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MAXWELL AND HELMHOLTZ AND THE BIRTH OF THE THEORY OF COLOUR

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Abstract

The present research study offers an overall picture of Maxwell and Helmholtz's fundamental contributions to the science of colour, which constitute the basis of modern colorimetry, colour photography and colour metrics.

The first main part of the dissertation, which embraces chapter 2 and chapter 3, is dedicated to a chronological account of their pivotal achievements on the subject, giving special focus on how they interpreted and elaborated over the course of the years Newton and Young's theories of colour, which represent a fundamental starting point for both Maxwell and Helmholtz's research in the field. Among their key contributions reported in the present work, it is worth mentioning that the two scientists clarified, for the first time and independently from one another, the distinction between additive and subtractive colour mixing, eliminating definitively Newton's confusion between optical and pigment mixture of colours (confusion still existing in Young's famous works and up to the solution provided by Forbes, Maxwell and Helmholtz). Moreover, they adopted Newton's analogy of the centre of gravity to predict the outcome of optical mixture of light and eliminated the arbitrary choice of the number of primary colours. Among Maxwell and Helmholtz's crucial experiments, deeply described and analysed in the text and from which they could obtain their colour diagrams (an equilateral triangle for Maxwell and a truncated hyperbola for Helmholtz), it is worth to cite the V-shaped slit experiment performed by Helmholtz, the famous Maxwell's *colour top*, Helmholtz's colour mixing experiment to detect directly pairs of complementary colours and Maxwell's *colour box*.

From an analysis of the obtained colour diagram, Helmholtz expressed pivotal considerations on the geometry of colour space: colour space could not be uniform, i.e. equal distances did not correspond to equal perceptual difference. Helmholtz's reflections introduce the second main part of the dissertation, composed of chapters 4 and 5, dedicated to the work undertaken by Helmholtz on the geometry of colour space, which he interpreted, such as Riemann before him, as non-Euclidian. This second part begins with a deep analysis of the studies on colour and colour perception carried out by Helmholtz's collaborators König, Dieterici and Brodhun, working in his laboratory in Berlin. Their crucial results constitute Helmholtz's starting point for the definition of his *line element* in colour space, which he elaborated following ideas borrowed from the new-born field of psychophysics. He extended the one-dimensional Weber-Fechner law of psychophysics to a complex of three dimensions, such as the colour space for subjects with normal colour vision. Helmholtz's mathematical treatment and related considerations are contained in three papers, published between 1891 and 1892, of which a partial English translation, with related analysis, is provided for the first

time. To conclude, in chapter 6 a short overview of the development of colorimetry, after Maxwell and Helmholtz's works, can be found. The aim, far from giving a complete and detailed review of all subsequent contributions to the field up to 1971 (when the "*Helmholtz Memorial Symposium on Color Metrics* of the *International Color Association"*, AIC, took place), is to highlight the pivotal role played by Helmholtz and Maxwell's studies in the development of colour science, as well as to illustrate some problems which have remained unsolved until the present time.

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Chapter 1

Introduction

James Clerk Maxwell and Hermann von Helmholtz created inter alia the foundation for quantitative colour measure, for practical colour photography and for colour metrics. From their first attempt of colour diagrams the model of CIE *Commission Internationale de l'Éclairage* (*International Commission on Illumination*), first published in 1931, has evolved. The CIE model is the most popular system of colour representation in use today and considered the basis of modern colorimetry.

The aim of the present research study is to provide an overall picture of Maxwell and Helmholtz's contributions to the field of colour science, which is still missing at the present time. Before exposing and analysing their crucial findings, the reasons that led them turn toward the study of colour and colour perception will be discussed. Particular emphasis will be given to the role played by the works of Newton and Young in shaping the development of their theories. Newton and Young's ideas on colour represent a fundamental starting point for both Maxwell and Helmholtz's research in the field. In the body of the text, indeed, we will bring to light fundamental aspects of Newton and Young's theories, which they accepted, elaborated and revised.

Among Maxwell and Helmholtz's key contributions, I would like to anticipate here that they clarified, as we will see in chapter 3, for the first time and independently from one another, the distinction between additive and subtractive colour mixing, eliminating definitively Newton's confusion between optical and pigment mixture of colours, confusion still existing in Young's famous works. Furthermore, they adopted Newton's analogy of the centre of gravity to predict the outcome of optical mixture of light and eliminated the arbitrary choice of the number of primary colours. Offering an overview of both their famous and minor works, their colour mixing experiments will be also presented. In particular the following crucial experiments are more deeply described and analysed: the V-shaped slit experiment performed by Helmholtz to obtain direct composition of coloured spectral lights, the famous Maxwell's *colour top*, that allowed him to determine the first quantitative expression of colour mixing in form of colorimetric equations, Helmholtz's colour mixing experiment to detect pairs of complementary colours and Maxwell's *colour box*, an experimental device built exploiting Newton's crucial experiences with lenses, prisms and mirrors. We will show how Maxwell and Helmholtz could obtain, from these decisive experiments, their colour diagrams, which took the form of an equilateral triangle (Maxwell) and the shape of a truncated hyperbola (Helmholtz). Helmholtz, analysing the obtained colour diagram, realized that colour space could not be uniform, i.e. equal distances did not correspond to equal perceptual difference. This motivated him to express some pivotal considerations on the geometry of colour space. He and Riemann before him, as shown in chapter 5, interpreted the geometry of colour space as non-Euclidean. In connection with this, a brief overview of the content of Riemann and Helmholtz's famous works will be provided, in which colour space was cited by both authors as an example of non-Euclidean geometry realized in nature. My intention will be, then, to highlight the way in which Helmholtz defined the first *line element* in colour space following ideas borrowed from the new-born field of psychophysics. He extended the one-dimensional Weber-Fechner law of psychophysics, which related physical stimuli and psychological experience, to three dimensions, because three is the number of independent variables in colour space in the case of normal vision. Helmholtz's mathematical treatment and related considerations are contained in three papers, published between 1891 and 1892, of which a partial English translation, with related analysis, is provided. Helmholtz's research on the subject was supported by the fundamental contributions of their assistants and collaborators, who furnished him with precious material, which he could collect and elaborate. Particular emphasis will be given to the works of König, who wrote most of his major papers in collaboration with Dieterici, and of Brodhun, who provided significant results for what concerns colour blind vision, being himself affected by green-blindness (*Grünblindheit*). Among König's major publications, I would like to cite here his paper of 1888, written in collaboration with Brodhun and entitled *Experimentelle Untersuchungen über die psycho-physische Fundamentalformel in Bezug auf den Gesichtssinn (Experimental investigation on the psycho-physical fundamental formula in relation to the sense of vision).* This paper is of crucial importance because it contains one of the first attempt to apply the Weber-Fechner law to the field of colour vision. For this reason, a replica of the experimental set-up used and described by König in the above-mentioned paper would be of historical relevance. The performance of this experiment, by using materials and technology of that time, would offer a more complete understanding of the subject, helping also to visualize strengths and weaknesses of the experimental technique. The first steps in this direction has already been done with the precious collaboration of Sara Magrin and Roberto Temporin, responsible of the Didactic Laboratories of the Department of Physics and Astronomy of the University of Padova and the complete performance of König's experiment, which, I would like to specify, belongs to the research area of Experimental History of Science, will be the object of future studies.

To conclude, a brief overview of the development of colorimetry, after Maxwell and Helmholtz's works, will be provided. My aim, far from giving a complete and detailed review of all subsequent contributions to the field up to 1971 (when the meeting "*Helmholtz Memorial Symposium on Color Metrics* of the *International Color Association"*, AIC, was held) will be to underline the pivotal role played by Helmholtz and Maxwell's studies in the development of colour science, as well as to discuss important aspects of colour metrics, and to illustrate some problems which have remained unsolved until the present time.

The work is arranged according to the plan illustrated below.

The intent of chapter 2 is to provide a brief overview of three significant breakthroughs, occurred between the seventeenth and nineteenth centuries, that inaugurated the modern approach to the science of colour and from which Maxwell and Helmholtz's research in the field started. The distinction between primary and secondary qualities, articulated by Galilei, will be presented, which represents the first turning point in the interpretation of colour; then, Newton's theory of light and colour will be described, before illustrating the third important breakthrough, embodied by Young's proposal of a wave theory of light and, in this connection, the idea of trichromatic colour vision.

In chapter 3 Maxwell and Helmholtz's key contributions to the theory of colours are described. Special focus is given to the influence of Newton and Young's ideas on the development of their theories, as well as on the set up of their colour mixing experiments, which allowed them to build their colour diagrams and to investigate colour blindness. Here, Grassmann's work on colour will be also analysed, constituting an important source of motivation for both Maxwell and Helmholtz. The last section of chapter 3 is dedicated to a brief overview of Maxwell and Helmholtz's laboratories, at the time they were appointed first Professors of Experimental Physics, Maxwell in Cambridge and Helmholtz in Berlin.

Chapter 4 offers a survey of the contributions of Helmholtz's collaborators and assistants to colour science, working at the Physical Institute of the University of Berlin. An analysis of the content of König's major papers is provided, most of which were written in collaboration with his colleague Dieterici. The experimental set-up used by König is described, which allowed him to obtain results of seminal importance, such as the determination of the curves of *elementary* and *fundamental* sensations for *monochromats*, for two kinds of *dichromats* and for *trichromats* and from which he could accurately represent the spectral trace inside Maxwell's triangle. Chapter 4 contains also König and Brodhun's study concerning the applicability of the Weber-Fechner law of psychophysics to light illumination for different colours. The last sections are dedicated to Brodhun's main achievements in the field of colour vision, starting with a presentation of his doctoral dissertation, published in 1887 and kindly supplied by the *Staatsbibliothek zu Berlin*.

Chapter 5 is dedicated to Helmholtz's construction of his *line element* in colour space, taking the one-dimensional Weber-Fechner law as starting point of the mathematical treatment. The first paragraph introduces the reader to Riemann and Helmholtz's papers containing their general reflection upon the foundation of geometry; particular emphasis is placed on their considerations relating the geometry of colour space, because they opened the doors to the pivotal work undertaken by Helmholtz in the subsequent years. The core of chapter 5 is, in fact, represented by Helmholtz's papers of 1891 and 1892, in which he elaborated the precious material offered by his collaborators to extend the psychophysical law to two dimensions (for the case of dichromatic colour vision), at first, and then to three dimensions (for the case of normal colour vision).

Chapter 6 contains a brief overview of the development of colorimetry, after Maxwell and Helmholtz's fundamental contributions, up to 1971, year in which the meeting "*Helmholtz Memorial Symposium on Color Metrics* of the *International Color Association"* was held for the 150th anniversary of the birth of Helmholtz, considered the founder of modern colour metrics. In this last chapter, a special focus is given to some implementations of Helmholtz's line element model, to Guild and Wright's pivotal works, whose experimental results were combined to give birth to the first CIE colour spaces and to Wright and MacAdam's pioneering attempts to measure just noticeable differences across the CIE (x, y) colour diagram. The main goal is to underline the role played by Helmholtz and Maxwell's findings in all subsequent studies on colorimetry. The last paragraph of chapter 6 is dedicated to a short overview of the above-mentioned symposium of 1971, of which some excerpts, published in the proceedings, are reported and briefly commented. During the conference important aspects and problems related to colour metrics are discussed. Although many implemented colour difference formulae and colour spaces have been elaborated over the course of the subsequent years up to the present time, it is worth to underline that some of the problems first raised by Helmholtz have remained unsolved until today.

At the end of the work, concluding remarks are given (chapter 7) and references are reported, for the benefit of those readers, who wish to deepen their knowledge on the discussed topics, and also for the benefit of historians of physics and more generally of historians of science, who will be interested in further developing the presented subject.

Appendix A contains a brief account of the method used by Maxwell to obtain the world's first colour photographic image, an accidental invention which can be, nevertheless, counted among his great achievements.

Appendix B provides an overview of the main types of colour deficiency, which can help the reader to better understand the reported pioneering work undertaken by König, Dieterici and Brodhun on the subject.

In Appendix C some precious manuscript material concerning experiments on colour and colour perception is reported, which is preserved at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften* in Berlin.

Chapter 2

Background of the study

Sensation of light and colour are only symbols for relations of reality. They have as much and as little connection or relation with it, as the name of a man, or the letters of his name, have to do with the man himself (Hermann von Helmholtz, Königsberger 1906, p. 98)

From classic Greece to the Renaissance, colour theories were closely linked to theories of vision. According to the Pythagoreans, the basic species of colour were white, black, red and yellow, four in number because four were the elements of nature. This view influenced strongly the Greek philosophy and art of that time. Aristoteles, about 200 years later, added three more colours, i.e. blue green and purple, to the Pythagorean colour categories and organized them in a linear order, from white to black, in which colours were sorted according to their brightness. All colours were created from light and darkness, from white and black (considered as colours) combined in different proportions. Aristoteles' colour system was influential for the following 2000 years.

My intention is to sketch briefly three significant breakthroughs, occurred between the seventeenth and nineteenth centuries, that inaugurated the modern approach to the science of colour and from which Maxwell and Helmholtz's research in the field began. I will start illustrating the first of these breakthroughs, i.e. the distinction between primary and secondary qualities introduced in the seventeenth century. I will proceed providing a brief account of Newton's theory of colour, which can be regarded as the second significant turning point in the interpretation of colours, before concluding this second chapter with a short description of Young's contribution to the field.

2.1 The distinction between primary and secondary qualities

The distinction between primary and secondary qualities was clearly articulated by Galileo Galilei in his work of 1623, *Il Saggiatore*, *The Assayer*, translated by Stillman Drake (Galilei 1957), and from which an excerpt is reported: *Now I say that whenever I conceive any material or corporeal* *substance, I immediately feel the need to think of it as bounded, and as having this or that shape; as being large or small in relation to other things, and in some specific place at any given time; as being in motion or at rest; as touching or not touching some other body; and as being one in number, or few, or many. From these conditions* [primary or objective qualities] *I cannot separate such a substance by any stretch of my imagination. But that it must be white or red, bitter or sweet, noisy or silent, and of sweet or foul odor, my mind does not feel compelled to bring in as a necessary accompaniment. Without the senses as our guides, reason or imagination unaided would probably never arrive at qualities like these. Hence I think that tastes, odors, colors, and so* [secondary or subjective qualities] *on are no more than mere names so far as the object in which we place them is concerned, and that they reside only in consciousness. Hence if the living creature were removed, all these qualities would be wiped away and annihilated. But since we have imposed upon them special names, distinct from those of the other and real qualities mentioned previously, we wish to believe that they really exist as actually different from those* (Galilei 1957, p. 274).

Galileo pointed out here that sounds, colours, smells and tastes, are only names and they exist only in relation to the consciousness of who perceives them. Without the consciousness, they are reduced to mere words. The concrete world of everyday experience is not real as matter and movements are, which occur according to determinable laws. The real world is a combination of quantitative and measurable data, of space and movements and relations in space and time. Galileo's reflections on this topic formed the background to Maxwell and Helmholtz's research.

Helmholtz, in his Inaugural Lecture *On the Nature of Human Sense-Perception (Über die Natur der menschlichen Sinnesempfindungen),* delivered at Königsberg on June 28, 1852, expressed with the following words his considerations on sensation of light and colour: *Sensation of light and colour are only symbols for relations of reality. They have as much and as little connection or relation with it, as the name of a man, or the letters of his name, have to do with the man himself* (Königsberger 1906, p. 99). From this excerpt it is evident that Helmholtz shared completely Galileo's point of view concerning the relation between sensation and reality and tried to explain it by means of a simile: he compared the relation between sensation of light and colour and the real world with that existing between the name of a man and the man himself. The thesis of the primary/secondary quality distinction was also supported by Descartes, Hobbes, Mersenne, Gassendi, Pascal and Locke[1] and constitutes one of the fundamental theoretical assumptions of mechanical philosophy. In this respect it is worth reporting Hobbes' interpretation of secondary qualities: *All which qualities called sensible are in the object that causeth them but so many several motions of the matter, by which it presseth our organs diversely. [...] So that sense in all cases is nothing else but original fancy caused (as I have said) by the pressure that is, by the motion of external things upon our eyes, ears, and other organs, thereunto ordained* (Hobbes 2005, p. 14). Our sensations are provoked by a determined state of the external body, that can be defined in geometrical-mechanical terms. From that point onwards, as P. Rossi clearly highlighted, the idea of extending the method of mechanical philosophy to the world of physiological and psychological phenomena began to take hold. Philosophy and psychology started to become natural sciences, interpretable using the same methods applied in the field of mechanics and physics and based on the same theoretical assumptions (Rossi 1976).

2.2 Newton's theory of colour

Newton's theory of colour can be considered as the second significant turning point in the interpretation of colours. It is worthwhile pointing out the essential points of Newton's theory of light and colours because it represents a fundamental starting point for Maxwell and Helmholtz's work in the field, having recognized the complexity and the richness of Newton's ideas and pivotal experiments on colours.

Newton was identified by his contemporaries and most of his epigones, the so-called Newtonians, as a supporter of the emission theory of light[2]. It is worth to underline that, even if he adopted the corpuscular theory to develop optical experiments and theories, he never considered it a demonstrated scientific truth. Newton's theory of light and colour underwent significant changes in both content

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[1] In his *Essay Concerning Human Understanding*, published in 1690, John Locke expounded on the differences between primary and secondary qualities, presenting the same arguments of Galileo: *The particular bulk, number, figure, and motion of the parts of fire, or snow, are really in them, whether any one's senses perceive them or no; and, therefore, they may be called real qualities, because they really exist in those bodies. But light, heat, whiteness, or coldness, are no more really in them, than sickness or pain is in manna. Take away the sensation of them; let not the eyes see light or colours, nor the ears hear sounds; let the palate not taste, nor the nose smell; and all colours, tastes, odours, and sounds, as they are such particular ideas, vanish and cease, and are reduced to their causes, i.e. bulk, figure, and motion of parts* (Locke 1836, p. 76-77).

[2] E. Bellone reported Einstein's original words regarding the complex movement of scientific ideas and theories, also known under the name of *Newtonianism* explicitly referring to the Newton's general conception of nature, diffused in Europe over the course of the 18th and 19th centuries: *I must emphasize that Newton himself was better aware of the weaknesses inherent in his intellectual edifice than the generations of learned scientists which followed him. This fact has always aroused my deep admiration* (Bellone 2007, p. 87).

and structure in its forty years' development. This long period of gestation began with his first optical experiments dated in 1664 (Hall 1993, p. 35), at the time he was an undergraduate at Trinity College, Cambridge University, and ended with the first publication of his *Opticks* in 1704[3]. After 1704, however, he continued speculating about the microphysics of light, as testified by the new *Queries*, open questions at the end of the book, that he introduced in the subsequent editions of the volume. We can briefly sketch the main phases of this complex evolution. Newton's first attempt to formulate his theory of light and colour is contained in the *Optical Lectures*, that he delivered between 1670 and 1672 as Lucasian Professor of Mathematics in the University of Cambridge. The rich material contained in these Lectures served as a starting point for the preparation of his first printed essay, that appeared in Philosophical Transactions of 1672 with the title *New Theory about Light and Colour*. In the *New Theory* we can find a review of Newton's first experiences conducted from 1666 onwards and focused on the decomposition of light through a prism. The most acclaimed result of the paper is that white light is a heterogenic composition of all the colours of the spectrum.

During these experiences, Newton observed that, when he transmitted a narrow beam of sunlight through a prism, the refracted image on the wall presented an oblong shape instead of a circular one, which was to be expected according to the law of refraction. He investigated the possible causes of this elongation of the spectrum and eventually solved the problem with the so-called *experimentum crucis*. The following figure depicts Newton's sketch of the *experimentum crucis*, whose description is to be found in the *New Theory* and in his *Opticks*.

In a darkened room, behind the prism denoted by *ABC*, Newton set up a board, *DE*, on which a small hole *G* can be found. In this way, a slightly rotation of the prism on its axis allowed him to project different portion of the spectrum through this small opening. He then placed at a given distance another board, denoted by *de,* equipped with a hole *g*, that allowed the beam to pass. The two boards defined a constant path for the beam and therefore a constant angle of incidence on a second prism, indicated by *abc,* placed behind the second board.

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[3] Rupert Hall in his book *All was Light* offered a complete account of Newton's achievements in the science of optics, starting with a short overview of his predecessors in the field, such as Descartes, Boyle and Hooke. Hall placed special emphasis on Newton's *Opticks*, highlighting its genesis, its long evolution and its reception in Britain and on the Continent (Hall 1993).

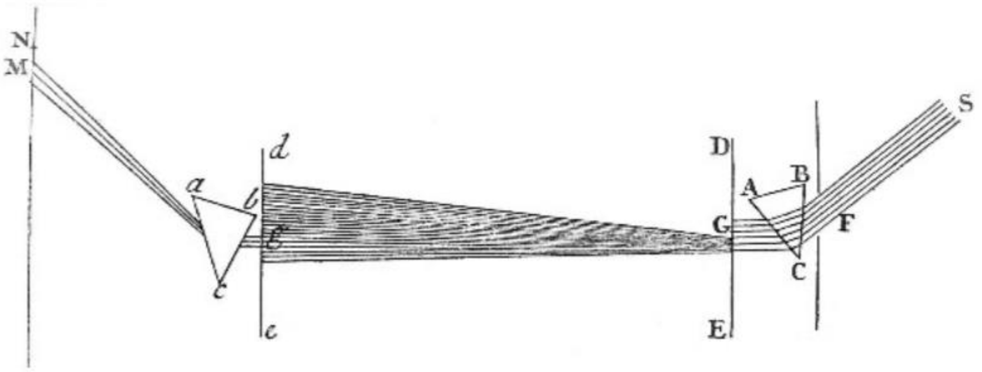

Fig. 2.1

Newton's experimentum crucis (Newton 1979, p. 47)

The crucial result that Newton achieved was that the elongation of the image was due to the different refrangibility of the light rays. In fact, he observed that light most refracted in the first prism was also most refracted in the second prism. Therefore, it is also worth to underline that when a coloured ray was refracted through the second prism it underwent no further change, preserving its identity.

All Newton's considerations and investigations of light and colour were consolidated in the *Opticks*. Newton began to prepare the book in 1687, after the publication of his *Philosophia naturalis principia mathematica*, and probably concluded it in 1692 but the volume was only published in 1704 (Hall 1993, p. 3). The reason of this delay could be due to a long work of revision and extension that the volume underwent over the course of the years. Newton wished to be able to dedicate a section for optics in the field of the mechanics of material point contained in his *Principia*. He did not succeed in realizing his original plan because of the broad and complex range of phenomena he wanted to deal with. Furthermore, another probable reason for the delay was that Newton retained his work incomplete. He had no original intention of concluding the book with open questions and conjectures, as he in fact did, given them the name of *Queries*, that we can find at the end of the *Opticks*. The *Queries* were originally sixteen in number in the first published edition of 1704, Newton subsequently enriched the Latin translation of 1706 with seven more, following sequentially the former sixteen. Eventually, the second English edition of *Opticks* of 1717 was enlarged with other eight *Queries*, that reached therefore the total number of thirty-one.

We can at this point summarize the essential points of Newton's theory of light and colour. The fundamental idea underlying his theory is that sunlight or white light is a mixture of coloured "rays" differing in degree of refrangibility. The colour of a ray is immutable and maintains its individuality when it undergoes reflection, refraction or transmission, as Newton reported in Prop. II. Theor. II. of

Book I, Part II of his *Opticks*: *All homogeneal light has its proper Colour answering to its Degree of Refrangibility, and that Colour cannot be changed by Reflexion or Refraction* (Newton 1979, p. 122). Newton was aware of the contributions to the field of light and colour made over the course of the 17th century by his predecessors, such as René Descartes, Robert Boyle and Robert Hooke. However, unlike Descartes, Boyle and Hooke, Newton refuted the theories that considered colours as modification of white light caused by reflection and refraction. He considered instead colours as eternal components of white light, connate properties of the rays that cannot be created, such as atoms constituting matter.

Descartes' *Dioptrique* and *Mètèores* (both published in 1637), that Newton probably read in one of the Latin collected editions of his writings, constituted the starting point of Newton's investigation of refraction and of colour. Newton also read Boyle's considerations on colours, as testified by his annotations, although he did not own a copy of Boyle's most significant book on optics, *Experiments and Considerations touching Colours* (1664). Furthermore, for what concerns the structure of his *Opticks*, Newton took inspiration from Boyle's books. Robert Hooke's *Micrographia* (1665) was also a source of inspiration for Newton. He read, in fact, Hooke's explanation of the colours of thin plates. Hooke observed a repetitive succession of colours in films of a non-uniform thickness, that made him deduce that the appearance of the colours was as a periodic phenomenon. He explained the colours produced by thin plates in terms of the distance between two *light pulses* reflected at the lower and at the upper surfaces of the plate. A combination of two streams of these *pulses*, one weaker (reflected from the lower surface) than the other, could reach the eye. Analysing the phenomenon, Newton also recognized this periodicity in light, as testified by the elaboration of his theory of "fits" in *Opticks*, clear evidence of Hooke's influence upon Newton's theory of light and colours. It is worth spending a few more words about Newton's "fits", that he proposed at the end of a long revision process that his theory underwent, to underline how it was more complex than the simple corpuscular model, traditionally attributed to him by his epigones. Newton developed his theory of "fits" during the autumn and winter of 1691-1692 (Shapiro 1989, p. 224). He borrowed the term "fits" from medical usage (use to define the intermittent fever called malaria) to indicate periodic perturbations that modulate the motion of beams of particles composing light, i.e. vibrations of different wavelengths. Newton baptized at first these "fits" as "Dispositions", in relation to his studies on thin plates, as we can read in Book II, Part II of *Opticks* (Newton 1979, pp. 242-243): *It is necessary therefore that every Ray have its proper and constant degree of Refrangibility connate with it, according to which its refraction is ever justly and regular perfor'd; and that several Rays have several of those degrees. And what is said of their Refrangibility may be also understood of their Disposition to be reflected, some at a greater, and other at a less thickness of thin Plates or Bubbles; namely, that those*

Dispositions are also connate with the Rays, and immutable [...]. The term “Dispositions” was then replaced by that of “fits”, that first appears in Book II, Part III, Proposition XII of *Opticks*: *The returns of the disposition of any Ray to be reflected I will call Fits of easy Reflection, and those of its disposition to be transmitted its Fits of easy Transmission, and the space it passes between every return and the next return, the Interval of its Fits* (Newton 1979, p. 281). Fig. 2.2 helps us to visualize Newton’s theory of “fits”. The semicircles correspond to “fits of transmission”. When the incoming light falling on the lens coincides with one of these “fits” it can be transmitted into the glass, whereas it undergoes reflection when it coincides with the “fits of reflection”, located half-way between those of transmission.

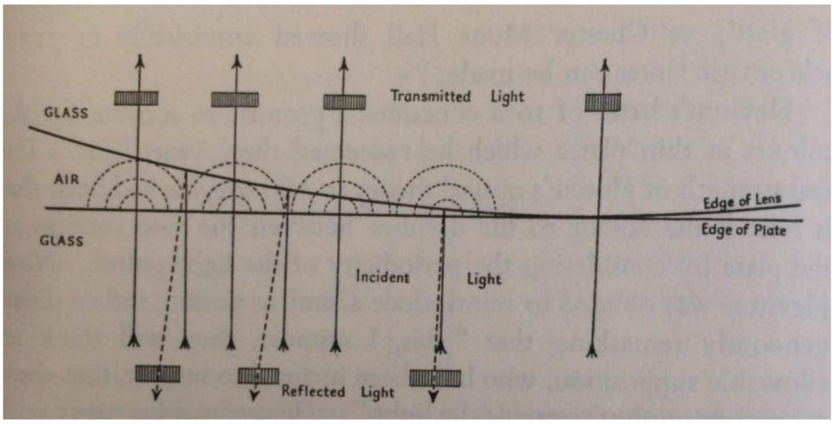

Fig. 2.2
Newton’s theory of “fits” (Hall 1963, p. 270)

Newton, however, was not able to make any definite physical model of his “fits”, that therefore earn a place in the *Queries* in Book III of *Opticks*: *Nothing more is requisite for putting the Ray of Light into Fits of easy Reflection and easy Transmission, than that they be small Bodies which, by their attractive Powers, or some other Force, stir up Vibrations in what they act upon, which Vibrations being swifter than the Ray, overtake them successively, and agitate them so as by turns to increase and decrease their Velocities, and thereby put them into those Fits* (Query 29, Newton 1979, p. 372-373).

In order to develop his theory further, Newton introduced the terms of *simple* and *compound* colours, as stated in Defin. VIII of Book I of *Opticks*: *The Colours of Homogeneal Lights, I call Primary, Homogeneal and Simple; and those of Heterogeneal Lights, Heterogeneal and Compound*. (Newton 1979, p. 4). Simple colours are rays of a single degree of refrangibility and compound colours consist of a mixture of rays of different refrangibility, that can be separated therefore by refraction. The numbers of the rays composing white light is extremely large if not infinite, nevertheless seven principal primary colours can be chosen arbitrarily, exploiting the analogy between colours and musical tones (seven were, in fact, the musical intervals contained in an octave), as depicted in Newton's colour circle (see Fig. 2.3 on the left). I will briefly describe his colour mixing rule, because it served as fundamental starting point for Maxwell and Helmholtz's research in the field of colour by the middle of the 19th century.

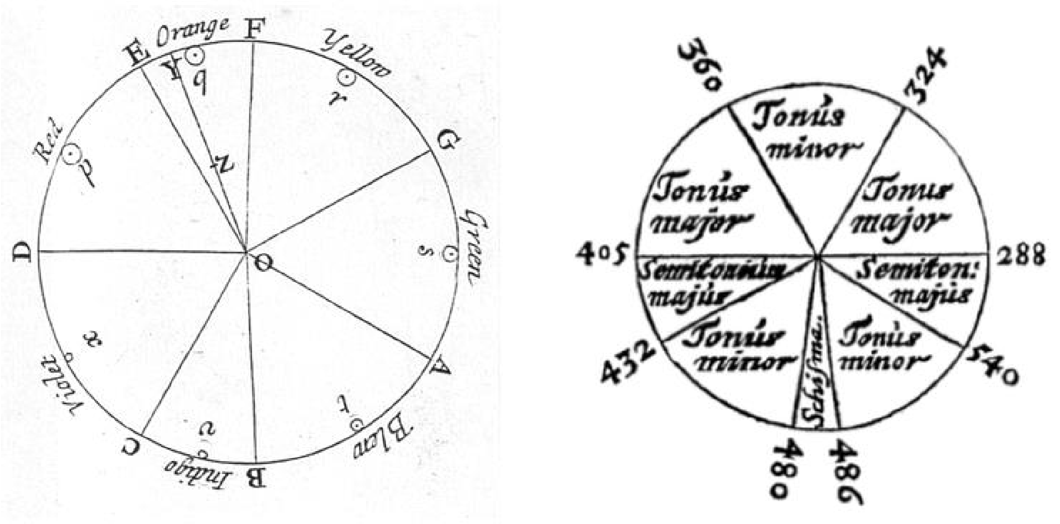

Fig. 2.3
Newton's colour mixing circle (left), indicating the method used to determine the compound colour resulting from any mixture of the others (Newton 1979) compared with Descartes' circle (right) of major and minor tones reported in his *Compendium musicae* (Descartes 1650)

Newton's colour circle can be found in his *Opticks*, Book I, Part II, Proposition VI. He laid down a clear method for predicting the compound colour which results from a combination of *seven* principal colours, *violet*, *indigo*, *blue*, *green*, *yellow*, *orange* and *red*, into which he divided the spectrum in line with the analogy between colours and musical harmony[4]. Newton, therefore, divided his circumference into seven parts, proportional to the musical intervals contained in an octave. The arches correspond to the above-mentioned principal colours, from which he derived all the others. At

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[4] The illustration of a circular diapason (Fig. 2.3 on the right), drawn by René Descartes and contained in his *Compendium musicae* (Descartes 1650) may have influenced Newton's colour circle. Newton also used the analogy between colours and musical tones to describe the thickness of thin films (Newton 1979, p. 211-212) and to explain the colour of soap-bubbles (Newton 1979, p. 225) and thick plates (although here he expressed this analogy in a different form, Newton 1979, p. 295). Moreover, we find again the comparison between colour and sound in *Query* 13 of the *Opticks*: *Do not several sort of Rays make Vibrations of several bignesses, which according to their bignesses excite Sensations of several Colours, much after the manner that the Vibrations of the Air, according to their several bignesses excite Sensations of several Sounds?[...]*(Newton 1979, p. 345).

the centre of gravity of these sections a small circle can be found. Being the area of each circle proportional to the number of rays of that colour that enter a given mixture, the common centre of gravity of these circles can be easily determined. Let that point be *Z*, the place of point *Y* in the circumference will represent the *spectral colour* of the mixture and the distance of point *Z* from the centre will show the *saturation* of the colour. Below are some extracts from Newton's proposition:

With the Center O [...] and Radius OD describe a Circle ADF, and distinguish its Circumference into seven Parts DE, EF, FG, GA, AB, BC, CD, proportional to the seven Musical Tones or Intervals of the eight Sounds, Sol, la, fa, sol, la, mi, fa, sol, *contained in an eight, that is, proportional to the Number 1/9, 1/16, 1/10, 1/9, 1/16, 1/16, 1/9. [...] Let* p *be the Center of Gravity of the Arch DE, and q, r, s, t, u, x, the Centers of Gravity of the Arches EF, FG, GA, AB, BC, and CD respectively, and about those Centers of Gravity let Circles proportional to the Number of Rays of each Colour in the given Mixture be describ'd: that is, the Circle p proportional to the Number of the red-making Rays in the Mixture, the Circle q proportional to the Number of the orange- making Rays in the Mixture, and so of the rest. Find the common Center of Gravity of all those Circles,* p, q, r, s, t, u, x. *Let that Center be Z; and from the Center of the Circle ADF, through Z to the Circumference, drawing the Right Line OY, the Place of the Point Y in the Circumference shall shew the Colour arising from the Composition of all the Colours in the given Mixture, and the Line OZ shall be proportional to the Fulness or Intenseness of the Colour, that is, to its distance from Whiteness. [...] This Rule I conceive accurate enough for practice, though not mathematically accurate;* (Newton 1979, 154-158).

Reading his original words, it is evident that Newton was aware of the lack of mathematical accuracy of his construction. As we will see in the next chapter, Maxwell responded to this challenge providing a quantitative expression of colour mixing in form of colorimetric equations. It is also worth to notice that Newton implicitly referred to the mixture of different quantities (masses) of pigments to describe spectral mixture of colours. This was one of the most debated topics in the field of optics over the course of the 18th century and finally solved, independently from one another, by Helmholtz and Maxwell, who clarified for the first time the difference between optical and pigment mixture of colours, as we will see in chapter 3.

In the same years Newton elaborated his theory of light and colour, Christian Huygens proposed a wave theory of light in his *Traitè de la lumière*, published in 1690. His wave theory belonged rather to geometrical than to physical optics. Huygens supposed that light propagates through an aether made of particles and the vibrations of each aether-particle spread a wave disturbance through its neighbours. A large number of these small waves reinforcing each other gives rise to a wave front, whose movement constitutes the radiation of light. Huygens' theory, however, was far from being complete, as demonstrated by the limited range of phenomena he attempted to explain. Huygens, in fact, did not treat diffraction, interference, colour or even polarization phenomena which were crucial for the rediscovery and affirmation of his theory in the 19th century.

