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Model Reduction for Multibody Systems

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Abstract

In the first part of this thesis we presents a computationally fast mathematical model of a motorcycle drivetrain, that can be implemented in a multibody motorcycle model for handling and maneuverability studies and long time-scale analysis. Given a mathematical model of the primary drive and the transmission, possessing an arbitrary number of shafts, we define a reduced model composed by only two counter-rotating shafts of suitable masses and inertias. Also, we show how forces and torques applied to the original model must be adapted to the compact one, in order for the two systems to be indistinguishable with respect to energy transfer and gyroscopic moments. Beyond the simplification of the equations due to the reduced number of bodies, another advantage of this model is that, once the equations of motion are derived, any n-shaft transmission can be simulated just changing the equation parameters, without the need of recompiling the whole multibody model.

In the second part of the dissertation the modeling of the drive chain is discussed. A multibody model of the rear half of a motorcycle is described, to reproduce the fast dynamics caused by the chain models with high stiffness. Applying a quasi-steady-state approximation of the extension of the chain, we effectively eliminates the undesired high frequencies vibrations, and preserving the slow signals that most influence the trajectory of the vehicle. The only drawback of this solution is the need of solving a nonlinear equation with two unknowns to evaluate the reduced equations of motion. The problem is overcome with another reduction method, that is enforcing holonomic constraints on the lengths of the upper and lower chain segments. In order to do that without loosing two degrees of freedom, the stiff nonlinear model is first described as a switched model where the chain segments are replaced by linear springs. Then, the submodels are reduced separately and merged together to obtain the overall reduced model. An intelligent formulation of the constraints let us derive the chain tension even if its extension is constantly zero.

Sommario

Nella prima parte di questa tesi presentiamo un modello matematico computazionalmente veloce della trasmissione di una motocicletta, che possa essere implementato in un modello multibody per studi di manovrabilità e analisi su scale di tempo elevate. Dato un modello matematico di una trasmissione con numero arbitrario di alberi, definiamo un modello ridotto composto solo da due alberi controrotanti di opportune masse e inerzie. Mostriamo anche come modificare forze e coppie applicate al modello originale per adattarle a quello compatto, con lo scopo di renderlo indistinguibile per quanto riguarda gli scambi energetici e i momenti giroscopici. Oltre ad avere un numero ridotto di corpi, un altro vantaggio di questo modello è che una volta che le equazioni del moto sono state calcolate, qualsiasi trasmissione può essere simulata solo cambiando i parametri delle equazioni, senza la necessità di ricompilarlo.

Nella seconda parte della dissertazione è trattata la modellazione della catena di trasmissione. Viene descritto un modello multibody della metà posteriore di una motocicletta per riprodurre la dinamica veloce causata dai modelli di catena con elevata rigidezza. Applicando un'approssimazione quasi-steady-state dell' estensione della catena abbiamo eliminato efficacemente le indesiderate vibrazioni ad alta frequenza, preservando inalterati i segnali lenti che maggiormente influenzano la traiettoria del veicolo. L'unico inconveniente di questa soluzione è la necessità di risolvere un'equazione non lineare in due incognite per valutare le equazioni del moto ridotte. Il problema viene risolto con un altro metodo di riduzione, cioè l'imposizione di vincoli olonomi sulle estensioni del ramo superiore ed inferiore della catena. Per fare questo senza perdere due gradi di libertà, innanzitutto il modello rigido non lineare viene descritto come un modello switched in cui i due rami della catena vengono sostituiti da due molle lineari. Poi, i sottomodelli vengono ridotti separatamente e infine fusi assieme per ottenere il modello ridotto complessivo. Una formulazione intelligente dei vincoli permette di calcolare la tensione della catena anche se la sua estensione è sempre zero.

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Contents

Introduction

The automotive industry benefits a lot from using virtual prototypes to accelerate the production of new vehicle models. A virtual prototype has been defined as a computer-based simulation of a system or subsystem with a degree of functional realism comparable to a physical prototype. Virtual prototyping is the process of using a virtual prototype, in lieu of a physical prototype, for test and evaluation of specific characteristics of a candidate design. The performances of new components and assemblies can be assessed before they are actually implemented, facilitating a high level of product maturity at conceptual stage of the development process. With the most modern techniques also fatigue durability and handling tests are carried out without the need of real prototypes, to the point that the International Standard Organization conceived specific standard maneuvers to test vehicle dynamics and road-holding ability both in open and closed-loop simulations.

Although motorcycle industry lags behind the automotive industry with respect to economic importance, it makes wide use of the same kind of tools and takes advantage of the technological transfer. The transferring is slowed down by the intrinsic differences between two and four wheel vehicle, for example open-loop instability of motorcycles makes hard the testing of nontrivial maneuvers also at moderate velocities. A great deal of solutions is also inherited from the racing, where specific pioneering technologies for motorcycle are tested in the first place.

Specific aspects of bikes dynamics represent also intriguing new challenges for control systems engineers, and new field of applications for existing methodologies. The scientific community has been recently very interested on new advanced techniques for motorcycle designing and control. Which is proved also by the special edition of IEEE Control System Magazine of October 2006, completely dedicated to the topic. Few years ago a deeper understanding of steering instabilities lead to the design of a steering compensator ([1]). Only recently ([2]) a satisfactory control strategy has been proposed to drive virtual motorcycles into aggressive maneuvers.

At the core of any virtual prototyping environment there is a mathematical model of the vehicle. The complexity of multibody mechanics requires the use of automatic generators of the equations of motion, called multibody programs. Generally they process a model definition file, where rigid bodies and their kinematic are described, and generate a code containing the equations of motion. In [3] you can find a rich list of commercial tools for simulating multibody systems that use either a numerical or a symbolic approach. A review of methods for the automatic generation of the equations of motion can be found in [4].

Literature on two-wheel vehicle mathematical models is quite rich. A complete historical review can be found in [5] and [6]. Depending on the purpose they are made for, they range from a very simple rigid plane model, to assemblies of several bodies with complex kinematic and road-tire interaction. In the first category we put the pioneering work [7], and the more modern [8], [9], [10], [11], and [12] that are suitable for designing control and optimization strategies. The second group accounts [13], [14], and [3], more suitable for mode analysis and linearization.

An important factor that determines the required modeling accuracy, is the time-scale of the analysis, that is the duration of the shortest phenomenon or the period of the fastest oscillation to recreate. Typical long time-scale employments of virtual bikes are equilibrium manifold analysis, optimal control, minimum time maneuvering, and maneuver regulation. In all these cases the vehicle is studied in a global fashion, and attention is given to slowly varying signals. On the other hand modal analysis, linearization, and specific subsystem testing generally benefit from or require more details.

In current literature great effort has been put in modeling tire dynamics ([14]), and suspension and steering geometry ([14] and [15]), but the drivetrain has not received much attention so far. A motorcycle drivetrain is composed by the primary drive, the transmission, and the chain drive. It is responsible for the generation of gyroscopic moments and interacts relevantly with the rear suspension affecting considerably the overall weight distribution and trimming.

Generally inertias of the internal rotating shafts are lumped into the counter shaft or, where the chain is absent, into the rear wheel. These simplifications lack the superposition of opposite gyroscopic moments arising from shafts with inverted spinning. As the engine speed can reach up to 19,000 rpm on a racing motorcycle, the error may be relevant. The first Chapter of this dissertation presents a fast and compact model of the internal drivetrain: two coaxial counter-rotating shafts reproduce correctly both the gyroscopic moments and the energy transfer of any type of transmission regardless for its number of shafts. Beyond the simplification of the equations due to the reduced number of bodies, another advantage of this model is that, once the equations of motion are derived, any n-shaft transmission can be simulated just changing the equation parameters, without the need of recompiling the whole multibody model.

The chain drive is the last part of the drivetrain that transmits the rotational movement of the counter shaft to the rear wheel. Its interaction with the rear suspension and its effects on the vertical load make it essential to the vehicle dynamics. Unfortunately, as remarked in [16], it is difficult to model for the high number of links and the fact that each link changes its function as it progresses through its duty cycle. Lumped parameters solutions, like that proposed in [16], replace the chain links with two nonlinear springs connecting the drive and driven sprockets on both sides, but introduce heavy nonlinearity, high frequency dynamics, and numerical problems when integrating the equations of motion. Only in [17] can be found a reduced model capable of reproducing the chain-suspension interaction without actually implement a chain. Nevertheless it is lacking of the internal drivetrain shafts, as their inertias vanished in the reduction process.

The second part of the dissertation takes inspiration from the approach used on the suspensions in [17], to simplify the chain dynamics without giving up the transmission shafts. A quasi-steady-state approximation proves to be effective eliminating the undesired high frequencies vibrations, and preserving the slow signals, that most influence the trajectory of the vehicle. A simple case study seems to indicate that the QSS model is actually the result of a singular perturbation. The only drawback of the solution presented is the need of solving a nonlinear equation with two unknowns to evaluate the reduced equations of motion. The problem is overcome with another reduction method, that is enforcing holonomic constraints on the lengths of the upper and lower chain segments. In order to do that without loosing two degrees of freedom, the stiff nonlinear model is first described as a switched model where the chain segments are replaced by linear springs. Then, the submodels are reduced separately and merged together to obtain the overall reduced model. An intelligent formulation of the constraints let us derive the chain tension even if its extension is constantly zero.

Chapter 1

Fast Model of the Internal Drivetrain

In this chapter we consider the part of the drivetrain that goes from the engine to the drive sprocket. First we give a description of a complete multibody model made of one rigid body for the center assembly plus one for each rotating shaft. This is the most spontaneous modeling choice that let us derive the equations of motion. Secondly a similar model with only two counter-rotating shafts is described. Though its dynamic and kinematic parameters are chosen to make it completely indistinguishable from the complete model, it is called fast model, as the absence of one of the bodies makes it computationally lighter.

A section on the equivalence of dynamic systems in free evolution clarifies the concept of indistinguishableness. We point out that if two functions representing two vector fields in $\mathbb{R}^n$ are equal up to a change of coordinates, they are the same dynamic system. This equivalence is proved for the two mechanical models we propose, using a Lagrangian approach. The proof shows that their two Lagrangian functions differ only for a change of coordinates, and exploit the intrinsic invariance of the Euler-Lagrange equations to coordinates transformations.

Imposing the equality of their kinetic and potential energies we find both the mapping between their generalized coordinates, and the formulae that define the kinematic and dynamic parameters of the fast model. These formulae are then generalized to define a fast model from a complete model with an arbitrary number of rotating shafts.

The presence of external inputs is treated in the last part of the chapter. Exploiting the cotangent lift of the coordinate transformation previously defined, we derive the formulae that transform any external torque and force acting on the complete model, into an equivalent torque and force to apply on the fast model, for maintaining its equivalence.

1.1 Notation

To ease the consultation we report here the main conventions, definitions and formulae used through the Chapter. Details and mathematical investigations on rigid body motion are collected in appendix A. Se also reference [18] for a complete treatment of the subject.

All vector quantities are written in boldface. When a quantity is expressed relative to a reference frame or belongs to a rigid body, this is indicated with a subscript. The letters k and j refer to a generic rotating shaft.

Reference frames are indicated with $\mathbf{\Sigma}$. they have an origin O and three axes $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$.

Rotation matrices $\mathbf{R}$ are elements of $SO(3)$ and are expressed as an ordered multiplication of basic rotations about the main axes: $\mathbf{R} = \mathbf{R}_z(\alpha)\mathbf{R}_x(\beta)\mathbf{R}_y(\gamma)$.

Rotation matrices together with translation vectors $\mathbf{r} \in \mathbb{R}^3$ define rigid transformations $\mathbf{g} = (\mathbf{r}, \mathbf{R})$, that are used to express position and orientation of the frames with respect to each other and with respect to the inertial frame $\mathbf{\Sigma}^G$. Rigid transformations have two subscripts, to specify the frames between whom they apply. In case the first reference frame is the ground, its subscript $_g$ will be omitted. So $\mathbf{g}_r$ gives the absolute position and orientation of R, while $\mathbf{g}_{r,ds}$ gives the position and orientation of body DS with respect to R.

Given a frame fixed to a rigid body, it is possible to define an inertia tensor

$$\mathbb{J} = \begin{bmatrix} \mathbb{J}^{xx} & \mathbb{J}^{yx} & \mathbb{J}^{zx} \\ \mathbb{J}^{xy} & \mathbb{J}^{yy} & \mathbb{J}^{zy} \\ \mathbb{J}^{xz} & \mathbb{J}^{yz} & \mathbb{J}^{zz} \end{bmatrix}$$

and a generalized inertia matrix

$$\mathbb{I} = \begin{bmatrix} mI_{3\times3} & m\hat{\mathbf{p}}^b \\ -m\hat{\mathbf{p}}^b & \mathbb{J} - m(\hat{\mathbf{p}}^b)^2 \end{bmatrix},$$

where $\mathbf{p}^b$ are the coordinates of the center of mass expressed in the body reference frame.

1.2 Complete model of the internal drivetrain

In this section we focus on the first part of the motorbike drivetrain, the one that is located inside the center assembly of the bike. The function of the drivetrain is two-fold: transmit power from the engine to the rear wheel and provide gear reduction. As shown in Figure 1.1, it is divided into three distinct assemblies: primary drive, transmission and secondary drive. The primary drive consists of an engine output sprocket, attached to the crankshaft, and the clutch assembly. The purpose of the clutch is to engage the primary drive to the transmission. The gear ratio between the crankshaft and the clutch is fixed. The transmission is made of the main shaft and the counter shaft, it reduces the rotational velocity and increases the torque according to the selected gear. The secondary drive includes a transmission output sprocket, roller chain, and rear wheel sprocket. It will not be considered in this Chapter but in the next one.

As we wrote in the introduction we want to develop a multibody model of drivetrain to include in a complete model of motorcycle. Our purpose is reproducing the influence of gyroscopic moments on the overall dynamics of the vehicle, and the energy transfer due to the acceleration of the rotating inertias. We believe this can be achieved reasonably accurately modeling each rotating shaft with a rigid body: one for the crankshaft CS, one for the main shaft MS, and one for the counter shaft DS. These three elements rotate inside the center assembly, which is modeled by another rigid body R free to move in space. In Figure 1.2 it is possible to see a schematic of the model. Each body has an inertia tensor, a mass, and a center of mass. Both the inertia tensors and the positions of the centers of mass are expressed with respect to reference frames fixed to each body. To simplify the calculations, we make a few assumptions on the positions of the reference frames and on the dynamic properties of the shafts:

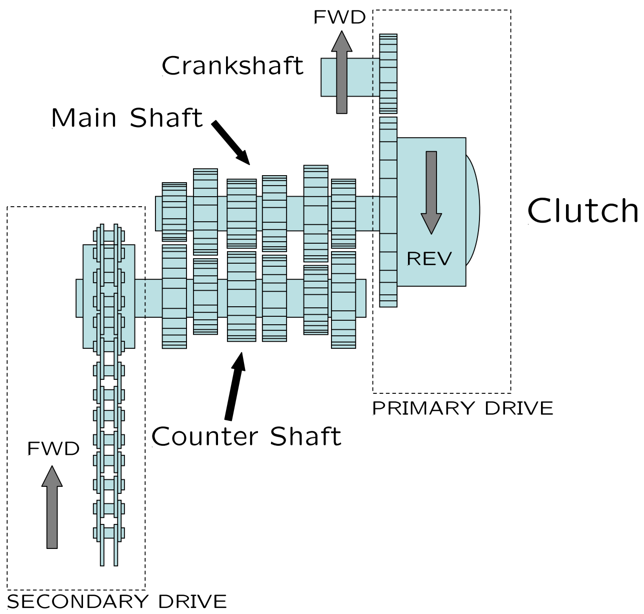


Figure 1.1: Schematic of the internal part of a bike drivetrain. It is divided into three distinct assemblies: primary drive, transmission and secondary drive.

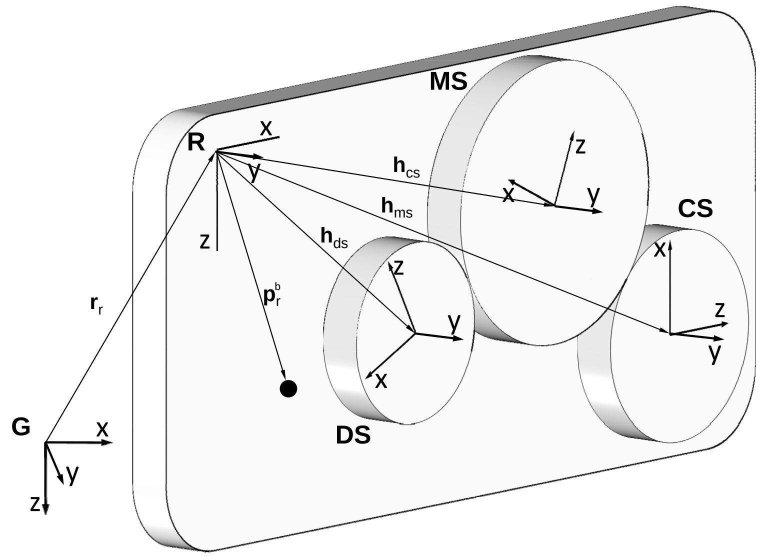


Figure 1.2: Complete multibody model of a motorbike internal drivetrain. Each rotating shaft of a real motorbike has an equivalent rigid body in the model. All reference frames and relevant quantities are shown.

1. the reference frame of the center assembly has its y-axis parallel to the spinning axes of the shafts.

2. The reference frames of the shafts have their y-axes coincident with their spinning axes.

3. All the origins of the frames of the shafts lay on the same plane $(\mathbf{e}_R^x, \mathbf{e}_R^z)$.

4. The rotating shafts are solid of revolutions, which implies that the axes of their reference frames are parallel to their main axes of inertia, and their inertia tensors are diagonal with $\mathbb{J}_k^{xx} = \mathbb{J}_k^{zz}$:

$$\mathbb{J}_k = \begin{bmatrix} \mathbb{J}_k^{xx} & 0 & 0 \\ 0 & \mathbb{J}_k^{yy} & 0 \\ 0 & 0 & \mathbb{J}_k^{xx} \end{bmatrix}. \tag{1.1}$$

5. The body coordinates of the centers of mass of all the shafts are zero. This does not represent a restriction, as any lateral displacement in the y direction can be compensated modifying the generalized inertia matrix of R.

As usual in multibody mechanics, to reduce the number of generalized coordinates, the positions of the bodies are not all given with respect to the ground frame, but with respect to other body frames, creating a tree structure of rigid transformations. In our case the root of the tree is $\mathbf{g}_r$, whose homogeneous coordinates are expressed using six variables: $(x, y, z, \varphi, \eta, \psi)$. The first three are the translational displacement of the origin O^R with respect to O^G, the second three are the angles (yaw, pitch, roll) that define the rotation matrix $\mathbf{R}_r$. The translational displacements of the shafts with respect to $\mathbf{\Sigma}^R$ are fixed parameters of the model indicated with the letter $\mathbf{h}$. Their rotational displacements are given through the angles between their x-axis and $\mathbf{e}_r^x$. Since the gear ratios between adjacent shafts is fixed (for each selected gear), two angular displacements can be taken as functions of the third one and can be considered as implicit variables. We chose the angles θ_{ms} and θ_{cs} as implicit variables and used θ_{ds} as the last generalized coordinate of the model. The reason for this is that the angular displacement of the drive sprocket plays a role also in the kinematic of the chain and will be used in the next chapter. The gear ratio ρ_{jk} between two not necessarily adjacent shafts is defined as

$$\rho_{jk} := \dot{\theta}_j / \dot{\theta}_k \,. \tag{1.2}$$

The position and orientation of the center assembly are defined by $\mathbf{g}_r = (\mathbf{r}_r, \mathbf{R}_r)$, with

$$\mathbf{r}_r = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1.3}$$

$$\mathbf{R}_r = \mathbf{R}_z(\psi)\mathbf{R}_x(\varphi)\mathbf{R}_y(\eta) \,. \tag{1.4}$$

The position of its center of mass in spatial coordinates is:

$$\bar{\mathbf{p}}_r^s = \begin{bmatrix} \mathbf{p}_r^s \\ 1 \end{bmatrix} = \mathbf{g}_r \begin{bmatrix} \mathbf{p}_r^b \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_r & \mathbf{r}_r \\ 0 & 1 \end{bmatrix} \bar{\mathbf{p}}_r^b. \tag{1.5}$$

The assumptions we made on the orientation of the reference frames of the shafts simplify the expressions of their rotation matrices and velocities. Firstly $\mathbf{R}_{rk}$ becomes a simple rotation about the y-axis $\mathbf{R}_{rk} = \mathbf{R}_y(\theta_k)$, secondly the rotational velocities of the shafts have the form

$$\boldsymbol{\omega}_{rk}^b = \begin{bmatrix} 0 \\ \dot{\theta}_k \\ 0 \end{bmatrix} \tag{1.6}$$

both in R's and K's reference frame:

$$\boldsymbol{\omega}_{rk}^s = \mathbf{R}_{rk}\boldsymbol{\omega}_{rk}^b = \mathbf{R}_y(\theta_k)\boldsymbol{\omega}_{rk}^b = \boldsymbol{\omega}_{rk}^b \,. \tag{1.7}$$

Also the inertia tensors of the shafts are invariant to left and right multiplications by a rotation matrix of the type $\mathbf{R}_y$. In particular

$$\mathbb{J}_k = \mathbf{R}_{rk}(\theta_k)\mathbb{J}_k\mathbf{R}_{rk}^T(\theta_k)\,. \tag{1.8}$$

All three shafts have similar expressions for $\mathbf{R}_k$ and $\mathbf{r}_k$:

$$\mathbf{r}_{ds} = \mathbf{r}_r + \mathbf{R}_r\mathbf{h}_{ds} \tag{1.9}$$
$$\mathbf{r}_{ms} = \mathbf{r}_r + \mathbf{R}_r\mathbf{h}_{ms} \tag{1.10}$$
$$\mathbf{r}_{cs} = \mathbf{r}_r + \mathbf{R}_r\mathbf{h}_{cs} \tag{1.11}$$

$$\mathbf{R}_{ds} = \mathbf{R}_z(\psi)\mathbf{R}_x(\varphi)\mathbf{R}_y(\eta + \theta_{ds}) \tag{1.12}$$
$$\mathbf{R}_{ms} = \mathbf{R}_z(\psi)\mathbf{R}_x(\varphi)\mathbf{R}_y(\eta + \rho_{ms,ds}\theta_{ds}) \tag{1.13}$$
$$\mathbf{R}_{cs} = \mathbf{R}_z(\psi)\mathbf{R}_x(\varphi)\mathbf{R}_y(\eta + \rho_{cs,ds}\theta_{ds}). \tag{1.14}$$

Each body of the model has a mass m, an inertia tensor $\mathbb{J}$, and a generalized inertia matrix. We list in table 1.1 the parameters of the complete model.

