes$ denotes the tensor product. For every 2×2 matrix Θ, we will write $\mathrm{vec}(\Theta)$ whenever we interpret it as a vector belonging to the tensor space just introduced. We can rewrite (1.24) as

$$\frac{d(\mathrm{vec}(\Sigma_t))}{dt} = 2[A_2(t)\otimes I + I \otimes A_2'(t)]\mathrm{vec}(\Sigma_t) + \mathrm{vec}(D(t)D'(t))\,, \tag{1.25}$$

where we used tensor product of matrices. Equation (1.25) is linear, so an explicit expression of its solutions can be given and it can be computed after having solved (1.15). The analysis of Σ_t for large t can be made explicitly and it is strictly related to the spectrum of the limiting matrix

$$A_* := \lim_{t\to+\infty} A_2(t)\,,$$

which is the drift matrix of the linearized system of (1.22) in the limit of stationarity and it is given by

$$A_* = \begin{bmatrix} \frac{\beta-\cosh^2(\beta(m_*^\sigma+h))}{2\cosh(\beta(m_*^\sigma+h))} + \frac{\beta-\cosh^2(\beta(m_*^\sigma-h))}{2\cosh(\beta(m_*^\sigma-h))} & -\sinh(\beta h)\sinh(\beta m_*^\sigma) \\ \\ \frac{\beta-\cosh^2(\beta(m_*^\sigma+h))}{2\cosh(\beta(m_*^\sigma+h))} - \frac{\beta-\cosh^2(\beta(m_*^\sigma-h))}{2\cosh(\beta(m_*^\sigma-h))} & -\cosh(\beta h)\cosh(\beta m_*^\sigma) \end{bmatrix},$$

once we have recalled that $\lim_{t\to+\infty}(m_t^\sigma, m_t^{\sigma\eta}) = (m_*^\sigma, m_*^{\sigma\eta})$, defined in (1.16a) and (1.16b), and we have made the due substitutions and computations.

1.4 Critical Dynamics ($\beta = \cosh^2(\beta h)$)

We are going to consider the critical dynamics of the system, in other words the long-time behavior of the fluctuations in the threshold case, when $\beta = \cosh^2(\beta h)$.

This condition of critical point for the parameters does not identify only the passage from unicity and non-unicity of the stationary solution for the limiting dynamics; but it also individuates the transition from 1 to 2 ferromagnetic solutions. Referring to Figure 1.1, we will describe the behavior of the fluctuations on the boundary between regions *0* and *1* and regions *1* and *2*, while we do not know what happens along the dotted separation curve between phases *0* and *2*.

In the previous section we proved that in a time interval $[0, T]$, where T is fixed, and in the infinite volume limit, we have Normal fluctuations for the system. Indeed, the infinitesimal generator of the rescaled process converges to the infinitesimal generator of a diffusion and the rescaled process itself converges weakly to that diffusion. It means we showed a Central Limit Theorem for all the values of β. But what does it change when we are in this critical case?
The particularity of this situation is that the Central Limit Theorem continues to be valid but there is an eigenvalue for the covariance matrix Σ_t which grows polynomially in t. This fact implies that the size of the Normal fluctuations must be further rescaled (in space and in time), because their size around the deterministic limit increases in time. In this case we will still obtain Normal fluctuations, solutions of a certain stochastic differential equation to be determined.

First of all, we need to locate the critical direction in the three-dimensional space of the order parameters. In the rest of the section, we will consider $\beta = \cosh^2(\beta h)$ and let us assume that the initial condition λ is a product measure such that

$$m_0^\sigma = 0, \qquad m_0^{\sigma\eta} = \tanh(\beta h)$$

and so

$$m_t^\sigma = 0, \qquad m_t^{\sigma\eta} = \tanh(\beta h),$$

for every value of $t \geq 0$, since it is an equilibrium solution.
Under these assumptions the drift matrix, A_2, of the linearized system of (1.22) and the infinitesimal generators $\mathcal{K}_N$, $\mathcal{H}_N$ and $\mathcal{H}$ are independent of t. In particular, equation (1.22) becomes

$$\begin{aligned} dr(t) &= 0 \\ \begin{bmatrix} dx(t) \\ dy(t) \end{bmatrix} &= 2\,\mathscr{H} \underbrace{\begin{bmatrix} \sinh(\beta h) \\ 0 \end{bmatrix}}_{:=A_1^{cr}} dt + 2 \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -\cosh(\beta h) \end{bmatrix}}_{:=A_2^{cr}} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} dt + D^{cr} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}, \end{aligned}$$

where the matrix D^{cr} satisfies

$$\frac{(D^{cr})(D^{cr})'}{2} = \begin{bmatrix} \frac{1}{\cosh(\beta h)} & 0 \\ 0 & \frac{1}{\cosh(\beta h)} \end{bmatrix},$$

and, as before, B_1, B_2 are independent Standard Brownian motions and $\mathscr{H}$ is a Standard Gaussian random variable.

As $t \longrightarrow +\infty$, the covariance matrix Σ_t^{cr}, solution of the Lyapunov equation

$$\frac{d\Sigma_t^{cr}}{dt} = 2\, A_2^{cr}\, \Sigma_t^{cr} + 2\, \Sigma_t^{cr} (A_2^{cr})' + (D^{cr})(D^{cr})',$$

becomes a diagonal matrix with an eigenvalue growing polynomially in t. Its (right) eigenvectors tend to the (right) eigenvectors of the drift matrix A_2^{cr}. The critical direction is determined by the (right) eigenvector corresponding to the eigenvalue increasing to infinity of the covariance matrix Σ_t^{cr}, which is also the (right) eigenvector corresponding to the null eigenvalue of the matrix A_2^{cr}. Hence, in our case, the critical direction is x.

Remark 1.4.1. Notice that the critical direction x does not depend explicitly on the random environment and it is one-dimensional.

Theorem 1.4.1. *For $t \in [0, T]$, if we consider the three-dimensional critical fluctuation process*

$$\begin{aligned} r_N(t) &:= N^{1/2} m^{\eta}_{\rho_N(t)} \\ \widetilde{x}_N(t) &:= N^{1/4} m^{\sigma}_{\rho_N(N^{1/4}t)} \\ \widetilde{y}_N(t) &:= N^{1/4} \left(m^{\sigma\eta}_{\rho_N(N^{1/4}t)} - \tanh(\beta h) \right), \end{aligned} \tag{1.26}$$

then, as $N \longrightarrow +\infty$, $r_N(t)$ converges to $\mathscr{H}$, a Standard Gaussian random variable, $\widetilde{y}_N(t) \longrightarrow 0$ in the sense of Proposition 1.4.1 and $\widetilde{x}_N(t)$ converges, in the sense of weak convergence of stochastic processes, to a limiting Gaussian process

$$\widetilde{x}(t) = 2\, \mathscr{H} \sinh(\beta h) t\,.$$

1.4.1 Proof of the Theorem 1.4.1

Before approaching the proof, we try to underline the main ideas developed in it. We make an attempt to explain what is going on.

When we have decided the right time-rescaling, we define a sequence of stopping times, allowing us to define a family of random time-interval, on which the processes $\widetilde{x}_N(t)$ and $\widetilde{y}_N(t)$ are bounded (we will see the process $r_N(t)$ is already bounded with high probability). For t fixed in one of such an interval, it happens that the non-critical direction $\widetilde{y}_N(t)$ vanishes or, more precisely, collapses. Then, we prove the probability that the stopping times exceed a given time T is very small, so we can deduce all the considerations we made are valid in the whole time-interval $[0, T]$. The last step is to verify the critical direction admits a limit and to compute it. It will be done using an argument of tightness, applied to a suitable martingale problem.

To have a complete picture of our setting, let us recall the characterization of *collapsing process*, in the sense described by Comets and Eisele (see [CE88]) and slightly generalized in [Sar07].

Proposition 1.4.1. *Let $\{X_n(t)\}_{n\geq 1}$ be a sequence of positive semimartingales on a probability space $(\Omega, \mathscr{A}, \mathscr{P})$, with*

$$dX_n(t) = S_n(t)dt + \int_{\mathbb{R}^+\times\mathscr{Y}} f_n(t^-, y)[\Lambda_n(dt, dy) - A_n(t, dy)dt]$$

Here, Λ_n is a Point Process of intensity $A_n(t, dy)dt$ on $\mathbb{R}^+ \times \mathscr{Y}$, where $\mathscr{Y}$ is a measurable space, and $S_n(t)$ and $f_n(t)$ are $\mathscr{A}_t$-adapted processes, if we consider $(\mathscr{A}_t)_{t\geq 0}$ a filtration on $(\Omega, \mathscr{A}, \mathscr{P})$ generated by Λ_n.
Let $d > 1$ and C_i constants independent of n and t. Suppose there exist $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$, increasing sequences with

$$n^{1/d}\alpha_n^{-1} \xrightarrow{n\to+\infty} 0,\ n^{-1}\alpha_n \xrightarrow{n\to+\infty} 0,\ n^{-1}\beta_n \xrightarrow{n\to+\infty} 0$$

and

$$E\Big[\Big(X_n(0)\Big)^d\Big] \leq C_1\alpha_n^{-d} \qquad \textit{for all } n\,.$$

Furthermore, let $\{\tau_n\}_{n\geq 1}$ be stopping times such that for $t \in [0, \tau_n]$ and $n \geq 1$,

$$S_n(t) \leq -n\delta X_n(t) + \beta_n C_2 + C_3 \qquad \textit{with } \delta > 0,$$

$$\sup_{\omega\in\Omega, y\in\mathscr{Y}, t\leq\tau_n} |f_n(t, y)| \leq C_4\alpha_n^{-1}\,.$$

Hence:

(a) if it holds

$$\int_{\mathscr{Y}} (f_n(t,y))^2 A_n(t,dy) \leq C_5\,, \tag{$\star\star$}$$

then, for any $\varepsilon > 0$, there exist $C_6 > 0$ and n_0 such that

$$\sup_{n\geq n_0} \mathscr{P}\left\{ \sup_{0\leq t\leq T\wedge\tau_n} X_n(t) > C_6(n^{1/d}\alpha_n^{-1} \vee \alpha_n n^{-1}) \right\} \leq \varepsilon\,; \tag{$\star\star\star$}$$

(b) if instead of $(\star\star)$ we have

$$\int_{\mathscr{Y}} (f_n(t,y))^2 A_n(t,dy) \leq C_5(X_n(t) + n^{-1})\,,$$

then, instead of $(\star\star\star)$, we get

$$\sup_{n\geq n_0} \mathscr{P}\left\{ \sup_{0\leq t\leq T\wedge\tau_n} X_n(t) > C_6(n^{1/d}\alpha_n^{-1} \vee \beta_n n^{-1}) \right\} \leq \varepsilon\,.$$

┘

Now, we can start to deal with the proof.

Phase 1: The identification of the time-rescaling. We first rescale the space by the "standard" critical factor $N^{1/4}$ and determine the evolution of the rescaled magnetizations. Later, we identify the right time-rescaling that leads to a nontrivial limit as $N \longrightarrow +\infty$.

Lemma 1.4.1. *For $t \in [0,T]$, if we consider only the space scaling*

$$\begin{aligned} r_N(t) &= N^{1/2} m^{\eta}_{\rho_N(t)} \\ \bar{x}_N(t) &= N^{1/4} m^{\sigma}_{\rho_N(t)} \\ \bar{y}_N(t) &= N^{1/4}\left(m^{\sigma\eta}_{\rho_N(t)} - \tanh(\beta h) \right)\,, \end{aligned} \tag{1.27}$$

then $(r_N(t), \bar{x}_N(t), \bar{y}_N(t))$ is a Markov process whose infinitesimal generator satisfies:

$$\begin{aligned} \mathcal{G}_N \psi(r,\bar{x},\bar{y}) &= 2\psi_{\bar{x}}\left[r\frac{\sinh(\beta h)}{N^{1/4}} - \beta\,\bar{x}\,\bar{y}\,\frac{\sinh(\beta h)}{N^{1/4}} \right] \\ &\quad + 2\psi_{\bar{y}}\left[\beta\, r\,\bar{x}\,\frac{\cosh(\beta h)}{N^{1/2}} - \bar{y}\,\cosh(\beta h) - \beta\bar{x}^{\,2}\,\frac{\sinh(\beta h)}{N^{1/4}} \right] \\ &\quad + o\left(\frac{1}{N^{1/4}}\right), \end{aligned} \tag{1.28}$$

where the remainders are continuous functions of $(r,\bar{x},\bar{y})$ and they are of order $o(\frac{1}{N^{1/4}})$ pointwise, but not uniformly in $(r,\bar{x},\bar{y})$.

Proof. We obtained from Theorem 1.3.2 that the triple $(r_N(t), x_N(t), y_N(t))$ is a Markov process with infinitesimal generator $\mathcal{H}_N$. Making the same considerations as before (see Lemma 1.3.2) and considering a function $\psi : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $\psi \in \mathcal{C}_b^3$, we know that

$$\mathcal{H}_N(\psi(r_N(t), \bar{x}_N(t), \bar{y}_N(t))) = (\mathcal{G}_N\psi)(r_N(t), \bar{x}_N(t), \bar{y}_N(t)) \,,$$

with

$$\mathcal{G}_N\psi(r, \bar{x}, \bar{y}) = \sum_{j,k\in\mathscr{S}} |A_N(j,k)|\, e^{-\beta j\left(\frac{\bar{x}}{N^{1/4}}+kh\right)} \cdot \\ \cdot \left[\psi\left(r, \bar{x} - j\frac{2}{N^{3/4}}, \bar{y} - jk\frac{2}{N^{3/4}}\right) - \psi(r, \bar{x}, \bar{y})\right]$$

and where

$$|A_N(j,k)| = \frac{N}{4}\left[1 + k\frac{r}{N^{1/2}} + j\frac{\bar{x}}{N^{1/4}} + jk\left(\frac{\bar{y}}{N^{1/4}} + \tanh(\beta h)\right)\right].$$

At least at the beginning of the standard computations we are going to perform, we consider the following representation for the Taylor expansions of the exponential functions

$$e^{\beta\frac{\bar{x}}{N^{1/4}}} = 1 + \frac{\beta\bar{x}}{N^{1/4}} + R_+ \quad \text{and} \quad e^{-\beta\frac{\bar{x}}{N^{1/4}}} = 1 - \frac{\beta\bar{x}}{N^{1/4}} + R_- \,. \tag{1.29}$$

Later on we will need more accurate estimates of R_+ and R_-. Moreover, we develop also ψ by a Taylor expansion stopped at second order.

$$\mathcal{G}_N\psi(r, \bar{x}, \bar{y}) = \sum_{j,k\in\mathscr{S}} |A_N(j,k)|\, e^{-\beta jkh}\left(1 - j\frac{\beta\bar{x}}{N^{1/4}} + R_{\mathrm{sgn}(-j)}\right)\left[-j\frac{2}{N^{3/4}}\psi_{\bar{x}} \right. \\ \left. - jk\frac{2}{N^{3/4}}\psi_{\bar{y}} + \frac{2}{N^{3/2}}\psi_{\bar{x}\bar{x}} + \frac{2}{N^{3/2}}\psi_{\bar{y}\bar{y}} + k\frac{4}{N^{3/2}}\psi_{\bar{x}\bar{y}} + o\left(\frac{1}{N^{3/2}}\right)\right]$$

$$= \psi_{\bar{x}}\,\frac{N^{1/4}}{2}\left[\left(1 - \frac{\beta\bar{x}}{N^{1/4}} + R_-\right)\left(-2\cosh(\beta h) + \frac{2r}{N^{1/2}}\sinh(\beta h) \right.\right. \\ \left. - \frac{2\bar{x}}{N^{1/4}}\cosh(\beta h) + \frac{2\bar{y}}{N^{1/4}}\sinh(\beta h) + 2\tanh(\beta h)\sinh(\beta h)\right) \\ + \left(1 + \frac{\beta\bar{x}}{N^{1/4}} + R_+\right)\left(2\cosh(\beta h) + \frac{2r}{N^{1/2}}\sinh(\beta h)\right.$$

$$- \frac{2\bar{x}}{N^{1/4}} \cosh(\beta h) - \frac{2\bar{y}}{N^{1/4}} \sinh(\beta h) - 2\tanh(\beta h)\sinh(\beta h)\Big)$$

$$+ \psi_{\bar{y}} \frac{N^{1/4}}{2} \Bigg[\Big(1 - \frac{\beta\bar{x}}{N^{1/4}} + R_-\Big)\Big(2\sinh(\beta h) - \frac{2r}{N^{1/2}} \cosh(\beta h)$$
$$+ \frac{2\bar{x}}{N^{1/4}} \sinh(\beta h) - \frac{2\bar{y}}{N^{1/4}} \cosh(\beta h) - 2\tanh(\beta h)\cosh(\beta h)\Big)$$
$$+ \Big(1 + \frac{\beta\bar{x}}{N^{1/4}} + R_+\Big)\Big(2\sinh(\beta h) + \frac{2r}{N^{1/2}} \cosh(\beta h)$$
$$- \frac{2\bar{x}}{N^{1/4}} \sinh(\beta h) - \frac{2\bar{y}}{N^{1/4}} \cosh(\beta h) - 2\tanh(\beta h)\cosh(\beta h)\Big)\Bigg]$$

$$+ \psi_{\bar{x}\bar{x}} \frac{1}{2N^{1/2}} \Bigg[\Big(1 - \frac{\beta\bar{x}}{N^{1/4}} + R_-\Big)\Big(2\cosh(\beta h) - \frac{2r}{N^{1/2}} \sinh(\beta h)$$
$$+ \frac{2\bar{x}}{N^{1/4}} \cosh(\beta h) - \frac{2\bar{y}}{N^{1/4}} \sinh(\beta h) - 2\tanh(\beta h)\sinh(\beta h)\Big)$$
$$+ \Big(1 + \frac{\beta\bar{x}}{N^{1/4}} + R_+\Big)\Big(2\cosh(\beta h) + \frac{2r}{N^{1/2}} \sinh(\beta h)$$
$$- \frac{2\bar{x}}{N^{1/4}} \cosh(\beta h) - \frac{2\bar{y}}{N^{1/4}} \sinh(\beta h) - 2\tanh(\beta h)\sinh(\beta h)\Big)$$

$$+ \psi_{\bar{y}\bar{y}} \frac{1}{2N^{1/2}} \Bigg[\Big(1 - \frac{\beta\bar{x}}{N^{1/4}} + R_-\Big)\Big(2\cosh(\beta h) - \frac{2r}{N^{1/2}} \sinh(\beta h)$$
$$+ \frac{2\bar{x}}{N^{1/4}} \cosh(\beta h) - \frac{2\bar{y}}{N^{1/4}} \sinh(\beta h) - 2\tanh(\beta h)\sinh(\beta h)\Big)$$
$$+ \Big(1 + \frac{\beta\bar{x}}{N^{1/4}} + R_+\Big)\Big(2\cosh(\beta h) + \frac{2r}{N^{1/2}} \sinh(\beta h)$$
$$- \frac{2\bar{x}}{N^{1/4}} \cosh(\beta h) - \frac{2\bar{y}}{N^{1/4}} \sinh(\beta h) - 2\tanh(\beta h)\sinh(\beta h)\Big)\Bigg]$$

$$+ \psi_{\bar{x}\bar{y}} \frac{1}{N^{1/2}} \Bigg[\Big(1 - \frac{\beta\bar{x}}{N^{1/4}} + R_-\Big)\Big(- 2\sinh(\beta h) + \frac{2r}{N^{1/2}} \cosh(\beta h)$$
$$- \frac{2\bar{x}}{N^{1/4}} \sinh(\beta h) + \frac{2\bar{y}}{N^{1/4}} \cosh(\beta h) + 2\tanh(\beta h)\cosh(\beta h)\Big)$$

$$
\begin{aligned}
&+\left(1+\frac{\beta\bar{x}}{N^{1/4}}+R_{+}\right)\left(2\sinh(\beta h)+\frac{2r}{N^{1/2}}\cosh(\beta h)\right.\\
&\left.\left.-\frac{2\bar{x}}{N^{1/4}}\sinh(\beta h)-\frac{2\bar{y}}{N^{1/4}}\cosh(\beta h)-2\tanh(\beta h)\cosh(\beta h)\right)\right]
\end{aligned}
$$

$$
+\,o\left(\frac{1}{N^{1/2}}\right)
$$

$$
\begin{aligned}
=\psi_{\bar{x}}\,2N^{1/4}\Bigg[&\frac{r}{N^{1/2}}\sinh(\beta h)-\frac{\bar{x}}{N^{1/4}}\cosh(\beta h)+\frac{\beta\bar{x}}{N^{1/4}}\cosh(\beta h)\\
&-\frac{\beta\bar{x}\bar{y}}{N^{1/2}}\sinh(\beta h)-\frac{\beta\bar{x}}{N^{1/4}}\tanh(\beta h)\sinh(\beta h)\\
&+\frac{1}{2}\left(\frac{r}{N^{1/2}}\sinh(\beta h)-\frac{\bar{x}}{N^{1/4}}\cosh(\beta h)\right)(R_{+}+R_{-})\\
&+\frac{1}{2}\left(\frac{1}{\cosh(\beta h)}-\frac{\bar{y}}{N^{1/4}}\sinh(\beta h)\right)(R_{+}-R_{-})\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
+\,\psi_{\bar{y}}\,2N^{1/4}\Bigg[&-\frac{\bar{y}}{N^{1/4}}\cosh(\beta h)+\frac{\beta r\bar{x}}{N^{3/4}}\cosh(\beta h)-\frac{\beta\bar{x}^{2}}{N^{1/2}}\sinh(\beta h)\\
&-\frac{\bar{y}}{2N^{1/4}}\cosh(\beta h)(R_{+}+R_{-})+\frac{1}{2}\left(\frac{r}{N^{1/2}}\cosh(\beta h)\right.\\
&\left.-\frac{\bar{x}}{N^{1/4}}\sinh(\beta h)\right)(R_{+}-R_{-})\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
+\,\psi_{\bar{x}\bar{x}}\,\frac{2}{N^{1/2}}\Bigg[&\cosh(\beta h)-\frac{\bar{y}}{N^{1/4}}\sinh(\beta h)-\tanh(\beta h)\sinh(\beta h)\\
&+\frac{\beta r\bar{x}}{N^{3/4}}\sinh(\beta h)-\frac{\beta\bar{x}^{2}}{N^{1/2}}\cosh(\beta h)+o\left(\frac{1}{N^{1/4}}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
+\,\psi_{\bar{y}\bar{y}}\,\frac{2}{N^{1/2}}\Bigg[&\cosh(\beta h)-\frac{\bar{y}}{N^{1/4}}\sinh(\beta h)-\tanh(\beta h)\sinh(\beta h)\\
&+\frac{\beta r\bar{x}}{N^{3/4}}\sinh(\beta h)-\frac{\beta\bar{x}^{2}}{N^{1/2}}\cosh(\beta h)+o\left(\frac{1}{N^{1/4}}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ \psi_{\bar{x}\bar{y}} \frac{4}{N^{1/2}} \left[\frac{r}{N^{1/2}} \cosh(\beta h) - \frac{\bar{x}}{N^{1/4}} \sinh(\beta h) - \frac{\beta \bar{x}\bar{y}}{N^{1/2}} \cosh(\beta h) \right. \\
&\qquad \left. + o\left(\frac{1}{N^{1/4}}\right) \right] + o\left(\frac{1}{N^{1/2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \psi_{\bar{x}}\, 2 \left[r \frac{\sinh(\beta h)}{N^{1/4}} + \underbrace{\bar{x} \frac{[-\cosh^2(\beta h) + \beta]}{\cosh(\beta h)}}_{\text{it vanishes in the critical case}} - \frac{\beta \bar{x}\bar{y}}{N^{1/4}} \sinh(\beta h) \right. \\
&\qquad + \frac{1}{2}\left(r \frac{\sinh(\beta h)}{N^{1/4}} - \bar{x} \cosh(\beta h) \right)(R_+ + R_-) \\
&\qquad \left. + \frac{1}{2}\left(\frac{N^{1/4}}{\cosh(\beta h)} - \bar{y} \sinh(\beta h) \right)(R_+ - R_-) \right]
\end{aligned}
$$

$$
+ \psi_{\bar{y}}\, 2 \left[-\bar{y}\cosh(\beta h) + \frac{\beta r \bar{x}}{N^{1/2}} \cosh(\beta h) - \frac{\beta \bar{x}^2}{N^{1/4}} \sinh(\beta h) \right. \tag{1.30}
$$

$$
- \bar{y} \frac{\cosh(\beta h)}{2}(R_+ + R_-) + \frac{1}{2}\left(r \frac{\cosh(\beta h)}{N^{1/4}} \right. \tag{1.31}
$$

$$
\left. \left. - \bar{x}\sinh(\beta h) \right)(R_+ - R_-) \right] + o\left(\frac{1}{N^{1/2}}\right) \tag{1.32}
$$

$$
= \psi_{\bar{x}}\, 2 \left[r \frac{\sinh(\beta h)}{N^{1/4}} - \frac{\beta \bar{x}\bar{y}}{N^{1/4}} \sinh(\beta h) \right]
$$

$$
+ \psi_{\bar{y}}\, 2 \left[-\bar{y}\cosh(\beta h) + \frac{\beta r \bar{x}}{N^{1/2}} \cosh(\beta h) - \frac{\beta \bar{x}^2}{N^{1/4}} \sinh(\beta h) \right]
$$

$$
+ o\left(\frac{1}{N^{1/4}}\right)
$$

which is just (1.28). ■

Phase 2: The process $\tilde{y}_N(t)$ collapses. Let us denote by $\{\tau_N^M\}_{N \geq 1}$ a family of stopping times, defined as

$$
\tau_N^M := \inf_{t \geq 0} \{ |\tilde{x}_N(t)| \geq M \quad \text{or} \quad |\tilde{y}_N(t)| \geq M \},
$$

where M is a positive constant. We are interested in introducing such sequence of stopping times because in this way the processes $\widetilde{x}_N(t)$ and $\widetilde{y}_N(t)$ result to be bounded in the time interval $[0, T \wedge \tau_N^M]$; $r_N(t)$ is still bounded for $t \in [0,T]$. In fact we can prove

Lemma 1.4.2. *For every $\varepsilon > 0$ there exists $M > 0$ such that the process $r_N(t)$, defined as in (1.26), satisfies*

$$P\left\{|r_N(t)| \geq M\right\} \leq \varepsilon$$

for $t \in [0,T]$.

Proof. The process $r_N(t)$ is a constant process and then, for every $t \in [0,T]$, we have $r_N(t) \equiv r_N(0)$, which is a sample average of independent, identically distributed Bernoulli random variables multiplied by $N^{1/2}$. So, by Central Limit Theorem, for any $\varepsilon > 0$, for every N and for a sufficiently large M

$$P\{|r_N(0)| \geq M\} \leq \varepsilon\,.$$

■

We consider the infinitesimal generator, $\mathcal{J}_N = N^{1/4}\mathcal{G}_N$, subject to the time-rescaling and we apply it to the particular function $\psi(r_N(t), \widetilde{x}_N(t), \widetilde{y}_N(t)) = (\widetilde{y}_N(t))^2$. We choose this kind of function since $(\widetilde{y}_N(t))^2$ is a sequence of positive semimartingales on a suitable probability space $(\Omega, \mathcal{A}, P)$ and then the following decomposition holds:

$$d(\widetilde{y}_N(t))^2 = \mathcal{J}_N(\widetilde{y}_N(t))^2\,dt + d\mathcal{M}^t_{N,\widetilde{y}^2}\,,$$

with $\mathcal{M}^t_{N,\widetilde{y}^2}$ the local martingale given by

$$\mathcal{M}^t_{N,\widetilde{y}^2} = \int_0^t \sum_{j,k\in\mathscr{S}} \overline{\nabla}^{(j)}[(\widetilde{y}_N(s))^2]\,\widetilde{\Lambda}^\sigma_N(j,k,ds)\,,$$

where we have defined

$$\overline{\nabla}^{(j)}[(\widetilde{y}_N(t))^2] := \left(\widetilde{y}_N(t) - jk\frac{2}{N^{3/4}}\right)^2 - (\widetilde{y}_N(t))^2 \tag{1.33}$$

and

$$\widetilde{\Lambda}^\sigma_N(j,k,dt) := \Lambda^\sigma_N(j,k,dt) - \underbrace{N^{1/4}\left|A(j,k,N^{1/4}t)\right| e^{-\beta j\left(\frac{\widetilde{x}_N(t)}{N^{1/4}}+kh\right)}dt}_{:=\lambda(j,k,t)\,dt}\,. \tag{1.34}$$

The counter $\left|A(j,k,N^{1/4}t)\right|$ is, similarly to previous cases, expressed by

$$\left|A(j,k,N^{1/4}t)\right| = \frac{N}{4}\left[1 + k\frac{r_N(t)}{N^{1/2}} + j\frac{\widetilde{x}_N(t)}{N^{1/4}} + jk\left(\frac{\widetilde{y}_N(t)}{N^{1/4}} + \tanh(\beta h)\right)\right]$$

and, as we can evidently see, $\widetilde{\Lambda}^{\sigma}_N(j,k,dt)$ is the difference between the point process $\Lambda^{\sigma}_N(j,k,dt)$, defined on $\mathscr{S}^2 \times \mathbb{R}^+$, and its intensity $\lambda(j,k,t)\,dt$.

Remark 1.4.2. If we call $(\mathcal{A}_t)_{t\geq 0}$ a filtration generated by Λ^{σ}_N on $(\Omega, \mathcal{A}, P)$, then the processes $\mathscr{J}_N(\widetilde{y}_N(t))^2$ and $\overline{\nabla}^{(j)}[(\widetilde{y}_N(t))^2]$ are $\mathcal{A}_t-$adapted processes.

As a consequence of the considerations just explained, we are in the proper situation to use the result about collapsing processes and we can easily adapt Proposition 1.4.1 to our specific case. We obtain we need to prove the following

Lemma 1.4.3. *Consider $d > 2$, $\delta > 0$ and $\kappa_N := \kappa(N)$, such that $\kappa_N \xrightarrow{N\to+\infty} +\infty$. For $t \in [0, \tau^M_N]$ and $N \geq 1$, there exist constants $C.$'s independent of N and t and two increasing sequences $\{\alpha_N\}_{N\geq1}$ and $\{\beta_N\}_{N\geq1}$ which satisfy*

$$\kappa_N^{1/d}\alpha_N^{-1} \xrightarrow{N\to+\infty} 0,\ \kappa_N^{-1}\alpha_N \xrightarrow{N\to+\infty} 0,\ \kappa_N^{-1}\beta_N \xrightarrow{N\to+\infty} 0\,, \tag{1.35}$$

$$E\left[(\widetilde{y}_N(0))^{2d}\right] \leq C_1\,\alpha_N^{-2d} \quad \textit{for all } N, \tag{1.36}$$

$$\mathscr{J}_N(\widetilde{y}_N(t))^2 \leq -\kappa_N\delta(\widetilde{y}_N(t))^2 + \beta_N C_2 + C_3\,, \tag{1.37}$$

$$\sup_{\omega\in\Omega, j\in\mathscr{S}, t\leq\tau^M_N}\left|\overline{\nabla}^{(j)}[(\widetilde{y}_N(t))^2]\right| \leq C_4\,\alpha_N^{-1}\,, \tag{1.38}$$

$$\sum_{j,k\in\mathscr{S}}\left[\overline{\nabla}^{(j)}[(\widetilde{y}_N(t))^2]\right]^2\lambda(j,k,t) \leq C_5\left((\widetilde{y}_N(t))^2 + \kappa_N^{-1}\right) \tag{1.39}$$

and such that, for every $\varepsilon > 0$, the following estimate holds

$$\sup_{N\geq N_0} P\left\{\sup_{0\leq t\leq T\wedge\tau^M_N}(\widetilde{y}_N(t))^2 \overset{(*)}{>} C_6\left(\kappa_N^{1/d}\alpha_N^{-1} \vee \kappa_N^{-1}\beta_N\right)\right\} \leq \varepsilon\,. \tag{1.40}$$

Proof. We aim to prove these sequences $\{\alpha_N\}_{N\geq1}$, $\{\beta_N\}_{N\geq1}$ and constants $C.$'s exist and to give a characterization of them. We show that the properties (1.35)-(1.39) hold true. The estimate (1.40) then follows from Proposition 1.4.1.

(1.36) From (1.26) we get

$$\widetilde{y}_N(0) = N^{1/4}(m^{\sigma\,\eta}_{\rho_N(0)} - \tanh(\beta h))\,.$$

The random variables $(\sigma_j(0), \eta_j)_{j=1}^N$ are independent, so a Central Limit Theorem applies: in the limit as $N \longrightarrow +\infty$,

$$N^{1/4}\widetilde{y}_N(0) = N^{1/2}(m_{\rho_N(0)}^{\sigma\,\eta} - \tanh(\beta h))$$

converges to a Gaussian random variable and, since $m_{\rho_N(0)}^{\sigma\,\eta} \in [-1,+1]$, there is convergence of all the moments. Thus,

$$E\left[N^d(m_{\rho_N(0)}^{\sigma\,\eta} - \tanh(\beta h))^{2d}\right] \leq C_1$$

and we obtain the following estimate for the $2d$-th moments of $\widetilde{y}_N(0)$:

$$\begin{aligned} E[(\widetilde{y}_N(0))^{2d}] &= E\left[N^{d/2}(m_{\rho_N(0)}^{\sigma\,\eta} - \tanh(\beta h))^{2d}\right] \\ &= N^{-d/2}\, E\left[N^d(m_{\rho_N(0)}^{\sigma\,\eta} - \tanh(\beta h))^{2d}\right] \leq C_1\, N^{-d/4}\,. \end{aligned}$$

Thus (1.36) holds.

(1.37) For $t \in [0, \tau_N^M]$ we consider the Taylor expansions of exponential functions defined in (1.29) and we give an estimate of their Lagrangian expression of the remainders:

$$|R_+| \leq \frac{1}{2}\sup\left\{e^z : z \in \left[0, \frac{\beta\,\widetilde{x}(t)}{N^{1/4}}\right]\right\}\frac{\beta^2\,\widetilde{x}(t)^2}{N^{1/2}} \leq \frac{\beta^2 M^2}{2N^{1/2}}e^{\frac{\beta M}{N^{1/4}}}$$

$$|R_-| \leq \frac{1}{2}\sup\left\{e^z : z \in \left[-\frac{\beta\,\widetilde{x}(t)}{N^{1/4}}, 0\right]\right\}\frac{\beta^2\,\widetilde{x}(t)^2}{N^{1/2}} \leq \frac{\beta^2 M^2}{2N^{1/2}}\,.$$

Now, we derive the particular characterization of $\mathcal{J}_N(\widetilde{y}_N(t))^2$, adapting the explicit expression of $\mathcal{G}_N\psi(r_N(t), \widetilde{x}_N(t), \widetilde{y}_N(t))$ given by (1.28) (in other words, setting $\psi(r_N(t), \widetilde{x}_N(t), \widetilde{y}_N(t)) = (\widetilde{y}_N(t))^2$ and taking into account the time-rescaling). Then, we proceed to find an upper bound for this quantity. By (1.30), (1.31) and (1.32),

$$\begin{aligned} \mathcal{J}_N(\widetilde{y}_N(t))^2 = 4N^{1/4}\widetilde{y}_N(t)\bigg[&-\widetilde{y}_N(t)\cosh(\beta h) + \frac{\beta\,\widetilde{x}_N(t) r_N(t)}{N^{1/2}}\cosh(\beta h) \\ &- \frac{\beta(\widetilde{x}_N(t))^2}{N^{1/4}}\bigg] + 2N^{1/4}\widetilde{y}_N(t)\bigg[-\widetilde{y}_N(t)\cosh(\beta h)(R_+ + R_-) \\ &+ \left(r_N(t)\,\frac{\cosh(\beta h)}{N^{1/4}} - \widetilde{x}_N(t)\sinh(\beta h)\right)(R_+ - R_-)\bigg] \end{aligned}$$

$$
\begin{aligned}
&\leq -4N^{1/4}(\widetilde{y}_N(t))^2\cosh(\beta h) + 4\beta|\widetilde{x}_N(t)||\widetilde{y}_N(t)||r_N(t)|\cosh(\beta h)\\
&\quad + 4\beta(\widetilde{x}_N(t))^2|\widetilde{y}_N(t)| + 2N^{1/4}|\widetilde{y}_N(t)|\Big[|\widetilde{x}_N(t)|\sinh(\beta h)\\
&\quad + \Big(|r_N(t)| + |\widetilde{y}_N(t)|\Big)\cosh(\beta h)\Big](|R_+| + |R_-|)\\
&\leq -4N^{1/4}(\widetilde{y}_N(t))^2\cosh(\beta h) + 4\beta M^3\cosh(\beta h) + 4\beta M^3\\
&\quad + \frac{\beta^2 M^4}{N^{1/4}}\Big(e^{\frac{\beta M}{N^{1/4}}} + 1\Big)[2\cosh(\beta h) + \sinh(\beta h)]\\
&\leq -4N^{1/4}(\widetilde{y}_N(t))^2\cosh(\beta h) + 4\beta M^3[\cosh(\beta h) + 1]\\
&\quad + \beta^2 M^4(e^{\beta M} + 1)[2\cosh(\beta h) + \sinh(\beta h)]\\
&= -4N^{1/4}(\widetilde{y}_N(t))^2\cosh(\beta h) + C_2 + C_3\,.
\end{aligned}
$$

Hence, we have obtained the desired inequality if we choose: $\kappa_N := N^{1/4}$, $\delta := 4\cosh(\beta h)$ (which is a positive constant as required), $\beta_N \equiv 1$ and $C_2 + C_3 := 4\beta M^3[\cosh(\beta h) + 1] + \beta^2 M^4(e^{\beta M} + 1)[2\cosh(\beta h) + \sinh(\beta h)]$.

(1.38) Now, we evaluate the supremum of the modulus of $\overline{\nabla}^{(j)}[(\widetilde{y}_N(t))^2]$, defined as in (1.33). It easily yields

$$
\begin{aligned}
\sup_{\omega\in\Omega, j\in\mathscr{S}, t\in[0,\tau_N^M]}\left|\overline{\nabla}^{(j)}[(\widetilde{y}_N(t))^2]\right| &= \sup_{\omega\in\Omega, j\in\mathscr{S}, t\in[0,\tau_N^M]}\left|\frac{4}{N^{3/2}} - jk\frac{4\widetilde{y}_N(t)}{N^{3/4}}\right|\\
&\leq \frac{4}{N^{5/8}}(1+M)N^{-1/8}\\
&\leq C_4\, N^{-1/8}\,,
\end{aligned}
$$

where we set $C_4 = 4(1+M)$ and $\alpha_N = N^{1/8}$.

(1.39) Recalling the definition of $\overline{\nabla}^{(j)}[(\widetilde{y}_N(s))^2]$ and $\lambda(j,k,t)$, which we can find in (1.33) and in (1.34), we have

$$N^{1/4}\sum_{j,k\in\mathscr{S}}|A(j,k,N^{1/4}t)|e^{-\beta j\left(\frac{\widetilde{x}_N(t)}{N^{1/4}}+kh\right)}\left[\left(\widetilde{y}_N(t)-j\frac{2}{N^{3/4}}\right)^2-(\widetilde{y}_N(t))^2\right]^2=$$

$$=\frac{N^{5/4}}{4}\sum_{j,k\in\mathscr{S}}\left[1+k\frac{r_N(t)}{N^{1/2}}+j\frac{\widetilde{x}_N(t)}{N^{1/4}}+jk\left(\frac{\widetilde{y}_N(t)}{N^{1/4}}+\tanh(\beta h)\right)\right]\cdot$$
$$\cdot e^{-\beta j\left(\frac{\widetilde{x}_N(t)}{N^{1/4}}+kh\right)}\left[\frac{16}{N^3}+\frac{16}{N^{3/2}}(\widetilde{y}_N(t))^2-j\frac{32}{N^{9/4}}\widetilde{y}_N(t)\right]$$

$$\leq\frac{N^{5/4}}{4}\sum_{j,k\in\mathscr{S}}\left[1+k\frac{r_N(t)}{N^{1/2}}+j\frac{\widetilde{x}_N(t)}{N^{1/4}}+jk\left(\frac{\widetilde{y}_N(t)}{N^{1/4}}+\tanh(\beta h)\right)\right]\cdot$$
$$\cdot e^{-\beta j\left(\frac{\widetilde{x}_N(t)}{N^{1/4}}+kh\right)}\frac{16}{N^{3/2}}(\widetilde{y}_N(t))^2$$

$$+N^{-1/4}\left\{\frac{N^{3/2}}{4}\sum_{j,k\in\mathscr{S}}\left[1+k\frac{r_N(t)}{N^{1/2}}+j\frac{\widetilde{x}_N(t)}{N^{1/4}}+jk\left(\frac{\widetilde{y}_N(t)}{N^{1/4}}+\tanh(\beta h)\right)\right]\cdot\right.$$
$$\left.\cdot e^{-\beta j\left(\frac{\widetilde{x}_N(t)}{N^{1/4}}+kh\right)}\left(\frac{16}{N^3}+\frac{32}{N^{9/4}}M\right)\right\}$$

(by the Taylor expansion of the exponential functions given by (1.29), and evaluating the remainders as before)

$$\leq\frac{4}{N^{1/4}}(\widetilde{y}_N(t))^2\left[\frac{4}{\cosh(\beta h)}+\frac{4|\widetilde{y}_N(t)|}{N^{1/4}}\sinh(\beta h)+\frac{4\beta\,|\widetilde{x}_N(t)||r_N(t)|}{N^{3/4}}\sinh(\beta h)\right.$$
$$+\frac{4\beta(\widetilde{x}_N(t))^2}{N^{1/2}}\cosh(\beta h)+(|R_+|+|R_-|)\left(\frac{2|\widetilde{y}_N(t)|}{N^{1/4}}\sinh(\beta h)\right.$$
$$\left.\left.+\frac{2|r_N(t)|}{N^{1/2}}\sinh(\beta h)+\frac{2|\widetilde{x}_N(t)|}{N^{1/4}}\cosh(\beta h)+\frac{2}{\cosh(\beta h)}\right)\right]$$

$$+N^{-1/4}\left\{4\left(\frac{1}{N^{3/2}}+\frac{2M}{N^{3/4}}\right)\left[\frac{4}{\cosh(\beta h)}+\frac{4|\widetilde{y}_N(t)|}{N^{1/4}}\sinh(\beta h)\right.\right.$$
$$+\frac{4\beta\,|\widetilde{x}_N(t)||r_N(t)|}{N^{3/4}}\sinh(\beta h)+\frac{4\beta(\widetilde{x}_N(t))^2}{N^{1/2}}\cosh(\beta h)$$
$$+(|R_+|+|R_-|)\left(\frac{2|\widetilde{y}_N(t)|}{N^{1/4}}\sinh(\beta h)+\frac{2|r_N(t)|}{N^{1/2}}\sinh(\beta h)\right.$$

$$+ \frac{2|\widetilde{x}_N(t)|}{N^{1/4}} \cosh(\beta h) + \frac{2}{\cosh(\beta h)}\Big)\Big]\Big\}$$

(by the definitions (1.26) we have that $N^{-1/4}\widetilde{x}_N(t)$, $N^{-1/2}r_N(t) \in [-1,+1]$ and $N^{-1/4}\widetilde{y}_N(t) \in [-2,+2]$)

$$\le 16(\widetilde{y}_N(t))^2\Big[\frac{1}{\cosh(\beta h)} + \beta\cosh(\beta h) + (2+\beta)\sinh(\beta h)\Big] + \frac{4\beta^2 M^4}{N^{1/4}}(e^{\beta M}+1)\Big(\frac{1}{\cosh(\beta h)} + 3\sinh(\beta h) + \cosh(\beta h)\Big)$$

$$+ N^{-1/4}\Big\{16(1+2M)\Big[\frac{1}{\cosh(\beta h)} + \beta\cosh(\beta h) + (2+\beta)\sinh(\beta h) + \beta^2 M^2 (e^{\beta M}+1)\Big(3\sinh(\beta h) + \cosh(\beta h) + \frac{1}{\cosh(\beta h)}\Big)\Big]\Big\}$$

$$= 16(\widetilde{y}_N(t))^2\Big[\frac{1}{\cosh(\beta h)} + \beta\cosh(\beta h) + (2+\beta)\sinh(\beta h)\Big]$$

$$+ N^{-1/4}\Big\{16(1+2M)\Big[\frac{1}{\cosh(\beta h)} + \beta\cosh(\beta h) + (2+\beta)\sinh(\beta h) + \beta^2 M^2 (e^{\beta M}+1)\Big(3\sinh(\beta h) + \cosh(\beta h) + \frac{1}{\cosh(\beta h)}\Big)\cdot \cdot\Big(1 + \frac{4M^2}{16(1+2M)}\Big)\Big]\Big\}$$

$$\le C_5\left((\widetilde{y}_N(t))^2 + N^{-1/4}\right),$$

which is what we need, with $C_5 := 16(1+2M)\Big[\frac{1}{\cosh(\beta h)} + \beta\cosh(\beta h) + (2+\beta)\sinh(\beta h) + \beta^2 M^2(e^{\beta M}+1)\Big(3\sinh(\beta h)+\cosh(\beta h)+\frac{1}{\cosh(\beta h)}\Big)\Big(1+\frac{4M^2}{16(1+2M)}\Big)\Big]$

(1.35) It remains to show that the sequences we have found satisfy the conditions about the convergence to zero. But,

$$\lim_{N\to+\infty}(N^{1/4})^{1/d}(N^{1/8})^{-1} = \lim_{N\to+\infty} N^{1/4d-1/8} = 0 \quad \Longleftrightarrow \quad d > 2\,,$$

$$\lim_{N\to+\infty} N^{1/8}N^{-1/4} = \lim_{N\to+\infty} N^{-1/8} = 0\,,$$
$$\lim_{N\to+\infty} N^{-1/4} = 0$$

and hence we have completed the proof, since by Proposition 1.4.1 we can now assure (1.40) holds.

■

Corollary 1.4.1. *We consider the same setting as in Lemma 1.4.3. For every* $\varepsilon > 0$ *there exist constants* C_7 *and* N_0 *such that*

$$\sup_{N\geq N_0} P\left\{\sup_{0\leq t\leq T\wedge\tau_N^M} |\widetilde{y}_N(t)| > C_7\left(\kappa_N^{1/2d}\alpha_N^{-1/2} \vee \kappa_N^{-1/2}\beta_N^{1/2}\right)\right\} \leq \varepsilon\,. \tag{1.41}$$

Proof. We set $C_7 = (C_6)^{1/2}$ and we extract the square root of the inequality $(*)$ in the previous Lemma to obtain an equivalent set, described in (1.41), for which the same property holds. ■

Remark 1.4.3. Notice that if we insert the quantities we choose during the proof of Lemma 1.4.3 into (1.41), we have shown that the following inequality holds

$$\sup_{N\geq N_0} P\left\{\sup_{0\leq t\leq T\wedge\tau_N^M} |\widetilde{y}_N(t)| > C_7\left(N^{1/8d-1/16} \vee N^{-1/8}\right)\right\} \leq \varepsilon\,. \tag{1.42}$$

The results we proved in this subparagraph show that the process $\widetilde{y}_N(t)$ is a collapsing process in the sense of Proposition 1.4.1, when $t \in [0, T\wedge\tau_N^M]$. The next step is the proof of the fact that, for every $\varepsilon > 0$ and $N \geq 1$, there exists a constant $M > 0$ such that it is true

$$P\left\{\tau_N^M \leq T\right\} \leq \epsilon\,.$$

This fact implies the process $\widetilde{y}_N(t)$ converges to zero in probability, as N is growing to infinity, for all $t \in [0, T]$.

Phase 3: Proof of $\mathbf{P}\left\{\tau_{\mathbf{N}}^{\mathbf{M}} \leq \mathbf{T}\right\} \leq \boldsymbol{\epsilon}$. As before, we consider the infinitesimal generator $\mathcal{J}_N = N^{1/4}\mathcal{G}_N$ and we apply it to the function $\psi(r_N(t), \widetilde{x}_N(t), \widetilde{y}_N(t)) = |\widetilde{x}_N(t)|$. The following decomposition holds

$$|\widetilde{x}_N(t)| = |\widetilde{x}_N(0)| + \int_0^t \mathcal{J}_N(|\widetilde{x}_N(s)|)ds + \mathcal{M}^t_{N,|\widetilde{x}|}$$

$$\leq |\widetilde{x}_N(0)| + \int_0^t |\mathcal{J}_N(|\widetilde{x}_N(s)|)|ds + \mathcal{M}^t_{N,|\widetilde{x}|}\,,$$

with

$$\mathcal{M}^t_{N,|\widetilde{x}|} = \int_0^t \sum_{j,k\in\mathscr{S}} \overline{\nabla}^{(j)}[|\widetilde{x}_N(s)|]\widetilde{\Lambda}^\sigma_N(j,k,ds)$$

and where in analogy to (1.33) we have defined

$$\overline{\nabla}^{(j)}[|\widetilde{x}_N(t)|] := \left|\widetilde{x}_N(t) - j\frac{2}{N^{3/4}}\right| - |\widetilde{x}_N(t)|\,. \tag{1.43}$$

$\widetilde{\Lambda}^\sigma_N(j,k,ds)$ is the same as in (1.34). We recall that the expression of $\mathcal{G}_N$ is given by (1.28). We always need (and use) the usual expansions (1.29) and the estimates of their remainders. For $t \in [0, \tau^M_N]$ we get

$$\begin{aligned}|\mathcal{J}_N(|\widetilde{x}_N(t)|)| =& \bigg|2N^{1/4}\mathrm{sgn}(\widetilde{x}_N(t))\bigg[-\frac{\beta\widetilde{x}_N(t)\widetilde{y}_N(t)}{N^{1/4}}\sinh(\beta h) + \frac{r_N(t)}{N^{1/4}}\sinh(\beta h)\\ &+ (R_+ + R_-)\bigg(-\widetilde{x}_N(t)\cosh(\beta h) + \frac{r_N(t)}{N^{1/4}}\sinh(\beta h)\bigg)\\ &+ (R_+ - R_-)\bigg(-\widetilde{y}_N(t)\sinh(\beta h) + \frac{N^{1/4}}{\cosh(\beta h)}\bigg)\bigg]\bigg|,\end{aligned}$$

but thanks to Central Limit Theorem, for every $\varepsilon > 0$ and sufficiently large M, $P\{|r_N(t)| \geq M\} \leq \varepsilon$, then

$$\begin{aligned}\leq& 2\bigg\{(M + \beta M^2)\sinh(\beta h) + \beta^2 M^2(e^{\beta M} + 1)\bigg[\frac{1}{\cosh(\beta h)} +\\ &+ 2M\sinh(\beta h) + M\cosh(\beta h)\bigg]\bigg\} =: C_8\,,\end{aligned}$$

with C_8 positive constant independent of N. Since the following inclusions are valid

$$
\begin{aligned}
\{\tau_N^M \le T\} &\subseteq \left\{ \sup_{0\le t\le T\wedge\tau_N^M} \{|\widetilde{x}_N(t)|, |\widetilde{y}_N(t)|\} \ge M \right\} \\
&\subseteq \left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{x}_N(t)| \ge M \right\} \cup \left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{y}_N(t)| \ge M \right\} \\
&\subseteq \left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{y}_N(t)| \ge M \right\} \cup \{|\widetilde{x}_N(0)| \ge C_9\} \cup \Bigg[\{|\widetilde{x}_N(0)| \le C_9\} \cap \\
&\qquad \cap \left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{x}_N(t)| \ge C_9 + TC_8 + C_{10} \right\} \Bigg] \\
&\subseteq \left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{y}_N(t)| \ge M \right\} \cup \{|\widetilde{x}_N(0)| \ge C_9\} \cup \\
&\qquad \cup \left\{ \sup_{0\le t\le T\wedge\tau_N^M} \mathcal{M}^t_{N,|\widetilde{x}|} \ge C_{10} \right\},
\end{aligned}
$$

we obtain the following inequality

$$
\begin{aligned}
P\{\tau_N^M \le T\} \le P\left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{y}_N(t)| \ge M \right\} + P\{|\widetilde{x}_N(0)| \ge C_9\} \\
+ P\left\{ \sup_{0\le t\le T\wedge\tau_N^M} \mathcal{M}^t_{N,|\widetilde{x}|} \ge C_{10} \right\}.
\end{aligned}
$$

We estimate the three terms of the right-hand side of the inequality.

- For any $\varepsilon > 0$, thanks to (1.42), we have

$$
P\left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{y}_N(t)| \ge M \right\} \le \varepsilon\,,
$$

  where $M := C_7 \left(N^{1/8d-1/16} \vee N^{-1/8} \right)$.

- From (1.26) we get $E[|\widetilde{x}_N(0)|] = N^{1/4} E[|m^{\sigma}_{\rho_N(0)}|]$. Since at time $t = 0$ the spins are distributed according to a product measure, $\widetilde{x}_N(0)$ is a sample average of independent, identically distributed Bernoulli random variables multiplied by $N^{1/4}$. So, we can conclude

$$
E[|\widetilde{x}_N(0)|] \le \sqrt{\mathrm{Var}(\sigma_1(0))} N^{-1/4}
$$

  and in the limit as $N \longrightarrow +\infty$, we have convergence to zero in L^1 and then in probability. Therefore

$$
P\{|\widetilde{x}_N(0)| \ge C_9\} \le \varepsilon
$$

for any $\varepsilon > 0$, for every N and for a sufficiently large C_9.

► We reduce to deal with $E[(\mathcal{M}^T_{N,|\widetilde{x}|})^2]$; in fact, Doob's "maximal inequality in L^p" (case $p = 2$) for martingales (we refer to Chapter VII, Section 3 of [Shi96]) tells us that $P\Big\{ \sup_{0 \le t \le T \wedge \tau_N^M} \mathcal{M}^t_{N,|\widetilde{x}|} \ge C_{10} \Big\} \le \frac{E[(\mathcal{M}^T_{N,|\widetilde{x}|})^2]}{(C_{10})^2}$.
We use the following Proposition about stochastic integrals with respect to point processes (see Chapter II, Section 3 of [IW81]).

Proposition 1.4.2. *Let $(\Omega, \mathscr{A}, \mathscr{P})$ be a complete probability space with a right-continuous increasing family $(\mathscr{A}_t)_{t\ge 0}$ of sub-σ-fields of $\mathscr{A}$ each containing all $\mathscr{P}$-null sets. Let X_n be a martingale of the form*

$$X_n(t) = \int_{\mathbb{R}^+ \times \mathscr{Y}} f_n(t,y) \widetilde{\Lambda}_n(dt, dy)\,,$$

where

$$\widetilde{\Lambda}_n(dt, dy) := \Lambda_n(dt, dy) - A_n(t, dy)dt$$

and Λ_n is a Point Process of intensity $A_n(t, dy)dt$ on $\mathbb{R}^+ \times \mathscr{Y}$, with $\mathscr{Y}$ measurable space. If f_n is $(\mathscr{A}_t)$-predictable and for every $t > 0$

$$E\bigg[\int_{\mathbb{R}^+ \times \mathscr{Y}} |f_n(t,y)| A_n(t, dy)dt \bigg] < \infty$$

and

$$E\bigg[\int_{\mathbb{R}^+ \times \mathscr{Y}} |f_n(t,y)|^2 A_n(t, dy)dt \bigg] < \infty\,,$$

then

$$X_n^2(t) = \bigg[\int_{\mathbb{R}^+ \times \mathscr{Y}} f_n(t,y) \widetilde{\Lambda}_n(dt, dy) \bigg]^2 = \int_{\mathbb{R}^+ \times \mathscr{Y}} f_n^2(t,y) A_n(t, dy)dt\,.$$

Hence, by (1.34) and (1.43), we obtain

$$
\begin{aligned}
E[(\mathcal{M}^T_{N,|\widetilde{x}|})^2] &= E\Bigg[\int_0^T \sum_{j,k\in\mathscr{S}} \Big[\overline{\nabla}^{(j)}[|\widetilde{x}_N(t)|]\Big]^2 \lambda(j,k,t)dt\Bigg] \\
&\leq E\Bigg[\int_0^T \frac{16}{N^{3/2}}\, N^{1/4} \sup_{j,k\in\mathscr{S}} |A(j,k,N^{1/4}t)| e^{\beta(1+h)} dt\Bigg] \\
&\leq E\Bigg[\int_0^T \frac{4}{N^{5/4}}\, 5N\, e^{\beta(1+h)} dt\Bigg] \\
&\leq 20 e^{\beta(1+h)} T =: C_{11} \qquad \text{(independent of } N \text{ and } M\text{)}
\end{aligned}
$$

We have established that, if we choose $C_{10} \geq \sqrt{\frac{C_{11}}{\varepsilon}}$, then

$$
P\Bigg\{ \sup_{0\leq t\leq T\wedge\tau_N^M} \mathcal{M}^t_{N,|\widetilde{x}|} \geq C_{10} \Bigg\} \leq \varepsilon\,.
$$

In summary, we proved the inequality we were looking for; in fact

$$
P\left\{\tau_N^M \leq T\right\} \leq 3\varepsilon := \epsilon\,.
$$

We have just concluded the proof of the first part of the statement of Theorem 1.4.1, concerning the collapse of the process $\widetilde{y}_N(t)$ in the limit as $N \longrightarrow +\infty$ and for $t \in [0,T]$. Now, we are going to show that in the same setting, i.e. the limit of infinite volume and $t \in [0,T]$, the process $\widetilde{x}_N(t)$ admits a limiting process and we are going to compute it.

Phase 4: The limit of $\widetilde{x}_N(t)$. First, we need to prove the tightness of the sequence $\{\widetilde{x}_N(t)\}_{N\geq 1}$. This property implies the existence of convergent subsequences. Secondly, we will verify that all the convergent subsequences have the same limit and hence also the sequence $\{\widetilde{x}_N(t)\}_{N\geq 1}$ must converge to that limit. We recall the definition of tightness for a family of probability measures.

Definition 1.4.1. A sequence $\{\mathscr{P}_n\}_{n\geq 1}$ of probability measures on $\mathscr{X}$ is tight in $\mathscr{X}$ if for each positive ε there exists a compact subset $\mathscr{K}$ of $\mathscr{X}$ such that $\mathscr{P}_n(\mathscr{K}) \geq 1-\varepsilon$ for all $n \geq 1$.

In the case we are working with processes with laws on $\mathcal{D}[0,T]$, we can give a characterization of the tightness in terms of those processes (through their distributions). In fact, as we can read in [CE88], an immediate consequence of Theorem 4.1 and Remark 1 in [Mit83] is the following *tightness criterion*:

A sequence of processes $\{\widetilde{x}_N(t)\}_{N\geq 1}$ with laws $\{\mathcal{P}_N\}_{N\geq 1}$ on $\mathcal{D}[0,T]$ is tight if

(a) for every $\varepsilon > 0$ there exists $M > 0$ such that

$$\sup_N P\left\{ \sup_{t\in[0,T]} |\widetilde{x}_N(t)| \geq M \right\} \leq \varepsilon\,, \tag{1.44}$$

(b) for every $\varepsilon > 0$ and $\alpha > 0$ there exists $\delta > 0$ such that

$$\sup_N \sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\delta}} P\{|\widetilde{x}_N(t)-\widetilde{x}_N(s)| \geq \alpha\} \leq \varepsilon\,. \tag{1.45}$$

Lemma 1.4.4. *The sequence $\{\widetilde{x}_N(t)\}_{N\geq 1}$ is tight.*

Proof. Since we have already proved that for every $\epsilon > 0$ the inequality $P\{\tau_N^M \leq T\} \leq \epsilon$ is true for M sufficiently large and uniformly in N, it is enough to show (a) and (b) for the stopped processes

$$\left\{\widetilde{x}_N(t\wedge\tau_N^M)\right\}_{N\geq 1}\,.$$

We showed before the validity of the following inclusion

$$\left\{ \sup_{0\leq t\leq T\wedge\tau_N^M} |\widetilde{x}_N(t)| \geq M \right\} \subseteq \{|\widetilde{x}_N(0)| \geq C_9\} \cup \left\{ \sup_{0\leq t\leq T\wedge\tau_N^M} \mathcal{M}^t_{N,|\widetilde{x}|} \geq C_{10} \right\},$$

therefore

$$\sup_N P\left\{ \sup_{0\leq t\leq T\wedge\tau_N^M} |\widetilde{x}_N(t)| \geq M \right\} \leq 2\varepsilon$$

and so we obtained the statement (a). Let us deal with (b) now. We notice that

$$|\widetilde{x}_N(t) - \widetilde{x}_N(s)| = \left| \int_s^t \mathcal{J}_N(\widetilde{x}_N(u))du + \mathcal{M}^{s,t}_{N,|\widetilde{x}|} \right|,$$

where we have denoted

$$\mathcal{M}^{s,t}_{N,|\widetilde{x}|} = -\frac{2}{N^{3/4}} \int_s^t \sum_{j,k \in \mathscr{S}} j\ \widetilde{\Lambda}^{\sigma}_N(j,k,du)$$

$$= -\frac{2}{N^{3/4}} \int_s^t \sum_{j,k \in \mathscr{S}} j \left[\Lambda^{\sigma}_N(j,k,du) - N^{1/4} |A(j,k,N^{1/4}u)| e^{-\beta j \left(\frac{\widetilde{x}_N(u)}{N^{1/4}} + kh \right)} du \right]$$

and $\widetilde{\Lambda}^{\sigma}_N$ is as in (1.34). Thus,

$$\{|\widetilde{x}_N(t) - \widetilde{x}_N(s)| \geq \alpha\} \subseteq \left\{ \underbrace{\left| \int_s^t \mathcal{J}_N |\widetilde{x}_N(u)| du \right|}_{\leq C_8(t-s)} + |\mathcal{M}^{s,t}_{N,|\widetilde{x}|}| \geq \alpha \right\} \subseteq \{|\mathcal{M}^{s,t}_{N,|\widetilde{x}|}| \geq \overline{C}_{10}\}$$

and then, applying Chebyscev inequality to the last right-handside of the previous inclusions, we get

$$\sup_{\substack{0 \leq s \leq t \leq T \\ t-s \leq \delta}} P\{|\mathcal{M}^{s,t}_{N,|\widetilde{x}|}| \geq \overline{C}_{10}\} \leq (\overline{C}_{10})^{-2} \sup_{\substack{0 \leq s \leq t \leq T \\ t-s \leq \delta}} E[(\mathcal{M}^{s,t}_{N,|\widetilde{x}|})^2]$$

$$\leq (\overline{C}_{10})^{-2} \sup_{\substack{0 \leq s \leq t \leq T \\ t-s \leq \delta}} 5e^{\beta(1+h)}(t-s)$$

$$\leq (\overline{C}_{10})^{-2} \underbrace{5e^{\beta(1+h)}}_{:=\overline{C}_{11}} \delta =: (\overline{C}_{10})^{-2} \overline{C}_{11}\, \delta\,.$$

Finally, we can conclude that

$$\sup_N \sup_{\substack{0 \leq s \leq t \leq T \\ t-s \leq \delta}} P\{|\widetilde{x}_N(t) - \widetilde{x}_N(s)| \geq \alpha\} \leq \sup_N \sup_{\substack{0 \leq s \leq t \leq T \\ t-s \leq \delta}} P\{|\mathcal{M}^{s,t}_{N,|\widetilde{x}|}| \geq \overline{C}_{10}\}$$

$$\leq (\overline{C}_{10})^{-2} \overline{C}_{11}\, \delta = O(\delta)$$

and the proof is complete. ■

Lemma 1.4.4 implies that there exist convergent subsequences for the sequence $\{\widetilde{x}_N(t)\}_{N\geq 1}$. Let $\{\widetilde{x}_n(t)\}_{n\geq 1}$ denote one of such a subsequences and let $\psi \in \mathcal{C}_b^3$ be a function of the type $\psi(r_n(t), \widetilde{x}_n(t), \widetilde{y}_n(t)) = \psi(\widetilde{x}_n(t))$. The following decomposition holds

$$\psi(\widetilde{x}_n(t)) - \psi(\widetilde{x}_n(0)) = \int_0^t \mathcal{J}_n\psi(\widetilde{x}_n(u))du + \mathcal{M}_{n,\psi}^t\,, \tag{1.46}$$

where

$$\mathcal{J}_n\psi(\widetilde{x}_n(t)) = 2\psi_{\widetilde{x}}\Big[r_n(t)\sinh(\beta h) - \beta\widetilde{x}_n(t)\widetilde{y}_n(t)\sinh(\beta h)\Big] + o_M(1)$$

which, as usual, is $\mathcal{G}_N$ (see (1.28)) rescaled of a power $n^{1/4}$ and applied to the particular function $\psi(r_n(t), \widetilde{x}_n(t), \widetilde{y}_n(t)) = \psi(\widetilde{x}_n(t))$. The remainder $o_M(1)$ goes to zero as $n \longrightarrow +\infty$, uniformly in M. If we compute the limit as $n \longrightarrow +\infty$, recalling that the process $\widetilde{y}_n(t)$ collapses and a Central Limit Theorem applies to $r_n(t)$, we have:

$$\mathcal{J}_n\psi(\widetilde{x}_n(t)) \xrightarrow[w]{n\to+\infty} \mathcal{J}\psi(\widetilde{x}(t))\,,$$

with

$$\mathcal{J}\psi(\widetilde{x}(t)) = 2\,\mathscr{H}\,\sinh(\beta h)\,\psi_{\widetilde{x}}$$

and $\mathscr{H}$ is a Standard Gaussian random variable. Then, because of (1.46), we obtain

$$\mathcal{M}_{n,\psi}^t \xrightarrow[w]{n\to+\infty} \mathcal{M}_\psi^t := \psi(\widetilde{x}(t)) - \psi(\widetilde{x}(0)) - \int_0^t \mathcal{J}\psi(\widetilde{x}(u))du\,.$$

We must prove the following Lemma:

Lemma 1.4.5. *M_ψ^t is a martingale (with respect to t); in other words, for all $s,t \in [0,T]$, $s \leq t$ and for all measurable and bounded functions $g(\widetilde{x}([0,s]))$ the following identity holds:*

$$E[\mathcal{M}_\psi^t g(\widetilde{x}([0,s]))] = E[\mathcal{M}_\psi^s g(\widetilde{x}([0,s]))]\,. \tag{1.47}$$

Proof. It is sufficient to prove $\{\mathcal{M}_{n,\psi}^t\}_{n\geq 1}$ is an uniformly integrable sequence of random variables. Let us suppose we have already proved this property holds and see that (1.47) is satisfied.