2.3 Naturphilosophie

Between 1790 and 1830, a vast movement of ideas took off in Germany, and then spread across Europe, under the name of *Naturphilosphie*, term that indicates a rich intertwining of philosophical ideas and scientific perspectives, expression of the common purpose of reaching an overall and unitary understanding of nature[5]. Among the various debate topics, a pivotal role was played by the nature of colours and their perception, topics that increasingly fascinated not only natural philosophers but also painters, poets and writers, such as Johann Wolfgang von Goethe, who published in 1810 his work *Zur Farbenlehre*. Here Goethe expressed his criticism toward Newton's theory of light and colour. Having experienced directly Newton's experiments with the prism, Goethe, in the light of the observed phenomena, came to strongly disagree with him. He believed that colours are the result of the interaction of light and shadow, i.e. of white and black, according to the idea of polarity of every natural process, that was one of the main leitmotiv of *Naturphilosophie*. Goethe also elaborated his six-colour circle, based on two triangles, following this principle: the left semicircle, in fact, is considered the *plus* side, whereas the right one corresponds to the *minus* side (see Fig. 2.4).

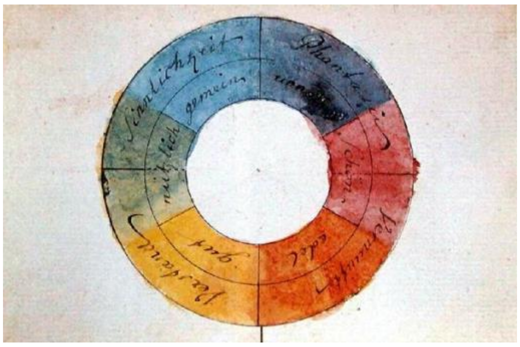

Fig. 2.4
Goethe's colour circle (Goethe 1810)

In the same year, the German Romantic painter Phillip Otto Runge published *Die Farben-Kugel oder Construction des Verhältnisses aller Mischungen der Farben zueinander, und ihrer vollständigen*

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[5] Stefano Poggi illustrated in his book *Il genio e l'unità della natura. La scienza della Germania romantica (1790-1830)* the main features of this current and its influences on the developments of science in that time (Poggi 2000).

Affinität, mit angehängtem Versuch einer Ableitung der Harmonie in den Zusammenstellungen der Farben (Colour sphere or construction of the relationship of all mixtures of colours, and their complete affinity, with an attached essay on the derivation of harmony in colour compositions). Runge exchanged a lively correspondence with Goethe. In his book a description of his colour sphere can be found. This sphere was constructed to contain all the colours obtained from the five chosen primary colours red, yellow, and blue, black and white. In order to obtain his sphere model, Runge started considering an equilateral triangle with yellow, red and blue (the painter's primary colours) as vertices and placed a second equilateral triangle midway on the sides of the first triangle containing the secondary colours, orange, violet and green (that was in agreement with Goethe). He expanded the whole into a circle that was, in turn, expanded to a double cone by the addition of white and black as vertical axis perpendicular to the plane of the colour circle. Runge, eventually, expanded this double cone to a sphere, depicted in Fig. 2.5[6]. Runge's sphere was an ideal solid, where all chromaticity dissolves in the centre.

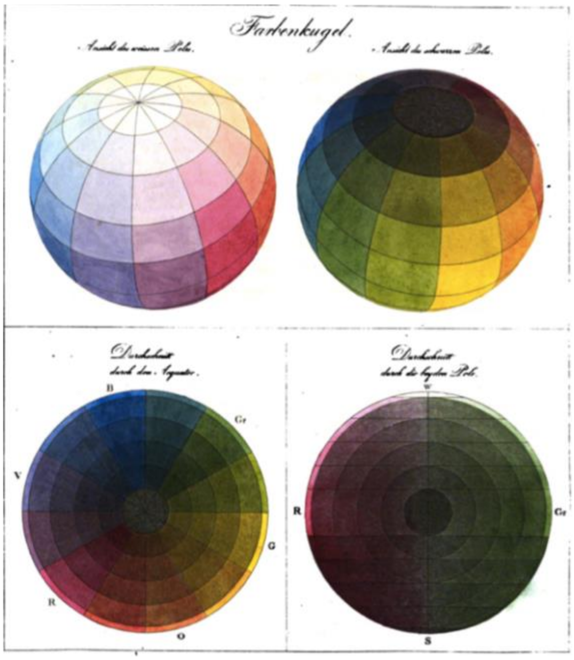

Fig. 2.5
Views of Runge's colour sphere from the white to the black poles (on the top) and a horizontal cross section along the equator and a vertical one through the poles (on the bottom) (Runge 1810)

Interested in the debate about colours and their perception was also the English Romantic painter Joseph Mallord William Turner. In his paintings, Turner replaced forms with colours (the main

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[6] A complete description of Runge's colour model can be found in *Color Order, A Survey of Color Order Systems from Antiquity to the Present* of R. G. Kuehni and A. Schwarz (Kuehni and Schwarz 2008).

colours he used were red, yellow and blue). He was strongly influence by Goethe's *Farbenlehre* and adopted his theory of light and darkness, depicting their relationship in most of his paintings.

Fig. 2.6
Joseph Mallord William Turner, Light and Colour (Goethe's Theory) - the Morning after the Deluge - Moses Writing the Book, 1843, Tate Britain, London.This painting illustrates Turner's relationship to *Naturphilosophie*

Animated by the same intent of deepening the connections between art and science were the chemists and aestheticians George Field and Michel Eugène Chevreul, the design theorists Owen Jones and Richard Redgrave, and the decorative painter David Ramsey Hay[7]. My intention here is to spend a few more words about Hay, because of his connection with Maxwell.

The first set of coloured papers for his colour top experiments (see chapter 3) were, in fact, furnished by David Ramsey Hay. Maxwell was introduced by the father, John Clerk Maxwell, and the uncle, John Cay, to the *Royal Scottish Society of Arts*, of which they were prominent members and in that occasions he came into contact with Hay, a decorative painter and colour theorist living in Edinburgh in that time, who expressed his ideas about the possibility to define laws of symmetry underlying the primary beauty of forms in his volume *First Principles of Symmetrical Beauty* (Hay 1846) and his treatise *Nomenclature of Colours* (Hay 1845), a colour sample collection, was considered as landmark for painters.

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[7] For more details, refer to Keyser, *Science and Sensibility: Chemistry and the Aesthetics of Color in the Early Nineteenth Century* (Keyser 1996).

2.4 Young's contributions

Thomas Young was the first who, starting from the well-known fact that there are three primary colours, sought for the explanation of this fact, not in the nature of light, but in the constitution of man (James Clerk Maxwell, Niven 1890, vol. II, p. 267)

It is not by chance that the modern theory of colour was inaugurated in the 19th century. In November 1801, Thomas Young delivered a lecture *On the theory of light and colours* at the Royal Society of London, in which he presented his arguments in favour of a wave theory of light and proposed, in this connection, the idea of trichromatic colour vision (Young 1802). Young's work led to the third key breakthrough in colour theory. He suggested the existence of three kinds of particles placed in the human retina responsible for colour vision. In this Bakerian lecture, Young was able to explain with the wave theory of light all the phenomena, whose treatment was left aside by Huygens. Young described the object of his dissertation with the following words: *The object of the present dissertation is not so much to propose any opinions which are absolutely new, as to refer some theories, which have been already advanced, to their original inventors, to support them by additional evidence, and to apply them to a great number of diversified facts, which have hitherto been buried in obscurity* (Young 1802, p. 12). Before introducing his three-receptor theory of vision, Young reported some key passages of Newton, from which his considerations started. It is worth to underline that it is to Newton we owe the idea that the various colours derive from the different sensations stimulated by the different vibrations associated with "rays" of different colours[8]. Young limited to three the number of particles in the retina responsible to colour vision, because three were the primary colours used by painters: *Now, as it is almost impossible to conceive each sensitive point of the retina to contain an infinite number of particles, each capable of vibrating in perfect unison with every possible undulation, it becomes necessary to suppose the number limited, for instance to the three principal colors, red, yellow, and blue, of which the undulations are related in magnitude nearly as*

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[8] Young reported Newton's words in this regard: *To explain colours, I suppose, that as bodies of various sizes, densities, or sensations, do by percussion or other action excite sounds of various tones, and consequently vibrations in the air of different bigness; so the rays of light, by impinging on the stiff refracting superficies, excite vibrations in the ether, of various bigness; the biggest, strongest, or most potent rays, the largest vibrations; and others shorter, according to their bigness, strength, or power: and therefore the ends of the capillamenta of the optic nerve, which pave or face the retina, being such refracting superficies, when rays impinge upon them, they must there excite these vibrations, which vibrations (like those of sound in a trunk or trumpet) will run along the aqueous pores or of the capillamenta through the optic nerves, into the sensorium; - and there, I suppose, affect the sense with various colours, according to their bigness and mixture;* (Young 1802, pp. 19-20). See also *Query* 13 of *Opticks*, reported in note 4.

the numbers 8, 7, and 6; and that each of the particles is capable of being put in vibration less or more forcibly, by undulations differing less or more from a perfect unison (Young 1802, pp. 20-21). In a second Bakerian lecture delivered in the same year, however, he substituted the three chosen primaries with *red*, *green* and *violet*[9], that he also considered at the vertices of his triangular-shaped colour diagram. Young's correction provides clear evidence of the confusion still existing at the beginning of the 19th century between additive and subtractive colour mixture. Between the most debate topics in the field of light and colour of the eighteenth century was, in fact, the relation subsisting between the spectral colours, that mixed together furnish white light, and those obtained by painters mixing different pigments. Young's theory constitutes the basis of modern colour vision, although it encountered, at first, many resistances, especially in the British scientific environment at that time still strongly influenced by the Newtonian tradition. These resistances were due not only to the wave theory of light proposed by Young, in opposition to the corpuscular theory attributed to Newton, but also to Young's choice of the primary colours, three in number instead of the seven reported by Newton in his colour circle.

Also Young's theory of colour constituted a starting point for Maxwell and Helmholtz's research in the field, as illustrated in the next chapters. The wave theory of light highlighted not only the limits of Newton's theory but also the richness of ideas, that were the background to Maxwell and Helmholtz's work on colour.

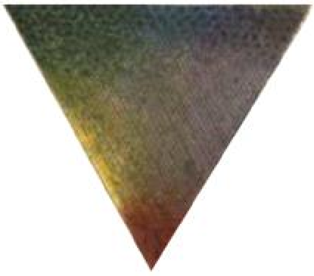

Fig. 2.7
Young's triangular representation of colours, whose description can be found in Plate XXIX of his *A course of lectures on natural philosophy and the mechanical arts*: *A triangular figure, exhibiting in theory all possible shades of colours. The red, the green, and the violet, are single at their respective angles, and are gradually shaded off toward the opposite sides: a little yellow and blue only are added in their places, in order to supply the want of brilliancy in the colours which ought to compose them. The centre is grey and the lights of any two colours, which are found at equal distances on opposite sides of it, would always very nearly make up together white light, as yellow and violet, greenish blue and red, or blue and orange* (Young 1845, vol. II, p. 786)

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[9] In Young's original words: *In consequence of Dr. Wollaston's correction of the description of the prismatic spectrum, compared with these observations, it becomes necessary to modify the supposition that I advanced in the last Bakerian lecture, respecting the proportions of the sympathetic fibres of the retina; substituting red, green, and violet, for red, yellow, and blue, and the numbers 7, 6, and 5, for 8, 7, and 6* (Young 1802b, p. 395).

2.5 Yellow, blue and red as primary colours: a brief historical account

The origin of the idea of *yellow*, *red* and *blue* as primary colours is unknown but it might be developed from dying and painting technology. The Belgian Jesuit Francois d'Aguilon expressed in his *Opticorum libri sex* (1613) the classical idea that all colours can be generated from white and black and he mentioned blue, yellow and red as chromatic primary colours. The same primaries appear in Robert Boyle's *Experiments and Considerations Touching Colours* (1664). Johann Christoffel Le Blon, the inventor of four-colour printing, considered yellow, blue and red as key colours: he discovered that using three and at most four colours a wide variety of hues can be obtained in printing. The concept of a three-dimensional logical arrangement of colour began to take shape in the eighteen century. The first proposal for colour solids had simple geometrical forms, such as double tetrahedron, cone and sphere. In 1758 Tobias Mayer, a German geographer, astronomer and physicist, realized the first model of a three-dimensional colour system, with the purpose of developing all possible hues from mixtures of yellow, red and blue, located at the vertices of a triangle. The basis triangle was then expanded toward white, upward, and toward black, downward, giving rise to a double tetrahedron.

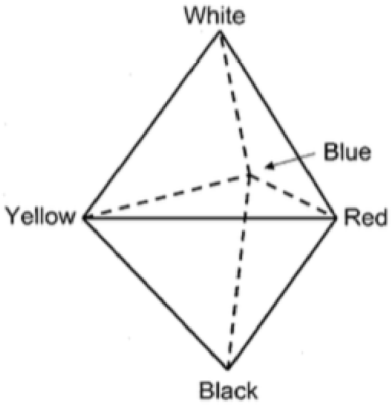

Fig. 2.8
Schematic representation of Mayer's double tetrahedron

The triangular pyramid, an iconic document in the history of colour systems, was introduced by the Alsatian mathematician, physicist and astronomer Johann Heinrich Lambert in his book *Beschreibung einer mit dem Calauischen Wachse ausgemalten Farbenpyramide (Description of a colour pyramid painted with Calau's wax)* published in 1772. Lambert developed his own colour system using Mayer's double tetrahedron as a basis. Lambert found out that mixing the three primaries yellow, red and blue in full concentration, a black was obtained, and as a result, he concluded that there was no need for the lower tetrahedron and built his colour pyramid with yellow, red, blue and white as vertices, as depicted in Fig. 2.9.

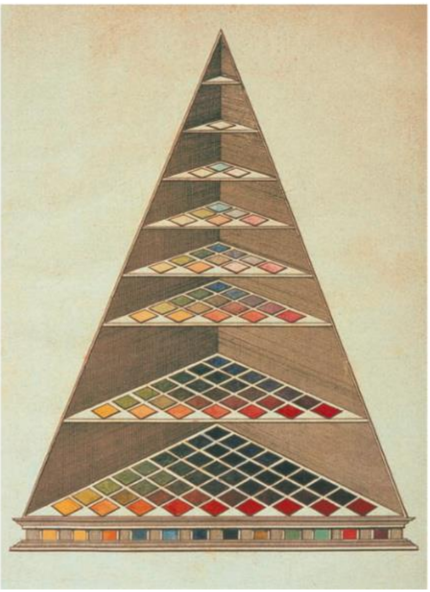

Fig. 2.9
Lambert's colour pyramid

Also the German painter Otto Runge (see section 2.3) developed his sphere model starting from Mayer triangle of the basic colours yellow, red and blue.

At the beginning of the 19th century, the idea of three primary colours received a significant boost from Young, as described in the previous section, who reported in the first volume of his writing *A course of lectures on natural philosophy and the mechanical arts* that *we may consider white light as composed of a mixture of red, green and violet only [...]* (Young 1845, p. 344). Young proposed, in fact, for the first time three principal colours different from those used by dyers and painters.

The Scottish physicist James David Forbes, Professor of Natural Philosophy at the University of Edinburgh, was interested in the problem of colour classification and developed his own version of Mayer's triangle in his paper *Hints towards a classification of colours* (Forbes 1849). We will see in the next chapter that Maxwell began his research in the field of colour under the mentoring guidance of Forbes. In his paper, Forbes, using the experimental data of Lambert, showed the possibility to match all colours in Mayer's triangle with angle segments of the three primaries yellow, red and blue: *It is to Mayer, the mathematician, that we owe a complete and perfect diagram of mixed colours, starting from red, yellow and blue, as constituents. Let the extreme corners of a triangle be painted of these colours, and let the periphery of the triangle be composed of graduating colours between each pair of these respectively; then the centres of the sides of the triangle will be occupied by perfect* *orange, perfect green and perfect purple, each of which will pass in each direction towards the predominating primary colour* (Forbes 1849, p. 168). He also exploited Newton's gravimetric rule for the mixture of lights, as we can read in the following excerpt: *if the three colours, red, yellow and blue, be mixed in equal proportions, the resulting colour, which will be neutral gray, will be found at the centre of gravity of the triangle W* (Forbes 1849, p. 168).

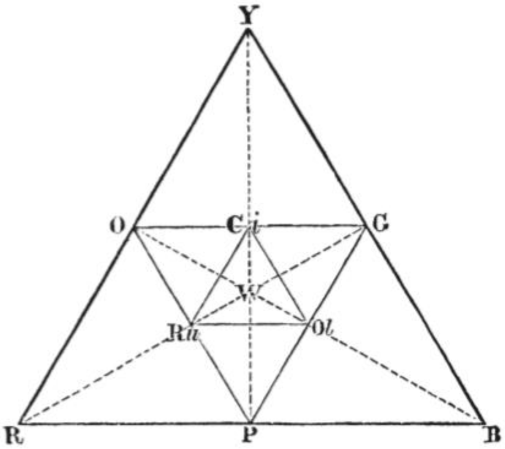

Fig. 2.10
Forbes' colour triangle based on Mayer's model. The primary colours yellow (Y), red (R) and blue (B) form, in binary mixture, orange (O), green (G) and purple (P). These three form, in turn, rubine (Ru), citrine (Ci) and olive (Ol). At the centre neutral grey (W) can be found (Forbes 1849)

Chapter 3

Maxwell and Helmholtz's key contributions to the theory of colour

All vision is colour vision, for it is only by observing differences of colour that we distinguish the forms of objects. I include differences of brightness or shade among differences of colour (James Clerk Maxwell, Niven 1890, vol. II, p. 267)

This chapter provides an overview of Maxwell and Helmholtz's pivotal achievements in the theory of colour. As a matter of fact, the two polymaths created inter alia the foundation for quantitative colour measure, for practical colour photography and for colour metrics. From their first attempt of colour diagrams the CIE, *Commission Internationale de l'Éclairage* (*International Commission on Illumination*), model has evolved, first published in 1931, which is the most popular system of colour representation in use today and considered the basis of modern colorimetry. My intention is to offer here a chronological account, as much as possible, of Helmholtz and Maxwell's contributions trying to highlight how they interpreted and elaborated Newton and Young's theory of colour. The ideas of Newton and Young, in fact, were especially important in shaping the development of their work, representing, therefore, a fundamental starting point for both Maxwell and Helmholtz's research in the field. Here Newton's influence on their work on colour will be discussed, underlying the aspects that they accepted and those they refined of his colour theory, in order to better understand the evolution of their own work.

Among Maxwell and Helmholtz's key contributions to the theory of colour it is worth mentioning that they clarified, for the first time and independently from one another, the distinction between additive and subtractive colour mixing, eliminating definitively Newton's confusion between optical and pigment mixture of colours. Furthermore, they adopted Newton's analogy of the centre of gravity to predict the outcome of optical mixture of light and eliminated the arbitrary choice of the number of primary colours. It is also worthwhile to spend some words about the work of the mathematician Hermann Günther Grassmann published in 1853, which was his first and only real contribution to the field of colour but, nevertheless, of pivotal importance for the development of modern colorimetry and constituted a significant source of inspiration for both the scientists.

Maxwell and Helmholtz colour mixing experiments will be also described, such as, in chronological order, the V-shaped slit experiment performed by Helmholtz to obtain direct composition of coloured spectral lights, the famous Maxwell's *colour top*, that allowed him to determine the first quantitative expression of colour mixing in form of colorimetric equations, Helmholtz's colour mixing experiment to detect pairs of complementary colours and Maxwell's *colour box*, an experimental device built exploiting Newton's crucial experiences with lenses, prisms and mirrors in order to obtain more accurate and reproducible colour equations.

I will also try to sketch the reasons that led Helmholtz, who at first expressed scepticism about Young's three-receptor hypothesis, and Maxwell to modify and adopt Young's theory of colour. This theory, in its revised form, assumed a central role in the second part of Helmholtz's monumental work, *Handbuch der physiologischen Optik (Treatise on Physiological Optics),* published in 1860. I will conclude this chapter briefly presenting Maxwell and Helmholtz's laboratories, at the time they were appointed first Professors of Experimental Physics, Maxwell in Cambridge, where he gave birth to the *Cavendish Laboratory*, and Helmholtz in Berlin.

3.1 Hermann von Helmholtz's first essays

Born in Potsdam in 1821, Hermann von Helmholtz stands for the whole diversity of scientific research. Helmholtz reflected a natural science which spanned the fields of physiology, physics, mathematics, chemistry and psychophysics. From 1838 to 1842, Helmholtz was student at the Royal Friedrich-Wilhelm Institute for Medicine and Surgery in Berlin, where he came into contact to Müller, who was Professor of Physiology and consolidated friendship with other Müller's students, physiologists Brücke and du Bois-Reymond. Helmholtz, inspired by the work of Müller, began here to find connections between physiology and physics, aim which he pursued throughout his life, as we will see in the next chapters. From 1843 to 1848, Helmholtz was army surgeon in Potsdam. After the military service, he became Lecturer at the Academy of Arts and assistant in the Anatomical Museum in Berlin. He then moved to Königsberg in 1849, where he was appointed Professor of Physiology; here he began systematically his research on colour[10].

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[10] For a complete biography of Helmholtz, the reader is referred to the text of the German mathematician and historian of science Leo Königsberger (Königsberger 1906).

From that moment onwards, he never abandoned his studies on colour and colour perception. He developed new simple apparatus through which he could offer new observations of pivotal importance for colour research.

Helmholtz's first essays on colours were published both in the *Annalen der Physik* in 1852 with the titles *Über die Theorie der zusammengesetzen Farben (On the Theory of Compound Colours)* (von Helmholtz 1852a) and *Über Hrn. D. Brewster's neue Analyse des Sonnenlichts (On Sir David Brewster New Analysis of Solar Light)* (von Helmholtz 1852c). In the latter, Helmholtz expressed his criticism toward the theory of colour of the Scottish physicist and inventor David Brewster.

Brewster, like Goethe, had stated that there were three different kinds of light, red, yellow, and blue exhibiting every degree of refrangibility and arranged in a way that the red light contains a preponderance of rays of less refrangibility, the yellow more rays of mean refrangibility, and the blue more rays of greater refrangibility. Brewster also supposed that all the remaining colours of the spectrum could be obtained by mixing these three primary colours. He then proposed three hypothetical intensity curves for his primaries. Brewster's theory was widely accepted until Helmholtz and Maxwell demonstrated its errors. The main object of Helmholtz's essay was to prove Brewster's view untenable and he succeeded in this purpose (von Helmholtz 1852c).

In his paper *On the Theory of Compound Colours,* Helmholtz exhibited his research on colour mixing and clarified definitively important topics of colour science, that until that time had remained confused. He opened his essay clarifying an important difference between the physiology of the visual system, which acts by synthesis of stimuli, and that of the auditory system, which acts by analysis of stimuli: *LUMINOUS rays of different wavelength and colour distinguish themselves in their physiological action from tones of different times of vibration, by the circumstance that every two of the former, acting simultaneously upon the same nervous fibres, give rise to a simple sensation in which the most practised organ cannot detect the single composing elements, while two tones, though exciting by their united action the peculiar sensation of harmony or discord, are nevertheless always capable of being distinguished singly by the ear. The union of the impressions of two different colours to a single one is evidently a physiological phenomenon, which depends solely upon the peculiar reaction of the visual nerves* (von Helmholtz 1852b, p. 519). He then provided a brief account of the history of the theory of three primary colours, underlying that this theory was based, until that moment, on a single mode of experiment, i.e. the composition of coloured pigments. Helmholtz offered in this paper a clear explanation of the difference between the composition of coloured lights and that of coloured pigments, as we can see in the following pages. He organized his essay around the question of the connotation of *primary colours*, reporting the threefold meaning that they had acquired:

1. That the primitive colours were such as permitted of the formation of all others from their combinations. 2. Or, as supposed by Mayer and Brewster, that the primitive colours correspond to three different kinds of objective light. 3. Or, as supposed by Thomas Young, that they correspond to three primitive modes of sensation experienced by the visual nerves, and from which the remaining sensations of colour are composed (von Helmholtz 1852b, p. 522).

For what concerns the second point, I would like to add a short remark about Brewster's theory of colour. Brewster, embracing the Newtonian tradition, was convinced of the superiority of the corpuscular conception of light over the undulatory interpretation; on the basis of the corpuscular theory, he believed to have experimentally demonstrated the existence of three fundamental kinds of light, distributed in various proportions in the spectrum. Therefore, colour would not derive, according to Brewster, from a physiological function of the human eye but it would be an objective property of light.

Excluding the second view, Helmholtz's aim was to test the other two by using the prismatic colours. For this purpose, he built a simple experimental device that allowed him to obtain binary mixtures of spectral colours, in all possible combination and at different intensities. By observing through a prism, located vertically, the spectra produced by a V-shaped slit (see Fig. 3.1) he could obtain all the combinations produced by the optical mixture of two spectral colours because each coloured band from the one spectrum intersect all the bands of the other. The intensity of the two colours could be varied by shifting the prism from its originally position to one more or less oblique. All the combinations could be investigated using a telescope.

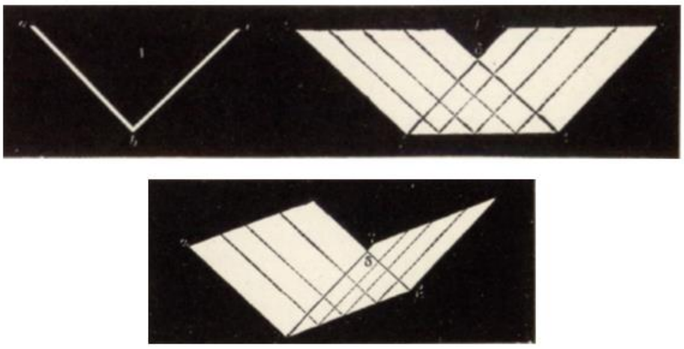
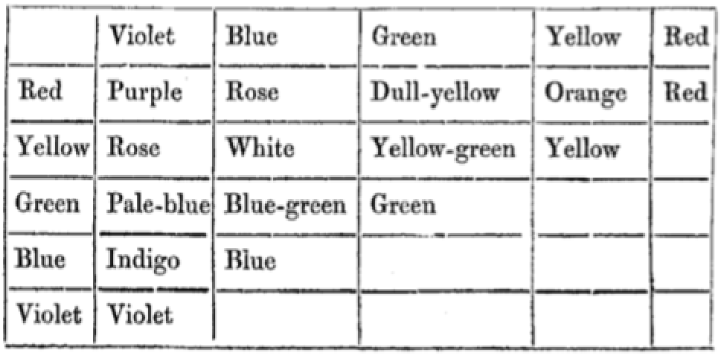


Fig. 3.1
Helmholtz's V-shaped slit experiment, as reprinted in his *Handbuch* (von Helmholtz 1962, p. 158)

In order to assure the purity of the coloured rays, Helmholtz used a flint-glass prism that permitted to see a great number of the finer Fraunhofer lines when direct sunlight was applied. The presence of the Fraunhofer lines assured him that in the spectra formed by the two limbs of the V-shaped slit the differently coloured rays could not overlap each other. He viewed the V-spectrum directly with a telescope, placed at suitable distance from the prism. Helmholtz presented the obtained results using the following words: *My observations, the principal points of which I have had corroborated by the testimony of several other persons practised in the judgement of colours, thus avoiding whatever error the subjective defects of my own eyes might occasion, have furnished the following results, some of which differ, in a surprising manner, from the views on this subject heretofore held.* [...] *The most surprising fact, and that which deviates most from the views hitherto entertained, is, that of the colours of the spectrum there are only two which together give pure white, that is, which are complementary to each other* (von Helmholtz 1852b, pp. 525-526). As we can read in this excerpt, Helmholtz was aware of the subjective aspect of colour perception; in order to limit errors due to subjective defects, in fact, he asked the testimony of several people with previous experience in colour judgment. As we will see in the next pages, Maxwell also recognized that the scientific approach to the problems of colour perception required careful consideration of the subjective dimension.

Examining separately each pair of superimposed spectral colours, Helmholtz detected only one pair of complementary colours, namely yellow and indigo-blue, that mixed together gave a pure white.



	Violet	Blue	Green	Yellow	Red
Red	Purple	Rose	Dull-yellow	Orange	Red
Yellow	Rose	White	Yellow-green	Yellow	
Green	Pale-blue	Blue-green	Green		
Blue	Indigo	Blue			
Violet	Violet				

Table 3.1
Table illustrating the obtained results of binary mixtures of spectral colours by using the V-shaped slit. Yellow and blue was the only pair that furnished a neutral tint (von Helmholtz 1852b)

This was in contradiction to Newton's theory of colour mixing, that required an infinite number of pairs of complementary colours, as the German mathematician Hermann Günther Grassmann recognized. Furthermore, Helmholtz, analysing the obtained results, found out that mixing red and green, and green and violet lights, he could not obtain colours as saturated as spectral yellow and blue respectively. Hence, he put forward the idea that a minimum of five simple colours (red, yellow, green, blue and violet) was necessary to produce all the possible hues by paired mixture: *Hence if we propose to ourselves the problem of imitating the colours of the spectrum by the union of the smallest possible number of simple colours, we find at least five of the latter necessary for this purpose, namely red, yellow, green, blue, violet. [...] According to the above we must also abandon the theory of three primitive colours, which, according to Thomas Young, are three fundamental qualities of sensation* (von Helmholtz 1852b, pp. 532-533). He considered at first Young's three-receptor theory untenable. We will see (see section 3.15) that he soon recognized the incorrectness of this conclusion. In the following years, in fact, he reversed his position on Young's hypothesis, that assumed, although in a revised form, a central role in the second part of his *Handbuch der physiologischen Optik (Treatise on Physiological Optics)* published in 1860 (von Helmholtz 1962).

In this paper of 1852, Helmholtz also clarified the difference between additive and subtractive mixture of colours. I will return to this point in section 3.5, in order to offer a more complete analysis, also in the light of Maxwell's contributions.

After the publication of his first papers, however, some scientists expressed their criticism. Joseph Plateau complained that already in 1829 he had demonstrated the difference of the two phenomena using a colour top (Plateau 1853). Plateau had already verified that mixing yellow and blue on a spinning top yielded a greyish tint instead of a green one, as Maxwell recognized. Grassmann, furthermore, in his paper *Zur Theorie der Farbenmischung (On the Theory of Compound Colours)*, published in German in 1853 and translated into English in 1854, proved that every spectral colour must have a complementary, as Newton had asserted in his theory of colour.

3.2 Grassmann's contributions

Grassmann came to the study of colour with the aim to find connections between the laws of nature and the forms of geometrical analysis he had developed in his first major work of 1844, *die Lineale Ausdehnungslehre (Extension Theory).* Also for Grassmann, Newton's work on colour represented a fundamental starting point for his research in the field. It is worth reporting the key elements of Grassmann's theory of colours, not only because his contributions constitute the basis for modern colorimetry, but also because his work represented an important source of motivation for Helmholtz. In this article, which was his first and only contribution to the field of colour, he applied the form of geometrical analysis to colour space. Grassmann began his essay reporting Helmholtz's results contained in the article of 1852 and declaring his aim: *Helmholtz published a series of observations, some of which were new and ingenious, and from which he comes to the conclusion, that the theory of compound colours, universally admitted since the time of Newton, is erroneous in its most essential points. […] It may consequently not be regarded as superfluous to show that the Newtonian theory of compound colours is correct to a certain point, and especially that the opinion that every colour has its complementary colour, with which when mixed it gives white, is founded upon mathematically* *incontestable facts, so that this opinion must be regarded as one of the most tenable in physics* (Grassmann 1854, p. 254).

The paper contains Grassmann's four rules, nowadays called Grassmann's laws, simple relationships among colours and their composition. The first rule regards the variables that characterize colour vision, that he called *mathematically determinable elements*, i.e. the *tint*, the *intensity of the colour* and the *intensity of the intermixed white*. He considered the nature of a colour as dependent on three variables, such as Maxwell did, as we will see, although calling them with different names (*greeness*, *redness* and *blueness*). We can anticipate here that Maxwell showed that these two methods of considering colour can be deduced one from the other: it becomes, therefore, a matter of geometry. Grassmann exposed then the second law with the following words: *In the second place we assume, "that if one of two mingling lights be continuously altered (whilst the other remains unchanged), the impression of the mixed light also is continuously changed."* (Grassmann 1854, p. 256). Grassmann derived then his colour circle from Newton's, embracing the idea of a spectrum in which colours change continuously along the circumference of the circle. This continuity condition required that the impression of colour of the extreme violet of the spectrum was identical to that of the extreme red, in order to close the circle. He also called the transition from red through orange, yellow, green, blue, violet and purple *positive* and that in the reverse direction *negative*, as depicted in his colour circle. He tried to offer a more quantitative version of Newton colour circle indicating the location of Fraunhofer lines in it, as we can see in Fig. 3.2.

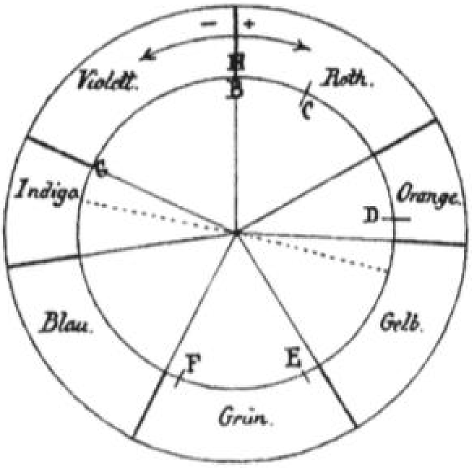


Fig. 3.2
Grassmann's version of Newton's colour circle, with indication of the Fraunhofer lines (capital letters) and the only pair of complementary colours detected by Helmholtz (dotted line segments) (Grassmann 1854)

From the first two laws, Grassmann deduced mathematically that *"To every colour belongs another homogeneous colour, which, when mixed with it, gives colourless light."* (Grassmann 1854, p.257) or, in modern terms, that for each colour there is a complementary spectral colour.

Grassmann expressed then the third law: *That two colours, each of which has a constant tint and a constant intensity of the intermixed white, also give constant mixed colours, no matter of what homogeneous colours they may be composed* (Grassmann 1854, p. 260). In other words, the result of an additive mixture of colours depends only on their appearance and not on their physical composition. The third postulate is of great importance because it allows us to abstract from the physical characteristics of light and to speak simply of colours. Grassmann could in this way interpret colours as vectors and their composition in terms of vector addition. He proposed, in fact, a mathematical method to predict the outcome of additive mixture of spectral colours based on vector addition: *Suppose the intensities of the two colours to be mixed to be signified by the length of the lines representing them, so that if a colour has the tone* a, *and its intensity is in the same proportion to that of* a *as* m *to 1, then the colour may be represented by a line having the same direction as* a, *but* m *times its length. Having represent in this manner the two colours geometrically, let us construct from these lines the geometrical sum, that is, the diagonal of the parallelogram which has the two lines for its side, and assume that this sum or diagonal shall represent the colour of the mixture, its direction showing the tint, and its length the intensity of the colour. This done, the tint and intensity of any mixture of colours may be found by mere construction* (Grassmann 1854, pp. 261-262).