Table 1.1: Parameters of the complete model.

$m_r, m_{cs}, m_{ms}, m_{ds}$	Masses of the four bodies.
$\mathbb{J}_r, \mathbb{J}_{ds}, \mathbb{J}_{ms}, \mathbb{J}_{cs}$	Inertia tensors of the four bodies.
$\mathbf{p}_r^b$	Position of the center of mass of R in body coordinates.
$\mathbf{h}_{ds}, \mathbf{h}_{ms}, \mathbf{h}_{cs}$	Position of the origins of O_k with respect to $\mathbf{\Sigma}_r$.
$\rho_{ds,ms} = \dot{\theta}_{ds}/\dot{\theta}_{ms}$	Gear ratio of the transmission (depends on the selected gear).
$\rho_{ms,cs} = \dot{\theta}_{ms}/\dot{\theta}_{cs}$	Gear ratio of the primary drive (unique).

For the moment we neglect the presence of external forces and torques. The model can only evolve freely and fall indefinitely under the effect of a conservative gravitational force field.

1.3 Fast model of the internal drivetrain

The second model we introduce is very similar to the complete one. It is called *fast* because it has only two rotating shafts instead of three, and this gives simpler and faster equations of motion. Despite one of the shaft being absent is a relevant structural difference, the dynamic and geometric properties of the remaining bodies are chosen to guarantee a dynamic equivalence of the two models. This concept will be defined precisely in the next sections both for free and forced evolution. Here we describe only the structure of the fast model and define its parameters.

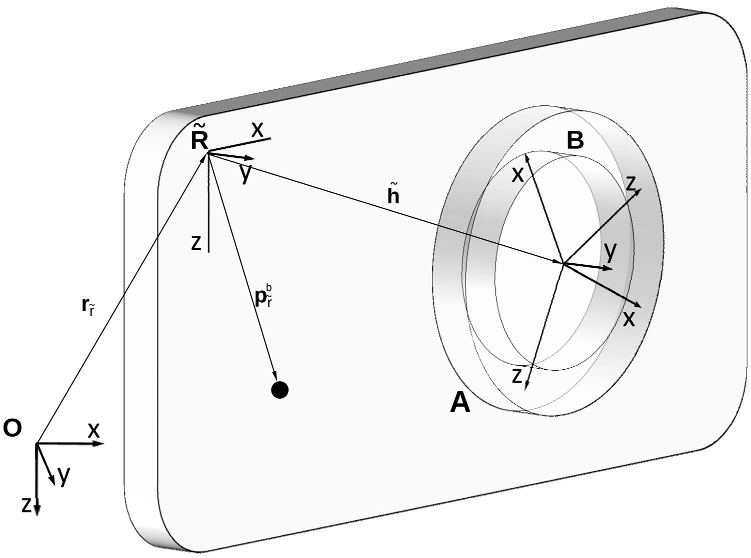


Figure 1.3: Fast multibody model of a motorbike internal drivetrain. The three shafts of the complete model have been replaced with an equivalent couple of counter-rotating shafts.

As the two models are strictly related many quantities are counterparts of one another. When this is the case they preserve the same name, but are differentiated by the presence of a tilde (˜) sign. In Figure 1.3 it is depicted a schematic of the fast model with some relevant parameters. Note that the two shafts are called A and B. Their names do not recall any part of the drivetrain as there is not a one-to-one correspondence between any of them and a particular shaft. They should rather be considered as a new couple of coaxial discs. They have a fixed gear ratio equal to $\rho_{a,b} = -1$ independently of the gear ratios of the complete model.

All assumptions of the list on page 10 about the parallelism of the y-axes of the reference frames, and on the form of the inertia tensors of the shafts still hold. Also the two counter-rotating shafts are solids of revolution with an inertia tensor of the form (1.1). As before their angular displacements are measured with the angles between $\mathbf{e}_k^x$ and $\mathbf{e}_r^x$, which implies that $\dot{\theta}_b = -\dot{\theta}_a$ and $\theta_b = -\theta_a$ up to a constant. Without loss of generality we pick θ_a as the generalized coordinate for the shafts. We list in table 1.2 the parameters of the fast model.

Table 1.2: *Parameters of the fast model.*

m_r, m_b, m_a	Masses of the three bodies.
$\mathbb{J}_{\tilde{r}}, \mathbb{J}_a, \mathbb{J}_b$	Inertia tensors of the three bodies.
$\mathbf{p}^b_{\tilde{r}}$	Position of the center of mass of $\tilde{R}$ in body coordinates.
$\tilde{\mathbf{h}} = \begin{bmatrix}\tilde{h}_x & 0 & \tilde{h}_z\end{bmatrix}^T$	Position of $O_{\tilde{k}}$ with respect to $\boldsymbol{\Sigma}_{\tilde{r}}$.

As before we temporary neglect the presence of external inputs and defer it to the last part of the chapter.

1.4 Equivalence in free evolution

Coordinate transformations are very useful tools to study differential equations. New coordinates can, for example, reveal unexpected features of the equations or simplify the calculations. Consider the case of two generic dynamic systems of the form

$$\dot{\mathbf{x}} = f(\mathbf{x}) \tag{1.15}$$

and

$$\dot{\mathbf{y}} = g(\mathbf{y})\,, \tag{1.16}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are the state vectors and $f, g : \mathbb{R}^n \to \mathbb{R}^n$ smooth vector fields. If f and g are related by a global change of coordinates[1] $\mathbf{y} = \Gamma(\mathbf{x})$, that is

$$f(\mathbf{x}) = D\Gamma^{-1} \circ g \circ \Gamma(\mathbf{x}) \qquad \forall \mathbf{x} \in \mathbb{R}^n\,, \tag{1.17}$$

then f and g are in fact the same dynamic system. The map Γ creates a correspondence between each point $\mathbf{x}_0$ of the first state space and a point $\mathbf{y}_0 = \Gamma(\mathbf{x}_0)$ in the second state space, and this correspondence is extended also to trajectories. If $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are two solutions of f and g passing respectively through $\mathbf{x}_0$ and $\mathbf{y}_0$ at a certain time, then $\mathbf{y}(t) = \Gamma(\mathbf{x}(t))$ $\forall t$ where they are defined.

A dramatic example of the possibilities given by a change of coordinates is given by the ([19]) *Flow Box theorem* (also called *straightening out theorem* or *rectification lemma*) that states that in a neighborhood of any regular point[2] all autonomous differential equations are the same, up to a change of coordinates. In particular there always exist a particular mapping such that the trajectories of any system can even become straight lines.

Though extremely powerful, this simplification is not cost-free, in fact the change of coordinates can be very complex and is not in general defined in all the

[1] A change of coordinates is diffeomorphism: smooth invertible mapping with smooth inverse.
[2] That is a point where the equation is not null.

state space, which may force to switch repeatedly among many maps during a simulation. Moreover you loose any meaningful structure of the system and any physical interpretation of the state variables. It could be more useful to renounce some simplicity in exchange of some structure in the equations.

Our work does not aim to drastically simplify the system, but to make its equations faster while maintaining its mechanical structure. To do so we prove that the equations of the complete model differ from the equations of the fast model, that is indeed simpler and mechanical, only for a change of coordinates. We will provide a simple mapping Γ between the two sets of generalized coordinates, valid in all the state space. Since Γ is a simple multiplication by a constant, the inverse mapping to retrieve the original state variables is a very fast operation. We will also give the formulae to calculate the parameters of the fast model from those of the complete model. They do not produce any overhead during simulations as they need to be computed only once during the initialization of the algorithm.

When considering mechanical systems, coordinate transformations are first defined between the generalized coordinates $\mathbf{q}$ and $\tilde{\mathbf{q}}$, and then naturally extended to their time derivatives $\dot{\mathbf{q}}$ and $\dot{\tilde{\mathbf{q}}}$. Be $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ the diffeomorphism transforming $\mathbf{q}$ in $\tilde{\mathbf{q}}$. The generalized velocities are related by the derivative

$$\dot{\tilde{\mathbf{q}}} = \mathbf{D}\Phi(\mathbf{q})\dot{\mathbf{q}}\,,$$

and the overall mapping Γ between state spaces is defined as

$$(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) = \Gamma(\mathbf{q}, \dot{\mathbf{q}}) := (\Phi(\mathbf{q}), \mathbf{D}\Phi(\mathbf{q})\dot{\mathbf{q}})\,. \tag{1.18}$$

With the new state variables the equivalence of systems (1.17) becomes

$$f(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{D}\Gamma^{-1} \circ g \circ \Gamma(\mathbf{q}, \dot{\mathbf{q}}) \qquad \forall (\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{2n}\,. \tag{1.19}$$

In Lagrangian mechanics a system is defined by a Lagrangian function $L : \mathbb{R}^{2n} \to \mathbb{R}$. The Lagrangian is in the form of kinetic minus potential energy and the dynamics satisfies Hamilton's principle

$$\delta \int_{t_0}^{t_1} L(\mathbf{q}(t), \dot{\mathbf{q}}(t))dt = 0,$$

with arbitrary variations vanishing at the end points. The equations of motion are given by the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = 0.$$

and depend only on the form of $L(\mathbf{q}, \dot{\mathbf{q}})$. Before proceeding to show the equivalence of the fast and complete models we give the

Lemma 1.1 *Be μ the vector of parameters defining the complete model and ν the vector of parameters defining the fast model. For any choice of μ with physical meaning*[3]*, there exist a vector $\nu(\mu)$ such that the two Lagrangian functions L and $\tilde{L}$ associated to the models are equal up to a change of coordinates:*

$$\tilde{L}(\Phi(\boldsymbol{q}), \boldsymbol{D}\Phi(\boldsymbol{q})\dot{\boldsymbol{q}}) = L(\boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{1.20}$$

We prove constructively this lemma by imposing separately the equality of kinetic and potential energies of the two systems, and providing $\nu(\mu)$, Φ, and $\mathbf{D}\Phi$.

The potential energy V of the complete model is only due to gravity and depends on the height of the centers of mass of each body. Let M be equal to the total mass of the system $M = m_r + m_{ds} + m_{ms} + m_{cs}$, then

$$\begin{aligned} V &= g(m_r \mathbf{p}_r^s \mathbf{e}_g^z + m_{ds}\mathbf{p}_{ds}^s \mathbf{e}_g^z + m_{ms}\mathbf{p}_{ms}^s \mathbf{e}_g^z + m_{cs}\mathbf{p}_{cs}^s \mathbf{e}_g^z) \\ &= g[m_r(\mathbf{r}_r + \mathbf{R}_r \mathbf{p}_r^b) + m_{ds}(\mathbf{r}_r + \mathbf{R}_r \mathbf{h}_{ds}) + \\ &+ m_{ms}(\mathbf{r}_r + \mathbf{R}_r \mathbf{h}_{ms}) + m_{cs}(\mathbf{r}_r + \mathbf{R}_r \mathbf{h}_{cs})]\mathbf{e}_g^z \\ &= g[M\mathbf{r}_r + \mathbf{R}_r(m_r \mathbf{p}_r^b + m_{ds}\mathbf{h}_{ds} + m_{ms}\mathbf{h}_{ms} + m_{cs}\mathbf{h}_{cs})]\mathbf{e}_g^z. \end{aligned} \tag{1.21}$$

The kinetic energy T of the complete model can be computed as the sum of the kinetic energies of its bodies:

$$T = \frac{1}{2}\sum_l (\mathbf{V}_l^b)^T \mathbb{I}_l \mathbf{V}_l^b \qquad l \in \{r, cs, ms, ds\}\,. \tag{1.22}$$

For the center assembly

$$\begin{aligned} T_R &= \frac{1}{2}(\mathbf{V}_r^b)^T \mathbb{I}_r \mathbf{V}_r^b \\ &= \frac{1}{2} m_r \|\mathbf{v}_r^b\|^2 + \frac{1}{2}(\boldsymbol{\omega}_r^b)^T \mathbb{J}_r \boldsymbol{\omega}_r^b + m_r (\mathbf{v}_r^b)^T \hat{\boldsymbol{\omega}}_r^b \mathbf{p}_r^b \end{aligned} \tag{1.23}$$

and for the generic rotating shaft K

$$\begin{aligned} T_K &= \frac{1}{2}(\mathbf{V}_k^b)^T \mathbb{I}_k \mathbf{V}_k^b \qquad (1.24) \\ &= \frac{1}{2} m_k \|\mathbf{v}_k^b\|^2 + \frac{1}{2}(\boldsymbol{\omega}_k^b)^T \mathbb{J}_k \boldsymbol{\omega}_k^b\,. \end{aligned}$$

[3] The parameters cannot be taken randomly, masses must be positive and the inertia tensors must be positive definite.

The body velocity of each shaft $\mathbf{V}_k^b$ is related to the body velocity of the main body $\mathbf{V}_r^b$ through

$$\mathbf{V}_k^b = Ad_{\mathbf{g}_{rk}^{-1}}\mathbf{V}_r^b + \mathbf{V}_{rk}^b = \begin{bmatrix} \mathbf{R}_{rk}^T & -\mathbf{R}_{rk}^T\hat{\mathbf{h}}_k \\ 0 & \mathbf{R}_{rk}^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_r^b \\ \boldsymbol{\omega}_r^b \end{bmatrix} + \begin{bmatrix} \mathbf{v}_{rk}^b \\ \boldsymbol{\omega}_{rk}^b \end{bmatrix}. \tag{1.25}$$

Since $\mathbf{v}_{rk}^b = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, $\boldsymbol{\omega}_{rk}^b = \begin{bmatrix} 0 & \dot{\theta}_k & 0 \end{bmatrix}^T$, and the rotation matrix $\mathbf{R}_{rk}^T$ leaves the y component unchanged,

$$\mathbf{V}_k^b = \begin{bmatrix} \mathbf{R}_{rk}^T\mathbf{v}_r^b - \mathbf{R}_{rk}^T\hat{\mathbf{h}}_k\boldsymbol{\omega}_r^b \\ \mathbf{R}_{rk}^T\boldsymbol{\omega}_r^b + \boldsymbol{\omega}_{rk}^b \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{rk}^T & 0 \\ 0 & \mathbf{R}_{rk}^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_r^b - \hat{\mathbf{h}}_k\boldsymbol{\omega}_r^b \\ \boldsymbol{\omega}_r^b + \boldsymbol{\omega}_{rk}^b \end{bmatrix}. \tag{1.26}$$

Substituting (1.26) in (1.24) and using the invariance (1.8) we obtain

$$T_K = \frac{1}{2} \begin{bmatrix} \mathbf{v}_r^b - \hat{\mathbf{h}}_k\boldsymbol{\omega}_r^b \\ \boldsymbol{\omega}_r^b + \boldsymbol{\omega}_{rk}^b \end{bmatrix}^T \mathbb{I}_k \begin{bmatrix} \mathbf{v}_r^b - \hat{\mathbf{h}}_k\boldsymbol{\omega}_r^b \\ \boldsymbol{\omega}_r^b + \boldsymbol{\omega}_{rk}^b \end{bmatrix}.$$

Since $\mathbb{I}_k$ is diagonal, T_K can be written as the sum of the translational part

$$T_K^{tr} = \frac{1}{2}m_k(\mathbf{v}_r^b - \hat{\mathbf{h}}_k\boldsymbol{\omega}_r^b)^T(\mathbf{v}_r^b - \hat{\mathbf{h}}_k\boldsymbol{\omega}_r^b) \tag{1.27}$$

and rotational part

$$T_K^{rot} = \frac{1}{2}(\boldsymbol{\omega}_r^b + \boldsymbol{\omega}_{rk}^b)^T\mathbb{J}_k(\boldsymbol{\omega}_r^b + \boldsymbol{\omega}_{rk}^b)\,. \tag{1.28}$$

It is possible to define the generalized inertia matrix of a point mass m_k placed at the center of mass of K (that is fixed relatively to R), with respect to $\boldsymbol{\Sigma}_r$ as

$$\mathbb{I}_{k,r}^{pm} := m_k \begin{bmatrix} I_{3\times3} & -\hat{\mathbf{h}}_k \\ \hat{\mathbf{h}}_k & -\hat{\mathbf{h}}_k^2 \end{bmatrix}. \tag{1.29}$$

The term T_K^{tr} can be thought of as its kinetic energy:

$$T_K^{tr} = \frac{1}{2}(\mathbf{V}_r^b)^T\mathbb{I}_{k,r}^{pm}\mathbf{V}_r^b\,. \tag{1.30}$$

On the other hand, the rotational kinetic energy (1.28) does not depend either on $\mathbf{v}_r^b$ nor on $\mathbf{h}_k$, but only on the rotational velocities of R and K. Making use of the previously defined gear ratios, we write $\boldsymbol{\omega}_{rms}^b$ and $\boldsymbol{\omega}_{rcs}^b$ in terms of $\boldsymbol{\omega}_{rds}^b$, obtaining

$$T_K^{rot} = \frac{1}{2}(\boldsymbol{\omega}_r^b + \rho_{k,ds}\boldsymbol{\omega}_{rds}^b)^T \mathbb{J}_k (\omega_r^b + \rho_{k,ds}\boldsymbol{\omega}_{rds}^b) \tag{1.31}$$

$$= \frac{1}{2}(\boldsymbol{\omega}_r^b)^T \mathbb{J}_k \boldsymbol{\omega}_r^b + \rho_{k,ds}(\boldsymbol{\omega}_r^b)^T \mathbb{J}_k \boldsymbol{\omega}_{rds}^b + \frac{1}{2}\rho_{k,ds}^2(\boldsymbol{\omega}_{rds}^b)^T \mathbb{J}_k \boldsymbol{\omega}_{rds}^b\,. \tag{1.32}$$

Finally, by using (1.30) and (1.32), we rewrite the total kinetic energy (1.22) as

$$\begin{aligned}
T &= T_R + T_{DS}^{tr} + T_{MS}^{tr} + T_{CS}^{tr} + T_{DS}^{rot} + T_{MS}^{rot} + T_{CS}^{rot} && (1.33)\\
&= \frac{1}{2}(\mathbf{V}_r^b)^T(\mathbb{I}_r + \mathbb{I}_{ds,r}^{pm} + \mathbb{I}_{ms,r}^{pm} + \mathbb{I}_{cs,r}^{pm})\mathbf{V}_r^b \\
&+ \frac{1}{2}(\boldsymbol{\omega}_r^b)^T(\mathbb{J}_{ds} + \mathbb{J}_{ms} + \mathbb{J}_{cs})\boldsymbol{\omega}_r^b \\
&+ (\boldsymbol{\omega}_r^b)^T(\mathbb{J}_{ds} + \rho_{ds,ms}\mathbb{J}_{ms} + \rho_{ds,cs}\mathbb{J}_{cs})\boldsymbol{\omega}_{rds}^b \\
&+ \frac{1}{2}(\boldsymbol{\omega}_{rds}^b)^T(\mathbb{J}_{ds} + \rho_{ds,ms}^2\mathbb{J}_{ms} + \rho_{ds,cs}^2\mathbb{J}_{cs})\boldsymbol{\omega}_{rds}^b. && (1.34)
\end{aligned}$$