Since $\mathcal{M}_{n,\psi}^t$ is a martingale (with respect to t) for every n, we have that for all $s,t \in [0,T]$, $s \leq t$ and for all measurable and bounded functions $g(\widetilde{x}([0,s]))$

$$E[\mathcal{M}_{n,\psi}^t g(\widetilde{x}([0,s]))] = E[\mathcal{M}_{n,\psi}^s g(\widetilde{x}([0,s]))]$$

and then

$$\lim_{n\to+\infty} E[\mathcal{M}^t_{n,\psi} g(\widetilde{x}([0,s]))] = \lim_{n\to+\infty} E[\mathcal{M}^s_{n,\psi} g(\widetilde{x}([0,s]))]\,.$$

But $\{\mathcal{M}^t_{n,\psi}\}_{n\geq 1}$ is a sequence of uniformly integrable random variables, hence it converges in L^1 (for instance, see [Shi96]). Moreover, we know the distribution of its L^1-limit, since we already know its weak-limit. Thus,

$$E[\mathcal{M}^t_{\psi} g(\widetilde{x}([0,s]))] = E\left[\lim_{n\to+\infty} \mathcal{M}^t_{n,\psi} g(\widetilde{x}([0,s]))\right] = \lim_{n\to+\infty} E[\mathcal{M}^t_{n,\psi} g(\widetilde{x}([0,s]))]$$

$$= \lim_{n\to+\infty} E[\mathcal{M}^s_{n,\psi} g(\widetilde{x}([0,s]))] = E\left[\lim_{n\to+\infty} \mathcal{M}^s_{n,\psi} g(\widetilde{x}([0,s]))\right]$$

$$= E[\mathcal{M}^s_{\psi} g(\widetilde{x}([0,s]))]$$

and the conclusion follows.

It remains to check that $\{\mathcal{M}^t_{n,\psi}\}_{n\geq 1}$ is an uniformly integrable family. A sufficient condition for uniform integrability is the existence of $p > 1$ such that $\sup_n E[|\mathcal{M}^t_{n,\psi}|^p] < +\infty$ (see again [Shi96]).

If we define

$$\overline{\nabla}^{(j)}[\psi(x_n(t))] := \psi\left(\widetilde{x}_n(t) - j\frac{2}{n^{3/4}}\right) - \psi(\widetilde{x}_n(t))\,,$$

it yields

$$E[(\mathcal{M}^t_{n,\psi})^2] = E\left[\int_0^t \sum_{j,k\in\mathscr{S}} \left[\overline{\nabla}^{(j)}[\psi(x_n(s))]\right]^2 \lambda(j,k,s)ds\right]$$

$$\leq 5n^{5/4}e^{\beta(1+h)} E\left[\int_0^t \sum_{j\in\mathscr{S}} \left[\psi\left(\widetilde{x}_n(s) - j\frac{2}{n^{3/4}}\right) - \psi(\widetilde{x}_n(s))\right]^2 ds\right]$$

we expand the function ψ around $\widetilde{x}_n(t)$ with the Taylor expansion stopped at first order and with remainder R such that $|R| \leq \frac{1}{2}\sup\left\{|\psi_{\widetilde{x}\widetilde{x}}(y)| : y \in \left[\widetilde{x}_n(t), \widetilde{x}_n(t) - j\frac{2}{n^{3/4}}\right]\right\}\frac{4}{n^{3/2}}$ and, moreover, we recall that $\psi \in \mathcal{C}^3_b$, so $|\psi_{\widetilde{x}}| \leq K_1$ and $|\psi_{\widetilde{x}\widetilde{x}}| \leq K_2$; therefore,

$$= 5n^{5/4}e^{\beta(1+h)} E\left[\int_0^t \sum_{j\in\mathscr{S}} \left[-j\frac{2}{n^{3/4}}\,\psi_{\widetilde{x}} + R\right]^2 ds\right]$$

$$\leq 5n^{5/4}e^{\beta(1+h)}E\left[\int_0^t \sup_{j\in\mathscr{S}}\left(\frac{4}{n^{3/2}}\,\psi_{\widetilde{x}}^2 - j\frac{4}{n^{3/4}}\,\psi_{\widetilde{x}}R + R^2\right)ds\right]$$

$$\leq 5n^{5/4}e^{\beta(1+h)}E\left[\int_0^t \left(\frac{4}{n^{3/2}}\,K_1^2 + \frac{8}{n^{9/4}}\,K_1K_2 + \frac{4}{n^3}\,K_2^2\right)ds\right]$$

$$\leq 20Te^{\beta(1+h)}(K_1+K_2)^2$$

since $t < T$; then $\mathcal{M}^t_{n,\psi}$ is uniformly integrable. ■

Now, the proof is easy to conclude. $\mathcal{M}^t_{n,\psi}$ solves the martingale problem with infinitesimal generator $\mathcal{J}$, admitting a unique solution, and hence we have shown all convergent subsequences have the same limit and so the sequence itself converges to that limit.

1.5 Conclusions

It remains to compare the behaviors of the homogeneous and inhomogeneous system. Using the same notation as before, we briefly sketch the main results on the Curie-Weiss model.

The stochastic process $\underline{\sigma}(t) = (\sigma_j(t))_{j=1}^N$, with t belonging to a generic time interval $[0,T]$, where T is fixed, describes a N-spin system evolving as a Markov process on its state space $\mathscr{S}^N$. The dynamics are specified by the requirement that the rates of transition are of the form

$$\sigma_j \longrightarrow -\sigma_j \qquad \text{at rate} \qquad e^{-\beta\sigma_j m_N^{\underline{\sigma}}}\,.$$

We reduce this system to be finite dimensional. A one-dimensional order parameter is necessary to describe the system: the magnetization $m_N^{\underline{\sigma}}$. We can recover the study of the limiting dynamics (Theorem 1.2.2 and Lemma 1.2.3) and of the Normal fluctuations (Theorem 1.3.2) as particular case of the inhomogeneous model, setting $h = 0$. The McKean-Vlasov limit ($N \longrightarrow +\infty$) for the dynamics of the magnetization is given by the ordinary differential equation

$$\dot{m}_t^\sigma = -2\,m_t^\sigma\cosh(\beta m_t^\sigma) + 2\sinh(\beta m_t^\sigma) \tag{1.48}$$

and any equilibrium solution of this equation is of the form $m_*^\sigma = \tanh(\beta m_*^\sigma)$. Depending on the parameters, we can see there exists phase transition; in fact

Theorem 1.5.1. *Consider the equation* (1.48).

- *For $\beta \leq 1$, it has 0 as a unique equilibrium solution and it is globally asymptotically stable, i.e. for every initial condition m_0^σ*

$$\lim_{t \to +\infty} m_t^\sigma = 0\,.$$

- *For $\beta > 1$, the point 0 is still an equilibrium and, moreover, two further equilibria arise:*

$$m_*^\sigma \qquad \text{and} \qquad -m_*^\sigma\,,$$

where m_*^σ is the unique positive solution of $x = \tanh(\beta x)$. In this case, the phase space $[-1,+1]$ is bi-partitioned by the origin in two domains of attraction. Given an initial condition m_0^σ,

$$\lim_{t \to +\infty} m_t^\sigma = \begin{cases} m_*^\sigma & \text{if } m_0^\sigma \in (0,1] \\ -m_*^\sigma & \text{if } m_0^\sigma \in [-1,0) \\ 0 & \text{if } m_0^\sigma = 0\,. \end{cases}$$

Moreover, with regard to the Normal fluctuations, it remains proved the following Theorem.

Theorem 1.5.2. *In the limit as $N \longrightarrow +\infty$, the fluctuation process $x_N(t)$, defined by*

$$x_N(t) := N^{1/2}\left(m^\sigma_{\rho_N(t)} - m_t^\sigma\right)\,,$$

converges (in the sense of weak convergence of stochastic processes) to a limiting Gaussian process $x(t)$, which is the unique solution of the linear stochastic differential equation

$$\begin{aligned} dx(t) = 2[(\beta-1)\cosh(\beta m_t^\sigma) - \beta m_t^\sigma \sinh(\beta m_t^\sigma)]\, x(t)\, dt + \\ + 2\sqrt{\cosh(\beta m_t^\sigma) - m_t^\sigma \sinh(\beta m_t^\sigma)}\, dB(t)\,, \end{aligned} \tag{1.49}$$

where B is a Standard Brownian motion and $x(0)$ has a centered Gaussian distribution with covariance $1 - (m_\lambda^\sigma)^2$.

Remark 1.5.1. We can notice that there is no constant drift in (1.49); drift which, on the contrary, is present in (1.22). It arises because of the disorder.

We focus on the critical dynamics of the system. The critical direction coincides with the magnetization, since the order parameter is one-dimensional. We construct the fluctuations in the threshold case, when $\beta = 1$, and we look at their long-time behavior. The size of the Normal fluctuations must be further rescaled (in space and in time), because their size around the deterministic limit increases in time. In this case we will obtain non-Normal fluctuations.
In the rest of the section, we will consider $\beta = 1$ and let us assume that the initial condition λ is a product measure such that $m_0^\sigma = 0$ and so $m_t^\sigma = 0$ for every value of $t \geq 0$, since it is an equilibrium solution.

Theorem 1.5.3. *For $t \in [0, T]$, if we consider the critical fluctuation process*

$$\widetilde{x}_N(t) := N^{1/4} m^\sigma_{\rho_N(N^{1/2}t)}\,, \tag{1.50}$$

then, as $N \longrightarrow +\infty$, $\widetilde{x}_N(t)$ converges, in the sense of weak convergence of stochastic processes, to a limiting non-Gaussian process $\widetilde{x}(t)$, which is the unique solution of the following stochastic differential equation:

$$\begin{cases} d\widetilde{x}(t) &= -\dfrac{2}{3}\,\widetilde{x}^3(t)\,dt + 2\,dB(t) \\ \\ \widetilde{x}(0) &= 0 \end{cases}$$

where B is a standard Brownian motion.

Concluding, we point out the fact that the inhomogeneous critical fluctuation process exists in a shorter time-scale than the homogeneous one, in fact in (1.26) we can amplify the time only by a factor $N^{1/4}$, instead of the usual scale $N^{1/2}$, as in (1.50). The reason of this difference is the constant drift, appearing in the dynamics of the Normal fluctuations. It obliges us to amplify the time by a smaller power of N than the one “permitted” by the linearized operator driving the diffusion equation. Besides, the limit of disordered critical fluctuations is Gaussian, since solution of a deterministic equation with constant (but random) drift given by a Gaussian random variable; while, it is not when there is no added field.

Chapter 2

A Non-Reversible Model Motivated by Credit Risk in Finance

Part of the results obtained in this chapter is due to a joint work with Elena Sartori.

We are interested in analyzing another interacting particle system embedded in a site-dependent random environment, applying the techniques explained in the previous chapter. We start considering the mean-field interacting spin-flip system described in [DPRST09] and we introduce an inhomogeneity in the model.
We consider N sites, indicated with j, and we associate with each of them a pair of spin values (σ_j, ω_j) and a random environment η_j, that we choose to be a dichotomic random variable. We start with a Markovian, but non-reversible dynamics, where the rates of transition are of the form

$$\begin{array}{llll} \sigma_j \longrightarrow -\sigma_j & \text{at rate} & e^{-\beta\sigma_j\omega_j} & \beta > 0, \\ \omega_k \longrightarrow -\omega_k & \text{at rate} & e^{-\gamma\omega_k\left(m_N^{\sigma}+h\eta_k\right)} & \gamma, h > 0. \end{array} \tag{2.1}$$

Also this model has no spatial geometry in the space of the configurations, since the interaction continues to be of the mean-field type.
Seven order parameters (magnetization field) are necessary to describe this system. Being based on a Large Deviation Principle, we compute the differential equations which drive their evolution in the infinite volume limit (McKean-Vlasov equations) and we derive a Law of Large Numbers they obey. Depending on the

parameters, we can see there exists phase transition to ferromagnetic states with constant magnetizations.
We then consider the fluctuation processes. We can provide a Central Limit Theorem for the non-critical seven-dimensional fluctuation process, but we skip the proof of this fact since it is completely analogous to the case of the random Curie-Weiss Model discussed in Chapter 1 and we focus on the infinite volume limit of the critical fluctuation process, which represents our main result. As in the previous case, we need an appropriate time-space rescaling to keep track of the critical slowing down and we obtain, in the limit, that only the critical structure survives and it is a lower dimensional process with respect to the non-critical fluctuation process. The fluctuations are one-dimensional at the critical point. In fact, when the size of the system grows to infinity, six order parameters collapse, while the other converges (in the sense of weak convergence of stochastic processes) to a deterministic process with constant (but random) drift given by a Gaussian variable with parameters depending on the environment.

The reason why we treat this system, which is a slight generalization of the homogeneous one in [DPRST09], is that the latter is interpreted in a financial contest. In fact, it is applied to describe the propagation of financial distress in a network of firms facing credit risk, i.e. the possibility of experiencing default. The default may be contagious, so there might be clustering of defaults. Hence, the phenomenon of a credit crisis is investigated and the losses of a financial institution, holding a large portfolio with positions issued by firms, are quantified.
Consider N firms active on the market, linked by business relationships.
The credit state of each firm j is represented by the couple of binary variables $(\sigma_j, \omega_j) \in \{-1, +1\}$: σ_j can be viewed as its rating indicator (we mean that $\sigma_j = -1$ is a bad rating class, in other words, the firm is not able to pay obligations back with higher probability; vice versa for $\sigma_j = +1$) and ω_j is its strength. The indicator ω_j is more important, but it is not directly observable from the market; so, it is reasonable to suppose that ω_j does not directly influence σ_k for $j \neq k$, while the interaction between ω_j and σ_j is strong. Introducing m_N^σ as a global health indicator and recalling that the mean-field assumption allows to suppose that the interaction depends only on the value of this quantity, the

contagion can be schematized as follows

$$\underset{\text{strength of firm } j}{\omega_j} \longrightarrow \underset{\text{rating class of firm } j}{\sigma_j} \longrightarrow \underset{\text{global health indicator}}{m_N^{\sigma}} \longrightarrow \underset{\text{strength of firm } k}{\omega_k}.$$

This reasoning justifies the choice of rates of transitions of the form (2.1), with $h = 0$.
Our aim is to extend the study of this system in the case when the portfolio is heterogeneous, fact which is modeled by the addition of the random field.

2.1 Description of the Model

Let $\mathscr{S} = \{-1, +1\}$ and $\underline{\eta} = (\eta_j)_{j=1}^N \in \mathscr{S}^N$ be a sequence of independent, identically distributed, symmetric, Bernoulli random variables defined on some probability space $(\Omega, \mathcal{F}, P)$. That is, $P(\eta_j = -1) = P(\eta_j = +1) = \frac{1}{2}$, for any j. We indicate by μ their common law. Given a configuration $(\underline{\sigma}, \underline{\omega}) = (\sigma_j, \omega_j)_{j=1}^N \in \mathscr{S}^{2N}$ and a realization of the random medium $\underline{\eta}$, we construct a $2N$-spin system evolving as a continuous time Markov chain on $\mathscr{S}^{2N}$, with infinitesimal generator L_N acting on functions $f : \mathscr{S}^{2N} \longrightarrow \mathbb{R}$ as follows:

$$L_N f(\underline{\sigma}, \underline{\omega}) = \sum_{j=1}^{N} e^{-\beta\sigma_j\omega_j} \nabla_j^{\sigma} f(\underline{\sigma}, \underline{\omega}) + \sum_{j=1}^{N} e^{-\gamma\omega_j(m_N^{\sigma} + h\eta_j)} \nabla_j^{\omega} f(\underline{\sigma}, \underline{\omega}), \tag{2.2}$$

where $\nabla_j^{\sigma} f(\underline{\sigma}, \underline{\omega}) = f(\underline{\sigma}^j, \underline{\omega}) - f(\underline{\sigma}, \underline{\omega})$ and $\nabla_j^{\omega} f(\underline{\sigma}, \underline{\omega}) = f(\underline{\sigma}, \underline{\omega}^j) - f(\underline{\sigma}, \underline{\omega})$. The k-th component of $\underline{\sigma}^j$, which has the meaning of a σ-spin flip at site j, is

$$\sigma_k^j = \begin{cases} \sigma_k \text{ for } k \neq j \\ -\sigma_k \text{ for } k = j \end{cases}$$

and the ω-spin flip at site j is defined similarly. The quantities $c_N^{\eta,\sigma}(j, \underline{\sigma}) = e^{-\beta\sigma_j\omega_j}$ and $c_N^{\eta,\omega}(j, \underline{\omega}) = e^{-\gamma\omega_j(m_N^{\sigma} + h\eta_j)}$ represent the jump rates of the spins; the rate at which the transition $\sigma_j \longrightarrow -\sigma_j$ and $\omega_j \longrightarrow -\omega_j$ occur respectively for some j. The parameters β, γ and h are positive and such that $h \neq \frac{\beta}{\gamma}$.
The expression (2.2) describes a system of mean-field coupled pairs of spins, each with its own random environment. It is subject to an inhomogeneous mean-field interaction (of intensity h) parametrized by the components η_j. As usual, with the expression "mean-field" we mean the sites interact all each other in the same

way and this assumption allows us to suppose that the interaction depends on the value of the magnetization

$$m_N^{\sigma}(t) = \frac{1}{N}\sum_{j=1}^{N} \sigma_j(t). \tag{2.3}$$

For simplicity, the initial condition $(\underline{\sigma}(0), \underline{\omega}(0))$ is assumed to have product distribution $\lambda^{\otimes N}$, with λ probability measure on $\mathscr{S}^2$. The quantity $(\sigma_j(t), \omega_j(t))$ represents the time evolution on $[0, T]$, T fixed, of j-th pair of spin values; it is the trajectory of the single j-th pair of spin values in time. The space of all these paths is $(\mathcal{D}[0, T])^2$, where $\mathcal{D}[0, T]$ is the space of the right-continuous, piecewise-constant function from $[0, T]$ to $\mathscr{S}$, endowed with the Skorohod topology, which provides a metric and a Borel σ-field (as we can see in [EK86]).

2.2 Non-reversibility of the system

The operator L_N given in (2.2) defines an irriducible, finite-state Markov chain. It follows the process admits a unique stationary distribution.
Simpler conditions for stationarity are the *Detailed Balance Conditions*. We say that a probability measure ν on $\{-1, +1\}^{2N}$ satisfies the Detailed Balance Conditions for the generator L_N if

$$\nu(\underline{\sigma}^j, \underline{\omega})e^{\beta\sigma_j\omega_j} = \nu(\underline{\sigma}, \underline{\omega})e^{-\beta\sigma_j\omega_j} \tag{2.4}$$

and

$$\nu(\underline{\sigma}, \underline{\omega}^j)e^{\gamma\omega_j(m_N^{\sigma}+h\eta_j)} = \nu(\underline{\sigma}, \underline{\omega})e^{-\gamma\omega_j(m_N^{\sigma}+h\eta_j)} \tag{2.5}$$

for every $(\underline{\sigma}, \underline{\omega})$. When these conditions hold, the system is *reversible*: the stationary Markov chain with infinitesimal generator L_N and marginal law ν has a distribution which is left *invariant by time-reversal*. In the case (2.4) and (2.5) admit a solution, they usually allow to derive the stationary distribution explicitly. This is not the case in our model. We have in fact

Proposition 2.2.1. *The Detailed Balance Conditions* (2.4), (2.5) *admit no solution, except at most for a specific value of* N.

Proof. By contradiction, let us assume a solution ν of (2.4) and (2.5) exists; so the equalities (2.4) and (2.5) are satisfied and we can deduce

$$\nabla_j^\sigma \log \nu(\underline{\sigma}, \underline{\omega}) = \log \frac{\nu(\underline{\sigma}^j, \underline{\omega})}{\nu(\underline{\sigma}, \underline{\omega})} = -2\beta\sigma_j\omega_j$$

$$\nabla_j^\omega \log \nu(\underline{\sigma}, \underline{\omega}) = -2\gamma\omega_j(m_N^\sigma + h\eta_j)\,,$$

which imply

$$\nabla_j^\omega \nabla_j^\sigma \log \nu(\underline{\sigma}, \underline{\omega}) = 4\beta\sigma_j\omega_j$$

$$\nabla_j^\sigma \nabla_j^\omega \log \nu(\underline{\sigma}, \underline{\omega}) = \frac{4\gamma\sigma_j\omega_j}{N}$$

that, being $\nabla_j^\omega \nabla_j^\sigma \log \nu(\underline{\sigma}, \underline{\omega}) \equiv \nabla_j^\sigma \nabla_j^\omega \log \nu(\underline{\sigma}, \underline{\omega})$, can hold true for at most one value of N. ■

2.3 Limiting Dynamics

We now derive the dynamics of the process (2.2), in the limit as $N \longrightarrow +\infty$, in a fixed time interval $[0,T]$, via a Large Deviation approach. Later, the large time behavior of the limiting dynamics will be studied.

So, let $(\sigma_j[0,T], \omega_j[0,T])_{j=1}^N \in (\mathcal{D}[0,T])^{2N}$ denote a path of the system in the time interval $[0,T]$, with T positive and fixed. If $f(\sigma_j[0,T], \omega_j[0,T])$ is a function of the trajectory of a single pair of spins, we are interested in the asymptotic behavior of *empirical averages* of the form

$$\frac{1}{N}\sum_{j=1}^N f(\sigma_j[0,T], \omega_j[0,T]) =: \int f d\rho_N\,,$$

where $\{\rho_N\}_{N\geq 1}$ is the sequence of *empirical measures*

$$\rho_N := \frac{1}{N}\sum_{j=1}^N \delta_{(\sigma_j[0,T], \omega_j[0,T], \eta_j)}\,.$$

We may think of ρ_N as a random element of $\mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$, the space of probability measures on $(\mathcal{D}[0,T])^2 \times \mathscr{S}$ endowed with the weak convergence topology.

First, we want to determine the weak limit of ρ_N in $\mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$ as N grows to infinity; i.e. for $f \in \mathcal{C}_b$ we look for $\lim_{N\to+\infty} \int f d\rho_N$. It corresponds to a Law of Large Numbers with the limit being a deterministic measure. Being an element of $\mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$, such a limit can be viewed as a stochastic process, which describes the dynamics of the system in the infinite volume limit.

2.3.1 Empirical Measure and Large Deviations

Let $W \in \mathcal{M}_1((\mathcal{D}[0,T])^2)$ denote the law of the $\mathscr{S}^2$-valued process $(\sigma(t), \omega(t))_{t\in[0,T]}$ such that the initial condition $(\sigma(0), \omega(0))$ has distribution λ and both spin signs change with constant rate equal to 1. By $W^{\otimes N}$ we mean the product of N copies of W, which represents the law of the $2N$-spin system whose generator is (2.2) where we have set $c_N^{\eta,\underline{\sigma}} = c_N^{\eta,\underline{\omega}} \equiv 1$; in other words, the law of our system in absence of interaction. Moreover, we shall write $P_N^{\underline{\eta}}$ the law of $(\underline{\sigma}([0,T]), \underline{\omega}([0,T])) = (\underline{\sigma}(t), \underline{\omega}(t))_{t\in[0,T]}$, the process with infinitesimal generator (2.2) and initial distribution $\lambda^{\otimes N}$, for a given $\underline{\eta}$.

Consider $Q \in \mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$, if $\Pi_t Q$ indicates the marginal distribution of Q at time t, we have

$$m^{\sigma}_{\Pi_t Q} := \int_{\mathscr{S}^3} \sigma \Pi_t Q(d\sigma, d\omega, d\eta).$$

For a given path $(\sigma([0,T]), \omega([0,T])) \in (\mathcal{D}[0,T])^2$ and being $\mathcal{N}_t^{\sigma}$, $\mathcal{N}_t^{\omega}$ the processes counting respectively the jumps of $\sigma(\cdot)$ and $\omega(\cdot)$, we define

$$\begin{aligned} F(Q) := \int \Bigg[\int_0^T \left(1 - e^{-\beta\sigma(t)\omega(t)}\right) dt + \beta \int_0^T \sigma(t)\omega(t) d\mathcal{N}_t^{\sigma} \\ + \int_0^T \left(1 - e^{-\gamma\omega(t)\left(m^{\sigma}_{\Pi_t Q} + h\eta\right)}\right) dt + \gamma \int_0^T \omega(t) \left(m^{\sigma}_{\Pi_t Q} + h\eta\right) d\mathcal{N}_t^{\sigma} \Bigg] dQ\,, \end{aligned} \tag{2.6}$$

whenever

$$\int \left(\mathcal{N}_T^{\sigma} + \mathcal{N}_T^{\omega}\right) dQ < +\infty,$$

otherwise $F(Q) \equiv 0$.

Remark 2.3.1. The function $F(Q)$ is neither continuous nor bounded.

We can obtain a representation of $P_N^{\underline{\eta}}$ in terms of ρ_N, as follows:

Lemma 2.3.1. *For a fixed realization $\underline{\eta}$,*

$$\frac{dP_N^{\underline{\eta}}}{dW^{\otimes N}}(\underline{\sigma}([0,T]),\underline{\omega}([0,T])) = \exp[NF(\rho_N(\underline{\sigma}([0,T]),\underline{\omega}([0,T]),\underline{\eta}))]$$

where, for $Q \in \mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$, $F(Q)$ is expressed by (2.6).

Proof. We apply the Girsanov's Formula for point processes (see Lemma 1.2.1). Let $(\mathcal{N}_t^{\sigma}(j))_{j=1}^N$, $(\mathcal{N}_t^{\omega}(j))_{j=1}^N$ be the multivariate Poisson processes counting the jumps of σ_j and ω_j, for $j = 1, \dots, N$. If we read $\sigma_j(t-) = \lim_{s \to t-} \sigma_j(s)$ (analogously for ω_j) and $m^{\sigma}_{\rho_N(t-)} = \lim_{s\to t-} m^{\sigma}_{\rho_N(s)}$, it yields

$$\frac{dP_N^{\underline{\eta}}}{dW^{\otimes N}}(\underline{\sigma}([0,T]),\underline{\omega}([0,T])) =$$

$$= \exp\left\{\sum_{j=1}^N \left[\int_0^T \log e^{-\beta\sigma_j(t-)\omega_j(t-)} d\mathcal{N}_t^{\sigma}(j) - \int_0^T \left(e^{-\beta\sigma_j(t)\omega_j(t)} - 1\right) dt \right.\right.$$
$$\left.\left. + \int_0^T \log e^{-\gamma\omega_j(t-)\left(m^{\sigma}_{\rho_N(t-)}+h\eta_j\right)} d\mathcal{N}_t^{\omega}(j) - \int_0^T \left(e^{-\gamma\omega_j(t)\left(m^{\sigma}_{\rho_N(t)}+h\eta_j\right)} - 1\right) dt\right]\right\}$$

but $\underline{\sigma}$ and $\underline{\omega}$ have no simultaneous jumps $W^{\otimes N}$–almost surely, therefore

$$= \exp\left\{\sum_{j=1}^N \left[\int_0^T \left(1 - e^{-\beta\sigma_j(t)\omega_j(t)}\right) dt - \beta\int_0^T (-\sigma_j(t))\omega_j(t) d\mathcal{N}_t^{\sigma}(j) \right.\right.$$
$$\left.\left. + \int_0^T \left(1 - e^{-\gamma\omega_j(t)\left(m^{\sigma}_{\rho_N(t)}+h\eta_j\right)}\right) dt - \gamma\int_0^T (-\omega_j(t))\left(m^{\sigma}_{\rho_N(t)} + h\eta_j\right) d\mathcal{N}_t^{\omega}(j)\right]\right\}$$

$$= \exp\left\{\sum_{j=1}^N \left[\int_0^T \left(1 - e^{-\beta\sigma_j(t)\omega_j(t)}\right) dt + \beta\int_0^T \sigma_j(t)\omega_j(t)) d\mathcal{N}_t^{\sigma}(j) \right.\right.$$
$$\left.\left. + \int_0^T \left(1 - e^{-\gamma\omega_j(t)\left(m^{\sigma}_{\rho_N(t)}+h\eta_j\right)}\right) dt + \gamma\int_0^T \omega_j(t)\left(m^{\sigma}_{\rho_N(t)} + h\eta_j\right) d\mathcal{N}_t^{\omega}(j)\right]\right\}$$

and, because $\int (\mathcal{N}_T^{\sigma} + \mathcal{N}_T^{\omega})\, d\rho_N < +\infty$ almost surely with respect to $W^{\otimes N}$, this leads us to the conclusion. ■

Lemma 2.3.1 allows us to deduce a Large Deviation Principle for ρ_N, from which we can derive its asymptotic behavior as $N \longrightarrow +\infty$.
Define

$$\mathcal{P}_N(\cdot) := \int \mu^{\otimes N}(d\underline{\eta}) P_N^{\underline{\eta}}(\rho_N \in \cdot)\,,$$

which is an element of $\mathcal{M}_1(\mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S}))$ and represents the law of ρ_N under the joint distribution of the process and the environment.
If $Q \in \mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$ we denote by

$$H(Q|W \otimes \mu) := \begin{cases} \int dQ \log \frac{dQ}{d(W\otimes\mu)} & \text{if} \quad Q \ll W \otimes \mu \quad \text{and} \quad \log \frac{dQ}{d(W\otimes\mu)} \in L^1(Q) \\ +\infty & \text{otherwise} \end{cases}$$

the relative entropy between Q and $W \otimes \mu$.

Proposition 2.3.1. *The laws $\{\mathcal{P}_N\}_{N\geq 1}$ of ρ_N (under the joint distribution of the process and the medium) obey a Large Deviation Principle with rate function*

$$I(Q) := H(Q|W \otimes \mu) - F(Q)$$

(mind Definitions 1.2.1 and 1.2.2).

Proof. The main problem to prove Proposition 2.3.1 is related to the fact that the function F defined in (2.6) is neither continuous nor bounded and it does not admit a characterization analogous to (1.3) (because of the non-reversibility). So, we need some technicalities to circumvent this problem.
First, we set

$$\mathcal{I} := \left\{ Q \in \mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S}) : \int (\mathcal{N}_T^\sigma + \mathcal{N}_T^\omega)\, dQ < +\infty \right\}$$

and we define, for $a > 0$ and $Q \in \mathcal{I}$,

$$\begin{aligned} F_a(Q) := \int \Bigg[\int_0^T \left(a - e^{-\beta\sigma(t)\omega(t)}\right) dt \\ + \int_0^T [\beta\sigma(t)\omega(t) - \log a]\, d\mathcal{N}_t^\sigma + \int_0^T \left(a - e^{-\gamma\omega(t)\left(m^\sigma_{\Pi_t Q} + h\eta\right)}\right) dt \\ + \int_0^T \left[\gamma\omega(t)\left(m^\sigma_{\Pi_t Q} + h\eta\right) - \log a\right] d\mathcal{N}_t^\omega \Bigg] dQ. \end{aligned} \tag{2.7}$$

Note that $F_1 = F$. Furthermore, Lemma 2.3.1 can be extended to show that

$$\frac{dP_N^{\underline{\eta}}}{dW_a^{\otimes N}}(\underline{\sigma}([0,T]), \underline{\omega}([0,T])) = \exp[N F_a(\rho_N(\underline{\sigma}([0,T]), \underline{\omega}([0,T]), \underline{\eta}))]\,, \tag{2.8}$$

where W_a is the law of the $\mathscr{S}^2$-valued process $(\sigma(t), \omega(t))$, which has distribution λ at time $t = 0$ and spins flip with constant rates a. Now, we can start the proof.

We divide it into several steps.

<u>STEP 1:</u> *F_a is lower semi-continuous on $\mathcal{I}$ for $0 < a \le \min\left(e^{-\beta}, e^{-\gamma(1+h)}\right)$ and it is upper semi-continuous on $\mathcal{I}$ for $a \ge \max\left(e^{\beta}, e^{\gamma(1+h)}\right)$.*

By definition of weak topology, the map

$$Q \longmapsto \int \left[\int_0^T \left(a - e^{-\beta\sigma(t)\omega(t)}\right) dt + \int_0^T \left(a - e^{-\gamma\omega(t)\left(m^{\sigma}_{\Pi_t Q} + h\eta\right)}\right) dt\right] dQ$$

is continuous in Q, since it is a Q-expectation of bounded and continuous functions in $(\mathcal{D}[0,T])^2$. Thus, we only have to deal with the term

$$\int \left[\int_0^T [\beta\sigma(t)\omega(t) - \log a]\, d\mathcal{N}^{\sigma}_t + \int_0^T \left[\gamma\omega(t)\left(m^{\sigma}_{\Pi_t Q} + h\eta\right) - \log a\right] d\mathcal{N}^{\omega}_t\right] dQ. \tag{2.9}$$

We prove that for $0 < a \le \min\left(e^{-\beta}, e^{-\gamma(1+h)}\right)$ the expression in (2.9) is lower semi-continuous in $Q \in \mathcal{I}$. This implies that F_a is lower semi-continuous. The case $a \ge \max\left(e^{\beta}, e^{\gamma(1+h)}\right)$ can be treated analogously.

For $\varepsilon > 0$, we consider the function $\upsilon_\varepsilon : \mathcal{D}[0,T] \longrightarrow \mathbb{R}$ defined, for $\xi \in \mathcal{D}[0,T]$, by

$$\upsilon_\varepsilon(\xi) := \begin{cases} \frac{1}{\varepsilon} \text{ if } \xi(t) \text{ jumps for some } t \in (0,\varepsilon] \\ 0 \text{ otherwise}. \end{cases}$$

We define $\xi(t)$ for $t > T$ by letting $\xi(t) \equiv \xi(T)$. Then, if we denote by θ_t the shift operator, we have that, for $t \in [0,T]$, $\theta_t\xi$ is an element of $\mathcal{D}[0,T]$ too and it is given by $\theta_t\xi(s) := \xi(t+s)$. Let now $f, g : \mathscr{S}^2 \longrightarrow \mathbb{R}$ be two functions and define $f_\varepsilon, g_\varepsilon : (\mathcal{D}[0,T])^2 \longrightarrow \mathbb{R}$ by

$$f_\varepsilon(\sigma[0,T], \omega[0,T]) := \inf\{f(\sigma(t), \omega(t)) : t \in (0,\varepsilon)\}$$

and similarly for g_ε. Then, we set

$$\Upsilon_\varepsilon(\sigma[0,T], \omega[0,T]) := \int_0^T f_\varepsilon(\theta_t\sigma, \theta_t\omega)\upsilon_\varepsilon(\theta_t\sigma) dt + \int_0^T g_\varepsilon(\theta_t\sigma, \theta_t\omega)\upsilon_\varepsilon(\theta_t\omega) dt.$$

The key facts are the following two properties of Υ_ε, whose demonstrations are omitted, since rather straightforward.

- $\Upsilon_\varepsilon(\sigma[0,T], \omega[0,T])$ is continuous and bounded on the set $\{(\sigma[0,T], \omega[0,T]) : \mathcal{N}^{\sigma}_T + \mathcal{N}^{\omega}_T < +\infty\}$.

- Suppose $f, g \geq 0$. Then, for $\delta_{(\sigma[0,T],\omega[0,T],\eta)} \in \mathcal{I}$, $\Upsilon_\varepsilon(\sigma[0,T], \omega[0,T])$ increases to
$$\Upsilon(\sigma[0,T], \omega[0,T]) := \int_0^T f(\sigma(t-), \omega(t-)) d\mathcal{N}_t^\sigma + \int_0^T g(\sigma(t-), \omega(t-)) d\mathcal{N}_t^\omega$$
as $\varepsilon \downarrow 0$. Therefore, by Monotone Convergence, we get
$$\int \Upsilon(\sigma[0,T], \omega[0,T]) dQ = \sup_{\varepsilon>0} \int \Upsilon_\varepsilon(\sigma[0,T], \omega[0,T]) dQ$$
and, in particular, the map
$$Q \longmapsto \int \Upsilon(\sigma[0,T], \omega[0,T]) dQ$$
is lower semi-continuous on $\mathcal{I}$.

Now, for $0 < a \leq \min\left(e^{-\beta}, e^{-\gamma(1+h)}\right)$, the function $f(\sigma, \omega) = \beta\sigma\omega - \log a$ is non-negative. As for the function g, that should be $g(\sigma, \omega) = \gamma\omega\left(m^\sigma_{\Pi_t Q} + h\eta\right) - \log a$, we notice it is not a function of the only variables (σ, ω), but rather a function of $(\sigma, \Pi_t Q)$, thus depending explicitly on t and Q. However, due to its boundness and the fact that $m^\sigma_{\Pi_t Q}$ is continuous in Q uniformly in t and σ, the argument above applies with minor modifications leading to the conclusion of the proof.

STEP 2: *Let $Q \in \mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$ be such that $H(Q|W \otimes \mu) < +\infty$. Then $Q \in \mathcal{I}$. The same result applies if W_a replaces W.*

Since $\mathcal{N}_T^\sigma$ is bounded, by the entropy equality
$$\log \int e^{\mathcal{N}_T^\sigma} d(W \otimes \mu) = \sup_Q \left[\int \mathcal{N}_T^\sigma dQ - H(Q|W \otimes \mu)\right]$$
(see (6.2.14) in [DZ93]), we can deduce
$$\int \mathcal{N}_T^\sigma dQ \leq \log \int e^{\mathcal{N}_T^\sigma} d(W \otimes \mu) + H(Q|W \otimes \mu)$$
But $\mathcal{N}_T^\sigma$ has Poisson distribution under $W \otimes \mu$ (since it has Poisson distribution under W), so $\int e^{\mathcal{N}_T^\sigma} dW < +\infty$. By applying the same argument to $\mathcal{N}_T^\omega$ we conclude. This proof extends to the case $a \neq 1$.

Remark 2.3.2. Note that whenever $\int \mathcal{N}_T^\sigma dQ = +\infty$ (or $\int \mathcal{N}_T^\omega dQ = +\infty$), then we obtain $H(Q|W \otimes \mu) = +\infty$.

STEP 3: *The function* $I(Q) := H(Q|W \otimes \mu) - F(Q)$ *is lower semi-continuous on* $\mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$.

We already know that the entropy $H(Q|W \otimes \mu)$ is lower semi-continuous in the whole space $\mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$ (recall Remark 1.2.3). Moreover, by definition, $F(Q) < +\infty$ for every Q and so we have $H(Q|W \otimes \mu) = I(Q)$ whenever $H(Q|W \otimes \mu) = +\infty$. Since, by STEP 2, $H(Q|W \otimes \mu) = +\infty$ for $Q \notin \mathcal{I}$, we are left to prove the following two statements:

(a) $I(Q)$ is lower semi-continuous in $\mathcal{I}$;

(b) if $H(Q|W \otimes \mu) = +\infty$ and $Q_n \xrightarrow{n \to +\infty} Q$ weakly, then $I(Q_n) \xrightarrow{n \to +\infty} +\infty$.

For $a > 0$ the following identity holds, which is a consequence of the definition of relative entropy and of the Girsanov's Formula for point processes (or, more precisely, furthermore adapted to Markov Chains):

$$H(Q|W_a \otimes \mu) = H(Q|W \otimes \mu) + \int dQ \log \frac{d(W \otimes \mu)}{d(W_a \otimes \mu)}$$

$$= H(Q|W \otimes \mu) + 2T(1-a) + \log a \int (\mathcal{N}_T^\sigma + \mathcal{N}_T^\omega)\, dQ\,. \quad (2.10)$$

Combining STEP 2 and (2.10), we obtain that

$$H(Q|W \otimes \mu) = +\infty \Longleftrightarrow H(Q|W_a \otimes \mu) = +\infty$$

and hence we can deduce

$$I(Q) = H(Q|W_a \otimes \mu) - F_a(Q)\,, \quad (2.11)$$

where the difference in (2.11) is meant to be $+\infty$ whenever $H(Q|W_a \otimes \mu) = +\infty$ (or, equivalently, $H(Q|W \otimes \mu) = +\infty$).
Now, we are ready to verify (a) and (b). To prove (a) it is enough to choose $a \geq \max\left(e^\beta, e^{\gamma(1+h)}\right)$ and use STEP 1. Moreover, for the same choice of a, the stochastic integral in (2.7) is nonpositive, so $F_a(Q) \leq 2Ta$. Therefore, if $H(Q|W \otimes \mu) = +\infty$ and if $Q_n \longrightarrow Q$,

$$\liminf_{n \to +\infty} I(Q_n) \geq \liminf_{n \to +\infty} H(Q_n|W_a \otimes \mu) - 2Ta = +\infty$$

where the last equality follows from the lower semi-continuity of $H(\cdot|W_a \otimes \mu)$ and $H(Q|W_a \otimes \mu) = +\infty$. Thus, (b) is proved as well.

<u>STEP 4:</u> *The function $I(Q)$ has compact level-sets.*

Choosing, $a \geq \max\left(e^{\beta}, e^{\gamma(1+h)}\right)$, we have that $F_a(Q) \leq 2Ta$ for every Q. Then, by (2.11), the following inclusion remains valid

$$\{Q : I(Q) \leq k\} \subseteq \{Q : H(Q|W_a \otimes \mu) \leq k + 2Ta\}.$$

Since the relative entropy has compact level-sets (see again Chapter VI, Section 2 in [DZ93]), the set $\{Q : I(Q) \leq k\}$ is contained in a compact. But, it is closed, thanks to the lower semi-continuity of $I(\cdot)$, and then the proof is completed.

<u>STEP 5:</u> *For every $a > 0$ there exists $\delta > 1$ such that*

$$\limsup_{N \to +\infty} \frac{1}{N} \log \int \exp[\delta N F_a(\rho_N)] d(W_a^{\otimes N} \otimes \mu^{\otimes N}) < +\infty\,. \tag{2.12}$$

We check the statement for $a = 1$. The modifications for the general case ($a \neq 1$) are straightforward. The proof consists of algebraic manipulations. The idea can be summarized as follows. First of all, we consider the integral with respect only to the measure $W^{\otimes N}$, because if $\delta = 1$, Lemma 2.3.1 implies that $\exp[\delta N F(\rho_N)]$ is the Radon-Nikodym derivative of P_N^{η} with respect to $W^{\otimes N}$ and, therefore, its expectation is equal to 1. For $\delta > 1$, we split $\delta F(\rho_N)$ into the sum of two terms: $\delta F(\rho_N) = F^{(1)}(\rho_N) + F^{(2)}(\rho_N)$, in such a way that $F^{(2)}$ is bounded and $\exp[N F^{(1)}(\rho_N)]$ is a Radon-Nikodym derivative of a probability with respect to $W^{\otimes N}$. Finally, we will integrate with respect to the environment. Let us start.
Using (2.6), we obtain

$$\begin{aligned}\delta N F(\rho_N) = \sum_{j=1}^{N} &\left\{ \delta \int_0^T \left(1 - e^{-\beta \sigma_j(t) \omega_j(t)}\right) dt + \delta\beta \int_0^T \sigma_j(t) \omega_j(t) d\mathcal{N}_t^{\sigma}(j) \right\} \\ &+ \sum_{j=1}^{N} \left\{ \delta \int_0^T \left(1 - e^{-\gamma \omega_j(t)\left(m^{\sigma}_{\rho_N(t)} + h\eta_j\right)}\right) dt \right. \\ &\qquad \left. + \delta\gamma \int_0^T \omega_j(t) \left(m^{\sigma}_{\rho_N(t)} + h\eta_j\right) d\mathcal{N}_t^{\omega}(j) \right\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} \Bigg\{ \int_0^T \left[\delta - \delta e^{-\gamma\omega_j(t)\left(m^{\sigma}_{\rho_N(t)}+h\eta_j\right)} - \left(1 - e^{-\delta\gamma\omega_j(t)\left(m^{\sigma}_{\rho_N(t)}+h\eta_j\right)}\right)\right] dt \\
&\qquad + \int_0^T \left[\delta - \delta e^{-\beta\sigma_j(t)\omega_j(t)} - \left(1 - e^{-\delta\beta\sigma_j(t)\omega_j(t)}\right)\right] dt \Bigg\} \\
&\qquad + \sum_{j=1}^{N} \Bigg\{ \int_0^T \left(1 - e^{-\delta\beta\sigma_j(t)\omega_j(t)}\right) dt + \delta\beta \int_0^T \sigma_j(t)\omega_j(t) d\mathcal{N}^{\sigma}_t(j) \Bigg\} \\
&\qquad + \sum_{j=1}^{N} \Bigg\{ \int_0^T \left(1 - e^{-\delta\gamma\omega_j(t)\left(m^{\sigma}_{\rho_N(t)}+h\eta_j\right)}\right) dt \\
&\qquad\qquad + \delta\gamma \int_0^T \omega_j(t) \left(m^{\sigma}_{\rho_N(t)} + h\eta_j\right) d\mathcal{N}^{\omega}_t(j) \Bigg\} \\
&= NF^{(2)}(\rho_N) + NF^{(1)}(\rho_N)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
NF^{(1)}(\rho_N) := &\sum_{j=1}^{N} \left[\int_0^T \left(1 - e^{-\delta\beta\sigma_j(t)\omega_j(t)}\right) dt + \delta\beta \int_0^T \sigma_j(t)\omega_j(t) d\mathcal{N}^{\sigma}_t(j) \right] \\
&+ \sum_{j=1}^{N} \Bigg[\int_0^T \left(1 - e^{-\delta\gamma\omega_j(t)\left(m^{\sigma}_{\rho_N(t)}+h\eta_j\right)}\right) dt \\
&\qquad + \delta\gamma \int_0^T \omega_j(t) \left(m^{\sigma}_{\rho_N(t)} + h\eta_j\right) d\mathcal{N}^{\omega}_t(j) \Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
NF^{(2)}(\rho_N) := &\sum_{j=1}^{N} \int_0^T \left[\delta - \delta e^{-\gamma\omega_j(t)\left(m^{\sigma}_{\rho_N(t)}+h\eta_j\right)} - \left(1 - e^{-\delta\gamma\omega_j(t)\left(m^{\sigma}_{\rho_N(t)}+h\eta_j\right)}\right)\right] dt \\
&+ \sum_{j=1}^{N} \int_0^T \left[\delta - \delta e^{-\beta\sigma_j(t)\omega_j(t)} - \left(1 - e^{-\delta\beta\sigma_j(t)\omega_j(t)}\right)\right] dt\,.
\end{aligned}
$$

We can see that $\exp[NF^{(1)}(\rho_N)]$ has the same form of $\exp[NF(\rho_N)]$ (given in Lemma 2.3.1) after having replace β with $\delta\beta$ and that its $W^{\otimes N}$–expectation, i.e. $\int \exp[NF^{(1)}(\rho_N)]dW^{\otimes N}$, is equal to 1. Then, also the $(W^{\otimes N} \otimes \mu^{\otimes N})$–expectation is equal to 1. Besides, we easily estimate

$$
F^{(2)}(\rho_N) \le T\left[2\delta - \delta\left(e^{-\beta} + e^{-\gamma(1+h)}\right) - 2 + e^{\delta\beta} + e^{\delta\gamma(1+h)}\right] := C(\beta,\gamma,\delta,h,T)\,.
$$

Putting all together, it yields

$$\int e^{\delta N F(\rho_N)} d(W^{\otimes N} \otimes \mu^{\otimes N}) \leq e^{NC(\beta,\gamma,\delta,h,T)} \int e^{NF^{(1)}(\rho_N)} d(W^{\otimes N} \otimes \mu^{\otimes N}) = e^{NC(\beta,\gamma,\delta,h,T)}$$

from which the conclusion follows.

STEP 6: it remains to show an upper and a lower bound of type (1.4b) and (1.4a) respectively. We prove them separately. The key tool is Varadhan's Lemma in the version given by Lemmas 4.3.4 and 4.3.6 in [DZ93]. We give here the statements in the form we need for completeness and convenience.

Lemma 2.3.2. *Consider the sequence of probability measures* $(\mathscr{P}_n)_{n\geq 1}$ *on* $\mathscr{X}$.

(a) If $f : \mathscr{X} \longrightarrow \mathbb{R}$ *is a lower semi-continuous function and the Large Deviation lower bound, i.e.*

$$\liminf_{n\to+\infty} \frac{1}{n} \log \mathscr{P}_n(\mathscr{O}) \geq - \inf_{x\in\mathscr{O}} i(x) \qquad \mathscr{O} \text{ open subset of } \mathscr{X},$$

holds with $i : \mathscr{X} \longrightarrow [0, +\infty]$, *then*

$$\liminf_{n\to+\infty} \frac{1}{n} \log \int \exp[nf] d\mathscr{P}_n \geq \sup_{x\in\mathscr{X}} [f(x) - i(x)] .$$

(b) Let us suppose that $f : \mathscr{X} \longrightarrow \mathbb{R}$ *is an upper semi-continuous function and that there exists a constant* $\delta > 1$ *such that*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \int \exp[\delta n f] d\mathscr{P}_n \leq +\infty .$$

If the Large Deviation upper bound, i.e.

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathscr{P}_n(\mathscr{C}) \leq - \inf_{x\in\mathscr{C}} i(x) \qquad \mathscr{C} \text{ closed subset of } \mathscr{X},$$

holds with $i : \mathscr{X} \longrightarrow [0, +\infty]$, *then*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \int \exp[nf] d\mathscr{P}_n \leq \sup_{x\in\mathscr{X}} [f(x) - i(x)] .$$

We deal first with the Large Deviation upper bound (type (1.4b)). Take $a \geq \max\left(e^{\beta}, e^{\gamma(1+h)}\right)$, so that the function F_a defined in (2.7) is upper semi-continuous. Denote by R_N the distribution of ρ_N under $W_a^{\otimes N} \times \mu^{\otimes N}$; in other words, if $A \in \mathcal{B}((\mathcal{D}[0,T])^2 \times \mathscr{S})$ is a Borelian set, then $R_N(A) = (W_a^{\otimes N} \times \mu^{\otimes N})(\rho_N^{-1}(A))$. Under R_N, the triples $(\sigma_j[0,T], \omega_j[0,T], \eta_j)$ are independent, identically distributed random variables.

Now, because of the result proved in Lemma 2.3.1 and its extension (2.8), we have

$$
\begin{aligned}
\mathcal{P}_N(\cdot) &= \int \mu^{\otimes N}(d\underline{\eta}) P_N^{\underline{\eta}}(\rho_N(d\underline{\sigma}[0,T], d\underline{\omega}[0,T], \underline{\eta}) \in \ \cdot) \\
&= \int \mu^{\otimes N}(d\underline{\eta}) \int W_a^{\otimes N}(d\underline{\sigma}[0,T], d\underline{\omega}[0,T]) \frac{dP_N^{\underline{\eta}}}{dW_a^{\otimes N}}(\underline{\sigma}[0,T], \underline{\omega}[0,T]) \mathbf{1}_{\{\rho_N \in \ \cdot\}} \\
&= \int d(W_a^{\otimes N} \times \mu^{\otimes N}) \exp[N F_a(\rho_N)] \mathbf{1}_{\{\rho_N \in \ \cdot\}} \\
&= \int R_N(dQ) \exp[N F_a(Q)] \mathbf{1}_{\{Q \in \ \cdot\}}\,, \qquad (2.13)
\end{aligned}
$$

with $Q = \rho_N$. The last identity (2.13) means that

$$
\frac{d\mathcal{P}_N}{dR_N}(Q) = \exp[N F_a(Q)]\,. \qquad (2.14)
$$

Since $(\mathcal{D}[0,T])^2 \times \mathscr{S}$ is a Polish space, by Sanov's Theorem (see Theorem 1.2.1) we can deduce that $\{R_N\}_{N \geq 1}$ satisfies a Large Deviation Principle with rate function $H(Q|W_a \otimes \mu)$. Therefore, F_a is upper semi-continuous and satisfies the superexponential estimate (2.12) and thus we can apply Varadhan's Lemma (Lemma 2.3.2) to obtain the upper bound of type (1.4b). In fact, if $C \in \mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$ is a closed set,

$$
\begin{aligned}
\limsup_{N \to +\infty} \frac{1}{N} \log \mathcal{P}_N(C) &= \limsup_{N \to +\infty} \frac{1}{N} \log \int R_N(dQ) \exp[N F_a(Q)] \mathbf{1}_{\{Q \in C\}} \\
&\leq \sup_{Q \in C} [F(Q) - H(Q|W_a \otimes \mu)] = -\inf_{Q \in C} I(Q)\,,
\end{aligned}
$$

where the definition of $I(Q)$ is in (2.11).
The Large Deviation lower bound (type (1.4a)) is proved similarly, by taking $0 < a \leq \min\left(e^{-\beta}, e^{-\gamma(1+h)}\right)$, so that F_a becomes lower semi-continuous, using (2.14) and Varadhan's Lemma again.

■

2.3.2 McKean-Vlasov Equation

Given $Q \in \mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$ and $\eta \in \mathscr{S}$, we can associate with Q a Markov process on $\mathscr{S}^2$ with law $P^{\eta,Q}$, initial distribution λ and time-dependent infinitesimal generator

$$\mathcal{L}_t^{\eta,Q} f(\sigma,\omega) = e^{-\beta\sigma\omega}\nabla^\sigma f(\sigma,\omega) + e^{-\gamma\omega(m^\sigma_{\Pi_t Q}+h\eta)}\nabla^\omega f(\sigma,\omega)\,,$$

acting on $f : \mathscr{S}^2 \longrightarrow \mathbb{R}$.

Proposition 2.3.2. *For every $Q \in \mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$ such that $I(Q) < +\infty$,*

$$I(Q) = H(Q|P^Q)\,, \tag{2.15}$$

where $P^Q \in \mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$ is defined by

$$P^Q(d\sigma[0,T], d\omega[0,T], d\eta) = P^{\eta,Q}(d\sigma[0,T], d\omega[0,T])\mu(d\eta).$$

Proof. First we need to verify that the following representation for $F(Q)$ (defined in (2.6)) holds

$$F(Q) = \int Q(d\sigma[0,T], d\omega[0,T], d\eta) \log \frac{dP^{\eta,Q}}{dW}(\sigma[0,T], \omega[0,T])\,.$$

We begin by observing that, since by assumption $I(Q) < +\infty$, we have also $H(Q|W \otimes \mu) < +\infty$ and so, by the proof of Proposition 2.3.1, it follows that $Q \in \mathcal{I}$, which implies the integrals below are well defined. Using again the Girsanov's Formula for Markov Chains, we get

$$\int dQ \log \frac{dP^{\eta,Q}}{dW}(\sigma[0,T], \omega[0,T]) =$$

$$= \int dQ \Bigg[\int_0^T \left(1 - e^{-\beta\sigma(t)\omega(t)}\right) dt + \int_0^T \left(1 - e^{-\gamma\omega(t)\left(\int \sigma \Pi_t Q(d\sigma, d\omega, d\eta) + h\eta\right)}\right) dt$$

$$- \beta \int_0^T \sigma(t-)\omega(t-) d\mathcal{N}_t^\sigma - \gamma \int_0^T \omega(t-) \left(\int \sigma \Pi_{t-} Q(d\sigma, d\omega, d\eta) + h\eta \right) d\mathcal{N}_t^\omega \Bigg]$$

$$= \int dQ \Bigg[\int_0^T \left(1 - e^{-\beta\sigma(t)\omega(t)}\right) dt + \int_0^T \left(1 - e^{-\gamma\omega(t)\left(\int \sigma \Pi_t Q(d\sigma, d\omega, d\eta) + h\eta\right)}\right) dt$$
$$+ \beta \int_0^T \sigma(t)\omega(t) d\mathcal{N}_t^\sigma + \gamma \int_0^T \omega(t) \left(\int \sigma \Pi_t Q(d\sigma, d\omega, d\eta) + h\eta \right) d\mathcal{N}_t^\omega \Bigg]$$

$$= \int dQ \Bigg[\int_0^T \left(1 - e^{-\beta\sigma(t)\omega(t)}\right) dt + \int_0^T \left(1 - e^{-\gamma\omega(t)\left(m^\sigma_{\Pi_t Q} + h\eta\right)}\right) dt$$
$$+ \beta \int_0^T \sigma(t)\omega(t) d\mathcal{N}_t^\sigma + \gamma \int_0^T \omega(t) \left(m^\sigma_{\Pi_t Q} + h\eta \right) d\mathcal{N}_t^\omega \Bigg]$$

$$= F(Q)$$

By combining what we obtained, we can compute

$$I(Q) = H(Q|W \otimes \mu) - F(Q) = \int dQ \log \frac{dQ}{d(W \otimes \mu)} - \int dQ \log \frac{dP^{\eta,Q}}{dW}$$

$$= \int dQ \log \frac{dQ}{d(P^{\eta,Q} \otimes \mu)} = \int dQ \log \frac{dQ}{dP^Q} = H(Q|P^Q)\,.$$

■

Theorem 2.3.1. *Let us suppose that the initial distribution of the Markov process* $(\underline{\sigma}(t), \underline{\omega}(t))_{t \geq 0}$ *with generator* (2.2) *is such that the random variables* $(\sigma_j(0), \omega_j(0))_{j=1}^N$ *are independent and identically distributed with law* λ. *Then the equation* $I(Q) = 0$ *admits a unique solution* $Q_* \in \mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$, *such that its marginals* $q_t^\eta = \Pi_t Q_*^\eta \in \mathcal{M}_1(\mathscr{S}^2)$ *are weak solutions of the nonlinear McKean-Vlasov equation*

$$\begin{cases} \frac{\partial q_t^\eta}{\partial t} = \mathcal{L}^\eta q_t^\eta & (t \in [0,T],\ \eta \in \mathscr{S}) \\ q_0^\eta = \lambda \end{cases} \tag{2.16}$$

where, for all the triples $(\sigma, \omega, \eta) \in \mathscr{S}^3$, *the operator* $\mathcal{L}^\eta$ *acts*

$$\mathcal{L}^\eta q_t^\eta(\sigma, \omega) = \nabla^\sigma \left[e^{-\beta\sigma\omega} q_t^\eta(\sigma,\omega) \right] + \nabla^\omega \left[e^{-\gamma\omega(m^\sigma_{q_t} + h\eta)} q_t^\eta(\sigma,\omega) \right] \tag{2.17}$$

and q_t is defined by

$$q_t(\sigma, \omega) = \int_{\mathscr{S}} q_t^\eta(\sigma, \omega)\mu(d\eta).$$

Moreover, with respect to a metric $d(\cdot,\cdot)$ inducing the weak topology, $\rho_N \longrightarrow Q_*$ in probability with exponential rate, i.e. $\mathcal{P}_N\{d(\rho_N, Q_*) > \varepsilon\}$ is exponentially small in N, for each $\varepsilon > 0$.