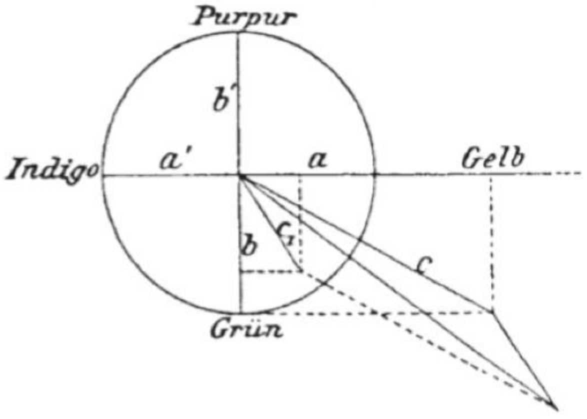


Fig. 3.3
Vector addition exploited by Grassmann to predict the result of the composition of spectral colours (Grassmann 1853)

He eventually formulated the fourth law: the total intensity of an additive mixture of colours is the sum of the intensities of the mixed colours, i.e. the light intensity of a mixture is additive.

It is important to highlight that Grassmann, in order to predict the outcome of optical mixtures of colours, adopted Newton's analogy with the calculation of the centre of gravity. Maxwell also adopted this analogy, as we will see in the next section, providing a clearer and general quantitative definition. Grassmann wrote, referring to Fig. 3.4 and applying his fourth law: *Let the colours* αA *and* βB *be mixed, i.e. two homogeneous colours, whose intensities are* α *and* β*, and their tints* A *and* B*, then the sum of the intensities is* $\alpha + \beta$*. In order now to determine the colour of the mixture, we have [...] to find the centre of gravity of the points* A *and* B*, furnished with the weight* α *and* β*. Let this be* C*, the centre of the circle* O*; then if the radius of the circle be supposed =1, the intensity of the colour is* $= (\alpha + \beta)OC$*. Let the point at which* OC*, if produced, strikes the periphery be* D*, then the total intensity is* $(\alpha + \beta)OD$*. The total intensity, according to the supposition, will be equal to the intensity of the colour plus the intensity of the intermixed white* (Grassmann 1854, pp. 263-264). Prolonging from the centre to the circumference the segment on which point *C* lies, the hue of the composed colour can be determined, at point *D*; the intensity of the colour is equal to *OC* multiplied by its weight $(\alpha + \beta)$, and the distance from the circumference, *CD*, multiplied by the weight represents the intensity of the intermixed white.

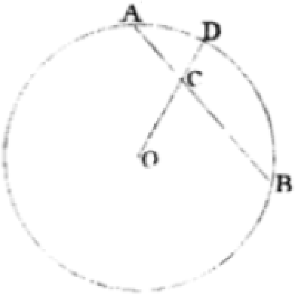


Fig. 3.4
Grassmann's geometrical construction to detect the position of a mixture of two colours calculating the centre of gravity (Grassmann 1853)

Grassmann closed the essay stating that from his four laws, each confirmed by experience, it was possible to deduce results in line with Newton's empirical rule. However, according to Grassmann, the way in which Newton distributed the homogeneous colours on the circumference of his circle required a total revision. This revision was undertaken, a few years later, by Helmholtz and his collaborators König, Dieterici and Brodhun and, in 1931, it led to the definition of the chromaticity diagram as we know it today. Both Maxwell and Helmholtz acknowledged in their works the influence of Grassmann's essay on colour. Maxwell, in particular, in his paper of 1860 reported Grassmann's vector addition to predict the composition of colours, as described in the following lines. In paragraph IV of Maxwell's paper of 1860, entitled *Method of representing Colours by Straight*

Lines drawn from a Point, he reported a geometrical method to represent the outcome of a mixture of colours developing Grassmann's idea. He considered the plane containing his colour triangle and established the colour intensity equal to 1. He identified a point 0, corresponding to the origin, outside this plane. The *direction* of a line joining the origin with any point of the triangle indicated the quality of the colour that occupied that position on the chart. In order to indicate the *quantity*, he extended it beyond the plane of the diagram in the same ratio as the colour exceeded in illumination the unit. Another parallel plane was then defined for another certain amount of colour. The intersection points of these planes with a given vector represented a colour and the plane to which it belonged determined the brightness. This could be applied to all the colours inside the triangle, outside of it and also on the edge. *In this way every colour in nature will be represented by a line drawn through the origin, whose direction indicates the quality of the colour, while its length indicates its quantity* (Clerk Maxwell 1860, p. 63). In Fig. 3.5 we can see the resultant of two colours obtained by using the parallelogram law of vectors. Maxwell, indeed, following Grassmann, exploited the analogy with the composition of forces in mechanics for the composition of colours.

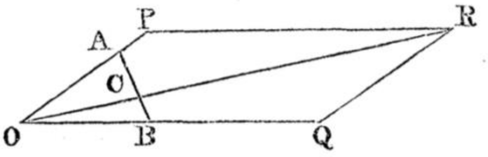


Fig. 3.5

Parallelogram law of vectors used to predict the outcome of the composition of two colours. AB represents a section of the plane of the diagram, OP and OQ represent two colours, the quantity of P can be expressed as $p = OP / OA$ and the quantity of Q is $q = OQ / OB$. The resultant is represented by C, where $AC : CB = q : p$ and its quantity can be expressed as $p + q$ so that $OR = OC\ (p + q)$. OR represents the diagonal of the parallelogram of which OP and OQ are two sides (Clerk Maxwell, p. 63)

This constitutes an example of Maxwell's use of analogy to create connections between different branches of scientific knowledge. In Maxwell's paper *On Faraday Lines of Force* published in 1856, he explained the meaning of physical analogy, method that he exploited throughout all his researches: *In order to obtain physical ideas without adopting a physical theory we must make ourselves familiar with the existence of physical analogies. By a physical analogy I mean that partial similarity between the laws of one science and those of another which makes each of them illustrate the other. Thus all the mathematical sciences are founded on relations between physical laws and laws of numbers, so that the aim of exact science is to reduce the problem of nature to the determination of quantities by operations with number* (Niven 1890, vol. I, p. 156).

3.3 James Clerk Maxwell and his colour top

Colour analysis and perception was one of Maxwell's main research topic. Born in Edinburgh in 1831, James Clerk Maxwell always showed, even as a child, his curiosity toward colour[11]: *that (sand) is red; this (whin) stone is blue. But how d'ye know its blue?* (Campbell and Garnett 1882, p. 31). In Edinburgh he attended school at the Edinburgh Academy and he progressed to the Edinburgh University until 1850, before moving to Cambridge University. Maxwell's curiosity led him to never abandon his studies on colours throughout his life, creating inter alia the foundation for quantitative colour measure and for practical colour photography. The ideas expressed by Maxwell related to colour can be counted among the most advanced of the nineteenth century. It is worth, therefore, reconstructing through which steps and with the aid of which experimental set-ups Maxwell came to their elaboration. Before proceeding, I would like to point out that at that time Edinburgh hosted a rich group of scholars dealing with colours: besides Forbes and Hay, already mentioned in chapter 2, I would like to cite William Swan, a physicist dealing with optics and spectroscopy, David Brewster (see section 3.1) and George Wilson, a chemist interested in the problem of colour deficiencies, as we will see in the following pages. This flourishing scientific background gave Maxwell a significant boost to the development of his own theory of colour. His approach to colour vision was shaped by the attempts of Hay and Forbes to provide a nomenclature of colours and by Forbes' use of the colour triangle as colour diagram to represent the combination of colours.

Maxwell began systematically his research on colours in the spring of 1849 in Edinburgh under the mentoring guidance of Forbes. Their first aim was to obtain a quantitative expression for the mixing of colours, more fulfilling than those available at that time. To reach this purpose, Maxwell built an experimental device upon suggestion of Forbes, the so called *colour top*.

Forbes had already experimented the mixing of coloured papers using a rotating disc. He endeavoured to form a neutral tint by the combination of three colours only and he found out, contrary to expectations, that using as primary colours those of painters, *yellow, blue* and *red* the resulting tint

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[11] A complete biography of Maxwell can be found in the text of Campbell and Garnett (Campbell and Garnett 1882). Lewis Campbell was one of Maxwell's classmates at the Edinburgh Academy and lifelong friend. Campbell, together with William Garnett, Maxwell's demonstrator at the Cavendish Laboratory (see section 3.16), wrote, in fact, a biography of James Clerk Maxwell, published three years after Maxwell's death, which represents the principal source of information for all subsequent biographies.

could not be rendered neutral by any combination of the three. For their experiments, therefore, Maxwell and Forbes used *ultramarine, vermilion* and *emerald green* as primaries[12].

The first set of coloured papers for their experiments were furnished by David Ramsey Hay. Maxwell was introduced by the father, John Clerk Maxwell, and the uncle, John Cay, to the *Royal Scottish Society of Arts*, of which they were prominent members and in that occasions he came into contact with Hay, a decorative painter and colour theorist living in Edinburgh in that time, as we have seen in chapter 2.

In his paper of 1855 entitled *Experiments on Colour, as perceived by the Eye, with Remarks on Colour-Blindness,* Maxwell described in detail his spinning top[13] and explained his colour mixing experiments. His results were presented at the Royal Society of Edinburgh in the same year.

Maxwell's colour top consists of three coloured papers, *ultramarine*, *vermilion* and *emerald green*, as we can see in Fig. 3.6, put at the top of a wooden disc. The circumference of the plate was divided into 100 equal parts in order to quantify the amount of colours that entered in a given mixture. Maxwell modified the first version of the colour top, realized by Forbes, inserting a central disc, of smaller dimensions, containing *black* and *white* paper discs. The top was then set in rapid rotation and an observer should compare the colour of the outer with that of the inner circle. The arrangement should be than changed to render the resultant tint of the outer and of the inner circle alike. The number of divisions occupied by the different colours must be read and registered in form of an equation. It is worth reading Maxwell's original words used to describe his colour mixing experiment:

When this system of discs is set in rapid rotation, the sectors of the different colours become indistinguishable, and the hole appears of one uniform tint. The resultant tints of two different combinations of colours may be compared by using a second set of discs of a smaller size, and placing these over the centre of the first set, so as to leave the outer portion of the larger discs exposed. The resultant tint of the combination will then appear a ring round that of the second, and may be very carefully compared with it. [...] As an example of the method of experimenting, let us endeavour to form a neutral gray by the combination of vermilion, ultramarine and emerald green.

The most perfect results are obtained by two persons acting in concert, when the operator arranges the colours and spins the top, leaving the eye of the observer free from the distracting effect of the bright colours of the papers when at rest.

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[12] *It may be asked, why some variety of yellow was not chosen in place of green, which is commonly placed among the secondary colours, while yellow ranks as a primary?* (Clerk Maxwell 1855, p. 276). Maxwell answered to this question remarking the impossibility to obtain a neutral tint using the painters' primaries, that disappointed Forbes' expectations.

[13] Another detailed description of Maxwell's colour top, or *chromatic teetotum*, can be found in *Manuscript on the comparison of colours using a spinning top*, dated 27 February 1855 (Harman 1990, pp. 284-286).

After placing discs of these three colours on the circular plate on the top, and smaller discs of white and black above them, the operator must spin the top and demand the opinion of the observer respecting the relation of the outer ring to the inner circle.
[…] The arrangement must then be changed, so as to render the resultant tint of the outer and inner circles more nearly alike (Clerk Maxwell 1855, pp. 275-276).

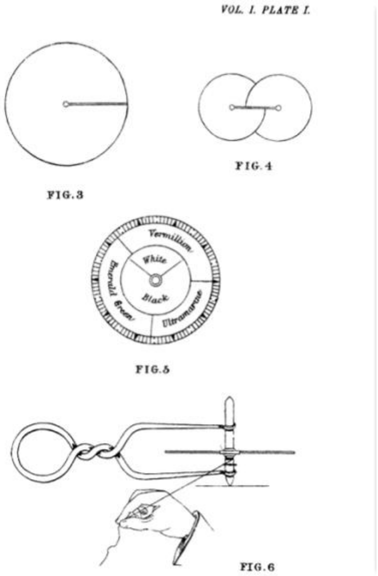

Fig. 3.6
Illustration of the working principle of Maxwell's colour top, indicating also the correct disposition of the five discs of paper. The diagram can be found at the end of his paper of 1855 (Clerk Maxwell 1855)

Maxwell registered the obtained results in the form of colorimetric equations. Here the typical form of his equation is reported, referring to an experiment conducted "on the 6th March 1855, in daylight without sun":

$$0.37\ V + 0.27\ U + 0.36\ EG = 0.28\ SW + 0.72\ Bk$$

The colours are labelled using letters *V* for vermillion, *U* for ultramarine, *EG* for emerald green, *SW* for snow white and *Bk* for black and the coefficients before each letter represent the number of divisions occupied by the different colours that can be read on the edge of the plate.
Maxwell also substituted for each chosen standard colour one of the colour standing under it, obtaining in this way equations that contained two standard colours and one of the remaining colours. Studying pale chrome (*PC*), for instance, the matching was:

$$0.34\,PC + 0.55\,U + 0.12\,EG = 0.37\,SW + 0.63\,Bk$$

He also made experiments in which the resulting tint was a certain colour instead of a neutral grey and experimented with different illumination conditions, reporting the obtained results by day-light and by gas-light. He came to the following conclusions: *1st, That the human eye is capable of estimating the likeness of colours with a precision which in some cases is very great. 2d, That the judgment thus formed is determined, not by the real identity of the colours, but by a cause residing in the eye of the observer. 3d, That the eyes of different observers vary in accuracy, but agree with each other so nearly as to leave no doubt that the law of colour-vision is identical for all ordinary eyes* (Clerk Maxwell 1855, p. 278)[14].

Helmholtz was aware of Maxwell's research in the field and reported in his *Handbuch* a clear description of the top, underlying also its pivotal role in verifying the correctness of Newton's law of colour mixing in analogy with the construction of the centre of gravity: *NEWTON's device of exhibiting the laws of colour mixture by the method used for constructing centres of gravity was intended originally simply as a kind of mathematical picture for expressing graphically a large mass of facts [...]. However, quite recently the method has been still further supported by careful quantitative measurements made by MAXWELL* (von Helmholtz 1962, p. 140). Reading Newton's contributions, in fact, Maxwell found a valid indication of the method to predict the outcome of optical mixtures of light.

Newton reported in his *Opticks* his famous colour circle and laid down a clear method for predicting the compound colour which results from a combination of *seven* principal colours, as we have shown in chapter 2. It is worth at this point reporting the description of Newton's colour circle in Maxwell's words. Maxwell adopted Newton's analogy with the centre of gravity for the composition of colours, having recognized the key role of analogy as a powerful method to conduct in parallel research in different fields of knowledge, as anticipated in section 3.2[15]. In a note of this paper Maxwell detailed in his own words Newton's method of colour mixing. It is worth reporting Maxwell's original words in order to better understand all his future considerations:

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[14] In *Manuscript on the comparison of colours using a spinning top*, in which Maxwell described in detail the construction of the colour top by J. M. Bryson, an optician from Edinburgh, he expressed his reflection upon the accuracy of the human eye: *[...] the human eye is an instrument constructed so as to give uniform results, and is the same in persons of all climates, ages and degree of cultivation and it is found that even those defects which sometimes occur may be accounted for by supposing a part of the complete organisation wanting* (Harman 1990a, p. 286).

[15] This analogy is clearly expressed in his paper of 1860: *This method of determining the position of the resultant colour is mathematically identical with that of finding the centre of gravity of two weights, and placing a weight equal to their sum at the point so found. We shall therefore speak of the resultant tint as the sum of its components placed at the centre of gravity* (Clerk Maxwell 1860, p.62).

NEWTON’S theorem on the mixture of colours is to be found in his Opticks, Book I, Pt. II., Prop. VI. In a mixture of primary colours, the quantity and quality of each being given, to know the colour of the compound.

He divides the circumference of a circle into parts proportional to the seven musical intervals, in accordance with his opinion of the division of the spectrum. He then conceives the colours of the spectrum arranged round the circle, and at the centre of gravity of each of the seven arcs he places a little circle, the area of which represents the number of rays of the corresponding colour which enter in the given mixture. […]

NEWTON, by this construction (for which he gives no reasons), plainly shows that he considers it possible to find a place within the circle for every possible colour, and that the entire nature of any compound colour may be known from its place in the circle. It will be seen that the same colour may be compounded from the colours of the spectrum in an infinite variety of ways. The apparent identity of all these mixtures, which are optically different, as may be shown by the prism, implies some laws of vision not explicitly stated by NEWTON.

The only safe method of completing NEWTON’s construction is by an examination of the colours of the spectrum and their mixtures, and subsequent calculation by the method used in the experiments with coloured papers (Clerk Maxwell 1855, pp. 294-295).

Using the colour top, Maxwell could obtain the first colorimetric equations, after a work of renormalization that takes into account the total intensity of the white. From these equations he built his triangular colour diagram[16], eliminating definitively Newton’s arbitrary choice of the numbers of primary colours. To render it evident, on Newton’s colour circle, Maxwell superimposed the colour triangle, as depicted in Fig. 3.7.

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[16] From Maxwell’s colour triangle has evolved the CIE, *Commission International d’Eclairage* (*International Commission on Illumination*), model, first published in 1931, which is the most popular system of colour representation in use today and considered the basis of modern colorimetry (see chapter 6).

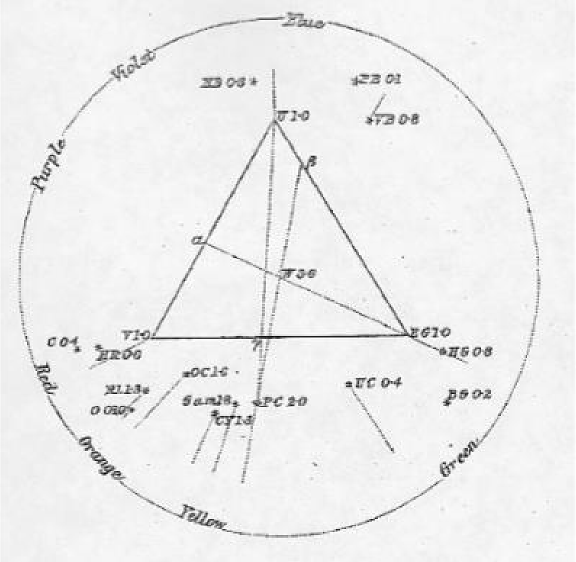

Fig. 3.7
Maxell's equilateral triangle of colours inside Newton's colour circle. The number of principal colours of Newton's model appears redundant. This diagram can be found at the end of his paper of 1855 (Clerk Maxwell 1855)

Maxwell came, therefore, to the conclusion that the nature of a colour is dependent on three things, the *greeness*, the *redness* and the *blueness,* and in general dependent on three variables. In the same work of 1855, Maxwell reported: *[...] in shade, hue, and tint, we have another mode of reducing the elements of colour to three* (Clerk Maxwell 1855, p. 279), adapting terms from Hay, read in Hay's *Nomenclature of Colours,* and Grassmann, who used the words *intensity* (*Intensität der Farbe*), *tint* (*Farbenton*) and *intensity of the intermixed white* (*Intensität des beigemischten Weißs*) to denote the three colour variables in his seminal article on colour mixture. Maxwell showed that these methods of considering colour could be deduced one from the other: it became, therefore, a matter of geometry. Although the best way to represent the space of colour should be a three-dimensional space, with coordinates proportional to the three elements of the colour, Maxwell preferred to consider a two-dimensional equilateral triangle (therefore maintaining the intensity fixed) for reason of simplicity. In this way, a direct comparison with Newton's colour circle was possible.

Fig. 3.8
Forbes' original set of Maxwell's paper discs made by James Mackay Bryson of Edinburgh (1824-94) and preserved at the *National Museum of Scotland*, Edinburgh

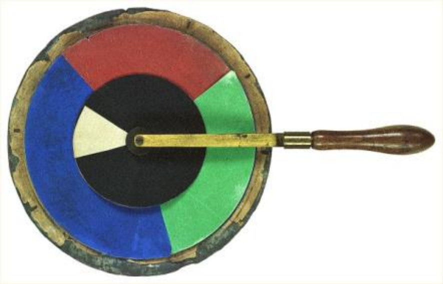

Fig. 3.9
Maxwell's original colour top preserved at the *Cavendish Laboratory*, Cambridge

3.4 Brief history of the rotating disc

Before proceeding, I would like to offer a brief account of the history of the rotating disc, from antiquity to the nineteenth century. This way to mix colours is known since antiquity. In the 2nd century AD the Greco-Roman natural philosopher Claudius Ptolemaeus gave the first written record of disc mixture illusion in his second book of *Optics*, describing the effect of a potter's wheel in motion, painted with several colours. Another clear description of disc mixture is to be found in Alhazen's Book II of *Optics*. Abu Ali al-Hasan ibn al-Haytham (965-1039), Latinized as Alhazen, was an Arab mathematician, astronomer and natural philosopher, who studied Ptolemaeus' *Book of Optics* and wrote in his own *Book of Optics* a description of a revolving top: *A clear and visible proof that perception of the quiddity of colour must take place in time is furnished by what can be observed* *in a revolving top. If the top is painted in different colours forming lines that extend from the middle of its visible surface, close to its neck, | to the limit of its circumference, then forcefully made to revolve, it will turn round with great speed. Looking at it the observer will now see one colour that differs from all the colours in it, as if this colour were composed of all the colours of those lines; he will neither perceive the lines nor their different colours; if the top moves with great speed, he will also perceive it as if it were stationary. Now if the top is moving fast, then no point in it will remain fixed in any one place for a sensible interval of time but rather traverse, in the smallest amount of time, the whole circle on which it moves; the form of any point will therefore trace out in the eye the circumference of a circle in the smallest amount of time* (Sabra 1989, p.144).

We have to wait about 700 years before finding another written description of the colour top, namely the Dutch natural philosopher Pieter van Musschenbroek (1692-1761) mentioned it in his book *Introductio ad philosophiam naturalem*, published in 1762. Helmholtz was aware of van Muschenbroek's experiments with the top, as he stated in his *Handbuch der physiologischen Optik* published in 1860: *The simplest contrivance of rotating disc, mentioned first by MUSSCHENBROEK, is the top. In most of his experiments the author uses a simple top turned out of brass [...]. It is set spinning by hand; and so it can be easily started at any time without preparation, and its velocity can be regulated at will, although the greatest velocity that can be imparted by hand is not more than about six revolutions per second; which is enough to keep it going three or four minutes. [...] In order to make the top spin faster, a cord can be wound around the stem and pulled;* (von Helmholtz 1962, pp. 215-216).

Fig. 3.10
Colour top with chord, that allowed it to spin faster (von Helmholtz 1962, p. 216)

Between the end of the 18th century and the beginning of the 19th century, not only scientists but also artists, writers and philosophers made use of rotating disc for studies concerning perception, successive contrast and persistence of visual stimuli in the human eye, animated by the same intent of deepening the connections between art and science. With the emergence of *Naturphilosophie*, in fact, a central role among the most debate topics was played by the nature of colours and their perception, topic that increasingly fascinated not only natural philosophers but also painters, poets and writers, such as Goethe and Runge (see chapter 2, section 2.3). Runge made a number of experiments using the colour top that allowed him to compare a given grey tint, resulting from the mixture of three colours, with a combination of white and black, as he wrote in a letter addressed to his friend Goethe and dated April 1808[17].

3.5 Colour top to show the difference between additive and subtractive colour mixture

Among Maxwell and Helmholtz's key contributions to the theory of colour it is worth mentioning that they eliminated definitively Newton's confusion between optical and pigment mixture of colours (confusion still existing in Young's works). In the first lines of Maxwell's paper *On the theory of compound colours with reference to mixtures of blue and yellow light* (Clerk Maxwell 1856) we can read: *When we mix together blue and yellow paint, we obtain green paint. This fact is well known to all who have handled colours; and it is universally admitted that blue and yellow make green. Red, yellow and blue, being the primaries among painters, green is regarded as a secondary colour, arising from the mixture of blue and yellow.* Then, Maxwell reported the crucial result obtained by the Belgian physicist Joseph Plateau, that anticipated Forbes' observations: *the first experiment on the subject is that of M. Plateau, who, before 1819, made a disc with alternate sectors of Prussian blue and gamboge and observed that, when spinning, the resultant tint was not green but a neutral gray, inclining sometimes to yellow or blue, but never to green. Prof. J. D. Forbes made similar experiments in 1849, with the same result. Prof Helmholtz of Königsberg, to whom we owe the most*

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[17] See Maltzahn H. *Philipp Otto Runge's Briefwechsel mit Göthe*. Weimar: Verlag der Göthe-Gesellschaft; 1940.

complete investigation on visible colour, has given the true explanation of the phaenomenon[18] (Clerk Maxwell 1856, p. 12).

Helmholtz offered, indeed, a clear and complete explanation of the difference between additive and subtractive colour mixture in his paper *Über die Theorie der zusammengesetzen Farben*, published in 1852 (von Helmholtz 1852a). On the basis of these observations, he underlined here that mixture of pigments, *Mischung der Farbstoffe*, and composition of spectral colours, *Zusammensetzung des gefärbten Lichtes*, are two different phenomena. The mixture of pigments determines a unique colour stimulus that reaches directly the eye and the retina. In the case of composition of lights, instead, different stimuli come separate and unchanged to the eye and their fusion take place in the visual system. They are, therefore, phenomena that obey completely different rules and belong to different domains: the mixture of pigments belongs to the *physical,* while the composition of lights belongs to the *physiological* domain[19]. We report an excerpt from Helmholtz's essay: *The theory of pigmentary colours here presented is simply derived from the generally recognized laws of physics; it explains the phenomena, so far as I am able to see, completely; showing that the mixture of the substances and the combination of their colours are two processes altogether distinct, and hence that the results obtained from the former furnish no conclusion regarding the latter* (von Helmholtz 1852b, pp. 529-

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[18] Forbes anticipated Helmholtz's more complete explanation in his paper *Hints toward a Classification of colours,* in which he tried to clarify the difference existing between mixture of lights and mixture of pigments with these words: *There will therefore be always this essential difference between compounding rays of the spectrum and compounding pigments; that in the former case, by throwing light of two or more colours upon a white screen, each of these colours being reflected with equal vividness, the brightness of the screen will be the sum of the brightness due to several rays [...]; but, on the other hand, by combining pigments we do not add together lights, but merely construct a ground or screen capable of scattering a greater number of the constituents of a beam of white light which falls upon it* (Forbes 1849, p. 165). In 1849, Forbes undertook experiments using colour discs, which Maxwell witnessed: under these circumstances he found out that a mixture of yellow and blue did not furnish green, contradicting the experience of all painters, but rather a pinkish tint. This discovery is reported in a letter written by Maxwell to Forbes, dated 12 May 1855: *I am glad to hear that you think blue and yellow were not known to produce pink when combined till you discovered it. I always considered it a new thing while I was with you but I was told of it as an established thing afterwards. I have since found, that most optical writers speak confidently of the same mixture as a fine green. [...] So that I must consider you as the first person who ever attempted to make green by spinning blue and yellow [...]* (Harman 1990a, p. 302). At the same conclusion came the Belgian physicist Joseph A. F. Plateau before 1819, by means of a spinning disc with alternate sectors of Prussian blue and gamboge. He observed that the resultant tint was not green but a neutral grey (compare Clerk Maxwell 1856).

[19] In this regard, Maxwell expressed his considerations on the different interpretations of the nature of colour given by physiologist, painters and opticians: *The Physiologists tell us that vision is performed by the optic nerve or nerves and that one nerve can only give us one sensation in greater or less intensity. The Painters tell us that all colours may be made by properly mixing three primary colours. Are we to suppose that all the rays of the spectrum except three are compound, and that Opticians are wrong in saying they are simple? or are we to ascribe to the nerve of vision the power of distinguishing the quality as well as the quantity of the agent which affects it contrary to the opinion of the Physiologist? By adopting Young's theory, that in the organ of vision there is a three fold sensitive apparatus of nerve fibrils, capable of different kinds of colour-sensation we reconcile the physical discoveries of optics with the practical experience of colourists in a way consistent with the conditions of nervous action* (Harman 1990a, p. 679).

530). Using an apparatus composed basically of prisms and lenses and designed to mix two spectral colours without other interference, Helmholtz experimentally verified that the model used for pigment mixtures could not be applied to composition of lights: a yellow light and a blue light, in fact, if combined in right proportion, produce an achromatic colour, while yellow pigments and blue pigments, when mixed, give a green colour. This, explained Helmholtz, is due to the fact that pigments consist of series of layers of semi-transparent particles that act as filters for the incoming light, that is reflected by the underlying layer. In a mixture of two pigments, the first absorbs part of the spectral radiation and part of the remaining radiation is then absorbed by the second. Only the radiation not absorbed by any of the two pigments is reflected. He also described two methods of composing light coming from pigments that furnish the same results obtained by mixing prismatic colours. One of these methods consists in observing the composition of one colour transmitted through a glass and another colour reflected by the glass at the same time: *Let a glass plate, with plane and parallel surfaces, be placed perpendicular to the leaf of a table, and let a coloured wafer be placed before it. The image of the wafer is reflected by the glass plate; the apparent place of the image is at the other side of the plate, and also on the surface of the table. Let another wafer of a different colour be placed upon the exact spot where the image is observed, this second wafer being seen through the glass. The observer's eye will thus be affected by two descriptions of rays, both of which appear to proceed from one and the same body, one of which however belongs to the transmitted and the other to the reflected light* (von Helmholtz 1852b, p. 530).

The other method regards the use of the colour top: *There are, however, two other methods of combining the light proceeding from pigments, which yield results altogether in harmony with those obtained from the combination of similar prismatic colours. The first of these methods is the union of the colours upon the rotating disc. It has been long noticed, that results thus obtained are different from those derived from the mixture of the pigments. I repeated the experiment with yellow and blue* (von Helmholtz 1852b, p.530). With these words, Helmholtz anticipated the fundamental proof to show the difference between additive and subtractive colour mixing contained in his *Handbuch der physiologischen Optik*. He was the first to use a colour top to exhibit in a very simple and clear manner the above-mentioned difference. We can find a clear description of the method he used in his *Handbuch: A convenient way of exhibiting the distinction here is to paint the sectors* a *and* b *on the edge of a colour disc in two different colours, while the central portion* c *is painted with a mixture of the two pigments. Thus, if the sectors are cobalt-blue and chrome-yellow, the appearance is pale grey when the disc is rotated so as to get the impression of both colours at once; but the mixture of the two pigments in the centre is a much darker green. Evidently, therefore, the result of mixing pigments cannot be used to deduce conclusions as to the effect of combining different kinds of light. The* *statement that yellow and blue make green is perfectly correct in speaking of the mixture of pigments; but it is not true at all as applied to mixture of these lights* (von Helmholtz 1962, pp. 124-125).

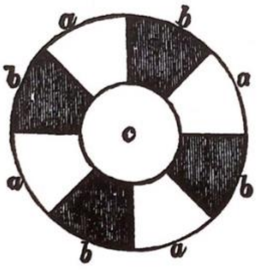


Fig. 3.11
Helmholtz's colour top used to show the difference between additive and subtractive colour mixing (von Helmholtz 1962, p. 124)

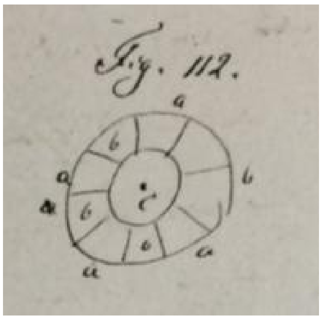


Fig. 3.12
Helmholtz's original sketch of his experiment with the colour top taken from the manuscript version of his *Handbuch* preserved at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften* (Nachlass Hermann von Helmholtz, 572 Über physiologische Optik, ∫∫20. Die zusammengesetzen Farben)

3.6 Rood's colour top

Disc-mixture technology became very popular. Ogden Nicholas Rood, a Scottish-American physicist, described in his book *Modern Chromatics, with Applications to Art and Industry*, published in 1879, some experiments conducted using the colour top. He was aware of Maxwell and Helmholtz's results in the field and made use of this device to obtain quantitative results to exhibit clearly the difference between additive and subtractive colour mixture, inspired by Helmholtz's method.

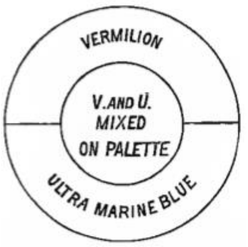
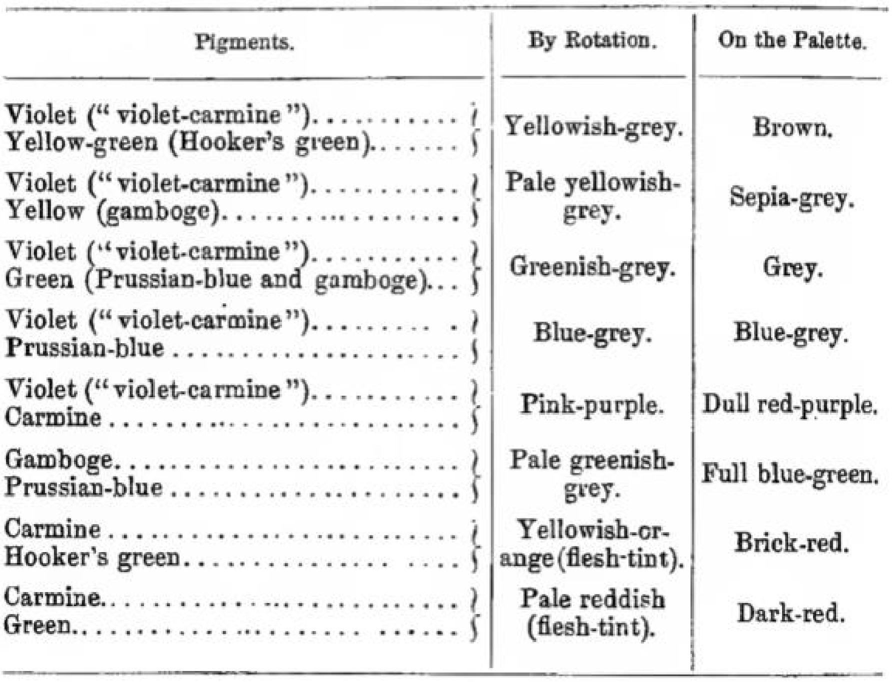


Fig. 3.13
Rood's colour top to exhibit the difference between coloured lights and coloured pigments (Rood 1879, p. 146)

Two water-colour pigments were prepared and used to paint the two outer sectors of the disc. Then the same amount of drops of the two colours was mixed on the palette and used to paint the inner circle. When the top was set in rapid rotation, the two colours placed at the outer circle underwent additive mixture and the resultant tint could be directly compared with that furnished by the palette, placed at the inner circle. Rood repeated this process using eight pairs of colours and reported in a table the experimental results.



Pigments.	By Rotation.	On the Palette.
Violet ("violet-carmine") Yellow-green (Hooker's green)	Yellowish-grey.	Brown.
Violet ("violet-carmine") Yellow (gamboge)	Pale yellowish-grey.	Sepia-grey.
Violet ("violet-carmine") Green (Prussian-blue and gamboge)	Greenish-grey.	Grey.
Violet ("violet-carmine") Prussian-blue	Blue-grey.	Blue-grey.
Violet ("violet-carmine") Carmine	Pink-purple.	Dull red-purple.
Gamboge Prussian-blue	Pale greenish-grey.	Full blue-green.
Carmine Hooker's green	Yellowish-orange (flesh-tint).	Brick-red.
Carmine Green	Pale reddish (flesh-tint).	Dark-red.

