Università degli Studi di Padova

University of Padova
Department of Mathematics
Doctoral School in Mathematical Sciences
Degree in Computational Mathematics
XXVIII cycle

Ph.D. thesis

# Topics in stochastic control and differential game theory, with application to mathematical finance

Director of the School: *Prof. Pierpaolo Soravia*

Coordinator of the Degree: *Prof. Michela Redivo Zaglia*

Supervisor: *Prof. Tiziano Vargiolu*

Candidate: *Matteo Basei*

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# Abstract

We consider three problems in stochastic control and differential game theory, arising from practical situations in mathematical finance and energy markets.

First, we address the problem of optimally exercising swing contracts in energy markets. Our main result consists in characterizing the value function as the unique viscosity solution of a Hamilton-Jacobi-Bellman equation. The case of contracts with penalties is straightforward. Conversely, the case of contracts with strict constraints gives rise to stochastic control problems where a non-standard integral constraint is present: we get the anticipated characterization by considering a suitable sequence of unconstrained problems. The approximation result is proved for a general class of problems with an integral constraint on the controls.

Then, we consider a retailer who has to decide when and how to intervene and adjust the price of the energy he sells, in order to maximize his earnings. The intervention costs can be either fixed or depending on the market share. In the first case, we get a standard impulsive control problem and we characterize the value function and the optimal price policy. In the second case, classical theory cannot be applied, due to the singularities of the penalty function; we then outline an approximation argument and we finally consider stronger conditions on the controls to characterize the optimal policy.

Finally, we focus on a general class of non-zero-sum stochastic differential games with impulse controls. After defining a rigorous framework for such problems, we prove a verification theorem: if a couple of functions is regular enough and satisfies a suitable system of quasi-variational inequalities, it coincides with the value functions of the problem and a characterization of the Nash equilibria is possible. We conclude by a detailed example: we investigate the existence of equilibria in the case where two countries, with different goals, can affect the exchange rate between the corresponding currencies.

# Riassunto

In questa tesi vengono considerati tre problemi relativi alla teoria del controllo stocastico e dei giochi differenziali; tali problemi sono legati a situazioni concrete nell'ambito della finanza matematica e, più precisamente, dei mercati dell'energia.

Innanzitutto, affrontiamo il problema dell'esercizio ottimale di opzioni *swing* nel mercato dell'energia. Il risultato principale consiste nel caratterizzare la funzione valore come unica soluzione di viscosità di un'opportuna equazione di Hamilton-Jacobi-Bellman. Il caso relativo ai contratti con penalità può essere trattato in modo standard. Al contrario, il caso relativo ai contratti con vincoli stretti porta a problemi di controllo stocastico in cui è presente un vincolo non standard sui controlli: la suddetta caratterizzazione è allora ottenuta considerando un'opportuna successione di problemi non vincolati. Tale approssimazione viene dimostrata per una classe generale di problemi con vincolo integrale sui controlli.

Successivamente, consideriamo un fornitore di energia che deve decidere quando e come intervenire per cambiare il prezzo che chiede ai suoi clienti, al fine di massimizzare il suo guadagno. I costi di intervento possono essere fissi o dipendere dalla quota di mercato del fornitore. Nel primo caso, otteniamo un problema standard di controllo stocastico impulsivo, in cui caratterizziamo la funzione valore e la politica ottimale di gestione del prezzo. Nel secondo caso, la teoria classica non può essere applicata a causa delle singolarità nella funzione che definisce le penalità. Delineiamo quindi una procedura di approssimazione e consideriamo infine condizioni più forti sui controlli, così da caratterizzare, anche in questo caso, il controllo ottimale.

Infine, studiamo una classe generale di giochi differenziali a somma non nulla e con controlli di tipo impulsivo. Dopo aver definito rigorosamente tali problemi, forniamo la dimostrazione di un teorema di verifica: se una coppia di funzioni è sufficientemente regolare e soddisfa un opportuno sistema di disequazioni quasi-variazionali, essa coincide con le funzioni valore del problema ed è possibile caratterizzare gli equilibri di Nash. Concludiamo con un esempio dettagliato: indaghiamo l'esistenza di equilibri nel caso in cui due nazioni, con obiettivi differenti, possono condizionare il tasso di cambio tra le rispettive valute.

# Contents

# Introduction

Due to recent liberalizations, in the last years many new operators entered the market of energy. As a consequence, the competition is fierce and new financial products appeared. Hence, there is the need to refine the existing pricing techniques in order to adapt them to a continuously evolving framework and to design new models to address the recent issues of the market: for a researcher in mathematical finance, energy markets represent a fruitful source of new and stimulating problems.

Of course, it is not possible to briefly describe the active research areas in energy finance, so we now just outline a couple of examples to provide a flavour of the problems one can deal with. First, consider a contract which gives the holder the right to buy energy at a fixed price, but with some local and global constraints; the seller of such contract wants to estimate the arbitrage-free price, whereas the buyer is interested in the optimal exercise policy: both these issues correspond to a stochastic control problem with some constraints on the controls. As another example, we can consider an energy retailer who has to decide when and how to intervene to adjust the final price he asks to his customers; we still deal with an optimization problem, but only a discrete number of interventions is here possible, so that impulse controls represent the best model in this case. Finally, as the market is highly competitive, in order to bring the models closer to reality one may also consider the presence of other players; consequently, we no longer deal with one-player control problems and we rather consider two-player stochastic games.

From a mathematical point of view, as anticipated in the previous examples, we often deal with topics in stochastic control and differential game theory. However, due to the particular constraints of the problems, the classical theory sometimes does not apply and suitable adaptations (or, if not possible, different procedures) are needed. In particular, in this thesis we focus on three problems, which we now briefly introduce.

**Chapter 1: optimal exercise of swing contracts.** To hedge against the risk of sudden rises in the price of energy, swing options are traded in the market: these contracts fix the price of energy and allow a certain amount of flexibility to the buyer, but they also give the seller the guarantee of a minimal purchase.

More in detail, the holder of a *swing contract with penalties* has the right, for each $s \in [0, T]$, to buy energy at a fixed unitary price $K$; however, the instantaneous purchase intensity $u_s$ has to belong to a fixed interval $[0, \bar{u}]$ and a penalty must be paid if the total bought quantity $Z_T = \int_0^T u_s ds$ does not belong to a fixed range $[m, M]$. The penalty usually depends on $Z_T$ and $P_T$, where $P$ is a continuous-time process which models the market price of energy. The problem of pricing such contracts, as well as the problem of optimally exercising them, relies in the following continuous-time stochastic control problem:

$$\sup_{u\in\mathcal{A}} \mathbb{E}\left[\int_0^T e^{-rs}(P_s - K)u_s\, ds - e^{-rT}\widetilde{\Phi}(P_T, Z_T)\right], \tag{0.1}$$

where $\mathcal{A}$ is the set of $[0,\bar{u}]$-valued progressively measurable processes $u = \{u_s\}_{s\in[0,T]}$. Classical theory - see, for example, [20] - here applies and we can characterize the value function as the unique viscosity solution with quadratic growth of the corresponding Hamilton-Jacobi-Bellman (HJB) equation, with suitable boundary conditions. Moreover, some regularity and monotonicity results can also be proved.

A slightly different version of the option in (0.1) is also present in the market: *swing contracts with strict constraints* are similar to the previous ones, but now the holder is forced to respect the global condition $Z_T \in [m, M]$. In this case, the optimization problem reads

$$\sup_{u\in\mathcal{A}^{\mathrm{adm}}} \mathbb{E}\left[\int_0^T e^{-rs}(P_s - K)u_s\, ds\right], \tag{0.2}$$

where $\mathcal{A}^{\mathrm{adm}}$ is the set of controls satisfying the final condition $Z_T \in [m, M]$. Unlike the case in (0.1), here classical theory does not apply, due to the non-standard integral constraint on the controls. Indeed, even if problems with an upper bound on the integral of the controls have been studied in the literature, we here have both an upper and a lower bound: to the best of our knowledge, such double constraint has never been considered before.

This led us to consider a general class of integral-constrained stochastic control problems in the form

$$\sup_{u\in\mathcal{A}^{\mathrm{adm}}} \mathbb{E}\left[\int_0^T e^{-rs}L(s, P_s, Z_s, u_s)ds + e^{-rT}\Phi(P_T, Z_T)\right], \tag{0.3}$$

where $Z_s = \int_0^s g(r, u_r)dr$ and $\mathcal{A}^{\mathrm{adm}}$ is the set of controls $u$ such that $Z_T \in [m, M]$. Since a simply adaptation of classical proofs is not possible, we use a penalty method: the problem in (0.3) is the limit of appropriate unconstrained problems, where the constraint has been substituted by a penalization in the objective functional. We finally apply these general results to the problem in (0.2) and characterize the value function as the unique solution of the HJB equation with quadratic growth.

**Chapter 2: optimal price management in retail energy markets.** We consider a retailer who buys energy in the wholesale market and re-sells it to final consumers. While the wholesale price can be modelled as a continuous-time process, the final price is a piecewise constant process, due to binding clauses in the contracts. Hence, the retailer has to decide when to intervene in order to change the price he asks to his customers and how to set the new price.

Denote by $X_t$ the retailer's unitary income form the sale of energy at time $t$, i.e. the spread between the retail price and the wholesale price. In our model $X_t$

is a Brownian motion with drift and the retailer's market share is given by $\Phi(X_t)$, where $\Phi \in [0,1]$ is a truncated linear function: for every $x \in \mathbb{R}$ we set

$$\Phi(x) = \min\big\{1, \, \max\{0, \, -1/\Delta(x-\Delta)\}\big\},$$

with $\Delta > 0$ a fixed parameter. When setting his price management policy, the retailer has to consider several elements: the profit by the sale of energy (corresponding to $X_t\Phi(X_t)$), the operational costs (which we assume to be a quadratic function of the market share) and the intervention penalties (here assumed to consist in a fixed part and in a variable part, directly proportional to the market share). If $u = \{(\tau_k, \delta_k)\}_{k\geq 1}$ denotes the retailer's policy, i.e. the intervention times and the corresponding shifts in the price process, we deal with the following stochastic impulsive control problem, where $b$, $c$, $\lambda$ are non-negative parameters:

$$\sup_u \mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\Big(X_t\Phi(X_t) - \frac{b}{2}\Phi(X_t)^2\Big)dt - \sum_{k\geq 1} e^{-\rho\tau_k}\Big(c + \lambda\Phi\big(X_{(\tau_k)^-}\big)\Big)\bigg].$$

In the case $\lambda = 0$, classical results - see, for example, [32] - can be applied. More precisely, we show that the standard formulation of the verification theorem can here be used: we start from the classical quasi-variational inequalities for the value function $V$, we define a candidate $\tilde{V}$ for $V$ and we finally show that $\tilde{V}$ actually corresponds to the value function, provided that a solution to a system of algebraic equations exists. In the case where $X_t$ is modelled by a (scaled) Brownian motion, we show that a solution to such system actually exists and we prove some properties for the limit case $c \to 0^+$. In particular, this is the first time, to our knowledge, that an asymptotic estimate for the continuation region is provided in the case of impulse control problems.

The previous procedure does not apply in the case $\lambda > 0$: the candidate value function presents some singularities, so that the standard verification theorem, which requires functions of class $C^1(\mathbb{R})$, cannot be used. Hence, we try to approximate the value function by a sequence of problems with smooth coefficients. However, even if the approximation procedure is possible, this is still an open problem, due to the concavity of the market share function in a singular point. Nevertheless, we finally outline a way to circumvent the problem, provided that stronger conditions on the controls are required.

**Chapter 3: non-zero-sum stochastic differential games with impulsive controls.** Let us come back, for a moment, to the problem in Chapter 2. As energy markets are highly competitive, we introduce a second retailer in our model and assume that the market share of each player depends on the difference between the final prices they ask. In so doing, we now consider a two-player stochastic game. More precisely, let $X^i$ denote the unitary income of player $i \in \{1,2\}$ from the sale of energy. Then, player $i$ wants to maximize

$$\mathbb{E}\bigg[\int_0^\infty e^{-\rho t}\Big(X_t^i\Phi\big(X_t^i - X_t^j\big) - \frac{b_i}{2}\Phi\big(X_t^i - X_t^j\big)^2\Big)dt - \sum_{k\geq 1} e^{-\rho\tau_k^i}\Big(c_i + \lambda_i\Phi\Big(X^i_{(\tau_k^i)^-} - X^j_{(\tau_k^i)^-}\Big)\Big)\bigg], \tag{0.4}$$

where $j \in \{1,2\}$ with $j \neq i$ and $\{\tau^i_k\}_k$ are the intervention times of player $i$. Hence, we deal with a non-zero-sum game with impulse controls.

Even if control problems and differential games have been widely studied in the last decades, the case of non-zero-sum impulsive games has never been considered, to the best of our knowledge. Indeed, related former works only address zero-sum stopping games [21], the corresponding non-zero-sum problems [4] (with only one, very recent, explicit example in [13]) and zero-sum impulsive games [14]. Hence, motivated by the example above and the corresponding lack in the literature, we here consider the non-zero-sum impulsive case: we provide a rigorous framework, introduce Nash equilibria and prove a verification theorem for this class of problems.

More in detail, we consider a problem where two players can affect a continuous-time stochastic process $X$ by discrete-time interventions which consist in shifting $X$ to a new state (when none of the players intervenes, we assume $X$ to diffuse according to a standard SDE). Each intervention corresponds to a cost for the intervening player and to a gain for the opponent. The strategy of player $i \in \{1,2\}$ is determined by a couple $\varphi_i = (A_i, \xi_i)$, where $A_i$ is a fixed subset of $\mathbb{R}^d$ and $\xi_i$ is a continuous function: player $i$ intervenes if and only if the process $X$ exits form $A_i$ and, when this happens, he shifts the process from state $x$ to state $\xi_i(x)$. Once the strategies $\varphi_i = (A_i, \xi_i)$ and a starting point $x$ have been chosen, a couple of impulse controls $u_i(x;\varphi_1,\varphi_2) = \{(\tau_{i,k}, \delta_{i,k})\}_{1\le k\le M_i}$ is uniquely defined: $\tau_{i,k}$ is the $k$-th intervention time of player $i$ and $\delta_{i,k}$ is the corresponding impulse. Each player aims at maximizing his payoff, defined as follows: for every $x \in S \subseteq \mathbb{R}^n$ and every couple of strategies $(\varphi_1, \varphi_2)$ we set

$$\begin{aligned} J^i(x;\varphi_1,\varphi_2) := \mathbb{E}_x\bigg[\int_0^{\tau_S} e^{-\rho_i s} f_i(X_s)ds + \sum_{1\le k\le M_i\,:\,\tau_{i,k}<\tau_S} e^{-\rho_i \tau_{i,k}} \phi_i\Big(X_{(\tau_{i,k})^-}, \delta_{i,k}\Big) \\ + \sum_{1\le k\le M_j\,:\,\tau_{j,k}<\tau_S} e^{-\rho_i \tau_{j,k}} \psi_i\Big(X_{(\tau_{j,k})^-}, \delta_{j,k}\Big) + e^{-\rho_i \tau_S} h_i\big(X_{(\tau_S)^-}\big)\mathbb{1}_{\{\tau_S<+\infty\}}\bigg], \end{aligned} \tag{0.5}$$

where $i,j \in \{1,2\}$, $i \neq j$ and $\tau_S$ is the exit time from $S$. The couple $(\varphi^*_1, \varphi^*_2)$ is a Nash equilibrium if $J^1(x;\varphi^*_1,\varphi^*_2) \ge J^1(x;\varphi_1,\varphi^*_2)$ and $J^2(x;\varphi^*_1,\varphi^*_2) \ge J^2(x;\varphi^*_1,\varphi_2)$, for every strategies $\varphi_1, \varphi_2$.

Let us consider the following quasi-variational inequalities (QVI), where $i,j \in \{1,2\}$ with $j \neq i$ and $\mathcal{M}_i, \mathcal{H}_i$ are suitable operators:

$$\begin{aligned} &V_i = h_i, && \text{in } \partial S, \\ &\mathcal{M}_j V_j - V_j \le 0, && \text{in } S, \\ &\mathcal{H}_i V_i - V_i = 0, && \text{in } \{\mathcal{M}_j V_j - V_j = 0\}, \\ &\max\big\{\mathcal{A}V_i - \rho_i V_i + f_i, \mathcal{M}_i V_i - V_i\big\} = 0, && \text{in } \{\mathcal{M}_j V_j - V_j < 0\}. \end{aligned} \tag{0.6}$$

The main result of this chapter is the Verification Theorem 3.8: if two functions $V_i$, with $i \in \{1,2\}$, are a solution to (0.6), have polynomial growth and satisfy the regularity condition

$$V_i \in C^2(D_j \setminus \partial D_i) \cap C^1(D_j) \cap C(S), \tag{0.7}$$

where $j \in \{1,2\}$ with $j \neq i$ and $D_j = \{\mathcal{M}_j V_j - V_j < 0\}$, then they coincide with the value functions of the game and a characterization of the Nash strategy is possible.

We remark the importance of (0.7). When imposing the regularity conditions, one usually needs to solve a system of algebraic equations, as the candidate solutions to the QVI problem are piecewise defined. In general, if the regularity conditions are too strong, the system may have more equations than parameters, with no possibility to apply the verification theorem. This is the main weakness of [4]: indeed, the only application of [4] is [13], where relaxed conditions are needed. In our case, the system is formally solvable (same number of equations and parameters, as we will see): an important contribution in this chapter consists in providing regularity conditions which allow to practically apply the verification theorem.

In the final part of the chapter we apply the verification theorem in a detailed example. The problem in (0.4) presents, as a matter of fact, a complicated structure and is currently subject of ongoing research. So, we here provide a simpler one-dimensional example. More precisely, we consider two countries which can affect the exchange rate between their currencies: the countries have different goals and we investigate the existence of Nash equilibria for this problem.

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# Chapter 1

# Optimal exercise of swing contracts in energy markets

Based on [3]: M. Basei, A. Cesaroni, T. Vargiolu, *Optimal exercise of swing contracts in energy markets: an integral constrained stochastic optimal control problem*, SIAM J. Finan. Math. 5 (2014), no 1, 581–608.

**Abstract.** We characterize the value of swing contracts in continuous time as the unique viscosity solution of a Hamilton-Jacobi-Bellman equation with suitable boundary conditions. The case of contracts with penalties is straightforward, and in that case only a terminal condition is needed. Conversely, the case of contracts with strict constraints gives rise to a stochastic control problem with a nonstandard state constraint. We approach this problem by a penalty method: we consider a general constrained problem and approximate the value function with a sequence of value functions of appropriate unconstrained problems with a penalization term in the objective functional. Coming back to the case of swing contracts with strict constraints, we finally characterize the value function as the unique viscosity solution with polynomial growth of the HJB equation subject to appropriate boundary conditions.

**Keywords:** swing contracts, energy markets, stochastic control, state constraints, penalty methods, dynamic programming, HJB equation, viscosity solutions.

## 1.1 Introduction

Energy is traded in financial markets, in its various forms (electricity, coal, gas, oil, etc.), mainly through two types of contracts, namely forwards and swings. Forward contracts are obligations between two parts to exchange some amount of energy, in a specified form (electricity or some fuel) and for a prespecified amount of money: once settled, this contract is strictly binding for both the parts, giving no flexibility to them. Conversely, swing contracts give a certain amount of flexibility to the buyer, while also giving the seller a certain guarantee that a minimum quantity of energy will be bought. This is due to the fact that energy storage is costly in the case of fuels and almost impossible in the case of electricity; moreover, energy markets are influenced by many elements (peaks in consumes related to sudden weather changes, breakdowns in power plants, financial crises, etc.). As a consequence, the price of energy is subject to remarkable fluctuations, so that flexibility is much welcomed by contract buyers.

The flexibility in swing contracts is implemented in this way (we here follow the approach in [5] and model the contract in continuous time): for a fixed contract maturity $T$ (usually one or several years), the buyer can choose, at each time $s \in [0,T]$, to buy a marginal amount of energy $u(s) \in [0,\bar{u}]$ at a prespecified strike price $K$, thus realizing a marginal profit (or loss) equal to $(P(s)-K)u(s)$, where $P(s)$ is the spot price of that kind of energy. This gives to the buyer the potential profit (or loss)

$$\int_0^T e^{-rs}(P(s)-K)u(s)\ ds,$$

with $r>0$ the risk-free interest rate.

However, the energy seller usually wants the total amount of energy $Z(T) = \int_0^T u(s)\ ds$ to lie between a minimum and a maximum quantity, that is $Z(T) \in [m,M]$. This is implemented in two main ways. The first way is to impose penalties when $Z(T) \notin [m,M]$, i.e. to make the buyer pay a penalty $\widetilde{\Phi}(P(T),Z(T))$, where $\widetilde{\Phi}(p,z)$ is a contractually fixed function, null for $z \in [m,M]$ and convex in $z$. The second way is to impose the constraint $Z(T) \in [m,M]$ to be satisfied strictly, i.e. to force the buyer to withdraw the minimum cumulative amount of energy $m$ and to stop giving the energy when the maximum $M$ has been reached.

We are interested in the problem of optimally exercising a swing contract in both the cases. This problem can be modelled as a continuous time stochastic control problem: our aim is to study the corresponding value function and to characterize it as the unique viscosity solution of the related Hamilton-Jacobi-Bellman (HJB) equation. Swing contracts are treated either in discrete time [1, 2, 17, 24] via the dynamic programming principle and Bellman equations or in continuous time [5, 19] only by reporting a verification theorem for a smooth solution of the HJB equation, without reporting existence or uniqueness results for that. Besides, we also extend the approach in [5, 19], which only treat the case $m=0$, to the case when $m>0$, which is the most relevant case in practical applications (in fact, [28] reports that typically $m \in [0.8M, M]$). We also refer to [12], where swing contracts in continuous time are treated in continuous time with multiple stopping techniques, and [6], where swings are priced using a discrete-time backward scheme for solving BSDEs with jumps.

In the case of swing contracts with penalties, we get a standard stochastic control problem, as the maximization of the final expected payoff for a buyer entering in the contract at a generic time $t \in [0,T]$ is given by

$$\widetilde{V}(t,p,z) = \sup_{u\in\mathcal{A}_t} \mathbb{E}_{tpz}\left[\int_t^T e^{-r(s-t)}(P(s)-K)u(s)\ ds - e^{-r(T-t)}\widetilde{\Phi}(P(T),Z(T))\right], \tag{1.1}$$

with $(t,p,z) \in [0,T]\times\mathbb{R}^2$, where $\mathcal{A}_t$ is the set of $[0,\bar{u}]$-valued progressively measurable processes $u = \{u(s)\}_{s\in[t,T]}$. Thus, in this case classical theory (see [16, 18, 20]) can be applied: see Section 1.2.

Conversely, swing contracts with strict constraints give rise to a stochastic control problem with integral constraints in the control:

$$V(t,p,z) = \sup_{u\in\mathcal{A}_{tz}^{\text{adm}}} \mathbb{E}_{tpz}\left[\int_t^T e^{-r(s-t)}(P(s)-K)u(s)\ ds\right], \tag{1.2}$$

with $(t,p,z)$ in a suitable domain $\mathcal{D} \subseteq [0,T]\times\mathbb{R}^2$, where $\mathcal{A}_{tz}^{\text{adm}}$ is the set of processes $u \in \mathcal{A}_t$ such that $\mathbb{P}_{tpz}$-a.s. $Z^{t,z;u}(T) = z + \int_t^T u(s)\ ds \in [m,M]$. Due to the presence of the constraint on $Z^{t,z;u}(T)$, here classical theory does not apply.

This motivates us to consider in Section 1.3 a more general class of integral constrained stochastic problems in the form

$$V(t,p,z)= \sup_{u\in\mathcal{A}_{tz}^{\text{adm}}} \mathbb{E}_{tpz}\left[\int_t^T e^{-r(s-t)}L(s,P(s),Z(s),u(s))ds+e^{-r(T-t)}\Phi(P(T),Z(T))\right]. \tag{1.3}$$

with $(t,p,z)$ in a suitable domain $\mathcal{D} \subseteq [0,T]\times\mathbb{R}^n\times\mathbb{R}$, where $\mathcal{A}_{tz}^{\text{adm}}$ is the set of processes $u \in \mathcal{A}_t$ such that, $\mathbb{P}_{tpz}$-a.s., $Z(T) = z + \int_t^T g(s,u(s))\ ds \in [m,M]$.

Control problems with integral constraints are classical in control theory, for instance they naturally arise in applications: e.g. control problems with bounded $L^p$ norm of the controls, control problems with prescribed bounded total variation or total energy of the trajectories, control systems with design uncertainties. However, the dynamic programming approach presents several technical difficulties. The main one relies on the fact that the dynamic programming principle is not satisfied directly by the value function and the problem has to be attacked differently. As for the case of deterministic systems, we refer to [30, 34] and references therein. As for the case of stochastic controls, the upper bound $Z(T) \le M$ is analogous to the constraint of the so-called finite fuel problems, which are optimal control problems with an upper bound on the integral of the absolute value of the controls (see e.g. [20, Chapter VIII] for an introduction to the problem, [29] and references therein). Instead, the lower bound $Z(T) \ge m$ is nonstandard. In the particular case of Equation (1.2), and only with $m = 0$, such a bound has been studied (treated in [5] and generalized in [19], still with $m = 0$). However, we already said that this case is quite unrealistic, as the seller wants to be sure to sell some amount of energy, so typically $m > 0$.

Note that the control problem in (1.3) can be interpreted as a state constraint control problem in the following way. First of all, the integral constraint on the control can be written as a terminal constraint on the state variables $(P(s),Z(s))$: they have to satisfy a.s. $(P(T),Z(T)) \in G$, where $G$ is a closed set in $\mathbb{R}^{n+1}$. In our case $G = \mathbb{R}^n\times[m,M]$. Then, we introduce the set of points such that $G$ is reachable from $(t,p,z)$, i.e.

$$\mathcal{D} = \{(t,p,z) \in [0,T]\times\mathbb{R}^n\times\mathbb{R} : \mathcal{A}_{tz}^{\text{adm}} \neq \emptyset\}.$$

Thus, we can rewrite the value function as

$$V(t,p,z)= \sup_{u\in\mathcal{A}_{tz}^{\text{adm}}} \mathbb{E}_{tpz}\left[\int_t^T e^{-r(s-t)}L(s,P(s),Z(s),u(s))ds+e^{-r(T-t)}\Phi(P(T),Z(T))\right],$$

with the constraints that $(P(s),Z(s)) \in \mathcal{D}$ a.s. for all $s \in [t,T]$.

Note that here $\mathcal{D}$ is not given explicitly, but it is defined by a stochastic target problem. In [8], a similar stochastic control problem has been considered, with $G = \{(p,z) \in \mathbb{R}^n \times \mathbb{R} \mid g(p,z) \geq 0\}$ and $g(p,\cdot)$ increasing and right continuous. In this case, though, differently from our case, the set $\mathcal{D}$ can be described as the epigraph of a continuous function, that is $\mathcal{D} = \{(t,p,z) \mid w(t,p) \geq z\}$. We also refer to [7] where reachable sets for state-constrained controlled stochastic systems have been studied.

To study our problem, we adopt a classical penalization method. We introduce the set $\widetilde{\mathcal{D}} \subseteq \mathcal{D}$ (which is the set of points such that the interior of $G$ is reachable from $(t,p,z)$, for precise definition we refer to Section 1.3.1) and we show that in $\widetilde{\mathcal{D}}$ the function $V$ in Equation (1.3) is the limit of the value functions $V^c$ of suitable unconstrained problems, where the constraint has been substituted by an appropriate penalization in the objective functional. This convergence result is obtained under a technical assumption, see Assumption 1.9, ensuring that, roughly speaking, given a control in $\mathcal{A}_{tz}^{\text{adm}}$ we can modify it in order to steer the trajectory in the interior of $G$, not paying too much in the cost functional. This result is contained in Theorem 1.11 and Corollary 1.12. In Propositions 1.15 and 1.16 we prove that, under suitable assumptions, the function $V(t,\cdot,z)$ is Lipschitz continuous and a.e. twice differentiable. In Section 1.3.4 we show that Assumption 1.9 is satisfied in the cases $g(s,v) = v$ and $g(s,v) = |v|^p$ $(p \geq 1)$, if $f$ and $\sigma$ satisfy appropriate conditions.

In Section 1.4 we apply these general results to the problem in (1.2). In this case stronger results will be achieved, since it can be proved directly that the value function is continuous not only in $\widetilde{\mathcal{D}} \subsetneq \mathcal{D}$ but in the whole domain $\mathcal{D}$ (Proposition 1.19 and 1.21). Thus, $V$ can be characterized as the unique continuous viscosity solution with polynomial growth of the HJB equation under suitable boundary conditions (Theorem 1.22). As for the regularity of the value function, besides the above cited general results about the variable $p$ (Proposition 1.23), we prove that $V(t,p,\cdot)$ is concave and study its monotonicity (Proposition 1.24).

The structure of the paper is as follows. In Section 1.2 the evaluation problem for a swing contract with penalty is studied. Section 1.3 deals with a general class of constrained control problems, as in Equation (1.3). Finally, in Section 1.4 we deeply analyze the problem, outlined in (1.2), of the optimal exercise of swing contracts with strict constraints.

**Notations.** By $\|\cdot\|_\infty$ we denote the sup-norm. If $B \in M_{ij}(\mathbb{R})$ (i.e. a real $i \times j$ matrix), $B^t$ denotes the transpose of $B$ and $\mathrm{tr}(B)$ denotes its trace. By $\overline{\mathrm{B}(x,R)}$ we mean the closed ball in $\mathbb{R}^n$ with center $x$ and radius $R$. If $O \subseteq \mathbb{R}^n$ and $k \in \mathbb{N}$, we denote by $C_b^k(O)$ (resp. $C_p^k(O)$) the set of functions of class $C^k(O)$ whose derivatives up to order $k$ are bounded (resp. are polynomially growing). If $\psi$ is a function from $(t,p,z) \in A \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}$, by $\psi_t, \psi_z$ we mean the derivatives with respect to $t$ and $z$ and by $D_p\psi, D_p^2\psi$ we mean the Jacobian and the Hessian matrix with respect to the variable $p$.

## 1.2 Swing contracts with penalties

In this section we consider the problem of the optimal exercise of swing contracts with penalties described in the Introduction: to this purpose, we formalize a continuous time model to which we apply classical results in stochastic control.

Let $T > 0$ and fix a filtered probability space $(\Omega,\, \mathcal{F}_T,\, \{\mathcal{F}_s\}_{s\in[0,T]},\, \mathbb{P})$ and a real $\{\mathcal{F}_s\}_s$-adapted Brownian motion $W = \{W(s)\}_{s\in[0,T]}$. Let $t \in [0,T]$ and $p \geq 0$. We model the price of energy through a stochastic process $\{P^{t,p}(s)\}_{s\in[t,T]}$ which satisfies the SDE

$$dP^{t,p}(s) = f(s, P^{t,p}(s))ds + \sigma(s, P^{t,p}(s))dW(s), \qquad s \in [t,T], \tag{1.4}$$

with initial condition $P^{t,p}(t) = p$. We assume

$$\begin{gathered} f, \sigma \in C([0,T]\times\mathbb{R};\mathbb{R}), \\ |f(t,p) - f(t,q)| + |\sigma(t,p) - \sigma(t,q)| \leq \hat{C}|p-q| \quad \forall p,q \in \mathbb{R},\ \forall t \in [0,T], \end{gathered} \tag{1.5}$$

where $\hat{C} > 0$ is a constant.

In each $s \in [t,T]$, the holder can buy energy at a fixed unitary price $K > 0$ and with purchase intensity $u(s) \in [0,\bar{u}]$, where $\bar{u} > 0$ is a constant: this gives a net instantaneous profit (or loss) of $(P^{t,p}(s) - K)u(s)$. Let $\mathcal{A}_t$ be the set of all $[0,\bar{u}]$-valued progressively measurable processes $u = \{u(s)\}_{s\in[t,T]}$ (i.e. all the possible usage strategies of the contract). Let $z$ be the amount of energy purchased until time $t$ and let $u \in \mathcal{A}_t$ be an exercise strategy from time $t$ on; for each $s \in [t,T]$ we denote by $Z^{t,z;u}(s)$ the energy bought up to time $s$:

$$Z^{t,z;u}(s) = z + \int_t^s u(\tau)d\tau, \qquad s \in [t,T].$$

If the globally purchased energy $Z^{t,z;u}(T)$ does not fall within a fixed range $[m,M]$ ($m, M \in \mathbb{R}$, with $m \leq M$), the holder must pay a penalty $\widetilde{\Phi}(P^{t,p}(T), Z^{t,z;u}(T))$, where $\widetilde{\Phi}$ is a function from $\mathbb{R}^2$ to $\mathbb{R}$. In the typical case (see, for example, [1, 2, 18]) the penalty is directly proportional to $P^{t,p}(T)^+$ and to the entity of the overrunning or underrunning: this is obtained by setting

$$\widetilde{\Phi}(p,z) = -Ap^+(z-M)^+ - Bp^+(m-z)^+,$$

for all $(p,z) \in \mathbb{R}^2$, where $A, B > 0$ are suitable constants. In several practical cases, $A = B$. However, other kind of penalties are possible (see e.g. [24]): typically $p^+$, representing the spot price at the end $T$ of the contract, is replaced either by an arithmetic mean of spot prices (thus requiring another state variable in the problem) or by a fixed (high) penalty. In the light of the above discussion, we assume that, for all $p \in \mathbb{R}$,

$$\begin{gathered} \widetilde{\Phi}(p,z) = 0, \ \ \forall z \in [m,M], \\ \widetilde{\Phi}(p,\cdot) \text{ is concave}, \\ |\widetilde{\Phi}(p+h,z) - \widetilde{\Phi}(p,z)| \leq Ch(1+|z|), \ \ \forall z \in \mathbb{R}, h > 0, \\ |\widetilde{\Phi}(p,z+h) - \widetilde{\Phi}(p,z)| \leq Ch(1+|p|), \ \ \forall z \in \mathbb{R}, h > 0, \end{gathered} \tag{1.6}$$

where $C > 0$ is a constant.

Let $r \geq 0$ be the risk-free rate. We get a stochastic optimal control problem, with the following value function:

$$\widetilde{V}(t,p,z) = \sup_{u\in\mathcal{A}_t} \widetilde{J}(t,p,z;u), \tag{1.7}$$

for each $(t, p, z) \in [0, T] \times \mathbb{R}^2$, where

$$\widetilde{J}(t, p, z; u) = \mathbb{E}_{tpz}\left[\int_t^T e^{-r(s-t)}(P^{t,p}(s) - K)u(s)ds + e^{-r(T-t)}\widetilde{\Phi}(P^{t,p}(T), Z^{t,z;u}(T))\right]$$

and by $\mathbb{E}_{tpz}$ we denote the mean value with respect to the probability $\mathbb{P}$ (subscripts recall initial conditions).

Problem (1.7) belongs to a widely studied class of control problems: by well-known classical results, summarized in Theorem 1.1, the value function is the unique viscosity solution of the corresponding Hamilton-Jacobi-Bellman equation subject to appropriate conditions (we refer to [15] for the definition of viscosity solutions).

**Theorem 1.1.** *Under assumptions* (1.5) *and* (1.6), *the function* $\widetilde{V}$ *is the unique viscosity solution of*

$$\begin{aligned} &- \widetilde{V}_t(t, p, z) + r\widetilde{V}(t, p, z) - f(t, p)\widetilde{V}_p(t, p, z) - \frac{1}{2}\sigma^2(t, p)\widetilde{V}_{pp}(t, p, z) \\ &\qquad + \min_{v \in [0, \bar{u}]}[-v(\widetilde{V}_z(t, p, z) + p - K)] = 0, \qquad \forall (t, p, z) \in [0, T[\times \mathbb{R}^2, \quad (1.8) \end{aligned}$$

*with final condition*

$$\widetilde{V}(T, p, z) = \widetilde{\Phi}(p, z), \qquad \forall (p, z) \in \mathbb{R}^2, \tag{1.9}$$

*and such that*

$$|\widetilde{V}(t, p, z)| \leq \check{C}(1 + |p|^2 + |z|^2), \qquad \forall (t, p, z) \in [0, T] \times \mathbb{R}^2,$$

*for some constant* $\check{C} > 0$.

*Proof.* See Theorem 1.10, of which this theorem is a particular case. □

We now list some properties of the function $\widetilde{V}$ with respect to the variables $p$ and $z$. Let us start by proving regularity results with respect to the variable $p$.

**Proposition 1.2.** *Under assumptions* (1.5) *and* (1.6), *for each* $(t, z) \in [0, T] \times \mathbb{R}$ *the function* $\widetilde{V}(t, \cdot, z)$ *is Lipschitz continuous, uniformly in* $t$. *Moreover, the derivative* $\widetilde{V}_p(t, p, z)$ *exists for a.e.* $(t, p, z) \in [0, T] \times \mathbb{R}^2$ *and we have* $|\widetilde{V}_p(t, p, z)| \leq M_1(1 + |z|)$, *for some constant* $M_1 > 0$ *depending only on* $\bar{u}$, $T$, $C$ *and on the constants in (1.18) and (1.21).*

*Proof.* Let $(t, p, z) \in [0, T] \times \mathbb{R}^2$, $h > 0$ and $u \in \mathcal{A}_t$. By estimate (D.8) in [20, Appendix D] we have

$$\begin{aligned} &|\widetilde{J}(t, p + h, z; u) - \widetilde{J}(t, p, z; u)| \\ &\leq \mathbb{E}_{tpz}\left[\int_t^T |P^{t,p+h}(s) - P^{t,p}(s)|\,|u(s)|ds + C|P^{t,p+h}(T) - P^{t,p}(T)|\,(1 + |Z^{t,z;u}(T)|)\right] \\ &\leq T\bar{u}\,\mathbb{E}_{tpz}[\|P^{t,p+h}(\cdot) - P^{t,p}(\cdot)\|_\infty] + C\mathbb{E}_{tpz}[\|P^{t,p+h}(\cdot) - P^{t,p}(\cdot)\|_\infty](1 + |z| + \bar{u}T) \\ &\leq M_1(1 + |z|)h, \end{aligned} \tag{1.10}$$

where $M_1 > 0$ is a constant. Since (1.10) holds for each $u \in \mathcal{A}_t$, we get

$$|\widetilde{V}(t, p + h, z) - \widetilde{V}(t, p, z)| \leq M_1(1 + |z|)h. \tag{1.11}$$

The function $\widetilde{V}(t,\cdot,z)$ is therefore Lipschitz continuous, uniformly in $t$, and then a.e. differentiable by the Rademacher theorem (see [23]). By standard arguments (see [23]) it is possible to prove that the set of points where $\widetilde{V}_p(t,p,z)$ does not exist is measurable in $[0,T]\times\mathbb{R}^2$, then by Fubini theorem it follows that $\widetilde{V}_p(t,p,z)$ exists for a.e. $(t,p,z)\in[0,T]\times\mathbb{R}^2$. The estimate on the derivative immediately follows by (1.11). □

In the following proposition we collect some results about smoothness and monotonicity of the function $\widetilde{V}$ with respect to $z$.