Using analogous arguments we find the potential and kinetic energies for the fast model:

$$\begin{aligned}
\tilde{V} &= g(m_{\tilde{r}}\mathbf{p}_{\tilde{r}}^s\mathbf{e}_g^z + m_a\mathbf{p}_a^s\mathbf{e}_g^z + m_b\mathbf{p}_b^s\mathbf{e}_g^z) \\
&= g[m_{\tilde{r}}(\mathbf{r}_{\tilde{r}} + \mathbf{R}_{\tilde{r}}\mathbf{p}_{\tilde{r}}^b) + (m_a + m_b)(\mathbf{r}_{\tilde{r}} + \mathbf{R}_{\tilde{r}}\tilde{\mathbf{h}})]\mathbf{e}_g^z \\
&= g[\tilde{M}\mathbf{r}_{\tilde{r}} + \mathbf{R}_{\tilde{r}}(m_{\tilde{r}}\mathbf{p}_{\tilde{r}}^b + (m_a + m_b)\tilde{\mathbf{h}})]\mathbf{e}_g^z\,. && (1.35)
\end{aligned}$$

$$\begin{aligned}
\tilde{T} &= T_{\tilde{R}} + T_A^{tr} + T_B^{tr} + T_A^{rot} + T_B^{rot} \\
&= \frac{1}{2}(\mathbf{V}_{\tilde{r}}^b)^T(\mathbb{I}_{\tilde{r}} + \mathbb{I}_{a,\tilde{r}}^{pm} + \mathbb{I}_{b,\tilde{r}}^{pm})\mathbf{V}_{\tilde{r}}^b + \\
&+ \frac{1}{2}(\boldsymbol{\omega}_{\tilde{r}}^b)^T(\mathbb{J}_a + \mathbb{J}_b)\boldsymbol{\omega}_{\tilde{r}}^b \\
&+ (\boldsymbol{\omega}_{\tilde{r}}^b)^T(\mathbb{J}_a - \mathbb{J}_b)\boldsymbol{\omega}_{\tilde{r}a}^b \\
&+ \frac{1}{2}(\boldsymbol{\omega}_{\tilde{r}a}^b)^T(\mathbb{J}_a + \mathbb{J}_b)\boldsymbol{\omega}_{\tilde{r}a}^b\,. && (1.36)
\end{aligned}$$

We now impose to Φ a linear structure like

$$\tilde{\mathbf{q}} = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\varphi}, \tilde{\theta}, \tilde{\psi}, \theta_a)^T = \Phi(\mathbf{q}) = (x, y, z, \varphi, \theta, \psi, \rho_e\theta_{ds})^T \tag{1.37}$$

(which is trivially a diffeomorphism as long as $\rho_e \neq 0$) with constant Jacobian

$$J_\Phi := \mathbf{D}\Phi = \begin{bmatrix} I_{6\times 6} & 0 \\ 0 & \rho_e \end{bmatrix}. \tag{1.38}$$

Note that the first six coordinates, that define position and orientation of the center assembly R, are left unchanged, and that the mass distributions of the models do not depend on the values of θ_{ds} and θ_a as the shafts are all solids of revolutions. This means that $\mathbf{r}_{\tilde{r}}(\tilde{\mathbf{q}}) = \mathbf{r}_{\tilde{r}}(\mathbf{q})$ and $\mathbf{R}_{\tilde{r}}(\tilde{\mathbf{q}}) = \mathbf{R}_{\tilde{r}}(\mathbf{q})$ $\forall \mathbf{q}$, and that when evaluating the potential energy of the fast model we can write $\tilde{V}(\mathbf{q})$ instead of $\tilde{V}(\Phi(\mathbf{q}))$. This said, by imposing the equality of the potential enegies (1.21) and (1.35) for any possible $\mathbf{q}$, we derived a first group of formulae that define five parameters of the fast model:

$$m_{\tilde{r}} = m_r \tag{1.39}$$

$$\mathbf{p}^b_{\tilde{r}} = \mathbf{p}^b_r \tag{1.40}$$

$$m_a = m_b = \frac{1}{2}(m_{ds} + m_{ms} + m_{cs}) \tag{1.41}$$

$$\tilde{\mathbf{h}} = \frac{m_{ds}\mathbf{h}_{ds} + m_{ms}\mathbf{h}_{ms} + m_{cs}\mathbf{h}_{cs}}{m_{ds} + m_{ms} + m_{cs}} \,. \tag{1.42}$$

These formulae imply that $\tilde{M} = M$, and can be verified by direct substitution into (1.35). Equation (1.42) means that the centers of mass of the two counter-rotating shafts coincides with the center of mass of the set of the three original shafts. The center assemblies, beyond having the same coordinates, also share the same mass and the same center of mass.

We now impose the equality of the kinetic energies: $T(\mathbf{q}, \dot{\mathbf{q}}) = \tilde{T}(\Gamma(\mathbf{q}, \dot{\mathbf{q}}))$. Note that the first terms of (1.34) and (1.36) (that account for the kinetic energy of the center assemblies plus the translational part of the kinetic energy of the shafts) depend only on the body velocities of R and $\tilde{R}$, that are not affected by the coordinates change. We define the generalized inertia matrix of the new center assembly in such a way to eliminate the consequences of grouping the three shafts together on the point $\tilde{\mathbf{h}}$:

$$\mathbb{I}_{\tilde{r}} = \mathbb{I}_r - \mathbb{I}^{pm}_{a,\tilde{r}} - \mathbb{I}^{pm}_{b,\tilde{r}} + \mathbb{I}^{pm}_{ds,r} + \mathbb{I}^{pm}_{ms,r} + \mathbb{I}^{pm}_{cs,r} \,. \tag{1.43}$$

When replaced in expression (1.36), the parameter $\mathbb{I}_{\tilde{r}}$ cancels the contributions of the translational kinetic energies of the the counter-rotating shafts and restore those of the three original shafts. There is no guarantee that $\mathbb{I}_{\tilde{r}}$ preserves the structure of a generalized inertia matrix. This check has to be done case by case when using the formula (1.43), nevertheless it seems likely that adding and subtracting some small terms to $\mathbb{I}_r$ does not spoil its nature.

To equate the remaining terms of the kinetic energies, we need to impose that $T^{rot}_A(\Gamma(\mathbf{q}, \dot{\mathbf{q}})) + T^{rot}_B(\Gamma(\mathbf{q}, \dot{\mathbf{q}})) = T^{rot}_{DS}(\mathbf{q}, \dot{\mathbf{q}}) + T^{rot}_{MS}(\mathbf{q}, \dot{\mathbf{q}}) + T^{rot}_{CS}(\mathbf{q}, \dot{\mathbf{q}})$. We remind the reader that

$$\boldsymbol{\omega}^b_{\tilde{r}}(\Gamma(\mathbf{q}, \dot{\mathbf{q}})) = \boldsymbol{\omega}^b_{\tilde{r}}(\mathbf{q}, \dot{\mathbf{q}}) = \boldsymbol{\omega}^b_r(\mathbf{q}, \dot{\mathbf{q}}) \,,$$

and that, being $\theta_a = \rho_e \theta_{ds}$,

$$\boldsymbol{\omega}^b_{\tilde{r}a}(\Gamma(\mathbf{q}, \dot{\mathbf{q}})) = \boldsymbol{\omega}^b_{\tilde{r}a}(\rho_e \dot{\theta}_{ds}) = \begin{bmatrix} 0 \\ \rho_e \dot{\theta}_{ds} \\ 0 \end{bmatrix} = \rho_e \boldsymbol{\omega}^b_{rds} \,.$$

We set a system with the last three terms of (1.34) and in (1.36) replacing $\boldsymbol{\omega}^b_{\tilde{r}}$ with $\boldsymbol{\omega}^b_r$ and $\boldsymbol{\omega}^b_{\tilde{r}a}$ with $\rho_e \boldsymbol{\omega}^b_{rds}$:

$$\begin{aligned}
(\boldsymbol{\omega}^b_r)^T(\mathbb{J}_a + \mathbb{J}_b)\boldsymbol{\omega}^b_r &= (\boldsymbol{\omega}^b_r)^T(\mathbb{J}_{ds} + \mathbb{J}_{ms} + \mathbb{J}_{cs})\boldsymbol{\omega}^b_r \\
(\boldsymbol{\omega}^b_r)^T(\mathbb{J}_a - \mathbb{J}_b)\rho_e \boldsymbol{\omega}^b_{rds} &= (\boldsymbol{\omega}^b_r)^T(\mathbb{J}_{ds} + \rho_{ms,ds}\mathbb{J}_{ms} + \rho_{cs,ds}\mathbb{J}_{cs})\boldsymbol{\omega}^b_{rds} \\
(\boldsymbol{\omega}^b_{rds})^T(\rho_e^2(\mathbb{J}_a + \mathbb{J}_b))\boldsymbol{\omega}^b_{rds} &= (\boldsymbol{\omega}^b_{rds})^T(\mathbb{J}_{ds} + \rho^2_{ms,ds}\mathbb{J}_{ms} + \rho^2_{cs,ds}\mathbb{J}_{cs})\boldsymbol{\omega}^b_{rds}
\end{aligned}$$

The equalities have to hold for all possible $\boldsymbol{\omega}^b_r$ and all possible $\boldsymbol{\omega}^b_{rds}$ with the form (1.6). Remembering that all inertia tensors are diagonal with $\mathbb{J}^{xx}_k = \mathbb{J}^{zz}_k$, the system above becomes

$$\mathbb{J}^{xx}_a + \mathbb{J}^{xx}_b = \mathbb{J}^{xx}_{ds} + \mathbb{J}^{xx}_{ms} + \mathbb{J}^{xx}_{cs} =: \alpha \tag{1.44}$$

$$\mathbb{J}^{yy}_a + \mathbb{J}^{yy}_b = \mathbb{J}^{yy}_{ds} + \mathbb{J}^{yy}_{ms} + \mathbb{J}^{yy}_{cs} =: \beta \tag{1.45}$$

$$\rho_e(\mathbb{J}^{yy}_a - \mathbb{J}^{yy}_b) = \mathbb{J}^{yy}_{ds} + \rho_{ms,ds}\mathbb{J}^{yy}_{ms} + \rho_{cs,ds}\mathbb{J}^{yy}_{cs} =: \gamma \tag{1.46}$$

$$\rho_e^2(\mathbb{J}^{yy}_a + \mathbb{J}^{yy}_b) = \mathbb{J}^{yy}_{ds} + \rho^2_{ms,ds}\mathbb{J}^{yy}_{ms} + \rho^2_{cs,ds}\mathbb{J}^{yy}_{cs} =: \delta \,, \tag{1.47}$$

that is solved by

$$\rho_e = \sqrt{\frac{\mathbb{J}^{yy}_{ds} + \rho^2_{ms,ds}\mathbb{J}^{yy}_{ms} + \rho^2_{bd}\mathbb{J}^{yy}_{cs}}{\mathbb{J}^{yy}_{ds} + \mathbb{J}^{yy}_{ms} + \mathbb{J}^{yy}_{cs}}} \tag{1.48}$$

$$\mathbb{J}^{xx}_a = \mathbb{J}^{zz}_a = \frac{1}{2}(1+\lambda)(\mathbb{J}^{xx}_{ds} + \mathbb{J}^{xx}_{ms} + \mathbb{J}^{xx}_{cs}) \tag{1.49}$$

$$\begin{aligned}
\mathbb{J}^{yy}_a = &\frac{1}{2}(\mathbb{J}^{yy}_{ds} + \mathbb{J}^{yy}_{ms} + \mathbb{J}^{yy}_{cs}) + \frac{1}{2}(\mathbb{J}^{yy}_{ds} + \mathbb{J}^{yy}_{ms} + \mathbb{J}^{yy}_{cs})^{\frac{1}{2}} \\
&(\mathbb{J}^{yy}_{ds} + \rho_{ms,ds}\mathbb{J}^{yy}_{ms} + \rho_{cs,ds}\mathbb{J}^{yy}_{cs}) \\
&(\mathbb{J}^{yy}_{ds} + \rho^2_{ms,ds}\mathbb{J}^{yy}_{ms} + \rho^2_{cs,ds}\mathbb{J}^{yy}_{cs})^{-\frac{1}{2}}
\end{aligned} \tag{1.50}$$

$$\mathbb{J}^{xx}_b = \mathbb{J}^{zz}_b = \frac{1}{2}(1-\lambda)(\mathbb{J}^{xx}_{ds} + \mathbb{J}^{xx}_{ms} + \mathbb{J}^{xx}_{cs}) \tag{1.51}$$

$$\begin{aligned}
\mathbb{J}^{yy}_b = &\frac{1}{2}(\mathbb{J}^{yy}_{ds} + \mathbb{J}^{yy}_{ms} + \mathbb{J}^{yy}_{cs}) - \frac{1}{2}(\mathbb{J}^{yy}_{ds} + \mathbb{J}^{yy}_{ms} + \mathbb{J}^{yy}_{cs})^{\frac{1}{2}} \\
&(\mathbb{J}^{yy}_{ds} + \rho_{ms,ds}\mathbb{J}^{yy}_{ms} + \rho_{ms,ds}\mathbb{J}^{yy}_{cs}) \\
&(\mathbb{J}^{yy}_{ds} + \rho^2_{ms,ds}\mathbb{J}^{yy}_{ms} + \rho^2_{ms,ds}\mathbb{J}^{yy}_{cs})^{-\frac{1}{2}} \,,
\end{aligned} \tag{1.52}$$

where λ can be any real number of the open interval $(-1, 1)$. In order for the values of $\mathbb{J}^{xx}_{\tilde{k}}$, $\mathbb{J}^{yy}_{\tilde{k}}$, and $\mathbb{J}^{zz}_{\tilde{k}}$ to be elements of the diagonal of an inertia tensor, they have to be positive and satisfy

$$\mathbb{J}^{xx}_{\tilde{k}} \leq \mathbb{J}^{yy}_{\tilde{k}} + \mathbb{J}^{zz}_{\tilde{k}} \tag{1.53}$$

$$\mathbb{J}^{yy}_{\tilde{k}} \leq \mathbb{J}^{xx}_{\tilde{k}} + \mathbb{J}^{zz}_{\tilde{k}} = 2\mathbb{J}^{xx}_{\tilde{k}}. \tag{1.54}$$

Condition (1.53) is satisfied once positiveness is proved, because $\mathbb{J}^{xx}_{\tilde{k}} = \mathbb{J}^{zz}_{\tilde{k}}$. Defining the right-hand sides of equations (1.44)-(1.47) as α, β, γ, and δ, we can rewrite the solutions as

$$\rho_e = \sqrt{\frac{\delta}{\beta}} \tag{1.55}$$

$$\mathbb{J}^{xx}_{a} = \frac{1}{2}(1+\lambda)\alpha \tag{1.56}$$

$$\mathbb{J}^{xx}_{b} = \frac{1}{2}(1-\lambda)\alpha \tag{1.57}$$

$$\mathbb{J}^{yy}_{a} = \frac{\beta}{2} + \frac{\gamma}{2}\sqrt{\frac{\beta}{\delta}} \tag{1.58}$$

$$\mathbb{J}^{yy}_{b} = \frac{\beta}{2} - \frac{\gamma}{2}\sqrt{\frac{\beta}{\delta}}\,. \tag{1.59}$$

The positiveness of $\mathbb{J}^{yy}_{a}$ and $\mathbb{J}^{yy}_{b}$ can be written as a single inequality:

$$(\mathbb{J}^{yy}_{ds} + \rho_{ms,ds}\mathbb{J}^{yy}_{ms} + \rho_{cs,ds}\mathbb{J}^{yy}_{cs})^2 < (\mathbb{J}^{yy}_{ds} + \mathbb{J}^{yy}_{ms} + \mathbb{J}^{yy}_{cs})(\mathbb{J}^{yy}_{ds} + \rho^2_{ms,ds}\mathbb{J}^{yy}_{ms} + \rho^2_{cs,ds}\mathbb{J}^{yy}_{cs})\,,$$

and, exploiting the forms (1.58) and (1.59), it becomes

$$\beta\delta - \gamma^2 > 0\,. \tag{1.60}$$

If we define

$$\mathbf{J} := \begin{bmatrix}\mathbb{J}^{yy}_{ds} & \mathbb{J}^{yy}_{ms} & \mathbb{J}^{yy}_{cs}\end{bmatrix}^T, \tag{1.61}$$

$$\boldsymbol{\rho} := \begin{bmatrix}1 & \rho_{ds,ms} & \rho_{ds,ms}\end{bmatrix}^T, \tag{1.62}$$

$$\mathbf{S} := \begin{bmatrix}1 & \rho^2_{ds,ms} & \rho^2_{ds,cs}\end{bmatrix}^T, \tag{1.63}$$

$$\mathbf{1} := \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T, \tag{1.64}$$

$$\mathbf{P} := \mathbf{1}\mathbf{S}^T - \boldsymbol{\rho}\boldsymbol{\rho}^T, \tag{1.65}$$

we can transform the left-hand side of condition (1.60) into a quadratic form:

$$\beta\delta - \gamma^2 = \mathbf{J}^T(\mathbf{1}\mathbf{S}^T - \boldsymbol{\rho}\boldsymbol{\rho}^T)\mathbf{J} = \mathbf{J}^T\mathbf{P}\mathbf{J} = \mathbf{J}^T(\frac{\mathbf{P}+\mathbf{P}^T}{2})\mathbf{J}\,. \tag{1.66}$$

One can show that the generic element (i,j) of the matrix $\frac{P+P^T}{2}$ has the form $\frac{1}{2}(\rho_i - \rho_j)^2$, which is always greater than zero as long as $\rho_i \neq \rho_j$. Hence, if there are at least two different ρ_i, since $\mathbf{J}$ is made of positive numbers, condition (1.60) is satisfied. Condition (1.54) has to hold for both $\mathbb{J}_a$ and $\mathbb{J}_b$. Using the forms (1.56) - (1.59) the two constraints

$$\begin{aligned} \mathbb{J}_a^{yy} &\leq 2\mathbb{J}_a^{xx} && (1.67)\\ \mathbb{J}_b^{yy} &\leq 2\mathbb{J}_b^{xx} && (1.68) \end{aligned}$$

become

$$\begin{aligned} \frac{\beta}{2} + \frac{\beta}{2}\frac{\gamma}{\sqrt{\beta\delta}} &\leq (1+\lambda)\alpha && (1.69)\\ \frac{\beta}{2} - \frac{\beta}{2}\frac{\gamma}{\sqrt{\beta\delta}} &\leq (1-\lambda)\alpha\,, && (1.70) \end{aligned}$$

which lead to the solution

$$\frac{\beta}{2\alpha}\frac{\gamma}{\sqrt{\beta\delta}} - (1-\frac{\beta}{2\alpha}) \leq \lambda \leq \frac{\beta}{2\alpha}\frac{\gamma}{\sqrt{\beta\delta}} + (1-\frac{\beta}{2\alpha})\,. \tag{1.71}$$

Since the constraints on the elements of the inertia tensors must hold also for the original rotating bodies, α must be greater than $\beta/2$, hence $\beta/2\alpha \in (0,1]$. Equation (1.60) implies that $\gamma/\sqrt{\beta\delta} \in (-1,1)$. These two facts means that the intersection of solution (1.71) with the interval of definition of λ, $(-1,1)$, is never empty. In the worst case it contains a unique value:

$$\lambda = \frac{\beta}{2\alpha}\frac{\gamma}{\sqrt{\beta\delta}} = \frac{1}{2}\frac{\mathbb{J}_{ds}^{yy} + \rho_{ms,ds}\mathbb{J}_{ms}^{yy} + \rho_{cs,ds}\mathbb{J}_{cs}^{yy}}{\mathbb{J}_{ds}^{xx} + \mathbb{J}_{ms}^{xx} + \mathbb{J}_{cs}^{xx}}\sqrt{\frac{\mathbb{J}_{ds}^{yy} + \mathbb{J}_{ms}^{yy} + \mathbb{J}_{cs}^{yy}}{\mathbb{J}_{ds}^{yy} + \rho_{ms,ds}^2\mathbb{J}_{ms}^{yy} + \rho_{cs,ds}^2\mathbb{J}_{cs}^{yy}}}\,. \tag{1.72}$$

Equations (1.39) - (1.43), (1.49) - (1.52), (1.72) determine all parameters of the fast model as functions of those of the complete model, in such a way that using a change of coordinate like (1.37) and (1.38) with ρ_e defined in (1.48), the equivalence (1.20) holds. This conclude the proof.

Proposition 1 *Given any complete model C of an internal transmission defined by a set of parameters μ, the fast model defined by $\nu(\mu)$ by mean of formulae* (1.39) - (1.43)*,* (1.49) - (1.52)*,* (1.72)*, is equivalent to C in free evolution in the sense specified by* (1.19)*, with Γ given by* (1.37) *and its derivative* (1.38)*, and ρ_e defined in* (1.48)*.*

Lemma 1.1 guarantees that the Lagrangians of the two systems are related by the coordinate change Γ. The thesis derive from the invariance of the Euler-Lagrange equations to coordinate changes.