Table 3.2
Table showing Rood's results obtained using the colour top. He compared the effects of mixing pigments by rotation and on the palette (Rood 1879, p. 148)

It is worth noting that the two methods of composition of colour furnished the same result only in one case, namely when violet and Prussian-blue were mixed, result that Helmholtz had already anticipated in his paper of 1852: *Only when we have to deal with colours which stand but slightly separated from each other in the spectrum does the composition of the coloured light give nearly the same results as the mixture of the pigments, for then the compound colour is similar to the tones of the spectrum* *which lie between both the simple ones* (von Helmholtz 1852b, p. 530). For the eight cases under examinations, Rood also obtained colour-equations, expressed in the form shown below:

$$50\ violet + 50\ Hooker's\ green = 21\ violet + 22.5\ Hooker's\ green + 4\ vermillion + 52.5\ black$$

The left side represents the mixture of two colours on the palette and the right side the mixture by rotation. Here Rood gave indication of the quantities of black and of other colours necessary to be introduced on the right side in order to match the mixed pigments on the palette, offering a quantitative relation between the additive and the subtractive mixture of colours. He found out that the amount of black required to match the left side of the equation was a very variable quantity, fluctuating from four to fifty-two per cent. Rood explained that *It is for this reason that artists are so careful in their selection of pigments for the production of definite tones, particularly when they are to be luminous in quality* (Rood 1879, p. 146).

3.7 Helmholtz's paper of 1855: *Über die Zusammensetzung von Spectralfarben*

To face criticism aroused after the publication of his article of 1852, Helmholtz carried out new experiments on the composition of coloured light and during the summer of 1854 he sent his new essay on colour to Poggendorff with the title *Über die Zusammensetzung von Spectralfarben (On the Composition of Spectral Colours)*, which was published in the following year. Here he described the new experimental set-up that allowed him to obtain only a small area covered simultaneously with the two mixed colours. He admitted, in fact, the limitations of the V-shaped slit experiment. It was difficult to judge with accuracy the mixtures, especially the paler ones, because the single colours occupied too much space and contrast effects altered the appearance of the less saturated colours.

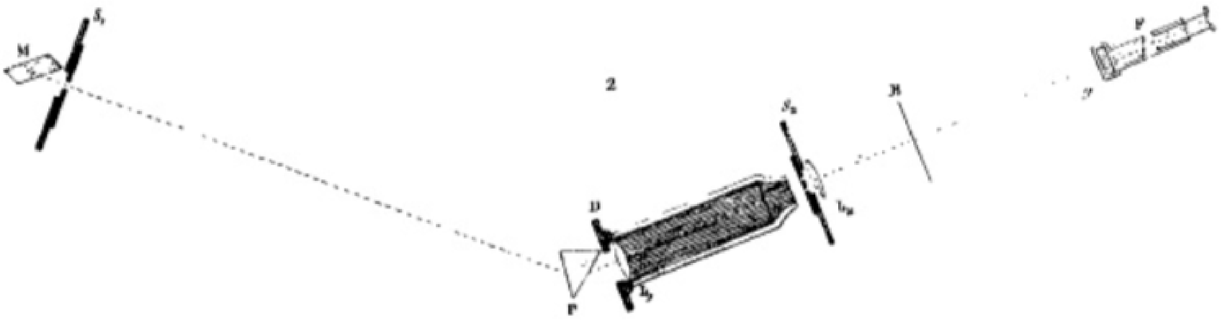


Fig. 3.14
First sketch of Helmholtz's new experimental set-up for colour mixing, as it appeared in his article of 1855 (von Helmholtz 1855, Tafel I. Fig. 2)

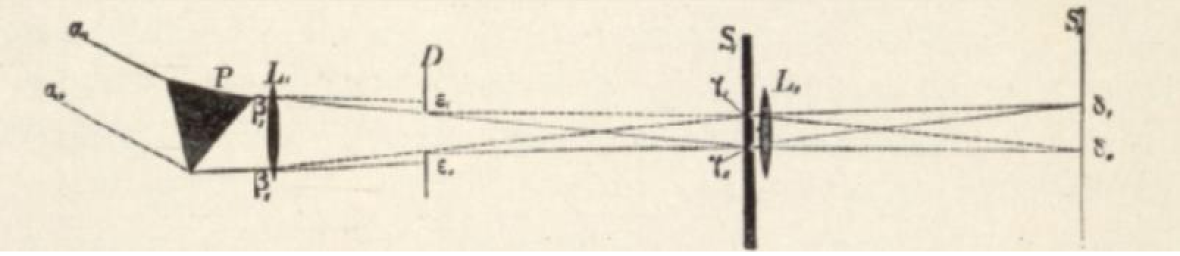


Fig. 3.15
Second sketch of Helmholtz's apparatus, as reprinted in his *Handbuch* (von Helmholtz 1962, p. 158)

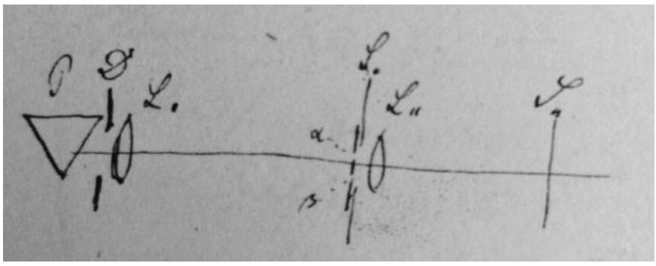


Fig. 3.16
Helmholtz's original sketch of the apparatus, taken from the manuscript version of his *Handbuch* preserved at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften* (Nachlass Hermann von Helmholtz, 572 Über physiologische Optik, ∫∫20. Die zusammengesetzen Farben)

My intention here is to offer a brief description of this colour mixing apparatus (Fig. 3.15). A heliostat allowed sunlight to be reflected through a vertical slit into a dark room. The beam passed through a prism, denoted as P, and a lens, denoted as $L_,$, and its width is indicated by $\alpha_,\alpha_{,,}$. A screen $S_,$ was located in the focal plane of this lens and between the screen and the lens was placed a rectangular diaphragm D. In $S_,$ there were two vertical slits, γ_1 and γ_2, which allowed to pass only two coloured bands of the spectrum (the dashed lines in Fig. 3.15 correspond to the more refrangible light and the dotted lines to the less refrangible). The double slits at $S_,$ were adjustable both in lateral and in width, in this way pairs of spectral lights could be combined by another achromatic lens, denoted as $L_{,,}$, beyond $S_,$ which projected an image of the diaphragm on a second screen, $S_{,,}$. The screen $S_,$ was so constructed in order to allow the investigator to vary the hue and the intensity of the mixture gradually[20]. The diaphragm was adjusted in order to render the two lights uniformly distributed over the image $\delta_{,}\delta_{,,}$ on the screen $S_{,,}$. This area $\delta_{,}\delta_{,,}$, therefore, appeared uniformly coloured by a mixture of the two coloured lights. It is also possible, screening one of the slits, to have an image of only one colour on the screen.

By using this new experimental set-up, Helmholtz found different pairs of complementary colours, not only one as he stated in his previous essay, confirming Grassmann's prediction. For instance, the complementary colours he observed were red and greenish blue; orange and cyan-blue; yellow and indigo-blue; greenish yellow and violet. Helmholtz could also measure the wavelengths of seven different pairs of colours using a simple method, that he described with the following words: *The way the writer determined the wavelengths of the pairs of complementary colours was by removing the lens* $L_{,,}$ *and the screen* $S_{,,}$ *and using a telescope to see the slit in the screen* $S_{,}$ *at some distance away. A glass plate with fine, equidistant vertical lines ruled on it was inserted in front of the object-glass. Then diffraction spectra of the slits will be seen; their apparent distances from the corresponding slits being proportional to the wavelengths. Therefore, all that is necessary is to measure in the same way the distances of the diffraction spectra for one of the dark lines in the spectrum, for which the wavelength has been found by Fraunhofer, in order to calculate immediately the wavelengths of the two mixed colours concerned* (von Helmholtz 1962, p. 160). He plotted, then, the measured wavelengths by pair, as depicted in Fig. 3.17, in order to investigate the connections between the wavelengths of the simple complementary colours. Helmholtz found out a strong irregularity in the distribution of the complementary colours in the spectrum. Proceeding from the violet to the red on the horizontal axis, in fact, we can see that the curve proceeds almost horizontally until the greenish blue colour. Then, the curve proceeds almost vertically downwards. The same thing happens in the yellow. In the red the change is again very gradual, giving rise to an almost horizontal curve. Helmholtz explained this irregularity considering hue changes in function of the wavelength: *at the ends of the spectrum the hue changes exceedingly slowly as compared with the wave-length, whereas it changes very fast in the middle of the spectrum. The result is that there is no simple or constant connection between the wavelength of the pairs of complementary colours* (von Helmholtz 1962, p. 127).

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[20] In order to allow precise variations of hue and intensity, Helmholtz made use of a special screen containing two pairs of 's Gravesande slits. A detailed description with illustration of its construction is given in his *Handbuch* (von Helmholtz 1962, pp. 159-160).

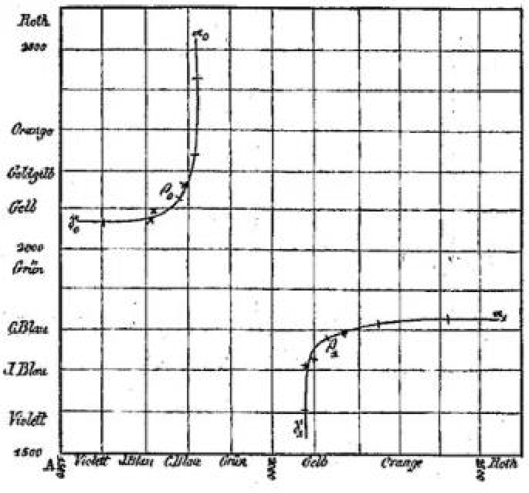

Fig. 3.17
Plot by pair of the wavelengths of the complementary colours. The axes indicate the wavelengths of the complementary colours in the spectrum from 400 to 700 nm (von Helmholtz 1855, Tafel I. Fig. 3)

Although he was able to find different pairs of complementary colours, the most surprising thing, that was in contradiction to Newton and Grassmann's theory of colour, was that Helmholtz could not detect a simple colour that was complementary to green, as we can see in Fig. 3.18. Green had to be mixed with red and violet to furnish white. Its complementary colour was, in fact, purple. This led to an important revision of Newton's colour circle, as Grassmann had predicted: *the manner in which Newton distributes the homogeneous colours on the circumference of his discs requires a thorough revision, towards which only the first steps are made by the researches of Helmholtz*[21] (Grassmann 1854, p. 264). Helmholtz introduced a correction in the colour diagram, the so called *purple-line*, a line connecting red to violet. The shape of the curve was determined from the intensity measurements of the lights of each complementary pair, that could be obtained by adjusting the respective slit widths[22].

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[21] Grassmann drew his colour circle, so that homogeneous colours of unit intensity were ranged along the circumference and with the idea that colours change continuously along it. Exploiting the centre-of-gravity construction, he could detect the hue, the totally intensity of the mixed colour and the intensity of the intermixed white (see section 3.2). However, he was aware that the disposition of colours along the circumference, as proposed by Newton, required a revision.

[22] This represents the first attempt to define quantitatively relative intensities of complementary lights.

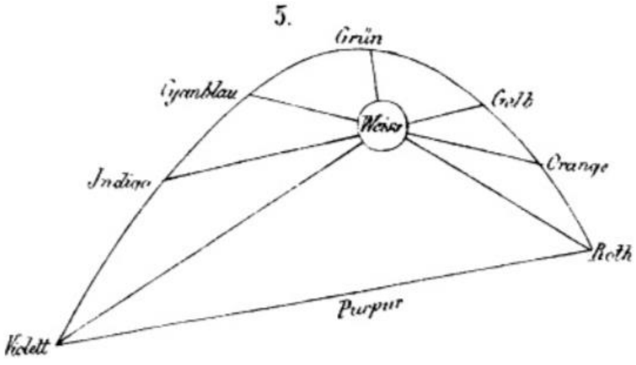

Fig. 3.18
Helmholtz’s colour diagram (von Helmholtz 1855, Tafel I. Fig. 5)

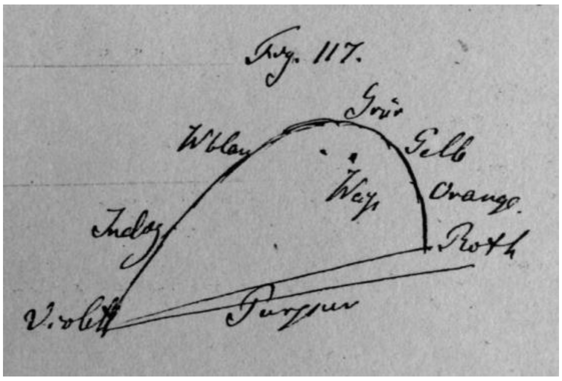

Fig. 3.19
Helmholtz’s original drawing of his colour diagram taken from the manuscript version of his *Handbuch* preserved at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften* (Nachlass Hermann von Helmholtz, 572 Über physiologische Optik, ∫∫20. Die zusammengesetzen Farben)

On the ground of these measurements, Helmholtz was also able to compare quantitatively the brightness of the different pairs of complementary colours. He combined two complementary colours in order to obtain a white field, and measured the width of the slit through which the brighter of the two colours was allowed to pass. Then, he narrowed the slit until a little stick held in front of the mixed field drew two equal darkly coloured shadows, and the width of the slit was measured again. The amount of light of the considered colour was then reduced in the same proportion as the width of the slit. The ratio of the two measured widths gave thus approximately the ratio of the brightness of both colours. Helmholtz reported then the following brightness ratios under two different light conditions:

Pair of complementary colours	Brightness ratio in high light	Brightness ratio in low light
Violet and yellow-green	1:10	1:5
Indigo and yellow	1:4	1:3
Cyan-blue and orange	1:1	1:1
Green-blue and red	1:0.44	1:0.44

Table. 3.3
Brightness ratios obtained by Helmholtz under two different light conditions (von Helmholtz 1855)

Helmholtz came to the conclusion that there must be differences of saturation in the spectral colours: he indicated violet as the most saturated and yellow as the least. This meant that spectral colours could not be equidistant from white in the colour diagram, as in Newton's colour circle: *Saturated violet and red must be farther from white than their less saturated complementary colours, because, as the eye estimates it, it takes less violet than yellow-green when we mix these two complementary colours together to get white. Hence, if white is to be in the position of their centre of gravity, the smaller amount of violet must have a longer lever-arm than the larger amount of yellow-green* (von Helmholtz 1962, pp. 139-140). Helmholtz's experimental results, reported in the table above, gave him indication of the relative distance from white to the edge of the colour surface as a function of wavelength. He offered, thus, a new version of the gross structure of colour space. As we will see in chapter 5, Helmholtz, from this experiment of colour mixing, recognized that colour space, as define by the colour mixing laws, could not be uniform: equal distances do not correspond to equal perceptual differences. This motivated him to turn toward non-Euclidian geometry to define a line element in colour space.

3.8 Helmholtz's and the construction of the colour chart

I would like to show in this paragraph how Helmholtz adopted Newton's analogy with the calculation of the centre of gravity, in the light of Maxwell's results, to build the colour chart. He reported, in fact, in a very clear manner the geometrical construction of the colour chart, exploiting Newton's method, followed by its mathematical proof[23]. He denoted with A, B and C the three colours from which he started the construction, and with a, b, and c their position in the chart. He considered then

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[23] This is contained in paragraph 20 of the second volume of the *Handbuch der physiologischen Optik* entitled *Construction of the Colour Chart* (von Helmholtz 1962, pp. 134-138).

an amount α and β of the colours A and B and found the mixed colour at the centre of gravity of the weight α and β, which was located at point d in the straight line ab, in order to have:

$$\alpha \cdot ad = \beta \cdot bd \qquad (3.1)$$

All the mixtures of colours A and B were thus located on the line ab. Adding another quantity γ of another colour C, he considered the quantity $(\alpha + \beta)$ of the mixed colour corresponding to point d and found the position of the centre of gravity e of the weights $(\alpha + \beta)$ at d and γ at c.

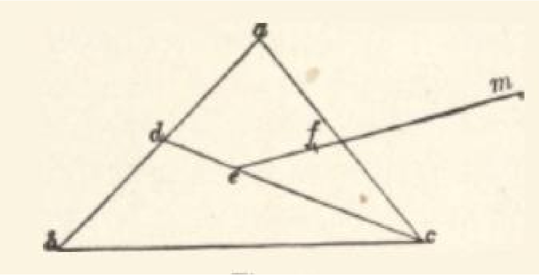

Fig. 3.20
Helmholtz's construction of the colour chart (von Helmholtz 1962, p. 134)

The quantity of the mixed colour was then: $\varepsilon = \alpha + \beta + \gamma$. Helmholtz could determine also the unit luminosity of this colour from the following equation: $1 = \frac{\varepsilon}{\alpha+\beta+\gamma}$.

Furthermore, he described the way of proceeding for all colours that could not be produced by a composition of A, B and C. He indicated such a colour by letter M and considered at first a quantity μ of this colour so small that the outcome of the mixture of this colour with one of the colours of the triangle could be imagined as a colour lying inside the triangle. Helmholtz then stated: *For example, it may be mixed with an amount ε of the colour defined by the point e, where ε is measured in terms of the unit found above. If the amount of M is supposed to be infinitesimal at first and then to be increased gradually up to the value μ, the mixture will begin by having the same colour as E itself and will continually change into the adjacent colours, according to the fundamental axioms assumed above* (von Helmholtz 1962, p. 135). Once the amount of M reaches the value of μ, and if we suppose that f indicates the position of the resultant colour still inside the triangle, the amount of the colour F will be: $\varphi = \varepsilon + \mu$. Furthermore, point f should be located at the same position at the centre of gravity of μ at m and ε at e. The place and amount of colour M can be determined by the following equation:

$$\frac{mf}{ef} = \frac{\varepsilon}{\mu} \tag{3.2}$$

The point *m* must lie in the prolongation of the line *ef*.

3.9 Helmholtz's *Handbuch der physiologischen Optik*

Helmholtz's *Handbuch* is a monumental work, of more than 1000 pages, which constitutes a systematic anthology of all knowledge at that time about vision. It summarizes and establishes, in fact, the science of physiological optics. The three volumes of the book appeared starting from 1856 to 1866. The first book was published in 1856, after Helmholtz's arrival in Bonn, where he occupied the Chair of Anatomy and Physiology, and dealt with the dioptrics of the eye. It was dedicated to results obtained by using optical apparatus invented by him, the *ophthalmoscope* and especially the *ophthalmometer*, an instrument built to obtain minute measurements of the eye. Here we can find the theory of ocular accommodation and measurements of optical constants of the eye, such as the radius of curvature of the cornea and the variations of distance between the cornea and the iris during accommodation. The second part was published in 1860, at the time he was Professor of Physiology at Heidelberg[24] (1858-1871), and dealt with responses of the organ of vision to light, simple and compound colours, intensity and duration of the light sensation, residual images and contrast. The third volume was published in 1866 and dealt principally with perception of vision. Among the various topics of the third volume, it is worth mentioning eye movements, the monocular field of vision, direction of vision, perception of depth and binocular vision. The first English edition of the *Handbuch*, *Treatise on Physiological Optics*, translated from the third German edition, appeared in 1924. It was edited by James P. C. Southall, Professor of Physics in Columbia University and published by the Optical Society of America.

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[24] At the end of September 1858, in fact, Helmholtz moved to Heidelberg, inaugurating here, together with Kirchhoff and Bunsen, an era of brilliancy. In 1859, Robert Bunsen and Gustav Kirchhoff founded spectral analysis. They found, indeed, a relationship between the emission (bright-line) spectrum and the absorption (dark-line) spectrum of different elements. In 1860 they discovered two alkali metals, caesium and rubidium, by using their spectroscope, inaugurating therefore a new era in the discovery of new chemical elements.

3.10 From Maxwell's colour top to Maxwell's colour box

Although Maxwell's results were at first based on disc mixture that allowed him to obtain precise colorimetric equations and to sketch his triangular diagram, he also experimented with spectral lights. His aim was to obtain even more accurate and reproducible equations[25] and that was possible mixing directly spectral lights. By using the colour top, in fact, the observations could be influenced by variations in the rotating velocity of the plate. Furthermore, in a letter addressed to William Thomson on September 13, 1855, Maxwell wrote: *Since I last wrote to you I have been making mixtures of colours by weight and finding the resulting colour by means of the colour-top. I find that very few mixtures lie in the line joining their components, but they form a curve generally pretty regular from the one to the other. In the case of Chrome Yellow & Mineral Blue (Copper I believe) the curve goes away among the greens. I have a good many other pairs of colours, but this is the most remarkable* (Harman 1990a, p. 324).

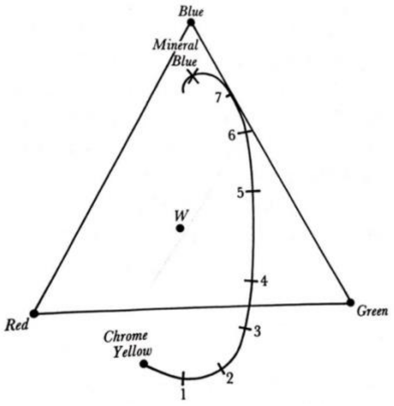

Fig. 3.21
Maxwell's colour triangle with indication of the curve connecting blue and yellow, based on his experiments with the colour top (Harman 1990a, p. 323)

Maxwell adopted, then, Newton's ideas of using lenses, prisms and mirrors to build a visual spectrometer, named *colour box*. He exploited Newton's crucial experiences to realize dispersion, recombination and reflection of light. In his conference *On colour vision* held at the Royal Institution in 1871, Maxwell described his colour box with the following words: *I have therefore constructed an*

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[25] In a lecture held at the Royal Institution in May 1861, Maxwell expressed clearly that experiments of spectrum mixing can provide more precise information in the determination of the three primaries: *The experiments with pigments do not indicate what colours are to be considered as primary; but experiments on the prismatic spectrum shew that all the colours of the spectrum, and therefore all the colours in nature, are equivalent to mixtures of three colours of the spectrum itself, namely, red, green (near the E line), and blue (near the line G)* (Niven 1890, vol. I, p. 449).

instrument which I may call a colour box, for the purpose of making matches between two colours. [...] It is nothing but the realization of the construction of one of Newton’s propositions in his Lectiones Opticae*, where he shows how to take a beam of light, to separate it into its components, to deal with these components as we please by means of slits, and afterwards to unite them into a beam again* (Niven 1890, vol. II, pp. 273-274). Newton reported also in his *Opticks* an experiment to demonstrate that the recombination of different coloured rays furnished white light again simply by means of prisms or lenses. Fig. 3.22 shows the dispersion of light through a prism and the recombination of light of different colours on the screen *G*, using other two prisms. Originally Newton used a lens instead, as he stated: *You may instead of the Lens use two Prisms HIK and LMN, which by refracting the coloured Light the contrary Way to that of the first Refraction, may make the diverging Rays converge and meet again in G, as you see represented in the seventh Figure. For where they meet and mix, they will compose a white Light, as when a Lens is used* (Exp. 10. Prop V. Theor. IV: *Whiteness and all grey Colours between white and black, may be compounded of Colours and the whiteness of the Sun’s Light is compounded of all primary Colours mix’d in a due Proportion*, Newton 1979, p. 142).

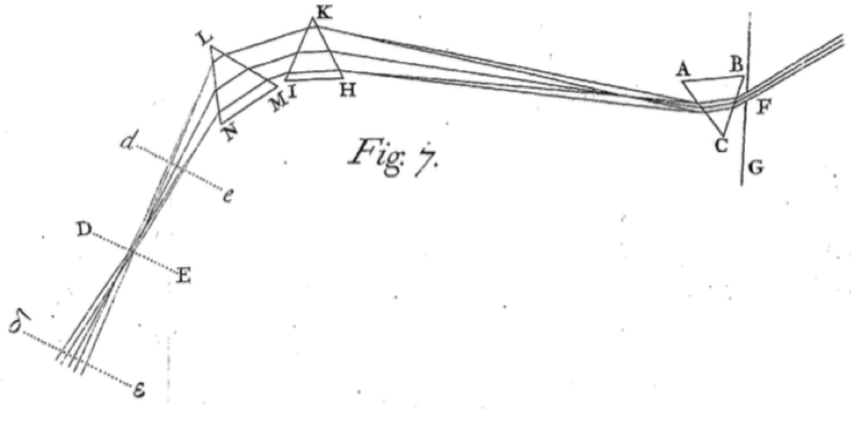

Fig. 3.22
Newton’s experiment to show the recombination of different colour rays (Newton 1979, p. 143)

Maxwell’s *colour box* is a visual spectrometer that allows direct comparison, in colour and intensity, of white light with a mixture of the chosen three primary spectral lights. To achieve this aim, Maxwell based his observations on the principle of optical reversibility. One of the first version of the apparatus was presented at the annual meeting of the British Association for the Advancement of Science held in 1856 at Cheltenham, although Maxwell began working on its realization starting from 1852 and constructed over that period a large number of adjustments and variations, decreasing its dimension to render it portable[26] (from nearly 3 m to the final portable version 1 m long). A detailed description of the colour box can be found in a paper to the Royal Society of London in 1860 with the title *On the Theory of Compound Colours, and the Relations of the Colours of the Spectrum* (Clerk Maxwell 1860) and from which an excerpt is reported: *The experimental method which I have used consists in forming a combination of three colours belonging to different portions of the spectrum, the quantity of each being so adjusted that the mixture shall be white, and equal in intensity to a given white* (Clerk Maxwell 1860, p. 65).

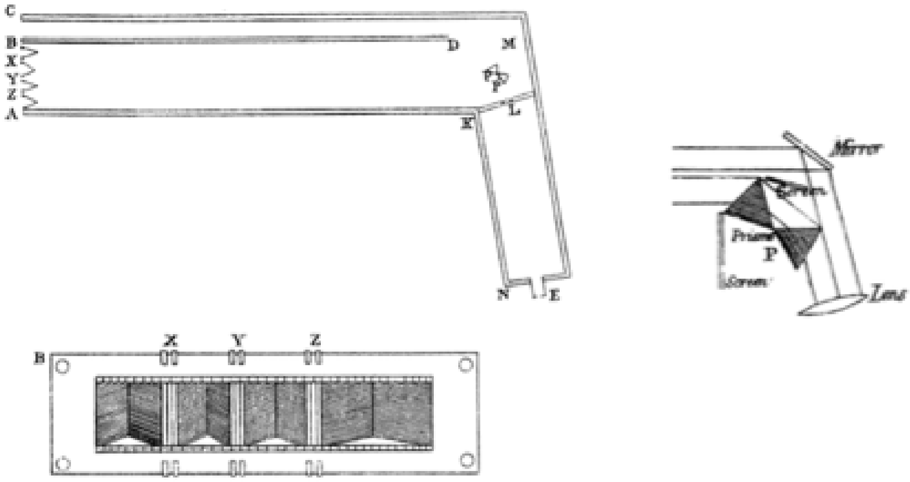

Fig. 3.23
Structure of Maxwell's colour box with details regarding the system of lens, prisms and mirror (on the right) and the sets of tuneable slits labelled X, Y and Z (on the left) (Clerk Maxwell 1860, Plate I)

The system, as shown in Fig. 3.23, consists of two wooden boxes fixed together at an angle of 100° with rectangular section and the inside was painted black (Maxwell reported also the dimension of the apparatus: the longest box was about 152 cm long, 18 cm broad and 10 cm deep, while the shortest was about 70 cm long, 13 cm broad and 10 cm deep). The only apertures were located at point *E* and at the end *AC*. A lens was placed at *L,* two 45° prisms to obtain dispersion were labelled with letter *P* and at *M* was located a mirror to reflect light. When white light was allowed to enter from slit *E*, a pure spectrum could be formed at the end panel of the box *AB,* while the mirror at *M* reflected white

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[26] In a letter to Forbes dated 26 November 1857, Maxwell detailed the succession of his experiments of spectrum mixing over that period, starting from 1852: *I have had some experience of spectrum mixing, as I have made 3 different sets of apparatus which all succeeded partially. I began in 1852 with one 3 feet long and a water prism. In 1855 I made one 7 feet long with a glass prism and arrangements for seeing two mixtures of pure colours spread uniformly over two contiguous fields. In 1856 I made a reflecting portable apparatus for showing the phenomena roughly to strangers* (Harman 1990a, p. 569). In the same volume of Harman also a complete report of the construction details of the colour box can be found with the title *Description of an instrument for comparing mixture of the colours of the spectrum, circa 1858* (Harman 1990a, pp. 600-602).

light from E to the portion BC. Three tuneable slits were positioned at AB across the extent of the spectrum. The slits widths could be adjusted in order to select specific ranges of the *red*, *green* and *blue* light. Wavelength calibration was performed referring to selected Fraunhofer absorption lines. Exploiting the reversibility of optical path, when white light was admitted through the end of the box AC, the eye of an observer placed at E could see, through the lens at L, a field consisting of two portions, separated by the edge of the prism. On the left, the composition of the three selected spectral colours could be viewed, while the right section appeared white, allowing a direct comparison. Thus the breadth of the slits must be tuned until the two portions of the field seen at E became equal, in white and in intensity.

Maxwell succeeded to determine the wavelength corresponding to any point of the graduated scale AB. It is worth offering a brief description of the method he used in order to better understand his subsequent considerations. Maxwell's exploited two plane surfaces of glass that enclosed a layer of air. White light was reflected from this layer and allowed to pass through opening E. The spectrum formed at the end AB was then analysed by means of a lens. This spectrum consisted of alternation of bright and dark bands, distributed at nearly uniform intervals. Maxwell's aim was, in fact, to determine the wavelength from a simple equation describing the theory of Newton's rings, as he wrote: *By the theory of NEWTON'S rings, the light reflected from a stratum of air consists of two parts, one of which has traversed a path longer than that of the other, by an interval depending on the thickness of the stratum and the angle of incidence. Whenever the interval of retardation is an exact multiple of a wave-length, these two portions of light destroy each other by interference; and when the interval is an odd number of half wave-lengths, the resultant light is a maximum. [...] If, then, we observe the positions of the dark bands on the scale AB, the wave-lengths corresponding to these positions will be a series of submultiples of the retardation* (Clerk Maxwell 1860, p. 67). Maxwell numbered the dark bands starting from 0, corresponding to the one on the red portion of the spectrum, and the numbers increased toward the violet end of the spectrum. He denoted with N the number of *ondulations* corresponding to the band 0 contained in the *retardation R*. The number of any other band was then indicated by n in order to express the number of the corresponding wavelengths in the retardation as $(N+n)$. From the simple equation reported below, Maxwell could calculate the wavelength corresponding to any dark band n, once he had found the value of R and of N:

$$R = (N+n)\lambda \tag{3.3}$$

$$\lambda = \frac{R}{N+n} \tag{3.4}$$

Maxwell could divide, in this way, the scale *AB* in sixteen equidistant positions obtaining, therefore, sixteen different wavelengths. He selected then three standard points at position 24, 44, and 68 of the scale, that correspond respectively to *scarlet*, *green* and *blue*, as we can read in his paper of 1860: *Having thus selected sixteen distinct points of the spectrum on which to operate, and determined their wave-lengths and apparent colours, I proceeded to ascertain the mathematical relations between these colours in order to lay them down on NEWTON'S diagram. For this purpose I selected three of these as point of reference, namely those at 24, 44 and 68 of the scale.* He also exposed the reason of his choice: *I chose these points because they are well separated from each other on the scale, and because the colour of the spectrum at these points does not appear to the eye to vary very rapidly, either in hue or brightness, in passing from one point to another* (Clerk Maxwell 1860, p. 69). Maxwell's equations were expressed in this form:

$$18.5(24) + 27(44) + 37(68) = W^*$$

This equation was obtained from Maxwell's observations on 18 October 1859. The coefficients represent here the width of the slits and the numbers 24, 44 and 68 the positions of the scale corresponding to a bright scarlet about one third of the distance from Fraunhofer line C to D, a green near the line E and a blue about one third of the distance from F to G respectively.

He then varied the width of the slits, one at the time, obtaining other colour equations that he compared with the standard equations marked with the asterisk, in which all the three standard colours appeared. In this way, by combining each colour with two chosen standard colours, he could obtain a white equal to the standard one, marked with the asterisk. Having then determined the average error in the different observations, Maxwell made also some considerations about the degree of accuracy in the detection of hue and brightness: *Now the hue of the resultant depends on the ratios of the components, while its brightness depends on their sum. Since, therefore, the difference of two colours is always more easily detected than variations in brightness, and the eye appears to be a more accurate judge of the identity of colour of the two parts of the field than of their equal illumination*

(Clerk Maxwell 1860, p. 72)[27]. My intention is to underline, with the reported excerpt, Maxwell's attention toward problems related to the estimation accuracy of the human eye, topic that attracted also the attention of Helmholtz and of his collaborators in the following years, as we will see in the next pages.

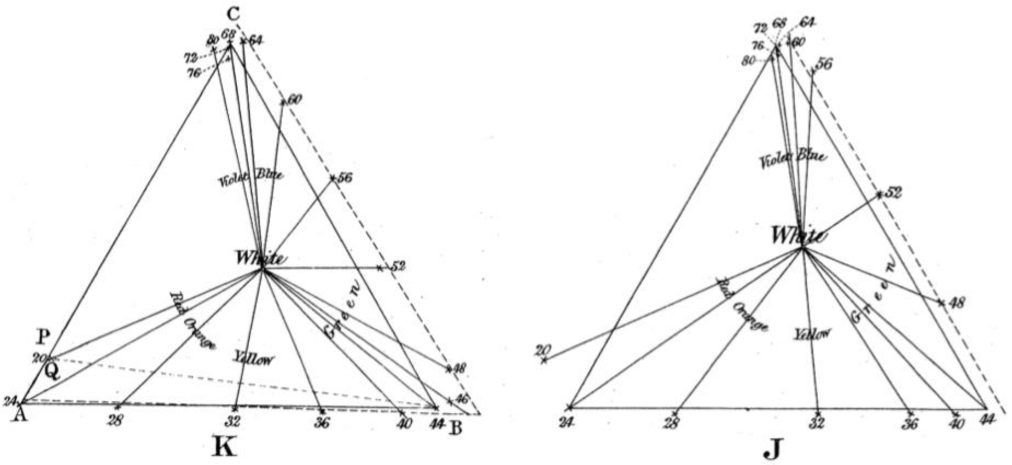

Fig. 3.24
Colour triangles obtained from analysis of data taken by Maxwell (on the right) and his wife Katherine Mary Dewar (on the left). Little variations between the two diagrams are shown. In both cases, however, all the points representing colours lay approximately on straight lines (Clerk Maxwell, 1860, Plate I)

Analysing the colorimetric equations based on his (J) and his wife's (K) observations, Maxwell could confirm his expectation that a triangular portion of Newton diagram could represent effectively all the colours of the spectrum. As shown in Fig. 3.24, all the points corresponding to colours lie approximately on straight lines. The results hence were in line with those obtained by means of his colour top. Using coloured lights coming directly from the spectrum instead of coloured pigments, he succeeded in his aim of obtaining more accurate information, as he expressed in a lecture held at the Royal Institution in May 1861: *The experiments with pigments do not indicate what colours are to be considered as primary; but experiments on the prismatic spectrum shew that all the colours of the spectrum, and therefore all the colours in nature, are equivalent to mixtures of three colours of the spectrum itself, namely, red, green (near the E line), and blue (near the line G)* (Niven 1890, vol.

