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CICLO XXII

A general method of weights in the $\bar{\partial}$-Neumann problem

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Abstract

This thesis works in partial differential equations and several complex variables that concentrates on a general estimate for $\bar{\partial}$-Neumann problem on domain which is q-pseudoconvex or q-pseudoconcave at the boundary point. Generalization of the Property (P) in [C84], we define the Property $(f\text{-}\mathcal{M}\text{-}P)^k$ at the boundary point. The Property $(f\text{-}\mathcal{M}\text{-}P)^k$ is a sufficient condition to get following estimate

$$(f\text{-}\mathcal{M})^k \qquad \|f(\Lambda)\mathcal{M}u\|^2 \leq c(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 + \|u\|^2) + C_{\mathcal{M}}\|u\|^2_{-1}$$

for any $u \in C_c^\infty(U \cap \bar{\Omega})^k \cap \mathrm{Dom}(\bar{\partial}^*)$. We want to point our attention that by the choice of f and $\mathcal{M}$, $(f\text{-}\mathcal{M})^k$ will be subelliptic estimate, superlogarithmic estimate, compactness estimate, subelliptic multiplier estimate...

Moreover, the thesis contains some applications of $(f\text{-}\mathcal{M})^k$ and constructions of the Property $(f\text{-}\mathcal{M}\text{-}P)^k$ on some class of domains.

Riassunto

La presente tesi ha come argomento lo studio di equazioni alle derivate parziali nell'ambito dell'analisi in piú variabili complesse. Un primo risultato è stato quello di trovare alcune stime, abbastanza generali, per il problema del $\bar{\partial}$-Neumann su domini q-pseudoconvessi o q-pseduconcavi. Un altro soggetto di studio è stato quello di generalizzare la proprietà (P) (vedi [C84]) tramite la definizione di una nuova nozione che chiameremo proprietà $(f\text{-}\mathcal{M}\text{-}P)^k$. Quest'ultima consente di ottenere una stima del tipo

$$(f\text{-}\mathcal{M})^k \qquad \|f(\Lambda)\mathcal{M}u\|^2 \leq c(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 + \|u\|^2) + C_{\mathcal{M}}\|u\|^2_{-1}$$

la quale, con oppurtune scelte di f e$\mathcal{M}$, $(f\text{-}\mathcal{M})^k$ produce stime subellittiche, superlogaritmiche, di compattezza o stime con moltiplicatori subellettici...

Inoltre mostreremo alcune applicazioni della proprietà $(f\text{-}\mathcal{M})^k$ e la costruzione di alcuni domini che soddisfano alla stessa.

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Contents

Chapter 1

Introduction

The $\bar{\partial}$-Neumann problem is probably the most important and natural example of a non-elliptic boundary value problem, arising as it does from the Cauchy-Riemann system. The main tools to prove regularity of solution of this problem are various L_2-estimates such as subelliptic estimates, superlogarithmic estimates, compactness estimates... In this thesis, we introduce a general estimate is called $(f\text{-}\mathcal{M})^k$. The principal result proved in this thesis are Theorem 1.10 and Theorem 1.13. To introduce the thesis, we first give a brief description of the $\bar{\partial}$-Neumann problem, for a detail account see [FK72].

1.1 The $\bar{\partial}$-Neumann problem

Let Ω be a bounded domain of $\mathbb{C}^n$ with smooth boundary denoted by $b\Omega$. Let $L_2^{h,k}(\Omega)$ be the space of square-integrable (h,k)-forms on Ω. Then we have

$$L_2^{h,k-1}(\Omega) \underset{\bar{\partial}^*}{\overset{\bar{\partial}}{\rightleftarrows}} L_2^{h,k}(\Omega) \underset{\bar{\partial}^*}{\overset{\bar{\partial}}{\rightleftarrows}} L_2^{h,k+1}(\Omega) \tag{1.1}$$

by $\bar{\partial}$ we mean the closed operator which is the maximal extension of the differential operator and by $\bar{\partial}^*$ we mean the L_2-adjoint of $\bar{\partial}$. We define $\mathcal{H}^{h,k} \subset L_2^{h,k}(\Omega)$ by

$$\mathcal{H}^{h,k} = \{u \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \big| \bar{\partial}u = 0 \quad \text{and} \quad \bar{\partial}^* u = 0\}. \tag{1.2}$$

The $\bar\partial$-*Neumann problem* for (h,k)-forms can them be stated as follow : given $\alpha \in L_2^{h,k}(\Omega)$ with $\alpha \perp \mathcal{H}^{h,k}$, does there exist $u \in L^{h,k}(\Omega)$ such that

$$\begin{cases} (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)u = \alpha \\ u \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*) \\ \bar\partial u \in \mathrm{Dom}(\bar\partial^*), \bar\partial^* u \in \mathrm{Dom}(\bar\partial). \end{cases} \tag{1.3}$$

Observe that if a solution of (1.3) exists then there is a unique solution of u of (1.3) such that $u \perp \mathcal{H}^{h,k}$. We will denote this solution by $N\alpha$. If a solution to (1.3) exists for all $\alpha \perp \mathcal{H}^{h,k}$, then we extend the operator N to a linear operator on $L_2^{h,k}(\Omega)$ by setting

$$N\alpha = \begin{cases} 0 & \text{if } \alpha \in \mathcal{H}^{h,k} \\ u & \text{if } \alpha \perp \mathcal{H}^{h,k}. \end{cases} \tag{1.4}$$

Then N is bounded self adjoint. Furthermore, if $\bar\partial\alpha = 0$, then form (1.3) we obtain $\bar\partial\bar\partial^*\bar\partial N\alpha = 0$, taking inner product with $\bar\partial N\alpha$ we get $\|\bar\partial^*\bar\partial N\alpha\|^2 = 0$ hence $\bar\partial^*\bar\partial N\alpha = 0$. Thus we see from (1.3) that if $\bar\partial\alpha = 0$ and $\alpha \perp \mathcal{H}^{h,k}$ then $\alpha = \bar\partial\bar\partial^* N\alpha$. It then follows that $v = \bar\partial^* N\alpha$ is a unique solution to the $\bar\partial$-*problem*

$$\begin{cases} \bar\partial v = \alpha, \\ v \text{ is orthogonal to } \mathrm{Ker}\,\bar\partial. \end{cases} \tag{1.5}$$

The $\bar\partial$-Neumann problem is a non-elliptic boundary value problem; in fact, the Laplacian $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ itself is elliptic but the boundary conditions which are imposed by the membership to $\mathrm{Dom}(\Box)$ are not. The main interest relies in the *regularity* at the boundary for these problems, that is, in stating under which condition u inherits from α the smoothness at the boundary $b\Omega$ (it certainly does in the interior). The regularity of the $\bar\partial$-Neumann operator is defined as follows

Definition 1.1.
1. *Global regularity :* if $\alpha \in C^\infty(\bar\Omega)$ then $N\alpha \in C^\infty(\bar\Omega)$.
2. *Local regularity :* if $\alpha \in C_c^\infty(U \cap \bar\Omega)$ then $N\alpha \in C^\infty(U' \cap \bar\Omega)$ where $U' \subset U$ is the neighborhoods of given point $z_0 \in \bar\Omega$.

One of the main tools used in investigating the local (resp. global) regularity at the boundary of the solutions of the $\bar\partial$-Neumann problem consist in the certain priori estimates such as subelliptic, superlogarithmic (resp. compactness) estimates. In this thesis we introduce the $(f\text{-}\mathcal{M})^k$ estimates which are general of those estimates.

1.2 The $(f\text{-}\mathcal{M})^k$ estimate

In order to define $(f\text{-}\mathcal{M})^k$, some preliminary material is required.

For quantifies A and B we use the notion $A \lesssim B$ to mean $A \leq cB$ for some constant $c > 0$, which independent of relevant parameters. We write $A \cong B$ to mean $A \lesssim B$ and $B \lesssim A$. For functions f and g we use the notion $f \gg g$ to mean that $\lim\limits_{t \to +\infty} \frac{f(t)}{g(t)} = +\infty$.

Let Ω be a smooth domain with local defining in a neighborhood of boundary point z_0. Throughout this thesis we assume z_0 is the origin point. For a neighborhood U of z_0, fix a smooth real-valued function r

$$\Omega \cap U = \{z \in U : r(z) < 0\} \tag{1.6}$$

such that $|\partial r| = 1$ on $b\Omega$. We take an local orthonormal basis of $(1,0)$ forms $\omega_1, ..., \omega_n = \partial r$ and dual basis of (1,0) vector fields $L_1, ..., L_n$; thus $L_1, ..., L_{n-1}$ generate $T^{1,0}(U \cap b\Omega)$. For $\phi \in C^2(U)$, we denote by ϕ_{ij} the coefficients of $\partial\bar{\partial}\phi$ in this basis.

Let $\lambda_1(z) \leq ... \leq \lambda_{n-1}(z)$ be the eigenvalues of $(r_{jk}(z))_{j,k=1}^{n-1}$. We take a pair of indices $1 \leq q \leq n-1$ and $0 \leq q_o \leq n-1$ such that $q \neq q_o$. We assume that there is a bundle $\mathcal{V}^{q_o} \in T^{1,0}b\Omega$ of rank q_o with smooth coefficients, by reordering we may suppose $\mathcal{V}^{q_o} = \text{ span } \{L_1, ..., L_{q_o}\}$ such that

$$\sum_{j=1}^{q} \lambda_j - \sum_{j=1}^{q_o} r_{jj}|u|^2 \geq 0 \text{ on } U \cap b\Omega; \tag{1.7}$$

here we conventionally set $\sum_{j=1}^{q_o} \cdot \equiv 0$ if $q_o = 0$.

Definition 1.2. (i) If $q > q_o$ we say that Ω is *q-pseudoconvex* at z_0.

(ii) If $q < q_o$ we say that Ω is *q-pseudoconcave* at z_0.

The q-pseudoconvexity/concavity is said to be *strong* when (1.7) holds as strict inequality.

The notion of q-pseudoconvexity was used in [Ah07] and [Za00] to prove the existence of $C^\infty(\bar{\Omega})^k$ solutions to the equation $\bar{\partial}u = f$. Though the notion of q-pseudoconcavity is formally symmetric to q-pseudoconvexity, it is useless in the existence problem. The reason is intrinsic. Existence is a "global" problem but bounded domains are never globally q-pseudoconcave. Owing to the local nature of estimates and the related local regularity of $\bar{\partial}$-Neumann problem, this is the first occurrence where q-pseudoconcavity comes successfully into play. Moreover, the local estimates on pseudoconcave domain play the leading role in study the $\bar{\partial}/\bar{\partial}_b$-Neumann problem on annuli/hypersurfaces.

Remark 1.3. If Ω is q-pseudoconvex at z_0 then it implies Ω is also k-pseudoconvex for any $k \geq q$. Similarly, if Ω is q-pseudoconcave at z_0 then it implies Ω is also k-pseudoconvex for any $k \leq q$.

Remark 1.4. Definition 1.2 is generalized from usual pseudoconvexity, pseudoconcavity and condition $Z(q)$. In fact, for $q_o = 0, q = 1$ then 1-pseudoconvex is usual pseudoconvex. Similarly, $q_o = n-1, q = n-2$ then $(n-1)$-pseudoconcave is usual pseudoconcave. Moreover, if Ω satisfies condition $Z(q)$ at the point z_0, that is, the Levi form has at least $n-q$ positive eigenvalues or at least $q+1$ negative eigenvalue at each point $z \in b\Omega \cap U$, then Ω is strongly q-pseudoconvex or strongly q-pseudoconcave at z_0.

We denote by $\mathcal{A}^{h,k}$ the space of smooth (h,k)-forms in $\bar{\Omega}$. Throughout this thesis we only deal with $(0,k)$-form since the extension form type $(0,k)$ to type (h,k) is trivial. Denote by

$$C_c^\infty(U \cap \bar{\Omega})^k = \{u \in \mathcal{A}^{0,k} \big| \mathrm{supp}(u) \subset U\}.$$

If $u \in C_c^\infty(U \cap \bar{\Omega})^k$, then u can be written as follows:

$$u = \sum_{|J|=k}{}' u_J \bar{\omega}_J, \tag{1.8}$$

where $\bar{\omega}_J = \bar{\omega}_{j_1} \wedge ... \wedge \bar{\omega}_{j_k}$ here $J = \{j_1, ..., j_k\}$ are multiindices and $\sum'$ denotes summation over strictly increasing index sets. If J decomposes as $J = jK$, then we write $u_{jK} = \epsilon_J^{jK} u_J$ where $\epsilon_J^{jK} u_J$ is the sign of the permutation $iK \to J$. We check ready that $u \in \mathrm{Dom}(\bar{\partial}^*)$ if and only if $u_{nK}|_{b\Omega} = 0$ for any K.

We define the multiplier $\mathcal{M}$ such that if $\mathcal{M} = \mathcal{A}^{0,0}$, we define

$$\mathcal{M}u := |\mathcal{M}||u|. \tag{1.9}$$

If $\mathcal{M} = \sum_j \mathcal{M}_j \omega_j \in \mathcal{A}^{1,0}$ and when Ω is q-pseudoconvex/concave at z_0, we define

$$\mathcal{M}u := \sqrt{\Big|\sum_{|K|=k-1}{}^{'} |\sum_{j=1}^{n} \bar{\mathcal{M}}_j u_{jK}|^2 - \sum_{j=1}^{q_o} |\mathcal{M}|^2 |u|^2\Big|}. \tag{1.10}$$

Let $z_0 \in b\Omega$, we choose *special boundary coordinates* $(x, r) \in \mathbb{R}^{2n-1} \times \mathbb{R}$ defined in a neigborhood U of z_0 where x_j's are the *tangential coordinates* and r, the defining function, is called *normal coordinate*. Denote by ξ the dual varialbes of x; and define $x \cdot \xi = \sum x_i \xi_j$; $|\xi|^2 = \sum \xi_j^2$.

For $\varphi \in C_c^\infty(U \cap \bar{\Omega})$ we define $\tilde{\varphi}$, the *tangential Fourier transforms* of φ, by

$$\tilde{\varphi}(\xi, r) = \int_{\mathbb{R}^{2n-1}} e^{-ix\cdot\xi} \varphi(x, r) dx.$$

We denote by $\Lambda_\xi = (1+|\xi|^2)^{\frac{1}{2}}$ the standard "tangential" elliptic symbol of order 1 and by Λ the operator with symbol Λ_ξ; We define a class of functions $\mathcal{F}$ by

$$\mathcal{F} = \{f \in C^\infty([1, +\infty)\Big| f(t) \lesssim t^{\frac{1}{2}}; f'(t) \geq 0 \text{ and } \left|f^{(m)}(t)\right| \lesssim \Big|\frac{f^{m-1}(t)}{t}\Big|, \forall m \in \mathbb{Z}^+\}.$$

Notice that with $f(t) = 1; \log(t)^s$ or t^ϵ, $\epsilon \leq \frac{1}{2}$, then $f \in \mathcal{F}$.

For $f \in \mathcal{F}$, we define the operator $f(\Lambda)$ by

$$f(\Lambda)\varphi(x, r) = (2\pi)^{-2n+1} \int_{\mathbb{R}^{2n-1}} e^{ix\cdot\xi} f(\Lambda_\xi) \tilde{\varphi}(\xi, r) d\xi. \tag{1.11}$$

where $\varphi \in C_c^\infty(U \cap \bar{\Omega})$.

We define the energy form Q on $C_c^\infty(U \cap \bar{\Omega})^k \cap \mathrm{Dom}(\bar{\partial}^*)$ by

$$Q(u, v) = (\bar{\partial}u, \bar{\partial}v) + (\bar{\partial}^* u, \bar{\partial}^* v) + (u, v)$$

for $u, v \in C_c^\infty(U \cap \bar{\Omega})^k \cap \mathrm{Dom}(\bar{\partial}^*)$. Now we are ready to define $(f\text{-}\mathcal{M})^k$ estimate.

Definition 1.5. Let Ω be q-pseudoconvex (resp. q-pseudoconcave) at $z_0 \in b\Omega$. Then the $\bar{\partial}$-Neumann problem is said to satisfy $(f\text{-}\mathcal{M})^k$ *estimate* at the boundary point $z_0 \in b\Omega$ if there exists a positive constant $C_{\mathcal{M}}$ and a neighborhood U of z_0 such that

$$(f\text{-}\mathcal{M})^k \qquad ||f(\Lambda)\mathcal{M}u||^2 \lesssim Q(u,u) + C_{\mathcal{M}}\|u\|^2_{-1} \tag{1.12}$$

holds for any $u \in C_c^\infty(U \cap \bar{\Omega})^k \cap \mathrm{Dom}(\bar{\partial}^*)$ where $k \geq q$ (resp. $k \leq q$).

Remark 1.6. Remark that $C_{\mathcal{M}}$ is only depend on the constant M so that we write C_M for $C_{\mathcal{M}}$. Then if M is bounded constant then we can assume that $C_{\mathcal{M}} = 0$.

Remark 1.7. If Ω is q-pseudoconvex at z_0 and $(f\text{-}\mathcal{M})^k$ holds for $k \geq q$ then $(f\text{-}\mathcal{M})^l$ holds for any $l \geq k$. Similarly, If Ω is q-pseudoconcave at z_0 and $(f\text{-}\mathcal{M})^k$ holds for $k \leq q$ then $(f\text{-}\mathcal{M})^l$ holds for any $l \leq k$.

We usually say simply $(f\text{-}\mathcal{M})^k$ holds when the definition applies. We remark that $f(|\xi|) \lesssim |\xi|^{\frac{1}{2}}$, and if $f(|\xi|) \cong |\xi|^{\frac{1}{2}}$ then the operator $\mathcal{M}$ have to bounded (that is, $\mathcal{M}$ independent on M) since the best estimate on boundary point is $\frac{1}{2}$-subelliptic estimates.

1.3 Some relations with $(f\text{-}\mathcal{M})^k$

We want to point our attention to the choice of f and $\mathcal{M}$ in relevant cases and review some results concerning those estimates :

1) For $f(|\xi|) = |\xi|^\epsilon$ for $0 < \epsilon \leq \frac{1}{2}$ and $\mathcal{M} = 1$, then $(f\text{-}\mathcal{M})^k$ become subellipitic estimate

$$|||u|||^2_\epsilon \lesssim Q(u,u). \tag{1.13}$$

When the domain Ω is pseudoconvex, a great deal of work has been done about subelliptic estimates. The most general results concerning this problem have been obtained in [Koh79] and [Cat87].

- In [Koh79], Kohn gave a sufficient condition for subellipticity over pseudoconvex domains with real analytic boundary by introducing a sequence of ideals of subelliptic multipliers.

- In [Cat87], Catlin proved, regardless whether $b\Omega$ is real analytic or not, that subelliptic estimates hold for k-forms at z_0 if and only if the D'Angelo type $D_k(z_0)$ is finite. Catlin applies the method of weight functions used earlier by Hormander [Ho66]. One step in Catlins proof is the following reduction:

  Theorem 1.8. *Suppose that $\Omega \subset\subset \mathbb{C}^n$ is a pseudoconvex domain defined by $\Omega = \{r < 0\}$, and that $z_0 \in b\Omega$. Let U is a neighborhood of z_0. Suppose that for all $\delta > 0$ there is a smooth real-valued function Φ^δ satisfying the properties:*

$$\begin{cases} |\Phi^\delta| \leq 1 \text{ on } U, \\ \Phi^\delta_{z_i\bar{z}_j} \geq 0 \text{ on } U, \\ \sum_{ij=1}^n \Phi^\delta_{z_i\bar{z}_j} u_i \bar{u}_j \gtrsim \delta^{-2\epsilon}|u|^2 \text{ on } U \cap \{-\delta < r \leq 0\} \end{cases} \tag{1.14}$$

  Then there is a subelliptic estimate of order ϵ at z_0.

However, not much is known in the case when the domain is not necessarily pseudoconvex except from the results related to the celebrated $Z(k)$ condition which characterizes the existence of subelliptic estimates for $\epsilon = \frac{1}{2}$ according to Hörmander [Hor65] and Folland-Kohn [FK72]. Some further results, mainly related to the case of forms of top degree $n-1$ are due to Ho [Ho85].

The basic theorem of Kohn and Nirenberg [1965] shows that local regularity is consequence of a subelliptic estimates. In fact, if subelliptic estimate of order ϵ holds for the $\bar{\partial}$-Neumann problem on a neighborhood U of the given point in $b\Omega$, then $\alpha|_U \in H_s(U)^k$ implies $N\alpha|_{U'} \in H_{s+2\epsilon}(U')^k$ for $U' \subset\subset V$; here $H_s(U)^k$ denote the L^2-Sobolev space of order s on k-forms.

2) For $f(|\xi|) = \log|\xi|$ and $\mathcal{M} = \frac{1}{\epsilon}$ for any $\epsilon > 0$, then $(f\text{-}\mathcal{M})^k$ implies superlogarithmic estimate

$$||\log(\Lambda)u||^2 \lesssim \epsilon(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2) + C_\epsilon\|u\|^2. \tag{1.15}$$

Superlogarithmic estimate was first introduced by Kohn in [Koh02]. He proved superlogarithmic estimate for the operator $\Box_b$ on pseudoconvex CR manifolds and using them to establish local regularity of $\Box_b$ and of the $\bar{\partial}$-Neumann problem. These estimates are established under the assumption that subellipticity degenerates in certain specified ways.

3) For $f(|\xi|) \equiv 1$ and $\mathcal{M} = \frac{1}{\epsilon}$ for any $\epsilon > 0$, then $(f\text{-}\mathcal{M})^k$ implies compactness estimate

$$||u||^2 \lesssim \epsilon(\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2) + C_\epsilon \|u\|^2_{-1}. \tag{1.16}$$

By definition $(f\text{-}\mathcal{M})^k$, the compactness estimate in the line (1.16) is local property. Roughly speaking, the $\bar{\partial}$-Neumann operator N on Ω is compact if and only if every boundary point has a neighborhood U such that the corresponding $\bar{\partial}$-Neumann operator on $U \cap \Omega$ is compact. A classical theorem of Kohn and Nirenberg [KN65] asserts that compactness of N (as an operator form $L_2(\Omega)$ to itsefl) implies global regularity in the sense of preservation of Sobolve space.

Catlin [Cat84] introduced Property (P) and showed that it implies a compactness estimate for $\bar{\partial}$-Neumann problem. A pseudoconvex domain Ω has Property (P) if for every positive number M there exists a plurisubhamonic function Φ^M in $C^\infty(\Omega)$, bounded between 0 and 1, whose complex Hessian has all its eigenvalues bounded below by M on $b\Omega$:

$$\sum_{ij=1}^{n} \frac{\partial^2 \Phi^M}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j \geq M|w|^2, \text{ for} z \in b\Omega, w \in \mathbb{C}^n. \tag{1.17}$$

Compactness is completely understood on (bounded) locally convexifiable domains. On such domains, the following are equivalent [FS98], [FS01] :

(i) N is compact,

(ii) the boundary of the domain satisfies property (P),

(iii) the boundary contains no q-dimensional analytic variety.

In general, however, the situation is not understood at all.

5) For $f(\xi) \equiv 1$ and $\mathcal{M} = \frac{1}{\epsilon}\sum_{j=1}^n r^\epsilon_{z_i\bar{z}_j} r_{\bar{z}_i} dz_j$ is a 1-form for any $\epsilon > 0$ here r^ϵ, r are defining functions with $|\nabla r^\epsilon| \cong 1$ on $b\Omega$, then $(f\text{-}\mathcal{M})^k$ can be written as

$$\sum_{|K|=k-1}{}' \|\sum_{i,j=1}^{n} \overline{r^\epsilon_{z_i\bar{z}_j} r_{\bar{z}_i}} u_{jK}\|^2 \lesssim \epsilon(\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2) + C_\epsilon \|u\|^2_{-1}. \tag{1.18}$$

for any $u = \sum_{|J|=k}' u_J d\bar{z}_J \in C^\infty(\bar{\Omega})^k \cap \text{Dom}(\bar{\partial}^*)$.

The estimate (1.18) was introduced by Straube in [Str08]. In [Str08], he showed that if (1.18) holds for all $u \in C^\infty(\Omega)^k \cap \text{Dom}(\bar{\partial}^*)$ then the $\bar{\partial}$-Neumann operator N_k on k-forms is exactly regular in Sobolev norms, that is

$$\|N_k u\|_s \leq C_s \|u\|_s$$

for any integer $s \geq 0$ and all $u \in H_s(\Omega)^k$. Notice that the estimate (1.18) is weaker than compactness estimate.

6) For $f(|\xi|) = |\xi|^\epsilon$ and $\mathcal{M} \in \mathcal{A}^{0,0}$ (resp. $\mathcal{M} \in \mathcal{A}^{1,0}$), then $(f\text{-}\mathcal{M})^k$-estimates will be

$$|||\mathcal{M}u|||_\epsilon^2 \lesssim Q(u,u). \tag{1.19}$$

One calls $\mathcal{M}$ be a subelliptic multiplier (resp. subelliptic vector-multiplier). Kohn [Koh79] used the subelliptic multiplier and subelliptic vector-multiplier to obtian the subelliptic estimate on real analytic domain. In particular, he showed there is a nonzero constant function belong to his collection of all multipliers.

1.4 The main theorems

The first goal in this thesis, we exploit here the full strength of Catlin's method to study $(f\text{-}\mathcal{M})^k$ estimate on a q-pseudoconvex or q-pseudoconcave domain. These results are related to works of my joint work with G. Zampieri in [KZ1],[KZ2] and [KZ3].

Denote by S_δ the strip set $\{z \in \Omega | -\delta < r < 0\}$, generalize conditions (1.14) in Theorem 1.8 and Property (P) in (1.17), we define

Definition 1.9. We call that Ω is satisfied Property $(f\text{-}\mathcal{M}\text{-}P)^k$ at the boundary point $z_0 \in b\Omega$ if there is a neighborhood U of z_0 and for all $\delta > 0$ sufficiently small there exists a real-valued function $\Phi := \Phi^{\delta,\mathcal{M}} \in C^2(U)$ such that:

$$(f\text{-}\mathcal{M}\text{-}P)^k \qquad \begin{cases} |\Phi| \lesssim 1 \\ \sum\nolimits'_{|K|=k-1} \sum_{ij=1}^{n} \Phi_{ij} u^\tau_{iK} \bar{u}^\tau_{jK} & -\sum_{j=1}^{q_o} \Phi_{jj} |u^\tau|^2 \\ & \gtrsim f(\delta^{-1})^2 |\mathcal{M} u^\tau|^2 + \sum_{j=1}^{q_o} |L_j(\Phi) u^\tau|^2 \end{cases} \tag{1.20}$$

on $S_\delta \cap U$ for any $u \in C_c^\infty(\bar{\Omega} \cap U)^k$, where u^τ is the tangetial component of u.

Our result is the following :

Theorem 1.10. *Let $\Omega \subset \mathbb{C}^n$ be q-pseudoconvex (resp. q-pseudoconcave) and satisfy Property $(f\text{-}\mathcal{M}\text{-}P)^k$ at boundary point $z_0 \in b\Omega$ then $(f\text{-}\mathcal{M})^k$-estimate holds at z_0 with $k \geq q$ (resp. $k \leq q$).*

Observe that conditions of the family $\{\Phi^{\delta,\mathcal{M}}\}$ in our Property $(f\text{-}\mathcal{M}\text{-}P)^k$ is simpler than conditions (1.14) in Theorem 1.8 and Property (P) in (1.17).

The main idea for proving Theorem 1.10 stems from a paper of Catlin [Cat87], combining with some modifications in our papers [KZ1] and [KZ2].

We remark that our Property $(f\text{-}\mathcal{M}\text{-}P)^k$ is restriction only on u^τ-tangential component of u. we firstly get $(f\text{-}\mathcal{M})^k$-estimate for u replaced by u^τ. However, for the normal component u^ν,of u, one can reasonably get the estimate $\|u^\nu\|_1^2 \lesssim Q(u,u)$. This supply completely u to get our estimates.

On the real hypersurface M of $\mathbb{C}^n$, $\bar\partial$ induces the tangetial Cauchy-Riemann operator $\bar\partial_b$. Let $\bar\partial_b^*$ be the L_2-adjoint of $\bar\partial_b$ and $\Box_b = \bar\partial_b\bar\partial_b^* + \bar\partial_b^*\bar\partial_b$, the Kohn Laplacian. We denote by

$$Q_b(u,v) = (\bar\partial u, \bar\partial u)_b + (\bar\partial^* u, \bar\partial^* v)_b + (u,v)_b$$

for any $u, v \in C_c^\infty(U \cap M)^k$ where $C_c^\infty(U \cap M)^k$ the space of tangential (0,k)-forms on M with support in U and $(\ ,\ )_b$ denotes the L_2-inner product on M.

Suppose that $u \in C_c^\infty(U \cap M)^k$. Write

$$u = u^+ + u^- + u^0,$$

where $\tilde{u}^+$ is supported in a conical neigborhood of 0 with $\xi_{2n-1} > 0$, $\tilde{u}^-$ is supported in a canonical neigborhood of 0 with $\xi_{2n-1} < 0$, and $\tilde{u}^0$ is supported outside of such neighborhoods.

Our estimates on M are defined as follows:

Definition 1.11. If M is a hypersurface and $z_0 \in M$ then a $(f\text{-}\mathcal{M})^k_b$ estimate holds for $(\bar\partial_b, \bar\partial_b^*)$ on at x_0 if there exists a neighborhood U of z_0 such that

$$(f\text{-}\mathcal{M})^k_b \qquad \|f(\Lambda)\mathcal{M}u\|^2_b \le cQ_b(u,u) + C_{\mathcal{M}}\|u\|^2_{b,-1}$$

for all $u \in C^\infty_c(U \cap M)^k$. And a $(f\text{-}\mathcal{M})^k_{b,+}$-estimate (resp. $(f\text{-}\mathcal{M})^k_{b,-}$) holds for $(\bar\partial_b, \bar\partial_b^*)$ at z_0 if the above holds with u replaced by u^+ (resp. u^-), that is,

$$(f\text{-}\mathcal{M})^k_{b,+} \qquad \|f(\Lambda)\mathcal{M}u^+\|^2_b \le cQ_b(u^+,u^+) + C_{\mathcal{M}}\|u^+\|^2_{b,-1}$$

(resp.

$$(f\text{-}\mathcal{M})^k_{b,-} \qquad \|f(\Lambda)\mathcal{M}u^-\|^2_b \le cQ_b(u^-,u^-) + C_{\mathcal{M}}\|u^-\|^2_{b,-1}. \quad)$$

Definition 1.12. The hypersurface M is called to be q-pseudoconvex at z_o if one of two sides divided by M is q-pseudoconvex at z_o.

We denote by $\Omega^+ = \{z \in U | r(z) < 0\}$ the q-pseudoconvex side at z_o which is divied by M; and another side denoted by Ω^-. By Remark 2.3, $\Omega^- = \{z \in U | -r(z) < 0\}$ is $(n-q-1)$-pseudoconcave at z_o. Here U is a neighborhood of $z_0 \in M$. We denote by $(f\text{-}\mathcal{M})^k_{\Omega^+}$ the $(f\text{-}\mathcal{M})^k$ on Ω^+; by $(f\text{-}\mathcal{M})^k_{\Omega^-}$ the $(f\text{-}\mathcal{M})^k$ on Ω^- .