Proof. We know that the relative entropy between two measures is zero then the two measures must be equal (see Remark 1.2.3). By this property, from (2.15) we have $I(Q) = 0$ translates into $Q = P^Q$. Let us suppose Q_* is a solution of this last equation. Then, in particular, for a given η, $q_t^\eta := \Pi_t Q_*^\eta = \Pi_t P^{Q_*^\eta}$. The marginals of a Markov process are solutions of the corresponding forward equation that, in this case, leads to the fact that q_t^η is a solution of (2.16). This differential equation, being an equation in finite dimension with locally Lipschitz coefficients, has at most one solution in $[0, T]$. Since $P^{Q_*^\eta}$ is totally determined by the flow q_t^η, it follows that equation $Q = P^Q$ has at most one solution. The existence of a solution derives from the fact that $I(Q)$ is the rate function of a Large Deviation Principle and therefore it has to have at least one zero: indeed, by the bound of type (1.4a) with $O = \mathcal{M}_1((\mathcal{D}[0,T])^2 \times \mathscr{S})$, we get $\inf_{Q \in O} I(Q) = 0$. Since I is lower semi-continuous, it attains this null value and so this infimum is actually a minimum.

It remains to prove the Law of Large Numbers for ρ_N: with respect to a metric $d(\cdot,\cdot)$ inducing the weak topology, $\rho_N \xrightarrow{N \to +\infty} Q_*$ in probability with exponential rate, i.e. $\mathcal{P}_N\{d(\rho_N, Q_*) > \varepsilon\}$ is exponentially small in N, for each $\varepsilon > 0$.
Let Q_* be the unique solution of equation $Q = P^Q$ and let B_{Q_*} be an arbitrary open neighborhood of Q_*. By the Large Devition upper bound (type (1.4b)), we have

$$\limsup_{N \to +\infty} \frac{1}{N} \log \mathcal{P}_N(\rho_N \notin B_{Q_*}) \leq - \inf_{Q \notin B_{Q_*}} I(Q) < 0\,,$$

where the last inequality comes from the lower semi-continuity of I, the compactness of its level-sets and the fact that $I(Q) > 0$ for $Q \neq Q_*$. Indeed, if $\inf_{Q \notin B_{Q_*}} I(Q) = 0$, then there exists a sequence $(Q_n)_n \notin B_{Q_*}$ such that $I(Q_n) \xrightarrow{n \to +\infty} 0$. By the compactness of level-sets, the sequence $(Q_n)_n$ admits a subsequence $(Q_{n_k})_{n_k}$ converging to $\overline{Q} \notin B_{Q_*}$, when $n_k \longrightarrow +\infty$. Thanks to the lower semi-continuity of I, it follows $I(\overline{Q}) \leq \liminf_{n_k \longrightarrow +\infty} I(Q_{n_k}) = 0$, which contradicts $I(Q) > 0$ for $Q \neq Q_*$. Thus, from the above inequality, we deduce that there exists a positive constant A such that

$$\mathcal{P}_N(\rho_N \notin B_{Q_*}) \leq A e^{-N \inf_{Q \notin B_{Q_*}} I(Q)}.$$

It means that, if we denote with $d(\cdot,\cdot)$ any metric which induces the weak topology on $\mathcal{M}_1$, for every $\varepsilon > 0$, the probability $\mathcal{P}_N(\rho_N \notin B_{Q_*}) = \mathcal{P}_N\{d(\rho_N, Q_*) \geq \varepsilon\}$ converges toward zero exponentially fast with respect to N and this concludes the proof of the Law of Large Numbers. ∎

2.3.3 Stationary Solution(s)

The equation (2.16) describes the behavior of the system governed by generator (2.2) in the infinite volume limit. We are interested in the detection of the t-stationary solution(s) of this equation and in the study of the large time dynamics of it (them). We recall that to be t-stationary solution for (2.16) means to satisfy the equation $\mathcal{L}^\eta q^\eta = 0$ for every t.
First of all, we proceed to reformulate the "original" McKean-Vlasov equation (2.16) in terms of $m^\eta_{q_t}$, $m^\sigma_{q_t}$, $m^\omega_{q_t}$, $m^{\sigma\omega}_{q_t}$, $m^{\sigma\eta}_{q_t}$, $m^{\omega\eta}_{q_t}$ and $m^{\sigma\omega\eta}_{q_t}$ defined as follows:

$$m^\eta_t := \frac{1}{2} \sum_{\sigma,\omega \in \mathscr{S}} \sum_{\eta \in \mathscr{S}} \eta\, q^\eta_t(\sigma,\omega) \tag{2.18}$$

$$m^\sigma_t := \frac{1}{2} \sum_{\sigma,\omega \in \mathscr{S}} \sum_{\eta \in \mathscr{S}} \sigma\, q^\eta_t(\sigma,\omega) \qquad m^{\sigma\eta}_t := \frac{1}{2} \sum_{\sigma,\omega \in \mathscr{S}} \sum_{\eta \in \mathscr{S}} \sigma\eta\, q^\eta_t(\sigma,\omega) \tag{2.19}$$

$$m^\omega_t := \frac{1}{2} \sum_{\sigma,\omega \in \mathscr{S}} \sum_{\eta \in \mathscr{S}} \omega\, q^\eta_t(\sigma,\omega) \qquad m^{\omega\eta}_t := \frac{1}{2} \sum_{\sigma,\omega \in \mathscr{S}} \sum_{\eta \in \mathscr{S}} \omega\eta\, q^\eta_t(\sigma,\omega) \tag{2.20}$$

$$m^{\sigma\omega}_t := \frac{1}{2} \sum_{\sigma,\omega \in \mathscr{S}} \sum_{\eta \in \mathscr{S}} \sigma\omega\, q^\eta_t(\sigma,\omega) \qquad m^{\sigma\omega\eta}_t := \frac{1}{2} \sum_{\sigma,\omega \in \mathscr{S}} \sum_{\eta \in \mathscr{S}} \sigma\omega\eta\, q^\eta_t(\sigma,\omega)\,, \tag{2.21}$$

where q^η_t has the meaning explained in Theorem 2.3.1 and we have written $m^\cdot_t$ instead of $m^\cdot_{q_t}$. We introduce these expectations because the probability measure q_t on $\mathscr{S}^3$ is completely determined by them.

Lemma 2.3.3. *Equations* (2.16) *can be rewritten in the following form:*

$$
\begin{aligned}
\dot{m}_t^{\eta} =&0\\
\dot{m}_t^{\sigma} =&-2\,m_t^{\sigma}\cosh(\beta)+2\,m_t^{\omega}\sinh(\beta)\\
\dot{m}_t^{\omega} =&-2\,m_t^{\omega}\cosh(\gamma h)\cosh(\gamma m_t^{\sigma})-2\,m_t^{\omega\eta}\sinh(\gamma h)\sinh(\gamma m_t^{\sigma})\\
&+2\,\cosh(\gamma h)\sinh(\gamma m_t^{\sigma})\\
\dot{m}_t^{\sigma\omega} =&2\,m_t^{\sigma}\cosh(\gamma h)\sinh(\gamma m_t^{\sigma})-2\,m_t^{\sigma\omega}\left[\cosh(\beta)+\cosh(\gamma h)\cosh(\gamma m_t^{\sigma})\right]\\
&+2\,m_t^{\sigma\eta}\sinh(\gamma h)\cosh(\gamma m_t^{\sigma})-2\,m_t^{\sigma\omega\eta}\sinh(\gamma h)\sinh(\gamma m_t^{\sigma})+2\sinh(\beta)\\
\dot{m}_t^{\sigma\eta} =&-2\,m_t^{\sigma\eta}\cosh(\beta)+2\,m_t^{\omega\eta}\sinh(\beta)\\
\dot{m}_t^{\omega\eta} =&-2\,m_t^{\omega}\sinh(\gamma h)\sinh(\gamma m_t^{\sigma})-2\,m_t^{\omega\eta}\cosh(\gamma h)\cosh(\gamma m_t^{\sigma})\\
&+2\,\sinh(\gamma h)\cosh(\gamma m_t^{\sigma})\\
\dot{m}_t^{\sigma\omega\eta} =&2\,m_t^{\sigma}\sinh(\gamma h)\cosh(\gamma m_t^{\sigma})-2\,m_t^{\sigma\omega}\sinh(\gamma h)\sinh(\gamma m_t^{\sigma})\\
&+2\,m_t^{\sigma\eta}\cosh(\gamma h)\sinh(\gamma m_t^{\sigma})-2\,m_t^{\sigma\omega\eta}\left[\cosh(\beta)+\cosh(\gamma h)\cosh(\gamma m_t^{\sigma})\right]
\end{aligned}
$$

with initial condition $m_0^{\eta}=m_{(\lambda,\mu)}^{\eta}=0$, $m_0^{\sigma}=m_{(\lambda,\mu)}^{\sigma}$, $m_0^{\omega}=m_{(\lambda,\mu)}^{\omega}$, $m_0^{\sigma\omega}=m_{(\lambda,\mu)}^{\sigma\omega}$, $m_0^{\sigma\eta}=m_{(\lambda,\mu)}^{\sigma\eta}$, $m_0^{\omega\eta}=m_{(\lambda,\mu)}^{\omega\eta}$ *and* $m_0^{\sigma\omega\eta}=m_{(\lambda,\mu)}^{\sigma\omega\eta}$.

Proof. By definition (2.19) and Theorem 2.3.1 we deduce

$$
\begin{aligned}
\dot{m}_t^{\sigma} =&\sum_{\sigma,\omega\in\mathscr{S}}\sigma\,\dot{q}_t(\sigma,\omega)=\frac{1}{2}\sum_{\sigma,\omega\in\mathscr{S}}\sum_{\eta\in\mathscr{S}}\sigma\,\dot{q}_t^{\eta}(\sigma,\omega)=\frac{1}{2}\sum_{\sigma,\omega\in\mathscr{S}}\sum_{\eta\in\mathscr{S}}\sigma\,\mathcal{L}^{\eta}q_t^{\eta}(\sigma,\omega)\\
=&\frac{1}{2}\sum_{\sigma,\omega\in\mathscr{S}}\sum_{\eta\in\mathscr{S}}\sigma\left\{\nabla^{\sigma}\left[e^{-\beta\sigma\omega}q_t^{\eta}(\sigma,\omega)\right]+\nabla^{\omega}\left[e^{-\gamma\omega(m_t^{\sigma}+h\eta)}q_t^{\eta}(\sigma,\omega)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{2}\sum_{\sigma,\omega\in\mathscr{S}}\sum_{\eta\in\mathscr{S}}\sigma\left\{e^{\beta\sigma\omega}q_t^{\eta}(-\sigma,\omega)-e^{-\beta\sigma\omega}q_t^{\eta}(\sigma,\omega)+\nabla^{\omega}\left[e^{-\gamma\omega(m_t^{\sigma}+h\eta)}q_t^{\eta}(\sigma,\omega)\right]\right\}\\
=&-\sum_{\sigma,\omega\in\mathscr{S}}\sum_{\eta\in\mathscr{S}}\sigma e^{-\beta\sigma\omega}q_t^{\eta}(\sigma,\omega)+\underbrace{\frac{1}{2}\sum_{\sigma,\omega\in\mathscr{S}}\sum_{\eta\in\mathscr{S}}\sigma\nabla^{\omega}\left[e^{-\gamma\omega(m_t^{\sigma}+h\eta)}q_t^{\eta}(\sigma,\omega)\right]}_{=0}\\
=&-\sum_{\sigma,\omega\in\mathscr{S}}\sum_{\eta\in\mathscr{S}}\sigma\left[\cosh(\beta)-\sigma\omega\sinh(\beta)\right]q_t^{\eta}(\sigma,\omega)\\
=&-\cosh(\beta)\sum_{\sigma,\omega\in\mathscr{S}}\sum_{\eta\in\mathscr{S}}\sigma\, q_t^{\eta}(\sigma,\omega)+\sinh(\beta)\sum_{\sigma,\omega\in\mathscr{S}}\sum_{\eta\in\mathscr{S}}\omega\, q_t^{\eta}(\sigma,\omega)\\
=&-2\,m_t^{\sigma}\cosh(\beta)+2\,m_t^{\omega}\sinh(\beta)\,,
\end{aligned}
$$

where the last equality holds thanks to (2.19) and (2.20). So the first equation of Lemma 2.3.3 is proved. Similarly, we can obtain all the others. ■

Also in this case m_t^{η} is a static variable, thus any equilibrium solution of the system in Lemma 2.3.3 is of the form

$$
\begin{aligned}
m_*^{\sigma}&=\tanh(\beta)\frac{\sinh(\gamma m_*^{\sigma})\cosh(\gamma m_*^{\sigma})}{\cosh^2(\gamma m_*^{\sigma})+\sinh^2(\gamma h)}\\
m_*^{\omega}&=\frac{\sinh(\gamma m_*^{\sigma})\cosh(\gamma m_*^{\sigma})}{\cosh^2(\gamma m_*^{\sigma})+\sinh^2(\gamma h)}\\
m_*^{\sigma\omega}&=\dots\\
m_*^{\sigma\eta}&=\tanh(\beta)\tanh(\gamma h)\frac{1+\sinh^2(\gamma h)}{\cosh^2(\gamma m_*^{\sigma})+\sinh^2(\gamma h)}\\
m_*^{\omega\eta}&=\tanh(\gamma h)\frac{1+\sinh^2(\gamma h)}{\cosh^2(\gamma m_*^{\sigma})+\sinh^2(\gamma h)}
\end{aligned}
\tag{2.22}
$$

$$m_*^{\sigma\omega\eta} = \dots$$

To discover the presence of phase transition(s) (multiple equilibria) and the stability of equilibria, it is sufficient studying the first equation of (2.22):

$$m_*^\sigma = \tanh(\beta)\frac{\sinh(\gamma m_*^\sigma)\cosh(\gamma m_*^\sigma)}{\cosh^2(\gamma m_*^\sigma) + \sinh^2(\gamma h)},$$

because all the remaining $m_*^{\cdot} = m_*^{\cdot}(m_*^\sigma)$ and hence $\lim_{t\to+\infty} m_t^{\cdot} = m_*^{\cdot}$ when $\lim_{t\to+\infty} m_t^\sigma = m_*^\sigma$. The stationary system we are dealing with is essentially one-dimensional.

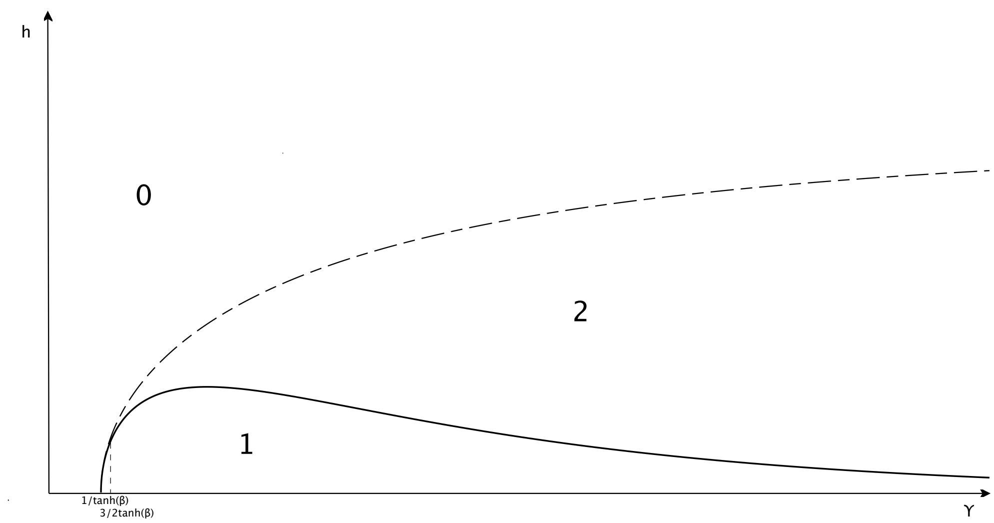

Figure 2.1: Phase diagram for a fixed value of β

For a fixed value of β, the phase diagram is qualitatively drawn in Figure 2.1. There are three phases, corresponding to 0, 1 and 2 ferromagnetic solutions respectively. The continuous separation curve is

$$h = h(\beta,\gamma) = \frac{1}{\gamma}\operatorname{arccosh}(\sqrt{\gamma\tanh(\beta)}) \qquad \gamma \in \left[\frac{1}{\tanh(\beta)}, +\infty\right), \tag{2.23}$$

while the dotted one is obtained numerically and it is due to the fact that the function

$$m_*^\sigma \longmapsto \tanh(\beta)\frac{\sinh(\gamma m_*^\sigma)\cosh(\gamma m_*^\sigma)}{\cosh^2(\gamma m_*^\sigma) + \sinh^2(\gamma h)}$$

is not always concave. The two curves coincide for $\gamma \in \left[\frac{1}{\tanh(\beta)}, \frac{3}{2\tanh(\beta)}\right]$ and separate at the "tricritical" point $\left(\frac{3}{2\tanh(\beta)}, h\left(\beta, \frac{3}{2\tanh(\beta)}\right)\right)$.

Theorem 2.3.2. *Consider* (2.22) *and fix a value for* β.

(a) If (γ, h) *belongs to the region* 0 *of Figure 2.1, then the only solution is*

$$m_*^0 := \left(0, 0, \frac{\text{th}(\beta)\text{th}(\gamma h)\text{sh}(\gamma h) + \text{sh}(\beta)}{\text{ch}(\beta) + \text{ch}(\gamma h)}, \text{th}(\beta)\text{th}(\gamma h), \text{th}(\gamma h), 0\right).$$

(b) If (γ, h), *with* $\gamma \in [\frac{1}{\tanh(\beta)}, +\infty)$, *is below the curve* (2.23), *then there are three solutions:* m_*^0, $(m_*, m_*^\omega(m_*), m_*^{\sigma\omega}(m_*), m_*^{\sigma\eta}(m_*), m_*^{\omega\eta}(m_*), m_*^{\sigma\omega\eta}(m_*))$ *and* $(-m_*, -m_*^\omega(m_*), m_*^{\sigma\omega}(-m_*), m_*^{\sigma\eta}(m_*), m_*^{\omega\eta}(m_*), m_*^{\sigma\omega\eta}(-m_*))$, *where* m_* *is the unique positive solution of the first equation of* (2.22).

(c) If we choose the parameters above the curve (2.23) *and* h *is small enough, in other words* (γ, h) *belongs to the region* 2 *of Figure 2.1, then two further solutions arise.*

Proof. To deepen the analysis of stationary solution(s) and phase transition(s), it is sufficient to study the behavior of the self-consistency relation satisfied by m_*^σ. Looking at the first expression in (2.22), we can write

$$\begin{aligned} m_*^\sigma &= \Gamma_{\beta,\gamma,h}(m_*^\sigma) \\ \Gamma_{\beta,\gamma,h}(m_*^\sigma) &= \tanh(\beta)\tfrac{\sinh(\gamma m_*^\sigma)\cosh(\gamma m_*^\sigma)}{\cosh^2(\gamma m_*^\sigma)+\sinh^2(\gamma h)}\,. \end{aligned} \tag{2.24}$$

It follows from (2.24) that

- $m_*^\sigma \longmapsto \Gamma_{\beta,\gamma,h}(m_*^\sigma)$ is a continuous function for all the values of β, γ and h;
- $\lim_{m_*^\sigma \to \pm\infty} \Gamma_{\beta,\gamma,h}(m_*^\sigma) = \pm\tanh(\beta)$;
- $\Gamma'_{\beta,\gamma,h}(m_*^\sigma) = \gamma\tanh(\beta)\dfrac{[1 + 2\sinh^2(\gamma h)]\cosh^2(\gamma m_*^\sigma) - \sinh^2(\gamma h)}{[\cosh^2(\gamma m_*^\sigma) + \sinh^2(\gamma h)]^2} > 0$ for every β, γ and h.

Since $\Gamma_{\beta,\gamma,h}(m_*^\sigma)$ is an odd function with respect to m_*^σ, we have $\Gamma_{\beta,\gamma,h}(0) = 0$ for all β, γ and h, so that (2.24) always has the paramagnetic solution $m_*^\sigma = 0$. Now, we investigate under what conditions ferromagnetic solutions $m_*^\sigma > 0$ may occur. We restrict to work in the positive half-plane.
If

$$\Gamma'_{\beta,\gamma,h}(0) = \gamma\frac{\tanh(\beta)}{\cosh^2(\gamma h)} > 1\,, \tag{2.25}$$

then there is at least one ferromagnetic solution. However, since $\Gamma_{\beta,\gamma,h}(m_*^\sigma)$ is not always concave, there may be a ferromagnetic solution even when (2.25) fails. In this case, there must be at least two ferromagnetic solutions (corresponding to the curve $m_*^\sigma \longmapsto \Gamma_{\beta,\gamma,h}(m_*^\sigma)$ crossing the diagonal first from below and then from above).
The regime defined by (2.25) lies under the curve (2.23). An idea of when two ferromagnetic solutions arise may be obtained from the Taylor expansion of $\Gamma_{\beta,\gamma,h}(m_*^\sigma)$ for small m_*^σ; in fact,

$$\Gamma_{\beta,\gamma,h}(m_*^\sigma) = \gamma \frac{\tanh(\beta)}{\cosh^2(\gamma h)} m_*^\sigma + \gamma^3 \frac{\tanh(\beta)[2\cosh^2(\gamma h) - 3]}{3\cosh^4(\gamma h)} (m_*^\sigma)^3 + O\left((m_*^\sigma)^5\right)$$

and on the curve defined by (2.23) it reduces to

$$\Gamma_{\beta,\gamma,h}(m_*^\sigma) = m_*^\sigma + \gamma \left(\frac{2}{3}\gamma - \frac{1}{\tanh(\beta)} \right) (m_*^\sigma)^3 + O\left((m_*^\sigma)^5\right)$$

from which we can see that $\overline{\gamma} = \frac{3}{2\tanh(\beta)}$ is a critical value. Indeed, if $\gamma > \overline{\gamma}$, then as h increases through $h(\beta, \gamma)$ (i.e. $\Gamma'_{\beta,\gamma,h}(0)$ decreases through 1) at least two positive ferromagnetic solutions occur, because $m_*^\sigma \longmapsto \Gamma_{\beta,\gamma,h}(m_*^\sigma)$ is convex for small m_*^σ.

■

The phase diagram is analogous to the random Curie-Weiss Model one. We are going to focus on the critical regime corresponding to the critical values for the parameters $\gamma = \frac{\cosh^2(\gamma h)}{\tanh(\beta)}$, meaning that we are on the curve (2.23). In this region of parameters, the equilibrium m_*^0 is neutrally stable for the linearized system. In fact, denoting by

$$\begin{aligned} V: \quad & [-1,+1]^6 \longrightarrow \mathbb{R}^6 \\ \underline{x} := (x_1, x_2, x_3, x_4, x_5, x_6) & \longmapsto (V_1(\underline{x}), V_2(\underline{x}), V_3(\underline{x}), V_4(\underline{x}), V_5(\underline{x}), V_6(\underline{x})) \end{aligned}$$

with

$$V_1(\underline{x}) := -2\, x_1 \cosh(\beta) + 2\, x_2 \sinh(\beta)$$

$$\begin{aligned} V_2(\underline{x}) := & -2\, x_2 \cosh(\gamma h) \cosh(\gamma x_1) - 2\, x_5 \sinh(\gamma h) \sinh(\gamma x_1) \\ & + 2\, \cosh(\gamma h) \sinh(\gamma x_1) \end{aligned}$$

$$V_3(\underline{x}) := 2\,x_1 \cosh(\gamma h) \sinh(\gamma x_1) - 2\,x_3\,[\cosh(\beta) + \cosh(\gamma h) \cosh(\gamma x_1)] \\ + 2\,x_4 \sinh(\gamma h) \cosh(\gamma x_1) - 2\,x_6 \sinh(\gamma h) \sinh(\gamma x_1) + 2 \sinh(\beta)$$

$$V_4(\underline{x}) := -2\,x_4 \cosh(\beta) + 2\,x_5 \sinh(\beta)$$

$$V_5(\underline{x}) := -2\,x_2 \sinh(\gamma h) \sinh(\gamma x_1) - 2\,x_5 \cosh(\gamma h) \cosh(\gamma x_1) \\ + 2\ \sinh(\gamma h) \cosh(\gamma x_1)$$

$$V_6(\underline{x}) := 2\,x_1 \sinh(\gamma h) \cosh(\gamma x_1) - 2\,x_3 \sinh(\gamma h) \sinh(\gamma x_1) \\ + 2\,x_4 \cosh(\gamma h) \sinh(\gamma x_1) - 2\,x_6\,[\cosh(\beta) + \cosh(\gamma h) \cosh(\gamma x_1)]$$

the vector field of the system in Lemma 2.3.3, we obtain the linearized matrix evaluated in the stationary solution is $DV(m_*^0)$:

$$2\begin{bmatrix} -\mathrm{ch}(\beta) & \mathrm{sh}(\beta) & 0 & 0 & 0 & 0 \\ \frac{\gamma}{\mathrm{ch}(\gamma h)} & -\mathrm{ch}(\gamma h) & 0 & 0 & 0 & 0 \\ 0 & 0 & -[\mathrm{ch}(\beta)+\mathrm{ch}(\gamma h)] & \mathrm{sh}(\gamma h) & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{ch}(\beta) & \mathrm{sh}(\beta) & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{ch}(\gamma h) & 0 \\ \mathrm{sh}(\gamma h) + \gamma \frac{\mathrm{th}(\beta)\mathrm{th}(\gamma h)}{\mathrm{ch}(\beta)+\mathrm{ch}(\gamma h)} & 0 & 0 & 0 & 0 & -[\mathrm{ch}(\beta)+\mathrm{ch}(\gamma h)] \end{bmatrix}.$$

Its eigenvalues are given by

$$\lambda_1 = -\cosh(\beta) - \cosh(\gamma h) + \sqrt{[\cosh(\beta) - \cosh(\gamma h)]^2 + 4\gamma \frac{\sinh(\beta)}{\cosh(\gamma h)}}$$

$$\lambda_2 = -\cosh(\beta) - \cosh(\gamma h) - \sqrt{[\cosh(\beta) - \cosh(\gamma h)]^2 + 4\gamma \frac{\sinh(\beta)}{\cosh(\gamma h)}}$$

$$\lambda_3 = \lambda_4 = -2[\cosh(\beta) + \cosh(\gamma h)]$$

$$\lambda_5 = -2\cosh(\gamma h)$$

$$\lambda_6 = -2\cosh(\beta);$$

they all are real and it is easy to see that $\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 < 0$ for every value of β, γ, h; instead, the value of λ_1 depends on the parameters:

- if $\gamma < \frac{\cosh^2(\gamma h)}{\tanh(\beta)}$, then $\lambda_1 < 0$ and thus m^0_* is linearly stable;
- if $\gamma = \frac{\cosh^2(\gamma h)}{\tanh(\beta)}$, then $\lambda_1 = 0$ and thus $DV(m^0_*)$ has a neutral direction;
- if $\gamma > \frac{\cosh^2(\gamma h)}{\tanh(\beta)}$, then $\lambda_1 > 0$ and thus the linearized system admits a direction which is unstable.

2.4 Normal Fluctuations and Central Limit Theorem

Thanks to Theorem 2.3.1 we established a Law of Large Numbers for the empirical measure ρ_N: $\rho_N \longrightarrow Q_*$. We are going to analyze the Normal fluctuations around the limit Q_*. We are also interested in the N-asymptotic distribution of $\rho_N - Q_*$.
Using a weak convergence-type approach based on uniform convergence of the infinitesimal generators, deeply explained in [EK86], it is possible to provide a dynamical interpretation of the recalled Law of Large Numbers.

Let $f : \mathscr{S}^2 \longrightarrow \mathbb{R}$ be a function and define $\rho_N(t)$, the marginal distribution of ρ_N at time t, by

$$\int f(\sigma, \omega)\, d\rho_N(t) = \frac{1}{N}\sum_{j=1}^{N} f(\sigma_j(t), \omega_j(t)).$$

We have $m^{\sigma}_N(t) = m^{\sigma}_{\rho_N(t)}$. For each fixed t, $\rho_N(t)$ is a probability on $\mathscr{S}^2$ and so, by the considerations which led as to introduce the expectations (2.18), (2.19), (2.20) and (2.21), we can proceed similarly saying $\rho_N(t)$ is completely determined by the vector $(m^{\eta}_{\rho_N(t)}, m^{\sigma}_{\rho_N(t)}, m^{\omega}_{\rho_N(t)}, m^{\sigma\,\omega}_{\rho_N(t)}, m^{\sigma\,\eta}_{\rho_N(t)}, m^{\omega\,\eta}_{\rho_N(t)}, m^{\sigma\,\omega\,\eta}_{\rho_N(t)})$ and seeing it as a seven-dimensional object. Thus $(\rho_N(t))_{t\in[0,T]}$ is a seven-dimensional flow. A simple consequence of Theorem 2.3.1 is the following convergence of flows:

$$(\rho_N(t))_{t\in[0,T]} \longrightarrow (q_t)_{t\in[0,T]}\,, \tag{2.26}$$

where the convergence is meant in probability, with respect to the weak topology for measure-valued processes. Since the flow of marginals contains less information than the full measure of paths, the Law of Large Numbers in (2.26) is weaker than the one in Theorem 2.3.1. However, the corresponding fluctuation flow

$$(N^{1/2}(\rho_N(t) - q_t))_{t\in[0,T]}$$

is also a finite-dimensional flow whose limiting distribution can be explicitly determined.

Lemma 2.4.1. *The stochastic process* $(m^{\eta}_{\rho_N(t)},\ m^{\sigma}_{\rho_N(t)},\ m^{\omega}_{\rho_N(t)},\ m^{\sigma\omega}_{\rho_N(t)},\ m^{\sigma\eta}_{\rho_N(t)},\ m^{\omega\eta}_{\rho_N(t)},\ m^{\sigma\omega\eta}_{\rho_N(t)})$ *is an* order parameter *for the model; it means its evolution is Markovian.*

Proof. To prove that $(m^{\eta}_{\rho_N(t)},\ m^{\sigma}_{\rho_N(t)},\ m^{\omega}_{\rho_N(t)},\ m^{\sigma\omega}_{\rho_N(t)},\ m^{\sigma\eta}_{\rho_N(t)},\ m^{\omega\eta}_{\rho_N(t)},\ m^{\sigma\omega\eta}_{\rho_N(t)})$ is a Markov process, one must write down the expression of the infinitesimal generator $\mathcal{K}_N$ whose dynamics are driven by. We apply Lemma 1.3.1.
The process $\{(\underline{\sigma}(t), \underline{\omega}(t)\}_{t\geq 0}$ is a continuous time Markov chain on the finite state space $\mathscr{S}^{2N}$, with infinitesimal generator L_N, defined by (2.2). Consider the function

$$\begin{aligned} \zeta :\ \mathscr{S}^{2N} &\longrightarrow [-1,+1]^7 \\ (\underline{\sigma},\underline{\omega}) &\longmapsto (m^{\eta}_{\rho_N}, m^{\sigma}_{\rho_N}, m^{\omega}_{\rho_N}, m^{\sigma\omega}_{\rho_N}, m^{\sigma\eta}_{\rho_N}, m^{\omega\eta}_{\rho_N}, m^{\sigma\omega\eta}_{\rho_N})\,, \end{aligned}$$

it plays the role of g in Lemma 1.3.1; then, for every $\phi : \mathscr{S}^{2N} \longrightarrow \mathbb{R}$, we have

$$L_N(\phi\circ\zeta) = (\mathcal{K}_N\phi)\circ\zeta$$

and $\zeta(\underline{\sigma},\underline{\omega})$ is a Markov process with generator $\mathcal{K}_N$ given by

$$\begin{aligned} &\mathcal{K}_N\phi(m^{\eta}_{\rho_N}, m^{\sigma}_{\rho_N}, m^{\omega}_{\rho_N}, m^{\sigma\omega}_{\rho_N}, m^{\sigma\eta}_{\rho_N}, m^{\omega\eta}_{\rho_N}, m^{\sigma\omega\eta}_{\rho_N}) = \\ &= \sum_{i,j,k\in\mathscr{S}} |A_{\rho_N}(i,j,k)| e^{-\beta ij}\cdot \\ &\quad\cdot\Bigg[\phi\Bigg(m^{\eta}_{\rho_N}, m^{\sigma}_{\rho_N} - i\frac{2}{N}, m^{\omega}_{\rho_N}, m^{\sigma\omega}_{\rho_N} - ij\frac{2}{N}, m^{\sigma\eta}_{\rho_N} - ik\frac{2}{N}, m^{\omega\eta}_{\rho_N}, m^{\sigma\omega\eta}_{\rho_N} - ijk\frac{2}{N}\Bigg) \\ &\qquad\qquad - \phi(m^{\eta}_{\rho_N}, m^{\sigma}_{\rho_N}, m^{\omega}_{\rho_N}, m^{\sigma\omega}_{\rho_N}, m^{\sigma\eta}_{\rho_N}, m^{\omega\eta}_{\rho_N}, m^{\sigma\omega\eta}_{\rho_N})\Bigg] \\ &+ \sum_{i,j,k\in\mathscr{S}} |A_{\rho_N}(i,j,k)| e^{-\gamma j(m^{\sigma}_{\rho_N}+kh)}\cdot \\ &\quad\cdot\Bigg[\phi\Bigg(m^{\eta}_{\rho_N}, m^{\sigma}_{\rho_N}, m^{\omega}_{\rho_N} - j\frac{2}{N}, m^{\sigma\omega}_{\rho_N} - ij\frac{2}{N}, m^{\sigma\eta}_{\rho_N}, m^{\omega\eta}_{\rho_N} - jk\frac{2}{N}, m^{\sigma\omega\eta}_{\rho_N} - ijk\frac{2}{N}\Bigg) \end{aligned}$$

$$- \phi(m^{\eta}_{\rho_N}, m^{\sigma}_{\rho_N}, m^{\omega}_{\rho_N}, m^{\sigma\omega}_{\rho_N}, m^{\sigma\eta}_{\rho_N}, m^{\omega\eta}_{\rho_N}, m^{\sigma\omega\eta}_{\rho_N})\Big], \qquad (2.27)$$

where $A_{\rho_N}(i,j,k)$ is the set of all triples $(\sigma_d, \omega_d, \eta_d)$, $d \in \{1, \ldots, N\}$, such that $\sigma_d = i$, $\omega_d = j$ and $\eta_d = k$, with $i, j, k \in \mathscr{S}$; hence

$$|A_{\rho_N}(i,j,k)| = \frac{N}{8}\left[1 + km^{\eta}_{\rho_N} + im^{\sigma}_{\rho_N} + jm^{\omega}_{\rho_N} + ijm^{\sigma\omega}_{\rho_N} + ikm^{\sigma\eta}_{\rho_N} + jkm^{\omega\eta}_{\rho_N} + ijkm^{\sigma\omega\eta}_{\rho_N}\right].$$

■

Theorem 2.4.1. *In the limit as $N \longrightarrow +\infty$, the seven-dimensional fluctuation process $(r_N(t), x_N(t), y_N(t), z_N(t), u_N(t), v_N(t), w_N(t))$, defined by*

$$r_N(t) := N^{1/2} m^{\eta}_{\rho_N(t)}$$

$$\begin{aligned} x_N(t) &:= N^{1/2}\left(m^{\sigma}_{\rho_N(t)} - m^{\sigma}_t\right) & u_N(t) &:= N^{1/2}\left(m^{\sigma\eta}_{\rho_N(t)} - m^{\sigma\eta}_t\right) \\ y_N(t) &:= N^{1/2}\left(m^{\omega}_{\rho_N(t)} - m^{\omega}_t\right) & v_N(t) &:= N^{1/2}\left(m^{\omega\eta}_{\rho_N(t)} - m^{\omega\eta}_t\right) \\ z_N(t) &:= N^{1/2}\left(m^{\sigma\omega}_{\rho_N(t)} - m^{\sigma\omega}_t\right) & w_N(t) &:= N^{1/2}\left(m^{\sigma\omega\eta}_{\rho_N(t)} - m^{\sigma\omega\eta}_t\right), \end{aligned}$$

converges (in the sense of weak convergence of stochastic processes) to a limiting seven-dimensional Gaussian process $(r(t), x(t), y(t), z(t), u(t), v(t), w(t))$, which is the unique solution of the linear stochastic differential equation

$$dr(t) = 0$$

$$\begin{bmatrix} dx(t) \\ dy(t) \\ dz(t) \\ du(t) \\ dv(t) \\ dw(t) \end{bmatrix} = 2\,\mathscr{H}\, A_1(t)dt + 2A_2(t) \begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ u(t) \\ v(t) \\ w(t) \end{bmatrix} dt + D(t) \begin{bmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t) \\ dB_4(t) \\ dB_5(t) \\ dB_6(t) \end{bmatrix}, \qquad (2.28)$$

where $B_1, B_2, B_3, B_4, B_5, B_6$ are independent Standard Brownian motions, $\mathscr{H}$ is a Standard Gaussian random variable, $A_1(t)$ and $A_2(t)$ are respectively

$$\begin{bmatrix} 0 \\ \sinh(\gamma h)\cosh(\gamma m^{\sigma}_t) \\ 0 \\ 0 \\ \cosh(\gamma h)\sinh(\gamma m^{\sigma}_t) \\ \sinh(\beta) \end{bmatrix},$$

$$
\begin{bmatrix}
-\mathrm{ch}(\beta) & \mathrm{sh}(\beta) & 0 & 0 & 0 & 0 \\
\begin{array}{c} -\gamma m_t^{\omega}\mathrm{ch}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) \\ -\gamma m_t^{\omega\eta}\mathrm{sh}(\gamma h)\mathrm{ch}(\gamma m_t^{\sigma}) \\ +\gamma\mathrm{ch}(\gamma h)\mathrm{ch}(\gamma m_t^{\sigma}) \end{array} & -\mathrm{ch}(\gamma h)\mathrm{ch}(\gamma m_t^{\sigma}) & 0 & 0 & -\mathrm{sh}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) & 0 \\
\begin{array}{c} \mathrm{ch}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) \\ +\gamma m_t^{\sigma}\mathrm{ch}(\gamma h)\mathrm{ch}(\gamma m_t^{\sigma}) \\ -\gamma m_t^{\sigma\omega}\mathrm{ch}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) \\ +\gamma m_t^{\sigma\eta}\mathrm{sh}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) \\ -\gamma m_t^{\sigma\omega\eta}\mathrm{sh}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) \end{array} & 0 & \begin{array}{c} -\mathrm{ch}(\beta) \\ -\mathrm{ch}(\gamma h)\mathrm{ch}(\gamma m_t^{\sigma}) \end{array} & \mathrm{sh}(\gamma h)\mathrm{ch}(\gamma m_t^{\sigma}) & 0 & -\mathrm{sh}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) \\
0 & 0 & 0 & -\mathrm{ch}(\beta) & \mathrm{sh}(\beta) & 0 \\
\begin{array}{c} \gamma\mathrm{sh}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) \\ -\gamma m_t^{\omega}\mathrm{sh}(\gamma h)\mathrm{ch}(\gamma m_t^{\sigma}) \\ -\gamma m_t^{\omega\eta}\mathrm{ch}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) \end{array} & -\mathrm{sh}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) & 0 & 0 & -\mathrm{ch}(\gamma h)\mathrm{ch}(\gamma m_t^{\sigma}) & 0 \\
\begin{array}{c} \mathrm{sh}(\gamma h)\mathrm{ch}(\gamma m_t^{\sigma}) \\ +\gamma m_t^{\sigma}\mathrm{sh}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) \\ -\gamma m_t^{\sigma\omega}\mathrm{sh}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) \\ +\gamma m_t^{\sigma\eta}\mathrm{ch}(\gamma h)\mathrm{ch}(\gamma m_t^{\sigma}) \\ -\gamma m_t^{\sigma\omega\eta}\mathrm{ch}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) \end{array} & 0 & -\mathrm{sh}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) & \mathrm{ch}(\gamma h)\mathrm{sh}(\gamma m_t^{\sigma}) & 0 & \begin{array}{c} -\mathrm{ch}(\beta) \\ -\mathrm{ch}(\gamma h)\mathrm{ch}(\gamma m_t^{\sigma}) \end{array}
\end{bmatrix},
$$

$D(t)$ is a suitable 6×6 matrix and $(r(0), x(0), y(0), z(0), u(0), v(0), w(0))$ has a centered Gaussian distribution with covariance matrix

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-(m^{\sigma}_{(\lambda,\mu)})^2 & m^{\sigma\omega}_{(\lambda,\mu)} - m^{\sigma}_{(\lambda,\mu)}m^{\omega}_{(\lambda,\mu)} & m^{\omega}_{(\lambda,\mu)} - m^{\sigma}_{(\lambda,\mu)}m^{\sigma\omega}_{(\lambda,\mu)} & -m^{\sigma}_{(\lambda,\mu)}m^{\sigma\eta}_{(\lambda,\mu)} & -m^{\sigma}_{(\lambda,\mu)}m^{\omega\eta}_{(\lambda,\mu)} & -m^{\sigma}_{(\lambda,\mu)}m^{\sigma\omega\eta}_{(\lambda,\mu)} \\
0 & m^{\sigma\omega}_{(\lambda,\mu)} - m^{\sigma}_{(\lambda,\mu)}m^{\omega}_{(\lambda,\mu)} & 1-(m^{\omega}_{(\lambda,\mu)})^2 & m^{\sigma}_{(\lambda,\mu)} - m^{\omega}_{(\lambda,\mu)}m^{\sigma\omega}_{(\lambda,\mu)} & -m^{\omega}_{(\lambda,\mu)}m^{\sigma\eta}_{(\lambda,\mu)} & -m^{\omega}_{(\lambda,\mu)}m^{\omega\eta}_{(\lambda,\mu)} & -m^{\omega}_{(\lambda,\mu)}m^{\sigma\omega\eta}_{(\lambda,\mu)} \\
0 & m^{\omega}_{(\lambda,\mu)} - m^{\sigma}_{(\lambda,\mu)}m^{\sigma\omega}_{(\lambda,\mu)} & m^{\sigma}_{(\lambda,\mu)} - m^{\omega}_{(\lambda,\mu)}m^{\sigma\omega}_{(\lambda,\mu)} & 1-(m^{\sigma\omega}_{(\lambda,\mu)})^2 & -m^{\sigma\omega}_{(\lambda,\mu)}m^{\sigma\eta}_{(\lambda,\mu)} & -m^{\sigma\omega}_{(\lambda,\mu)}m^{\omega\eta}_{(\lambda,\mu)} & -m^{\sigma\omega}_{(\lambda,\mu)}m^{\sigma\omega\eta}_{(\lambda,\mu)} \\
0 & -m^{\sigma}_{(\lambda,\mu)}m^{\sigma\eta}_{(\lambda,\mu)} & -m^{\omega}_{(\lambda,\mu)}m^{\sigma\eta}_{(\lambda,\mu)} & -m^{\sigma\omega}_{(\lambda,\mu)}m^{\sigma\eta}_{(\lambda,\mu)} & 1-(m^{\sigma\eta}_{(\lambda,\mu)})^2 & -m^{\sigma\eta}_{(\lambda,\mu)}m^{\omega\eta}_{(\lambda,\mu)} & -m^{\sigma\eta}_{(\lambda,\mu)}m^{\sigma\omega\eta}_{(\lambda,\mu)} \\
0 & -m^{\sigma}_{(\lambda,\mu)}m^{\omega\eta}_{(\lambda,\mu)} & -m^{\omega}_{(\lambda,\mu)}m^{\omega\eta}_{(\lambda,\mu)} & -m^{\sigma\omega}_{(\lambda,\mu)}m^{\omega\eta}_{(\lambda,\mu)} & -m^{\sigma\eta}_{(\lambda,\mu)}m^{\omega\eta}_{(\lambda,\mu)} & 1-(m^{\omega\eta}_{(\lambda,\mu)})^2 & -m^{\omega\eta}_{(\lambda,\mu)}m^{\sigma\omega\eta}_{(\lambda,\mu)} \\
0 & -m^{\sigma}_{(\lambda,\mu)}m^{\sigma\omega\eta}_{(\lambda,\mu)} & -m^{\omega}_{(\lambda,\mu)}m^{\sigma\omega\eta}_{(\lambda,\mu)} & -m^{\sigma\omega}_{(\lambda,\mu)}m^{\sigma\omega\eta}_{(\lambda,\mu)} & -m^{\sigma\eta}_{(\lambda,\mu)}m^{\sigma\omega\eta}_{(\lambda,\mu)} & -m^{\omega\eta}_{(\lambda,\mu)}m^{\sigma\omega\eta}_{(\lambda,\mu)} & 1-(m^{\sigma\omega\eta}_{(\lambda,\mu)})^2
\end{bmatrix}.
$$

Proof. Omitted, since it goes on analogously to the proof of Theorem 1.3.2. ■

2.5 Critical Dynamics ($\gamma = \frac{\cosh^2(\gamma h)}{\tanh(\beta)}$)

We are going to consider the critical dynamics of the system, in other words the long-time behavior of the fluctuations in the threshold case, when $\gamma = \frac{\cosh^2(\gamma h)}{\tanh(\beta)}$. In the previous section we told that in a time interval $[0, T]$, where T is fixed, and in the infinite volume limit, we have Normal fluctuations for the system. Indeed, the infinitesimal generator of the rescaled process converges to the infinitesimal generator of a diffusion and the rescaled process itself converges weakly to that diffusion. It means we can provide a Central Limit Theorem for all the values of β and γ. This Central Limit Theorem continues to be valid in the critical case, but there is an eigenvalue of the covariance matrix which grows polynomially in t

and identifies the critical direction. This fact implies that the size of the Normal fluctuations must be further rescaled (in space and in time), because their size around the deterministic limit increases in time. In this case we will still obtain Normal fluctuations, solutions of a certain stochastic differential equation to be determined.

In the rest of the section, we will consider $\gamma = \frac{\cosh^2(\gamma h)}{\tanh(\beta)}$ and let us assume that the initial condition λ is a product measure such that

$$m_0^\sigma = 0, \quad m_0^\omega = 0, \quad m_0^{\sigma\omega} = \frac{\tanh(\beta)\tanh(\gamma h)\sinh(\gamma h) + \sinh(\beta)}{\cosh(\beta) + \cosh(\gamma h)},$$

$$m_0^{\sigma\eta} = \tanh(\beta)\tanh(\gamma h), \quad m_0^{\omega\eta} = \tanh(\gamma h), \quad m_0^{\sigma\omega\eta} = 0,$$

and so

$$m_t^\sigma = 0, \quad m_t^\omega = 0, \quad m_t^{\sigma\omega} = \frac{\tanh(\beta)\tanh(\gamma h)\sinh(\gamma h) + \sinh(\beta)}{\cosh(\beta) + \cosh(\gamma h)},$$

$$m_t^{\sigma\eta} = \tanh(\beta)\tanh(\gamma h), \quad m_t^{\omega\eta} = \tanh(\gamma h), \quad m_t^{\sigma\omega\eta} = 0,$$

for every value of $t \geq 0$, since it is an equilibrium solution.
First of all, we need to locate the critical direction in the seven-dimensional space of the order parameters. If we recall what we explained at the beginning of setion 1.4, we can deduce it corresponds to the direction identified by the right eigenvector corresponding to the null eigenvalue of the matrix $DV(m_*^0)$ and thus it is

$$\widetilde{x} = \cosh(\gamma h) m_{\rho_N}^\sigma + \sinh(\beta) m_{\rho_N}^\omega$$

Remark 2.5.1. Notice that the critical direction $\widetilde{x}$ does not depend on the random environment and it is one-dimensional.

Theorem 2.5.1. *For $t \in [0, T]$, if we consider the seven-dimensional critical fluctuation process*

$$r_N(t) = N^{1/2} m_{\rho_N(t)}^\eta$$

$$\widetilde{x}_N(t) = N^{1/4}\Big(\cosh(\gamma h) m_{\rho_N(N^{1/4}t)}^\sigma + \sinh(\beta) m_{\rho_N(N^{1/4}t)}^\omega\Big)$$

$$\widetilde{y}_N(t) = N^{1/4}\Big([\cosh(\gamma h) - \cosh(\beta)]m^{\sigma\eta}_{\rho_N(N^{1/4}t)} + \sinh(\beta)m^{\omega\eta}_{\rho_N(N^{1/4}t)} \\ - [\cosh(\gamma h) - \cosh(\beta)]\tanh(\beta)\tanh(\gamma h) - \sinh(\beta)\tanh(\gamma h)\Big)$$

$$\widetilde{z}_N(t) = N^{1/4}\Big(m^{\omega\eta}_{\rho_N(N^{1/4}t)} - \tanh(\gamma h)\Big)$$

$$\widetilde{u}_N(t) = N^{1/4}\Big(m^{\sigma\omega}_{\rho_N(N^{1/4}t)} - \tanh(\gamma h)m^{\sigma\eta}_{\rho_N(N^{1/4}t)} + \tanh(\beta)\tanh(\gamma h)m^{\omega\eta}_{\rho_N(N^{1/4}t)} \\ - \frac{[\tanh(\beta)\tanh(\gamma h)\sinh(\gamma h) + \sinh(\beta)]}{\cosh(\beta) + \cosh(\gamma h)}\Big)$$

$$\widetilde{v}_N(t) = N^{1/4}\Big(2\cosh(\beta)\sinh(\gamma h)[\cosh(\beta) + 2\cosh(\gamma h)]m^{\sigma}_{\rho_N(N^{1/4}t)} \\ - 2\sinh(\beta)\sinh(\gamma h)[\cosh(\beta) + 2\cosh(\gamma h)]m^{\omega}_{\rho_N(N^{1/4}t)}\Big)$$

$$\widetilde{w}_N(t) = N^{1/4}\Big(- \tanh(\beta)\sinh(\gamma h)[\cosh(\beta) + 2\cosh(\gamma h)]m^{\omega}_{\rho_N(N^{1/4}t)} \\ + [\cosh(\beta) + \cosh(\gamma h)]^2\, m^{\sigma\omega\eta}_{\rho_N(N^{1/4}t)}\Big)\,,$$

then, as $N \longrightarrow +\infty$, $r_N(t)$ converges to $\mathscr{H}$, a Standard Gaussian random variable, $\widetilde{y}_N(t)$, $\widetilde{z}_N(t)$, $\widetilde{u}_N(t)$, $\widetilde{v}_N(t)$, $\widetilde{w}_N(t) \longrightarrow 0$ in the sense of Proposition 1.4.1 and $\widetilde{x}_N(t)$ converges, in the sense of weak convergence of stochastic processes, to a limiting Gaussian process

$$\widetilde{x}(t) = 2\,\mathscr{H}\sinh(\beta)\sinh(\gamma h)t\,.$$

2.5.1 Proof of the Theorem 2.5.1

Let us denote by $\{\tau_N^M\}_{N\geq 1}$ a family of stopping times, defined as

$$\tau_N^M := \inf_{t\geq 0}\{|\widetilde{x}_N(t)| \geq M \quad \text{or} \quad |\widetilde{y}_N(t)| \geq M \quad \text{or} \quad |\widetilde{z}_N(t)| \geq M \\ \text{or} \quad |\widetilde{u}_N(t)| \geq M \quad \text{or} \quad |\widetilde{v}_N(t)| \geq M \quad \text{or} \quad |\widetilde{w}_N(t)| \geq M\}\,,$$

where M is a positive constant. We are interested in introducing such a sequence of stopping times because in this way the processes $\widetilde{x}_N(t)$, $\widetilde{y}_N(t)$, $\widetilde{z}_N(t)$, $\widetilde{u}_N(t)$,

$\widetilde{v}_N(t)$, $\widetilde{w}_N(t)$ result to be bounded in the time interval $[0, T \wedge \tau_N^M]$; $r_N(t)$ is still bounded for $t \in [0, T]$, as we proved in Lemma 1.4.2.

Lemma 2.5.1. *For $t \in [0, T \wedge \tau_N^M]$, if we consider only the space scaling*

$$r_N(t) = N^{1/2} m^{\eta}_{\rho_N(t)}$$

$$\bar{x}_N(t) = N^{1/4}\Big(\cosh(\gamma h) m^{\sigma}_{\rho_N(t)} + \sinh(\beta) m^{\omega}_{\rho_N(t)} \Big)$$

$$\begin{aligned}\bar{y}_N(t) = N^{1/4}\Big(&[\cosh(\gamma h) - \cosh(\beta)] m^{\sigma\,\eta}_{\rho_N(t)} + \sinh(\beta) m^{\omega\,\eta}_{\rho_N(t)} \\ &- [\cosh(\gamma h) - \cosh(\beta)] \tanh(\beta) \tanh(\gamma h) - \sinh(\beta) \tanh(\gamma h) \Big)\end{aligned}$$

$$\bar{z}_N(t) = N^{1/4}\Big(m^{\omega\,\eta}_{\rho_N(t)} - \tanh(\gamma h) \Big)$$

$$\begin{aligned}\bar{u}_N(t) = N^{1/4}\Big(&m^{\sigma\,\omega}_{\rho_N(t)} - \tanh(\gamma h) m^{\sigma\,\eta}_{\rho_N(t)} + \tanh(\beta) \tanh(\gamma h) m^{\omega\,\eta}_{\rho_N(t)} \\ &- \frac{[\tanh(\beta) \tanh(\gamma h) \sinh(\gamma h) + \sinh(\beta)]}{\cosh(\beta) + \cosh(\gamma h)} \Big)\end{aligned}$$

$$\begin{aligned}\bar{v}_N(t) = N^{1/4}\Big(&2\cosh(\beta) \sinh(\gamma h)[\cosh(\beta) + 2\cosh(\gamma h)] m^{\sigma}_{\rho_N(t)} \\ &- 2\sinh(\beta) \sinh(\gamma h)[\cosh(\beta) + 2\cosh(\gamma h)] m^{\omega}_{\rho_N(t)} \Big)\end{aligned}$$

$$\begin{aligned}\bar{w}_N(t) = N^{1/4}\Big(&- \tanh(\beta) \sinh(\gamma h)[\cosh(\beta) + 2\cosh(\gamma h)] m^{\omega}_{\rho_N(t)} \\ &+ [\cosh(\beta) + \cosh(\gamma h)]^2\, m^{\sigma\,\omega\,\eta}_{\rho_N(t)} \Big),\end{aligned}$$

then the process $(r_N(t), \bar{x}_N(t), \bar{y}_N(t), \bar{z}_N(t), \bar{u}_N(t), \bar{v}_N(t), \bar{w}_N(t))$ is a Markov process.

Proof. To prove that $(r_N(t), \bar{x}_N(t), \bar{y}_N(t), \bar{z}_N(t), \bar{u}_N(t), \bar{v}_N(t), \bar{w}_N(t))$ is a Markov process, one must write down the expression of the infinitesimal generator $\mathcal{G}_N$

whose dynamics are driven by. We apply Lemma 1.3.1.
The process $\{(\underline{\sigma}(t), \underline{\omega}(t)\}_{t\geq 0}$ is a continuous time Markov chain on the finite state space $\mathscr{S}^{2N}$, with infinitesimal generator L_N, defined by (2.2). Consider the function

$$\begin{aligned} \zeta:\ \mathscr{S}^{2N} &\xrightarrow{\zeta_1} \qquad [-1,+1]^7 \qquad \xrightarrow{\zeta_2} \qquad \mathbb{R}^7 \\ (\underline{\sigma}, \underline{\omega}) &\longmapsto (m^{\eta}_{\rho_N}, m^{\sigma}_{\rho_N}, \dots, m^{\sigma\,\omega\,\eta}_{\rho_N}) \longmapsto (r_N(t), \bar{x}_N(t), \dots, \bar{w}_N(t)), \end{aligned}$$

it plays the role of g in Lemma 1.3.1; then, for every $\psi : \mathscr{S}^{2N} \longrightarrow \mathbb{R}$, we have

$$L_N(\psi \circ \zeta) = (\mathcal{G}_N \psi) \circ \zeta$$

and $\zeta(\underline{\sigma}, \underline{\omega})$ is a Markov process with generator $\mathcal{G}_N$ given by

$$\begin{aligned} &\mathcal{G}_N \psi(r, \bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}) = \\ &= \sum_{i,j,k\in\mathscr{S}} |A_N(i,j,k)| e^{-\beta ij} \Bigg[\psi \Bigg(r, \bar{x} - i\frac{2}{N^{3/4}} \mathrm{ch}(\gamma h), \bar{y} - ik \frac{2}{N^{3/4}} [\mathrm{ch}(\gamma h) - \mathrm{ch}(\beta)], \\ &\qquad \bar{z}, \bar{u} - ij\frac{2}{N^{3/4}} + ik\frac{2}{N^{3/4}} \mathrm{th}(\gamma h), \bar{v} - i\frac{4}{N^{3/4}} \mathrm{sh}(\gamma h)\mathrm{ch}(\beta)[2\mathrm{ch}(\gamma h) + \mathrm{ch}(\beta)], \\ &\qquad\qquad \bar{w} - ijk\frac{2}{N^{3/4}}[\mathrm{ch}(\gamma h) + \mathrm{ch}(\beta)]^2 \Bigg) - \psi(r, \bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}) \Bigg] \\ &+ \sum_{i,j,k\in\mathscr{S}} |A_N(i,j,k)| e^{-\gamma j \left(\frac{\bar{x}}{N^{1/4} \mathrm{ch}(\gamma h) + \mathrm{ch}(\beta)} + \frac{\bar{v}}{2N^{1/4} \mathrm{sh}(\gamma h)[\mathrm{ch}(\gamma h) + \mathrm{ch}(\beta)][2\mathrm{ch}(\gamma h) + \mathrm{ch}(\beta)]} + kh \right)} \cdot \\ &\quad \cdot \Bigg[\psi \Bigg(r, \bar{x} - j\frac{2}{N^{3/4}}, \bar{y} - jk\frac{2}{N^{3/4}} \mathrm{sh}(\beta), \bar{z} - jk\frac{2}{N^{3/4}}, \bar{u} - ij\frac{2}{N^{3/4}} \\ &\qquad - jk\frac{2}{N^{3/4}} \mathrm{th}(\gamma h)\mathrm{th}(\beta), \bar{v} + j\frac{4}{N^{3/4}} \mathrm{sh}(\gamma h)\mathrm{sh}(\beta)[2\mathrm{ch}(\gamma h) + \mathrm{ch}(\beta)], \\ &\qquad \bar{w} + j\frac{2}{N^{3/4}} \mathrm{sh}(\gamma h)\mathrm{th}(\beta)[2\mathrm{ch}(\gamma h) + \mathrm{ch}(\beta)] - ijk\frac{2}{N^{3/4}}[\mathrm{ch}(\gamma h) + \mathrm{ch}(\beta)]^2 \Bigg) \\ &\qquad\qquad\qquad - \psi(r, \bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}) \Bigg], \end{aligned} \tag{2.29}$$

where $A_N(i,j,k)$ is the set of all triples $(\sigma_d, \omega_d, \eta_d)$, $d \in \{1, \dots, N\}$, such that $\sigma_d = i$, $\omega_d = j$ and $\eta_d = k$, with $i,j,k \in \mathscr{S}$; hence

$$
\begin{aligned}
|A_N(i,j,k)| = \frac{N}{8}\Bigg[& 1 + jk\tanh(\gamma h) + ik\tanh(\beta)\tanh(\gamma h) \\
& + ij\frac{\tanh(\beta)\tanh(\gamma h)\sinh(\gamma h) + \sinh(\beta)}{\cosh(\beta) + \cosh(\gamma h)} + k\frac{r}{N^{1/2}} \\
& + i\frac{\bar{x}}{N^{1/4}[\cosh(\beta) + \cosh(\gamma h)]} + j\frac{\bar{x}\cosh(\beta)}{N^{1/4}\sinh(\beta)[\cosh(\beta) + \cosh(\gamma h)]} \\
& + ijk\frac{\bar{x}\sinh(\gamma h)[\cosh(\beta) + 2\cosh(\gamma h)]}{N^{1/4}[\cosh(\beta) + \cosh(\gamma h)]^3} + ij\frac{\tanh(\gamma h)\bar{y} - \sinh(\gamma h)\tanh(\beta)\bar{z}}{N^{1/4}[\cosh(\gamma h) - \cosh(\beta)]} \\
& + ik\frac{\bar{y} - \sinh(\beta)\bar{z}}{N^{1/4}[\cosh(\gamma h) - \cosh(\beta)]} + jk\frac{\bar{z}}{N^{1/4}} + ij\frac{\bar{u}}{N^{1/4}} \\
& + i\frac{\bar{v}}{2N^{1/4}\sinh(\gamma h)[\cosh(\gamma h) + \cosh(\beta)][2\cosh(\gamma h) + \cosh(\beta)]} \\
& - j\frac{\bar{v}}{2N^{1/4}\tanh(\gamma h)\sinh(\beta)[\cosh(\gamma h) + \cosh(\beta)][2\cosh(\gamma h) + \cosh(\beta)]} \\
& - ijk\frac{\cosh(\gamma h)\bar{v}}{2N^{1/4}\cosh(\beta)[\cosh(\gamma h) + \cosh(\beta)]^3} + ijk\frac{\bar{w}}{N^{1/4}[\cosh(\gamma h) + \cosh(\beta)]^2}\Bigg].
\end{aligned}
$$

■

By standard argument on collapsing processes (see Proposition 1.4.1 and Lemma 1.4.3), it is easy to prove that for $t \in [0, T \wedge \tau_N^M]$ the directions $\widetilde{y}_N(t)$, $\widetilde{z}_N(t)$, $\widetilde{u}_N(t)$, $\widetilde{v}_N(t)$, $\widetilde{w}_N(t)$ collapse. It means that, if we consider $\widetilde{y}_N(t)$, for instance, then there exist constants $N_{0,y}$, C, $d > 2$, $\kappa_N := \kappa(N)$ and two increasing sequences $\{\alpha_N\}_{N\geq 1}$, $\{\beta_N\}_{N\geq 1}$ satisfying (1.35)–(1.39) and such that for every $\varepsilon > 0$ the following property is true

$$
\sup_{N \geq N_{0,y}} P\left\{ \sup_{0\leq t\leq T\wedge\tau_N^M} |\widetilde{y}_N(t)| > C\left(\kappa_N^{1/2d}\alpha_N^{-1/2} \vee \kappa_N^{-1/2}\beta_N^{1/2}\right)\right\} \leq \varepsilon\,. \tag{2.30}
$$

The same property holds for each of the processes $\widetilde{z}_N(t)$, $\widetilde{u}_N(t)$, $\widetilde{v}_N(t)$, $\widetilde{w}_N(t)$, with specific and adapted constants. Hence, $\widetilde{y}_N(t), \widetilde{z}_N(t), \widetilde{u}_N(t), \widetilde{v}_N(t), \widetilde{w}_N(t) \longrightarrow 0$, as $N \longrightarrow +\infty$.