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[27] Similar considerations can be found in a delayed passage of his paper of 1860: *In order to determine whether the eye is more sensitive to variations in brightness or to variations in colour I have determined also the average error in the sum of the three terms and also that of their differences. [...] a variation in hue between red and green is more easily detected than a variation of brightness due to an increase or diminution of both simultaneously* (Harman 1990a, p. 639).

I, p. 449). However, the two diagrams show that there was an evident difference in the eyes of the two observers, producing measurable differences in the colour perception.

In *On the Theory of Colours in relation to Colour-Blindness*, a letter written to George Wilson on January 4, 1855, Maxwell stated that every possible colour must be located inside his equilateral triangle but he was aware of the fact that, although the homogeneous rays of the spectrum were completely pure in themselves, they gave rise to all three sensations (which he called *redness*, *greenness* and *blueness*) in different proportions: *hence the position of the colours of the spectrum is not at the boundary of the triangle , but in some curve CRYGBV considerably within the triangle* (Niven 1890, vol. I, p. 122). His aim was hence to determine the nature of this curve, that he provisionally sketched in a circular shape, as we can see in Fig. 3.25.

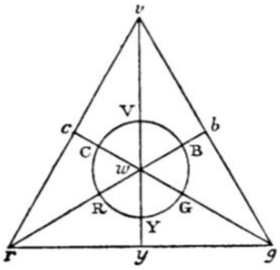

Fig. 3.25
Maxwell’s colour triangle containing the curve CRYGBV of provisional circular shape (Niven 1890, vol. I, p. 121)

By using his colour box, Maxwell succeeded to determine the curves of intensity of the three chosen standard colours, corresponding to values 24, 44 and 68 of the scale, reporting the values of fourteen selected tints expressed in terms of the primaries[28]. The dashed curves denoted by R, G and B in Fig. 3.26 indicate respectively the intensity of the red, green and blue and represent the relative values of the coefficients of the colorimetric equations. Maxwell used letter S to label the sum of the three curves. As reference, he also marked the position of the Fraunhofer lines with the upper letter C, D, E, F and G.

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[28] Comparing Katherine’s with his own intensity curves, small differences can be noticed, as he wrote in a letter to Gabriel Stoke, written on 10 September 1862: *By choosing a standard eye, or taking an average I could express the facts of colour vision in two diagrams. 1st a triangle of colour giving relations in the quality of colours and the same for all eyes. 2nd a curve representing the intensity of colour at each point of the spectrum. This curve is different for different eyes and for different parts of the same eye. These differences may arise from absorption of certain rays before they reach the retina. Irregularities in the curve extending over small spaces are probably of this sort. There may also be differences in the sensibility to the nerves to particular sensations. This would account for inequalities extending over entire regions of colour in a regular manner* (Harman 1990b, pp. 58-59).

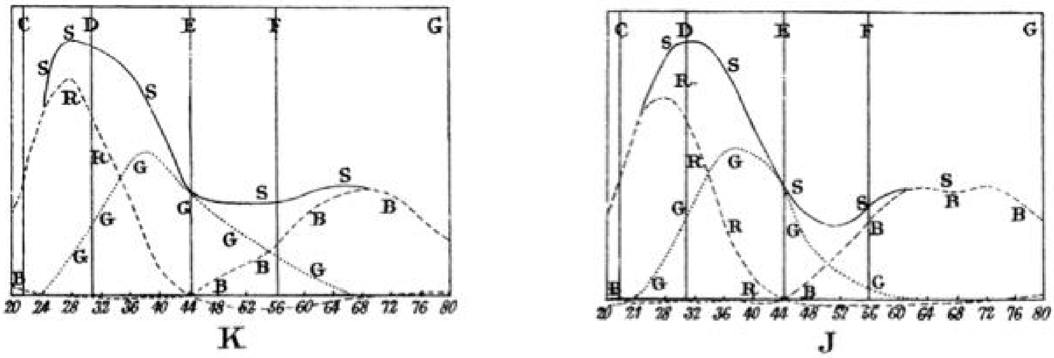

Fig. 3.26
Curves of intensities of the primaries for two different observers, Maxwell (J) and his wife Katherine (K). From these graphs it becomes evident that all the colours of the spectrum can be obtained from a combination of the three standard colours. Maxwell interpreted negative coefficients as quantities of the specific standard colour that has to be add to the spectral colour in order to obtain a result equivalent to the combination of the other two primaries (in quantities specified by their coefficients). The maxima of the curves indicate the sensitivity maxima of the three receptors in the retina for specific wavelengths, corresponding to red, green and blue (Clerk Maxwell 1860, Plate I)

If we focus our attention on the shape of curve *S* in both the plots, we observe that it has a maximum in proximity of Fraunhofer line D, which corresponds to wavelengths 589,625 and 589,024 nm, according to the first determinations of the wavelengths of light of different colours made by Young[29], therefore in the yellow region of the spectrum. What we know today, in fact, is that the relative human sensitivity in function of the wavelength presents a peak at 555 nm, in the yellow-green region of the spectrum (Malacara 2011).

Maxwell continued experimenting with his colour box during the time he spent in London, as he was appointed to the vacant chair of Natural Philosophy in King's College, from 1860 to 1865. He lived in 8 Palace Gardens Terrace and when he made his observations at the window with the colour box aroused the curiosity of his neighbours, as testified by Lewis Campbell and William Garnett's words: *When experimenting at the window with the colour-box (which was painted black, and nearly eight feet long), he excited the wonder of his neighbours, who thought him mad to spend so many hours in staring into a coffin* (Campbell and Garnett 1882, p. 158).

During that time, Maxwell also wrote a letter to Helmholtz, inviting him at his place in order to conduct some experiments together using the colour box: *Dear Sir, I have been a long time getting my instrument for mixing colours put right but it is now ready. Can you come and take lunch with us on Saturday* [12 April 1864] *about half past one o Clock and then we can have light to analyze. Yours truly J. Clerk Maxwell* (Harman 1990b, p.146). Helmholtz spent, in fact, in the Easter holidays some

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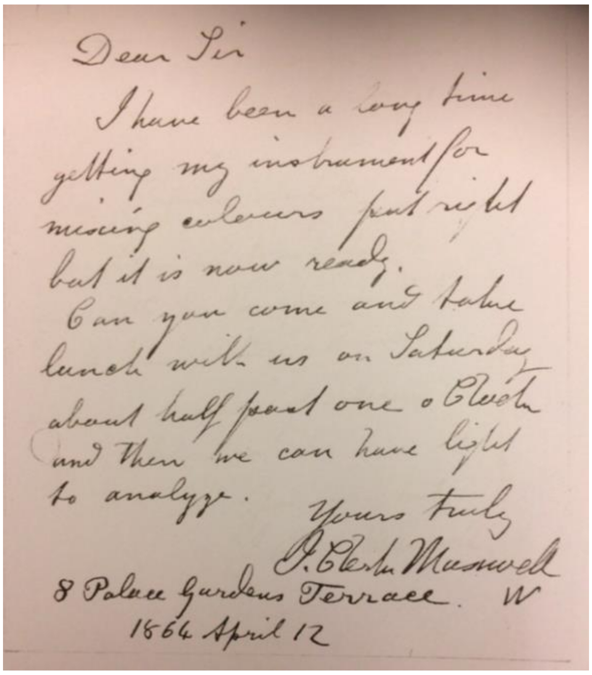
[29] Helmholtz reported in his *Handbuch* Young's results in a table containing the hues corresponding to the Fraunhofer lines together with their wavelengths expressed in millionths of a millimetre (von Helmholtz 1962, p. 75).

weeks in England, where he met Maxwell, as he wrote to his wife: *Then I went with an old Berlin friend to Kensington, to see Prof. Clerk Maxwell, the physicist at King's College, a keen mathematician, who showed me some fine apparatus for the Theory of Colours which I used to work at; he had invited a colour-blind colleague, on whom we experimented* (Königsberger 1906, p. 224). In fact, between 1862 and 1865, Maxwell was able to record about 200 observations each year because all guests at his house experimented with the colour box (Mahon 2003, p.111).



Dear Sir

I have been a long time
getting my instrument for
mixing colours put right
but it is now ready.
Can you come and take
lunch with us on Saturday
about half past one o'Clock
and then we can have light
to analyze.

Yours truly
J. Clerk Maxwell

8 Palace Gardens Terrace. W
1864 April 12

Fig. 3.27
Original letter preserved at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften*, Berlin (Nachlass Hermann von Helmholtz, 305 Briefe James Clerk Maxwell)

3.11 Colour blindness and its key role in Maxwell and Helmholtz's investigations

The first clear and detailed account of colour blindness was provided by John Dalton, who read his paper before the Manchester Literary and Philosophical Society on 31 October 1794. This essay was published in the Memoirs of the Society in 1798 with the title *Extraordinary facts relating to the vision of colours: with observations* (Dalton 1798). Dalton described here his own colour vision deficiency, which we nowadays call *protanopia* or *red-blindness*. The paper is divided in three sections entitled *I. An account of my own vision, II. An account of others whose vision has been found similar to mine and III. Observations on the probable cause of our anomalous vision*. In the first section, he reported his observations of the solar spectrum obtained using a glass prism. He could only see two or three different colours of the spectrum (namely yellow and blue, and in some cases yellow, blue and purple), whereas other people were able to distinguish six different colours (red, orange, yellow, green, blue and purple). Dalton then reported the colours seen by himself corresponding to *red*, *orange and yellow*, *green*, *blue* and *purple* seen by normal individuals, by both day-light and candle-light. At the end of the second section, in light of the observations, he summarized his conclusions: *1. In the solar spectrum three colours appear, yellow, blue, and purple; the two former make a contrast; the two latter seem to differ more in degree than in kind. 2. Pink appears, by day-light, to be sky-blue a little faded; by candle-light it assumes an orange or yellowish appearance, which forms a strong contrast to blue. [...] 9. In all points where we* (referring to people affected by the same vision defect as himself) *differ from other persons, the difference is much less by candle-light than by day-light* (Dalton 1798, pp. 40-41). In the last section of his paper, Dalton put forward a hypothesis about the cause of his anomaly. His loss of sensitivity could be due to one of the humours of his eye that absorbed the red: *I was led to conjecture that one of the humours of my eye must be a transparent, but coloured, medium, so constituted as to absorb red and green rays principally, because I obtain no proper ideas of these in the solar spectrum; and to transmit blue and other colours more perfectly* (Dalton 1798, p. 42).

Thomas Young and John Herschel, his contemporaries, disagreed with this idea. Young found here a key argument in favour of his trichromatic theory, that led to its diffusion.

Young attributed Dalton's deficit to the absence or paralysis of one of the receptor in the retina, as he remarked in the bibliography of *Lectures of Natural Philosophy*, listing Dalton's paper: *He* (Dalton) *thinks it probable that the vitreous humour is of a deep blue tinge: but this has never been observed by anatomists, and it is much more simple to suppose the absence or paralysis of those fibres of the retina, which are calculated to perceive red* (Young 1807, p. 514). Herschel also shared Young's idea, as he wrote in a letter to Dalton dated 20 May 1833: *It seems to me, that we have three primary sensations when you have only two. [...] Now to eyes of your kind it seems to me that all your tints are referable to two, which I shall call A and B; the equilibrium of A and B producing your white, their negation your black, and their mixture in various proportions your compound tints* (Henry 1854, p. 26). Herschel proposed to call Dalton's vision *dichromic*, because it depends only on two primary colours instead of three, on which depends the colour vision of normal subjects.

George Wilson, Professor of Technology at the University of Edinburgh, gave an account of a wide variety of cases of colour blindness in his *Researches on colour blindness* (Wilson 1855), becoming an authority on the subject. He distinguished three main cases of colour blindness or *chromato-*

pseudopsis[30] (i.e. *false vision of colours*): the first concerns the inability to discern any colour; individuals that do not discriminate between the *nicer shades of the more composite colours* belong to the second case, and the third case, the most frequent variety, concerns the inability to distinguish between the primary colours. Considering the last case, Wilson wrote that *red is the primary colour most distracting to the subjects of colour-blindness: for some it has absolutely no existence* (Wilson 1855, p. 13). Thus, Maxwell, who was familiar with Wilson's work, concentrated on colour blind subjects with eyes insensible to red light. He used his colour top and later his colour box, as we can see in the next paragraphs, to verify Young and Herschel's hypothesis of the absence or inhibition of one receptor in the eye. Maxwell's results gave valuable clues to the three receptors theory of human vision. He could conclude that one of the receptors should have a particularly high sensitivity to the red portion of the spectrum, confirming *red* as one of the primary colours[31]. It is worth emphasising with Maxwell's own words the key role of colour blindness in the investigation of the true nature of colour: *the most value evidence which we have as to the true nature of colour vision is furnished by the colour blind* (Harman 1990, vol. II, p. 621).

Studies on colour blindness played also a key role in Helmholtz's investigations on the geometry of colour space, as we will see in chapter 5. Helmholtz's collaborators and assistants took another important step forward in understanding the nature of different kinds of colour blind vision, providing definite proof of Young's three-receptor theory.

In fact, the period of greatest activity in the study of colour blindness was between 1870 and 1890, when Helmholtz's collaborators provided for the first time what we today call the cone spectral sensitivity curves for monochromats, dichromats and trichromats, as I will describe in chapter 4, and the interest has continued arising up to the present time.

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[30] *Daltonism* was the original name attributed to this kind of defect, because Dalton was the first to provide a detailed account of it. George Wilson explained in his volume the reasons why this name was then substituted: *The countrymen of Dalton have protested against the immortalising of his name, in connection with a personal defect; and the chemists of all countries might claim that the term Daltonian is needed by them to distinguish those who adopt the famous opinions of Dalton, concerning the atomic constitution of matter* (Wilson 1855, p. 5). Dalton, in fact, suggested that all matter was made of indivisible and indestructible atoms and explained chemical phenomena with his theory, which became the theoretical foundation in modern chemistry.

[31] In a note of his *Researches on colour blindness,* G. Wilson wrote: *[...]as Mr. Maxwell has very forcibly illustrated, any three colours may be regarded as primary, provided by their union they constitute white light. The phenomena of colour-blindness, nevertheless, assign to red a speciality and prominence among colours which warrant its being made a primary, and it is the only colour which all opticians are willing to consider such* (Wilson 1855, p. 176).

3.12 Maxwell's colour top to study colour blindness

Using his colour top, Maxwell found a mathematical formulation of the difference between ordinary and colour blind eyes, expressed in term of equations, as he wrote in his paper of 1855 (Clerk Maxwell 1855, p. 285):

The equations thus obtained do not require five colours including black, but four only. For instance, the mean of several observations gives

$$0.19\,G + 0.05\,B + 0.76\,Bk = 1.00\,R$$

Maxwell used his colour top with four coloured papers only instead of five and the mean of his observations gave the expression reported above, where letters *G*, *B*, *Bk* and *R* denote respectively the green, blue, black and red papers prepared by D. R. Hay. A mixture of *green* and *blue* and *black* on the left side of the equation gave rise to a full *red* on the right side. Maxwell was then able to identify point *D* on his colour diagram, which is invisible to the colour blind, as we can see in the Fig. 3.28, and equivalent to black. All lines passing through point *D* pointing in different directions differed only in intensity for a subject affected by colour blindness and could be rendered equal to each other by the addition of black only.

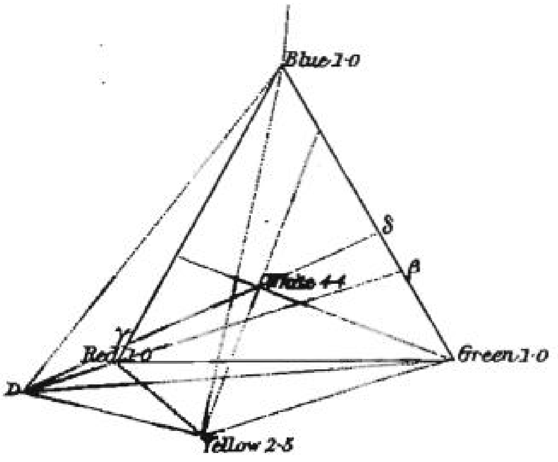

Fig. 3.28

Maxwell's colour diagram, with indication of point *D*, from which he could derive all his results of colour blindness. Point *D* corresponds to a colour that should be added to a colour blind eye to obtained an ordinary vision. From this representation, furthermore, we can see that *White* lies outside the triangle *Blue-Red-Yellow*, further evidence of the fact that white cannot be obtained from a mixture of the three primaries used by painters, blue, red and yellow. This colour diagram can be found at the end of his paper of 1855 (Clerk Maxwell 1855)

Maxwell's colour diagram can be portioned in two sectors separated by the boundary line that passes through point *D* (invisible to the colour blind subject and located in the red region) and point *W* (white). Maxwell gave a full exposition of the twofold nature of colour space for colour blind subjects: all colours for a colour blind person appear as if composed of *blue* and *yellow*, while for an ordinary eye, colour is a function of three independent variables, that he identified with *redness*, *greenness* and *blueness*: *[…] all colours on the upper side of DW will be varieties of blue and all colours on the under side varieties of yellow* (Clerk Maxwell 1855, p. 286). In a letter addressed to George Wilson, he described the method used to obtain lines that appeared identical to a colour blind person inside his colour triangle: *If we find two combinations of colours which appear identical to a Colour-Blind person, and mark their position on the triangle of colours, then the straight line passing through these points will pass through all points corresponding to other colours, which to such a person, appear identical with the first two. We may in the same way find other lines passing through the series of colours which appear alike to the Colour-Blind* (Niven 1890, vol. I, pp. 123-124).

Helmholtz also conducted experiments using the colour top with a colour blind student at the Polytechnic Institute. The results of his investigation were in line with those obtained by Maxwell, i.e. all hues for a colour blind subject could be obtained by mixing yellow and blue (von Helmholtz 1962, p. 148).

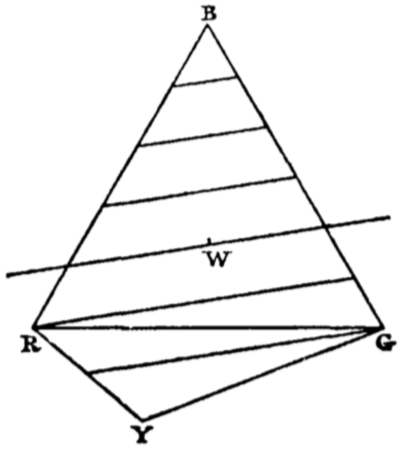

Fig. 3.29
Maxwell colour triangle with indication of the parallel lines that appeared of the same colour to a colour blind subject (Niven 1890, vol. I, p. 124)

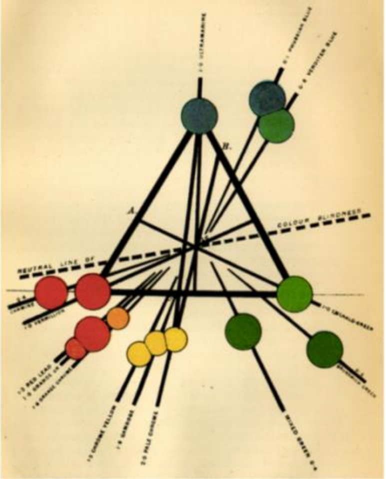

Fig. 3.30
Maxwell colour triangle illustrating the chromatic relations of coloured papers and showing the neutral line of colour blindness (dotted line) (Campbell and Garnett 1882, p. 228)

Maxwell's results using the top gave valuable clues to the three-receptor theory of human vision. He could conclude, furthermore, that one of the receptors should have a particularly high sensitivity to the red portion of the spectrum, confirming *red* as one of the primary colours. The straight lines in Fig. 3.29 on Maxwell's triangle indicate the chromaticity confusions of colour blind subjects. Maxwell and Helmholtz restricted the investigation to what they called red-blindness because as Maxwell stated: *I am not aware of any method of determining by a legitimate process the nature of the other two sensations, although Young's reasons for adopting something like green and violet appear to me worthy of attention* (Clerk Maxwell 1855, p. 287).

3.13 Following Maxwell: the work of William Pole

Many scientists were aware of Maxwell's method to investigate colour blind subjects, which combined effectiveness with simplicity. I would like to cite the work of William Pole. Maxwell, in his conference *On colour vision*, speaking about colour blindness, praised the work of Professor William Pole for his clear description of his own case. Pole was Professor of Civil Engineering at the University College, London, and was affected by red-blindness. He made use of Maxwell's colour top to build colorimetric equations, taken as reference for his investigations the colour circle of the chemist Michel Eugène Chevreul (see Fig. 3.31), who was, together with George Field, one of the most important colour theorist of the early nineteenth century and whose work on colour was intimately connected with his chemical researches.

In his paper *On colour blindness* published in 1859, Pole described the experimental procedure: one circle of the colour top contained the two primaries according to his anomalous vision, namely, *ultramarine* and *pale or lemon chrome,* identified as primaries for a colour blind subject, while in the other circle he placed the colour to be matched. In this way, he could match every colour by simply varying the proportions of *blue* and *yellow*. He also took into consideration the intensity; since the intensity of the combination of blue and yellow was generally different from that of the matched colour, he used in addition *white* or *black*. He reported then the obtained colour equations (Pole 1859).

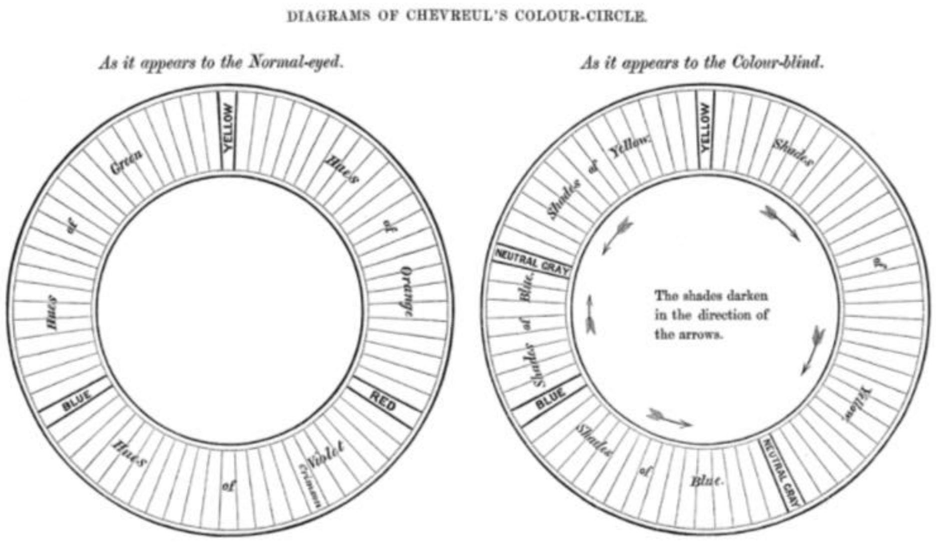

Fig. 3.31
Chevreul's colour circle as it appears to a subject with normal colour vision (left) and to a subject affected by red-blindness (right) (Pole 1859)

Analysing his results, he could describe most of the colours of the spectrum perceived by a normal trichromatic subject according to his perception. I would like to report here one key consideration; Pole, recognizing that he was totally blind to green as to red, stated: *I am, in fact, as totally blind to green as to red; an element of the malady which, I think, has not hitherto received the attention it deserves. But here a curious question arises, whether these two kinds of blindness have any connexion with each other? We have seen that insensibility to red induces also a want of perception of its compounds, orange and violet; but can this in any way affect the vision of green, a colour into which red can scarcely be supposed to enter?* (Pole 1859). These questions found soon an answer in the modified Young's three-receptor theory.

3.14 Maxwell's colour box to study colour blindness

Although most of his results were obtained using the colour box described in section 3.10 (Fig. 3.23), Maxwell used the final portable colour box version described in the postscript of the paper of 1860 to study colour blindness. The box was made by Messrs. Smith and Ramage of Aberdeen. The method of operation was the same, white light coming from sunlight was directed into the box at point *E* from a sheet of white paper. Then light was dispersed by the two 45° prisms, through which it passed two times. After the first pass, in fact, light was reflected back by a concave mirror *S* (radius 34 inches) and then reflected again by a small mirror, *e*, to the opening *E*, where the eye of the observer was located. At the same time, a portion of the white light could enter the colour box at *BC* and it underwent reflection by a mirror M, then passed through the lens L before being reflected again at the mirror M' to the opening E. *In this way*, wrote Maxwell, *the compound colour is compared with a constant white light in optical juxtaposition with it* (Clerk Maxwell 1860, p. 78).

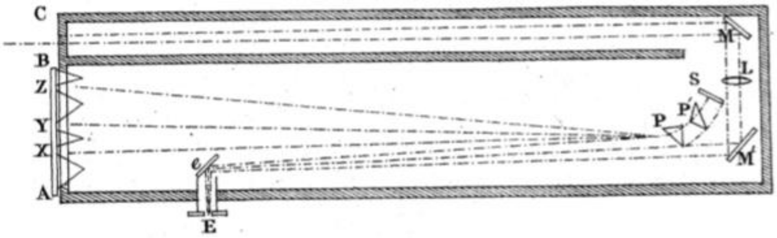

Fig. 3.32
Maxwell's portable colour box with indication of the light paths (Clerk Maxwell 1860)

Fig. 3.33
Replica of the portable colour box preserved in the exhibition room of the James Clerk Maxwell Foundation in Edinburgh

To investigate colour blindness, only two tuneable slits were required to produce white, the nature of colours, in fact, depends only on two independent variables for colour blind subjects (at least the most common cases), as he stated in his paper of 1855. Maxwell could, furthermore, identify a point of the spectrum at which the composition of two colours appeared identical with white, namely near Fraunhofer line F (486 nm corresponding to cyan-blue for a normal trichromat). The new colorimetric equations were obtained taken as standard colours green and blue, corresponding to position 88 and 68 respectively on the new scale. He then summarized his results with the following considerations: *1. That the colour-blind observations were consistent among themselves, on the hypothesis of* two *elements of colour. 2. That the colour-blind observations were consistent with the author's observations, on the hypothesis that the two elements of colour in dichromic vision are identical with two of the three elements of colour in normal vision. 3. That the element of colour, by which the two*

types of vision differ, is a red, whose relations to vermillion, ultramarine and emerald-green are expressed by the equation $D = 1.198V + 0.078U - 0.276G$
where D is the defective element, and V, U and G the three colours named above (Harman 1990a, p. 656).

Maxwell also determined the intensity curves of the two primaries, as we can see in Fig. 3.34.

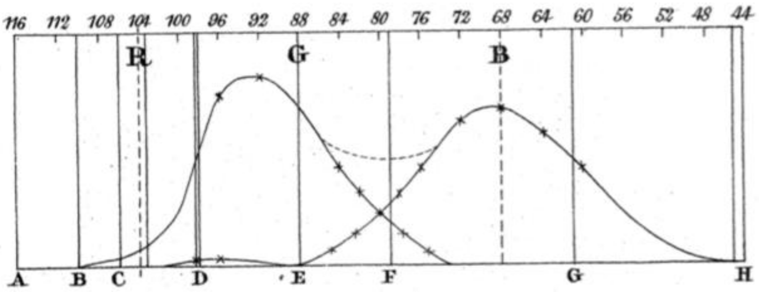

Fig. 3.34
Curves of intensities of the primaries for a red blind subject obtained from Mr. James Simpson's observations, a student of Natural Philosophy in Maxwell's class. The curve on the left represents the intensity of the *yellow* element and the curve on the right that of the *blue* element as it appeared to Mr. Simpson (Harmann 1990a, p. 651)

Maxwell described the appearance of the spectrum to the colour blind subject with the following words: *From A to E the colour is pure "yellow" very faint up to D, and reaching a maximum between D and E. From E to one-third beyond F towards G the colour is mixed, varying from "yellow" to "blue" and becoming neutral or white at point near F* (Clerk Maxwell 1860, p. 81). In the next chapter we will offer a comparison between Maxwell and Helmholtz's collaborators' intensity curves.

3.15 Helmholtz, Maxwell and Young

Thomas Young's theory of colour-vision, one of the most beautiful twigs in his laurel crown, after lying as it were buried in the darkness of oblivion for more than half of a century, was brought to light again by Maxwell and Helmholtz (König and Dieterici 1886, p. 438)

Helmholtz in his paper of 1855 did not repudiate his previous assertion regarding the number of primary colours. He probably still believed that a minimum of five elementary colours was required to produce all the spectral hues. We can suppose that he still doubted in that period on the validity of Young's three-receptor hypothesis. During the summer of 1855, Helmholtz dedicated himself to writing the first part of the *Handbuch der physiologischen Optik (Treatise of Physiological Optics)*, published in the following year, 1856. Between 1855 and 1860 Helmholtz changed his position on Young's hypothesis. In fact, in his second part of the *Handbuch*, published in 1860, Young's hypothesis assumed a key role. He elaborated it into a complete theory of vision that could unify until that time unrelated phenomena. Helmholtz, in fact, mentioned here all the phenomena that Young's theory could explain, such as colour blindness, total colour blindness, colour harmony and the Purkinje phenomenon[32].

Maxwell, on the contrary, accepted from the beginning Young's three-receptor hypothesis. In the paragraph entitled *Theory of the Perception of Colours* of his paper of 1855, Maxwell reported with the following words Young's theory: *In order to fully understand YOUNG'S theory, the function which he attributes to each system of nerves must be carefully borne in mind. Each nerve acts, not, as some have thought, by conveying to the mind the knowledge of the length of an ondulation of light, or of its periodic time, but simply by being more or less affected by the rays which fall on it. The sensation of each elementary nerve is capable only of increase and diminution, and of no other change* (Clerk Maxwell 1855, p. 283). Moreover, by investigating the nature of colour for a colour blind eye he obtained crucial results that corroborated Young's theory (see section 3.14).

Maxwell measured (see section 3.10) the shape of the three response curves which Helmholtz only postulated, as we shall see in the following lines. Maxwell considered it necessary to consider Young's intensity curves as coterminal, as he wrote at the end of his paper of 1855: […] *I would conclude, that every ray of the spectrum is capable of producing all three pure sensations, though in different degrees* (Clerk Maxwell 1855, p. 296). In fact, if each point inside the colour triangle corresponds to a colour which can be obtained mixing red, green and blue, then every colour sensation can be considered as the simultaneous excitation of the three receptors in the retina.

In the lecture held at the Royal Institution on 17 May 1861, *On the Theory of three Primary Colours*, Maxwell expressed clearly Young's theory in his revised form (coterminal response curves): *Young supposes that the eye is provided with three distinct sets of nervous fibres, each set extending over the whole sensitive surface of the eye. Each of these three systems of nerves, when excited, gives us a different sensation. One of them, which gives us the sensation we call red, is excited most by the red*

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[32] In 1823 the Czech anatomist and physiologist Johann Evangelist Purkinje reported a change in relative brightness of the long wavelengths and short wavelengths as the illumination decreased. He found out that at high illumination red appeared brighter than blue but at low illumination conditions blue appeared brighter than red. Helmholtz reported Purkinje's original description of the phenomenon, considering the appearance of colours in the morning twilight: *"Blue is what I saw first. The shades of red that are usually brightest in daylight, namely, carmine, vermilion, and orange, are for a long time the darkest, and to be compared to their ordinary brightness."* (von Helmholtz 1962, p. 358). Helmholtz also used his spectrophotometer to investigate the phenomenon.

rays, but also by the orange and yellow, and slightly by the violet; another is acted on by the green rays, but also by the orange and yellow and part of the blue; while the third is acted on by the blue and violet rays (Niven 1890, vol. I, p. 447). This excerpt helps us to better understand Maxwell's revision of Young's originally theory. Young, in fact, excluded the possibility that a single homogeneous ray could stimulate at the same time all three receptors.

Helmholtz accepted Young's theory some years after the publication of his paper of 1855[33], as we can read in the second volume of his *Handbuch*, and it is worth to notice that he reported it in the revised form proposed by Maxwell: *Objective homogeneous light excites these three kinds of fibres in various degrees, depending on its wave-length. The red-sensitive fibres are stimulated most by light of longest wave-length, and the violet-sensitive fibres by light of shortest wave-length. But this does not mean that each colour of the spectrum does not stimulate all three kinds of fibres, some feebly and other strongly; on the contrary, in order to explain a series of phenomena, it is necessary to assume that that is exactly what does happen* (von Helmholtz 1962, p. 143). This extract is followed by Helmholtz's sketch of his hypothetical coterminal response curves for Young's receptors. He made no attempt to measure the actual shape of the curves; it was a challenge for his collaborators, as we will see in the next chapters.

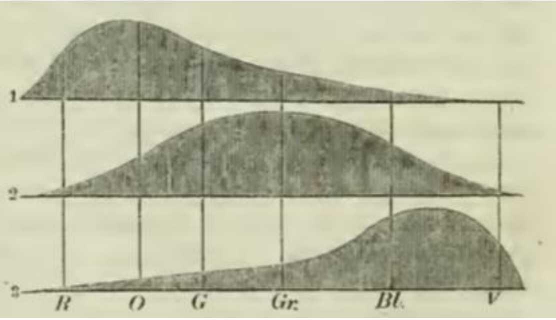

Fig. 3.35
Helmholtz's three hypothetical intensity curves: *Pure* red *light stimulates the red-sensitive fibres strongly and the two other kind of fibres feebly; given the sensation red. Pure* yellow *light stimulates the red-sensitive and green-sensitive fibres moderately and the violet-sensitive fibres feebly; giving the sensation yellow. Pure* green *light stimulates the green-sensitive fibres strongly, and the two other kinds more feebly; giving the sensation green. Pure* blue *light stimulates the green-sensitive and violet-sensitive fibres moderately, and the red-sensitive fibres feebly; giving the sensation of blue. Pure* violet *light stimulates the violet-sensitive fibres strongly, and the other fibres feebly; giving the sensation violet. When all the fibres are stimulated about equally, the sensation is that of white or pale hues* (von Helmholtz 1962, p. 144)

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[33] On July 3, 1858, Helmholtz read the paper *Über die subjectiven Nachbilder im Auge (On Subjective After-images of the Eye),* subsequently extended in his *Handbuch*, in which we can find the first mention of Young's coterminal response curves (Königsberger 1906).

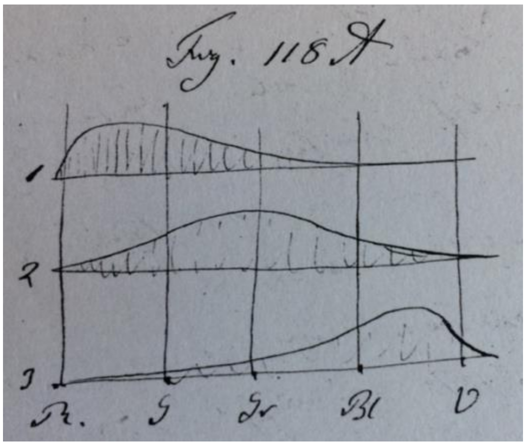

Fig. 3.36
Helmholtz's original illustration of the three hypothetical intensity curves taken from the manuscript version of his *Handbuch* preserved at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften* (Nachlass Hermann von Helmholtz, 572 Über physiologische Optik, ʃʃ20. Die zusammengesetzen Farben)

Some of Helmholtz's students and assistants, in fact, conducted in the 1870s, 1880s and 1890s researches on colour vision in Helmholtz's *Physikalische Institut* obtaining crucial results, that were path breaking for the development of modern colour science (see chapter 4).