1.4.1 Fast model of an n-shaft internal drivetrain

In case the motorbike to model has other rotating elements inside the transmission, it is possible to define a fast model starting from a drivetrain with more than three shafts. But it is necessary that any further rigid body respects the assumptions we made on position, orientation and mass distribution of the other shafts. They have to be solids of revolutions with axes parallel to all other shafts, and their angular displacements have to be related by constant gear ratios. The structure of the fast model is the same as for the three shaft case, but the formulae for its parameters are the generalized versions of those given in the previous Section. In particular, the result of lemma 1.1 still holds for an arbitrary number n of rotating shafts, it is sufficient to substitute the formulae (1.41) (1.42) (1.43) with

$$\begin{aligned}
m_a = m_b &= \frac{1}{2}\sum_k^n m_k \\
\mathbf{h} &= \frac{\sum_k^n m_k \mathbf{h}_k}{\sum_k^n m_k} \\
\mathbb{I}_{\tilde{r}} &= \mathbb{I}_r - \mathbb{I}_{a,r}^{pm} - \mathbb{I}_{b,r}^{pm} + \sum_k^n \mathbb{I}_{k,r}^{pm} \,,
\end{aligned}$$

and the definition of α, β, γ and δ with

$$\begin{aligned}
\alpha &= \sum_k^n \mathbb{J}_k^{xx} \\
\beta &= \sum_k^n \mathbb{J}_k^{yy} \\
\gamma &= \mathbb{J}_{ds}^{yy} + \sum_k^n \rho_{k,ds} \mathbb{J}_k^{yy} \\
\delta &= \mathbb{J}_{ds}^{yy} + \sum_k^n \rho_{k,ds}^2 \mathbb{J}_k^{yy} \,.
\end{aligned}$$

The values of ρ_e, $\mathbb{J}_a^{xx}$, $\mathbb{J}_a^{yy}$, $\mathbb{J}_b^{xx}$, $\mathbb{J}_b^{yy}$ and λ are still given by (1.55) - (1.59), (1.72). Also the verifications of positiveness of the elements of the inertia tensors, and of conditions (1.53) and (1.54) still hold.

1.5 Equivalence in forced evolution

Inputs of multibody models are generally torques $\boldsymbol{\tau}$ and forces $\mathbf{f}$. Given a body, for example R, and a reference frame $\boldsymbol{\Sigma}_r$ rigidly attached to it, forces and torques are represented as column vectors in $\mathbb{R}^3$. While torques have no point of applications, forces are applied on the origin of the frame. A couple force-toque, both expressed in the same body reference frame, is called wrench $\mathbf{F}$ and belongs to $\mathbb{R}^6$:

$$\mathbf{F}_r = \begin{bmatrix} \mathbf{f}_r \\ \boldsymbol{\tau}_r \end{bmatrix} = \begin{bmatrix} \mathrm{f}_r^x & \mathrm{f}_r^y & \mathrm{f}_r^z & \tau_r^x & \tau_r^y & \tau_r^z \end{bmatrix}^T .$$

The subscript indicates the frame on which $\mathbf{F}$ is applied, in case the frame and the body have different names, for example the body is R and the frame is K, a second subscript will be used: $\mathbf{F}_{k,r}$.

The power injected by $\mathbf{F}$ is

$$P = \left\langle \mathbf{F}_r, \mathbf{V}_r^b \right\rangle ,$$

where the body velocity $\mathbf{V}_r^b$ is a linear function of the generalized velocities $\dot{\mathbf{q}}$. The configuration dependent matrix describing this linear relationship is the body jacobian, that we indicate with $\mathbf{J}_r^b(\mathbf{q})$ (we refer to [18] for further details), so the body velocity of R is

$$\mathbf{V}_r^b = \mathbf{J}_r^b(\mathbf{q})\dot{\mathbf{q}} . \tag{1.73}$$

The infinitesimal work allows the computation of the generalized forces $\boldsymbol{\Upsilon}_r$ relative to a wrench applied on $\boldsymbol{\Sigma}_r$. Indeed, we have

$$\left\langle \mathbf{F}_r, \mathbf{V}_r^b \right\rangle = \left\langle \mathbf{F}_r, \mathbf{J}_r^b(\mathbf{q})\dot{\mathbf{q}} \right\rangle = \left\langle (\mathbf{J}_r^b)^T \mathbf{F}_r, \dot{\mathbf{q}} \right\rangle = \langle \boldsymbol{\Upsilon}_r, \dot{\mathbf{q}} \rangle , \tag{1.74}$$

where we defined

$$\boldsymbol{\Upsilon}_r := (\mathbf{J}_r^b)^T \mathbf{F}_r . \tag{1.75}$$

In general there are many wrenches that produce the same generalized force, in fact if $Ker(\mathbf{J}_r^b)$ is non trivial, only the part of $\mathbf{F}_r$ that is orthogonal to it, contributes to the value of $\boldsymbol{\Upsilon}_r$, while the parallel component is uninfluential.

According to Lagrange-D'Alembert principle the presence of external forces and torques is modeled in Lagrangian mechanics just by mean of generalized forces:

$$\delta \int_{t_0}^{t_1} L(\mathbf{q}(t), \dot{\mathbf{q}}(t))dt + \int_{t_0}^{t_1} \langle \boldsymbol{\Upsilon}(\mathbf{q}, \dot{\mathbf{q}}, t), \delta\mathbf{q} \rangle \, dt = 0 , \tag{1.76}$$

where $\boldsymbol{\Upsilon}$ is the sum of the generalized forces relative to all wrenches acting on the model. Using integration by part shows that this is equivalent to the forced Euler-Lagrange equations, which have expression

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\Upsilon}(\mathbf{q}, \dot{\mathbf{q}}, t) .$$

The computation of the power injected on the system can be done indifferently using both body velocities and wrenches, and generalized velocities and forces.

The power produced is also invariant to coordinate transformations.

If $\dot{\tilde{\mathbf{q}}} = \mathbf{D}\Phi(\mathbf{q})\dot{\mathbf{q}}$, then defining

$$\tilde{\boldsymbol{\Upsilon}} := [\mathbf{D}\Phi^*]^{-1}(\mathbf{q})\boldsymbol{\Upsilon}\,, \tag{1.77}$$

results in

$$\left\langle \tilde{\boldsymbol{\Upsilon}}, \dot{\tilde{\mathbf{q}}} \right\rangle = \left\langle [\mathbf{D}\Phi^*]^{-1}(\mathbf{q})\boldsymbol{\Upsilon}, \mathbf{D}\Phi(\mathbf{q})\dot{\mathbf{q}} \right\rangle = \left\langle \mathbf{D}\Phi^*(\mathbf{q})[\mathbf{D}\Phi^*]^{-1}(\mathbf{q})\boldsymbol{\Upsilon}, \dot{\mathbf{q}} \right\rangle = \left\langle \boldsymbol{\Upsilon}, \dot{\mathbf{q}} \right\rangle$$

The mapping $[\mathbf{D}\Phi^*]^{-1}(\mathbf{q}) : \mathbb{R}^n \to \mathbb{R}^n$ is the inverse of the adjoint of $\mathbf{D}\Phi(\mathbf{q})$, and if Φ is defined like in (1.37) it coincides with the inverse transpose of J_Φ:

$$[\mathbf{D}\Phi^*]^{-1}(\mathbf{q}) := \begin{bmatrix} I_{6\times 6} & 0 \\ 0 & 1/\rho_e \end{bmatrix} = J_\Phi^{-1}\,.$$

The systems equivalence (1.19) becomes

$$f(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\Upsilon}) = \mathbf{D}\Gamma^{-1} g(\Phi(\mathbf{q}), \mathbf{D}\Phi\dot{\mathbf{q}}, [\mathbf{D}\Phi*]^{-1}\boldsymbol{\Upsilon}) \qquad \forall(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\Upsilon}) \in \mathbb{R}^{3n}\,. \tag{1.78}$$

Since Proposition 1 states that the equations of complete and fast model can be regarded as two versions of the same system, also the new generalized forces obtained by mean of (1.77) have a double nature: they are both an alternative representation of $\boldsymbol{\Upsilon}$, and the generalized forces of some set of wrenches $\tilde{\mathbf{F}}$ acting on the fast model.

We furnish expressions to transform wrenches $\mathbf{F}$ acting on the complete model, into equivalent wrenches $\tilde{\mathbf{F}}$ of the fast model, such that their associate generalized forces are related by

$$\tilde{\boldsymbol{\Upsilon}} = J_\Phi^{-1}\boldsymbol{\Upsilon}\,.$$

First of all we analyze the case of a generic wrench $\mathbf{F}_r$ applied on the center assembly. The body jacobians of $\mathbf{V}_r^b$ and $\mathbf{V}_{\tilde{r}}^b$ are equal, and their last column is equal to zero. This means that equal wrenches applied on $\boldsymbol{\Sigma}_r$ and $\boldsymbol{\Sigma}_{\tilde{r}}$ have the same generalized forces, whose last component is null. Considering that J_Φ^{-1} does not modify any of the first six components of $\boldsymbol{\Upsilon}_r$, we can conclude that all wrenches applied on $\boldsymbol{\Sigma}_r$ can be applied on $\boldsymbol{\Sigma}_{\tilde{r}}$ without any changes, producing the same power for any choice of the body velocities.

$$\mathbf{F}_r \to \mathbf{F}_{\tilde{r}} := \mathbf{F}_r\,. \tag{1.79}$$

To treat the case of wrenches applied on the shafts, we will need to distinguish between wrenches with only the fifth component different from zero $\mathbf{F}^5 := \begin{bmatrix} 0 & 0 & 0 & 0 & \tau_y & 0 \end{bmatrix}^T$, and those with the fifth component equal to zero $\mathbf{F}^* :=$

$\begin{bmatrix} \mathrm{f}_x & \mathrm{f}_y & \mathrm{f}_z & \tau_z & 0 & \tau_z \end{bmatrix}^T$.

Let $\mathbf{F}_k^*$ be a wrench acting on the shaft K, the power produced is

$$\begin{aligned} P &= \left\langle \mathbf{F}_k^*, \mathbf{V}_k^b \right\rangle \\ &= \left\langle \mathbf{F}_k^*, Ad_{\mathbf{g}_{rk}^{-1}} \mathbf{V}_r^b + \mathbf{V}_{rk}^b \right\rangle \\ &= \left\langle \mathbf{F}_{k,r}^*, Ad_{\mathbf{g}_{rk}^{-1}} \mathbf{V}_r^b \right\rangle + \left\langle \mathbf{F}_k^*, \mathbf{V}_{rk}^b \right\rangle \\ &= \left\langle Ad_{\mathbf{g}_{rk}^{-1}}^T \mathbf{F}_{k,r}^*, \mathbf{V}_r^b \right\rangle \\ &= \left\langle \mathbf{F}_r, \mathbf{V}_r^b \right\rangle . \end{aligned} \tag{1.80}$$

The second term of (1.80) is zero because it is a scalar product of two orthogonal vectors. The main point is that $\mathbf{F}_k^*$ can be applied indifferently on the generic shaft K or on R (becoming formally $\mathbf{F}_{k,r}^*$) as long as its application point is the origin of $\mathbf{\Sigma}_k$. In the latter case it can be rewritten in $\mathbf{\Sigma}_r$ coordinates and follows the rule (1.79).

$$\mathbf{F}_k^* \rightarrow \mathbf{F}_{\tilde{r}} := Ad_{\mathbf{g}_{rk}^{-1}}^T \mathbf{F}_k^* . \tag{1.81}$$

Consider now a wrench $\mathbf{F}_k^5$ representing a pure torque acting on the shaft $K \neq DS$ along its y-axis. The power produced is

$$\begin{aligned} P =& \left\langle \mathbf{F}_k^5, \mathbf{V}_k^b \right\rangle \\ =& \left\langle \mathbf{F}_k^5, Ad_{\mathbf{g}_{rk}^{-1}} \mathbf{V}_r^b + \mathbf{V}_{rk}^b \right\rangle \\ =& \left\langle Ad_{\mathbf{g}_{rk}^{-1}}^T \mathbf{F}_{k,r}^5, \mathbf{V}_r^b \right\rangle + \left\langle \mathbf{F}_k^5, \mathbf{V}_{rk}^b \right\rangle \end{aligned}$$

using the equalities $\mathbf{F}_r^5 = Ad_{\mathbf{g}_{rk}^{-1}}^T \mathbf{F}_{k,r}^5 = \mathbf{F}_{k,r}^5$ and $\mathbf{V}_{rk}^b = \rho_{k,ds} \mathbf{V}_{rds}^b$

$$\begin{aligned} =& \left\langle \mathbf{F}_r^5, \mathbf{V}_r^b \right\rangle + \rho_{k,ds} \left\langle \mathbf{F}_k^5, \mathbf{V}_{rds}^b \right\rangle + \left(\rho_{k,ds} \left\langle Ad_{\mathbf{g}_{rk}^{-1}}^T \mathbf{F}_{k,r}^5, \mathbf{V}_r^b \right\rangle - \rho_{k,ds} \left\langle \mathbf{F}_r^5, \mathbf{V}_r^b \right\rangle \right) \\ =& (1 - \rho_{k,ds}) \left\langle \mathbf{F}_r^5, \mathbf{V}_r^b \right\rangle + \rho_{k,ds} \left\langle \mathbf{F}_k^5, Ad_{\mathbf{g}_{rk}^{-1}} \mathbf{V}_r^b + \mathbf{V}_{rds}^b \right\rangle \\ =& (1 - \rho_{k,ds}) \left\langle \mathbf{F}_r^5, \mathbf{V}_r^b \right\rangle + \rho_{k,ds} \left\langle \mathbf{F}_k^5, \mathbf{V}_{ds}^b \right\rangle \end{aligned} \tag{1.82}$$

Equation (1.82) states that a pure torque τ_y acting on a generic shaft K can be replaced by a torque on DS, as long as it is scaled of a factor equal to $\rho_{k,ds}$, and another compensation torque is applied on R. This reduces case three to case one and case four.

Note that in case $V_r^b = 0$ (center assembly fixed to the ground), from (1.82) derives the classic expression of power preservation during gear reduction:

$$\tau_{ds} = \frac{1}{\rho_{ds,k}} \tau_k ,$$

where τ is a torque acting on the y direction.

The forth and last case treats the application of a wrench $\mathbf{F}^5_{ds}$ exerted on DS. The fifth rows of $\mathbf{J}^b_r$, $\mathbf{J}^b_{\tilde{r}}$, $\mathbf{J}^b_{ds}$ and $\mathbf{J}^b_a$ are

$$[\mathbf{J}^b_r]^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & \sin\varphi & 0 \end{bmatrix} \tag{1.83}$$

$$[\mathbf{J}^b_{\tilde{r}}]^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & \sin\tilde{\varphi} & 0 \end{bmatrix} \tag{1.84}$$

$$[\mathbf{J}^b_{ds}]^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & \sin\varphi & 1 \end{bmatrix} \tag{1.85}$$

$$[\mathbf{J}^b_a]^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & \sin\tilde{\varphi} & 1 \end{bmatrix} \tag{1.86}$$

The generalized force corresponding to the pure torque $\mathbf{F}^5_{ds}$ is

$$\boldsymbol{\Upsilon}_{ds} = [\mathbf{J}^b_{ds}]^T \mathbf{F}^5_{ds} = \begin{bmatrix} 0 & 0 & 0 & 0 & \tau^y_{ds} & (\tau^y_{ds}\sin\varphi) & \tau^y_{ds} \end{bmatrix}^T$$

and using $\boldsymbol{\Upsilon}_a = J_\Phi^{-1}\boldsymbol{\Upsilon}_{ds}$ we can calculate its equivalent version:

$$\boldsymbol{\Upsilon}_a = \begin{bmatrix} 1_{6\times6} & 0 \\ 0 & 1/\rho_e \end{bmatrix} \boldsymbol{\Upsilon}_{ds} = \begin{bmatrix} 0 & 0 & 0 & 0 & \tau^y_{ds} & (\tau^y_{ds}\sin\varphi) & \tau^y_{ds}/\rho_e \end{bmatrix}^T . \tag{1.87}$$

If we apply on A the torque $\mathbf{F}^5_a := \frac{1}{\rho_e}\mathbf{F}^5_{ds}$ and on $\tilde{R}$ the torque $\mathbf{F}^5_{\tilde{r}} := (1 - \frac{1}{\rho_e})\mathbf{F}^5_{ds}$, their combined generalized force is

$$\boldsymbol{\Upsilon}_{\tilde{r}+a} = \boldsymbol{\Upsilon}_{\tilde{r}} + \boldsymbol{\Upsilon}_a = \left[\left(1 - \frac{1}{\rho_e}\right)[\mathbf{J}^b_{\tilde{r}}]^T + \frac{1}{\rho_e}[\mathbf{J}^b_a]^T\right]\mathbf{F}^5_{ds}$$

that is equal to (1.87). This means that the couple of torques $(\mathbf{F}^5_a, \mathbf{F}^5_{\tilde{r}})$ as we defined it here, is equivalent to $\mathbf{F}^5_{ds}$ for the fast model.

$$\mathbf{F}^5_{ds} \to (\mathbf{F}^5_{\tilde{r}}, \mathbf{F}^5_a) := \left((1 - \frac{1}{\rho_e})\mathbf{F}^5_{ds}, \frac{1}{\rho_e}\mathbf{F}^5_{ds}\right) . \tag{1.88}$$

Formulae (1.79), (1.81) and (1.88), together with equivalence (1.82), cover all possible wrench acting on the complete model and make it utterly equivalent to the fast model.

Chapter 2

Reduced Models of the Drive Chain

In this Chapter two non-stiff models of the drive chain are presented. Both of them are part of a planar multibody system with three degrees of freedom simulating the rear half of a motorbike. Three moving bodies (the counter shaft, the swing arm, and the rear wheel) are connected to a center assembly that is fixed to the ground. External torques reproduce the engine, the rear brake and the load on the wheel.

One starting model, called full model, is intended to recreate the drawbacks that affect systems with stiff nonlinear dynamics. The chain is simulated by a couple of massless elastic thread placed on both the upper and lower sides of the sprockets. The characteristic function of the threads is such to give very high tension when they are stretched and no force during compression. This induces high frequency dynamics and numerical problems.

The first reduced model derives from the application of a quasi-steady-state approximation of the chain and rear suspension extension. This approach consists in replacing their fast stable dynamics with quasi-steady-state values, making the transient to the equilibrium point instantaneous. The reduced equations are obtained setting to zero both the velocities and accelerations of the chain length and the swing arm. Numerical simulations show that in the reduced model the high frequency vibrations are absent, while the slow signals are left unaltered.

In the last Section the inextensible chain model is derived. First the full model is presented as a switched system made of three submodels, that are selected on the base of which side of the chain is active, the upper, the lower, or none. Then the two submodels with active chain segments are reduced separately imposing a holonomic constraint on the chain extension. The inextensible chain model is the switched system originate merging the two reduced models, which are selected on the base of the sign of the holonomic constraint force.

2.1 Half-bike model

The chain is the last part of the drivetrain, that transmits the rotational movement of the counter shaft to the rear wheel. It also provides further gear reduction, as the wheel sprocket (also called driven sprocket) is bigger than the drive sprocket. The chain tension has an important effect on the trimming of the bike and contributes enormously in the weight distribution and the compression of the suspensions. For this reason the most suitable workbench to test different chain models would be a complete model of motorbike, where all valuable quantities, like the normal loads, the steering angle, the roll angle, etc., could be compared. However, to avoid the designing of a complete motorcycle, only the rear half of it has been implemented and simulated. In the opinion of the author, comparisons made in this framework are meaningful and likely to give similar results if repeated on complete vehicle models.