The computations we should do to prove these processes converge to zero in probability are similar to those we did in Phase 2 of Subsection 1.4.1 to prove the process representing the non-critical direction of the random Curie-Weiss Model collapses. Thus, we omit this proof and *we focus only on the critical direction* $\widetilde{x}_N(t)$*, assuming all the others vanish.* We apply the generator (2.29) to a function of the only critical direction, leaving all the terms coming from those processes we know collapsing in the infinite volume limit.

Lemma 2.5.2. *For $t \in [0, T \wedge \tau_N^M]$, if we consider a function of the pair of processes*

$$\begin{aligned} r_N(t) &= N^{1/2} m^{\eta}_{\rho_N(t)} \\ \bar{x}_N(t) &= N^{1/4}\Big(\cosh(\gamma h) m^{\sigma}_{\rho_N(t)} + \sinh(\beta) m^{\omega}_{\rho_N(t)}\Big) \end{aligned} \tag{2.31}$$

only rescaled in space, then (2.29) *reduces to*

$$\begin{aligned} \mathcal{G}_N\psi(r,\bar{x}) =& 2\Bigg[r\,\frac{\sinh(\beta)\sinh(\gamma h)}{N^{1/4}} - \frac{\gamma^2\bar{x}^3}{2}\frac{\cosh(\beta)\cosh(\gamma h)}{N^{1/2}[\cosh(\beta)+\cosh(\gamma h)]^3} \\ &+ \frac{\gamma^3\bar{x}^3}{6}\frac{\sinh(\beta)[\cosh(\gamma h)-\sinh(\gamma h)\tanh(\gamma h)]}{N^{1/2}[\cosh(\beta)+\cosh(\gamma h)]^3} \\ &+ \frac{\gamma^2 r\bar{x}^2}{2}\frac{\sinh(\beta)\sinh(\gamma h)}{N^{3/4}[\cosh(\beta)+\cosh(\gamma h)]^2}\Bigg]\psi_{\bar{x}} + o\left(\frac{1}{N^{1/4}}\right) \end{aligned} \tag{2.32}$$

where the remainder is a continuous function of $\bar{x}$ and it is of order $o(\frac{1}{N^{1/4}})$ pointwise, but not uniformly in $\bar{x}$.

Proof. We recall that we are leaving all the term collapsing in the limit as $N \longrightarrow +\infty$. By (2.29), considering a function $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\psi \in \mathcal{C}^3_b$, we deduce

$$\begin{aligned} \mathcal{G}_N\psi(r,\bar{x}) = \sum_{i,j,k\in\mathscr{S}} |A_N(i,j,k)| \Bigg\{ & e^{-\beta ij}\left[\psi\left(r, \bar{x} - i\frac{2}{N^{3/4}}\cosh(\gamma h)\right) - \psi(r,\bar{x})\right] \\ &+ e^{-\gamma j\left(\frac{\bar{x}}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}+kh\right)}\left[\psi\left(r, \bar{x} - j\frac{2}{N^{3/4}}\sinh(\beta)\right) - \psi(r,\bar{x})\right]\Bigg\}, \end{aligned}$$

where

$$\begin{aligned} |A_N(i,j,k)| = \frac{N}{8}\Bigg[& 1 + k\frac{r}{N^{1/2}} + ik\tanh(\beta)\tanh(\gamma h) + jk\tanh(\gamma h) \\ &+ ij\frac{\tanh(\beta)\tanh(\gamma h)\sinh(\gamma h)+\sinh(\beta)}{\cosh(\beta)+\cosh(\gamma h)} + i\frac{\bar{x}}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]} \\ &+ j\frac{\bar{x}\cosh(\beta)}{N^{1/4}\sinh(\beta)[\cosh(\beta)+\cosh(\gamma h)]} + ijk\frac{\bar{x}\sinh(\gamma h)[\cosh(\beta)+2\cosh(\gamma h)]}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]^3}\Bigg]. \end{aligned} \tag{2.33}$$

We develop ψ with a Taylor expansion stopped at second order.

$$\mathcal{G}_N\psi(r,\bar{x}) = \sum_{i,j,k\in\mathscr{S}} |A_N(i,j,k)| \Bigg\{ e^{-\beta ij}\Bigg[- i\frac{2}{N^{3/4}}\cosh(\gamma h)\psi_{\bar{x}} + \frac{2}{N^{3/2}}\cosh^2(\gamma h)\psi_{\bar{x}\bar{x}}$$

$$+ o\left(\frac{1}{N^{3/2}}\right)\Bigg] + e^{-\gamma j\left(\frac{\bar{x}}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}+kh\right)}\Bigg[- j\frac{2}{N^{3/4}}\sinh(\beta)\psi_{\bar{x}} + \frac{2}{N^{3/2}}\sinh^2(\beta)\psi_{\bar{x}\bar{x}} + o\left(\frac{1}{N^{3/2}}\right)\Bigg]\Bigg\}$$

$$= 2N^{1/4}\Bigg[\sinh\left(\frac{\gamma\bar{x}}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}\right)\sinh(\beta)\cosh(\gamma h) + \frac{r}{N^{1/2}}\cosh\left(\frac{\gamma\bar{x}}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}\right)\sinh(\beta)\sinh(\gamma h) - \frac{\bar{x}}{N^{1/4}}\cosh\left(\frac{\gamma\bar{x}}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}\right)\frac{\cosh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]} - \sinh\left(\frac{\gamma\bar{x}}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}\right)\sinh(\beta)\sinh(\gamma h)\tanh(\gamma h)\Bigg]\psi_{\bar{x}}$$

$$+ \frac{2}{N^{1/2}}\Bigg[- \sinh(\beta)\cosh^2(\gamma h)\frac{\tanh(\beta)\tanh(\gamma h)\sinh(\gamma h)+\sinh(\beta)}{\cosh(\beta)+\cosh(\gamma h)} + \cosh\left(\frac{\gamma\bar{x}}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}\right)\sinh^2(\beta)\cosh(\gamma h) + \frac{r}{N^{1/2}}\sinh\left(\frac{\gamma\bar{x}}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}\right)\sinh^2(\beta)\sinh(\gamma h) - \frac{\bar{x}}{N^{1/4}}\sinh\left(\frac{\gamma\bar{x}}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}\right)\frac{\sinh(\beta)\cosh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]} - \cosh\left(\frac{\gamma\bar{x}}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}\right)\sinh^2(\beta)\sinh(\gamma h)\tanh(\gamma h) + \cosh(\beta)\cosh^2(\gamma h)\Bigg]\psi_{\bar{x}\bar{x}} + o\left(\frac{1}{N^{1/2}}\right)$$

considering the Taylor expansions, stopped at third order, of the hyperbolic sine and cosine functions

$$= 2N^{1/4}\Bigg[\frac{\gamma\bar{x}}{N^{1/4}}\frac{\sinh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]} + \frac{\gamma^3\bar{x}^3}{6N^{3/4}}\frac{\sinh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^3} + \frac{r}{N^{1/2}}\sinh(\beta)\sinh(\gamma h) + \frac{\gamma^2 r\bar{x}^2}{2N}\frac{\sinh(\beta)\sinh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^2} - \frac{\bar{x}}{N^{1/4}}\frac{\cosh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]} - \frac{\gamma^2\bar{x}^3}{2N^{3/4}}\frac{\cosh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^3} - \frac{\gamma^3\bar{x}^3}{6N^{3/4}}\frac{\sinh(\beta)\sinh(\gamma h)\tanh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^3}$$

$$- \frac{\gamma\bar{x}}{N^{1/4}} \frac{\sinh(\beta)\sinh(\gamma h)\tanh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]} + o\left(\frac{1}{N^{3/4}}\right)\Bigg]\psi_{\bar{x}} + O\left(\frac{1}{N^{1/2}}\right)\psi_{\bar{x}\bar{x}} + o\left(\frac{1}{N^{1/2}}\right)$$

$$= 2N^{1/4}\Bigg[\frac{\bar{x}}{N^{1/4}} \underbrace{\frac{\gamma\sinh(\beta)[\cosh(\gamma h)-\sinh(\gamma h)\tanh(\gamma h)]-\cosh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]}}_{\text{it vanishes in the critical case}} + \frac{r}{N^{1/2}}\sinh(\beta)\sinh(\gamma h) - \frac{\gamma^2\bar{x}^3}{2N^{3/4}}\frac{\cosh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^3} + \frac{\gamma^3\bar{x}^3}{6N^{3/4}}\frac{\sinh(\beta)[\cosh(\gamma h)-\sinh(\gamma h)\tanh(\gamma h)]}{[\cosh(\beta)+\cosh(\gamma h)]^3} + \frac{\gamma^2 r\bar{x}^2}{2N}\frac{\sinh(\beta)\sinh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^2}\Bigg]\psi_{\bar{x}} + o\left(\frac{1}{N^{1/4}}\right)$$

$$= 2\Bigg[\frac{r}{N^{1/4}}\sinh(\beta)\sinh(\gamma h) - \frac{\gamma^2\bar{x}^3}{2N^{1/2}}\frac{\cosh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^3} + \frac{\gamma^3\bar{x}^3}{6N^{1/2}}\frac{\sinh(\beta)[\cosh(\gamma h)-\sinh(\gamma h)\tanh(\gamma h)]}{[\cosh(\beta)+\cosh(\gamma h)]^3} + \frac{\gamma^2 r\bar{x}^2}{2N^{3/4}}\frac{\sinh(\beta)\sinh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^2}\Bigg]\psi_{\bar{x}} + o\left(\frac{1}{N^{1/4}}\right)$$

which is just (2.32) and so we have concluded. ■

The next step is to prove, for every $\varepsilon > 0$ and $N \geq 1$, the existence of a constant $M > 0$ such that

$$P\left\{\tau_N^M \leq T\right\} \leq \epsilon\,.$$

This fact implies the processes $\widetilde{y}_N(t)$, $\widetilde{z}_N(t)$, $\widetilde{u}_N(t)$, $\widetilde{v}_N(t)$ and $\widetilde{w}_N(t)$ converge to zero in probability, as N is growing to infinity, for $t \in [0, T]$ and thus it follows that Lemma 2.5.2 is valid for t belonging to the whole time interval $[0, T]$.

We consider the infinitesimal generator, $\mathcal{J}_N = N^{1/4}\mathcal{G}_N$, subject to the time-rescaling and we apply it to the particular function $\psi(r_N(t), \widetilde{x}_N(t)) = |\widetilde{x}_N(t)|$. The following decomposition holds

$$|\widetilde{x}_N(t)| = |\widetilde{x}_N(0)| + \int_0^t \mathcal{J}_N(|\widetilde{x}_N(s)|)ds + \mathcal{M}^t_{N,|\widetilde{x}|}$$

$$\leq |\widetilde{x}_N(0)| + \int_0^t |\mathcal{J}_N(|\widetilde{x}_N(s)|)|ds + \mathcal{M}^t_{N,|\widetilde{x}|}\,,$$

with

$$\mathcal{M}^t_{N,|\widetilde{x}|} = \int_0^t \sum_{i,j,k\in\mathscr{S}} \left\{ \overline{\nabla}^{(i)}[|\widetilde{x}_N(s)|]\widetilde{\Lambda}^{\sigma}_N(i,j,k,ds) + \overline{\nabla}^{(j)}[|\widetilde{x}_N(s)|]\widetilde{\Lambda}^{\omega}_N(i,j,k,ds) \right\} ,$$

where we have defined

$$\begin{aligned} \overline{\nabla}^{(i)}[|\widetilde{x}_N(t)|] &:= \left| \widetilde{x}_N(t) - i\frac{2}{N^{3/4}}\cosh(\gamma h) \right| - |\widetilde{x}_N(t)| \\ \overline{\nabla}^{(j)}[|\widetilde{x}_N(t)|] &:= \left| \widetilde{x}_N(t) - j\frac{2}{N^{3/4}}\sinh(\beta) \right| - |\widetilde{x}_N(t)| \end{aligned} \tag{2.34}$$

and

$$\begin{aligned} &\widetilde{\Lambda}^{\sigma}_N(i,j,k,dt) := \Lambda^{\sigma}_N(i,j,k,dt) - \underbrace{N^{1/4}\left|A(i,j,k,N^{1/4}t)\right| e^{-\beta ij}dt}_{:=\lambda^{\sigma}(i,j,k,t)\,dt} \\ &\widetilde{\Lambda}^{\omega}_N(i,j,k,dt) := \Lambda^{\omega}_N(i,j,k,dt) - \underbrace{N^{1/4}\left|A(i,j,k,N^{1/4}t)\right| e^{-\gamma j\left(\frac{\widetilde{x}_N(t)}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}+kh\right)}dt}_{:=\lambda^{\omega}(i,j,k,t)dt} \end{aligned} \tag{2.35}$$

As we can clearly see, the quantities $\widetilde{\Lambda}^{\cdot}_N(i,j,k,dt)$ are the differences between the point processes $\Lambda^{\cdot}_N(i,j,k,dt)$, defined on $\mathscr{S}^3 \times \mathbb{R}^+$, and their intensities $\lambda^{\cdot}(i,j,k,t)\,dt$.
The counter $\left|A(i,j,k,N^{1/4}t)\right|$ is given in analogy with (2.33), replacing the variables r and $\bar{x}$ with the stochastic processes $r_N(t)$ and $\widetilde{x}_N(t)$, defined in Theorem 2.5.1.
We recall that the expression of $\mathcal{G}_N$ is given by (2.32). We consider the following Taylor expansions stopped at second order

$$\sinh\left(\frac{\gamma\widetilde{x}_N(t)}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}\right) = \frac{\gamma\widetilde{x}_N(t)}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]} + R_s$$

$$\cosh\left(\frac{\gamma\widetilde{x}_N(t)}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}\right) = 1 + \frac{\gamma^2\widetilde{x}^2_N(t)}{N^{1/2}[\cosh(\beta)+\cosh(\gamma h)]^2} + R_c\,,$$

where

$$
\begin{aligned}
|R_s| &\leq \sup\left\{\cosh(z) : z \in \left[0, \frac{\gamma\widetilde{x}_N(t)}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}\right]\right\} \cdot \\
&\qquad\qquad\qquad\qquad \cdot \frac{\gamma^3\widetilde{x}_N^3(t)}{6N^{3/4}[\cosh(\beta)+\cosh(\gamma h)]^3} \\
&\leq \cosh\left(\frac{\gamma M}{\cosh(\beta)+\cosh(\gamma h)}\right)\frac{\gamma^3 M^3}{6N^{3/4}[\cosh(\beta)+\cosh(\gamma h)]^3}
\end{aligned}
$$

and

$$
\begin{aligned}
|R_c| &\leq \sup\left\{\sinh(z) : z \in \left[0, \frac{\gamma\widetilde{x}_N(t)}{N^{1/4}[\cosh(\beta)+\cosh(\gamma h)]}\right]\right\} \cdot \\
&\qquad\qquad\qquad\qquad \cdot \frac{\gamma^3\widetilde{x}_N^3(t)}{6N^{3/4}[\cosh(\beta)+\cosh(\gamma h)]^3} \\
&\leq \sinh\left(\frac{\gamma M}{\cosh(\beta)+\cosh(\gamma h)}\right)\frac{\gamma^3 M^3}{6N^{3/4}[\cosh(\beta)+\cosh(\gamma h)]^3}\,.
\end{aligned}
$$

For $t \in [0, \tau_N^M]$ we can estimate

$$
\begin{aligned}
|\mathcal{J}_N(|\widetilde{x}_N(t)|)| = \Bigg| 2N^{1/4}\operatorname{sgn}(\widetilde{x}_N(t))\Bigg\{ &\frac{r_N(t)}{N^{1/4}}\sinh(\beta)\sinh(\gamma h) \\
&- \frac{\gamma^2|\widetilde{x}_N(t)|^3}{2N^{1/2}}\frac{\cosh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^3} \\
&+ \frac{\gamma^3|\widetilde{x}_N(t)|^3}{6N^{1/2}}\frac{\sinh(\beta)[\cosh(\gamma h)-\sinh(\gamma h)\tanh(\gamma h)]}{[\cosh(\beta)+\cosh(\gamma h)]^3} \\
&+ \frac{\gamma^2 r_N(t)|\widetilde{x}_N(t)|^2}{2N^{3/4}}\frac{\sinh(\beta)\sinh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^2} \\
&+ N^{1/4}R_s[\sinh(\beta)\cosh(\gamma h)-\sinh(\beta)\sinh(\gamma h)\tanh(\gamma h)] \\
&+ N^{1/4}R_c\left[\frac{r_N(t)}{N^{1/2}}\sinh(\beta)\sinh(\gamma h) - \frac{|\widetilde{x}_N(t)|}{N^{1/4}}\frac{\cosh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]}\right]\Bigg\}\Bigg|
\end{aligned}
$$

and thanks to Lemma 1.4.2 and the stopping times we have introduced,

$$
\begin{aligned}
\leq 2\Bigg\{ M\sinh(\beta)\sinh(\gamma h) &+ \frac{\gamma^2 M^3}{2N^{1/4}}\frac{\cosh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^3} \\
&+ \frac{\gamma^3 M^3}{6N^{1/4}}\frac{\sinh(\beta)[\cosh(\gamma h)+\sinh(\gamma h)\tanh(\gamma h)]}{[\cosh(\beta)+\cosh(\gamma h)]^3} \\
&+ \frac{\gamma^2 M^2}{2}\frac{\sinh(\beta)\sinh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^2} \\
&+ \frac{\gamma^3 M^3}{6N^{1/4}}\cosh\left(\frac{\gamma M}{\cosh(\beta)+\cosh(\gamma h)}\right)\frac{\sinh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^3}
\end{aligned}
$$

$$+\frac{\gamma^3 M^3}{6N^{1/4}}\cosh\left(\frac{\gamma M}{\cosh(\beta)+\cosh(\gamma h)}\right)\frac{\sinh(\beta)\sinh(\gamma h)\tanh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^3}$$
$$+\frac{\gamma^3 M^3}{6N^{1/4}}\sinh\left(\frac{\gamma M}{\cosh(\beta)+\cosh(\gamma h)}\right)\frac{\sinh(\beta)\sinh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^3}$$
$$+\frac{\gamma^3 M^4}{6N^{1/2}}\sinh\left(\frac{\gamma M}{\cosh(\beta)+\cosh(\gamma h)}\right)\frac{\cosh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^4}\Bigg\}$$

$$\leq 2\Bigg\{M\sinh(\beta)\sinh(\gamma h)+\frac{\gamma^2 M^3}{2}\frac{\cosh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^3}$$
$$+\frac{\gamma^3 M^3}{6}\frac{\sinh(\beta)[\cosh(\gamma h)+\sinh(\gamma h)\tanh(\gamma h)]}{[\cosh(\beta)+\cosh(\gamma h)]^3}$$
$$+\frac{\gamma^2 M^2}{2}\frac{\sinh(\beta)\sinh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^2}$$
$$+\cosh\left(\frac{\gamma M}{\cosh(\beta)+\cosh(\gamma h)}\right)\frac{\gamma^3 M^3}{6}\frac{\sinh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^3}$$
$$+\cosh\left(\frac{\gamma M}{\cosh(\beta)+\cosh(\gamma h)}\right)\frac{\gamma^3 M^3}{6}\frac{\sinh(\beta)\sinh(\gamma h)\tanh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^3}$$
$$+\sinh\left(\frac{\gamma M}{\cosh(\beta)+\cosh(\gamma h)}\right)\frac{\gamma^3 M^3}{6}\frac{\sinh(\beta)\sinh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^3}$$
$$+\sinh\left(\frac{\gamma M}{\cosh(\beta)+\cosh(\gamma h)}\right)\frac{\gamma^3 M^4}{6}\frac{\cosh(\beta)\cosh(\gamma h)}{[\cosh(\beta)+\cosh(\gamma h)]^4}\Bigg\}:=C_8\,,$$

with C_8 positive constant independent of N. Since the following inclusions are valid

$$\{\tau_N^M\leq T\}\subseteq\left\{\sup_{0\leq t\leq T\wedge\tau_N^M}\{|\widetilde{x}_N(t)|,|\widetilde{y}_N(t)|,|\widetilde{z}_N(t)|,|\widetilde{u}_N(t)|,|\widetilde{v}_N(t)|,|\widetilde{w}_N(t)|\}\geq M\right\}$$
$$\subseteq\left\{\sup_{0\leq t\leq T\wedge\tau_N^M}|\widetilde{x}_N(t)|\geq M\right\}\cup\left\{\sup_{0\leq t\leq T\wedge\tau_N^M}|\widetilde{y}_N(t)|\geq M\right\}\cup$$
$$\cup\left\{\sup_{0\leq t\leq T\wedge\tau_N^M}|\widetilde{z}_N(t)|\geq M\right\}\cup\left\{\sup_{0\leq t\leq T\wedge\tau_N^M}|\widetilde{u}_N(t)|\geq M\right\}\cup$$
$$\cup\left\{\sup_{0\leq t\leq T\wedge\tau_N^M}|\widetilde{v}_N(t)|\geq M\right\}\cup\left\{\sup_{0\leq t\leq T\wedge\tau_N^M}|\widetilde{w}_N(t)|\geq M\right\}$$
$$\subseteq\left\{\sup_{0\leq t\leq T\wedge\tau_N^M}|\widetilde{y}_N(t)|\geq M\right\}\cup\left\{\sup_{0\leq t\leq T\wedge\tau_N^M}|\widetilde{z}_N(t)|\geq M\right\}\cup$$
$$\cup\left\{\sup_{0\leq t\leq T\wedge\tau_N^M}|\widetilde{u}_N(t)|\geq M\right\}\cup\left\{\sup_{0\leq t\leq T\wedge\tau_N^M}|\widetilde{v}_N(t)|\geq M\right\}\cup$$

$$
\begin{aligned}
&\cup \left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{w}_N(t)| \ge M \right\} \cup \{|\widetilde{x}_N(0)| \ge C_9\} \cup \\
&\cup \left[\{|\widetilde{x}_N(0)| \le C_9\} \cap \left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{x}_N(t)| \ge C_9 + TC_8 + C_{10} \right\} \right] \\
\subseteq &\left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{y}_N(t)| \ge M \right\} \cup \left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{z}_N(t)| \ge M \right\} \cup \\
&\cup \left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{u}_N(t)| \ge M \right\} \cup \left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{v}_N(t)| \ge M \right\} \cup \\
&\cup \left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{w}_N(t)| \ge M \right\} \cup \{|\widetilde{x}_N(0)| \ge C_9\} \cup \\
&\cup \left\{ \sup_{0\le t\le T\wedge\tau_N^M} \mathcal{M}^t_{N,|\widetilde{x}|} \ge C_{10} \right\},
\end{aligned}
$$

we obtain the following inequality for the probability of the interested set

$$
\begin{aligned}
P\{\tau_N^M \le T\} \le\, & P\left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{y}_N(t)| \ge M \right\} + P\left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{z}_N(t)| \ge M \right\} \\
&+ P\left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{u}_N(t)| \ge M \right\} + P\left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{v}_N(t)| \ge M \right\} \\
&+ P\left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{w}_N(t)| \ge M \right\} + P\{|\widetilde{x}_N(0)| \ge C_9\} \\
&+ P\left\{ \sup_{0\le t\le T\wedge\tau_N^M} \mathcal{M}^t_{N,|\widetilde{x}|} \ge C_{10} \right\}.
\end{aligned}
$$

We estimate the seven terms of the right-hand side of the inequality.

- For any $\varepsilon > 0$, thanks to the fact that the process $\widetilde{y}_N(t)$ collapses we have

  $$
  P\left\{ \sup_{0\le t\le T\wedge\tau_N^M} |\widetilde{y}_N(t)| \ge M \right\} \le \varepsilon ,
  $$

  where we set $M := C\left(\kappa_N^{1/2d}\alpha_N^{-1/2} \vee \kappa_N^{-1/2}\beta_N^{1/2}\right)$ (see (2.30)) and analogous relations hold for all the other processes $\widetilde{z}_N(t)$, $\widetilde{u}_N(t)$, $\widetilde{v}_N(t)$, $\widetilde{w}_N(t)$, with proper constants.

- From (2.31) we get $E[\widetilde{x}_N(0)] = N^{1/4}E\Big[\cosh(\gamma h)m^{\sigma}_{\rho_N(0)} + \sinh(\beta)m^{\omega}_{\rho_N(0)}\Big]$. Since at time $t = 0$ the spins are distributed according to a product measure,

$\widetilde{x}_N(0)$ is a linear combination of sample average of independent, identically distributed Bernoulli random variables multiplied by $N^{1/4}$. So, we can conclude

$$E[|\widetilde{x}_N(0)|] \leq \left[\cosh(\gamma h)\sqrt{\mathrm{Var}(\sigma_1(0))} + \sinh(\beta)\sqrt{\mathrm{Var}(\omega_1(0))}\right] N^{-1/4}$$

and in the limit as $N \longrightarrow +\infty$, we have convergence to zero in L^1 and then in probability. Therefore

$$P\{|\widetilde{x}_N(0)| \geq C_9\} \leq \varepsilon$$

for any $\varepsilon > 0$, for every N and for a sufficiently large C_9.

- We reduce to deal with $E[(\mathcal{M}^T_{N,|\widetilde{x}|})^2]$; in fact, Doob's "maximal inequality in L^p" (case $p = 2$) for martingales (we refer to Chapter VII, Section 3 of [Shi96]) tells us that $P\Big\{\sup_{0\leq t\leq T\wedge\tau_N^M} \mathcal{M}^t_{N,|\widetilde{x}|} \geq C_{10}\Big\} \leq \frac{E[(\mathcal{M}^T_{N,|\widetilde{x}|})^2]}{(C_{10})^2}$. Hence, remembering (2.34) and (2.35), we are able to compute

$$\begin{aligned}
E[(\mathcal{M}^T_{N,|\widetilde{x}|})^2] &= E\bigg[\int_0^T \sum_{i,j,k\in\mathscr{S}} \bigg\{\Big[\overline{\nabla}^{(i)}[|\widetilde{x}_N(t)|]\Big]^2 \lambda^\sigma(i,j,k,t)dt \\
&\qquad\qquad + \Big[\overline{\nabla}^{(j)}[|\widetilde{x}_N(t)|]\Big]^2 \lambda^\omega(i,j,k,t)dt\bigg\}\bigg] \\
&\leq E\bigg[\int_0^T \frac{4}{N^{3/2}}\cosh^2(\gamma h)N^{1/4} \sup_{i,j,k\in\mathscr{S}} |A(i,j,k,N^{1/4}t)|e^{\beta}dt \\
&\qquad + \frac{4}{N^{3/2}}\sinh^2(\beta)N^{1/4} \sup_{i,j,k\in\mathscr{S}} |A(i,j,k,N^{1/4}t)|e^{\gamma(1+h)}dt\bigg] \\
&\leq E\bigg[\int_0^T \frac{4}{N^{5/4}}N\left[\cosh^2(\gamma h)e^{\beta} + \sinh^2(\beta)e^{\gamma(1+h)}\right]dt\bigg] \\
&\leq 4T\left[\cosh^2(\gamma h)e^{\beta} + \sinh^2(\beta)e^{\gamma(1+h)}\right] =: C_{11}\,,
\end{aligned}$$

with C_{11} positive constant independent of N and M. We have established that, if we choose $C_{10} \geq \sqrt{\frac{C_{11}}{\varepsilon}}$, then

$$P\bigg\{\sup_{0\leq t\leq T\wedge\tau_N^M} \mathcal{M}^t_{N,|\widetilde{x}|} \geq C_{10}\bigg\} \leq \varepsilon\,.$$

In summary, we proved the inequality we were looking for; in fact

$$P\left\{\tau_N^M \leq T\right\} \leq 7\varepsilon := \epsilon\,.$$

We have just concluded the proof of the first part of the statement of Theorem 2.5.1, concerning the collapse of the processes $\widetilde{y}_N(t)$, $\widetilde{z}_N(t)$, $\widetilde{u}_N(t)$, $\widetilde{v}_N(t)$ and $\widetilde{w}_N(t)$ in the limit as $N \longrightarrow +\infty$ and for $t \in [0,T]$. Now, we are going to show that in the same setting, i.e. the limit of infinite volume and $t \in [0,T]$, the process $\widetilde{x}_N(t)$ admits a limiting process and we are going to compute it.
First, we need to prove the tightness of the sequence $\{\widetilde{x}_N(t)\}_{N\geq 1}$. This property implies the existence of convergent subsequences. Secondly, we will verify that all the convergent subsequences have the same limit and hence also the sequence $\{\widetilde{x}_N(t)\}_{N\geq 1}$ must converge to that limit.

Lemma 2.5.3. *The sequence* $\{\widetilde{x}_N(t)\}_{N\geq 1}$ *is tight.*

Proof. We must verify the conditions (1.44) and (1.45) hold. Since we have already proved that for every $\epsilon > 0$ the inequality $P\{\tau_N^M \leq T\} \leq \epsilon$ is true for M sufficiently large and uniformly in N, it is enough to show tightness for the stopped processes

$$\left\{\widetilde{x}_N(t \wedge \tau_N^M)\right\}_{N\geq 1}.$$

We showed before the validity of the following inclusion

$$\left\{\sup_{0\leq t\leq T\wedge\tau_N^M} |\widetilde{x}_N(t)| \geq M\right\} \subseteq \{|\widetilde{x}_N(0)| \geq C_9\} \cup \left\{\sup_{0\leq t\leq T\wedge\tau_N^M} \mathcal{M}^t_{N,|\widetilde{x}|} \geq C_{10}\right\},$$

therefore

$$\sup_N P\left\{\sup_{0\leq t\leq T\wedge\tau_N^M} |\widetilde{x}_N(t)| \geq M\right\} \leq 2\varepsilon$$

and so we obtained (1.44). Let us deal with (1.45) now. We notice that

$$|\widetilde{x}_N(t) - \widetilde{x}_N(s)| = \left|\int_s^t \mathcal{J}_N(\widetilde{x}_N(u))du + \mathcal{M}^{s,t}_{N,|\widetilde{x}|}\right|,$$

where we have denoted

$$\mathcal{M}^{s,t}_{N,|\widetilde{x}|} = -\frac{2}{N^{3/4}}\int_s^t \sum_{i,j,k\in\mathscr{S}} \left[i\cosh(\gamma h)\widetilde{\Lambda}^\sigma_N(i,j,k,du) + j\sinh(\beta)\widetilde{\Lambda}^\omega_N(i,j,k,du)\right]$$

and $\widetilde{\Lambda}^{\sigma}_N$, $\widetilde{\Lambda}^{\omega}_N$ are as in definition (2.35). Thus,

$$\{|\widetilde{x}_N(t)-\widetilde{x}_N(s)| \geq \alpha\} \subseteq \left\{ \underbrace{\left| \int_s^t \mathcal{J}_N |\widetilde{x}_N(u)| du \right|}_{\leq C_8(t-s)} + |\mathcal{M}^{s,t}_{N,|\widetilde{x}|}| \geq \alpha \right\} \subseteq \{|\mathcal{M}^{s,t}_{N,|\widetilde{x}|}| \geq \overline{C}_{10}\}$$

and then, applying Chebyscev inequality to the last right-handside of the previous inclusions, we get

$$\begin{aligned}
\sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\delta}} P\{|\mathcal{M}^{s,t}_{N,|\widetilde{x}|}| \geq \overline{C}_{10}\} &\leq (\overline{C}_{10})^{-2} \sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\delta}} E[(\mathcal{M}^{s,t}_{N,|\widetilde{x}|})^2] \\
&\leq (\overline{C}_{10})^{-2} \sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\delta}} 4(t-s) \Big[\cosh^2(\gamma h) e^{\beta} \\
&\qquad\qquad + \sinh^2(\beta) e^{\gamma(1+h)} \Big] \\
&\leq (\overline{C}_{10})^{-2} \underbrace{4 \Big[\cosh^2(\gamma h) e^{\beta} + \sinh^2(\beta) e^{\gamma(1+h)} \Big]}_{:=\overline{C}_{11}} \delta \,.
\end{aligned}$$

Finally, we can conclude that

$$\begin{aligned}
\sup_N \sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\delta}} P\{|\widetilde{x}_N(t)-\widetilde{x}_N(s)| \geq \alpha\} &\leq \sup_N \sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\delta}} P\{|\mathcal{M}^{s,t}_{N,|\widetilde{x}|}| \geq \overline{C}_{10}\} \\
&\leq (\overline{C}_{10})^{-2} \overline{C}_{11}\, \delta = O(\delta)
\end{aligned}$$

and the proof is complete. ■

Lemma 2.5.3 implies that there exist convergent subsequences for the sequence $\{\widetilde{x}_N(t)\}_{N\geq 1}$. Let $\{\widetilde{x}_n(t)\}_{n\geq 1}$ denote one of such a subsequence and let $\psi \in \mathcal{C}^3_b$ be a function of the type $\psi(r_n(t), \widetilde{x}_n(t)) = \psi(\widetilde{x}_n(t))$. The following decomposition holds

$$\psi(\widetilde{x}_n(t)) - \psi(\widetilde{x}_n(0)) = \int_0^t \mathcal{J}_n \psi(\widetilde{x}_n(u)) du + \mathcal{M}^t_{n,\psi}\,, \tag{2.36}$$

where

$$\begin{aligned}\mathcal{J}_n\psi(\widetilde{x}_n(t)) =2\Bigg[& r_n(t)\sinh(\beta)\sinh(\gamma h) - \frac{\gamma^2(\widetilde{x}_n(t))^3}{2}\frac{\cosh(\beta)\cosh(\gamma h)}{n^{1/4}[\cosh(\beta)+\cosh(\gamma h)]^3} \\ & + \frac{\gamma^3(\widetilde{x}_n(t))^3}{6}\frac{\sinh(\beta)[\cosh(\gamma h)-\sinh(\gamma h)\tanh(\gamma h)]}{n^{1/4}[\cosh(\beta)+\cosh(\gamma h)]^3} \\ & + \frac{\gamma^2 r_n(t)(\widetilde{x}_n(t))^2}{2}\frac{\sinh(\beta)\sinh(\gamma h)}{n^{3/4}[\cosh(\beta)+\cosh(\gamma h)]^2}\Bigg]\psi_{\widetilde{x}} + o_M(1)\end{aligned}$$

which, as usual, is $\mathcal{G}_N$ (see (2.32)) rescaled of a power $n^{1/4}$ and applied to the particular function $\psi(r_n(t),\widetilde{x}_n(t)) = \psi(\widetilde{x}_n(t))$. The remainder $o_M(1)$ goes to zero as $n \longrightarrow +\infty$, uniformly in M. If we compute the limit as $n \longrightarrow +\infty$, remembering a Central Limit Theorem applies to $r_n(t)$, we have:

$$\mathcal{J}_n\psi(\widetilde{x}_n(t)) \xrightarrow[w]{n\to+\infty} \mathcal{J}\psi(\widetilde{x}(t))\,,$$

with

$$\mathcal{J}\psi(\widetilde{x}(t)) = 2\,\mathscr{H}\,\sinh(\beta)\sinh(\gamma h)\;\psi_{\widetilde{x}}$$

and $\mathscr{H}$ is a Standard Gaussian random variable. Then, because of (2.36), we obtain

$$\mathcal{M}^t_{n,\psi} \xrightarrow[w]{n\to+\infty} \mathcal{M}^t_{\psi} := \psi(\widetilde{x}(t)) - \psi(\widetilde{x}(0)) - \int_0^t \mathcal{J}\psi(\widetilde{x}(u))du\,.$$

We must prove the following Lemma:

Lemma 2.5.4. *M^t_ψ is a martingale (with respect to t); in other words, for all $s,t\in[0,T]$, $s\le t$ and for all measurable and bounded functions $g(\widetilde{x}([0,s]))$ the following identity holds:*

$$E[\mathcal{M}^t_\psi g(\widetilde{x}([0,s]))] = E[\mathcal{M}^s_\psi g(\widetilde{x}([0,s]))]\,. \tag{2.37}$$

Proof. The reasoning we explained in Lemma 1.4.5 applies in this case too, so it is sufficient to prove $\{\mathcal{M}^t_{n,\psi}\}_{n\ge1}$ is an uniformly integrable sequence of random variables.

If we define

$$\overline{\nabla}^{(i)}[\psi(\widetilde{x}_n(t))] := \psi\left(\widetilde{x}_n(t) - i\frac{2}{n^{3/4}}\cosh(\gamma h)\right) - \psi(\widetilde{x}_n(t))$$

$$\overline{\nabla}^{(j)}[\psi(\widetilde{x}_n(t))] := \psi\left(\widetilde{x}_n(t) - j\frac{2}{n^{3/4}}\sinh(\beta)\right) - \psi(\widetilde{x}_n(t))\,,$$

it yields

$$E[(\mathcal{M}^t_{n,\psi})^2] = E\bigg[\int_0^t \sum_{i,j,k\in\mathscr{S}} \bigg\{\Big[\overline{\nabla}^{(i)}[\psi(\widetilde{x}_n(s))]\Big]^2 \lambda^\sigma(i,j,k,s)ds + \Big[\overline{\nabla}^{(j)}[\psi(\widetilde{x}_n(s))]\Big]^2 \lambda^\omega(i,j,k,s)ds\bigg\}\bigg]$$

$$\leq n^{5/4} E\bigg[\int_0^t \sum_{i,j\in\mathscr{S}} \bigg\{\Big[\overline{\nabla}^{(i)}[\psi(\widetilde{x}_n(s))]\Big]^2 e^\beta + \Big[\overline{\nabla}^{(j)}[\psi(\widetilde{x}_n(s))]\Big]^2 e^{\gamma(1+h)}\bigg\}ds\bigg]$$

we expand the function ψ around $\widetilde{x}_n(t)$ with the Taylor expansion stopped at first order and with remainder R, $\overline{R}$ such that

$$|R| \leq \frac{1}{2}\sup\Big\{|\psi_{\widetilde{x}\widetilde{x}}(z)| : z \in \Big[\widetilde{x}_n(t), \widetilde{x}_n(t) - i\frac{2}{n^{3/4}}\cosh(\gamma h)\Big]\Big\}\frac{4}{n^{3/2}}\cosh^2(\gamma h)$$

$$|\overline{R}| \leq \frac{1}{2}\sup\Big\{|\psi_{\widetilde{x}\widetilde{x}}(z)| : z \in \Big[\widetilde{x}_n(t), \widetilde{x}_n(t) - j\frac{2}{n^{3/4}}\sinh(\beta)\Big]\Big\}\frac{4}{n^{3/2}}\sinh^2(\beta)$$

and moreover, we recall that $\psi \in \mathcal{C}^3_b$, so $|\psi_{\widetilde{x}}| \leq K_1$ and $|\psi_{\widetilde{x}\widetilde{x}}| \leq K_2$; therefore,

$$= n^{5/4} E\bigg[\int_0^t \bigg\{\sum_{i\in\mathscr{S}}\Big[-i\frac{2}{n^{3/4}}\cosh(\gamma h)\psi_{\widetilde{x}} + R\Big]^2 e^\beta + \sum_{j\in\mathscr{S}}\Big[-j\frac{2}{n^{3/4}}\sinh(\beta)\psi_{\widetilde{x}} + \overline{R}\Big]^2 e^{\gamma(1+h)}\bigg\}ds\bigg]$$

$$\leq n^{5/4} E\bigg[e^\beta\int_0^t \sup_{i\in\mathscr{S}}\Big(\frac{4}{n^{3/2}}\cosh^2(\gamma h)\psi^2_{\widetilde{x}} - i\frac{4}{n^{3/4}}\cosh(\gamma h)\psi_{\widetilde{x}}R + R^2\Big)ds + e^{\gamma(1+h)}\int_0^t \sup_{j\in\mathscr{S}}\Big(\frac{4}{n^{3/2}}\sinh^2(\beta)\psi^2_{\widetilde{x}} - \frac{4}{n^{3/4}}\sinh(\beta)\psi_{\widetilde{x}}\overline{R} + \overline{R}^2\Big)ds\bigg]$$

$$\leq n^{5/4} E\bigg[e^\beta\int_0^t \Big(\frac{4}{n^{3/2}}K_1^2\cosh^2(\gamma h) + \frac{8}{n^{9/4}}K_1K_2\cosh^3(\gamma h) + \frac{4}{n^3}K_2^2\cosh^4(\gamma h)\Big)ds + e^{\gamma(1+h)}\int_0^t \Big(\frac{4}{n^{3/2}}K_1^2\sinh^2(\beta)$$

$$+\frac{8}{n^{9/4}}K_1K_2\sinh^3(\beta)+\frac{4}{n^3}K_2^2\sinh^4(\beta)\Big)ds\Big]$$

$$\leq 4T\Big\{e^{\beta}\cosh^2(\gamma h)[K_1+\cosh(\gamma h)K_2]^2$$
$$+e^{\gamma(1+h)}\sinh^2(\beta)[K_1+\sinh(\beta)K_2]^2\Big\}$$

since $t < T$; then $\mathcal{M}^t_{n,\psi}$ is uniformly integrable. ■

Now, the proof is easy to complete. $\mathcal{M}^t_{n,\psi}$ solves the martingale problem with infinitesimal generator $\mathcal{J}$, admitting a unique solution, and hence we have shown all convergent subsequences have the same limit and so the sequence itself converges to that limit.

2.6 Conclusions

It remains to compare the behaviors of the homogeneous and inhomogeneous system. Using the same notation as before, we briefly sketch the main results of the homogeneous model.

The stochastic process $(\underline{\sigma}(t), \underline{\omega}(t)) = (\sigma_j(t), \omega_j(t))_{j=1}^N$, with t belonging to a generic time interval $[0, T]$, where T is fixed, describes a $2N$-spin system evolving as a Markov process on its state space $\mathscr{S}^{2N}$. The dynamics are specified by the requirement that the rates of transition are of the form

$$\begin{array}{llll}\sigma_j \longrightarrow -\sigma_j & \text{at rate} & e^{-\beta\sigma_j\omega_j} & \beta > 0\,, \\ \omega_k \longrightarrow -\omega_k & \text{at rate} & e^{-\gamma\omega_k m_N^{\sigma}}\,. & \end{array}$$

We reduce this system to be finite dimensional. A three-dimensional order parameter is necessary to describe the system: $(m_N^{\sigma}, m_N^{\omega}, m_N^{\sigma\omega})$. The study of the limiting dynamics (Theorem 2.3.1 and Lemma 2.3.3) and of the Normal fluctuations (Theorem 2.4.1) is completely developed in [DPRST09]. The McKean-Vlasov limit ($N \longrightarrow +\infty$) for the dynamics of the order parameter is given by the the system of ordinary differential equations

$$
\begin{aligned}
\dot{m}_t^\sigma =& -2\, m_t^\sigma \cosh(\beta) + 2\, m_t^\omega \sinh(\beta) \\
\dot{m}_t^\omega =& -2\, m_t^\omega \cosh(\gamma m_t^\sigma) + 2\, \sinh(\gamma m_t^\sigma) \\
\dot{m}_t^{\sigma\omega} =& \, 2\, m_t^\sigma \sinh(\gamma m_t^\sigma) - 2\, m_t^{\sigma\omega} \left[\cosh(\beta) + \cosh(\gamma m_t^\sigma)\right] + 2\, \sinh(\beta)\,.
\end{aligned}
\tag{2.38}
$$

Note that $m_t^{\sigma\omega}$ does not appear in the first and in the second equation in (2.38); this means that the differential system (2.38) is essentially two-dimensional: first one solves the two-dimensional system (on $[-1,+1]^2$)

$$
(\dot{m}_t^\sigma, \dot{m}_t^\omega) = V(m_t^\sigma, m_t^\omega)\,, \tag{2.39}
$$

with $V(x,y) = (2\sinh(\beta)y - 2\cosh(\beta)x, 2\sinh(\gamma x) - 2y\cosh(\gamma x))$, and then one solves the third equation in (2.38), which is linear in $m_t^{\sigma\omega}$. Note also that to any (m_*^σ, m_*^ω) satisfying $V(m_*^\sigma, m_*^\omega) = 0$, there corresponds a unique

$$
m_*^{\sigma\omega} = \frac{\sinh(\beta) + m_*^\sigma \sinh(\gamma m_*^\sigma)}{\cosh(\beta) + \cosh(\gamma m_*^\sigma)}
$$

such that $(m_*^\sigma, m_*^\omega, m_*^{\sigma\omega})$ is an equilibrium of (2.38). Moreover, if $m_t^\sigma \longrightarrow m_*^\sigma$ as $t \longrightarrow +\infty$, then $m_t^{\sigma\omega} \longrightarrow m_*^{\sigma\omega}$. Depending on the parameters, we can see there exists phase transition; in fact

Theorem 2.6.1. *Consider the equation* (2.39)*.*

- *For $\gamma \leq \frac{1}{\tanh(\beta)}$, it has $(0,0)$ as a unique equilibrium solution and it is globally asymptotically stable, i.e. for every initial condition (m_0^σ, m_0^ω)*

$$
\lim_{t \to +\infty} (m_t^\sigma, m_t^\omega) = (0,0)\,.
$$

- *For $\gamma < \frac{1}{\tanh(\beta)}$ the equilibrium $(0,0)$ is linearly stable. For $\gamma = \frac{1}{\tanh(\beta)}$ the linearized system has a neutral direction, i.e. $DV(0,0)$ has one zero eigenvalue.*

- *For $\gamma > \frac{1}{\tanh(\beta)}$ the point $(0,0)$ is still an equilibrium for* (2.39)*, but it is a saddle point for the linearized system, i.e. $DV(0,0)$ has two nonzero real eigenvalues of opposite sign. Moreover,* (2.39) *has two linearly stable solutions (m_*^σ, m_*^ω), $(-m_*^\sigma, -m_*^\omega)$, where m_*^σ is the unique strictly positive solution of the equation*

$$
x = \tanh(\beta)\tanh(\gamma x)\,,
$$

and

$$m_*^\omega = \frac{1}{\tanh(\beta)} m_*^\sigma \, .$$

- ► *For* $\gamma > \frac{1}{\tanh(\beta)}$*, the phase space* $[-1,+1]^2$ *is bi-partitioned by a smooth curve* Γ *containing* $(0,0)$ *such that* $[-1,+1]^2 \setminus \Gamma$ *is the union of two disjoint sets* Γ^+*,* Γ^- *that are open in the induced topology of* $[-1,+1]^2$*. Moreover, given an initial condition* (m_0^σ, m_0^ω)*,*

$$\lim_{t \to +\infty} (m_t^\sigma, m_t^\omega) = \begin{cases} (m_*^\sigma, m_*^\omega) & \text{if } (m_0^\sigma, m_0^\omega) \in \Gamma^+ \\ (-m_*^\sigma, -m_*^\omega) & \text{if } (m_0^\sigma, m_0^\omega) \in \Gamma^- \\ (0,0) & \text{if } (m_0^\sigma, m_0^\omega) \in \Gamma \, . \end{cases}$$

Moreover, with regard to the Normal fluctuations, it holds true the following Theorem.

Theorem 2.6.2. *In the limit as* $N \longrightarrow +\infty$*, the three-dimensional fluctuation process* $(x_N(t), y_N(t), z_N(t))$*, defined by*

$$\begin{aligned} x_N(t) &:= N^{1/2} \left(m^\sigma_{\rho_N(t)} - m_t^\sigma \right) \\ y_N(t) &:= N^{1/2} \left(m^\omega_{\rho_N(t)} - m_t^\omega \right) \\ z_N(t) &:= N^{1/2} \left(m^{\sigma\omega}_{\rho_N(t)} - m_t^{\sigma\omega} \right) , \end{aligned}$$

converges (in the sense of weak convergence of stochastic processes) to a limiting three-dimensional Gaussian process $(x(t), y(t), z(t))$*, which is the unique solution of the linear stochastic differential equation*

$$\begin{bmatrix} dx(t) \\ dy(t) \\ dz(t) \end{bmatrix} = 2A(t) \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} dt + D(t) \begin{bmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t) \end{bmatrix} , \tag{2.40}$$

where B_1, B_2, B_3 *are independent Standard Brownian motions,* $A(t)$ *and* $\frac{D(t)D'(t)}{2}$ *are respectively*

$$\begin{bmatrix} -\cosh(\beta) & \sinh(\beta) & 0 \\ -\gamma m_t^\omega \sinh(\gamma m_t^\sigma) + \gamma \cosh(\gamma m_t^\sigma) & -\cosh(\gamma m_t^\sigma) & 0 \\ \sinh(\gamma m_t^\sigma) + \gamma m_t^\sigma \cosh(\gamma m_t^\sigma) - \gamma m_t^{\sigma\omega} \sinh(\gamma m_t^\sigma) & 0 & -\cosh(\beta) - \cosh(\gamma m_t^\sigma) \end{bmatrix} ,$$

$$\begin{bmatrix} -m_t^{\sigma\omega} \sinh(\beta) + \cosh(\beta) & 0 & -m_t^\sigma \sinh(\beta) + m_t^\omega \cosh(\beta) \\ 0 & -m_t^\omega \sinh(\gamma m_t^\sigma) + \cosh(\gamma m_t^\sigma) & m_t^\sigma \cosh(\gamma m_t^\sigma) - m_t^{\sigma\omega} \sinh(\gamma m_t^\sigma) \\ -m_t^\sigma \sinh(\beta) + m_t^\omega \cosh(\beta) & m_t^\sigma \cosh(\gamma m_t^\sigma) - m_t^{\sigma\omega} \sinh(\gamma m_t^\sigma) & -m_t^{\sigma\omega} \sinh(\beta) + \cosh(\beta) - m_t^\omega \sinh(\gamma m_t^\sigma) + \cosh(\gamma m_t^\sigma) \end{bmatrix}$$

and $(x(0), y(0), z(0))$ has a centered Gaussian distribution with covariance matrix

$$\begin{bmatrix} 1-(m_\lambda^\sigma)^2 & m_\lambda^{\sigma\omega} - m_\lambda^\sigma m_\lambda^\omega & m_\lambda^\omega - m_\lambda^\sigma m_\lambda^{\sigma\omega} \\ \\ m_\lambda^{\sigma\omega} - m_\lambda^\sigma m_\lambda^\omega & 1-(m_\lambda^\omega)^2 & m_\lambda^\sigma - m_\lambda^\omega m_\lambda^{\sigma\omega} \\ \\ m_\lambda^\omega - m_\lambda^\sigma m_\lambda^{\sigma\omega} & m_\lambda^\sigma - m_\lambda^\omega m_\lambda^{\sigma\omega} & 1-(m_\lambda^{\sigma\omega})^2 \end{bmatrix}.$$

Remark 2.6.1. We can notice that there is no constant drift in (2.40); drift which, on the contrary, is present in (2.28). It arises because of the disorder.

We focus on the critical dynamics of the system (all the results are proved in [Sar07]). We construct the fluctuations in the threshold case, when $\gamma = \frac{1}{\tanh(\beta)}$, and we look at their long-time behavior. The size of the Normal fluctuations must be further rescaled (in space and in time), because their size around the deterministic limit increases in time. In this case we will obtain non-Normal fluctuations.
In the rest of the section, we will consider $\gamma = \frac{1}{\tanh(\beta)}$ and let us assume that the initial condition λ is a product measure such that

$$m_0^\sigma = 0, \qquad m_0^\omega = 0, \qquad m_0^{\sigma\omega} = \frac{\sinh(\beta)}{\cosh(\beta)+1}$$

and so

$$m_t^\sigma = 0, \qquad m_t^\omega = 0, \qquad m_t^{\sigma\omega} = \frac{\sinh(\beta)}{\cosh(\beta)+1},$$

for every value of $t \geq 0$, since it is an equilibrium solution.

Theorem 2.6.3. *For $t \in [0,T]$, if we consider the critical fluctuation process*

$$\begin{aligned} \widetilde{x}_N(t) &:= N^{1/4}\left(m^\sigma_{\rho_N(N^{1/2}t)} - \tanh(\beta) m^\omega_{\rho_N(N^{1/2}t)}\right) \\ \widetilde{y}_N(t) &:= N^{1/4}\left(m^\sigma_{\rho_N(N^{1/2}t)} + \sinh(\beta) m^\omega_{\rho_N(N^{1/2}t)}\right) \\ \widetilde{z}_N(t) &:= N^{1/4}\left(m^{\sigma\omega}_{\rho_N(N^{1/2}t)} - \frac{\sinh(\beta)}{\cosh(\beta)+1}\right), \end{aligned} \tag{2.41}$$

then, as $N \longrightarrow +\infty$, $\widetilde{x}_N(t), \widetilde{z}_N(t) \longrightarrow 0$ in the sense of Proposition 1.4.1 and $\widetilde{y}_N(t)$ converges, in the sense of weak convergence of stochastic processes, to a limiting non-Gaussian process $\widetilde{y}(t)$, which is the unique solution of the following *stochastic differential equation:*

$$\begin{cases} d\widetilde{y}(t) &= -\dfrac{2\cosh^3(\beta)}{3\sinh^2(\beta)[\cosh(\beta)+1]^3}\,\widetilde{y}^3(t)\,dt + 2\cosh(\beta)\,dB(t) \\ \\ \widetilde{y}(0) &= 0 \end{cases}$$

where B is a standard Brownian motion.

Concluding, we point out the fact that the inhomogeneous critical fluctuation process exists in a shorter time-scale than the homogeneous one; in fact when we construct this process, see Theorem 2.5.1, we can amplify the time only by a factor $N^{1/4}$, instead of the usual scale $N^{1/2}$, as in (2.41). The reason of this difference is the constant drift, appearing in the dynamics of the Normal fluctuations. It obliges us to amplify the time by a smaller power of N than the one "permitted" by the linearized operator driving the diffusion equation. Besides, the limit of disordered critical fluctuations is Gaussian, since solution of a deterministic equation with constant (but random) drift given by a Gaussian random variable; while, it is not when there is no added field.

Part II

Diffusion Systems

Chapter 3

The Kuramoto Model

The model under consideration is a system of nonlinearly coupled rotators subject to an attractive interaction. It was introduced by Kuramoto ([Kur75] and [Kur84]) to describe synchronization phenomena observable in nature.

We consider N sites and we associate with each of them a rotator on $[0, 2\pi]$. We start with a reversible Markovian dynamics for the N-particle system, where the rotators evolve depending on the gradient of the Hamiltonian felt by the particle. It is an interacting diffusion system with a *mean-field* Hamiltonian. This model is space-independent because it is subject to a mean-field interaction, in other words each particle interacts with all the others in the same way; thus, there is no spatial geometry.

An infinite dimensional order parameter is necessary to describe the system. Being based on a Large Deviation Principle, we compute the differential equations which drive its evolution in the infinite particle limit (McKean-Vlasov equations) and we derive a Law of Large Number it obeys. Depending on the parameters, we can see there exists phase transition. We state these results for completeness; they are already known in literature. They can be deduced from the analogous ones for the inhomogeneous system, studied in [DPdH95] and [dH00].

Our main result is the infinite particle limit of the critical fluctuation flow. With regard to the critical fluctuation flow – besides an appropriate scaling of the space – it require a rescaling of the time in order to keep track of long time fluctuations of the critical direction (critical slowing down). As a result, only the critical structure survives the new scaling, and in the limit, the critical fluctuation process is a lower dimensional process compared with the non-critical one. The fluctuations are two-dimensional at the critical point, while they are infinite dimensional for non-critical values. In fact, we prove that, when the size of the system grows towards infinity, a two-dimensional process converges (in the sense of weak convergence of stochastic processes) to a non-Gaussian process, while all the others collapse.

3.1 Description of the Model

Given a configuration $\underline{x} = (x_j)_{j=1}^N \in [0, 2\pi]^N$, we can define the Hamiltonian $H_N(\underline{x}) : [0, 2\pi]^N \longrightarrow \mathbb{R}$ as

$$H_N(\underline{x}) = -\frac{\theta}{2N} \sum_{j,k=1}^{N} \cos(x_k - x_j)\,, \tag{3.1}$$

where x_j is the position of the rotator at site j. Let θ, positive parameter, be the coupling strength. Think of $\underline{x} \longrightarrow H_N(\underline{x})$ as a mean-field Hamiltonian in the components x_j. With the expression "mean-field" we mean the sites interact all each other in the same way.
Let us define the dynamics we consider: $\underline{x}(t) = (x_j(t))_{j=1}^N$, with t belonging to a generic time interval $[0, T]$, where T is fixed, describes a N-rotator system evolving as a continuous time Markov chain on $[0, 2\pi]^N$, with infinitesimal generator L_N acting on functions $f : [0, 2\pi]^N \longrightarrow \mathbb{R}$ as follows:

$$L_N f(\underline{x}) = \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_j^2}(\underline{x}) + \sum_{j=1}^{N} \left\{ \frac{\theta}{N} \sum_{k=1}^{N} \sin(x_k - x_j) \right\} \frac{\partial f}{\partial x_j}(\underline{x})\,. \tag{3.2}$$

Consider the complex quantity

$$r_N e^{i\Psi_N} = \frac{1}{N} \sum_{j=1}^{N} e^{i x_j}\,, \tag{3.3}$$

where $0 \leq r_N \leq 1$ measures the phase coherence of the rotators and Ψ_N measures the average phase. We can reformulate the expression of the infinitesimal generator (3.2) in terms of (3.3):

$$L_N f(\underline{x}) = \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_j^2}(\underline{x}) + \sum_{j=1}^{N} \{\theta r_N \sin(\Psi_N - x_j)\} \frac{\partial f}{\partial x_j}(\underline{x})\,. \tag{3.4}$$

The expressions (3.1) and (3.4) describe a system of mean-field coupled rotators, each with its own frequency and subject to diffusive dynamics. The interaction tends to synchronize the rotators.

Remark 3.1.1. The system described by (3.2) has a reversible stationary distribution proportional to $\exp[-H_N(\underline{x})]$.

For simplicity, the initial condition $\underline{x}(0)$ is assumed to have product distribution $\lambda^{\otimes N}$, with λ probability measure on $[0, 2\pi]$ with finite second moment. The quantity $x_j(t)$ represents the time evolution on $[0, T]$ of j-th rotator; it is the trajectory of the single j-th rotator in time. The space of all these paths is $\mathcal{C}[0, T]$, which is the space of the continuous function from $[0, T]$ to $[0, 2\pi]$, endowed with the uniform topology.

The process $\underline{x}(t) = (x_j(t))_{j=1}^N$ turns out to be the system of N interacting diffusions evolving according to the Itô differential equations

$$dx_j(t) = [\theta r_N \sin(\Psi_N - x_j)]\, dt + dB_j(t)\,, \tag{3.5}$$

where $\{B_j(t) : t > 0, j = 1, \dots, N\}$ is a system of independent Standard Brownian motions on $[0, 2\pi]$.

3.2 Limiting Dynamics

We now derive the dynamics of the process (3.2), in the limit as $N \longrightarrow +\infty$, in a fixed time interval $[0, T]$, via a Large Deviation approach. Later, the large time behavior of the limiting dynamics will be studied.

For completeness, we report all the statements that allow us to deduce the dynamics of the model in the infinite volume limit, but we omit their proofs, since they are a particular application of a more general study on interacting diffusions developed in [DPdH96] and [dH00].

So, let $(x_j([0, T]))_{j=1}^N \in (\mathcal{C}[0, T])^N$ denote a path of the system in the time interval $[0, T]$, with T positive and fixed. If $f(x_j([0, T]))$ is a function of the trajectory of a single rotator, we are interested in the asymptotic behavior of *empirical averages* of the form

$$\frac{1}{N}\sum_{j=1}^{N} f(x_j([0,T])) =: \int f d\rho_N\,,$$

where $\{\rho_N\}_{N\geq 1}$ is the sequence of *empirical measures*

$$\rho_N := \frac{1}{N}\sum_{j=1}^{N}\delta_{(x_j([0,T]))}\,.$$

We may think of ρ_N as a random element of $\mathcal{M}_1(\mathcal{C}[0,T])$, the space of probability measures on $\mathcal{C}[0,T]$ endowed with the weak convergence topology.
First, we want to determine the weak limit of ρ_N in $\mathcal{M}_1(\mathcal{C}[0,T])$ as N grows to infinity; i.e. for $f\in\mathcal{C}_b$ we look for $\lim_{N\to+\infty}\int f d\rho_N$. It corresponds to a Law of Large Number with the limit being a deterministic measure. Being an element of $\mathcal{M}_1(\mathcal{C}[0,T])$, such a limit can be viewed as a stochastic process, which represents the dynamics of the system in the infinite volume limit.

3.2.1 Empirical Measure and Large Deviations

Let $W\in\mathcal{M}_1(\mathcal{C}[0,T])$ denote the law of a standard Brownian motion starting with initial condition λ. By $W^{\otimes N}$ we mean the product of N copies of W, which represents the law of the solution of the system (3.5) when $H_N(\underline{x},\underline{\omega})\equiv 0$. Moreover, we shall write P_N the law of $\underline{x}([0,T])=(\underline{x}(t))_{t\in[0,T]}$, the process with infinitesimal generator (3.2) and initial distribution $\lambda^{\otimes N}$.

Consider $Q\in\mathcal{M}_1(\mathcal{C}[0,T])$, if $\Pi_t Q$ indicates the marginal distribution of Q at time t, we have

$$r_{\Pi_t Q}e^{i\Psi_{\Pi_t Q}} := \int_{[0,2\pi]} e^{ix}\,\Pi_t Q(dx).$$

For a given path $x([0,T])\in\mathcal{C}[0,T]$, we define

$$\begin{aligned} F(Q) = \int Q(dx[0,T])\Bigg\{ -\frac{1}{2}\int_0^T dt\,\Bigg[\Big(\int Q(dy[0,T])\sin(y(t)-x(t))\Big)^2 \\ + \int Q(dy[0,T])\cos(y(t)-x(t))\Bigg] \\ -\frac{1}{2}\int Q(dy[0,T])[\cos(y(T)-x(T))-\cos(y(0)-x(0))]\Bigg\} \end{aligned} \quad (3.6)$$

We can obtain a representation of P_N in terms of ρ_N, as follows:

Lemma 3.2.1. *It holds that*

$$\frac{dP_N}{dW^{\otimes N}}(\underline{x}([0,T])) = \exp[NF(\rho_N(\underline{x}([0,T])))]$$

where, for $Q \in \mathcal{M}_1(\mathcal{C}[0,T])$, $F(Q)$ *is expressed by* (3.6).

Lemma 3.2.1 allows us to deduce a Large Deviation Principle for ρ_N, from which we can derive its asymptotic behavior as $N \longrightarrow +\infty$.
Define

$$\mathcal{P}_N(\cdot) := \int P_N(\rho_N \in \cdot)\,,$$

which is an element of $\mathcal{M}_1(\mathcal{M}_1(\mathcal{C}[0,T]))$ and represents the law of ρ_N under the distribution of the process.
If $Q \in \mathcal{M}_1(\mathcal{C}[0,T])$ we denote by

$$H(Q|W) := \begin{cases} \int dQ \log \frac{dQ}{dW} \text{ if } \quad Q \ll W \quad \text{and} \quad \log \frac{dQ}{dW} \in L^1(Q) \\ \\ +\infty \qquad\qquad \text{otherwise} \end{cases}$$

the relative entropy between Q and W.