The second goal in this thesis is the equivalents of $(f\text{-}\mathcal{M})^k$ estimates on the domain and on its boundary.

Theorem 1.13. *Let M be a q-pseudoconvex hypersurface at z_0. Asumme that M divides a neighborhood U of z_0 to be Ω^+ and Ω^- as above. Then*

$$(f\text{-}\mathcal{M})^k_{\Omega^+} \Longleftrightarrow (f\text{-}\mathcal{M})^k_{b,+} \Longleftrightarrow (f\text{-}\mathcal{M})^{n-1-k}_{b,-} \Longleftrightarrow (f\text{-}\mathcal{M})^{n-1-k}_{\Omega^-} \tag{1.21}$$

for any $k \ge q$.

Since $Q_b(u^0, u^0) \gtrsim \|u^0\|^2_{b,1}$, Theorem 1.13 implies that $(f\text{-}\mathcal{M})^k_b$ on M holds at z_0 if we have Property $(f\text{-}\mathcal{M}\text{-}P)^k$ and Property $(f\text{-}\mathcal{M}\text{-}P)^{n-1-k}$ hold on Ω^+ at z_0.

The thesis also contains the constructions of the Property $(f\text{-}\mathcal{M}\text{-}P)^k$ on some class of domains such that : $Z(k)$, decoupled, regular coordinate; and the discussion about global and local regularity.

The thesis is structured as follows. In chapter 2 we give the background of the $\bar\partial$-Neumann problem. In chapter 3 we prove theorem 1.10. The proof of theorem 1.13 is presented in chapter 4. In chapter 5, we construct the Property $(f\text{-}\mathcal{M}\text{-}P)^k$ in some class of domains. The global and local regularity is discussed in chapter 6.

Chapter 2

Background

In this chapter, we provide almost of the background for reading of the $(f\mathcal{M})^k$ estimate of the $\bar\partial$-Neumann problem on q-pseudoconvex/concave domains at given point z_0.

2.1 The terminology and notations

For $z \in \mathbb{C}^n$, we denote by $\mathbb{C}T_z$ the complex-valued tangent vectors to $\mathbb{C}^n$ at z and we have the direct sum decomposition $\mathbb{C}T_z = T_z^{1,0} \oplus T_z^{0,1}$, where $T_z^{1,0}$ and $T_z^{0,1}$ denote the holomorphic and anti-holomorphic vectors at z respectively.

Denote $A_z^{0,k}$ the space of $(0,k)$-forms at z and by $\langle\ ,\ \rangle_z$ the pairing of $A_z^{0,k}$ with its dual space, we will also denote by $\langle\ ,\ \rangle_z$ the inner product induced on $A_z^{0,k}$ by the hermitian metric and by $|\ |_z$ the associated norm.

We fix the point $z_0 \in \mathbb{C}^n$. Then there exists a neighborhood U of z_0 such that we can choose C^∞ vector fields with value in $T^{1,0}$, which at each point $z \in U$ are an orthonormal basis of $T^{1,0}$. Let $L_1, \cdots, L_n$ be such a basis, then for z we have $\langle (L_i)_z, (L_j)_z \rangle_z = \delta_{ij}$.

Let $\omega_1, \cdots, \omega_n$ be the dual basis of (1,0)-forms on U, so for each $z \in U$ we have $\langle (\omega_i)_z, (L_j)_z \rangle_z = \delta_{ij}$. We denote by $\bar L_1, \cdots, \bar L_n$ the conjugates of $L_1, \cdots, L_n$, respectively; these form an orthonormal basis of $T^{0,1}$ on U. Denote by $\bar{\omega}_1, \cdots, \bar{\omega}_n$, the conjugates of $\omega_1, \cdots, \omega_n$ respectively; then they are the local basis of (0,1)-forms on U which dual to $\bar{L}_1, \cdots, \bar{L}_n$. In this basis, for any $\phi \in C^\infty(U)$, we can write

$$d\phi = \sum_{j=1}^{n} L_j(\phi)\omega_j + \sum_{j=1}^{n} \bar{L}_j(\phi)\bar{\omega}_j.$$

Then one defines

$$\partial\phi = \sum_{j=1}^{n} L_j(\phi)\omega_j \quad \text{and} \quad \bar{\partial}\phi = \sum_{j=1}^{n} \bar{L}_j(\phi)\bar{\omega}_j.$$

We set ϕ_{ij} to be the coefficients of $\partial\bar{\partial}\phi$, i.e.

$$\partial\bar{\partial}\phi = \sum_{ij} \phi_{ij}\omega_i \wedge \bar{\omega}_j. \tag{2.1}$$

For each $k = 1, ..., n$, let $\bar{c}^k_{ij}$ be smooth functions such that

$$\partial\bar{\omega}_k = \sum_{ij} \bar{c}^k_{ij}\omega_i \wedge \bar{\omega}_j.$$

Then ϕ_{ij} can be calculated as follows:

$$\begin{aligned} \partial\bar{\partial}\phi &= \partial\Big(\sum_k \bar{L}_k(\phi)\bar{\omega}_k\Big) \\ &= \sum_{i,k} L_i\bar{L}_k(\phi)\omega^i \wedge \bar{\omega}^k + \sum_k \bar{L}_k(\phi) \sum_{i,j} \bar{c}^k_{ij}\omega_i \wedge \bar{\omega}_j \\ &= \sum_{i,j} \Big(L_i\bar{L}_j(\phi) + \sum_k \bar{c}^k_{ij}\bar{L}_k(\phi)\Big)\omega_i \wedge \bar{\omega}_j. \end{aligned}$$

Form the fact that $\partial\bar{\partial} + \bar{\partial}\partial = 0$, we have

$$\phi_{ij} = L_i\bar{L}_j(\phi) + \sum_k \bar{c}^k_{ij}\bar{L}_k(\phi) = \bar{L}_jL_i(\phi) + \sum_k c^k_{ji}L_k(\phi). \tag{2.2}$$

With our notions of ϕ_{ij}'s, we say that (ϕ_{ij}) is Levi matrix of ϕ under basis $\omega_1, ..., \omega_n$. Moreover, from (2.2) we obtain

$$[L_i, \bar{L}_j] = \sum_k c^k_{ji} L_k - \sum_k \bar{c}^k_{ij} \bar{L}_k. \tag{2.3}$$

where $[L_i, \bar{L}_j] = L_i \bar{L}_j - \bar{L}_j L_i$, as usual.

Let $\Omega \subset \mathbb{C}^n$ be an open subset of $\mathbb{C}^n$ and let $b\Omega$ denote the boundary of Ω. Throughout this thesis we will restrict ourselves to domain Ω such that $b\Omega$ is smooth in the following sense. We assume that in a neighborhood U of $b\Omega$ there exist a C^∞ real-valued function r such that $dr \neq 0$ in U and $r(z) = 0$ if and only if $z \in b\Omega$. Without loss of generality, we shall assume that $r > 0$ outside of $\bar{\Omega}$ and $r < 0$ in Ω.

For $z_0 \in b\Omega$, we fix r so that $|\partial r|_z = 1$ in a neighborhood U of z_0. We choose $\omega_1, \cdots, \omega_n$ to be (1,0)-forms on U such that $\omega_n = \partial r$ and such that $\langle \omega_i, \omega_j \rangle = \delta_{ij}$ for $z \in U$. We then define $L_1, \cdots, L_n, \bar{L}_1, \cdots, \bar{L}_n, \bar{\omega}_1, \cdots, \bar{\omega}_n$ as above. Note that on $U \cap b\Omega$, we have

$$L_j(r) = \bar{L}_j(r) = \delta_{jn}.$$

Thus $L_1, \cdots, L_{n-1}$ and $\bar{L}_1, \cdots, \bar{L}_{n-1}$ are local bases of $T^{1,0}(U \cap b\Omega) := \mathbb{C}T(U \cap b\Omega) \cap T^{1,0}$ and $T^{0,1}(U \cap b\Omega) := \mathbb{C}T(U \cap b\Omega) \cap T^{0,1}$ respectively, where $\mathbb{C}T(U \cap b\Omega)$ the space of complex-valued tangent vector to $U \cap b\Omega$. We define a vector filed T on $U \cap b\Omega$ with values in $\mathbb{C}T(U \cap b\Omega)$ by:

$$T = L_n - \bar{L}_n.$$

Observe that $L_1, \cdots, L_{n-1}, \bar{L}_1, \cdots, \bar{L}_{n-1}, T$ are local basis of $\mathbb{C}T(U \cap b\Omega)$. Using integration by parts, we get the proof of following lemma:

Lemma 2.1. *Let $\varphi, \psi \in C^\infty_c(U \cap \bar{\Omega})$, then we have*

$$(\bar{L}_j \varphi, \psi) = -(\varphi, L_j \psi) + \int_{b\Omega} \bar{L}_j(r) \varphi \bar{\psi} dS + (\varphi, a_j \psi)$$

where $a_j \in C^\infty(\bar{U} \cap \bar{\Omega})$.

Substituting r for ϕ in (2.2), we get $r_{ij} = \bar{c}^n_{ij} = c^n_{ij}$ and by (2.3) hence

$$[L_i, \bar{L}_j] = r_{ij}T + \sum_{k=1}^{n-1} c^k_{ji} L_k - \sum_{k=1}^{n-1} \bar{c}^k_{ji} \bar{L}_k. \tag{2.4}$$

Let $\lambda_1(z) \leq ... \leq \lambda_{n-1}(z)$ be the eigenvalues of $(r_{jk}(z))^{n-1}_{j,k=1}$ and denote $s^+_{b\Omega}(z)$, $s^-_{b\Omega}(z)$, $s^0_{b\Omega}(z)$ their number according to the different sign.

We take a pair of indices $1 \leq q \leq n-1$ and $0 \leq q_o \leq n-1$ such that $q \neq q_o$. We assume that there is a bundle $\mathcal{V}^{q_o} \in T^{1,0}b\Omega$ of rank q_o with smooth coefficients, by reordering we may suppose $\mathcal{V}^{q_o} =$ span $\{L_1, ..., L_{q_o}\}$ such that

$$\sum_{j=1}^{q} \lambda_j(z) - \sum_{j=1}^{q_o} r_{jj}(z) \geq 0 \qquad z \in U \cap b\Omega. \tag{2.5}$$

Remember that we have defined Ω to be q-pseudoconvex or q-pseudoconcave according to $q > q_o$ or $q < q_o$ follows when (2.5) holds.

Lemma 2.2. *The domain Ω is q-pseudoconvex at $z_0 \in b\Omega$ if and only if*

$$\sum_{|K|=k-1}{}' \sum_{ij=1}^{n-1} r_{ij} u_{iK} \bar{u}_{jK} - \sum_{j=1}^{q_o} r_{jj} |u|^2 \geq 0 \text{ on } U \cap b\Omega \tag{2.6}$$

holds for any $u \in C^\infty_c(\bar{\Omega} \cap U)^q \cap Dom(\bar{\partial}^*)$.

Proof. The proof of Lemma immediately follows by the choice $C = \sum_{j=1}^{q_o} r_{jj}$ in the equivalence of two facts:

(i) $\sum_{|K|=k-1}{}' \sum_{i,j=1}^{n-1} r_{ij} u_{iK} \bar{u}_{jK} \geq C|u|^2$ for all $u \in C^\infty_c(\Omega \cap U)^k \cap \mathrm{Dom}(\bar{\partial}^*)$

(ii) The sum of any q eigenvalue of the matrix (r_{ij}) is greater than or equal to C.

A proof of the equivalence of (i) and (ii) follows by diagonalizing the matrix (r_{ij}); see [Hor65] and [Cat87].

□

As it has already been noticed, (2.5) for $q > q_o$ implies $\lambda_q \geq 0$; hence (2.5) is still true if we replace the first sum $\sum_{j=1}^{q} \cdot$ by $\sum_{j=1}^{k} \cdot$ for any k such that $q \leq k \leq n-1$. Similarly, if it holds for $q < q_o$, then $\lambda_{q+1} \leq 0$ and hence it also holds with q replaced by $k \leq q$ in the first sum.

We notice that q-pseudoconvexity/concavity is invariant under a change of an orthonormal basis but not of an adapted frame. In fact, not only the number, but also the size of the eigenvalues comes into play. Thus, when we say that $b\Omega$ is q-pseudoconvex/concave, we mean that there is an adapted frame in which (2.5) is fulfilled. Sometimes, it is more convenient to put our calculatiuons in an orthonormal frame. In this case, it is meant that the metric has been changed so that the adapted frame has become orthonormal.

Example 2.1. Let $s^-(z)$ be constant for $z \in b\Omega$ close to z_0; then (2.5) holds for $q_o = s^-$ and $q = s^-+1$. In fact, we have $\lambda_{s^-} < 0 \leq \lambda_{s^-+1}$, and therefore the negative eigenvectors span a bunlde $\mathcal{V}^{q_o}$ for $q_o = s^-$ that, identified with the span of the first q_o coordinate vector fields, yields $\sum_{j=1}^{q_o+1} \lambda_j(z) \geq \sum_{j=1}^{q_o} r_{jj}(z)$. Note that a pseudoconvex domain is characterized by $s^-(z) \equiv 0$, thus, it is 1-pseudoconvex in our terminology.

In the same way, if $s^+(z)$ is constant for $z \in b\Omega$ close to z_0, then $\lambda_{s^-+s^0} \leq 0 < \lambda_{s^-+s^0+1}$. Then, the eigenspace of the eigenvectors ≤ 0 is a bundle which, identified to that of the first $q_o = s^- + s^0$ coordinate vector fields yields (2.5) for $q = q_o - 1$. In particular a pseudoconcave domain, that is a domain which satisfies $s^+ \equiv 0$, is $(n-2)$-pseudoconcave in our terminology.

Example 2.2. Let Ω satisfy $Z(q)$ condition at z_0, that is, $s^+(z) \geq n-q$ or $s^-(z) \leq q+1$ for $z \in b\Omega \cap U$. Thus Ω is strongly q-pseudoconvex or strongly q-pseudoconcave at z_0

Example 2.3. Let Ω be a domain which defining function r defined in a neighborhood z_0 by

$$r = 2\mathrm{Re} z_n - Q(z_1, ..., z_{q_o}) + P(z_q, ..., z_{n-1})$$

where Q, P is real function such that $(Q_{z_i\bar{z}_j})_{ij=1}^{q_o}$ and $(P_{z_i\bar{z}_j})_{ij=q}^{n-1}$ are semipositive matrices. Then we can check that Ω is q-pseudoconvex at z_0.

Similarly, Let Ω be a domain which defining function r defined in a neighborhood z_0 by

$$r = 2\mathrm{Re} z_n - P(z_1, ..., z_{q+1}) + Q(z_{q_o+1}, ..., z_{n-1})$$

where P, Q is real function such that $(P_{z_i\bar{z}_j})_{ij=1}^{q+1}$ and $(Q_{z_i\bar{z}_j})_{ij=q_o+1}^{n-1}$ are semi-positive matrices. Then we can check that Ω is q-pseudoconcave at z_0 (see Proposition 5.5).

Remark 2.3. Since the fact $\sum_{j=1}^{n-1}\lambda_j(z) = \sum_{j=1}^{n-1} r_{jj}(z)$ for $z \in U \cap b\Omega$, it follows

$$\sum_{j=1}^{q}\lambda_j - \sum_{j=1}^{q_o} r_{jj} = \sum_{j=q+1}^{n-1}(-\lambda_j) - \sum_{q_o+1}^{j=n-1}(-r_{jj}).$$

Therefore if Ω defined by $r < 0$ is q-pseudoconvex (q-pseudoconcave) at z_0, then $\mathbb{C}^n \backslash \bar{\Omega} = \{-r < 0\}$ is $(n-q-1)$-pseudoconcave ($(n-q-1)$-pseudoconvex) at z_0.

2.2 The basic estimate

Remember that we have already denoted by $\mathcal{A}^{0,k}$ the space of $(0,k)$-forms in $C^\infty(X)$ restricted to $\bar{\Omega}$ and

$$C_c^\infty(U \cap \bar{\Omega})^k = \{u \in \mathcal{A}^{0,k} \big| \operatorname{supp}(u) \subset U\}.$$

If $u \in C_c^\infty(U \cap \bar{\Omega})^k$, then u can be written as follows:

$$u = {\sum_{|J|=k}}' u_J \bar{\omega}_J, \tag{2.7}$$

where $\bar{\omega}_J = \bar{\omega}_{j_1} \wedge ... \wedge \bar{\omega}_{j_k}$ here $J = \{j_1, ..., j_k\}$ are multiindices and $\sum'$ denotes summation over strictly increasing index sets. When the multiindices are not ordered, the coefficients are assumed to be alternant. Thus, if J decomposes as $J = jK$, then $u_{jK} = \epsilon_J^{jK} u_J$ where ϵ_J^{jK} is the sign if the permutation $jK \to J$. Then Cauchy-Riemann operator, $\bar{\partial}$, acts usual on a $(0,k)$-forms via

$$\bar{\partial} u = {\sum_{|J|=k}}' \sum_{j=1}^{n} \bar{L}_j u_J \bar{\omega}_j \wedge \bar{\omega}_J + ... \tag{2.8}$$

where the dots refer to terms of order zero in u. Thus we have a complex $\mathcal{A}^{0,k} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,k+1}$.

We extend this complex to $L_2^{0,k}(\Omega)$ the L_2 space of $(0,k)$-forms, so that the Hilbert space techniques may be applied to analyze the complex. For each $x \in \bar{\Omega}$ de denote by $(dV)_x$ the unique positive (n,n)-form such that : $|(dV)_x| = 1$. We call dV the volume element. We define the inner products and the norms

$$(u,v) = \int_\Omega \langle u,v\rangle_x (dV)_x; \quad \|u\|^2 = (u,u), \quad u,v \in L_2^{0,k}(\Omega).$$

For each form degree $(0,k)$, we define

$$\text{Dom}(\bar\partial) = \{v \in L_2^{0,k}(\Omega) : \bar\partial v(\text{as distribution}) \in L_2^{0,k+1}(\Omega)\}.$$

Then the operator $\bar\partial : \text{Dom}(\bar\partial) \to L_2^{0,k+1}(\Omega)$ is well-defined, and we have $\bar\partial : L_2^{0,k}(\Omega) \to L_2^{0,k+1}(\Omega)$ as a densely defined operator by noting that $\mathcal{A}^{0,k} \subset \text{Dom}(\bar\partial)$. Thus, the operator $\bar\partial$ has an L_2-adjoint, $\bar\partial^*$, defined as follows : if $u \in \text{Dom}(\bar\partial^*)$ and $\bar\partial^* u = \alpha$ if

$$(v,\alpha) = (\bar\partial v, u) \text{ for all } v \in \text{Dom}(\bar\partial).$$

We have

$$\begin{aligned}
(\bar\partial v, u) &= \sum_{|K|=k-1}{}' \sum_{j=1} (\bar L_j v_K, u_{jK}) + (v,\cdots) \\
&= \sum_{|K|=k-1}{}' \sum_{j=1} \Big(-(v_K, L_j u_{jK}) + \delta_{jn} \int_{b\Omega} v_K \bar u_{jK} dS \Big) + (v,\cdots) \qquad (2.9) \\
&= (v, -\sum_{|K|=k-1}{}' L_j u_{jK} \bar\omega_K) + \delta_{jn} \sum_{|K|=k-1}{}' \int_{b\Omega} v_K \bar u_{jK} dS + (v,\cdots)
\end{aligned}$$

where dots denote an error term in which u is not differentiated. Here the second inequality in (2.9) follows by Lemma 2.1. By (2.9), we have the proof of following lemma :

Lemma 2.4.

$$u \in Dom(\bar\partial^*) \textit{ if and only if } u_{jK}|_{b\Omega} = 0 \textit{ for any } K. \qquad (2.10)$$

Over such a form in (2.7) the action of the Hilbert adjoint of $\bar\partial$, coindides with of its "formal adjoint" and is therefore expressed by a "divergence operator":

$$\bar\partial^* u = -\sum_{|K|=k-1}{}' \sum_j L_j u_{jK} \bar\omega_K + ... \qquad (2.11)$$

for any $u \in \mathrm{Dom}(\bar\partial^*)$.

For a real function ϕ in class C^2, let the weighted L^2_ϕ-norm be defined by

$$\|u\|^2_\phi = (u,u)_\phi := \|ue^{-\frac{\phi}{2}}\|^2 = \int_\Omega \langle u,u\rangle e^{-\phi}dV.$$

Let $\bar\partial^*_\phi$ be the L^2_ϕ-adjoint of $\bar\partial$. It is easy to see that $\mathrm{Dom}(\bar\partial^*) = \mathrm{Dom}(\bar\partial^*_\phi)$ and

$$\bar\partial^*_\phi u = -\sum_{|K|=k-1}{}' \sum_{j=1}^n \delta_j^\phi u_K \bar\omega_K + \cdots \tag{2.12}$$

where $\delta_j^\phi \varphi = e^\phi L_j(e^{-\phi}\varphi)$ and where dots denote an error term in which u not differentiated and ϕ does not occur.

By developing the equalities (2.8) and (2.12), the key technical result is contained in the following proposition :

Proposition 2.5. *Let $z_0 \in b\Omega$ and fix an index q_o with $0 \le q_o \le n-1$, then there exists a neighborhood U of z_0 and suitable constant C such that*

$$\begin{aligned} &2\|\bar\partial u\|^2_\phi + 2\|\bar\partial^*_\phi u\|^2_\phi + C\|u\|^2_\phi \\ &\quad \ge \sum_{|K|=k-1}{}' \sum_{i,j=1}^n (\phi_{ij}u_{iK}, u_{jK})_\phi - \sum_{|J|=k}{}' \sum_{j=1}^{q_o} (\phi_{jj}u_J, u_J)_\phi \\ &\qquad + \sum_{|K|=k-1}{}' \sum_{i,j=1}^{n-1} \int_{b\Omega} e^{-\phi} r_{ij} u_{iK}\bar u_{jK} dS - \sum_{|J|=q}{}' \sum_{j=1}^{q_o} \int_{b\Omega} e^{-\phi} r_{jj}|u_J|^2 dS \\ &\qquad + \frac{1}{2}\Big(\sum_{j=1}^{q_o} \|\delta_j^\phi u\|^2_\phi + \sum_{j=q_o+1}^n \|\bar L_j u\|^2_\phi\Big) \end{aligned} \tag{2.13}$$

for any $u \in C_c^\infty(U\cap\bar\Omega)^k \cap Dom(\bar\partial^*)$.

Proof. Let Au denote the sum in (2.8), we obtain that

$$\|Au\|^2_\phi = \sum_{|J|=k}{}' \sum_{j=1}^n \|\bar L_j u_J\|^2_\phi - \sum_{|K|=k-1}{}' \sum_{ij} (\bar L_i u_{jK}, \bar L_j u_{iK})_\phi. \tag{2.14}$$

Let Bu denote the sum in (2.12), we obtain that

$$\|Bu\|^2_\phi = \sum_{|K|=k-1}{}' \sum_{ij} (\delta_i^\phi u_{iK}, \delta_j^\phi u_{jK})_\phi. \tag{2.15}$$

Since Au and Bu differ from $\bar\partial u$ and $\bar\partial^* u$ by terms of order zero. It follows from (2.14) and (2.15) that

$$\begin{aligned} 2(\|\bar\partial u\|_\phi^2 + \|\bar\partial_\phi^* u\|_\phi^2) + C\|u\|_\phi^2 & \\ \geq \|Au\|_\phi^2 + \|Bu\|_\phi^2 & \\ = \sum_{|J|=k}{}' \sum_{j=1}^n \|\bar L_j u_J\|_\phi^2 + \sum_{|K|=k-1}{}' \sum_{i,j=1}^n (\delta_i^\phi u_{iK}, \delta_j^\phi u_{jK})_\phi - (\bar L_j u_{iK}, \bar L_i u_{jK})_\phi & \end{aligned} \tag{2.16}$$

where C is constant independent of ϕ.

Now we want to apply integration by parts to the term $(\delta_i^\phi u_{iK}, \delta_j^\phi u_{jK})_\phi$ and $(\bar L_j u_{iK}, \bar L_i u_{jK})_\phi$. Notice that for each $\varphi, \psi \in C_c^1(U\cap\bar\Omega)$, similar to Lemma 2.1, we have

$$\begin{cases} (\varphi, \delta_j^\phi \psi)_\phi & = -(\bar L_j \varphi, \psi)_\phi + (a_j\varphi, \psi)_\phi + \delta_{jn}\int_{b\Omega} e^{-\phi}\varphi\bar\psi dS \\ -(\varphi, \bar L_i \psi)_\phi & = (\delta_i^\phi \varphi, \psi)_\phi - (b_i\varphi, \psi)_\phi - \delta_{in}\int_{b\Omega} e^{-\phi}\varphi\bar\psi dS \end{cases}$$

and for some $a_j, b_i \in C^1(\bar\Omega\cap U)$ independent on ϕ.

This immediately implies that

$$\begin{cases} (\delta_i^\phi u_{iK}, \delta_j^\phi u_{jK})_\phi & = -(\bar L_j \delta_i^\phi u_{iK}, u_{jK})_\phi + \delta_{jn}\int_{b\Omega} e^{-\phi}\delta_i^\phi(u_{iK})\bar u_{jK} dS + R \\ -(\bar L_j u_{iK}, \bar L_i u_{jK})_\phi & = (\delta_i^\phi \bar L_j u_{iK}, u_{jK})_\phi - \delta_{in}\int_{b\Omega} e^{-\phi} L_j(u_{iK})\bar u_{jK} dS + R. \end{cases} \tag{2.17}$$

From here on, we denote terms involving product of u by $\delta_j^\phi u$ for $j\leq n-1$ or $\bar L_j u$ for $j \leq n$ by R.

Recall that $\bar u_{nK} = L_j(u_{nK}) = 0$ on $b\Omega$ if $j \leq n-1$. We thus conclude that the boundary integrals vanish in both equalities of (2.17). Now by taking the sum of two terms in the right side of (2.17), after discarding the boundary integrals and we put in evidence the commutator $[\delta_i^\phi, \bar L_j]$, we get

$$(\delta_i^\phi u_{iK}, \delta_j^\phi u_{jK})_\phi - (\bar L_j u_{iK}, \bar L_i u_{jK})_\phi = ([\delta_i^\phi, \bar L_j] u_{iK}, u_{jK})_\phi + R \tag{2.18}$$

for $i, j = 1, ..., n$ and any K.

Notice that (2.17) is also true if we replace both u_{iK} and u_{jK} by u_J for indices $i = j \leq q_0$. Then we obtain

$$\|\bar L_j u_J\|_\phi^2 = \|\delta_j^\phi u_J\|_\phi^2 - ([\delta_j^\phi, \bar L_j] u_J, u_J)_\phi + R. \tag{2.19}$$

Applying (2.18) and (2.19) to the last line in (2.16), we have

$$
\begin{aligned}
&2(\|\bar\partial u\|_\phi^2 + \|\bar\partial_\phi^* u\|_\phi^2) + C\|u\|_\phi^2 \\
&\qquad \geq \sum_{|K|=k-1}{}' \sum_{i,j=1}^{n} ([\delta_i^\phi, \bar L_j] u_{iK}, u_{jK})_\phi - \sum_{|J|=k}{}' \sum_{j=1}^{q_0} ([\delta_j^\phi, \bar L_j] u_J, u_J)_\phi \\
&\qquad + \sum_{|J|=k}{}' \Big(\sum_{j=1}^{q_0} \|\delta_j^\phi u_J\|_\phi^2 + \sum_{j=q_0+1}^{n} \|\bar L_j u_J\|_\phi^2 \Big) + R.
\end{aligned} \tag{2.20}
$$

Now we calculate the commutator $[\delta_i^\phi, \bar L_j]$,

$$
\begin{aligned}
[\delta_i^\phi, \bar L_j] &= L_i \bar L_j \phi + [L_i, \bar L_j] \\
&= \phi_{ij} + \sum_{k}^{n} c_{ji}^k \delta_k^\phi - \sum_{j=1}^{n} \bar c_{ij}^k \bar L_k \\
&= \phi_{ij} + r_{ij}\delta_n^\phi + \sum_{k}^{n-1} c_{ji}^k \delta_k^\phi - \sum_{j=1}^{n} \bar c_{ij}^k \bar L_k
\end{aligned} \tag{2.21}
$$

here we use the formula in (2.2) and (2.4).

Since $L_n(r) = 1$, we have

$$
(r_{ij}\delta_n^\phi u_{iK}, u_{jK})_\phi = \int_{b\Omega} r_{ij} e^{-\phi} u_{iK} u_{jK} dS + R. \tag{2.22}
$$

Substituting (2.21) in (2.20) and combine with (2.22), we get

$$
\begin{aligned}
&2(\|\bar\partial u\|_\phi^2 + \|\bar\partial_\phi^* u\|_\phi^2) + C\|u\|_\phi^2 \\
&\qquad \geq \sum_{|K|=k-1}{}' \sum_{i,j=1}^{n} (\phi_{ij} u_{iK}, u_{jK})_\phi - \sum_{|J|=k}{}' \sum_{j=1}^{q_0} (\phi_{jj} u_J, u_J)_\phi \\
&\qquad + \sum_{|K|=k-1}{}' \sum_{i,j=1}^{n} \int_{b\Omega} r_{ij} u_{iK} \bar u_{jK} e^{-\phi} dS - \sum_{|J|=k}{}' \sum_{j=1}^{q_0} \int_{b\Omega} r_{jj} |u_J| e^{-\phi} dS \\
&\qquad + \sum_{|J|=k}{}' \Big(\sum_{j=1}^{q_0} \|\delta_j^\phi u_J\|_\phi^2 + \sum_{j=q_0+1}^{n} \|\bar L_j u_J\|_\phi^2 \Big) + R.
\end{aligned} \tag{2.23}
$$

We denote by S the sum in the last line in (2.23). To conclude our proof, we only need to prove that

$$R \leq \frac{1}{2}\sum_{|J|=k}{}' \Big(\sum_{j=1}^{q_0} \|\delta_j^\phi u_J\|_\phi^2 + \sum_{j=q_0+1}^{n} \|\bar{L}_j u_J\|_\phi^2 \Big) + C\|u\|_\phi^2. \tag{2.24}$$

In fact, if we point our attention at term which involve $\delta_j^\phi u$ for $j \leq q_0$ or $\bar{L}_j u$ for $q_0+1 \leq j \leq n$, then (2.24) is clear since S carries the corresponding square $\|\delta_j^\phi u\|_\phi^2$ and $\|\bar{L}_j u\|_\phi^2$. Otherwise, we note that for $j \leq n-1$ we may interchange $\bar{L}_j$ and δ_j^ϕ by means of integration by parts : boundary integrals do not occur because $L_j(r) = 0$ on $b\Omega$ for $j \leq n-1$. As for δ_n^ϕ, notice that it only hits coefficients whose index contains n and hence $u_{nK} = 0$ on $b\Omega$. So $\delta_n^\phi(u_{nK})\bar{u}$ is also interchangeable with $u_{nK}\bar{L}_n\bar{u}$ by integration by parts. This concludes the proof of Proposition 2.5.