**Proposition 1.3.** *Under assumptions* (1.5) *and* (1.6), *for each* $(t,p)\in[0,T]\times\mathbb{R}$ *the function* $\widetilde{V}(t,p,\cdot)$ *is*

- *Lipschitz continuous, uniformly in* $t$. *Moreover, the derivative* $\widetilde{V}_z(t,p,z)$ *exists for a.e.* $(t,p,z)\in[0,T]\times\mathbb{R}^2$ *and we have* $|\widetilde{V}_z(t,p,z)|\le M_2(1+|p|)$, *for some constant* $M_2>0$ *depending only on* $\bar{u}$, $T$, $C$ *and on the constants in (1.18) and (1.21).*
- *concave and a.e. twice differentiable;*
- *non-decreasing in* $]-\infty, M-(T-t)\bar{u}]$ *and non-increasing in* $[m,+\infty[$. *In particular, if* $M-(T-t)\bar{u}\ge m$ *then the function* $\widetilde{V}(t,p,\cdot)$ *is constant in* $[m, M-(T-t)\bar{u}]$ *(they all are maximum points).*

*Proof. Item 1.* Let $(t,p,z)\in[0,T]\times\mathbb{R}^2$, $h>0$ and $u\in\mathcal{A}_t$. Recall the following estimate from [20, Appendix D]: for each $k\ge 0$ there exists a constant $B_k\ge 0$, depending only on $\bar{u}$, $T$, $C$ and on the constants in (1.18) and (1.21), such that

$$\mathbb{E}_{tpz}\left[\|P^{t,p;u}(\cdot)\|_\infty^k\right]\le B_k(1+|p|^k). \tag{1.12}$$

By this and the Lipschitzianity of $\widetilde{\Phi}(P^{t,p}(T),\cdot)$ we have

$$\begin{aligned}|\widetilde{J}(t,p,z+h;u)-\widetilde{J}(t,p,z;u)| &\le C\mathbb{E}_{tpz}[(1+|P^{t,p}(T)|)\,|Z^{t,z+h;u}(T)-Z^{t,z;u}(T)|]\\ &\le Ch(1+\mathbb{E}_{tpz}[|P^{t,p}(T)|])\le M_2(1+|p|)h,\end{aligned}$$

where $M_2>0$ is a constant. Then argue as in Proposition 1.2.

*Item 2.* Let $(t,p)\in[0,T]\times\mathbb{R}$, $z_1,z_2\in\mathbb{R}$ and $u_1,u_2\in\mathcal{A}_t$. Notice that $(u_1+u_2)/2\in\mathcal{A}_t$ and that

$$Z^{t,\frac{z_1+z_2}{2};\frac{u_1+u_2}{2}}(T)=\frac{Z^{t,z_1;u_1}(T)+Z^{t,z_2;u_2}(T)}{2}. \tag{1.13}$$

By the concavity of the function $\widetilde{\Phi}(P^{t,p}(T),\cdot)$ and by (1.13) we have

$$\frac{\widetilde{J}(t,p,z_1;u_1)+\widetilde{J}(t,p,z_2;u_2)}{2}\le\widetilde{J}\left(t,p,\frac{z_1+z_2}{2};\frac{u_1+u_2}{2}\right)\le V\left(t,p,\frac{z_1+z_2}{2}\right). \tag{1.14}$$

Since (1.14) holds for each $u_1,u_2\in\mathcal{A}_t$, we get

$$\frac{\widetilde{V}(t,p,z_1)+\widetilde{V}(t,p,z_2)}{2}\le\widetilde{V}\left(t,p,\frac{z_1+z_2}{2}\right),$$

and then the concavity of the function $\widetilde{V}(t,p,\cdot)$. The a.e. existence of the second derivative follows from the Alexandrov theorem.

*Item 3.* Let $(t,p) \in [0,T] \times \mathbb{R}$, $z_1 \le z_2 \le M - (T-t)\bar{u}$ (the case $m \le z_1 \le z_2$ is similar) and $u \in \mathcal{A}_t$. Since

$$Z^{t,z_1;u}(T) \le Z^{t,z_2;u}(T) = z_2 + \int_t^T u(s)ds \le z_2 + (T-t)\bar{u} \le M$$

and since the function $\widetilde{\Phi}(P^{t,p}(T),\cdot)$ is non-decreasing in $]-\infty, M]$ (as it is concave and null in $[m,M]$), we have that

$$\widetilde{J}(t,p,z_1;u) \le \widetilde{J}(t,p,z_2;u). \tag{1.15}$$

As inequality (1.15) holds for each $u \in \mathcal{A}_t$, we get

$$\widetilde{V}(t,p,z_1) \le \widetilde{V}(t,p,z_2).$$

The second part immediately follows, since $]-\infty, M-(T-t)\bar{u}] \cap [m,+\infty[ = [m, M-(T-t)\bar{u}]$. □

The monotonicity result in Proposition 1.3 is described in Figure 1.1.

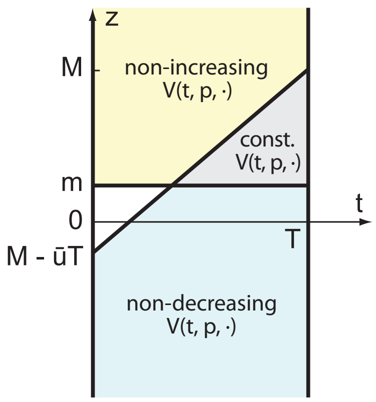


Figure 1.1: monotonicity of $\widetilde{V}(t,p,\cdot)$

The third part of Proposition 1.3 implies in particular that for suitable $t$ and for all $p$, the function $V(t,p,\cdot)$ is constant in an interval. As a matter of fact, this was foreseeable: it is easy to check that if $M-(T-t)\bar{u} \ge m$ and $z \in [m, M-(T-t)\bar{u}]$ then $Z^{t,z;u}(T) \in [m,M]$ for each $u \in \mathcal{A}_t$, so that the penalization term in the objective functional vanishes and the initial value $z$ does not influence the value function.

**Remark 1.4.** *As observed in [5, Equation (3.9)], by (1.8) a candidate optimal control policy is*

$$\underline{u}(t,p,z) = \begin{cases} \bar{u} & \text{if } \widetilde{V}_z(t,p,z) \ge p - K, \\ 0 & \text{if } \widetilde{V}_z(t,p,z) < p - K. \end{cases} \tag{1.16}$$

*Notice that by Proposition 1.3 the candidate in (1.16) is a.e. well-defined. Moreover, since $\widetilde{V}$ is concave in $z$, for each fixed $(t,p)$ there exists $\bar{z}(t,p) \in [-\infty,+\infty]$ such* *that* $\widetilde{V}_z(t,p,z) < p-K$ *if and only if* $z > \bar{z}(t,p)$*: for* $t$ *fixed, the function* $\bar{z}(t,\cdot)$ *(which in [5] is called exercise curve) can be used to write* $\underline{u}$ *as*

$$\underline{u}(t,p,z) = \begin{cases} \bar{u} & \textit{if } z \le \bar{z}(t,p), \\ 0 & \textit{if } z > \bar{z}(t,p). \end{cases} \tag{1.17}$$

## 1.3 Integral constrained stochastic optimal control

Let us now consider the problem, outlined in the Introduction, of optimally exercising swing contracts with strict constraints. Due to the presence of the constraint, in this case it is not possible to argue as in Section 1.2 and use classical results in control theory. This motivates us to study a more general class of stochastic optimal control problems with integral constraints, of which swing contracts with strict constraints will be a particular case.

### 1.3.1 Formulation of the problem

Let $d,l,n \in \mathbb{N}$, $r \ge 0$, $T > 0$ and $m, M \in \mathbb{R}$ with $m < M$. Let $U \subseteq \mathbb{R}^l$ be nonempty and $f, \sigma, g, L, \Phi$ be functions satisfying the following assumptions:

**Assumption 1.5.** *i)* $U$ *is a compact subset of* $\mathbb{R}^l$*;*

*ii)* $f \in C([0,T]\times\mathbb{R}^n\times U;\mathbb{R}^n)$, $\sigma \in C([0,T]\times\mathbb{R}^n\times U; M_{nd}(\mathbb{R}))$ *and there exists a constant* $C > 0$ *such that*

$$\begin{aligned} |f(t,p,v) - f(t,q,v)| &\le C|p-q|, \quad \forall p,q \in \mathbb{R}^n, \ \forall (t,v) \in [0,T]\times U, \\ |\sigma(t,p,v) - \sigma(t,q,v)| &\le C|p-q|, \quad \forall p,q \in \mathbb{R}^n, \ \forall (t,v) \in [0,T]\times U; \end{aligned} \tag{1.18}$$

*iii)* $g \in C([0,T]\times U;\mathbb{R})$*;*

*iv)* $L \in C([0,T]\times\mathbb{R}^n\times\mathbb{R}\times U;\mathbb{R})$, $\Phi \in C(\mathbb{R}^n\times\mathbb{R};\mathbb{R})$ *and there exist constants* $\tilde{C}, k > 1$ *such that*

$$\begin{aligned} &|L(t,p,z,v)| \le \tilde{C}(1+|p|^k+|z|^k), \quad \forall (t,p,z,v) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}\times U, \\ &|\Phi(p,z)| \le \tilde{C}(1+|p|^k+|z|^k), \quad \forall (p,z) \in \mathbb{R}^n\times\mathbb{R}. \end{aligned} \tag{1.19}$$

*Moreover, for each compact subset* $A \subseteq \mathbb{R}^{n+1}$ *there exists a modulus of continuity* $\omega_A$ *such that*

$$|L(t,p,z,v) - L(t,q,y,v)| \le \omega_R(|p-q|+|z-y|), \tag{1.20}$$

*for all* $(t,v) \in [0,T]\times U$ *and for all* $(p,z),(q,y) \in A$.

Notice that conditions (1.18) implies that

$$\begin{aligned} |f(t,p,v)| &\le \hat{C}(1+|p|), \quad \forall (t,p,v) \in [0,T]\times\mathbb{R}^n\times U, \\ |\sigma(t,p,v)| &\le \hat{C}(1+|p|), \quad \forall (t,p,v) \in [0,T]\times\mathbb{R}^n\times U, \end{aligned} \tag{1.21}$$

where $\hat{C} > 0$ is a constant.

Let $(\Omega, \mathcal{F}_T, \{\mathcal{F}_s\}_{s\in[0,T]}, \mathbb{P}, W)$ be a fixed filtered probability space where a $d$-dimensional $\{\mathcal{F}_s\}_s$-adapted Brownian motion $W = \{W(s)\}_{s\in[0,T]}$ is defined. If $t \in [0,T]$, let $\mathcal{A}_t$ denote the set of all $U$-valued progressively measurable processes $u = \{u(s)\}_{s\in[t,T]}$ (*controls*) such that for each $p \in \mathbb{R}^n$ the $n$-dimensional stochastic differential equation.

$$dP^{t,p;u}(s) = f(s, P^{t,p;u}(s), u(s))ds + \sigma(s, P^{t,p;u}(s), u(s))dW(s), \qquad s \in [t,T], \tag{1.22}$$

with initial condition

$$P^{t,p;u}(t) = p, \tag{1.23}$$

has a pathwise unique strong solution.

Let $t \in [0,T]$, $z \in \mathbb{R}$, $u \in \mathcal{A}_t$ and let

$$Z^{t,z;u}(s) = z + \int_t^s g(\tau, u(\tau))d\tau, \qquad s \in [t,T]. \tag{1.24}$$

A control $u \in \mathcal{A}_t$ is called *admissible* if the process $Z^{t,z;u}$ a.s. reaches the interval $[m,M]$ at the final time $T$:

$$\mathcal{A}_{tz}^{\mathrm{adm}} = \left\{u \in \mathcal{A}_t : Z^{t,z;u}(T) \in [m,M] \quad \mathbb{P}_{tpz}\text{-a.s.}\right\}.$$

We will often write $P^u$ and $Z^u$, in order to shorten the notations.

Given $(t,z) \in [0,T] \times \mathbb{R}$ and $A \subseteq \mathbb{R}$, we say that $A$ *is reachable from* $(t,z)$ if there exists a Borel measurable function $u$ from $[t,T]$ to $U$ (notice that then $u \in \mathcal{A}_t$) such that $Z^{t,z;u}(T) \in A$. Let $\mathcal{D}, \widetilde{\mathcal{D}}, \mathcal{D}^\rho$ (for $0 < \rho < (M-m)/2$) denote the subsets of $[0,T] \times \mathbb{R}^n \times \mathbb{R}$ defined by

$$\begin{aligned}
\mathcal{D} &= \{(t,p,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} : [m,M] \text{ is reachable from } (t,z)\}, \\
\widetilde{\mathcal{D}} &= \{(t,p,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} : ]m,M[ \text{ is reachable from } (t,z)\}, \\
\mathcal{D}^\rho &= \{(t,p,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} : [m+\rho, M-\rho] \text{ is reachable from } (t,z)\}.
\end{aligned}$$

Notice that $\bigcup_\rho \mathcal{D}^\rho = \widetilde{\mathcal{D}} \subseteq \mathcal{D}$. It is easy to prove that these sets are nonempty.

**Lemma 1.6.** *The sets $\mathcal{D}, \widetilde{\mathcal{D}}, \mathcal{D}^\rho$ are nonempty.*

*Proof.* Let $0 < \rho < (M-m)/2$. As $\mathcal{D}^\rho \subseteq \widetilde{\mathcal{D}} \subseteq \mathcal{D}$, it suffices to show that $\mathcal{D}^\rho \neq \varnothing$. Since $g([0,T] \times U) = [\xi_1, \xi_2]$, for suitable $(\tilde{t}, \tilde{z}) \in [0,T] \times \mathbb{R}$ we have that $Z^{\tilde{t},\tilde{z},u}(T) \in [\tilde{z} + \xi_1(T-\tilde{t}), \tilde{z} + \xi_2(T-\tilde{t})] \subseteq [m+\rho, M-\rho]$ for each Borel measurable function $u$ from $[\tilde{t}, T]$ to $U$, and thus $(\tilde{t}, \tilde{p}, \tilde{z}) \in \mathcal{D}^\rho$ (arbitrary $\tilde{p} \in \mathbb{R}$). $\square$

If $(t,p,z) \in \mathcal{D}$, by $\mathbb{E}_{tpz}$ we denote the mean value with respect to the probability $\mathbb{P}_{tpz} = \mathbb{P}$ (subscripts recall initial data). We can now define the *value function.*

**Definition 1.7.** *We set*

$$V(t,p,z) = \sup_{u \in \mathcal{A}_{tz}^{\mathrm{adm}}} J(t,p,z;u), \tag{1.25}$$

*for each $(t,p,z) \in \mathcal{D}$, where*

$$J(t,p,z;u) = \mathbb{E}_{tpz}\left[\int_t^T e^{-r(s-t)} L(s, P^{t,p;u}(s), Z^{t,z;u}(s), u(s))ds \right.$$
$$\left. + e^{-r(T-t)}\Phi(P^{t,p;u}(T), Z^{t,z;u}(T))\right].$$

Let us prove that the value function (1.25) is well defined.

**Lemma 1.8.** *The expectations in (1.25) are well posed, $V(t,p,z) < \infty$ and*

$$|V(t,p,z)| \leq \Gamma(1 + |p|^k + |z|^k), \tag{1.26}$$

*for each $(t,p,z) \in \mathcal{D}$, where $k$ is as in (1.19) and $\Gamma \geq 0$ is a constant depending only on $U$, $T$, $C$, $\tilde{C}$, $\hat{C}$ and $\max g$. Moreover, $\mathcal{D}$ is the maximal set in which expression (1.25) makes sense.*

*Proof.* First of all, notice that $\mathcal{A}_{tz}^{\mathrm{adm}} \neq \varnothing$ if and only if $(t,p,z) \in \mathcal{D}$ for each $p \in \mathbb{R}^n$. Recall estimate (1.12): it can be shown that $B_k$ depends only on the set $U$ and on constants $T$, $C$, $\hat{C}$ (see [20, Appendix D]. By (1.19) and (1.12) we have

$$\begin{aligned}
&\mathbb{E}_{tpz}\bigg[\bigg|\int_t^T e^{-r(s-t)} L(s, P^{t,p;u}(s), Z^{t,z;u}(s), u(s))ds + e^{-r(T-t)}\Phi(P^{t,p;u}(T), Z^{t,z;u}(T))\bigg|\bigg] \\
&\leq \tilde{C}\mathbb{E}_{tpz}\bigg[\int_t^T (1 + |P^{t,p;u}(s)|^k + |Z^{t,z;u}(s)|^k)ds + (1 + |P^{t,p;u}(T)|^k + |Z^{t,z;u}(T)|^k)\bigg] \\
&\leq C_1\mathbb{E}_{tpz}[1 + \|P^{t,p;u}(\cdot)\|^k + \|Z^{t,z;u}(\cdot)\|^k] \\
&\leq C_2(1 + |p|^k + |z|^k),
\end{aligned} \tag{1.27}$$

for suitable constants $C_1, C_2 > 0$. □

We also require the following assumption to hold.

**Assumption 1.9.** *Given $0 < \rho < (M-m)/2$ and a compact subset $A \subseteq \mathcal{D}^\rho$, there exist $\bar{\varepsilon} > 0$ and a function $0 < \eta \leq (M-m)/2$, both depending only on $\rho$, $A$, $T$ and $U$, with the following property: for each $0 < \varepsilon < \bar{\varepsilon}$, $(t,p,z) \in A$, $u \in \mathcal{A}_{tz}^{\mathrm{adm}}$ there exists $\tilde{u} \in \mathcal{A}_t$ such that $|J(t,p,z;u) - J(t,p,z;\tilde{u})| \leq \varepsilon$ and that a.s. $Z^{t,z;\tilde{u}}(T) \in [m + \eta(\varepsilon), M - \eta(\varepsilon)]$.*

In Section 1.3.4 we will give two examples of wide classes of problems satisfying Assumption 1.9.

### 1.3.2 Approximating problems

We would like to obtain for the problems of Section 1.3.1 the standard results in unconstrained control theory: continuity of the value function and characterization of the value function by the Hamilton-Jacobi-Bellman (HJB) equation. A straightforward approach is not possible, since condition $u \in \mathcal{A}_{tz}^{\mathrm{adm}}$ prevents from simply adapting classical proofs.

The idea is then the following: to define suitable unconstrained problems which approximate our constrained problem and then to obtain the properties of the value function (1.25) through a limiting procedure. The construction of the approximating problems is based on the idea of penalizing the case $Z^{t,z;u}(T) \notin [m,M]$ by adding a suitable term in the objective functional. In particular, we need, as a key point in the proof of Theorem 1.11, that $\lim_c \Phi^c(m) = \lim_c \Phi^c(M) = -\infty$ and that definitely $\Phi^c(z) = 0$ for each $z \in ]m,M[$; for this reason we choose the following penalization functions.

Given $c > 0$, let $\Phi^c$ be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\Phi^c(z) = -c\left[\left(z - \left(M - \frac{1}{\sqrt{c}}\right)\right)^+ + \left(\left(m + \frac{1}{\sqrt{c}}\right) - z\right)^+\right], \tag{1.28}$$

for each $z \in \mathbb{R}$. Let the assumptions of Section 1.3.1 hold and consider the following unconstrained problem:

$$V^c(t,p,z) = \sup_{u \in \mathcal{A}_t} J^c(t,p,z;u), \tag{1.29}$$

where $(t,p,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}$,

$$J^c(t,p,z;u) = \mathbb{E}_{tpz}\left[\int_t^T e^{-r(s-t)} L(s, P^{t,p;u}(s), Z^{t,z;u}(s), u(s))ds \right.$$
$$\left. + e^{-r(T-t)}\Phi(P^{t,p;u}(T), Z^{t,z;u}(T)) + e^{-r(T-t)}\Phi^c(Z^{t,z;u}(T))\right]$$

and this time the maximization is performed over the set $\mathcal{A}_t$ of all controls.

Problem (1.29) is a classical unconstrained stochastic control problem; therefore, by classical results, the function $V^c$ is characterized by the HJB equation. Here is the precise statement.

**Theorem 1.10.** *Let the assumptions of Section 1.3.1 hold and let $c > 0$ and $k$ as in (1.19). Then $V^c$ is the unique continuous viscosity solution of*

$$-V_t^c(t,p,z) + rV^c(t,p,z) + \min_{v \in U}\left[-f(t,p,v) \cdot D_p V^c(t,p,z) - g(t,v)V_z^c(t,p,z)\right.$$
$$\left. -\frac{1}{2}\mathrm{tr}\big(\sigma(t,p,v)\sigma^t(t,p,v)D_p^2 V^c(t,p,z)\big) - L(t,p,z,v)\right] = 0, \ \forall (t,p,z) \in [0,T[\times\mathbb{R}^n\times\mathbb{R}, \tag{1.30}$$

*with final condition*

$$V^c(T,p,z) = \Phi(p,z) + \Phi^c(z), \qquad \forall (p,z) \in \mathbb{R}^{n+1}, \tag{1.31}$$

*and such that*

$$|V^c(t,p,z)| \leq \check{C}(1 + |p|^k + |z|^k), \qquad \forall (t,p,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R},$$

*for some constant $\check{C} > 0$.*

*Proof.* The value function is a viscosity solution of (1.30) by a standard result in unconstrained control theory (the proof, for instance, can be achieved by slightly modifying the arguments in [20, Chapter IV]). As for uniqueness, see [16, Thm. 3.1]. □

### 1.3.3 Properties of the value function

We now prove the central result of this paper: the value functions $V^c$ in (1.29) converge, uniformly on the compact subsets of each $\mathcal{D}^\rho$, to the value function $V$ in (1.25). On one hand, this results provides an approximation of $V$ (recall the characterization of the functions $V^c$ in Theorem 1.10); on the other hand, in such a way $V$ inherits continuity from the functions $V^c$.

**Theorem 1.11.** *Let the assumptions of Section 1.3.1 hold. Then, as $c \to +\infty$, the functions $V^c$ converge to $V$ uniformly on compact subsets of $\mathcal{D}^\rho$, for each $0 < \rho < (M-m)/2$.*

*Proof.* Let $0 < \rho < (M-m)/2$, $A$ be a compact subset of $\mathcal{D}^\rho$ and $R > 0$ be such that $\overline{\mathrm{B}(0,R)} \supseteq A$. For each $\varepsilon > 0$, we have to prove that there exists $\delta > 0$ such that

$$\left| \sup_{u \in \mathcal{A}_t} J^c(t,p,z;u) - \sup_{u \in \mathcal{A}_{tz}^{\mathrm{adm}}} J(t,p,z;u) \right| \leq \varepsilon, \tag{1.32}$$

for each $c \geq \delta$ and $(t,p,z) \in A$.

*Step 1: lower bound for $V^c$ in $A$.* By definition of $\mathcal{D}^\rho$, for each $(t,p,z) \in A$ let $u_{tpz}$ be a Borel measurable function from $[t,T]$ to $U$ such that $Z^{u_{tpz}}(T) \in [m+\rho, M-\rho]$. Since $[m+\rho, M-\rho] \subseteq [m + c^{-\frac{1}{2}}, M - c^{-\frac{1}{2}}]$ for $c \geq \rho^{-2}$, notice that $J^c(t,p,z;u_{tpz}) \equiv K_{tpz}$ for a suitable constant $K_{tpz}$ for all $c \geq \rho^{-2}$. By estimates as in (1.27), it is easy to show that, for a constant $C_1 > 0$ and for $k \geq 0$ as in (1.19), we have $|K_{tpz}| \leq C_1(1 + |p|^k + |z|^k) \leq C_1(1 + 2R^k)$ for each $(t,p,z) \in A$, so that $K := \inf_{(t,p,z) \in A} K_{tpz} \in \mathbb{R}$. Therefore,

$$V^c(t,p,z) \geq J^c(t,p,z;u_{tpz}) = K_{tpz} \geq K, \tag{1.33}$$

for each $(t,p,z) \in A$ and $c \geq \rho^{-2}$.

*Step 2: new formulation of (1.32).* Let $(t,p,z) \in A$. For each $n \in \mathbb{N}$ we set

$$B_n^{tpz} = \left\{ u \in \mathcal{A}_t : \frac{1}{n+1} < \mathbb{P}_{tpz}(Z^u(T) \notin [m,M]) \leq \frac{1}{n} \right\}.$$

Let $c \geq \rho^{-2}$, $n \in \mathbb{N}$ and $u \in B_n^{tpz}$. By noting that $\Phi^c \leq 0$ and that $\Phi^c(x) \leq -\sqrt{c}$ for $x \notin [m,M]$ and by estimates as in (1.27), we have

$$\begin{aligned} J^c(t,p,z;u) &= J(t,p,z;u) + \mathbb{E}_{tpz}\left[ e^{-r(T-t)} \Phi^c(Z^u(T)) \right] \\ &\leq J(t,p,z;u) + \mathbb{E}_{tpz}\left[ e^{-r(T-t)} \Phi^c(Z^u(T)) \mathbb{1}_{\{Z^u(T) \notin [m,M]\}} \right] \\ &\leq C_2(1 + |p|^k + |z|^k) - e^{-rT} \sqrt{c}\, \mathbb{P}_{tpz}(Z^u(T) \notin [m,M]) \\ &< C_2(1 + 2R^k) - e^{-rT} \frac{\sqrt{c}}{n+1}, \end{aligned} \tag{1.34}$$

for a suitable constant $C_2 > 0$. By (1.34) it follows that for each $n \in \mathbb{N}$ there exists $c(n) \geq \rho^{-2}$ such that

$$J^c(t,p,z;u) < K,$$

for each $c \geq c(n)$ and $u \in B_n^{tpz}$, with $K$ as in Step 1. By (1.33) we thus get

$$\sup_{u \in \mathcal{A}_t} J^c(t,p,z;u) = \sup_{u \in \mathcal{A}_t \setminus \bigcup_{\{i \in \mathbb{N} : c(i) \leq c\}} B_i^{tpz}} J^c(t,p,z;u), \tag{1.35}$$

for each $c \geq \rho^{-2}$. The sequence $\{c(n)\}_n$ is obviously increasing; hence, there exists a function $m$ from $[\rho^{-2}, +\infty[$ to $\mathbb{N}$ such that $\{i \in \mathbb{N} : c(i) \leq c\} = \{1, \dots, m(c)\}$ for each $c \geq \rho^{-2}$. As a consequence, we can rewrite (1.35) as follows:

$$\sup_{u \in \mathcal{A}_t} J^c(t,p,z;u) = \sup_{u \in \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}} J^c(t,p,z;u). \tag{1.36}$$

Notice that $m(\cdot)$ is increasing and that $m(c) \to +\infty$.

Let $\varepsilon > 0$. By (1.36), for $c \geq \rho^{-2}$ inequality (1.32) is equivalent to

$$-\varepsilon \leq \sup_{u \in \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}} J^c(t,p,z;u) - \sup_{u \in \mathcal{A}_{tz}^{\mathrm{adm}}} J(t,p,z;u) \leq \varepsilon. \tag{1.37}$$

Therefore, we have to prove that there exists $\delta \geq \rho^{-2}$ such that (1.37) holds for each $c \geq \delta$ and for each $(t,p,z) \in A$. In Step 3 we will prove the right inequality in (1.37), while in Step 4 the left inequality will be proved, thus concluding the proof.

*Step 3: right inequality in (1.37).* Let us show that there exists $\delta_1 \geq \rho^{-2}$ independent of $(t,p,z) \in A$ such that

$$\sup_{u \in \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}} J(t,p,z;u) \leq \sup_{u \in \mathcal{A}_{tz}^{\mathrm{adm}}} J(t,p,z;u) + \varepsilon, \tag{1.38}$$

for each $c \geq \delta_1$. Since $J^c \leq J$, by (1.38) we get the right inequality in (1.37).

Let $c \geq \rho^{-2}$, $(t,p,z) \in A$, $u \in \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}$. We set

$$\Pi^u = \{Z^u(T) \notin [m, M]\};$$

notice that $0 \leq \mathbb{P}_{tpz}(\Pi^u) \leq 1/(m(c)+1)$. Let $\tilde{u}$ be the process defined in the following way: $\tilde{u}$ coincides in $\Pi^u$ with the process which assures the reachability of $[m+\rho, M-\rho]$ (see the definition of $\mathcal{D}^\rho$), and $\tilde{u} \equiv u$ in $\Omega \setminus \Pi^u$. A simple check shows that $\tilde{u} \in \mathcal{A}_t$ and that

$$Z^{\tilde{u}}(T) \in [m, M] \qquad \mathbb{P}_{tpz}\text{-a.s.} \tag{1.39}$$

By recalling that $\tilde{u} \equiv u$ in $\Omega \setminus \Pi^u$, by the Hölder inequality (twice) and by estimates as in (1.27), we obtain that

$$\big|J(t,p,z;u) - J(t,p,z;\tilde{u})\big| \leq C_3(1+|p|^k+|z|^k)\,\mathbb{P}_{tpz}(\Pi^u)^{\frac{1}{2}} \leq \frac{C_3(1+2R^k)}{(m(c)+1)^{\frac{1}{2}}}, \tag{1.40}$$

for some constant $C_3 > 0$. Then, by (1.40) and (1.39) it follows that

$$J(t,p,z;u) \leq J(t,p,z;\tilde{u}) + \frac{C_3(1+2R^k)}{(m(c)+1)^{\frac{1}{2}}} \leq \sup_{u \in \mathcal{A}_{tz}^{\mathrm{adm}}} J(t,p,z;u) + \frac{C_3(1+2R^k)}{(m(c)+1)^{\frac{1}{2}}}.$$

This inequality holds for each $(t,p,z) \in A$ and $u \in \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}$. Since $m(c) \to +\infty$, for sufficiently large $c$ (and this choice is independent of $(t,p,z)$ and $u$), we have that $C_3(1+2R^k)/(m(c)+1)^{\frac{1}{2}} \leq \varepsilon$, thus obtaining (1.38).

*Step 4: left inequality in (1.37).* We still have to prove the left inequality in (1.37), i.e.

$$\sup_{u \in \mathcal{A}_{tz}^{\mathrm{adm}}} J(t,p,z;u) \leq \sup_{u \in \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}} J^c(t,p,z;u) + \varepsilon, \tag{1.41}$$

for $c \geq \delta_2$, with $\delta_2 \geq \rho^{-2}$ independent of $(t,p,z) \in A$.

Let $c \geq \rho^{-2}$, $(t,p,z) \in A$ and $u \in \mathcal{A}_{tz}^{\mathrm{adm}}$. By Assumption 1.9, let $\tilde{u} \in \mathcal{A}_t$ be such that

$$|J(t,p,z;u) - J(t,p,z;\tilde{u})| \leq \varepsilon \tag{1.42}$$

and with the property

$$Z^{\tilde{u}}(T) \in [m + \eta(\varepsilon), M - \eta(\varepsilon)] \qquad \mathbb{P}_{tpz}\text{-a.s.} \tag{1.43}$$

First of all notice that

$$\tilde{u} \in \mathcal{A}_{tz}^{\mathrm{adm}} \subseteq \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}. \tag{1.44}$$

By (1.42) we obtain that

$$\begin{aligned} &\left|J(t,p,z;u) - J^c(t,p,z;\tilde{u})\right| \\ &= \left|J(t,p,z;u) - J(t,p,z;\tilde{u}) - \mathbb{E}_{tpz}\left[e^{-r(T-t)}\Phi^c(Z^{\tilde{u}}(T))\right]\right| \leq \varepsilon + \mathbb{E}_{tpz}\left[|\Phi^c(Z^{\tilde{u}}(T))|\right]. \end{aligned}$$

Notice that by (1.43) the second term equals zero for $c \geq \eta(\varepsilon)^{-2}$ (in fact $\Phi^c \equiv 0$ in $[m + c^{-\frac{1}{2}}, M - c^{-\frac{1}{2}}]$); by recalling (1.44), we therefore have that

$$J(t,p,z;u) \leq J^c(t,p,z;\tilde{u}) + \varepsilon \leq \sup_{u \in \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}} J^c(t,p,z;u) + \varepsilon,$$

for each $c \geq \max\{\eta(\varepsilon)^{-2}, \rho^{-2}\}$. Since this inequality holds for each $(t,p,z) \in A$ and $u \in \mathcal{A}_{tz}^{\mathrm{adm}}$, we get (1.41). □

**Corollary 1.12.** *Let the assumptions of Section 1.3.1 hold. Then the functions $V^c$ converge pointwise to $V$ in $\widetilde{\mathcal{D}}$ and $V$ is continuous on $\widetilde{\mathcal{D}}$.*

*Proof.* It follows immediately from Theorem 1.11 (recall that $\bigcup_\rho \mathcal{D}^\rho = \widetilde{\mathcal{D}}$). □

**Corollary 1.13.** *Let the assumptions of Section 1.3.1 hold. Then the function $V$ is a viscosity solution of Equation (1.30) in $\widetilde{\mathcal{D}}$.*

*Proof.* Due to Theorem 1.10, the functions $V^c$ are viscosity solutions of the same equation, that is

$$\begin{aligned} &- V_t^c(t,p,z) + rV^c(t,p,z) + \min_{v \in U}\Big[-f(t,p,v) \cdot D_p V^c(t,p,z) - g(t,v)V_z^c(t,p,z) \\ &-\frac{1}{2}\mathrm{tr}\big(\sigma(t,p,v)\sigma^t(t,p,v)D_p^2 V^c(t,p,z)\big) - L(t,p,z,v)\Big] = 0, \ \ \forall (t,p,z) \in [0,T[\times\mathbb{R}^n\times\mathbb{R}. \end{aligned} \tag{1.45}$$

Moreover by Theorem 1.11 for each $0 < \rho < (M-m)/2$ the functions $V^c$ locally uniformly converge in $\mathcal{D}^\rho$ to the function $V$. So the conclusion follows by the stability property of viscosity solutions with respect to the uniform convergence and the fact that $\widetilde{\mathcal{D}} = \bigcup_\rho \mathcal{D}^\rho$. □

**Remark 1.14.** *We have proved that the value function in the set $\widetilde{\mathcal{D}}$ is the locally uniform limit of the functions $V^c$. In some particular cases, stronger conclusions can be achieved: the value function is characterized in its whole domain $\mathcal{D}$ by the HJB equation. See Section 1.4.*

We now face the problem of the regularity of the function $V$. In control theory, regularity results are usually achieved by passing to the supremum in estimates on quantities such as $|J(t,p',z;u)-J(t,p'',z;u)|$ or $|J(t,p,z';u)-J(t,p,z'';u)|$, so as to obtain the corresponding inequality for $V$. In the case of constrained problems, this approach cannot be applied to $V(t,p,\cdot)$. In fact, consider $|J(t,p,z';u)-J(t,p,z'';u)|$: on one hand such a quantity is defined only for $u \in \mathcal{A}^{\mathrm{adm}}_{tz'} \cap \mathcal{A}^{\mathrm{adm}}_{tz''}$, on the other hand the supremum should be with respect to different sets (precisely, $\mathcal{A}^{\mathrm{adm}}_{tz'}$ and $\mathcal{A}^{\mathrm{adm}}_{tz''}$). Of course, in particular cases some regularity results can be achieved also for $V(t,p,\cdot)$, see Section 1.4. The only case when that approach still works regards estimates on $V(t,\cdot,z)$, given that, fixed $t$ and $z$, the set of admissible controls does not depend on $p$. Hence, as for $V(t,\cdot,z)$ we can follow this approach.