Proposition 3.2.1. *The laws* $\{\mathcal{P}_N\}_{N\geq 1}$ *of* ρ_N *(under the distribution of the process) obey a Large Deviation Principle with rate function*

$$I(Q) := H(Q|W) - F(Q)$$

(mind Definitions 1.2.1 and 1.2.2).

3.2.2 McKean-Vlasov Equation

Given $Q \in \mathcal{M}_1(\mathcal{C}[0,T])$, we can associate with Q a Markov process on $[0,2\pi]$ with law P^Q, initial distribution λ and time-dependent infinitesimal generator

$$\mathcal{L}_t^Q f(x) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x) + [\theta r_{\Pi_t Q} \sin(\Psi_{\Pi_t Q} - x)]\frac{\partial f}{\partial x}(x)\,,$$

acting on $f : [0,2\pi] \longrightarrow \mathbb{R}$.
It can be proved

Proposition 3.2.2. *For every* $Q \in \mathcal{M}_1(\mathcal{C}[0,T])$ *such that* $I(Q) < +\infty$, *we have*

$$I(Q) = H(Q|P^Q)\,.$$

Theorem 3.2.1. *Suppose that the initial distribution of the Markov process* $(\underline{x}(t))_{t\geq 0}$ *with generator* (3.4) *is such that the random variables* $(x_j(0))_{j=1}^N$ *are independent and identically distributed with law* λ. *Then the equation* $I(Q) = 0$ *admits a unique solution* $Q_* \in \mathcal{M}_1(\mathcal{C}[0,T])$, *such that its marginals* $q_t = \Pi_t Q_* \in \mathcal{M}_1([0,2\pi])$ *are weak solutions of the nonlinear McKean-Vlasov equation*

$$\begin{cases} \frac{\partial q_t}{\partial t} = \mathcal{L}q_t & (t \in [0,T]) \\ \\ q_0 = \lambda \end{cases} \tag{3.7}$$

where, for $x \in [0,2\pi]$, *the operator* $\mathcal{L}$ *acts*

$$\mathcal{L}q_t(x) = \frac{1}{2}\frac{\partial^2 q_t}{\partial x^2}(x) - \frac{\partial}{\partial x}\{[\theta r_{q_t}\sin(\Psi_{q_t} - x)]q_t(x)\}\,, \tag{3.8}$$

with $q_t(0) = q_t(2\pi)$. *Moreover, with respect to a metric* $d(\cdot,\cdot)$ *inducing the weak topology,* $\rho_N \longrightarrow Q_*$ *in probability with exponential rate, i.e.* $\mathcal{P}_N\{d(\rho_N, Q_*) > \varepsilon\}$ *is exponentially small in* N, *for each* $\varepsilon > 0$.

Remark 3.2.1. Q_* is the law of a time-inhomogeneous diffusion process on $[0,2\pi]$ with generator

$$\mathcal{L}_t^{q_t} f(x) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x) + [\theta r_{q_t}\sin(\Psi_{q_t} - x)]\frac{\partial f}{\partial x}(x)\,.$$

3.2.3 Stationary Solution(s)

The equation (3.7) describes the behavior of the system governed by generator (3.4) in the infinite volume limit. We are interested in the detection of the t-stationary solution(s) of this equation and in the study of the large time dynamics of it (them). We recall that to be t-stationary solution for (3.7) means to satisfy the equation $\mathcal{L}q = 0$ for every t.

Since the operator $\mathcal{L}$ preserves evenness, we can suppose the average phase $\Psi_{q_t} \equiv 0$, without loss of generality.

Hence, every equilibrium probability distribution is the solution of

$$\frac{1}{2}\frac{\partial^2 q}{\partial x^2}(x) - \frac{\partial}{\partial x}\{[-\theta r_q \sin x]q(x)\} = 0\,, \tag{3.9}$$

with the boundary condition $q(0) = q(2\pi)$ and for our model is characterized as follows.

Lemma 3.2.2. *Every equilibrium distribution for the nonlinear Markov process given by* (3.7) *is of the form:*

$$q_*(x) = (Z_*^{-1}) \cdot 2\theta r_* \cos x \left[\int_0^{2\pi} e^{-2\theta r_* \cos x} dx + \int_0^x e^{-2\theta r_* \cos y} dy \right], \tag{3.10}$$

where Z_* *is a normalizing factor and the variable* r_* *must satisfy the self-consistency relation*

$$r_{q_*} := r_* = \int_0^{2\pi} e^{ix}\, q_*(dx)\,. \tag{3.11}$$

Remark 3.2.2. There is a one-to-one correspondence between equilibrium distributions and solutions of the self-consistency equation (3.11).

Remark 3.2.3. Note that $r_* \equiv 0$ is always a solution of (3.11), for all the choices of θ. In this case the stationary distribution reduces to:

$$q_*^0(x) := \frac{1}{2\pi} \qquad \text{for all } x \in [0, 2\pi]. \tag{3.12}$$

Solutions with $r_* = 0$ are called *incoherent*, while those with $r_* > 0$ are called *synchronized*. The next theorem shows that if θ exceeds a threshold a synchronized solution is always possible.

Theorem 3.2.2. *Consider the equation* (3.11) *and define* $\theta_c = 1$. *Then,*

(a) if $\theta \leq \theta_c$, *the unique solution is* $r_* = 0$;

(b) if $\theta > \theta_c$, *at least one synchronized solution is possible.*

Proof. We refer to [DPdH95] and [dH00] for a detailed proof, concerning the complete phase diagram of the system. ∎

3.3 Critical Dynamics ($\theta = 1$)

We are going to consider the critical dynamics of the system, in other words the long-time behavior of the fluctuations in the threshold case, when $\theta = 1$. The size of Normal fluctuations must be further rescaled (in space and time), because their size around the deterministic limit increases in time. In this case we will obtain non-Normal fluctuations, solutions of a certain stochastic differential equation to be determined.

First of all, we need to locate the critical direction in the infinite dimensional space of the order parameters. In the rest of the section, we will consider $\theta = 1$ and let us assume that the initial condition λ is a product measure such that

$$q_0(dx) = q_*^0(dx) = \frac{1}{2\pi}\, dx$$

and so

$$q_t(dx) = q_*^0(dx) = \frac{1}{2\pi}\, dx\,,$$

for every value of $t \geq 0$, since we are in stationary conditions.

We consider the linearization of the operator $\mathcal{L}$, given by (3.8), at the equilibrium distribution $q_*^0(x)$, which is

$$\mathfrak{L}\phi(x) = \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(x) + \cos x \langle \cos y\, \phi(y), q_*^0(dy)\rangle + \sin x \langle \sin y\, \phi(y), q_*^0(dy)\rangle\,, \quad (3.13)$$

where we have denoted $\langle f_1, f_2\rangle := \int_0^{2\pi} f_1(x) f_2(x) dx$.

Lemma 3.3.1. *The operator $\mathfrak{L}$, defined by* (3.13)*, is self-adjoint in $L^2(q_*^0)$.*

Proof. Obviously $\mathfrak{L}$ is a linear and continuous operator. If we mean $\langle f_1, f_2\rangle_{L^2(q_*^0)} := \int_0^{2\pi} f_1(x) f_2(x) q_*^0(dx)$, we have to prove the following: if $\phi_1, \phi_2 \in L^2(q_*^0)$, then $\langle \mathfrak{L}\phi_1, \phi_2\rangle_{L^2(q_*^0)} = \langle \phi_1, \mathfrak{L}\phi_2\rangle_{L^2(q_*^0)}$. Thus,

$$\langle \mathfrak{L}\phi_1, \phi_2\rangle_{L^2(q_*^0)} =$$

$$= \left\langle \frac{1}{2}\frac{\partial^2 \phi_1}{\partial x^2}(x) + \cos x\, \frac{1}{2\pi}\int_0^{2\pi} \cos y\, \phi_1(y)\, dx + \sin x\, \frac{1}{2\pi}\int_0^{2\pi} \sin y\, \phi_1(y)\, dx, \phi_2 \right\rangle_{L^2(q_*^0)}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{\partial^2 \phi_1}{\partial x^2}(x)\, \phi_2(x)\, q_*^0(dx) + \int_0^{2\pi} \cos x\, \phi_1(x)\, q_*^0(dx) \int_0^{2\pi} \cos x\, \phi_2(x)\, q_*^0(dx) + \int_0^{2\pi} \sin x\, \phi_1(x)\, q_*^0(dx) \int_0^{2\pi} \sin x\, \phi_2(x)\, q_*^0(dx)$$

$$= -\frac{1}{2}\int_0^{2\pi} \frac{\partial \phi_1}{\partial x}(x)\, \frac{\partial \phi_2}{\partial x}(x)\, q_*^0(dx) + \Big\langle \phi_1, \cos x \, \frac{1}{2\pi}\int_0^{2\pi} \cos y\, \phi_2(y)\, dy + \sin x\, \frac{1}{2\pi}\int_0^{2\pi} \sin y\, \phi_2(y)\, dy \Big\rangle_{L^2(q_*^0)}$$

$$= \Big\langle \phi_1, \frac{1}{2}\int_0^{2\pi} \frac{\partial^2 \phi_2}{\partial x^2}(x) + \cos x\, \frac{1}{2\pi}\int_0^{2\pi} \cos y\, \phi_2(y)\, dy + \sin x\, \frac{1}{2\pi}\int_0^{2\pi} \sin y\, \phi_2(y)\, dy \Big\rangle_{L^2(q_*^0)}$$

and the proof is concluded. ■

Lemma 3.3.2. *The null space of the operator $\mathfrak{L}$, defined by* (3.13)*, is spanned by the functions* $\sin x$ *and* $\cos x$*.*

Proof. If $\varphi(\cdot)$ belongs to the null space of $\mathfrak{L}$, then $\mathfrak{L}\varphi = 0$. Therefore, we require that

$$\frac{1}{2}\frac{\partial^2 \varphi}{\partial x^2}(x) + \cos x\, \frac{1}{2\pi}\int_0^{2\pi} \cos y\, \varphi(y)\, q_*^0(dy) + \sin x\, \frac{1}{2\pi}\int_0^{2\pi} \sin y\, \varphi(y)\, q_*^0(dy) = 0\,. \tag{3.14}$$

We solve the ordinary differential equation (3.14). Having defined

$$A := \frac{1}{2\pi}\int_0^{2\pi} \cos y\, \varphi(y)\, q_*^0(dy) \quad \text{and} \quad B := \frac{1}{2\pi}\int_0^{2\pi} \sin y\, \varphi(y)\, q_*^0(dy)\,, \tag{3.15}$$

the solution is $\varphi(x) = 2B\sin x + 2A\cos x$; this function yields a solution of (3.14) provided that it satisfies the self-consistency relations (3.15), but it does for every value of A and B. ■

Remark 3.3.1. In the case that $\theta \neq 1$, the unique value for which the self-consistency relations in (3.15) are satisfied is $A = B = 0$, meaning that at the critical point the kernel of the operator $\mathfrak{L}$ is two-dimensional, while it is a trivial set for all the other values of the parameter θ.

Remark 3.3.2. The null space of the operator $\mathfrak{L}$ represents the critical direction for our model.

Lemma 3.3.3. *The spectrum of the operator $\mathfrak{L}$, defined by* (3.13)*, is described by the set* $\mathrm{Spec}(\mathfrak{L}) = \{0\} \cup \{-k^2,\, k \geq 2\}$*.*

Proof. The zero eigenvalue corresponds to the eigenvectors $\sin x$ and $\cos x$, generating the null space of the operator $\mathfrak{L}$. To find the rest of the spectrum means solving the problem

$$\phi''(x) = -\lambda\phi(x), \qquad \phi(0) = \phi(2\pi),$$

with $\lambda > 0$. The set of the solutions is $\phi(x) = a\sin(\sqrt{\lambda}x) + b\cos(\sqrt{\lambda}x)$, with $a, b \in \mathbb{R}$.
The boundary condition translates to $b = a\sin(2\pi\sqrt{\lambda}) + b\cos(2\pi\sqrt{\lambda})$ and it is satisfied for $a, b \neq 0$ if and only if $\sqrt{\lambda} = k$, $k \geq 2$. Finally, we have obtained a complete set of eigenvectors; therefore the corresponding eigenvalues cover the whole spectrum. ∎

We want to describe the action of the infinitesimal generator of the critical fluctuation flow

$$\widetilde{\rho}_N(t, dx) = N^{1/4}\left[\rho_N(N^{1/2}t, dx) - \frac{1}{2\pi}\,dx\right]$$

on the family of functions of the form $\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle)$, where

$$\psi : \mathbb{R}^m \longrightarrow \mathbb{R}, \qquad \psi \in \mathcal{C}_b^3(\mathbb{R}^m)$$

and

$$\phi_j : [0, 2\pi] \longrightarrow \mathbb{R}, \qquad \phi_j \in \mathcal{C}^2([0, 2\pi])\,,$$

for $j = 1, \dots, m$. Since we must consider fluctuations around $q_*^0(\cdot)$, that is, we must consider the "centered" process, we restrict our attention to functions ϕ_j with

$$\int_0^{2\pi} \phi_j(x) q_*^0(dx) = 0, \qquad j = 1, \dots, m\,.$$

Then, for this kind of functions it yields

$$\langle\phi_j, \widetilde{\rho}_N(t)\rangle = N^{1/4}\langle\phi_j, \rho_N(N^{1/2}t)\rangle\,,$$

for $j = 1, \dots, m$.

Lemma 3.3.4. *For $t \in [0, T]$, the critical fluctuation flow*

$$\widetilde{\rho}_N(t, dx) = N^{1/4}\left[\rho_N(N^{1/2}t, dx) - \frac{1}{2\pi}\,dx\right] \tag{3.16}$$

is a Markov process whose infinitesimal generator $\mathcal{J}_N$ satisfies:

$$\mathcal{J}_N\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle) = \\ = \left[N^{1/2}L_1 + N^{1/4}L_2 + L_3 + N^{-1/4}L_4\right]\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle), \tag{3.17}$$

where

$$L_1\psi(\langle\phi_1,\widetilde{\rho}_N\rangle,\ldots,\langle\phi_m,\widetilde{\rho}_N\rangle) = \sum_{j=1}^{m}\frac{\partial\psi}{\partial y_j}\langle\mathfrak{L}\phi_j,\widetilde{\rho}_N\rangle \tag{3.18}$$

$$\begin{aligned} L_2\psi(\langle\phi_1,\widetilde{\rho}_N\rangle,\ldots,\langle\phi_m,\widetilde{\rho}_N\rangle) = \sum_{j=1}^{m}\frac{\partial\psi}{\partial y_j}[&\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x\,\phi_j'(x),\widetilde{\rho}_N\rangle \\ &- \langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x\,\phi_j'(x),\widetilde{\rho}_N\rangle] \end{aligned} \tag{3.19}$$

$$L_3\psi(\langle\phi_1,\widetilde{\rho}_N\rangle,\ldots,\langle\phi_m,\widetilde{\rho}_N\rangle) = \frac{1}{2}\sum_{h,j=1}^{m}\frac{\partial^2\psi}{\partial y_h\partial y_j}\langle\phi_h'(x)\phi_j'(x),q_*^0\rangle \tag{3.20}$$

$$L_4\psi(\langle\phi_1,\widetilde{\rho}_N\rangle,\ldots,\langle\phi_m,\widetilde{\rho}_N\rangle) = \frac{1}{2}\sum_{h,j=1}^{m}\frac{\partial^2\psi}{\partial y_h\partial y_j}\langle\phi_h'(x)\phi_j'(x),\widetilde{\rho}_N\rangle \tag{3.21}$$

and the operator $\mathfrak{L}$ *is the linear operator given by* (3.13).

Proof. Just a very long and tedious computation. ■

Theorem 3.3.1. *For* $t\in[0,T]$, *if we consider the infinite-dimensional critical fluctuation process*

$$\langle\sin x,\widetilde{\rho}_N(t)\rangle,\langle\cos x,\widetilde{\rho}_N(t)\rangle,\{\langle\sin kx,\widetilde{\rho}_N(t)\rangle\}_{k\geq 2},\{\langle\cos kx,\widetilde{\rho}_N(t)\rangle\}_{k\geq 2}\,,$$

then, as $N\longrightarrow+\infty$, $\{\langle\sin kx,\widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$, $\{\langle\cos kx,\widetilde{\rho}_N(t)\rangle\}_{k\geq 2}\longrightarrow 0$ *in the sense of Proposition 1.4.1 and* $(\langle\sin x,\widetilde{\rho}_N(t)\rangle,\ \langle\cos x,\widetilde{\rho}_N(t)\rangle)$ *converges, in the sense of weak convergence of stochastic processes, to a limiting non-Gaussian process* $(X(t),Y(t))$, *which is the unique solution of the following stochastic differential equation:*

$$\begin{cases} dX(t) = -\dfrac{1}{8}\,X(t)\,[X^2(t)+Y^2(t)]\,dt + \dfrac{1}{\sqrt{2}}\,dB^{(1)}(t) \\[2ex] dY(t) = -\dfrac{1}{8}\,Y(t)\,[X^2(t)+Y^2(t)]\,dt + \dfrac{1}{\sqrt{2}}\,dB^{(2)}(t) \end{cases}$$

with initial condition $X(0)=Y(0)=0$ *and where* $B^{(1)}$ *and* $B^{(2)}$ *are two independent Standard Brownian motions.*

3.3.1 Proof of the Theorem 3.3.1

Let us denote by $\{\tau_N^M\}_{N\geq 1}$ a family of stopping times, defined as

$$\begin{aligned}\tau_N^M := \inf_{t\geq 0}\{&|\langle \sin x, \widetilde{\rho}_N(t)\rangle| \geq M \quad \text{or} \quad |\langle \cos x, \widetilde{\rho}_N(t)\rangle| \geq M \\ &\text{or} \quad |\langle \sin kx, \widetilde{\rho}_N(t)\rangle| \geq M \quad \text{for at least a value of } k = 2,3,\ldots \\ &\quad \text{or} \quad |\langle \cos kx, \widetilde{\rho}_N(t)\rangle| \geq M \quad \text{for at least a value of } k = 2,3,\ldots\},\end{aligned}$$

where M is a positive constant. We are interested in introducing such sequence of stopping times because in this way the processes $\langle \sin x, \widetilde{\rho}_N(t)\rangle$, $\langle \cos x, \widetilde{\rho}_N(t)\rangle$, $\{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$ and $\{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$ result to be bounded in the time interval $[0, T \wedge \tau_N^M]$.
We consider the infinitesimal generator $\mathcal{J}_N$, subject to the time-rescaling $N^{1/2}$, and we apply it to the particular function

$$\psi\Big(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle, \{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}\Big) = \|\widetilde{\rho}_N(t)\|_r^2,$$

where, for $r > 0$, we have defined the norm

$$\|\widetilde{\rho}_N\|_r^2 := \sum_{k=2}^{+\infty} \frac{1}{(1+k^2)^r}[\langle \sin kx, \widetilde{\rho}_N\rangle^2 + \langle \cos kx, \widetilde{\rho}_N\rangle^2]\,.$$

$\|\widetilde{\rho}_N(t)\|_r^2$ is a sequence of positive semimartingales on a suitable probability space $(\Omega, \mathcal{A}, P)$ and then the following decomposition holds:

$$d\|\widetilde{\rho}_N(t)\|_r^2 = \mathcal{J}_N(\|\widetilde{\rho}_N(t)\|_r^2)\, dt + d\mathcal{M}^t_{N,\|\widetilde{\rho}_N\|_r^2}\,,$$

with $\mathcal{M}^t_{N,\|\widetilde{\rho}_N\|_r^2}$ the local martingale given by

$$\begin{aligned}\mathcal{M}^t_{N,\|\widetilde{\rho}_N\|_r^2} &= \frac{2}{N^{3/4}}\int_0^t \sum_{j=1}^N \Bigg[\sum_{k=2}^{+\infty} \frac{n}{(1+k^2)^r}[\langle \sin kx, \widetilde{\rho}_N(s)\rangle \cos kx_j \\ &\qquad\qquad - \langle \cos kx, \widetilde{\rho}_N(s)\rangle \sin kx_j]\Bigg] dB_j(s) \quad (3.22) \\ &:= \int_0^t \sum_{j=1}^N \Big(\nabla_x \|\widetilde{\rho}_N(s)\|_r^2\Big)_j\, dB_j(s)\,,\end{aligned}$$

where $\{B_j(t) : t > 0, j = 1, \ldots, N\}$ is a system of independent Standard Brownian motions on $[0, 2\pi]$ and $(\nabla_x \|\widetilde{\rho}_N\|_r^2)_j$ is the j-th component of the gradient computed with respect to the processes $(x_j)_{j=1}^N$.

As a consequence of the considerations just explained, we are in the proper situation to adapt Proposition 1.4.1 to our specific case. Before stating the result about collapsing processes we need to prove the following technical Lemma.

Lemma 3.3.5. *Given two sequences $\{a_n\}_{n\geq 2}$, $\{b_n\}_{n\geq 2}$ of positive, real numbers and $r \in \mathbb{R}$, $r > 0$, it holds*

$$\sum_{n=2}^{+\infty} \frac{n}{(1+n^2)^r} a_n b_{n+1} \leq C \sum_{n=2}^{+\infty} \frac{n^2}{(1+n^2)^r} (a_n^2 + b_n^2)\,,$$

with C positive constant.

Proof. Just write $a_n b_{n+1} \leq a_n^2 + b_{n+1}^2$, and observe that $\frac{n+1}{n}$ is bounded from above and below for $n \geq 2$. ■

Lemma 3.3.6. *Consider $d > 2$, $\delta > 0$ and $\kappa_N := \kappa(N)$, such that $\kappa_N \xrightarrow{N\to+\infty} +\infty$. For $t \in [0, \tau_N^M]$ and $N \geq 1$, there exist constants C.'s independent of N and t and two increasing sequences $\{\alpha_N\}_{N\geq 1}$ and $\{\beta_N\}_{N\geq 1}$ which satisfy*

$$\kappa_N^{1/d}\alpha_N^{-1} \xrightarrow{N\to+\infty} 0,\ \kappa_N^{-1}\alpha_N \xrightarrow{N\to+\infty} 0,\ \kappa_N^{-1}\beta_N \xrightarrow{N\to+\infty} 0\,, \tag{3.23}$$

$$E\left[\|\widetilde{\rho}_N(0)\|_r^{2d}\right] \leq C_1\,\alpha_N^{-2d} \quad \textit{for all } N, \tag{3.24}$$

$$\mathcal{J}_N(\|\widetilde{\rho}_N(t)\|_r^2) \leq -\kappa_N\delta\|\widetilde{\rho}_N(t)\|_r^2 + \beta_N C_2 + C_3\,, \tag{3.25}$$

$$\sup_{\omega\in\Omega, x\in[0,2\pi], t\leq\tau_N^M} \left|\nabla_x\|\widetilde{\rho}_N(t)\|_r^2\right| \leq C_4\,\alpha_N^{-1}\,, \tag{3.26}$$

and such that, for every $\varepsilon > 0$, the following estimate holds

$$\sup_{N\geq N_0} P\left\{\sup_{0\leq t\leq T\wedge\tau_N^M} \|\widetilde{\rho}_N(t)\|_r^2 > C_5\left(\kappa_N^{1/d}\alpha_N^{-1} \vee \kappa_N^{-1}\alpha_N\right)\right\} \leq \varepsilon\,. \tag{3.27}$$

Proof. We aim to prove these sequences $\{\alpha_N\}_{N\geq 1}$, $\{\beta_N\}_{N\geq 1}$ and constants C.'s exist and to give a characterization of them. We show that properties (3.23)-(3.26) hold true. The estimate (3.27) then follows from Proposition 1.4.1.

(3.24) We start noticing that a Central Limit Theorem applies to the process $\langle \sin kx, \rho_N(0)\rangle$, since the random variables $(x_j(0))_{j=1}^N$ are independent; so, in the limit as $N \longrightarrow +\infty$, $N^{1/4}\langle \sin kx, \widetilde{\rho}_N(0)\rangle$ converges to a Gaussian random variable and, since $(\sin kx_j)_{j=1}^N$ are bounded random variables, there is convergence of all the moments. Thus,

$$E\left[N^d\langle \sin kx, \rho_N(0)\rangle^{2d}\right] \leq C_1^{(1)}$$

and the process $\langle \cos kx, \rho_N(0)\rangle$ obeys an analogous result. We obtain the following estimate for the $2d$-th moments of $\|\widetilde{\rho}_N(0)\|_r^{2d}$:

$$E[\|\widetilde{\rho}_N(0)\|_r^{2d}] = E\left[\left(\sum_{k=2}^{+\infty}\frac{1}{(1+k^2)^r}[\langle \sin kx, \widetilde{\rho}_N(0)\rangle^2 + \langle \cos kx, \widetilde{\rho}_N(0)\rangle^2]\right)^d\right]$$

$$\leq CE\left[\sum_{k=2}^{+\infty}\frac{1}{(1+k^2)^r}[\langle \sin kx, \widetilde{\rho}_N(0)\rangle^{2d} + \langle \cos kx, \widetilde{\rho}_N(0)\rangle^{2d}]\right]$$

$$= CN^{d/2}\sum_{k=2}^{+\infty}\frac{1}{(1+k^2)^r}\Big(E[\langle \sin kx, \rho_{N(0)}\rangle^{2d}] + E[\langle \cos kx, \rho_{N(0)}\rangle^{2d}]\Big)$$

$$\leq CN^{d/2}\left(C_1^{(1)} + C_1^{(2)}\right)N^{-d}S_1 := C_1 N^{-d/2}\,,$$

where S_1 is the sum of the series $\sum_{k=2}^{+\infty}\frac{1}{(1+k^2)^r}$, which is finite whenever $r > \frac{1}{2}$. Thus (3.24) holds.

(3.25) For $t \in [0, \tau_N^M]$, we derive the particular characterization of $\mathcal{J}_N(\|\widetilde{\rho}_N\|_r^2)$, adapting the explicit expression of $\mathcal{J}_N$ given by (3.17), and then we proceed to find an upper bound for this quantity.

$$\begin{aligned}\mathcal{J}_N\|\widetilde{\rho}_N\|_r^2 = \sum_{k=2}^{+\infty}\frac{1}{(1+k^2)^r}\Big\{&2N^{1/2}[\langle \sin kx, \widetilde{\rho}_N\rangle\langle \mathfrak{L}\sin kx, \widetilde{\rho}_N\rangle \\ &+ \langle \cos kx, \widetilde{\rho}_N\rangle\langle \mathfrak{L}\cos kx, \widetilde{\rho}_N\rangle] \\ &+ 2kN^{1/4}\{\langle \sin kx, \widetilde{\rho}_N\rangle[\langle \sin x, \widetilde{\rho}_N\rangle\langle \cos x\cos kx, \widetilde{\rho}_N\rangle \\ &- \langle \cos x, \widetilde{\rho}_N\rangle\langle \sin x\cos kx, \widetilde{\rho}_N\rangle] + \langle \cos kx, \widetilde{\rho}_N\rangle[\langle \cos x, \widetilde{\rho}_N\rangle\langle \sin x\sin kx, \widetilde{\rho}_N\rangle \\ &- \langle \sin x, \widetilde{\rho}_N\rangle\langle \cos x\sin kx, \widetilde{\rho}_N\rangle]\} + k^2[\langle \cos^2 kx, q_*\rangle + \langle \sin^2 kx, q_*\rangle] \\ &- \frac{2k^2}{N^{1/4}}\langle \sin kx, \widetilde{\rho}_N\rangle\langle \cos kx, \widetilde{\rho}_N\rangle\langle \sin kx\cos kx, \widetilde{\rho}_N\rangle\Big\}\end{aligned}$$

(by using Prosthaphaeresis formulas)

$$\begin{aligned}= \sum_{k=2}^{+\infty}\frac{1}{(1+k^2)^r}\Big\{&-2k^2N^{1/2}[\langle \sin kx, \widetilde{\rho}_N\rangle^2 + \langle \cos kx, \widetilde{\rho}_N\rangle^2] \\ &+ kN^{1/4}\{\langle \sin kx, \widetilde{\rho}_N\rangle[\langle \sin x, \widetilde{\rho}_N\rangle\langle \cos(k+1)x + \cos(k-1)x, \widetilde{\rho}_N\rangle\end{aligned}$$

$$
\begin{aligned}
&- \langle \cos x, \widetilde{\rho}_N \rangle \langle \sin(k+1)x - \sin(k-1)x, \widetilde{\rho}_N \rangle] \\
&+ \langle \cos kx, \widetilde{\rho}_N \rangle [\langle \cos x, \widetilde{\rho}_N \rangle \langle \cos(k-1)x - \cos(k+1)x, \widetilde{\rho}_N \rangle \\
&- \langle \sin x, \widetilde{\rho}_N \rangle \langle \sin(k+1)x + \sin(k-1)x, \widetilde{\rho}_N \rangle]\} \\
&+ \frac{k^2}{2} - \frac{2k^2}{N^{1/4}} \langle \sin kx, \widetilde{\rho}_N \rangle \langle \cos kx, \widetilde{\rho}_N \rangle \langle \sin kx \cos kx, \widetilde{\rho}_N \rangle \Big\}
\end{aligned}
$$

(by observing that $N^{-1/4}\langle \sin kx \cos kx, \widetilde{\rho}_N \rangle = \langle \sin kx \cos kx, \rho_N \rangle$ and hence $|\langle \sin kx \cos kx, \rho_N \rangle| \leq 1$)

$$
\begin{aligned}
\leq \sum_{k=2}^{+\infty} \frac{1}{(1+k^2)^r} \Big\{ &- 2k^2 N^{1/2} \left(\langle \sin kx, \widetilde{\rho}_N \rangle^2 + \langle \cos kx, \widetilde{\rho}_N \rangle^2 \right) \\
&+ kN^{1/4} M \Big[|\langle \sin kx, \widetilde{\rho}_N \rangle| \Big(|\langle \cos(k+1)x, \widetilde{\rho}_N \rangle| + |\langle \cos(k-1)x, \widetilde{\rho}_N \rangle| \\
&+ |\langle \sin(k+1)x, \widetilde{\rho}_N \rangle| + |\langle \sin(k-1)x, \widetilde{\rho}_N \rangle| \Big) \\
&+ |\langle \cos kx, \widetilde{\rho}_N \rangle| \Big(|\langle \sin(k+1)x, \widetilde{\rho}_N \rangle| + |\langle \sin(k-1)x, \widetilde{\rho}_N \rangle| \\
&+ |\langle \cos(k-1)x, \widetilde{\rho}_N \rangle| + |\langle \cos(k+1), \widetilde{\rho}_N \rangle| \Big) \Big] + \frac{k^2}{2} \\
&+ 2k^2 |\langle \sin kx, \widetilde{\rho}_N \rangle| |\langle \cos kx, \widetilde{\rho}_N \rangle| \Big\}
\end{aligned}
$$

(by Lemma 3.3.5)

$$
\begin{aligned}
\leq -2N^{1/2} &\sum_{k=2}^{+\infty} \frac{k^2}{(1+k^2)^r} \left(\langle \sin kx, \widetilde{\rho}_N \rangle^2 + \langle \cos kx, \widetilde{\rho}_N \rangle^2 \right) \\
&+ 4N^{1/4} MC \Big\{ \sum_{k=2}^{+\infty} \frac{k^2}{(1+k^2)^r} \left(\langle \sin kx, \widetilde{\rho}_N \rangle^2 + \langle \cos kx, \widetilde{\rho}_N \rangle^2 \right) \\
&+ \sum_{k=3}^{+\infty} \frac{(k-1)^2}{[1+(k-1)^2]^r} \left(\langle \sin(k-1)x, \widetilde{\rho}_N \rangle^2 + \langle \cos(k-1)x, \widetilde{\rho}_N \rangle^2 \right) \\
&+ \frac{4}{5^r} \left(\langle \sin x, \widetilde{\rho}_N \rangle^2 + \langle \cos x, \widetilde{\rho}_N \rangle^2 \right) \Big\} \\
&+ \sum_{k=2}^{+\infty} \frac{k^2}{(1+k^2)^r} \left(\langle \sin kx, \widetilde{\rho}_N \rangle^2 + \langle \cos kx, \widetilde{\rho}_N \rangle^2 \right) + \sum_{k=2}^{+\infty} \frac{k^2}{(1+k^2)^r}
\end{aligned}
$$

$$
\begin{aligned}
&\leq -2N^{1/2} \sum_{k=2}^{+\infty} \frac{1}{(1+k^2)^r} \left(\langle \sin kx, \widetilde{\rho}_N \rangle^2 + \langle \cos kx, \widetilde{\rho}_N \rangle^2 \right) \\
&+ 16N^{1/4} M^3 C \Big\{ \sum_{k=2}^{+\infty} \frac{k^2}{(1+k^2)^r} + \frac{2}{5^r} \Big\} + 2M^2 \sum_{k=2}^{+\infty} \frac{k^2}{(1+k^2)^r} + \sum_{k=2}^{+\infty} \frac{k^2}{(1+k^2)^r}
\end{aligned}
$$

$$\leq -2N^{1/2}\|\widetilde{\rho}_N\|_r^2 + N^{1/4}M^3C(S_2+2) + S_2(2M^2+1)\,,$$

where S_2 is the sum of the series $\sum_{k=2}^{+\infty}\frac{k^2}{(1+k^2)^r}$, which is finite whenever $r > \frac{3}{2}$.
Hence, we have obtained the desired inequality if we choose: $\kappa_N := N^{1/2}$, $\delta := 2$ (which is positive as required), $\beta_N := N^{1/4}$, $C_2 := M^3C(S_2+2)$ and $C_3 := S_2(2M^2+1)$.

(3.26) Now, we evaluate the supremum of the modulus of $\nabla_x\|\widetilde{\rho}_N(t)\|_r^2$, whose components are defined in (3.22). Since

$$\left[\sum_{k=2}^{+\infty}\frac{k}{(1+k^2)^r}[\langle\sin kx,\widetilde{\rho}_N(t)\rangle\cos kx_j - \langle\cos kx,\widetilde{\rho}_N(t)\rangle\sin kx_j]\right]^2 \leq$$

$$\leq C\sum_{k=2}^{+\infty}\frac{k^2}{(1+k^2)^{2r}}\left[\langle\sin kx,\widetilde{\rho}_N(t)\rangle^2|\cos kx_j|^2 + \langle\cos kx,\widetilde{\rho}_N(t)\rangle^2|\sin kx_j|^2\right]$$

$$\leq 2CM^2S_3\,,$$

it easily yields

$$\sup_{\omega\in\Omega,x\in[0,2\pi],t\leq\tau_N^M}\left|\nabla_x\|\widetilde{\rho}_N(t)\|_r^2\right| = \sup_{\omega\in\Omega,x\in[0,2\pi],t\leq\tau_N^M}\sqrt{\sum_{j=1}^N\left[(\nabla_x\|\widetilde{\rho}_N(t)\|_r^2)_j\right]^2}$$

$$\leq \frac{2}{N^{3/4}}M\sqrt{2CS_3}N^{1/2} = C_4N^{-1/4}\,,$$

where S_3 is the sum of the series $\sum_{k=2}^{+\infty}\frac{k^2}{(1+k^2)^{2r}}$, which is finite whenever $r > \frac{3}{4}$ and we set $C_4 := 2CM\sqrt{2S_3}$, $\alpha_N := N^{1/4}$.

(3.23) It remains to show that the sequences we have found satisfy the conditions about the convergence to zero. But,

$$\lim_{N\to+\infty}(N^{1/2})^{1/d}(N^{1/4})^{-1} = \lim_{N\to+\infty}N^{1/2d-1/4} = 0 \quad\Longleftrightarrow\quad d > 2,$$

$$\lim_{N\to+\infty}N^{1/4}N^{-1/2} = \lim_{N\to+\infty}N^{-1/4} = 0,$$

$$\lim_{N\to+\infty} N^{1/4}N^{-1/2} = \lim_{N\to+\infty} N^{-1/4} = 0$$

and hence we have completed the proof, since by Proposition 1.4.1 we can now assure (3.27) holds.

■

Remark 3.3.3. Notice that if we insert the quantities we choose during the proof of Lemma 3.3.6 into (3.27), we have shown that the following inequality holds

$$\sup_{N\geq N_0} P\left\{ \sup_{0\leq t\leq T\wedge\tau_N^M} \|\widetilde{\rho}_N(t)\|_r^2 > C_5\left(N^{1/2d-1/4}\vee N^{-1/4}\right)\right\} \leq \varepsilon\,. \tag{3.28}$$

The results we proved in this subparagraph show that the processes $\{\langle\sin kx,\widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$ and $\{\langle\cos kx,\widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$ are collapsing processes in the sense of Proposition 1.4.1, when $t\in[0,T\wedge\tau_N^M]$.

We want to find the expression of the limiting operator of the infinitesimal generator $\mathcal{J}_N$, as N grows to infinity. We choose

$$\psi\Big(\langle\sin x,\widetilde{\rho}_N(t)\rangle,\langle\cos x,\widetilde{\rho}_N(t)\rangle,\{\langle\sin kx,\widetilde{\rho}_N(t)\rangle\}_{k\geq 2},\{\langle\cos kx,\widetilde{\rho}_N(t)\rangle\}_{k\geq 2}\Big) = \\ = \psi(\langle\sin x,\widetilde{\rho}_N(t)\rangle,\langle\cos x,\widetilde{\rho}_N(t)\rangle)$$

and we apply the operator $\mathcal{J}_N$. Since $\ker\mathfrak{L} = \operatorname{span}\{\sin x,\cos x\}$, referring to (3.17)-(3.21), we obtain

$$\mathcal{J}_N\psi(\langle\sin x,\widetilde{\rho}_N\rangle,\langle\cos x,\widetilde{\rho}_N\rangle) =$$

$$\begin{aligned}
&= N^{1/4}\partial_1\psi(\cdot,\cdot)[\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos^2 x,\widetilde{\rho}_N\rangle - \langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x\cos x,\widetilde{\rho}_N\rangle] \\
&\quad + N^{1/4}\partial_2\psi(\cdot,\cdot)[\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin^2 x,\widetilde{\rho}_N\rangle - \langle\sin x,\widetilde{\rho}_N\rangle\langle\sin x\cos x,\widetilde{\rho}_N\rangle] \\
&\quad + \frac{1}{2}\partial_{11}\psi(\cdot,\cdot)\langle\cos^2 x,q_*^0\rangle + \frac{1}{2}\partial_{22}\psi(\cdot,\cdot)\langle\sin^2 x,q_*^0\rangle + \frac{1}{2N^{1/4}}\partial_{11}\psi(\cdot,\cdot)\langle\cos^2 x,\widetilde{\rho}_N\rangle \\
&\quad + \frac{1}{2N^{1/4}}\partial_{22}\psi(\cdot,\cdot)\langle\sin^2 x,\widetilde{\rho}_N\rangle + \frac{1}{N^{1/4}}\partial_{12}\psi(\cdot,\cdot)\langle\sin x\cos x,\widetilde{\rho}_N\rangle
\end{aligned}$$

(by using Bisection formulas and the fact that the measure $\widetilde{\rho}_N$ is centered, in other words that $\langle 1,\widetilde{\rho}_N\rangle = 0$)

$$= \frac{N^{1/4}}{2}\partial_1\psi(\cdot,\cdot)[\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle - \langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle]$$

$$+ \frac{N^{1/4}}{2}\partial_2\psi(\cdot,\cdot)[-\langle \sin x, \widetilde{\rho}_N\rangle\langle \sin 2x, \widetilde{\rho}_N\rangle - \langle \cos x, \widetilde{\rho}_N\rangle\langle \cos 2x, \widetilde{\rho}_N\rangle] \\ + \frac{1}{4}\partial_{11}\psi(\cdot,\cdot) + \frac{1}{4}\partial_{22}\psi(\cdot,\cdot) + o(1) \quad (3.29)$$

where $o(1)$ includes the terms coming from $L_4\psi$, which are of order $N^{-1/4}$. The operator L_4 is defined by (3.21).

Now, to determine the limiting generator, we apply the first order perturbation theory. The methodology for treating a perturbation problem has been developed in the paper [PSV77] and extends the earlier works done in [Kur73] and [Pap77]. The idea is the following. If we look at the expression of $\mathcal{J}_N$ in (3.17), we see that, in the limit as $N \longrightarrow +\infty$, the operators L_1 and L_2 can explode. Since we have just proved that, for $t \in [0, T \wedge \tau_N^M]$, the process surviving the critical time-space scaling lives in the kernel of the operator L_1, we restrict to work with $\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle) \in \ker L_1$, the term $L_1\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle)$ vanishes and then we need to control the operator L_2. So, we think of $N^{-1/4}$ as a perturbative parameter and we use a first order perturbation of ψ to introduce some negligible (in the limit as $N \longrightarrow +\infty$) terms in the expression (3.17), which provide that the operator L_2 does not diverge.
More precisely, we consider $\psi_N(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle)$, a first order perturbation of $\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle)$,

$$\psi_N(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle) = \psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle) \\ + N^{-1/4}\psi_1(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle) \quad (3.30)$$

and we apply the generator $\mathcal{J}_N$ to this function:

$$\mathcal{J}_N\psi_N(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle) =$$

$$= N^{1/2}L_1\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle) \\ + N^{1/4}[L_2\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle) + L_1\psi_1(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle)] \\ + L_3\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle) + L_2\psi_1(\langle\phi_1, \widetilde{\rho}_N\rangle, \dots, \langle\phi_m, \widetilde{\rho}_N\rangle) + o(1)\,, \quad (3.31)$$

where $o(1)$ includes the terms coming from $L_4\psi$, $L_3\psi_1$ and $L_4\psi_1$, which are of order $N^{-1/4}$ and $N^{-1/2}$.

The first term vanishes, since $\psi(\langle \phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle \phi_m, \widetilde{\rho}_N\rangle) \in \ker L_1$. To eliminate the $N^{1/4}$ term, we require that

$$L_2\psi(\langle \phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle \phi_m, \widetilde{\rho}_N\rangle) + L_1\psi_1(\langle \phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle \phi_m, \widetilde{\rho}_N\rangle) = 0\,,$$

that is

$$\psi_1(\langle \phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle \phi_m, \widetilde{\rho}_N\rangle) := -L_1^{-1}L_2\psi(\langle \phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle \phi_m, \widetilde{\rho}_N\rangle)\,, \tag{3.32}$$

where formally

$$L_1^{-1}\nu = -\int_0^{+\infty} \exp(L_1 t)\,\nu\,dt \quad \text{provided that} \quad \nu \in \ker L_1\,. \tag{3.33}$$

If we substitute the expression we found for $\psi_1(\langle \phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle \phi_m, \widetilde{\rho}_N\rangle)$, (3.32), in the two terms left of (3.31), we obtain

$$\begin{aligned}&L_3\psi(\langle \phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle \phi_m, \widetilde{\rho}_N\rangle) + L_2\psi_1(\langle \phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle \phi_m, \widetilde{\rho}_N\rangle) = \\ &= [L_3 - L_2L_1^{-1}L_2]\psi(\langle \phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle \phi_m, \widetilde{\rho}_N\rangle) := \widetilde{\mathcal{J}}_N\psi(\langle \phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle \phi_m, \widetilde{\rho}_N\rangle)\,,\end{aligned}$$

where $\widetilde{\mathcal{J}}_N$ satisfies

$$\lim_{N\to+\infty} \mathcal{J}_N\psi_N(\langle \phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle \phi_m, \widetilde{\rho}_N\rangle) = \lim_{N\to+\infty} \widetilde{\mathcal{J}}_N\psi(\langle \phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle \phi_m, \widetilde{\rho}_N\rangle)\,.$$

We apply the method to our case. The expression of $\mathcal{J}_N\psi(\langle \sin x, \widetilde{\rho}_N\rangle, \langle \cos x, \widetilde{\rho}_N\rangle)$ is described by (3.29). We need to compute ψ_1, which allows us to introduce the terms necessary to the convergence of the infinitesimal generator: $L_1\psi_1$ and $L_2\psi_1$. By the definitions (3.32) and (3.33), it yields

$$\begin{aligned}&\psi_1(\langle \sin x, \widetilde{\rho}_N\rangle, \langle \cos x, \widetilde{\rho}_N\rangle, \langle \sin 2x, \widetilde{\rho}_N\rangle, \langle \cos 2x, \widetilde{\rho}_N\rangle) = \\ &\quad = -L_1^{-1}L_2\psi(\langle \sin x, \widetilde{\rho}_N\rangle, \langle \cos x, \widetilde{\rho}_N\rangle) \\ &\quad = -\frac{1}{2}L_1^{-1}\Big[\partial_1\psi(\cdot,\cdot)\Big(\langle \sin x, \widetilde{\rho}_N\rangle\langle \cos 2x, \widetilde{\rho}_N\rangle - \langle \cos x, \widetilde{\rho}_N\rangle\langle \sin 2x, \widetilde{\rho}_N\rangle\Big) \\ &\qquad\qquad - \partial_2\psi(\cdot,\cdot)\Big(\langle \sin x, \widetilde{\rho}_N\rangle\langle \sin 2x, \widetilde{\rho}_N\rangle + \langle \cos x, \widetilde{\rho}_N\rangle\langle \cos 2x, \widetilde{\rho}_N\rangle\Big)\Big]\end{aligned}$$

$$= -\frac{1}{8}\Big[\partial_1\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle - \langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle\Big)$$
$$- \partial_2\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle + \langle\cos x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle\Big)\Big]. \quad (3.34)$$

Thus,

$$\mathcal{J}_N[\psi(\langle\sin x,\widetilde{\rho}_N\rangle,\langle\cos x,\widetilde{\rho}_N\rangle)$$
$$+ N^{-1/4}\psi_1(\langle\sin x,\widetilde{\rho}_N\rangle,\langle\cos x,\widetilde{\rho}_N\rangle,\langle\sin 2x,\widetilde{\rho}_N\rangle,\langle\cos 2x,\widetilde{\rho}_N\rangle)] =$$

$$= N^{1/4}[L_2\psi(\langle\sin x,\widetilde{\rho}_N\rangle,\langle\cos x,\widetilde{\rho}_N\rangle)$$
$$+ L_1\psi_1(\langle\sin x,\widetilde{\rho}_N\rangle,\langle\cos x,\widetilde{\rho}_N\rangle,\langle\sin 2x,\widetilde{\rho}_N\rangle,\langle\cos 2x,\widetilde{\rho}_N\rangle)]$$
$$+ L_3\psi(\langle\sin x,\widetilde{\rho}_N\rangle,\langle\cos x,\widetilde{\rho}_N\rangle)$$
$$+ L_2\psi_1(\langle\sin x,\widetilde{\rho}_N\rangle,\langle\cos x,\widetilde{\rho}_N\rangle,\langle\sin 2x,\widetilde{\rho}_N\rangle,\langle\cos 2x,\widetilde{\rho}_N\rangle) + o(1)\,.$$

Since $L_2\psi + L_1\psi_1 = 0$ by construction and $L_3\psi = \frac{1}{4}\left[\partial_{11}\psi + \partial_{22}\psi\right]$, it remains to compute the term $L_2\psi_1$.

$$L_2\psi_1(\langle\sin x,\widetilde{\rho}_N\rangle,\langle\cos x,\widetilde{\rho}_N\rangle,\langle\sin 2x,\widetilde{\rho}_N\rangle,\langle\cos 2x,\widetilde{\rho}_N\rangle) =$$

$$= -\frac{1}{8}L_2\Big[\partial_1\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle - \langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle\Big)$$
$$+ \partial_2\psi(\cdot,\cdot)\Big(- \langle\sin x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle - \langle\cos x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle\Big)\Big]$$

$$= -\frac{1}{8}\Bigg\{\Big[\partial_{11}\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle - \langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle\Big)$$
$$+ \partial_1\psi(\cdot,\cdot)\langle\cos 2x,\widetilde{\rho}_N\rangle\Big]\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos^2 x,\widetilde{\rho}_N\rangle - \langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x\cos x,\widetilde{\rho}_N\rangle\Big)$$
$$+ \Big[\partial_{12}\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle - \langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle\Big)$$
$$+ \partial_1\psi(\cdot,\cdot)\langle\sin 2x,\widetilde{\rho}_N\rangle\Big]\Big(\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin^2 x,\widetilde{\rho}_N\rangle - \langle\sin x,\widetilde{\rho}_N\rangle\langle\sin x\cos x,\widetilde{\rho}_N\rangle\Big)$$
$$-2\partial_1\psi(\cdot,\cdot)\Big[\langle\cos x,\widetilde{\rho}_N\rangle\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x\cos 2x,\widetilde{\rho}_N\rangle - \langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x\cos 2x,\widetilde{\rho}_N\rangle\Big)$$
$$+ \langle\sin x,\widetilde{\rho}_N\rangle\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x\sin 2x,\widetilde{\rho}_N\rangle - \langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x\sin 2x,\widetilde{\rho}_N\rangle\Big)\Big]$$
$$+ \Big[\partial_{22}\psi(\cdot,\cdot)\Big(- \langle\sin x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle - \langle\cos x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle\Big)$$
$$- \partial_2\psi(\cdot,\cdot)\langle\cos 2x,\widetilde{\rho}_N\rangle\Big]\Big(\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin^2 x,\widetilde{\rho}_N\rangle - \langle\sin x,\widetilde{\rho}_N\rangle\langle\sin x\cos x,\widetilde{\rho}_N\rangle\Big)$$

$$+\Big[\partial_{12}\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle\Big)$$
$$+\partial_2\psi(\cdot,\cdot)\langle\sin 2x,\widetilde{\rho}_N\rangle\Big]\Big(\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x\cos x,\widetilde{\rho}_N\rangle-\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos^2 x,\widetilde{\rho}_N\rangle\Big)$$
$$-2\partial_2\psi(\cdot,\cdot)\Big[\langle\cos x,\widetilde{\rho}_N\rangle\Big(\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x\sin 2x,\widetilde{\rho}_N\rangle-\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x\sin 2x,\widetilde{\rho}_N\rangle\Big)$$
$$+\langle\sin x,\widetilde{\rho}_N\rangle\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x\cos 2x,\widetilde{\rho}_N\rangle-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x\cos 2x,\widetilde{\rho}_N\rangle\Big)\Big]\Bigg\}$$

(by using Prosthaphaeresis formulas)

$$=-\frac{1}{8}\Bigg\{\frac{1}{2}\Big[\partial_{11}\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle\Big)$$
$$+\partial_1\psi(\cdot,\cdot)\langle\cos 2x,\widetilde{\rho}_N\rangle\Big]\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle\Big)$$
$$-\frac{1}{2}\Big[\partial_{12}\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle\Big)$$
$$+\partial_1\psi(\cdot,\cdot)\langle\sin 2x,\widetilde{\rho}_N\rangle\Big]\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle+\langle\cos x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle\Big)$$
$$-\partial_1\psi(\cdot,\cdot)\Big[\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x,\widetilde{\rho}_N\rangle^2+\langle\sin x,\widetilde{\rho}_N\rangle^3\Big]$$
$$-\frac{1}{2}\Big[\partial_{22}\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle+\langle\cos x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle\Big)$$
$$-\partial_2\psi(\cdot,\cdot)\langle\cos 2x,\widetilde{\rho}_N\rangle\Big]\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle+\langle\cos x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle\Big)$$
$$+\frac{1}{2}\Big[\partial_{12}\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle\Big)$$
$$+\partial_2\psi(\cdot,\cdot)\langle\sin 2x,\widetilde{\rho}_N\rangle\Big]\Big(\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle-\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle\Big)$$
$$-\partial_2\psi(\cdot,\cdot)\Big[\langle\cos x,\widetilde{\rho}_N\rangle^3+\langle\sin x,\widetilde{\rho}_N\rangle^2\langle\cos x,\widetilde{\rho}_N\rangle\Big]\Bigg\}.$$

Summing up all the terms we have obtained, recalling that ψ_1 is defined by (3.34), we get

$$\mathcal{J}_N[\psi(\langle\sin x,\widetilde{\rho}_N\rangle,\langle\cos x,\widetilde{\rho}_N\rangle)$$
$$+N^{-1/4}\psi_1(\langle\sin x,\widetilde{\rho}_N\rangle,\langle\cos x,\widetilde{\rho}_N\rangle,\langle\sin 2x,\widetilde{\rho}_N\rangle,\langle\cos 2x,\widetilde{\rho}_N\rangle)]=$$

$$=-\frac{1}{8}\Bigg\{\frac{1}{2}\Big[\partial_{11}\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle\Big)$$
$$+\partial_1\psi(\cdot,\cdot)\langle\cos 2x,\widetilde{\rho}_N\rangle\Big]\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle\Big)$$
$$-\frac{1}{2}\Big[\partial_{12}\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x,\widetilde{\rho}_N\rangle-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x,\widetilde{\rho}_N\rangle\Big)$$

$$
\begin{aligned}
&+ \partial_1\psi(\cdot,\cdot)\langle \sin 2x, \widetilde{\rho}_N\rangle\Big]\Big(\langle \sin x, \widetilde{\rho}_N\rangle\langle \sin 2x, \widetilde{\rho}_N\rangle + \langle \cos x, \widetilde{\rho}_N\rangle\langle \cos 2x, \widetilde{\rho}_N\rangle\Big) \\
&\qquad - \partial_1\psi(\cdot,\cdot)\Big[\langle \sin x, \widetilde{\rho}_N\rangle\langle \cos x, \widetilde{\rho}_N\rangle^2 + \langle \sin x, \widetilde{\rho}_N\rangle^3\Big] \\
&- \frac{1}{2}\Big[\partial_{22}\psi(\cdot,\cdot)\Big(\langle \sin x, \widetilde{\rho}_N\rangle\langle \sin 2x, \widetilde{\rho}_N\rangle + \langle \cos x, \widetilde{\rho}_N\rangle\langle \cos 2x, \widetilde{\rho}_N\rangle\Big) \\
&- \partial_2\psi(\cdot,\cdot)\langle \cos 2x, \widetilde{\rho}_N\rangle\Big]\Big(\langle \sin x, \widetilde{\rho}_N\rangle\langle \sin 2x, \widetilde{\rho}_N\rangle + \langle \cos x, \widetilde{\rho}_N\rangle\langle \cos 2x, \widetilde{\rho}_N\rangle\Big) \\
&+ \frac{1}{2}\Big[\partial_{12}\psi(\cdot,\cdot)\Big(\langle \sin x, \widetilde{\rho}_N\rangle\langle \cos 2x, \widetilde{\rho}_N\rangle - \langle \cos x, \widetilde{\rho}_N\rangle\langle \sin 2x, \widetilde{\rho}_N\rangle\Big) \\
&+ \partial_2\psi(\cdot,\cdot)\langle \sin 2x, \widetilde{\rho}_N\rangle\Big]\Big(\langle \cos x, \widetilde{\rho}_N\rangle\langle \sin 2x, \widetilde{\rho}_N\rangle - \langle \sin x, \widetilde{\rho}_N\rangle\langle \cos 2x, \widetilde{\rho}_N\rangle\Big) \\
&\qquad - \partial_2\psi(\cdot,\cdot)\Big[(\langle \cos x, \widetilde{\rho}_N\rangle^3 + \langle \sin x, \widetilde{\rho}_N\rangle^2\langle \cos x, \widetilde{\rho}_N\rangle\Big]\Big\} \\
&\qquad\qquad + \frac{1}{4}\partial_{11}\psi(\cdot,\cdot) + \frac{1}{4}\partial_{22}\psi(\cdot,\cdot) + o(1)\,. \qquad (3.35)
\end{aligned}
$$

The next step is the proof of the fact that, for every $\varepsilon > 0$ and $N \geq 1$, there exists a constant $M > 0$ such that it is true $P\left\{\tau_N^M \leq T\right\} \leq \epsilon$. This fact implies that, as N is growing to infinity, the processes $\{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$ and $\{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$ converge to zero in probability for all $t \in [0, T]$.

We consider the infinitesimal generator $\mathcal{J}_N$, subject to the time-rescaling $N^{1/2}$ and we apply it to the particular function

$$
\begin{aligned}
&\psi\Big(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle, \{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}\Big) \\
&\qquad + N^{-1/4}\psi_1(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle, \langle \sin 2x, \widetilde{\rho}_N(t)\rangle, \langle \cos 2x, \widetilde{\rho}_N(t)\rangle) = \\
&= \langle \sin x, \widetilde{\rho}_N(t)\rangle^2 + \langle \cos x, \widetilde{\rho}_N(t)\rangle^2 - \frac{1}{4N^{1/4}}\Big\{\langle \sin x, \widetilde{\rho}_N(t)\rangle^2\langle \cos 2x, \widetilde{\rho}_N(t)\rangle \\
&\quad - 2\langle \sin x, \widetilde{\rho}_N(t)\rangle\langle \cos x, \widetilde{\rho}_N(t)\rangle\langle \sin 2x, \widetilde{\rho}_N(t)\rangle + \langle \cos x, \widetilde{\rho}_N(t)\rangle^2\langle \cos 2x, \widetilde{\rho}_N(t)\rangle\Big\} \\
&:= \psi_N^{(3)}(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle, \langle \sin 2x, \widetilde{\rho}_N(t)\rangle, \langle \cos 2x, \widetilde{\rho}_N(t)\rangle)\,,
\end{aligned}
$$

meaning we have chosen the function ψ to be of the form

$$
\begin{aligned}
&\psi\Big(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle, \{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}\Big) = \\
&\qquad\qquad = \langle \sin x, \widetilde{\rho}_N(t)\rangle^2 + \langle \cos x, \widetilde{\rho}_N(t)\rangle^2\,.
\end{aligned}
$$

The following decomposition holds

$$\psi_N^{(3)}(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle, \langle \sin 2x, \widetilde{\rho}_N(t)\rangle, \langle \cos 2x, \widetilde{\rho}_N(t)\rangle) =$$

$$= \mathcal{M}^t_{N,\psi_N^{(3)}} + \psi_N^{(3)}(\langle \sin x, \widetilde{\rho}_N(0)\rangle, \langle \cos x, \widetilde{\rho}_N(0)\rangle, \langle \sin 2x, \widetilde{\rho}_N(0)\rangle, \langle \cos 2x, \widetilde{\rho}_N(0)\rangle)$$
$$+ \int_0^t \mathcal{J}_N\Big[\psi_N^{(3)}(\langle \sin x, \widetilde{\rho}_N(s)\rangle, \langle \cos x, \widetilde{\rho}_N(s)\rangle, \langle \sin 2x, \widetilde{\rho}_N(s)\rangle, \langle \cos 2x, \widetilde{\rho}_N(s)\rangle)\Big] ds$$

$$\leq \mathcal{M}^t_{N,\psi_N^{(3)}} + \psi_N^{(3)}(\langle \sin x, \widetilde{\rho}_N(0)\rangle, \langle \cos x, \widetilde{\rho}_N(0)\rangle, \langle \sin 2x, \widetilde{\rho}_N(0)\rangle, \langle \cos 2x, \widetilde{\rho}_N(0)\rangle)$$
$$+ \int_0^t \Big|\mathcal{J}_N\Big[\psi_N^{(3)}(\langle \sin x, \widetilde{\rho}_N(s)\rangle, \langle \cos x, \widetilde{\rho}_N(s)\rangle, \langle \sin 2x, \widetilde{\rho}_N(s)\rangle, \langle \cos 2x, \widetilde{\rho}_N(s)\rangle)\Big]\Big| ds,$$

with $\mathcal{M}^t_{N,\psi_N^{(3)}}$ a martingale given by

$$\mathcal{M}^t_{N,\psi_N^{(3)}} = \int_0^t \sum_{j=1}^N \Bigg\{ \frac{2}{N^{3/4}}\Big(\langle \sin x, \widetilde{\rho}_N(s)\rangle \cos x_j - \langle \cos x, \widetilde{\rho}_N(s)\rangle \sin x_j\Big)$$
$$+ \frac{1}{2N}\Bigg[\Big(\langle \sin x, \widetilde{\rho}_N(s)\rangle\langle \cos 2x, \widetilde{\rho}_N(s)\rangle - \langle \cos x, \widetilde{\rho}_N(s)\rangle\langle \sin 2x, \widetilde{\rho}_N(s)\rangle\Big) \cos x_j$$
$$+ \Big(\langle \sin x, \widetilde{\rho}_N(s)\rangle\langle \sin 2x, \widetilde{\rho}_N(s)\rangle + \langle \cos x, \widetilde{\rho}_N(s)\rangle\langle \cos 2x, \widetilde{\rho}_N(s)\rangle\Big) \sin x_j$$
$$+ \Big(- \langle \sin x, \widetilde{\rho}_N(s)\rangle^2 + \langle \cos x, \widetilde{\rho}_N(s)\rangle^2\Big) \sin 2x_j$$
$$- 2\langle \sin x, \widetilde{\rho}_N(s)\rangle\langle \cos x, \widetilde{\rho}_N(s)\rangle \cos 2x_j\Bigg]\Bigg\} dB_j(s)$$

$$:= \int_0^t \sum_{j=1}^N (\nabla_x \psi_N^{(3)})_j \, dB_j(s)\,,$$

where $\{B_j(t) : t > 0, j = 1, \dots, N\}$ is a system of independent Standard Brownian motions on $[0, 2\pi]$ and $\Big(\nabla_x \psi_N^{(3)}\Big)_j$ is the j-th component of the gradient computed with respect to the processes $(x_j)_{j=1}^N$. We recall that the expression of $\mathcal{J}_N$ is given by (3.35). For $t \in [0, \tau_N^M]$ we get

$$\Big|\mathcal{J}_N \Big[\psi_N^{(3)}(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle, \langle \sin 2x, \widetilde{\rho}_N(t)\rangle, \langle \cos 2x, \widetilde{\rho}_N(t)\rangle)\Big]\Big| =$$

$$= \frac{1}{8}\Bigg|\Bigg\{ \langle \sin x, \widetilde{\rho}_N\rangle\langle \cos 2x, \widetilde{\rho}_N\rangle\Big[2\langle \sin x, \widetilde{\rho}_N\rangle\langle \cos 2x, \widetilde{\rho}_N\rangle - \langle \cos x, \widetilde{\rho}_N\rangle\langle \sin 2x, \widetilde{\rho}_N\rangle\Big]$$

$$
\begin{aligned}
&- \langle \cos x, \widetilde{\rho}_N\rangle\langle \sin 2x, \widetilde{\rho}_N\rangle\Big[3\langle \sin x, \widetilde{\rho}_N\rangle\langle \cos 2x, \widetilde{\rho}_N\rangle - 2\langle \cos x, \widetilde{\rho}_N\rangle\langle \sin 2x, \widetilde{\rho}_N\rangle\Big]\\
&- 2\langle \sin x, \widetilde{\rho}_N\rangle\langle \sin 2x, \widetilde{\rho}_N\rangle\Big[\langle \sin x, \widetilde{\rho}_N\rangle\langle \sin 2x, \widetilde{\rho}_N\rangle + \langle \cos x, \widetilde{\rho}_N\rangle\langle \cos 2x, \widetilde{\rho}_N\rangle\Big]\\
&\qquad - \langle \sin x, \widetilde{\rho}_N\rangle\Big[\langle \sin x, \widetilde{\rho}_N\rangle\langle \cos x, \widetilde{\rho}_N\rangle^2 + \langle \sin x, \widetilde{\rho}_N\rangle^3\Big]\\
&\qquad - \langle \cos x, \widetilde{\rho}_N\rangle\Big[\langle \cos x, \widetilde{\rho}_N\rangle^3 + \langle \sin x, \widetilde{\rho}_N\rangle^2\langle \cos x, \widetilde{\rho}_N\rangle\Big]\Bigg\} + 1 + o(1)\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
\le \frac{1}{8}\Bigg\{&|\langle \sin x, \widetilde{\rho}_N\rangle||\langle \cos 2x, \widetilde{\rho}_N\rangle|\Big[2|\langle \sin x, \widetilde{\rho}_N\rangle||\langle \cos 2x, \widetilde{\rho}_N\rangle|\\
&\qquad + |\langle \cos x, \widetilde{\rho}_N\rangle||\langle \sin 2x, \widetilde{\rho}_N\rangle|\Big]\\
&+ |\langle \cos x, \widetilde{\rho}_N\rangle||\langle \sin 2x, \widetilde{\rho}_N\rangle|\Big[3|\langle \sin x, \widetilde{\rho}_N\rangle||\langle \cos 2x, \widetilde{\rho}_N\rangle|\\
&\qquad + 2|\langle \cos x, \widetilde{\rho}_N\rangle||\langle \sin 2x, \widetilde{\rho}_N\rangle|\Big]\\
&+ 2|\langle \sin x, \widetilde{\rho}_N\rangle||\langle \sin 2x, \widetilde{\rho}_N\rangle|\Big[|\langle \sin x, \widetilde{\rho}_N\rangle||\langle \sin 2x, \widetilde{\rho}_N\rangle|\\
&\qquad + |\langle \cos x, \widetilde{\rho}_N\rangle||\langle \cos 2x, \widetilde{\rho}_N\rangle|\Big]\\
&+ |\langle \sin x, \widetilde{\rho}_N\rangle|\Big[|\langle \sin x, \widetilde{\rho}_N\rangle|\langle \cos x, \widetilde{\rho}_N\rangle^2 + |\langle \sin x, \widetilde{\rho}_N\rangle|^3\Big]\\
&\qquad + |\langle \cos x, \widetilde{\rho}_N\rangle|\Big[|\langle \cos x, \widetilde{\rho}_N\rangle|^3 + \langle \sin x, \widetilde{\rho}_N\rangle^2|\langle \cos x, \widetilde{\rho}_N\rangle|\Big]\Bigg\} + 2
\end{aligned}
$$

$$
\le 2(M^4 + 1) =: C_8\,,
$$

with C_8 positive constant independent on N. Since the following inclusions are valid

$$
\{\tau_N^M \le T\} \subseteq
$$

$$
\begin{aligned}
&\subseteq \left\{\sup_{0\le t\le T\wedge\tau_N^M} \|\widetilde{\rho}_N\|_r^2 \ge C_5(N^{1/2d-1/4} \vee N^{-1/4})\right\} \cup \left\{\left|\psi_N^{(3)}(\dots)\Big|_{t=0}\right| \ge C_9\right\}\\
&\quad \cup \left[\left\{\left|\psi_N^{(3)}(\dots)\Big|_{t=0}\right| \le C_9\right\} \cap \left\{\sup_{0\le t\le T\wedge\tau_N^M} \left|\psi_N^{(3)}(\dots)\Big|_{t}\right| \ge C_9 + TC_8 + C_{10}\right\}\right]
\end{aligned}
$$

$$\subseteq \left\{ \sup_{0 \le t \le T \wedge \tau_N^M} \|\widetilde{\rho}_N\|_r^2 \ge C_5 (N^{1/2d-1/4} \vee N^{-1/4}) \right\} \cup \left\{ \left| \psi_N^{(3)}(\dots)\Big|_{t=0} \right| \ge C_9 \right\}$$
$$\cup \left\{ \sup_{0 \le t \le T \wedge \tau_N^M} \mathcal{M}^t_{N,\psi_N^{(3)}} \ge C_{10} \right\},$$

we obtain the following inequality

$$P\{\tau_N^M \le T\} \le P\left\{ \sup_{0 \le t \le T \wedge \tau_N^M} \|\widetilde{\rho}_N(t)\|_r^2 \ge C_5 (N^{1/2d-1/4} \vee N^{-1/4}) \right\}$$
$$+ P\left\{ \left| \psi_N^{(3)}(\dots)\Big|_{t=0} \right| \ge C_9 \right\} + P\left\{ \sup_{0 \le t \le T \wedge \tau_N^M} \mathcal{M}^t_{N,\psi_N^{(3)}} \ge C_{10} \right\}$$

We estimate the three terms of the right-hand side of the inequality.