□

For the choice $\phi = 0$, we can rewrite the estimate (2.13) as

$$\begin{aligned} Q(u,u) \gtrsim & \sum_{|K|=k-1}{}' \sum_{i,j=1}^{n-1} \int_{b\Omega} u_{iK}\bar{u}_{jK} dS - \sum_{|J|=q}{}' \sum_{j=1}^{q_o} \int_{b\Omega} |u_J|^2 dS \\ & + \sum_{j=1}^{q_o} \|L_j u\|^2 + \sum_{j=q_o+1}^{n} \|\bar{L}_j u\|^2 \end{aligned} \tag{2.25}$$

for any $u \in C\infty_c(U \cap \bar{\Omega})^k \mathrm{Dom}(\bar{\partial}^*)$.

Observe that if $\varphi \in C_c^\infty(U \cap \bar{\Omega})$ with $\varphi = 0$ on $U \cap b\Omega$, then each $\|L_j\varphi\|^2$ can be interchanged with $\|\bar{L}_j\varphi\|^2 \pm (\epsilon\|\varphi\|_1^2 + C_\epsilon\|\varphi\|^2)$ even for $j = n$ due to the vanishing of the boundary integral because φ vanish on the boundary. Thus

$$\begin{aligned} \sum_{j=1}^{q_o} \|L_j\varphi\|^2 + & \sum_{j=q_o+1}^{n} \|\bar{L}_j\varphi\|^2 + \|\varphi\|^2 \\ & \geq \frac{1}{2} \sum_{j=1}^{n} \big(\|L_j\varphi\|^2 + \|\bar{L}_j\varphi\|^2\big) - \epsilon\|\varphi\|_1^2 + \|\varphi\|^2 \\ & \gtrsim \|\varphi\|_1^2 \end{aligned} \tag{2.26}$$

where $\|.\|_1$ is the Sobolev norm of index 1.

So that we get an estimate which fully expresses the innterior elliptic regularity of the system $(\bar{\partial}, \bar{\partial}^*)$

$$Q(u,u) \gtrsim \|u\|_1^2 \tag{2.27}$$

for any $u \in C_c^\infty(U \cap \bar{\Omega})^k$ with $u|_{U\cap b\Omega} = 0$. Using observation (2.26) for φ replaced by u_{nK} for any K, we get

$$\begin{aligned} Q(u,u) \gtrsim & \sum_{|K|=k-1}{}' \sum_{i,j=1}^{n-1} \int_{b\Omega} r_{ij} u_{iK} \bar{u}_{jK} dS - \sum_{|J|=q}{}' \sum_{j=1}^{q_o} \int_{b\Omega} r_{jj} |u_J|^2 dS \\ & + \sum_{j=1}^{q_o} \|L_j u\|^2 + \sum_{j=q_o+1}^{n} \|\bar{L}_j u\|^2 + \sum_{|K|=k-1}{}' \|u_{nK}\|_1^2 \end{aligned} \tag{2.28}$$

for any $u \in C_c^\infty(U \cap \bar{\Omega})^k \cap \mathrm{Dom}(\bar{\partial}^*)$.

Notice that conversely we have

$$\begin{aligned} Q(u,u) \lesssim & \Big| \sum_{|K|=k-1}{}' \sum_{i,j=1}^{n-1} \int_{b\Omega} r_{ij} u_{iK} \bar{u}_{jK} dS - \sum_{|J|=q}{}' \sum_{j=1}^{q_o} \int_{b\Omega} r_{jj} |u_J|^2 dS \Big| \\ & + \sum_{j=1}^{q_0} \|L_j u\|_\phi^2 + \sum_{j=q_o+1}^{n} \|\bar{L}_j u\|_\phi^2 + \|u\|_\phi^2 \end{aligned} \tag{2.29}$$

for any $u \in C_c^\infty(U \cap \bar{\Omega})^k \cap \mathrm{Dom}(\bar{\partial}^*)$. This inequality is a consequence of the calculation in the Proposition 2.5 and holds without the assumption of pseudoconvexity.

2.3 The tangential operators

In our study of $(f\text{-}\mathcal{M})^k$-estimates, we will use tangential pseudo-differential operators on $U \cap \bar{\Omega}$ with U is a neighborhood of $z_0 \in b\Omega$. These will be expressed on the terms of boundary coordinates which are defined as follows

Definition 2.6. If $z_0 \in b\Omega$ we will call a system of real C^∞ coordinates, defined in a neighborhood U of z_0, *boundary coordinates* if one of coordinates function is defining function r. We will denote such a system by $(x,r) = (x_1,...,x_{2n-1},r) \in \mathbb{R}^{2n-1}\times\mathbb{R}$ and call the x_j's *tangential* coordinates and r the *normal coordinate*.

We denote the dual variables of x by ξ, and define $x\cdot\xi = \sum x_i\xi_j$; $|\xi|^2 = \sum \xi_j^2$. For $\varphi \in C_c^\infty(U\cap\bar\Omega)$ we define $\tilde\varphi$, the *tangential Fourier transforms* of φ, by

$$\tilde\varphi(\xi,r) = \int_{\mathbb{R}^{2n-1}} e^{-ix\cdot\xi}\varphi(t,r)dt.$$

Denote by $\Lambda_\xi = (1+|\xi|^2)^{\frac12}$ the standard "tangential" elliptic symbol of order 1 and by Λ the operator with symbol Λ_ξ. For $f \in C^\infty([1,+\infty))$ we define $f(\Lambda)\varphi$ by

$$f(\Lambda)\varphi(\xi,r) = \int e^{ix\cdot\xi} f(\Lambda_\xi)\tilde\varphi(\xi,r)d\xi. \tag{2.30}$$

Hence

$$\|f(\Lambda)\varphi\|^2 = \int_{-\infty}^0\int_{\mathbb{R}^{2n-1}} f(\Lambda_\xi)^2|\tilde\varphi(\xi,r)|^2 drd\xi. \tag{2.31}$$

In the case $f(t)=t^s$, $s\in\mathbb{R}$, we define tangential Sobolev norms by

$$|||\varphi|||_s = \|\Lambda^s\varphi\| \tag{2.32}$$

Lemma 2.7. *Let $f,g \in C^\infty([1,+\infty))$ satisfy $\lim\limits_{t\to+\infty}\frac{f(t)}{g(t)} = +\infty$. Then for any $\epsilon>0$ and $s\in\mathbb{R}^+$ there exists the constant $C_{\epsilon,s}$ such that*

$$\|g(\Lambda)\varphi\|^2 \le \epsilon\|f(\Lambda)\varphi\|^2 + C_{\epsilon,s}|||u|||_{-s}^2$$

for any $\varphi\in C_c^\infty(U\cap\bar\Omega)$.

Proof. Since $\lim\limits_{t\to+\infty}\frac{f(t)}{g(t)} = +\infty$, then for any $\epsilon>0$, there exists the constant $t_\epsilon>0$ such that $g(\Lambda_\xi)\le\epsilon f(\Lambda_\xi)$ for $|\xi|\ge t_\epsilon$. Hence

$$\begin{aligned}\|g(\Lambda)\varphi\|^2 =& \int_{-\infty}^0\int_{|\xi|\ge t_\epsilon} g(\Lambda_\xi)^2|\tilde\varphi(\xi,r)|^2drd\xi + \int_{-\infty}^0\int_{|\xi|\le t_\epsilon} g(\Lambda_\xi)^2|\tilde\varphi(\xi,r)|^2drd\xi\\ \le&\epsilon\int_{-\infty}^0\int_{|\xi|\ge t_\epsilon} f(\Lambda_\xi)^2|\tilde\varphi(\xi,r)|^2drd\xi + \int_{-\infty}^0\int_{|\xi|\le t_\epsilon} g(\Lambda_\xi)^2|\tilde\varphi(\xi,r)|^2drd\xi\\ \le&\epsilon\|f(\Lambda)\varphi\|^2 + C_{\epsilon,s}|||u|||_{-s}^2.\end{aligned}$$

□

In $\mathbb{R}^{2n}_- = \{(x_1, ..., x_{2n})|x_{2n} = r \leq 0\}$, the Schwarz function are smooth functions which decrease rapidly at infinity:

$$\mathcal{S} = \{f \in C^\infty(\mathbb{R}^{2n}_-)|\forall\alpha, \beta \sup_{x\in\mathbb{R}^{2n}_-} |x^\alpha D^\beta f(x)| < \infty\}$$

where α and β are $2n$-indeces and $D^\beta = \frac{\partial^{\beta_1}}{\partial x^{\beta_1}} \cdots \frac{\partial^{\beta_{2n}}}{\partial x^{\beta_{2n}}}$. Recall that a class of functions $\mathcal{F}$ is defined by

$$\mathcal{F} = \{f \in C^\infty([1, +\infty)\Big| f(t) \lesssim t^{\frac{1}{2}}; f'(t) \geq 0 \text{ and } |f^{(m)}(t)| \lesssim |\frac{f^{m-1}(t)}{t}|, \forall m \in \mathbb{Z}^+\}.$$

Proposition 2.8. *Let $f \in \mathcal{F}$; $a \in \mathcal{S}(\mathbb{R}^{2n}_-)$; $s \in \mathbb{R}$; and M be the a vector field with coefficients in $\mathcal{S}(\mathbb{R}^{2n}_-)$. Then we have*

(i) $|||[f(\Lambda), a]\varphi|||_s \lesssim |||f(\Lambda)\varphi|||_{s-1}$;

(ii) $|||[f(\Lambda), M]\varphi|||_s \lesssim |||f(\Lambda)\varphi|||_s + |||D_r f(\Lambda)\varphi|||_{s-1}$;

for any $\varphi \in C^\infty_c(U \cap \bar{\Omega})$.

Proof. The proof of (i) of Proposition follows by Kohn-Nirenberg formula:

$$\sigma([A, B]) = \sum_{k>0} \frac{(\partial/\partial\xi)^k\sigma(A)D^k_x\sigma(B) - (\partial/\partial\xi)^k\sigma(B)D^k_x\sigma(A)}{k!} \tag{2.33}$$

(notice here that the $k = 0$ term cancels out). Using formula (2.33) for $A = f(\Lambda)$ where $f \in \mathcal{F}$ and $B = a \in \mathcal{S}(\mathbb{R}^{2n}_-)$, we obtain

$$|\sigma([f(\Lambda), a])| \lesssim \sigma(\Lambda^{-1}f(\Lambda)).$$

The second part of this proposition immediately follows by the first part with notice that $M = \sum_k a_k(x, r)\frac{\partial}{\partial x_k} + b(x, r)\frac{\partial}{\partial r}$.

□

Proposition 2.9. *For $f \in \mathcal{F}$, the estimate*

$$\begin{aligned} \|\Lambda^{-1}f(\Lambda)\varphi\|^2_1 \lesssim &\sum_{j=1}^{q_0} \|L_j\Lambda^{-1}f(\Lambda)\varphi\|^2 + \sum_{j=q_0+1}^{n} \|\bar{L}_j\Lambda^{-1}f(\Lambda)\varphi\|^2 \\ &+ \|\Lambda^{-1}f(\Lambda)\varphi\|^2 + \|\Lambda^{-1/2}f(\Lambda)\varphi_b\|^2_b \end{aligned} \tag{2.34}$$

holds for all $\varphi \in C^\infty_c(U \cap \bar{\Omega})$.

The above proposition is a variant of Theorem (2.4.5) of [FK72]. The key to proving this theorem is a passage between functions in $C^\infty(U\cap\bar\Omega)$ to $C^\infty(U\cap b\Omega)$. This is done by using an extension of ψ on $U\cap b\Omega$ to $U\cap\bar\Omega$. Suppose that $\psi\in C^\infty_c(U\cap b\Omega)$, we define $\psi^{(e)}\in C^\infty(\{(x,r)\in\mathbb{R}^{2n}\big|r\le 0\})$ by

$$\psi^{(e)}(x,r)=\int e^{ix\cdot\xi}e^{r(1+|\xi|^2)^{1/2}}\tilde\psi(\xi)d\xi$$

so that $\psi^{(e)}(x,0)=\psi(x)$.

Lemma 2.10. *For each $k\in\mathbb{Z},k\ge 0$; $s\in\mathbb{R}$; and $f\in\mathcal{F}$, then we have*

(i) $|||r^kf(\Lambda)\psi^{(e)}|||_s\cong C_k\|f(\Lambda)\psi\|_{b,s-k-\frac12}$

(ii) $|||D_rf(\Lambda)\psi^{(e)}|||_s\cong\|f(\Lambda)\psi\|_{b,s+\frac12}$

for any $\psi\in C^\infty_c(U\cap b\Omega)$.

Proof. (i): We observe that for every positive integer k, we have from integration by parts:

$$\int_{-\infty}^0 r^{2k}e^{r(1+|\xi|^2)^{1/2}}dr=C'_k(1+|\xi|^2)^{-k-\frac12}. \tag{2.35}$$

Hence,

$$\begin{aligned}|||r^kf(\Lambda)\psi^{(e)}|||^2_s\cong&\int_{-\infty}^0\int_{\mathbb{R}^{2n-1}}r^{2k}e^{2r(1+|\xi|^2)^{1/2}}(1+|\xi|^2)^sf((1+|\xi|^2)^{1/2})^2|\psi(\xi)|^2d\xi dr\\ \cong&C'_k\int_{\mathbb{R}^{2n-1}}(1+|\xi|^2)^{s-k-\frac12}f((1+|\xi|^2)^{1/2})^2|\psi(\xi)|^2d\xi\\ \cong&C'_k|||f(\Lambda)\psi|||^2_{b,s-k-\frac12}.\end{aligned} \tag{2.36}$$

(ii). Since the derivative D_r does not affect the variables in which we take the Fourier transform, the two operations commute. Hence

$$\begin{aligned}|||D_rf(\Lambda)\psi^{(e)}|||^2_s&=\int_{\mathbb{R}^{2n-1}}\int_{-\infty}^0(1+|\xi|^2)^sf((1+|\xi|^2)^{1/2})^2(1+|\xi|^2)e^{2r(1+|\xi|^2)^{1/2}}|\tilde\psi(\xi,0)|^2drd\xi\\ &=\frac12\int_{\mathbb{R}^{2n-1}}f((1+|\xi|^2)^{1/2})^2(1+|\xi|^2)^{s+1/2}|\tilde\psi(\xi,0)|^2d\xi\\ &=\frac12\|f(\Lambda)\psi\|^2_{b,s+\frac12}.\end{aligned}$$

Then lemma is proven.

□

Proof of Proposition 2.9. We prove the lemma for elliptic system $\{L_j\}_{j\le q_0}\cup\{\bar L_j\}_{q_0+1\le j\le n}\cup$ replaced by $\{S_j\}_{1\le j\le n}$ where $S_j\in\mathbb{C}TU$ satisfies there exist no $0\neq\eta\in T^*U$ such that $\langle S_j,\eta\rangle_x=0$ for all $j=1,...,n$.

Assume for the moment that $\varphi(x,0)=0$, (i.e, $\varphi|_{b\Omega}=0$), which implies that $\Lambda^{-1}g(\Lambda)\varphi(x,0)=0$ since the boundary condition is invariant under the action tangential operators. Using the observation (2.26) to S_j's, we get

$$\begin{aligned}\|\Lambda^{-1}f(\Lambda)\varphi\|_1^2 &\lesssim \sum_{j=1}^n\left(\|S_j\Lambda^{-1}f(\Lambda)\varphi\|^2+\|\bar S_j\Lambda^{-1}f(\Lambda)\varphi\|^2\right)+\|\Lambda^{-1}f(\Lambda)\varphi\|^2\\ &\lesssim \sum_{j=1}^n\|S_j\Lambda^{-1}f(\Lambda)\varphi\|^2+\|\Lambda^{-1}f(\Lambda)\varphi\|^2.\end{aligned}\tag{2.37}$$

Next suppose that φ may or may not vanish at the boundary. Let φ_b be the restriction of φ to boundary, that is $\varphi_b(x)=\varphi(x,0)$. We set $\varphi^{(0)}=\varphi-\varphi_b^{(e)}$. Then $\varphi^{(0)}$ vanishes on the boundary so that the previous result applies to $\varphi^{(0)}$. We then have

$$\|f(\Lambda)\varphi^{(0)}\|^2\lesssim\sum_{j=1}^n\|S_j\Lambda^{-1}f(\Lambda)\varphi^{(0)}\|^2+\|\Lambda^{-1}f(\Lambda)\varphi^{(0)}\|^2.\tag{2.38}$$

Therefore

$$\begin{aligned}\|\Lambda^{-1}f(\Lambda)\varphi\|_1^2 \lesssim&\|f(\Lambda)\varphi^{(0)}\|^2+\|f(\Lambda)\varphi_b^{(e)}\|^2\\ \lesssim&\sum_{j=1}^n\|S_j\Lambda^{-1}f(\Lambda)\varphi^{(0)}\|^2+\|\Lambda^{-1}f(\Lambda)\varphi^{(0)}\|^2+\|f(\Lambda)\varphi_b^{(e)}\|^2\\ \lesssim&\sum_{j=1}^n\left[\|S_j\Lambda^{-1}f(\Lambda)\varphi\|^2+\|S_j\Lambda^{-1}f(\Lambda)\varphi_b^{(e)}\|^2\right]\\ &+\|\Lambda^{-1}f(\Lambda)\varphi\|^2+\|\Lambda^{-1}f(\Lambda)\varphi_b^{(e)}\|^2+\|f(\Lambda)\varphi_b^{(e)}\|^2\\ \lesssim&\sum_{j=1}^n\|S_j\Lambda^{-1}f(\Lambda)\varphi\|^2+\|\Lambda^{-1}f(\Lambda)\varphi\|^2\\ &+\|\Lambda^{-1}f(\Lambda)\varphi_b^{(e)}\|^2+\|f(\Lambda)\varphi_b^{(e)}\|^2+\|D_r\Lambda^{-1}f(\Lambda)\varphi_b^{(e)}\|^2\end{aligned}\tag{2.39}$$

since the S_j's are linear combinations of $\frac{\partial}{\partial x_j}$'s and D_r . Using Lemma 2.10, the last line of (2.39) is estimated by $\|\Lambda^{-\frac{1}{2}}g(\Lambda)\varphi_b\|_b^2$. This concludes the proof.

□

Chapter 3

The $(f\text{-}\mathcal{M})^k$-estimates : a sufficient condition

In this chapter, we give the proof of Theorem 1.10.

3.1 Reduction to the boundary

Since the Property $(f\text{-}\mathcal{M}\text{-}P)^k$ act only on u^τ, the tangential component of u, so that we firstly show $(f\text{-}\mathcal{M})$-estimate for u^τ, that is

$$\|f(\Lambda)\mathcal{M}u^\tau\|^2 \lesssim Q(u^\tau, u^\tau) + C_\mathcal{M}|||u^\tau|||^2_{-1}. \tag{3.1}$$

One of our steps we reduce the estimate (3.1) to the boundary by following theorem.

Theorem 3.1. *Let Ω be the q-pseudoconvex (resp. q-pseudoconcave) at boundary point z_o. Then there is a neighborhood U of z_0 such that*

$$\|f(\Lambda)\mathcal{M}u^\tau\|^2 \lesssim Q(u^\tau, u^\tau) + C_\mathcal{M}|||u^\tau|||^2_{-1} + ||\Lambda^{-1/2}f(\Lambda)\mathcal{M}u^\tau_b||^2_b, \tag{3.2}$$

holds for any $u \in C^\infty_c(U \cap \bar{\Omega})^k \cap Dom(\bar{\partial}^*)$ for any $k \geq q$ (resp. $k \leq q$).

Before the proof of Theorem 3.1, we need following lemma :

Lemma 3.2. *Let Ω be the q-pseudoconvex (resp. q-pseudoconcave) at boundary point z_o. Then there is a neighborhood U of z_0 such that*

$$\sum_{|J|=k}{}' \Big(\sum_{j=1}^{q_o} \|L_j u_J\|^2 + \sum_{j=q_o+1}^{n} \|\bar{L}_j u_J\|^2 + \|D_r\Lambda^{-1}u_J\|^2 + \|u_J\|^2 \Big) \lesssim Q(u,u) \tag{3.3}$$

holds for any $u \in C_c^\infty(U\cap\bar\Omega)^k \cap Dom(\bar\partial^*)$ *for any* $k \geq q$ *(resp.* $k \leq q$ *).*

Proof. We only need to show

$$\sum_{|J|=k}{}' \|D_r\Lambda^{-1}u_J\|^2 \lesssim Q(u,u).$$

Since $D_r = a\bar L_n + bT$ where $a, b \in C^\infty(\bar\Omega)$ and T is tangential operator order 1, thus

$$\|D_r\Lambda^{-1}u_J\|^2 \lesssim \|\bar L_n u_J\|^2 + \|T\Lambda^{-1}u\|^2 \lesssim \|\bar L_n u_J\|^2 + \|u\|^2 \lesssim Q(u,u).$$

□

Proof of Theorem 3.1. Apply Proposition 2.9 for $\varphi = Mu^\tau \in C_c^\infty(U\cap\bar\Omega)$, it gives

$$\begin{aligned}\|f(\Lambda)\mathcal{M}u^\tau\|^2 \lesssim& \sum_{j=1}^{q_o}\|L_j\Lambda^{-1}f(\Lambda)\mathcal{M}u^\tau\|^2 + \sum_{j=q_o+1}^{n}\|\bar L_j\Lambda^{-1}f(\Lambda)\mathcal{M}u^\tau\|^2 \\ &+ \|\Lambda^{-1}f(\Lambda)\mathcal{M}u^\tau\|^2 + \|\Lambda^{-1/2}f(\Lambda)\mathcal{M}u_b^\tau\|_b^2.\end{aligned} \tag{3.4}$$

For any $S \in \{L_j\}_{j\leq q_0} \cup \{L_j\}_{q_0+1\leq j\leq n} \cap \{id\}$, we have :

Case 1. If $\mathcal{M} = M\rho$, where $\rho \in \mathcal{A}^{0,0}$ and M is either every positive number or 1, remember that

$$\mathcal{M}u^\tau := M\sqrt{\sum_{|J|=k}{}'|\rho|^2|u_J^\tau|^2};$$

then

$$\begin{aligned}\|S\Lambda^{-1}f(\Lambda)\mathcal{M}u^\tau\|^2 =& M^2\sum_{|J|=k}{}'\|S\Lambda^{-1}f(\Lambda)\rho u_J^\tau\|^2 \\ \lesssim& M^2\sum_{|J|=k}{}'\Big(\|\rho\Lambda^{-1}f(\Lambda)Su_J^\tau\|^2 \\ &+ \|\rho[S,\Lambda^{-1}f(\Lambda)]u_J\|^2 + \|[S\Lambda^{-1}f(\Lambda),\rho]u_J^\tau\|^2\Big) \\ \lesssim& M^2\sum_{|J|=k}{}'\Big(\|\Lambda^{-1}f(\Lambda)Su_J^\tau\|^2 + \|\Lambda^{-1}f(\Lambda)u_J^\tau\|^2 + \|D_r\Lambda^{-2}f(\Lambda)u_J^\tau\|^2\Big) \\ \lesssim& \sum_{|J|=k}{}'\Big(\|Su_J^\tau\|^2 + \|u_J^\tau\|^2 + \|D_r\Lambda^{-1}u_J^\tau\| + C_M|||u_J^\tau|||_{-1}^2\Big) \\ \lesssim& Q(u^\tau,u^\tau) + C_M|||u_J^\tau|||_{-1}^2\end{aligned} \tag{3.5}$$

here the second inequality follows by Proposition 2.8; the fourth inequality follows by Lemma 2.7 and the last follows by Lemma 3.2.

Case 2. If $\mathcal{M} = M\theta$, where $\theta \in \mathcal{A}^{1,0}$ and M is either every positive number or 1, remember that

$$\mathcal{M}u^\tau := M\sqrt{\Big|\sum_{|K|=k-1}{}^{'}|\sum_{j=1}^{n}\theta_j u^\tau_{jK}|^2 - \sum_{|J|=k}{}^{'}\sum_{j=1}^{q_o}|\theta_j|^2|u^\tau_J|^2\Big|} \tag{3.6}$$

In the same way in case 1, we get

$$\|S\Lambda^{-1}f(\Lambda)\mathcal{M}u^\tau\|^2 \lesssim Q(u^\tau,u^\tau) + C_M|||u^\tau_J|||^2_{-1}$$

This completes the proof of Theorem 3.1.

□

Remark 3.3. We notice again that the constant $C_{\mathcal{M}} := C_M$ if M is every positve number and $C_{\mathcal{M}} := 0$ if $M = 1$.

3.2 Estimate on strip

In this section, we show the Property $(f - \mathcal{M} - P)^k$ implies the estimate on the δ-strip near the boundary for each $\delta > 0$.

Recall that $S_\delta := \{z \in \mathbb{C}^n : -\delta < r < 0\}$.

Theorem 3.4. *Let Ω be q-pseudoconvex (resp. q-pseudoconcave) at the boundary point z_0. Assume that property $(f\text{-}\mathcal{M}\text{-}P)^k$ holds at z_0 with $k \geq q$ (resp. $k \leq q$). Then there is a neighborhood U of z_0 such that for any $\delta > 0$*

$$f(\delta^{-1})^2\int_{S_{\delta/2}}|\mathcal{M}u^\tau|^2dV \lesssim Q(u^\tau,u^\tau) \tag{3.7}$$

holds for any $u \in C^\infty_c(U\cap\bar\Omega)^k \cap Dom(\bar\partial^*)$.

To simplify, define the quadratic form $H^k_{q_o}(\phi,u)$ by

$$H^k_{q_o}(\phi,u) = \sum_{|K|=k-1}{}^{'}\sum_{ij=1}^{n}\phi_{ij}u_{iK}\bar u_{jK} - \sum_{j=1}^{q_o}\phi_{jj}|u|^2$$

Proof of Theorem 3.4. The proof is divided into two steps. In step 1, we shall modify Φ^δ to ϕ^δ which has property not only on strip. In step 2, we shall prove the estimate (3.7).

Step 1. By our assumption, for any $\delta > 0$ sufficiently small, there is a function $\Phi^{\delta,\mathcal{M}}$ such that

$$\begin{cases} H^k_{q_o}(\Phi^{\delta,\mathcal{M}}, u^\tau) \geq c\Big(f(\delta^{-1})^2|\mathcal{M}u^\tau|^2 + \sum_{j=1}^{q_o}|\Phi^{\delta,\mathcal{M}})u^\tau|^2\Big) \\ |\Phi^{\delta,\mathcal{M}}| \leq 1 \end{cases} \quad \text{on } U\cap S_\delta \tag{3.8}$$

where $\Phi^{\delta,\mathcal{M}}_j = L_j(\Phi^{\delta,\mathcal{M}})$.

The support of u^τ on U but the weighted function has the properties only on the strip S_δ. So that we have to modify $\Phi^{\delta,\mathcal{M}}$ to $\phi^{\delta,\mathcal{M}}$ to get the properties on the whole U.

We define

$$\phi^{\delta,\mathcal{M}} := \Phi^{\delta,\mathcal{M}}\chi(-\frac{r}{\delta}), \tag{3.9}$$

where χ is the cut off function which satisfies $\chi(t) = \begin{cases} 1 & \text{for } t \leq \frac{1}{2} \\ 0 & \text{for } t \geq 1 \end{cases}$ and $\dot\chi \leq 0$.

Computation of $\partial\bar\partial\phi^{\delta,\mathcal{M}}$ show that

$$\partial\bar\partial\phi^{\delta,\mathcal{M}} = \chi\partial\bar\partial\Phi^{\delta,\mathcal{M}} - \frac{\dot\chi\Phi^{\delta,\mathcal{M}}}{\delta}\partial\bar\partial r + 2\ \mathrm{Re}\ \frac{\dot\chi}{\delta}\partial\Phi^{\delta,\mathcal{M}}\otimes\bar\partial r + \frac{\Phi^{\delta,\mathcal{M}}\ddot\chi}{\delta^2}\partial r\otimes\bar\partial r \tag{3.10}$$

and notice that for all $j \leq q_0$ with $q_0 \leq n-1$

$$\phi^{\delta,\mathcal{M}}_{jj} = \chi\Phi^{\delta,\mathcal{M}}_{jj} - \frac{\dot\chi\Phi^{\delta,\mathcal{M}}}{\delta}r_{jj}.$$

We remark that $u^\tau_{nK} = 0$ for any K on U. So that

$$H^k_{q_o}(\phi^{\delta,\mathcal{M}}, u^\tau) = \chi H^k_{q_o}(\Phi^{\delta,\mathcal{M}}, u^\tau) - \frac{\dot\chi\Phi^{\delta,\mathcal{M}}}{\delta}H^k_{q_o}(r, u^\tau). \tag{3.11}$$

We note that we can write $r = 2Rez_n + h(z_1, ..., z_{n-1}, y_n)$ is a graphing local defining function and denote by $z \to z^*$ the projection $\mathbb{C}^n \to b\Omega$

in a neigborhood of z_o along the x_n-axis. We have the evident equality $(r_{ij}(z))_{ij}^{n-1} = (r_{ij}(z^*))_{ij=1}^{n-1}$. Thus the second term in right hand side of (3.11) can be discard since Ω is q-pseudoconvex.

Combining with (3.8), we have

$$\begin{aligned} H^k_{q_o}(\phi^{\delta,\mathcal{M}}, u^\tau) &\geq \chi H^k_{q_o}(\Phi^{\delta,\mathcal{M}}, u^\tau) \\ &\geq c\chi\Big(f(\delta^{-1})^2|\mathcal{M}u^\tau|^2 + \sum_{j=1}^{q_o}|\Phi_j^{\delta,\mathcal{M}}u^\tau|^2\Big) \\ &\geq c\Big(\chi f(\delta^{-1})^2|\mathcal{M}u^\tau|^2 + \sum_{j=1}^{q_o}|\phi_j^{\delta,\mathcal{M}}u^\tau|^2\Big) \end{aligned} \tag{3.12}$$

hold for $z \in U \cap \Omega$. Here last inequality follow form $(\phi^{\delta,\mathcal{M}})_j = \chi(\Phi^{\delta,\mathcal{M}})_j$ for $j \leq q_o$ and $\chi \geq \chi^2$.