The three rigid bodies that play an active role in the kinematics of the chain are: the counter shaft DS, the rear wheel RW, and the swing arm SA. While the first two are directly in contact with the chain by mean of the sprockets, the swing arm affects and is affected by the chain force through the wheel joint constraint force. A schematic of the half-bike model is depicted in Figure 2.1. The center

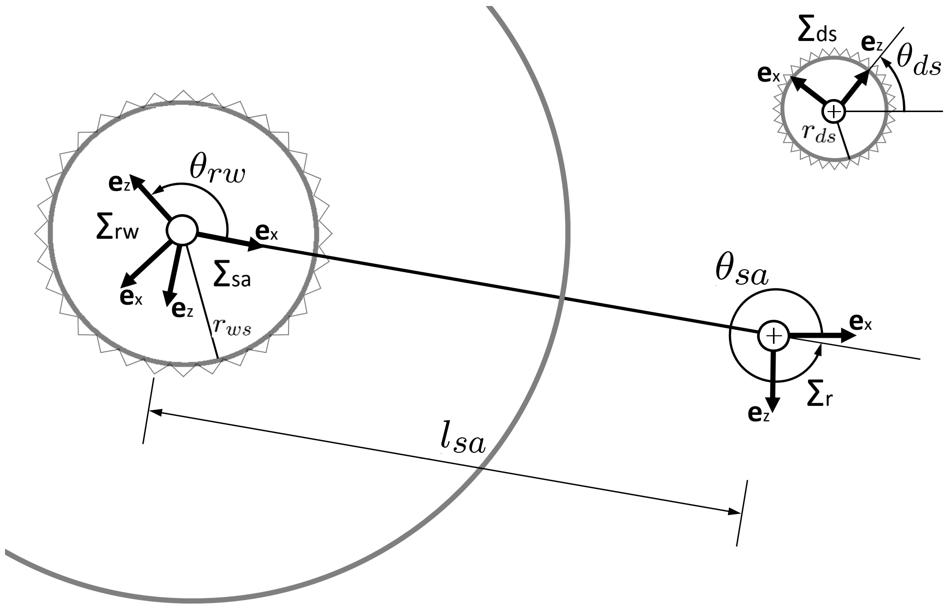


Figure 2.1: The half-bike model is composed of the center assembly R, fixed to the ground, the counter shaft, the swing arm and the rear wheel.

assembly R on which are pivoted DS and SA, is considered fixed to the ground, and its reference frame $\mathbf{\Sigma}_r$ is inertial. Its $z-axis$, $\mathbf{e}^r_z$, points downward, and the plane $(\mathbf{e}^r_x, \mathbf{e}^r_z)$ coincides with the symmetry plane of the bike. As the bodies are rigid, this three dimensional model is studied as it were planar.

Drive sprocket. The origin O_{ds} is fixed in position $\mathbf{r}_{ds} = \mathbf{h}_{ds} = \begin{bmatrix} h_{ds}^x & 0 & h_{ds}^z \end{bmatrix}^T$, where h_{ds}^x and h_{ds}^z are parameters of the model. As in Chapter 1, its rotational displacement is defined by the angle θ_{ds} between $\mathbf{e}_x^{ds}$ and $\mathbf{e}_x^r$. The rotation matrix is defined as

$$\mathbf{R}_{ds} = \mathbf{R}_y(\theta_{ds}).$$

The body coordinates of the center of mass are null, and the spatial coordinates coincide with $\mathbf{r}_{ds}$.

Without considering the chain tension, on DS are applied two forces: an external torque $\tau_{eng}(t)$ that represents the engine action, and a friction force $\tau_{fri}^{ds}(\dot{\theta}_{ds}) = -\gamma_{ds}\dot{\theta}_{ds}$ with $\gamma_{ds} > 0$. The overall wrench is

$$\mathbf{F}_{ds}(\dot{\theta}_{ds}, t) = \begin{bmatrix} 0 & 0 & 0 & 0 & (\tau_{eng} + \tau_{fri}^{ds}) & 0 \end{bmatrix}^T.$$

Swing arm. The reference frame of the swing arm $\mathbf{\Sigma}_{sa}$ is located at the wheel joint with the $x - axis$ pointing towards O_r. The swing arm pivot point is located at the origin of $\mathbf{\Sigma}_r$, with axis $\mathbf{e}_y^r$; its angular displacement is measured by the angle θ_{sa} between $\mathbf{e}_x^r$ and $\mathbf{e}_x^{sa}$ as indicated in Figure 2.1. Its rotation matrix and translation vector are

$$\mathbf{R}_{sa} = \mathbf{R}_y(\theta_{sa})$$

$$\mathbf{r}_{sa} = \mathbf{R}_{sa} \begin{bmatrix} -l_{sa} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_{sa}\cos(\theta_{sa}) \\ 0 \\ l_{sa}\sin(\theta_{sa}) \end{bmatrix}.$$

The coordinates of its center of mass are

$$\bar{\mathbf{p}}_{sa}^s = \begin{bmatrix} \mathbf{p}_{sa}^s \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{sa} & \mathbf{r}_{sa} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{sa}^b \\ 1 \end{bmatrix} = \mathbf{g}_{sa}\bar{\mathbf{p}}_{sa}^b.$$

The forces acting of the swing arm are due to the rear suspension, and to the interaction with the rear wheel. The rear suspension of a bike is generally made of a set of levers that relate θ_{sa} to the length of a spring and a damper. We assume that the spring length l_{susp} is given by an invertible function $l_{susp}(\theta_{sa})$ of class C^2 accounting for the lever ratio. The torque τ_{spr} applied on the joint is given by a characteristic smooth function

$$\tau_{spr} := \tau_{spr}(l_{susp}(\theta_{sa})).$$

A simple positive damping coefficient has been used to reproduce the damper:

$$\tau_{dam}(\dot{\theta}_{sa}) := -\gamma_{sa}\dot{\theta}_{sa} \,.$$

At the end of the swing arm there is the wheel brake that, when operated, exerts a differential torque on the wheel and the arm itself. The braking torque is proportional to the speed of the wheel and to a time-varying positive coefficient:

$$\tau_{bra}(\dot{\theta}_{rw}, t) := \gamma_{bra}(t)\dot{\theta}_{rw} \,.$$

We also include a friction torque due to the dissipation of the bearing:

$$\tau_{fri}^{rw} := \gamma_{rw}\dot{\theta}_{rw} \,.$$

The overall wrench acting on SA is

$$\mathbf{F}_{sa}(\theta_{sa}, \dot{\theta}_{sa}, \dot{\theta}_{rw}, t) = \begin{bmatrix} 0 & 0 & 0 & 0 & (\tau_{spr} + \tau_{dam} + \tau_{bra} + \tau_{fri}^{rw}) & 0 \end{bmatrix}^T$$

Rear wheel. $\mathbf{\Sigma}_{rw}$ is located at the center of the wheel, that rotates about $\mathbf{e}_y^{rw}$. The rotation is measured with the angle θ_{rw} between $\mathbf{e}_x^{sa}$ and $\mathbf{e}_x^{rw}$. The rotation matrix and the translation vector are

$$\mathbf{R}_{rw} = \mathbf{R}_{sa}\mathbf{R}_y(\theta_{rw}) = \mathbf{R}_y(\theta_{sa} + \theta_{rw}) \tag{2.1}$$

$$\mathbf{r}_{rw} = \mathbf{r}_{sa} \,. \tag{2.2}$$

The body coordinates of the center of mass are null, and the spatial coordinates coincide with $\mathbf{r}_{rw}$. Beyond the friction torque τ_{fri}^{rw}, also an external time-varying load $\tau_{load}(t)$ is applied. The the expression of the overall wrench is:

$$\mathbf{F}_{rw}(\dot{\theta}_{rw}, t) = \begin{bmatrix} 0 & 0 & 0 & 0 & (\tau_{load} - \tau_{bra} - \tau_{fri}^{rw}) & 0 \end{bmatrix}^T .$$

All parameters fo the full model are collected in table 2.1.

Table 2.1: Parameters of the full model.

m_{sa}, m_{rw}	Masses of SAand RW.
$\mathbb{J}_{ds}^{yy}, \mathbb{J}_{sa}^{yy}, \mathbb{J}_{rw}^{yy}$	Moments of inertia of the three bodies.
$\mathbf{p}_{sa}^{b}$	Position of the center of mass of SA in body coordinates.
l_{sa}	Length of the swing arm.
r_{ws}	Wheel sprocket radius.
r_{ds}	Drive sprocket radius.
$\mathbf{h} = \begin{bmatrix} h_x & 0 & h_z \end{bmatrix}^T$	Position of O_{ds} in spatial coordinates.

The vector of the three angles $\theta := \begin{bmatrix}\theta_{ds} & \theta_{sa} & \theta_{rw}\end{bmatrix}^T$ is the vector of the generalized coordinates. The body jacobians of the three moving bodies are

$$\mathbf{J}^b_{ds} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{J}^b_{sa} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & l_{sa} & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{J}^b_{rw} = \begin{bmatrix} 0 & -l_{sa}\sin(\theta_{rw}) & 0 \\ 0 & 0 & 0 \\ 0 & l_{sa}\cos(\theta_{rw}) & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and the equations of motion of the half-bike model have the general form

$$\begin{gathered} M(\theta)\ddot{\theta} + C(\theta, \ddot{\theta}) + G(\theta) = \\ = [\mathbf{J}^b_{ds}]^T \mathbf{F}_{ds}(\dot{\theta}_{ds}, t) + [\mathbf{J}^b_{sa}]^T \mathbf{F}_{sa}(\theta_{sa}, \dot{\theta}_{sa}, \dot{\theta}_{rw}, t) + [\mathbf{J}^b_{rw}]^T \mathbf{F}_{rw}(\dot{\theta}_{rw}, t) =: \mathbf{\Upsilon}'(\theta, \dot{\theta}, t) \end{gathered}$$

2.2 Full model

The chain is modeled as a massless elastic thread wrapped around the wheel and drive sprocket. Consider the center of the sprockets fixed as if the suspension were locked. As the sprockets rotate, they put in tension either the upper chain segment or the lower one. When the upper (lower) segment is straight, it becomes tangent to both the sprockets, and defines the tangent points A and C (B and D) like in figure 2.2. We assume that the tangent points separate the stretched

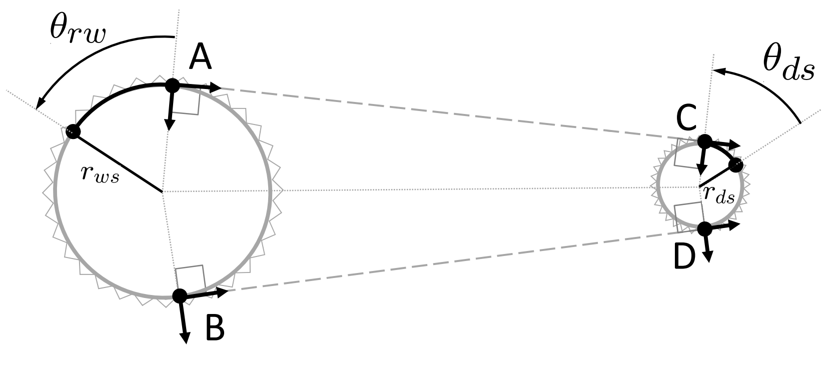


Figure 2.2: Tangent points coordinate frames and chain length. When the upper part of the chain is tight, the chain lays on the segment AC. Similarly, when the bottom part is tight, the chain is tangent on B and D. We define four reference frames Σ_A, Σ_B, Σ_C, and Σ_D as depicted, where we will apply the chain tension.

part of the chain (the one under tension) from the parts that are attached to the gears and move solidly with them (arcs $\widehat{AB}$ and $\widehat{CD}$). Imagine to mark with a sign the points of the chain p_A and p_C that occupy the tangent points at a certain time, when the chain is straight but the tension is zero. We define the length l_u of the upper chain segment as the distance $d_u(p_A, p_C)$ along the chain, that, in this configuration, is equal to the length of the segment $\overline{AC}$. Let the sprockets rotate of angles $\theta_{ds} < 0$ (clockwise in the Figure 2.2) and $\theta_{rw} > 0$ (counterclockwise); p_A and p_C rotate about the sprockets along two arcs. The upper segment length becomes $l_u = r_{ws}\theta_{rw} + \overline{AC} - r_{ds}\theta_{ds}$. This formula holds also for arbitrary rotations even if p_A or p_C get off the sprockets. The extension of the upper segment e_u is defined as the difference between the upper length and the upper rest length l_u^0, that is a parameter of the model:

$$e_u := l_u - l_u^0 = r_{ws}\theta_{rw} + \overline{AC} - r_{ds}\theta_{ds} - l_u^0$$

An analogue reasoning let us define the lower segment extension as

$$e_l := l_l - l_l^0 = -r_{ws}\theta_{rw} + \overline{BD} + r_{ds}\theta_{ds} - l_l^0$$

If the sum of the two rest lengths is sufficiently big, the two segments are never both tense at the same time, but there is a dead zone.

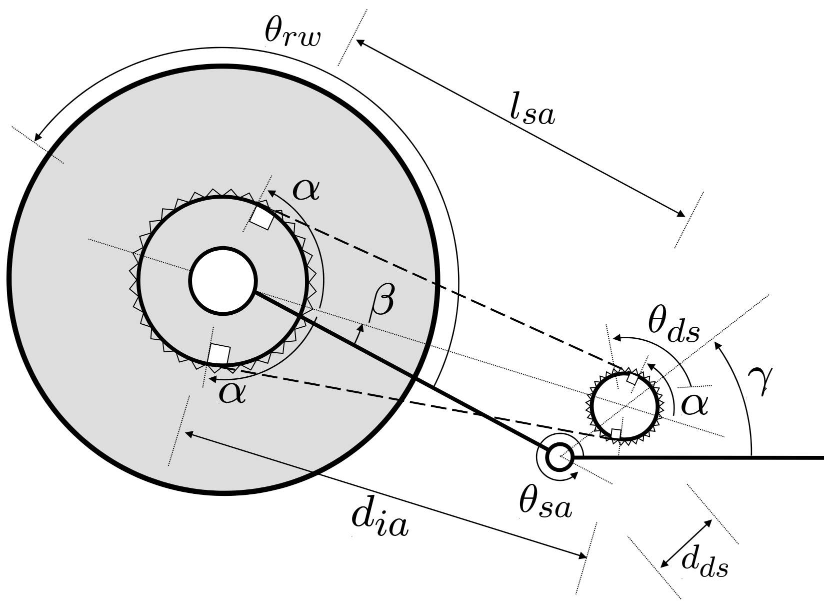


Figure 2.3: Full chain model. When the swing arm is free to move the distance of the sprockets changes, and the positions of the tangent points become a function of θ_{sa}.

When the swing arm is free to move the distance of the sprockets changes, and the positions of the tangent points become a function of θ_{sa}. We use some geometrical arguments to express this dependence. The two angles α and β, reported in Figure 2.3, are implicitly defined as

$$\begin{aligned}
\cos(\alpha) &= (r_{ws} - r_{ds})/d_{ia} && (2.3)\\
\tan(\beta) &= d_{ds}\sin(\gamma - \theta_{sa})/[l_{sa} + d_{ds}\cos(\gamma - \theta_{sa})] && (2.4)\\
d_{ia} &= [l_{sa} + d_{ds} cos(\gamma - \theta_{sa})]/cos(\beta)\,, && (2.5)
\end{aligned}$$

where d_{ds} is the distance between the drive sprocket and the swing arm joint, while γ is the elevation of the drive sprocket center from the center assembly. The quantity d_{ia} is the distance between the centers of the two sprockets. The angle α specifies the angular displacement of the tangent points relative to the axis passing through the center of the two sprockets. The angle β represents the angle between the swing arm and the axis that goes through the center of the two sprockets. The distance of the tangent points is

$$\overline{AC} = \overline{BD} = \sqrt{d_{ia}^2 - (r_{ws} - r_{ds})^2}\,.$$

The modules of the chain tensions are

$$\begin{aligned}
T_u &:= \begin{cases} ke_u & e_u \geq 0 \\ 0 & e_u < 0 \end{cases} \\
T_l &:= \begin{cases} ke_l & e_l \geq 0 \\ 0 & e_l < 0 \end{cases}
\end{aligned} \quad . \qquad (2.6)$$

As the tangent points are located on the sprockets, they can be described in body coordinates relative to Σ_{ds} and Σ_{rw}. Is it worth noting that $\beta + \alpha$ is the angular displacement of A with respect to the x axis of the rear assembly Σ_{sa} and that similarly $\beta - \alpha$ describes the angular displacement of B.

We describe in detail how to model the interaction of the chain with the point A of the wheel sprocket, being the other cases for B, C, and D similar. See Figures 2.2 and 2.3 for a description of the angles and reference frames that we will make use of.

The reference frame $\mathbf{\Sigma}_{rw}$ is attached at the center rear wheel and rotates with it. A reference frame Σ_a is attached to the point A. Its position and orientation relative to the rear wheel frame are

$$\begin{aligned}
\mathbf{r}_A &= r_{ws}\mathbf{R}_y(\alpha + \beta - \theta_{rw})\mathbf{e}_x && (2.7)\\
\mathbf{R}_A &= \mathbf{R}_y(\alpha + \beta - \theta_{rw} - \pi/2)\,. && (2.8)
\end{aligned}$$

Notice that Σ_a is defined so that its $x - axis$ points along the segment $\overline{AC}$.

To compute the infinitesimal work done by the chain tension, we imagine that the chain applies a force oriented along AC of magnitude T_u. We may represent this situation using a wrench F_a applied at the origin of $\mathbf{\Sigma}_a$. In coordinates relative to $\mathbf{\Sigma}_a$, we have

$$\mathbf{F}_a = \begin{bmatrix} T_u & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T . \tag{2.9}$$

Since the wrench is applied on a moving frame, we need to rewrite it with respect to the rear wheel frame $\mathbf{\Sigma}_{rw}$. As described in the appendix Sections A.3 and A.4, the equivalent wrench can be computed as

$$\mathbf{F}^a_{rw} = Ad^{-T}_{\mathbf{g}_a} \mathbf{F}_a \,, \tag{2.10}$$

and the infinitesimal work is then given by

$$\left\langle \mathbf{F}^a_{rw}, \mathbf{V}^b_{rw} \right\rangle . \tag{2.11}$$

The rigid transformations that define the reference frames of the other three tangent points with respect to $\mathbf{\Sigma}_{rw}$ and $\mathbf{\Sigma}_{ds}$ are:

$$\begin{aligned}
\mathbf{R}_b &= \mathbf{R}_y(-\theta_{rw} + \beta - \alpha + \pi/2) \\
\mathbf{r}_b &= \mathbf{R}_b \begin{bmatrix} 0 \\ 0 \\ r_{ws} \end{bmatrix} \\
\mathbf{R}_c &= \mathbf{R}_y(-\theta_{ds} + \theta_{sa} + \beta + \alpha - \pi/2) \\
\mathbf{r}_c &= \mathbf{R}_c \begin{bmatrix} 0 \\ 0 \\ -r_{ds} \end{bmatrix} \\
\mathbf{R}_d &= \mathbf{R}_y(-\theta_{ds} + \theta_{sa} + \beta - \alpha + \pi/2) \\
\mathbf{r}_d &= \mathbf{R}_d \begin{bmatrix} 0 \\ 0 \\ r_{ds} \end{bmatrix} ,
\end{aligned}$$

the overall generalized force due to the chain tensions is

$$\mathbf{\Upsilon}_{chain}(\theta) = [\mathbf{J}^b_{rw}]^T(\mathbf{F}_{A,rw}(\theta) + \mathbf{F}_{B,rw}(\theta) + [\mathbf{J}^b_{rw}]^T(\mathbf{F}_{C,ds}(\theta) + \mathbf{F}_{D,ds}(\theta) \tag{2.12}$$

The equation of motions of the half-bike model endowed with a full chain become

$$M(\theta)\ddot{\theta} + C(\theta, \ddot{\theta}) + G(\theta) = \mathbf{\Upsilon}'(\theta, \dot{\theta}, t) + \mathbf{\Upsilon}_{chain}(\theta) \,, \tag{2.13}$$

and will be called full model equations.

2.3 Quasi-steady-state model

The quasi-steady-state approximation of a dynamic system consists briefly in replacing the dynamics of a certain subsystem, with its equilibrium configuration, transforming one or more first order differential equation into algebraic equations.