- For any $\varepsilon > 0$, thanks to (3.28), we have

$$P\left\{ \sup_{0 \le t \le T \wedge \tau_N^M} \|\widetilde{\rho}_N(t)\|_r^2 \ge C_5 (N^{1/2d-1/4} \vee N^{-1/4}) \right\} \le \varepsilon\,.$$

- We start noticing that a Central Limit Theorem applies to the process $\langle \sin x, \rho_N(0) \rangle$, since the random variables $(x_j(0))_{j=1}^N$ are independent; so, in the limit as $N \longrightarrow +\infty$, $N^{1/4} \langle \sin x, \widetilde{\rho}_N(0) \rangle$ converges to a Gaussian random variable and, since $(\sin x_j)_{j=1}^N$ are bounded random variables, there is convergence of all the moments. Thus,

$$E\left[N^d \langle \sin x, \rho_N(0) \rangle^{2d} \right] \le \overline{C}^{(1)}$$

and the process $\langle \cos x, \rho_N(0) \rangle$ obeys an analogous result. So, we can estimate

$$E\Bigg[\Bigg| \langle \sin x, \widetilde{\rho}_N(0) \rangle^2 + \langle \cos x, \widetilde{\rho}_N(0) \rangle^2 - \frac{1}{4N^{1/4}} \Big\{ \langle \sin x, \widetilde{\rho}_N(0) \rangle^2 \langle \cos 2x, \widetilde{\rho}_N(0) \rangle$$
$$- 2 \langle \sin x, \widetilde{\rho}_N(0) \rangle \langle \cos x, \widetilde{\rho}_N(0) \rangle \langle \sin 2x, \widetilde{\rho}_N(0) \rangle$$
$$+ \langle \cos x, \widetilde{\rho}_N(0) \rangle^2 \langle \cos 2x, \widetilde{\rho}_N(0) \rangle \Big\} \Bigg| \Bigg]$$

$$\le E\Bigg[\Bigg| \langle \sin x, \widetilde{\rho}_N(0) \rangle^2 + \langle \cos x, \widetilde{\rho}_N(0) \rangle^2$$

$$- \frac{1}{4N^{1/4}}\Big\{\langle\cos 2x, \widetilde{\rho}_N(0)\rangle\Big(\langle\sin x, \widetilde{\rho}_N(0)\rangle^2 + \langle\cos x, \widetilde{\rho}_N(0)\rangle^2\Big)$$

$$+ \langle\sin 2x, \widetilde{\rho}_N(0)\rangle\Big(\langle\sin x, \widetilde{\rho}_N(0)\rangle^2 + \langle\cos x, \widetilde{\rho}_N(0)\rangle^2\Big)\Big|\Big]$$

$$\leq E\Big[\Big|\langle\sin x, \widetilde{\rho}_N(0)\rangle^2 + \langle\cos x, \widetilde{\rho}_N(0)\rangle^2$$

$$- \frac{1}{4}\Big\{|\langle\cos 2x, \rho_{N(0)}\rangle|\Big(\langle\sin x, \widetilde{\rho}_N(0)\rangle^2 + \langle\cos x, \widetilde{\rho}_N(0)\rangle^2\Big)$$

$$+ |\langle\sin 2x, \rho_{N(0)}\rangle|\Big(\langle\sin x, \widetilde{\rho}_N(0)\rangle^2 + \langle\cos x, \widetilde{\rho}_N(0)\rangle^2\Big)\Big|\Big]$$

$$\leq \frac{3}{2}\left(\overline{C}^{(1)} + \overline{C}^{(2)}\right) N^{-1/2} = \overline{C} N^{-1/2}$$

and in the limit as $N \longrightarrow +\infty$, we have convergence to zero in L^1 and then in probability. Therefore

$$P\left\{\left|\psi_N^{(3)}(\dots)\Big|_{t=0}\right| \geq C_9\right\} \leq \varepsilon$$

for any $\varepsilon > 0$, for every N and for a sufficiently large C_9.

► We reduce to deal with $E\left[\left(\mathcal{M}^t_{N,\psi_N^{(3)}}\right)^2\right]$; in fact, Doob's "maximal inequality in L^p" (case $p = 2$) for martingales (we refer to Chapter VII, Section 3 of [Shi96]) tells us that

$$P\left\{\sup_{0\leq t\leq T\wedge\tau_N^M}\left|\mathcal{M}^t_{N,\psi_N^{(3)}}\right| \geq C_{10}\right\} \leq \frac{E\left[\left(\mathcal{M}^T_{N,\psi_N^{(3)}}\right)^2\right]}{(C_{10})^2}.$$

Hence, we obtain

$$E\left[\left(\mathcal{M}^T_{N,\psi_N^{(3)}}\right)^2\right] =$$

$$= E\Big[\Big[\int_0^T \sum_{j=1}^N \Big\{\frac{2}{N^{3/4}}\Big(\langle\sin x, \widetilde{\rho}_N(s)\rangle\cos x_j - \langle\cos x, \widetilde{\rho}_N(s)\rangle\sin x_j\Big)$$

$$+ \frac{1}{2N}\Big[\Big(\langle\sin x, \widetilde{\rho}_N(s)\rangle\langle\cos 2x, \widetilde{\rho}_N(s)\rangle - \langle\cos x, \widetilde{\rho}_N(s)\rangle\langle\sin 2x, \widetilde{\rho}_N(s)\rangle\Big)\cos x_j$$

$$+ \Big(\langle \sin x, \widetilde{\rho}_N(s)\rangle\langle \sin 2x, \widetilde{\rho}_N(s)\rangle + \langle \cos x, \widetilde{\rho}_N(s)\rangle\langle \cos 2x, \widetilde{\rho}_N(s)\rangle\Big) \sin x_j$$
$$+ \Big(- \langle \sin x, \widetilde{\rho}_N(s)\rangle^2 + \langle \cos x, \widetilde{\rho}_N(s)\rangle^2\Big) \sin 2x_j$$
$$- 2\langle \sin x, \widetilde{\rho}_N(s)\rangle\langle \cos x, \widetilde{\rho}_N(s)\rangle \cos 2x_j\Big]\Big\} dB_j(s)\Big]^2\Big]$$

(thanks to Itô's isometry)

$$= \int_0^T \sum_{j=1}^N E\Big[\Big\{ \frac{2}{N^{3/4}}\Big(\langle \sin x, \widetilde{\rho}_N(s)\rangle \cos x_j - \langle \cos x, \widetilde{\rho}_N(s)\rangle \sin x_j\Big)$$
$$+ \frac{1}{2N}\Big[\Big(\langle \sin x, \widetilde{\rho}_N(s)\rangle\langle \cos 2x, \widetilde{\rho}_N(s)\rangle - \langle \cos x, \widetilde{\rho}_N(s)\rangle\langle \sin 2x, \widetilde{\rho}_N(s)\rangle\Big) \cos x_j$$
$$+ \Big(\langle \sin x, \widetilde{\rho}_N(s)\rangle\langle \sin 2x, \widetilde{\rho}_N(s)\rangle + \langle \cos x, \widetilde{\rho}_N(s)\rangle\langle \cos 2x, \widetilde{\rho}_N(s)\rangle\Big) \sin x_j$$
$$+ \Big(- \langle \sin x, \widetilde{\rho}_N(s)\rangle^2 + \langle \cos x, \widetilde{\rho}_N(s)\rangle^2\Big) \sin 2x_j$$
$$- 2\langle \sin x, \widetilde{\rho}_N(s)\rangle\langle \cos x, \widetilde{\rho}_N(s)\rangle \cos 2x_j\Big]\Big\}^2\Big] ds$$

$$\leq \int_0^T \sum_{j=1}^N E\left[\left(\frac{4}{N^{1/2}} + \frac{1}{N^{1/2}}\right)^2\right] ds = 25T =: C_{11}\,,$$

with C_{11} independent of N and M. We have established that, if we choose $C_{10} \geq \sqrt{\frac{C_{11}}{\varepsilon}}$, then

$$P\Big\{ \sup_{0\leq t\leq T\wedge \tau_N^M} \mathcal{M}^t_{N,\psi_N^{(3)}} \geq C_{10}\Big\} \leq \varepsilon\,.$$

In summary, we proved the inequality we were looking for; in fact,

$$P\left\{\tau_N^M \leq T\right\} \leq 3\varepsilon := \epsilon\,.$$

We have just concluded the proof of the first part of the statement of Theorem 3.3.1, concerning the collapse of the processes $\{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$ and $\{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$ in the limit as $N \longrightarrow +\infty$ and for $t \in [0,T]$. Now, we are going to show that in the same setting, i.e. the limit of infinite volume and $t \in [0,T]$, the process $(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle)$ admits a limiting process and we are going to compute it.

First, we need to prove the tightness of the sequence $\{\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle\}_{N\geq 1}$, for $t \in [0,T]$. This property implies the existence of convergent subsequences. Secondly, we will verify that all the convergent subsequences have the same limit and hence also the sequence $\{\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle\}_{N\geq 1}$ must converge to that limit for $t \in [0,T]$.

Lemma 3.3.7. *The sequence* $\{\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle\}_{N\geq 1}$ *is tight.*

Proof. We must verify the conditions (1.44) and (1.45) hold. Since we have shown already that, for every $\epsilon > 0$ the inequality $P\{\tau_N^M \leq T\} \leq \epsilon$ is true for M sufficiently large and uniformly in N, it is enough to show tightness for the stopped processes

$$\{\langle \sin x, \widetilde{\rho}_N(t\wedge\tau_N^M)\rangle, \langle \cos x, \widetilde{\rho}_N(t\wedge\tau_N^M)\rangle\}_{N\geq 1}.$$

We consider the function ψ of the form

$$\psi\Big(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle, \{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}\Big) = \\ = \langle \sin x, \widetilde{\rho}_N(t)\rangle .$$

(1.44) The decomposition

$$\langle \sin x, \widetilde{\rho}_N(t)\rangle = \langle \sin x, \widetilde{\rho}_N(0)\rangle + \int_0^t \mathcal{J}_N(\langle \sin x, \widetilde{\rho}_N(u)\rangle)\, du + \mathcal{M}^t_{N,\langle \sin x, \widetilde{\rho}_N(t)\rangle},$$

with

$$\mathcal{M}^t_{N,\langle \sin x, \widetilde{\rho}_N(t)\rangle} := \frac{1}{N^{3/4}}\int_0^t \sum_{j=1}^N \cos x_j \, dB_j(s),$$

holds true and we get the following inclusion

$$\left\{ \sup_{0\leq t\leq T\wedge\tau_N^M} |\langle \sin x, \widetilde{\rho}_N(t)\rangle| \geq M \right\} \subseteq \\ \subseteq \{|\langle \sin x, \widetilde{\rho}_N(0)\rangle| \geq M\} \cup \left\{ \sup_{0\leq t\leq T\wedge\tau_N^M} \left|\mathcal{M}^t_{N,\langle \sin x, \widetilde{\rho}_N(t)\rangle}\right| \geq M \right\}.$$

We estimate the probability of its right-hand side.
A Central Limit Theorem applies to the process $\langle \sin x, \rho_N(0)\rangle$, since the random variables $(x_j(0))_{j=1}^N$ are independent; so, in the limit as $N \longrightarrow +\infty$, $N^{1/4}\langle \sin x, \widetilde{\rho}_N(0)\rangle$ converges to a Gaussian random variable and, since $(\sin x_j)_{j=1}^N$

are bounded random variables, there is convergence of all the moments. Thus,

$$E\left[N^d\langle \sin x, \rho_N(0)\rangle^{2d}\right] \leq \overline{C}^{(1)}\,.$$

So, we have that

$$E\Big[\Big|\langle \sin x, \widetilde{\rho}_N(0)\rangle\Big|\Big] \leq \overline{C}^{(1)} N^{-1/4}$$

and in the limit as $N \longrightarrow +\infty$, we have convergence to zero in L^1 and then in probability. Therefore

$$P\left\{\Big|\langle \sin x, \widetilde{\rho}_N(0)\rangle\Big| \geq M\right\} \leq \varepsilon$$

for any $\varepsilon > 0$, for $N \geq \overline{N}$ and for a sufficiently large M.
Secondly, by Doob's inequality, it yields

$$P\Bigg\{\sup_{0\leq t\leq T\wedge\tau_N^M}\Big|\mathcal{M}^t_{N,\langle \sin x, \widetilde{\rho}_N(t)\rangle}\Big| \geq M\Bigg\} \leq \frac{E\left[\left(\mathcal{M}^t_{N,\langle \sin x, \widetilde{\rho}_N(t)\rangle}\right)^2\right]}{M^2}$$

and

$$E\left[\left(\mathcal{M}^t_{N,\langle \sin x, \widetilde{\rho}_N(t)\rangle}\right)^2\right] = \frac{1}{N^{3/2}}E\left[\left(\int_0^t \sum_{j=1}^N \cos x_j \, dB_j(s)\right)^2\right]$$

$$= \frac{1}{N^{3/2}}E\left[\int_0^t \sum_{j=1}^N \cos^2 x_j \, ds\right] \leq \frac{T}{N^{3/2}} \leq T =: C_{15}\,.$$

We have established that, if we choose $M \geq \sqrt{\frac{C_{15}}{\varepsilon}}$, then

$$P\Bigg\{\sup_{0\leq t\leq T\wedge\tau_N^M}\Big|\mathcal{M}^t_{N,\langle \sin x, \widetilde{\rho}_N(t)\rangle}\Big| \geq M\Bigg\} \leq \varepsilon\,;$$

hence, for $N \geq \overline{N}$,

$$P\left\{\Big|\langle \sin x, \widetilde{\rho}_N(0)\rangle\Big| \geq M\right\} + P\Bigg\{\sup_{0\leq t\leq T\wedge\tau_N^M}\Big|\mathcal{M}^t_{N,\langle \sin x, \widetilde{\rho}_N(t)\rangle}\Big| \geq M\Bigg\} \leq 2\varepsilon$$

and so we obtained (1.44). Let us deal with (1.45) now.

(1.45) Recalling the expression of ψ_1 given by (3.34), we construct

$$\psi\Big(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle, \{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}\Big)$$
$$+ N^{-1/4}\psi_1(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle, \langle \sin 2x, \widetilde{\rho}_N(t)\rangle, \langle \cos 2x, \widetilde{\rho}_N(t)\rangle) =$$

$$= \langle \sin x, \widetilde{\rho}_N(t)\rangle - \frac{1}{8N^{1/4}}[\langle \sin x, \widetilde{\rho}_N(t)\rangle\langle \cos 2x, \widetilde{\rho}_N(t)\rangle - \langle \cos x, \widetilde{\rho}_N(t)\rangle\langle \sin 2x, \widetilde{\rho}_N(t)\rangle]$$

$$:= \psi_N^{(1)}(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle, \langle \sin 2x, \widetilde{\rho}_N(t)\rangle, \langle \cos 2x, \widetilde{\rho}_N(t)\rangle)\,.$$

The following decomposition holds

$$\psi_N^{(1)}(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle, \langle \sin 2x, \widetilde{\rho}_N(t)\rangle, \langle \cos 2x, \widetilde{\rho}_N(t)\rangle) =$$

$$= \psi_N^{(1)}(\langle \sin x, \widetilde{\rho}_N(0)\rangle, \langle \cos x, \widetilde{\rho}_N(0)\rangle, \langle \sin 2x, \widetilde{\rho}_N(0)\rangle, \langle \cos 2x, \widetilde{\rho}_N(0)\rangle)$$
$$+ \int_0^t \mathcal{J}_N\psi_N^{(1)}(\langle \sin x, \widetilde{\rho}_N(s)\rangle, \langle \cos x, \widetilde{\rho}_N(s)\rangle, \langle \sin 2x, \widetilde{\rho}_N(s)\rangle, \langle \cos 2x, \widetilde{\rho}_N(s)\rangle)ds + \mathcal{M}^t_{N,\psi_N^{(1)}}$$

where $\mathcal{J}_N\psi_N^{(1)}$ is deduced by adapting the expansion (3.35) and with $\mathcal{M}^t_{N,\psi_N^{(1)}}$ a martingale given by

$$\mathcal{M}^t_{N,\psi_N^{(1)}} = \int_0^t \sum_{j=1}^N \Bigg\{ \frac{1}{N^{3/4}}\cos x_j - \frac{1}{8N}\Big[\langle \cos 2x, \widetilde{\rho}_N(s)\rangle \cos x_j + \langle \sin 2x, \widetilde{\rho}_N(s)\rangle \sin x_j - 2\langle \sin x, \widetilde{\rho}_N(s)\rangle \sin 2x_j - 2\langle \cos x, \widetilde{\rho}_N(s)\rangle \cos 2x_j\Big]\Bigg\} dB_j(s)$$

$$:= \int_0^t \sum_{j=1}^N \left(\nabla_x \psi_N^{(1)}\right)_j \, dB_j(s)\,,$$

where $\{B_j(t) : t > 0, j = 1, \ldots, N\}$ is a system of independent Standard Brownian motions on $[0, 2\pi]$ and $\left(\nabla_x \psi_N^{(1)}\right)_j$ is the j-th component of the

gradient computed with respect to the processes $(x_j)_{j=1}^N$.
We notice that

$$|\langle \sin x, \widetilde{\rho}_N(t)\rangle - \langle \sin x, \widetilde{\rho}_N(s)\rangle| =$$

$$= \Bigg| \frac{1}{8N^{1/4}}\Big[\langle \sin x, \widetilde{\rho}_N(t)\rangle\langle \cos 2x, \widetilde{\rho}_N(t)\rangle - \langle \cos x, \widetilde{\rho}_N(t)\rangle\langle \sin 2x, \widetilde{\rho}_N(t)\rangle\Big] - \frac{1}{8N^{1/4}}\Big[\langle \sin x, \widetilde{\rho}_N(s)\rangle\langle \cos 2x, \widetilde{\rho}_N(s)\rangle - \langle \cos x, \widetilde{\rho}_N(s)\rangle\langle \sin 2x, \widetilde{\rho}_N(s)\rangle\Big] + \int_s^t \mathcal{J}_N\psi_N^{(1)}(\langle \sin x, \widetilde{\rho}_N(u)\rangle, \langle \cos x, \widetilde{\rho}_N(u)\rangle, \langle \sin 2x, \widetilde{\rho}_N(u)\rangle, \langle \cos 2x, \widetilde{\rho}_N(u)\rangle)du + \mathcal{M}^{s,t}_{N,\psi_N^{(1)}}\Bigg|,$$

where we have denoted

$$\mathcal{M}^{s,t}_{N,\psi_N^{(1)}} = \int_s^t \sum_{j=1}^N \left(\nabla_x \psi_N^{(1)}\right)_j \, dB_j(u)\,.$$

Thus,

$$\{|\langle \sin x, \widetilde{\rho}_N(t)\rangle - \langle \sin x, \widetilde{\rho}_N(s)\rangle| \geq \alpha\} \subseteq$$

$$\subseteq \Bigg\{\Bigg| \frac{1}{8N^{1/4}}\Big[\langle \sin x, \widetilde{\rho}_N(t)\rangle\langle \cos 2x, \widetilde{\rho}_N(t)\rangle - \langle \cos x, \widetilde{\rho}_N(t)\rangle\langle \sin 2x, \widetilde{\rho}_N(t)\rangle - \langle \sin x, \widetilde{\rho}_N(s)\rangle\langle \cos 2x, \widetilde{\rho}_N(s)\rangle + \langle \cos x, \widetilde{\rho}_N(s)\rangle\langle \sin 2x, \widetilde{\rho}_N(s)\rangle\Big]\Bigg|$$

$$+ \left| \int_s^t \mathcal{J}_N\psi_N^{(1)}(\langle \sin x, \widetilde{\rho}_N(u)\rangle, \langle \cos x, \widetilde{\rho}_N(u)\rangle, \langle \sin 2x, \widetilde{\rho}_N(u)\rangle, \langle \cos 2x, \widetilde{\rho}_N(u)\rangle)du \right| + \left|\mathcal{M}^{s,t}_{N,\psi_N^{(1)}}\right| \geq \alpha\Bigg\}$$

$$\subseteq \Bigg\{\Bigg| \frac{1}{8N^{1/4}}\Big[\langle \sin x, \widetilde{\rho}_N(t)\rangle\langle \cos 2x, \widetilde{\rho}_N(t)\rangle - \langle \cos x, \widetilde{\rho}_N(t)\rangle\langle \sin 2x, \widetilde{\rho}_N(t)\rangle - \langle \sin x, \widetilde{\rho}_N(s)\rangle\langle \cos 2x, \widetilde{\rho}_N(s)\rangle + \langle \cos x, \widetilde{\rho}_N(s)\rangle\langle \sin 2x, \widetilde{\rho}_N(s)\rangle\Big]\Bigg| \geq \alpha\Bigg\} \cup$$

$$\cup \left\{ \left| \int_s^t \mathcal{J}_N \psi_N^{(1)} (\langle \sin x, \widetilde{\rho}_N(u) \rangle, \langle \cos x, \widetilde{\rho}_N(u) \rangle, \langle \sin 2x, \widetilde{\rho}_N(u) \rangle, \right. \right.$$
$$\left. \left. \langle \cos 2x, \widetilde{\rho}_N(u) \rangle) du \right| \geq \alpha \right\} \cup \left\{ \left| \mathcal{M}^{s,t}_{N,\psi_N^{(1)}} \right| \geq \alpha \right\}.$$

We need to estimate the probability of the three sets of the right-hand side of the previous inclusion.

- Since

$$\left| \frac{1}{8N^{1/4}} \Big[\langle \sin x, \widetilde{\rho}_N(t) \rangle \langle \cos 2x, \widetilde{\rho}_N(t) \rangle - \langle \cos x, \widetilde{\rho}_N(t) \rangle \langle \sin 2x, \widetilde{\rho}_N(t) \rangle \right.$$
$$\left. - \langle \sin x, \widetilde{\rho}_N(s) \rangle \langle \cos 2x, \widetilde{\rho}_N(s) \rangle + \langle \cos x, \widetilde{\rho}_N(s) \rangle \langle \sin 2x, \widetilde{\rho}_N(s) \rangle \Big] \right|$$

$$\leq \frac{1}{8N^{1/4}} \Big[|\langle \sin x, \widetilde{\rho}_N(t) \rangle| |\langle \cos 2x, \widetilde{\rho}_N(t) \rangle|$$
$$+ |\langle \cos x, \widetilde{\rho}_N(t) \rangle| |\langle \sin 2x, \widetilde{\rho}_N(t) \rangle| + |\langle \sin x, \widetilde{\rho}_N(s) \rangle| |\langle \cos 2x, \widetilde{\rho}_N(s) \rangle|$$
$$+ |\langle \cos x, \widetilde{\rho}_N(s) \rangle| |\langle \sin 2x, \widetilde{\rho}_N(s) \rangle| \Big] \leq \frac{M^2}{2} N^{-1/4}.$$

Hence,

$$P \left\{ \left| \frac{1}{8N^{1/4}} \Big[\langle \sin x, \widetilde{\rho}_N(t) \rangle \langle \cos 2x, \widetilde{\rho}_N(t) \rangle \right. \right.$$
$$- \langle \cos x, \widetilde{\rho}_N(t) \rangle \langle \sin 2x, \widetilde{\rho}_N(t) \rangle - \langle \sin x, \widetilde{\rho}_N(s) \rangle \langle \cos 2x, \widetilde{\rho}_N(s) \rangle$$
$$\left. \left. + \langle \cos x, \widetilde{\rho}_N(s) \rangle \langle \sin 2x, \widetilde{\rho}_N(s) \rangle \Big] \right| \geq \alpha \right\} \leq \varepsilon$$

for $N \geq \overline{N}$ and α sufficiently large.

- Adapting the expansion (3.35), we get

$$\left| \int_s^t \mathcal{J}_N \psi_N^{(1)} (\langle \sin x, \widetilde{\rho}_N(u) \rangle, \langle \cos x, \widetilde{\rho}_N(u) \rangle, \langle \sin 2x, \widetilde{\rho}_N(u) \rangle, \right.$$
$$\left. \langle \cos 2x, \widetilde{\rho}_N(u) \rangle) du \right| \leq$$

$$
\begin{aligned}
\leq \Bigg| \int_s^t \Bigg\{ -\frac{1}{8}\Bigg[\frac{1}{2}\Big(&\langle \sin x, \widetilde{\rho}_N(u)\rangle \langle \cos 2x, \widetilde{\rho}_N(u)\rangle^2 \\
&- \langle \sin x, \widetilde{\rho}_N(u)\rangle \langle \sin 2x, \widetilde{\rho}_N(u)\rangle^2\Big) \\
&- \langle \cos x, \widetilde{\rho}_N(u)\rangle \langle \sin 2x, \widetilde{\rho}_N(u)\rangle \langle \cos 2x, \widetilde{\rho}_N(u)\rangle \\
&- \langle \sin x, \widetilde{\rho}_N(u)\rangle \langle \cos x, \widetilde{\rho}_N(u)\rangle^2 - \langle \sin x, \widetilde{\rho}_N(u)\rangle^3\Bigg] + o(1)\Bigg\} du\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
\leq \int_s^t \Bigg\{\frac{1}{8}\Bigg[\frac{1}{2}\Big(&|\langle \sin x, \widetilde{\rho}_N(u)\rangle| \langle \cos 2x, \widetilde{\rho}_N(u)\rangle^2 \\
&+ |\langle \sin x, \widetilde{\rho}_N(u)\rangle| \langle \sin 2x, \widetilde{\rho}_N(u)\rangle^2\Big) \\
&+ \langle \cos x, \widetilde{\rho}_N(u)\rangle || \langle \sin 2x, \widetilde{\rho}_N(u)\rangle || \langle \cos 2x, \widetilde{\rho}_N(u)\rangle| \\
&+ |\langle \sin x, \widetilde{\rho}_N(u)\rangle| \langle \cos x, \widetilde{\rho}_N(u)\rangle^2 + |\langle \sin x, \widetilde{\rho}_N(u)\rangle|^3\Bigg] + o(1)\Bigg] du\Bigg\}
\end{aligned}
$$

$$
\leq \left(\frac{M^3}{2} + 1\right)(t-s) \leq C_{12}\delta\,,
$$

where we have defined $C_{12} := M^3 + 1$. Hence,

$$
P\left\{\left| \int_s^t \mathcal{J}_N \psi_N^{(1)}(\langle \sin x, \widetilde{\rho}_N(u)\rangle, \langle \cos x, \widetilde{\rho}_N(u)\rangle, \langle \sin 2x, \widetilde{\rho}_N(u)\rangle, \langle \cos 2x, \widetilde{\rho}_N(u)\rangle) du \right| \geq \alpha \right\} \leq \varepsilon
$$

for α sufficiently large.

► Applying Chebyscev inequality, we obtain

$$
P\left\{ \left| \mathcal{M}^{s,t}_{N,\psi_N^{(1)}} \right| \geq \alpha \right\} \leq \alpha^{-2} E\left[\left(\mathcal{M}^{s,t}_{N,\psi_N^{(1)}} \right)^2 \right]
$$

$$
\begin{aligned}
= \alpha^{-2} E\Bigg[\Bigg(\int_0^t \sum_{j=1}^N \Bigg\{ \frac{1}{N^{3/4}} \cos x_j - \frac{1}{8N}\Big[&\langle \cos 2x, \widetilde{\rho}_N(s)\rangle \cos x_j \\
&+ \langle \sin 2x, \widetilde{\rho}_N(s)\rangle \sin x_j - 2\langle \sin x, \widetilde{\rho}_N(s)\rangle \sin 2x_j \\
&- 2\langle \cos x, \widetilde{\rho}_N(s)\rangle \cos 2x_j\Big]\Bigg\} dB_j(s)\Bigg)^2\Bigg]
\end{aligned}
$$

$$\leq C\alpha^{-2}\int_s^t \sum_{j=1}^N E\bigg[\frac{1}{N^{3/2}}|\cos x_j|^2 + \frac{1}{64N^2}\Big[\langle \cos 2x, \widetilde{\rho}_N(u)\rangle^2 |\cos x_j|^2 + \langle \sin 2x, \widetilde{\rho}_N(u)\rangle^2 |\sin x_j|^2 + 4\langle \sin x, \widetilde{\rho}_N(u)\rangle^2 |\sin 2x_j|^2 + 4\langle \cos x, \widetilde{\rho}_N(u)\rangle^2 |\cos 2x_j|^2\Big]\bigg] du$$

$$\leq C\alpha^{-2}\int_s^t \sum_{j=1}^N E\bigg[\frac{1}{N^{3/2}} + \frac{5}{32N^2}M^2\bigg] du \leq C_{14}\alpha^{-2}\delta\,,$$

with $C_{14} := C(1+5M^2)$. Hence,

$$P\bigg\{ \Big|\mathcal{M}^{s,t}_{N,\psi_N^{(1)}}\Big| \geq \alpha \bigg\} \leq \varepsilon$$

for α sufficiently large.

Finally, we can conclude that

$$\sup_N \sup_{\substack{0\leq s\leq t\leq T \\ t-s\leq\delta}} P\{|\langle \sin x, \widetilde{\rho}_N(t)\rangle - \langle \sin x, \widetilde{\rho}_N(s)\rangle| \geq \alpha\} \leq 3\varepsilon$$

and (1.45) follows. Choosing

$$\psi\Big(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle, \{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}\Big) = \langle \cos x, \widetilde{\rho}_N(t)\rangle$$

we can analogously prove the tightness for the sequence $\{\langle \cos x, \widetilde{\rho}_N(t)\rangle\}_{N\geq 1}$.

■

Lemma 3.3.7 implies there exist convergent subsequences for the sequence of processes $\{\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle\}_{N\geq 1}$. Let $\{\langle \sin x, \widetilde{\rho}_n(t)\rangle, \langle \cos x, \widetilde{\rho}_n(t)\rangle\}_{n\geq 1}$ denote one of such a subsequence and let $\psi \in \mathcal{C}_b^3$ be a function of the type

$$\psi\Big(\langle \sin x, \widetilde{\rho}_n(t)\rangle, \langle \cos x, \widetilde{\rho}_n(t)\rangle, \{\langle \sin kx, \widetilde{\rho}_n(t)\rangle\}_{k\geq 2}, \{\langle \cos kx, \widetilde{\rho}_n(t)\rangle\}_{k\geq 2}\Big) = \psi(\langle \sin x, \widetilde{\rho}_n(t)\rangle, \langle \cos x, \widetilde{\rho}_n(t)\rangle)\,.$$

Recalling the expression of ψ_1 given by (3.34), we construct

$$\psi\Big(\langle \sin x, \widetilde{\rho}_n(t)\rangle, \langle \cos x, \widetilde{\rho}_n(t)\rangle, \{\langle \sin kx, \widetilde{\rho}_n(t)\rangle\}_{k\geq 2}, \{\langle \cos kx, \widetilde{\rho}_n(t)\rangle\}_{k\geq 2}\Big)$$
$$+ n^{-1/4}\psi_1(\langle \sin x, \widetilde{\rho}_n(t)\rangle, \langle \cos x, \widetilde{\rho}_n(t)\rangle, \langle \sin 2x, \widetilde{\rho}_n(t)\rangle, \langle \cos 2x, \widetilde{\rho}_n(t)\rangle)$$

$$= \psi(\langle \sin x, \widetilde{\rho}_n(t)\rangle, \langle \cos x, \widetilde{\rho}_n(t)\rangle)$$
$$- \frac{1}{8n^{1/4}}\partial_1\psi(\cdot,\cdot)[\langle \sin x, \widetilde{\rho}_n(t)\rangle\langle \cos 2x, \widetilde{\rho}_n(t)\rangle - \langle \cos x, \widetilde{\rho}_n(t)\rangle\langle \sin 2x, \widetilde{\rho}_n(t)\rangle]$$
$$- \frac{1}{8n^{1/4}}\partial_2\psi(\cdot,\cdot)[-\langle \sin x, \widetilde{\rho}_n(t)\rangle\langle \sin 2x, \widetilde{\rho}_n(t)\rangle - \langle \cos x, \widetilde{\rho}_n(t)\rangle\langle \cos 2x, \widetilde{\rho}_n(t)\rangle]$$

$$:= \psi_n^{(2)}(\langle \sin x, \widetilde{\rho}_n(t)\rangle, \langle \cos x, \widetilde{\rho}_n(t)\rangle, \langle \sin 2x, \widetilde{\rho}_n(t)\rangle, \langle \cos 2x, \widetilde{\rho}_n(t)\rangle)\,.$$

The following decomposition holds

$$\psi_n^{(2)}(\langle \sin x, \widetilde{\rho}_n(t)\rangle, \langle \cos x, \widetilde{\rho}_n(t)\rangle, \langle \sin 2x, \widetilde{\rho}_n(t)\rangle, \langle \cos 2x, \widetilde{\rho}_n(t)\rangle) =$$

$$= \mathcal{M}^t_{n,\psi_n^{(2)}} + \psi_n^{(2)}(\langle \sin x, \widetilde{\rho}_n(0)\rangle, \langle \cos x, \widetilde{\rho}_n(0)\rangle, \langle \sin 2x, \widetilde{\rho}_n(0)\rangle, \langle \cos 2x, \widetilde{\rho}_n(0)\rangle)$$
$$+ \int_0^t \mathcal{J}_n\psi_n^{(2)}(\langle \sin x, \widetilde{\rho}_n(s)\rangle, \langle \cos x, \widetilde{\rho}_n(s)\rangle, \langle \sin 2x, \widetilde{\rho}_n(s)\rangle, \langle \cos 2x, \widetilde{\rho}_n(s)\rangle)\, ds \tag{3.36}$$

where

$$\mathcal{J}_n\psi_n^{(2)}(\langle \sin x, \widetilde{\rho}_n(t)\rangle, \langle \cos x, \widetilde{\rho}_n(t)\rangle, \langle \sin 2x, \widetilde{\rho}_n(t)\rangle, \langle \cos 2x, \widetilde{\rho}_n(t)\rangle) =$$

$$= -\frac{1}{8}\Bigg\{\frac{1}{2}\Big[\partial_{11}\psi(\cdot,\cdot)\Big(\langle \sin x, \widetilde{\rho}_n(t)\rangle\langle \cos 2x, \widetilde{\rho}_n(t)\rangle - \langle \cos x, \widetilde{\rho}_n(t)\rangle\langle \sin 2x, \widetilde{\rho}_n(t)\rangle\Big)$$
$$+\partial_1\psi(\cdot,\cdot)\langle \cos 2x, \widetilde{\rho}_n(t)\rangle\Big]\Big(\langle \sin x, \widetilde{\rho}_n(t)\rangle\langle \cos 2x, \widetilde{\rho}_n(t)\rangle - \langle \cos x, \widetilde{\rho}_n(t)\rangle\langle \sin 2x, \widetilde{\rho}_n(t)\rangle\Big)$$
$$- \frac{1}{2}\Big[\partial_{12}\psi(\cdot,\cdot)\Big(\langle \sin x, \widetilde{\rho}_n(t)\rangle\langle \cos 2x, \widetilde{\rho}_n(t)\rangle - \langle \cos x, \widetilde{\rho}_n(t)\rangle\langle \sin 2x, \widetilde{\rho}_n(t)\rangle\Big)$$
$$+\partial_1\psi(\cdot,\cdot)\langle \sin 2x, \widetilde{\rho}_n(t)\rangle\Big]\Big(\langle \sin x, \widetilde{\rho}_n(t)\rangle\langle \sin 2x, \widetilde{\rho}_n(t)\rangle + \langle \cos x, \widetilde{\rho}_n(t)\rangle\langle \cos 2x, \widetilde{\rho}_n(t)\rangle\Big)$$
$$- \partial_1\psi(\cdot,\cdot)\Big[\langle \sin x, \widetilde{\rho}_n(t)\rangle\langle \cos x, \widetilde{\rho}_n(t)\rangle^2 + \langle \sin x, \widetilde{\rho}_n(t)\rangle^3\Big]$$
$$- \frac{1}{2}\Big[\partial_{22}\psi(\cdot,\cdot)\Big(\langle \sin x, \widetilde{\rho}_n(t)\rangle\langle \sin 2x, \widetilde{\rho}_n(t)\rangle + \langle \cos x, \widetilde{\rho}_n(t)\rangle\langle \cos 2x, \widetilde{\rho}_n(t)\rangle\Big)$$

$$-\partial_2\psi(\cdot,\cdot)\langle\cos 2x,\widetilde{\rho}_n(t)\rangle\Big]\Big(\langle\sin x,\widetilde{\rho}_n(t)\rangle\langle\sin 2x,\widetilde{\rho}_n(t)\rangle+\langle\cos x,\widetilde{\rho}_n(t)\rangle\langle\cos 2x,\widetilde{\rho}_n(t)\rangle\Big)$$
$$+\frac{1}{2}\Big[\partial_{12}\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_n(t)\rangle\langle\cos 2x,\widetilde{\rho}_n(t)\rangle-\langle\cos x,\widetilde{\rho}_n(t)\rangle\langle\sin 2x,\widetilde{\rho}_n(t)\rangle\Big)$$
$$+\partial_2\psi(\cdot,\cdot)\langle\sin 2x,\widetilde{\rho}_n(t)\rangle\Big]\Big(\langle\cos x,\widetilde{\rho}_n(t)\rangle\langle\sin 2x,\widetilde{\rho}_n(t)\rangle-\langle\sin x,\widetilde{\rho}_n(t)\rangle\langle\cos 2x,\widetilde{\rho}_n(t)\rangle\Big)$$
$$-\partial_2\psi(\cdot,\cdot)\Big[(\langle\cos x,\widetilde{\rho}_n(t)\rangle^3+\langle\sin x,\widetilde{\rho}_n(t)\rangle^2\langle\cos x,\widetilde{\rho}_n(t)\rangle\Big]\Big\}$$
$$+\frac{1}{4}\partial_{11}\psi(\cdot,\cdot)+\frac{1}{4}\partial_{22}\psi(\cdot,\cdot)+o_M(1)$$

which, as usual, is deduced by adapting the expansion (3.35). The remainder $o_M(1)$ goes to zero as $n\longrightarrow+\infty$, uniformly in M. If we compute the limit as $n\longrightarrow+\infty$, we have:

$$\mathcal{J}_n\psi_n^{(2)}(\langle\sin x,\widetilde{\rho}_n(t)\rangle,\langle\cos x,\widetilde{\rho}_n(t)\rangle,\langle\sin 2x,\widetilde{\rho}_n(t)\rangle,\langle\cos 2x,\widetilde{\rho}_n(t)\rangle)$$
$$\xrightarrow[w]{n\to+\infty}\mathcal{J}\psi^{(2)}(X(t),Y(t))\,,$$

with

$$\mathcal{J}\psi^{(2)}(X(t),Y(t))=$$
$$=-\frac{1}{8}\Big\{\partial_1\psi(\cdot,\cdot)X(t)[X^2(t)+Y^2(t)]+\partial_2\psi(\cdot,\cdot)Y(t)[X^2(t)+Y^2(t)]\Big\}$$
$$+\frac{1}{4}\left[\partial_{11}\psi(\cdot,\cdot)+\partial_{22}\psi(\cdot,\cdot)\right]\,.$$

Then, because of (3.36), we obtain

$$\mathcal{M}^t_{n,\psi_n^{(2)}}\xrightarrow[w]{n\to+\infty}\mathcal{M}^t_{\psi^{(2)}}\,,$$

once we have defined

$$\mathcal{M}^t_{\psi^{(2)}}:=\psi^{(2)}(X(t),Y(t))-\psi^{(2)}(X(0),Y(0))-\int_0^t\mathcal{J}\psi^{(2)}(X(u),Y(u))\,du\,.$$

We must prove the following Lemma:

Lemma 3.3.8. $\mathcal{M}^t_{\psi_n^{(2)}}$ *is a martingale (with respect to* t*); in other words, for all* $s,t\in[0,T]$, $s\le t$ *and for all measurable and bounded functions* $g(X([0,s]),Y([0,s]))$ *the following identity holds:*

$$E[\mathcal{M}^t_{\psi_n^{(2)}}g(X([0,s]),Y([0,s]))]=E[\mathcal{M}^s_{\psi_n^{(2)}}g(X([0,s]),Y([0,s]))]\,. \qquad (3.37)$$

Proof. The reasoning we explained in Lemma 1.4.5 applies in this case too, so it is sufficient to prove $\left\{\mathcal{M}^t_{n,\psi_n^{(2)}}\right\}_{n\geq 1}$ is an uniformly integrable sequence of random variables.
So, since

$$\begin{aligned}
\mathcal{M}^{s,t}_{n,\psi_n^{(2)}} = \int_s^t \sum_{j=1}^n \Bigg\{ & \frac{1}{n^{3/4}}\Big(\partial_1\psi(\cdot,\cdot)\cos x_j - \partial_2\psi(\cdot,\cdot)\sin x_j\Big) \\
& - \frac{1}{8n}\Big(\partial_{11}\psi(\cdot,\cdot)\cos x_j - \partial_{12}\psi(\cdot,\cdot)\sin x_j\Big)\Big(\langle \sin x, \widetilde{\rho}_n(u)\rangle\langle \cos 2x, \widetilde{\rho}_n(u)\rangle \\
& \qquad - \langle \cos x, \widetilde{\rho}_n(u)\rangle\langle \sin 2x, \widetilde{\rho}_n(u)\rangle\Big) \\
& + \frac{1}{8n}\Big(\partial_{12}\psi(\cdot,\cdot)\cos x_j - \partial_{22}\psi(\cdot,\cdot)\sin x_j\Big)\Big(\langle \sin x, \widetilde{\rho}_n(u)\rangle\langle \sin 2x, \widetilde{\rho}_n(u)\rangle \\
& \qquad + \langle \cos x, \widetilde{\rho}_n(u)\rangle\langle \cos 2x, \widetilde{\rho}_n(u)\rangle\Big) \\
& - \frac{1}{8n}\partial_1\psi(\cdot,\cdot)\Big(\langle \cos 2x, \widetilde{\rho}_n(u)\rangle\cos x_j - 2\langle \sin x, \widetilde{\rho}_n(u)\rangle \sin 2x_j \\
& \qquad + \langle \sin 2x, \widetilde{\rho}_n(u)\rangle \sin x_j - 2\langle \cos x, \widetilde{\rho}_n(u)\rangle\cos 2x_j\Big) \\
& - \frac{1}{8n}\partial_2\psi(\cdot,\cdot)\Big(- \langle \sin 2x, \widetilde{\rho}_n(u)\rangle\cos x_j - 2\langle \sin x, \widetilde{\rho}_n(u)\rangle \cos 2x_j \\
& \qquad + \langle \cos 2x, \widetilde{\rho}_n(u)\rangle \sin x_j + 2\langle \cos x, \widetilde{\rho}_n(u)\rangle\sin 2x_j\Big)\Bigg\}\, dB_j(u)\,,
\end{aligned}$$

by using Itô's isometry, we get

$$\begin{aligned}
E\left[\left(\mathcal{M}^{s,t}_{n,\psi_n^{(2)}}\right)^2\right] = \int_s^t \sum_{j=1}^n E\Bigg[\Bigg\{ & \frac{1}{n^{3/4}}\Big(\partial_1\psi(\cdot,\cdot)\cos x_j - \partial_2\psi(\cdot,\cdot)\sin x_j\Big) \\
& - \frac{1}{8n}\Big(\partial_{11}\psi(\cdot,\cdot)\cos x_j - \partial_{12}\psi(\cdot,\cdot)\sin x_j\Big)\Big(\langle \sin x, \widetilde{\rho}_n(u)\rangle\langle \cos 2x, \widetilde{\rho}_n(u)\rangle \\
& \qquad - \langle \cos x, \widetilde{\rho}_n(u)\rangle\langle \sin 2x, \widetilde{\rho}_n(u)\rangle\Big) \\
& + \frac{1}{8n}\Big(\partial_{12}\psi(\cdot,\cdot)\cos x_j - \partial_{22}\psi(\cdot,\cdot)\sin x_j\Big)\Big(\langle \sin x, \widetilde{\rho}_n(u)\rangle\langle \sin 2x, \widetilde{\rho}_n(u)\rangle \\
& \qquad + \langle \cos x, \widetilde{\rho}_n(u)\rangle\langle \cos 2x, \widetilde{\rho}_n(u)\rangle\Big) \\
& - \frac{1}{8n}\partial_1\psi(\cdot,\cdot)\Big(\langle \cos 2x, \widetilde{\rho}_n(u)\rangle\cos x_j - 2\langle \sin x, \widetilde{\rho}_n(u)\rangle \sin 2x_j \\
& \qquad + \langle \sin 2x, \widetilde{\rho}_n(u)\rangle \sin x_j - 2\langle \cos x, \widetilde{\rho}_n(u)\rangle\cos 2x_j\Big) \\
& - \frac{1}{8n}\partial_2\psi(\cdot,\cdot)\Big(- \langle \sin 2x, \widetilde{\rho}_n(u)\rangle\cos x_j - 2\langle \sin x, \widetilde{\rho}_n(u)\rangle \cos 2x_j \\
& \qquad + \langle \cos 2x, \widetilde{\rho}_n(u)\rangle \sin x_j + 2\langle \cos x, \widetilde{\rho}_n(u)\rangle\sin 2x_j\Big)\Bigg\}^2\Bigg]\, du
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_s^t \sum_{j=1}^n E\Bigg[\frac{1}{n^{3/2}}\Big(|\partial_1\psi(\cdot,\cdot)||\cos x_j| + |\partial_2\psi(\cdot,\cdot)||\sin x_j|\Big)^2 \\
&\qquad\qquad + \frac{1}{64n^2}\Big(|\partial_{11}\psi(\cdot,\cdot)||\cos x_j| + |\partial_{12}\psi(\cdot,\cdot)||\sin x_j|\Big)^2\cdot \\
&\quad \cdot\Big(|\langle\sin x,\widetilde{\rho}_n(u)\rangle||\langle\cos 2x,\widetilde{\rho}_n(u)\rangle| + |\langle\cos x,\widetilde{\rho}_n(u)\rangle||\langle\sin 2x,\widetilde{\rho}_n(u)\rangle|\Big)^2 \\
&\qquad\qquad + \frac{1}{64n^2}\Big(|\partial_{12}\psi(\cdot,\cdot)||\cos x_j| + |\partial_{22}\psi(\cdot,\cdot)||\sin x_j|\Big)^2\cdot \\
&\quad \cdot\Big(|\langle\sin x,\widetilde{\rho}_n(u)\rangle||\langle\sin 2x,\widetilde{\rho}_n(u)\rangle| + |\langle\cos x,\widetilde{\rho}_n(u)\rangle||\langle\cos 2x,\widetilde{\rho}_n(u)\rangle|\Big)^2 \\
&\quad + \frac{1}{64n^2}|\partial_1\psi(\cdot,\cdot)|\Big(|\langle\cos 2x,\widetilde{\rho}_n(u)\rangle||\cos x_j| + 2|\langle\sin x,\widetilde{\rho}_n(u)\rangle||\sin 2x_j| \\
&\qquad\quad + |\langle\sin 2x,\widetilde{\rho}_n(u)\rangle||\sin x_j| + 2|\langle\cos x,\widetilde{\rho}_n(u)\rangle||\cos 2x_j|\Big)^2 \\
&\quad + \frac{1}{64n^2}|\partial_2\psi(\cdot,\cdot)|\Big(|\langle\sin 2x,\widetilde{\rho}_n(u)\rangle||\cos x_j| + 2|\langle\sin x,\widetilde{\rho}_n(u)\rangle||\cos 2x_j| \\
&\qquad\qquad + |\langle\cos 2x,\widetilde{\rho}_n(u)\rangle||\sin x_j| + 2|\langle\cos x,\widetilde{\rho}_n(u)\rangle||\sin 2x_j|\Big)^2\Bigg]\,du
\end{aligned}
$$

since $\psi \in \mathcal{C}_b^3$ and so $|\partial_1\psi| \le c_1$, $|\partial_2\psi| \le c_2$, $|\partial_{11}\psi| \le c_3$, $|\partial_{22}\psi| \le c_4$ and $|\partial_{12}\psi| \le c_5$, it yields

$$
\le C\int_s^t \sum_{j=1}^n E\left[\frac{1}{n^{3/2}}(c_1+c_2)^2 + \frac{1}{16n}\left[(c_3+c_5)^2 + (c_4+c_5)^2\right] + \frac{9}{16n^{3/2}}(c_1+c_2)\right]du
$$

$$
\le CT\left[(c_1+c_2)^2 + (c_3+c_5)^2 + (c_4+c_5)^2 + 9(c_1+c_2)\right]
$$

since $t < T$; then $\mathcal{M}^t_{n,\psi_n^{(2)}}$ is uniformly integrable. ■

Now, the proof is easy to complete. $\mathcal{M}^t_{n,\psi_n^{(2)}}$ solves the martingale problem with infinitesimal generator $\mathcal{J}$, admitting a unique solution, and hence we have shown all convergent subsequences have the same limit and so the sequence itself converges to that limit.

Chapter 4

The Random Kuramoto Model

In this chapter we consider the Kuramoto model with the addition of a random site-dependent field, which acts as random environment.

We consider N sites and we associate with each of them a rotator on $[0, 2\pi]$ and a frequency value, that we choose to be a dichotomic random variable. Although the limiting dynamics and the critical point are known for a general distribution of the field, the analysis of critical fluctuations involves technical issues that we do not fully control in the general case. For this reason we restrict to the simplest case of a field with two values. We start with a non-reversible Markovian dynamics for the N-particle system, where the rotators evolve depending on the gradient of the Hamiltonian felt by the particles. It is an interacting diffusion system with a *mean-field* Hamiltonian that depends on the random medium we introduced. In this model there is no spatial geometry in the space of the configurations, since it is subject to a mean-field interaction, meaning that each particle interacts with all the others in the same way.

An infinite dimensional order parameter is necessary to describe the system. Being based on a Large Deviation Principle, we compute the differential equations which drive its evolution in the infinite particle limit (McKean-Vlasov equations) and we derive a Law of Large Number it obeys. Depending on the parameters, we can see there exists phase transition. We state these results for completeness; they are already known in literature. The statements about the McKean-Vlasov limit of the dynamics and the existence of a phase transition are got by [DPdH95]

and [dH00].
Our main result is the infinite particle limit of the critical fluctuation flow. With regard to the critical fluctuation flow – besides an appropriate scaling of the space – it require a rescaling of the time in order to keep track of long time fluctuations of the critical direction (critical slowing down). As a result, only the critical structure survives the new scaling, and in the limit, the critical fluctuation process is a lower dimensional process compared with the non-critical one. The fluctuations are two-dimensional at the critical point. In fact, we prove that, when the size of the system grows towards infinity, a two-dimensional process converges (in the sense of weak convergence of stochastic processes) to a non-Gaussian process, while all the others collapse.

4.1 Description of the Model

Let $\mathscr{S} = \{-1, +1\}$ and $\underline{\eta} = (\eta_j)_{j=1}^N \in \mathscr{S}^N$ be a sequence of independent, identically distributed, symmetric, Bernoulli random variables defined on some probability space $(\Omega, \mathcal{F}, P)$. That is, $P(\eta_j = -1) = P(\eta_j = +1) = \frac{1}{2}$, for any j. We indicate with μ their common law. Given a configuration $\underline{x} = (x_j)_{j=1}^N \in [0, 2\pi]^N$ and a realization of the random environment $\underline{\eta}$, we can define the Hamiltonian $H_N(\underline{x}, \underline{\eta}) : [0, 2\pi]^N \times \mathbb{R}^N \longrightarrow \mathscr{S}$ as

$$H_N(\underline{x}, \underline{\eta}) = -\frac{\theta}{2N} \sum_{j,k=1}^{N} \cos(x_k - x_j) + \omega \sum_{j=1}^{N} \eta_j x_j \,, \tag{4.1}$$

where x_j is the position of the rotator at site j and $\omega\eta_j$, with $\omega > 0$, can be interpreted as its own frequency. Let θ, positive parameter, be the coupling strength. For a fixed realization of $\underline{\eta}$, think of $\underline{x} \longrightarrow H_N(\underline{x}, \underline{\eta})$ as a Hamiltonian in the components x_j with an inhomogeneous mean-field interaction parametrized by the components η_j. With the expression "mean-field" we mean the sites interact all each other in the same way.
Let us define the dynamics we consider. For given $\underline{\eta}$, $\underline{x}(t) = (x_j(t))_{j=1}^N$, with t belonging to a generic time interval $[0, T]$, where T is fixed, describes a N-rotator system evolving as a continuous time Markov chain on $[0, 2\pi]^N$, with infinitesimal

generator L_N acting on functions $f : [0, 2\pi]^N \longrightarrow \mathbb{R}$ as follows:

$$L_N f(\underline{x}) = \frac{1}{2}\sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_j^2}(\underline{x}) + \sum_{j=1}^{N} \frac{\partial H_N}{\partial x_j}(\underline{x}, \underline{\eta})$$

$$= \frac{1}{2}\sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_j^2}(\underline{x}) + \sum_{j=1}^{N} \left\{ \omega\eta_j + \frac{\theta}{N}\sum_{k=1}^{N} \sin(x_k - x_j) \right\} \frac{\partial f}{\partial x_j}(\underline{x}) . \tag{4.2}$$

Consider the complex quantity

$$r_N e^{i\Psi_N} = \frac{1}{N}\sum_{j=1}^{N} e^{ix_j} , \tag{4.3}$$

where $0 \leq r_N \leq 1$ measures the phase coherence of the rotators and Ψ_N measures the average phase. We can reformulate the expression of the infinitesimal generator (4.2) in terms of (4.3):

$$L_N f(\underline{x}) = \frac{1}{2}\sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_j^2}(\underline{x}) + \sum_{j=1}^{N} \{\omega\eta_j + \theta r_N \sin(\Psi_N - x_j)\} \frac{\partial f}{\partial x_j}(\underline{x}) . \tag{4.4}$$

The expressions (4.1) and (4.4) describe a system of mean-field coupled rotators, each with its own frequency and subject to diffusive dynamics. The two terms in the Hamiltonian have different effects: the first one tends to synchronize the rotators, while the second one tends to make each of them rotate at its own frequency.

For simplicity, the initial condition $\underline{x}(0)$ is assumed to have product distribution $\lambda^{\otimes N}$, with λ probability measure on $[0, 2\pi]$ with finite second moment. The quantity $x_j(t)$ represents the time evolution on $[0, T]$ of j-th rotator; it is the trajectory of the single j-th rotator in time. The space of all these paths is $\mathcal{C}[0, T]$, which is the space of the continuous function from $[0, T]$ to $[0, 2\pi]$, endowed with the uniform topology.

For given $\underline{\eta}$, $\underline{x}(t) = (x_j(t))_{j=1}^N$ turns out to be the system of N interacting diffusions evolving according to the Itô differential equations

$$dx_j(t) = [\omega\eta_j + \theta r_N \sin(\Psi_N - x_j)]\, dt + dB_j(t) , \tag{4.5}$$

where $\{B_j(t) : t > 0, j = 1, \ldots, N\}$ is a system of independent Standard Brownian motions on $[0, 2\pi]$.

4.2 Limiting Dynamics

We now derive the dynamics of the process (4.2), in the limit as $N \longrightarrow +\infty$, in a fixed time interval $[0,T]$, via a Large Deviation approach. Later, the large time behavior of the limiting dynamics will be studied.

For completeness, we report all the statements that allow us to deduce the dynamics of the model in the infinite volume limit, but we omit their proofs, since they are a particular application of a more general study on interacting diffusions developed in [DPdH96] and [dH00].

So, let $(x_j([0,T]))_{j=1}^N \in (\mathcal{C}[0,T])^N$ denote a path of the system in the time interval $[0,T]$, with T positive and fixed. If $f(x_j([0,T]))$ is a function of the trajectory of a single rotator, we are interested in the asymptotic behavior of *empirical averages* of the form

$$\frac{1}{N}\sum_{j=1}^{N} f(x_j([0,T])) =: \int f d\rho_N \,,$$

where $\{\rho_N\}_{N\geq 1}$ is the sequence of *empirical measures*

$$\rho_N := \frac{1}{N}\sum_{j=1}^{N} \delta_{(x_j([0,T]),\eta_j)} \,.$$

Remark 4.2.1. The measure ρ_N is a joint measure of the process and the environment.

We may think of ρ_N as a random element of $\mathcal{M}_1(\mathcal{C}[0,T]\times\mathscr{S})$, the space of probability measures on $\mathcal{C}[0,T]\times\mathscr{S}$ endowed with the weak convergence topology. First, we want to determine the weak limit of ρ_N in $\mathcal{M}_1(\mathcal{C}[0,T]\times\mathscr{S})$ when N grows to infinity; i.e. for $f \in \mathcal{C}_b$ we look for $\lim_{N\to+\infty}\int f d\rho_N$. It corresponds to a Law of Large Number with the limit being a deterministic measure. Being an element of $\mathcal{M}_1(\mathcal{C}[0,T]\times\mathscr{S})$, such a limit can be viewed as a stochastic process, which describes the dynamics of the system in the infinite volume limit.