Step 2. We apply Proposition 2.5 for $\phi = \psi(\phi^\delta)$ and $u = u^\tau$. First we remark that

$$\begin{aligned} H^k_{q_0}(\psi(\phi^{\delta,\mathcal{M}}), u^\tau) =&\dot{\psi} H^k_{q_0}(\phi^{\delta,\mathcal{M}}, u^\tau) \\ &+ \ddot{\psi}\Big(\sum_{|K|=k-1}{}' |\sum_{j=1}^{n-1}\phi_j^{\delta,\mathcal{M}}u^\tau_{jK}|^2 - \sum_{j=1}^{q_0}|\phi_j^{\delta,\mathcal{M}}|^2|u^\tau|^2\Big). \end{aligned} \tag{3.13}$$

We also have

$$\begin{aligned} \|\bar{\partial}^*_{\psi(\phi^{\delta,\mathcal{M}})}u^\tau\|^2_{\psi(\phi^{\delta,\mathcal{M}})} \leq& 2\|\bar{\partial}^* u^\tau\|^2_{\psi(\phi^{\delta,\mathcal{M}})} + 2\|\sum_{|K|=k-1}{}' \sum_{j=1}^{n-1}\psi(\phi^{\delta,\mathcal{M}})_j u^\tau_{jK}\bar{\omega}_K\|^2_{\psi(\phi^{\delta,\mathcal{M}})} \\ =& 2\|\bar{\partial}^* u^\tau\|^2_{\psi(\phi^{\delta,\mathcal{M}})} + 2\sum_{|K|=k-1}{}' \|\sum_{j=1}^{n-1}\dot{\psi}\phi_j^{\delta,\mathcal{M}}u^\tau_{jK}\|^2_{\psi(\phi^{\delta,\mathcal{M}})}. \end{aligned} \tag{3.14}$$

Thus we get from (2.13), under the choice of the weight $\psi(\phi^\delta)$ and k-form

u^τ, and taking into account (3.13) and (3.14):

$$\begin{aligned} 2\|\bar\partial u^\tau\|^2_{\psi(\phi^{\delta,\mathcal{M}})} &+ 4\|\bar\partial^* u^\tau\|^2_{\psi(\phi^{\delta,\mathcal{M}})} + C\|u^\tau\|^2_{\psi(\phi^{\delta,\mathcal{M}})} \\ &\geq \int_\Omega \dot\psi e^{-\psi(\phi^{\delta,\mathcal{M}})} H^k_{q_o}(\phi^{\delta,\mathcal{M}}, u^\tau) dV \\ &+ \int_\Omega (\ddot\psi - 4\dot\psi^2) e^{-\psi(\phi^{\delta,\mathcal{M}})} \sum_{|K|=k-1}{}' |\sum_{j=1}^{n-1} \phi_j^{\delta,\mathcal{M}} u^\tau_{jK}|^2 dV \\ &- \int_\Omega \ddot\psi e^{-\psi(\phi^{\delta,\mathcal{M}})} \sum_{j=1}^{q_o} |\phi_j^{\delta,\mathcal{M}}|^2 |u^\tau|^2 dV. \end{aligned} \tag{3.15}$$

We now specify our choice of ψ. First, we want $\ddot\psi \geq 4\dot\psi^2$ so that the term in the third line of (3.15) can be disregarded. Keeping this condition, we need an opposite estimate which assures that the absolute value of the second negative term in the last line of (3.15) is controlled by one half of the term in the second line. In fact, from (3.12), we have

$$\begin{aligned} \frac{1}{2}\int_\Omega \dot\psi e^{-\psi(\phi^{\delta,\mathcal{M}})} H_{q_o}(\phi^{\delta,\mathcal{M}}, u^\tau) dV - \int_\Omega \ddot\psi e^{-\psi(\phi^{\delta,\mathcal{M}})} \sum_{j=1}^{q_o} |\phi_j^{\delta,\mathcal{M}}|^2 |u^\tau|^2 dV \\ \geq \int_\Omega e^{-\psi(\phi^{\delta,\mathcal{M}})} \Big(\frac{1}{2} c\dot\psi - \ddot\psi\Big) \sum_{j=1}^{q_o} |\phi_j^{\delta,\mathcal{M}}|^2 |u^\tau|^2 dV. \end{aligned} \tag{3.16}$$

The above term is nonnegative as soon as $\ddot\psi \leq \frac{c}{2}\dot\psi$. If we then set $\psi := \frac{1}{2}e^{\frac{c}{2}(t-1)}$ then both requests are satisfied. Thus the inequality of (3.15) continues as

$$\begin{aligned} &\geq \frac{1}{2}\int_\Omega \dot\psi e^{-\psi(\phi^\delta)} H^k_{q_o}(\phi^\delta, u^\tau) dV \\ &\geq \int_\Omega \frac{1}{2}\dot\psi e^{-\psi(\phi^\delta)} c f(\delta^{-1})^2 \chi(-\frac{r}{\delta}) |\mathcal{M}u^\tau|^2 dV \\ &\geq \frac{c}{2} f(\delta^{-1})^2 \int_{S_{\delta/2}} \dot\psi e^{-\psi(\phi^\delta)} |\mathcal{M}u^\tau|^2 dV. \end{aligned} \tag{3.17}$$

Here the first inequality come from (3.12) and the last equality follows the fact $\chi(\frac{-r}{\delta}) = 1$ on $S_{\delta/2}$.

Now we want to remove the weight from the resulting inequality. The weight in the first line of (3.15) can be handled owing to $e^{-\psi(\phi^{\delta,\mathcal{M}})} \leq 1$ on

$\bar{\Omega} \cap U$. Furthermore, since $|\phi^{\delta,\mathcal{M}}| < 1$ on U then the term $\dot{\psi} e^{-\psi(\phi^{\delta,\mathcal{M}})}$ in the last line of (3.17) can be greater than a positive constant. We end up with the unweighted estimate

$$\|\bar{\partial} u^\tau\|^2 + \|\bar{\partial}^* u^\tau\|^2 + \|u^\tau\|^2 \gtrsim f(\delta^{-1})^2 \int_{S_{\delta/2}} |\mathcal{M} u^\tau|^2 dV. \tag{3.18}$$

This concludes the proof of the theorem.

□

3.3 The proof of Theorem 1.10

In this section, we give the proof of Theorem 1.10. We firstly show $(f\text{-}\mathcal{M})^k$ estimate for u^τ.

Theorem 3.5. *Let Ω be q-pseudoconvex (resp. q-pseudoconcave) at the boundary point z_0. Assume that Property $(f\text{-}\mathcal{M}\text{-}P)^k$ holds at z_0. Then, for a suitable neighborhood U of z_0 , we have*

$$\|f(\Lambda)\mathcal{M} u^\tau\|^2 \lesssim Q(u^\tau, u^\tau) + C_{\mathcal{M}} |||u^\tau|||^2_{-1} \tag{3.19}$$

for any $u \in C_c^\infty(U \cap \bar{\Omega})^k \cap Dom(\bar{\partial}^*)$ with $k \geq q$ (resp. $k \leq q$).

For the proof of Theorem 3.5, we use a method derived from [Cat87].

Let $\{p_k\}$ with $k = 0, 1, \cdot$ be a sequence of cutoff functions with properties

1. $\sum_{k=0}^\infty p_k^2(t) \cong 1$; for any $t \geq 0$
2. $p_k(t) \equiv 0$ if $t \notin (2^{k-1}, 2^{k+1})$ with $k \geq 1$ and $p_0(t) \equiv 0, t \geq 2$.

We can also choose p_k so that $p_k'(t) \lesssim 2^{-k}$

Let P_k denote the operator defined by

$$(\widetilde{P_k \varphi})(\xi, r) = p_k(|\xi|) \varphi(\xi, r)$$

for any $\varphi \in C_0^\infty(U \cap \bar{\Omega})$. To show the inequality (3.19) ,we need following lemma

Lemma 3.6. *Let $z_0 \in b\Omega$; $f \in \mathcal{F}$; $a \in S(\mathbb{R}^{2n}_-)$ and S is the operator of order 1. Then there is a neighborhood U of z_0 such that*

(i) $\|f(\Lambda)\varphi\|^2 \cong \sum_{k=0}^{\infty} f(2^k)^2\|P_k\varphi\|^2$;

(ii) $\sum_{k=0}^{\infty} f(2^k)^2||[P_k, a]\varphi||^2 \lesssim ||\Lambda^{-1}f(\Lambda)\varphi||^2$;

(iii) $\sum_{k=0}^{\infty} \|[P_k, S]\varphi\|^2 \lesssim \|D_r\Lambda^{-1}\varphi\|^2 + \|\varphi\|$;

holds for any $\varphi \in C_c^\infty(U \cap \bar{\Omega})$.

Proof. (i): We have

$$\begin{aligned}\|f(\Lambda)\varphi\|^2 &= \int_{-\infty}^0 \int_{\mathbb{R}^{2n-1}} f(\Lambda_\xi)^2|\tilde{\varphi}(\xi, r)|^2 d\xi dr\\ &= \int_{-\infty}^0 \int_{\mathbb{R}^{2n-1}} f(\Lambda_\xi)^2\Big(\sum_{k=0}^{\infty} p_k^2(|\xi|)\Big)|\tilde{\varphi}(\xi, r)|^2 d\xi dr\end{aligned}$$

since $\sum_{k=0}^{\infty} p_k^2 = 1$. We notice that $\Lambda_\xi = (1 + |\xi|^2)^{1/2} \cong 2^k$ as long as $|\xi|$ is in the support of p_k. Thus, it follow

$$\begin{aligned}\|f(\Lambda)u\|^2 &\cong \sum_{k=0}^{\infty} \int_{-\infty}^0 \int_{\mathbb{R}^{2n-1}} f(2^k)^2|p_k(\tau)\tilde{\varphi}(\xi, r)|^2 d\xi dr\\ &= \sum_{k=0}^{\infty} f(2^k)^2\|P_k u\|^2.\end{aligned}$$

(ii): We can choose another sequence of cutoff functions $\{q_k\}_{k=0}^{\infty}$ such that

1. $q_k \equiv 1$ on $\mathrm{supp}(p_k)$;
2. $q_k(t) \equiv 0$ if $t \notin (2^{k-2}, 2^{k+3})$ with $k \geq 1$ and $q_0(t) \equiv 0, t \geq 4$.

Then

$$|p_k(x) - p_k(y)| \lesssim 2^{-k}q_k(y)|x - y| \tag{3.20}$$

for any $x, y \geq 0$ and $\sum_{k=0}^{\infty} q_k^2(t) \cong 1$. Observe that

$$\mathcal{F}([P_k, a]\varphi)(\xi, r) = \int_{\mathbb{R}^{2n-1}} \big[p_k(|\xi|) - p_k(|\tau|)\big]\tilde{a}(\xi - \tau, r)\tilde{\varphi}(\tau, r)d\tau. \tag{3.21}$$

Using (3.20), and Plancherel Theorem and Young's Inequality, it follows that

$$\|[P_k, a]\varphi\|^2 \lesssim 2^{-2k}\|Q_k u\|^2 \tag{3.22}$$

where Q_k defined by

$$\mathcal{F}_t(P_k\varphi)(\xi, r) = q_k(|\xi|)u(\xi, r).$$

Multiply (3.22) by $f(2^k)^2$; and take sum over $k = 0, 1, ...,$ using result of (i) for $f(|\xi|)$ replaced by $|\xi|^{-1}f(|\xi|)$ and P_k by Q_k, we get conclusion of (ii).

(iii): The proof of (iii) follows immediately by (ii) with observing that $S = \sum a_j \frac{\partial}{\partial x_j} + b\frac{\partial}{\partial r}$. □

Proof of Theorem 3.5. By Theorem 3.1, we only need to estimate $\|\Lambda^{-1/2}f(\Lambda)(\mathcal{M}u^\tau)_b\|_b^2$. Let $\chi_k \in C_c^\infty(-2^{-k}, 0]$ with $0 \le \chi_k \le 1$ and $\chi_k(0) = 1$.

We have the elementary inequality

$$|g(0)|^2 \le \frac{2^k}{\eta}\int_{-2^{-k}}^0 |g(r)|^2 dr + 2^{-k}\eta\int_{-2^{-k}}^0 |g'(r)|^2 dr,$$

which holds for any g such that $g(-2^{-k}) = 0$. If we apply it for $g(r) = \|\chi_k(r)P_k\mathcal{M}u^\tau(\cdot, r)\|_b$, we get

$$\begin{aligned}
\|\Lambda^{-1/2}f(\Lambda)(\mathcal{M}u^\tau)_b\|_b^2 &\simeq \sum_{k=0}^\infty f(2^k)^2 2^{-k}\|\chi_k(0)P_k\mathcal{M}u^\tau(\cdot, 0)\|_b^2 \\
&\le \eta^{-1}\underbrace{\sum_{k=0}^\infty f(2^k)^2\int_{-2^{-k}}^0 \|\chi_k P_k\mathcal{M}u^\tau(., r)\|^2 dr}_{I} \\
&\quad +\eta\underbrace{\sum_{k=0}^\infty f(2^k)^2 2^{-2k}\int_{-2^{-k}}^0 \|D_r\Big(\chi_k P_k\mathcal{M}u^\tau(., r)\Big)\|^2 dr}_{II}.
\end{aligned}$$

Observe that $\chi_k \le 1$ and recall Theorem 3.4 that we apply for $P_k\mathcal{M}u^\tau$ and $\delta = 2^{-k}$. Thus the first sums above can be estimated by

$$
\begin{aligned}
(I) &\le \sum_{k=0}^{\infty} f(2^{2k}) \int_{-2^{-k}}^{0} \|P_k \mathcal{M} u^\tau(.,r)\|^2 dr \\
&\lesssim \sum_{k=0}^{\infty} Q(P_k u^\tau, P_k u^\tau) \\
&\lesssim \sum_{k=0}^{\infty} \|P_k \bar\partial u^\tau\|^2 + \|P_k \bar\partial^* u^\tau\|^2 + \|[P_k,\bar\partial] u^\tau\|^2 + \|[P_k,\bar\partial^*] u^\tau\|^2 \\
&\lesssim Q(u^\tau,u^\tau) + \|\Lambda^{-1} D_r u^\tau\|^2
\end{aligned}
\tag{3.23}
$$

where the estimates on the commutator terms follow by Lemma 3.6. we remark that $D_r u^\tau$ can be expressed as a linear combination of $\bar L_n u^\tau$ and $T u^\tau$ for some tangential vector field T. Then

$$
\begin{aligned}
||\Lambda^{-1} D_r(u^\tau)||^2 &\lesssim ||\Lambda^{-1} \bar L_n u^\tau||^2 + ||\Lambda^{-1} T u^\tau||^2 \\
&\lesssim \|\bar L_n u^\tau\|^2 + ||u^\tau||^2 \\
&\lesssim Q(u^\tau, u^\tau).
\end{aligned}
$$

Therefore,

$$(I) \lesssim Q(u^\tau, u^\tau).$$

We now estimate (II). Since $D_r(\chi_k) \le 2^k$, $D_r P_k = P_k D_r$ and $\chi_k \le 1$, we get

$$
\begin{aligned}
(II) \le &\sum_{k=0}^{\infty} f(2^k)^2 2^{-2k} \Big(\int_{-2^{-k}}^{0} \|D_r(\chi_k) P_k \mathcal{M} u^\tau(.,r)\|^2 dr \\
&+ \int_{-2^{-k}}^{0} \|\chi_k D_r(P_k \mathcal{M} u^\tau(.,r))\|^2 dr \Big) \\
\le &\sum_{k=0}^{\infty} f(2^k)^2 \int_{-2^{-k}}^{0} \|P_k \mathcal{M} u^\tau(.,r)\|^2 dr \\
&+ \sum_{k=0}^{\infty} f(2^k)^2 2^{-2k} \int_{-2^{-k}}^{0} \|P_k D_r(\mathcal{M} u^\tau(.,r))\|^2 dr \\
\le &\|f(\Lambda)\mathcal{M} u^\tau\|^2 + \|f(\Lambda)\Lambda^{-1} D_r(\mathcal{M} u^\tau)\|^2.
\end{aligned}
\tag{3.24}
$$

where the last inequality follows by Lemma 3.6. We now estimate the second term in last line in (3.24), as before $D_r = aL_n + bT$, we obtain

$$\begin{aligned}\|f(\Lambda)\Lambda^{-1}D_r(\mathcal{M}u^\tau)\|^2 &\lesssim ||f(\Lambda)\Lambda^{-1}\bar{L}_n\mathcal{M}u^\tau||^2 + ||f(\Lambda)\Lambda^{-1}T\mathcal{M}u^\tau||^2\\ &\lesssim \|\bar{L}_nu^\tau\| + C_{\mathcal{M}}|||u^\tau|||^2_{-1} + ||f(\Lambda)\mathcal{M}u^\tau||^2\\ &\lesssim Q(u^\tau,u^\tau) + C_{\mathcal{M}}|||u^\tau|||^2_{-1} + ||f(\Lambda)Mu^\tau||^2.\end{aligned}$$

Combining all our estimates of $\|f(\Lambda)\Lambda^{-1/2}(\mathcal{M}u^\tau)_b\|^2_b$, we obtain

$$\|f(\Lambda)\Lambda^{-1/2}\mathcal{M}u^\tau_b\|^2_b \lesssim (\eta^{-1}+\eta)\Big(Q(u^\tau,u^\tau)+\|u^\tau\|^2+C_{\mathcal{M}}|||u^\tau|||^2_{-1}\Big)+\eta||f(\Lambda)\mathcal{M}u^\tau||^2.$$

Summarizing up, we have shown that

$$||f(\Lambda)\mathcal{M}u^\tau|| \lesssim \eta^{-1}Q(u^\tau,u^\tau) + C_{\mathcal{M}}\|u^\tau\|^2_{-1} + \eta||f(\Lambda)\mathcal{M}u^\tau||^2.$$

Choosing $\eta > 0$ sufficiently small, we can move the term $\eta||f(\Lambda)\mathcal{M}u^\tau||^2$ into the left-hand-side and get

$$||f(\Lambda)\mathcal{M}u^\tau||^2 \lesssim Q(u^\tau,u^\tau) + C_{\mathcal{M}}|||u^\tau|||^2_{-1}.$$

The proof is complete.

□

The proof of theorem 1.10 is proven by Theorem 3.5 and following lemma:

Lemma 3.7. *Let Ω be a q-pseudoconvex (resp. q-pseudoconcave) at z_0 and U be a neigborhood of z_0. For each $u \in C^\infty(U\cap\bar{\Omega})^k \cap Dom(\bar{\partial}^*)$ with $k \geq q$ (resp. $k \leq q$), assume that $(f\text{-}\mathcal{M})^k$ estimate holds for u^τ. Then $(f\text{-}\mathcal{M})^k$ holds for u.*

Proof. From (2.28), it follows

$$Q(u^\nu,u^\nu) \lesssim \|u^\nu\|^2_1 = \sum_{|K|=k-1}{}' \|u_{nK}\|^2_1 \lesssim Q(u,u).$$

On the other hand,

$$\begin{cases}\|\bar{\partial}u^\tau\| = \|\bar{\partial}(u-u^\nu)\| \leq \|\bar{\partial}u\| + \|\bar{\partial}u^\nu\|\\ \|\bar{\partial}^*u^\tau\| = \|\bar{\partial}^*(u-u^\nu)\| \leq \|\bar{\partial}^*u\| + \|\bar{\partial}^*u^\nu\|.\end{cases}$$

Hence

$$Q(u^\tau,u^\tau) \leq Q(u,u) + Q(u^\nu,u^\nu) \lesssim Q(u,u). \tag{3.25}$$

Notice that, we always have

$$\|f(\Lambda)\mathcal{M}u^\nu\|^2 \lesssim \|u^\nu\|_1^2 + C_\mathcal{M}|||u^\nu|||_{-1}^2 \lesssim Q(u,u) + C_\mathcal{M}|||u^\nu|||_{-1}^2.$$

Therefore,

$$\begin{aligned}\|f(\Lambda)\mathcal{M}u\| \lesssim& \|f(\Lambda)\mathcal{M}u^\tau\| + \|f(\Lambda)\mathcal{M}u^\nu\| \\ \lesssim& Q(u^\tau,u^\tau) + C_\mathcal{M}|||u^\tau|||_{-1}^2 + Q(u,u) + C_\mathcal{M}|||u^\nu|||_{-1}^2 \\ \lesssim& Q(u,u) + C_\mathcal{M}|||u|||_{-1}^2.\end{aligned} \tag{3.26}$$

Finally, we remark that

$$\begin{aligned}C_\mathcal{M}|||u|||_{-1}^2 \lesssim& \tilde{C}_\mathcal{M}\|u\|_{-1}^2 + \|D_r\Lambda^{-1}u\|^2 \\ \lesssim& Q(u,u) + \tilde{C}_\mathcal{M}\|u\|_{-1}^2.\end{aligned} \tag{3.27}$$

This completes the proof of Lemma 3.7.

□

3.4 Some remarks of $(f\text{-}\mathcal{M})^k$

In this section we give some remarks of $(f\text{-}\mathcal{M})^k$ estimate.

Lemma 3.8. *Let $f \gg g$. Assume that $(f\text{-}\mathcal{M})^k$ holds then for any $\epsilon > 0$, $(g\text{-}\frac{1}{\epsilon}M)^k$ also holds.*

The proof of lemma follows by Lemma 2.7.

For example, if $f \gg \log$, then $(f\text{-}1)^k$ estimate implies superlogarithmic estimate. Similarly, if $f \gg 1$, then $(f\text{-}1)^k$ estimate implies compactness estimate.

Lemma 3.9. *Let $\mathcal{M}' = \mathcal{M}$ on $b\Omega$. Assume that $(f\text{-}\mathcal{M})^k$ holds then $(f\text{-}\mathcal{M}')^k$ also holds.*

Using observation (2.28), we get the proof of this lemma.

Lemma 3.10. *If $(f\text{-}\mathcal{M})^k$ estimate holds then*

$$\|D_r\Lambda^{-1}f(\Lambda)(\mathcal{M}u)\|^2 \lesssim Q(u,u) + C_\mathcal{M}|||u|||_{-1}^2$$

for any $u \in C_c^\infty(U \cap \bar{\Omega})^k \cap Dom(\bar{\partial}^*)$.

Proof. Since $\bar{L}_n$ is non-characteristic operator respect to the surface $r = 0$, there is functions a and b such that

$$D_r = a\bar{L}_n + aT$$

where T is tangential operator of order one. Therefore

$$\begin{aligned}\|\Lambda^{-1}f(\Lambda)\frac{\partial}{\partial r}\mathcal{M}u\|^2 \lesssim& \|\Lambda^{-1}f(\Lambda)\bar{L}_n(\mathcal{M}u)\|^2 + \|\Lambda^{-1}f(\Lambda)T\mathcal{M}u\|^2\\ \lesssim& \|\bar{L}_n u\|^2 + \|f(\Lambda)\mathcal{M}u\|^2 + C_{\mathcal{M}}\|u\|^2_{-1}\\ \lesssim& Q(u,u) + C_{\mathcal{M}}\|u\|^2_{-1}.\end{aligned} \tag{3.28}$$

This is completely the proof of lemma.

□

It is interesting to remark that when Ω is pseudoconvex (resp. pseudoconcave) then if $(f\text{-}\mathcal{M})^k$ holds then $(f\text{-}\mathcal{M})^{k+1}$ (resp. $(f\text{-}\mathcal{M})^{k-1}$) also holds.

Lemma 3.11. *Let Ω be a pseudoconvex (resp. pseudoconcave) at the boundary point z_0. We assume that $(f\text{-}\mathcal{M})^k$ holds. Then $(f\text{-}\mathcal{M})^{k+1}$ holds (resp. $(f\text{-}\mathcal{M})^{k-1}$)*

Proof. **The pseudoconvexity case.** Let

$$u = \sum_{|L|=k+1}{}' u_L\bar{\omega}_L \in C_c^\infty(U\cap\bar{\Omega})^{k+1}\cap \mathrm{Dom}(\bar{\partial}^*).$$

We rewrite

$$u = \frac{1}{(k+1)!}\sum_{|L|=k+1} u_L\bar{\omega}_L = \frac{(-1)^k}{k+1}\sum_{l=1}^n\Big(\frac{1}{k!}\sum_{|J|=k}u_{lJ}\bar{\omega}_J\Big)\wedge\bar{\omega}_l.$$

For $l = 1,\dots,n$, we define k-forms v_l by $v_l := \sum'_{|J|=k} u_{lJ}\bar{\omega}_J$. It is easy to see that $v_l \in C_c^\infty(U\cap\bar{\Omega})^k\cap\mathrm{Dom}(\bar{\partial}^*)$; $\sum_{l=1}^n |v_l|^2 = (k+1)|u|^2$ and $\sum_{l=1}^n\sum'_{|K|=k-1}(v_l)_{iK}\overline{(v_l)}_{jK} = k\sum'_{|J|=k} u_{iJ}\bar{u}_{jJ}$.

Using formula (2.29) we have

$$\begin{aligned}\sum_{l=1}^{n} Q(v_l, v_l) \lesssim& \sum_{l=1}^{n} \Big(\sum_{j=1}^{n} \|\bar{L}_j v_l\|^2 + \sum_{ij} \sum_{|K|=k-1}{}' \int_{b\Omega} r_{ij}(v_l)_{iK}\overline{(v_l)}_{jK} dS \Big) \\ \lesssim&(k+1) \sum_{j=1}^{n} \|\bar{L}_j u\|^2 + k \sum_{ij} \sum_{|J|=k}{}' \int_{b\Omega} r_{ij} u_{iJ} \overline{u}_{jJ} dS \\ \lesssim&(k+1) Q(u,u).\end{aligned} \tag{3.29}$$

If $\mathcal{M} \in \mathcal{A}^{0,0}$, then

$$\sum_{l=1}^{n} \|f(\Lambda)\mathcal{M}v_l\|^2 = (k+1)\|f(\Lambda)\mathcal{M}u\|^2.$$

If $\mathcal{M} \in \mathcal{A}^{1,0}$ we have

$$\sum_{l=1}^{n} |\mathcal{M}v_l|^2 = \sum_{l=1}^{n} \sum_{|K|=k-1}{}' |\mathcal{M}_j (v_l)_{jK}|^2 = k \sum_{|J|=k}{}' |\mathcal{M}_j u_{jJ}|^2 = k|\mathcal{M}u|^2;$$

then

$$\sum_{l=1}^{n} \|f(\Lambda)\mathcal{M}v_l\|^2 = k\|f(\Lambda)\mathcal{M}u\|^2.$$

The pseudoconcavity case. Let $u = \sum_{|K|=k-1}{}' u_K \bar{\omega}_K \in C_c^\infty(U \cap \bar{\Omega})^{k-1} \cap \mathrm{Dom}(\bar{\partial}^*)$. For l=1,...,n, we define $v_l = \sum_{|K|=k-1}{}' u_K \bar{\omega}_K \wedge \bar{\omega}_l \in C_c^\infty(U \cap \bar{\Omega})^{k-1}$. Using the same argument in pseudoconvexity case we obtain the conclusion for this case.

$\square$

Chapter 4

The $(f\text{-}\mathcal{M})^k$ estimate on boundary

In this chapter we shall study the behavior of the boundary value of forms associated to the $\bar\partial$-Neumann problem. In fact, we establish a relation between the $(f\text{-}\mathcal{M})^k$-estimate on Ω and $b\Omega$.

4.1 Definitions and notations

Let M be a smoothly real hypersurface in $\mathbb{C}^n$. We start with the forms on the boundary, denoted by $\mathcal{A}_b^{0,k}$ the space of restriction of element of $\mathcal{A}^{0,k}\cap\text{Dom}(\bar\partial^*)$ to the boundary $b\Omega$. Then $\mathcal{A}_b^{0,k}$ is the space of smooth section of the vector bundle $\big(T^{*0,1}(M)\big)^k$ on M.

The tangential Cauchy-Riemann operator $\bar\partial_b : \mathcal{A}_b^{0,k} \to \mathcal{A}_b^{0,k+1}$ is defined as follows. If $u \in \mathcal{A}_b^{0,k}$ and let u' be a (0,k)-form which restricted to $\mathcal{A}_b^{0,k}$ equals u. Then $\bar\partial_b u$ is the restriction of du' to $\mathcal{A}_b^{0,k+1}$.

Let $z_0 \in M$ and U be a neigborhood of z_0, we fix a defining function r of M such that $|\partial r| = 1$ on $U \cap M$. We assume that Ω is one of two side divided by M defined in U. Let $L_1, ..., L_n$ be the local basis for (1,0) vector fields defined in U, associated with Ω, which are defined in Chapter 1.

We can define a Hermitian inner product on $\mathcal{A}_b^{0,k}$ by

$$(\varphi, \psi)_b = \int_M \langle \varphi, \psi \rangle_x dS$$

where dS is the volume element on M. The inner product gives rise to an L^2-norm $\|.\|_b$.

In analogy with development in Chapter 2, we also define $\bar{\partial}_b^*$ to be the L^2-adjoint of $\bar{\partial}_b$ in the standard way. Thus $\bar{\partial}_b^* : \mathcal{A}_b^{0,k+1} \to \mathcal{A}_b^{0,k}$ when $k \geq 0$. The Kohn-Laplacian is defined by

$$\Box_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b.$$

Remember that we have already defined

$$Q_b(u, v) = (\bar{\partial}_b u, \bar{\partial}_b v)_b + (\bar{\partial}_b^* u, \bar{\partial}_b^* v)_b + (u, v)_b.$$

Denote

$$C_c^\infty(U \cap M)^k = \mathcal{A}_b^{0,k} \cap C_c^\infty(U)$$

the space of smooth $(0, k)$-forms on the boundary with support compact in U. If $u \in C_c^\infty(U \cap M)^k$ given by

$$u = {\sum_{|J|=k}}' u_J \bar{\omega}_J.$$

Then on M, the operator $\bar{\partial}_b$ and $\bar{\partial}_b^*$ are expressed as follows

$$\bar{\partial}_b u = {\sum_{|J|=k}}' \sum_{j=1}^{n-1} \bar{L}_j u_J \bar{\omega}_j \wedge \bar{\omega}_J + ... \tag{4.1}$$

and

$$\bar{\partial}_b^* u = - {\sum_{|K|=k-1}}' \sum_{j}^{n-1} L_j u_{jK} \bar{\omega}_K + ... \tag{4.2}$$

where dots refer the error term in which u is not differentiated.

In U, we choose special boundary coordinate $(x_1, ..., x_{2n-1}, r)$. Let $\xi = (\xi_1, ..., x_{2n-1}) = (\xi', \xi_{2n-1})$ be the dual coordinates to $\{x_1, ..., x_{2n-1}\}$. Let

ψ^+, ψ^-, ψ^0 be nonnegative functions in $C^\infty(\{\xi \in \mathbb{R}^{2n-1} \big| |\xi| = 1\})$, with range in [0,1], such that

$$\psi^+(\xi) = 1 \text{ when } \xi_{2n-1} \geq \frac{3}{4}|\xi'| \text{ and } \mathrm{supp}\psi^+ \subset\subset \{\xi \big| \xi_{2n-1} \geq \frac{1}{2}|\xi'|\};$$
$$\psi^-(\xi) = \psi^+(-\xi);$$
$$\psi^0(\xi) \text{ satisfies } \psi^0(\xi) = 1 - \psi^+(\xi) - \psi^-(\xi).$$

We extend these functions to $\mathbb{R}^{2n-1}$ so that

$$\psi^+(\xi) = \psi^+(\frac{\xi}{|\xi|+1}), \psi^-(\xi) = \psi^-(\frac{\xi}{|\xi|+1}), \psi^0(\xi) = \psi^0(\frac{\xi}{|\xi|+1}).$$

Define

$$\begin{aligned} \mathcal{C}^+ =& \{\xi \big| \xi_{2n-1} \geq \frac{1}{2}|\xi'|\}; \\ \mathcal{C}^- =& \{\xi \big| -\xi \in \mathcal{C}^+\}; \\ \mathcal{C}^0 =& \{\xi \big| -\frac{3}{4}|\xi'| \leq \xi_{2n-1} \leq \frac{3}{4}|\xi'|\}. \end{aligned} \tag{4.3}$$

Then $\mathrm{supp}\psi^+ \subset\subset \mathcal{C}^+$; $\mathrm{supp}\psi^- \subset\subset \mathcal{C}^-$; and $\mathrm{supp}\psi^0 \subset\subset \mathcal{C}^0$.