To give an example of the quasi-steady-state dynamics of a spring, we study the simple case of a linear mechanical model (see Figure 2.4). Two masses m_1 and

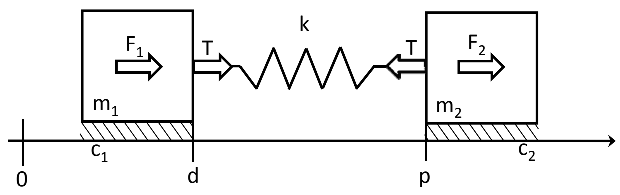


Figure 2.4: Mass-spring-mass model. Two masses m_1 and m_2, free to move on a straight line, are linked by a spring with stiffness k. F_1 and F_2 are external forces, T is the spring tension, c_1 and c_2 are the friction coefficients.

m_2 are connected by a spring. The spring tension T is equal to the product of a high positive stiffness k and the spring extension $l := p - d - l_0$. The symbol l_0 is the rest length of the spring, while d and p indicate the longitudinal displacements of the masses. Two positive parameters c_1 and c_2 model a linear friction with the ground. The equations of motion are

$$\begin{aligned}
\dot{d} &= d_d && (2.14)\\
\dot{d}_d &= -\frac{c_1}{m_1} d_d + \frac{k}{m_1}(p - d - l_0) + \frac{F_1(t)}{m_1} && (2.15)\\
\dot{p} &= p_d && (2.16)\\
\dot{p}_d &= -\frac{c_2}{m_2} p_d - \frac{k}{m_2}(p - d - l_0) + \frac{F_2(t)}{m_2} && (2.17)
\end{aligned}$$

where the subscript $_d$ is used for the velocities. As for the chain, the extension of the spring depends on the difference of two state variables and can be taken as a generalized coordinate instead of one of them. Defining as new coordinates the spring extension l, and $l_d := p_d - d_d$, we put in evidence the dynamics of the spring:

$$\begin{aligned}
\dot{d} &= d_d && (2.18)\\
\dot{d}_d &= -\frac{c_1}{m_1} d_d + \frac{k}{m_1} l + \frac{F_1(t)}{m_1} && (2.19)\\
\dot{l} &= l_d && (2.20)\\
\dot{l}_d &= -k\frac{m_1 + m_2}{m_1 m_2} l - \frac{c_2}{m_2} l_d + \frac{c_1 m_2 - c_2 m_1}{m_1 m_2} d_d - \frac{F_1(t) m_2 - F_2(t) m_1}{m_1 m_2} && (2.21)
\end{aligned}$$

The spring shows the behavior of an asymptotically stable oscillator with inputs. The equilibrium point is found setting $\dot{l}$ and $\dot{l}_d$ to zero and inverting 2.21. The quasi-steady-state function depends on the reduced state variables and the input:

$$l_{qs} = \frac{1}{k}\frac{c_1 + c_2}{m_1 + m_2} d_d - \frac{1}{k}\frac{F_1(t) + F_2(t)}{m_1 + m_2} \,. \tag{2.22}$$

Substituting 2.22 in 2.15 gives the reduced dynamic equations

$$\begin{aligned} \dot{d} &= d_d & (2.23) \\ \dot{d}_d &= -\frac{c_1 + c_2}{m_1 + m_2} d_d + \frac{F_1(t) + F_2(t)}{m_1 + m_2} \,. & (2.24) \end{aligned}$$

The state space of the reduced model has now dimension 2.

For the mass-spring-mass model, the QSSA is equivalent to the application of a singular perturbation. To apply the singular perturbation theory ([20], [21], [22], [23], [24]) the system must be written in standard form

$$\begin{aligned} \dot{x} &= f(t, x, z, \epsilon) \\ \epsilon \dot{z} &= g(t, x, z, \epsilon) \,, \end{aligned}$$

where the state variables are partitioned in slow and fast ones, called respectively x and z, and ϵ is a small parameter of the system. If we apply the further change of variables

$$\begin{aligned} x_1 &:= d \\ x_2 &:= d_d & (2.25) \\ z_1 &:= kl & (2.26) \\ z_2 &:= kl_d \,, \end{aligned}$$

to the mass-spring-mass model, and define $\epsilon := 1/k$, the standard form is

$$\begin{aligned} \dot{x}_1 &= x_2 & (2.27) \\ \dot{x}_2 &= -\frac{c_1}{m_1} x_2 + \frac{z_1}{m_1} + \frac{F_1(t)}{m_1} & (2.28) \\ \epsilon \dot{z}_1 &= \epsilon z_2 & (2.29) \\ \epsilon \dot{z}_2 &= \frac{c_1 m_2 - c_2 m_1}{m_1 m_2} x_2 - \frac{m_1 + m_2}{m_1 m_2} z_1 + \epsilon \frac{c_2}{m_2} z_2 - \frac{F_1(t) m_2 - F_2(t) m_1}{m_1 m_2} & (2.30) \end{aligned}$$

Setting $\epsilon = 0$ the dimension of the state space decreases as two equations become algebraic. Inverting 2.30 we find $z_1 = h(t, x)$. Unfortunately equation 2.29 collapses when $\epsilon = 0$, and any choice of z_2 is admissible, which means that the roots of $0 = g(t, x, z, 0)$ are not isolated. This fact prevent us from applying Tikhonov's theorem and proving the convergence of the reduced dynamics to the full one as ϵ goes to zero. Nevertheless, picking $z_2 = 0$, we can still write a quasi-steady-state for the fast variable z

$$\begin{aligned} z_1 &= \frac{c_1 m_2 + c_2 m_1}{m_1 + m_2} x_2 - \frac{F_1(t) m_2 + F_2(t) m_1}{m_1 + m_2} \\ z_2 &= 0\,, \end{aligned} \tag{2.31}$$

and a slow model $\dot{x} = f(t, x, \begin{bmatrix} h(t,x) & 0 \end{bmatrix}^T, 0)$, that is indeed equal to 2.23 - 2.24. We now pass to describe the application of the QSSA to the full chain model.

In most cases the equations derived from multibody codes are not tractable, so, though the equations of the full model are not extremely complex, we will operate as if they were a black-box. To increase the generality of the discourse, let us consider the more general case of a mechanical system representing a complete motorbike with also the front steering, the fork, etc. Its equations are

$$M(q)\ddot{q} + C(q, \dot{q}) + G(q) = \boldsymbol{\Upsilon}(q, \dot{q}, t)\,.$$

Assume that the rear half of this model is equal to the full model; the vector q and its time derivatives can be partitioned in

$$q =: \begin{bmatrix} \theta \\ q_* \end{bmatrix}\,,$$

where q_* collects all state variables involving the front half of the bike. Inverting the kinetic matrix M we write the equations of motion in explicit form, like they are usually returned by multibody codes:

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{q}_* \end{bmatrix} = M^{-1}(\boldsymbol{\Upsilon} - C - G) =: f(\theta, q_*, \dot{\theta}, \dot{q}_*, t) =: \begin{bmatrix} f_\theta(\theta, q_*, \dot{\theta}, \dot{q}_*, t) \\ f_{q_*}(\theta, q_*, \dot{\theta}, \dot{q}_*, t) \end{bmatrix}\,.$$

The vector function f_θ can be further subdivided in

$$f_\theta = \begin{bmatrix} f_{ds} & f_{sa} & f_{rw} \end{bmatrix}^T\,.$$

With respect to the time-scale of the simulations we are interested in, also the dynamics of the suspensions is considered fast. As it has been proved in [25], the QSSA is effective also on the rear suspension. We impose the first and second derivatives of the rear suspension spring length l_{susp} equal to zero and obtain two conditions

$$\begin{aligned} \dot{\theta}_{sa} &= 0 & (2.32)\\ \ddot{\theta}_{sa} &= 0\,. & (2.33) \end{aligned}$$

This is due to the fact that the length of the spring suspension is a smooth invertible function of θ_{sa} only.

Exploiting 2.32 and 2.33, the derivatives of the chain length become

$$\begin{aligned} \dot{l}_u &= r_{rw}\dot{\theta}_{rw} - r_{ds}\dot{\theta}_{ds} & (2.34)\\ \dot{l}_l &= -r_{rw}\dot{\theta}_{rw} + r_{ds}\dot{\theta}_{ds} & (2.35)\\ \ddot{l}_u &= r_{rw}\ddot{\theta}_{rw} - r_{ds}\ddot{\theta}_{ds} & (2.36)\\ \ddot{l}_l &= -r_{rw}\ddot{\theta}_{rw} + r_{ds}\ddot{\theta}_{ds}\,. & (2.37) \end{aligned}$$

and setting them to zero yields

$$\begin{aligned} \dot{\theta}_{ds} &= \frac{r_{rw}}{r_{ds}}\dot{\theta}_{rw} & (2.38)\\ \ddot{\theta}_{ds} &= \frac{r_{rw}}{r_{ds}}\ddot{\theta}_{rw}\,. & (2.39) \end{aligned}$$

While conditions 2.32 and 2.38 involve directly state variables of the full model, imposing 2.33 and 2.39 would require the possibility of manipulating the function $f(q, \dot{q}, t)$. As we assumed that it is not possible, they must be imposed numerically. Given the values of the reduced state space variables $(\bar{q}, \dot{\bar{q}}) := (\theta_{rw}, q_*, \dot{\theta}_{rw}, \dot{q}_*)$, the quasi-steady-state is calculated solving

$$\begin{aligned} r_{ds} f_{ds}(\theta_{ds}, \theta_{sa}, \bar{q}, \frac{r_{ws}}{r_{ds}}\dot{\theta}_{rw}, 0, \dot{\bar{q}}, t) - r_{ws} f_{rw}(\theta_{ds}, \theta_{sa}, \bar{q}, \frac{r_{ws}}{r_{ds}}\dot{\theta}_{rw}, 0, \dot{\bar{q}}, t) &= 0\\ f_{sa}(\theta_{ds}, \theta_{sa}, \bar{q}, \frac{r_{ws}}{r_{ds}}\dot{\theta}_{rw}, 0, \dot{\bar{q}}, t) &= 0 \end{aligned}$$

for $(\theta_{ds}, \theta_{sa})$. The solution $(\theta_{ds}^{qs}, \theta_{sa}^{qs}) = h(\theta_{rw}, q_*, \dot{\theta}_{rw}, \dot{q}_*, t)$ is replaced into the full equations to derive the reduced dynamics:

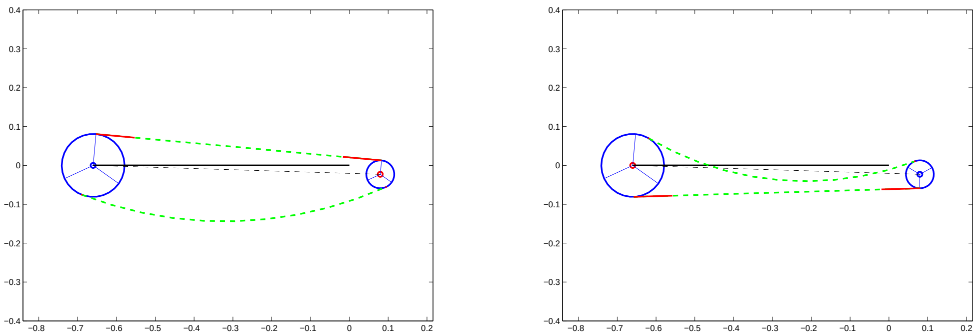
$$\ddot{\bar{q}} = \bar{f}(\theta_{ds}^{qs}, \theta_{sa}^{qs}, \bar{q}, \frac{r_{ws}}{r_{ds}}\dot{\theta}_{rw}, 0, \dot{\bar{q}}, t)\,, \qquad (2.40)$$

where $\bar{f}$ are the last $n-2$ component of $f(q, \dot{q}, t)$ and n is the dimension of the state space of the complete motorcycle model.

We made the arbitrary choice of reducing θ_{ds}, because the angular displacement of the counter shaft is a minor quantity. In principle θ_{rw} could be chosen as well, in which case the reduced state space would contain θ_{ds}. No assumption can be made on the structure of the reduced system, which in general does not present a mechanical structure. The sequence of quasi-steady-state values for θ_{ds} and θ_{sa} can be numerically derived to obtain an estimate of their velocities. Numerical results presented in Section 2.4 prove the goodness of the approximation adopted.

2.4 Numerical results

In this section we present a series of simulation results, obtained comparing the full model and the quasi-steady-state model, that have been implemented in Matlab as a Symulink level-2 S-Functions. In Figure 2.5 there are 2 views of the full chain model subject to the engine and braking torque. The values of the model parameters are listed in table 2.2. The simulation step is $10^{-3}s$.



(a) Traction (b) Braking

Figure 2.5: Full chain model. During traction and braking one segment at a time is tense.

Table 2.2: Parameters of the Half-bike model.

m_{sa}	$=$	7	$\mathbb{J}_{ds}^{yy}$	$=$	0.005	h_x	$=$	0.08	p_{sa}^x	$=$	0.35
m_{rw}	$=$	15	$\mathbb{J}_{sa}^{yy}$	$=$	0.4	h_y	$=$	0	p_{sa}^y	$=$	0
l_{sa}	$=$	0.7	$\mathbb{J}_{rw}^{yy}$	$=$	0.7	h_z	$=$	0.02	p_{sa}^z	$=$	0
r_{ds}	$=$	0.03	r_{ws}	$=$	0.08						

2.4.1 QSS approximation

The first set of simulation results is obtained applying an engine torque ramp during the first five seconds, letting the systems evolve freely for other five seconds and applying a brake torque step at $t = 10$. Positive engine torques accelerate the counter shaft counterclockwise. The velocity of the drive sprocket in the QSS model is obtained through finite differences.

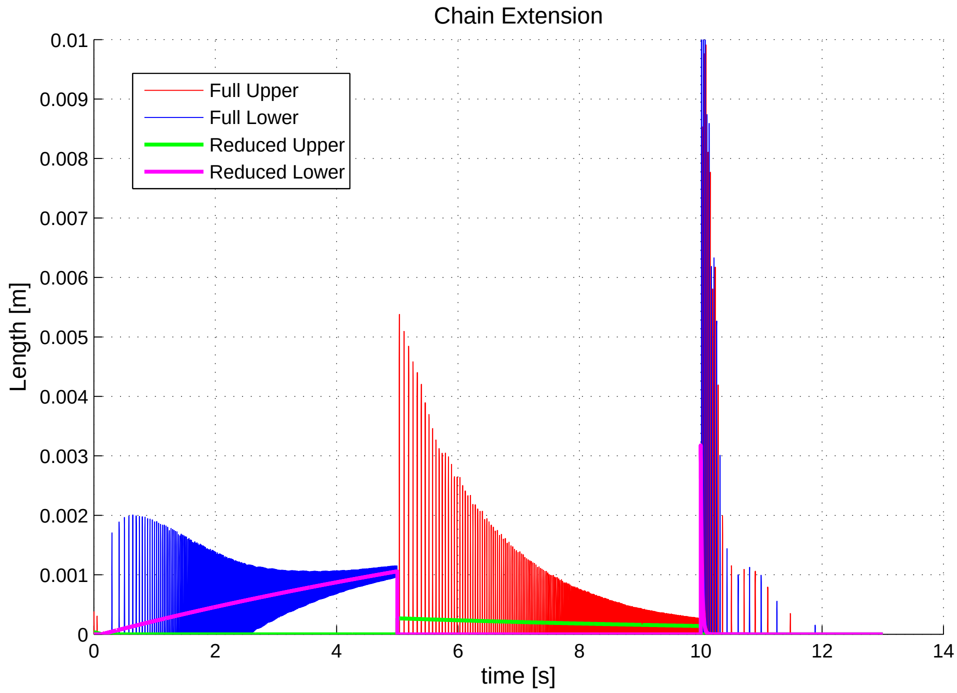


Figure 2.6: Chain extension.

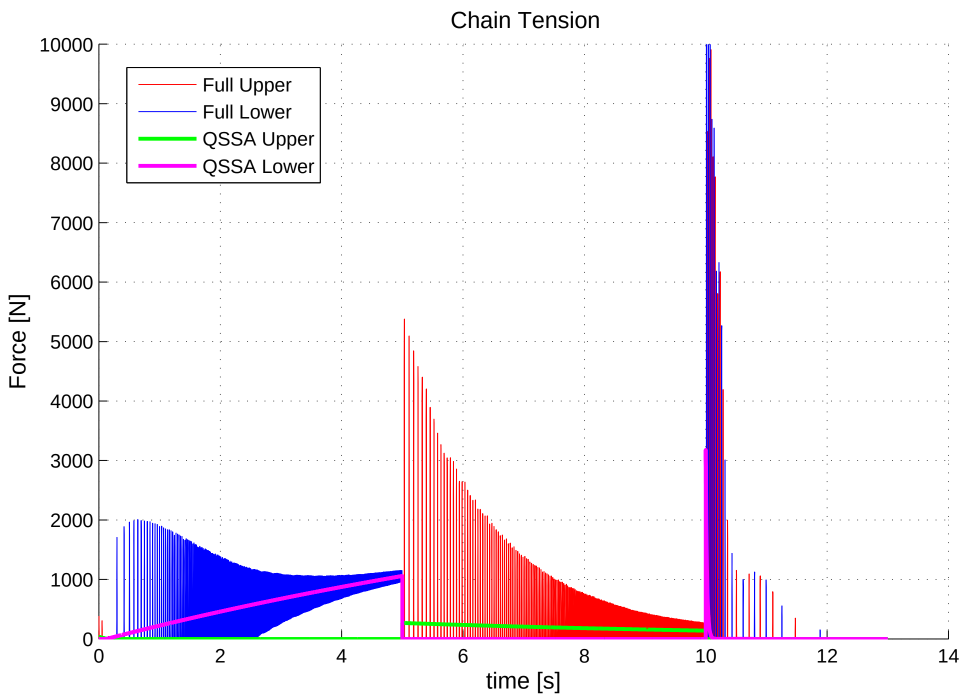


Figure 2.7: Chain tension.

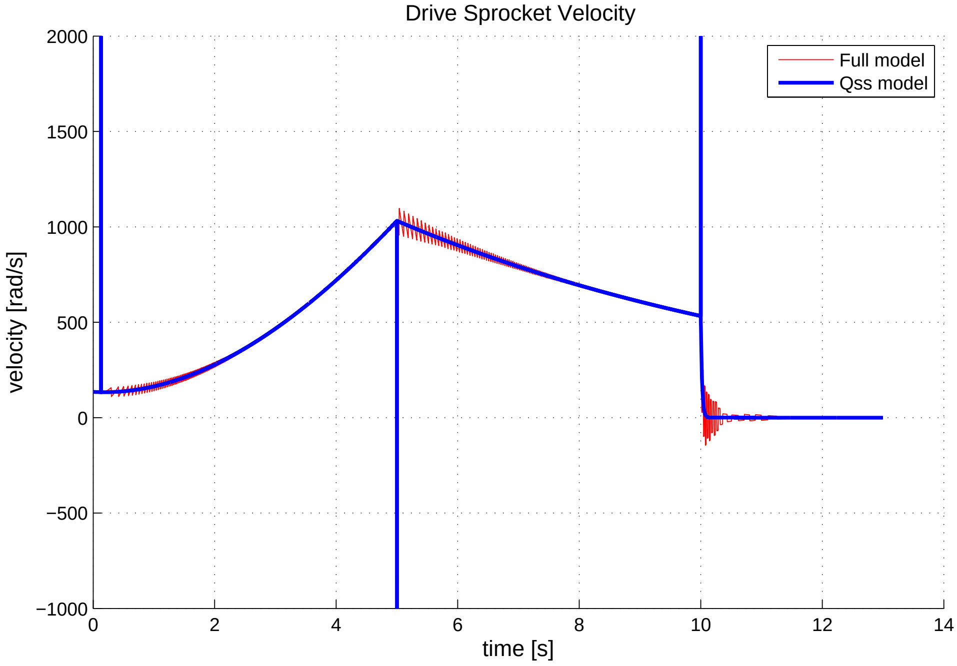


Figure 2.8: Drive sprocket velocity.

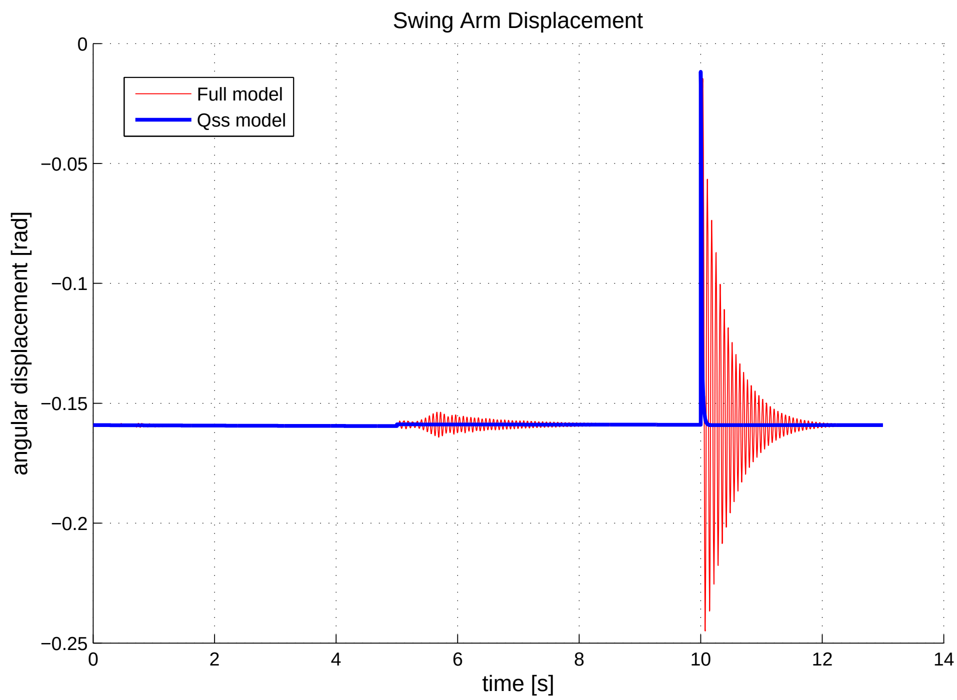


Figure 2.9: Swing arm displacement.