3.16 Maxwell, Helmholtz and their laboratories

Maxwell and Helmholtz were appointed in the same year first Professor of Experimental Physics. Until the second half of the nineteenth century no teaching laboratories and no courses of experimental physics were known in both Cambridge and in Berlin. However, it is worth to point out that, before that time, institutions in Europe had begun to place an increasing emphasis on practical research, particularly in Scotland and in Germany, where the first physical research laboratories already existed[34]. In England the first steps in this direction were taken in the 1860s in Oxford and London. At the University of Glasgow, furthermore, William Thomson became Director of the University Laboratory.

In 1866, 1867, 1869 and 1870 Maxwell was invited to be an examiner for the *Mathematical Tripos*, the final written examinations of undergraduate students which was considered as an honour degree

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[34] I would like to highlight that, starting already from the second decade of the 18th century, the first *cabinets of physics* were created. Professors of experimental physics were expected to provide their equipment in order to conduct the lessons. Therefore, large private collections resulted, such as those of 's Gravesande and Musschenbroek at Leyden, Winkler at Leipzig, Lichtenberg at Göttingen, Bose at Wittenberg, Jallabert and Saussure at Geneva, Poleni at Padua and Guadagni at Pisa. Further details on the subject can be found in the text of Heilbron, *Electricity in the 17th and 18th centuries: a study of early modern physics* (Heilbron 1979, pp. 140-166).

in Cambridge. Maxwell played a pivotal role in the initiation of a reform of the exams. Between 1868 and 1869, in fact, thanks to Maxwell's work, new regulations saw the introduction of topics related to heat, electricity and magnetism. The University Senate approved recommendations to establish a chair of Experimental Physics in order to teach the new subjects. They failed, however, to find a way to finance the project. The University's Chancellor William Cavendish, Seventh Duke of Devonshire was not only a landowner and industrialist, but also Second Wrangler, i.e. the second highest-scoring student and among his ancestors it is worth mentioning Henry Cavendish and Robert Boyle. He was sensitive to the new social and economic needs of that time. He provided 6300 pounds from his own resources to meet the costs of both building and equipping the new laboratory. At this point, in order to explore the availability of possible candidates for the new position, the first informal contacts were taken. William Thomson was the first to be contacted; at that time, in fact, was one of the most eminent British physicist. He was forced to refuse the offer, preferring to maintain his position at Glasgow, where he directed the University laboratory, perfectly suited to pursue his aims. In 1871 Helmholtz received a letter from William Thomson, in which he asked him if he was disposed to accept the Professorship of Experimental Physics at Cambridge, that he was forced to refuse. Helmholtz preferred, however, to stay in Berlin, because, as we will see in the following lines, he became Director of the Physical Institute in Berlin in the same year. The position, after Helmholtz's renounce, was then offered to James Clerk Maxwell, who was elected on March 8, 1871, the first Professor of Experimental Physics and Director of the annexed laboratory, then called the *Devonshire*. Maxwell dedicated several years to its design and construction until its inauguration, that took place in 1874; it was in that occasion that the laboratory was baptized as the *Cavendish*.
Maxwell visited Oxford and Glasgow, before receiving plans for the building from the syndicate, in order to become acquainted with the arrangements made in these universities for the teaching of experimental physics[35].

In Maxwell's inaugural lecture in 1871, *Introductory Lecture on Experimental Physics* (Niven 1890, vol. II, pp. 241-255), Maxwell presented the task of the new-born laboratory for experimental physics.

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[35] A detailed account of the history of the Cavendish can be found in *A History of the Cavendish Laboratory, 1871–1910* written by J. J. Thomson et al. (Thomson et al. 1910). Moreover, a sketch of the plans for the Cavendish is contained in vol. II edited by Harman (Harman 1990b, p. 630-633). I would also like to mention that, after Maxwell's death, occurred in 1879, he was succeeded by Lord Rayleigh. Among his research topics, it is worth to remember that Lord Rayleigh devoted himself to colour mixing experiments, refining Maxwell's colour box, as we can read in his paper *Experiments on Colour*: *For systematic observations on compound colours nothing probably can be better than Maxwell's colour box in its original form; but it seemed to me that for the examination of certain special questions a more portable arrangement would be convenient* (Rayleigh 1881, p. 64). His aim was that of directly comparing a yellow produced mixing pure red and pure green with the yellow of the spectrum and he could express the results in form of accurate colorimetric equations.

This lecture acted as a guide for his future achievements and we can also find here Maxwell's reflections on the methodology and aims of scientific research. He described the course with the following words: *This course of study, while it requires us to maintain in action all those powers of attention and analysis which have been so long cultivated in the University, calls on us to exercise our senses in observation, and our hands in manipulation. The familiar apparatus of pen, ink and paper will no longer be sufficient for us, and we shall require more room than that afforded by a seat at a desk, and a wider area than that of the black board* (Niven 1890, vol. II, p. 241).

And for what concerns the role of experimental physics, he stated: *When we shall be able to employ in scientific education, not only the trained attention of the student, and his familiarity with symbols, but the keenness of his eye, the quickness of his ear, the delicacy of his touch, and the adroitness of his fingers, we shall not only extend our influence over a class of men who are not fond of cold abstractions, but, by opening at once all the gateways of knowledge, we shall ensure the association of the doctrines of science with those elementary sensations which form the obscure background of all our conscious thoughts, and which lend a vividness and relief to ideas, which, when presented as mere abstract terms, are apt to fade entirely from the memory* (Niven 1890, vol. II, p. 241). The Cavendish Laboratory was expected, in fact, to attract students who had interest in practical work with the aim of combining mental and sensory activity. In this lecture, furthermore, Maxwell underlined the importance of what he called *cross-fertilization* of the sciences: *There is no powerful method for introducing knowledge into the mind than that of presenting it in as many different ways as we can* (Niven 1890, vol. II, p. 247). The cross-fertilization of the sciences was the background of all his investigations in the different research fields; as discussed in the previous sections, all his work on colour was in line with this conception. Moreover, in a review paper on Helmholtz's scientific work, published in 1877, Maxwell reported Helmholtz's contributions in different research fields, such as mathematics, physics, physiology, psychology and aesthetics as a case in point of cross-fertilization: *The time, however, must sooner or later arrive when two or more departments of knowledge can no longer remain independent of each other, but must be fused into a consistent whole* (Niven 1890, vol. II, p. 592).

Fig. 3.37
Entrance of the Cavendish Laboratory (Thomson et al. 1910)

In the same year Maxwell was appointed Professor of Experimental Physics and Director of the Cavendish, Helmholtz became Director of the Physical Institute in Berlin. Before moving from Heidelberg to Berlin, Helmholtz had to work under fairly limited circumstances. Apparatus and assistants were available only in insufficient numbers; in Berlin, on the other hand, he found a *Tempel der Physik (temple of physics),* as it was called at the moment of its inauguration.

In Berlin, the University Professor Gustav Magnus built his *Privat-Laboratorium (private laboratory)* located at 'Kupfergraben 7', where young scientists could work independently on their own topics and discuss together weekly about them. In 1863 Magnus arranged the first physical laboratory of the University of Berlin; some rooms of the laboratory were dedicated to students' physical exercises. In 1868 Magnus wrote a plan for the building of a new Physical Institute. After his death occurred in 1870, the realization of this plan was reserved to his successor Helmholtz, from 1873 to 1878[36]. At the beginning of the year 1870, in fact, Helmholtz was elected external member

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[36] In 1877 Maxwell dedicated a paper to Helmholtz, which appeared in *Nature*, Vol. XV, entitled *Hermann Ludwig Ferdinand Helmholtz*. He concluded the work with the following words: *Helmholtz is now in Berlin, directing the labours of able men of science in his splendid laboratory. Let us hope that from his present position he will again take a comprehensive view of the waves and ripples of our intellectual progress, and give us from time to time his idea of the meaning of it all* (Niven 1890, vol. II, p. 598).

of the Academy of Berlin. On April of the same year, his friend Emil Heinrich du Bois-Reymond[37], at that time Professor of Physiology in Berlin and Rector of the University, informed him of the death of Magnus. The Philosophical Faculty of the University at Berlin proposed Kirchhoff as successor to Magnus but he preferred to remain in Heidelberg. Du Bois-Reymond could, therefore, open negotiation with Helmholtz, who explained in detail the conditions of his employment.

Fig. 3.38
The new Berlin *Physikalische Institut* built according to Magnus' and Helmholtz's plan in the years from 1873 to 1878. It was destroyed during World War II (Schreier et al. 1995)

A state-of-the-art Physical Institute with a large auditorium for 300 people was built on a surface of 1350 m^2 on the banks of the Spree river (*Reichstagsufer*). With 4.5 million marks of construction costs, it was one of the largest and most expensive in Germany. From 1878 it became the permanent conference venue of the Physical Society (Schreier et al. 1995)[38].

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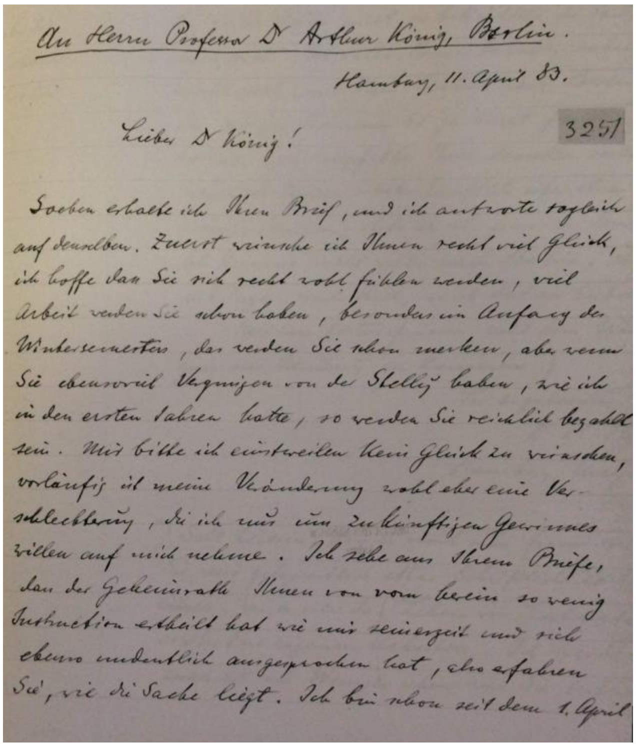
[37] Helmholtz met du Bois-Reymond at the time they were students at the Royal-Wilhelm Institute for Medicine and Surgery in Berlin, where Johannes Müller held the course of Physiology. In 1841 Helmholtz consolidated friendship with physiologist du Bois-Reymond and Ernst Wilhelm von Brücke, who lived in the circle of Müller's students. It should be mentioned that Müller's students had a common aim, i.e. to create connection between physiology and physics. Helmholtz succeeded in this aim, as testified by his studies on colour.

[38] I would like to report the words used by Stanley Hall, who gave a realistic picture of Helmholtz's activity in the laboratory: *In seinem Laboratorium befanden sich neben dem großen Raum, in dem viele Studenten praktisch tätig sein und die wichtigsten Experimente wiederholen konnten, viele kleinere Räume für fortgeschrittene Studenten. In diesen pflegte Helmholtz mehrere Stunden am Tag hintereinander bei den Experimentatoren zu verbringen. Er regte neue Verfahren und neue Formen des Experiments an, er wiederholte, berichtigte, kontrollierte die schon fertigen Arbeiten, zeichnete wohl auch einmal flüchtige Skizzen von neuen Apparaten auf oder bestellte neue Instrumente, die in der Werkstätte hergestellt werden sollten (In his laboratory, there were many smaller rooms for advanced students next to the large room, where many students could practice and repeat the most important experiments. In these rooms, Helmholtz used to spend several hours a day with the experimenters. He stimulated new procedures and new forms of experiment, repeating, correcting, controlling already completed works, sketching new apparatus or ordering new instruments to be built in the workshop.)* (Hall 1914, p. 128).

It should be mentioned that, among Helmholtz's students and assistants, there was Heinrich Hertz, who was his assistant from 1880 to 1883, when he accepted a position as a lecturer in theoretical physics at the University of Kiel. Arthur König was his successor, as testified by a letter written on April 1883 by Hertz and addressed to König, which contains Hertz's elucidation about the position of assistant and from which an excerpt is reported: *Zuerst wünsche ich Ihnen recht viel Glück, ich hoffe, dass Sie sich recht wohl fühlen werden, viel Arbeit werden Sie schon haben, besonders im Anfang des Wintersemesters, das werden Sie schon merken (At first, I wish you good luck, I hope you will feel comfortable, you will already have a lot of work, especially in the beginning of the winter semester, as you will notice)* (Deutsches Museum-Archive, Handschriften-Bestand 03251 Brief Hertz, Heinrich an König, Arthur).



An Herrn Professor Dr Arthur König, Berlin.
Hamburg, 11. April 83.

3251

Lieber Dr König!

Soeben erhalte ich Ihren Brief, und ich antworte sogleich
auf denselben. Zuerst wünsche ich Ihnen recht viel Glück,
ich hoffe dass Sie sich recht wohl fühlen werden, viel
Arbeit werden Sie schon haben, besonders im Anfang des
Wintersemesters, das werden Sie schon merken, aber wenn
Sie ebensoviel Vergnügen von der Stellung haben, wie ich
in den ersten Jahren hatte, so werden Sie reichlich bezahlt
sein. Mir bitte ich einstweilen kein Glück zu wünschen,
vorläufig ist meine Veränderung wohl eher eine
Verschlechterung, die ich nur um zukünftigen Gewinnes
willen auf mich nehme. Ich sehe aus Ihrem Briefe,
dass der Geheimrath Ihnen von vorn herein so wenig
Instruction ertheilt hat wie mir seinerzeit und sich
ebenso undeutlich ausgesprochen hat, also erfahren
Sie, wie die Sache liegt. Ich bin schon seit dem 1. April

Fig. 3.39
Excerpt from a letter written by Hertz to König, preserved at the *Deutsches Museum Archive* (Handschriften-Bestand 03251 Brief Hertz, Heinrich an König, Arthur)

In 1887 Werner von Siemens[39] founded the new Physico-technical Institute (*Physikalisch-Technsiche Reichsanstalt*) in Potsdam, devoted to fundamental and applied research, of which Helmholtz became

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[39] Siemens, a symbol of Prussian industrial power, was a family friend because his son Arnold became the husband of Helmholtz's daughter by his second marriage, Ellen von Helmholtz. After the death of his first wife, Olga von Velten, occurred in 1859, Helmholtz met Anna von Mohl, coming from a cultured and respectable Heidelberg family, who also helped him in his research career. Siemens financed a major part of the new Physico-technical Institute, that represents a perfect example of collaboration between public and private sectors.

President. He moved, therefore, in 1889 to the residence assigned to the President in ‘Marchstrasse 25b, Charlottenburg’.

Chapter 4

Helmholtz’s collaborators in Berlin: König and Brodhun’s contributions to colour science

This chapter is dedicated to the crucial contributions to colour vision of Helmholtz’s collaborators and assistants working at his laboratory in Berlin. I will focus on the work of Arthur König, who wrote most of his major works in collaboration with Conrad Dieterici, in turn Helmholtz’s assistant at the Physical Institute of the University of Berlin, and of Eugen Brodhun, who provided significant results for what concerns colour blind vision, being himself affected by green-blindness (*Grünblindheit*). König, Dieterici and Brodhun offered Helmholtz precious material, which he could collect and elaborate to lay down the first line element in colour space, as we will see in chapter 5. I will begin presenting König’s investigations and results contained in his most famous works, starting from the determination of the intensity curves for the three elementary sensations. Particular focus will be also placed on his paper *Experimentelle Untersuchungen über die psycho-physische Fundamentalformel in Bezug auf den Gesichtssinn (Experimental investigation on the psycho-physical fundamental formula in relation to the sense of vision),* published in 1888 and written in collaboration with Brodhun. This paper is of historical importance because it contains one of the first attempt to apply a psychophysical law, i.e. the Weber-Fechner law, to the field of colour vision. It is noteworthy to underline that Helmholtz’s starting point for his research on the geometry of colour space was exactly this one-dimensional psychophysical law.

I will proceed showing Brodhun’s main achievements in the field, starting with an analysis of the content of his doctoral dissertation, *Beiträge zur Farbenlehre* of 1887, kindly supplied by the *Staatsbibliothek zu Berlin*, which has not as yet been translated into English. The last section is dedicated to Brodhun’s investigation on the validity of Newton’s law of colour mixing for subjects affected by his own deficiency, in which he provided an accurate analysis of the position of the neutral point.

4.1 Arthur König

König was fortunate to be present in the right place at the right time, to be able to produce a series of key measurements of the human color vision system that were path breaking for the development of color science in the 20th century (Rolf G. Kuehni, Kuehni 2001, p. 339)

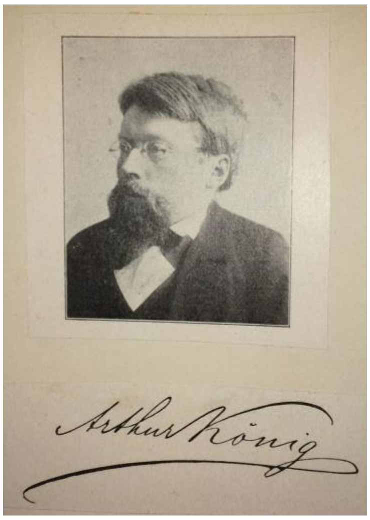

Fig. 4.1
Photo portrait of Arthur König with autograph signature (*Staatsbibliothek zu Berlin*. Handschriftenabteilung; Signatur: Slg. Darmstaedter F1c 1880: König, Arthur, Blatt 1)

Arthur Peter König was Helmholtz's assistant and co-worker and he contributed significantly to colour science. Born in Krefeld in 1856, after studying in Bonn and Heidelberg he moved to Berlin in the fall of 1879, where he remained for the rest of his life. Here he met Professor Hermann von Helmholtz, who suggested a doctoral dissertation to him. In 1882 König obtained his Ph.D. degree and in the same year, he became Helmholtz's assistant. König was named firstly lecturer and in 1889 Director of the physical division of the Physico-technical Institute, with the special task of teaching physiological optics. It should be mentioned that in 1890 he founded together with the psychologist Hermann Ebbinghaus, the *Zeitschrift für Psychologie und Physiologie der Sinnesorgane (Journal for Psychology and Physiology of Sensory Organs),* the oldest psychological journal in Europe and the second oldest in the world. He was also co-editor, together with Helmholtz, of the second edition of the *Handbuch* and, after Helmholtz's death occurred in 1894, sole editor of the third edition. He authored fifteen papers dealing with physics and in 1883 he focused his attention on physiological optics, as testified by the large number of papers (over thirty) of seminal importance, that he published. Among these, as we will see in the next section, König published a paper in 1883, in which he compared Maxwell and Helmholtz's intensity curves. He calculated the intersection wavelengths between the fundamental curves and found them relatively similar. In 1884 he reported the results obtained by an investigation made, together with Helmholtz's student Conrad Dieterici[40], of the sensitivity of a normal eye to wavelengths differences. In 1886 König read in England a paper, *Über die neuere Entwicklung von Thomas Young's Farbentheorie (The Modern Development of Thomas Young's Theory of Colour-vision)*, in which he reported the new development of Young's theory of colour and the measured spectral trace sketched inside Maxwell's triangle. Here we can also find König's determination of the elementary curves for people with monochromatic vision and with two different kinds of dichromatic vision (today these different anomalous conditions are known as *protanomaly*, which is a reduced sensitivity to red light, and *deuteranomaly*, which is a reduced sensitivity to green light). Furthermore, König described the experiments conducted by Eugen Brodhun, student at the time in their laboratory and affected by green-blindness, aimed to detect the positions in the spectrum at which the sensation of colour changes most rapidly. In the same year, another paper was published in collaboration with Conrad Dieterici, with the title *Die Grundempfindungen und ihre Intensitäts-Vertheilung im Spectrum (Fundamental sensations and their intensity distribution in the spectrum),* in which the fundamental sensitivity curves of a *monochromat*, of two kinds of *dichromats* and of normal and anomalous *trichromats* are reported. This paper was the first part of a second one, published in 1892, which completed and refined the first one. König and Dieterici's measurements of the intensity curves of the fundamentals remained the standard ones until 1928[41].

König, in a paper published in 1887 with the title *Über Newton's Gesetz der Farbenmischung und darauf bezügliche Versuche des Hrn. Eugen Brodhun (On Newton's law of colour mixing and related experiments of Mr. Eugen Brodhun)*, examined whether Newton's laws of colour mixing could still be regarded as fulfilling at his time. He took into account only trichromatic and dichromatic colour systems. All the results obtained by Brodhun were then published in 1893 in a paper with the title

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[40] Conrad Dieterici studied physics at the University of Berlin and received his doctorate there in 1882. From 1885 to 1890, he was assistant at the Physical Institute of the University of Berlin and also librarian of the Physikalische Gesellschaft. In 1887, Dieterici became *Privatdozent* before receiving, in 1890, a chair at the University of Breslau.

[41] In 1931, the International Commission on Illumination (abbreviated with the acronym CIE, that stands for its French name, *Commission internationale de l'éclairage*) was established to develop international standard methods in lighting, colour, colour spaces and related subjects. Since its foundation, the CIE has been the worldwide preeminent organization dealing with standardization of colour stimuli measurements. In 1931 a trichromatic colour system, valid for all human observers with normal colour vision, was proposed by the CIE based on the work of Newton, Maxwell, Grassmann, Helmholtz and its collaborators. The first ideal observer, known as The Standard Colorimetric Observer, was defined by three independent functions of wavelength, based on experimental colour-matching data of English physicists John Guild and William Wright (see chapter 6), from which they obtained colour matching functions (Kuehni and Schwarz 2008).

Die Gültigkeit des NEWTON'schen Farbenmischungsgesetzes bei dem sog. Grünblinden Farbensystem (The validity of Newton's law of colour mixture in green-blind colour system), in which he restricted the analysis to the green-blind colour vision. Moreover, it should be mentioned that König, together with Brodhun, investigated the validity of the Weber-Fechner law of psychophysics for light illumination for different colours. They calculated the visual differential threshold with several intensities. All the obtained results were published in two papers published in 1888 with the title *Experimentelle Untersuchungen über die psycho-physische Fundamentalformel in Bezug auf den Gesichtssinn (Experimental investigation on the psycho-physical fundamental formula in relation to the sense of vision).* In the second paper, they summarized the content of the first one and extended the results in the case of white light. König's 32 papers on physiological optics were collected in a book published in 1903 (König and Engelmann 1903).

4.2 König's first steps toward the determination of the intensity curves for the three elementary sensations

Maxwell and Helmholtz's work on colour represent the starting point for König's investigation in the field. In his first paper on physiological optics, *Über den Ort der Schnittpunkte der Intensitätscurven für die drei Grundempfindungen im normalen Auge (On the intersection points of the intensity curves for the three elementary sensations in the normal eye)* (König and Engelmann 1903, pp. 1-3) published in 1883, he calculated the intersection wavelengths between the intensity curves obtained by Maxwell and Helmholtz[42] in order to offer a direct comparison. He opened his work with the following words: *Bei der Wiederaufnahme der YOUNG'schen Farbentheorie hat Hr. v. HELMHOLTZ die Form der drei Grundempfindung zuerst in schematischer Weise construirt. Später suchte MAXWELL ihre Gestalt auf Grund experimenteller Untersuchungen zu bestimmen und gelangte zu Formen, welche mit den HELMHOLTZ'schen ziemlich übereinstimmen (In resuming Young's colour theory, Hr. v. Helmholtz first constructs the form of the three elementary sensation curves in a schematic way. Later, Maxwell tried to determine their form on the basis of experimental investigations, and obtained forms, which quite agree with Helmholtz's)* (König and Engelmann, p. 1).

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[42] As reported in chapter 3, Helmholtz made no attempt to measure the actual shape of the curves. König therefore based his considerations on Helmholtz's determination of complementary colours (von Helmholtz 1855).

He proceeded then in the determination of two intersection points, the first between the red and the green curve and the second between the green and the violet curve on the basis of complementary spectral colour determinations by Helmholtz and Maxwell.

He denoted the intensity of the red, green and violet sensations at two complementary points of the spectrum by $R(\lambda_1)$ and $R(\lambda_2)$, $G(\lambda_1)$ and $G(\lambda_2)$, $V(\lambda_1)$ and $V(\lambda_2)$ respectively. In this way, indicating with c a factor dependent only on λ_1 and λ_2, the following double equation must be used:

$$R(\lambda_1) + cR(\lambda_2) = G(\lambda_1) + cG(\lambda_2) = V(\lambda_1) + V(\lambda_2) \tag{4.1}$$

If $R(\lambda_1) > G(\lambda_1) > V(\lambda_1)$ (4.2), then $R(\lambda_2) < G(\lambda_2) < V(\lambda_2)$ (4.3)

If $G(\lambda_1) > R(\lambda_1) > V(\lambda_1)$ (4.4), then $G(\lambda_2) < R(\lambda_2) < V(\lambda_2)$ (4.5)

If $G(\lambda_1) > V(\lambda_1) > R(\lambda_1)$ (4.6), then $G(\lambda_2) < V(\lambda_2) < R(\lambda_2)$ (4.7)

Since the values of λ_1 and λ_2 could be interchanged, with these six relations König could obtain all the possible relations between the functions *R*, *G*, and *V*. Relations (4.4) and (4.6) corresponded to green colour tones, complementary to the purple colour tones given by relations (4.5) and (4.7). Since the latter did not occur in the spectrum, he excluded from his investigation relations (4.4), (4.5), (4.6) and (4.7). Only those parts of the spectrum were taken into consideration, which corresponded to relations expressed by (4.2) and (4.3), corresponding to the two intersections mentioned above toward the ends of the spectrum.

At this point, all the spectral colours that have a complementary colour in the spectrum could be considered lying in these two portions. The boundaries of these portions, which lie at the centre of the spectrum, correspond to the two intersection points. According to Helmholtz's investigating on complementary spectral colours, the portion located at the red end of the spectrum extends to wavelength 563,5 nm, and that at the blue end extends to 492,1 nm, furnishing the wavelengths corresponding to the two intersection points. From Maxwell's measurements, the respective wavelengths were of 566 nm and 489 nm. He found, therefore, their intensity curves relatively similar[43]. König gave a more detailed explanation of this subject, supported by further experiments, in his papers published in 1886 and in 1892.

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[43] Starting from Maxwell and Helmholtz's work, more than a century was required to discover the sensitivities of human cones by using several methods. A good review of this subject, with related bibliography, can be found in *Color vision and colorimetry: theory and applications* written by Malacara (Malacara 2011, pp. 51-58).

4.3 König and the Modern Development of Young's Theory of Colour

In September 1886, König read in Birmingham, for the fifty-sixth meeting for the advancement of science, his paper *Über die neue Entwicklungen von Thomas Young's Farbentheorie*[44] (König and Engelmann, pp. 88-107). He showed here, for the first time, the curves of *elementary* and *fundamental* sensations for *monochromats*, two kinds of *dichromats* and for *trichromats* by using a colour matching technique, and from which he could obtain the spectral trace accurately represented inside Maxwell's triangle. He also investigated the places in the spectrum at which the change of colour was the most rapid and presented the obtained results in a graph. It should be mentioned that the investigations were carried out in collaboration with his colleague Dieterici and Brodhun, at the time student in their laboratory. All this renders the paper historically important. My intention is, therefore, to underline König's crucial results taking as starting point Young, Maxwell and Helmholtz's theory of colour.

In the first section of the paper, König reported a brief overview of Young's theory of colour vision remarking the pivotal work undertaken by Maxwell and Helmholtz in its quantitative demonstration, as he stated, referring to Young's ideas: *It was not until thirty years ago that Maxwell and Helmholtz saved them from utter oblivion* (König and Dieterici 1886, p. 432). Before introducing his own work, König acknowledged the contributions of other scientists, such as Kries, Frey, Donders and Lord Rayleigh, who carried out more exact measurements with spectral light twenty years after Maxwell's investigations. The second part was dedicated to a detailed description of the experimental set-up used to obtain colour equations and, consequently, the curves of elementary sensations. König and Dieterici used Helmholtz colour mixer. Before proceeding with the analysis of their results, I would like to offer a description of the apparatus in order to highlight its simplicity of realization and its effectiveness. In fact, over the course of the years, it has been used to conduct colour matching experiments by many celebrated colour vision scientists, such as Selig Hecht and Simon Schlaer from the Laboratory of Biophysics of Columbia University, New York (Hecht and Schlaer 1936).

Helmholtz colour mixing apparatus, built by Schmidt and Haensch, for comparison of mixed colours is described in detail in *Bericht über die Wissenschaftlichen Instrumente auf der Berliner Gewerbeausstellung im Jahre 1879* (Loewenherz 1880).

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[44] The English edition of the paper was published in the same year with the title *The Modern Development of Thomas Young's Theory of Colour-vision* in *Report of the British Association for the Advancement of Science* (König and Dieterici 1886, pp. 431-439).

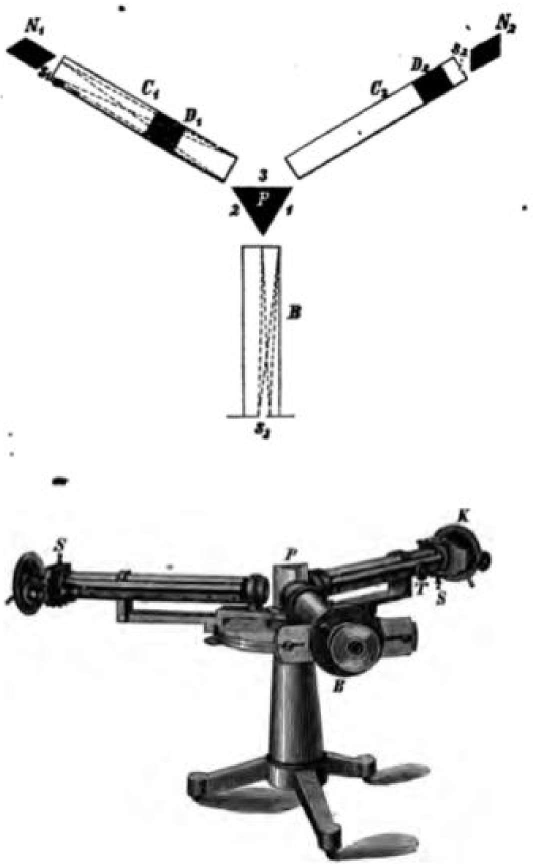

Fig. 4.2
Helmholtz original Colour Mixer: it is essentially a spectrometer having one telescope and two collimators each with a light source (Loewenherz 1880, p. 521)

König and Dieterici colour mixer is sketched in Fig. 4.3.

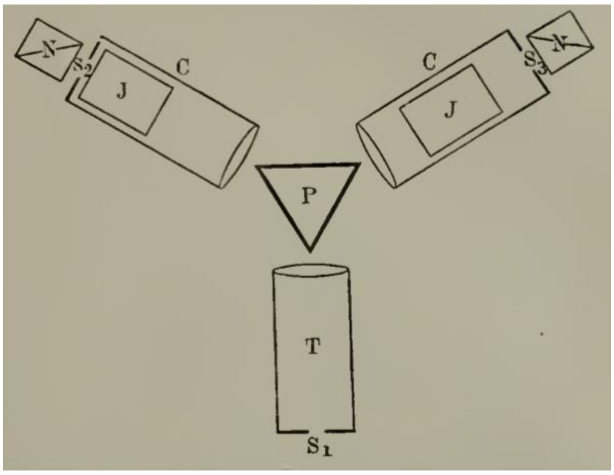

Fig. 4.3
Schematic representation of König and Dieterici's apparatus with indication of the optical arrangements (König and Dieterici 1886, p. 433)

The apparatus consists of a spectroscope with two collimators indicated by *C* and an equilateral prism *P*. *T* is a telescope with a slit denoted by S_1. In front of the slits of the collimators two Nicol prisms *N* are to be found, and each collimator contains two achromatic Iceland spars, *J*. An observer placed at the slit S_1 will see, when slit S_2 and S_3 are illuminated, two coloured fields next to each other, as shown in Fig. 4.4.

Fig. 4.4
Two parts of the field supposed to be differently coloured, as seen by an observer placed at slit S_1 (König and Dieterici 1886, p. 433)

The Iceland Spars, producing the phenomenon of birefringence, allow the formation of two pairs of spectra at S_1, one pair due to the slit S_2 and the other one to the slit S_3. The lights of each pair will be polarised perpendicularly to each other, according to a property of birefringent materials. The presence of the Nicol prism between the slit and the light source allows to vary, for each spectra, the amount of the components of light that enters in a mixture. If the Iceland spar is located a short distance away from the slit, such as depicted in Fig. 4.3 (left collimator), we will have monochromatic light in the corresponding half of the picture. This construction renders it possible to produce the same colour, in tone, shade and intensity, in each half of the field. The comparison will yield to a colorimetric equation. The coefficients and variables of the equation will be given by the position of the collimators, the distances between the Iceland spars and the slits, the position of the Nicol prisms and the width of the slits (measured in micrometres)[45].

König and Dieterci used a gas lamp as light source for their measurements (of which they did not furnish lighting specifications).

By using this apparatus, they could obtain the curves depicted in Fig. 4.5.

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[45] A more detailed picture and description of König and Dieterci's apparatus can be found in their work of 1892, *Die Grundempfindungen in normalen und anomalen Farbensystemen und ihre Intensitätsvertheilung im Spectrum (The fundamental sensation functions in normal and anomalous colour systems and their intensity distribution in the spectrum)* (König and Engelmann 1903, pp. 214-321).

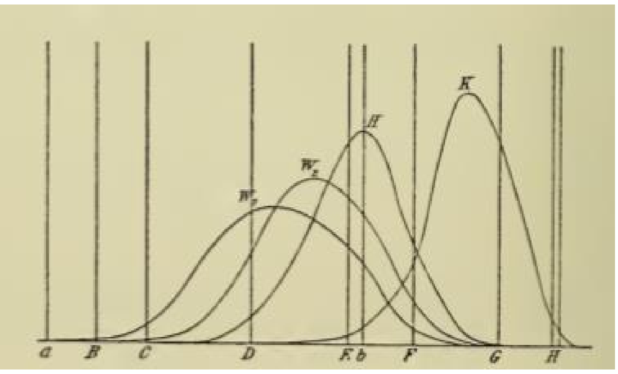

Fig. 4.5
König's elementary curves for people with monochromatic vision (*H*) and people with two different kinds of dichromatic vision (K, W_1 and K, W_2). In the abscissa are shown Fraunhofer lines as reference (König and Dieterici 1886, p. 434)

The curve denoted by *H* represents the elementary curve for monochromats, people that have only one elementary sensation. The curves indicated by *K* and W_1 refer to one class of colour blind people and the other two curves *K* and W_2 represent the elementary curves of a second class of colour blind subjects. It is worth underlying that König and Dieterici first made systematic examinations of the colour mixtures of colour blind subjects with spectral colours. In Fig. 4.6 we can see two different group of curves, one group is composed of curves *R*, *G* and *V* and correspond to the normal class of trichromatic people and the second one, formed by the two dotted curves and the one denoted by *V*, refers to a second type of trichromatic vision, defined by König as *abnormal*[46].