The operator Ψ is defined by

$$\widehat{\Psi\varphi}(\xi) = \psi(\xi)\hat{\varphi}(\xi) \qquad \text{for} \qquad \varphi \in C_c^\infty(U \cap M);$$
$$\widetilde{\Psi\varphi}(\xi, r) = \psi(\xi)\tilde{\varphi}(\xi, r) \qquad \text{for} \qquad \varphi \in C_c^\infty(U \cap \Omega).$$

The operator Ψ^+, Ψ^-, Ψ^0 are defined as above with substitution of ψ^+, ψ^-, ψ^0 for ψ, respectively. The microlocal decomposition $\varphi = \varphi^+ + \varphi^- + \varphi^0$ is interpreted as follows

$$\varphi = \zeta\Psi^+\varphi + \zeta\Psi^-\varphi + \zeta\Psi^0\varphi$$

for all $\varphi \in C_0^\infty(U)$, where $\zeta \in C^\infty(U'), \bar{U} \subset U'$ and $\zeta = 1$ on U.

We recall some definitions in Chapter 1:

Definition 4.1. If M is a hypersurface and $z_0 \in M$ then a $(f\text{-}\mathcal{M})^k_b$ estimate holds for $(\bar\partial_b, \bar\partial_b^*)$ on at x_0 if there exists a neighborhood U of z_0 such that

$$(f\text{-}\mathcal{M})^k_b \qquad \|f(\Lambda)\mathcal{M}u\|^2_b \leq cQ_b(u,u) + C_\mathcal{M}\|u\|^2_{b,-1}$$

for all $u \in C_c^\infty(U \cap M)^k$. And a $(f\text{-}\mathcal{M})^k_{b,+}$-estimate (resp. $(f\text{-}\mathcal{M})^k_{b,-}$) holds for $(\bar\partial_b, \bar\partial_b^*)$ at z_0 if the above holds with u replaced by u^+ (resp. u^-), that is,

$$(f\text{-}\mathcal{M})^k_{b,+} \qquad \|f(\Lambda)\mathcal{M}u^+\|_b^2 \le cQ_b(u^+,u^+) + C_\mathcal{M}\|u^+\|^2_{b,-1}$$

(resp.

$$(f\text{-}\mathcal{M})^k_{b,-} \qquad \|f(\Lambda)\mathcal{M}u^-\|_b^2 \le cQ_b(u^-,u^-) + C_\mathcal{M}\|u^-\|^2_{b,-1}. \quad)$$

Definition 4.2. The hypersurface M is called to be q-pseudoconvex at z_o if one of two parts divided by M is q-pseudoconvex at z_0.

Denoted by $\Omega^+ = \{z \in U | r(z) < 0\}$ the q-pseudoconvex part at z_0 which is divided by M; and another part denoted by Ω^-. By Remark 2.3, $\Omega^- = \{z \in U| -r(z) < 0\}$ is $(n-q-1)$-pseudoconcave at z_0. Here U is a neighborhood of $z_0 \in M$. Remember that $\omega_n = \partial r$, then $\omega_1, \dots, \omega_{n-1}, \omega_n^+ = \omega_n$ are the orthonormal (1,0)-forms on $U \cap \bar\Omega^+$ and $\omega_1, \dots, \omega_{n-1}, \omega_n^- = -\omega_n$ are the orthonormal (1,0)-forms on $U \cap \bar\Omega^-$. So that $L_n^+ = -L_n^- = L_n$ where L_n^+ ; L_n^- ; L_n are the duals of ω_n^+ ; ω_n^- ; ω_n, respectively. We define $T = \frac{1}{2}(L_n - \bar L_n)$ and $\frac{\partial}{\partial r} = \frac{1}{2}(L_n + \bar L_n)$. So that

$$L_n = \frac{\partial}{\partial r} + T; \qquad \text{and} \qquad \bar L_n = \frac{\partial}{\partial r} - T. \tag{4.4}$$

4.2 Basic microlocal estimates on M

This section we show the basic microlocal estimates on M.

In a way similar to Proposition 2.5, it follows

Lemma 4.3. *For two indices q_1, q_2 ; $(1 \le q_1 \le q_2 \le n-1)$, then there is a constant C such that*

$$\begin{aligned} 2\|\bar\partial_b u\|_b^2 + &2\|\bar\partial_b^* u\|_b^2 + C\|u\|_b^2 \\ &\ge \sideset{}{'}\sum_{|K|=k-1} \sum_{ij=1}^{n-1} (r_{ij} T u_{iK}, u_{iK})_b - \sideset{}{'}\sum_{|J|=k} \sum_{j=q_1}^{q_2} (r_{jj} T u_J, u_J)_b^2 \\ &\quad + \frac{1}{2} \sideset{}{'}\sum_{|J|=k} \Big(\sum_{j=1}^{n-1} \|\bar L_j u_J\|_b^2 - \sum_{j=q_1}^{q_2} \|L_j u_J\|_b^2 \Big) \end{aligned} \tag{4.5}$$

and conversely,

$$\begin{aligned}\|\bar\partial_b u\|_b^2+&\|\bar\partial_b^* u\|_b^2\\ \le & 2\sideset{}{'}\sum_{|K|=k-1}\sum_{ij=1}^{n-1}(r_{ij}Tu_{iK},u_{iK})_b-\sideset{}{'}\sum_{|J|=k}\sum_{j=q_1}^{q_2}(r_{jj}Tu_J,u_J)_b^2\\ &+3\sideset{}{'}\sum_{|J|=k}\Big(\sum_{j=1}^{n-1}\|\bar L_j u_J\|_b^2-\sum_{j=q_1}^{q_2}\|L_j u_J\|_b^2\Big)+C\|u\|_b^2\end{aligned}\tag{4.6}$$

holds for all $u\in C_c^\infty(U\cap M)^k$ for any k.

The following lemma is the estimate for u^0:

Lemma 4.4. *Let M be a hypersurface defined near $z_o\in M$. Then there is a neighborhood U of z_0 such that*

$$Q_b(u^0,u^0)\cong\|u^0\|_{b,1}^2$$

holds for all $u\in C_c^\infty(U\cap M)^k$ with any k.

Proof. Using inequality (4.5) twice times for $q_1=q_2=0$ and $q_1=0; q_2=n-1$ after that taking sum of them, we get

$$\begin{aligned}4\|\bar\partial_b u\|_b^2+&4\|\bar\partial_b^* u\|_b^2+2C\|u\|_b^2\\ \ge & 2\sideset{}{'}\sum_{|K|=k-1}\sum_{ij=1}^{n-1}(r_{ij}Tu_{iK},u_{iK})_b-\sideset{}{'}\sum_{|J|=k}\sum_{j=1}^{n-1}(r_{jj}Tu_J,u_J)_b^2\\ &+\frac12\Big(\sideset{}{'}\sum_{|J|=k}\sum_{j=1}^{n-1}\|L_j u_J\|_b^2+\sideset{}{'}\sum_{|J|=k}\sum_{j=1}^{n-1}\|\bar L_j u_J\|_b^2\Big)\\ \ge & \frac14\|\Lambda' u\|_b^2-\big(\epsilon+\operatorname{diam}(U)\big)\|Tu\|_b^2-C_\epsilon\|u\|_b^2\end{aligned}\tag{4.7}$$

where Λ' is the pseudodifferential operator of order 1 whose symbol is $(1+\sum_{j=1}^{2n-2}|\xi_j|^2)^{\frac12}$. Choose U and ϵ sufficiently small, substituting $u=u^0$ in (4.7) with notice that $\|\Lambda' u^0\|_b\gtrsim\|Tu^0\|_b$, we get

$$Q_b(u^0,u^0)\gtrsim\|\Lambda' u^0\|_b^2\gtrsim\|\Lambda u^0\|_b^2.$$

The conversely inequality is always true.

$\square$

Lemma 4.5. *Let M be a q-pseudoconvex hypersurface at z_0. Then, there is a neighborhood U of z_0, such that*

(i).

$$\begin{aligned}
&Q_b(u^+,u^+)+\|\Psi^+u\|^2_{b,-\infty}\\
&\qquad\cong\sum_{|K|=k-1}{}'\sum_{ij=1}^{n-1}(r_{ij}\zeta'(T^+)^{\frac{1}{2}}u^+_{iK},\zeta'(T^+)^{\frac{1}{2}}u^+_{iK})_b\\
&\qquad\quad-\sum_{|J|=k}{}'\sum_{j=1}^{q_0}(r_{jj}\zeta'(T^+)^{\frac{1}{2}}u^+_{J},\zeta'(T^+)^{\frac{1}{2}}u^+_{J})^2_b\\
&\qquad\quad+\sum_{|J|=k}{}'\sum_{j=1}^{q_0}\|L_ju^+_J\|^2_b+\sum_{|J|=k}{}'\sum_{j=q_0+1}^{n-1}\|\bar{L}_ju^+_J\|^2_b+\|u^+\|^2_b+\|\Psi^+u\|^2_{b,-\infty}
\end{aligned}\tag{4.8}$$

holds for all $u\in C_c^\infty(U\cap M)^k$ with any $k\geq q$, where $(T^+)^{\frac{1}{2}}$ is the pseudodifferential operator of order 1 whose symbol is $\xi_{2n-1}^{\frac{1}{2}}\psi^+(\xi)$ and ζ' is canonical cutoff function with $\zeta'=1$ on supp (u^+).

(ii).

$$\begin{aligned}
&Q_b(u^-,u^-)+\|\Psi^-u\|^2_{b,-1}\\
&\qquad\cong\sum_{|J|=k}{}'\sum_{j=q_0+1}^{n-1}(r_{jj}\zeta'(\bar{T}^-)^{\frac{1}{2}}u^-_{J},\zeta'(\bar{T}^-)^{\frac{1}{2}}u^-_{J})_b\\
&\qquad\quad-\sum_{|K|=k-1}{}'\sum_{ij=1}^{n-1}(r_{ij}\zeta'(\bar{T}^-)^{\frac{1}{2}}u^-_{iK},\zeta'(\bar{T}^-)^{\frac{1}{2}}u^-_{iK})_b\\
&\qquad\quad+\sum_{|J|=k}{}'\sum_{j=1}^{q_0}\|\bar{L}_ju^-_J\|^2_b+\sum_{|J|=k}{}'\sum_{j=q_0+1}^{n-1}\|L_ju^-_J\|^2_b+\|u^-\|^2_b+\|\Psi^-u\|^2_{b,-\infty}
\end{aligned}\tag{4.9}$$

holds for all $u\in C_c^\infty(U\cap M)^k$ with any $k\leq n-1-q$, where $(\bar{T}^-)^{\frac{1}{2}}$ is the pseudodifferential operator of order 1 whose symbol is $(-\xi_{2n-1})^{\frac{1}{2}}\psi^-(\xi)$ and ζ' is canonical cutoff function with $\zeta'=1$ on supp (u^-).

Recall that M is q-pseudoconvex at z_0 then there is a defining function of M such that

$$\sum'_{|K|=k-1}\sum_{ij=1}^{n-1} r_{ij}u_{iK}\bar{u}_{jK} - \sum_{j=1}^{q_o} r_{jj}|u|^2 \geq 0 \quad \text{on} \quad M$$

for any $u \in C_c^\infty(U \cap M)^k$ with $k \geq q$, and

$$\sum_{j=q_o+1}^{n-1} r_{jj}|u|^2 - \sum'_{|K|=k-1}\sum_{ij=1}^{n-1} r_{ij}u_{iK}\bar{u}_{jK} \geq 0 \quad \text{on} \quad M$$

for any $u \in C_c^\infty(U \cap M)^k$ with $k \leq n-q-1$, where U is a neighborhood of z_0.

Proof. (i): Since $\tilde{\psi}^+$ be a cutoff function with $\text{supp}(\tilde{\psi}^+) \subset \mathcal{C}^+$ and $\tilde{\psi}^+ = 1$ on $\text{supp}\psi^+$, we have

$$\varphi^+ = \zeta\Psi^+\varphi = \zeta(\tilde{\Psi}^+)^2\Psi^+\varphi = (\tilde{\Psi}^+)^2\zeta\Psi^+\varphi + [\zeta, (\tilde{\Psi}^+)^2]\Psi^+\varphi.$$

Then, the supports of symbols of Ψ^+ and $[\zeta, (\tilde{\Psi}^+)^2]$ are disjoint, the operator $[\zeta, (\tilde{\Psi}^+)^2]\Psi^+$ is order $-\infty$ and we have

$$\begin{aligned}(r_{ij}T\varphi^+, \varphi^+)_b =&(r_{ij}T\zeta\Psi^+\varphi, \zeta\Psi^+\varphi)_b \\ =&(r_{ij}T(\tilde{\Psi}^+)^2\zeta\Psi^+\varphi, \zeta\Psi^+\varphi)_b + O(\|\Psi^+\varphi\|^2_{b,-\infty}) \\ =&((\tilde{\zeta})^2 r_{ij}(T^+)^{\frac{1}{2}*}(T^+)^{\frac{1}{2}}\zeta\Psi^+\varphi, \zeta\Psi^+\varphi)_b + O(\|\Psi^+\varphi\|^2_{b,-\infty}) \\ =&(r_{ij}\tilde{\zeta}(T^+)^{\frac{1}{2}}\zeta\Psi^+\varphi, \tilde{\zeta}(T^+)^{\frac{1}{2}}\zeta\Psi^+\varphi)_b \\ &+ ([(\tilde{\zeta})^2 r_{ij}, (T^+)^{\frac{1}{2}*}](T^+)^{\frac{1}{2}}\zeta\Psi^+\varphi, \zeta\Psi^+\varphi)_b + O(\|\Psi^+\varphi\|^2_{b,-\infty}).\end{aligned} \tag{4.10}$$

From the pseudodifferential operator calculus we get

$$([(\tilde{\zeta})^2 r_{ij}, (T^+)^{\frac{1}{2}*}](T^+)^{\frac{1}{2}}\zeta\Psi^+\varphi, \zeta\Psi^+\varphi)_b| \lesssim \|\varphi^+\|^2_b \tag{4.11}$$

Substituting u_{jK} or u_J for φ in (4.10), we obtain

$$\begin{aligned}&\sum'_{|K|=k-1}\sum_{ij=1}^{n-1}(r_{ij}Tu^+_{iK}, u^+_{iK})_b - \sum'_{|J|=k}\sum_{j=1}^{q_0}(r_{jj}Tu^+_J, u^+_J)^2_b \\ =&\sum'_{|K|=k-1}\sum_{ij=1}^{n-1}(r_{ij}\zeta'(T^+)^{\frac{1}{2}}u^+_{iK}, \zeta'(T^+)^{\frac{1}{2}}u^+_{iK})_b - \sum'_{|J|=k}\sum_{j=1}^{q_0}(r_{jj}\zeta'(T^+)^{\frac{1}{2}}u^+_J, \zeta'(T^+)^{\frac{1}{2}}u^+_J)^2_b \\ &+ O(\|u^+\|^2_b) + O(\|\Psi^+u\|^2_{b,-\infty}).\end{aligned} \tag{4.12}$$

Since M is q-pseudoconvex then the sum in second line in (4.12) is nonnegative if $k \geq q$ where k is the degree of u . Thus the first part of Lemma 4.5 is proven by applying Lemma 4.3. with $q_1 = 0; q_2 = q_0$ and u for u^+.

(ii): The second part is proven analogously with notice

$$\begin{aligned}
&\sum_{|K|=k-1}{}' \sum_{ij=1}^{n-1} (r_{ij} T u^-_{iK}, u^-_{iK})_b - \sum_{|J|=k}{}' \sum_{j=q_0+1}^{n-1} (r_{jj} T u^-_J, u^-_J)^2_b \\
&= - \sum_{|K|=k-1}{}' \sum_{ij=1}^{n-1} (r_{ij} \zeta'(\bar{T}^-)^{\frac{1}{2}} u^-_{iK}, \zeta'(\bar{T}^-)^{\frac{1}{2}} u^-_{iK})_b + \sum_{|J|=k}{}' \sum_{j=q_0+1}^{n-1} (r_{jj} \zeta'(\bar{T}^-)^{\frac{1}{2}} u^-_J, \zeta'(\bar{T}^-)^{\frac{1}{2}} u^-_J)^2_b \\
&\quad + O(\|u^-\|^2_b) + O(\|\Psi^- u\|^2_{b,-\infty}),
\end{aligned} \tag{4.13}$$

and by Remark (2.3), the second line is nonnegative for any k-form u with $k \leq n - q - 1$. □

4.3 Basic microlocal estimates on Ω^+ and Ω^-

In this section, we show the basic microlocal estimates on Ω^+ and Ω^-. We begin with the harmonic extension. This extension was introduced by Kohn in [Ko86]; [Ko01].

In terms of special boundary coordinate (x, r) the operator L_j can be written as

$$L_j = \delta_{jn} \frac{\partial}{\partial r} + \sum_k a^k_j(x, r) \frac{\partial}{\partial x_k}$$

for $i = 1, ..., n$. We define the tangential symbols of L_j, $1 \leq j \leq n - 1$, by

$$\sigma^{L_j}(x, r, \xi) = \frac{1}{\sqrt{-1}} \sum_k a^k_j(x, r) \xi_k$$

and

$$\sigma^T(x, r, \xi) = \frac{1}{2\sqrt{-1}} \sum_k \left(a^k_n(x, r) - \bar{a}^k_n(x, r) \right) \xi_k$$

Note that σ^T is real. We set $\sigma_b^{L_j}(x,\xi) = \sigma^{L_j}(x,0,\xi)$; $\sigma_b^T(x,\xi) = \sigma^T(x,0,\xi)$ and

$$\sigma_b(x,\xi) = \sqrt{\sum_j |\sigma_b^{L_j}(x,\xi)|^2 + |\sigma_b^T(x,\xi)|^2 + 1}.$$

We see that if U is sufficiently small, then $\sigma_b(x,\xi) \cong (1+|\xi|^2)^{\frac{1}{2}}$.

Harmonic extension is defined as follows: Suppose that $\varphi \in C_c^\infty(U \cap M)$ we define $\varphi^{(h)} \in C^\infty(\{(x,r) \in \mathbb{R}^{2n} | r \le 0\})$ by

$$\varphi^{(h)}(x,r) = (2\pi)^{-2n+1} \int_{\mathbb{R}^{2n-1}} e^{ix\cdot\xi} e^{r\sigma_b(x,\xi)} \tilde{\varphi}(\xi) d\xi,$$

so that $\varphi(x,0) = \varphi(x)$. This extension is called "harmonic" since $\triangle\varphi^{(h)}(x,r)$ has order 1 on M. In fact, we have

$$\begin{aligned}\triangle &= -\sum_{j=1}^n \frac{\partial^2}{\partial_{z_j}\partial\bar{z}_j} \\ &= -\sum_{j=1}^n L_j\bar{L}_j + \sum_{k=1}^{2n-1} a^k(x,r)\frac{\partial}{\partial x_k} + a(x,r)\frac{\partial}{\partial r} \qquad (4.14) \\ &= -\frac{\partial^2}{\partial^2 r} + T^2 - \sum_{j=1}^{n-1} L_j\bar{L}_j + \sum_{k=1}^{2n-1} b^k(x,r)\frac{\partial}{\partial x_k} + b(x,r)\frac{\partial}{\partial r}\end{aligned}$$

since by (4.6) implies that $L_n\bar{L}_n = \frac{\partial^2}{\partial^2 r} - T^2 + D$, where D is a first order operator. Hence if $(x,r) \in U \cap \bar{\Omega}^+$,

$$\triangle(\varphi^{(h)})(x,r) = \int e^{ix\cdot\xi} e^{r\sigma_b(x,\xi)}\big(p^1(x,r,\xi) + rp^2(x,r,\xi)\tilde{\varphi}(\xi)\big)d\xi. \qquad (4.15)$$

Further more

$$L_j\varphi^{(h)}(x,r) = (L_j\varphi)^h(x,r) + E_j\varphi(x,r)$$

where

$$E_j\varphi(x,r) = \int e^{ix\cdot\xi} e^{r\sigma_b(x,\xi)}\big(p_j^0(x,r,\xi) + rp_j^1(x,r,\xi)\tilde{\varphi}(\xi)\big)d\xi$$

and

$$\bar{L}_j\varphi^{(h)}(x,r) = (\bar{L}_j\varphi)^h(x,r) + \bar{E}_j\varphi(x,r)$$

for $j = 1, ..., n-1$.

Lemma 4.6. *For each $k \in \mathbb{Z}, k \geq 0$; $s \in \mathbb{R}$; and $f \in \mathcal{F}$, then we have*

(i) $|||r^k f(\Lambda)\varphi^{(h)}|||_s \underset{\sim}{<} \|f(\Lambda)\varphi\|_{b,s-k-\frac{1}{2}}$;

(i) $|||D_r f(\Lambda)\varphi^{(h)}|||_s \underset{\sim}{<} \|f(\Lambda)\varphi\|_{b,s+\frac{1}{2}}$

for any $\varphi \in C_c^\infty(U \cap \bar{\Omega}^+)$.

Proof. We notice again that $\sigma_b(x,\xi) \cong (1+|\xi|^2)^{\frac{1}{2}}$, then the proof of this lemma is similar with the proof of Lemma 2.10.

□

Moreover, if $\varphi \in C_c^\infty(U \cap \bar{\Omega}^+)$ we define φ_b to be the restriction of φ to boundary, we have

$$\|\varphi_b\|_{b,s}^2 \lesssim |||\varphi|||_{s+\frac{1}{2}}^2 + |||D_r\varphi|||_{s-\frac{1}{2}}^2. \tag{4.16}$$

The following lemma is the basic microlocal estimates on Ω^+.

Lemma 4.7. *Let Ω^+ be a q-pseudoconvex at the boundary z_0. If U is a sufficiently small neighborhood of z_0 then*

(i)

$$|||\Psi^0\varphi|||_1^2 \lesssim \sum_{j=1}^{q_o} \|L_j\Psi^0\varphi\|^2 + \sum_{q_o+1}^{n} \|\bar{L}_j\Psi^0\varphi\| + \|\Psi^0\varphi\|^2 \tag{4.17}$$

holds for all $\varphi \in C_c^\infty(U \cap \bar{\Omega}^+)$.

(ii)

$$|||\Psi^-\varphi|||_1^2 \lesssim \sum_{j=1}^{q_o} \|L_j\Psi^-\varphi\|^2 + \sum_{q_o+1}^{n} \|\bar{L}_j\Psi^-\varphi\| + \|\Psi^-\varphi\|^2 \tag{4.18}$$

holds for all $\varphi \in C_c^\infty(U \cap \bar{\Omega}^+)$.

(iii)

$$|||\bar{L}_n\Psi^+\varphi^{(h)}|||_{\frac{1}{2}}^2 \lesssim \sum_{j=1}^{q_o} \|L_j\Psi^+\varphi\|_b^2 + \sum_{q_o+1}^{n-1} \|\bar{L}_j\Psi^+\varphi\|_b^2 + \|\Psi^+\varphi\|_b^2 \tag{4.19}$$

holds for all $\varphi \in C_c^\infty(U \cap b\Omega^+)$.

Proof. (i): Since supp$\psi^0 \in \mathcal{C}^0$, then we have

$$\begin{aligned}(1+|\xi|^2)|\psi^0(\xi)|^2 &\lesssim \sum_{j=1}^{2n-2} |\xi_j|^2|\psi^0(\xi)|^2 + \psi^0 \\ &\lesssim \sum_{j=1}^{n-1} \mu_j(0,0,\xi)^2|\psi^0(\xi)|^2 + \psi^0 \\ &\lesssim \sum_{j=1}^{n-1} \mu_j(x,r,\xi)^2|\psi^0(\xi)|^2 + \sum_{j=1}^{n-1} \big(\mu_j(0,0,\xi) - \mu_j(x,r,\xi)\big)^2|\psi^0(\xi)|^2 + \psi^0 \\ &\lesssim \sum_{j=1}^{n-1} \mu_j(x,r,\xi)^2|\psi^0(\xi)|^2 + \mathrm{diam}(U\cap\bar\Omega) \sum_{j=1}^{2n-1} \xi_j^2|\psi^0(\xi)|^2 + \psi^0.\end{aligned} \tag{4.20}$$

Hence

$$\begin{aligned}\|\Lambda\Psi^0\varphi\|_1^2 \lesssim &\sum_{j=1}^{q_0} \|L_j\Psi^0\varphi\|^2 + \sum_{j=q_0+1}^{n} \|\bar L_j\Psi^0\varphi\|^2 \\ &+ \|\Psi^0\varphi\|^2 + \mathrm{diam}(U\cap\bar\Omega) \sum_{j=1}^{2n-1} \|D_j\Psi^0\varphi\|^2.\end{aligned} \tag{4.21}$$

The estimate (4.17) follows by (4.21) by taking U with a sufficiently small diameter.

(ii): For all $\varphi \in C_c^\infty(U\cap\bar\Omega)$, let $\varphi_b^{(h)}$ be the harmonic extension of $\varphi_b = \varphi|_{U\cap b\Omega^+}$.

$$\begin{aligned}|||\Psi^-\varphi|||_1^2 \lesssim& |||\Psi^-(\varphi - \varphi_b^{(h)_b})|||_1^2 + |||\Psi^-\varphi_b^{(h)}|||_1^2 \\ \lesssim& \|\Psi^-(\varphi - \varphi_b^{(h)})\|_1^2 + |||\Psi^-\varphi_b^{(h)}|||_1^2.\end{aligned} \tag{4.22}$$

Now we estimate $|||\Psi^-\varphi_b^{(h)}|||_1^2$, we have

$$\bar L_n\Psi^-\varphi_b^{(h)}(x,r) = \int e^{ix\cdot\xi} e^{r(\sigma_b(x,\xi))}\Big(\sigma_b(x,\xi) - \sigma_b^T(x,\xi) + rp_1(x,\xi)\Big)\psi^-(\xi)\tilde\varphi_b(\xi,0)d\xi \tag{4.23}$$

where $p_1(x,\xi)$ is a symbol of the tangential operator P_1 of order 1. Choosing U sufficiently small we have $\sigma_b^T(x,\xi) \le 0$ when $\xi \in \mathrm{supp}(\psi^-) \subset \mathcal{C}^-$. Then

$$\sigma_b(x,\xi) - \sigma_b^T(x,\xi) \gtrsim |\xi| + 1.$$

So that

$$|||\Psi^-\varphi_b^{(h)}|||_1^2 \lesssim \|\bar{L}_n\Psi^-\varphi_b^{(h)}\|^2 + \|rP_1\Psi^-\varphi_b^{(h)}\|^2. \tag{4.24}$$

Applying Lemma 4.6 and inequality (4.16) for the second term in (4.24), we get

$$\begin{aligned}\|rP_1\Psi^-\varphi_b^{(h)}\|^2 \lesssim& \|\Lambda^{-1/2}\Psi^-\varphi_b\|_b^2\\ \lesssim& \|\Lambda^{-1}D_r\Psi^-\varphi\|^2 + \|\Psi^-\varphi\|^2\\ \lesssim& \|\bar{L}_n\Psi^-\varphi\|^2 + \|\Psi^-\varphi\|^2.\end{aligned} \tag{4.25}$$

For the first term in (4.24), we have

$$\begin{aligned}\|\bar{L}_n\Psi^-\varphi_b^{(h)}\|^2 \lesssim& \|\bar{L}_n\Psi^-(\varphi-\varphi_b^{(h)})\|^2 + \|\bar{L}_n\Psi^-\varphi\|^2\\ \lesssim& \|\Psi^-(\varphi-\varphi_b^{(h)})\|_1^2 + \|\bar{L}_n\Psi^-\varphi\|^2.\end{aligned} \tag{4.26}$$

Combining (4.22), (4.24), (4.25) and (4.26), we get

$$|||\Psi^-\varphi|||_1^2 \lesssim \|\Psi^-(\varphi-\varphi_b^{(h)})\|_1^2 + \|\bar{L}_n\Psi^-\varphi\|^2 + \|\Psi^-\varphi\|^2. \tag{4.27}$$

Finally, we estimate $\|\Psi^-(\varphi-\varphi_b^{(h)})\|_1^2$. Since $\Psi^-(\varphi-\varphi_b^{(h)}) = 0$ on $U\cap b\Omega$ then

$$\begin{aligned}\|\Psi^-(\varphi-\varphi_b^{(h)})\|_1^2 \lesssim& \|\Delta\Psi^-(\varphi-\varphi^{(h)})\|_{-1}^2\\ \lesssim& \|\Delta\Psi^-\varphi\|_{-1}^2 + \|\Delta\Psi^-\varphi^{(h)}\|_{-1}^2\\ \lesssim& \sum_{j=1}^{q_0}\|\bar{L}_jL_j\Psi^-\varphi\|_{-1}^2 + \sum_{j=q_0+1}^{n}\|L_j\bar{L}_j\Psi^-\varphi\|_{-1}^2\\ &+\|P_1\Psi^-\varphi\|_{-1}^2 + \|(rP_2+P_1)\Psi^-\varphi^{(h)}\|_{-1}^2\\ \lesssim& \sum_{j=1}^{q_0}\|L_j\Psi^-\varphi\|^2 + \sum_{j=q_0+1}^{n}\|\bar{L}_j\Psi^-\varphi\|^2 + \|\Psi^-\varphi\|^2.\end{aligned} \tag{4.28}$$

Here the third inequality in (4.28) follows by (4.15). This is completed the proof of (ii).

(iii): For any $\varphi \in C_c^\infty(U\cap b\Omega^+)$, we have

$$\bar{L}_n\Psi^+\varphi^{(h)}(x,r) = \int e^{ix\cdot\xi}e^{r(\sigma_b(x,\xi))}\Big(\sigma_b(x,\xi)-\sigma_b^T(x,\xi)+rp_1(x,\xi)\Big)\psi^+(\xi)\tilde{\varphi}(\xi,0)d\xi.$$

Choosing U sufficiently small we have $\sigma_b^T(x,\xi) > 0$ when $\xi \in \mathrm{supp}\psi^+ \subset \mathcal{C}^+$. So that

$$\sigma_b - \sigma_b^T = \sum_{j=1}^{q_o} \frac{\sigma_b^{\bar{L}_j}}{\sigma_b + \sigma_b^T}\sigma_b^{L_j} + \sum_{j=q_o+1}^{n-1} \frac{\sigma_b^{L_j}}{\sigma_b + \sigma_b^T}\sigma_b^{\bar{L}_j}.$$

Since

$$\left\{\frac{\sigma_b^{\bar{L}_j}}{\sigma_b + \sigma_b^T}\right\}_{j\le q_o} \quad \text{and} \quad \left\{\frac{\sigma_b^{L_j}}{\sigma_b + \sigma_b^T}\right\}_{q_o+1\le j\le n-1}$$

are absolutely bounded. Hence

$$\begin{aligned} |||\bar{L}_n\Psi^+\varphi^{(h)}|||_{\frac{1}{2}}^2 &\lesssim \sum_{j=1}^{q_o} |||(L_j\Psi^+\varphi)^h|||_{\frac{1}{2}} + \sum_{j=q_o+1}^{n-1} |||(L_j\Psi^+\varphi)^h|||_{\frac{1}{2}} + |||rP_1\Psi^+\varphi^{(h)}|||_{\frac{1}{2}}^2 \\ &\lesssim \sum_{j=1}^{q_o} ||L_j\Psi^+\varphi||_b^2 + \sum_{j=q_o+1}^{n-1} ||L_j\Psi^+\varphi||_b^2 + \|\Psi^+\varphi\|_b^2. \end{aligned} \tag{4.29}$$

□

Using Lemma 4.7 for each coefficient of a form we obtain :

Lemma 4.8. *Let Ω^+ be a q-pseudoconvex at the boundary z_0. Then, there is a neighborhood U of z_0, such that*

(i)

$$|||\Psi^0 u|||_1^2 + |||\Psi^- u|||_1^2 \lesssim Q(u,u)$$

holds for all $u \in C_c^\infty(U\cap\bar{\Omega}^+)^k \cap Dom(\bar{\partial}^*)$ with any $k \ge q$.