**Proposition 1.15.** *Let the assumptions of Section 1.3.1 hold. Assume that there exists a constant $\bar{C} > 0$ such that*

$$|L(t,p,z,v) - L(t,q,z,v)| \leq \bar{C}|p-q|, \qquad |\Phi(p,z) - \Phi(q,z)| \leq \bar{C}|p-q|, \tag{1.46}$$

*for each $p,q \in \mathbb{R}^n$, $t \in [0,T]$, $v \in U$ and $z \in \mathbb{R}$. Then the function $V(t,\cdot,z)$ is Lipschitz continuous, uniformly in $(t,z)$. Moreover, the gradient $D_pV(t,p,z)$ exists for a.e. $(t,p,z) \in \mathcal{D}$ and we have $|D_pV(t,p,z)| \leq M_1$ for some constant $M_1 > 0$ depending only on $U$, $T$, $\bar{C}$ and on the constants in (1.18) and (1.21).*

*Proof.* Let $(t,p,z) \in \mathcal{D}$, $h > 0$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$ and $u \in \mathcal{A}_t$. In order to avoid ambiguity, we will omit the subscripts in the notation of the mean value (initial data are different, but the probability is obviously the same). By (1.46) and estimates (D.8) in [20, Appendix D] we have

$$\begin{aligned}
&|J(t,p,z;u) - J(t,p+h\xi,z;u)| \\
&\leq \bar{C}\mathbb{E}\left[\int_t^T \left|P^{t,p;u}(s) - P^{t,p+h\xi;u}(s)\right| ds + \left|P^{t,p;u}(T) - P^{t,p+h\xi;u}(T)\right|\right] \\
&\leq \bar{C}(T-t+1)\mathbb{E}[\|P^{t,p;u}(\cdot) - P^{t,p+h\xi;u}(\cdot)\|_\infty] \\
&\leq C_1\bar{C}(T+1)|p-(p+h\xi)| \\
&= C_1\bar{C}(T+1)h,
\end{aligned} \tag{1.47}$$

for some constant $C_1 > 0$. Estimate (1.47) holds for each $u \in \mathcal{A}_t$; thus, it follows that

$$|V(t,p,z) - V(t,p+h\xi,z)| \leq M_0 h, \tag{1.48}$$

where $M_0 := C_1\bar{C}(T+1)$. The function $V(t,\cdot,z)$ is therefore Lipschitz continuous, uniformly in $(t,z)$, and then a.e. differentiable by the Rademacher theorem. By classical results it follows that $D_pV(t,p,z)$ exists for a.e. $(t,p,z) \in \mathcal{D}$. Finally, if the gradient exists and $e_i \in \mathbb{R}^n$ is a vector of the canonical basis ($i = 1,\dots,n$), by (1.48) we get

$$|(D_pV(t,p,z))_i| = \lim_{h\to 0^+} \frac{|V(t,p,z) - V(t,p+he_i,z)|}{h} \leq M_0,$$

and then the estimate on the gradient immediately follows. □

**Proposition 1.16.** *Let the assumptions of Section 1.3.1 hold. Assume that* $\Phi \in C^2(\mathbb{R}^{n+1})$*, that the functions* $f(t,\cdot,v)$*,* $\sigma(t,\cdot,v)$*,* $L(t,\cdot,\cdot,v)$ *are of class* $C^2$ *for each* $(t,v) \in [0,T] \times U$ *and that there exist constants* $\bar{C} \geq 0$*,* $j \in \mathbb{N}$ *such that*

$$|D_p f(t,p,v)| + |D_p^2 f(t,p,v)| + |D_p \sigma(t,p,v)| + |D_p^2 \sigma(t,p,v)| \leq \bar{C},$$
$$|D_{(p,z)} L(t,p,z,v)| + |D_{(p,z)}^2 L(t,p,z,v)| \leq \bar{C}(1 + |p|^j + |z|^j),$$
$$|D_{(p,z)} \Phi(p,z)| + |D_{(p,z)}^2 \Phi(p,z)| \leq \bar{C}(1 + |p|^j + |z|^j),$$

*for each* $p \in \mathbb{R}^n$*,* $t \in [0,T]$*,* $v \in U$ *and* $z \in \mathbb{R}$*. The function* $V(t,\cdot,z)$ *is then locally semiconvex, uniformly in* $t$*, and a.e. twice differentiable.*

*Proof.* Since $\Phi \in C_p^2(\mathbb{R}^{n+1})$, it is possible to rewrite the problem so that $\Phi \equiv 0$ (see [20, Remark IV.6.1]). By arguing as in the proof of [20, Lemma IV.9.1] (with minor modifications: the assumptions are slightly different), we get

$$V(t,p+h\xi,z) + V(t,p-h\xi,z) - 2V(t,p,z) \geq -M_2(1+|p|^j)h^2,$$

for each $(t,p,z) \in \mathcal{D}$, $h > 0$ and $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, where $M_2 > 0$ is a constant. The function $V(t,\cdot,z)$ is therefore locally semiconvex, uniformly in $(t,z)$, and then a.e. twice differentiable by the Alexandrov theorem. □

### 1.3.4 Examples

We now show two wide classes of problems satisfying Assumption 1.9. We first consider problems where $U$ is a compact interval of $\mathbb{R}$ and $g(s,v) = v$, so that the constraint is $z + \int_t^T u(s) \in [m,M]$.

**Proposition 1.17.** *Let* $a, b \in \mathbb{R}$*, with* $a < b$*. Let the assumptions of Section 1.3.1 hold, with* $U = [a,b]$ *and* $g(s,v) = v$*. Moreover, assume that there exist* $\Gamma, l > 0$ *such that for* $\xi = f, \sigma$ *the following condition holds:*

$$|\xi(s,p,v') - \xi(s,p,v'')| \leq \Gamma(1+|p|)|v'-v''|^l, \quad \forall (s,p) \in [0,T] \times \mathbb{R}^n, \ \forall v', v'' \in U. \tag{1.49}$$

*Then Assumption 1.9 is satisfied.*

*Proof.* For the sake of simplicity, in this proof we assume $l = 1$ (for the general case, in the definition of $\tilde{u}$ it suffices to substitute $\delta^2$ by $\delta^i$, where $i > 1/l$).

Let $0 < \rho < (M-m)/2$, $A$ be a compact subset of $\mathcal{D}^\rho$, $R > 0$ be such that $\overline{B(0,R)} \supseteq A$, $\varepsilon > 0$. Fix $(t,p,z) \in A$ and $u \in \mathcal{A}_{tz}^{\mathrm{adm}}$.

Let $\gamma > 0$ (it will be afterwards precisely defined). Since the functions $L$ and $\Phi$ are continuous, there exists $\delta = \delta(\varepsilon,\gamma) > 0$ such that

$$|L(s,p',z',v') - L(s,p'',z'',v'')| \leq \frac{\varepsilon}{4T} \quad \text{and} \quad |\Phi(p',z') - \Phi(p'',z'')| \leq \frac{\varepsilon}{4}, \tag{1.50}$$

for each $s \in [0,T]$, for each $p', p'' \in \overline{B(0,\gamma)}$ with $|p'-p''| \leq \delta$, for each $z', z'' \in [m,M]$ with $|z'-z''| \leq T\delta$ and for each $v', v'' \in U$ with $|v'-v''| \leq \delta$.

We now define, starting from $u$, a suitable process $\tilde{u}$. Let $\Pi^M = \{\omega \in \Omega : Z^u(T) \in ]M-\rho/2, M]\}$ and in $\Pi^M$ let $\tilde{u}$ be defined in the following way:

$$\tilde{u}(s) = \begin{cases} u(s) - \delta^2, & \text{if } s \in E, \\ u(s), & \text{if } s \in [t,T] \setminus E, \end{cases}$$

where $E = E(\omega) = \{s \in [t,T] : u(s) - \delta^2 \in ]a,b[\} = \{s \in [t,T] : u(s) > a + \delta^2\}$. Let $\Pi_m = \{\omega \in \Omega : Z^u(T) \in [m, m + \rho/2[\}$ and in $\Pi_m$ let $\tilde{u}$ be defined in the following way:

$$\tilde{u}(s) = \begin{cases} u(s) + \delta^2, & \text{if } s \in F, \\ u(s), & \text{if } s \in [t,T] \setminus F, \end{cases}$$

where $F = F(\omega) = \{s \in [t,T] : u(s) + \delta^2 \in ]a,b[\} = \{s \in [t,T] : u(s) < b - \delta^2\}$. Finally, in $\Omega \setminus (\Pi^M \cup \Pi_m)$ let

$$\tilde{u} \equiv u.$$

We will show that such a process $\tilde{u}$ satisfies the required properties.

*Step 1.* We prove that

$$Z^{t,z;\tilde{u}}(T) \in [m + \eta(\varepsilon), M - \eta(\varepsilon)] \qquad \mathbb{P}_{tpz}\text{-a.s.}, \tag{1.51}$$

for a suitable function $0 < \eta \leq (M - m)/2$ depending only on $\rho, T, R, a, b$.

Consider the case $\omega \in \Pi^M$, i.e.

$$Z^u(T) \in ]M - \rho/2, M]. \tag{1.52}$$

Let us first of all notice that

$$Z^{\tilde{u}}(T) = z + \int_t^T \tilde{u}(s)ds = z + \int_t^T u(s)ds - \delta^2\mu(E) = Z^u(T) - \delta^2\mu(E), \tag{1.53}$$

where $\mu$ denotes the Lebesgue measure in $\mathbb{R}$. We now look for an estimate for $\mu(E)$. By definition of $E$, we have

$$\int_t^T u(s)ds = \int_E u(s)ds + \int_{[t,T]\setminus E} u(s)ds \leq b\mu(E) + (a + \delta^2)(T - t - \mu(E))$$

and then

$$\mu(E) \in \left[\frac{\int_t^T u(s)ds - (a + \delta^2)(T - t)}{b - a - \delta^2}, T - t\right] \subseteq \left[\frac{\rho/2 - \delta T}{b - a}, T\right], \tag{1.54}$$

where the inclusion follows by $z + \int_t^T u(s)ds \geq M - \rho/2$ (since $\omega \in \Pi^M$) and $z \leq M - \rho - a(T - t)$ (since $(t, p, z) \in \mathcal{D}^\rho$). By possibly decreasing $\delta$ (and the choice depends only on $a, b, \rho, T$), we can assume that the lower bound in (1.54) is positive. Recall (1.53): by (1.52) and (1.54) we get

$$Z^{\tilde{u}}(T) \in \left]M - \frac{\rho}{2} - \delta^2 T, M - \frac{\delta^2\rho/2 - \delta^3 T}{b - a}\right] \subseteq \left]m + \frac{\rho}{2}, M - \frac{\delta^2\rho/2 - \delta^3 T}{b - a}\right],$$

where the inclusion follows by $M - \rho/2 > m + \rho/2$ and by assuming $\delta$ sufficiently small. This estimate holds for each $\omega \in \Pi^M$; by arguing in the same way, for each $\omega \in \Pi_m$ we get

$$Z^{\tilde{u}}(T) \in \left[m + \frac{\delta^2\rho/2 - \delta^3 T}{b - a}, M - \frac{\rho}{2}\right[.$$

Finally, in $\Omega\setminus(\Pi^M\cup\Pi_m)$ we have $Z^{\tilde u}\in[m+\rho/2, M-\rho/2]$. To summarize, condition (1.51) is verified with

$$\eta(\varepsilon)=\min\left\{\frac{\rho}{2},\frac{\delta(\varepsilon,\gamma)^2\rho/2-\delta(\varepsilon,\gamma)^3\,T}{b-a}\right\}.$$

*Step 2.* We still have to prove that

$$|J(t,p,z;u)-J(t,p,z;\tilde u)|\le\varepsilon. \tag{1.55}$$

Let $\Pi\subseteq\Omega$ be defined by

$$\Pi=\{\|P^u(\cdot)\|\le\gamma,\|P^{\tilde u}(\cdot)\|\le\gamma,\|P^u(\cdot)-P^{\tilde u}(\cdot)\|\le\delta\};$$

first of all, we set for brevity

$$\begin{aligned}\Gamma(t,p,z;u,\tilde u)=\int_t^T|L(s,P^u(s),Z^u(s),u(s))-L(s,P^{\tilde u}(s),Z^{\tilde u}(s),\tilde u(s))|ds\\+|\Phi(P^u(T),Z^u(T))-\Phi(P^{\tilde u}(T),Z^{\tilde u}(T))|,\end{aligned}$$

and notice that

$$|J(t,p,z;u)-J(t,p,z;\tilde u)|\le\mathbb{E}_{tpz}[\Gamma(t,p,z;u,\tilde u)\mathbb{1}_{\Pi}]+\mathbb{E}_{tpz}[\Gamma(t,p,z;u,\tilde u)\mathbb{1}_{\Pi^c}]. \tag{1.56}$$

As for the first term in (1.56), for $s\in[t,T]$ we have

$$|Z^u(s)-Z^{\tilde u}(s)|\le(s-t)\delta^2\le T\delta\quad\text{and}\quad|u(s)-\tilde u(s)|\le\delta^2\le\delta,$$

so that by (1.50) it follows that

$$\mathbb{E}_{tpz}[\Gamma(t,p,z;u,\tilde u)\mathbb{1}_{\Pi}]\le\left(\int_0^T\frac{\varepsilon}{4T}ds+\frac{\varepsilon}{4}\right)\mathbb{P}_{tpz}(\Pi)=\frac{\varepsilon}{2}\mathbb{P}_{tpz}(\Pi)\le\frac{\varepsilon}{2}. \tag{1.57}$$

We now consider the second term in (1.56). By (D.8) in [20, Appendix D], by (1.49) and (1.12) we get

$$\mathbb{E}_{tpz}[\|P^u(\cdot)-P^{\tilde u}(\cdot)\|]\le C_1(1+|p|)\delta^2, \tag{1.58}$$

for a suitable constant $C_1>0$. By the Markov inequality, (1.58) and (1.12) we then get

$$\begin{aligned}\mathbb{P}_{tpz}(\Pi^c)\le\mathbb{E}_{tpz}[\|P^u(\cdot)\|_\infty]\gamma^{-1}+\mathbb{E}_{tpz}[\|P^{\tilde u}(\cdot)\|_\infty]\gamma^{-1}+\mathbb{E}_{tpz}[\|P^u(\cdot)-P^{\tilde u}(\cdot)\|_\infty]\delta^{-1}\\\le C_2(1+|p|)(\gamma^{-1}+\delta),\end{aligned} \tag{1.59}$$

where $C_2>0$ is a constant. By the Hölder inequality (twice), estimates as in (1.27) and (1.59), we obtain that

$$\begin{aligned}\mathbb{E}_{tpz}[\Gamma(t,p,z;u,\tilde u)\mathbb{1}_{\Pi^c}]^2&\le\mathbb{E}_{tpz}[\Gamma(t,p,z;u,\tilde u)^2]\mathbb{P}_{tpz}(\Pi^c)\\&\le C_3(1+|p|^{2k}+|z|^{2k})(1+|p|)(\gamma^{-1}+\delta(\varepsilon,\gamma))\\&\le C_4(1+R^{2k+1})\gamma^{-1}+C_4(1+R^{2k+1})\delta(\varepsilon,\gamma),\end{aligned}$$

with $C_3,C_4>0$. First by choosing a suitable $\gamma$ and then by possibly taking a less $\delta$ (and these choices depends only on $R$ and $\varepsilon$), we get

$$\mathbb{E}_{tpz}\left[\Gamma(t,p,z;u,\tilde u)\mathbb{1}_{\Pi^c}\right]\le\frac{\varepsilon}{2}. \tag{1.60}$$

Estimates (1.56), (1.57) e (1.60) imply (1.55), thus ending the proof. □

Let us now consider problems where $U$ is a closed ball of $\mathbb{R}^l$ and $g(s,v) = |v|^p$, so that the constraint is $z + \int_t^T |u(s)|^p \in [m, M]$, with $p \geq 1$.

**Proposition 1.18.** *Let $p \geq 1$ and $b > 0$. Let the assumptions of Section 1.3.1 and (1.49) hold, with $U = \overline{B(0,b)} \subseteq \mathbb{R}^l$ and $g(s,v) = |v|^p$. Assumption 1.9 is then satisfied.*

*Proof.* The proof is similar to the one of Proposition 1.17, with the following modifications:

- In the definitions of $E$ and $F$ replace $u(s)$ by $|u(s)|$. Process $\tilde{u}$ in $E$ is now

$$\tilde{u}(s) = u(s) - \delta^2 \frac{u(s)}{|u(s)|}.$$

  Notice that $|\tilde{u}(s)| = |u(s)| - \delta^2$. Similarly in $F$.

- It is easy to check that

$$(\zeta - \delta^2)^p \leq \zeta^p - \delta^{2p}, \qquad \text{for } \zeta \geq \delta^2,$$
$$(\zeta + \delta^2)^p \geq \zeta^p + \delta^{2p}, \qquad \text{for } \zeta \geq 0.$$

  By the first estimate, in $\Pi^M$ we have that

$$\begin{aligned} Z^{\tilde{u}}(T) &= z + \int_t^T |\tilde{u}(s)|^p ds \\ &= z + \int_E (|u(s)| - \delta^2)^p ds + \int_{[t,T]\setminus E} |u(s)|^p ds \\ &\leq z + \int_t^T |u(s)|^p ds - \delta^{2p} \mu(E), \end{aligned}$$

  and then we can argue as in the proof of Proposition 1.17. As for $\Pi_m$, use the second estimate and the same argument.

□

## 1.4 Swing contracts with strict constraints

We now use the results of Section 1.3 to study the problem of optimally exercising swing contracts with strict constraints (see the Introduction). In this case we will obtain results stronger than the general ones proved in Section 1.3.3.

### 1.4.1 Formulation of the problem

Let $T > 0$, $(\Omega, \mathcal{F}_T, \{\mathcal{F}_s\}_{s\in[0,T]}, \mathbb{P})$, $U$, $\mathcal{A}_t$, $P^{t,p}$ and $Z^{t,z;u}$ be as in Section 1.2. If $(t,p,z) \in [0,T] \times \mathbb{R}^2$ and $s \in [t,T]$, recall, in particular, that $P^{t,p}(s)$ models the price of energy at time $s$ and that $Z^{t,z;u}(s)$ represents the energy bought up to time $s$, where $u \in \mathcal{A}_t$ is the usage strategy from time $t$ on.

Given $m, M \geq 0$ with $m < M$, here we ask the following constraint to hold:

$$Z^{t,z;u}(T) \in [m, M] \qquad \mathbb{P}_{tpz}\text{-a.s.}$$

The problem of the optimal exercise of this contract (i.e. to find a process $u$ satisfying all the conditions and providing the maximal expected earning) is clearly a constrained stochastic optimal control problem as described in Section 1.3.1 (here $P$ does not depend on $u$), whose value function is

$$V(t,p,z) = \sup_{u \in \mathcal{A}_{tz}^{\mathrm{adm}}} \mathbb{E}_{tpz}\left[\int_t^T e^{-r(s-t)}(P^{t,p}(s) - K)u(s)ds\right].$$

In this problem the sets $\mathcal{D}, \widetilde{\mathcal{D}}, \mathcal{D}^\rho$ are formed by all points $(t,p,z) \in [0,T] \times \mathbb{R}^2$ such that $(t,z)$ belongs, respectively, to the marked surfaces in Figure 1.2.

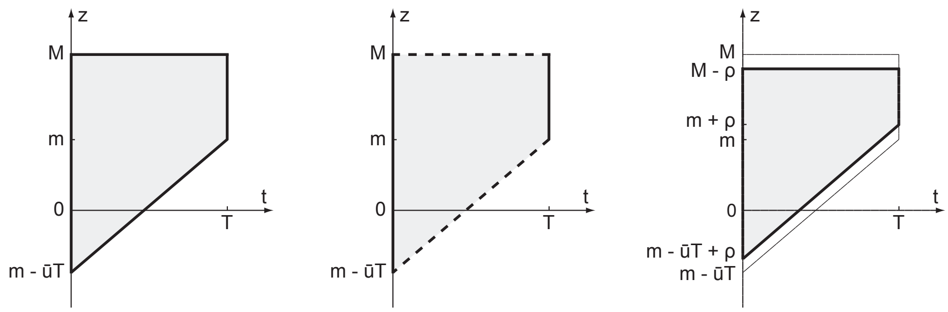


Figure 1.2: the sets $\mathcal{D}, \widetilde{\mathcal{D}}, \mathcal{D}^\rho$

More in details, we have

$$\begin{aligned} \mathcal{D} &= \{(t,p,z) \in [0,T] \times \mathbb{R}^2 : m - \bar{u}(T-t) \le z \le M\}, \\ \widetilde{\mathcal{D}} &= \{(t,p,z) \in [0,T] \times \mathbb{R}^2 : m - \bar{u}(T-t) < z < M\}, \\ \mathcal{D}^\rho &= \{(t,p,z) \in [0,T] \times \mathbb{R}^2 : m + \rho - \bar{u}(T-t) \le z \le M - \rho\}. \end{aligned}$$

Notice that these sets include initial data that are inconsistent with the practical problem: in fact, our mathematical formulation admits negative starting values for $p$ and $z$.

The functions $V^c$ are here defined by

$$V^c(t,p,z) = \sup_{u \in \mathcal{A}_t} \mathbb{E}_{tpz}\left[\int_t^T e^{-r(s-t)}(P^{t,p}(s) - K)u(s)ds + e^{-r(T-t)}\Phi^c(Z^{t,z;u}(T))\right],$$

for each $c > 0$ and $(t,p,z) \in [0,T] \times \mathbb{R}^2$, where $\Phi^c$ is defined in (1.28). This has also a nice economical interpretation: in fact, here we are approximating a swing contract with the strict constraint $Z(T) \in [m, M]$ with a sequence of suitable contracts with increasing penalties for $Z(T) \notin \left[m + \frac{1}{\sqrt{c}}, M - \frac{1}{\sqrt{c}}\right]$.

The HJB equation for the function $V^c$ is

$$\begin{aligned} &- V_t^c(t,p,z) + rV^c(t,p,z) - f(t,p)V_p^c(t,p,z) - \frac{1}{2}\sigma^2(t,p)V_{pp}^c(t,p,z) \\ &\qquad + \min_{v \in [0,\bar{u}]} [-v(V_z^c(t,p,z) + p - K)] = 0, \qquad \forall (t,p,z) \in [0,T[ \times \mathbb{R}^2, \quad (1.61) \end{aligned}$$

with final condition

$$V^c(T,p,z) = \Phi^c(z), \qquad \forall (p,z) \in \mathbb{R}^2.$$

### 1.4.2 Properties of the value function

The problem described in Section 1.4.1 belongs to the class treated in Proposition 1.17. Therefore Theorem 1.11, Corollary 1.12 and Corollary 1.13 hold, but it turns out that in this case we can strengthen such results.

We set for brevity

$$\alpha=\{(t,p,z)\in\mathcal{D} : z=M\},\quad \beta=\{(t,p,z)\in\mathcal{D} : z+\bar{u}(T-t)=m\},\quad \gamma=\{T\}\times\mathbb{R}\times[m,M],$$

so that $\mathcal{D}\setminus\widetilde{\mathcal{D}}=\alpha\cup\beta$.

Let us first consider Theorem 1.11 and adapt it to our problem, as here something about $\mathcal{D}\setminus\widetilde{\mathcal{D}}$ can also be said.

**Proposition 1.19.** *Let the assumptions of Section 1.4.1 hold. The functions* $V^c$ *converge to* $V$ *uniformly on compact subsets of* $\widetilde{\mathcal{D}}$ *. Moreover, if* $(t,p,z)\in\alpha$ *we have* $V(t,p,z)=0$*. Finally, if* $(t,p,z)\in\beta$ *we have*

$$V(t,p,z)=\bar{u}\,\mathbb{E}_{tpz}\left[\int_t^T e^{-r(s-t)}(P^{t,p}(s)-K)ds\right]=:\xi(t,p). \tag{1.62}$$

*Proof.* As for the first part, notice that each compact subset of $\widetilde{\mathcal{D}}$ is contained in some $\mathcal{D}^\rho$ and use Theorem 1.11. Second and third items: in $\alpha\cup\beta$ there exists a unique admissible control, respectively $u\equiv 0$ and $u\equiv\bar{u}$. □

Notice that the boundary condition $\xi$ in (1.62) is continuous and can be computed in many models used in practice (see [5]).

Corollary 1.12 assures continuity of $V$ on $\widetilde{\mathcal{D}}$. We now prove that in this case a stronger result holds, i.e. the value function is continuous on the whole domain $\mathcal{D}$. For this, we first need a technical lemma (see [20, Appendix D] or [27]), where we give a bound for the mean distance between solutions of (1.4) starting from different data.

**Lemma 1.20.** *Let the assumptions of Section 1.4.1 hold. Let* $t_1,t_2\in[0,T]$ *with* $t_1<t_2$ *and* $p_1,p_2\in\mathbb{R}^n$*. Then*

$$\mathbb{E}\big[\big|P^{t_1,p_1}(s)-P^{t_2,p_2}(s)\big|\big]\le M\big[|p_2-p_1|+(t_2-t_1)^{\frac{1}{2}}(1+|p_1|)\big],$$

*for each* $s\in[t_2,T]$*, where* $\mathbb{E}$ *denotes the mean value with respect to the probability* $\mathbb{P}$ *and* $M>0$ *is a constant depending only on* $T,U$ *and on the constants in (1.18) and (1.21).*

**Proposition 1.21.** *Let the assumptions of Section 1.4.1 hold. Then* $V$ *is continuous on* $\mathcal{D}$*.*

*Proof.* As Corollary 1.12 holds, we have to prove that $V$ is continuous on $\mathcal{D}\setminus\widetilde{\mathcal{D}}=\alpha\cup\beta$.

*Step 1: continuity on* $\alpha$. Let $(\tilde{t},\tilde{p},\tilde{z})\in\alpha$. Since in this case the only admissible control is $u\equiv 0$, we have to prove that

$$\lim_{\substack{(t,p,z)\to(\tilde{t},\tilde{p},\tilde{z})\\(t,p,z)\in\mathcal{D}}} V(t,p,z)=V(\tilde{t},\tilde{p},\tilde{z})=0. \tag{1.63}$$

Let $(t,p,z) \in \mathcal{D}$ and $u \in \mathcal{A}_{tz}^{\mathrm{adm}}$. Given arbitrary $\gamma > 0$, we first of all observe that

$$\mathbb{E}_{tpz}\left[\left(\int_t^T |P^{t,p}(s) - K|u(s)ds\right)\mathbb{1}_{\{\|P(\cdot)\| \le \gamma\}}\right] \le (\gamma + K)\mathbb{E}_{tpz}\left[\int_t^T u(s)ds\right] \le (\gamma + K)(M - z), \quad (1.64)$$

where in the last passage we have used condition $Z^u(T) \le M$. By the Hölder inequality (twice), estimates as in (1.27), the Markov inequality and (1.12) we get

$$\mathbb{E}_{tpz}\left[\left(\int_t^T |P^{t,p}(s) - K|u(s)ds\right)\mathbb{1}_{\{\|P^{t,p}(\cdot)\| > \gamma\}}\right] \le T\mathbb{E}_{tpz}\left[\int_t^T (P^{t,p}(s) - K)^2 u(s)^2 ds\right]^{\frac{1}{2}} \mathbb{P}_{tpz}(\|P^{t,p}(\cdot)\| > \gamma)^{\frac{1}{2}} \le C_1(1 + |p|)^{\frac{3}{2}}\gamma^{-\frac{1}{2}}, \quad (1.65)$$

for some constant $C_1 > 0$. By (1.64) and (1.65) it follows that

$$\left|\sup_{u \in \mathcal{A}_{tz}^{\mathrm{adm}}} \mathbb{E}_{tpz}\left[\int_t^T e^{-r(s-t)}(P^{t,p}(s) - K)u(s)ds\right]\right| \le (\gamma + K)(M - z) + C_1(1 + |p|)^{\frac{3}{2}}\gamma^{-\frac{1}{2}}. \quad (1.66)$$

Inequality (1.66) holds for each $\gamma > 0$ and for each $(t,p,z) \in \mathcal{D}$. We get (1.63) by passing to the limit first as $(t,p,z) \to (\tilde{t},\tilde{p},\tilde{z})$ (recall that $\tilde{z} = M$) and then as $\gamma \to \infty$.

*Step 2: continuity on $\beta$.* Let $(\tilde{t},\tilde{p},\tilde{z}) \in \beta$. Since in $\beta$ function $V$ is as in (1.62), we have to prove that

$$\lim_{\substack{(t,p,z) \to (\tilde{t},\tilde{p},\tilde{z}) \\ (t,p,z) \in \mathcal{D}}} V(t,p,z) = V(\tilde{t},\tilde{p},\tilde{z}) = \bar{u}\,\mathbb{E}_{\tilde{t}\tilde{p}\tilde{z}}\left[\int_{\tilde{t}}^T e^{-r(s-\tilde{t})}(P^{\tilde{t},\tilde{p}}(s) - K)ds\right]. \quad (1.67)$$

From now on, we will omit the subscripts in the notation of the mean value (the initial data are different, but the probability is clearly the same). Let $(t,p,z) \in \mathcal{D}$ (notice that necessarily $t \le \tilde{t}$) and fix $u \in \mathcal{A}_{tz}^{\mathrm{adm}}$; for simplicity we will write $P = P^{t,p}$ and $\tilde{P} = P^{\tilde{t},\tilde{p}}$. Since for $s \in [\tilde{t}, T]$ we have

$$\begin{aligned} e^{-r(s-t)}(P(s) - K)u(s) - e^{-r(s-\tilde{t})}(\tilde{P}(s) - K)\bar{u} &= e^{-r(s-t)}(P(s) - \tilde{P}(s))u(s) - e^{-r(s-t)}(\tilde{P}(s) - K)(\bar{u} - u(s)) \\ &\quad - (e^{-r(s-\tilde{t})} - e^{-r(s-t)})(\tilde{P}(s) - K)\bar{u}, \end{aligned}$$

let us first of all observe that

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg|\int_t^T e^{-r(s-t)}(P(s)-K)u(s)ds-\bar u\int_{\tilde t}^T e^{-r(s-\tilde t)}(\tilde P(s)-K)ds\bigg|\bigg]\\
&\leq \mathbb{E}\bigg[\int_t^{\tilde t}|P(s)-K|u(s)ds\bigg]+\mathbb{E}\bigg[\int_{\tilde t}^T\Big|e^{-r(s-t)}(P(s)-K)u(s)-e^{-r(s-\tilde t)}(\tilde P(s)-K)\bar u\Big|ds\bigg]\\
&\leq \mathbb{E}\bigg[\int_t^{\tilde t}|P(s)-K|u(s)ds\bigg]+\mathbb{E}\bigg[\int_{\tilde t}^T|P(s)-\tilde P(s)|u(s)ds\bigg]\\
&\quad+\mathbb{E}\bigg[\int_{\tilde t}^T|\tilde P(s)-K|(\bar u-u(s))ds\bigg]+\mathbb{E}\bigg[\int_{\tilde t}^T\big(e^{-r(s-\tilde t)}-e^{-r(s-t)}\big)|\tilde P(s)-K|\bar u\,ds\bigg].
\end{aligned} \tag{1.68}$$

Consider the first term in (1.68). By estimates as in (1.27) we have that

$$\mathbb{E}\bigg[\int_t^{\tilde t}|P(s)-K|u(s)ds\bigg]\leq C_2(\tilde t-t)(1+|p|), \tag{1.69}$$

for some constant $C_2>0$. As for the second term in (1.68), by the Fubini-Tonelli theorem and Lemma 1.20 we get

$$\mathbb{E}\bigg[\int_{\tilde t}^T|P(s)-\tilde P(s)|u(s)ds\bigg]\leq C_3\big[|p-\tilde p|+(\tilde t-t)^{\frac12}(1+|p|)\big], \tag{1.70}$$

where $C_3>0$ is a constant. Let us now estimate the third term in (1.68). Given arbitrary $\gamma>0$, we observe that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\tilde t}^T|\tilde P(s)-K|(\bar u-u(s))ds\right)\mathbb{1}_{\{\|\tilde P(\cdot)\|\leq\gamma\}}\right]\leq(\gamma+K)\mathbb{E}\left[\int_{\tilde t}^T(\bar u-u(s))ds\right]\\
&\qquad=(\gamma+K)\mathbb{E}\left[\bar u(T-t)-\int_t^T u(s)ds\right]\leq(\gamma+K)\big(\bar u(T-t)-m+z\big),
\end{aligned} \tag{1.71}$$

where in the last passage we have used condition $Z^u(T)\geq m$. By arguing as in (1.65), we get

$$\mathbb{E}\left[\left(\int_{\tilde t}^T|\tilde P(s)-K|(\bar u-u(s))ds\right)\mathbb{1}_{\{\|\tilde P(\cdot)\|>\gamma\}}\right]\leq C_4(1+|\tilde p|)^{\frac32}\gamma^{-\frac12}, \tag{1.72}$$

for some constant $C_4>0$. We finally consider the fourth term in (1.68). By local Lipschitzianity of the exponential function and by estimates as in (1.27) we obtain

$$\mathbb{E}\bigg[\int_{\tilde t}^T\big(e^{-r(s-\tilde t)}-e^{-r(s-t)}\big)|\tilde P(s)-K|\bar u ds\bigg]\leq C_5(\tilde t-t)(1+|\tilde p|), \tag{1.73}$$

where $C_5>0$ is constant.

By estimates from (1.69) to (1.73), it follows from (1.68) that

$$\begin{aligned}
&\bigg|\sup_{u\in\mathcal{A}_{tz}^{\mathrm{adm}}}\mathbb{E}\bigg[\int_t^T e^{-r(s-t)}(P(s)-K)u(s)ds\bigg]-\bar u\,\mathbb{E}\bigg[\int_{\tilde t}^T e^{-r(s-\tilde t)}(\tilde P(s)-K)ds\bigg]\bigg|\\
&\qquad\leq C_2(\tilde t-t)(1+|p|)+C_3\big[|p-\tilde p|+(\tilde t-t)^{\frac12}(1+|p|)\big]+C_5(\tilde t-t)(1+|\tilde p|)\\
&\qquad\qquad+(\gamma+K)\big(\bar u(T-t)-m+z\big)+C_4(1+|\tilde p|)^{\frac32}\gamma^{-\frac12}.
\end{aligned} \tag{1.74}$$

Estimate (1.74) holds for each $\gamma > 0$ and for each $(t, p, z) \in \mathcal{D}$. We get (1.67) by passing to the limit first as $(t, p, z) \to (\tilde{t}, \tilde{p}, \tilde{z})$ (recall that $\tilde{z} + \bar{u}(T - \tilde{t}) = m$) and then as $\gamma \to \infty$. □

Let us now consider the HJB equation and prove a result which is stronger than Corollary 1.13: in this case the value function is, in its whole domain $\mathcal{D}$, the unique viscosity solution of the HJB equation with polynomial growth and the boundary conditions given below. Thus, we get another characterization of the value function, in addition to the one of Proposition 1.19.

**Theorem 1.22.** *Let the assumptions of Section 1.4.1 hold. Then the function $V$ is the unique continuous viscosity solution of Equation (1.61) in the domain $\mathcal{D} \setminus (\alpha \cup \beta \cup \gamma)$, with boundary conditions*

$$\begin{aligned} &V(t, p, z) = 0, && \forall (t, p, z) \in \alpha, \\ &V(t, p, z) = \xi(t, z), && \forall (t, p, z) \in \beta, \\ &V(T, p, z) = 0, && \forall (p, z) \in \mathbb{R} \times [m, M], \end{aligned} \tag{1.75}$$

*such that*

$$|V(t, p, z)| \le \check{C}(1 + |p|^2 + |z|^2), \qquad \forall (t, p, z) \in \mathcal{D}, \tag{1.76}$$

*for some constant $\check{C} > 0$.*

*Proof.* In this problem, $k = 2$ in (1.26). Thus, by (1.26), Corollary 1.13 and Proposition 1.19, the function $V$ is a viscosity solution of problem (1.61)-(1.75)-(1.76). Moreover it satisfies in viscosity sense the boundary conditions (see [15]).

We now need a uniqueness result. By the following change of variables

$$t' = t, \qquad p' = p, \qquad z' = \frac{z - M}{M - m + \bar{u}(T - t)} + 1,$$

problem (1.61)-(1.75) becomes

$$\begin{aligned} - V_{t'}(t', p', z') &- \frac{\bar{u}(z' - 1)}{M - m + \bar{u}(T - t')} V_{z'}(t', p', z') \\ &+ rV(t', p', z') - f(t', p') V_{p'}(t', p', z') - \frac{1}{2}\sigma^2(t', p') V_{p'p'}(t', p', z') \\ &+ \min_{v \in [0, \bar{u}]} \left[ -v \left( \frac{1}{M - m + \bar{u}(T - t')} V_{z'}(t', p', z') + p' - K \right) \right] = 0, \\ &\qquad\qquad\qquad\qquad \forall (t', p', z') \in [0, T[ \times \mathbb{R} \times ]0, 1[, \end{aligned}$$

with boundary condition

$$\begin{aligned} &V(T, p', z') = 0, && \forall (p', z') \in \mathbb{R} \times [0, 1], \\ &V(t', p', 1) = 0, && \forall (t', p') \in [0, T] \times \mathbb{R}, \\ &V(t', p', 0) = \xi(t', p'), && \forall (t', p') \in [0, T] \times \mathbb{R}. \end{aligned}$$

Moreover the polynomial growth (1.76) is preserved and the domain is $[0, T[ \times \mathbb{R} \times ]0, 1[$. We can adapt to the case of bounded domain the comparison principle stated in [16, Thm. 2.1], which is based on the standard argument of doubling the variables in viscosity solution theory. This argument is easily extended to deal with boundary conditions in the viscosity sense. From this we get uniqueness of the solution. □

The previous result generalizes an analogous result in [5], valid in the case $m = 0$. We now turn to prove some properties of the value function with respect to the variables $p$ and $z$. As for $V(t, \cdot, z)$, Propositions 1.15 and 1.16 hold.

**Proposition 1.23.** *Let the assumptions of Section 1.4.1 hold. Let $(t, z) \in [0, T] \times \mathbb{R}$ be such that $(t, p, z) \in \mathcal{D}$ for each $p \in \mathbb{R}$. Then*

- *the function $V(t, \cdot, z)$ is Lipschitz continuous, uniformly in $(t, z)$. Moreover, the derivative $V_p(t, p, z)$ exists for a.e. $(t, p, z) \in \mathcal{D}$ and we have $|V_p(t, p, z)| \leq M_1$, for some constant $M_1 > 0$ depending only on $T$, $\bar{u}$ and on the constants in (1.18), (1.21) and (1.46).*
- *if $f(s, \cdot), \sigma(s, \cdot) \in C_b^2(\mathbb{R})$, uniformly in $s \in [0, T]$, the function $V(t, \cdot, z)$ is locally semiconvex, uniformly in $t$, and a.e. twice differentiable.*

*Proof.* The first part follows from Proposition 1.15 (notice that function $p \mapsto (p - K)v$ is Lipschitz continuous). As for the second item, it suffices to rewrite Proposition 1.16 (notice that the function $p \mapsto (p - K)v$ is of class $C^\infty(\mathbb{R})$, with bounded derivatives). □

Let us now consider the function $V(t, p, \cdot)$. Recall that its domain is $[m - \bar{u}(T - t), M]$.