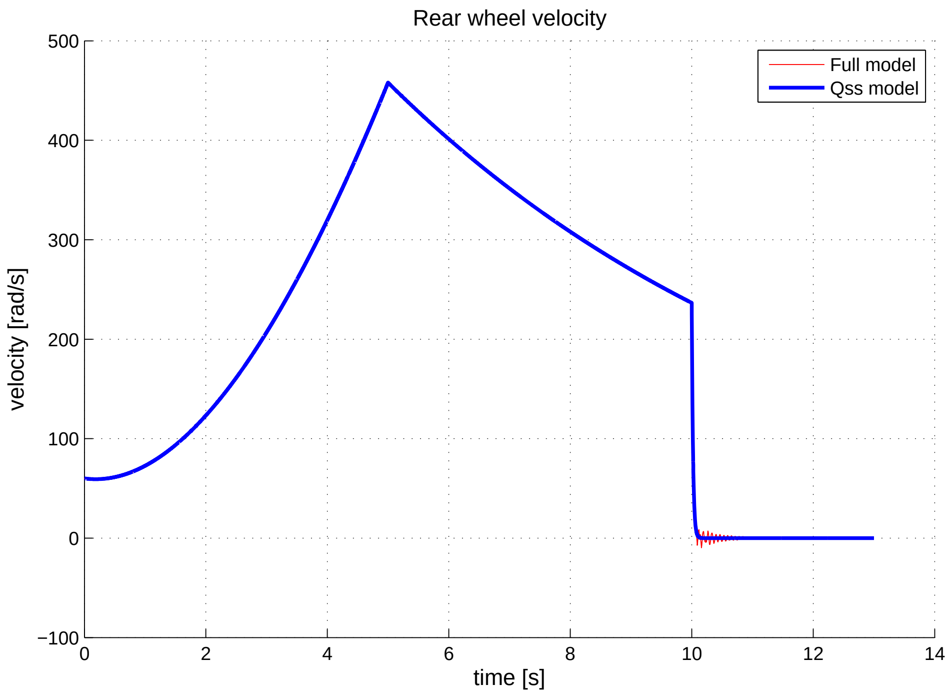


Figure 2.10: Rear wheel velocity.

2.4.2 Tendency for high stiffness

We ran three simulations with the same inputs: the engine torque function is has the triangular form of a saw tooth with a ten second wide base and $50Nm$ height; also the load torque function is shaped like a triangle with the same base, but opposite height, and it is shifted ten seconds forward. The chain stiffnesses are 10^4, 10^5, and 10^6 for the first, second, and third simulation respectively. As expected, when the chain stiffness increases the the chain extension reduces.

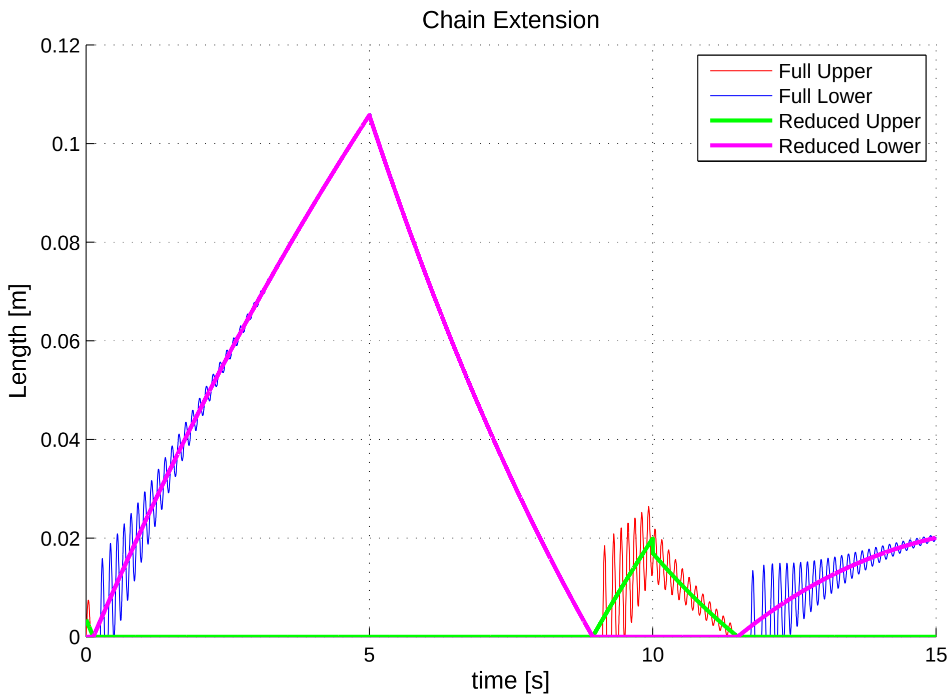


Figure 2.11: Chain extension.

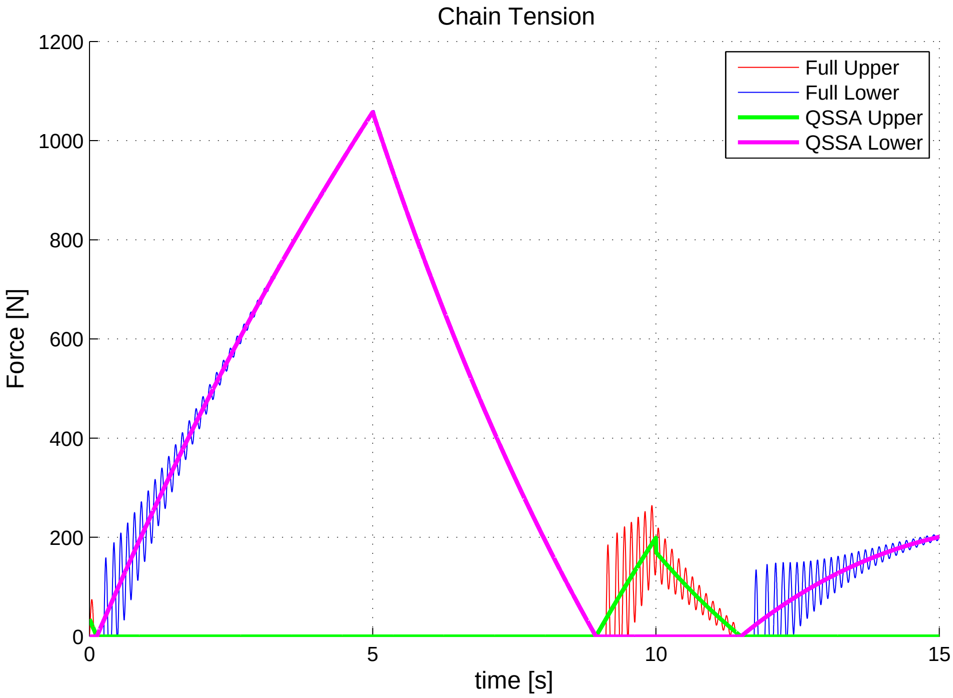


Figure 2.12: Chain tension.

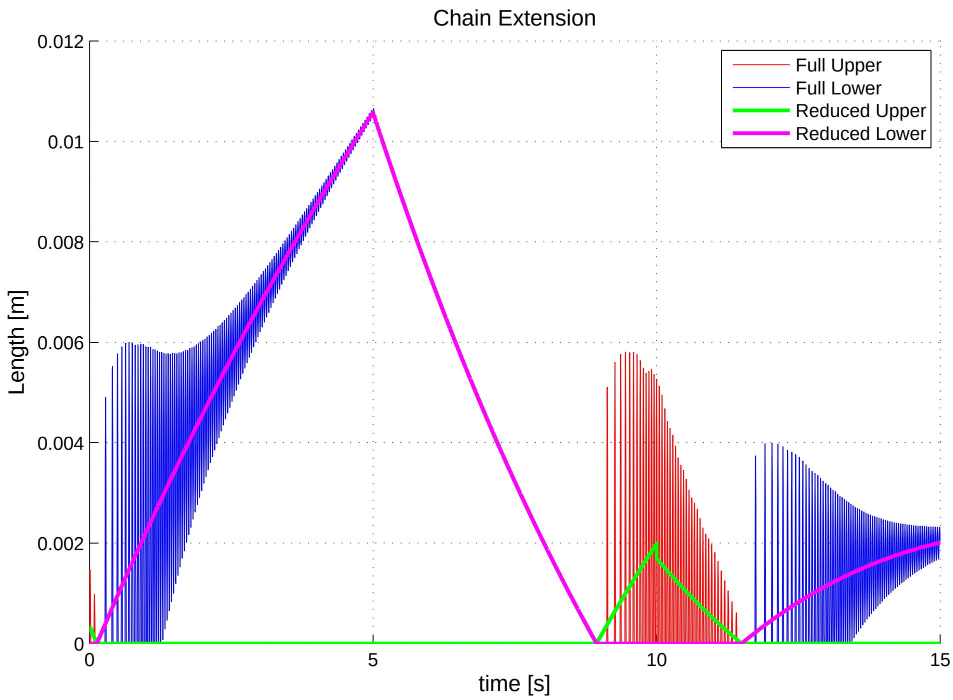


Figure 2.13: Chain extension.

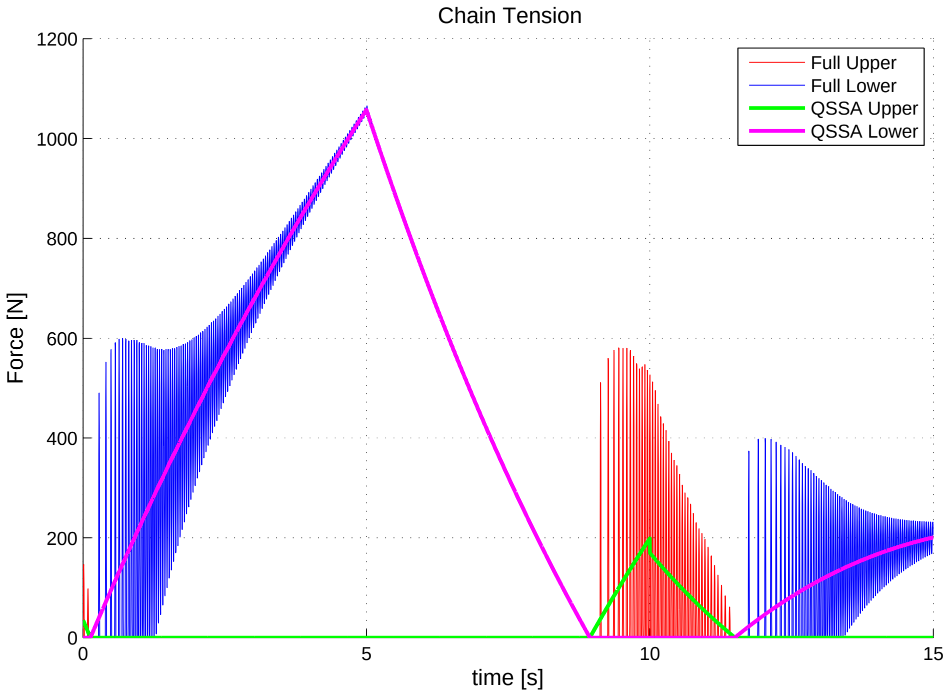


Figure 2.14: Chain tension.

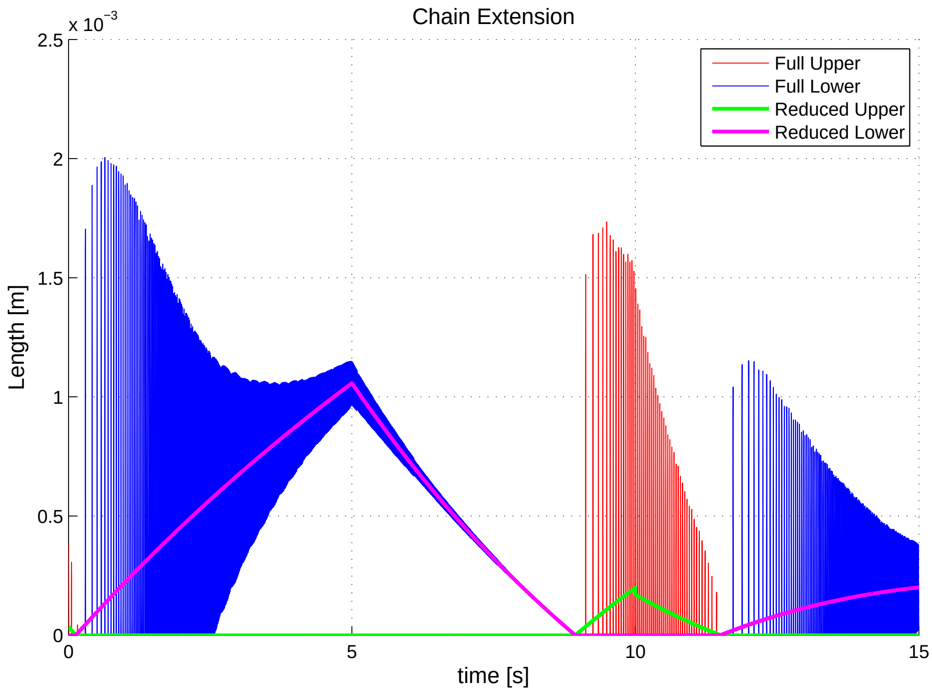


Figure 2.15: Chain extension.

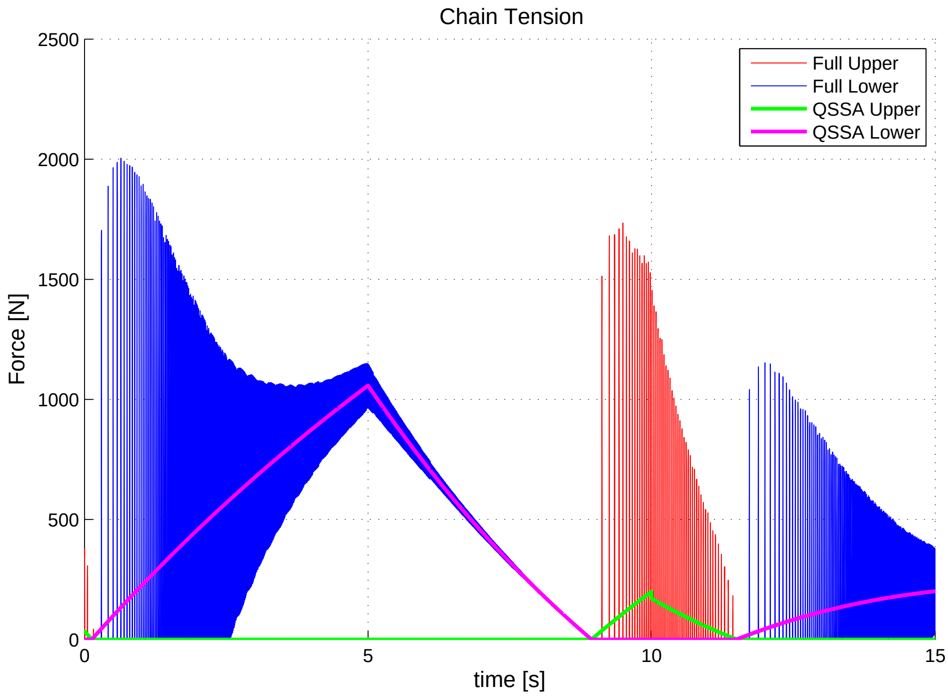


Figure 2.16: Chain tension.

2.5 Inextensible model

The QSS model solved the problems due to high gain and nonlinearities of the chain, nevertheless for each evaluation of the reduced system, it is necessary to solve numerically a nonlinear equation in two unknowns, which wastes part of the advantage of having non-stiff dynamics. In this section we present another reduced model without this drawback.

To express the concept behind the inextensible chain model, we refer again to the mass-spring-mass model introduced in Section 2.3. Note that the reduced dynamics 2.23 - 2.24 does not depend on the stiffness k, whereas the quasi-steady-state 2.22 does. For k tending to infinity the quasi-steady-state extension of the spring becomes the constant zero. Simulations suggest that this is also true for the full model, which let us think that the QSS model can be further simplified. Ideally each chain segment should be modeled with a unilateral constraint that allows the chain compression but prevents the stretching. An alternative, would be the simultaneous enforcement of two bilateral holonomic constraints like

$$l_u - l_u^0 = 0 \tag{2.41}$$

$$l_l - l_l^0 = 0\,, \tag{2.42}$$

but they would force the total chain length to be constant and would not allow any segment to compress itself. This, in turn, would fix the distance of the sprockets and consequently lock the swing arm. Before presenting our solution we need to do an intermediate step.

Let us consider a variation of the full model called upper spring model, endowed with the upper chain only. Its chain characteristic function is fully linear, that is when it is compressed it is capable of pushing like a spring. This system reproduces exactly the original full model only in the region of the state space where the upper chain is stretched, but it is different elsewhere. The points of application of the spring forces are A and C, while the lower chain pulls on B and D. Such a situation is not rare, it always occurs when the an engine brake torque[1] is applied on the counter shaft, which tends to decelerate the rear wheel. Analogously we define the lower spring model, which correctly reproduces the full one only when the lower chain tension is positive. Relationships between models are sketched in Figure 2.17.

We could recreate the full model switching among three systems: the upper spring model, the lower spring model, and a free-wheel model, where the wheel and the counter shaft are independent. The values of the upper and lower chain

[1] The engine brake torque is exerted by the engine when the throttle is closed and tends to decelerate the vehicle.

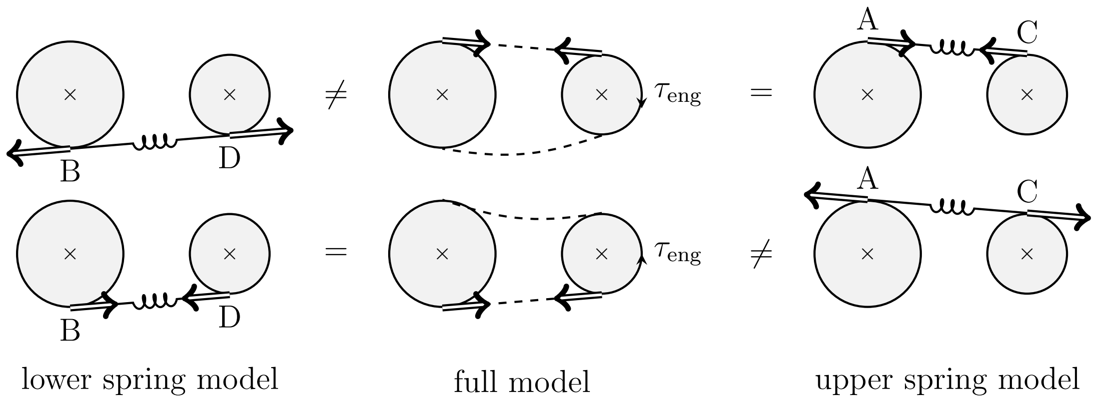


Figure 2.17: The upper and lower spring models reproduce correctly the full model only when their omonimous chain segment is active.

extension define the regions of validity of each model. The choice is always unique as in the full model at most one segment at a time is active. Note that the free-wheel model (equivalent to the dead-zone of the chain) is hardly active when running, as generally the chain is always tense; it is rather a transient state between two working conditions.

Both the upper and lower spring models can be effectively reduced with constraints 2.41 and 2.42 respectively. This idea comes from the considerations done for the mass-spring-mass model: as the stiffness goes to infinity, the reduced dynamics does not change while the quasi-steady-state function converge to zero. With the application of a holonomic (hence bilateral) constraint the spring force become the constraint force. To calculate it we suggest a technique that can be found in the second Chapter of [26], and exploit an intelligent formulation of the constraint function.

If we suppose that the constraint is ideal, that is the constraint force does no work, it can be proved that it is a linear combination of the gradient of the constraint function:

$$\left(\frac{\partial h(\theta)}{\partial \theta}\right)^T \lambda =: A^T(\theta)\lambda \,. \tag{2.43}$$

$\lambda \in \mathbb{R}$ is called Lagrange multiplier and represents the relative magnitude of the constraint force.

Instead of using the explicit form of the constraint to express one generalized coordinate as a function of the others (in our case it would be θ_{ds} to be eliminated), the constraint force is added to the generalized forces of the equations of motion and 2.13 becomes:

$$
\begin{aligned}
M(\theta)\ddot{\theta} + C(\theta, \ddot{\theta}) + G(\theta) = \boldsymbol{\Upsilon}'(\theta, \dot{\theta}, t) + A^T(\theta)\lambda \\
h(q) = 0\,,
\end{aligned}
\tag{2.44}
$$

As A is full rank, the Lagrange multiplier can be explicitly computed differentiating twice the constraint (2.41), or (2.42), obtaining

$$
A(\theta)\ddot{\theta} + \dot{A}(\theta)\dot{\theta} = 0. \tag{2.45}
$$

Solving for $\ddot{\theta}$ from (2.44), one finally gets

$$
\lambda = \left(A(\theta)M^{-1}(\theta)A^T(\theta)\right)^{-1}\left[-A(\theta)M^{-1}(\theta)\left(\boldsymbol{\Upsilon}'(\theta, \dot{\theta}, t) - C(\theta, \ddot{\theta}) - G(\theta)\right) - \dot{A}(\dot{\theta})\dot{\theta}\right]
$$

note that $\left(A(\theta)M^{-1}(\theta)A^T(\theta)\right) \neq 0$ as $M(\theta)$ is positive definite and $A(\theta)$ is not the null vector: this assures that a unique solution (θ, λ) exists for (2.44). Note also that λ depends smoothly on the inputs.