(ii)

$$|||\bar{L}_n\Psi^+(u^+)^{(h)}|||_{\frac{1}{2}}^2 \lesssim Q_b(u^+,u^+)$$

holds for all $u \in C_c^\infty(U\cap b\Omega^+)^k$ with any $k \ge q$.

Similarly, we get the basic microlocal estimates for Ω^-.

Lemma 4.9. *Let Ω^- be a $n-1-q$-pseudoconvex at the boundary z_0. Then, there is a neighborhood U of z_0, such that*

(i)

$$|||\Psi^0 u|||_1^2 + |||\Psi^+ u|||_1^2 \lesssim Q(u,u)$$

holds for all $u \in C_c^\infty(U\cap\bar{\Omega}^-)^k \cap Dom(\bar{\partial}^*)$ with any $k \le n-1-q$.

(ii)

$$|||\bar{L}_n\Psi^-(u^-)^{(h)}|||^2_{\frac{1}{2}} \lesssim Q_b(u^-,u^-)$$

holds for all $u \in C^\infty_c(U\cap b\Omega^-)^k$ with any $k \le n-1-q$.

4.4 The equivalent of $(f\text{-}\mathcal{M})$ estimate on Ω and $b\Omega$

In this section, we give the proof of Theorem 1.13. Theorem 1.13 immediate consequence of the Theorem follows by Theorem 4.10, Theorem 4.11 and Theorem 4.12.

Theorem 4.10. *Let Ω^+ be a smoothly q-pseudoconvex domain at $z_0 \in b\Omega$ in a hermitian manifold X with the boundary M. Then*

(i) If $(f\text{-}\mathcal{M})^k_{\Omega^+}$ holds then $(f\text{-}\mathcal{N})^k_{b,+}$ holds where $\mathcal{N}$ is the restriction of $\mathcal{M}$ to M

(ii) If $(f\text{-}\mathcal{N})^k_{b,+}$ holds then then $(f\text{-}\mathcal{M})^k_{\Omega^+}$ holds where $\mathcal{M}$ is any extension of $\mathcal{N}$ from M to Ω such that $\mathcal{M}\big|_M = \mathcal{N}$.

Proof. (i): We need to show that

$$\|f(\Lambda)\mathcal{N}u^+\|^2_b \lesssim Q(u^+,u^+) + C_{\mathcal{N}}\|u^+\|^2_{b,-1} + \|\Psi^+u\|^2_{b,-\infty}$$

for any $u \in C^\infty_c(U\cap M)^k$ where U is a neighborhood of z_0. Set $u^{(h)} := \sum'_{|J|=k} u^{(h)}_J\bar{\omega}_J$. Let χ be the cutoff function on r with $\chi(0)=1$. Applying inequality (4.16), we have

$$\begin{aligned}\|f(\Lambda)\mathcal{N}u^+\|^2_b \lesssim&\|\Lambda^{\frac{1}{2}}\chi f(\Lambda)\mathcal{M}u^{(h)+}\|^2 + \|\Lambda^{-\frac{1}{2}}D_r(\chi f(\Lambda)\mathcal{M}u^{(h)+})\|^2\\ \lesssim&\|f(\Lambda)\mathcal{M}\chi\zeta'(T^+)^{\frac{1}{2}}u^{(h)+}\|^2\\ &+\|\Lambda^{-1}f(\Lambda)D_r(\mathcal{M}\chi\zeta'(T^+)^{\frac{1}{2}}u^{(h)+})\|^2 + error\end{aligned} \tag{4.30}$$

where $\zeta' = 1$ on $\text{supp}(u^+)$ and $\text{supp}(\zeta') \subset U'$. The way to insert $\zeta'(T^+)^{\frac{1}{2}}$ is similar Lemma (4.5) and the error term is

$$error \lesssim \|\Lambda^{\frac{1}{2}}u^{(h)+}\|^2 + C_{\mathcal{M}}\|\Lambda^{\frac{1}{2}}u^{(h)+}\|^2_{-1} + \|\Psi^+u^{(h)}\|^2_{-\infty}.$$

We can choose χ such that $\chi\zeta' T^{\frac{1}{2}} u^{(h)+} \in C_c^\infty(U'\cap\bar\Omega)^k$. Notice that $\chi\zeta' T^{\frac{1}{2}} u^{(h)+} \in \mathrm{Dom}(\bar\partial^*)$ is always true. Using hypothesis for the term in the second line in (4.30) and applying Lemma 3.10 for the second term in last line in (4.30), continuing (4.30), we have

$$\begin{aligned}
&\lesssim Q(\chi\zeta' T^{\frac{1}{2}} u^{(h)+}, \chi\zeta' T^{\frac{1}{2}} u^{(h)+}) + C_{\mathcal{M}}\|\chi\zeta' T^{\frac{1}{2}} u^{(h)+}\|^2_{-1} + error\\
&\lesssim \sum_{|K|=k-1}{}' \sum_{ij}^{n-1} (r_{ij}\zeta' T^{\frac{1}{2}} u^+, \zeta' T^{\frac{1}{2}} u^+)_b - \sum_{|J|=k}{}' (r_{jj}\zeta' T^{\frac{1}{2}} u^+_J, \zeta' T^{\frac{1}{2}} u^+_J)_b\\
&\quad + \sum_{|J|=k}{}' \Big(\sum_{j=1}^{q_0} \|L_j\chi\zeta' T^{\frac{1}{2}} u_J^{(h)+}\|^2 + \sum_{j=1}^{n} \|\bar L_j\chi\zeta' T^{\frac{1}{2}} u_J^{(h)+}\|^2 \Big)\\
&\quad + \|\chi\zeta' T^{\frac{1}{2}} u^{(h)+}\|^2 + C_{\mathcal{M}}\|\chi\zeta' T^{\frac{1}{2}} u^{(h)+}\|^2_{-1} + error\\
&\lesssim Q_b(u^+,u^+) + C_{\mathcal{M}}\|u^+\|^2_{b,-1} + \|\Psi^+ u\|^2_{b,-\infty}\\
&\quad + \sum_{|J|=k}{}' \Big(\sum_{j=1}^{q_0} \|\Lambda^{\frac{1}{2}} (L_j u_J^+)_b^{(h)}\|^2 + \sum_{j=1}^{n-1} \|\Lambda^{\frac{1}{2}} (\bar L_j u_J^+)_b^{(h)}\|^2 \Big) + \|\Lambda^{\frac{1}{2}} \bar L_n \Psi^+ (u^+)^{(h)}\|^2\\
&\lesssim Q_b(u^+,u^+) + C_{\mathcal{M}}\|u^+\|^2_{b,-1} + \|\Psi^+ u\|^2_{b,-\infty}
\end{aligned} \tag{4.31}$$

here the second inequality follows by (2.29), the third one follows by Lemma 4.5, the last one by Lemma 4.8.

Proof of (ii). For any $u \in C_c^\infty(U\cap\bar\Omega)^k \cap \mathrm{Dom}(\bar\partial^*)$, we decompose $u = u^\tau + u^\nu$; $u^\tau = u^{\tau+} + u^{\tau-} + u^{\tau 0}$. By Lemma ?? and Lemma 4.8, we have

$$\begin{aligned}
\|f(\Lambda)\mathcal{M}u^\nu\|^2 &\le |||u^\nu|||_1^2 + C_{\mathcal{M}}\|u^\nu\|^2_{-1} \lesssim Q(u,u) + C_{\mathcal{M}}\|u\|^2_{-1};\\
\|f(\Lambda)\mathcal{M}u^{\tau 0}\|^2 &\le |||u^{\tau 0}|||_1^2 + C_{\mathcal{M}}\|u^{\tau 0}\|^2_{-1} \lesssim Q(u,u) + C_{\mathcal{M}}\|u\|^2_{-1};\\
\|f(\Lambda)\mathcal{M}u^{\tau -}\|^2 &\le |||u^{\tau -}|||_1^2 + C_{\mathcal{M}}\|u^{\tau -}\|^2_{-1} \lesssim Q(u,u) + C_{\mathcal{M}}\|u\|^2_{-1}.
\end{aligned} \tag{4.32}$$

Moreover, by Theorem 3.1 and (3.25), we have

$$\begin{aligned}
\|f(\Lambda)\mathcal{M}u^{\tau+}\|^2 &\lesssim \|\Lambda^{-\frac{1}{2}} f(\Lambda)\mathcal{N}u_b^{\tau+}\|_b^2 + Q(u^\tau,u^\tau) + C_{\mathcal{M}}\|u^\tau\|^2_{-1}\\
&\lesssim \|\Lambda^{-\frac{1}{2}} f(\Lambda)\mathcal{N}u_b^{\tau+}\|_b^2 + Q(u,u) + C_{\mathcal{M}}\|u\|^2_{-1}.
\end{aligned} \tag{4.33}$$

Thus, we obtain

$$\begin{aligned}
\|f(\Lambda)\mathcal{M}u\|^2 &\lesssim \|f(\Lambda)\mathcal{M}u^{\tau+}\|^2 + \|f(\Lambda)\mathcal{M}u^{\tau-}\|^2 + \|f(\Lambda)\mathcal{M}u^{\tau 0}\|^2 + \|f(\Lambda)\mathcal{M}u^{\nu}\|^2\\
&\lesssim \|\Lambda^{-\frac{1}{2}} f(\Lambda)\mathcal{N}u_b^{\tau+}\|_b^2 + Q(u,u) + C_{\mathcal{M}}\|u\|^2_{-1}.
\end{aligned} \tag{4.34}$$

So that we only need to estimate $\|\Lambda^{-\frac{1}{2}}f(\Lambda)\mathcal{N}u_b^{\tau+}\|_b^2$, we have

$$\|\Lambda^{-\frac{1}{2}}f(\Lambda)\mathcal{N}u_b^{\tau+}\|_b^2 \lesssim \|f(\Lambda)\mathcal{N}\tilde{\zeta}\Lambda^{-\frac{1}{2}}u_b^{\tau+}\|_b^2 + error. \tag{4.35}$$

Then $\tilde{\zeta}\Lambda^{-\frac{1}{2}}u_b^{\tau+} \in C^\infty(U\cap b\Omega)^k$, using hypothesis and continuing (4.35), we get

$$\begin{aligned}
&\lesssim Q_b(\tilde{\zeta}\Lambda^{-\frac{1}{2}}u_b^{\tau+}, \tilde{\zeta}\Lambda^{-\frac{1}{2}}u_b^{\tau+}) + C_{\mathcal{M}}\|\tilde{\zeta}\Lambda^{-\frac{1}{2}}u_b^{\tau+}\|_{b,-1}^2 + error\\
&\lesssim \sum_{|K|=k-1}{}' \sum_{ij=1}^{n-1}(r_{ij}\zeta' T^{\frac{1}{2}}(\tilde{\zeta}\Lambda^{-\frac{1}{2}}u_b^{\tau+})_{iK}, \zeta' T^{\frac{1}{2}}(\tilde{\zeta}\Lambda^{-\frac{1}{2}}u_b^{\tau+})_{jK})_b\\
&\quad \sum_{|J|=k}{}' \Big(\sum_{j=1}^{q_0}\|L_j(\tilde{\zeta}\Lambda^{-\frac{1}{2}}u_b^{\tau+})_J\|_b^2 + \sum_{j=1}^{n-1}\|\bar{L}_j(\tilde{\zeta}\Lambda^{-\frac{1}{2}}u_b^{\tau+})_J\|_b^2\Big)\\
&\quad + \|\tilde{\zeta}\Lambda^{-\frac{1}{2}}u_b^{\tau+}\|_b^2 + C_{\mathcal{M}}\|\tilde{\zeta}\Lambda^{-\frac{1}{2}}u_b^{\tau+}\|_{b,-1}^2 + \|\Psi^+\Lambda^{-\frac{1}{2}}u_b^{\tau}\|_{b,-\infty}^2 + error\\
&\lesssim Q(\zeta' T^{\frac{1}{2}}\tilde{\zeta}\Lambda^{-\frac{1}{2}}\zeta\Psi^+u^\tau, \zeta' T^{\frac{1}{2}}\tilde{\zeta}\Lambda^{-\frac{1}{2}}\zeta\Psi^+u^\tau)\\
&\quad \sum_{|J|=k}{}' \Big(\sum_{j=1}^{q_0}\|\tilde{\zeta}L_j\Lambda^{-\frac{1}{2}}(u_b^{\tau+})_J\|_b^2 + \sum_{j=1}^{n-1}\|\tilde{\zeta}\bar{L}_j\Lambda^{-\frac{1}{2}}(u_b^{\tau+})_J\|_b^2\Big)\\
&\quad + \|\tilde{\zeta}\Lambda^{-\frac{1}{2}}u_b^{\tau+}\|_b^2 + \tilde{C}_{\mathcal{M}}\|\Lambda^{-\frac{1}{2}}u_b^{\tau+}\|_{b,-1}^2.
\end{aligned} \tag{4.36}$$

Since $\zeta' T^{\frac{1}{2}}\tilde{\zeta}\Lambda^{-\frac{1}{2}}\zeta\Psi^+$ is tangential pseudodifferential operators of order zero. So that

$$Q(\zeta' T^{\frac{1}{2}}\tilde{\zeta}\Lambda^{-\frac{1}{2}}\zeta\Psi^+u^\tau, \zeta' T^{\frac{1}{2}}\tilde{\zeta}\Lambda^{-\frac{1}{2}}\zeta\Psi^+u^\tau) \lesssim Q(u^\tau, u^\tau).$$

To estimate the last two line above we proceed as follows. For $j \le q_0$, since $\tilde{\zeta}L_j\Lambda^{-\frac{1}{2}}(u_b^{\tau+})_J \in C_c^\infty(U\cap\bar{\Omega})$, then using inequality (4.16), we have

$$\begin{aligned}
\|\tilde{\zeta}L_j\Lambda^{-\frac{1}{2}}(u_b^{\tau+})_J\|_b^2 &\lesssim \|\Lambda^{\frac{1}{2}}\tilde{\zeta}L_j\Lambda^{-\frac{1}{2}}(u^{\tau+})_J\|^2 + \|\Lambda^{-\frac{1}{2}}\frac{\partial}{\partial r}\tilde{\zeta}L_j\Lambda^{-\frac{1}{2}}u_J^{\tau+}\|^2\\
&\lesssim \|L_ju_J^{\tau+}\|^2 + \|\frac{\partial}{\partial r}\Lambda^{-1}L_ju_J^{\tau+}\|^2 + error\\
&\lesssim \|L_ju_J^{\tau+}\|^2 + \|T\Lambda^{-1}L_ju_J^{\tau+}\|^2 + \|\bar{L}_n\Lambda^{-1}L_ju_J^{\tau+}\|^2 + error\\
&\lesssim \|L_ju_J^{\tau+}\|^2 + \|\bar{L}_nu_J^{\tau+}\|^2 + error\\
&\lesssim Q(u^{\tau+}, u^{\tau+}) \lesssim Q(u^\tau, u^\tau).
\end{aligned} \tag{4.37}$$

By the same way for the term $\|\tilde{\zeta}\bar{L}_j\Lambda^{-\frac{1}{2}}(u_b^{\tau+})_J\|_b^2$ with $q_0+1 \le j \le n-1$ and for $\|\tilde{\zeta}\Lambda^{-\frac{1}{2}}u_b^{\tau+}\|_b^2$. This concludes the proof of Theorem 4.10.

□

Similarly, we get the equivalent of $(f\text{-}\mathcal{M})^k$ on Ω^- and $\mathcal{M}$:

Theorem 4.11. *Let Ω^- be a smoothly q-pseudoconcave domain at $z_0 \in b\Omega$ in a hermitian manifold X. Then*

1. *If $(f\text{-}\mathcal{M})^k_{\Omega^-}$ holds then $(f\text{-}\mathcal{N})^k_{b,-}$ holds where $\mathcal{N}$ is the restriction of $\mathcal{M}$ to $b\Omega$*

2. *If $(f\text{-}\mathcal{N})^k_{b,-}$ holds then then $(f\text{-}\mathcal{M})^k_{\Omega-}$ holds where $\mathcal{M}$ is any extension of $\mathcal{N}$ to Ω.*

Finall,y we show that

Theorem 4.12. *Let M be a q-pseudoconvex hypersurface at z_0. Then $(f\text{-}\mathcal{N})^k_{b,+}$ holds if and only if $(f\text{-}\mathcal{N})^{n-1-k}_{b,-}$ holds.*

Proof. We define the local conjugate-linear duality map $F^k : \mathcal{A}^{0,k}_b \to \mathcal{A}^{0,n-1-k}$ as follows. If $u = \sum'_{|J|=k} u_J\bar{\omega}_J$ then

$$F^k u = \sum \epsilon^{\{J,J'\}}_{\{1,\dots,n-1\}} \bar{u}_J \bar{\omega}_{J'},$$

where J' denotes the strictly increasing $(n-k-1)$-tuple consiting if all integers in $[1, n-1]$ which do not belong to J and $\epsilon^{J,J'}_{\{1,n-1\}}$ is the sign of the permutation $\{J, J'\} \to \{1, \dots, n-1\}$.

Since $\overline{(\varphi^+)} = (\overline{\varphi})^-$ Then

$$F^k u^+ = \sum \epsilon^{\{J,J'\}}_{\{1,\dots,n-1\}} (\bar{u})^-_J \bar{\omega}_{J'},$$

$F^{n-1-k}F^k u^+ = u^+$; $\|F^k u^+\| = \|u^-\|$;

$$\bar{\partial}_b F^k u^+ = F^{k-1}\bar{\partial}^*_b u^+ + \cdots$$

and

$$\bar{\partial}^*_b F^k u^+ = F^{k+1}\bar{\partial}_b u^+ + \cdots$$

where dots refers the term in which u is not differentiated. Hence

$$Q_b(F^k u^+, F^k u^+) \cong Q_b(u^+, u^+).$$

On the other hand, we also have $\|f(\Lambda)\mathcal{N}F^k u^+\|^2 = \|f(\Lambda)\mathcal{N}u^+\|^2$. In fact, this inequality follows by the definition of $\mathcal{N}u^+$; $\mathcal{N}u^-$ and F^k .

Corollary 4.13. *Let M be a pseudoconvex hypersurface at z_0 and let $\mathcal{N} \in C^\infty(M)$. Then for $1 \le k \le n-2$, the estimate $(f\text{-}\mathcal{N})^k_b$ holds if M has one of following conditions:*

1. $(f\text{-}\mathcal{N})^k_{b,+}$ *and* $(f\text{-}\mathcal{N})^k_{b,-}$ *hold*
2. $(f\text{-}\mathcal{N})^k_{b,+}$ *and* $(f\text{-}\mathcal{N})^{n-1-k}_{b,+}$ *hold*
3. $(f\text{-}\mathcal{N})^l_{b,+}$ *holds for* $l \le \min\{k, n-1-k\}$
4. $(f\text{-}\mathcal{N})^l_{b,-}$ *holds for* $l \ge \max\{k, n-1-k\}$
5. $(f\text{-}\mathcal{M})^k_{\Omega^+}$ *and* $(f\text{-}\mathcal{M})^k_{\Omega^-}$ *hold*
6. $(f\text{-}\mathcal{M})^k_{\Omega^+}$ *and* $(f\text{-}\mathcal{M})^{n-1-k}_{\Omega^+}$ *hold*
7. $(f\text{-}\mathcal{M})^l_{\Omega^+}$ *holds for* $l \le \min(k, n-1-k)$
8. $(f\text{-}\mathcal{M})^l_{\Omega^-}$ *holds for* $l \ge \max(k, n-1-k)$
9. *Property* $(f\text{-}\mathcal{M}\text{-}P)^k$ *holds on the both side* Ω^+ *and* Ω^-
10. *Property* $(f\text{-}\mathcal{M}\text{-}P)^k$ *holds on the side* Ω^+ *and Property* $(f\text{-}\mathcal{M}\text{-}P)^{n-k-1}$ *holds on the side* Ω^-
11. *Property* $(f\text{-}\mathcal{M}\text{-}P)^l$ *holds on the side* Ω^+ *for* $l \le \min\{k, n-k-1\}$
12. *Property* $(f\text{-}\mathcal{M}\text{-}P)^l$ *holds on the side* Ω^- *for* $l \ge \max\{k, n-k-1\}$

where $\mathcal{M}$ is any extension of $\mathcal{N}$ form M to Ω^+ or Ω^-.

Proof. By Theorem 1.10, Theorem 1.13, Lemma 3.11 , we see that

$$(9) \Rightarrow (5) \Leftrightarrow (1)$$
$$(10) \Rightarrow (6) \Leftrightarrow (2)$$
$$(11) \Rightarrow (7) \Leftrightarrow (3)$$
$$(12) \Rightarrow (8) \Leftrightarrow (4)$$
$$(4) \Leftrightarrow (3) \Rightarrow (2) \Leftrightarrow (1)$$

Thus we only need to show (1) implies $(f\text{-}\mathcal{M})^k_b$ holds. It follows from that fact that

$$\|\Lambda u^0\|^2_b \lesssim Q_b(u,u)$$

holds for all $u \in C^\infty_c(U \cap M)^k$. We get the conclusion.

□

Chapter 5

Property $(f\text{-}\mathcal{M}\text{-}P)^k$ in some class of domains

5.1 Domain satisfies $Z(k)$ condition

In this section, we consider domain satisfying $Z(k)$ condition at the boundary point. This class of domains is probably the simplest of non-pseudoconvex domains.

Theorem 5.1. *Let Ω be a domain of $\mathbb{C}^n$ then Ω satisfies $Z(k)$ condition at $z_0 \in b\Omega$ if and only if*

$$|||u|||^2_{1/2} \lesssim Q(u,u)$$

holds for any $u \in C_c^\infty(U \cap \bar{\Omega})^k \cap Dom(\bar{\partial}^*)$, where U is a neighborhood of z_0

This is classical result of non-pseudoconvex domain. Theorem 5.1 can be found in [Hö65], [FK72],... In this section, we give the new way to get $\frac{1}{2}$-subelliptic estimates by construction the family weight functions in the Property $(f\text{-}\mathcal{M}\text{-}P)^k$ when $\mathcal{M} = 1$.

Proof. We assume Ω satisfies $Z(k)$ condition at $z_0 \in b\Omega$. Then Ω is strongly k-pseudoconvex or strongly k-pseudoconcave. There is a number $q_o \neq k$ and neighborhood U of z_0 such that

$$\sum_{|K|=k-1}{}' \sum_{ij=1}^{n-1} r_{ij} u_{iK} \bar{u}_{jK} - \sum_{j=1}^{q_o} r_{jj} |u|^2 \gtrsim |u|^2 \text{ on } U \cap \bar{\Omega} \tag{5.1}$$

for any $u \in C_0^\infty(U \cap \bar{\Omega})^k \cap \mathrm{Dom}(\bar{\partial}^*)$. So that we only need to show Ω satisfies Property $(f\text{-}1\text{-}P)^k$ at z_0 with $f(\delta^{-1}) = \delta^{-1/2}$.

We define $\Phi^\delta = -\frac{r}{\delta}$. For $z \in S_\delta$, we see that Φ^δ is absolutely bounded and

$$\begin{aligned} H^k_{q_o}(\Phi^\delta, u^\tau) &\gtrsim \delta^{-1} \sideset{}{'}\sum_{|K|=k-1} \sum_{ij=1}^{n-1} r_{ij} u^\tau_{iK} \bar{u}^\tau_{jK} - \sum_{j=1}^{q_o} r_{jj} |u^\tau|^2 \\ &\gtrsim \delta^{-1} |u^\tau|^2 \end{aligned} \tag{5.2}$$

Moreover, $L_j(\Phi^\delta) = 0$. Then the family $\{\Phi^\delta\}_{\delta > 0}$ satisfies Property $(f\text{-}1\text{-}P)^k$. □

Corollary 5.2. *Let M be a smooth hypersurface in $\mathbb{C}^n$. Assume that M satisfies $Z(k)$ and $Z(n-1-k)$ condition at z_0. Then there is a neigborhood U of z_0 such that*

$$\|u\|^2_{b,1/2} \lesssim Q_b(u,u)$$

for any $u \in C_c^\infty(U \cap M)^k$.

The proof of Corollary 5.2 follows by Theorem 5.1 and Corrolary 4.13.

5.2 q-decoupled-pseudoconvex/concave domain

Let $\Omega \subset \mathbb{C}^n$ be defined in a neighborhood of z_0 by

$$r = 2\mathrm{Re} z_n - h_1(z_1, \dots, z_{q_1+1}) + h_2(z_{q_2}, \dots, z_{n-1}) < 0 \tag{5.3}$$

where $1 \le q_1 + 1 < q_2 \le n-1$ and h_l's with $l = 1, 2$ are the real functions satisfying $\partial\bar{\partial} h_l$ be semipositive.

Definition 5.3. Ω is said to be q_2-decoupled-pseudoconvex (resp. q_1-decoupled-pseudoconcave) at z_0 if there are functions P_j such that $h_2(z_{q_2}, \dots, z_{n-1}) = \sum_{j=q_2}^{n-1} P_j(z_j)$ (resp. $h_1(z_1, \dots, z_{q_1+1}) = \sum_{j=1}^{q_1+1} P_j(z_j)$)

Remark 5.4. Since $\partial\bar{\partial} h_j \ge 0$ then $\frac{\partial^2 P_j}{\partial z_j \partial \bar{z}_j} \ge 0$.

Proposition 5.5. *The domain Ω, defined by (5.3), is q_2-pseudoconvex and q_1-pseudoconcave at z_0.*

Proof. We consider the basis of vector fields

$$L_i = \frac{\partial}{\partial z_j} - r_{z_j}\frac{\partial}{\partial z_n}, j = 1...n-1 \text{ and } L_n = \frac{\partial}{\partial z_n}.$$

Let $\omega_1, ..., \omega_n = \partial r$ be the dual (1,0) forms of these vector fields. We may choose the Hermitian metric in which $\omega_1, ..., \omega_n$ are orthonormal. Then

$$\partial\bar{\partial} r = -\sum_{ij=1}^{q_1+1}(h_1)_{ij}\omega_i \wedge \bar{\omega}_j + \sum_{ij=q_2}^{n-1}(h_2)_{ij}\omega_i \wedge \bar{\omega}_j$$

where $(h_k)_{ij} = \frac{\partial^2 h_k}{\partial z_i \partial \bar{z}_j}$ for $k = 1, 2$. Hence

$$H^k_{q_1+1}(r, u^\tau) = \sum_{j=1}^{q_1+1}(h_1)_{jj}|u^\tau|^2 - \sum_{|K|=k-1}{}' \sum_{ij=1}^{q_1+1}(h_1)_{ij}u^\tau_{iK}\bar{u}^\tau_{jK} + \sum_{|K|=k-1}{}' \sum_{ij=q_2}^{n-1}(h_2)_{ij}u^\tau_{iK}\bar{u}^\tau_{jK} \quad (5.4)$$

for any k-form u. If $H^k_{q_1+1}(r, u^\tau) \geq 0$, then by Definition 1.2, Ω is q_2-pseudoconvex and q_1-pseudoconcave.

Let F^k be the operator defined as in the proof of Theorem 4.12, then

$$\sum_{j=1}^{q_1+1}(h_1)_{jj}|u^\tau|^2 = \sum_{|K|=k-1}{}' \sum_{ij=1}^{q_1+1}(h_1)_{ij}u^\tau_{iK}\bar{v}_{jK} + \sum_{|K'|=n-k-2}{}' \sum_{ij=1}^{q_1+1}(h_1)_{ij}(F^k u^\tau)_{iK'}\overline{(F_j u^\tau)}_{jK'} \quad (5.5)$$

holds for any k, $1 \leq k \leq n-2$. In fact, the left hand side of (5.5) can be rewritten as following

$$\begin{aligned} =&\sum_{j=1}^{q_1+1}(h_1)_{jj}\Big(\sum_{|K|=k-1}{}' |u^\tau_{jK}|^2 + \sum_{|K'|=n-k-2}{}' |(F^k u^\tau)_{jK'}|^2\Big) \\ &+ \sum_{i,j=1;i\neq j}^{q_1+1}(h_1)_{ij}\Big(\sum_{|K|=k-1}{}' v_{iK}\bar{u}^\tau_{jK} + \sum_{|K'|=n-k-2}{}' (F^k u^\tau)_{iK'}\overline{(F^k u^\tau)}_{jK'}\Big). \end{aligned} \quad (5.6)$$

It is easily to check that the term in the first line in (5.6) equals to the right hand side of (5.5) and the term in second line equals to 0.

Therefore,

$$H^k_{q_1+1}(r, u^\tau) = \sum_{|K'|=n-k-2}{}' \sum_{ij}(h_1)_{ij}(F^k u^\tau)_{iK'}\overline{(F_j u^\tau)}_{jK'} + \sum_{|K|=k-1}{}' \sum_{ij}(h_2)_{ij}u^\tau_{iK}\bar{u}^\tau_{jK} \quad (5.7)$$

is semipositive on $U \cap \bar{\Omega}$ for any k.

□

Remark 5.6. If Ω is q_2-decoupled-pseudoconvex then

$$H^k_{q_1+1}(r, u^\tau) \gtrsim \sum_{j=k}^{n-1} \frac{\partial^2 P_j}{\partial z_j \partial \bar{z}_j} \sum'_{|K|=k-1} |u^\tau_{jK}|^2 \geq 0 \text{ on } U \cap \bar{\Omega} \tag{5.8}$$

for all $k \geq q_2$. Similarly, if Ω is q_1-decoupled-pseudoconcave then

$$H^k_{q_1+1}(r, u^\tau) \gtrsim \sum_{j=1}^{k+1} \frac{\partial^2 P_j}{\partial z_j \partial \bar{z}_j} \big(|u^\tau|^2 - \sum'_{|K|=k-1} |u^\tau_{jK}|^2\big) \geq 0 \text{ on } U \cap \bar{\Omega} \tag{5.9}$$

for all $k \leq q_1$.

Theorem 5.7. *Let Ω be q_2-decoupled-pseudoconvexity (resp. q_1-decoupled pseudoconvity) at z_0. Further suppose that for each j there is a invertible function F_j with $\frac{F_j(|t|)}{|t|^2}$ increasing for any t near 0 such that either*

$$\frac{\partial^2 P_j}{\partial z_j \partial \bar{z}_j}(z_j) \gtrsim \frac{F_j(|x_j|)}{x_j^2} \quad \textit{or} \quad \frac{\partial^2 P_j}{\partial z_j \partial \bar{z}_j}(z_j) \gtrsim \frac{F_j(|y_j|)}{y_j^2} \tag{5.10}$$

By reordering, we may assume to be increasing ...$F_j \lesssim F_{j+1}$... (resp. decreasing ...$F_j \gtrsim F_{j+1}$...). Then $(f\text{-}1)^k$ estimate holds in degree $k \geq q_2$ with $f(\delta^{-1}) = (F^*_k(\delta))^{-1}$ (resp. $k \leq q_1$ with $f(\delta^{-1}) = (F^*_{k+1}(\delta))^{-1}$) where F^*_j is inverse function of F_j .