**Proposition 1.24.** *Let the assumptions of Section 1.4.1 hold. For each $(t, p) \in [0, T] \times \mathbb{R}$ the function $V(t, p, \cdot)$ is*

- *concave, Lipschitz continuous and a.e. twice differentiable;*
- *non-decreasing in $[m - (T - t)\bar{u}, M - (T - t)\bar{u}]$ and non-increasing in $[m, M]$. In particular, if $M - (T - t)\bar{u} \geq m$ then the function $V(t, p, \cdot)$ is constant in $[m, M - (T - t)\bar{u}]$ (they all are maximum points).*

*Proof. Item 1* (this is an adaptation of [5, Prop. 3.4], which takes into account only an upper bound on $Z^{t,z;u}(T)$). Let $(t, p) \in [0, T] \times \mathbb{R}$, $z_1, z_2 \in [m - \bar{u}(T - t), M]$, $u_1 \in \mathcal{A}_{tz_1}^{\text{adm}}$ e $u_2 \in \mathcal{A}_{tz_2}^{\text{adm}}$. By (1.13) the process $(u_1 + u_2)/2$ belongs to the set of admissible controls for initial point $(t, (z_1 + z_2)/2)$. By the linearity of the function $v \mapsto (P^{t,p}(s) - K)v$ we have

$$\frac{J(t, p, z_1; u_1) + J(t, p, z_2; u_2)}{2} = J\left(t, p, \frac{z_1 + z_2}{2}; \frac{u_1 + u_2}{2}\right) \leq V\left(t, p, \frac{z_1 + z_2}{2}\right). \tag{1.77}$$

Since (1.77) holds for each $u_1 \in \mathcal{A}_{tz_1}^{\text{adm}}$ and $u_2 \in \mathcal{A}_{tz_2}^{\text{adm}}$, it follows that

$$\frac{V(t, p, z_1) + V(t, p, z_2)}{2} \leq V\left(t, p, \frac{z_1 + z_2}{2}\right),$$

which implies the concavity of the function $V(t, p, \cdot)$. Local Lipschitzianity is a well-known property of concave functions (and here the domain is a compact set), while the a.e. existence of the second derivative follows from the Alexandrov theorem.

*Item 2.* Let $(t, p) \in [0, T] \times \mathbb{R}$. If $m \leq z_1 \leq z_2 \leq M$, it is easy to check that $\mathcal{A}_{tz_2}^{\text{adm}} \subseteq \mathcal{A}_{tz_1}^{\text{adm}}$, so that $V(t, p, z_1) \geq V(t, p, z_2)$. Similarly, if $m - (T - t)\bar{u} \leq z_1 \leq z_2 \leq M - (T - t)\bar{u}$, we have $\mathcal{A}_{tz_1}^{\text{adm}} \subseteq \mathcal{A}_{tz_2}^{\text{adm}}$ and then $V(t, p, z_2) \leq V(t, p, z_1)$. The second part immediately follows, since $[m - (T - t)\bar{u}, M - (T - t)\bar{u}] \cap [m, M] = [m, M - (T - t)\bar{u}]$. □

The monotonicity result in Proposition 1.24 is described in Figure 1.3.

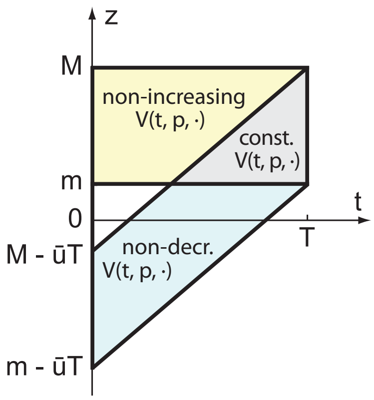


Figure 1.3: monotonicity of $V(t, p, \cdot)$

As in Section 1.2, it was foreseeable that the function $V(t, p, \cdot)$ is constant in an interval: if $M - (T - t)\bar{u} \geq m$ and $z \in [m, M - (T - t)\bar{u}]$ then $\mathcal{A}_{tz}^{\text{adm}} = \mathcal{A}_t$ (i.e. all controls satisfies the constraint), which implies that the initial value $z$ does not influence the value function. This generalizes an intuitive result in [5, Lemma 3.2]: for $(t, z)$ such that the volume constraint is *de facto* absent, the value function $V$ does not depend on $z$.

Finally, also in this case Remark 1.4 holds: by Proposition 1.24 the candidate in (1.16-1.17) is well-defined.

## 1.5 Conclusions

We characterize the value of swing contracts in continuous time as the unique viscosity solution of a Hamilton-Jacobi-Bellman equation with suitable boundary conditions. More in details, swings can be divided in two broad contract classes, those with penalties on the cumulated quantity of energy $Z(T)$ at the end $T$ of the contract, and those with strict constraints on the same quantity: usually these constraints and penalties are meant to make $Z(T)$ belong to an interval $[m, M]$ with $m > 0$ (in real contracts usually $m > 0.8M$, see [28]).

In Section 2 we treat the case of contracts with penalties, which results in a straightforward application of classical optimal control theory, and in that case only a terminal condition is needed. For swing contracts with penalties, we prove that their value is the unique viscosity solution of the HJB equation (1.8), and that is Lipschitz both in $p$ (spot price of energy) as in $z$ (current cumulated quantity), with first weak derivatives with sublinear growth. We also prove that the value function is also concave with respect to $z$, non-increasing for $z \leq M - (T - t)\bar{u}$, where $t$ is the current time and $\bar{u}$ is the maximum marginal energy that can be purchased, and non-decreasing for $z \geq m$. In this, we extend and generalize previous results of [5], which were proved only for swing contracts with strict penalties. These results make the candidate optimal exercise policy in Equations (1.16–1.17) well defined.

Conversely, the case of contracts with strict constraints gives rise to a stochastic control problem with a nonstandard state constraint in $Z(T)$. In Section 3 we approach a suitable generalization of this problem by a penalty method: we consider a general constrained problem and approximate the value function with a sequence of value functions of appropriate unconstrained problems with a penalization term in the objective functional, showing that they converge uniformly on compact sets to the value function of the constrained problem.

In Section 4 we come back to the case of swing contracts with strict constraints: in this case the penalty functions used in Section 3 turn out to be penalties of suitable swing contracts, so that we also have the economic interpretation that a swing contract with strict constraints can be approximated by swing contracts with suitable penalties. In this context we succeed in strengthening the results of Section 3, by characterizing the value function as the unique viscosity solution with polynomial growth of the HJB equation (1.8) subject to the boundary conditions in Equation (1.75). As for the smoothness of the value function with respects to $p$ and $z$, we find exactly the same results as in Section 2, extending previous results of [5] to the case $m > 0$. These results make the candidate optimal exercise policy in Equations (1.16–1.17), i.e. the same as in Section 2, again well defined.

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# Chapter 2

# Optimal price management in retail energy markets

## 2.1 Introduction

In energy markets, retailers first buy energy on the wholesale market and then resell it to final consumers. The wholesale price is assumed to be a continuous-time process, as energy is traded almost instantaneously. Conversely, it is reasonable to model the final price by a piecewise-constant process, since all the customers have to be informed before each price adjustment, due to specific clauses in the contracts. The retailer has to decide when and how to intervene to change the price he asks, in order to maximize his earnings; in this chapter we address this problem by means of impulse control theory.

When setting the final price to ask to his customers, a retailer has to consider several elements. For example, he has to decide if he prefers to set a high price, so as to have high unitary incomes from a small market share, or to keep low prices, so as to have many customers but low unitary incomes. Moreover, he has to consider the operational costs, which are proportional to the market share and can be greater than the incomes from selling energy, to the point that acquiring new customers may correspond to a loss. Finally, if the retailer realizes that the price he is currently asking is too low, he can clearly intervene and raise it, but this implies the payment of a possibly high intervention cost, so that he has to carefully decide whether it is actually worth intervening.

More in detail, we denote by $X_t$ the spread between the final price and the wholesale price of the commodity (i.e. the unitary income of the retailer when selling energy) and we assume the market share to be a function of $X_t$, which we denote by $\Phi = \Phi(X_t)$. The retailer's payoff consists in the income from the sale of energy and in the operational cost to be paid, here assumed to be a quadratic function of $\Phi(X_t)$. Moreover, we assume the intervention penalty to be the sum of a fixed part and a variable part, the latter being directly proportional to the market share. Hence, if $u = \{(\tau_k, \delta_k)\}_{k \geq 1}$ denotes the retailer's intervention policy (the intervention times and the corresponding shifts in the price process), we deal with the following impulse control problem:

$$\sup_u \mathbb{E}_x \left[ \int_0^\infty e^{-\rho t} \left( X_t \Phi(X_t) - \frac{b}{2} \Phi(X_t)^2 \right) dt - \sum_{1 \le k < M} e^{-\rho \tau_k} \left( c + \lambda \Phi\big(X_{(\tau_k)^-}\big) \right) \right].$$

Our procedure to study this problem is the following one: we start from the corresponding quasi-variational problem (classical results are here recalled in Section 2.2.2), we build a candidate and we finally apply the verification theorem: we give a semi-explicit expression for the value function and we characterize the optimal controls, provided that a solution to a system of algebraic equations actually exists. In the particular case where the penalty is constant and the process is a scaled Brownian motion, we can prove an existence result for the system and provide some properties of the limit case $c \to 0^+$. In particular, we prove an asymptotic estimate for the continuation region; to the best of our knowledge, this is the first time that such an estimate is provided for an impulse control problem.

We separately consider the case with fixed penalty (i.e. $\lambda = 0$) and the case with variable penalties (i.e. $\lambda > 0$). In the first case we can use standard results to tackle the problem, whereas in the general case an approximation procedure will be necessary, as the penalty function presents some singularities which prevent us from applying the standard theorems. In the latter case the procedure still presents some open problems and we present the current state of our work; however, we also show how the problem can be circumvented by imposing stronger conditions on the controls.

Impulsive problems represent quite a recent development in control theory. The main advantage of impulse controls, which consist in discrete-time interventions by the controller, is that they provide models which can be, in some cases, closer to reality with respect to classical controls, which are based on continuous-time interventions. A complete introduction to impulse control theory is [32, Chap. 6]. Classical examples of impulsive problems are [11], [22], [25], [26], [31] and [35]. More recent papers are [9], [10] and [33].

The structure of this chapter is as follows. In Section 2.2 we give a precise definition of the problem and recall the formulation of the classical verification theorem. In Section 2.3 we study the problem in the case with constant penalty, whereas in Section 2.4 we outline the open problem of adapting the procedure if a variable penalty is also present.

## 2.2 The price management problem

We here describe the problem we are going to consider (Section 2.2.1) and recall the classical verification theorem for impulsive control problems (Section 2.2.2).

### 2.2.1 Formulation of the problem

Let us consider a retailer who buys energy (electricity, gas, gasoline) on the wholesale market and resells it to final consumers. We address the problem of investigating the retailer's optimal strategy in setting the final price and we model it as an impulsive stochastic control problem.

As anticipated, the retailer buys the commodity in the wholesale market. We assume that the continuous-time price of the commodity is modelled by a Brownian motion with drift:

$$S_t = s + \mu t + \sigma W_t, \tag{2.1}$$

for $t \geq 0$, where $s \in ]0, +\infty[$ is the initial price and $\mu \geq 0, \sigma > 0$ are fixed constants. Notice that the retailer has no control on the wholesale price: in most of the cases, i.e. when the company is not too big, this is a reasonable assumption. After buying the energy, the retailer sells it to final consumers. According to the most common contracts in energy markets, the retailer can change the price only after a written communication to all his customers. Then, we model the final price by a piecewise-constant process $P$. More precisely, we consider an initial price $p > 0$ and a sequence $\{\tau_k\}_{k\geq 1}$ of non-negative random times, which correspond to the retailer's interventions to adjust the price and move $P$ to a new state. If we denote by $\{\delta_k\}_{k\geq 1}$ the corresponding impulses, i.e. $\delta_k = P_{\tau_k} - P_{(\tau_k)^-}$, we have

$$P_t = p + \sum_{\tau_k \leq t} \delta_k, \tag{2.2}$$

for every $t \geq 0$. Let us denote by $X$ the difference between the final price and the wholesale price. In other words, $X$ represents the retailer's unitary income when selling energy (we do not consider, for the moment, the operational costs he faces). By (2.1) and (2.2), we have

$$X_t = P_t - S_t = x - \mu t - \sigma W_t + \sum_{\tau_k \leq t} \delta_k, \tag{2.3}$$

for every $t \geq 0$, where we have set $x = p - s$. We remark that, when the player does not intervene, the process $X$ satisfies the following stochastic equation:

$$dX_t = -\mu dt - \sigma dW_t. \tag{2.4}$$

We assume that the retailer's market share at time $t \geq 0$ is a function of $X_t$, which we denote by $\Phi = \Phi(X_t)$. In our model, we set

$$\Phi(x) = \begin{cases} 1, & x \leq 0, \\ -\frac{1}{\Delta}(x - \Delta), & 0 < x < \Delta, \\ 0, & x \geq \Delta, \end{cases} \tag{2.5}$$

for every $x \in \mathbb{R}$, where $\Delta > 0$ is a fixed constant. In other words, the market share is a truncated linear function of $X_t$ with two thresholds: if $X_t \leq 0$ all the customers buy energy from the retailer, whereas if $X_t \geq \Delta$ the retailer has lost all his customers.

At each $t \geq 0$, the retailer's income from the sell of the energy is given by $X_t \Phi(X_t)$, but he also has to pay an operational cost, which we assume to be a quadratic function of the market share $\Phi(X_t)$; hence, the instantaneous payoff is given by

$$R(x) = x\Phi(x) - \frac{b}{2}\Phi(x)^2, \tag{2.6}$$

where $x$ is the current state of the process. Moreover, there is a penalty to pay when the retailer intervenes to adjust $P$. We are going to consider two situations: the case when the penalty simply consists in a fixed cost $c$ and the case where, besides $c$, we also have a variable cost, directly proportional to the market share. To include both the cases in the same notation, we set

$$K(x) = c + \lambda\Phi(x), \tag{2.7}$$

where $x \in \mathbb{R}$ is the state of the process before the intervention and $c > 0, \lambda \geq 0$ are fixed constants. Finally, let $\rho > 0$ be the discount rate.

To sum up, we here consider the following impulsive stochastic control problem.

**Definition 2.1.** *Throughout the chapter, the following definitions hold.*

- *A control is a sequence $u = \{(\tau_k, \delta_k)\}_{1\leq k<M}$, where $M \in \mathbb{R}\cup\{+\infty\}$, $\{\tau_k\}_{1\leq k<M}$ are non-decreasing non-negative stopping times (the intervention times) and $\{\delta_k\}_{1\leq k<M}$ are real random variables (the corresponding impulses).*
- *For each $x \in \mathbb{R}$ and $u$ control, we denote by $X^{x;u}$ the process defined in* (2.3)*.*
- *Let $x \in \mathbb{R}$ and $K$ as in* (2.7)*; we say that a control $u$ is admissible in $x$ if*

$$\mathbb{E}_x\Bigg[\sum_{1\leq k<M} e^{-\rho\tau_k} K\big(X^{x;u}_{(\tau_k)^-}\big)\Bigg] < \infty \tag{2.8}$$

*and we denote by $\mathcal{U}_x$ the set of the controls which are admissible in $x$.*

**Definition 2.2.** *The function $V$ (value function) is defined, for each $x \in \mathbb{R}$, by*

$$V(x) = \sup_{u\in\mathcal{U}_x} J(x;u),$$

*where, for every $u \in \mathcal{U}_x$, we have set*

$$J(x;u) = \mathbb{E}_x\Bigg[\int_0^\infty e^{-\rho t} R(X^{x;u}_t)dt - \sum_{1\leq k<M} e^{-\rho\tau_k} K\big(X^{x;u}_{(\tau_k)^-}\big)\Bigg], \tag{2.9}$$

*where the functions $R$ and $K$ have been defined in* (2.6) *and* (2.7)*. If there exists $u^* \in \mathcal{U}_x$ such that $V(x) = J(x;u^*)$, we say that $u^*$ is an optimal control in $x$.*

Notice that the functional $J$ in (2.9) is well-defined, as the sum of the penalties is integrable by (2.8). Since $R$ is bounded, a corresponding integrability condition on the payoff is not necessary. To shorten the notations, we will often omit the dependence on the control and simply write $X$.

We conclude this section with some remarks about the payoff and the penalty of our problem: these properties will be useful in the next sections.

- An explicit expression for the running cost $R$ and the penalty $K$ is

$$R(x) = \begin{cases} x - b/2, & \text{if } x < 0, \\ f(x), & \text{if } 0 \leq x \leq \Delta, \\ 0, & \text{if } x > \Delta, \end{cases} \qquad K(x) = \begin{cases} \lambda + c, & \text{if } x < 0, \\ -\frac{\lambda}{\Delta}x + \lambda + c, & \text{if } 0 \leq x \leq \Delta, \\ c, & \text{if } x > \Delta, \end{cases}$$

for every $x \in \mathbb{R}$, where we have set

$$f(x) = -\alpha x^2 + \beta x - \gamma, \qquad \alpha = \frac{1}{\Delta} + \frac{b}{2\Delta^2}, \qquad \beta = 1 + \frac{b}{\Delta}, \qquad \gamma = \frac{b}{2}. \quad (2.10)$$

In particular, we remark that we have $R(x) \geq f(x)$, for every $x \in \mathbb{R}$.

- The function $f$ in (2.10) is a concave parabola:

$$f(x) = -\alpha(x - x_v)^2 + y_v, \quad (2.11)$$

where $\alpha$ is as in (2.10) and the vertex $v = (x_v, y_v)$ has the following expression:

$$x_v = \frac{\Delta(\Delta + b)}{2\Delta + b}, \qquad y_v = f(x_v) = \frac{\Delta^2}{2(2\Delta + b)}. \quad (2.12)$$

From the retailer's point of view, Equation (2.11) says that $x_v$ is the state which maximizes the payoff $R(x)$, the optimal income being $y_v$. Notice that the optimal share $\Phi_v = \Phi(x_v)$ is given by

$$\Phi_v = \Phi(x_v) = \frac{\Delta}{2\Delta + b}. \quad (2.13)$$

In particular, if $b = 0$ the optimal share is $1/2$.

- Moreover, we notice that

$$f(x) \geq 0 \text{ if and only if } x \in [x_z, \Delta], \text{ where } x_z = \frac{b\Delta}{2\Delta + b}. \quad (2.14)$$

Equivalently, the payoff $R(X_t)$ is positive if and only if $X_t \in [x_z, \Delta]$. In other words, if we want the income from the sale of energy to be higher than the operational costs, we need the spread between the wholesale price and the final price to be greater than $x_z$.

- Finally, if we consider $x_v, y_v, x_z, \Phi_v$ as functions of $b$, we notice that

$$\begin{array}{llll} x_v(b) \in [\Delta/2, \Delta[, & x_v(0) = \Delta/2, & x_v(+\infty) = \Delta, & x_v' > 0, \\ y_v(b) \in ]0, \Delta/4], & y_v(0) = \Delta/4, & y_v(+\infty) = 0, & y_v' < 0, \\ x_z(b) \in ]0, \Delta[, & x_z(0) = 0, & x_z(+\infty) = \Delta, & x_z' > 0, \\ \Phi_v(b) \in ]0, 1/2[, & \Phi_v(0) = 1/2, & \Phi_v(+\infty) = 0, & \Phi_v' < 0. \end{array} \quad (2.15)$$

Some intuitive properties of the model are formalized in (2.15): as the operational costs increases, the optimal spread $x_v$ increases, the maximal instantaneous income $y_v$ decreases, the region where the payoff is positive gets smaller and the optimal share decreases. In particular, we remark that $\Phi_v \in ]0, 1/2[$: for any value of $b$, it is never optimal to have a market share greater than $1/2$.

### 2.2.2 Classical verification theorem

In control theory, verification theorems provide sufficient conditions for the value function by considering suitable differential problems. The main drawback of such theorems is that they require strong regularity assumptions.

We now recall the statement, in our particular case, of the classical verification theorem for impulsive stochastic control problems.

**Definition 2.3.** *Let $V$ be a function from $\mathbb{R}$ to $\mathbb{R}$ with $\sup V \in \mathbb{R}$. The function $\mathcal{M}V$ is defined, for every $x \in \mathbb{R}$, by*

$$\mathcal{M}V(x) = \sup_{\delta\in\mathbb{R}}\{V(x+\delta) - c - \lambda\Phi(x)\} = \sup V - c - \lambda\Phi(x). \tag{2.16}$$

**Proposition 2.4 (Verification Theorem).** *Let the assumptions and notations of Section 2.2.1 hold. Let $V$ be a function from $\mathbb{R}$ to $\mathbb{R}$ satisfying the following conditions:*

- *$V$ is bounded and there exists $x^* \in \mathbb{R}$ such that $V(x^*) = \max_{x\in\mathbb{R}} V(x)$;*
- *$D = \{\mathcal{M}V - V < 0\}$ is a finite union of intervals;*
- *$V \in C^2(\mathbb{R}\setminus\partial D)\cap C^1(\mathbb{R})$ and the second derivative of $V$ is bounded near $\partial D$;*
- *$V$ is a solution to*

$$\max\{\mathcal{A}V - \rho V + R, \mathcal{M}V - V\} = 0, \tag{2.17}$$

*where $\mathcal{A}V = (\sigma^2/2)V'' - \mu V'$ is the generator associated to Equation (2.4).*

*Let $x \in \mathbb{R}$ and let $u^*(x) = \{(\tau_k^*(x), \delta_k^*(x))\}_{1\le k<\infty}$, where the variables $(\tau_k^*, \delta_k^*)$ (we omit the dependence on $x$ to shorten the notations) are recursively defined by*

$$\tau_k^* = \inf\left\{t > \tau_{k-1}^* : (\mathcal{M}V - V)\big(X_t^{x;u_k^*}\big) = 0\right\},$$
$$\delta_k^* = x^* - X_{\tau_k^*}^{x;u_k^*},$$

*for $k \ge 1$, where we have set $\tau_0^* = \delta_0^* = 0$ and $u_k^*(x) = \{(\tau_j^*, \delta_j^*)\}_{1\le j\le k}$. Assume that $u^*(x) \in \mathcal{U}_x$. Then,*

$$u^*(x) \text{ is an optimal control in } x \text{ and } V(x) = J(x; u^*(x)).$$

*Proof.* See [32, Thm. 6.2]. □

Practically, when dealing with a control problem, one first guesses the form of the continuation region and gets a candidate for the value function by solving Equation (2.17); the final step consists in trying to actually apply the verification theorem to such candidate.

**Remark 2.5.** *If the parameter $c$ is very high, the retailer may lose all his customers without intervening, as the intervention cost would be higher than the loss he is experiencing. However, such situation is clearly not practically admissible (if the costs are too big, a retailer does not even enter the market). So, in order to keep the* *model close to reality, we will always require the continuation region to be a subset of* $]0,\Delta[$*:*

$$\{\mathcal{M}V - V < 0\} \subseteq ]0,\Delta[. \tag{2.18}$$

*As a consequence, when dealing with the continuation region, we can consider* $R_{|]0,\Delta[} = f$ *as the running cost of the problem (clearly, we cannot substitute* $R$ *with* $f$ *in* (2.17)*, as such equation must hold for each* $x \in \mathbb{R}$*).*

The key-stone of Proposition 2.4 is Equation (2.17), which implies

$$\mathcal{A}V - \rho V + R = 0, \qquad \text{in } \{\mathcal{M}V - V < 0\}.$$

We now provide an explicit solution to such equation. By (2.18) we can replace $R$ with $f$; hence, we are interested in solving

$$\mathcal{A}\varphi - \rho\varphi + f = \frac{\sigma^2}{2}\varphi'' - \mu\varphi' - \rho\varphi + f = 0. \tag{2.19}$$

The general solution to (2.19) is given by

$$\varphi_{A_1,A_2}(x) = A_1 e^{m_1 x} + A_2 e^{m_2 x} - k_2 x^2 + k_1 x - k_0, \tag{2.20}$$

where $A_1, A_2 \in \mathbb{R}$ and we have set

$$\begin{gathered} m_{1,2} = \frac{\mu \pm \sqrt{\mu^2 + 2\rho\sigma^2}}{\sigma^2}, \\ k_2 = \frac{\alpha}{\rho}, \qquad k_1 = \frac{\beta}{\rho} + \frac{2\alpha\mu}{\rho^2}, \qquad k_0 = \frac{\gamma}{\rho} + \frac{\beta\mu + \alpha\sigma^2}{\rho^2} + \frac{2\alpha\mu^2}{\rho^3}, \end{gathered} \tag{2.21}$$

with $\alpha, \beta, \gamma$ as in (2.10). Notice that, when $\mu = 0$, we have

$$-k_2 x^2 + k_1 x - k_0 = \frac{f(x)}{\rho} - \frac{\alpha\sigma^2}{\rho^2}.$$

Hence, the polynomial part in (2.20) is, in this case, a concave parabola with vertex in $x_v$, with $x_v$ as in (2.12); as a consequence, by (2.11) we also have the following representation:

$$\varphi_{A_1,A_2}(x) = A_1 e^{\theta x} + A_2 e^{-\theta x} - k_2 (x - x_v)^2 + k_3, \tag{2.22}$$

where, to shorten the notations, we have set

$$\theta = \sqrt{\frac{2\rho}{\sigma^2}}, \qquad k_3 = \frac{f(x_v)}{\rho} - \frac{\alpha\sigma^2}{\rho^2} = \frac{f(x_v)}{\rho} - \frac{2k_2}{\theta^2}. \tag{2.23}$$

We underline that the representation in (2.22) holds only in the case $\mu = 0$.

## 2.3 The case with fixed penalty

In this section we consider the problem in Section 2.2 in the case of fixed intervention costs, i.e. with $\lambda = 0$. In particular, the problem now reads

$$V(x) = \sup_{u \in \mathcal{U}_x} J(x;u) = \sup_{u \in \mathcal{U}_x} \mathbb{E}_x \left[ \int_0^\infty e^{-\rho t} \left( X_t \Phi(X_t) - \frac{b}{2}\Phi(X_t)^2 \right) dt - c \sum_k e^{-\rho\tau_k} \right].$$

We will show that the classical verification theorem (Proposition 2.4 above) can be applied, so that a semi-explicit expression for the value function and the optimal control is possible.

More in detail, in Section 2.3.1 we consider the case $\mu = 0$ and build a candidate $\tilde{V}$ for the value function $V$, whereas in Section 2.3.2 we apply Proposition 2.4 to such candidate; finally, in Section 2.3.3 we consider the case with a non-zero $\mu$.

### 2.3.1 Looking for a candidate for the value function

We here consider the following case:

$$\lambda = 0, \qquad \mu = 0, \qquad c \leq \bar{c}, \tag{2.24}$$

where $\bar{c}$ will be specified later. Since our goal is to use the Verification Theorem 2.4, we first try to find a solution to (2.17), in order to get a candidate $\tilde{V}$ for $V$.

It is reasonable to assume that the retailer's continuation region (i.e. when he does not intervene) is in the form $C = ]\underline{x}, \bar{x}[$ and included in $]0, \Delta[$ by (2.18). As a consequence, the real line is heuristically divided into:

$$\mathbb{R} \backslash ]\underline{x}, \bar{x}[ = \{\mathcal{M}V - V = 0\}, \text{ where the retailer intervenes,}$$
$$]\underline{x}, \bar{x}[ = \{\mathcal{M}V - V < 0\}, \text{ where the retailer does not intervene.}$$

Then, the QVI problem in (2.17) suggests the following candidate for $V$:

$$\tilde{V}(x) = \begin{cases} \varphi(x), & \text{if } x \in ]\underline{x}, \bar{x}[, \\ \mathcal{M}\tilde{V}(x), & \text{if } x \in \mathbb{R} \backslash ]\underline{x}, \bar{x}[, \end{cases}$$

where $\varphi$ is a solution to the equation (recall that $]\underline{x}, \bar{x}[ \subseteq ]0, \Delta[$, where $R = f$)

$$\mathcal{A}\varphi - \rho\varphi + f = 0,$$

and the function $\mathcal{M}\tilde{V}$ (see Definition 2.16) is given by

$$\mathcal{M}\tilde{V}(x) = \sup_{\delta \in \mathbb{R}} \{\tilde{V}(x + \delta) - c\} = \sup_{y \in \mathbb{R}} \{\tilde{V}(y)\} - c.$$

Heuristically, it is reasonable to assume that the function $\tilde{V}$ has a unique maximum point $x^*$, which belongs to the continuation region $]\underline{x}, \bar{x}[$ (where $\tilde{V} = \varphi$):

$$\max_{y \in \mathbb{R}} \{\tilde{V}(y)\} = \max_{y \in ]\underline{x}, \bar{x}[} \{\varphi(y)\} = \varphi(x^*), \qquad \text{where} \qquad \varphi'(x^*) = 0, \;\; \varphi''(x^*) \leq 0, \;\; \underline{x} < x^* < \bar{x}.$$

We recall that an explicit formula for $\varphi$ has been provided in Section 2.2.2: in particular, since we are considering the case $\mu = 0$, we can use the formula in (2.22). Moreover, we recall that the parameters in $\tilde{V}$ must be chosen so as to satisfy the regularity assumptions of the verification theorem: $\tilde{V}$ has to be continuous and differentiable in $\underline{x}, \bar{x}$. To sum up, the candidate is as follows.

**Definition 2.6.** *For every $x \in \mathbb{R}$, we set*

$$\tilde{V}(x) = \begin{cases} \varphi_{A_1, A_2}(x), & \text{in } ]\underline{x}, \bar{x}[, \\ \varphi_{A_1, A_2}(x^*) - c, & \text{in } \mathbb{R} \backslash ]\underline{x}, \bar{x}[, \end{cases}$$

*where* $\varphi_{A_1,A_2}$ *is as in* (2.22) *and the five parameters* $(A_1, A_2, \underline{x}, \bar{x}, x^*)$ *satisfy*

$$0 < \underline{x} < x^* < \bar{x} < \Delta, \tag{2.25}$$

*and the following conditions:*

$$\begin{cases} \varphi'_{A_1,A_2}(x^*) = 0 \text{ and } \varphi''_{A_1,A_2}(x^*) < 0, & \text{(optimality of } x^*\text{)} \\ \varphi'_{A_1,A_2}(\underline{x}) = 0, & (C^1\text{-pasting in } \underline{x}) \\ \varphi'_{A_1,A_2}(\bar{x}) = 0, & (C^1\text{-pasting in } \bar{x}) \\ \varphi_{A_1,A_2}(\underline{x}) = \varphi_{A_1,A_2}(x^*) - c, & (C^0\text{-pasting in } \underline{x}) \\ \varphi_{A_1,A_2}(\bar{x}) = \varphi_{A_1,A_2}(x^*) - c. & (C^0\text{-pasting in } \bar{x}) \end{cases} \tag{2.26}$$

In order to have a well-posed definition, we first need to prove that a solution to (2.26) actually exists.

Since the system cannot be solved directly, we try to make some guesses to simplify it. Consider the structure of the problem: the running cost is symmetric with respect to $x_v$ (see Section 2.2.1), the penalty is constant (as $\lambda = 0$ here), the uncontrolled process is a scaled Brownian motion (recall that $\mu = 0$). Then, we expect the value function to be symmetric with respect to $x_v$, which corresponds to the choice $A_1 e^{\theta x_v} = A_2 e^{-\theta x_v}$. The same argument suggests to set $(\underline{x} + \bar{x})/2 = x_v$. Finally, as a symmetry point is always a local maximum or minimum point, we expect $x^* = x_v$. In short, our guess is

$$A_1 = Ae^{-\theta x_v}, \qquad A_2 = Ae^{\theta x_v}, \qquad (\underline{x} + \bar{x})/2 = x_v, \qquad x^* = x_v, \tag{2.27}$$

with $A \in \mathbb{R}$. In particular, we now consider functions in the form

$$\varphi_A(x) = Ae^{\theta(x-x_v)} + Ae^{-\theta(x-x_v)} - k_2(x-x_v)^2 + k_3,$$

where $A \in \mathbb{R}$ and the coefficients have been defined in (2.21) and (2.23).

Indeed, an easy check shows that $x^* = x_v$ is a local maximum for $\varphi_A$ (so that the first condition in (2.26) is satisfied) if and only if $A > 0$. Then, under our guess (2.27), we can equivalently rewrite (2.26) as

$$\begin{cases} \varphi'_A(\bar{x}) = 0, \\ \varphi_A(\bar{x}) = \varphi_A(x_v) - c, \end{cases}$$

with $A > 0$ and $\bar{x} > x_v$. Explicitly, we have to solve

$$\begin{cases} A\theta e^{\theta(\bar{x}-x_v)} - A\theta e^{-\theta(\bar{x}-x_v)} - 2k_2(\bar{x}-x_v) = 0, \\ Ae^{\theta(\bar{x}-x_v)} + Ae^{-\theta(\bar{x}-x_v)} - k_2(\bar{x}-x_v)^2 - 2A + c = 0. \end{cases}$$

In order to simplify the notations, we operate a change of variable and set $\bar{y} = \bar{x} - x_v$. We now deal with

$$\begin{cases} A\theta e^{\theta\bar{y}} - A\theta e^{-\theta\bar{y}} - 2k_2\bar{y} = 0, & \text{(2.28a)} \\ Ae^{\theta\bar{y}} + Ae^{-\theta\bar{y}} - k_2\bar{y}^2 - 2A + c = 0, & \text{(2.28b)} \end{cases}$$

where $A > 0$ and $\bar{y} > 0$. Finally, recall the order condition (2.25), which now reads

$$\bar{y} < \Delta - x_v. \tag{2.29}$$

So, to prove that $\tilde{V}$ is well-defined it is enough to show that a solution to (2.28a)-(2.28b)-(2.29) exists and is unique.

In Lemma 2.7 we focus on the first two conditions, whereas in Lemma 2.8 we consider the third one.

**Lemma 2.7.** *A solution* $(A, \bar{y}) \in\, ]0, +\infty[^2$ *to* (2.28a)-(2.28b) *exists and is unique.*

*Proof. First step.* Let us start by Equation (2.28a). For a fixed $A > 0$, we are looking for the strictly positive zeros of the function $h_A$ defined by

$$h_A(y) = A\theta e^{\theta y} - A\theta e^{-\theta y} - 2k_2 y, \tag{2.30}$$

for each $y > 0$. The derivative is

$$h'_A(y) = A\theta^2 e^{\theta y} + A\theta^2 e^{-\theta y} - 2k_2 = \frac{A\theta^2 \big(e^{\theta y}\big)^2 - 2k_2 \big(e^{\theta y}\big) + A\theta^2}{e^{\theta y}}.$$

We need to consider two cases, according to the value of $A$. Let

$$\bar{A} = \frac{k_2}{\theta^2} = \frac{\sigma^2(2\Delta + b)}{4\rho^2\Delta^2}. \tag{2.31}$$

If $A \geq \bar{A}$ we have $h'_A > 0$ in $]0, \infty[$; hence, since $h_A(0) = 0$, Equation (2.28a) does not have any solution in $[0, +\infty[$. On the contrary, if $A < \bar{A}$ we have $h'_A < 0$ in $]0, \tilde{y}[$ and $h'_A > 0$ in $]\tilde{y}, \infty[$, for a suitable $\tilde{y} = \tilde{y}(A) > 0$; hence, since $h_A(0) = 0$ and $h_A(+\infty) = +\infty$, Equation (2.28a) has exactly one solution $\bar{y} = \bar{y}(A) > 0$ (notice that $\bar{y}(A) > \tilde{y}(A)$). In short, we have proved that, for a fixed $A > 0$, Equation (2.28a) admits a solution $\bar{y} \in ]0, \infty[$ if and only if $A \in ]0, \bar{A}[$; in this case the solution is unique and we denote it by $\bar{y} = \bar{y}(A)$.

Finally, we remark that

$$\lim_{A \to 0^+} \bar{y}(A) = +\infty, \qquad \lim_{A \to \bar{A}^-} \bar{y}(A) = 0. \tag{2.32}$$

The first limit follows by $\bar{y}(A) > \tilde{y}(A)$ and $\lim_{A \to 0^+} \tilde{y}(A) = +\infty$ (this one by a direct computation of $\tilde{y}$), whereas the second limit is immediate.

*Second step.* We now consider Equation (2.28b). For each $A \in ]0, \bar{A}[$, we define

$$g(A) = -Ae^{\theta \bar{y}(A)} - Ae^{-\theta \bar{y}(A)} + k_2 \bar{y}^2(A) + 2A, \tag{2.33}$$

where $\bar{y}(A)$ is well-defined by the first step. We are going to prove that

$$\lim_{A \to 0^+} g(A) = +\infty, \qquad \lim_{A \to \bar{A}^-} g(A) = 0, \qquad g' < 0. \tag{2.34}$$

This concludes the proof: indeed, if we assume (2.34), it follows that the equation $g(A) = c$, which is just a rewriting of (2.28b), has exactly one solution $A \in ]0, \bar{A}[$. It is then clear that the couple $(A, \bar{y}(A))$ is a solution to (2.28a)-(2.28b) (the unique one, since uniqueness holds for (2.28b)).