The differentiated version of 2.41 can be seen as an equation on the velocities of the material points that occupy the tangent points A and C. Making use of the frames $\boldsymbol{\Sigma}_a$ and $\boldsymbol{\Sigma}_c$ it becomes

$$
[\mathbf{V}^b_{a,rw}]^x = [\mathbf{V}^b_{c,ds}]^x\,, \tag{2.46}
$$

where the apex ^{x} indicates the first component of the velocity vector, the one parallel to the segment $\overline{AC}$; the subscripts $_{rw}$ and $_{ds}$ reminds that the points belong to these two bodies. Using arguments treated in the Appendix, it is possible to show that 2.46 can be written as

$$
\left[Ad_{\mathbf{g}^{-1}_{rw,a}}\mathbf{J}^b_{rw} - Ad_{\mathbf{g}^{-1}_{ds,c}}\mathbf{J}^b_{ds}\right]^x\dot{\theta} = 0
$$

that must be equal to $A(\theta)\dot{\theta} = 0$. From the formulation of the instantaneous work done by the constraint force we get

$$
\left\langle \boldsymbol{\Upsilon}_{chain}, \dot{\theta}\right\rangle = \left\langle A(\theta)^T\lambda, \dot{\theta}\right\rangle = \left\langle \lambda, A(\theta)\dot{\theta}\right\rangle = \left\langle \lambda, [\mathbf{V}^b_{a,rw}]^x\right\rangle - \left\langle \lambda, [\mathbf{V}^b_{c,ds}]^x\right\rangle
$$

which means that λ is the x component of the wrenches enforcing the constraint, that is the spring force module. If it is positive the constraint is pulling, otherwise it is pushing, or at rest. Using the same technique for 2.42 leads to the knowledge of the lower constraint force.

The two reduced models can be merged to define a switched system that reduces the full model. The switching function determines the active model depending on which one has a pulling constraint force. If both are zero, the two reduced models coincide and any can be chosen. Figure 2.18 sketches the relationships among all models we introduced.

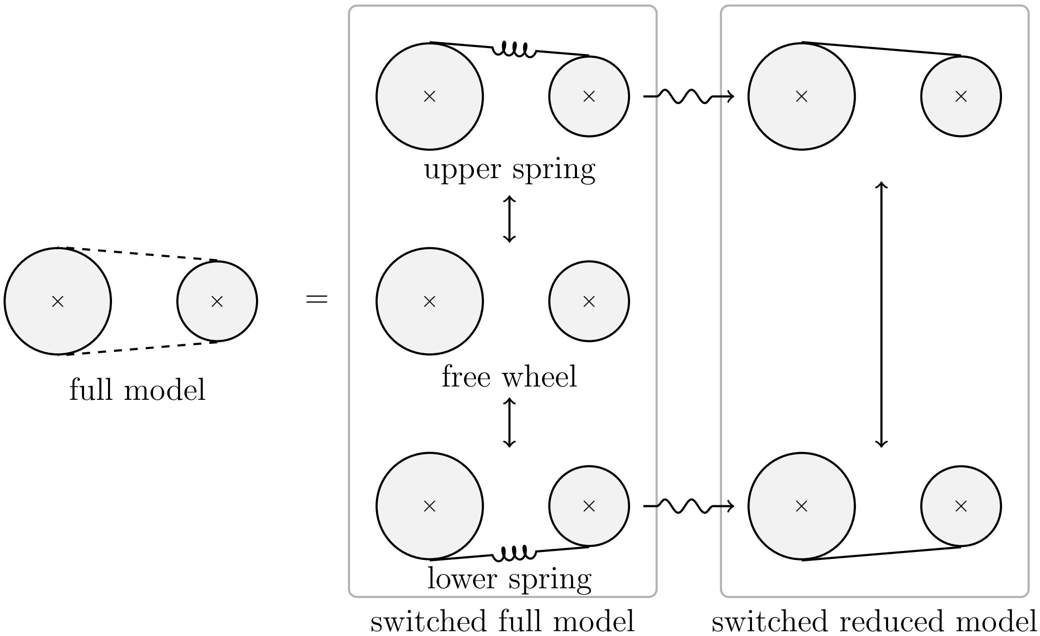


Figure 2.18: The full model is thought of as a switched model with three dynamics. The reduction is applied separately on the upper and lower spring models. The switched model made by the union of these two, reduces the full model.

Appendix A

Representation of Rigid Motion

A.1 Introduction

In this section, we briefly introduce the representation of points, vector quantities and rigid transformations that are used throughout this dissertation. For a more thorough presentation, the reader is referred to [18, Chapter 2 and Appendix A] and [27, Chapter 9].

A.2 Points, vectors and rigid motion

Rigid transformations have a simple representation in terms of matrices and vectors in $\mathbb{R}^4$. Consider a point $\mathbf{p}$ given in coordinates by $[p_x, p_y, p_z]^T$. Appending 1 to the coordinate vector, we get the *homogeneous representation* of the point $\mathbf{p}$, obtaining

$$\bar{\mathbf{p}} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} .$$

A rigid transformation is represented with the linear mapping

$$\bar{\mathbf{p}}_a = \begin{bmatrix} \mathbf{p}_a \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{ab} & \mathbf{r}_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_b \\ 1 \end{bmatrix} = \bar{\mathbf{g}}_{ab}\, \bar{\mathbf{p}}_b. \tag{A.1}$$

The 4×4 matrix $\mathbf{g}_{ab}$ is the homogeneous representation of $(\mathbf{r}_{ab}, \mathbf{R}_{ab}) \in SE(3)$. The matrix $\mathbf{R}_{ab}$ and vector $\mathbf{r}_{ab}$ represent the position and orientation of a coordinate frame B relative to another coordinate frame A, assumed as reference. In this context, $\mathbf{p}_b$ are the coordinates of a point measured with respect to the first coordinate system B, while $\mathbf{p}_a$ are the coordinates of the same point with respect to the second reference frame A.

The reference frame B defined by $\mathbf{r}_{ab}$ and $\mathbf{R}_{ab}$ may be interpreted as the position and orientation of a rigid body in the Euclidean space $\mathbb{R}^3$. In this context, p_b defines the coordinates (relative to B) of a material point attached to the body.

A rigid motion of the body may be expressed as curve in $SE(3)$ given by $\mathbf{g}_{ab}(t) = (\mathbf{r}_{ab}(t), \mathbf{R}_{ab}(t))$. To express the velocity of a rigid body one may attempt to compute directly $\dot{\mathbf{g}}_{ab}$. However, this would imply using $3 + 3 \times 3 = 12$ coordinates to express it. We know from basic physic course that a rigid body has 6 degree of freedom, so we expect to use 6 independent variables to express its velocity. The solution to the problem is given by Lie group theory and is to study the velocity vector transporting it at the group identity. Since there are right and left translations, we may define two different kind of velocities. These are the spatial and body velocities. The body velocity is more natural to understand in terms of physical intuition. In this Appendix, we will mainly concentrate on it. We refer to [18, Chapter 2] for a more detailed discussion.

The *spatial velocity* of a rigid motion is defined as

$$\mathbf{V}^s = \begin{bmatrix} \mathbf{v}^s \\ \boldsymbol{\omega}^s \end{bmatrix} = \begin{bmatrix} -\dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^T\mathbf{r}_{ab} + \dot{\mathbf{r}}_{ab} \\ (\dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^T)^\vee \end{bmatrix}$$

and the *body velocity* is

$$\mathbf{V}^b = \begin{bmatrix} \mathbf{v}^b \\ \boldsymbol{\omega}^b \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{ab}^T\dot{\mathbf{r}}_{ab} \\ (\mathbf{R}_{ab}^T\dot{\mathbf{R}}_{ab})^\vee \end{bmatrix}.$$

In homogeneous representation, the above expressions are given by

$$\begin{aligned} \hat{\mathbf{V}}^s &= \dot{\mathbf{g}}_{ab}\mathbf{g}_{ab}^{-1}, \\ \hat{\mathbf{V}}^b &= \mathbf{g}_{ab}^{-1}\dot{\mathbf{g}}_{ab}. \end{aligned}$$

The linear part of the body velocity, $\mathbf{R}_{ab}^T\dot{\mathbf{r}}_{ab}$, is interpreted as the linear velocity of the origin of B relative to A but written in coordinates of B. The angular velocity $(\mathbf{R}_{ab}^T\dot{\mathbf{R}}_{ab})^\vee$ has a similar interpretation: It represents the rotation velocity of the coordinate system B with respect to A expressed in coordinates of B.

Spatial and body velocities may defined for a generic Lie Group. $\mathbf{V}^s$ and $\mathbf{V}^b$ should be interpreted as vectors laying on the tangent space at the identity of $SE(3)$. In the context of rigid motion, they are also referred to as *twists*.

The relationship between the body and spatial velocities is captured by the adjoint matrix

$$\mathrm{Ad}_{\mathbf{g}} = \begin{bmatrix} \mathbf{R} & \hat{\mathbf{r}}\mathbf{R} \\ 0 & \mathbf{R} \end{bmatrix}. \tag{A.2}$$

It may be shown that

$$\mathbf{V}^s = \mathrm{Ad}_{\mathbf{g}_{ab}} \mathbf{V}^b.$$

The composition of rigid transformations is simply given by a multiplication between elements of $SE(3)$. Now, suppose to place in the Euclidean space three coordinates systems and denote them A, B, and C. We call $\mathbf{g}_{ab}$ the rigid transformation from B to A and $\mathbf{g}_{bc}$ that from C to B. A useful formula to compute the body velocity of frame C relative to frame A is given by

$$\mathbf{V}^b_{ac} = \mathrm{Ad}_{\mathbf{g}_{bc}^{-1}} \mathbf{V}^b_{ab} + \mathbf{V}^b_{bc}. \tag{A.3}$$

The above formula is simple to obtain by direct calculation and may be found also in [18, Chapter 2]. The body velocity seen from frame C is a compositions of two terms. The first involves the velocity of B relative to A and the second relates the velocity of C with that of B. Note how the adjoint representation $\mathrm{Ad}_{g_{bc}^{-1}}$ in this case maps body velocities measured with respect to B into body velocities relative to C.

A.3 Force/moment representation

Forces and moments are expressed relative to a reference frame that is attached to the rigid body. In this body coordinate system, the force/moment pair is represented like a vector in $\mathbb{R}^6$ as

$$\mathbf{F}^b = \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\tau} \end{bmatrix} \qquad \begin{array}{ll} \mathbf{f} \in \mathbb{R}^3 & \text{linear component} \\ \boldsymbol{\tau} \in \mathbb{R}^3 & \text{rotational component} \end{array}$$

A force/moment pair is also called a *wrench*.

Wrenches combine naturally with twists to define instantaneous work. Let $\mathbf{g}_{ab}(t) \in SE(3)$ parameterize the motion of a rigid body, where A is an inertial frame and B is a frame attached to the rigid body. Let $\mathbf{V}^b_{ab}$ represent the instantaneous body velocity of the rigid body and let $\mathbf{F}^b_b$ represent the body coordinates of a wrench applied to the origin of B. The pairing (the scalar product in $\mathbb{R}^6$) between this two objects defines the infinitesimal work (or power)

$$\left\langle \mathbf{F}_b, \mathbf{V}^b_{ab} \right\rangle = \mathbf{f} \cdot \mathbf{v} + \boldsymbol{\tau} \cdot \boldsymbol{\omega} \ . \tag{A.4}$$

The infinitesimal work may be also expressed in spatial coordinates. The key for obtaining this equivalent formulation is the coadjoint representation Ad^*. As a matrix, the coadjoint representation is just given by the transpose of adjoint matrix $Ad_{\mathbf{g}}$, that is

$$Ad_g^* = Ad_{\mathbf{g}}^T = \begin{bmatrix} \mathbf{R}^T & 0 \\ -\mathbf{R}^T\hat{\mathbf{r}} & \mathbf{R}^T \end{bmatrix}.$$

The relationship between the body and spatial coordinates of a wrench is given by

$$\mathbf{F}^s = \mathrm{Ad}^*_{\mathbf{g}_{ab}^{-1}} \mathbf{F}^b \ . \tag{A.5}$$

and the infinitesimal work is spatial coordinates is computed as

$$\left\langle \mathbf{F}_b^s, \mathbf{V}_{ab}^s \right\rangle = \mathbf{f}^s \cdot \mathbf{v}^s + \boldsymbol{\tau}^s \cdot \boldsymbol{\omega}^s. \tag{A.6}$$

To prove the equivalence between (A.4) and (A.6), just notice that

$$\left\langle \mathbf{F}^b, \mathbf{V}_{ab}^b \right\rangle = \left\langle \mathbf{F}^b, \mathrm{Ad}_{g_{ab}^{-1}} \mathbf{V}_{ab}^s \right\rangle = \left\langle \mathrm{Ad}^*_{g_{ab}^{-1}} \mathbf{F}^b, \mathbf{V}_{ab}^s \right\rangle = \left\langle \mathbf{F}^s, \mathbf{V}_{ab}^s \right\rangle.$$

A.4 Applying a wrench on a moving frame

Let A be the inertial frame and B the body frame attached to a rigid body. As previously explained, the motion of the rigid body can be described by a curve in $SE(3)$ as

$$\mathbf{g}_{ab}(t) = \begin{bmatrix} \mathbf{R}_{ab}(t) & \mathbf{r}_{ab}(t) \\ 0 & 1 \end{bmatrix}. \tag{A.7}$$

Consider the problem of computing the infinitesimal work due to a wrench applied to the rigid body on a reference frame C that changes is position relative to the body frame B. For our purposes, this problem is encounter when modeling the chain drive and tire-road interaction forces.

The transformation relating the frame C to inertial frame A is expressed by the product of two elements of $SE(3)$ as

$$\mathbf{g}_{ac}(t) = \mathbf{g}_{ab}(t)\mathbf{g}_{bc}(t) = \begin{bmatrix} \mathbf{R}_{ab}(t) & \mathbf{r}_{ab}(t) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_{bc}(t) & \mathbf{r}_{bc}(t) \\ 0 & 1 \end{bmatrix} \tag{A.8}$$

A graphical representation of the described frames is shown in Figure A.1.

The wrench in body coordinates is $\mathbf{F}_c^b = (\mathbf{f}_c^b, \boldsymbol{\tau}_c^b)$ and is applied at to the origin of C. We want to compute the infinitesimal work done by the wrench. Unfortunately, the expression

$$\left\langle \mathbf{F}_c, \mathbf{V}_{ac}^b \right\rangle \tag{A.9}$$

fails to be the right expression for the infinitesimal work and the main reason is that the frame C is not rigidly attached to the body. Otherwise stated, the material point at which the wrench is applied changes at any instant of time.

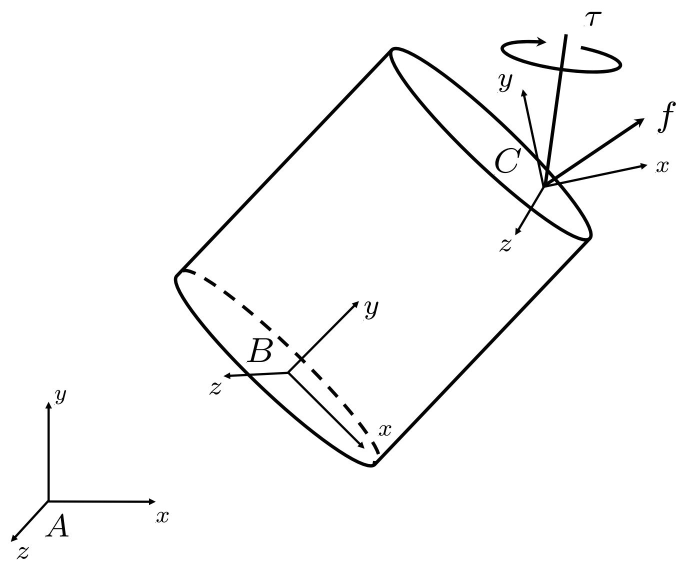


Figure A.1: The application of a wrench on a moving frame. The coordinate frame A is inertial, while the coordinate system B is the body reference frame. The wrench $(\mathbf{f}, \boldsymbol{\tau})$ is applied to the material point located at the origin of frame C. The material point to which the wrench is applied changes time by time since the position and orientation of the frame C is not *fixed with respect to the rigid body.*

There are then two equivalent ways to compute properly the infinitesimal work. The first is to consider an equivalent wrench applied at the origin of a different coordinate system rigidly attached to the body (cf. [18]). As an example, we may choose to write the wrench in terms of the body frame B. The expressions of the two equivalent wrenches are related by the coadjoint representation

$$\mathbf{F}_b = \mathrm{Ad}^*_{\mathbf{g}_{bc}^{-1}} \mathbf{F}_c. \tag{A.10}$$

Explicitly, from (A.5) we get

$$\begin{bmatrix} \mathbf{f}_b \\ \boldsymbol{\tau}_b \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{bc} & 0 \\ \hat{\mathbf{r}}_{bc}\mathbf{R}_{bc} & \mathbf{R}_{bc} \end{bmatrix} \begin{bmatrix} \mathbf{f}_c \\ \boldsymbol{\tau}_c \end{bmatrix}.$$

The inverse of the transformation expressing the wrench $\mathbf{F}_b$ as a function of the wrench $\mathbf{F}_c$ is given by

$$\mathbf{F}_c = \mathrm{Ad}^*_{\mathbf{g}_{bc}} \mathbf{F}_b,$$

that is

$$\begin{bmatrix} \mathbf{f}_c \\ \boldsymbol{\tau}_c \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{bc}^T & 0 \\ -\mathbf{R}_{bc}^T\hat{\mathbf{r}}_{bc} & \mathbf{R}_{bc}^T \end{bmatrix} \begin{bmatrix} \mathbf{f}_b \\ \boldsymbol{\tau}_b \end{bmatrix}.$$

There is a well known physical interpretation for the transformation (A.10). Force and torque vectors that were written relative to C frame are rotated into vectors expressed in the B frame using the matrix $\mathbf{R}_{bc}$. Moreover, an additional torque of the form $\mathbf{r}_{bc} \times \mathbf{f}_b$ shows up and that is the equivalent torque due to the force $\mathbf{f}_b$. This contribution is necessary since the force is applied at a distance $\mathbf{r}_{bc}$ from the origin of B.

Once we have expressed the applied wrench relatively to the B frame, the infinitesimal work may be computed with the expression

$$\left\langle \mathbf{F}_b, \mathbf{V}^b_{ab} \right\rangle = \left\langle \mathrm{Ad}^*_{\mathbf{g}^{-1}_{bc}} \mathbf{F}_c, \mathbf{V}^b_{ab} \right\rangle. \tag{A.11}$$

This expression also introduces the second way of computing the infinitesimal work. Indeed, due to the pairing between tangent and cotangent vectors, the equation (A.11) is equivalent to

$$\left\langle \mathbf{F}_c, \mathrm{Ad}_{\mathbf{g}^{-1}_{bc}} \mathbf{V}^b_{ab} \right\rangle =: \left\langle \mathbf{F}_c, \mathbf{V}^b_{a\bar{c}} \right\rangle. \tag{A.12}$$

The body velocity $\mathbf{V}^b_{a\bar{c}}$, defined as $\mathrm{Ad}_{\mathbf{g}^{-1}_{bc}} \mathbf{V}^b_{ab}$, will be called *material body velocity.* It may be computed from the rigid motion

$$\mathbf{g}_{ac}(t) = \mathbf{g}_{ab}(t)\ \mathbf{g}_{bc}(t)$$

treating the rigid transformation $\mathbf{g}_{bc}(t)$ from C to B as if were constant (i.e., time independent). Formally, we obtain

$$\hat{\mathbf{V}}^b_{a\bar{c}} = \mathbf{g}^{-1}_{ac}\,\dot{\mathbf{g}}_{a\bar{c}} = \mathbf{g}^{-1}_{bc}\mathbf{g}^{-1}_{ab}\dot{\mathbf{g}}_{ab}\,\mathbf{g}_{bc} \ = \mathbf{g}^{-1}_{bc}\hat{\mathbf{V}}^b_{ab}\,\mathbf{g}_{bc} \ \ ,$$

that is

$$\mathbf{V}^b_{a\bar{c}} = \mathrm{Ad}^{-1}_{\mathbf{g}_{bc}} \mathbf{V}^b_{ab} \ . \tag{A.13}$$

The transformation (A.13) is similar to (A.3) except that we treat $\mathbf{g}_{bc}$ as it were independent of time, i.e., as if $V^b_{bc} = 0$. From a physical point of view, this expression underline the fact that power is related to the velocity of the material points and not to the velocity of the points where the wrench is applied.

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