4.2.1 Empirical Measure and Large Deviations

Let $W \in \mathcal{M}_1(\mathcal{C}[0,T])$ denote the law of a standard Brownian motion starting with initial condition λ. By $W^{\otimes N}$ we mean the product of N copies of W, which represents the law of the solution of the system (4.5) when $H_N(\underline{x},\underline{\eta}) \equiv 0$. Moreover, we shall write $P_N^{\underline{\eta}}$ the law of $\underline{x}([0,T]) = (\underline{x}(t))_{t\in[0,T]}$, the process with infinitesimal generator (4.2) and initial distribution $\lambda^{\otimes N}$, for a given $\underline{\eta}$.

Consider $Q \in \mathcal{M}_1(\mathcal{C}[0,T]\times\mathscr{S})$, if $\Pi_t Q$ indicates the marginal distribution of Q at time t, we have

$$r_{\Pi_t Q} e^{i\Psi_{\Pi_t Q}} := \int_{[0,2\pi]\times\mathscr{S}} e^{ix}\, \Pi_t Q(dx, d\eta).$$

For a given path $x([0,T]) \in \mathcal{C}[0,T]$, we define

$$\begin{aligned} F(Q) = \int Q(dx[0,T], d\eta)\Bigg\{ -\frac{1}{2}\int_0^T dt \Bigg[\Big(\omega + \int Q(dy[0,T], d\varsigma)\sin(y(t)-x(t))\Big)^2 \\ + \int Q(dy[0,T], d\varsigma)\cos(y(t)-x(t))\Bigg] \\ -\frac{1}{2}\int Q(dy[0,T], d\varsigma)[\cos(y(T)-x(T)) - \cos(y(0)-x(0))]\Bigg\} \end{aligned} \tag{4.6}$$

We can obtain a representation of $P_N^{\underline{\eta}}$ in terms of ρ_N, as follows:

Lemma 4.2.1. *For a fixed realization* $\underline{\eta}$,

$$\frac{dP_N^{\underline{\eta}}}{dW^{\otimes N}}(\underline{x}([0,T])) = \exp[NF(\rho_N(\underline{x}([0,T]), \underline{\eta}))]$$

where, for $Q \in \mathcal{M}_1(\mathcal{C}[0,T]\times\mathscr{S})$, $F(Q)$ *is expressed by* (4.6).

Lemma 4.2.1 allows us to deduce a Large Deviation Principle for ρ_N, from which we can derive its asymptotic behavior as $N \longrightarrow +\infty$.

Define

$$\mathcal{P}_N(\cdot) := \int \mu^{\otimes N}(d\underline{\eta}) P_N^{\underline{\omega}}(\rho_N \in \cdot)\,,$$

which is an element of $\mathcal{M}_1(\mathcal{M}_1(\mathcal{C}[0,T]\times\mathscr{S}))$ and represents the law of ρ_N under the joint distribution of the process and the environment.

If $Q \in \mathcal{M}_1(\mathcal{C}[0,T]\times\mathscr{S})$ we denote by

$$H(Q|W\otimes\mu) := \begin{cases} \int dQ \log\frac{dQ}{d(W\otimes\mu)} & \text{if} \quad Q \ll W\otimes\mu \quad \text{and} \quad \log\frac{dQ}{d(W\otimes\mu)} \in L^1(Q) \\ \\ +\infty & \text{otherwise} \end{cases}$$

the relative entropy between Q and $W\otimes\mu$.

Proposition 4.2.1. *The laws* $\{\mathcal{P}_N\}_{N\geq 1}$ *of* ρ_N *(under the joint distribution of the process and the medium) obey a Large Deviation Principle with rate function*

$$I(Q) := H(Q|W \otimes \mu) - F(Q)$$

(mind Definitions 1.2.1 and 1.2.2).

4.2.2 McKean-Vlasov Equation

Given $Q \in \mathcal{M}_1(\mathcal{C}[0,T] \times \mathscr{S})$ and $\eta \in \mathscr{S}$, we can associate with Q a Markov process on $\mathscr{S}$ with law $P^{\eta,Q}$, initial distribution λ and time-dependent infinitesimal generator

$$\mathcal{L}_t^{\eta,Q} f(x) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x) + [\omega\eta + \theta r_{\Pi_t Q} \sin(\Psi_{\Pi_t Q} - x)]\frac{\partial f}{\partial x}(x)\,,$$

acting on $f : [0, 2\pi] \longrightarrow \mathbb{R}$.
It can be proved

Proposition 4.2.2. *For every* $Q \in \mathcal{M}_1(\mathcal{C}[0,T] \times \mathscr{S})$ *such that* $I(Q) < +\infty$*,*

$$I(Q) = H(Q|P^Q)\,,$$

where $P^Q \in \mathcal{M}_1(\mathcal{C}[0,T] \times \mathscr{S})$ *is defined by*

$$P^Q(dx[0,T], d\eta) = P^{\eta,Q}(dx[0,T])\mu(d\eta).$$

Theorem 4.2.1. *Suppose that the initial distribution of the Markov process* $(\underline{x}(t))_{t\geq 0}$ *with generator* (4.4) *is such that the random variables* $(x_j(0))_{j=1}^N$ *are independent and identically distributed with law* λ*. Then the equation* $I(Q) = 0$ *admits a unique solution* $Q_* \in \mathcal{M}_1(\mathcal{C}[0,T] \times \mathscr{S})$*, such that its marginals* $q_t^\eta = \Pi_t Q_*^\eta \in \mathcal{M}_1([0, 2\pi])$ *are weak solutions of the nonlinear McKean-Vlasov equation*

$$\begin{cases} \frac{\partial q_t^\eta}{\partial t} = \mathcal{L}^\eta q_t^\eta & (t \in [0,T],\ \eta \in \mathscr{S}) \\ \\ q_0^\eta = \lambda \end{cases} \tag{4.7}$$

where, for all the pairs $(x,\eta) \in [0, 2\pi] \times \mathscr{S}$*, the operator* $\mathcal{L}^\eta$ *acts*

$$\mathcal{L}^\eta q_t^\eta(x) = \frac{1}{2}\frac{\partial^2 q_t^\eta}{\partial x^2}(x) - \frac{\partial}{\partial x}\{[\omega\eta + \theta r_{q_t} \sin(\Psi_{q_t} - x)]q_t^\eta(x)\}\,, \tag{4.8}$$

with $q_t^\eta(0) = q_t^\eta(2\pi)$ *and* q_t *defined by*

$$q_t(x) = \int_{\mathscr{S}} q_t^\eta(x)\mu(d\eta).$$

Moreover, with respect to a metric $d(\cdot,\cdot)$ *inducing the weak topology,* $\rho_N \longrightarrow Q_*$ *in probability with exponential rate, i.e.* $\mathcal{P}_N\{d(\rho_N, Q_*) > \varepsilon\}$ *is exponentially small in* N*, for each* $\varepsilon > 0$.

Remark 4.2.2. For μ-almost surely all η, Q_*^η is the law of a time-inhomogeneous diffusion process on $[0, 2\pi]$ with generator

$$\mathcal{L}_t^{\eta,q_t} f(x) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x) + [\omega\eta + \theta r_{q_t}\sin(\Psi_{q_t} - x)]\frac{\partial f}{\partial x}(x)\,.$$

4.2.3 Stationary Solution(s)

The equation (4.7) describes the behavior of the system governed by generator (4.4) in the infinite volume limit. We are interested in the detection of the t-stationary solution(s) of this equation and in the study of the large time dynamics of it (them). We recall that to be t-stationary solution for (4.7) means to satisfy the equation $\mathcal{L}^\eta q^\eta = 0$ for every t.
Since μ is symmetric and the operator $\mathcal{L}^\eta$ preserves evenness, we can suppose the average phase $\Psi_{q_t} \equiv 0$, without loss of generality.
Hence, every equilibrium probability distribution is the solution of

$$\frac{1}{2}\frac{\partial^2 q^\eta}{\partial x^2}(x) - \frac{\partial}{\partial x}\{[\omega\eta - \theta r_q \sin x]q^\eta(x)\} = 0\,, \tag{4.9}$$

with the boundary condition $q^\eta(0) = q^\eta(2\pi)$ and for our model is characterized as follows.

Lemma 4.2.2. *Every equilibrium distribution for the nonlinear Markov process given by* (4.7) *is of the form:*

$$\begin{aligned} q_*^\eta(x) = (Z_*^\eta)^{-1} \cdot 2(\omega\eta x + \theta r_* \cos x) \Big[& e^{4\pi\omega\eta} \int_0^{2\pi} e^{-2(\omega\eta x + \theta r_* \cos x)} dx \\ & + (1 - e^{4\pi\omega\eta}) \int_0^x e^{-2(\omega\eta y + \theta r_* \cos y)} dy \Big]\,, \end{aligned} \tag{4.10}$$

where Z_*^η *is a normalizing factor and the variable* r_* *must satisfy the self-consistency relation*

$$r_{q_*} := r_* = \int_{[0,2\pi]\times\mathscr{S}} e^{ix}\, q_*^\eta(dx)\mu(d\eta)\,. \tag{4.11}$$

Remark 4.2.3. There is a one-to-one correspondence between equilibrium distributions and solutions of the self-consistency equation (4.11).

Remark 4.2.4. Note that $r_* \equiv 0$ is always a solution of (4.11), for all the choices of θ and μ. In this case the stationary distribution reduces to:

$$q_*^{\eta,0}(x) := \frac{1}{2\pi} \qquad \text{for all } x \in [0, 2\pi]. \tag{4.12}$$

Solutions with $r_* = 0$ are called *incoherent*, while those with $r_* > 0$ are called *synchronized*. The next theorem shows that if θ exceeds a μ-dependent threshold a synchronized solution is always possible.

Theorem 4.2.2. *Consider the equation* (4.11) *and define* $\theta_c = 1 + 4\omega^2$*. Then,*

(a) if $\theta \leq \theta_c$*, the unique solution is* $r_* = 0$*;*

(b) if $\theta > \theta_c$*, at least one synchronized solution is possible.*

Proof. We refer to [DPdH95] and [dH00] for a detailed proof, concerning the complete phase diagram of the system. ■

4.3 Critical Dynamics ($\theta = 1 + 4\omega^2$)

We are going to consider the critical dynamics of the system, in other words the long-time behavior of the fluctuations in the threshold case, when $\theta = 1 + 4\omega^2$. The size of Normal fluctuations must be further rescaled (in space and time), because their size around the deterministic limit increases in time. In this case we will obtain non-Normal fluctuations, solutions of a certain stochastic differential equation to be determined.

First of all, we need to locate the critical direction in the infinite dimensional space of the order parameters. In the rest of the section, we will consider $\theta = 1 + 4\omega^2$ and let us assume that the initial condition λ is a product measure such that

$$q_0(dx, d\eta) = q_*^0(dx, d\eta) = \frac{1}{2\pi}\, dx\, \mu(d\eta)$$

and so

$$q_t(dx, d\eta) = q_*^0(dx, d\eta) = \frac{1}{2\pi}\, dx\, \mu(d\eta)\,,$$

for every value of $t \geq 0$, since we are in stationary conditions.

We consider the linearization of the operator $\mathcal{L}^\eta$, given by (4.8), at the equilibrium distribution $q_*^{\eta,0}(x)$, which is

$$\begin{aligned}\mathfrak{L}^\eta \phi(x,\eta) = \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(x,\eta) + \omega\eta\frac{\partial \phi}{\partial x}(x,\eta) \\ + (1+4\omega^2)\Big[\cos x \langle \cos y\, \phi(y,\eta), q_*^0(dy,d\eta)\rangle \\ + \sin x \langle \sin y\, \phi(y,\eta), q_*^0(dy,d\eta)\rangle\Big]\,, \quad (4.13)\end{aligned}$$

where we have denoted $\langle f_1, f_2\rangle := \int_{[0,2\pi]\times\mathscr{S}} f_1(x,\eta) f_2(x,\eta) dx\, d\eta$.

Remark 4.3.1. The operator $\mathfrak{L}^\eta$, defined by (4.13), is not self-adjoint with respect to the measure $q_*^{\eta,0}$.

Lemma 4.3.1. *The null space of the operator $\mathfrak{L}^\eta$, defined by* (4.13)*, is spanned by the functions $\alpha_1(x,\eta) := \sin x + 2\omega\eta \cos x$ and $\alpha_2(x,\eta) := \cos x - 2\omega\eta \sin x$.*

Proof. If $\varphi(\cdot,\cdot)$ belongs to the null space of $\mathfrak{L}^\eta$, then $\mathfrak{L}^\eta \varphi = 0$. Therefore, we require that

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 \varphi}{\partial x^2}(x,\eta) + \omega\eta\frac{\partial \varphi}{\partial x}(x,\eta) \\ + (1+4\omega^2)\Bigg[\cos x\, \frac{1}{2\pi}\int_{[0,2\pi]\times\mathscr{S}} \cos y\, \varphi(y,\varsigma)\, q_*^0(dy,d\varsigma) \\ + \sin x\, \frac{1}{2\pi}\int_{[0,2\pi]\times\mathscr{S}} \sin y\, \varphi(y,\varsigma)\, q_*^0(dy,d\varsigma)\Bigg] = 0\,. \quad (4.14)\end{aligned}$$

We solve the ordinary differential equation (4.14). Having defined

$$A := \frac{1}{2\pi}\int_{[0,2\pi]\times\mathscr{S}} \cos y\, \varphi(y,\varsigma)\, q_*^0(dy,d\varsigma) \quad (4.15a)$$

and

$$B := \frac{1}{2\pi}\int_{[0,2\pi]\times\mathscr{S}} \sin y\, \varphi(y,\varsigma)\, q_*^0(dy,d\varsigma)\,, \quad (4.15b)$$

the solution is $\varphi(x,\eta) = 2(B - 2A\omega\eta)\sin x + 2(A + 2B\omega\eta)\cos x$; this function yields a solution of (4.14) provided that it satisfies the self-consistency relations (4.15a) and (4.15b), but it does for every value of A and B. Then the two directions which generate the kernel are $\alpha_1(x,\eta) := \sin x + 2\omega\eta\cos x$ and $\alpha_2(x,\eta) := \cos x - 2\omega\eta \sin x$. ■

Remark 4.3.2. In the case that $\theta \neq 1 + 4\omega^2$, the unique value for which the self-consistency relations in (4.15a) and (4.15b) are satisfied is $A = B = 0$, meaning that at the critical point the kernel of the operator $\mathfrak{L}^\eta$ is two-dimensional, while it is a trivial set for all the other values of the parameter θ.

Remark 4.3.3. The null space of the operator $\mathfrak{L}^\eta$ represents the critical direction for our model.

We want to analyze the spectrum of the operator $\mathfrak{L}^\eta$, defined by (4.13). Let us consider the linearization of the operator (4.8) for general values of the parameter θ:

$$\begin{aligned}\mathfrak{L}^\eta_\theta \phi(x,\eta) = \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(x,\eta) + \omega\eta\frac{\partial \phi}{\partial x}(x,\eta) \\ + \theta\Big[\cos x\langle \cos y\, \phi(y,\eta), q^0_*(dy,d\eta)\rangle \\ + \sin x\langle \sin y\, \phi(y,\eta), q^0_*(dy,d\eta)\rangle\Big]\,. \quad (4.16)\end{aligned}$$

Its spectral properties have been investigated in [SM91], [BNS92] and [DPdH95] for several classes of distributions of the random field. We recall some results of these general cases, by which the description of the spectrum of operator $\mathfrak{L}^\eta$ follows.

Lemma 4.3.2. *The spectrum of the operator $\mathfrak{L}^\eta_\theta$, defined by* (4.16)*, is described by the set*

$$\mathrm{Spec}(\mathfrak{L}^\eta_\theta) = \left\{-\frac{1}{2} + \frac{\theta}{4} \pm \frac{1}{4}\sqrt{\theta^2 - 16\omega^2}\right\} \cup \left\{-\frac{k^2}{2} + ik\omega\eta,\; k \in \mathbb{Z}\setminus\{-1,0,+1\}\right\}.$$

Proof. If we denote by $\mathcal{F}[\phi(x,\eta)]$ the Fourier transform of $\phi(x,\eta)$ in $L^2(q^{\eta,0}_*)$, we have to solve

$$\mathcal{F}[\mathfrak{L}^\eta_\theta \phi(x,\eta)] = \lambda \mathcal{F}[\phi(x,\eta)]\,,$$

which translates into

$$\begin{aligned}\left(-\frac{k^2}{2} + ik\omega\eta - \lambda\right)\hat{\phi}(k,\eta) = \\ = \frac{\theta}{2}\left[\delta_1(k)\int_{\mathscr{S}} \hat{\phi}(1,\eta)\mu(d\eta) + \delta_{-1}(k)\int_{\mathscr{S}} \hat{\phi}(-1,\eta)\mu(d\eta)\right], \quad (4.17)\end{aligned}$$

where $\hat{\phi}(k,\eta) := \frac{1}{2\pi}\int_0^{2\pi} \phi(x,\eta)\, e^{-ikx}\, dx$.

Equation (4.17) shows that, for a fixed value of η and for $k \in \mathbb{Z}\setminus\{-1,0,+1\}$, the numbers $\lambda = -\frac{k^2}{2} + ik\omega\eta$ are eigenvalues for the operator $\mathfrak{L}^\eta_\theta$, defined by (4.16). In the case $k = \pm 1$ the right-hand side of (4.16) does not vanish and the equation has two solutions

$$\lambda_\pm = -\frac{1}{2} + \frac{\theta}{4} \pm \frac{1}{4}\sqrt{\theta^2 - 16\omega^2}\,. \tag{4.18}$$

■

From (4.17), we can deduce that

$$\begin{array}{lll} \text{if } \theta \le 1 : & \Re e\,\lambda^+, \Re e\,\lambda^- < 0 & \text{for all the values of } \omega \\ \text{if } 1 < \theta < \theta_1 = 2 : & \Re e\,\lambda^+ < 0 & \text{if and only if } \theta < \theta_c = 1 + 4\omega^2 \\ \text{if } \theta = \theta_1 = 2 : & \Re e\,\lambda^+ = \Re e\,\lambda^- = 0 & \text{for all the values of } \omega \\ \text{if } \theta > \theta_1 = 2 : & \Re e\,\lambda^+ > 0 & \text{for all the values of } \omega \end{array}$$

and then, the inchoerent solution $r_* \equiv 0$ is linearly stable when $\theta < \theta_1 \wedge \theta_c$, neutrally stable when $\theta = \theta_1 \wedge \theta_c$ and unstable when $\theta > \theta_1 \wedge \theta_c$.

We choose ω such that $\theta_c = \theta_1 \wedge \theta_c$, in other words the critical temperature for the system is really θ_c. We assume $\omega < \frac{1}{2}$.

Corollary 4.3.1. *Let $\omega < \frac{1}{2}$ and fix a value of η. The spectrum of the operator $\mathfrak{L}^\eta$, defined by* (4.13)*, is described by the set* $\mathrm{Spec}(\mathfrak{L}^\eta) = \left\{0, -\frac{1}{2} + 2\omega^2\right\} \cup \left\{-\frac{k^2}{2} + ik\omega\eta,\ k \in \mathbb{Z} \setminus \{-1, 0, +1\}\right\}$.

Proof. The statement follows by Lemma 4.3.2 setting $\theta = 1 + 4\omega^2$. ■

We want to describe the action of the infinitesimal generator of the critical fluctuation flow

$$\widetilde{\rho}_N(t, dx, d\eta) = N^{1/4}\left[\rho_N(N^{1/2}t, dx, d\eta) - \frac{1}{2\pi}\,dx\,\mu(d\eta)\right]$$

on the family of functions of the form $\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle\phi_m, \widetilde{\rho}_N\rangle)$, where

$$\psi : \mathbb{R}^m \longrightarrow \mathbb{R}, \qquad \psi \in \mathcal{C}^3_b(\mathbb{R}^m)$$

and

$$\phi_j : [0, 2\pi] \times \mathscr{S} \longrightarrow \mathbb{R}, \qquad \phi_j \in \mathcal{C}^2([0, 2\pi] \times \mathscr{S})\,,$$

for $j = 1, \ldots, m$. Since we must consider fluctuations around $q^0_*(\cdot, \cdot)$, that is, we must consider the "centered" process, we restrict our attention to functions ϕ_j with

$$\int_{[0,2\pi]\times\mathscr{S}} \phi_j(x, \eta) q^0_*(dx, d\eta) = 0, \qquad j = 1, \ldots, m\,.$$

Then, for this kind of functions it yields

$$\langle \phi_j, \widetilde{\rho}_N(t)\rangle = N^{1/4}\langle \phi_j, \rho_N(N^{1/2}t)\rangle \,,$$

for $j = 1, \ldots, m$.

Lemma 4.3.3. *For $t \in [0, T]$, the critical fluctuation flow*

$$\widetilde{\rho}_N(t, dx, d\eta) = N^{1/4}\left[\rho_N(N^{1/2}t, dx, d\eta) - \frac{1}{2\pi}\, dx\, \mu(d\eta)\right] \tag{4.19}$$

is a Markov process whose infinitesimal generator $\mathcal{J}_N$ satisfies:

$$\begin{aligned}\mathcal{J}_N\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle\phi_m, \widetilde{\rho}_N\rangle) &= \\ &= \left[N^{1/2}L_1 + N^{1/4}L_2 + L_3 + N^{-1/4}L_4\right]\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle\phi_m, \widetilde{\rho}_N\rangle),\end{aligned} \tag{4.20}$$

with

$$L_1\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle\phi_m, \widetilde{\rho}_N\rangle) = \sum_{j=1}^{m} \frac{\partial\psi}{\partial y_j}\langle \mathfrak{L}^\eta \phi_j, \widetilde{\rho}_N\rangle \tag{4.21}$$

$$\begin{aligned}L_2\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle\phi_m, \widetilde{\rho}_N\rangle) = (1 + 4\omega^2)\sum_{j=1}^{m} \frac{\partial\psi}{\partial y_j}[&\langle \sin x, \widetilde{\rho}_N\rangle\langle \cos x\, \phi_j'(x,\eta), \widetilde{\rho}_N\rangle \\ &- \langle \cos x, \widetilde{\rho}_N\rangle\langle \sin x\, \phi_j'(x,\eta), \widetilde{\rho}_N\rangle]\end{aligned} \tag{4.22}$$

$$L_3\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle\phi_m, \widetilde{\rho}_N\rangle) = \frac{1}{2}\sum_{h,j=1}^{m} \frac{\partial^2\psi}{\partial y_h \partial y_j}\langle \phi_h'(x,\eta)\phi_j'(x,\eta), q_*^0\rangle \tag{4.23}$$

$$L_4\psi(\langle\phi_1, \widetilde{\rho}_N\rangle, \ldots, \langle\phi_m, \widetilde{\rho}_N\rangle) = \frac{1}{2}\sum_{h,j=1}^{m} \frac{\partial^2\psi}{\partial y_h \partial y_j}\langle \phi_h'(x,\eta)\phi_j'(x,\eta), \widetilde{\rho}_N\rangle \,, \tag{4.24}$$

where by the notation ϕ' we mean the derivation with respect to the variable x and the operator $\mathfrak{L}^\eta$ is the linear operator given by (4.13).

Proof. Just a very long and tedious computation. ∎

Theorem 4.3.1. *Assume $\omega < \frac{1}{2}$. For $t \in [0, T]$, if we consider the infinite-dimensional critical fluctuation process*

$$\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle, \{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2},$$

$$\{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \{\langle \eta \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \{\langle \eta \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2},$$

then, as $N \longrightarrow +\infty$*,* $\{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$*,* $\{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$*,* $\{\langle \eta \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$*,* $\{\langle \eta \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2} \longrightarrow 0$ *in the sense of Proposition 1.4.1 and the process* $(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle)$ *converges, in the sense of weak convergence of stochastic processes, to a limiting non-Gaussian process* $(X(t), Y(t))$*, which is the unique solution of the following stochastic differential equation:*

$$\begin{cases} dX(t) = -\dfrac{1}{8}\dfrac{(1+4\omega^2)^2}{(1-4\omega^2)^3}\, X(t)\, [X^2(t)+Y^2(t)]\, dt + \sqrt{\dfrac{1+4\omega^2}{2}}\, dB^{(1)}(t) \\ \\ dY(t) = -\dfrac{1}{8}\dfrac{(1+4\omega^2)^2}{(1-4\omega^2)^3}\, Y(t)\, [X^2(t)+Y^2(t)]\, dt + \sqrt{\dfrac{1+4\omega^2}{2}}\, dB^{(2)}(t) \end{cases}$$

with initial condition $X(0) = Y(0) = 0$ *and where* $B^{(1)}$ *and* $B^{(2)}$ *are two independent Standard Brownian motions.*

4.3.1 Proof of the Theorem 4.3.1

Let us denote by $\{\tau_N^M\}_{N\geq 1}$ a family of stopping times, defined as

$$\begin{aligned} \tau_N^M := \inf_{t\geq 0} \{ & |\langle \sin x, \widetilde{\rho}_N(t)\rangle| \geq M \quad \text{or} \quad |\langle \cos x, \widetilde{\rho}_N(t)\rangle| \geq M \\ & \text{or} \quad |\langle \eta \sin x, \widetilde{\rho}_N(t)\rangle| \geq M \quad \text{or} \quad |\langle \eta \cos x, \widetilde{\rho}_N(t)\rangle| \geq M \\ & \text{or} \quad |\langle \sin kx, \widetilde{\rho}_N(t)\rangle| \geq M \quad \text{for at least a value of } k = 2,3,\dots \\ & \text{or} \quad |\langle \cos kx, \widetilde{\rho}_N(t)\rangle| \geq M \quad \text{for at least a value of } k = 2,3,\dots \\ & \text{or} \quad |\langle \eta \sin kx, \widetilde{\rho}_N(t)\rangle| \geq M \quad \text{for at least a value of } k = 2,3,\dots \\ & \text{or} \quad |\langle \eta \cos kx, \widetilde{\rho}_N(t)\rangle| \geq M \quad \text{for at least a value of } k = 2,3,\dots \}, \end{aligned}$$

where M is a positive constant. We are interested in introducing such sequence of stopping times because in this way the processes $\langle \sin x, \widetilde{\rho}_N(t)\rangle$, $\langle \cos x, \widetilde{\rho}_N(t)\rangle$ and $\langle \eta \sin x, \widetilde{\rho}_N(t)\rangle$, $\langle \eta \cos x, \widetilde{\rho}_N(t)\rangle$ and $\{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$, $\{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$ and $\{\langle \eta \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$, $\{\langle \eta \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$ result to be bounded in the time interval $[0, T\wedge \tau_N^M]$.

By standard argument on collapsing processes (see Proposition 1.4.1 and Lemma 3.3.6), it is easy to prove that for $t \in [0, T\wedge\tau_N^M]$ the directions $\{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$,

$\{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$, $\{\langle \eta \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$ and $\{\langle \eta \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$ collapse. It means that, if we consider the norm $\|\widetilde{\rho}_N\|_r$, defined by

$$\|\widetilde{\rho}_N\|_r^2 := \sum_{k=2}^{+\infty} \frac{1}{(1+k^2)^r} [\langle \sin kx, \widetilde{\rho}_N\rangle^2 + \langle \cos kx, \widetilde{\rho}_N\rangle^2 + \langle \eta \sin kx, \widetilde{\rho}_N\rangle^2 + \langle \eta \cos kx, \widetilde{\rho}_N\rangle^2],$$

where $r > 0$, then there exist constants N_0, C, $d > 2$, $\kappa_N := \kappa(N)$ and two increasing sequences $\{\alpha_N\}_{N\geq 1}$, $\{\beta_N\}_{N\geq 1}$ satisfying (3.23)–(3.26) and such that for every $\varepsilon > 0$ the following property is true

$$\sup_{N\geq N_0} P\left\{ \sup_{0\leq t\leq T\wedge \tau_N^M} \|\widetilde{\rho}_N(t)\|_r^2 > C\left(\kappa_N^{1/d}\alpha_N^{-1} \vee \kappa_N^{-1}\alpha_N\right)\right\} \leq \varepsilon\,. \tag{4.25}$$

Hence, we obtain $\{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$, $\{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$, $\{\langle \eta \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}$, $\{\langle \eta \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2} \longrightarrow 0$, as $N \longrightarrow +\infty$.
The computations we should do to prove these processes converge to zero in probability are similar to those we did in Subsection 3.3.1 to prove the process representing the non-critical directions of the homogeneous Kuramoto Model collapses. Thus, we omit this proof and *we focus only on the critical directions* $\alpha_1(x,\eta) := \sin x + 2\omega\eta \cos x$ *and* $\alpha_2(x,\eta) := -2\omega\eta \sin x + \cos x$, *assuming all the others vanish*. We apply the generator (4.20) to a function of the only critical directions, leaving all the terms coming from those processes we know collapsing in the infinite volume limit.

We want to find the expression of the limiting operator of the infinitesimal generator $\mathcal{J}_N$, as N grows to infinity. We choose

$$\begin{aligned}
&\psi\Big(\langle \sin x, \widetilde{\rho}_N(t)\rangle, \langle \cos x, \widetilde{\rho}_N(t)\rangle, \{\langle \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2},\\
&\langle \eta \sin x, \widetilde{\rho}_N(t)\rangle, \langle \eta \cos x, \widetilde{\rho}_N(t)\rangle, \{\langle \eta \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \{\langle \eta \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}\Big) =\\
&\qquad = \psi(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle)
\end{aligned}$$

and we apply the operator $\mathcal{J}_N$. Since $\ker \mathfrak{L}^\eta = \operatorname{span}\{\alpha_1(x,\eta), \alpha_2(x,\eta)\}$, referring to (4.20)-(4.24), we obtain

$$\mathcal{J}_N \psi(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle) =$$

$$= N^{1/4}(1+4\omega^2)\Big\{\partial_1\psi(\cdot,\cdot)[\langle \sin x, \widetilde{\rho}_N\rangle\langle \alpha_2(x,\eta)\cos x, \widetilde{\rho}_N\rangle - \langle \cos x, \widetilde{\rho}_N\rangle\langle \alpha_2(x,\eta)\sin x, \widetilde{\rho}_N\rangle] + \partial_2\psi(\cdot,\cdot)[\langle \cos x, \widetilde{\rho}_N\rangle\langle \alpha_1(x,\eta)\sin x, \widetilde{\rho}_N\rangle - \langle \sin x, \widetilde{\rho}_N\rangle\langle \alpha_1(x,\eta)\cos x, \widetilde{\rho}_N\rangle]\Big\}$$

$$+ \frac{1}{2}\partial_{11}\psi(\cdot,\cdot)\langle[\alpha_2(x,\eta)]^2, q_*^0\rangle + \frac{1}{2}\partial_{22}\psi(\cdot,\cdot)\langle[\alpha_1(x,\eta)]^2, q_*^0\rangle - \partial_{12}\psi(\cdot,\cdot)\langle \alpha_1(x,\eta)\alpha_2(x,\eta), q_*^0\rangle$$

$$+ \frac{1}{2N^{1/4}}\partial_{11}\psi(\cdot,\cdot)\langle[\alpha_2(x,\eta)]^2, \widetilde{\rho}_N\rangle + \frac{1}{2N^{1/4}}\partial_{22}\psi(\cdot,\cdot)\langle[\alpha_1(x,\eta)]^2, \widetilde{\rho}_N\rangle - \frac{1}{N^{1/4}}\partial_{12}\psi(\cdot,\cdot)\langle \alpha_1(x,\eta)\alpha_2(x,\eta), \widetilde{\rho}_N\rangle$$

(by using Bisection formulas and the fact that the measure $\widetilde{\rho}_N$ is centered, in other words that $\langle 1, \widetilde{\rho}_N\rangle = 0$)

$$= \frac{N^{1/4}}{2}(1+4\omega^2)\Big\{\partial_1\psi(\cdot,\cdot)[\langle \sin x, \widetilde{\rho}_N\rangle\langle \cos 2x - 2\omega\eta\sin 2x, \widetilde{\rho}_N\rangle - \langle \cos x, \widetilde{\rho}_N\rangle\langle \sin 2x + 2\omega\eta\cos 2x, \widetilde{\rho}_N\rangle] - \partial_2\psi(\cdot,\cdot)[\langle \sin x, \widetilde{\rho}_N\rangle\langle \sin 2x + 2\omega\eta\cos 2x, \widetilde{\rho}_N\rangle + \langle \cos x, \widetilde{\rho}_N\rangle\langle \cos 2x + 2\omega\eta\sin 2x, \widetilde{\rho}_N\rangle]\Big\} + \frac{1}{4}(1+4\omega^2)[\partial_{11}\psi(\cdot,\cdot) + \partial_{22}\psi(\cdot,\cdot)] + o(1)\,, \quad (4.26)$$

where $o(1)$ includes the terms coming from $L_4\psi$, which are of order $N^{-1/4}$. The operator L_4 is defined by (4.24).

Now, we apply the first order perturbation theory explained in the previous chapter. The expression of $\mathcal{J}_N\psi(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle)$ is described by (4.26). We need to compute ψ_1, which allows us to introduce the terms necessary to the convergence of the infinitesimal generator: $L_1\psi_1$ and $L_2\psi_1$ of the expansion (3.31).

By the definitions (3.32) and (3.33), it yields

$$\psi_1(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \sin 2x, \widetilde{\rho}_N\rangle, \langle \cos 2x, \widetilde{\rho}_N\rangle, \langle \eta\sin 2x, \widetilde{\rho}_N\rangle, \langle \eta\cos 2x, \widetilde{\rho}_N\rangle) =$$

$$= -L_1^{-1}L_2\psi(\langle\alpha_1(x,\eta),\widetilde{\rho}_N(t)\rangle,\langle\alpha_2(x,\eta),\widetilde{\rho}_N(t)\rangle)$$

$$= -\frac{1}{2}(1+4\omega^2)L_1^{-1}\Big[\partial_1\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos 2x - 2\omega\eta\sin 2x,\widetilde{\rho}_N\rangle - \langle\cos x,\widetilde{\rho}_N\rangle\langle\sin 2x+\omega\eta\cos 2x,\widetilde{\rho}_N\rangle\Big)$$

$$- \partial_2\psi(\cdot,\cdot)\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\sin 2x+\omega\eta\cos 2x,\widetilde{\rho}_N\rangle + \langle\cos x,\widetilde{\rho}_N\rangle\langle\cos 2x-\omega\eta\sin 2x,\widetilde{\rho}_N\rangle\Big)\Big]$$

$$\begin{aligned}= -\frac{1}{16}(1+4\omega^2)\Big\{\partial_1\psi(\cdot,\cdot)\Big[&\langle\cos 2x - 2\omega\eta\sin 2x,\widetilde{\rho}_N\rangle\langle 2\omega^2\sin x+\omega\eta\cos x,\widetilde{\rho}_N\rangle\\ &+ 2\langle\sin x,\widetilde{\rho}_N\rangle\langle-(1+2\omega^2)\cos 2x+\omega\eta\sin 2x,\widetilde{\rho}_N\rangle\\ &- \langle\sin 2x+2\omega\eta\cos 2x,\widetilde{\rho}_N\rangle\langle 2\omega^2\cos x-\omega\eta\sin x\widetilde{\rho}_N\rangle\\ &+ 2\langle\cos x,\widetilde{\rho}_N\rangle\langle(1+2\omega^2)\sin 2x+\omega\eta\cos 2x,\widetilde{\rho}_N\rangle\Big]\end{aligned}$$

$$\begin{aligned}+ \partial_2\psi(\cdot,\cdot)\Big[&-\langle\sin 2x+2\omega\eta\cos 2x,\widetilde{\rho}_N\rangle\langle 2\omega^2\sin x+\omega\eta\cos x,\widetilde{\rho}_N\rangle\\ &+ 2\langle\sin x,\widetilde{\rho}_N\rangle\langle(1+2\omega^2)\sin 2x+\omega\eta\cos 2x,\widetilde{\rho}_N\rangle\\ &- \langle\cos 2x-2\omega\eta\sin 2x,\widetilde{\rho}_N\rangle\langle 2\omega^2\cos x-\omega\eta\sin x\widetilde{\rho}_N\rangle\\ &- 2\langle\cos x,\widetilde{\rho}_N\rangle\langle-(1+2\omega^2)\cos 2x+\omega\eta\sin 2x,\widetilde{\rho}_N\rangle\Big]\Big\}. \quad (4.27)\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{J}_N[\psi(&\langle\alpha_1(x,\eta),\widetilde{\rho}_N(t)\rangle,\langle\alpha_2(x,\eta),\widetilde{\rho}_N(t)\rangle)\\ &+ N^{-1/4}\psi_1(\langle\alpha_1(x,\eta),\widetilde{\rho}_N(t)\rangle,\langle\alpha_2(x,\eta),\widetilde{\rho}_N(t)\rangle,\langle\sin 2x,\widetilde{\rho}_N\rangle,\\ &\langle\cos 2x,\widetilde{\rho}_N\rangle,\langle\eta\sin 2x,\widetilde{\rho}_N\rangle,\langle\eta\cos 2x,\widetilde{\rho}_N\rangle)] =\end{aligned}$$

$$\begin{aligned}= N^{1/4}[&L_2\psi(\langle\alpha_1(x,\eta),\widetilde{\rho}_N(t)\rangle,\langle\alpha_2(x,\eta),\widetilde{\rho}_N(t)\rangle)\\ &+ L_1\psi_1(\langle\alpha_1(x,\eta),\widetilde{\rho}_N(t)\rangle,\langle\alpha_2(x,\eta),\widetilde{\rho}_N(t)\rangle,\langle\sin 2x,\widetilde{\rho}_N\rangle,\\ &\langle\cos 2x,\widetilde{\rho}_N\rangle,\langle\eta\sin 2x,\widetilde{\rho}_N\rangle,\langle\eta\cos 2x,\widetilde{\rho}_N\rangle)]\end{aligned}$$

$$
\begin{aligned}
&+ L_3\psi(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle) \\
&\qquad + L_2\psi_1(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \sin 2x, \widetilde{\rho}_N\rangle, \\
&\qquad\qquad \langle \cos 2x, \widetilde{\rho}_N\rangle, \langle \eta \sin 2x, \widetilde{\rho}_N\rangle, \langle \eta\cos 2x, \widetilde{\rho}_N\rangle) + o(1)\,,
\end{aligned}
$$

where $o(1)$ includes the terms coming from $L_4\psi$, $L_3\psi_1$ and $L_4\psi_1$, which are of order $N^{-1/4}$ and $N^{-1/2}$.

Since $L_2\psi + L_1\psi_1 = 0$ by construction and $L_3\psi = \frac{1}{4}(1+4\omega^2)\,[\partial_{11}\psi + \partial_{22}\psi]$, it remains to compute the term $L_2\psi_1$.

$$
\begin{aligned}
&L_2\psi_1(\langle \alpha_1(x,\eta), \widetilde{\rho}_N\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N\rangle, \langle \sin 2x, \widetilde{\rho}_N\rangle, \\
&\qquad\qquad \langle \cos 2x, \widetilde{\rho}_N\rangle, \langle \eta \sin 2x, \widetilde{\rho}_N\rangle, \langle \eta\cos 2x, \widetilde{\rho}_N\rangle) =
\end{aligned}
$$

$$
\begin{aligned}
= -\frac{1}{16}(1+4\omega^2)^2\Big\{\partial_{11}\psi(\cdot,\cdot)\Big[&\langle \cos 2x - 2\omega\eta\sin 2x, \widetilde{\rho}_N\rangle\langle 2\omega^2\sin x + \omega\eta\cos x, \widetilde{\rho}_N\rangle \\
&+ 2\langle \sin x, \widetilde{\rho}_N\rangle\langle -(1+2\omega^2)\cos 2x + \omega\eta\sin 2x, \widetilde{\rho}_N\rangle \\
&- \langle \sin 2x + 2\omega\eta\cos 2x, \widetilde{\rho}_N\rangle\langle 2\omega^2\cos x - \omega\eta\sin x, \widetilde{\rho}_N\rangle \\
&+ 2\langle \cos x, \widetilde{\rho}_N\rangle\langle (1+2\omega^2)\sin 2x + \omega\eta\cos 2x, \widetilde{\rho}_N\rangle\Big]^2
\end{aligned}
$$

$$
\begin{aligned}
+ 2\partial_{12}\psi(\cdot,\cdot)\Big[&\langle \cos 2x - 2\omega\eta\sin 2x, \widetilde{\rho}_N\rangle\langle 2\omega^2\sin x + \omega\eta\cos x, \widetilde{\rho}_N\rangle \\
&+ 2\langle \sin x, \widetilde{\rho}_N\rangle\langle -(1+2\omega^2)\cos 2x + \omega\eta\sin 2x, \widetilde{\rho}_N\rangle \\
&- \langle \sin 2x + 2\omega\eta\cos 2x, \widetilde{\rho}_N\rangle\langle 2\omega^2\cos x - \omega\eta\sin x, \widetilde{\rho}_N\rangle \\
&+ 2\langle \cos x, \widetilde{\rho}_N\rangle\langle (1+2\omega^2)\sin 2x + \omega\eta\cos 2x, \widetilde{\rho}_N\rangle\Big]\cdot \\
\cdot\Big[&- \langle \sin 2x + 2\omega\eta\cos 2x, \widetilde{\rho}_N\rangle\langle 2\omega^2\sin x + \omega\eta\cos x, \widetilde{\rho}_N\rangle \\
&+ 2\langle \sin x, \widetilde{\rho}_N\rangle\langle (1+2\omega^2)\sin 2x + \omega\eta\cos 2x, \widetilde{\rho}_N\rangle \\
&- \langle \cos 2x - 2\omega\eta\sin 2x, \widetilde{\rho}_N\rangle\langle 2\omega^2\cos x - \omega\eta\sin x, \widetilde{\rho}_N\rangle \\
&- 2\langle \cos x, \widetilde{\rho}_N\rangle\langle -(1+2\omega^2)\cos 2x + \omega\eta\sin 2x, \widetilde{\rho}_N\rangle\Big]
\end{aligned}
$$

$$
\begin{aligned}
+ \partial_{22}\psi(\cdot,\cdot)\Big[&- \langle \sin 2x + 2\omega\eta\cos 2x, \widetilde{\rho}_N\rangle\langle 2\omega^2\sin x + \omega\eta\cos x, \widetilde{\rho}_N\rangle \\
&+ 2\langle \sin x, \widetilde{\rho}_N\rangle\langle (1+2\omega^2)\sin 2x + \omega\eta\cos 2x, \widetilde{\rho}_N\rangle \\
&- \langle \cos 2x - 2\omega\eta\sin 2x, \widetilde{\rho}_N\rangle\langle 2\omega^2\cos x - \omega\eta\sin x, \widetilde{\rho}_N\rangle \\
&- 2\langle \cos x, \widetilde{\rho}_N\rangle\langle -(1+2\omega^2)\cos 2x + \omega\eta\sin 2x, \widetilde{\rho}_N\rangle\Big]^2
\end{aligned}
$$

$$
\begin{aligned}
&+\partial_1\psi(\cdot,\cdot)\Big[2\langle 2\omega^2\sin x+\omega\eta\cos x,\widetilde{\rho}_N\rangle\Big(-\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x[\sin 2x+2\omega\eta\cos 2x],\widetilde{\rho}_N\rangle\\
&\qquad+\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x[\sin 2x+2\omega\eta\cos 2x],\widetilde{\rho}_N\rangle\Big)\\
&\quad+\langle\cos 2x-2\omega\eta\sin 2x,\widetilde{\rho}_N\rangle\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x[2\omega^2\cos x-\omega\eta\sin x],\widetilde{\rho}_N\rangle\\
&\qquad-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x[2\omega^2\cos x-\omega\eta\sin x],\widetilde{\rho}_N\rangle\Big)\\
&\quad+2\langle-(1+2\omega^2)\cos 2x+\omega\eta\sin 2x,\widetilde{\rho}_N\rangle\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos^2 x,\widetilde{\rho}_N\rangle\\
&\qquad-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x\cos x,\widetilde{\rho}_N\rangle\Big)\\
&\quad+4\langle\sin x,\widetilde{\rho}_N\rangle\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x[(1+2\omega^2)\sin 2x+\omega\eta\cos 2x,\widetilde{\rho}_N\rangle\\
&\qquad-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x[(1+2\omega^2)\sin 2x+\omega\eta\cos 2x],\widetilde{\rho}_N\rangle\Big)\\
&\quad-2\langle 2\omega^2\cos x-\omega\eta\sin x,\widetilde{\rho}_N\rangle\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x[\cos 2x-2\omega\eta\sin 2x],\widetilde{\rho}_N\rangle\\
&\qquad-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x[\cos 2x-2\omega\eta\sin 2x],\widetilde{\rho}_N\rangle\Big)\\
&\quad+\langle\sin 2x+2\omega\eta\cos 2x,\widetilde{\rho}_N\rangle\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x[2\omega^2\sin x+\omega\eta\cos x,\widetilde{\rho}_N\rangle\\
&\qquad-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x[2\omega^2\sin x+\omega\eta\cos x,\widetilde{\rho}_N\rangle\Big)\\
&\quad+2\langle(1+2\omega^2)\sin 2x+\omega\eta\cos 2x,\widetilde{\rho}_N\rangle\Big(-\langle\sin x,\widetilde{\rho}_N\rangle\langle\sin x\cos x,\widetilde{\rho}_N\rangle\\
&\qquad+\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin^2 x,\widetilde{\rho}_N\rangle\Big)\\
&\quad+4\langle\cos x,\widetilde{\rho}_N\rangle\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x[(1+2\omega^2)\cos 2x-\omega\eta\sin 2x],\widetilde{\rho}_N\rangle\\
&\qquad-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x[(1+2\omega^2)\cos 2x-\omega\eta\sin 2x],\widetilde{\rho}_N\rangle\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\partial_2\psi(\cdot,\cdot)\Big[-2\langle 2\omega^2\sin x+\omega\eta\cos x,\widetilde{\rho}_N\rangle\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x[\cos 2x-2\omega\eta\sin 2x],\widetilde{\rho}_N\rangle\\
&\qquad-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x[\cos 2x-2\omega\eta\sin 2x],\widetilde{\rho}_N\rangle\Big)\\
&\quad-\langle\sin 2x+2\omega\eta\cos 2x,\widetilde{\rho}_N\rangle\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x[2\omega^2\cos x-\omega\eta\sin x],\widetilde{\rho}_N\rangle\\
&\qquad-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x[2\omega^2\cos x-\omega\eta\sin x],\widetilde{\rho}_N\rangle\Big)\\
&\quad+2\langle(1+2\omega^2)\sin 2x+\omega\eta\cos 2x,\widetilde{\rho}_N\rangle\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos^2 x,\widetilde{\rho}_N\rangle\\
&\qquad-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x\cos x,\widetilde{\rho}_N\rangle\Big)\\
&\quad+4\langle\sin x,\widetilde{\rho}_N\rangle\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x[(1+2\omega^2)\cos 2x-\omega\eta\sin 2x,\widetilde{\rho}_N\rangle\\
&\qquad-\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x[(1+2\omega^2)\cos 2x-\omega\eta\cos 2x],\widetilde{\rho}_N\rangle\Big)\\
&\quad-2\langle 2\omega^2\cos x-\omega\eta\sin x,\widetilde{\rho}_N\rangle\Big(-\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x[\sin 2x+2\omega\eta\cos 2x],\widetilde{\rho}_N\rangle\\
&\qquad+\langle\cos x,\widetilde{\rho}_N\rangle\langle\sin x[\sin 2x+2\omega\eta\cos 2x],\widetilde{\rho}_N\rangle\Big)\\
&\quad+\langle\cos 2x+2\omega\eta\sin 2x,\widetilde{\rho}_N\rangle\Big(\langle\sin x,\widetilde{\rho}_N\rangle\langle\cos x[2\omega^2\sin x+\omega\eta\cos x,\widetilde{\rho}_N\rangle
\end{aligned}
$$

$$
\begin{aligned}
&- \langle \cos x, \widetilde{\rho}_N\rangle\langle \sin x[2\omega^2 \sin x + \omega\eta\cos x, \widetilde{\rho}_N\rangle\Big) \\
&- 2\langle -(1+2\omega^2)\cos 2x + \omega\eta \sin 2x, \widetilde{\rho}_N\rangle\Big(- \langle \sin x, \widetilde{\rho}_N\rangle\langle \sin x\cos x, \widetilde{\rho}_N\rangle \\
&\quad + \langle \cos x, \widetilde{\rho}_N\rangle\langle \sin^2 x, \widetilde{\rho}_N\rangle\Big) \\
&- 4\langle \cos x, \widetilde{\rho}_N\rangle\Big(\langle \sin x, \widetilde{\rho}_N\rangle\langle \cos x[(1+2\omega^2)\sin 2x + \omega\eta\cos 2x], \widetilde{\rho}_N\rangle \\
&\quad - \langle \cos x, \widetilde{\rho}_N\rangle\langle \sin x[(1+2\omega^2)\sin 2x + \omega\eta\cos 2x], \widetilde{\rho}_N\rangle\Big)\Big]\Big\}
\end{aligned}
$$

(by using Prosthaphaeresis formulas and leaving all the terms we know collapsing)

$$
\begin{aligned}
\text{“}=\text{”}& -\frac{1}{16}(1+4\omega^2)^2\Big\{\partial_1\psi(\cdot,\cdot)\Big[-\langle \alpha_1(x,\eta), \widetilde{\rho}_N\rangle\Big(\langle \sin x, \widetilde{\rho}_N\rangle\langle 2\omega^2\sin x + \omega\eta\cos x, \widetilde{\rho}_N\rangle \\
&+ \langle \cos x, \widetilde{\rho}_N\rangle\langle 2\omega^2\cos x - \omega\eta\sin x, \widetilde{\rho}_N\rangle\Big) \\
&- \langle \alpha_2(x,\eta), \widetilde{\rho}_N\rangle\Big(- \langle \cos x, \widetilde{\rho}_N\rangle\langle 2\omega^2\sin x + \omega\eta\cos x, \widetilde{\rho}_N\rangle \\
&+ \langle \sin x, \widetilde{\rho}_N\rangle\langle 2\omega^2\cos x - \omega\eta\sin x, \widetilde{\rho}_N\rangle\Big) \\
&+ \langle (1+2\omega^2)\sin x + \omega\eta\cos x, \widetilde{\rho}_N\rangle\Big(\langle \sin x, \widetilde{\rho}_N\rangle^2 + \langle \cos x, \widetilde{\rho}_N\rangle^2\Big)\Big] \\
&+ \partial_2\psi(\cdot,\cdot)\Big[- \langle \alpha_1(x,\eta), \widetilde{\rho}_N\rangle\Big(\langle \sin x, \widetilde{\rho}_N\rangle\langle 2\omega^2\cos x - \omega\eta\sin x, \widetilde{\rho}_N\rangle \\
&- \langle \cos x, \widetilde{\rho}_N\rangle\langle 2\omega^2\sin x + \omega\eta\cos x, \widetilde{\rho}_N\rangle\Big) \\
&- \langle \alpha_2(x,\eta), \widetilde{\rho}_N\rangle\Big(- \langle \cos x, \widetilde{\rho}_N\rangle\langle 2\omega^2\cos x - \omega\eta\sin x, \widetilde{\rho}_N\rangle \\
&+ \langle \sin x, \widetilde{\rho}_N\rangle\langle 2\omega^2\sin x + \omega\eta\cos x, \widetilde{\rho}_N\rangle\Big) \\
&+ \langle (1+2\omega^2)\cos x - \omega\eta\sin x, \widetilde{\rho}_N\rangle\Big(\langle \sin x, \widetilde{\rho}_N\rangle^2 + \langle \cos x, \widetilde{\rho}_N\rangle^2\Big)\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
= &-\frac{1}{8}(1+4\omega^2)^2\Bigg\{\partial_1\psi(\cdot,\cdot)\Bigg[- \frac{1}{4}\langle \sin x, \widetilde{\rho}_N\rangle\Big(\langle \alpha_1(x,\eta), \widetilde{\rho}_N\rangle^2 + \langle \alpha_2(x,\eta), \widetilde{\rho}_N\rangle^2\Big) \\
&+ \frac{3-4\omega^2}{4}\langle \alpha_1(x,\eta), \widetilde{\rho}_N\rangle\Big(\langle \sin x, \widetilde{\rho}_N\rangle^2 + \langle \cos x, \widetilde{\rho}_N\rangle^2\Big) \\
&+ \frac{1+4\omega^2}{2}\langle \sin x, \widetilde{\rho}_N\rangle\Big(\langle \sin x, \widetilde{\rho}_N\rangle^2 + \langle \cos x, \widetilde{\rho}_N\rangle^2\Big)\Bigg] \\
&+ \partial_2\psi(\cdot,\cdot)\Bigg[- \frac{1}{4}\langle \cos x, \widetilde{\rho}_N\rangle\Big(\langle \alpha_1(x,\eta), \widetilde{\rho}_N\rangle^2 + \langle \alpha_2(x,\eta), \widetilde{\rho}_N\rangle^2\Big) \\
&+ \frac{3-4\omega^2}{4}\langle \alpha_2(x,\eta), \widetilde{\rho}_N\rangle\Big(\langle \sin x, \widetilde{\rho}_N\rangle^2 + \langle \cos x, \widetilde{\rho}_N\rangle^2\Big) \\
&+ \frac{1+4\omega^2}{2}\langle \cos x, \widetilde{\rho}_N\rangle\Big(\langle \sin x, \widetilde{\rho}_N\rangle^2 + \langle \cos x, \widetilde{\rho}_N\rangle^2\Big)\Bigg]\Bigg\}
\end{aligned}
$$

by replacing

$$\sin x = \frac{4\omega^2}{16\omega^4 - 1}[(\eta + 4\omega^2)\sin x + 2\omega(\eta - 1)\cos x] + \frac{2\omega}{16\omega^4 - 1}[(\eta + 4\omega^2)\cos x - 2\omega(\eta - 1)\sin x] - \frac{4\omega^2 + 1}{16\omega^4 - 1}\alpha_1(x,\eta)$$

and

$$\cos x = \frac{4\omega^2}{16\omega^4 - 1}[(\eta + 4\omega^2)\cos x - 2\omega(\eta - 1)\sin x] - \frac{2\omega}{16\omega^4 - 1}[(\eta + 4\omega^2)\sin x + 2\omega(\eta - 1)\cos x] - \frac{1}{4\omega^2 - 1}\alpha_2(x,\eta)\,,$$

which are the expression of $\sin x$ and $\cos x$ as a linear combination of the elements of the subspace generated by the Fourier components corresponding to $k = \pm 1$, in other words $\alpha_1(x,\eta)$, $\alpha_2(x,\eta)$, $(\eta+4\omega^2)\sin x+2\omega(\eta-1)\cos x$ and $(\eta+4\omega^2)\cos x - 2\omega(\eta - 1)\sin x$,

$$= -\frac{1}{8}\frac{(1+4\omega^2)^2}{(1-4\omega^2)^3}\langle\alpha_1(x,\eta),\widetilde{\rho}_N\rangle\Big[\langle\alpha_1(x,\eta),\widetilde{\rho}_N\rangle^2 + \langle\alpha_2(x,\eta),\widetilde{\rho}_N\rangle^2\Big]\partial_1\psi(\cdot,\cdot) - \frac{1}{8}\frac{(1+4\omega^2)^2}{(1-4\omega^2)^3}\langle\alpha_2(x,\eta),\widetilde{\rho}_N\rangle\Big[\langle\alpha_1(x,\eta),\widetilde{\rho}_N\rangle^2 + \langle\alpha_2(x,\eta),\widetilde{\rho}_N\rangle^2\Big]\partial_2\psi(\cdot,\cdot)\,,$$

where $(1 - 4\omega^2)^3 > 0$, since we have chosen $\omega < \frac{1}{2}$.

Proceeding as in the previous chapter, we can show which are the limiting dynamics of the fluctuation processes. We skip the details and we state the main result.

Let $\psi \in \mathcal{C}_b^3$ be a function of the type

$$\psi\Big(\langle\sin x,\widetilde{\rho}_N(t)\rangle, \langle\cos x,\widetilde{\rho}_N(t)\rangle, \{\langle\sin kx,\widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \{\langle\cos kx,\widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \langle\eta\sin x,\widetilde{\rho}_N(t)\rangle, \langle\eta\cos x,\widetilde{\rho}_N(t)\rangle, \{\langle\eta\sin kx,\widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \{\langle\eta\cos kx,\widetilde{\rho}_N(t)\rangle\}_{k\geq 2}\Big) = = \psi(\langle\alpha_1(x,\eta),\widetilde{\rho}_N(t)\rangle, \langle\alpha_2(x,\eta),\widetilde{\rho}_N(t)\rangle)$$

recalling the expression of ψ_1 given by (4.27), we construct

$$\psi\Big(\langle\alpha_1(x,\eta),\widetilde{\rho}_N(t)\rangle, \langle\alpha_2(x,\eta),\widetilde{\rho}_N(t)\rangle, \{\langle\sin kx,\widetilde{\rho}_N(t)\rangle\}_{k\geq 2},$$

$$\{\langle \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \{\langle \eta \sin kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}, \{\langle \eta \cos kx, \widetilde{\rho}_N(t)\rangle\}_{k\geq 2}\Big)$$
$$+ N^{-1/4}\psi_1(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \sin 2x, \widetilde{\rho}_N(t)\rangle,$$
$$\langle \cos 2x, \widetilde{\rho}_N(t)\rangle, \langle \eta \sin 2x, \widetilde{\rho}_N(t)\rangle, \langle \eta \cos 2x, \widetilde{\rho}_N(t)\rangle) =$$

$$= \psi(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle)$$

$$- \frac{1+4\omega^2}{16N^{1/4}}\Big\{\partial_1\psi(\cdot,\cdot)\Big[\langle \cos 2x - 2\omega\eta \sin 2x, \widetilde{\rho}_N(t)\rangle\langle 2\omega^2 \sin x + \omega\eta \cos x, \widetilde{\rho}_N(t)\rangle$$
$$+ 2\langle \sin x, \widetilde{\rho}_N(t)\rangle\langle -(1+2\omega^2)\cos 2x + \omega\eta \sin 2x, \widetilde{\rho}_N(t)\rangle$$
$$- \langle \sin 2x + 2\omega\eta \cos 2x, \widetilde{\rho}_N(t)\rangle\langle 2\omega^2 \cos x - \omega\eta \sin x, \widetilde{\rho}_N(t)\rangle$$
$$+ 2\langle \cos x, \widetilde{\rho}_N(t)\rangle\langle (1+2\omega^2)\sin 2x + \omega\eta \cos 2x, \widetilde{\rho}_N(t)\rangle\Big]$$

$$+ \partial_2\psi(\cdot,\cdot)\Big[- \langle \sin 2x + 2\omega\eta \cos 2x, \widetilde{\rho}_N(t)\rangle\langle 2\omega^2 \sin x + \omega\eta \cos x, \widetilde{\rho}_N(t)\rangle$$
$$+ 2\langle \sin x, \widetilde{\rho}_N(t)\rangle\langle (1+2\omega^2)\sin 2x + \omega\eta \cos 2x, \widetilde{\rho}_N(t)\rangle$$
$$- \langle \cos 2x - 2\omega\eta \sin 2x, \widetilde{\rho}_N(t)\rangle\langle 2\omega^2 \cos x - \omega\eta \sin x, \widetilde{\rho}_N(t)\rangle$$
$$- 2\langle \cos x, \widetilde{\rho}_N(t)\rangle\langle -(1+2\omega^2)\cos 2x + \omega\eta \sin 2x, \widetilde{\rho}_N(t)\rangle\Big]\Big\}$$

$$:= \psi_N^{(2)}(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \sin 2x, \widetilde{\rho}_N(t)\rangle,$$
$$\langle \cos 2x, \widetilde{\rho}_N(t)\rangle, \langle \eta \sin 2x, \widetilde{\rho}_N(t)\rangle, \langle \eta \cos 2x, \widetilde{\rho}_N(t)\rangle)\,.$$

The following decomposition holds

$$\psi_N^{(2)}(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \sin 2x, \widetilde{\rho}_N(t)\rangle,$$
$$\langle \cos 2x, \widetilde{\rho}_N(t)\rangle, \langle \eta \sin 2x, \widetilde{\rho}_N(t)\rangle, \langle \eta \cos 2x, \widetilde{\rho}_N(t)\rangle) =$$

$$= \psi_N^{(2)}(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(0)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(0)\rangle, \langle \sin 2x, \widetilde{\rho}_N(0)\rangle,$$
$$\langle \cos 2x, \widetilde{\rho}_N(0)\rangle, \langle \eta \sin 2x, \widetilde{\rho}_N(0)\rangle, \langle \eta \cos 2x, \widetilde{\rho}_N(0)\rangle)$$

$$+ \int_0^t \mathcal{J}_N\psi_N^{(2)}(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(u)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(u)\rangle, \langle \sin 2x, \widetilde{\rho}_N(u)\rangle,$$

$$\langle \cos 2x, \widetilde{\rho}_N(u)\rangle, \langle \eta \sin 2x, \widetilde{\rho}_N(u)\rangle, \langle \eta \cos 2x, \widetilde{\rho}_N(u)\rangle)\, du + \mathcal{M}^t_{N,\psi_N^{(2)}}\,, \quad (4.28)$$

where $\mathcal{M}^t_{N,\psi_N^{(2)}}$ is a martingale and

$$\mathcal{J}_N \psi_N^{(2)}(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \sin 2x, \widetilde{\rho}_N(t)\rangle,$$
$$\langle \cos 2x, \widetilde{\rho}_N(t)\rangle, \langle \eta \sin 2x, \widetilde{\rho}_N(t)\rangle, \langle \eta \cos 2x, \widetilde{\rho}_N(t)\rangle)\text{“} = \text{”}$$

$$\text{“} = \text{”} -\frac{1}{8}\frac{(1+4\omega^2)^2}{(1-4\omega^2)^3}\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle\Big[\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle^2 + \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle^2\Big]\partial_1\psi(\cdot,\cdot)$$
$$-\frac{1}{8}\frac{(1+4\omega^2)^2}{(1-4\omega^2)^3}\langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle\Big[\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle^2 + \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle^2\Big]\partial_2\psi(\cdot,\cdot)$$
$$+\frac{1}{4}(1+4\omega^2)[\partial_{11}\psi(\cdot,\cdot) + \partial_{22}\psi(\cdot,\cdot)] + o_M(1)\,.$$

The remainder $o_M(1)$ goes to zero as $N \longrightarrow +\infty$. If we compute the limit as $N \longrightarrow +\infty$, we have:

$$\mathcal{J}_N \psi_N^{(2)}(\langle \alpha_1(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \alpha_2(x,\eta), \widetilde{\rho}_N(t)\rangle, \langle \sin 2x, \widetilde{\rho}_N(t)\rangle,$$
$$\langle \cos 2x, \widetilde{\rho}_N(t)\rangle, \langle \eta \sin 2x, \widetilde{\rho}_N(t)\rangle, \langle \eta \cos 2x, \widetilde{\rho}_N(t)\rangle)$$
$$\xrightarrow[w]{N\to+\infty} \mathcal{J}\psi^{(2)}(X(t),Y(t))\,,$$

with

$$\mathcal{J}\psi^{(2)}(X(t),Y(t)) =$$

$$= -\frac{1}{8}\frac{(1+4\omega^2)^2}{(1-4\omega^2)^3}\Big\{X(t)\Big[X(t)^2+Y(t)^2]\partial_1\psi(\cdot,\cdot) + Y(t)\Big[X(t)^2+Y(t)^2\Big]\partial_2\psi(\cdot,\cdot)\Big\}$$
$$+\frac{1}{4}(1+4\omega^2)[\partial_{11}\psi(\cdot,\cdot)+\partial_{22}\psi(\cdot,\cdot)]\,.$$

Then, because of (4.28), we obtain

$$\mathcal{M}^t_{n,\psi_n^{(2)}} \xrightarrow[w]{n\to+\infty} \mathcal{M}^t_{\psi^{(2)}}\,,$$

once we have defined

$$\mathcal{M}^t_{\psi^{(2)}} := \psi^{(2)}(X(t),Y(t)) - \psi^{(2)}(X(0),Y(0)) - \int_0^t \mathcal{J}\psi^{(2)}(X(u),Y(u))\,du\,.$$

4.4 Conclusions

Concluding, we point out the fact that the inhomogeneous critical fluctuation process exists in same time-scale than the homogeneous one; in fact, if we compare the flows (3.16) and (4.19), we can see that, when we construct these fluctuations, we are allowed to amplify the time by a factor $N^{1/2}$ in both cases. Besides, they are a two-dimensional process at the critical point, converging (in the sense of weak convergence of stochastic processes) to a non-Gaussian process, solution of a cubic stochastic differential equation. These limiting equations have the same form.