For the first claim in (2.34), by (2.28a) we can write $A$ as a function of $\bar{y}$,

$$A = \frac{2k_2}{\theta} \frac{\bar{y}(A)}{e^{\theta \bar{y}(A)} - e^{-\theta \bar{y}(A)}}, \tag{2.35}$$

so that $g$ also reads

$$g(A) = k_2\bar{y}^2(A) - \frac{2k_2}{\theta}\frac{e^{\theta\bar{y}(A)} + e^{-\theta\bar{y}(A)} - 2}{e^{\theta\bar{y}(A)} - e^{-\theta\bar{y}(A)}}\bar{y}(A), \tag{2.36}$$

which we rewrite as

$$g(A) = k_2\bar{y}^2(A) - \frac{2k_2}{\theta}\frac{(e^{\theta\bar{y}(A)} - 1)^2}{(e^{\theta\bar{y}(A)})^2 - 1}\bar{y}(A); \tag{2.37}$$

then, by (2.32) we have

$$\lim_{A\to 0^+} g(A) = \lim_{z\to+\infty}\left(k_2 z^2 - \frac{2k_2}{\theta}\frac{(e^{\theta z} - 1)^2}{(e^{\theta z})^2 - 1}z\right) = +\infty.$$

As for the second claim in (2.34), it is immediate by the definition of $g$ and by (2.32). We finally show that the third claim in (2.34) holds. Notice that

$$g'(A) = -e^{\theta\bar{y}(A)} - e^{-\theta\bar{y}(A)} + 2 - \left(A\theta e^{\theta\bar{y}} - A\theta e^{-\theta\bar{y}} - 2k_2\bar{y}\right)\bar{y}'(A).$$

By (2.28a), the coefficient of $\bar{y}'(A)$ is zero; thus, we have

$$g'(A) = -e^{\theta\bar{y}(A)} - e^{-\theta\bar{y}(A)} + 2 = -\frac{(e^{\theta\bar{y}(A)} - 1)^2}{e^{\theta\bar{y}(A)}} < 0, \tag{2.38}$$

which concludes the proof. □

We now focus on the order condition (2.29). As already noticed in (2.35), by (2.28a) we can write $A$ as a function of $\bar{y}$: for every $\bar{y} > 0$ we have

$$A(\bar{y}) = \frac{2k_2}{\theta}\frac{\bar{y}}{e^{\theta\bar{y}} - e^{-\theta\bar{y}}}. \tag{2.39}$$

We are going to consider the function $\xi := g \circ A$, where $g$ has been defined in (2.33) and $A$ is as in (2.39). In (2.37) we have already computed an expression for $\xi$, which we here recall: for every $\bar{y} > 0$ we have

$$\xi(\bar{y}) = (g \circ A)(\bar{y}) = k_2\bar{y}^2 - \frac{2k_2}{\theta}\frac{(e^{\theta\bar{y}} - 1)^2}{(e^{\theta\bar{y}})^2 - 1}\bar{y}. \tag{2.40}$$

**Lemma 2.8.** *Let $(A, \bar{y})$ be as in Lemma 2.7 and let $\bar{c} = \xi(\Delta^2/(2\Delta + b))$, with $\xi$ as in (2.40). Then, the condition in (2.29) is satisfied if and only if $c \leq \bar{c}$.*

*Proof.* Let $g$ be as in (2.33) and assume, for the moment, that the function $A$ in (2.39) is decreasing. Then, since $g$ is decreasing by (2.34), we deduce that $\xi = g \circ A$ is increasing. Hence, we have $\bar{y} < \Delta - x_v$ if and only if $\xi(\bar{y}) < \xi(\Delta - x_v)$. The conclusion follows since $\xi(\bar{y}) = g(A(\bar{y})) = c$ by (2.28b) and since $\Delta - x_v = \Delta^2/(2\Delta + b)$ by (2.12).

So, we just need to prove that $\bar{y} \mapsto A(\bar{y})$ is decreasing. A direct differentiation in (2.39) leads to an expression whose sign is not easy to estimate. Then, we write $A = A(\bar{y})$ in (2.28a) and differentiate with respect to $\bar{y}$. We get

$$A'(\bar{y})\theta\big(e^{\theta\bar{y}} - e^{-\theta\bar{y}}\big) + A(\bar{y})\theta^2\big(e^{\theta\bar{y}} + e^{-\theta\bar{y}}\big) - 2k_2 = 0,$$

so that, after rearranging, we have

$$A'(\bar{y}) = -\frac{A(\bar{y})\theta^2 e^{\theta\bar{y}} + A(\bar{y})\theta^2 e^{-\theta\bar{y}} - 2k_2}{\theta(e^{\theta\bar{y}} - e^{-\theta\bar{y}})} = -\frac{h'_{A(\bar{y})}(\bar{y})}{\theta(e^{\theta\bar{y}} - e^{-\theta\bar{y}})} < 0, \tag{2.41}$$

where in the numerator we have recognized $h'_{A(\bar{y})}(\bar{y})$, with $h_{A(\bar{y})}$ as in (2.30), and we have $h'_{A(\bar{y})}(\bar{y}) > 0$ since $h_{A(\bar{y})}$ is increasing in $[\tilde{y}, +\infty[ \ni \bar{y}$ (see Lemma 2.7). □

All the previous results are summarized in the next proposition: we have proved that our candidate $\tilde{V}$ is well-defined.

**Proposition 2.9.** *Assume $c < \bar{c}$, with $\bar{c}$ as in Lemma 2.8. Then, the function $\tilde{V}$ in Definition 2.6 is well-defined. More precisely, there exists a solution*

$$(A_1, A_2, \underline{x}, \bar{x}, x^*)$$

*to System (2.26), which is given by*

$$A_1 = Ae^{-\theta x_v}, \qquad A_2 = Ae^{\theta x_v},$$
$$x^* = x_v, \qquad \underline{x} = x_v - \bar{y}, \qquad \bar{x} = x_v + \bar{y},$$

*where $x_v$ is as in (2.12) and $(A, \bar{y})$ is the unique solution to (2.28a)-(2.28b)-(2.29).*

### 2.3.2 Application of the verification theorem

In this section we apply the verification theorem (Proposition 2.4) and show that the candidate $\tilde{V}$ defined in the previous section actually corresponds to the value function. Moreover, we characterize the the optimal price management policy: the retailer has to intervene if and only if the process hits $\underline{x}$ or $\bar{x}$ and, when this happens, he has to shift $X$ to the state $x^*$.

We emphasize the importance of carefully checking all the assumptions: this passage is often omitted, but it can be no trivial at all, as we will see here and, above all, in the next section.

**Lemma 2.10.** *Let (2.24) hold and let $\tilde{V}$ be as in Definition 2.6. Then, for every $x \in \mathbb{R}$ we have*

$$\mathcal{M}\tilde{V}(x) = \varphi_A(x^*) - c.$$

*In particular, we have*

$$\{\mathcal{M}\tilde{V} - \tilde{V} < 0\} = ]\underline{x}, \bar{x}[, \qquad \{\mathcal{M}\tilde{V} - \tilde{V} = 0\} = \mathbb{R} \setminus ]\underline{x}, \bar{x}[. \tag{2.42}$$

*Proof.* First of all, recall that $\tilde{V}$ is symmetric with respect to $x^*$ and notice that:

- $\tilde{V}$ is strictly decreasing in $]x^*, \bar{x}[$ (since we have $\tilde{V} = \varphi_A$ by definition and $\varphi'_A < 0$ in $]x^*, \bar{x}[$ by the proof of Lemma 2.7);

- $\tilde{V}$ is constant in $[\bar{x}, +\infty[$ by definition of $\tilde{V}$, with $\tilde{V} \equiv \varphi_A(x^*) - c$.

Then, we deduce that

$$\max_{y\in\mathbb{R}} \tilde{V}(y) = \tilde{V}(x^*) = \varphi_A(x^*), \qquad \min_{y\in\mathbb{R}} \tilde{V}(y) = \varphi_A(x^*) - c. \tag{2.43}$$

As a consequence, for every $x \in \mathbb{R}$ we have

$$\mathcal{M}\tilde{V}(x) = \max_{\delta\in\mathbb{R}}\{\tilde{V}(x+\delta) - c\} = \max_{y\in\mathbb{R}} \tilde{V}(y) - c = \varphi_A(x^*) - c.$$

By the definition of $\tilde{V}$, we have

$$\mathcal{M}\tilde{V}(x) - \tilde{V}(x) = 0, \qquad \text{in } \mathbb{R}\backslash]\underline{x}, \bar{x}[.$$

Moreover, as $\varphi_A(\bar{x}) = \varphi_A(x^*) - c$ by (2.26) and $\varphi_A(\bar{x}) = \min_{[\underline{x},\bar{x}]} \varphi_A$ by the previous arguments, we have

$$\mathcal{M}\tilde{V}(x) - \tilde{V}(x) = \varphi_A(x^*) - c - \varphi_A(x) = \varphi_A(\bar{x}) - \varphi_A(x) < 0, \qquad \text{in } ]\underline{x}, \bar{x}[,$$

which concludes the proof. □

**Proposition 2.11.** *Let* (2.24) *hold and let* $\tilde{V}$ *be as in Definition 2.6. For every* $x \in \mathbb{R}$*, an optimal control for the problem in Section 2.2 exists and is given by* $u^*(x) = \{(\tau_k^*, \delta_k^*)\}_{1\leq k<\infty}$*, where the variables* $(\tau_k^*, \delta_k^*)$ *are recursively defined by*

$$\begin{aligned} \tau_k^* &= \inf\left\{t > \tau_{k-1}^* : X_t^{x;u_k^*} \in \{\underline{x}, \bar{x}\}\right\}, \\ \delta_k^* &= x^* - X_{\tau_k^*}^{x;u_k^*}, \end{aligned} \tag{2.44}$$

*for* $k \geq 1$*, where we have set* $\tau_0^* = \delta_0^* = 0$ *and* $u_k^* = \{(\tau_j^*, \delta_j^*)\}_{1\leq j\leq k}$*. Moreover,* $\tilde{V}$ *coincides with the value function: for every* $x \in \mathbb{R}$ *we have*

$$\tilde{V}(x) = V(x) = J(x; u^*(x)).$$

*Proof.* We have to check that the candidate $\tilde{V}$ satisfies all the assumptions of Proposition 2.4. For the reader's convenience, we briefly report the conditions we have to check:

(i) $\tilde{V}$ is bounded and $\max_{x\in\mathbb{R}} \tilde{V}(x)$ exists;

(ii) $\tilde{V} \in C^2(\mathbb{R} \setminus \{\underline{x}, \bar{x}\}) \cap C^1(\mathbb{R})$;

(iii) $\tilde{V}$ satisfies $\max\{\mathcal{A}\tilde{V} - \rho\tilde{V} + R, \mathcal{M}\tilde{V} - \tilde{V}\} = 0$;

(iv) the optimal control is admissible, i.e. $u^*(x) \in \mathcal{U}_x$ for every $x \in \mathbb{R}$.

*Condition (i) and (ii).* The first condition holds by (2.43), whereas the second condition follows by the definition of $\tilde{V}$.

*Condition (iii).* We have to prove that for every $x \in \mathbb{R}$ we have

$$\max\{\mathcal{A}\tilde{V}(x) - \rho\tilde{V}(x) + R(x), \mathcal{M}\tilde{V}(x) - \tilde{V}(x)\} = 0. \tag{2.45}$$

In $]\underline{x}, \bar{x}[$ the claim is true, as $\mathcal{M}\tilde{V} - \tilde{V} < 0$ by (2.42) and $\mathcal{A}\tilde{V} - \rho\tilde{V} + R = 0$ by definition (recall that here we have $R = f$ and $\tilde{V} = \varphi_A$, with $\mathcal{A}\varphi_A - \rho\varphi_A + f = 0$).

As for $\mathbb{R}\backslash]\underline{x}, \bar{x}[$, we already know by (2.42) that $\mathcal{M}\tilde{V} - \tilde{V} = 0$. Then, to conclude we have to prove that

$$\mathcal{A}\tilde{V}(x) - \rho\tilde{V}(x) + R(x) \leq 0, \qquad \forall x \in \mathbb{R}\backslash]\underline{x}, \bar{x}[.$$

By symmetry, it is enough to prove the claim for $x \in [\bar{x}, +\infty[$. By the definition of $\tilde{V}(x)$ and (2.26), in the interval $[\bar{x}, +\infty[$ we have $\tilde{V} \equiv \varphi_A(x^*) - c = \varphi_A(\bar{x})$; hence, the inequality reads

$$-\rho\varphi_A(\bar{x}) + R(x) \leq 0, \qquad \forall x \in [\bar{x}, +\infty[.$$

As $R$ is decreasing in $[x_v, +\infty[ \supseteq [\bar{x}, +\infty[$, it is enough to prove the claim in $x = \bar{x}$:

$$-\rho\varphi_A(\bar{x}) + R(\bar{x}) \leq 0.$$

Since $\mathcal{A}\varphi_A(\bar{x}) - \rho\varphi_A(\bar{x}) + f(\bar{x}) = 0$ and $f(\bar{x}) = R(\bar{x})$, we can rewrite as

$$-\frac{\sigma^2}{2}\varphi_A''(\bar{x}) \leq 0,$$

which is true as $\bar{x}$ is a local minimum of $\varphi_A \in C^\infty(\mathbb{R})$, so that $\varphi_A''(\bar{x}) \geq 0$.

*Condition (iv).* Let $x \in \mathbb{R}$; recall (2.8): we have to show that

$$\mathbb{E}_x\left[\sum_{k\geq 1} e^{-\rho\tau_k^*}\right] < \infty.$$

When acting according to the optimal control $u^*$, the retailer intervenes when the process hits $\underline{x}$ or $\bar{x}$ and shifts the process to $x^* \in ]\underline{x}, \bar{x}[$. As a consequence, we can decompose each variable $\tau_k^*$ as a sum of suitable exit times from $]\underline{x}, \bar{x}[$. Given $y \in \mathbb{R}$, let $\zeta^y$ denote the exit time of the process $y + \sigma W$, where $W$ is a real Brownian motion, from the interval $]\underline{x}, \bar{x}[$; then, we have $\tau_1^* = \zeta^x$ and

$$\tau_k^* = \zeta^x + \sum_{l=1}^{k-1} \zeta_l^{x^*},$$

for every $k \geq 2$, where the variables $\zeta_l^{x^*}$ are independent and distributed as $\zeta^{x^*}$. As a consequence, we have

$$\mathbb{E}_x\left[\sum_{k\geq 2} e^{-\rho\tau_k^*}\right] = \mathbb{E}_x\left[\sum_{k\geq 2} e^{-\rho\left(\zeta^x + \sum_{l=1}^{k-1}\zeta_l^{x^*}\right)}\right] = \mathbb{E}_x\left[e^{-\rho\zeta^x}\sum_{k\geq 2}\prod_{l=1,\dots,k-1} e^{-\rho\zeta_l^{x^*}}\right].$$

By the Fubini-Tonelli theorem and the independence of the variables:

$$\mathbb{E}_x\left[e^{-\rho\zeta^x}\sum_{k\geq 2}\prod_{l=1,\dots,k-1} e^{-\rho\zeta_l^{x^*}}\right] = \mathbb{E}_x\left[e^{-\rho\zeta^x}\right]\sum_{k\geq 2}\prod_{l=1,\dots,k-1}\mathbb{E}_x\left[e^{-\rho\zeta_l^{x^*}}\right].$$

As the variables $\zeta_l^{x^*}$ are identically distributed with $\zeta_l^{x^*} \sim \zeta^{x^*}$, we can conclude:

$$\sum_{k\geq 2}\prod_{l=1,\dots,k-1}\mathbb{E}_x\left[e^{-\rho\zeta_l^{x^*}}\right] = \sum_{k\geq 2}\mathbb{E}_x\left[e^{-\rho\zeta^{x^*}}\right]^{k-1} < \infty,$$

which is a converging geometric series. □

The value function in Definition 2.6 clearly depends on the parameter $c$. Up to now we have always assumed $c$ to be a constant; we now consider $V$ as a function of $c$ and investigate the limit case $c \to 0^+$. To stress the dependence on $c$, we will write $V^c$, $\underline{x}(c)$, $\bar{x}(c)$, $u^*(x, c)$.

When the fixed cost $c$ decreases, it is clear that the player intervenes more frequently and that the continuation region $]\underline{x}, \bar{x}[$ gets smaller. In the limit case $c = 0$ (which corresponds to a problem with no intervention cost) we guess that $\underline{x} = \bar{x} = x^* = x_v$: the continuation region collapses in the singleton $\{x_v\}$. Hence, we expect the value function to equal the value of the game in the static case (i.e. when the process is constant), that is

$$V^{\text{static}} = \max_{x \in \mathbb{R}} \left[ \int_0^\infty e^{-\rho t} R(x) dt \right] = \max_{x \in \mathbb{R}} R(x)/\rho = f(x_v)/\rho, \tag{2.46}$$

where $\max R = f(x_v)$ by (2.11). Proposition 2.13 makes this guess rigorous.

We start by proving a stronger result: we provide an estimate for $\underline{x}(c), \bar{x}(c)$ as $c \to 0^+$. To our knowledge, this is the first time that an asymptotic estimate for the continuation region of an impulse control problem is provided.

**Proposition 2.12.** *Let* (2.24) *hold and let* $\underline{x}(c), \bar{x}(c)$ *be as in Definition 2.6, for* $c > 0$. *The following asymptotic estimates hold:*

$$\underline{x}(c) \sim_{c \to 0^+} x_v - C \sqrt[4]{c}, \qquad \bar{x}(c) \sim_{c \to 0^+} x_v + C \sqrt[4]{c}, \tag{2.47}$$

*where we have set* $C = \sqrt[4]{6/(k_2 \theta^2)}$.

*Proof.* Recall by Proposition 2.9 that for each $c > 0$ we have

$$\underline{x}(c) = x_v - \bar{y}(A(c)), \qquad \bar{x}(c) = x_v + \bar{y}(A(c)), \tag{2.48}$$

where the function $\bar{y}$ has been defined in the proof of Lemma 2.7 and $A(c) \in ]0, \bar{A}[$ is the unique solution to $g(A) = c$, with $g$ as in (2.33) and $\bar{A}$ as in (2.31). Hence, we have to estimate $\bar{y}(A(c))$ as $c \to 0^+$.

Let us start by the expression of $g$ in (2.36):

$$g(A) = \frac{\theta k_2 \bar{y}^2(A)\big(e^{\theta \bar{y}(A)} - e^{-\theta \bar{y}(A)}\big) - 2k_2 \bar{y}(A)\big(e^{\theta \bar{y}(A)} + e^{-\theta \bar{y}(A)} - 2\big)}{\theta\big(e^{\theta \bar{y}(A)} - e^{-\theta \bar{y}(A)}\big)},$$

for every $A \in ]0, \bar{A}[$. Recall by (2.32) that $\bar{y}(A) \to 0$ as $A \to \bar{A}^-$; hence, by the Taylor series we have

$$e^{\theta \bar{y}(A)} - e^{-\theta \bar{y}(A)} = 2\big(\theta \bar{y}(A)\big) + \frac{2}{3!}\big(\theta \bar{y}(A)\big)^3 + o\big(\bar{y}(A)^4\big),$$
$$e^{\theta \bar{y}(A)} + e^{-\theta \bar{y}(A)} = 2 + \frac{2}{2!}\big(\theta \bar{y}(A)\big)^2 + o\big(\bar{y}(A)^3\big),$$

which leads to the following approximation:

$$g(A) \sim_{A \to \bar{A}^-} \frac{\theta k_2 \bar{y}^2(A)\Big(2\theta \bar{y}(A) + \theta^3 \bar{y}^3(A)/3\Big) - 2k_2 \bar{y}(A)\Big(\theta^2 \bar{y}^2(A)\Big)}{\theta\big(2\theta \bar{y}(A)\big)};$$

after rearranging the terms, we get

$$g(A) \sim_{A \to \bar{A}^-} \frac{k_2 \theta^2}{6} \bar{y}^4(A). \tag{2.49}$$

Now, since $A$ is the inverse function of $g$ (as reminded above) and since $g(\bar{A}^-) = 0$ by (2.34), we deduce that

$$\lim_{c \to 0^+} A(c) = \bar{A}. \tag{2.50}$$

Hence, by (2.49) we have

$$g(A(c)) \sim_{c \to 0^+} \frac{k_2 \theta^2}{6} \bar{y}^4(A(c)).$$

But $g(A(c)) \equiv c$ by the definition of $A(c)$, so that we deduce that

$$\bar{y}(A(c)) \sim_{c \to 0^+} \sqrt[4]{\frac{6}{k_2 \theta^2}} \sqrt[4]{c},$$

which concludes the proof. □

**Proposition 2.13.** *Let* (2.24) *hold and let* $V^c$ *be as in Definition 2.6, for* $c > 0$. *Then, we have*

$$\underline{x}'(c) < 0, \qquad \bar{x}'(c) > 0, \qquad \lim_{c \to 0^+} \underline{x}(c) = \lim_{c \to 0^+} \bar{x}(c) = x_v, \tag{2.51}$$

*for* $c > 0$. *As a consequence, the following punctual limits hold:*

$$\lim_{c \to 0^+} V^c(x) = V^{static}, \qquad \lim_{c \to 0^+} X_t^{x; u^*(x,c)} = x_v. \tag{2.52}$$

*for every* $x \in \mathbb{R}$ *and* $t \geq 0$, *where* $V^{static}$ *is the constant defined in* (2.46).

*Proof.* Let us start with (2.51). The limits immediately follow by Proposition 2.12; moreover, by symmetry it is enough to prove that $\bar{x}'(c) > 0$ for $c > 0$. By (2.48) we have

$$\bar{x}'(c) = \bar{y}'\big(A(c)\big) A'(c), \tag{2.53}$$

for every $c > 0$, where the functions $\bar{y}, A$ are as in Proposition 2.12. The function $A \mapsto \bar{y}(A)$ is the inverse function of $\tilde{A}$, with $\tilde{A}$ as in (2.39) (we have written $\tilde{A}$ instead of $A$ not to create confusion with the map $c \mapsto A(c)$ we have previously used). Since $\tilde{A}' < 0$ by (2.41), it follows that

$$\bar{y}'\big(A(c)\big) = \frac{1}{\tilde{A}'\big(\bar{y}(A(c))\big)} < 0, \tag{2.54}$$

for every $c > 0$. Similarly, the function $c \mapsto A(c)$ is the inverse function of $g$, with $g$ as in (2.33). Since $g' < 0$ by (2.34), it follows that

$$A'(c) = \frac{1}{g'(A(c))} < 0, \tag{2.55}$$

for every $c > 0$. By (2.53), (2.54) and (2.55) we conclude that $\bar{x}'(c) > 0$.

Let us now consider (2.52). Recall that, thanks to Proposition 2.11, we have a semi-explicit expression for $V^c$; in the notations of this proof, it reads

$$V^c(x) = \begin{cases} \varphi_{A(c)}(x), & x \in \, ]\underline{x}(c), \bar{x}(c)[, \\ \varphi_{A(c)}(x_v) - c, & x \in \mathbb{R} \backslash ]\underline{x}(c), \bar{x}(c)[, \end{cases} \tag{2.56}$$

for $x \in \mathbb{R}$ and a fixed $c > 0$. We now rewrite $V^c$ so as to focus on the parameter $c$. Since we have proved that $\underline{x}(c), \bar{x}(c)$ are monotone and converge to $x_v$, by (2.56) it follows that for every fixed $x \in \mathbb{R} \setminus \{x_v\}$ we have

$$V^c(x) = \begin{cases} \varphi_{A(c)}(x_v) - c, & c \in \, ]0, \tilde{c}(x)[, \\ \varphi_{A(c)}(x), & c \in [\tilde{c}(x), +\infty[, \end{cases} \tag{2.57}$$

where $\tilde{c} > 0$ is a suitable function, while in the case $x = x_v$ we simply have

$$V^c(x_v) = \varphi_{A(c)}(x_v), \qquad c \in \, ]0, +\infty[. \tag{2.58}$$

By (2.57) and (2.58) it follows that for every $x \in \mathbb{R}$ and $c > 0$ we have

$$\lim_{c \to 0^+} V^c(x) = \lim_{c \to 0^+} \varphi_{A(c)}(x_v) = \varphi_{\bar{A}}(x_v),$$

where $A(0^+) = \bar{A}$, as proved in (2.50). Recall now the expression of $\varphi_{\bar{A}}$ in (2.22), the definition of $\bar{A}$ in (2.31) and the value of $k_3$ in (2.23): we have

$$\varphi_{\bar{A}}(x_v) = 2\bar{A} + k_3 = \frac{2k_2}{\theta^2} + \frac{f(x_v)}{\rho} - \frac{2k_2}{\theta^2} = \frac{f(x_v)}{\rho},$$

which proves the first claim in (2.52).

Finally, as $X^{x;u^*(x,c)}$ lies by definition in the continuation region, i.e.

$$\underline{x}(c) < X^{x;u^*(x,c)} < \bar{x}(c),$$

the second claim in (2.52) immediately follows by passing to the limit as $c \to 0^+$. □

Clearly, even if the value function is well-defined in the case $c = 0$, an impulse optimal policy does not exist in this case, as it would consist in continuous interventions by the player. Indeed, the situation would be as follows: at the beginning of the game the process is immediately shifted (with no penalty, as $c = 0$) to the optimal state $x_v$ and then the retailer instantaneously intervenes to keep the process constant.

We now investigate the monotonicity of the value function with respect to $c$. When the intervention cost decreases, the retailer can intervene more frequently to shift the process to the optimal state $x_v$, so that we expect a bigger value for the problem. In other words, given a fixed $x \in \mathbb{R}$, we expect $V^c(x)$ to increase as $c \to 0^+$, which is equivalent to $(dV^c/dc)(x) < 0$ (since $c$ approaches 0 from the right, a maximal value in 0 corresponds to a decreasing function). In particular, the value function should be always smaller than the static value of the game.

**Proposition 2.14.** *Let* (2.24) *hold and let* $V^c$ *be as in Definition 2.6, for* $c > 0$. *For every* $x \in \mathbb{R}$ *and* $c > 0$ *we have*

$$\frac{d}{dc}V^c(x) < 0; \tag{2.59}$$

*in particular, the value functions are always smaller than the static maximum:*

$$V^c(x) < V^{static}, \tag{2.60}$$

*for every* $x \in \mathbb{R}$ *and* $c > 0$, *where* $V^{static}$ *is the constant defined in* (2.46).

*Proof.* First of all, the inequality in (2.60) is an immediate consequence of (2.59) and Proposition 2.13. Then, let us use the notations of Proposition 2.13 and focus on (2.59). Recall the definition of $\varphi_{A(c)}$:

$$\varphi_{A(c)}(x) = A(c)e^{\theta(x-x_v)} + A(c)e^{-\theta(x-x_v)} - k_2(x-x_v)^2 + k_3,$$

for every $c > 0$ and $x \in \mathbb{R}$. Then, by (2.57) and (2.58) we have

$$\frac{dV^c}{dc}(x) = \begin{cases} 2A'(c) - 1, & c \in \,]0, \tilde{c}(x)[, \\ A'(c)\big(e^{\theta(x-x_v)} + e^{-\theta(x-x_v)}\big), & c \in [\tilde{c}(x), +\infty[, \end{cases} \tag{2.61}$$

for every $x \in \mathbb{R} \setminus \{x_v\}$, whereas in $x = x_v$ we have

$$\frac{dV^c}{dc}(x_v) = 2A'(c), \qquad c \in \,]0, +\infty[. \tag{2.62}$$

The inequality in (2.59) follows since $A' < 0$, as proved in (2.55). □

Finally, we study the robustness of the value function with respect to $c$ in a (right) neighbourhood of zero. The problem is as follows: how sensitive is the value function to small changes in $c$ around zero? Clearly, we have to estimate $(dV^c/dc)(x)$ as $c \to 0^+$, with $x \in \mathbb{R}$.

It has been shown that intervention costs in the form $c + \tilde{\lambda}|\delta|$ often imply a value function $V^c$ which is not robust with respect to $c = 0$, i.e. such that $(dV^c/dc)(x)$ diverges as $c \to 0$, for every $x \in \mathbb{R}$. Practically, given a very small value of $c$, a slight change in the intervention cost does not correspond to a proportionally slight change in the value function: the difference explodes as $c \to 0^+$. Clearly, such a behaviour is extremely problematic when performing numerical experiments. This property has been first noticed in [31], in a specific case, and then generalized in [33] for a class of problems with quadratic payoff and $\tilde{\lambda} > 0$. Our problem does not belong to the the class studied in [33], as we here have $\tilde{\lambda} = 0$; however, we can prove that such behaviour is present in our case as well.

**Corollary 2.15.** *Let* (2.24) *hold and let* $V^c$ *be as in Definition 2.6, for* $c > 0$. *For every* $x \in \mathbb{R}$, *we have*

$$\lim_{c \to 0^+} \frac{dV^c}{dc}(x) = -\infty.$$

*Proof.* By (2.61) and (2.62) we just have to prove that $A'(0^+) = -\infty$. As already noticed in (2.55), for each $c > 0$ we have $A'(c) = 1/g'(A(c)) < 0$, with $g$ as in (2.33); by using the expression for $g'$ in (2.38), it is immediate to deduce that $g'(\bar{A}^-) = 0$, so that $A'(0^+) = -\infty$. □

### 2.3.3 Extension to processes with non-zero drift

We now extend the results of the previous sections to the case where the process has a non-zero drift term. Hence, we no longer assume $\mu = 0$ and we consider the problem in Section 2.2 with

$$\lambda = 0. \tag{2.63}$$

By arguing as in Section 2.3.1, in this case the candidate for the value function is as follows.

**Definition 2.16.** *For every $x \in \mathbb{R}$, we set*

$$\tilde{V}(x) = \begin{cases} \varphi_{A_1,A_2}(x), & \text{in } ]\underline{x}, \bar{x}[, \\ \varphi_{A_1,A_2}(x^*) - c, & \text{in } \mathbb{R} \backslash ]\underline{x}, \bar{x}[, \end{cases}$$

*where $\varphi_{A_1,A_2}$ is as in* (2.20) *and the five parameters $(A_1, A_2, \underline{x}, \bar{x}, x^*)$ satisfy the conditions* (2.25)-(2.26).

The only difference with respect to Definition 2.6 relies in the function $\varphi_{A_1,A_2}$: as we now consider a generic value for $\mu$, we can no longer use the expression in (2.22) and we have to use the generic expression in (2.20).

In Section 2.3.1 we managed, by using a symmetry argument, to simplify the conditions in (2.26) and to deduce an existence result for (2.25)-(2.26). This procedure cannot be replicated here, due to the presence of a drift: the process is no longer symmetric, so that the value function cannot be symmetric either. Hence, it is not possible to simplify (2.26) and a general existence result is not available for this system. So, we have to assume that a solution actually exists.

**Assumption 2.17.** *We assume that a solution to* (2.25)-(2.26) *exists. Moreover, we assume that there exist $\tilde{x}_1, \tilde{x}_2$, with $\underline{x} < \tilde{x}_1 < x^* < \tilde{x}_2 < \bar{x}$, such that $\varphi''_{A_1,A_2} < 0$ in $]\tilde{x}_1, \tilde{x}_2[$ and $\varphi''_{A_1,A_2} > 0$ in $]\underline{x}, \tilde{x}_1[ \cup ]\tilde{x}_2, \bar{x}[$.*

We remark that the conditions on the second derivative are, heuristically, always satisfied by a solution to (2.25)-(2.26): since $x^*$ is a local maximum and we need a $C^1$-pasting (with slope equal to zero) in $\underline{x}$ and $\bar{x}$, we expect to have two changes in the convexity of $\varphi_{A_1,A_2}$. Moreover, we do not need to require uniqueness for the solution: this is an immediate consequence of the verification theorem.

If a candidate solution actually exists, i.e. if Assumption 2.17 holds, the same arguments as the ones in Proposition 2.11 show that it coincides with the value function; moreover, we can characterize the optimal control.

**Lemma 2.18.** *Let* (2.63) *and Assumption 2.17 hold and let $\tilde{V}$ be as in Definition 2.16. Then, for every $x \in \mathbb{R}$ we have*

$$\mathcal{M}\tilde{V}(x) = \varphi_{A_1,A_2}(x^*) - c.$$

*In particular, we have*

$$\{\mathcal{M}\tilde{V} - \tilde{V} < 0\} = ]\underline{x}, \bar{x}[, \qquad \{\mathcal{M}\tilde{V} - \tilde{V} = 0\} = \mathbb{R} \setminus ]\underline{x}, \bar{x}[.$$

*Proof.* The proof is similar to the one of Lemma 2.10. First of all, let us study the monotonicity of $\tilde{V}$:

- $\tilde{V}$ is constant in $]-\infty, \underline{x}]$ and $[\bar{x}, +\infty[$ by definition, with $\tilde{V} \equiv \varphi_{A_1,A_2}(x^*) - c$;
- $\tilde{V}$ is strictly increasing in $]\underline{x}, x^*[$ (by (2.26) we have $\varphi'_{A_1,A_2}(\underline{x}) = \varphi'_{A_1,A_2}(x^*) = 0$ and by Assumption 2.17 the function $\varphi'_{A_1,A_2}$ is first increasing and then decreasing in $]\underline{x}, x^*[$, so that $\tilde{V}' = \varphi'_{A_1,A_2} > 0$ in $]\underline{x}, x^*[$) and strictly decreasing in $]x^*, \bar{x}[$ (by similar arguments).

From now on, the proof is like in Lemma 2.10; for the reader's convenience, we briefly recall the arguments. By the previous arguments, we deduce that

$$\max_{y\in\mathbb{R}} \tilde{V}(y) = \tilde{V}(x^*) = \varphi_{A_1,A_2}(x^*), \qquad \min_{y\in\mathbb{R}} \tilde{V}(y) = \varphi_{A_1,A_2}(x^*) - c,$$

As a consequence, for every $x \in \mathbb{R}$ we have

$$\mathcal{M}\tilde{V}(x) = \max_{\delta\in\mathbb{R}}\{\tilde{V}(x+\delta) - c\} = \max_{y\in\mathbb{R}} \tilde{V}(y) - c = \varphi_{A_1,A_2}(x^*) - c.$$

By the definition of $\tilde{V}$, we have

$$\mathcal{M}\tilde{V}(x) - \tilde{V}(x) = 0, \qquad \text{in } \mathbb{R}\backslash]\underline{x}, \bar{x}[.$$

Moreover, as $\varphi_{A_1,A_2}(\bar{x}) = \varphi_{A_1,A_2}(x^*) - c$ by (2.26) and $\varphi_{A_1,A_2}(\bar{x}) = \min_{[\underline{x},\bar{x}]} \varphi_{A_1,A_2}$ by the previous arguments, we have

$$\mathcal{M}\tilde{V}(x) - \tilde{V}(x) = \varphi_{A_1,A_2}(x^*) - c - \varphi_{A_1,A_2}(x) = \varphi_{A_1,A_2}(\bar{x}) - \varphi_{A_1,A_2}(x) < 0, \quad \text{in } ]\underline{x}, \bar{x}[,$$

which concludes the proof. □

**Proposition 2.19.** *Let* (2.63) *and Assumption 2.17 hold. Then, for every $x \in \mathbb{R}$ an optimal control $u^*(x)$ for the problem in Section 2.2 exists and is given by* (2.44)*. Moreover, $\tilde{V}$ coincides with the value function: for every $x \in \mathbb{R}$ we have*

$$\tilde{V}(x) = V(x) = J(x; u^*(x)).$$

*Proof.* The same as Proposition 2.11: adding a drift term does not change any passage in the proof. □

In short, if a drift term is present in the process, we no longer have an analytical existence result for (2.26), so that we need to assume it. Under this further assumption, we can argue as in Sections 2.3.1 and 2.3.2 and characterize the value function and the optimal control.

## 2.4 The case with variable penalty

In Section 2.3 we have studied the case $\lambda = 0$. In this section we study the problem in the case $\lambda > 0$, which means that, besides the fixed cost $c$, a variable intervention penalty is also present.

### 2.4.1 A generalization of the procedure

Exactly as we did in Section 2.3.1 and Section 2.3.3, let us start from the QVI problem (2.17) to define a candidate for the value function. In this case, the function $\tilde{V}$ has the following form:

$$\tilde{V}(x) = \begin{cases} \varphi_{A_1,A_2}(x), & \text{if } x \in ]\underline{x}, \bar{x}[, \\ \varphi_{A_1,A_2}(x^*) - c - \lambda\Phi(x), & \text{if } x \in \mathbb{R}\backslash]\underline{x}, \bar{x}[, \end{cases}$$

where $\varphi_{A_1,A_2}$ is a solution to (2.19) and the parameters satisfy the usual conditions, which will be detailed later. First of all, we notice that in this case we face a regularity problem: the function $\Phi$, defined in (2.5), is not differentiable in $\{0, \Delta\} \in \mathbb{R}\backslash]\underline{x}, \bar{x}[$ (recall that $0 < \underline{x} < \bar{x} < \Delta$), so that here we cannot apply Proposition 2.4 (recall that it requires the candidate $\tilde{V}$ to be everywhere differentiable).

Since classical results cannot be used in this case, we need another approach to tackle the problem. The procedure we are going to use is as follows.

1. We approximate $\Phi$ by smooth functions $\Phi_n$ and we show that, by substituting $\Phi$ with $\Phi_n$ in the penalty, we get value functions $V_n$ which converge to the original value function $V$ (and such that classical results can now be applied);
2. We look for candidates $\tilde{V}_n$ for $V_n$ and $\tilde{V}$ for $V$, with the property $\tilde{V}_n \to \tilde{V}$;
3. by using the classical verification theorem, we prove that actually $\tilde{V}_n = V_n$ and we deduce that $\tilde{V} = V$.

We now develop in detail these steps. We anticipate that this procedure presents, at the moment, some open problems which are currently subject of ongoing study: we here summarize the current state of the work.

**First step.** By a standard procedure, we can approximate the function $\Phi$ by smooth functions $\Phi_n$; consequently, we define a sequence of control problems with smooth penalty functions.