Example 5.1. If $P_j(z_j) = |z_j|^{2m_j}$ or $|x_j|^{2m_j}$ then we get ϵ-subelliptic estimate with $\epsilon = \frac{1}{2\max\{m_j\}}$. If $P_j(z_j) = \exp(-\frac{1}{|z_j|^{m_j}})$ or $\exp(-\frac{1}{|x_j|^{m_j}})$ then we get $(f\text{-}1)^k$ estimate with $f(t) = (\log t)^{\frac{1}{\max\{m_j\}}}$.

Example 5.2. For indices (q_o, q) with $q_o < q \leq \frac{n-1}{2}$, let

$$r = 2\text{Re}z_n - h(z_1, ..., z_{q_o}) + \sum_{j=q}^{n-1} P_j(z_j).$$

where $\partial\bar{\partial}h \geq 0$ and $P_j(z_j)$ is defined in Example 5.1. Then we get $(f\text{-}1)^k$ estimate at z_0 for domain $\Omega^+ = \{+r \leq 0\}$ (resp. $\Omega^- = \{-r \leq 0\}$) for any degree $k \geq q$ (resp. $k \leq n - q - 1$) of forms. By Theorem 1.13 , $(f\text{-}1)^k$ estimate for system $(\bar{\partial}_b, \bar{\partial}^*_b)$ on $M = \{r = 0\}$ holds for degree k between q and $n - q - 1$.

Proof of Theorem 5.7. We may assume that $\frac{\partial^2 P_j}{\partial z_j \partial \bar{z}_j}(z_j) \gtrsim \frac{F_j(|x_j|)}{x_j^2}$ for each j because if $\frac{\partial^2 P_j}{\partial z_j \partial \bar{z}_j}(z_j) \gtrsim \frac{F_j(|y_j|)}{y_j^2}$ we change coordinate by $z_j := iz_j$. Let C be the postive constant such that $C\frac{\partial^2 P_j}{\partial z_j \partial \bar{z}_j}(z_j) \geq \frac{F_j(|x_j|)}{x_j^2}$.

The case q-decoupled-pseudoconvexity : For each degree k of forms ($k \geq q_2$), we define the family of weights by

$$\Phi_k^\delta = C\frac{r}{\delta} - 4\sum_{j=k}^{n-1} \exp(-\frac{x_j^2}{4F_k^*(\delta)^2}) \tag{5.11}$$

then Φ_k^δ's are absolutely bounded on S_δ. Computation the Levi form of Φ_k^δ shows that

$$\partial\bar{\partial}\Phi_k^\delta = C\frac{\partial\bar{\partial}r}{\delta} + \frac{1}{2F_k^*(\delta)^2}\sum_{j=k}^{n-1}\Big(1 - \frac{x_j^2}{2F_k^*(\delta)^2}\Big)\exp(-\frac{x_j^2}{4F_k^*(\delta)^2})\omega_i \wedge \bar{\omega}_i.$$

Then

$$\begin{aligned} H_{q_1+1}^k(\Phi_k^\delta, u^\tau) =& \frac{C}{\delta}H_{q_1+1}^k(r, u^\tau) \\ &+ \frac{1}{2F_k^*(\delta)^2}\sum_{j=k}^{n-1}\Big(1 - \frac{x_j^2}{2F_k^*(\delta)^2}\Big)\exp(-\frac{x_j^2}{4F_k^*(\delta)^2})\sum_{|K|=k-1}{}' |u_{jK}^\tau|^2. \end{aligned} \tag{5.12}$$

Combining with (5.8). (5.10) and (5.12), we obtain

$$\begin{aligned} H_{q_1+1}^k(\Phi_k^\delta, u^\tau) \geq& \sum_{j=k}^{n-1}\Big(\frac{1}{\delta}\frac{F_j(|x_j|)}{x_j^2} + \frac{1}{2F_k^*(\delta)^2}\Big(1 - \frac{x_j^2}{2F_k^*(\delta)^2}\Big)\exp(-\frac{x_j^2}{4F_k^*(\delta)^2})\Big)\sum_{|K|=k-1}{}' |u_{jK}^\tau|^2 \\ =& \sum_{j=k}^{n-1}(A_j + B_j)\sum_{|K|=k-1}{}' |u_{jK}^\tau|^2 \end{aligned} \tag{5.13}$$

where

$$A_j = \frac{1}{\delta}\frac{F(|x_j|)}{x_j^2}; B_j = \frac{1}{2F_k^*(\delta)^2}\Big(1 - \frac{x_j^2}{2F_k^*(\delta)^2}\Big)\exp(-\frac{x_j^2}{4F_k^*(\delta)^2}).$$

Notice that $A_j \geq 0$ for any j. For each j ($k \leq j \leq n-1$), we consider two cases of $|x_j|$:

Case 1. If $|x_j| \leq F_k^*(\delta)$, we have

$$B_j \geq \frac{1}{4F_k^*(\delta)^2} e^{-1/4} \geq cF_k^*(\delta)^{-2};$$

hence

$$A_j + B_j \gtrsim F_k^*(\delta)^{-2}.$$

Case 2. Otherwise. we assume $|x_j| \geq F_k^*(\delta)$. Using our assumption $\frac{F(|x_j|)}{x_j^2}$ increasing, it follows

$$A_j = \frac{1}{\delta}\frac{F_j(|x_j|)}{x_j^2} \geq \frac{1}{\delta}\frac{F_k(F_k^*(\delta))}{F_k^*(\delta)^2} = \frac{1}{\delta}\frac{\delta}{F_k^*(\delta)^2} = F_k^*(\delta)^{-2}.$$

But in this case, B_j can get negative values; however, by using the fact $\min_{t\geq\frac{1}{2}}\Big\{(1-t)e^{-t/2}\Big\} = -2e^{-3/2}$ for $t = \frac{x_j^2}{2F_k^*(\delta)^2} \geq \frac{1}{2}$ we have

$$B_j \geq -e^{-3/2}F_k^*(\delta)^{-2}.$$

This imply

$$A_j + B_j \gtrsim F_k^*(\delta)^{-2}.$$

Therefore, continuous our estimate in (5.13), we obtain

$$\begin{aligned} H^k_{q_1+1}(\Phi_k^\delta, u^\tau) &\gtrsim \sum_{j=k}^{n-1} F_k^*(\delta)^{-2} {\sum_{|K|=k-1}}' |u^\tau_{jK}|^2 \\ &\gtrsim f(\delta)^2 |u^\tau|^2 \end{aligned} \tag{5.14}$$

here the last inequality follows by $\sum\limits_{j=k}^{n-1}{\sum\limits_{|K|=k-1}}' |u^\tau_{jK}|^2 \geq {\sum\limits_{|J|=k}}' |u^\tau_J|^2 = |u^\tau|^2$. Moreover, we see that $(\Phi_k^\delta)_j = \frac{r_j}{\delta} = 0$ for any $j \leq q_1+1$. Hence $\sum_{j=1}^{q_1+1} |(\Phi_k^\delta)_j u^\tau|^2 = 0$. Thus the weights Φ_k^δ satisfy Property $(f\text{-}1\text{-}P)^k$. Applying Theorem 1.10, we get

$$\|f(\Lambda)u\|^2 \lesssim Q(u,u)$$

for any $u \in C_c^\infty(\bar\Omega \cap U)^k \cap \mathrm{Dom}(\bar\partial^*)$ with $f(\delta^{-1}) = F_k^*(\delta)^{-1}$.

The case q-decoupled-pseudoconcavity : For each $k \leq q_1$, we define the family of weights by

$$\Phi_k^\delta = C\frac{r}{\delta} + 4\sum_{j=1}^{k+1} \exp(-\frac{x_j^2}{4F_{k+1}^*(\delta)^2}), \tag{5.15}$$

In the same way to above argument , we get Φ_k^δ's are absolutely bounded on S_δ and

$$\begin{aligned} H_{q_1+1}^k(\Phi_k^\delta, u^\tau) &\gtrsim F_{k+1}^*(\delta)^{-2}\sum_{j=1}^{k+1}\Big(|u^\tau|^2 - \sideset{}{'}\sum_{|K|=k-1} |u_{jK}^\tau|^2\Big) \\ &\gtrsim F_{k+1}^*(\delta)^{-2}\Big((k+1)|u^\tau|^2 - \sum_{j=1}^{k+1}\sideset{}{'}\sum_{|K|=k-1} |u_{jK}^\tau|^2\Big) \\ &\gtrsim F_{k+1}^*(\delta)^{-2}|u^\tau|^2. \end{aligned} \tag{5.16}$$

Now we need to show

$$H_{q_1+1}^k(\Phi_k^\delta, u^\tau) \gtrsim \Big(\sum_{j=1}^{q_1+1} |(\Phi_k^\delta)_j|^2\Big)|u^\tau|^2. \tag{5.17}$$

We note that $(\Phi_k^\delta)_j = 0$ for $k+2 \leq j \leq q_1+1$ and

$$|(\Phi_k^\delta)_j|^2 = 8F_j^*(\delta)^{-2}\Big(\frac{x_j^2}{2F_j^*(\delta)^2}\exp(-\frac{x_j^2}{2F_j^*(\delta)^2})\Big) \lesssim F_j^*(\delta)^{-2} \tag{5.18}$$

for $1 \leq j \leq k+1$. So that, from (5.16) and (5.18) we get (5.17).

□

5.3 Subelliptic estimates for regular coordinate domains

We state precise subelliptic estimates for the $\bar{\partial}$-Neumann problem over the class of *regular coordinate domains* of $\mathbb{C}^n$.

We consider a domain Ω defined by

$$2\text{Re}z_n+\sum_{j=1}^{n-1}|f_j(z)|^2<0,\ \text{with } f_j \text{ holomorphic, } f_j=f_j(z_1,...,z_j) \text{ and } \partial_{z_j}^{m_j}f_j\neq 0. \tag{5.19}$$

This is called a *regular coordinate domain*; the inequality which defines Ω is denoted by $r<0$. We discuss *subelliptic* estimates for the $\bar\partial$-Neumann problem on Ω:

$$|||u|||^2_\epsilon \underset{\sim}{<} ||\bar\partial u||^2_0+||\bar\partial^* u||^2_0+||u||^2_0 \quad \text{for any } u\in C^\infty_c(U\cap\bar\Omega)^1\cap \text{Dom}(\bar\partial^*). \tag{5.20}$$

Our problem is to find the *optimal* ϵ. There are several relevant numbers related to Ω:

- $m=m_1\cdot...\cdot m_{n-1}$ the *multiplicity.*
- D the *D'Angelo type* defined as the maximal order of contact of a complex curve with $\partial\Omega$. Note here that since $\sum_j|f_j|^2 \underset{\sim}{>} |z|^{2m}$ then necessarily $D\leq 2m$.
- ϵ the (optimal) index of subelliptic estimates. It satisfies

$$\epsilon\leq\frac{1}{D}\ \text{Catlin 1983 [C83]}, \qquad \epsilon\overset{?}{\geq}\frac{1}{2m}\ \text{D'Angelo conjecture 1992 [?]}.$$

We define a new number γ. For this, we write

$$f_j=g_j(z_1,...,z_j)+z_j^{m_j}+O(z_j^{m_j+1}) \quad \text{for } g_j=O(z_1^{\lambda^1_j},...,z_j^{\lambda^{j-1}_j}).$$

Let j_o, $j_o\geq 2$, $j\geq 2$, be the first index with the property that g_j is independent of z_j for any $j\leq j_o-1$ and write l^i_j for the minimum between λ^i_j and m_i (resp. $m_i-\eta$ for any $\eta>0$) when $i\leq j_o-1$ and $j\leq j_o-1$ (resp. $j\geq j_o$), and otherwise put $l^i_j=1$. Define

$$\gamma_j=\min_{i\leq j-1}\frac{l^i_j}{m_j}\gamma_i, \qquad \gamma=\min_j\gamma_j. \tag{5.21}$$

Note that $\frac{1}{2m}\leq\frac{\gamma}{2}\leq\frac{1}{D}$. Here is our main result (which is also presented in [?] with $\frac{\gamma}{2}$ replaced by $\frac{1}{2m}$).

Theorem 5.8. *Let Ω be a regular coordinate domain and let γ be the number defined by* (5.21)*; then we have ϵ-subelliptic estimates for $\epsilon = \frac{\gamma}{2}$.*

Example 5.3. For the domain of $\mathbb{C}^n$ defined by

$$2\text{Re}z_n + |z_1^{m_1}|^2 + \sum_{j=2}^{n-1} |z_j^{m_j} - z_{j-1}^{l_j}|^2 < 0, \quad l_j \leq m_{j-1} \leq m_j,$$

the number γ is given by $\gamma = \frac{l_2 \cdot \ldots \cdot l_{n-1}}{m_1 \cdot \ldots \cdot m_{n-1}}$. We claim that $\epsilon = \frac{\gamma}{2} = \frac{1}{D}$. The first equality follows from Theorem 5.8. As for the second, we can easily find the *critical curve* Γ with the maximal contact with $\partial\Omega$; this is parameterized over $\tau \in \Delta$ by

$$\tau \mapsto (\tau^{\frac{1}{m_1\gamma}}, \tau^{\frac{l_2}{m_2 m_1 \gamma}}, ..., \tau, 0).$$

In fact, we have $r|_\Gamma = |\tau^{\frac{1}{\gamma}}|^2 + |\not\tau^{\frac{l_2}{m_1\gamma}} - \not\tau^{\frac{l_2}{m_1\gamma}}| + ... = |\tau^{\frac{1}{\gamma}}|^2$ and therefore $D \geq \frac{2}{\gamma}$. On the other hand $\epsilon = \frac{\gamma}{2} < \frac{1}{D}$ by Catlin 1983 [C83].

Example 5.4. For the domain in $\mathbb{C}^3$ defined by

$$2\text{Re}z_3 + |z_1^4|^2 + |z_2^6 + \sum_{j=0}^{5} c_j z_2^j z_1^{\alpha_j} + O(z_2^7)|^2 < 0,$$

with $\alpha_j \geq 3$ for any j, we have $\gamma_1 = \frac{1}{4}$, $\gamma_2 = \frac{3}{6 \cdot 4}$ and $\gamma = \gamma_2$.

Example 5.5. Let us consider in $\mathbb{C}^4$ the domain defined by

$$2\text{Re}z_4 + |z_1^6|^2 + |z_2^4 + z_1^3|^2 + |z_3^4 + z_3 z_1^a + z_3 z_2^b|^2 < 0.$$

Here $\gamma_1 = \frac{1}{6}$, $\gamma_2 = \frac{3}{6 \cdot 4}$ and $\gamma_3 = \min(\frac{3 \cdot b}{6 \cdot 4 \cdot 4}, \frac{a}{6 \cdot 4}, \frac{3}{6 \cdot 4})$ and $\gamma = \gamma_3$; in particular, if $a \geq 3$, and $b \geq 4$, we have $\gamma = \frac{3}{6 \cdot 4}$.

Proof of Theorem 5.8.

In order to establish (5.20) it suffices to find a family of bounded weights $\{\phi^\delta\}$ whose Levi form satisfies over the strip $S_\delta := \{z \in \Omega : -r(z) < \delta\}$, the estimate $\partial\bar{\partial}\phi^\delta(z)(u, \bar{u}) \gtrsim \delta^{-\gamma}|u|^2$ for $u \in \mathbb{C}^n$. We choose $\alpha \geq 1$, put $\alpha_j = \alpha(m_{j+1}...m_n)$ and choose a smooth cut off function χ with $\chi \equiv 1$ in $[0, 1]$ and $\chi \equiv 0$ in $[2, +\infty)$. We define

$$\begin{aligned}\phi^\delta = -\log(\frac{-r+\delta}{\delta}) + \sum_{j=j_o}^{n} \sum_{h=1}^{m_j-1} \frac{1}{|\log *|} \log\left(|\partial_{z_j}^h f_j|^2 + \frac{\delta^{(m_j-h)\gamma_j}}{|\log\delta|^{(m_j-h)\alpha_j}}\right) \\ + c\sum_{j=1}^{n} \chi(\frac{|z_j|^2}{\delta^{\gamma_j}}) \log\left(\frac{|z_j|^2 + \delta^{\gamma_j}}{\delta^{\gamma_j}}\right),\end{aligned} \tag{5.22}$$

where $* = \frac{\delta^{\gamma_j(m_j-h)}}{|\log\delta|^{(m_j-h)\alpha_j}}$; notice that $\log * \sim \log\delta$. The weights ϕ^δ that we have defined are bounded in the strip S_δ. When calculating the Levi form, we observe that $\partial\bar\partial r(u,\bar u) = \sum_j |\partial f_j \cdot u|^2$ and $\partial\bar\partial|\partial^h_{z_j} f_j|^2(u,\bar u) = |\partial\partial^h_{z_j} f_j \cdot u|^2$ and finally $\partial\bar\partial|z_j|^2(u,\bar u) = |u_j|^2$. Thus, we have got a decomposition

$$\partial\bar\partial\phi^\delta(u,\bar u) = \sum_{j=1}^n A_j + \sum_{j=j_o}^n \sum_{h=1}^{m_j-1} B^h_j + \sum_{j=1}^n C_j, \tag{5.23}$$

with the estimates

$$\begin{cases} A_j \gtrsim \delta^{-1}|\partial f_j \cdot u|^2 & \text{for any } z \in S_\delta, \\ B^h_j \gtrsim \delta^{-(m_j-h)\gamma_j}\frac{|\log\delta|^{(m_j-h)\gamma_j}}{|\log *|}|\partial\partial^h_{z_j} f_j \cdot u|^2 & \text{if } |\partial^h_{z_j} f_j|^2 < \frac{\delta^{(m_j-h)\gamma_j}}{|\log\delta|^{(m_j-h)\alpha_j}}, \\ C_j \gtrsim c\delta^{-\gamma_j}|u_j|^2 & \text{if } |z_j|^2 < \delta^{\gamma_j}. \end{cases}$$

We note that the A_j's and B^h_j's are positive for any z but, instead, the C_j's can take negative values when $|z_j| > \delta^{\gamma_j}$; however, $|C_j| \lesssim c\delta^{-\gamma_j}|u_j|^2$ and thus the C_j's are controlled by the A_j's and B^h_j's. We define

$$D_j = A_j + C_j \quad \text{for } j \le j_o - 1, \qquad D_j = A_j + \sum_{h \le m_j-1} B^h_j + C_j \quad \text{for } j \ge j_o.$$

We wish to start by proving that, when $j \le j_o - 1$, then

$$\sum_{i \le j} D_i \gtrsim \sum_{i\le j} \delta^{-s_i\gamma_i}|z_i|^{2(s_i-1)}|u_i|^2 \text{ for any } s_i \le m_i. \tag{5.24}$$

We use induction and show how to pass from step $j-1$ to step j (the step $j=1$ being elementary). We fix our choice $s_i = l^i_j$ and remark that

$$\begin{aligned} A_j + \sum_{i\le j-1} D_i &\gtrsim \delta^{-1}\,|\partial f_j \cdot u|^2 + \sum_{i \le j-1}\delta^{-l^i_j\gamma_i}|z_i|^{2(l^i_j-1)}|u_i|^2 \\ &\gtrsim \sum_{i\le j-1}\delta^{-l^i_j\gamma_i}\left[|z_j|^{2m_j-1}|u_j|^2 - |z_i|^{2(l^i_j-1)}|u_i|^2\right] + \sum_{i\le j-1}\delta^{-l^i_j\gamma_i}|z_i|^{2(l^i_j-1)}|u_i|^2. \end{aligned} \tag{5.25}$$

This proves (5.24) for $s = m_j$. On the other hand, we have

$$C_j + \delta^{-m_j\gamma_j}|z_j|^{2(m_j-1)} \gtrsim (\delta^{-\gamma_j} + \delta^{-m_j\gamma_j}|z_j|^{2(m_j-1)})|u_j|^2. \tag{5.26}$$

This is clear for $|z_j|^2\leq\delta^{\gamma_j}$; otherwise, C_j gets negative but it is controlled by the second term in the left of (5.26) for small c. By combining (5.25) with (5.26) we get (5.24) for $s=m_j$ and $s=1$ and thus also for any $1\leq s\leq m_j$. This concludes the proof of our claim (5.24).

We pass to treat the terms $D_j=A_j+\sum\limits_h B_j^h+C_j\quad$ for $j\geq j_o$. We begin by an auxiliary statement: if for some i with $j_o\leq i\leq j-1$ and for any $1\leq s_{i'}\leq m_{i'}$, we have

$$\sum_{i'\leq i}D_{i'}\gtrsim\delta^{-\gamma_i}|\log\delta|^{\alpha_i-1}|u_i|^2+\sum_{j_o\leq i'\leq i-1}\delta^{-\gamma_{i'}}|u_{i'}|^2+\sum_{i'\leq j_o-1}\delta^{-s_{i'}\gamma_{i'}}|z_{i'}|^{2(s_{i'}-1)}|u_{i'}|^2, \tag{5.27}$$

then we also have

$$\sum_{i\leq j}D_i\gtrsim\delta^{-\gamma_j}|\log\delta|^{\alpha_j-1}|u_j|^2. \tag{5.28}$$

We prove the implication from step $j-1$ to j. By choosing $s_{i'}=l_j^{i'}$ in (5.27), and observing that, if $i\leq j-2$, then $\delta^{-\gamma_i}\geq\delta^{-\gamma_{j-1}}|\log\delta|^a$ for any a, we get

$$\begin{aligned}A_j+\sum_{i\leq j-1}D_i&\gtrsim\sum_{i\leq j_o-1}\delta^{-\gamma_i l_j^i}\left(|\partial_{z_j}f_j|^2|u_j|^2-|z_i|^{2(l_j^i-1)}|u_i|^2\right)\\&\quad+\sum_{i\geq j_o}\delta^{-\gamma_{j-1}}|\log\delta|^{\alpha_{j-1}-1}\left(|\partial_{z_j}f_j|^2|u_j|^2-|u_i|^2\right)+\sum_{i\leq j-1}D_i\\&\gtrsim\delta^{-\gamma_{j-1}}|\log\delta|^{\alpha_{j-1}-1}|\partial_{z_j}f_j|^2|u_j|^2.\end{aligned} \tag{5.29}$$

If now $|\partial_{z_j}f_j|^2\geq\frac{\delta^{(m_j-1)\gamma_j}}{|\log\delta|^{(m_j-1)\alpha_j}}$, then (5.29) can be continued by

$$\begin{aligned}&\geq\delta^{-\gamma_{j-1}+(m_j-1)\gamma_j}|\log\delta|^{(\alpha_{j-1}-1)-(m_j-1)\alpha_j}|u_j|^2\\&\geq\delta^{-\gamma_j}|\log\delta|^{\alpha_j-1}|u_j|^2.\end{aligned}$$

If not, we pass to use B_j^1 instead of A_j. In this way we jump from $\partial_{z_j}^h f_j$ to $\partial_{z_j}^{h+1}f_j$ until we reach $B_j^{m_j-1}$; since $|z_j|^2=|\partial_{z_j}^{m_j-1}f_j|^2$ is smaller than $\frac{\delta^{\gamma_j}}{|\log\delta|^{\alpha_j}}$ (otherwise we would have used the former term $B_j^{m_j-2}$), then $B_j^{m_j-1}$ is bigger than the right side of (5.28). This concludes the proof of the auxiliary statement.

We show that, for any value of $|z_j|^2$

$$\sum_{i\leq j}D_i\gtrsim\delta^{-\gamma_j}|u_j|^2, \tag{5.30}$$

whereas, when $|z_j|^2 \geq \delta^{\gamma_j}$

$$\sum_{i\leq j} D_i \underset{\sim}{>} \begin{cases} \text{either} & \delta^{-\gamma_j}|\log\delta|^{\alpha_j-1}|u_j|^2 \\ \text{or} & \delta^{-\gamma_{j-1}}|z_j|^{2(m_j-1)}|u_j|^2. \end{cases} \tag{5.31}$$

For $j \leq j_o - 1$, the claim has already been proved in (5.24): the second alternative in (5.31) holds. If $|z_j|^2 \leq \delta^{\gamma_j}$, then $C_j \underset{\sim}{>} c\delta^{-\gamma_j}|u_j|^2$. Assume therefore $|z_j|^2 \geq \delta^{\gamma_j}$ and suppose (5.31) true up to step $j-1$; we prove that it also holds for j (which implies (5.30)). First, if in the inductive statement it is the first of (5.31) which is fulfilled at some step i with $j_o \leq i \leq j-1$, the first is also fulfilled at step j; this follows from the auxiliary statement. Otherwise, assume we have the second for any $i \leq j-1$; (we surely do have for any $i \leq j_o - 1$). Now, if for some $i \leq j_o - 1$, we have $|z_i|^{l^i_j} \geq |z_j|^{m_j-1}$, or, for some $j_o \leq i \leq j-1$, we have $|z_i| \geq |z_j|^{m_j-1}$, then, owing to $|z_j| \geq \delta^{\gamma_j}$, we have

$$\begin{cases} \delta^{-1}|z_i|^{2(m_i-1)} \geq \delta^{-\gamma_i l^i_j - \eta}|z_i|^{2(l^i_j-1)}, & i \leq j_o - 1, \\ \delta^{-\gamma_{i-1}}|z_i|^{2(m_i-1)} \geq \delta^{-\gamma_i-\eta}, & i \geq j_o. \end{cases} \tag{5.32}$$

To prove the second of (5.32), it suffices to notice that $\delta^{-\gamma_{i-1}}|z_i|^{2(m_i-1)} \geq \delta^{-\gamma_{i-1}+\gamma_j(m_j-1)(m_i-1)} \geq \delta^{-\gamma_i-\eta}$. As for the first, we notice that

$$\begin{aligned} \delta^{-1}|z_i|^{2m_i-1} &\geq \delta^{-1+(m_i-l^i_j)\frac{m_j-1}{l^i_j}\gamma_j}|z_i|^{2(l^i_j-1)} \\ &\geq \delta^{-l^i_j\gamma_i-(\frac{m_i}{l^i_j}-1)\gamma_j}|z_i|^{2(l^i_j-1)} \geq \delta^{-l^i_j-\eta}|z_i|^{2(l^i_j-1)}, \end{aligned}$$

(because $l^i_j \leq m_i - \eta$). This proves (5.32). By (5.32), the second of (5.31) is converted into the first in the inductive statement for $i \leq j-1$ (and thus also for j owing to the auxiliary statement). Thus the only critical case occurs when both the inequalities

$$\begin{cases} |z_i|^{l^i_j} \leq |z_j|^{m_j-1}, & i \leq j_o - 1, \\ |z_i| \leq |z_j|^{m_j-1}, & i \geq j_o, \end{cases} \tag{5.33}$$

are fulfilled. But we have in this situation

$$\begin{aligned} |\partial_{z_j} f_j|^2 &\geq |z_j|^{2(m_j-1)} - \frac{1}{2}\left(\sum_{i=1}^{j_o-1}|z_i|^{l^i_j-1} + \sum_{i=j_o}^{j-1}|z_i|^2\right) \\ &\geq |z_j|^{2(m_j-1)}, \end{aligned}$$

which implies $A_j + \sum_i D_i \underset{\sim}{>} \delta^{-\gamma_{j-1}}|z_j|^{2(m_j-1)}|u_j|^2$. This yields the second of (5.31). Thus induction works and brings us to step $j = n$. At this point we can disregard (5.31) (though it did a great job for the inductive argument): (5.30) for any $j \leq n$ yields the conclusion of the proof.

□

Chapter 6

Global regularity and local regularity

6.1 Compactness estimates and global regularity

In this section, we will discuss about compactness estimate implies global regularity.

It is well-known that (global) compactness estimates implies global regularity. The globally compactness estimate can be defined as : for every positive number M, the estimate

$$M\|u\|^2 \lesssim Q(u,u) + C_M\|u\|^2_{-1} \tag{6.1}$$

holds for any $u \in C^\infty(\Omega)^k \cap \mathrm{Dom}(\bar\partial^*)$.

The idea of the proof is very simple. By using elliptic regularization one sees that the global regularity for the $\bar\partial$-Neumann operator holds if

$$\|u\|_s \lesssim \|\Box u\|_s \tag{6.2}$$

for any $u \in C^\infty(\bar\Omega)^k \cap \mathrm{Dom}(\Box)$ and for any integer s. Moreover, since the operator $\Box$, it is non-characteristic with respect to the boundary. Hence

$$\|u\|^2_s \lesssim \|\Box u\|^2_{s-2} + ||\Lambda^{s-1}Du||^2 \tag{6.3}$$

where D is the differential operator of order 1 and Λ is the tangential differential operator of order s. By Lemma 3.10, the estimate (6.1) implies that

$$M\|D\Lambda^{-1}u\|^2 \lesssim Q(u,u) + C_M\|u\|^2_{-1}. \tag{6.4}$$

In fact, it follows by the non-characteristic with respect to the boundary of $\bar{L}_n$; the operator D can be understood as D_r or Λ.

We now estimate the last term of (6.3), we have

$$\begin{aligned} M\|\Lambda^{s-1}Du\|^2 \lesssim & M\|D\Lambda^{-1}\Lambda^s u\|^2 + M\|u\|^2_{s-1} \\ \lesssim & Q(\Lambda^s u, \Lambda^s u) + C_M\|u\|^2_{s-1} \\ \lesssim & (\Lambda^s \Box u, \Lambda^s u) + \|[\bar\partial, \Lambda^s]u\|^2 + \|[\bar\partial, \Lambda^s]u\|^2 \\ & + \|[\bar\partial^*, [\bar\partial, \Lambda^s]u\|^2 + \|[\bar\partial, [\bar\partial^*, \Lambda^s]u\|^2 + C_M\|u\|^2_{s-1} \\ \lesssim & \|\Lambda^s \Box u\|^2 + \|\Lambda^{s-1}Du\|^2 + \|\Lambda^{s-2}D^2u\|^2 + C_M\|u\|^2_{s-1} \\ \lesssim & \|\Box u\|^2_s + \|\Lambda^{s-1}Du\|^2 + C_M\|u\|^2_{s-1} \end{aligned} \tag{6.5}$$

where the second inequality follows by (6.4). Then the term $\|\Lambda^{s-1}Du\|^2$ can be absorbed by the left-hand side term when M is sufficiently large. By induction method, we obtain the estimate (6.2).

In the chapter 1, we have introduced the locally compactness estimate at boundary point by

$$M\|u\|^2 \lesssim Q(u,u) + C_M\|u\|^2_{-1} \tag{6.6}$$

for any $u \in C_c^\infty(U \cap \Omega)^k \cap \mathrm{Dom}(\bar\partial^*)$, where U is a neiborhood of a given boundary point z_0. The following lemma will obtain global estimate from local estimate.