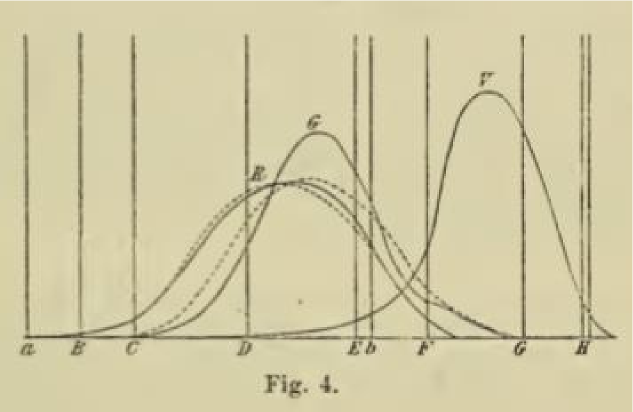

Fig. 4.6
König's elementary curves for people with trichromatic vision (normal: *R*, *G* and *V*; abnormal: dotted curves, *V*). In the abscissa are shown Fraunhofer lines as reference (König and Dieterici 1886, p. 435)

At this point, it is worth explaining the meaning of *elementary sensation curves*, which they called *Elementarempfindungskurven*, and the crucial distinction made by them between these and

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[46] For greater clarity, in Appendix B a diagnostic classification of all kinds of colour vision is provided.

fundamental sensation functions (Grundempfindungskurven). The elementary sensations were intended as the result of *the reduction of the infinitely large number of colour-sensations to the smallest possible number of elementary sensations, which by their intensity and mutual relation produce every possible kind of colour-sensations. This is a purely experimental problem, whose solution can and will be made independent of every theoretical hypothesis. This is the reason why we choose the expression 'elementary sensation', and not fundamental sensation, because the latter expression usually refers to a simple process going on at the terminal of the optical nerve* (König and Dieterici 1886, p. 432). And he continued stating that: *It is evident that for every person the number of fundamental sensations is equal to the number of elementary sensations, and we can speak of 'curve of fundamental sensation' just the same as we did before of curves of elementary sensation* (König and Dieterici 1886, p. 435). This distinction today is no longer made[47].

Since both the elementary and fundamental sensations were meant to be the solution of all colour equations, linear and homogeneous, they could consider the fundamental sensation as a linear combination of the elementary sensation. König reported their aim as follows: *The object of this superposition is to examine whether among the infinite number of possible curves of fundamental sensations we can find three such curves that a person of the first class will have some one of them, a person of the third class will have all three of them* (König and Dieterici 1886, p. 435). They were able to find such a relation only for the normal trichromats and dichromats, whose curves of fundamental sensations are reported in Fig. 4.7.

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[47] As Rolf G. Kuehni and Claudio Oleari pointed out in the retrospective introduction to the English translation of the paper 1886, *Die Grundempfindungen und ihre Intensitäts-Vertheilung im Spectrum (Fundamental sensations and their intensity distribution in the spectrum)* (Kuehni and Oleari 2010), König made a distinction between elementary and fundamental sensations because he was convinced that his experimental set-up was not able to furnish fundamental sensitivity curves directly. Today it is sufficient to measure colour matching functions in the laboratory-reference frame RGB, as Maxwell already did, and make a linear transformation toward the reference frame with axes defined by the confusion points of dichromats (points that correspond to the primary sensations which are missing). They also offered a comparison between König spectral sensitivities and the cone sensation curves defined by the International Commission on Illumination in 1931. They found out that the latter are closer to the König elementary sensation curves than to the fundamental sensation curves.

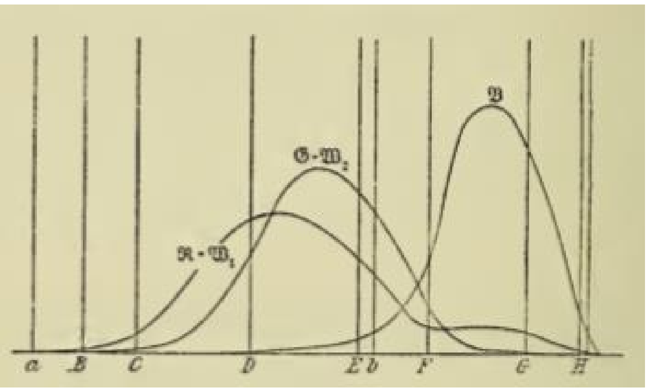

Fig. 4.7
König’s fundamental curves for people with normal trichromatic vision (ℜ𝔊𝔅) and for people with two different kind of dichromatic vision ($\mathfrak{W}_1\mathfrak{B}$ and $\mathfrak{W}_2\mathfrak{B}$). In the abscissa are shown Fraunhofer lines as reference (König and Dieterici 1886, p. 436)

They obtained a crucial result concerning the nature of these two different kinds of colour blind vision, i.e. the curves ℜ, 𝔅 and 𝔊, 𝔅 were found to be equal to the two curves representing the first class and the second class of colour blindness respectively. In the paper published in 1892 with the title *Die Grundempfindungen in normalen und anomalen Farbensystemen und ihre Intensitätsvertheilung im Spectrum (The fundamental sensation functions in normal and anomalous colour systems and their intensity distribution in the spectrum)*, König and Dieterici presented the subject of elementary and fundamental curves in great detail. They could conclude that three curves were sufficient to describe normal colour vision and in the case of their dichromatic observers, either *R* (red) or *G* (green) fundamental were missing, confirming definitely Young’s three-receptor theory and providing quantitative results also for what they called green-blindness. I would like to report here, for completeness, also the fundamental curves obtained by using a refined version of the colour mixture apparatus, which appeared in the paper of 1886, *Die Grundempfindungen und ihre Intensitäts-Vertheilung im Spectrum (Fundamental sensations and their intensity distribution in the spectrum)* (König and Engelmann 1903, pp. 60-87).

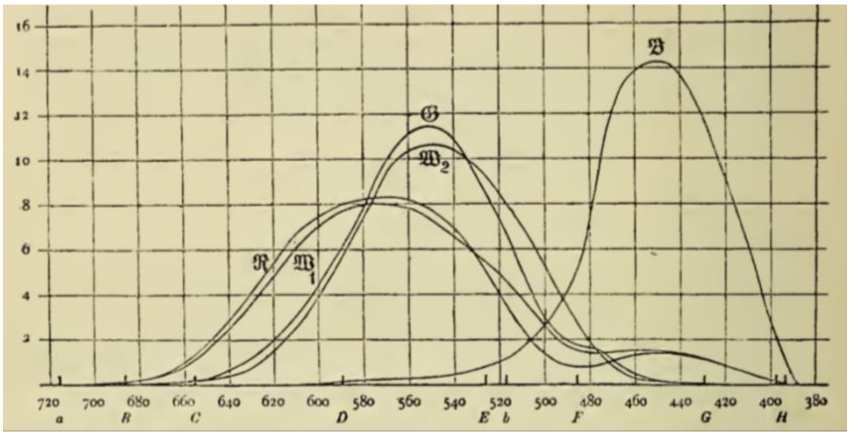

Fig. 4.8
Fundamental curves for people with normal trichromatic vision ($\mathfrak{RGB}$) and for people with two different kind of dichromatic vision ($\mathfrak{W}_1\mathfrak{B}$ and $\mathfrak{W}_2\mathfrak{B}$). In the abscissa are shown wavelengths and Fraunhofer lines as reference (König and Hengelmann 1903, p. 84)

After the determination of the curves of fundamental sensations, König represented the spectral trace in Newton's colour diagram based on Maxwell's triangle. For the first time, therefore, after Maxwell and Helmholtz's estimations of the spectral locus, a precise measurement of it was undertaken. In Fig. 4.9 capital letters denote Fraunhofer lines.

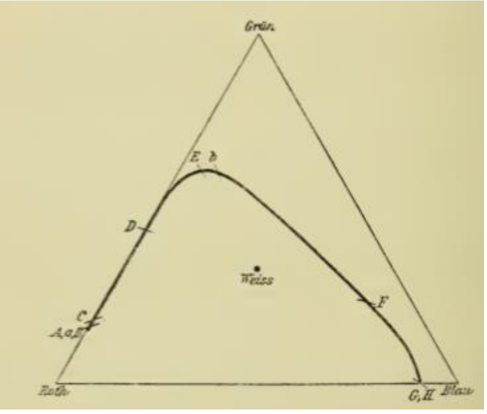

Fig. 4.9
König's measured spectral locus inside Maxwell's triangle (König and Dieterici 1886, p. 437)

König and Dieterici found that the colours of the three fundamental sensations were: *For* $\mathfrak{R}$, *red (somewhat more purple than the colour of the long-waved end of the spectrum). For* $\mathfrak{G}$, *green (about the wave-length 505* $\mu\mu$[48]*). For* $\mathfrak{B}$, *blue (about the wave-length 470* $\mu\mu$*)* (König and Dieterici 1886, p. 437). I would like to offer a direct comparison between König's measured spectral locus and Helmholtz's estimated spectral curve inside Maxwell's triangle. Helmholtz, in fact, as we have

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[48] $\mu\mu$ stands for *Milliontelmillimeter (millimicron)*; $1\ \mu\mu = 1\ nm$.

already seen in chapter 3 (Fig. 3.18), comparing quantitatively the brightness of the different pairs of complementary colours, proposed a modified version of Maxwell’s triangle. He was also aware of the fact that not every colour sensation can be excited directly by external light and stated that *the colours aroused in the normal eye by external light are comprised within the area bounded by the curve and the straight line* $V_I R_I$*. The rest of the triangle corresponds to colour sensations that cannot be excited directly by external light* (von Helmholtz 1962, p. 146). This was confirmed for the first time by König and Dieterici’s measurements.

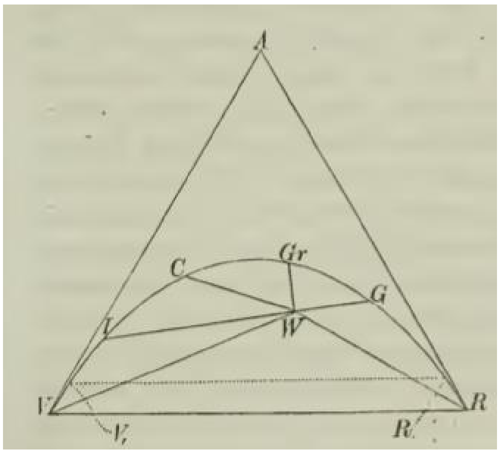

Fig. 4.10
Helmholtz’s estimated spectral curve depicted inside Maxwell’s triangle (von Helmholtz 1962, p. 145)

4.4 The discrimination of hue in the spectrum: first attempt to quantify the subjective

In a paper published in 1884 with the title *Über die Empfindlichkeit des normalen Auges für Wellenlängenunterschiede des Lichtes (In Gemeinschaft mit Conrad Dieterici) (On the Sensitivity of the Normal Eye to Wavelength Differences in Light (In Collaboration with Conrad Dieterici)),* König reported an investigation, made in collaboration with Dieterici, of the sensitivity to wavelength differences of persons with normal trichromatic colour vision. This was a starting point for a subsequent and more complete work on the subject published in 1888, as will be described later. Observations on the discrimination of hue in the spectrum had already been made in 1867 by Mandelstamm, who studied under Helmholtz in Heidelberg and was Professor of ophthalmology at the University of Kiev, and in 1872 by Dobrowolski, Professor at the University of Saint Petersburg. In this preliminary study, König and Dieterici used the same method employed by Dobrowolski, as we will discuss in more detail in section 4.5.2. They reported the experimental results obtained by the method of mean error[49] with both König and Dieterici as observers. Fifty matches were made, each at two different intensities. Fig. 4.11 illustrates the wavelength discrimination curves of König and Dieterici measured at low and high intensity, motivated by the intention to test the effect of light intensity on colour vision: *Die hohe Intensität wurde in dem lichtstärkeren Theile des Spectrums, also von 640 μμ, etwa der Linie C entsprechend, durch zwei gleiche Gaslampen mit Argandbrennern, die mit ihrer höchsten Intensität brannten, hergestellt und von 520 μμ ersetzt durch das intensive Licht einer Knallgaslampe (The high intensity was produced by two identical gas lamps with Argand burners burning at their highest intensity in the light-stronger part of the spectrum, i.e. 640 μμ, and replaced by 520 μμ by the intense light of a detonating gas lamp)*. […] *Für die niedrige Intensität wendeten wir dieselben Gas-Argand-brenner an, die jedoch passend gedämpft waren; diese Intensität, die etwa 1/2 der ersteren war, genügte bis 470 μμ (For the low intensity we used the same Argand burners, which were suitably damped; this intensity, which was about 1/2 of the former, sufficed until 470 μμ)* (König and Engelmann 1903, p. 28).

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[49] As a measure of the sensitivity, they considered the mean error, calculated from many equality relations, that indicated the degree of accuracy in a given portion of the spectrum with which the equality of the wavelengths of light could be estimated from the colour sensation.

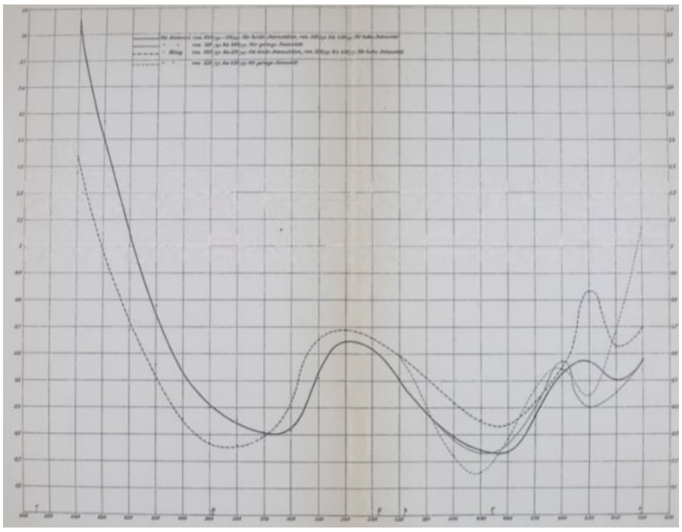

Fig. 4.11
Wavelength discrimination curves of König (dashed lines) and Dieterici (full lines) measured at low and high intensity (König and Engelmann 1903)

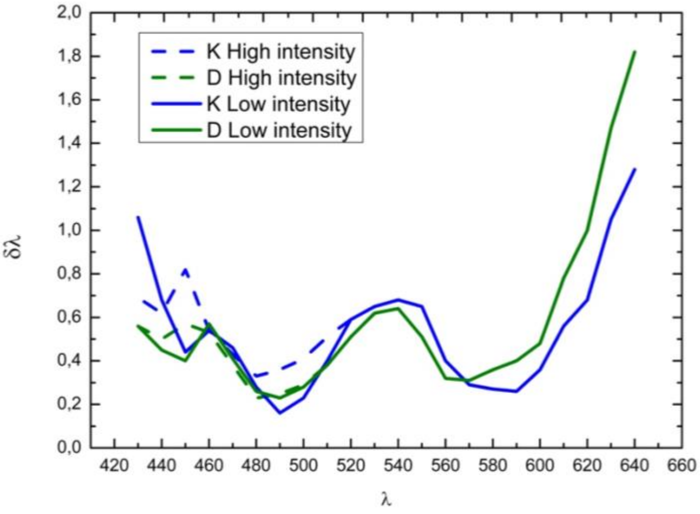

Fig. 4.12
Wavelength discrimination curves replotted by us, using König and Dieterici's experimental values. The blue and the green full lines represent König (K) and Dieterici's (D) curves of high intensity respectively, whereas the blue and the green dashed lines correspond to König and Dieterici's curves of low intensity

The curves illustrate the mean error $\delta\lambda$ in function of the wavelength λ for two different levels of light intensity for König and Dieterici. As we can see from Fig. 4.12, from 640 nm both curves undergo a decrement until the first minimum is reached. Here we can find, in fact, a first minimum, at 590 nm for König and at 570 nm for Dieterici, i.e. in the yellow region of the spectrum. Instead, in the green region of the spectrum the sensitivity decreases for both the observers and in the proximity of Fraunhofer line E the curves split for the two different intensities. Then, both the curves of low intensity reach a second minimum at 490 nm, near Fraunhofer line F, in the cyan. This place

corresponds to the highest sensitivity to wavelength differences for both observers. The curves of low intensity reach also another minimum at 450 nm, which corresponds to the transition point from indigo to violet, before increasing until 430 nm, at Fraunhofer line G. For what concerns the curves of high intensity, they show a similar behaviour for both observers, from the point at which they split from those of low intensity; in both cases, as in the first, two maxima emerge.
Summarizing the results, König and Dieterici could state that the wavelength differences in the red part of the spectrum to the Fraunhofer line C depend only slightly on the intensity. The maxima of sensitivity to wavelength differences in the yellow region of the spectrum (570-590 nm) lie, as we can see from the plot, at different position of the spectrum for the two observers. The other two maxima of sensitivity (at the blue-green and at the transition point between indigo and violet) are located at the same position for both observers. With increasing intensity, however, they move toward the violet end of the spectrum.

I would like to present another method used by König and his collaborators to estimate the discrimination sensibility for hues. König reported, as we have already mentioned, in his paper of 1886, *Über die neue Entwicklungen von Thomas Young's Farbentheorie,* the experiments conducted by Brodhun, who belonged to the second class of dichromats, aimed to detect the positions in the spectrum at which the sensation of colour changes most rapidly. The method was introduced by the following observation: *According to our theory the colour at any part of the spectrum is determined by the ratio between the fundamental sensations, whose resultant produces that colour. The change of this ratio determines the rate at which the colour changes* (König and Dieterici 1886, pp. 435-438). The smaller the rate of change of the colour tone at a given place, the larger will be the mean error of the judgment. In this way, Brodhun succeeded in the determination of the position in the spectrum at which the colour changed most rapidly, i.e. the position of maximum discrimination sensitivity for hue. The curve depicted in Fig. 4.13 and denoted by *AAA* represents the result of König and Dieterici's investigation on a normal trichromat. We can see that the places in which the colour changes most rapidly are about Fraunhofer lines D and F, that is in the yellow and blue region of the spectrum, in agreement with their previous results, whereas in the green region of the spectrum, in the proximity of Fraunhofer line E, we can see a slower change of colour. The curve *BBB* was obtained by the same procedure from a person (Brodhun) belonging to the second type of dichromat and this was in perfect agreement with the inferences which can be drawn from the inspection of the two curves of fundamental sensation for this type (see Fig. 4.7) (König and Dieterici 1886, p. 438).

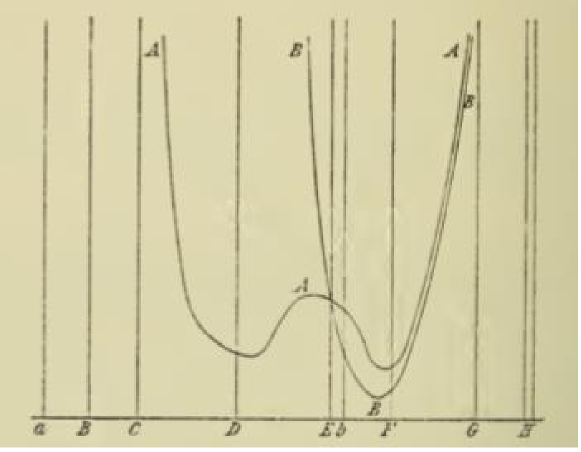

Fig. 4.13
Curves of discrimination sensibility for hues of the spectrum of gas light, *AAA* for a normal trichromat and *BBB* for a dichromat belonging to the second class (today deuteranope). On the y-axis is reported the ratio between the fundamental sensations and in the abscissa are shown Fraunhofer lines as reference (König and Dieterici 1886, p. 438)

4.5 The validity of the Weber-Fechner law for vision

In 1888 König, in collaboration with Brodhun, published a paper with the title *Experimentelle Untersuchungen über die psycho-physische Fundamentalformel in Bezug auf den Gesichtssinn (Experimental investigation on the psycho-physical fundamental formula in relation to the sense of vision)* (König and Engelmann 1903, pp. 116-134). I would like to offer a brief overview of the content in order to highlight its historical importance. The paper contains indeed one of the first attempt to apply a psychophysical law to the field of colour vision. It is the first of two: the second one, published in 1889, summarizes the content of the first and extends the results in the case of white light.

In this paper, König and Brodhun investigated the validity of the Weber-Fechner law of the new-born research field called psychophysics for light illumination for different colours. They calculated the visual differential threshold, or *just noticeable difference*, the fundamental concept at the basis of early psychophysics, with several intensities, from the barely perceptible to the intense brightness, for the six wavelengths, 670 nm, 575 nm, 505 nm, 470 nm, 430 nm (corresponding to the supposed fundamental colours of their vision) and 605 nm, that corresponds to the intensity maximum in the chosen spectrum. The first section of the paper is dedicated to the Weber-Fechner law and its possible applicability to light illumination. Therefore, before proceeding, I will offer a brief overview of Fechner's pioneering work in developing the field of psychophysics. A systematic survey of the literature on the subject is beyond the scope of the present work. However, my intention is to emphasize how a simple concept at the basis of psychophysics could be extended to other fields of knowledge and used, in this case, to shed light on human colour vision and colour perception. As we will see in chapter 5, Weber-Fechner law also represents Helmholtz's starting point for defining a line element in colour space.

4.5.1 The Weber-Fechner law and Helmholtz's revised formula

Several studies of subjective experience which contained the basic idea embodied in the Weber-Fechner law were described over the course of the years, starting from Bouger's pioneering work of 1760[50]. However, the first well defined experimental method to quantify relationships between any physical stimulus and the perceived response by individuals can be dated to the German physicist and experimental psychologist Gustav Theodor Fechner (1801-1887), presented in his *Elemente der Psychophysik (Elements of psychophysics)*, published in 1860. Fechner realized that it was impossible to measure a sensation directly but what could be measured was the stimuli or stimulus differences that produce equal differences between sensations. He could then consider the intensity of sensation as a function of stimulus strength. By varying, in fact, the stimulus strength, i.e. the independent variable, he could obtain the relative value of the intensity of the sensation, i.e. the dependent variable. In this way, for the first time a defined quantitative relation between physical stimuli and sensations could be obtained. The fundamental concept at the basis of Fechner's method is the so called *just noticeable difference*, as we have already mentioned. Fechner's work was heavily influenced by his mentor and collaborator Ernst Heinrich Weber (1795-1878), a German physiologist and anatomist. Weber conducted experiments asking subjects to compare the intensities of sensations caused by two physical stimuli. One stimulus was kept fixed, while some parameter of the second was systematically varied. In this way, he could find out the smallest amount of physical difference between stimuli that produced a *just noticeable difference* between sensations. Weber's experiments allowed him to find a systematic relationship between stimuli and sensations, that followed the same basic pattern in many different sensory modalities. He found out that the change in a physical stimulus required to produce a just noticeable difference in the perception of a subject stood in a constant proportion to the comparison stimulus. Therefore, he could write the following equation:

$$\frac{\Delta r}{r} = K \tag{4.8}$$

---

[50] A brief account of the historical development of the Weber-Fechner law can be found in *The visual discrimination of intensity and the Weber-Fechner law* by Hecht (Hecht 1924), in which Hecht also cited the works of Bouger (1760), Arago (1858), Masson (1845) and Steinheil (1837).

where r indicates the absolute value of the comparison stimulus, Δr represents the change of physical stimulus strength and K is a constant, that assumes a different value for each sensory modality.

At this point, I would like to introduce the Weber-Fechner law using König's original words: *FECHNER's Bedeutung für die von ihm „Psychophysik" benannte Wissenschaft von dem Zusammenhang und den Beziehungen zwischen Körper und Seele beruht im Wesentlichen darauf, daß er die Differentialgleichung des „WEBER'schen Gesetzes" integrierte und dadurch die Intensität der Empfindungen aus der physikalische meßbaren Stärke der Reize ableitete (Fechner's contribution to the field of the science that studies the relationships between body and soul, which he baptized as psychophysics, is based on the integration of the differential equation of Weber's law and the subsequent determination of the intensity of the sensations of a physical stimulus*) (König and Engelmann 1903, p. 116). Then, he continued explaining Fechner's procedure to obtain the expression known as Weber-Fechner law.

Let e denoted the sensation intensity, δe the sensation increment, r the physical stimulus, δr the physical stimulus increment and r_0 the force of the internal stimulus corresponding to the auto excitation of the nerves, then Fechner's law can be expressed by the following formula:

$$\delta e = K \cdot \frac{\delta r}{r + r_0} \tag{4.9}$$

where K indicates a constant. By integration, the following expression for e can be obtained:

$$e = K \cdot \log(r + r_0) + \text{Const.} \tag{4.10}$$

However, Helmholtz's investigations in the sensation of vision for high intensity revealed that this formula did not fit the facts, because it implied an increment of the sensitivity as long as the luminosity increased, as we can read in paragraph *On the Intensity of the Light Sensation* in the second volume of his *Handbuch*. From his observation, he found out that the intensity of the sensation reached a maximum value, that could not be exceeded even when the luminosity still increased. From this consideration, he proposed a revised formula:

$$dE = \frac{a dH}{(b + H)(H_0 + H)} \tag{4.11}$$

where a, d and b are constants, H denotes the luminosity, H_0 takes into account the stimulation due to 'internal causes', and dE indicates the difference of intensity of sensation E. The maximum of sensitivity could be reached for:

$$H = \sqrt{bH_0} \tag{4.12}$$

I would like to report Helmholtz's words used to summarize in a very clear manner the results of his investigation: *The connection between the intensity of the sensation and the luminosity which has thus been shown to exist explains a fact which the writer has often noticed, namely, that on dark nights bright objects are much brighter as compared with their environment than they are in daytime, so that sometimes it is hard not to believe they are self-luminous. When the luminosity is very low, the intensity of the sensation is considered as being proportional to the luminosity; whereas with greater illumination the sensation for brighter objects is relatively weaker. Now since we are accustomed to compare the brightness of familiar objects when they are highly illuminated, under feeble illumination bright objects are relatively too bright, and dark ones relatively too dark. Painters also make use of this circumstance in moonlight scenes, to produce the impression of faint illumination* (von Helmholtz 1963, pp. 180-181). During the summer of 1860, Helmholtz sent a copy of the second volume of the *Handbuch*, hot off the press, to his friend Fechner, with whom he exchanged a lively correspondence, followed by these words: *You will find the same subjects in this second part that you have dealt with lately in your own work. I had written the chapter on Intensity of Light, in all essentials, before I received your treatise on it. I therefore introduced some modifications afterwards*[51](Königsberger 1906, p. 192).

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[51] Fechner wrote in his *In Sachen der Psychophysik* of 1877 a chapter with the title *Übersichtliche Vorführung der Gegner (Clear presentation of the opponents),* in which he discussed about modifications made to his psychophysical law by Helmholtz, Aubert, Mach, Bernstein, Plateau, Brentano, Hering and Langer. He presented their works in detail, also reporting the mathematical derivation of the modified equation. Referring to Helmholtz, he introduced the experimental deviation of the law with these words: *Helmholtz und Aubert machen geltend, dass die experimentalen Abweichungen vom Weberschen Gesetze im Gebiete der Lichtempfindung nach ihren Versuchen erheblich weiter gehen, als ich nach allen vorgängigen Versuchen Grund hatte anzunehmen (Helmholtz and Aubert claim that the experimental deviation from Weber's Law in the field of sensation of light, after their experiments, goes considerably further than I had reason to believe after all previous attempts)* (Fechner 1877, p. 16).

4.5.2 The experiment of Dobrowolski

In *Monatsberichte der Königlich Preussischen Akademie der Wissenschaft zu Berlin* of the year 1872, Helmholtz shared with his class of physics and mathematics the important results obtained by Prof. Dobrowolski at the University of Saint Petersburg, who was the first to calculate the differential threshold, the just noticeable difference, in brightness for different spectral wavelengths (before then, experiments were conducted using only white light, offering under normal circumstances a value of 1/100 for the fraction, and under favourable circumstances of purity of the field and illumination, of 1/150-160). Dobrowolski's work was published in *Albrecht von Gräfe's Archiv für Ophthalmologie* of 1872, *Beiträge zur physiologischen Optik von Dr. W. Dobrowolski aus St. Petersburg (Contributions to physiological optics of Dr. W. Dobrowolsky from St. Petersburg)* (Dobrowolski 1872). In the second contribution, with the title *Über Empfindlichkeit des Auges gegen verschiedene Spectralfarben (On the sensitivity of the eye to different spectral colours)* are reported Dobrowolski's investigations on the sensitivity of the eye to different spectral colours conducted using a spectral apparatus. He found out that his eye was more sensitive to colour differences at Fraunhofer lines D (in the yellow region of the spectrum), F (in the blue region of the spectrum) and between E (green region) and F, referring to daylight illumination. It is worth spending a few more words about the third contribution, with the title *Über die Empfindlichkeit des Auges gegen die Lichtintensität verschiedener Spectralfarben (On the sensitivity of the eye to light intensity of different. spectral colours)*, because it introduced the work carried out by König. Dobrowolski began his essay with a description of Fechner psychophysical law. Then he gave a detailed account of the experimental method, suggested by Helmholtz. Dobrowolski maintained the input intensity fixed for all his investigations. White light, directed by a heliostat, was allowed to pass through two Nicol prisms with parallel principal sections. Between the two Nicol prisms a perpendicular slab of gypsum, a biaxial, birefringent crystal, was located, 7 mm thick, able to rotate around the optical axis of the Nicol prisms. The rotation angle could be measured and registered. Subsequently, the light passed through the slit of a spectroscope, so as to undergo dispersion and build a spectrum. The purpose of the whole arrangement was to create fields that the eyes could compare in terms of their light intensity.

Rotating the gypsum plate around the axis of the Nicol prism, four positions were to be found in which a series of spectral colours was extinguished, producing thereby a system of dark lines in the spectrum, four other intermediate positions in between, in which these lines completely disappeared.

The laws of birefringence and polarization allowed Dobrowolski to calculate, starting from the rotation angle of the gypsum plate, the ratio between the intensity of the light of the darker and lighter lines seen in the spectrum.

The difference between the maximum and the minimum, indicated as a fraction of the brightness of the maximum, was:

$$\sin^2 2\alpha$$

where α denoted the rotation angle of the gypsum plate starting from the position in which its principal sections coincided with those of the Nicol's prisms. An observer could indicate how much the gypsum should be rotated in one direction and in the other, from a position where the lines completely disappeared, before they begun to become visible again in one or the other part of the spectrum.

In Fig. 4.14 is sketched the experimental set-up. Letter *N* denotes the Nicol prisms, parallel to each other, *G* represents the gypsum plate between them, *A*, *A* and *B*, *B* are two perpendicular polarized rays in the plate and α indicates the angle between the vibration planes of the plate and the two Nicol prisms.

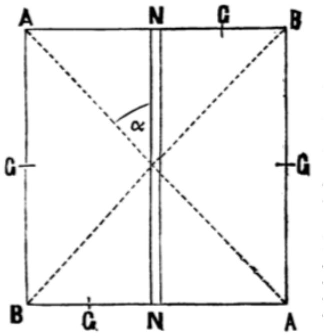

Fig. 4.14
Schematic representation of Dobrowolski's experimental set-up (Dobrowolski 1872, p. 77)

The results obtained from the observations of Dobrowolski himself and two other observers are reported in Table 4.1:

Colour and Fraunhofer lines	**Mr. D**	**Mr. D**	**Mr. B**	**Mr. B**	**Mr. G**	**Mr. G**
	Angle 2α	**Fraction**	**Angle 2α**	**Fraction**	**Angle 2α**	**Fraction**
Red at A	15° 30'	$\frac{1}{14}$				
Red at B	13°	$\frac{1}{19,76}$	14° 30'	$\frac{1}{15,9}$	17°	$\frac{1}{11,7}$
Red at C	11° 30'	$\frac{1}{25,16}$				
Orange between C and D	10°	$\frac{1}{33,16}$				
Gold yellow at D	8° 30'	$\frac{1}{45,77}$	9°	$\frac{1}{40,86}$	11°	$\frac{1}{27,46}$
Green between D and E	7° 30'	$\frac{1}{58,7}$				
Blue green between E and b	7°	$\frac{1}{67,23}$				
Cyan at F	5°	$\frac{1}{131,6}$				
Indigo near G	3° 30'	$\frac{1}{268}$				
Violet between G and H	3° 30'	$\frac{1}{268}$	4°	$\frac{1}{205,5}$	4°	$\frac{1}{205,5}$
Violet at H	7°	$\frac{1}{67,33}$				

Table 4.1
Experimental results obtained by Dobrowolski and other two observers, indicated by B and G. α denotes the rotation angle of the gypsum plate and *Fraction* stands for Weber fraction

The obtained results show that the differential sensitivity increases continuously from red to purple. For violet it is from 10 to 20 times larger than for red.

Dobrowolski's results were in line with Helmholtz's observations: with proportional weakening of the brightness of the different colours, red seemed to decrease much more and disappeared earlier than blue. Blue seemed, therefore, to contain more different brightness gradations than red[52].

4.5.3 Description of the experimental set-up and results

König and Brodhun, being aware of Dobrowolski's results, proposed an experiment to investigate the dependence of the sensibility to light illumination for different wavelengths also varying the input intensity. In this section a description of their experimental set-up is provided.

As light source, they used a *Triplex gas burner* and, for the greatest intensities, Zirconium light. A bilateral slit indicated as S_1, whose width could be determined using a micrometre, was located at the focal point of a 10 cm diameter lens denoted by L_1. The light beam passed through a prism P, filled with ethyl ether, and combined into a spectrum by an achromatic objective lens, L_2, of an astronomical telescope. The spectrum range was of 670 nm to 430 nm and extended about 20 cm. On the spectrum plane is located a slit, denoted by S_2, of 5 mm width and 7 mm height. Behind this slit an Iceland spar was to be found. The slit S_2 was then viewed through the spar using a short astronomical telescope in whose eyepiece a Nicol prism, N_2, is located.

The six chosen wavelengths were set taking Fraunhofer lines as reference and the slit was so adjusted in order to let it correspond to a given portion of the scale.