**Lemma 2.20.** *There exists a sequence* $\{\Phi_n\}_{n\in\mathbb{N}} \subseteq C^\infty(\mathbb{R})$ *such that*

$$\lim_{n\to\infty} \|\Phi_n - \Phi\|_\infty = 0, \tag{2.64}$$

$$\Phi_n \equiv \Phi \text{ in } \mathbb{R} \setminus K_n, \text{ where } K_n = [-1/n, 1/n] \cup [\Delta - 1/n, \Delta + 1/n]. \tag{2.65}$$

*Proof.* Standard mollification theory. The convergence is uniform everywhere, not only on compact subsets, as the functions $\Phi_n$ differ from $\Phi$ only in a bounded subset. Moreover, by choosing symmetric mollifiers we have (2.65). □

**Definition 2.21.** *Let the constants $c$, $\lambda$ be as in Section 2.2. For every $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we set*

$$K_n(x) = c + \lambda\Phi_n(x),$$

*with the function $\Phi_n$ as in Lemma 2.20, and*

$$V_n(x) = \sup_{u\in\mathcal{U}_x} J_n(x; u) = \sup_{u\in\mathcal{U}_x} \mathbb{E}_x \left[ \int_0^\infty e^{-\rho t} R(X_t)dt - \sum_k e^{-\rho\tau_k} K_n(X_{\tau_k}) \right],$$

*with the set $\mathcal{U}_x$, the constant $\rho$ and the function $R$ as in Section 2.2.*

Notice that in $V_n$ the mollified function $\Phi_n$ has been used only in the penalty function. Indeed, we have no need to use $\Phi_n$ also in the running cost $R$, as we have seen that the regularity problem just relies in the singularities of the penalty $K$.

We now show that the functions $V_n$ converge to $V$, as $n \to \infty$. The convergence result is preceded by a technical lemma, where some estimates are proved.

**Lemma 2.22.** *Let $V$ as in Definition 2.2 and let $V_n$ as in Definition 2.21, with $n \in \mathbb{N}$. The following estimates hold.*

1. *The functions $V_n$, $V$ are bounded from below by a polynomial: for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we have*
$$V_n(x), V(x) \geq p(x), \tag{2.66}$$
*where $p$ is a second-degree polynomial whose coefficients depend only on $\Delta, b, \rho$.*

2. *For every $n \in \mathbb{N}$, $x \in \mathbb{R}$ and every control $u \in \mathcal{U}_x$, we have*
$$|J_n(x;u) - J(x;u)| \leq \lambda \|\Phi_n - \Phi\|_\infty \mathbb{E}_x\left[\sum_k e^{-\rho\tau_k}\right], \tag{2.67}$$
*where the expectation in the right-hand side is finite.*

*Proof. Estimate* (2.66)*.* Let $x \in \mathbb{R}$ and let $u^0 \in \mathcal{U}_x$ be the control corresponding to the policy with no interventions. By the definition of $V$ and as $R \geq f$, see Section 2.2.1, we have

$$V(x) \geq J(x;u^0) = \mathbb{E}_x\left[\int_0^\infty e^{-\rho t} R(X_t)dt\right] \geq \mathbb{E}_x\left[\int_0^\infty e^{-\rho t} f(X_t)dt\right].$$

By the definition of $f$ in (2.10) and the Fubini-Tonelli theorem, we have

$$\mathbb{E}_x\left[\int_0^\infty e^{-\rho t} f(X_t)dt\right] = \int_0^\infty e^{-\rho t}(-\alpha\mathbb{E}_x[X_t^2] + \beta\mathbb{E}_x[X_t] - \gamma)dt.$$

As $X_t$ corresponds to the process with no interventions, we have

$$X_t = x - \mu t - \sigma W_t, \qquad \mathbb{E}_x[X_t] = x - \mu t, \qquad \mathbb{E}_x[X_t^2] = (x-\mu t)^2 + \sigma^2 t.$$

As a consequence, by integrating by parts it is immediate to see that

$$\int_0^\infty e^{-\rho t}(-\alpha\mathbb{E}_x[X_t^2] + \beta\mathbb{E}_x[X_t] - \gamma)dt = p(x),$$

where $p$ is a second-degree polynomial. The result holds for $V_n$ as well, since the functions $V$ and $V_n$ differ only in the penalty part, which plays no role in this proof.

*Estimate* (2.67)*.* Let $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $u \in \mathcal{U}_x$. By definition, we have

$$\begin{aligned}|J_n(x;u) - J(x;u)| &\leq \mathbb{E}_x\left[\sum_k e^{-\rho\tau_k}\left|K_n(X_{(\tau_k)^-}) - K(X_{(\tau_k)^-})\right|\right] \\ &= \lambda\mathbb{E}_x\left[\sum_k e^{-\rho\tau_k}\left|\Phi_n(X_{(\tau_k)^-}) - \Phi(X_{(\tau_k)^-})\right|\right] \leq \lambda\|\Phi_n - \Phi\|_\infty\mathbb{E}_x\left[\sum_k e^{-\rho\tau_k}\right].\end{aligned}$$

To conclude the proof we have to show that the expectation is finite. Since $u$ is admissible, by (2.8) we have

$$\mathbb{E}_x\Big[\sum_k e^{-\rho\tau_k}\big(\lambda\Phi\big(X_{(\tau_k)^-}\big)+c\big)\Big] = \lambda\,\mathbb{E}_x\Big[\sum_k e^{-\rho\tau_k}\Phi\big(X_{(\tau_k)^-}\big)\Big]+c\,\mathbb{E}_x\Big[\sum_k e^{-\rho\tau_k}\Big] < \infty,$$

so that both the terms in the right-hand side are finite. □

Notice that the expectation in (2.67) depends on $x$ and $u$: a key-point when dealing with (2.67) is then to remove such dependence. We now prove the convergence of the value functions $V_n$.

**Proposition 2.23.** *Let $V$ as in Definition 2.2 and let $V_n$ as in Definition 2.21, with $n \in \mathbb{N}$. The functions $V_n$ converge to $V$ as $n \to \infty$, uniformly on the compact subsets.*

*Proof.* Let $x \in \mathbb{R}$. By the definition of $V(x)$, for each $\varepsilon > 0$ there exists a control $u^\varepsilon = u^\varepsilon(x) \in \mathcal{U}_x$ such that $V(x) - \varepsilon < J(x, u^\varepsilon)$. Since the function $R$ is bounded from above and since $K(\cdot) > c$, we first notice that

$$J(x,u^\varepsilon) = \mathbb{E}_x\bigg[\int_0^\infty e^{-\rho t}R(X_t)dt - \sum_k e^{-\rho\tau_k}K(X_{(\tau_k)^-})\bigg] \le C - c\,\mathbb{E}_x\bigg[\sum_k e^{-\rho\tau_k}\bigg],$$

for a suitable constant $C > 0$. Then, by the definition of $u^\varepsilon$ and by Estimate (2.66) we have

$$\mathbb{E}_x\bigg[\sum_k e^{-\rho\tau_k}\bigg] \le \frac{1}{c}(C - J(x,u^\varepsilon)) \le \frac{1}{c}(C - V(x)+\varepsilon) \le \frac{1}{c}(C-p(x)+\varepsilon) = q(x)+\varepsilon/c,$$

where $q(x) = (C - p(x))/c$ is a second-degree polynomial. Hence, by estimate (2.67) we get

$$J(x,u^\varepsilon) \le J_n(x,u^\varepsilon)+\lambda\|\Phi_n-\Phi\|_\infty\mathbb{E}_x\bigg[\sum_k e^{-\rho\tau_k}\bigg] \le V_n(x)+\lambda\|\Phi_n-\Phi\|_\infty(q(x)+\varepsilon/c).$$

Finally, as $\varepsilon \to 0^+$ we have

$$V(x) \le V_n(x) + \lambda q(x)\|\Phi_n - \Phi\|_\infty. \tag{2.68}$$

Let now $n \in \mathbb{N}$. By arguing exactly as in the first step (we just switch $V_n$ with $V$ and $J_n$ with $J$), we get

$$V_n(x) \le V(x) + \lambda q(x)\|\Phi_n - \Phi\|_\infty. \tag{2.69}$$

As Estimates (2.68) and (2.69) hold for every $x \in \mathbb{R}$, we finally get

$$\|V_n - V\|_\infty \le \lambda q(x)\|\Phi_n - \Phi\|_\infty,$$

which concludes the proof. □

**Second step.** We now try to build candidates for $V$ and $V_n$. As usual, the starting point of this procedure is the QVI problem in (2.17).

A natural candidate for $V$ is

$$\tilde{V}(x) = \begin{cases} \varphi(x), & \text{in } ]\underline{x}, \bar{x}[, \\ \varphi(x^*) - c - \lambda\Phi(x), & \text{in } \mathbb{R}\backslash]\underline{x}, \bar{x}[, \end{cases} \quad \text{where} \quad \begin{cases} \varphi'(x^*) = 0, \quad \varphi''(x^*) < 0, \\ \varphi'(\underline{x}) = -\lambda\Phi'(\underline{x}), \\ \varphi'(\bar{x}) = -\lambda\Phi'(\bar{x}), \\ \varphi(\underline{x}) = \varphi(x^*) - c - \lambda\Phi(\underline{x}), \\ \varphi(\bar{x}) = \varphi(x^*) - c - \lambda\Phi(\bar{x}), \end{cases} \tag{2.70}$$

where the function $\varphi = \varphi_{A_1,A_2}$ is as in (2.20) and we ask $0 < \underline{x} < x^* < \bar{x} < \Delta$.

Similarly, a natural candidate for $V_n$ is

$$\tilde{V}_n(x) = \begin{cases} \varphi(x), & \text{in } ]\underline{x}_n, \bar{x}_n[, \\ \varphi(x^*) - c - \lambda\Phi_n(x), & \text{in } \mathbb{R}\backslash]\underline{x}_n, \bar{x}_n[, \end{cases} \quad \text{where} \quad \begin{cases} \varphi'(x_n^*) = 0, \quad \varphi''(x_n^*) < 0, \\ \varphi'(\underline{x}_n) = -\lambda\Phi_n'(\underline{x}_n), \\ \varphi'(\bar{x}_n) = -\lambda\Phi_n'(\bar{x}_n), \\ \varphi(\underline{x}_n) = \varphi(x^*) - c - \lambda\Phi_n(\underline{x}_n), \\ \varphi(\bar{x}_n) = \varphi(x^*) - c - \lambda\Phi_n(\bar{x}_n), \end{cases} \tag{2.71}$$

where the function $\varphi = \varphi_{A_1^n,A_2^n}$ is as in (2.20) and we ask $0 < \underline{x}_n < x_n^* < \bar{x}_n < \Delta$.

The conditions on the coefficients in (2.70) and (2.71) are very close to each other. Indeed, since $0 < \underline{x} < x^* < \bar{x} < \Delta$, for $n$ big enough (say $n \geq N$) we have

$$\underline{x},\, x^*,\, \bar{x} \;\in\; \big]1/n, \Delta - 1/n\big[ \;\subseteq\; \{y : \Phi(y) = \Phi_n(y)\}.$$

It follows that, for $n \geq N$, we have

$$\Phi(y) = \Phi_n(y) = -(y - \Delta)/\Delta, \qquad \Phi'(y) = \Phi_n'(y) = -1/\Delta, \qquad y \in \{\underline{x}, x^*, \bar{x}\}.$$

As a consequence, if the 5-uple $(A_1, A_2, x^*, \underline{x}, \bar{x})$ is a solution to (2.70), then it is also a solution to (2.71), for $n \geq N$. In short, we have seen that, given a candidate for $V$, we immediately have a candidate for $V_n$. We summarize these arguments in the following definition.

**Definition 2.24.** *For each $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we set*

$$\tilde{V}(x) = \begin{cases} \varphi_{A_1,A_2}(x), & \text{in } ]\underline{x}, \bar{x}[, \\ \varphi_{A_1,A_2}(x^*) - c - \lambda\Phi(x), & \text{in } \mathbb{R}\backslash]\underline{x}, \bar{x}[, \end{cases} \tag{2.72}$$

$$\tilde{V}_n(x) = \begin{cases} \varphi_{A_1,A_2}(x), & \text{in } ]\underline{x}, \bar{x}[, \\ \varphi_{A_1,A_2}(x^*) - c - \lambda\Phi_n(x), & \text{in } \mathbb{R}\backslash]\underline{x}, \bar{x}[, \end{cases} \tag{2.73}$$

*where $\varphi_{A_1,A_2}$ is as in* (2.20) *and the five parameters $(A_1, A_2, \underline{x}, \bar{x}, x^*)$ satisfy*

$$0 < \underline{x} < x^* < \bar{x} < \Delta \tag{2.74}$$

*and the following conditions:*

$$\begin{cases} \varphi'_{A_1,A_2}(x^*) = 0 \quad \textit{and} \quad \varphi''_{A_1,A_2}(x^*) < 0, & \textit{(optimality of } x^*\textit{)} \\ \varphi'_{A_1,A_2}(\underline{x}) = \lambda/\Delta, & \textit{(}C^1\textit{-pasting in } \underline{x}\textit{)} \\ \varphi'_{A_1,A_2}(\bar{x}) = \lambda/\Delta, & \textit{(}C^1\textit{-pasting in } \bar{x}\textit{)} \\ \varphi_{A_1,A_2}(\underline{x}) = \varphi_{A_1,A_2}(x^*) - c + \lambda/\Delta(\underline{x} - \Delta), & \textit{(}C^0\textit{-pasting in } \underline{x}\textit{)} \\ \varphi_{A_1,A_2}(\bar{x}) = \varphi_{A_1,A_2}(x^*) - c + \lambda/\Delta(\bar{x} - \Delta). & \textit{(}C^0\textit{-pasting in } \bar{x}\textit{)} \end{cases} \tag{2.75}$$

**Third step.** As anticipated, the problems $V_n$ have smooth penalty functions, so that it seems plausible to use the classical verification theorem. Assume for a moment that in this way one proves that $\tilde{V}_n$ actually corresponds to the value function, i.e. that $\tilde{V}_n = V_n$. Then, by passing to the limit, it immediately follows that $\tilde{V} = V$ and we also know the optimal control, as shown in the next lemma.

**Lemma 2.25.** *Assume that there exists $n \in \mathbb{N}$ such that $\tilde{V}_n = V_n$ for $n \geq N$, with $V_n$ as in Definition 2.21. Then, the function $\tilde{V}$ is the value function of the non-smooth problem: $\tilde{V} = V$, with $V$ as in Definition 2.2. Moreover, an optimal control exists and is given by* (2.44).

*Proof.* Since $V_n \to V$ by Proposition 2.23 and $\tilde{V}_n \to \tilde{V}$ by Definition 2.24, from the assumption $\tilde{V}_n = V_n$ it immediately follows that $\tilde{V} = V$.

As for the second part, let $x \in \mathbb{R}$ and let $u^*(x)$ be defined as in (2.44). Since $X^{x;u^*(x)} \in \,]\underline{x}, \bar{x}[$ by the definition of $u^*(x)$ and since $V = \tilde{V} \in C^\infty(]\underline{x}, \bar{x}[)$, one can apply Itô's formula and argue as in the standard verification theorem to prove that $u^*(x)$ is the optimal control. For the reader's convenience, we here briefly report such arguments. To shorten the notations, we write $u^* = u^*(x)$ and $X = X^{x;u^*(x)}$; by applying the Itô formula, we get

$$V(x) = \mathbb{E}_x\bigg[\int_0^\infty e^{-\rho t}(\mathcal{A}V - \rho V)(X_t)dt + \sum_k e^{-\rho\tau_k^*}\Big(V(X_{\tau_k^*}) - V(X_{(\tau_k^*)-})\Big)\bigg].$$

By the definition of $u^*$ we have

$$(\mathcal{A}V - \rho V + f)(X_t) = 0 \quad \text{and} \quad V\big(X_{(\tau_k^*)-}\big) = \mathcal{M}V\big(X_{(\tau_k^*)-}\big) = V(x^*) - c - \lambda\Phi\big(X_{(\tau_k^*)-}\big).$$

Since $X_{\tau_k^*} = x^*$ (the process has just been shifted to the optimal value), we finally have

$$V(x) = \mathbb{E}_x\bigg[\int_0^\infty e^{-\rho t}f(X_t)dt - \sum_k e^{-\rho\tau_k^*}\big(c + \lambda\Phi\big(X_{(\tau_k^*)-}\big)\big)\bigg] = J(x, u^*),$$

as $R(X_t) = f(X_t)$ since $X \in ]\underline{x}, \bar{x}[$, which concludes the proof. □

Finally, we just have to prove that $\tilde{V}_n = V_n$ by using the Verification Theorem 2.4. Unfortunately, this is still an open problem, as we explain in the next section.

### 2.4.2 Penalty functions and concavity: an open problem

As anticipated, in this section we explain why the claim $\tilde{V}_n = V_n$ is, at the moment, an open problem.

**Proposition 2.26 (Tentative).** *There exists $N \in \mathbb{N}$ such that $\tilde{V}_n = V_n$ for $n \geq N$.*

*Proof. (Tentative).* As seen in Proposition 2.11, the key-point is to show that $\tilde{V}_n$ is a solution to

$$\max\left\{\mathcal{A}\tilde{V}_n - \rho\tilde{V}_n + R,\ \mathcal{M}\tilde{V}_n - \tilde{V}_n\right\} = 0.$$

Let us focus on the first term: we need to prove that

$$\mathcal{A}\tilde{V}_n(x) - \rho\tilde{V}_n(x) + R(x) = \frac{\sigma^2}{2}\tilde{V}_n''(x) - \mu\tilde{V}_n'(x) - \rho\tilde{V}_n(x) + R(x) \leq 0,$$

for each $x \in \mathbb{R}$. Let us consider, in particular, the case $x \in [-1/n, 1/n]$. For $n$ big enough, by definition here we have $\tilde{V}_n(x) = \varphi_{A_1,A_2}(x^*) - c - \lambda\Phi_n(x)$, so that we have to prove that

$$-\frac{\lambda\sigma^2}{2}\Phi_n''(x) + \lambda\mu\Phi_n'(x) - \rho\big(\varphi_{A_1,A_2}(x^*) - c - \lambda\Phi_n(x)\big) + R(x) \leq 0, \tag{2.76}$$

for every $x \in [-1/n, 1/n]$. Now, the functions $\Phi_n$ have the following properties in $x = 0$:

$$\Phi_n(0) \in [1 - \Delta/n, 1], \qquad \Phi_n'(0) \in [-1/\Delta, 0], \qquad \lim_n \Phi_n''(0) = -\infty, \tag{2.77}$$

where the first two properties are immediate and the third one follows by $\Phi_n''(0) < 0$ (immediate, as $\Phi$ is concave in a neighbourhood of 0) and the Lagrange theorem:

$$\Delta = |\Phi_n'(1/n) - \Phi_n'(-1/n)| \leq \sup_{[-1/n,1/n]} |\Phi_n''|(2/n).$$

Since (2.77) holds, it is clear that (2.76) cannot be true in $x = 0$. Hence, in the present framework we cannot apply the verification theorem to our candidate. □

In short, our candidate $\tilde{V}_n$ does not satisfy the assumptions of the verification theorem. We now collect some remarks about this issue.

- *[Importance of a rigorous procedure]* When dealing with verification theorems in control theory, one first builds a candidate and then checks if it really satisfies all the assumptions of the theorem. Although this final passage is usually omitted and considered as obvious, we underline that it should not be underestimated. On the one hand, the proof presents, even in standard cases, some non-trivial parts and may require further conditions on the coefficients (see Proposition 2.11); on the other hand, we have provided an explicit example where the candidate value function, built by starting from the verification theorem, does *not* actually satisfy the assumptions of the theorem itself.

- *[Singularities successfully overcome]* We underline that our issue consists in applying the verification theorem to $V_n$, which is a problem with smooth penalty. Indeed, the approximating procedure has worked and we have been able to overcome the singularities in the penalty function.

- *[Research area: non-standard penalty function]* In the literature, all the explicit examples of stochastic impulsive problems have constant or linear penalty functions. However, it is clear that many practical models need penalties with a more complicated structure (for example, one can consider truncated penalties, like in our case). To the best of our knowledge, this is the first time that a non-standard penalty function is considered, and we have seen that the situation is not straightforward.

We are currently working on this open problem, according to the remarks above. In particular, we focus on the properties required to penalty functions.

### 2.4.3 A way out by stronger conditions on the controls

We here show a way to circumvent the problem outlined in the previous section, provided that stronger conditions on the admissible controls are required. More precisely, if we force the retailer to keep the process $X$ in the interval $]0, \Delta[$, the singularities of the penalty no longer belong to the set where the value function is defined and the standard arguments can be applied. We now detail this procedure.

From now on, let us assume that the following condition is also required to admissible controls $u \in \mathcal{U}_x$ (besides the ones in Definition 2.1), with $x \in \mathbb{R}$:

$$X_t^{x;u} \in \, ]0, \Delta[, \qquad \forall t \geq 0. \tag{2.78}$$

Practically, (2.78) forces the retailer to intervene (at least) every time his market share hits 0 or 1; in other words, we do not admit situations where the retailer has no customers or where he holds the monopoly of the market. Indeed, this assumption is quite strong, but realistic and reasonable as well.

In this new framework, it makes sense to define the value function $V$ only in the interval $]0, \Delta[$; explicitly, for each $x \in \, ]0, \Delta[$ we have

$$V(x) = \sup_{\substack{u \in \mathcal{U}_x \text{ s.t.} \\ X^{x;u} \in ]0,\Delta[}} \mathbb{E}_x \left[ \int_0^\infty e^{-\rho t} \left( X_t \Phi(X_t) - \frac{b}{2} \Phi(X_t)^2 \right) dt - \sum_k e^{-\rho \tau_k} \left( c - \frac{\lambda}{\Delta} \left( X_{(\tau_k)^-} - \Delta \right) \right) \right].$$

We underline that the second term in the intervention penalty is non-negative by (2.78). The candidate $\tilde{V}$ is now defined as follows.

**Definition 2.27.** *For each $x \in \, ]0, \Delta[$, we set*

$$\tilde{V}(x) = \begin{cases} \varphi_{A_1,A_2}(x), & \text{in } ]\underline{x}, \bar{x}[, \\ \varphi_{A_1,A_2}(x^*) - c + \lambda/\Delta(x - \Delta), & \text{in } ]0, \Delta[ \setminus ]\underline{x}, \bar{x}[, \end{cases}$$

*where $\varphi_{A_1,A_2}$ is as in* (2.20) *and the five parameters $(A_1, A_2, \underline{x}, \bar{x}, x^*)$ satisfy the conditions* (2.74)-(2.75)*.*

As in Section 2.3.3, the system in (2.74)-(2.75) is not tractable analitically; hence, we have to assume that a solution actually exists.

**Assumption 2.28.** *We assume that a solution to* (2.74)-(2.75) *exists. Moreover, we assume that there exist $\tilde{x}_1, \tilde{x}_2$, with $\underline{x} < \tilde{x}_1 < x^* < \tilde{x}_2 < \bar{x}$, such that $\varphi''_{A_1,A_2} < 0$ in $]\tilde{x}_1, \tilde{x}_2[$ and $\varphi''_{A_1,A_2} > 0$ in $]\underline{x}, \tilde{x}_1[ \cup ]\tilde{x}_2, \bar{x}[$. Finally, we assume $\underline{x} < \hat{x} < \bar{x}$, where we have set $\hat{x} = x_v - (\rho\lambda)/(2\alpha\Delta)$.*

As anticipated, now the penalty has no singularities in the set where $V$ is defined, so that apply the standard theory and argue as in Section 2.3.

**Lemma 2.29.** *Let* (2.78) *and Assumption 2.28 hold and let* $\tilde{V}$ *be as in Definition 2.27. Then, for every* $x \in \,]0, \Delta[$ *we have*

$$\mathcal{M}\tilde{V}(x) = \varphi_{A_1,A_2}(x^*) - c + \frac{\lambda}{\Delta}(x - \Delta).$$

*In particular, we have*

$$\{\mathcal{M}\tilde{V} - \tilde{V} < 0\} = ]\underline{x}, \bar{x}[, \qquad \{\mathcal{M}\tilde{V} - \tilde{V} = 0\} = ]0, \Delta[ \setminus ]\underline{x}, \bar{x}[.$$

*Proof.* The proof is similar to the one in Lemma 2.18, with some modifications due to the presence of the variable term in the penalty. First of all, let us study the monotonicity of $\tilde{V}$:

- $\tilde{V}$ is strictly increasing in $]0, \underline{x}[$ and $]\bar{x}, \Delta[$ by definition;
- $\tilde{V}$ is strictly increasing in $]\underline{x}, x^*[$ (by (2.75) we have $\varphi'_{A_1,A_2}(\underline{x}), \varphi'_{A_1,A_2}(x^*) \geq 0$ and by Assumption 2.28 the function $\varphi'_{A_1,A_2}$ is first increasing and then decreasing in $]\underline{x}, x^*[$, so that $\tilde{V}' = \varphi'_{A_1,A_2} > 0$ in $]\underline{x}, x^*[$) and strictly decreasing in $]x^*, \bar{x}[$ (by similar arguments).

It follows that $\sup_{]0,\Delta[} \tilde{V} \in \{\tilde{V}(x^*), \tilde{V}(\Delta^-)\}$ and that $\inf_{]0,\Delta[} \tilde{V} \in \{\tilde{V}(0^+), \tilde{V}(\bar{x})\}$. By (2.75) it is immediate to see that $\tilde{V}(x^*) > \tilde{V}(\Delta^-)$ and that $\tilde{V}(0^+) < \tilde{V}(\bar{x})$, so that we finally have

$$\max_{y \in ]0,\Delta[} \tilde{V}(y) = \tilde{V}(x^*) = \varphi_{A_1,A_2}(x^*), \qquad \inf_{y \in ]0,\Delta[} \tilde{V}(y) = \varphi_{A_1,A_2}(x^*) - c - \lambda.$$

As a consequence, for every $x \in \,]0, \Delta[$ we have

$$\mathcal{M}\tilde{V}(x) = \max_{\delta\,:\, x+\delta \in ]0,\Delta[} \{\tilde{V}(x + \delta) - c + \lambda/\Delta(x - \Delta)\} = \varphi_{A_1,A_2}(x^*) - c + \lambda/\Delta(x - \Delta).$$

By the definition of $\tilde{V}$, we have

$$\mathcal{M}\tilde{V}(x) - \tilde{V}(x) = 0, \qquad \text{in } ]0, \Delta[ \setminus ]\underline{x}, \bar{x}[.$$

Moreover, as $\varphi_{A_1,A_2}(\underline{x}) = \varphi_{A_1,A_2}(x^*) - c + \lambda/\Delta(\underline{x} - \Delta)$ by (2.75) and $\varphi_{A_1,A_2}(\underline{x}) = \min_{[\underline{x},\bar{x}]} \varphi_{A_1,A_2}$ by the previous arguments, we have

$$\begin{aligned}\mathcal{M}\tilde{V}(x) - \tilde{V}(x) &= \varphi_{A_1,A_2}(x^*) - c + \lambda/\Delta(x - \Delta) - \varphi_{A_1,A_2}(x) \\ &= \big(\varphi_{A_1,A_2}(\underline{x}) - \varphi_{A_1,A_2}(x)\big) - \lambda/\Delta(\underline{x} - x) < 0, \qquad \text{in } ]\underline{x}, \bar{x}[,\end{aligned}$$

which concludes the proof. □

**Proposition 2.30.** *Let* (2.78) *and Assumption 2.28 hold. Then, for every* $x \in ]0, \Delta[$ *an optimal control* $u^*(x)$ *for the problem in Section 2.2 exists and is given by* (2.44). *Moreover,* $\tilde{V}$ *coincides with the value function: foe every* $x \in \mathbb{R}$ *we have*

$$\tilde{V}(x) = V(x) = J(x; u^*(x)).$$

*Proof.* First of all, the condition in (2.78) does not change anything in the proof of the verification theorem, which still holds. Hence, we can use exactly the same arguments as the ones in Proposition 2.11, with minor modifications.

The only difference, due to the presence of the variable part in the penalty, is in the proof of

$$\mathcal{A}\tilde{V}(x) - \rho\tilde{V}(x) + R(x) \leq 0, \qquad \forall x \in ]0, \Delta[ \setminus ]\underline{x}, \bar{x}[.$$

We prove the claim for $x \in [\bar{x}, \Delta[$, the case $x \in ]0, \underline{x}]$ being similar. Since for $x \in [\bar{x}, \Delta[$ we have $\tilde{V}(x) = \varphi_{A_1,A_2}(x^*) - c + \lambda/\Delta(x - \Delta)$ by the definition of $\tilde{V}(x)$, the inequality reads

$$-\frac{\lambda\mu}{\Delta} - \rho\Big(\varphi_{A_1,A_2}(x^*) - c + \frac{\lambda}{\Delta}(x - \Delta)\Big) + R(x) \leq 0, \qquad \forall x \in [\bar{x}, \Delta[.$$

As $\varphi_{A_1,A_2}(\bar{x}) = \varphi_{A_1,A_2}(x^*) - c + \lambda/\Delta(\bar{x} - \Delta)$ by (2.75), we can rewrite as

$$-\frac{\lambda\mu}{\Delta} - \rho\Big(\varphi_{A_1,A_2}(\bar{x}) + \frac{\lambda}{\Delta}(x - \bar{x})\Big) + R(x) \leq 0, \qquad \forall x \in [\bar{x}, \Delta[.$$

The function $x \mapsto R(x) - \rho\lambda/\Delta$ is decreasing in $[\bar{x}, \Delta[$ by Assumption 2.28 (immediate check on the derivative); then, it is enough to prove the claim in $x = \bar{x}$:

$$-\frac{\lambda\mu}{\Delta} - \rho\varphi_{A_1,A_2}(\bar{x}) + R(\bar{x}) \leq 0.$$

Since $\mathcal{A}\varphi_{A_1,A_2}(\bar{x}) - \rho\varphi_{A_1,A_2}(\bar{x}) + f(\bar{x}) = 0$ and $f(\bar{x}) = R(\bar{x})$, we can rewrite as

$$-\frac{\lambda\mu}{\Delta} - \frac{\sigma^2}{2}\varphi''_{A_1,A_2}(\bar{x}) + \mu\varphi'_{A_1,A_2}(\bar{x}) \leq 0;$$

as $\varphi'_{A_1,A_2}(\bar{x}) = \lambda/\Delta$ by (2.75), we finally have

$$-\frac{\sigma^2}{2}\varphi''_{A_1,A_2}(\bar{x}) \leq 0,$$

which is true since $\varphi''_{A_1,A_2}(\bar{x}) \geq 0$ by Assumption 2.28. □

In short, if we impose the condition in (2.78) to admissible controls and if Assumption 2.28 holds, we can bypass the problem related to the singularities of $\Phi$ and characterize the value function and the optimal control.

# Chapter 3

# Non-zero-sum stochastic differential games with impulse controls

## 3.1 Introduction

In this chapter we provide a general framework for non-zero-sum stochastic games with impulse controls. Within this setting, we investigate the notion of Nash equilibrium through the corresponding quasi-variational inequalities. To the best of our knowledge, this class of games has not been addressed in the literature yet.

The theory of optimal stopping games dates back to the seventies: two players are present and each one decides when to intervene and stop the game. In the zero-sum case such problem has been studied in [21], whereas the non-zero-sum case was treated in [4]. To our knowledge, the only explicit application of the techniques in [4] is a very recent paper: see [13]. Conversely, if we consider games with iterate interventions by means of stopping times, i.e. with impulsive controls, the references in the literature are very few and exclusively related to the zero-sum case, see [14]. The goal of this chapter, as anticipated, is to address the non-zero-sum impulsive case: more precisely, we provide a rigorous framework for such problems, introduce Nash equilibria, define a suitable system of quasi-variational inequalities (QVI) and prove a verification theorem. We anticipate that the QVI problem in [14] can here be obtained as a particular case. Finally, a practical example will be provided.

More in detail, we consider problems where two players can affect a continuous-time stochastic process $X$ by discrete-time interventions. Each intervention, which consists in shifting $X$ to a new state, corresponds to a cost for the intervening player and to a gain for the opponent. When none of the players intervenes, we assume $X$ to diffuse according to a standard SDE. In our model, the action of player $i \in \{1, 2\}$ is determined by the couple $\varphi_i = (A_i, \xi_i)$, where $A_i$ is a subset of $\mathbb{R}^n$ and $\xi_i$ is a continuous function: player $i$ intervenes if and only if the controlled process exits from $A_i$ and, when this happens, he shifts the process from the state $x$ to the state $\xi_i(x)$. Let $S$ be a fixed subset of $\mathbb{R}^n$ and let $x \in S$ be the starting state of the game; the goal of player $i$ is to maximize his payoff $J^i$, defined as follows: for each $\varphi_1, \varphi_2$

and $i \in \{1,2\}$ we set

$$J^i(x;\varphi_1,\varphi_2) := \mathbb{E}_x\bigg[\int_0^{\tau_S} e^{-\rho_i s} f_i(X_s)ds + \sum_{1\le k\le M_i\,:\,\tau_{i,k}<\tau_S} e^{-\rho_i\tau_{i,k}}\phi_i\Big(X_{(\tau_{i,k})^-},\delta_{i,k}\Big) + \sum_{1\le k\le M_j\,:\,\tau_{j,k}<\tau_S} e^{-\rho_i\tau_{j,k}}\psi_i\Big(X_{(\tau_{j,k})^-},\delta_{j,k}\Big) + e^{-\rho_i\tau_S}h_i\big(X_{(\tau_S)^-}\big)\mathbb{1}_{\{\tau_S<+\infty\}}\bigg], \quad (3.1)$$

where $j \in \{1,2\}$ with $j \neq i$, the exit time from $S$ is denoted by $\tau_S$, the variable $\tau_{i,k}$ is the $k$-th intervention time of player $i$ and $\delta_{i,k}$ is the corresponding impulse (so that $u_i = \{(\tau_{i,k},\delta_{i,k})\}_{1\le k\le M_i}$ is the impulse control collecting player $i$'s intervention). The Nash equilibria for problem (3.1) are defined in the usual way.

Let $V_1(x)$ and $V_2(x)$ denote the value of the game with starting state $x \in \mathbb{R}$, in the case where a Nash equilibrium exists. We consider the following QVI problem, where $\mathcal{M}_i$ and $\mathcal{H}_i$ are suitable operators:

$$\begin{aligned}
&V_i = h_i, && \text{in } \partial S,\\
&\mathcal{M}_j V_j - V_j \le 0, && \text{in } S,\\
&\mathcal{H}_i V_i - V_i = 0, && \text{in } \{\mathcal{M}_j V_j - V_j = 0\},\\
&\max\big\{\mathcal{A}V_i - \rho_i V_i + f_i, \mathcal{M}_i V_i - V_i\big\} = 0, && \text{in } \{\mathcal{M}_j V_j - V_j < 0\}.
\end{aligned} \quad (3.2)$$

The main result of this chapter is the Verification Theorem 3.8: if two functions $V_i$, with $i \in \{1,2\}$, are a solution to (3.2), have polynomial growth and satisfy the regularity condition

$$V_i \in C^2(D_j \setminus \partial D_i) \cap C^1(D_j) \cap C(S), \quad (3.3)$$

where $j \in \{1,2\}$ with $j \neq i$ and $D_j = \{\mathcal{M}_j V_j - V_j < 0\}$, then they coincide with the value functions of the game and a characterization of the Nash strategy is possible.

Practically, one first tries to solve the last equation in (3.2), then sets the parameters so as to meet the regularity conditions in (3.3) (it will be clear during the chapter that this corresponds to a system of algebraic equations) and finally applies the verification theorem. In general, if the regularity conditions are too strong, the system may have more equations than parameters, so that there is (almost) no possibility to apply the verification theorem. In our opinion, this is the main weakness in [4]: indeed, we have found only one explicit application of [4], see [13], and relaxed conditions are needed. In our case, a practical example will show that the system is formally solvable. In short, an important contribution in this chapter consists in (3.3): it is the right condition for our problem, in the sense that it allows to prove the verification theorem but it also makes possible to practically apply this theorem.

As anticipated, besides developing a general theory for this class of problems, we show how to apply the verification theorem to a practical example. We consider a problem where two countries can affect the exchange rate between their currencies. The countries have different goals and we investigate the existence of Nash equilibria for this problem, which can be modelled in the form (3.1).

The chapter is structured as follows. In Section 3.2 we give a rigorous formalization to the problem in (3.1), define the equation of the QVI problem and prove the verification theorem. In Section 3.3 we apply the general theorem to the specific example outlined above.

## 3.2 Non-zero-sum impulsive games

In this section we consider a general class of two-player non-zero-sum stochastic differential games with impulse controls: after a rigorous formalization (Section 3.2.1), we define a suitable differential problem for the value functions of such games (Section 3.2.2) and prove a verification theorem (Section 3.2.3).

### 3.2.1 Formulation of the problem

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in[0,\infty[}, \mathbb{P})$ be a filtered probability space and let $\{W_s\}_{s\in[0,\infty[}$ be a $k$-dimensional $\{\mathcal{F}_s\}_s$-adapted Brownian motion; let $S$ be an open subset of $\mathbb{R}^d$. For every $t \in [0, \infty[$ and $y \in S$ we denote by $Y^{t,y}$ a solution to the problem

$$dY^{t,y}_s = b(Y^{t,y}_s)ds + \sigma(Y^{t,y}_s)dW_s, \qquad s \in [t, \infty[, \tag{3.4}$$

with initial condition $Y^{t,y}_t = y$, where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times k}$ are given continuous functions. We will later provide precise conditions ensuring that the process $Y^{t,y}$ is well-defined.

Two players, indexed by $i \in \{1, 2\}$, are present. Equation (3.4) models the underlying process when none of the players intervenes; conversely, if player $i$ intervenes with impulse $\delta \in Z_i$, the process is shifted from the present state $x$ to the new state $\Gamma^i(x, \delta)$, where $\Gamma^i : \mathbb{R}^d \times Z_i \to S$ is a continuous function and $Z_i$ is a fixed subset of $\mathbb{R}^{l_i}$, with $l_i \in \mathbb{N}$. Each intervention corresponds to a cost for the intervening player and to a gain for the opponent, both depending on the state $x$ and the impulse $\delta$.

The action of the players is modelled by discrete-time controls: an impulse control for player $i$ is a sequence

$$u_i = \{(\tau_{i,k}, \delta_{i,k})\}_{1\leq k\leq M_i}, \tag{3.5}$$

where $M_i \in \mathbb{N} \cup \{\infty\}$, $\{\tau_{i,k}\}_k$ are non-decreasing stopping times (the intervention times) and $\{\delta_{i,k}\}_k$ are $Z_i$-valued $\mathcal{F}_{\tau_{i,k}}$-measurable random variables (the corresponding impulses).