Lemma 6.1. *Let Ω be a bounded domain. Assume the the estimates* (6.6) *holds at any boundary point. Then* (6.1) *also holds.*

Proof. Let $\{\zeta_j\}_{j=0}^N$ be a partition of unity such that $\zeta_0 \in C_c^\infty(\Omega)$ and each ζ_j, $1 \le j \le N$, is supported in coordinate patch U_i, $\zeta_j \in C_c^\infty(U_j)$, $\bar\Omega \subset \Omega \cup \big(\cup U_j\big)$ and

$$\sum_{j=0}^N \zeta_j^2 = 1 \qquad \text{on } \bar\Omega.$$

Let $u \in C^\infty(\bar{\Omega})^k \cap \mathrm{Dom}(\bar{\partial}^*)$, we need to show (6.1). For $j = 0$, Form the interior elliptic regularity of the $\Box$ (see (2.27)), we have

$$\|\zeta_0 u\|_1^2 \lesssim Q(\zeta_0 u, \zeta_0 u).$$

For every positive constant M, using the Cauchy-Schwarz inequlity

$$M\|\zeta_0 u\|^2 \lesssim \|\zeta_0 u\|_1^2 + C_M\|\zeta_0 u\|_{-1}^2$$

Hence

$$\begin{aligned} M\|\zeta_0 u\|^2 \lesssim & Q(\zeta_0 u, \zeta_0 u) + C_M\|\zeta_0 u\|_{-1}^2 \\ \lesssim & Q(u,u) + C_M\|u\|_{-1}^2. \end{aligned} \tag{6.7}$$

Similarly, for $j = 1, ..., N$, using hypothesis, we have

$$\begin{aligned} M\|\zeta_j u\|^2 \lesssim & Q(\zeta_j u, \zeta_j u) + C_M\|\zeta_j u\|_{-1}^2 \\ \lesssim & Q(u,u) + C_M\|u\|_{-1}^2. \end{aligned} \tag{6.8}$$

Summing up over j, the lemma is proved.

$\Box$

We conclude the results without the proof in the following theorem.

Theorem 6.2. *1. Let Ω be the smoothly bounded q-pseudoconvex domain at any boundary point in $\mathbb{C}^n$, $n \geq 2$. Assume that compactness estimates holds on $(0,k)$-forms with $q \leq k \leq n-1$ (resp. $p \leq k \leq q$) in a neighborhood of any boundary point . Then the $\bar{\partial}$-Neumann operator N_k on $(0,k)$-form is global regularity.*

2. Let Ω be the smoothly bounded annulus in $\mathbb{C}^n$, $n \geq 3$, defined by $\Omega = \Omega_1 \setminus \Omega_2$ where $\bar{\Omega}_2 \subset \Omega_1$ and Ω_1 is p-pseudoconvex and Ω_2 is $(n-q-1)$-pseudoconvex. Assume that compactness estimates holds on $(0,k)$-forms with $p \leq k \leq q$ in a neighborhood of any boundary point . Then the $\bar{\partial}$-Neumann operator N_k on $(0,k)$-form is global regularity.

3. Let M be the smoothly compact q-pseudoconvex hypersurface at any local poin in $\mathbb{C}^n$, $n \geq 3$. Assume that compactness estimates holds on $(0,k)$-forms with $q \leq k \leq n-1-q$ in a neighborhood of any point in M . Then the Green operator G_k ($G_k := \Box_b^{-1}$) on $(0,k)$-form is global regularity.

6.2 "Weak" compactness estimates and global regularity

The purpose of this section is to present that global regularity follows by an estimate which is weaker than compactness estimate in a bounded pseudoconvex domain. In fact, we will show the following theorem

Theorem 6.3. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. Assume that for any positive constant $\epsilon > 0$ there exists a defining function r^ϵ of Ω with $\sum\limits_k |r^\epsilon_{z_j}|^2 \cong 1$ on $b\Omega$ such that*

$$\sum_{|K|=k-1}{}' \left\| \sum_{ij}^n r^\epsilon_{z_i\bar{z}_j} r^\epsilon_{\bar{z}_i} \bar{u}_{jK} \right\|^2 \leq \epsilon \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 + C_\epsilon \|u\|^2_{-1} \tag{6.9}$$

for any $u \in C^\infty(\bar\Omega)^k \cap Dom(\bar\partial^*)$. Then the Bergman projection P_{k-1} is exactly (global) regular, that is,

$$\|P_{k-1}\alpha\|_s \lesssim \|\alpha\|_s$$

for $s \geq 0$ and all $\alpha \in H_s(\Omega)^k$.

Let U be a neighborhood of $b\Omega$. For any $\epsilon > 0$, we may assume that the defining function $r := r^\epsilon$ of Ω satisfies $\sum_{k=1}^n |r_{z_k}|^2 \neq 0$ on U. We define (1,0) vector fields as follows

$$N = \frac{1}{\sum_{k=1}^n |r_{z_k}|^2} \sum_{k=1}^n r_{\bar{z}_k} \frac{\partial}{\partial z_k}; \quad T = N - \bar{N} \quad \text{and} \quad L_j = \frac{\partial}{\partial z_j} - r_{z_j} N$$

for $j = 1, ..., n$. Notice that T and $L_j, 1 \leq j \leq n$ are tangential; $\bar{T} = -T$.

Firstly, we consider $u \in C^\infty_c(U \cap \bar\Omega)^k$. Using integration by part we get

$$\begin{aligned} \|L_j u\|^2 &\lesssim ([\bar{L}_j, L_j]u, u) + \|\bar{L}_j u\|^2 + \|u\|^2 \\ &\lesssim \|u\|_1 . \|u\| + \|\bar{L}_j u\|^2 \end{aligned} \tag{6.10}$$

Denote $\mathcal{S} = \text{span}\{L_1, ..., L_n, \frac{\partial}{\partial \bar{z}_1}, ..., \frac{\partial}{\partial \bar{z}_1}, Id\}$.

Proposition 6.4. *Let $s \in \mathbb{N}$; $S \in \mathcal{S}$ then we have*

$$\|Su\|^2_{s-1} \lesssim \|\bar\partial u\|^2_{s-1} + \|\bar\partial^* u\|^2_{s-1} + \|u\|_{s-1} . \|u\|_s \tag{6.11}$$

and

$$\|u\|^2_s \lesssim \|\bar{\partial}u\|^2_{s-1} + \|\bar{\partial}^*u\|^2_{s-1} + \|u\|_{s-1}.\|u\|_s + \|T^s u\|^2 \tag{6.12}$$

for any $u \in C^\infty_c(U \cap \bar{\Omega})^k \cap Dom(\bar{\partial}^*)$. *Moreover, for* $u \in C^\infty_c(\Omega)^k \cap Dom(\bar{\partial}^*)$, *we have*

$$\|u\|^2_{s+1} \lesssim \|\bar{\partial}u\|^2_s + \|\bar{\partial}^*u\|^2_s + \|u\|^2_{s-1} \tag{6.13}$$

Proof. The proof of (6.11) follows by the fact that

$$\|\frac{\partial u}{\partial \bar{z}_j}\|^2_{s-1} \lesssim \|\bar{\partial}u\|^2_{s-1} + \|\bar{\partial}^*u\|^2_{s-1} + \|u\|^2_{s-1} \tag{6.14}$$

for $j = 1, ..., n$; and

$$\|L_j u\|^2_{s-1} \lesssim \|\bar{\partial}u\|^2_{s-1} + \|\bar{\partial}^*u\|^2_{s-1} + \|u\|_{s-1}.\|u\|_s \tag{6.15}$$

for $j = 1, ..., n-1$. (The inequalities (6.14) and (6.15) may be found in [BS91, p.83]; or in [CS01, Section 6.2]. The proof of (6.16) follows by the induction in j that

$$\|T^j u\|^2_{s-j} \lesssim \|\bar{\partial}u\|^2_{s-1} + \|\bar{\partial}^*u\|^2_{s-1} + \|u\|_{s-1}\|u\|_s + \|T^{j+1}u\|^2_{s-j-1}. \tag{6.16}$$

The last one follows by the elliptic holds in the interior of domain.

□

Lemma 6.5. *(i)*

$$[\frac{\partial}{\partial \bar{z}_j}, T] = \theta_j T + S_{j0} \tag{6.17}$$

where $\theta_j = -\frac{1}{\sum_k |r_{z_k}|^2} \sum_i r_{z_i \bar{z}_j} r_{\bar{z}_i}$, *and* $S_{j0}, \in \mathcal{S}$

(ii)

$$[\frac{\partial}{\partial \bar{z}_j}, T^{2s}] = 2s\theta_j T^{s+1} + A^{2s-1}_j S_j \tag{6.18}$$

where A^{2s-1}_j *are the tangential differential operator of order* $2s-1$, *and* $S_{js} \in \mathcal{S}$

Proof. By differentiating and using formula $\frac{\partial}{\partial z_j} = r_{z_j} T + L_j + r_{z_j} \bar{L}_n$, we get the proof of lemma.

□

Proof of Theorem 6.3. For any $s \in \mathbb{N}$, we will show by inductions that P_l is exactly continuous on $H^s(\Omega)^l$ for $l = n-1, ..., k-1$.

Since N_n is elliptic, it follows that $P_{n-1} = I - \bar{\partial}^* N_n \bar{\partial}$ is exactly continuous on $H^s(\Omega)^{n-1}$. The induction hypothesis is that P_k is exactly continuous on $H^s(\Omega)^k$. We need to show that P_{k-1} is exactly continuous on $H^s(\Omega)^k$ for any s.

We firstly prove that

$$\|P_{k-1}\alpha\|_s^2 \lesssim \|\alpha\|_s^2 + \|P_{k-1}\alpha\|_{s-1}^2 \tag{6.19}$$

for any $\alpha \in C_c^\infty(U \cap \bar{\Omega})^{k-1}$. We have

$$(I - P_{k-1})\alpha = \bar{\partial}^* N_k \bar{\partial}\alpha \in C_c^\infty(U \cap \bar{\Omega})^{k-1} \cap \mathrm{Dom}(\bar{\partial}^*).$$

Using (6.16), for u replaced by $(I - P_{k-1})\alpha$, it follows

$$\begin{aligned} \|(I - P_{k-1})\alpha\|_s^2 \lesssim & \|\bar{\partial}\bar{\partial}^* N_k \bar{\partial}\alpha\|_{s-1}^2 + \|(I - P_{k-1})\alpha\|_{s-1}^2 + \|T^s(I - P_{k-1})\alpha\|^2 \\ \lesssim & \|\alpha\|_s^2 + \|P_{k-1}\alpha\|_{s-1}^2 + \|T^s P_{k-1}\alpha\|^2 \end{aligned} \tag{6.20}$$

Hence

$$\|P_{k-1}\alpha\|_s^2 \lesssim \|\alpha\|_s^2 + \|P_{k-1}\alpha\|_{s-1}^2 + \|T^s P_{k-1}\alpha\|^2 \tag{6.21}$$

Similarly, by using (6.11), we get

$$\|SP_{k-1}\alpha\|_{s-1}^2 \lesssim \|\alpha\|_s^2 + \|P_{k-1}\alpha\|_s \|P_{k-1}\alpha\|_{s-1} \tag{6.22}$$

for any $S \in \mathcal{S}$.

Now, we estimate the last term of (6.21). We have

$$\begin{aligned} \|T^s P_{k-1}\alpha\|^2 = & (T^s P_{k-1}\alpha, T^s\alpha) - (T^s P_{k-1}\alpha, T^s \bar{\partial}^* N_k \bar{\partial}\alpha) \\ = & (T^s P_{k-1}\alpha, T^s\alpha) - ((T^s)^* T^s \bar{\partial} P_{k-1}\alpha, N_k \bar{\partial}\alpha) \\ & - ([\bar{\partial}, (T^s)^* T^s] P_{k-1}\alpha, N_k \bar{\partial}\alpha) \end{aligned} \tag{6.23}$$

Since $\bar\partial P_{k-1}\alpha = 0$ and

$$\begin{aligned}[\bar\partial, (T^s)^*T^s] =&[\bar\partial, T^{2s}] + [\bar\partial, ((T^s)^* - T^s)T^s]\\ =&2s\bar\Theta T^{2s} + \sum_j A_j^{2s} S_j \bar I_j\\ =&2s(T^s)^*\bar\Theta T^{2s} + \sum_j (C_j^s)^* B_j^{s-1} S_j \bar I_j.\end{aligned} \tag{6.24}$$

where A_j^{2s-1}, B_j^{s-1} and C_j^s are the tangential operator of order $2s-1$; $s-1$ and s, respectively; and $\bar I_j : \mathcal{A}^{0,k-1} \to \mathcal{A}^{0,k}$ such that

$$\bar I_j u = \sum_{|J|=k-1} u_J d\bar z_j \wedge d\bar z_J$$

Continuing (6.23):

$$\begin{aligned}=&(T^s P_{k-1}\alpha, T^s\alpha) - (\bar\Theta 2sT^s P_{k-1}\alpha, T^s N_k\bar\partial\alpha) + \sum_j (B_j^{s-1} S_j \bar I_j P_{k-1}\alpha, C_j^s X_j^s N_k\bar\partial\alpha)\\ \le&\sqrt{\epsilon}\Big(\|P_{k-1}\alpha\|_s^2 + \|N_k\bar\partial\alpha\|_s^2\Big) + \frac{1}{\sqrt{\epsilon}}\Big(\|\alpha\|_s^2 + \sum_j \|S_j P_{k-1}\alpha\|_{s-1}^2 + \|\bar\Theta^* T^s N_k\bar\partial\alpha\|^2\Big).\end{aligned} \tag{6.25}$$

Using hypothesis of this theorem, we have

$$\begin{aligned}\|\bar\Theta^* T^s N_k\bar\partial\alpha\|^2 \lesssim&\epsilon(\|\bar\partial T^s N_k\bar\partial\alpha\|^2 + \|\bar\partial^* T^s N_k\bar\partial\alpha\|^2) + C_\epsilon\|T^s N_k\bar\partial\alpha\|_{-1}^2\\ \lesssim&\epsilon(\|[\bar\partial, T^s]N_k\bar\partial\alpha\|^2 + \|T^s\bar\partial^* N_k\bar\partial\alpha\|^2 + \|[\bar\partial^*, T^s]N_k\bar\partial\alpha\|^2)\\ &+ C_\epsilon\|T^s N_k\bar\partial\alpha\|_{-1}^2\\ \lesssim&\epsilon(\|N_k\bar\partial\alpha\|_s^2 + \|P_{k-1}\alpha\|_s^2 + \|\alpha\|_s^2) + C_\epsilon\|N_k\bar\partial\alpha\|_{s-1}^2\end{aligned} \tag{6.26}$$

Combining (6.21);(6.22);(6.25) and (6.26), we obtain

$$\|P_{k-1}\alpha\|_s^2 \lesssim \|\alpha\|_s^2 + \|P_{k-1}\alpha\|_{s-1}^2 + \sqrt{\epsilon}\|N_k\bar\partial\alpha\|_s^2 + C_\epsilon\|N_k\bar\partial\alpha\|_{s-1}^2 \tag{6.27}$$

We use the Boas-Straube formula in [BS] for $\bar\partial\alpha$, $\alpha \in H^s(\Omega)^{k-1}$ we get

$$N_k\bar\partial\alpha = P_k w_t N_{t,k}\bar\partial w_{-t}(I - P_{k-1})\alpha$$

where $N_{t,k}$ is the solution operator to the weighted $\bar\partial$-Neumann problem with weight $w_t(z) = \exp(-t|z|^2)$. Using Kohn's theory (see [Ko65]) implies that $N_{t,k}\bar\partial$ is also exactly continuous on $H^s(\Omega)^{k-1}$. Therefore

$$\|N_k\bar\partial\alpha\|_s \lesssim \|\alpha\|_s + \|P_{k-1}\alpha\|_s \tag{6.28}$$

for any $\alpha \in H^s(\Omega)^{k-1}$. From (6.27) and (6.28), we get (6.19).

Finally, for any $\alpha \in C^\infty(\bar{\Omega})^k$ we can write $\alpha = \chi_1\alpha + \chi_2\alpha$ where $\chi_1 \in C_c^\infty(U \cap \bar{\Omega})^k$ and $\chi_2 \in C_c^\infty(\Omega)^k$

We have

$$\begin{aligned}\|P_{k-1}\alpha\|_s^2 \lesssim & \|P_{k-1}(\chi_1\alpha)\|_s^2 + \|P_{k-1}(\chi_2\alpha)\|_s^2 \\ \lesssim & \|\chi_1\alpha\|_s^2 + \|P_{k-1}(\chi_1\alpha)\|_{s-1}^2 + \|\chi_2\alpha\|_s + \|P_{k-1}(\chi_2\alpha)\|_{s-1}^2 \\ \lesssim & \|\alpha\|_s^2 + \|P_{k-1}(\chi_1\alpha)\|_{s-1}^2 + \|P_{k-1}(\chi_2\alpha)\|_{s-1}^2\end{aligned} \tag{6.29}$$

for any $\alpha \in H^s(\Omega)^{k-1}$.

Since $\|\bar{\partial}^* N_k\bar{\partial}\alpha\|^2 = (\bar{\partial}^* N_k\bar{\partial}\alpha, \bar{\partial}^* N_k\bar{\partial}\alpha) = (\bar{\partial}\alpha, N_k\bar{\partial}\alpha) = (\alpha, \bar{\partial}^* N_k\bar{\partial}\alpha)$, hence $\|P_{k-1}\alpha\| \lesssim \|\alpha\|$. We assume by inductive method that $\|P_{k-1}\alpha\|_{s-1}^2 \lesssim \|\alpha\|_{s-1}^2$ for any $\alpha \in C^\infty(\bar{\Omega})^k$, then by (6.29) we get

$$\|P_{k-1}\alpha\|_s^2 \lesssim \|\alpha\|_s^2 \tag{6.30}$$

for any $\alpha \in C^\infty(\bar{\Omega})^k$. Using the method of elliptic regularization as in [KN65]; [FK72], we past from the a priori estimate (6.30) to get the conclusion of Theorem 6.3

□

6.3 Superlogarithmic estimates and local regularity

In [Ko02], Kohn proved that superlogarithmic estimate of system $(\bar{\partial}_b, \bar{\partial}_b^*)$ implies local regularity of the operator $\Box_b^{-1}$; and superlogarithmic estimate on positive microlocal of system $(\bar{\partial}_b, \bar{\partial}_b^*)$ implies local regularity of the $\bar{\partial}$-Neumann operator. The purpose of this section is to prove the local regularity of the $\bar{\partial}$-Neumann operator and Bergman projection by using Kohn's technique.

Theorem 6.6. *Assume that $(f\text{-}\mathcal{M})^k$ holds in a neighborhood of $z_0 \in \bar{\Omega}$ with either $f \gg \log$; $\mathcal{M} = 1$ or $f = \log$; $\mathcal{M} = \frac{1}{\epsilon}$ for any $\epsilon > 0$. Then if $u \in L_2^{0,k}$ such that $\Box u = \alpha$ with $\alpha \in L_2^{0,k}$ whose restriction to U in C^∞ then the restriction of u to U is also in C^∞. Moreover, if χ_0, χ_1 are the canonical* *cutoff functions with supp*$\chi_0 \subset$ *supp*$\chi_1 \subset U$ *and* $\chi_1 = 1$ *on supp*χ_0*, then for each integer* $s \geq 0$*, we have*

$$\|f(\Lambda)\chi_0 u\|_s^2 \lesssim \|f(\Lambda)^{-1}\chi_1\alpha\|_s^2 + \|u\|^2.$$

Proof. For each integer s, we interpolate a sequence of cutoff functions $\{\zeta_j\}_{j=0}^{2s}$ such that $\zeta_0 = \chi_1$; $\zeta_{2s} = \chi_0$; and $\zeta_j = 1$ on a neighborhood of suppζ_{j+1}. For $m = 1, ..., s$, we define the operator

$$R^m\varphi(x,r) = \int_{\mathbb{R}^{2n-1}} e^{ix\cdot\xi}(1+|\xi|^2)^{\frac{s\zeta_{2m-1}(x,r)}{2}}\tilde{\varphi}(\xi,r)d\xi.$$

Since the symbol of $(\Lambda^m - R^m)\zeta_{2m}$ is zero,

$$\begin{aligned} |||f(\Lambda)\zeta_{2m}u|||_m^2 \lesssim & \|f(\Lambda)R^m\zeta_{2m}u\|^2 + \|f(\Lambda)\zeta_{2m}u\|^2 \\ \lesssim & \|f(\Lambda)\zeta_{2m}R^m\zeta_{2(m-1)}u\|^2 \\ & + \|f(\Lambda)[R^m,\zeta_{2m}]\zeta_{2(m-1)}u\|^2 + \|f(\Lambda)\zeta_{2m}u\|^2 \\ \lesssim & \|f(\Lambda)\zeta_{2(m-1)}R^m\zeta_{2(m-1)}u\|^2 \\ & + \|[f(\Lambda),\zeta_{2m}]R^m\zeta_{2(m-1)}u\|^2 + \|f(\Lambda)[R^m,\zeta_{2m}]\zeta_{2(m-1)}u\|^2 + \|f(\Lambda)\zeta_{2m}u\|^2. \end{aligned} \tag{6.31}$$

From the calculus of pseudodifferential operator we conclude that the last line of (6.31) is dominated by $|||f(\Lambda)\zeta_{2(m-1)u}|||_{m-1}^2$. So that

$$|||f(\Lambda)\zeta_{2m}u|||_m^2 \lesssim \|f(\Lambda)\zeta_{2(m-1)}R^m\zeta_{2(m-1)}u\|^2 + ||f(\Lambda)\zeta_{2(m-1)}u||_{m-1}^2. \tag{6.32}$$

Similarly,

$$|||D_r\Lambda^{-1}f(\Lambda)\zeta_{2m}u|||_m^2 \lesssim \|D_r\Lambda^{-1}f(\Lambda)\zeta_{2(m-1)}R^m\zeta_{2(m-1)}u\|^2 + ||f(\Lambda)\zeta_{2(m-1)}u||_{m-1}^2 \tag{6.33}$$

Denote $A^m = \zeta_{2(m-1)}R^m\zeta_{2(m-1)}$, then $(A^m)^* = A^m$ and $A^m u \in C_c^\infty(U \cap \bar{\Omega})^k \cap \mathrm{Dom}(\bar{\partial}^*)$ if $u \in \mathcal{A}^{0,k} \cap \mathrm{Dom}(\bar{\partial}^*)$. Using hypothesis and Lemma 3.10, we obtain

$$|||f(\Lambda)\zeta_{2m}u|||_m^2 + |||D_r\Lambda^{-1}f(\Lambda)\zeta_{2m}u|||_m^2 \lesssim Q(A^m u, A^m u) + \|f(\Lambda)\zeta_{2(m-1)}u\|_{m-1}^2 \tag{6.34}$$

Next, we estimate $Q(A^m u, A^m u)$, we have

$$\begin{aligned} \|\bar{\partial}A^m u\|^2 = & (A^m\bar{\partial}u, \bar{\partial}A^m u) + ([\bar{\partial}, A^m]u, \bar{\partial}A^m u) \\ = & (f(\Lambda)^{-1}A^m\bar{\partial}^*\bar{\partial}u, f(\Lambda)A^m u) + ([\bar{\partial}, A^m]u, \bar{\partial}A^m u) + ([\bar{\partial}, A^m]^*u, \bar{\partial}^*A^m u) \\ & + (f(\Lambda)^{-1}[A^m, \bar{\partial}^*]^*, \bar{\partial}]u, f(\Lambda)A^m u). \end{aligned} \tag{6.35}$$

Similarly,

$$\begin{aligned}\|\bar\partial^* A^m u\|^2 =&(f(\Lambda)^{-1}A^m\bar\partial\bar\partial^* u, f(\Lambda)A^m u) + ([\bar\partial^*, A^m]u, \bar\partial^* A^m u) + ([\bar\partial^*, A^m]^* u, \bar\partial A^m u)\\ &+ (f(\Lambda)^{-1}[A^m, \bar\partial]^*, \bar\partial^*]u, f(\Lambda)A^m u).\end{aligned} \tag{6.36}$$

Take over sum of (6.35) and (6.36), and using the (lc-sc) inequality, we obtain

$$\begin{aligned}Q(A^m u, A^m u) \lesssim& \|f(\Lambda)^{-1}A^m\Box u\|^2 + error\\ \lesssim& |||f(\Lambda)^{-1}\zeta_{2(m-1)}\Box u|||^2_m + error\end{aligned} \tag{6.37}$$

where

$$\begin{aligned}error =&\|[\bar\partial, A^m]u\|^2 + \|[\bar\partial^*, A^m]u\|^2 + \|[\bar\partial, A^m]^* u\| + \|[\bar\partial^*, A^m]^* u\|\\ &+ \|f(\Lambda)^{-1}[A^m, \bar\partial]^*, \bar\partial^*]u\|^2 + \|f(\Lambda)^{-1}[A^m, \bar\partial^*]^*, \bar\partial]u\|^2 + \|A^m u\|^2.\end{aligned} \tag{6.38}$$

Now we estimate the error terms. First, we consider $\|[\bar\partial, A^m]u\|^2$, by the Jacobi identity

$$\begin{aligned}[\bar\partial, A^m] =&[\bar\partial, \zeta_{2(m-1)}R^m\zeta_{2(m-1)}]\\ =&[\bar\partial, \zeta_{2(m-1)}]R^m\zeta_{2(m-1)} + \zeta_{2(m-1)}[\bar\partial, R^m]\zeta_{2(m-1)} + \zeta_{2(m-1)}R^m[\bar\partial, \zeta_{2(m-1)}].\end{aligned} \tag{6.39}$$

Since the support of derivatives of $\zeta_{2(m-1)}$ is disjoint from the support of ζ_{2m-1} in the operator R^m. Let D is $\frac{\partial}{\partial x_j}$ or D_r, we have

$$[aD, R^m] = mD(\zeta_{2m-1})\log\Lambda R^m + [a, R^m]D.$$

Hence

$$\begin{aligned}\|[\bar\partial, A^m]u\|^2 \lesssim&\|\log\Lambda A^m u\|^2 + \|\log\Lambda\zeta_{2(m-1)}u\|^2_{m-1} + \|u\|^2\\ \lesssim&\epsilon\|f(\Lambda)A^m u\|^2 + \|f(\Lambda)\zeta_{2(m-1)}u\|^2_{m-1} + C_\epsilon\|u\|^2.\end{aligned} \tag{6.40}$$

here we apply Lemma 2.7. Arguing similarly we can bound all the term in (6.38) we obtain

$$\begin{aligned}error \lesssim&\epsilon\|f(\Lambda)A^m u\|^2 + \|f(\Lambda)\zeta_{2(m-1)}u\|^2_{m-1} + C_\epsilon\|u\|^2\\ \lesssim&\epsilon Q(A^m u, A^m u) + \|f(\Lambda)\zeta_{2(m-1)}u\|^2_{m-1} + C_\epsilon\|u\|^2\end{aligned} \tag{6.41}$$

where the last inequality follows by using again the $(f\text{-}1)^k$ estimate. Therefore

$$Q(A^m u, A^m u) \lesssim |||f(\Lambda)^{-1}\zeta_{2(m-1)}\Box u|||^2_m + \|f(\Lambda)\zeta_{2(m-1)}u\|^2_{m-1} + \|u\|^2. \tag{6.42}$$

Combining (6.34), (6.37) and (6.42), we get

$$|||f(\Lambda)\zeta_{2m}u|||_m^2 + |||D_r\Lambda^{-1}f(\Lambda)\zeta_{2m}u|||_m^2 \lesssim |||f(\Lambda)^{-1}\zeta_{2(m-1)}\Box u|||_m^2 + \|f(\Lambda)\zeta_{2(m-1)}u\|_{m-1}^2 + \|u\|^2. \quad (6.43)$$

for $m = 1, 2, ..., s$.

For $m = 2, ..., s$, since the operator $\Box$ is elliptic, it is non-characteristic with respect to the boundary, we have

$$\|f(\Lambda)\zeta_{2m}u\|_m^2 \lesssim \|f(\Lambda)\zeta_{2m}\Box u\|_{m-2}^2 + |||f(\Lambda)\zeta_{2m}u|||_m^2 + |||D_r\Lambda^{-1}f(\Lambda)\zeta_{2m}u|||_m^2$$

hence

$$\|f(\Lambda)\zeta_{2m}u\|_m^2 \lesssim \|f(\Lambda)^{-1}\zeta_{2(m-1)}\Box u\|_m^2 + \|f(\Lambda)\zeta_{2(m-1)}u\|_{m-1}^2 + \|u\|^2. \quad (6.44)$$

For $m = 1$, from (6.43), we get

$$\|f(\Lambda)\zeta_2 u\|_1^2 \lesssim \|f(\Lambda)^{-1}\zeta_0\Box u\|_1^2 + \|f(\Lambda)\zeta_0 u\|^2 + \|u\|^2. \quad (6.45)$$

For $m = 0$, it is easy to get

$$\|f(\Lambda)\zeta_0 u\|^2 \lesssim \|f(\Lambda)^{-1}\zeta_0\Box u\|^2 + \|u\|^2. \quad (6.46)$$

Combining (6.44), (6.45) and (6.46), we obtain

$$\begin{aligned}\|f(\Lambda)\zeta_{2s}u\|_s^2 &\lesssim \sum_{m=0}^{s}\|f(\Lambda)^{-1}\zeta_{2m}\Box u\|_m^2 + \|u\|^2 \\ &\lesssim \|f(\Lambda)^{-1}\zeta_0\Box u\|_s^2 + \|u\|^2\end{aligned} \quad (6.47)$$

namely,

$$\|f(\Lambda)\chi_0 u\|_s^2 \lesssim \|f(\Lambda)^{-1}\chi_1\Box u\|_s^2 + \|u\|^2 \quad (6.48)$$

for any $u \in \mathcal{A}^{0,k} \cap \mathrm{Dom}(\Box)$. Using the method of elliptice regularization as in [KN65], we conclude the proof of theorem.

$\Box$

Using above technique and conbining the research about smoothness of Bergman kernel of Kerzman [Ke72], we obtian the following thoerem

Theorem 6.7. *Let Ω be a bounded q-pseudoconvex domain in $\mathbb{C}^n$ with class C^∞ boundary. Suppose further that $(f\text{-}\mathcal{M})^k$ estimate hold in a neighborhood U of a point $z_0 \in b\Omega$ with either $f \gg \log$, $\mathcal{M} = 1$ or $f = \log$, $\mathcal{M} = \frac{1}{\epsilon}$ for any $\epsilon > 0$. Let χ_0 and χ_1 be smooth cutoff functions supported in U with $\chi_1 = 1$ in a neighborhood of the support of χ_0. For every integer $s \geq 0$, we have the $\bar\partial$-Neumann operator N_k and the Bergman projection P_{k-1} satisfy the estimates*

$$\|f(\Lambda)\chi_0 N_k\alpha\|_s^2 \lesssim \|f(\Lambda)^{-1}\chi_1\alpha\|_s^2 + \|\alpha\|^2; \tag{6.49}$$

$$\|\chi_0\bar\partial^* N_k\alpha\|_s^2 + \|\chi_0\bar\partial N_k\alpha\|_s^2 \lesssim \|f(\Lambda)^{-1}\chi_1\alpha\|_s^2 + \|\alpha\|^2; \tag{6.50}$$

$$\|\chi_0 P_{k-1}\alpha\|_s^2 \lesssim \|\chi_1\alpha\|_s^2 + \|\alpha\|^2; \tag{6.51}$$

for any $\alpha \in H_s(\Omega)^{0,k}$. Moreover, if $w_0 \neq z_0$ is another point of $b\Omega$ such that $(f\text{-}\mathcal{M})^k$ estimate hold in a neighborhood V of w. Then the Bergman kernel function $K(z,w)$ extends smoothly to $(U \cap \bar\Omega) \times (V \cap \bar\Omega)$.

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