Part III

Some Generalizations

Chapter 5
Back to the Random Curie-Weiss Model

In this last chapter we consider again the Curie-Weiss model. We generalize the environment, precisely we choose it distributed according to an even distribution with finite support.
For such systems one can find a finite dimensional order parameter, whose dimension equals the dimension of the support of the distribution of the random field. Since we allow this cardinality to be arbitrary a low-dimensional analysis as in Chapter 1 and 2 is not appropriate. Therefore, we proceed to apply the method developed in Chapter 3.

5.1 Description of the Model

Let $\mathscr{D} := \{-h_i, -h_{i-1}, \dots, -h_1, 0, h_1, \dots, h_{i-1}, h_i\}$ be a finite subset of $\mathbb{R}$, with $0 < h_1 < \cdots < h_i$, and $\mathscr{S} = \{-1, +1\}$. Let $\underline{\eta} = (\eta_j)_{j=1}^N \in \mathscr{D}^N$ be a sequence of independent, identically distributed random variables defined on some probability space $(\Omega, \mathcal{F}, P)$ and distributed according to an even distribution μ.

Remark 5.1.1. The assumption on the support of μ is necessary so that the critical dynamics can be dealt with. All the results concerning the McKean-Vlasov limit for the system are true more in general.

Given a configuration $\underline{\sigma} = (\sigma_j)_{j=1}^N \in \mathscr{S}^N$ and a realization of the magnetic field

$\underline{\eta}$, we can define the Hamiltonian $H_N(\underline{\sigma},\underline{\eta}) : \mathscr{S}^N \times \mathscr{D}^N \longrightarrow \mathbb{R}$ as

$$H_N(\underline{\sigma},\underline{\eta}) = -\frac{\beta}{2N}\sum_{j,k=1}^{N}\sigma_j\sigma_k - \beta\sum_{j=1}^{N}\eta_j\sigma_j\,, \tag{5.1}$$

where σ_j is the spin value at site j and η_j the local magnetic field associated with the same site. Let β, positive parameter, be the inverse of the temperature. For a fixed realization of $\underline{\eta}$, think of $\underline{\sigma} \longrightarrow H_N(\underline{\sigma},\underline{\eta})$ as a Hamiltonian in the components σ_j with an inhomogeneous mean-field interaction parametrized by the components η_j. With the expression "mean-field" we mean the sites interact all each other in the same way.

Let us define the dynamics we consider. For given $\underline{\eta}$, $\underline{\sigma}(t) = (\sigma_j(t))_{j=1}^N$, with t belonging to a generic time interval $[0,T]$, where T is fixed, describes a N-spin system evolving as a continuous time Markov chain on $\mathscr{S}^N$, with infinitesimal generator L_N acting on functions $f : \mathscr{S}^N \longrightarrow \mathbb{R}$ as follows:

$$L_N f(\underline{\sigma}) = \sum_{j=1}^{N} e^{-\beta\sigma_j(m_N^{\underline{\sigma}}+\eta_j)}\nabla_j^{\sigma} f(\underline{\sigma}), \tag{5.2}$$

where $\nabla_j^{\sigma} f(\underline{\sigma}) = f(\underline{\sigma}^j) - f(\underline{\sigma})$ and the k-th component of $\underline{\sigma}^j$, which has the meaning of a spin flip at site j, is

$$\sigma_k^j = \begin{cases} \sigma_k \text{ for } k \neq j \\ -\sigma_k \text{ for } k = j \end{cases}.$$

The quantity $c_N^{\eta}(j,\underline{\sigma}) = e^{-\beta\sigma_j(m_N^{\underline{\sigma}}+\eta_j)}$ represents the jump rate of the spins; the rate at which the transition $\sigma_j \longrightarrow -\sigma_j$ occurs for some j. The mean-field assumption allows us to suppose that the interaction depends on the value of the magnetization

$$m_N^{\underline{\sigma}}(t) = \frac{1}{N}\sum_{j=1}^{N}\sigma_j(t). \tag{5.3}$$

The expressions (5.1) and (5.2) describe a system of mean-field ferromagnetically coupled spins, each with its own random magnetic field and subject to Glauber dynamics. The two terms in the Hamiltonian have different effects: the first one tends to align the spins, while the second one tends to point each of them in the direction of its local field.

Remark 5.1.2. For every value of $\underline{\eta}$, (5.2) has a reversible stationary distribution proportional to $\exp[-H_N(\underline{\sigma},\underline{\eta})]$.

For simplicity, the initial condition $\underline{\sigma}(0)$ is assumed to have product distribution $\lambda^{\otimes N}$, with λ probability measure on $\mathscr{S}$. The quantity $\sigma_j(t)$ represents the time evolution on $[0,T]$ of j-th spin value; it is the trajectory of the single j-th spin in time. The space of all these paths is $\mathcal{D}[0,T]$, which is the space of the right-continuous, piecewise-constant functions from $[0,T]$ to $\mathscr{S}$. We endow $\mathcal{D}[0,T]$ with the Skorohod topology, which provides a metric and a Borel σ-field (as we can see in [EK86]).

5.2 Limiting Dynamics

We now derive the dynamics of the process (5.2), in the limit as $N \longrightarrow +\infty$, in a fixed time interval $[0,T]$, via a Large Deviation approach. Later, the large time behavior of the limiting dynamics will be studied.

For completeness, we report all the statements that allow us to deduce the dynamics of the model in the infinite volume limit, but we omit their proofs, since they proceed as in Chapter 1, once we have changed the distribution of the environment.

So, let $(\sigma_j[0,T])_{j=1}^N \in (\mathcal{D}[0,T])^N$ denote a path of the system in the time interval $[0,T]$, with T positive and fixed. If $f(\sigma_j[0,T])$ is a function of the trajectory of a single spin, we are interested in the asymptotic behavior of *empirical averages* of the form

$$\frac{1}{N}\sum_{j=1}^{N} f(\sigma_j[0,T]) =: \int f d\rho_N \,,$$

where $\{\rho_N\}_{N\geq 1}$ is the sequence of *empirical measures*

$$\rho_N := \frac{1}{N}\sum_{j=1}^{N} \delta_{(\sigma_j[0,T],\eta_j)} \,.$$

We may think of ρ_N as a random element of $\mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{D})$, the space of probability measures on $\mathcal{D}[0,T]\times\mathscr{D}$ endowed with the weak convergence topology. First, we want to determine the weak limit of ρ_N in $\mathcal{M}_1(\mathcal{D}[0,T]\times\mathscr{D})$ as N grows to infinity; i.e. for $f \in \mathcal{C}_b$ we look for $\lim_{N\to+\infty}\int f d\rho_N$. It corresponds to a Law of Large Number with the limit being a deterministic measure. Being an element of $\mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{D})$, such a limit can be viewed as a stochastic process, which describes the dynamics of the system in the infinite volume limit.

5.2.1 Empirical Measure and Large Deviations

Let $W \in \mathcal{M}_1(\mathcal{D}[0,T])$ denote the law of the $\mathscr{S}$-valued process $(\sigma(t))_{t\in[0,T]}$ such that the initial condition $\sigma(0)$ has distribution λ and the spin signs change with constant rate equal to 1. By $W^{\otimes N}$ we mean the product of N copies of W, which represents the law of the N-spin system whose generator is (5.2) where we have set $c_N^{\eta} \equiv 1$; in other words, the law of our system in absence of interaction. Moreover, we shall write P_N^{η} the law of $\underline{\sigma}([0,T]) = (\underline{\sigma}(t))_{t\in[0,T]}$, the process with infinitesimal generator (5.2) and initial distribution $\lambda^{\otimes N}$, for a given $\underline{\eta}$.

Consider $Q \in \mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{D})$, if $\Pi_t Q$ indicates the marginal distribution of Q at time t, then we have

$$m^{\sigma}_{\Pi_t Q} := \int_{\mathscr{S}\times\mathscr{D}} \sigma \Pi_t Q(d\sigma, d\eta)$$

and for a given path $\sigma([0,T]) \in \mathcal{D}[0,T]$, we define

$$F(Q) := \int \Bigg\{ \int_0^T \left(1 - e^{-\beta\sigma(t)\left(m^{\sigma}_{\Pi_t Q}+\eta\right)}\right) dt \\ - \frac{\beta}{2}\left[\sigma(T) m^{\sigma}_{\Pi_T Q} - \sigma(0) m^{\sigma}_{\Pi_0 Q} + \eta\left(\sigma(T) - \sigma(0)\right)\right] \Bigg\} dQ\,. \quad (5.4)$$

We can obtain a representation of P_N^{η} in terms of ρ_N, as follows:

Lemma 5.2.1. *For a fixed realization* $\underline{\eta}$*,*

$$\frac{dP_N^{\eta}}{dW^{\otimes N}}(\underline{\sigma}([0,T])) = \exp[NF(\rho_N(\underline{\sigma}([0,T]), \underline{\eta})) + O(1)]$$

where, for $Q \in \mathcal{M}_1(\mathcal{D}[0,T] \times \mathscr{D})$*,* $F(Q)$ *is expressed by* (5.4)*.*

Lemma 5.2.1 allows us to deduce a Large Deviation Principle for ρ_N, from which we can derive its asymptotic behavior as $N \longrightarrow +\infty$.

Define

$$\mathcal{P}_N(\cdot) := \int \mu^{\otimes N}(d\underline{\eta}) P_N^{\eta}(\rho_N \in \cdot)\,,$$

which is an element of $\mathcal{M}_1(\mathcal{M}_1(\mathcal{D}[0,T]\times\mathscr{D}))$ and represents the law of ρ_N under the joint distribution of the process and the environment.
If $Q\in\mathcal{M}_1(\mathcal{D}[0,T]\times\mathscr{D})$, we denote by

$$H(Q|W\otimes\mu):=\begin{cases}\int dQ\log\frac{dQ}{d(W\otimes\mu)} & \text{if}\quad Q\ll W\otimes\mu\quad\text{and}\quad\log\frac{dQ}{d(W\otimes\mu)}\in L^1(Q)\\ +\infty & \text{otherwise}\end{cases}$$

the relative entropy between Q and $W\otimes\mu$.

Proposition 5.2.1. *The laws $\{\mathcal{P}_N\}_{N\geq 1}$ of ρ_N (under the joint distribution of the process and the medium) obey a Large Deviation Principle with rate function*

$$I(Q):=H(Q|W\otimes\mu)-F(Q)\,.$$

5.2.2 McKean-Vlasov Equation

Given $Q\in\mathcal{M}_1(\mathcal{D}[0,T]\times\mathscr{D})$ and $\eta\in\mathscr{D}$, we can associate with Q a Markov process on $\mathscr{S}$ with law $P^{\eta,Q}$, initial distribution λ and time-dependent infinitesimal generator

$$\mathcal{L}_t^{\eta,Q}f(\sigma)=e^{-\beta\sigma(m^{\sigma}_{\Pi_t Q}+\eta)}\nabla^{\sigma}f(\sigma)\,,$$

acting on $f:\mathscr{S}\longrightarrow\mathbb{R}$.

Proposition 5.2.2. *For every $Q\in\mathcal{M}_1(\mathcal{D}[0,T]\times\mathscr{D})$ such that $I(Q)<+\infty$,*

$$I(Q)=H(Q|P^Q)\,,\tag{5.5}$$

where $P^Q\in\mathcal{M}_1(\mathcal{D}[0,T]\times\mathscr{D})$ is defined by

$$P^Q(d\sigma[0,T],d\eta)=P^{\eta,Q}(d\sigma[0,T])\mu(d\eta).$$

Theorem 5.2.1. *Suppose that the initial distribution of the Markov process $(\underline{\sigma}(t))_{t\geq 0}$ with generator* (5.2) *is such that the random variables $(\sigma_j(0))_{j=1}^N$ are independent and identically distributed with law λ. Then the equation $I(Q)=0$ admits a unique solution $Q_*\in\mathcal{M}_1(\mathcal{D}[0,T]\times\mathscr{D})$, such that its marginals $q_t^\eta=\Pi_t Q_*^\eta\in\mathcal{M}_1(\mathscr{S})$ are weak solutions of the nonlinear McKean-Vlasov equation*

$$\begin{cases}\frac{\partial q_t^\eta}{\partial t}=\mathcal{L}^\eta q_t^\eta & (t\in[0,T],\ \eta\in\mathscr{D})\\ q_0^\eta=\lambda\end{cases}\tag{5.6}$$

where, for all the pairs $(\sigma, \eta) \in \mathscr{S} \times \mathscr{D}$*, the operator* $\mathcal{L}^\eta$ *acts*

$$\mathcal{L}^\eta q_t^\eta(\sigma) = \nabla^\sigma \left[e^{-\beta\sigma\left(m_{q_t}^\sigma + \eta\right)} q_t^\eta(\sigma) \right]$$

and q_t *is defined by*

$$q_t(\sigma) = \int_{\mathscr{D}} q_t^\eta(\sigma)\mu(d\eta).$$

Moreover, with respect to a metric $d(\cdot, \cdot)$ *inducing the weak topology,* $\rho_N \longrightarrow Q_*$ *in probability with exponential rate, i.e.* $\mathcal{P}_N\{d(\rho_N, Q_*) > \varepsilon\}$ *is exponentially small in* N*, for each* $\varepsilon > 0$*.*

5.2.3 Stationary Solution(s)

The equation (5.6) describes the behavior of the system governed by generator (5.2) in the infinite volume limit. We are interested in the detection of the t-stationary solution(s) of this equation. We recall that to be t-stationary solution for (5.6) means to satisfy the equation $\mathcal{L}^\eta q^\eta = 0$ for every t.

Hence, every equilibrium probability distribution is the solution of

$$\nabla^\sigma \left[e^{-\beta\sigma\left(m_{q_t}^\sigma + \eta\right)} q_t^\eta(\sigma) \right] = 0 \tag{5.7}$$

and for our model is characterized as follows.

Lemma 5.2.2. *Every equilibrium distribution for the nonlinear Markov process given by* (5.6) *is of the form:*

$$q_*^\eta(\sigma) = \frac{e^{\beta\sigma(m_*^\sigma + \eta)}}{2\cosh\left(\beta\left(m_*^\sigma + \eta\right)\right)}, \tag{5.8}$$

where $2\cosh\left(\beta\left(m_*^\sigma + \eta\right)\right)$ *is a normalizing factor and the variable* m_*^σ *must satisfy the self-consistency relation*

$$m_{q_*}^\sigma := m_*^\sigma = \int_{\mathscr{S}\times\mathscr{D}} \sigma\, q_*^\eta(d\sigma)\mu(d\eta). \tag{5.9}$$

Proof. As we already said, an equilibrium probability density for (5.6) must satisfy (5.7), which is equivalent to

$$e^{-\beta\sigma(m_*^\sigma + \eta)} q_*^\eta(\sigma) = e^{\beta\sigma(m_*^\sigma + \eta)} q_*^\eta(-\sigma), \tag{5.10}$$

where $m_*^\sigma = \int_{\mathscr{S}\times\mathscr{D}} \sigma q_*^\eta(d\sigma)\mu(d\eta)$. Solving (5.10), we obtain

$$q_*^\eta(\sigma) = e^{\beta\sigma(m_*^\sigma+\eta)}\,,$$

with the normalizing constant

$$Z_*^\eta = \int_{\mathscr{S}} e^{\beta\sigma(m_*^\sigma+\eta)} d\sigma = 2\cosh\left(\beta\left(m_*^\sigma+\eta\right)\right)$$

and the proof is complete. ■

Remark 5.2.1. There is a one-to-one correspondence between equilibrium distributions and solutions of the self-consistency equation (5.9).

Regarding to Remark 5.2.1, any stationary solution of the system (5.6) is of the form

$$\begin{aligned} m_*^\sigma &= \Gamma_{\beta,\mu}(m_*^\sigma) \\ \Gamma_{\beta,\mu}(m_*^\sigma) &= \int_{\mathscr{D}} \tanh(\beta(m_*^\sigma+\eta))\mu(d\eta)\,. \end{aligned} \tag{5.11}$$

$m_*^\sigma \equiv 0$ is always a solution of (5.11), for all the choices of β and μ. Whenever the parameter β take values such that

$$\Gamma'_{\beta,\mu}(0) = \beta\int_{\mathscr{D}} \frac{\mu(d\eta)}{\cosh^2(\beta\eta)} = 1\,, \tag{5.12}$$

this equilibrium results to be neutrally stable for the linearized system. Moreover, if $\Gamma'_{\beta,\mu}(0) > 1$, then there is at least one ferromagnetic solution.

Remark 5.2.2. If $m_*^\sigma \equiv 0$, the stationary distribution reduces to:

$$q_*^{\eta,0}(\sigma) := \frac{e^{\beta\sigma\eta}}{2\cosh(\beta\eta)} \qquad \text{for all } \sigma\in\mathscr{S}. \tag{5.13}$$

5.3 Critical Dynamics ($\beta\int_{\mathscr{D}} \frac{\mu(d\eta)}{\cosh^2(\beta\eta)} = 1$)

We are going to consider the critical dynamics of the system, in other words the long-time behavior of the fluctuations in the threshold case, when $\beta\int_{\mathscr{D}}\frac{\mu(d\eta)}{\cosh^2(\beta\eta)} = 1$. The size of Normal fluctuations must be further rescaled (in space and time), because their size around the deterministic limit increases in time. In this case we will still obtain Normal fluctuations, solutions of a certain stochastic differential equation to be determined.

First of all, we need to locate the critical direction in the infinite dimensional space of the order parameters. In the rest of the section, we will consider $\beta\int_{\mathscr{D}} \frac{\mu(d\eta)}{\cosh^2(\beta\eta)} = 1$ and let us assume that the initial condition λ is a product measure such that

$$q_0(d\sigma, d\eta) = q_*^0(d\sigma, d\eta) = \frac{e^{\beta\sigma\eta}}{2\cosh(\beta\eta)}\, d\sigma\, \mu(d\eta)$$

and so

$$q_t(d\sigma, d\eta) = q_*^0(d\sigma, d\eta) = \frac{e^{\beta\sigma\eta}}{2\cosh(\beta\eta)}\, d\sigma\, \mu(d\eta)\,,$$

for every value of $t \geq 0$, since we are in stationary conditions.

For our model all the observables are of the form $F(\sigma, \eta) = \gamma(\eta) + \sigma\phi(\eta)$ and we can assume, without loss of generality, $\gamma \equiv 0$, since the term $\gamma(\eta)$ does not contribute to the dynamics of the system. Thus, we want to describe the action of the infinitesimal generator of the critical fluctuation flow

$$\widetilde{\rho}_N(t, dx, d\eta) = N^{1/4}\left[\rho_N(N^{1/4}t, dx, d\eta) - \frac{e^{\beta\sigma\eta}}{2\cosh(\beta\eta)}\, d\sigma\, \mu(d\eta)\right]$$

on the family of functions of the form $\psi(\langle\sigma\phi(\eta), \widetilde{\rho}_N\rangle)$, with

$$\psi : \mathbb{R} \longrightarrow \mathbb{R},\ \psi \in \mathcal{C}_b^3(\mathbb{R}), \qquad \phi : \mathscr{D} \longrightarrow \mathbb{R}$$

and where we have denoted $\langle f_1, f_2\rangle := \int_{\mathscr{S}\times\mathscr{D}} f_1(\sigma, \eta) f_2(\sigma, \eta)\, d\sigma\, d\eta$.

Lemma 5.3.1. *For $t \in [0, T]$, if we consider only the space scaling*

$$\bar{\rho}_N(t, d\sigma, d\eta) = N^{1/4}\left[\rho_N(t, d\sigma, d\eta) - \frac{e^{\beta\sigma\eta}}{2\cosh(\beta\eta)}\, d\sigma\, \mu(d\eta)\right], \tag{5.14}$$

then the critical flow $\bar{\rho}_N(t, d\sigma, d\eta)$ is a Markov process whose infinitesimal generator $\mathcal{G}_N$ satisfies:

$$\mathcal{G}_N\psi(\langle\sigma\phi(\eta), \bar{\rho}_N\rangle) = \left[L_0 + L_1 + N^{-1/4}L_2\right]\psi(\langle\sigma\phi(\eta), \bar{\rho}_N\rangle) + o(N^{-1/4}), \tag{5.15}$$

where

$$L_0\psi(\langle\sigma\phi(\eta),\widetilde{\rho}_N\rangle) = 2\,\psi'(\cdot)\,\langle\sinh(\beta\eta)\phi(\eta),\bar{\rho}_N\rangle \tag{5.16}$$

$$L_1\psi(\langle\sigma\phi(\eta),\widetilde{\rho}_N\rangle) = -2\,\psi'(\cdot)\,\langle\sigma\mathfrak{L}^\eta\phi(\eta),\bar{\rho}_N\rangle$$

$$L_2\psi(\langle\sigma\phi(\eta),\widetilde{\rho}_N\rangle) = 2\beta\,\psi'(\cdot)\,\langle\sigma,\bar{\rho}_N\rangle\langle[\cosh(\beta\eta)-\sigma\sinh(\beta\eta)]\phi(\eta),\bar{\rho}_N\rangle$$

and the operator $\mathfrak{L}^\eta$ *is the linear operator given by*

$$\mathfrak{L}^\eta\phi(\eta) = \cosh(\beta\eta)\phi(\eta) - \beta\int_{\mathscr{D}}\frac{\phi(\eta)}{\cosh(\beta\eta)}\mu(d\eta)\,. \tag{5.17}$$

The remainders are continuous functions of $\langle\sigma\phi(\eta),\bar{\rho}_N\rangle$ *and they are of order* $o(N^{-1/4})$ *pointwise, but not uniformly on* $\langle\sigma\phi(\eta),\bar{\rho}_N\rangle$*.*

Proof. To prove that $\langle\sigma\phi(\eta),\bar{\rho}_N\rangle$ is a Markov process, one must write down the expression of the infinitesimal generator whose dynamics are driven by. We apply Lemma 1.3.1.
The process $\{\underline{\sigma}(t)\}_{t\geq 0}$ is a continuous time Markov chain on the finite state space $\mathscr{S}^N$, with infinitesimal generator (5.2). Consider the function

$$\begin{aligned}\zeta:\mathscr{S}&\longrightarrow\quad\mathbb{R}\\ \underline{\sigma}&\longmapsto\langle\sigma\phi(\eta),\bar{\rho}_N\rangle\,,\end{aligned}$$

it plays the role of g in Lemma 1.3.1; then, for every $\psi:\mathscr{S}^N\longrightarrow\mathbb{R}$, we have

$$L_N(\psi\circ\zeta) = (\mathcal{G}_N\psi)\circ\zeta$$

and $\zeta(\underline{\sigma})$ is a Markov process with generator $\mathcal{G}_N$ given by

$$\mathcal{G}_N\psi(\langle\sigma\phi(\eta),\bar{\rho}_N\rangle) =$$

$$= \sum_{j=1}^N\underbrace{[\cosh(\beta\eta_j)-\sigma_j\sinh(\beta\eta_j)]}_{=e^{-\beta\sigma_j\eta_j}}\left(1-\frac{\beta\langle\sigma,\bar{\rho}_N\rangle}{N^{1/4}}\sigma_j+\frac{\beta^2\langle\sigma,\bar{\rho}_N\rangle^2}{N^{1/2}}+o\left(\frac{1}{N^{1/2}}\right)\right)\cdot$$
$$\cdot\left[-\frac{2}{N^{3/4}}\sigma_j\phi(\eta_j)\psi'(\cdot)+\frac{2}{N^{3/2}}\phi^2(\eta_j)\psi''(\cdot)+o\left(\frac{1}{N^{3/2}}\right)\right]$$

$$= \psi'(\cdot)\frac{2}{N^{3/4}}\Bigg\{ -\sum_{j=1}^{N}\sigma_j\cosh(\beta\eta_j)\phi(\eta_j) + \sum_{j=1}^{N}\sinh(\beta\eta_j)\phi(\eta_j)$$
$$+\frac{\beta\langle\sigma,\bar{\rho}_N\rangle}{N^{1/4}}\sum_{j=1}^{N}\cosh(\beta\eta_j)\phi(\eta_j) - \frac{\beta\langle\sigma,\bar{\rho}_N\rangle}{N^{1/4}}\sum_{j=1}^{N}\sigma_j\sinh(\beta\eta_j)\phi(\eta_j)$$
$$-\frac{\beta^2\langle\sigma,\bar{\rho}_N\rangle^2}{N^{1/2}}\sum_{j=1}^{N}\sigma_j\cosh(\beta\eta_j)\phi(\eta_j) + \frac{\beta^2\langle\sigma,\bar{\rho}_N\rangle^2}{N^{1/2}}\sum_{j=1}^{N}\sinh(\beta\eta_j)\phi(\eta_j)\Bigg\}$$

$$+\psi''(\cdot)\frac{2}{N^{3/2}}\Bigg\{\sum_{j=1}^{N}\cosh(\beta\eta_j)\phi^2(\eta_j) - \sum_{j=1}^{N}\sigma_j\sinh(\beta\eta_j)\phi^2(\eta_j)$$
$$-\frac{\beta\langle\sigma,\bar{\rho}_N\rangle}{N^{1/4}}\sum_{j=1}^{N}\sigma_j\cosh(\beta\eta_j)\phi^2(\eta_j) + \frac{\beta\langle\sigma,\bar{\rho}_N\rangle}{N^{1/4}}\sum_{j=1}^{N}\sinh(\beta\eta_j)\phi^2(\eta_j)$$
$$+\frac{\beta^2\langle\sigma,\bar{\rho}_N\rangle^2}{N^{1/2}}\sum_{j=1}^{N}\cosh(\beta\eta_j)\phi^2(\eta_j) - \frac{\beta^2\langle\sigma,\bar{\rho}_N\rangle^2}{N^{1/2}}\sum_{j=1}^{N}\sigma_j\sinh(\beta\eta_j)\phi^2(\eta_j)\Bigg\}$$

$$+o\left(\frac{1}{N^{1/2}}\right)$$

Since $\bar{\rho}_N$ is a centered measure and we want to represent all the terms as integrals with respect to this measure, we need to center all of them; but

$$\langle[-\sigma\cosh(\beta\eta) + \sinh(\beta\eta)]\phi(\eta), q_*^0\rangle = 0$$

and

$$\langle[\cosh(\beta\eta) - \sigma\sinh(\beta\eta)]\phi(\eta), q_*^0\rangle = \int_{\mathscr{D}}\frac{\phi(\eta)}{\cosh(\beta\eta)}\mu(d\eta)\,,$$

hence we obtain

$$= 2\psi'(\cdot)\Bigg\{\langle\sinh(\beta\eta)\phi(\eta),\bar{\rho}_N\rangle - \langle\sigma\cosh(\beta\eta)\phi(\eta),\bar{\rho}_N\rangle$$
$$+\frac{\beta\langle\sigma,\bar{\rho}_N\rangle}{N^{1/4}}\langle\cosh(\beta\eta)\phi(\eta),\bar{\rho}_N\rangle - \frac{\beta\langle\sigma,\bar{\rho}_N\rangle}{N^{1/4}}\langle\sigma\sinh(\beta\eta)\phi(\eta),\bar{\rho}_N\rangle$$
$$-\beta\langle\sigma,\bar{\rho}_N\rangle\int_{\mathscr{D}}\frac{\phi(\eta)}{\cosh(\beta\eta)}\mu(d\eta) - \frac{\beta^2\langle\sigma,\bar{\rho}_N\rangle^2}{N^{1/2}}\langle\sigma\cosh(\beta\eta)\phi(\eta),\bar{\rho}_N\rangle$$
$$+\frac{\beta^2\langle\sigma,\bar{\rho}_N\rangle^2}{N^{1/2}}\langle\sinh(\beta\eta)\phi(\eta),\bar{\rho}_N\rangle\Bigg\} + o\left(\frac{1}{N^{1/4}}\right)$$

$$
\begin{aligned}
= 2\psi'(\cdot)\Bigg\{ - \Bigg\langle \sigma \left[\cosh(\beta\eta)\phi(\eta) - \beta \int_{\mathscr{D}} \frac{\phi(\eta)}{\cosh(\beta\eta)}\mu(d\eta)\right], \bar{\rho}_N \Bigg\rangle \\
+ \langle \sinh(\beta\eta)\phi(\eta), \bar{\rho}_N\rangle + \frac{\beta\langle\sigma, \bar{\rho}_N\rangle}{N^{1/4}} \langle \cosh(\beta\eta)\phi(\eta), \bar{\rho}_N\rangle \\
- \frac{\beta\langle\sigma, \bar{\rho}_N\rangle}{N^{1/4}} \langle \sigma \sinh(\beta\eta)\phi(\eta), \bar{\rho}_N\rangle \Bigg\} + o\left(\frac{1}{N^{1/4}}\right).
\end{aligned}
$$

which is just (5.15). ■

Remark 5.3.1. The term $\langle \sinh(\beta\eta)\phi(\eta), \bar{\rho}_N\rangle$ appearing in L_0, defined by (5.16), is of order $N^{1/4}$ and gives rise to the random drift.

Lemma 5.3.2. *The operator* $\mathfrak{L}^\eta$*, defined by* (5.17)*, is self-adjoint in* $L^2(\bar{\mu})$*, where* $\bar{\mu} := \frac{\mu(d\eta)}{\cosh(\beta\eta)}$.

Proof. Obviously $\mathfrak{L}^\eta$ is a linear and continuous operator. If we mean $\langle f_1, f_2\rangle_{L^2(\bar{\mu})} := \int_{\mathscr{D}} f_1(\eta) f_2(\eta) \frac{\mu(d\eta)}{\cosh(\beta\eta)}$, we have to prove the following: if $\phi_1, \phi_2 \in L^2(\bar{\mu})$, then $\langle \mathfrak{L}^\eta\phi_1, \phi_2\rangle_{L^2(\bar{\mu})} = \langle \phi_1, \mathfrak{L}^\eta\phi_2\rangle_{L^2(\bar{\mu})}$. Thus,

$$
\begin{aligned}
\langle \mathfrak{L}^\eta\phi_1, \phi_2\rangle_{L^2(\bar{\mu})} &= \int_{\mathscr{D}} \left[\cosh(\beta\eta)\phi_1(\eta) - \beta \int_{\mathscr{D}} \frac{\phi_1(\eta)}{\cosh(\beta\eta)}\mu(d\eta)\right] \phi_2(\eta)\bar{\mu}(d\eta) \\
&= \int_{\mathscr{D}} [\cosh(\beta\eta)\phi_2(\eta)]\phi_1(\eta)\bar{\mu}(d\eta) \\
&\qquad - \beta \int_{\mathscr{D}} \frac{\phi_1(\eta)}{\cosh(\beta\eta)}\mu(d\eta) \int_{\mathscr{D}} \frac{\phi_2(\eta)}{\cosh(\beta\eta)}\mu(d\eta) \\
&= \int_{\mathscr{D}} \left[\cosh(\beta\eta)\phi_2(\eta) - \beta \int_{\mathscr{D}} \frac{\phi_2(\eta)}{\cosh(\beta\eta)}\mu(d\eta)\right] \phi_1(\eta)\bar{\mu}(d\eta) \\
&= \langle \phi_1, \mathfrak{L}^\eta\phi_2\rangle_{L^2(\bar{\mu})}
\end{aligned}
$$

and the proof is concluded. ■

Lemma 5.3.3. *The operator* $\mathfrak{L}^\eta$*, defined by* (5.17)*, is positive and its kernel is spanned by the function* $\frac{1}{\cosh(\beta\eta)}$.

Proof. To prove positivity we have to show that $\langle \phi(\eta), \mathfrak{L}^\eta \phi(\eta)\rangle_{L^2(\bar{\mu})} \geq 0$; but,

$$\langle \phi(\eta), \mathfrak{L}^\eta \phi(\eta)\rangle_{L^2(\bar{\mu})} = \int_{\mathscr{D}} \left[\cosh(\beta\eta)\phi(\eta) - \beta \int_{\mathscr{D}} \frac{\phi(\eta)}{\cosh(\beta\eta)}\mu(d\eta)\right] \phi(\eta)\bar{\mu}(d\eta)$$

$$= \int_{\mathscr{D}} \phi^2(\eta)\mu(d\eta) - \beta \left(\int_{\mathscr{D}} \frac{\phi(\eta)}{\cosh(\beta\eta)}\mu(d\eta)\right)^2$$

$$= \frac{1}{\beta}\int_{\mathscr{D}} \cosh^2(\beta\eta)\phi^2(\eta)\frac{\beta}{\cosh^2(\beta\eta)}\mu(d\eta) - \frac{1}{\beta}\left(\int_{\mathscr{D}} \cosh(\beta\eta)\phi(\eta)\frac{\beta}{\cosh^2(\beta\eta)}\mu(d\eta)\right)^2 \geq 0$$

by Jensen's inequality, recalling that we are at the critical point, in other words $\beta \int_{\mathscr{D}} \frac{\mu(d\eta)}{\cosh^2(\beta\eta)} = 1$. Moreover, the equality holds true whenever $\cosh(\beta\eta)\phi(\eta)$ is constant, therefore the null space of the operator $\mathfrak{L}^\eta$ is generated by the functions of the form $\phi(\eta) = \frac{1}{\cosh(\beta\eta)}$. ■

Remark 5.3.2. The critical direction for our model is then $\left\langle \frac{\sigma}{\cosh(\beta\eta)}, \widetilde{\rho}_N(t)\right\rangle$.

The space $L^2\left(\frac{\mu(d\eta)}{\cosh(\beta\eta)}\right)$ is finite dimensional and then the spectrum of the operator $\mathfrak{L}^\eta$ is discrete and entirely composed by positive eigenvalues. Moreover, there exists an orthonormal basis of eigenvectors of $\mathfrak{L}^\eta$ in $L^2\left(\frac{\mu(d\eta)}{\cosh(\beta\eta)}\right)$.

Theorem 5.3.1. *Consider $\{\phi_m(\eta)\}_{m=0}^{2i+1}$, an orthonormal basis of eigenvectors of $\mathfrak{L}^\eta$ in $L^2\left(\frac{\mu(d\eta)}{\cosh(\beta\eta)}\right)$, and suppose that the function $\frac{1}{\cosh(\beta\eta)} = \phi_0(\eta)$.*
For $t \in [0,T]$, if we consider the critical fluctuation process

$$\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(t)\rangle\,, \quad \{\langle \sigma\phi_m(\eta), \widetilde{\rho}_N(t)\rangle\}_{m=1}^{2i+1}$$

then, as $N \longrightarrow +\infty$, $\{\langle \sigma\phi_m(\eta), \widetilde{\rho}_N(t)\rangle\}_{m=1}^{2i+1} \longrightarrow 0$ in the sense of Proposition 1.4.1 and $\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(t)\rangle$ converges, in the sense of weak convergence of stochastic processes, to a limiting Gaussian process

$$X(t) = 2\mathscr{H}t\,,$$

with $\mathscr{H}$ a Normal random variable.

5.4 Proof of the Theorem 5.3.1

Let us denote by $\{\tau_N^M\}_{N\geq 1}$ a family of stopping times, defined as

$$\tau_N^M := \inf_{t\geq 0}\Big\{ |\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(t)\rangle| \geq M \quad \text{or} \quad |\langle \sigma\phi_m(\eta), \widetilde{\rho}_N(t)\rangle| \geq M \\ \text{for at least a value of } m = 1, \dots, 2i+1 \Big\},$$

where M is a positive constant. We are interested in introducing such a sequence of stopping times because in this way the processes $\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(t)\rangle$ and $\{\langle \sigma\phi_m(\eta), \widetilde{\rho}_N(t)\rangle\}_{m=1}^{2i+1}$ result to be bounded in the time interval $[0, T \wedge \tau_N^M]$.

By standard argument on collapsing processes (see Proposition 1.4.1 and Lemma 3.3.6), it is easy to prove that for $t \in [0, T\wedge\tau_N^M]$ the directions $\{\langle \sigma\phi_m(\eta), \widetilde{\rho}_N(t)\rangle\}_{m=1}^{2i+1}$ collapse. It means that, if we consider the norm $\|\widetilde{\rho}_N\|_r$, defined by

$$\|\widetilde{\rho}_N\|_r^2 := \sum_{m=1}^{2i+1} \frac{1}{(1+m^2)^r} \langle \sigma\phi_m(\eta), \widetilde{\rho}_N\rangle^2 ,$$

where $r > 0$, then there exist constants N_0, C, $d > 2$, $\kappa_N := \kappa(N)$ and two increasing sequences $\{\alpha_N\}_{N\geq 1}$, $\{\beta_N\}_{N\geq 1}$ satisfying (3.23)–(3.26) and such that for every $\varepsilon > 0$ the following is true

$$\sup_{N\geq N_0} P\left\{ \sup_{0\leq t\leq T\wedge\tau_N^M} \|\widetilde{\rho}_N(t)\|_r^2 > C\left(\kappa_N^{1/d}\alpha_N^{-1} \vee \kappa_N^{-1}\alpha_N\right)\right\} \leq \varepsilon . \tag{5.18}$$

The property (5.18) allows us to deduce that $\{\langle \sigma\phi_m(\eta), \widetilde{\rho}_N(t)\rangle\}_{m=1}^{2i+1} \longrightarrow 0$, as $N \longrightarrow +\infty$. The computations we should do to prove these processes converge to zero in probability are similar to those we did in Subsection 3.3.1 to prove the process representing the non-critical directions of the homogeneous Kuramoto Model collapses. Thus, we omit this proof and *we focus only on the critical direction* $\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(t)\rangle$*, assuming all the others vanish.*

Remark 5.4.1. It is possible to prove (5.18) since all the eigenvalues corresponding to the eigenvectors $\{\phi_m(\eta)\}_{m=1}^{2i+1}$ are negative.

The next step is to prove, for every $\varepsilon > 0$ and $N \geq 1$, the existence of a constant $M > 0$ such that

$$P\left\{\tau_N^M \leq T\right\} \leq \epsilon .$$

This fact implies the processes $\{\langle\sigma\phi_m(\eta),\widetilde{\rho}_N(t)\rangle\}_{m=1}^{2i+1}$ converge to zero in probability, as N is growing to infinity, for t belonging to the whole time interval $[0,T]$.

We consider the infinitesimal generator, $\mathcal{J}_N = N^{1/4}\mathcal{G}_N$, subject to the time-rescaling and we apply it to the particular function

$$\psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(t)\rangle\right) = \left|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(t)\rangle\right| .$$

The following decomposition holds

$$\begin{aligned}\left|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(t)\rangle\right| &= \left|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(0)\rangle\right| + \int_0^t \mathcal{J}_N \left|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(s)\rangle\right| ds + \mathcal{M}^t_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|} \\ &\leq \left|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(0)\rangle\right| + \int_0^t \left|\mathcal{J}_N \left|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(s)\rangle\right|\right| ds + \mathcal{M}^t_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|} ,\end{aligned}$$

with

$$\mathcal{M}^t_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|} = \int_0^t \sum_{j\in\mathscr{S},k\in\mathscr{D}} \overline{\nabla}^{(j)}\left[\left|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(s)\rangle\right|\right] \widetilde{\Lambda}^\sigma_N(j,k,ds) ,$$

where we have defined

$$\begin{aligned}&\overline{\nabla}^{(j)}\left[\left|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(t)\rangle\right|\right] = \\ &\qquad = \left|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(t)\rangle - j\frac{2}{N^{3/4}\cosh(\beta k)}\right| - \left|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(t)\rangle\right| \end{aligned} \tag{5.19}$$

and

$$\widetilde{\Lambda}^\sigma_N(j,k,dt) := \Lambda^\sigma_N(j,k,dt) - \underbrace{N^{1/4}\left|A(j,k,N^{1/4}t)\right| e^{-\beta j\left(\frac{\langle\sigma,\widetilde{\rho}_N(t)\rangle}{N^{1/4}}+k\right)} dt}_{:=\lambda^\sigma(j,k,t)\,dt} . \tag{5.20}$$

As we can clearly see, the quantity $\widetilde{\Lambda}^\sigma_N(j,k,dt)$ is the difference between the point process $\Lambda^\sigma_N(j,k,dt)$, defined on $\mathscr{S}\times\mathscr{D}\times\mathbb{R}^+$, and its intensity $\lambda^\sigma(j,k,t)\,dt$. The counter $\left|A(j,k,N^{1/4}t)\right|$ is given by

$$\begin{aligned}\left|A(j,k,N^{1/4}t)\right| = \frac{N}{4}\Bigg[1 &+ \frac{1}{k}\frac{\langle\eta,\widetilde{\rho}_N\rangle}{N^{1/4}} + j\frac{\langle\sigma,\widetilde{\rho}_N\rangle}{N^{1/4}} \\ &+ \frac{j}{k}\left(\frac{\langle\sigma\eta,\widetilde{\rho}_N\rangle}{N^{1/4}} - \int_{\mathscr{D}} \eta\tanh(\beta\eta)\mu(d\eta)\right)\Bigg] .\end{aligned}$$

We recall that the expression of $\mathcal{G}_N$ is given by (5.15). We consider the following Taylor expansions stopped at second order

$$e^{\beta\frac{\langle\sigma,\widetilde{\rho}_N\rangle}{N^{1/4}}} = 1 + \beta\frac{\langle\sigma,\widetilde{\rho}_N\rangle}{N^{1/4}} + R_+ \quad \text{and} \quad e^{-\beta\frac{\langle\sigma,\widetilde{\rho}_N\rangle}{N^{1/4}}} = 1 - \beta\frac{\langle\sigma,\widetilde{\rho}_N\rangle}{N^{1/4}} + R_- \,,$$

where

$$|R_+| \le \sup\left\{e^z : z \in \left[0, \beta\frac{\langle\sigma,\widetilde{\rho}_N\rangle}{N^{1/4}}\right]\right\} \frac{\beta^2\langle\sigma,\widetilde{\rho}_N\rangle^2}{2N^{1/2}} \le \frac{\beta^2 M^2}{2N^{1/2}}\, e^{\frac{\beta M}{N^{1/4}}}$$

and

$$|R_-| \le \sup\left\{e^z : z \in \left[-\beta\frac{\langle\sigma,\widetilde{\rho}_N\rangle}{N^{1/4}}, 0\right]\right\} \frac{\beta^2\langle\sigma,\widetilde{\rho}_N\rangle^2}{2N^{1/2}} \le \frac{\beta^2 M^2}{2N^{1/2}}\,.$$

For $t \in [0, \tau_N^M]$ we can estimate

$$|\mathcal{J}_N\,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(t)\rangle|| =$$

$$\begin{aligned}
= \Bigg| 2N^{1/4}\mathrm{sgn}\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(t)\rangle\right) \Bigg\{ &\langle\tanh(\beta\eta),\widetilde{\rho}_N(t)\rangle \\
&+ \beta\frac{\langle\sigma,\widetilde{\rho}_N(t)\rangle}{N^{1/4}}\langle\coth(\beta\eta),\widetilde{\rho}_N(t)\rangle - \beta\frac{\langle\sigma,\widetilde{\rho}_N(t)\rangle}{N^{1/4}}\langle\sigma\tanh(\beta\eta),\widetilde{\rho}_N(t)\rangle \\
&+ R_+\Bigg[N^{1/4} + \langle\tanh(\beta\eta),\widetilde{\rho}_N(t)\rangle - \langle\sigma,\widetilde{\rho}_N(t)\rangle - \langle\sigma\tanh(\beta\eta),\widetilde{\rho}_N(t)\rangle \\
&+ \int_{\mathscr{D}}\tanh^2(\beta\eta)\mu(d\eta)\Bigg] - R_-\Bigg[N^{1/4} - \langle\tanh(\beta\eta),\widetilde{\rho}_N(t)\rangle + \langle\sigma,\widetilde{\rho}_N(t)\rangle \\
&- \langle\sigma\tanh(\beta\eta),\widetilde{\rho}_N(t)\rangle + \int_{\mathscr{D}}\tanh^2(\beta\eta)\mu(d\eta)\Bigg]\Bigg\}\Bigg|
\end{aligned}$$

and thanks to the stopping times we have introduced and the Central Limit Theorem applying to the processes of the form $\langle\phi(\eta),\widetilde{\rho}_N(t)\rangle$ (i.e. for every $\varepsilon > 0$ and sufficiently large M, $P\{N^{1/4}|\langle\phi(\eta),\widetilde{\rho}_N(t)\rangle| \ge M\} \le \varepsilon$),

$$\le 2\left\{M + \frac{2\beta M^2}{N^{1/4}} + \frac{\beta^2 M^2}{2N^{1/2}}\left(e^{\frac{\beta M}{N^{1/4}}} + 1\right)\left(N^{1/4} + \frac{M}{N^{1/4}} + 2M + 1\right)\right\}$$

$$\le 2\left\{M + 2\beta M^2 + \beta^2 M^2\left(e^{\beta M} + 1\right)(2 + 3M)\right\} =: C_8\,,$$

with C_8 positive constant independent of N. Since the following inclusions are valid

$$\begin{aligned}\{\tau_N^M \le T\} &\subseteq \left\{ \sup_{0\le t\le T\wedge\tau_N^M} \left\{ |\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(t)\rangle|\,, \|\widetilde{\rho}_N(t)\|_r^2 \right\} \ge M \right\} \\ &\subseteq \left\{ \sup_{0\le t\le T\wedge\tau_N^M} \|\widetilde{\rho}_N(t)\|_r^2 \ge M \right\} \cup \{ |\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(0)\rangle| \ge C_9 \} \cup \\ &\qquad \cup \left\{ \sup_{0\le t\le T\wedge\tau_N^M} \left| M^t_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|} \right| \ge C_{10} \right\},\end{aligned}$$

we obtain the following inequality for the probability of the interested set

$$\begin{aligned}P\{\tau_N^M \le T\} \le P\left\{ \sup_{0\le t\le T\wedge\tau_N^M} \|\widetilde{\rho}_N(t)\|_r^2 \ge M \right\} + P\left\{ |\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(0)\rangle| \ge C_9 \right\} \\ + P\left\{ \sup_{0\le t\le T\wedge\tau_N^M} \left| M^t_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|} \right| \ge C_{10} \right\}.\end{aligned}$$

We estimate the three terms of the right-hand side of the inequality.

- For any $\varepsilon > 0$, thanks to (5.18) we have

$$P\left\{ \sup_{0\le t\le T\wedge\tau_N^M} \|\widetilde{\rho}_N(t)\|_r^2 \ge C\left(\kappa_N^{1/2d}\alpha_N^{-1/2} \vee \kappa_N^{-1/2}\beta_N^{1/2}\right) \right\} \le \varepsilon\,.$$

- We get $E\left[\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(0)\rangle\right] = N^{1/4}E\left[\langle\sigma\phi_0(\eta),\rho_N(0)\rangle\right]$. Since at time $t = 0$ the spins are distributed according to a product measure, $\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(0)\rangle$ is a linear combination of sample average of independent, identically distributed Bernoulli random variables multiplied by $N^{1/4}$. So, we can conclude

$$E\left[|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(0)\rangle|\right] \le \sqrt{\operatorname{Var}\left(\frac{\sigma_1(0)}{\cosh(\beta\eta_1)}\right)}\, N^{-1/4}$$

  and in the limit as $N \longrightarrow +\infty$, we have convergence to zero in L^1 and then in probability. Therefore

$$P\left\{ |\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(0)\rangle| \ge C_9 \right\} \le \varepsilon$$

  for any $\varepsilon > 0$, for every N and for a sufficiently large C_9.

► We reduce to deal with $E\left[\left(\mathcal{M}^T_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|}\right)^2\right]$; in fact, Doob's "maximal inequality in L^p" (case $p=2$) for martingales (we refer to Chapter VII, Section 3 of [Shi96]) tells us that

$$P\left\{\sup_{0\le t\le T\wedge\tau_N^M}\left|\mathcal{M}^t_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|}\right|\ge C_{10}\right\}\le\frac{E\left[\left(\mathcal{M}^T_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|}\right)^2\right]}{(C_{10})^2}\,.$$

Hence, remembering (5.19) and (5.20), we are able to compute

$$E\left[\left(\mathcal{M}^T_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|}\right)^2\right]=$$

$$=E\left[\int_0^T\sum_{j\in\mathscr{S},k\in\mathscr{D}}\left[\overline{\nabla}^{(j)}\left[|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(t)\rangle|\right]\right]^2\lambda^\sigma(j,k,t)dt\right]$$

$$\le E\left[\int_0^T\frac{8(2i+1)}{N^{5/4}}\sup_{k\in\mathscr{D}}\left\{\frac{1}{\cosh(\beta k)}\right\}\sup_{j\in\mathscr{S},k\in\mathscr{D}}\left\{|A(j,k,N^{1/4}t)|e^{\beta(1+k)}\right\}dt\right]$$

$$\le E\left[\int_0^T\frac{2(2i+1)}{N^{1/4}}\left(2+3h_i\right)e^{\beta(1+h_i)}dt\right]$$

$$\le 2(2i+1)\left(2+3h_i\right)e^{\beta(1+h_i)}\,T=:C_{11}\,,$$

with C_{11} positive constant independent of N and M. We have established that, if we choose $C_{10}\ge\sqrt{\frac{C_{11}}{\varepsilon}}$, then

$$P\left\{\sup_{0\le t\le T\wedge\tau_N^M}\left|\mathcal{M}^t_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|}\right|\ge C_{10}\right\}\le\varepsilon\,.$$

In summary, we proved the inequality we were looking for; in fact

$$P\left\{\tau_N^M\le T\right\}\le 3\varepsilon:=\epsilon\,.$$

We have just concluded the proof of the first part of the statement of Theorem 5.3.1, concerning the collapse of the processes $\{\langle\sigma\phi_m(\eta),\widetilde{\rho}_N(t)\rangle\}_{m=1}^{2i+1}$ in the limit as $N\longrightarrow+\infty$ and for $t\in[0,T]$. Now, we are going to show that in the same setting, i.e. the limit of infinite volume and $t \in [0,T]$, the process $\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(t)\rangle$ admits a limiting process and we are going to compute it.

First, we need to prove the tightness of the sequence $\{\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(t)\rangle\}_{N\geq 1}$. This property implies the existence of convergent subsequences. Secondly, we will verify that all the convergent subsequences have the same limit and hence also the sequence $\{\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(t)\rangle\}_{N\geq 1}$ must converge to that limit.

Lemma 5.4.1. *The sequence* $\{\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(t)\rangle\}_{N\geq 1}$ *is tight.*

Proof. We must verify the conditions (1.44) and (1.45) hold. Since we have already proved that for every $\epsilon > 0$ the inequality $P\{\tau_N^M \leq T\} \leq \epsilon$ is true for M sufficiently large and uniformly in N, it is enough to show tightness for the stopped processes

$$\left\{\left\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(t\wedge\tau_N^M)\right\rangle\right\}_{N\geq 1}.$$

We showed before the validity of the following inclusion

$$\left\{\sup_{0\leq t\leq T\wedge\tau_N^M} |\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(t)\rangle| \geq M\right\} \subseteq \{|\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(0)\rangle| \geq C_9\} \cup \\ \cup \left\{\sup_{0\leq t\leq T\wedge\tau_N^M} \left|\mathcal{M}^t_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|}\right| \geq C_{10}\right\},$$

therefore

$$\sup_N P\left\{\sup_{0\leq t\leq T\wedge\tau_N^M} |\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(t)\rangle| \geq M\right\} \leq 2\varepsilon$$

and so we obtained (1.44). Let us deal with (1.45) now. We notice that

$$|\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(t)\rangle - \langle \sigma\phi_0(\eta), \widetilde{\rho}_N(s)\rangle| = \\ = \left|\int_s^t \mathcal{J}_N\left(\langle \sigma\phi_0(\eta), \widetilde{\rho}_N(u)\rangle\right) du + \mathcal{M}^{s,t}_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|}\right|,$$

where we have denoted

$$\mathcal{M}^{s,t}_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|} = -\frac{2}{N^{3/4}}\int_s^t \sum_{j\in\mathscr{S},k\in\mathscr{D}} \frac{j}{\cosh(\beta k)}\,\widetilde{\Lambda}_N^\sigma(j,k,du)$$

and $\widetilde{\Lambda}_N^\sigma$ is as in definition (5.20). Thus,

$$\{|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(t)\rangle-\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(s)\rangle|\geq\alpha\}\subseteq$$
$$\subseteq\left\{\underbrace{\left|\int_s^t\mathcal{J}_N\,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(u)\rangle|\,du\right|}_{\leq C_8(t-s)}+\left|\mathcal{M}^{s,t}_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|}\right|\geq\alpha\right\}$$
$$\subseteq\left\{\left|\mathcal{M}^{s,t}_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|}\right|\geq\overline{C}_{10}\right\}$$

and then, applying Chebyscev inequality to the last right-handside of the previous inclusions, we get

$$\sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\delta}}P\left\{\left|\mathcal{M}^{s,t}_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|}\right|\geq\overline{C}_{10}\right\}\leq(\overline{C}_{10})^{-2}\sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\delta}}E\left[\left(\mathcal{M}^{s,t}_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|}\right)^2\right]$$
$$\leq(\overline{C}_{10})^{-2}\sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\delta}}\underbrace{2(2i+1)(2+3h_i)e^{\beta(1+h_i)}}_{:=\overline{C}_{11}}(t-s) \tag{5.21}$$
$$\leq(\overline{C}_{10})^{-2}\,\overline{C}_{11}\,\delta\,.$$

Finally, we can conclude that

$$\sup_N\sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\delta}}P\{|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(t)\rangle-\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(s)\rangle|\geq\alpha\}\leq$$
$$\leq\sup_N\sup_{\substack{0\leq s\leq t\leq T\\ t-s\leq\delta}}P\left\{\left|\mathcal{M}^{s,t}_{N,|\langle\sigma\phi_0(\eta),\widetilde{\rho}_N\rangle|}\right|\geq\overline{C}_{10}\right\}$$
$$\leq(\overline{C}_{10})^{-2}\,\overline{C}_{11}\,\delta=O(\delta)$$

and the proof is complete. ■

Lemma 5.4.1 implies that there exist convergent subsequences for the sequence $\{\langle\sigma\phi_0(\eta),\widetilde{\rho}_N(t)\rangle\}_{N\geq1}$. Let $\{\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle\}_{n\geq1}$ denote one of such a subsequence and let $\psi\in\mathcal{C}_b^3$ be a function of the type

$$\psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle\,,\{\langle\sigma\phi_m(\eta),\widetilde{\rho}_n(t)\rangle\}_{m=1}^{2i+1}\right)=\psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle\right)\,.$$

The following decomposition holds

$$\psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle\right)-\psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(0)\rangle\right)= \\ =\int_0^t \mathcal{J}_n\psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(u)\rangle\right)du+\mathcal{M}^t_{n,\psi}\,, \quad (5.22)$$

where

$$\mathcal{J}_n\psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle\right)=2\psi'(\cdot)\Big\{n^{1/4}\langle\tanh(\beta\eta),\widetilde{\rho}_n(t)\rangle \\ +\beta\langle\sigma,\widetilde{\rho}_n(t)\rangle\langle 1,\widetilde{\rho}_n(t)\rangle-\beta\langle\sigma,\widetilde{\rho}_n(t)\rangle\langle\sigma\tanh(\beta\eta),\widetilde{\rho}_n(t)\rangle\Big\}+o_M(1),$$

which, as usual, is $\mathcal{G}_N$ (see (5.15)) rescaled of a power $n^{1/4}$ and applied to the particular function $\psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle,\{\langle\sigma\phi_m(\eta),\widetilde{\rho}_n(t)\rangle\}_{m=1}^{2i+1}\right)=\psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle\right)$. The remainder $o_M(1)$ goes to zero as $n\longrightarrow+\infty$, uniformly in M. If we compute the limit as $n\longrightarrow+\infty$, remembering that a Central Limit Theorem applies to the term $\langle\tanh(\beta\eta),\widetilde{\rho}_n(t)\rangle$, $\langle 1,\widetilde{\rho}_n(t)\rangle$ is zero since $\widetilde{\rho}_n$ is a centered measure and the process $\langle\sigma\tanh(\beta\eta),\widetilde{\rho}_n(t)\rangle$ collapse since $\tanh(\beta\eta)$ and $\frac{1}{\cosh(\beta\eta)}$ are perpendicular in $L^2\left(\frac{\mu(d\eta)}{\cosh(\beta\eta)}\right)$, we have:

$$\mathcal{J}_n\psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle\right)\xrightarrow[w]{n\to+\infty}\mathcal{J}\psi(X(t))\,,$$

with

$$\mathcal{J}\psi(X(t))=2\,\mathscr{H}\,\psi'(\cdot)$$

and $\mathscr{H}$ is a Standard Gaussian random variable. Then, because of (5.22), we obtain

$$\mathcal{M}^t_{n,\psi}\xrightarrow[w]{n\to+\infty}\mathcal{M}^t_{\psi}:=\psi(X(t))-\psi(X(0))-\int_0^t\mathcal{J}\psi(X(u))du\,.$$

We must prove the following Lemma:

Lemma 5.4.2. *M^t_ψ is a martingale (with respect to t); in other words, for all $s,t\in[0,T]$, $s\le t$ and for all measurable and bounded functions $g(X([0,s]))$ the following identity holds:*

$$E[\mathcal{M}^t_\psi g(X([0,s]))]=E[\mathcal{M}^s_\psi g(X([0,s]))]\,. \quad (5.23)$$

Proof. The reasoning we explained in Lemma 1.4.5 applies in this case too, so it is sufficient to prove $\{\mathcal{M}^t_{n,\psi}\}_{n\geq 1}$ is an uniformly integrable sequence of random variables.
If we define

$$\overline{\nabla}^{(j)}\left[\psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle\right)\right] := \\ := \psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle - j\frac{2}{n^{3/4}\cosh(\beta k)}\right) - \psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle\right),$$

it yields

$$E[(\mathcal{M}^t_{n,\psi})^2] = E\left[\int_0^t \sum_{j\in\mathscr{S},k\in\mathscr{D}} \left[\overline{\nabla}^{(j)}\left[\psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle\right)\right]\right]^2 \lambda^\sigma(j,k,s)ds\right]$$

$$\leq n^{5/4}\,\overline{C}_{11}\,E\left[\int_0^t \sum_{j\in\mathscr{S},k\in\mathscr{D}} \left[\overline{\nabla}^{(j)}\left[\psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle\right)\right]\right]^2 ds\right]$$

$$\leq n^{5/4}\,\overline{C}_{11}\,E\left[\int_0^t \sum_{j\in\mathscr{S},k\in\mathscr{D}} \left[\psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle - j\frac{2}{n^{3/4}\cosh(\beta k)}\right) \right.\right. \\ \left.\left. - \psi\left(\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle\right)\right]^2 ds\right]$$

where $\overline{C}_{11}$ is defined by (5.21). We expand the function ψ around $\langle\sigma\phi_0(\eta),\widetilde{\rho}_n(t)\rangle$ with the Taylor expansion stopped at first order and with remainder R such that

$$|R| \leq \sup\left\{|\psi''(z)| : z\in\left[\langle\sigma\phi_0,\widetilde{\rho}_n\rangle, \langle\sigma\phi_0,\widetilde{\rho}_n\rangle - j\frac{2}{n^{3/4}\cosh(\beta k)}\right]\right\}\frac{2}{n^{3/2}\cosh^2(\beta k)}$$

and moreover, we recall that $\psi\in\mathcal{C}^3_b$, so $|\psi'|\leq K_1$ and $|\psi''|\leq K_2$; therefore,

$$\leq n^{5/4}\,\overline{C}_{11}\,E\left[\int_0^t \sum_{j\in\mathscr{S},k\in\mathscr{D}} \left[-j\frac{2}{n^{3/4}\cosh(\beta k)}\psi'(\cdot) + R\right]^2 ds\right]$$

$$\leq n^{5/4}\,\overline{C}_{11}\,E\left[\int_0^t \sup_{j\in\mathscr{S},k\in\mathscr{D}} \left[\frac{4}{n^{3/2}\cosh^2(\beta k)}(\psi'(\cdot))^2 \right.\right. \\ \left.\left. -j\frac{4}{n^{3/4}\cosh(\beta k)}\psi'(\cdot)R + R^2\right] ds\right]$$

$$\leq n^{5/4}\,\overline{C}_{11}\,E\left[\int_0^t \sup_{j\in\mathscr{S},k\in\mathscr{D}}\left[\frac{4}{n^{3/2}\cosh^2(\beta k)}K_1^2 + \frac{8}{n^{9/4}\cosh^3(\beta k)}K_1K_2 + K_2^2\right]ds\right]$$

$$\leq 4T\,\overline{C}_{11}\,(K_1+K_2)^2$$

since $t < T$; then $\mathcal{M}^t_{n,\psi}$ is uniformly integrable. ■

Now, the proof is easy to complete. $\mathcal{M}^t_{n,\psi}$ solves the martingale problem with infinitesimal generator $\mathcal{J}$, admitting a unique solution, and hence we have shown all convergent subsequences have the same limit and so the sequence itself converges to that limit.

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