As usual with multiple-control games, we assume that the behaviour of the players, modelled by impulse controls, is driven by strategies. In this paper, the definition is as follows.

**Definition 3.1.** *A strategy for player $i \in \{1, 2\}$ is a couple $\varphi_i = (A_i, \xi_i)$, where $A_i$ is a fixed subset of $\mathbb{R}^d$ and $\xi_i$ is a continuous function from $\mathbb{R}^d$ to $Z_i$.*

Strategies determine the action of the players in the following sense. Once the couples $\varphi_i = (A_i, \xi_i)$ and a starting point $x \in S$ have been chosen, a couple of impulse controls, which we denote $u_i(x; \varphi_1, \varphi_2)$, is uniquely defined by the following procedure:

- player $i$ intervenes if and only if the process exits from $A_i$, in which case the impulse is given by $\xi_i(y)$, where $y$ is the state;
- a contemporary intervention is not possible: if both the players want to act, player 1 has the priority and player 2 does not intervene; (3.6)
- the game ends when the process exits from $S$.

In the following definition we provide a rigorous formalization of the controls associated to a couple of strategies and the corresponding controlled process, which we denote by $X^{x;\varphi_1,\varphi_2}$.

**Definition 3.2.** *Let $x \in S$ and let $\varphi_i = (A_i, \xi_i)$ be a strategy for player $i \in \{1,2\}$. Let $\tilde\tau_0 = 0, x_0 = x, \widetilde X^0 = Y^{\tilde\tau_0, x_0}, \alpha_0^S = \infty$ and consider the conventions $\inf\emptyset = \infty$ and $[\infty,\infty[ = \emptyset$; for every $k \in \mathbb{N}$ we define, by induction,*

$$\alpha_k^O = \inf\{s > \tilde\tau_{k-1} : \widetilde X_s^{k-1} \notin O\}, \qquad \text{[exit time from } O \subseteq \mathbb{R}^d\text{]}$$
$$\tilde\tau_k = (\alpha_k^{A_1} \wedge \alpha_k^{A_2} \wedge \alpha_k^S)\mathbb{1}_{\{\tilde\tau_{k-1} < \alpha_{k-1}^S\}} + \tilde\tau_{k-1}\mathbb{1}_{\{\tilde\tau_{k-1} = \alpha_{k-1}^S\}}, \qquad \text{[intervention time]}$$
$$m_k = \mathbb{1}_{\{\alpha_k^{A_1} \le \alpha_k^{A_2}\}} + 2\,\mathbb{1}_{\{\alpha_k^{A_2} < \alpha_k^{A_1}\}}, \qquad \text{[index of the player interv. at } \tilde\tau_k\text{]}$$
$$\tilde\delta_k = \xi_{m_k}\big(\widetilde X_{\tilde\tau_k}^{k-1}\big)\mathbb{1}_{\{\tilde\tau_k < \infty\}}, \qquad \text{[impulse]}$$
$$x_k = \Gamma^{m_k}\big(\widetilde X_{\tilde\tau_k}^{k-1}, \tilde\delta_k\big)\mathbb{1}_{\{\tilde\tau_k < \infty\}}, \qquad \text{[starting point for the next step]}$$
$$\widetilde X^k = \widetilde X^{k-1}\mathbb{1}_{[0,\tilde\tau_k[} + Y^{\tilde\tau_k, x_k}\mathbb{1}_{[\tilde\tau_k,\infty[}. \qquad \text{[contr. process up to the k-th interv.]}$$

*Let $\bar k \in \mathbb{N}\cup\{\infty\}$ be the index of the last significant intervention, and let $M_i \in \mathbb{N}\cup\{\infty\}$ be the number of interventions of player $i$:*

$$\bar k := \sup\big\{k \in \mathbb{N} : \mathbb{P}_x(\tilde\tau_k = \alpha_k^S) < 1 \text{ and } \mathbb{P}_x(\tilde\tau_k = \infty) < 1\big\},$$
$$M_i := \textstyle\sum_{1\le k\le \bar k} \mathbb{1}_{\{m_k = i\}}(k).$$

*For $i \in \{1,2\}$ and $1 \le k \le M_i$, let $\eta(i,k) = \min\{l \in \mathbb{N} : \sum_{1\le h\le l}\mathbb{1}_{\{m_h = i\}} = k\}$ (index of the k-th intervention of player i) and let*

$$\tau_{i,k} := \tilde\tau_{\eta(i,k)}, \qquad \delta_{i,k} := \tilde\delta_{\eta(i,k)}. \tag{3.7}$$

*Finally, the controls $u_i(x,\varphi_1,\varphi_2)$, $i \in \{1,2\}$, the controlled process $X^{x,\varphi_1,\varphi_2}$ and the exit time from $S$ are defined by (with the convention $\inf\emptyset = \infty$)*

$$u_i(x,\varphi_1,\varphi_2) := \{(\tau_{i,k},\delta_{i,k})\}_{1\le k\le M_i},$$
$$X^{x,\varphi_1,\varphi_2} := \widetilde X^{\bar k},$$
$$\tau_S^{x;\varphi_1,\varphi_2} = \inf\{s > 0 : X_s^{x;\varphi_1,\varphi_2} \notin S\}.$$

To shorten the notations, we will simply write $X$ and $\tau_S$. Notice that player 1 has priority in case of contemporary intervention (i.e. if $\alpha_k^{A_1} = \alpha_k^{A_2}$). In the following lemma we give a rigorous formulation to the properties outlined in (3.6).

**Lemma 3.3.** *Let $x \in S$ and let $\varphi_i = (A_i,\xi_i)$ be a strategy for player $i \in \{1,2\}$.*

- *The process $X$ admits the following representation (with convention $[\infty,\infty[=\emptyset$):*

$$X_s = \sum_{k=0}^{\bar k - 1} Y_s^{\tilde\tau_k, x_k}\mathbb{1}_{[\tilde\tau_k,\tilde\tau_{k+1}[}(s) + Y_s^{\tilde\tau_{\bar k}, x_{\bar k}}\mathbb{1}_{[\tilde\tau_{\bar k},\infty[}(s), \tag{3.8}$$

- *The process $X$ is right-continuous. In detail, $X$ is continuous and satisfies Equation (3.4) in $[0,\infty[\setminus\{\tau_{i,k} : \tau_{i,k} < \infty\}$, whereas $X$ is discontinuous in $\{\tau_{i,k} : \tau_{i,k} < \infty\}$, where we have*

$$X_{\tau_{i,k}} = \Gamma^i\big(X_{(\tau_{i,k})^-}, \delta_{i,k}\big), \qquad \delta_{i,k} = \xi_i\big(X_{(\tau_{i,k})^-}\big), \qquad X_{(\tau_{i,k})^-} \in \partial A_i. \tag{3.9}$$

- *The process $X$ never exits from the set $A_1 \cap A_2$.*

*Proof.* We just prove the first property in (3.9), the other ones being immediate. Let $i \in \{1,2\}$, $1 \leq k \leq M_i$ with $\tau_{i,k} < \infty$ and set $\sigma = \eta(i,k)$, with $\eta$ as in Definition 3.2. By (3.7), (3.8) and Definition 3.2, we have

$$\begin{aligned} X_{\tau_{i,k}} = X_{\tilde\tau_\sigma} = Y^{\tilde\tau_\sigma, x_\sigma}_{\tilde\tau_\sigma} = x_\sigma = \Gamma^i\big(\widetilde{X}^{\sigma-1}_{\tilde\tau_\sigma}, \tilde\delta_\sigma\big) \\ = \Gamma^i\big(\widetilde{X}^{\sigma-1}_{(\tilde\tau_\sigma)^-}, \tilde\delta_\sigma\big) = \Gamma^i\big(X_{(\tilde\tau_\sigma)^-}, \tilde\delta_\sigma\big) = \Gamma^i\big(X_{(\tau_{i,k})^-}, \delta_{i,k}\big), \end{aligned}$$

which concludes the proof (we have used the continuity of the process $\widetilde{X}^{\sigma-1}$ in $[\tilde\tau_{\sigma-1}, \infty[$ and the fact that $\widetilde{X}^{\sigma-1} \equiv X$ in $[0, \tilde\tau_\sigma[$). □

Each player aims at maximizing his payoff, consisting in four discounted terms: a running payoff, the costs due to the player's own interventions, the gains due to the opponent's interventions and a terminal payoff. More precisely, for each $i \in \{1,2\}$ we consider $\rho_i > 0$ (the discount rate) and continuous functions $f_i : \mathbb{R}^d \to \mathbb{R}$ (the running payoff), $h_i : \mathbb{R}^d \to \mathbb{R}$ (the terminal payoff) and $\phi_i : \mathbb{R}^d \times Z_i \to \mathbb{R}$, $\psi_i : \mathbb{R}^d \times Z_j \to \mathbb{R}$ (the intervention penalties/gains), where $j \in \{1,2\}$ with $j \neq i$; the payoff of player $i$ is defined as follows.

**Definition 3.4.** *Let $x \in S$ and let $(\varphi_1, \varphi_2)$ be a couple of strategies. For each $i \in \{1,2\}$, provided that the right-hand side exists and is finite, we set*

$$\begin{aligned} J^i(x; \varphi_1, \varphi_2) := \mathbb{E}_x\bigg[ \int_0^{\tau_S} e^{-\rho_i s} f_i(X_s) ds + \sum_{1 \leq k \leq M_i \,:\, \tau_{i,k} < \tau_S} e^{-\rho_i \tau_{i,k}} \phi_i\Big(X_{(\tau_{i,k})^-}, \delta_{i,k}\Big) \\ + \sum_{1 \leq k \leq M_j \,:\, \tau_{j,k} < \tau_S} e^{-\rho_i \tau_{j,k}} \psi_i\Big(X_{(\tau_{j,k})^-}, \delta_{j,k}\Big) + e^{-\rho_i \tau_S} h_i\big(X_{(\tau_S)^-}\big) \mathbb{1}_{\{\tau_S < +\infty\}} \bigg], \end{aligned} \quad (3.10)$$

*where $j \in \{1,2\}$ with $j \neq i$ and $\{(\tau_{i,k}, \delta_{i,k})\}_{1 \leq k \leq M_i}$ is the impulse controls of player $i$ associated to the strategies $\varphi_1, \varphi_2$.*

As usual in control theory, the subscript in the expectation recalls the starting point. Notice that we do not consider stopping times which equal $\tau_S$: indeed, since $\tau_S$ is the final time, we would pay exercise penalties without changing anything.

In order to have a good definition in (3.10), we now define the set of admissible strategies in $x \in S$.

**Definition 3.5.** *Let $x \in S$ and $\varphi_i = (A_i, \xi_i)$ be a strategy for player $i \in \{1,2\}$. We use the notations of Definition 3.2 and say that the couple $(\varphi_1, \varphi_2)$ is $x$-admissible if:*

1. *for every $k \in \mathbb{N} \cup \{0\}$, the process $Y^{\tilde\tau_k, x_k}$ exists and is uniquely defined;*
2. *for $i \in \{1,2\}$, the following variables are in $L^1(\Omega)$:*

$$\begin{aligned} \int_0^{\tau_S} e^{-\rho_i s} |f_i|(X_s) ds, \qquad e^{-\rho_i \tau_S} |h_i|(X_{(\tau_S)^-}), \\ \sum_{\tau_{i,k} < \tau_S} e^{-\rho_i \tau_{i,k}} |\phi_i|(X_{(\tau_{i,k})^-}, \delta_{i,k}), \qquad \sum_{\tau_{i,k} < \tau_S} e^{-\rho_i \tau_{i,k}} |\psi_i|(X_{(\tau_{i,k})^-}, \delta_{i,k}); \end{aligned} \quad (3.11)$$

*3. for each $k \in \mathbb{N}$, the process $\|X\|_\infty = \sup_{t\geq 0} |X_t|$ is in $L^k(\Omega)$:*

$$\mathbb{E}_x[\|X\|_\infty^k] < \infty; \tag{3.12}$$

*4. if $\tau_{i,k} = \tau_{i,k+1}$ for some $i \in \{1,2\}$ and $1 \leq k \leq M_i$, then $\tau_{i,k} = \tau_{i,k+1} = \tau_S$;*

*5. if there exists $\lim_k \tau_{i,k} =: \eta$ for some $i \in \{1,2\}$, then $\eta = \tau_S$.*

*We denote by $\mathcal{A}_x$ the set of the $x$-admissible couples.*

Thanks to the first and the second conditions in Definition 3.5, the controls $u_i(x; \varphi_1, \varphi_2)$ and the payoffs $J^i(x; \varphi_1, \varphi_2)$ are well-defined. The third condition will be used in the proof of the verification theorem. As for the fourth and the fifth conditions, they prevent each player to exercise twice at the same time and to accumulate the interventions before $\tau_S$.

We conclude the section with the definition of Nash equilibria.

**Definition 3.6.** *Given $x \in S$, we say that a couple $(\varphi_1^*, \varphi_2^*) \in \mathcal{A}_x$ is a Nash equilibrium of the game if*

$$\begin{aligned} J^1(x; \varphi_1^*, \varphi_2^*) &\geq J^1(x; \varphi_1, \varphi_2^*), \qquad \forall \varphi_1 \text{ s.t. } (\varphi_1, \varphi_2^*) \in \mathcal{A}_x, \\ J^2(x; \varphi_1^*, \varphi_2^*) &\geq J^2(x; \varphi_1^*, \varphi_2), \qquad \forall \varphi_2 \text{ s.t. } (\varphi_1^*, \varphi_2) \in \mathcal{A}_x. \end{aligned}$$

*Finally, the value functions of the game are defined as follows: if $x \in S$ and a Nash equilibrium $(\varphi_1^*, \varphi_2^*) \in \mathcal{A}_x$ exists, we set $V_i(x) = J^i(x; \varphi_1^*, \varphi_2^*)$ for $i \in \{1,2\}$.*

### 3.2.2 The quasi-variational inequality problem

We now introduce the differential problem satisfied by the value functions of our games: this will be the key-point of the verification theorem in the next section.

Let us consider an impulsive game as in Section 3.2.1. Assume that the corresponding value functions $V_1, V_2$ are defined for each $x \in S$ and that the following property is satisfied: for $i \in \{1,2\}$ there exists a function $\delta_i$ from $S$ to $Z_i$ such that

$$\{\delta_i(x)\} = \arg\max_{\delta \in Z_i} \big\{ V_i(\Gamma^i(x,\delta)) + \phi_i(x,\delta) \big\}, \tag{3.13}$$

for each $x \in S$. We define the four intervention operators by

$$\begin{aligned} \mathcal{M}_i V_i(x) &= V_i\big(\Gamma^i(x, \delta_i(x))\big) + \phi_i\big(x, \delta_i(x)\big), \\ \mathcal{H}_i V_i(x) &= V_i\big(\Gamma^j(x, \delta_j(x))\big) + \psi_i\big(x, \delta_j(x)\big), \end{aligned} \tag{3.14}$$

for $x \in S$ and $i, j \in \{1,2\}$, with $i \neq j$. Notice that $\mathcal{M}_i V_i = \max_\delta \{V_i(\Gamma^i(\cdot,\delta)) + \phi_i(\cdot,\delta)\}$.

The functions in (3.13) and (3.14) have an immediate practical interpretation. Let $x$ be the current state of the process; if player $i$ (resp. player $j$) intervenes with impulse $\delta$, the present value of the game for player $i$ can be written as $V_i(\Gamma^i(x,\delta)) + \phi_i(x,\delta)$ (resp. $V_i(\Gamma^j(x,\delta)) + \psi_i(x,\delta)$): we have considered the intervention cost (resp. gain) and the value in the new state. Hence, $\delta_i(x)$ is the impulse that player $i$ would use in case of intervention. Similarly, $\mathcal{M}_i V_i(x)$ (resp. $\mathcal{H}_i V_i(x)$) represents the value of the game for player $i$ under the additional assumption that player $i$ (resp. player $j$) immediately intervenes. Notice that, as a consequence, player $i$

should intervene (with impulse $\delta_i(x)$, as already seen) if and only if $\mathcal{M}_i V_i(x) = V_i(x)$: we have an heuristic formulation for the Nash equilibria, provided that an explicit expression for $V_i$ is available. Indeed, the verification theorem will give a rigorous proof to this argument. However, in order to practically apply this idea, we first need to characterize the value functions $V_i$.

Assume $V_1, V_2 \in C^2(S)$ (weaker conditions will be given later) and define

$$\mathcal{A}V_i = b \cdot \nabla V_i + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^t D^2 V_i\right),$$

where $b, \sigma$ are as in (3.4), $\sigma^t$ denotes the transpose of $\sigma$ and $\nabla V_i, D^2 V_i$ are the gradient and the Hessian matrix of $V_i$. We are interested in the following quasi-variational inequalities (QVI) for $V_1, V_2$, where $i, j \in \{1,2\}$ and $i \neq j$:

$$\begin{aligned}
&V_i = h_i, && \text{in } \partial S, && \text{(3.15a)}\\
&\mathcal{M}_j V_j - V_j \leq 0, && \text{in } S, && \text{(3.15b)}\\
&\mathcal{H}_i V_i - V_i = 0, && \text{in } \{\mathcal{M}_j V_j - V_j = 0\}, && \text{(3.15c)}\\
&\max\left\{\mathcal{A}V_i - \rho_i V_i + f_i, \mathcal{M}_i V_i - V_i\right\} = 0, && \text{in } \{\mathcal{M}_j V_j - V_j < 0\}. && \text{(3.15d)}
\end{aligned}$$

Indeed, there is a small abuse of notation in (3.15a), as $V_i$ is not defined in $\partial S$ (the domain is the open set $S$): we mean $\lim_{y \to x} V_i(y) = h_i(x)$, for each $x \in \partial S$.

We now heuristically explain why the conditions in (3.15a)-(3.15d) actually represent the right differential problem for $V_i$; once more, the verification theorem will provide a rigorous proof to this claim. First of all, the terminal condition is obvious. Moreover, as $\mathcal{M}_j V_j$ represents the value of the game under an additional assumption, we also expect $\mathcal{M}_j V_j - V_j \leq 0$: indeed, this is a standard condition in impulse control theory. Now, if player $j$ intervenes (i.e. $\mathcal{M}_j V_j - V_j = 0$), by the definition of Nash equilibrium we expect that player $i$ does not lose anything: this is modelled by $\mathcal{H}_i V_i - V_i = 0$. On the contrary, if player $j$ does not intervene (i.e. $\mathcal{M}_j V_j - V_j < 0$), then $V_i$ behaves according to the PDE of a standard one-player impulse problem, that is $\max\left\{\mathcal{A}V_i - \rho_i V_i + f_i, \mathcal{M}_i V_i - V_i\right\} = 0$. In short, the latter condition says that $\mathcal{A}V_i - \rho_i V_i + f_i \leq 0$, with equality in case of non-intervention (i.e. $\mathcal{M}_i V_i - V_i < 0$).

We remark that the functions $V_i$ can be unbounded. Indeed, this is the typical case when the penalties depend on the impulse: when the state diverges to infinity, one of the player has to pay a bigger and bigger cost to move the process to the continuation region, which corresponds to a strictly decreasing value function (whereas the value of the game is strictly increasing for the competitor, who gains from the opponent's intervention). As a comparison, we recall that in one-player impulsive problems the value function is usually bounded from above. Finally, we notice that the operator $\mathcal{A}V_i$ appears only in the region $\{\mathcal{M}_j V_j - V_j < 0\}$; as a consequence, the function $V_i$ needs to be of class $C^2$ only in such region (indeed, this assumption can be slightly relaxed, as we will see). Again, we remark the difference with the one-player case, where the value function is asked to be twice differentiable almost everywhere in $S$, see [32, Thm. 6.2].

A verification theorem will be provided in the next section. Here, as a preliminary check on the problem we propose, we show that we are indeed generalizing the sufficient condition provided in [14], where the zero-sum case is considered. To do so, we prove that, if we assume

$$\begin{aligned} f := f_1 = -f_2, \quad \phi := \phi_1 = -\psi_2, \quad \psi := \psi_1 = -\phi_2, \quad h := h_1 = -h_2, \\ Z := Z_1 = Z_2, \quad \Gamma := \Gamma^1 = \Gamma^2, \quad V := V_1 = -V_2, \end{aligned} \tag{3.16}$$

then the problem in (3.15) collapses into the one considered in [14]. To shorten the equations, we assume $\rho_1 = \rho_2 = 0$ (this is possible since in [14] a finite-horizon problem is considered). First of all, we define

$$\widetilde{\mathcal{M}}V(x) = \sup_{\delta \in Z} \big\{ V(\Gamma(x,\delta)) + \phi(x,\delta) \big\},$$
$$\widetilde{\mathcal{H}}V(x) = \inf_{\delta \in Z} \big\{ V(\Gamma(x,\delta)) + \psi(x,\delta) \big\},$$

for each $x \in S$. It is easy to see that, under the conditions in (3.16), we have

$$\mathcal{M}_1 V_1 = \widetilde{\mathcal{M}}V, \qquad \mathcal{M}_2 V_2 = -\widetilde{\mathcal{H}}V, \qquad \mathcal{H}_1 V_1 = \widetilde{\mathcal{H}}V, \qquad \mathcal{H}_2 V_2 = -\widetilde{\mathcal{M}}V,$$

so that problem (3.15) writes

$$V = h, \qquad \text{in } \partial S, \tag{3.17a}$$
$$\widetilde{\mathcal{M}}V \le V \le \widetilde{\mathcal{H}}V, \qquad \text{in } S, \tag{3.17b}$$
$$\mathcal{A}V + f \le 0, \qquad \text{in } \{V = \widetilde{\mathcal{M}}V\}, \tag{3.17c}$$
$$\mathcal{A}V + f = 0, \qquad \text{in } \{\widetilde{\mathcal{M}}V < V < \widetilde{\mathcal{H}}V\}, \tag{3.17d}$$
$$\mathcal{A}V + f \ge 0, \qquad \text{in } \{V = \widetilde{\mathcal{H}}V\}. \tag{3.17e}$$

Simple computations, reported below, show that problem (3.17) is equivalent to

$$V = h, \qquad \text{in } \partial S, \tag{3.18a}$$
$$\widetilde{\mathcal{M}}V - V \le 0, \qquad \text{in } S, \tag{3.18b}$$
$$\min\{\max\{\mathcal{A}V + f, \widetilde{\mathcal{M}}V - V\}, \widetilde{\mathcal{H}}V - V\} = 0, \qquad \text{in } S, \tag{3.18c}$$

which is exactly Cosso's problem, as anticipated. We conclude this section by proving the equivalence of (3.17) and (3.18).

**Lemma 3.7.** *Problems* (3.17) *and* (3.18) *are equivalent.*

*Proof. Step 1.* We prove that (3.17) implies (3.18). The only property to be proved is (3.18c). We consider three cases.

First, assume $V = \widetilde{\mathcal{M}}V$. Since $\mathcal{A}V + f \le 0$ and $\widetilde{\mathcal{M}}V - V = 0$, we have $\max\{\mathcal{A}V + f, \widetilde{\mathcal{M}}V - V\} = 0$, which implies (3.18c) since $\widetilde{\mathcal{H}}V - V \ge 0$. Then, assume $\widetilde{\mathcal{M}}V < V < \widetilde{\mathcal{H}}V$. Since $\mathcal{A}V + f = 0$ and $\widetilde{\mathcal{M}}V - V < 0$, we have $\max\{\mathcal{A}V + f, \widetilde{\mathcal{M}}V - V\} = 0$, which implies (3.18c) since $\widetilde{\mathcal{H}}V - V > 0$. Finally, assume $V = \widetilde{\mathcal{H}}V$. Since $\mathcal{A}V + f \ge 0$ and $\widetilde{\mathcal{M}}V - V \le 0$, we have $\max\{\mathcal{A}V + f, \widetilde{\mathcal{M}}V - V\} \ge 0$, which implies (3.18c) since $\widetilde{\mathcal{H}}V - V = 0$.

*Step 2.* We prove that (3.18) implies (3.17). The only properties to be proved are (3.17c), (3.17d) and (3.17e). We assume $\widetilde{\mathcal{M}}V < \widetilde{\mathcal{H}}V$ (the case $\widetilde{\mathcal{M}}V = \widetilde{\mathcal{H}}V$ being immediate) and consider three cases.

First, assume $V = \widetilde{\mathcal{M}}V$. Since $\widetilde{\mathcal{H}}V - V > 0$, from (3.18c) it follows that $\max\{\mathcal{A}V + f, 0\} = 0$, which implies $\mathcal{A}V + f \leq 0$. Then, assume $\widetilde{\mathcal{M}}V < V < \widetilde{\mathcal{H}}V$. Since $\min\{\max\{\alpha, \beta\}, \gamma\} \in \{\alpha, \beta, \gamma\}$ for every $\alpha, \beta, \gamma \in \mathbb{R}$, and since $\widetilde{\mathcal{M}}V - V < 0 < \widetilde{\mathcal{H}}V - V$, from (3.18c) it follows that $\mathcal{A}V + f = 0$. Finally, assume $V = \widetilde{\mathcal{H}}V$. From (3.18c) it follows that $\max\{\mathcal{A}V + f, \widetilde{\mathcal{M}}V - V\} \geq 0$, which implies $\mathcal{A}V + f \geq 0$ since $\widetilde{\mathcal{M}}V - V < 0$. □

### 3.2.3 A verification theorem

We finally prove a verification theorem for the problems formalized in Section 3.2.1.

**Theorem 3.8** (**Verification theorem**). *In the assumptions of Sections 3.2.1, let $i \in \{1, 2\}$ and let $V_i$ be a function from $S$ to $\mathbb{R}$. Assume that* (3.13) *holds and set $D_i = \{\mathcal{M}_i V_i - V_i < 0\}$, with $\mathcal{M}_i V_i$ as in* (3.14)*. Moreover, for $i \in \{1, 2\}$ assume that:*

- *$V_i$ is a solution to* (3.15)*;*
- *$V_i \in C^2(D_j \setminus \partial D_i) \cap C^1(D_j) \cap C(S)$ and has polynomial growth;*
- *$\partial D_i$ is a Lipschitz surface and $V_i$ has locally bounded derivatives near $\partial D_i$.*

*Finally, let $x \in S$ and assume that $(\varphi_1^*, \varphi_2^*) \in \mathcal{A}_x$, where*

$$\varphi_i^* = (D_i, \delta_i),$$

*with $i \in \{1, 2\}$, the set $D_i$ as above and the function $\delta_i$ as in* (3.13)*. Then,*

$$(\varphi_1^*, \varphi_2^*) \text{ is a Nash equilibrium and } V_i(x) = J^i(x; \varphi_1^*, \varphi_2^*) \text{ for } i \in \{1, 2\}.$$

**Remark 3.9.** *Basically, we are saying that the Nash strategy is characterized as follows: player i intervenes if and only if the controlled process exits from the region $\{\mathcal{M}_i V_i - V_i < 0\}$ (equivalently, if and only if $\mathcal{M}_i V_i(x) = V_i(x)$, where $x$ is the current state); when this happens, the impulse is $\delta_i(x)$.*

**Remark 3.10.** *In the case of such (candidate) optimal strategies, we notice that the properties in Lemma 3.3 read as follows (indeed, the notations are a bit heavy, but in the proof it is fundamental to have explicit indexes in every parameter):*

$$(\mathcal{M}_1 V_1 - V_1)\big(X_s^{x;\varphi_1^*,\varphi_2}\big) < 0, \tag{3.19a}$$

$$(\mathcal{M}_2 V_2 - V_2)\big(X_s^{x;\varphi_1,\varphi_2^*}\big) < 0, \tag{3.19b}$$

$$\delta_{1,k}^{x;\varphi_1^*,\varphi_2} = \delta_1\left(X^{x;\varphi_1^*,\varphi_2}_{\left(\tau_{1,k}^{x;\varphi_1^*,\varphi_2}\right)^-}\right), \tag{3.19c}$$

$$\delta_{2,k}^{x;\varphi_1,\varphi_2^*} = \delta_2\left(X^{x;\varphi_1,\varphi_2^*}_{\left(\tau_{2,k}^{x;\varphi_1,\varphi_2^*}\right)^-}\right), \tag{3.19d}$$

$$(\mathcal{M}_1 V_1 - V_1)\left(X^{x;\varphi_1^*,\varphi_2}_{\left(\tau_{1,k}^{x;\varphi_1^*,\varphi_2}\right)^-}\right) = 0, \tag{3.19e}$$

$$(\mathcal{M}_2 V_2 - V_2)\left(X^{x;\varphi_1,\varphi_2^*}_{\left(\tau_{2,k}^{x;\varphi_1,\varphi_2^*}\right)^-}\right) = 0, \tag{3.19f}$$

*for every $\varphi_1, \varphi_2$ strategies such that $(\varphi_1, \varphi_2^*), (\varphi_1^*, \varphi_2) \in \mathcal{A}_x$, every $s \in [0, \infty[$ and every $\tau_{i,k}^{x;\varphi_1,\varphi_2^*}$, $\tau_{i,k}^{x;\varphi_1^*,\varphi_2} < \infty$.*

*Proof.* By Definition 3.6, we have to prove that

$$V_i(x) = J^i(x;\varphi_1^*,\varphi_2^*), \qquad V_1(x) \geq J^1(x;\varphi_1,\varphi_2^*), \qquad V_2(x) \geq J^2(x;\varphi_1^*,\varphi_2),$$

for every $i \in \{1,2\}$ and $(\varphi_1,\varphi_2)$ strategies such that $(\varphi_1,\varphi_2^*) \in \mathcal{A}_x$ and $(\varphi_1^*,\varphi_2) \in \mathcal{A}_x$. We show the results for $V_1$ and $J^1$, the arguments for $V_2$ and $J^2$ being symmetric.

*Step 1:* $V_1(x) \geq J^1(x;\varphi_1,\varphi_2^*)$. Let $\varphi_1$ be a strategy for player 1 such that $(\varphi_1,\varphi_2^*) \in \mathcal{A}_x$. In this step, the shortened notations have the following meaning:

$$X = X^{x;\varphi_1,\varphi_2^*}, \qquad \tau_{i,k} = \tau_{i,k}^{x;\varphi_1,\varphi_2^*}, \qquad \delta_{i,k} = \delta_{i,k}^{x;\varphi_1,\varphi_2^*}.$$

Thanks to the regularity assumptions and by standard approximation arguments, it is not restrictive to assume $V_1 \in C^2(D_2) \cap C(S)$: see [32, Thm. 3.1]. For each $r > 0$ and $n \in \mathbb{N}$, we set

$$\tau_{r,n} = \tau_S \wedge \tau_r \wedge n,$$

where $\tau_r = \inf\{s > 0 : X_s \notin B(0,r)\}$ is the exit time from the ball with radius $r$. We apply Itô's formula to the function $(t,X_t) \mapsto e^{-\rho_1 t}V_1(X_t)$, integrate in the interval $[0,\tau_{r,n}]$ and take expectations (the initial point and the stochastic integral are integrable, so that the other terms are integrable too by equality): we get

$$\begin{aligned} V_1(x) = \mathbb{E}_x\bigg[ &-\int_0^{\tau_{r,n}} e^{-\rho_1 s}(\mathcal{A}V_1 - \rho_1 V_1)(X_s)ds - \sum_{\tau_{1,k}<\tau_{r,n}} e^{-\rho_1\tau_{1,k}}\Big(V_1\big(X_{\tau_{1,k}}\big) - V_1\big(X_{(\tau_{1,k})^-}\big)\Big) \\ &- \sum_{\tau_{2,k}<\tau_{r,n}} e^{-\rho_1\tau_{2,k}}\Big(V_1\big(X_{\tau_{2,k}}\big) - V_1\big(X_{(\tau_{2,k})^-}\big)\Big) + e^{-\rho_1\tau_{r,n}}V_1(X_{\tau_{r,n}})\bigg]. \end{aligned} \tag{3.20}$$

We now estimate each term in the right-hand side of (3.20). As for the first term, since $(\mathcal{M}_2V_2 - V_2)(X_s) < 0$ by (3.19b), from (3.15d) it follows that

$$(\mathcal{A}V_1 - \rho_1 V_1)(X_s) \leq -f_1(X_s), \tag{3.21}$$

for all $s \in [0,\tau_S]$. Let us now consider the second term: by (3.15b) and the definition of $\mathcal{M}_1V_1$ in (3.14), for every stopping time $\tau_{1,k} < \tau_S$ we have

$$\begin{aligned} V_1\big(X_{(\tau_{1,k})^-}\big) &\geq \mathcal{M}_1V_1\big(X_{(\tau_{1,k})^-}\big) \\ &= \sup_{\delta\in Z_1}\big\{V_1\big(\Gamma^1\big(X_{(\tau_{1,k})^-},\delta\big)\big) + \phi_1\big(X_{(\tau_{1,k})^-},\delta\big)\big\} \\ &\geq V_1\big(\Gamma^1\big(X_{(\tau_{1,k})^-},\delta_{1,k}\big)\big) + \phi_1\big(X_{(\tau_{1,k})^-},\delta_{1,k}\big) \\ &= V_1\big(X_{\tau_{1,k}}\big) + \phi_1\big(X_{(\tau_{1,k})^-},\delta_{1,k}\big). \end{aligned} \tag{3.22}$$

As for the third term, let us consider any stopping time $\tau_{2,k} < \tau_S$. By (3.19f) we have $(\mathcal{M}_2V_2 - V_2)\big(X_{(\tau_{2,k})^-}\big) = 0$; hence, the condition in (3.15c), the definition of $\mathcal{H}_1V_1$ in (3.14) and the expression of $\delta_{2,k}$ in (3.19d) imply that

$$\begin{aligned} V_1\big(X_{(\tau_{2,k})^-}\big) &= \mathcal{H}_1V_1\big(X_{(\tau_{2,k})^-}\big) \\ &= V_1\big(\Gamma^2\big(X_{(\tau_{2,k})^-},\delta_2\big(X_{(\tau_{2,k})^-}\big)\big)\big) + \psi_1\big(X_{(\tau_{2,k})^-},\delta_2\big(X_{(\tau_{2,k})^-}\big)\big) \\ &= V_1\big(\Gamma^2\big(X_{(\tau_{2,k})^-},\delta_{2,k}\big)\big) + \psi_1\big(X_{(\tau_{2,k})^-},\delta_{2,k}\big) \\ &= V_1\big(X_{\tau_{2,k}}\big) + \psi_1\big(X_{(\tau_{2,k})^-},\delta_{2,k}\big). \end{aligned} \tag{3.23}$$

By (3.20) and the estimates in (3.21)-(3.23) it follows that

$$V_1(x) \geq \mathbb{E}_x\bigg[\int_0^{\tau_{r,n}} e^{-\rho_1 s} f_1(X_s)ds + \sum_{\tau_{1,k}<\tau_{r,n}} e^{-\rho_1\tau_{1,k}}\phi_1\big(X_{(\tau_{1,k})^-}, \delta_{1,k}\big) + \sum_{\tau_{2,k}<\tau_{r,n}} e^{-\rho_1\tau_{2,k}}\psi_1\big(X_{(\tau_{2,k})^-}, \delta_{2,k}\big) + e^{-\rho_1\tau_{r,n}}V_1(X_{\tau_{r,n}})\bigg].$$

Thanks to the conditions in (3.11), (3.12) and the polynomial growth of $V_1$, we can use the dominated convergence theorem and pass to the limit, first as $r \to \infty$ and then as $n \to \infty$. In particular, for the fourth term we notice that

$$V_1(X_{\tau_{r,n}}) \leq C(1 + |X_{\tau_{r,n}}|^k) \leq C(1 + \|X\|_\infty^k) \in L^1(\Omega), \tag{3.24}$$

for suitable constants $C > 0$ and $k \in \mathbb{N}$; the corresponding limit immediately follows by the continuity of $V_1$ in the case $\tau_S < \infty$ and by (3.24) itself in the case $\tau_S = \infty$ (as a direct consequence of (3.12), we have $\|X\|_\infty^k < \infty$ a.s.). Hence, we finally get

$$V_1(x) \geq \mathbb{E}_x\bigg[\int_0^{\tau_S} e^{-\rho_1 s} f_1(X_s)ds + \sum_{\tau_{1,k}<\tau_S} e^{-\rho_1\tau_{1,k}}\phi_1\big(X_{(\tau_{1,k})^-}, \delta_{1,k}\big) + \sum_{\tau_{2,k}<\tau_S} e^{-\rho_1\tau_{2,k}}\psi_1\big(X_{(\tau_{2,k})^-}, \delta_{2,k}\big) + e^{-\rho_1\tau_S}h_1(X_{(\tau_S)^-})\mathbb{1}_{\{\tau_S<+\infty\}}\bigg] = J^1(x; \varphi_1, \varphi_2^*).$$

*Step 2:* $V_1(x) = J^1(x; \varphi_1^*, \varphi_2^*)$. We argue as in Step 1, but here all the inequalities are equalities by the properties of $\varphi_1^*$. □

## 3.3 An example: optimal interventions on the exchange rate

In this section we provide a detailed application of the Verification Theorem 3.8 by considering a practical problem: we look for the Nash equilibria in the case where two countries intervene to adjust the exchange rate between their currencies.

### 3.3.1 The problem

Let $X$ denote the exchange rate between two currencies. The central banks of the corresponding countries (denoted as players, from now on) have different targets for the rate: player 1 needs a high value for $X$ in order to gain, whereas the goal of player 2 is to have a low rate. More precisely, if $x$ denotes the current value of the rate, we assume that the payoff of the two players is given by

$$f_1(x) = (x - s_1)^3, \qquad f_2(x) = (s_2 - x)^3, \qquad s_1 < s_2,$$

where $s_1, s_2$ are fixed constants.

Each player can influence the rate according to its own preferences and intervene to shift $X$ from state $x$ to state $x + \delta$, with $\delta \in \mathbb{R}$. When none of the players intervenes, we assume that $X$ is modelled by a (scaled) Brownian motion. In short, if $x$ denotes the initial state and $u_i = \{(\tau_{i,k}, \delta_{i,k})\}_{k\geq 1}$ collects the intervention times and the corresponding impulses of player $i$ (with $i \in \{1,2\}$), we have

$$X_s^{x;u_1,u_2} = x + \sigma W_s + \sum_{k\,:\,\tau_{1,k}\leq s} \delta_{1,k} + \sum_{k\,:\,\tau_{2,k}\leq s} \delta_{2,k},$$

for $s \geq 0$, where $W$ is a standard one-dimensional Brownian motion and $\sigma > 0$ is a fixed parameter. To shorten the notation, we will simply write $X_s$. Finally, as player 2 aims at lowering the level, we can assume that his impulses are negative: $\delta_{2,k} < 0$, for every $k \in \mathbb{N}$. For the same reason, we require $\delta_{1,k} > 0$, for every $k \in \mathbb{N}$.

Affecting the exchange rate clearly implies a penalty to be paid by the intervening player and it is reasonable to assume that there is a corresponding gain for the opponent. In our model the intervention penalty consists in a fixed cost and in a variable cost, assumed to be proportional to the absolute value of the impulse. In other words, if $\phi_i$ denotes the intervention penalty for player $i$ and $\psi_j$ denotes the corresponding gain for player $j$, we have

$$\phi_i(\delta) = -\psi_j(\delta) = -(c + \lambda|\delta|),$$

where $c, \lambda > 0$ are fixed constants and $\delta \in \mathbb{R}$ is the impulse corresponding to the intervention of player $i$. Finally, we assume the discount rate, denoted by $\rho$, to be the same for both the players.

This problem clearly belongs to the class described in Section 3.2, with

$$d = 1, \quad S = \mathbb{R}, \quad \Gamma^i(x,\delta) = x + \delta, \quad \rho_i = \rho, \quad Z_1 = ]0,\infty[, \quad Z_2 = ]-\infty, 0[,$$

and with $f_i, \phi_i, \psi_i$ as above. In short, if $\varphi_i = (A_i, \xi_i)$ denotes the strategy of player $i$, we deal with the following functions:

$$J^1(x;\varphi_1,\varphi_2) := \mathbb{E}_x\left[\int_0^\infty e^{-\rho s}(X_s - s_1)^3 ds - \sum_{k\geq 1} e^{-\rho\tau_{1,k}}(c+\lambda|\delta_{1,k}|) + \sum_{k\geq 1} e^{-\rho\tau_{2,k}}(c+\lambda|\delta_{2,k}|)\right],$$

$$J^2(x;\varphi_1,\varphi_2) := \mathbb{E}_x\left[\int_0^\infty e^{-\rho s}(s_2 - X_s)^3 ds - \sum_{k\geq 1} e^{-\rho\tau_{2,k}}(c+\lambda|\delta_{2,k}|) + \sum_{k\geq 1} e^{-\rho\tau_{1,k}}(c+\lambda|\delta_{1,k}|)\right],$$

where $\{(\tau_{i,k}, \delta_{i,k})\}_{k\geq 1}$ denotes the impulse controls of player $i$ associated to the strategies $\varphi_1, \varphi_2$. As already outlined, the players have different goals: we are going to investigate if a Nash equilibrium for such a problem exists. Indeed, since $s_1 < s_2$ both the players gain in the interval $[s_1, s_2]$, so that it seems that there is room for a Nash configuration. If a Nash equilibrium exists, we denote by $V_1(x), V_2(x)$ the corresponding value of the game with starting state $x \in \mathbb{R}$.

### 3.3.2 Looking for candidates for the value functions

Our goal is to use the Verification Theorem 3.8. Hence, we now try to find a solution to the problem in (3.15), in order to get a couple of candidates $\tilde{V}_1, \tilde{V}_2$ for $V_1, V_2$.

We start from the equations in the QVI problem (3.15), which we recall for the reader's convenience ($S = \mathbb{R}$ in this problem):

$$\begin{aligned} &\mathcal{H}_i\tilde{V}_i - \tilde{V}_i = 0, && \text{in } \{\mathcal{M}_j\tilde{V}_j - \tilde{V}_j = 0\}, \\ &\max\{\mathcal{A}\tilde{V}_i - \rho\tilde{V}_i + f_i, \mathcal{M}_i\tilde{V}_i - \tilde{V}_i\} = 0, && \text{in } \{\mathcal{M}_j\tilde{V}_j - \tilde{V}_j < 0\}, \end{aligned}$$

for $i, j \in \{1, 2\}$, with $i \neq j$; this suggests the following representation for $\tilde{V}_i$:

$$\tilde{V}_i(x) = \begin{cases} \mathcal{M}_i\tilde{V}_i(x), & \text{in } \{\mathcal{M}_i\tilde{V}_i - \tilde{V}_i = 0\}, \\ \varphi_i(x), & \text{in } \{\mathcal{M}_i\tilde{V}_i - \tilde{V}_i < 0, \mathcal{M}_j\tilde{V}_j - \tilde{V}_j < 0\}, \\ \mathcal{H}_i\tilde{V}_i(x), & \text{in } \{\mathcal{M}_j\tilde{V}_j - \tilde{V}_j = 0\}, \end{cases}$$

for $i \in \{1, 2\}$ and $x \in \mathbb{R}$, where $\varphi_i$ is a solution to

$$\mathcal{A}\varphi_i - \rho\varphi_i + f_i = \frac{\sigma^2}{2}\varphi_i'' - \rho\varphi_i + f_i = 0.$$

Notice that an explicit formula for $\varphi_i$ is present: for each $x \in \mathbb{R}$, we have

$$\varphi_1(x) = \varphi_1^{A_{11},A_{12}}(x) = A_{11}e^{\theta x} + A_{12}e^{-\theta x} + k_3(x - s_1)^3 + k_1(x - s_1),$$
$$\varphi_2(x) = \varphi_2^{A_{21},A_{22}}(x) = A_{21}e^{\theta x} + A_{22}e^{-\theta x} + k_3(s_2 - x)^3 + k_1(s_2 - x),$$

where $A_{ij}$ are real parameters and we have set

$$\theta = \sqrt{\frac{2\rho}{\sigma^2}}, \qquad k_3 = \frac{1}{\rho}, \qquad k_1 = \frac{3\sigma^2}{\rho^2}.$$

In order to go on, we need to guess an expression for the intervention regions. As the goal of player 1 is to keep a high rate, it is reasonable to assume that his intervention region is in the form $] - \infty, \bar{x}_1]$. For a similar reason, we expect the intervention region of player 2 to be in the form $[\bar{x}_2, +\infty[$. Since $s_1 < s_2$, we guess that $\bar{x}_1 < \bar{x}_2$; as a consequence, the real line is heuristically divided into three intervals:

$$] - \infty, \bar{x}_1] = \{\mathcal{M}_1\tilde{V}_1 - \tilde{V}_1 = 0\}, \text{ where player 1 intervenes,}$$
$$]\bar{x}_1, \bar{x}_2[= \{\mathcal{M}_1\tilde{V}_1 - \tilde{V}_1 < 0\} \cap \{\mathcal{M}_2\tilde{V}_2 - \tilde{V}_2 < 0\}, \text{ where no one intervenes,}$$
$$[\bar{x}_2, +\infty[= \{\mathcal{M}_2\tilde{V}_2 - \tilde{V}_2 = 0\}, \text{ where player 2 intervenes.}$$

By the representation above, this leads to the following expression for $\tilde{V}_1$ and $\tilde{V}_2$:

$$\tilde{V}_1(x) = \begin{cases} \mathcal{M}_1\tilde{V}_1(x), & \text{if } x \in ] - \infty, \bar{x}_1], \\ \varphi_1(x), & \text{if } x \in ]\bar{x}_1, \bar{x}_2[, \\ \mathcal{H}_1\tilde{V}_1(x), & \text{if } x \in [\bar{x}_2, +\infty[, \end{cases} \qquad \tilde{V}_2(x) = \begin{cases} \mathcal{H}_2\tilde{V}_2(x), & \text{if } x \in ] - \infty, \bar{x}_1], \\ \varphi_2(x), & \text{if } x \in ]\bar{x}_1, \bar{x}_2[, \\ \mathcal{M}_2\tilde{V}_2(x), & \text{if } x \in [\bar{x}_2, +\infty[. \end{cases}$$

Let us now investigate the form of $\mathcal{M}_i\tilde{V}_i$ and $\mathcal{H}_i\tilde{V}_i$. Recall that the impulses of player 1 (resp. player 2) are positive (resp. negative); then, we have

$$\mathcal{M}_1\tilde{V}_1(x) = \sup_{\delta \geq 0}\{\tilde{V}_1(x + \delta) - c - \lambda\delta\} = \sup_{y \geq x}\{\tilde{V}_1(y) - c - \lambda(y - x)\},$$
$$\mathcal{M}_2\tilde{V}_2(x) = \sup_{\delta \leq 0}\{\tilde{V}_2(x + \delta) - c - \lambda(-\delta)\} = \sup_{y \leq x}\{\tilde{V}_2(y) - c - \lambda(x - y)\}.$$

It is reasonable to assume that the maximum point of the function $y \mapsto \tilde{V}_1(y) - \lambda y$ (resp. $y \mapsto \tilde{V}_2(y) + \lambda y$) exists, is unique and belongs to the common continuation region $]\bar{x}_1, \bar{x}_2[$, where we have $\tilde{V}_1 = \varphi_1$ (resp. $\tilde{V}_2 = \varphi_2$). As a consequence, if we denote by $x_i^*$ the maximum point of $\varphi_i$ in $]\bar{x}_1, \bar{x}_2[$, that is

$$\varphi_1(x_1^*) = \max_{y \in ]\bar{x}_1, \bar{x}_2[} \{\varphi_1(y) - \lambda y\}, \qquad \text{i.e.} \qquad \varphi_1'(x_1^*) = \lambda, \; \varphi_1''(x_1^*) \leq 0, \; \bar{x}_1 < x_1^* < \bar{x}_2,$$

$$\varphi_2(x_2^*) = \max_{y \in ]\bar{x}_1, \bar{x}_2[} \{\varphi_2(y) + \lambda y\}, \qquad \text{i.e.} \qquad \varphi_2'(x_2^*) = -\lambda, \; \varphi_2''(x_2^*) \leq 0, \; \bar{x}_1 < x_2^* < \bar{x}_2,$$

the functions $\mathcal{M}_i\tilde{V}_i$, $\mathcal{H}_i\tilde{V}_i$ have the following (heuristic, at the moment) expression:

$$\mathcal{M}_1\tilde{V}_1(x) = \varphi_1(x_1^*) - c - \lambda(x_1^* - x), \qquad \mathcal{M}_2\tilde{V}_2(x) = \varphi_2(x_2^*) - c - \lambda(x - x_2^*),$$
$$\mathcal{H}_1\tilde{V}_1(x) = \varphi_1(x_2^*) + c + \lambda(x - x_2^*), \qquad \mathcal{H}_2\tilde{V}_2(x) = \varphi_2(x_1^*) + c + \lambda(x_1^* - x).$$

As for the parameters involved in $\tilde{V}_1, \tilde{V}_2$, they must be chosen so as to satisfy the regularity assumptions in the verification theorem, which here write

$$\tilde{V}_1 \in C^2\big(]-\infty, \bar{x}_1[ \,\cup\, ]\bar{x}_1, \bar{x}_2[\big) \cap C^1\big(]-\infty, \bar{x}_2[\big) \cap C(\mathbb{R}),$$
$$\tilde{V}_2 \in C^2\big(]\bar{x}_1, \bar{x}_2[ \,\cup\, ]\bar{x}_2, +\infty[\big) \cap C^1\big(]\bar{x}_1, +\infty[\big) \cap C(\mathbb{R}).$$

Since $\tilde{V}_1$ and $\tilde{V}_2$ are, by definition, smooth in $]-\infty, \bar{x}_1[ \,\cup\, ]\bar{x}_1, \bar{x}_2[ \,\cup\, ]\bar{x}_2, +\infty[$, we have to set the parameters so that $\tilde{V}_i$ is continuous in $\bar{x}_1, \bar{x}_2$ and differentiable in $\bar{x}_i$ (we underline that $\tilde{V}_1$ and $\tilde{V}_2$ may be not differentiable in, respectively, $\bar{x}_2$ and $\bar{x}_1$).

Finally, to summarize all the previous arguments, our candidates for the value functions are defined as follows.

**Definition 3.11.** *For every $x \in \mathbb{R}$, we set*

$$\tilde{V}_1(x) = \begin{cases} \varphi_1(x_1^*) - c - \lambda(x_1^* - x), & \text{if } x \in ]-\infty, \bar{x}_1], \\ \varphi_1(x), & \text{if } x \in ]\bar{x}_1, \bar{x}_2[, \\ \varphi_1(x_2^*) + c + \lambda(x - x_2^*), & \text{if } x \in [\bar{x}_2, +\infty[, \end{cases}$$

$$\tilde{V}_2(x) = \begin{cases} \varphi_2(x_1^*) + c + \lambda(x_1^* - x), & \text{if } x \in ]-\infty, \bar{x}_1], \\ \varphi_2(x), & \text{if } x \in ]\bar{x}_1, \bar{x}_2[, \\ \varphi_2(x_2^*) - c - \lambda(x - x_2^*), & \text{if } x \in [\bar{x}_2, +\infty[, \end{cases}$$

*where $\varphi_1 = \varphi_1^{A_{11}, A_{12}}$, $\varphi_2 = \varphi_2^{A_{21}, A_{22}}$ and the eight parameters involved*

$$(A_{11}, A_{12}, A_{21}, A_{22}, \bar{x}_1, \bar{x}_2, x_1^*, x_2^*)$$

*satisfy the order conditions*

$$\bar{x}_1 < x_1^* < \bar{x}_2, \qquad \bar{x}_1 < x_2^* < \bar{x}_2, \tag{3.25}$$

*and the following conditions:*

$$\begin{cases} \varphi_1'(x_1^*) = \lambda \quad \textit{and} \quad \varphi_1''(x_1^*) \leq 0, & \textit{(optimality of } x_1^*\textit{)} \\ \varphi_1'(\bar{x}_1) = \lambda, & \textit{(}C^1\textit{-pasting in } \bar{x}_1\textit{)} \\ \varphi_1(\bar{x}_1) = \varphi_1(x_1^*) - c - \lambda(x_1^* - \bar{x}_1), & \textit{(}C^0\textit{-pasting in } \bar{x}_1\textit{)} \\ \varphi_1(\bar{x}_2) = \varphi_1(x_2^*) + c + \lambda(\bar{x}_2 - x_2^*), & \textit{(}C^0\textit{-pasting in } \bar{x}_2\textit{)} \end{cases} \tag{3.26}$$

$$\begin{cases} \varphi_2'(x_2^*) = -\lambda \quad \textit{and} \quad \varphi_2''(x_2^*) \leq 0, & \textit{(optimality of } x_2^*\textit{)} \\ \varphi_2'(\bar{x}_2) = -\lambda, & \textit{(}C^1\textit{-pasting in } \bar{x}_2\textit{)} \\ \varphi_2(\bar{x}_1) = \varphi_2(x_1^*) + c + \lambda(x_1^* - \bar{x}_1), & \textit{(}C^0\textit{-pasting in } \bar{x}_1\textit{)} \\ \varphi_2(\bar{x}_2) = \varphi_2(x_2^*) - c - \lambda(\bar{x}_2 - x_2^*). & \textit{(}C^0\textit{-pasting in } \bar{x}_2\textit{)} \end{cases} \tag{3.27}$$

In order to have a well-posed definition, we need to show that the equations in (3.25)-(3.26)-(3.27) actually admit a solution. To solve such a system is clearly very complicated and existence results are not easy to achieve. Hence, we first try to simplify the equations.

We notice that the running costs $f_i$ are symmetric with respect to $(s_1+s_2)/2$; indeed, the global structure of the problem seems to satisfy such a property. This suggests us to consider candidates with the following property: the couples $(x_1^*, x_2^*)$, $(\bar{x}_1, \bar{x}_2)$, $(\varphi_1, \varphi_2)$ are symmetric with respect to $(s_1+s_2)/2$, that is

$$x_1^* = s_1 + s_2 - x_2^*, \qquad \bar{x}_1 = s_1 + s_2 - \bar{x}_2, \qquad \varphi_1(x) = \varphi_2(s_1 + s_2 - x), \tag{3.28}$$

for each $x \in \mathbb{R}$. Notice that the last condition in (3.28) is clearly equivalent to

$$A_{11} = A_{22} e^{-(s_1+s_2)}, \qquad A_{12} = A_{21} e^{(s_1+s_2)}. \tag{3.29}$$

As $\bar{x}_1 < \bar{x}_2$, we remark that $\bar{x}_1 < (s_1+s_2)/2 < \bar{x}_2$. It is easy to see that, under condition (3.28), the systems in (3.26) and (3.27) are independent and equivalent. In other words, the 4-uple $(A_{11}, A_{12}, \bar{x}_1, x_1^*)$ solves (3.26) if and only if $(A_{21}, A_{22}, \bar{x}_2, x_2^*)$, defined by (3.28)-(3.29), is a solution to (3.27). Hence, we just need to solve one of the two systems of equations. We decide to focus on (3.27), which now reads

$$\begin{cases} \varphi_2'(x_2^*) = -\lambda \quad \text{and} \quad \varphi_2''(x_2^*) \leq 0, \\ \varphi_2'(\bar{x}_2) = -\lambda, \\ \varphi_2(s_1 + s_2 - \bar{x}_2) = \varphi_2(s_1 + s_2 - x_2^*) + c + \lambda(\bar{x}_2 - x_2^*), \\ \varphi_2(\bar{x}_2) = \varphi_2(x_2^*) - c - \lambda(\bar{x}_2 - x_2^*), \end{cases} \tag{3.30}$$

where $\varphi_2 = \varphi_2^{A_{21}, A_{22}}$. We now have a system of four equations in four variables (instead of the eight variables of the starting conditions). Also recall the order condition (3.25), which now reads

$$|x_2^* - (s_1+s_2)/2| < \bar{x}_2 - (s_1+s_2)/2. \tag{3.31}$$

As System (3.30)-(3.31) cannot be treated analytically, we need to assume that a solution actually exists.

**Assumption 3.12.** *We assume that a solution to* (3.30)-(3.31) *exists. Moreover, we assume that there exists* $\tilde{x} \in ]x_2^*, \bar{x}_2[$ *such that* $\varphi_2'' < 0$ *in* $]\bar{x}_1, \tilde{x}[$ *and* $\varphi_2'' > 0$ *in* $]\tilde{x}, \bar{x}_2[$. *Finally, we assume* $\bar{x}_2 < s_2 - \sqrt{\lambda\rho/3}$.

We remark that the conditions on $\varphi_2''$ (we are going to use them in the next section) are, heuristically, always satisfied by the solutions to (3.30). Indeed, as $x_2^*$ is a local maximum, $\varphi_2$ must be concave in a neighbourhood of $x_2^*$; moreover, as we need a $C^1$-pasting in $\bar{x}_2$ with a straight line with slope $-\lambda$, a change of concavity is needed in $]x_2^*, \bar{x}_2[$. Moreover, notice that we do not require the solution to be unique, as this would be a direct consequence of the verification theorem.

We summarize the previous arguments in the following proposition.

**Proposition 3.13.** *Let Assumption* (3.12) *hold. Then the functions* $\tilde{V}_1, \tilde{V}_2$ *in Definition 3.11 are well-defined. More precisely, let* $(A_{21}, A_{22}, x_2^*, \bar{x}_2)$ *be a solution to* (3.30)-(3.31)*; then, there exists a solution*

$$(A_{11}, A_{12}, A_{21}, A_{22}, \bar{x}_1, \bar{x}_2, x_1^*, x_2^*)$$

*to equations* (3.25)-(3.26)-(3.27)*, which is given by* (3.28)-(3.29).

### 3.3.3 Application of the verification theorem

We now apply the Verification Theorem 3.8 and prove that the candidates $\tilde{V}_1, \tilde{V}_2$ in Definition 3.11 actually coincide with the value functions $V_1, V_2$ of the problem described in Section 3.3.1. We refer the reader to Section 3.2.2 for the definition of the functions $\delta_1, \delta_2$, used in the following lemma.

**Lemma 3.14.** *Let $\tilde{V}_1, \tilde{V}_2$ be as in Definition 3.11. Then, for every $x \in \mathbb{R}$ we have*

$$\delta_1(x) = \begin{cases} x_1^* - x, & \text{in } ]-\infty, x_1^*], \\ 0, & \text{in } ]x_1^*, +\infty[, \end{cases} \qquad \delta_2(x) = \begin{cases} 0, & \text{in } ]-\infty, x_2^*[, \\ x - x_2^*, & \text{in } [x_2^*, +\infty[. \end{cases} \tag{3.32}$$

*Moreover, for every $x \in \mathbb{R}$ we have*

$$\mathcal{M}_1\tilde{V}_1(x) = \begin{cases} \tilde{V}_1(x), & \text{in } ]-\infty, \bar{x}_1], \\ \tilde{V}_1(x) - \xi_1(x), & \text{in } ]\bar{x}_1, +\infty[, \end{cases} \qquad \mathcal{M}_2\tilde{V}_2(x) = \begin{cases} \tilde{V}_2(x) - \xi_2(x), & \text{in } ]-\infty, \bar{x}_2[, \\ \tilde{V}_2(x), & \text{in } [\bar{x}_2, +\infty[, \end{cases}$$

*where $\xi_1, \xi_2$ are strictly positive functions. In particular, we have*

$$\begin{aligned} &\mathcal{M}_1\tilde{V}_1 - \tilde{V}_1 \leq 0, \quad \{\mathcal{M}_1\tilde{V}_1 - \tilde{V}_1 < 0\} = ]\bar{x}_1, +\infty[, \quad \{\mathcal{M}_1\tilde{V}_1 - \tilde{V}_1 = 0\} = ]-\infty, \bar{x}_1], \\ &\mathcal{M}_2\tilde{V}_2 - \tilde{V}_2 \leq 0, \quad \{\mathcal{M}_2\tilde{V}_2 - \tilde{V}_2 < 0\} = ]-\infty, \bar{x}_2[, \quad \{\mathcal{M}_2\tilde{V}_2 - \tilde{V}_2 = 0\} = [\bar{x}_2, +\infty[. \end{aligned} \tag{3.33}$$

*Proof.* We here consider $\delta_2$ and $\mathcal{M}_2\tilde{V}_2$, the arguments for $\delta_1$ and $\mathcal{M}_1\tilde{V}_1$ being the same. For every $x \in \mathbb{R}$, we have

$$\mathcal{M}_2\tilde{V}_2(x) = \max_{\delta_2 \leq 0}\{\tilde{V}_2(x+\delta_2) - c - \lambda|\delta_2|\} = \max_{y \leq x}\{\tilde{V}_2(y) - c - \lambda(x-y)\} = \max_{y \leq x}\{\Gamma_2(y)\} - c - \lambda x,$$

where for each $y \in \mathbb{R}$ we have set

$$\Gamma_2(y) = \tilde{V}_2(y) + \lambda y.$$

As we are interested in $\max_{y \leq x} \Gamma_2(y)$, we now study the monotonicity of $\Gamma_2$. By the definition of $\tilde{V}_2$, we have $\Gamma_2'(x_2^*) = \Gamma_2'(\bar{x}_2) = 0$; moreover, we notice that:

- $\Gamma_2' = 0$ in $]-\infty, \bar{x}_1[$, by the definition of $\tilde{V}_2$;
- $\Gamma_2' > 0$ in $]\bar{x}_1, x_2^*[$, as $\Gamma_2'(x_2^*) = 0$ and $\Gamma_2'$ is here decreasing (since, by Assumption (3.12), we have $\Gamma_2'' = \varphi_2'' < 0$ in $]\bar{x}_1, x_2^*[$);
- $\Gamma_2' < 0$ in $]x_2^*, \bar{x}_2[$, as $\Gamma_2'(x_2^*) = \Gamma_2'(\bar{x}_2) = 0$ and $\Gamma_2'$ is here first decreasing and then increasing (since, by Assumption (3.12), $\Gamma_2'' = \varphi_2''$ is negative in $]x_2^*, \tilde{x}[$ and positive in $]\tilde{x}, \bar{x}_2[$);
- $\Gamma_2' = 0$ in $]\bar{x}_2, +\infty[$, by the definition of $\tilde{V}_2$.

As a consequence, the function $\Gamma_2$ has a unique global maximum point in $x_2^*$, so that

$$\max_{y \leq x} \Gamma_2(y) = \begin{cases} \Gamma_2(x), & \text{in } ]-\infty, x_2^*], \\ \Gamma_2(x_2^*), & \text{in } ]x_2^*, +\infty[; \end{cases}$$

therefore, by the previous computations, we have

$$\mathcal{M}_2\tilde{V}_2(x) = \begin{cases} \tilde{V}_2(x) - c, & \text{in } ]-\infty, x_2^*], \\ \varphi_2(x_2^*) - c - \lambda(x - x_2^*), & \text{in } ]x_2^*, +\infty[, \end{cases}$$

as $\tilde{V}_2(x_2^*) = \varphi_2(x_2^*)$, since $x_2^* \in ]\bar{x}_1, \bar{x}_2[$. By the definition of $\tilde{V}_2$, this can be written as

$$\mathcal{M}_2\tilde{V}_2(x) = \begin{cases} \tilde{V}_2(x) - \xi_2(x), & \text{in } ]-\infty, \bar{x}_2[, \\ \tilde{V}_2(x), & \text{in } [\bar{x}_2, +\infty[, \end{cases}$$

where, for each $x \in ]-\infty, \bar{x}_2[$, we have set

$$\xi_2(x) = \begin{cases} c, & \text{in } ]-\infty, x_2^*[, \\ \varphi_2(x) - \varphi_2(x_2^*) + c + \lambda(x - x_2^*), & \text{in } [x_2^*, \bar{x}_2[, \end{cases}$$

To prove the positivity of $\xi_2$, recall by (3.27) that $\varphi_2(\bar{x}_2) = \varphi_2(x_2^*) - c - \lambda(\bar{x}_2 - x_2^*)$; then, if $x \in [x_2^*, \bar{x}_2[$ we have that

$$\varphi_2(x) - \varphi_2(x_2^*) + c + \lambda(x - x_2^*) = \varphi_2(x) - \varphi_2(\bar{x}_2) - \lambda(\bar{x}_2 - x) = \Gamma_2(x) - \Gamma_2(\bar{x}_2) > 0,$$

as $\Gamma_2$ is decreasing in $[x_2^*, \bar{x}_2[$. Finally, by the previous arguments it is clear that

$$\arg\max_{\delta_2 \le 0}\{\tilde{V}_2(x + \delta_2) - c - \lambda|\delta_2|\} = \begin{cases} \{0\}, & \text{in } ]-\infty, x_2^*[, \\ \{x - x_2^*\}, & \text{in } ]x_2^*, +\infty[, \end{cases}$$

which implies (3.32). □

**Proposition 3.15.** *A Nash equilibrium for the problem in Section 3.3.1 exists and is given by the strategies* $(A_1^*, \xi_1^*)$, $(A_2^*, \xi_2^*)$ *defined by*

$$A_1^* = ]-\infty, \bar{x}_1], \qquad \xi_1^*(y) = x_1^* - y,$$
$$A_2^* = [\bar{x}_2, +\infty[, \qquad \xi_2^*(y) = y - x_2^*,$$

*with* $y \in \mathbb{R}$ *and* $x_i^*, \bar{x}_i$ $(i \in \{1, 2\})$ *as in Definition 3.11. Moreover, the functions* $\tilde{V}_1, \tilde{V}_2$ *in Definition 3.11 coincide with the value functions* $V_1, V_2$*:*

$$V_1 \equiv \tilde{V}_1 \qquad \textit{and} \qquad V_2 \equiv \tilde{V}_2.$$

*Proof.* We have to check that the candidates $\tilde{V}_1, \tilde{V}_2$ satisfy all the assumptions of Theorem 3.8. We prove the claim for $\tilde{V}_2$, the arguments for $\tilde{V}_1$ being the same.

For the reader's convenience, we briefly report the conditions we have to check:

(i) $\tilde{V}_2 \in C^2(]\bar{x}_1, +\infty[\setminus\{\bar{x}_2\}) \cap C^1(]\bar{x}_1, +\infty[) \cap C(\mathbb{R})$ and has polynomial growth;

(ii) $\mathcal{M}_2\tilde{V}_2 - \tilde{V}_2 \le 0$;

(iii) in $\{\mathcal{M}_1\tilde{V}_1 - \tilde{V}_1 = 0\}$ we have $\tilde{V}_2 = \mathcal{H}_2\tilde{V}_2$;

(iv) in $\{\mathcal{M}_1\tilde{V}_1 - \tilde{V}_1 < 0\}$ we have $\max\{\mathcal{A}\tilde{V}_2 - \rho\tilde{V}_2 + f_2, \mathcal{M}_2\tilde{V}_2 - \tilde{V}_2\} = 0$;

(v) the optimal strategies are $x$-admissible (see Definition 3.5) for every $x \in \mathbb{R}$.

*Condition (i) and (ii).* The first condition holds by the definition of $\tilde{V}_2$, whereas the second condition has been proved in (3.33).

*Condition (iii).* Let $x \in \{\mathcal{M}_1\tilde{V}_1 - \tilde{V}_1 = 0\} = ]-\infty, \bar{x}_1]$. By the definition of $\mathcal{H}_2\tilde{V}_2$ in (3.14), by (3.32) and by the definition of $\tilde{V}_2$ we have

$$\mathcal{H}_2\tilde{V}_2(x) = \tilde{V}_2(x + \delta_1(x)) + c + \lambda|\delta_1(x)| = \tilde{V}_2(x_1^*) + c + \lambda(x_1^* - x) = \tilde{V}_2(x),$$

where we have used that $\tilde{V}_2(x_1^*) = \varphi_2(x_1^*)$, since $x_1^* \in ]\bar{x}_1, \bar{x}_2[$.

*Condition (iv).* We have to prove that

$$\max\{\mathcal{A}\tilde{V}_2 - \rho\tilde{V}_2 + f_2, \mathcal{M}_2\tilde{V}_2 - \tilde{V}_2\} = 0, \qquad \text{in } \{\mathcal{M}_1\tilde{V}_1 - \tilde{V}_1 < 0\} = ]\bar{x}_1, +\infty[.$$

In $]\bar{x}_1, \bar{x}_2[$ the claim is true, as $\mathcal{M}_2\tilde{V}_2 - \tilde{V}_2 < 0$ by (3.33) and $\mathcal{A}\tilde{V}_2 - \rho\tilde{V}_2 + f_2 = 0$ by definition (in $]\bar{x}_1, \bar{x}_2[$ we have $\tilde{V}_2 = \varphi_2$, which is a solution to the ODE). In $[\bar{x}_2, \infty[$ we already know by (3.33) that $\mathcal{M}_2\tilde{V}_2 - \tilde{V}_2 = 0$. Then, to conclude we have to check that

$$\mathcal{A}\tilde{V}_2(x) - \rho\tilde{V}_2(x) + f_2(x) \leq 0, \qquad \forall x \in [\bar{x}_2, \infty[.$$

As $\tilde{V}_2(x) = \varphi_2(x_2^*) - c - \lambda(x - x_2^*)$ by the definition of $\tilde{V}_2(x)$, the inequality can be written as

$$-\rho\big(\varphi_2(x_2^*) - c - \lambda(x - x_2^*)\big) + f_2(x) \leq 0, \qquad \forall x \in [\bar{x}_2, \infty[.$$

Since $\varphi_2(\bar{x}_2) = \varphi_2(x_2^*) - c - \lambda(\bar{x}_2 - x_2^*)$ by (3.27), we can rewrite the claim as

$$-\rho\big(\varphi_2(\bar{x}_2) - \lambda(x - \bar{x}_2)\big) + f_2(x) \leq 0, \qquad \forall x \in [\bar{x}_2, \infty[.$$

By Assumption (3.12) the function $x \mapsto \lambda\rho x + f_2(x) = \lambda\rho x + (s_2 - x)^3$ is decreasing in $[\bar{x}_2, +\infty[$ (immediate check on the derivative); then, it is enough to prove the claim in $x = \bar{x}_2$:

$$-\rho\varphi_2(\bar{x}_2) + f_2(\bar{x}_2) \leq 0.$$

Since $\mathcal{A}\varphi_2(\bar{x}_2) - \rho\varphi_2(\bar{x}_2) + f_2(\bar{x}_2) = 0$, we can rewrite as

$$-\frac{\sigma^2}{2}\varphi_2''(\bar{x}_2) \leq 0,$$

which is true since $\varphi_2''(\bar{x}_2) \geq 0$ by Assumption 3.12.

*Condition (v).* Let $x \in \mathbb{R}$. We have to show that all the conditions in Definition 3.5 are satisfied by the optimal strategies, which are basically described as follows: player $i$ intervenes when the controlled process $X$ hits $\bar{x}_i$ and shifts the process to the state $x_i^* \in ]\bar{x}_1, \bar{x}_2[$. As an immediate consequence, we have

$$X \in ]\bar{x}_1, \bar{x}_2[ \cup \{x\};$$

hence, condition (3.12) holds. It is easy to check that all the other conditions of Definition 3.5 are satisfied; the only non-trivial proof is the integrability of the intervention costs: as it is quite technical, we postpone it to Lemma 3.16. □

**Lemma 3.16.** *Let $x \in \mathbb{R}$. For $i \in \{1,2\}$, let $u_i^* = \{\tau_{i,k}^*, \delta_{i,k}^*\}_k$ be the controls corresponding to the optimal strategies defined in Proposition 3.15. Then, for $i \in \{1,2\}$ we have*

$$\mathbb{E}_x\Big[\sum_{k\geq 1} e^{-\rho\tau_{i,k}^*}(c + \lambda|\delta_{i,k}^*|)\Big] < \infty. \qquad (3.34)$$

*Proof.* Recall the optimal strategies: when the process hits $\bar{x}_i$, player $i$ shifts it to $x_i^*$. This suggests to re-write the times $\tau_{i,k}^*$ as sums of independent exit times.

To start, let us assume that the initial state is either $x_1^*$ or $x_2^*$. First of all, we re-label the indexes and write $\{\tau_{i,k}^*\}_{i,k}$ as $\{\sigma_j\}_j$, with $\sigma_j < \sigma_{j+1}$ for every $j \in \mathbb{N}$ (this is possible: see Definition 3.2). Denote by $\mu_i$ the exit time of the process $x_i^* + \sigma W$ from $]\bar{x}_1, \bar{x}_2[$, where $W$ is a real Brownian motion; then, each time $\sigma_k$ can be written as $\sigma_k = \sum_{j=1}^k \zeta_k$, where the $\zeta_k$ are independent variables which are distributed either as $\mu_1$ or as $\mu_2$.

We can now start with the estimates. As $\delta_{i,k}^* \in \{\bar{x}_2 - x_2^*, x_1^* - \bar{x}_1\}$, we have

$$\mathbb{E}_x\bigg[\sum_{i\in\{1,2\}}\sum_{k\geq 1} e^{-\rho\tau_{i,k}^*}(c+\lambda|\delta_{i,k}^*|)\bigg] \leq (c+\lambda\max\{\bar{x}_2-x_2^*, x_1^*-\bar{x}_1\})\,\mathbb{E}_x\bigg[\sum_{i\in\{1,2\}}\sum_{k\geq 1} e^{-\rho\tau_{i,k}^*}\bigg].$$

By the definition of $\{\sigma_j\}_j$ and the decomposition of $\sigma_j$ we have

$$\mathbb{E}_x\bigg[\sum_{i\in\{1,2\}}\sum_{k\geq 1} e^{-\rho\tau_{i,k}^*}\bigg] = \mathbb{E}_x\bigg[\sum_{j\geq 1} e^{-\rho\sigma_j}\bigg] = \mathbb{E}_x\bigg[\sum_{j\geq 1} e^{-\rho\sum_{l=1}^j \zeta_l}\bigg] = \mathbb{E}_x\bigg[\sum_{j\geq 1}\prod_{l=1,\dots,j} e^{-\rho\zeta_l}\bigg].$$

For $i \in \{1,2\}$ and $j \in \mathbb{N}$, let $m_i(j)$ be the cardinality of the set $\{1 \leq l \leq j : \zeta_l \sim \mu_i\}$. By the Fubini-Tonelli theorem and the independence of the variables $\zeta_j$ we get

$$\mathbb{E}_x\bigg[\sum_{j\geq 1}\prod_{l=1,\dots,j} e^{-\rho\zeta_l}\bigg] = \sum_{j\geq 1}\prod_{l=1,\dots,j}\mathbb{E}_x[e^{-\rho\zeta_l}] = \sum_{j\geq 1}\mathbb{E}_x[e^{-\rho\mu_1}]^{m_1(j)}\mathbb{E}_x[e^{-\rho\mu_2}]^{m_2(j)}.$$

Since $m_1(j) + m_2(j) = j$, we finally have

$$\sum_{j\geq 1}\mathbb{E}_x[e^{-\rho\mu_1}]^{m_1(j)}\mathbb{E}_x[e^{-\rho\mu_2}]^{m_2(j)} \leq \sum_{j\geq 1}\Big(\max\big\{\mathbb{E}_x[e^{-\rho\mu_1}], \mathbb{E}_x\big[e^{-\rho\mu_2}\big]\big\}\Big)^j,$$

which is a converging geometric series. To sum up, we have shown that

$$\mathbb{E}_x\bigg[\sum_{i\in\{1,2\}}\sum_{k\geq 1} e^{-\rho\tau_{i,k}^*}(c+\lambda|\delta_{i,k}^*|)\bigg] < \infty,$$

which clearly implies (3.34).

The general case with initial state $x \in \mathbb{R}$ can be treated similarly: we have $\sigma_j = \eta + \sum_{l=1}^j \zeta_l$, where $\eta$ is the exit time of $x + \sigma W$ from $[\bar{x}_1, \bar{x}_2]$, and the argument can be easily adapted. □

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