If the main section of the prism N_2, was parallel to that of the Iceland spar, then two equally luminous rectangles were to be seen on the screen; for a rotation of the Nicol prism of 90° the upper rectangle disappeared, while the lower one maintained its intensity unchanged. Between the slit S_2 and the lens L_2 a second Nicol prism, indicated as N_1, was located, whose polarization plane had the required inclination. In this way, by rotating the Nicol prism N_2, it was possible to darken the upper rectangle

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[52] Helmholtz referred to Dobrowolski's work on the subject also in chapter III, *die Farbe*, of his *Optisches über Malerei (1871-1873)* (von Helmholtz 1896, pp. 117-125). Here Helmholtz illustrated all the physiological aspects that painters should take into account, beginning with a consideration of the difference in the perception of colour tone with variation in brightness. He stated that the strength of the sensation for a given intensity and for a given coloured light depends entirely on the reaction mode of the nerve apparatus, which is excited by the action of that particular light. Then, referring to the work carried out by Dobrowolski, he reported: *Neuere Messungen haben gezeigt, dass die Empfindlichkeit unseres Auges für schwache Schatten im Blau am grössten ist, im Rot am kleinsten. Im Blau wird ein Unterschied von 1/205 bis 1/268 der Lichtstärke erkannt, im Roth vom unermüdeten Auge 1/16, bei Abstumpfung der Farbe durch längeres Betrachten 1/50 bis 1/70 (Recent measurements have shown that the sensitivity of our eye under weak illumination conditions is greatest in the blue and smallest in the red region of the spectrum. In the blue region, a difference in light intensity from 1/205 to 1/268 could be recognized, whereas in the red portion of the spectrum this difference's value goes from 1/16, for unwearied eyes, to 1/50, reaching even the value of 1/70 with prolonged viewing)* (von Helmholtz 1896, p. 118).

until it disappeared, while the lower one was maintained at constant intensity, and the angular positions at which it occurred could be registered.

König and Brodhun indicated with letter J the constant intensity of the lower rectangle and with α the angle between the polarization plane of the prism N_2 and the principal section of the spar. In this way, they could write the following relations:

$$r + \delta r = J \tag{4.13}$$

$$r = J \cdot \cos^2 \alpha \tag{4.14}$$

from which it follows:

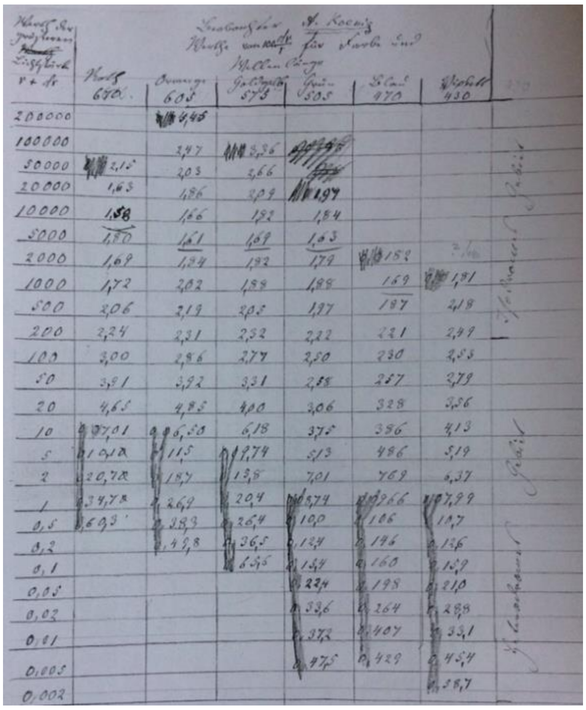
$$\delta r = J \cdot \sin^2 \alpha \tag{4.15}$$

Therefore, the *visual threshold* could be expressed as:

$$\frac{\delta r}{r} = \tan^2 \alpha \tag{4.16}$$

where r denoted the output intensity and δr the output intensity increment. The angle was registered ten times for each observations and then the average value of these was taken into account. In order to grant a proper isolation from the outside environment, the observations took place in a darkened room and it is worth underlying that, in the interval between these series of measurements, taken by an assistant, the observer was not aware of the values previously registered, in such a way as to circumscribe the subjective element. They reported also the chosen unit of intensity: *Als Einheit der Intensität J wurde die Helligkeit festgesetzt, mit welcher einem durch ein Diaphragma von 1 qmm blickenden Auge eine Magnesiumoxyd überzogene Fläche erscheint, die in einem Abstand von 1m durch eine parallel stehende 0.1 qcm große Fläche von schmelzendem Platin senkrecht bestrahlt wird. (The unit of the intensity J is the illumination seen by an eye looking through a diaphragm, whose opening measures 1* mm^2 *at a surface coated with magnesium oxide, the surface is located 1 m far from the eye and reflects the light from a platinum surface, whose area measures 0.1* cm^2 *standing parallel to it.)* (König and Engelmann 1903, p. 120).

The intensity of the illumination was varied for each wavelength upwards from 1 to 2, 5, 10, 20, 50, 100, 200, 500, 1000, 2000, 5000, 10.000, 20.000, 50.000 and 100.000 and downwards to 0.5, 0.2, 0.1, 0.05 and so on. Smoked glasses of known absorption coefficients were used to attenuate the intensity.



Beobachter A. Koenig
Werthe von 1000 [illegible] für Farbe und Wellenlänge

Lichtstärke	Roth 670	Orange 605	Gelbgrün 575	Grün 505	Blau 470	Violett 430
200000		4,45				
100000		3,47	3,36	[illegible]		
50000	2,15	2,03	2,66	[illegible]		
20000	1,63	1,86	2,09	1,97		
10000	1,58	1,66	1,92	1,84		
5000	1,80	1,61	1,69	1,63		
2000	1,69	1,24	1,92	1,79	152	
1000	1,72	2,02	1,88	1,88	169	1,91
500	2,06	2,19	2,03	1,97	187	2,18
200	2,24	2,31	2,32	2,22	221	2,49
100	3,00	2,86	2,77	2,50	230	2,53
50	3,91	3,92	3,31	2,58	257	2,79
20	4,65	4,85	4,00	3,06	328	3,56
10	7,01	6,50	6,18	3,75	386	4,13
5	10,1	11,5	9,74	5,13	486	5,19
2	20,7	18,7	13,8	7,01	769	6,37
1	34,7	26,9	20,4	8,74	966	7,99
0,5	69,3	38,3	26,4	10,0	106	10,7
0,2		49,8	36,5	12,4	146	12,6
0,1			65,5	15,4	160	15,9
0,05				22,4	198	21,0
0,02				33,6	264	28,8
0,01				37,2	407	33,1
0,005				47,5	429	45,4
0,002						58,7

Fig. 4.15
König's manuscript table preserved at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften* (Nachlass Hermann von Helmholtz, 569 Rechnungen zur Anwendung des psychophysischen Gesetzes im Farbensystem, Blatt 18)

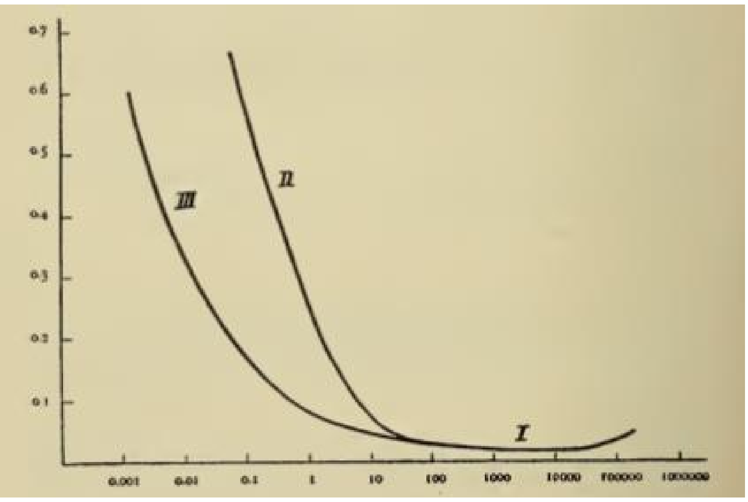


Fig. 4.16
Curves illustrating Weber fraction (y-axis) as a function of light intensity (x-axis) for König and Brodhun (König and Engelmann 1903, p. 128)

In Fig. 4.16 we can see the curve illustrating Weber fraction $\frac{\delta r}{r}$ vs. light intensity r in logarithmic scale for König and Brodhun. Group II applies to wavelengths 575, 605 and 670 nm, while Group III applies to wavelengths 430, 470 and 505 nm.

Analysing the graph, they could state that for the intensity interval 2000-20000 the curve runs horizontally, i.e. the ratio $\frac{\delta r}{r}$ has a constant value. An increase then occurs on both sides of this interval. The curves increase steadily as the intensity decreases, and, as we can notice, the increase of $\frac{\delta r}{r}$ in Group II is more rapid than in Group III. This indicates that at low intensity they could better discriminate intensity differences of wavelengths 430, 470 and 505 nm, i.e. of violet, cyan and green. In fact, the lower the Weber fraction, the better the discrimination of the differences in intensity. This result was in agreement with Helmholtz's previous investigation. In his *Handbuch*, in fact, he made use of the apparatus built to compare directly coloured lights, described in chapter 3 (section 3.7), to investigate the intensity of sensation for different kinds of light. He came to the following conclusion: the less refrangible red and yellow colours predominate when the illumination is high and the blue and violet colours, more refrangible, predominate when the illumination is low: *So also in picture galleries towards evening (supposing the sky is cloudy and twilight disappearing) the red colours are the first to fade away, and the blue ones continue visible longest* (von Helmholtz 1962, p. 182). After a description of the experimental method[53], he reported the figure depicted below, in which the abscissae are supposed to be proportional to the objective intensity of the light and the ordinates represent the intensity of the light sensation.

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[53] With the following words, Helmholtz described the experimental procedure: *If the apparatus described in the preceding section [...] is used for mixing the colours of the spectrum, and if a vertical rod is held in front of the field illuminated by the two colours, it casts two differently coloured shadows. Since the two coloured lights fall on the illuminated field from different direction, proceeding from the two slits in the last screen, they cast their shadows at different places. Thus, if, say, violet and yellow were mixed, there would be one shadow which would not be illuminated by violet but would be illuminated by yellow, so that it would look yellow; and there would be another shadow which would not be illuminated by yellow but would be illuminated by blue, so that it would look blue; whereas the rest of the field would be white or whitish. Now if the slit in the screen which lets the violet through is made wider, the violet becomes stronger, and the violet shadow therefore is more luminous. By regulating the two slits properly, the violet shadow can be made to look just as bright as the yellow one. Now if the single slit in the first screen, through which the light reflected from the heliostat passes to the prism, is opened more or closed more, the total amount of light that goes through the apparatus will be increased or diminished, all the different colours in the same proportion; and hence the amounts of light in the yellow and violet shadows will be changed in equal fashion also. The result is that even a slight increase of light will make the yellow come out brighter, and a slight decrease will make the violet look fainter. The difference here will be much less when the two colours are both in the less refrangible half of the spectrum, and much more when both colours are from the more refrangible half; it will be greatest of all when they are taken from two ends of the spectrum* (von Helmholtz 1962, pp. 182-183).

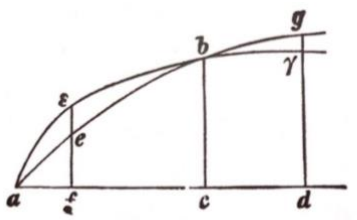


Fig. 4.17
Intensity of the light sensation in function of the objective intensity of the light for yellow light (curve aebg) and for violet light (curve aεbγ) (von Helmholtz 1962, p. 183)

He denoted by $a\varepsilon b\gamma$ and $aebg$ the curves indicating the sensation for violet light and yellow light respectively. The units for yellow and violet light were so chosen that for the quantity of light ac, as depicted in Fig. 4.17, the intensity of the sensation was the same for both lights. We can see that, for the lowest objective intensities, from a to c, the intensity of sensation for violet light is higher than that for yellow.

I would like to report, at this point, the graphs obtained by us plotting König and Brodhun's original data in order to offer a direct comparison between the two observers and to discern the curves related to the different coloured lights.

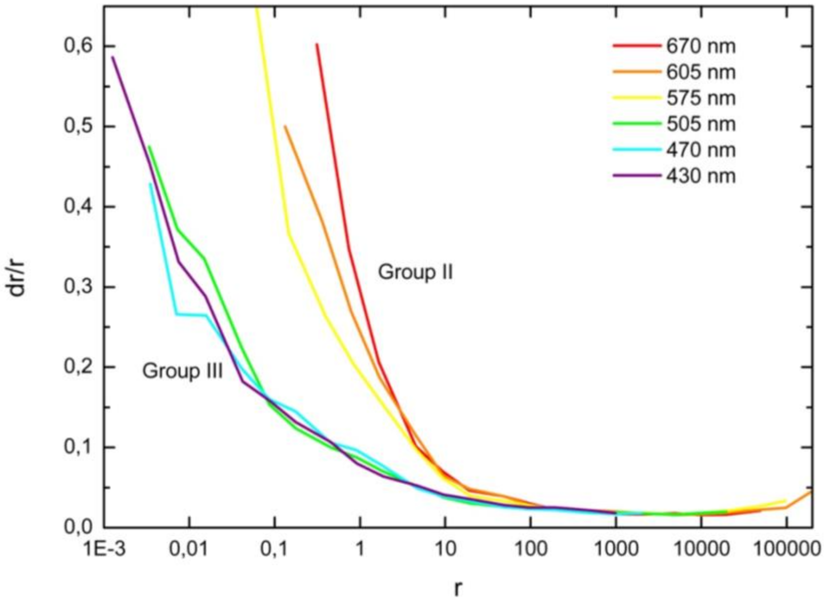


Fig. 4.18
König's (normal trichromat) curves illustrating Weber fraction (y-axis) as a function of light intensity (x-axis, on a logarithmic scale). Branch II applies to wavelengths 670, 605, 575 nm, while branch III applies to wavelengths 505, 470 and 430 nm

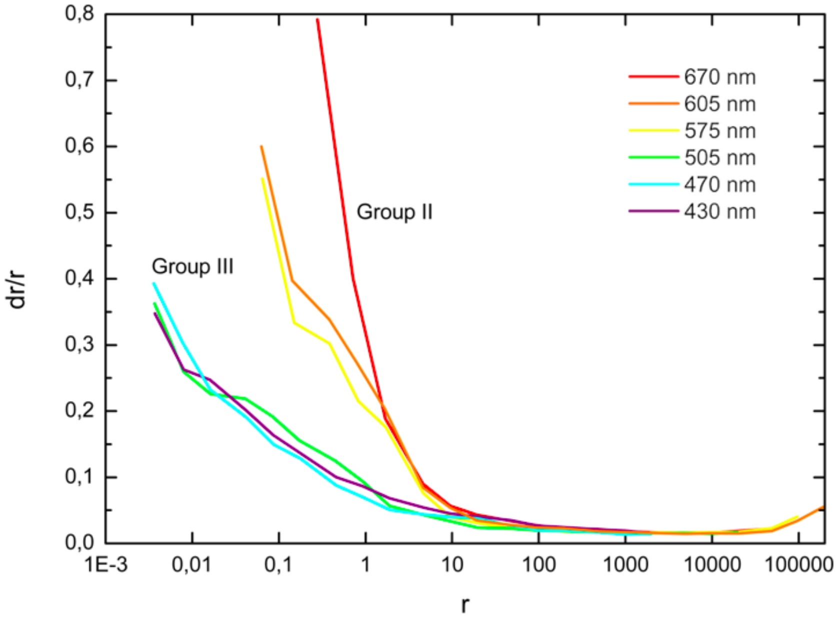


Fig. 4.19

Brodhun's (green-blind) curves illustrating Weber fraction (y-axis) as a function of light intensity (x-axis, on a logarithmic scale). Branch II applies to wavelengths 670, 605, 575 nm, while branch III applies to wavelengths 505, 470 and 430 nm

In order to compare directly the behaviour of each curve, we restricted the intensity range to 0.5-1000. In this way, in fact, for every wavelength under study we can see the corresponding Weber fraction and intensity value.

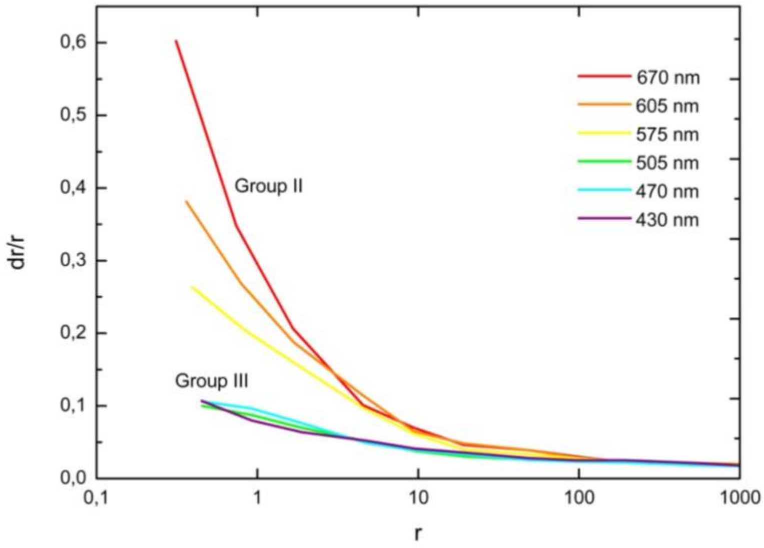


Fig. 4.20
König's (normal trichromat) curves illustrating Weber fraction (y-axis) as a function of light intensity (x-axis, on a logarithmic scale, restricted range: 0.5-1000)

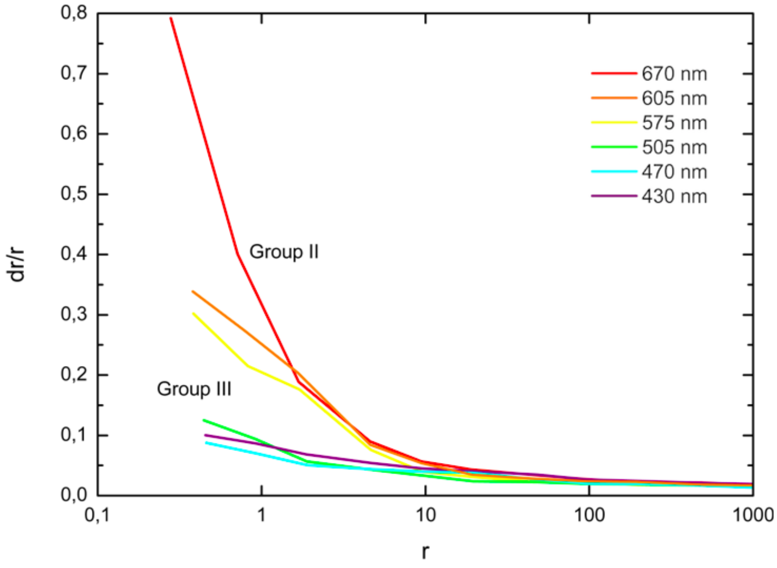


Fig. 4.21
Brodhun's (green-blind) curves illustrating Weber fraction (y-axis) as a function of light intensity (x-axis, on a logarithmic scale, restricted range: 0.5-1000)

Fig. 4.20 and Fig. 4.21 show the fraction $\frac{\delta r}{r}$ vs. light intensity r in logarithmic scale for the intensity interval 0.5-1000. König and Brodhun's show a similar behaviour. The only difference that we can notice is related to the region of lowest intensities; in fact, the two branches converge at an intensity of about 20 for Brodhun and at an intensity of about 200 for König.

Analysing these results, König and Brodhun came to the conclusion that the ratio $\frac{\delta r}{r}$ depends on the intensity of the light source and only in a limited way, for the lowest intensities, on the wavelength.

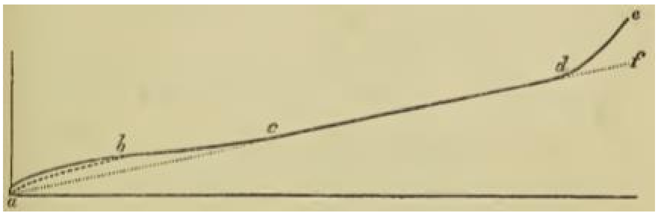


Fig. 4.22
Curve illustrating the increment δr (y-axis) vs. r (x-axis) for König and Dieterici (König and Engelmann 1903, p. 129)

In Fig. 4.22, the two curves running from *a* to *b* correspond to Group III (dashed line) and to Group II (straight line). Point *b* refers to the intensity of 20 for Brodhun and 200 for König. The curve *bcde* gives the same differential threshold for all the spectral wavelength. Point *c* and *d* correspond to an intensity of about 2000 and about 20000 respectively. If the ratio $\frac{\delta r}{r}$ had a constant value, then a straight line (denoted by *af* in Fig. 4.22) would appear instead of a curve. Here only the track *cd* lies on a straight line, indicating that $\frac{\delta r}{r}$ has a constant value only in the intensity interval 2000-20000, confirming therefore the validity of the Weber-Fechner law in this range. Deviations occur for the lowest intensities to about 2000 and for the highest intensity, starting from 20000 (in line with Helmholtz’s investigation). In order to better visualize the behaviour of the curves, we offer a graphic representation of König and Brodhun’s results, considering different intensity ranges. The first two plots offer a zoom of the region *ab* for both observers, i.e. from the lowest intensity to point *b*, which correspond to an intensity of about 20 for Brodhun and of about 200 for König.

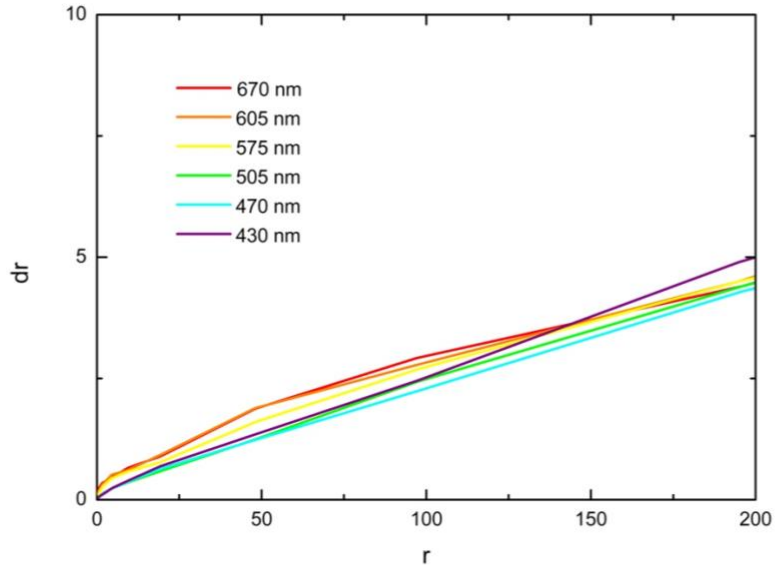


Fig. 4.23
König's curve illustrating δr (y-axis) vs. r (x-axis) in the range of intensity 0-200

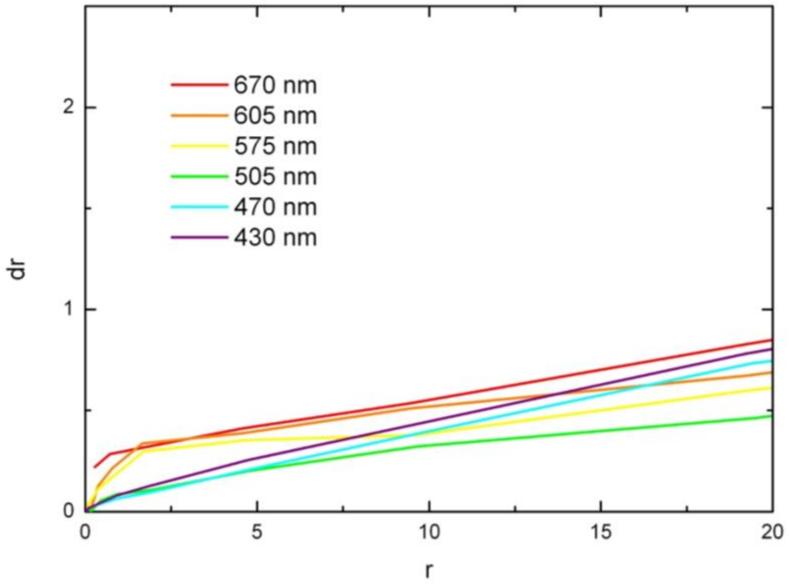


Fig. 4.24
Brodhun's curve illustrating δr (y-axis) vs. r (x-axis) in the range of intensity 0-20

In order to better visualize deviations occurring for the highest intensity, starting from an intensity value of 20000, we present here two plots in which the intensity range is extended to a value of intensity of 24000. Deviations occur for both observers at the same intensity. I would like to point out that, as we can see in our plots, for the highest intensities it was possible to measure only the values related to wavelengths 670, 605 and 575 nm.

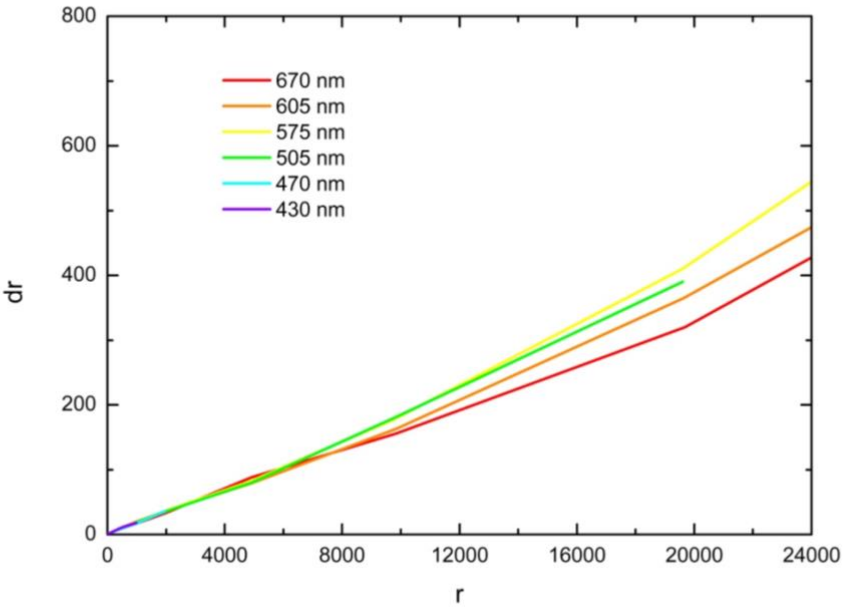


Fig. 4.25
König's curve δr (y-axis) vs. r (x-axis) for intensity reaching the value of 24000

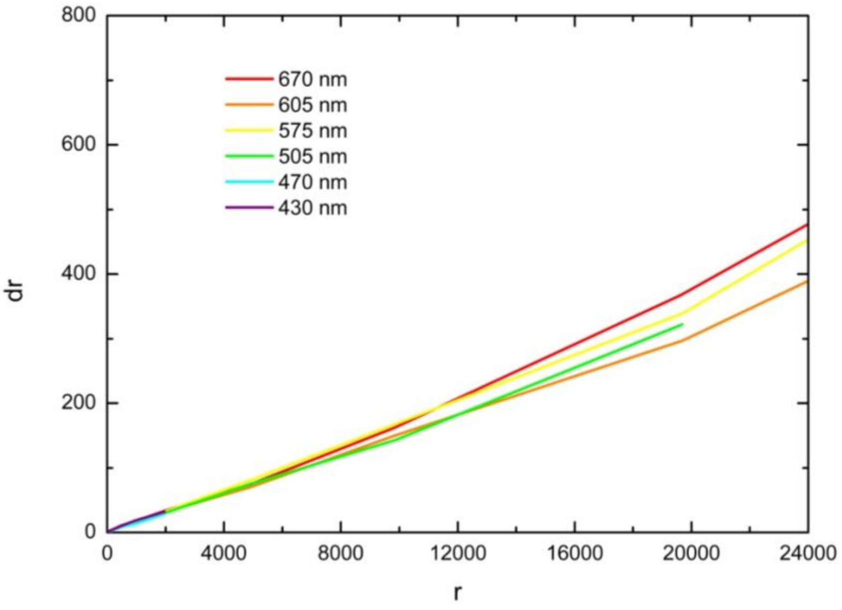
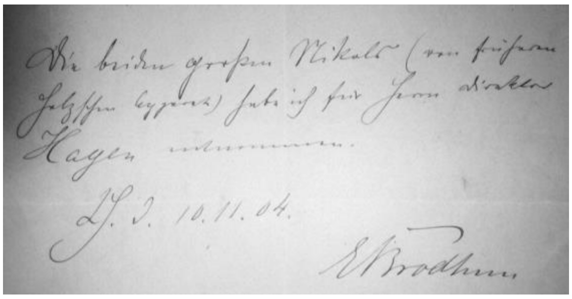


Fig. 4.26
Brodhun's curve δr (y-axis) vs. r (x-axis) for intensity reaching the value of 24000

To conclude, I would like to summarize König and Brodhun's crucial results. They found out that, for the same wavelengths, the variations between the two observers were insignificant and in general distributed so irregularly that they could be attributed only to observation errors or deviations of individual nature, therefore not to be considered as a consequence of their different colour vision system. From the highest intensity down to 200 (for König) or 20 (for Brodhun) the sensibility could be considered as a function of the intensity only and did not depend on the wavelength. For what concerns the lowest intensity values (from 0 to 20/200), the curve splits in two branches, denoted by

Group II and Group III, corresponding respectively to wavelengths 575, 605 and 670 nm and 430, 470, 505 nm. The average value of the sensibility between 20 and 100000 was found to be about 1/44.

4.6 Eugen Brodhun



Die beiden großen Nikols (von früheren
[illegible] [illegible]) habe ich für Herrn Direktor
Hagen entnommen.

Ch. d. 10.11.04.

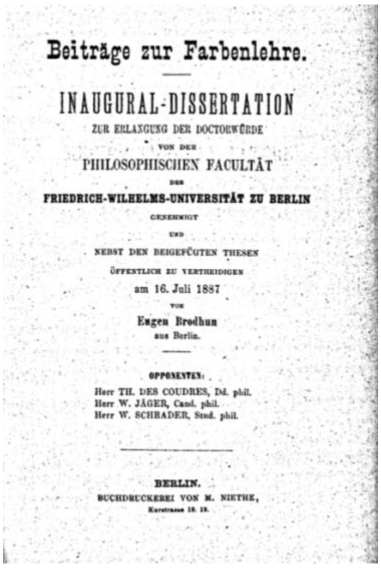
E. Brodhun

Fig. 4.27
Manuscript note of Eugen Brodhun (unknown recipient) dated 10.11.1904 with autograph signature (*Staatsbibliothek zu Berlin*. Handschriftenabteilung; Signatur: Slg. Darmstaedter F1c 1887: Brodhun, Eugen Heinrich Eduard Ernst Blatt 1)

Eugen Brodhun studied from 1880 to 1885 at the Humboldt University in Berlin, where he was a student of Helmholtz, and at the University of Heidelberg. Under the supervision of Helmholtz and König in Berlin, he wrote his doctoral thesis in 1887 with the title *Beiträge zur Farbenlehre (Contributions to the theory of colour).* From 1888 onwards, Brodhun was a research assistant (*wissenschaftliche Hilfskraft*) at the Institute and in 1923 he became Department Director. As we will see in the following lines, Brodhun provided crucial results for what concerns green-blind colour vision, being himself affected by green-blindness. He determined the elementary sensation curves, investigated the validity of the Newton's law of colour mixing and obtained the curves of discrimination sensibility for different hues of the spectrum for his own colour system.

In relation to Brodhun's studies of colour differences, it is worth to report that he experimented also with Maxwell's colour top. His manuscript colour equations, contained in Helmholtz's notebook of 1887 and preserved at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften*, are reported in Appendix C.

4.7 Brodhun's *Beiträge zur Farbenlehre*



Beiträge zur Farbenlehre.

INAUGURAL-DISSERTATION
ZUR ERLANGUNG DER DOCTORWÜRDE
VON DER
PHILOSOPHISCHEN FACULTÄT
DER
FRIEDRICH-WILHELMS-UNIVERSITÄT ZU BERLIN
GENEHMIGT
UND
NEBST DEN BEIGEFÜGTEN THESEN
ÖFFENTLICH ZU VERTHEIDIGEN
am 16. Juli 1887
VON
Eugen Brodhun
aus Berlin.

OPPONENTEN:
Herr TH. DES COUDRES, Dr. phil.
Herr W. JÄGER, Cand. phil.
Herr W. SCHRADER, Stud. phil.

BERLIN.
BUCHDRUCKEREI VON M. NIETHE,

Fig. 4.28
Front-page of Brodhun's dissertation preserved at the *Staatsbibliothek zu Berlin*

It is worth saying a few more words about the content of Brodhun's inaugural dissertation because it represents the starting point for his subsequent work on colour. The work consists of five sections[54]: *I. Über das Leukoskop (About the leucoscope). II. Über die Empfindlichkeit des Auges gegen Farbenänderung im Spectrum (About the sensitivity of the eye to colour changes in the spectrum). III. Über das Purkinjesche Phänomen (On the Purkinje phenomenon). IV. Über die Intensitätsvertheilung im Spectrum (On the intensity distribution in the spectrum). V. Über die Veränderlichkeit des neutralen Punktes der Farbenblinden und der Farbe monochromatischen Lichtes bei Veränderung der Intensität (On the variability of the neutral point of the colour blind subject and of the colour of monochromatic light with change in intensity).* I would like to remember that Brodhun himself was affected by colour blindness, he was a deuteranope, *grünblind (green-blind)* as he wrote. Therefore, for his investigation on colour blindness he used the results obtained by his own observations.

The introduction is dedicated to a detailed description of the Helmholtz colour mixer (see section 4.3).

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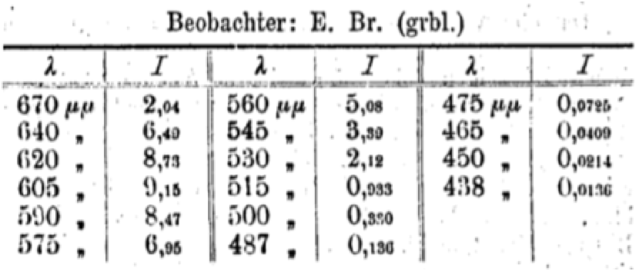
[54] As Brodhun explained in the first lines, he decided to report in the dissertation only sections III and IV, because the remaining sections were intended to be published elsewhere (Brodhun 1887, p. 5).

Brodhun used this apparatus to obtain the shape of the intensity curves of elementary sensations for his own colour system. He stated that the spectrum obtained for people affected by dichromacy could be portioned in three parts: two differently coloured end sections, in which only differences in intensity were to be noticed, and a middle part, which, in addition to this, showed differences in saturation.

The red end of the spectrum (denoted by *W* from the German *warm, warm* also in English) and the violet one (denoted by *K* from the German *kalt, cold* in English) furnished the elementary sensations, since every colour of the middle part of the spectrum could be mixed using them.

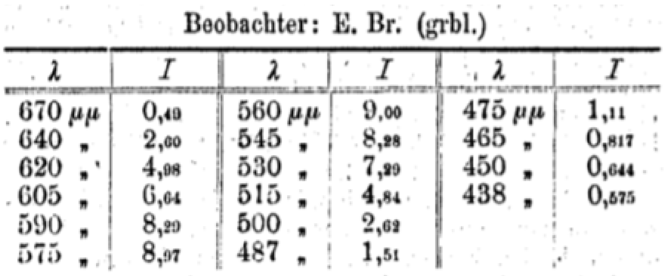
For what concerns section *IV. Über die Intensitätsvertheilung im Spectrum*, Brodhun described his experiment aimed to determine the intensity distribution in the spectrum by using Helmholtz colour mixer as experimental set-up. Measurements were conducted using a single homogeneous colour for the whole spectrum for comparison. This colour was introduced in the left collimator, where the right one was set to the wavelength under investigation. The right slit was maintained fixed, whereas the left one was set in order to ensure the same brightness for every measurement. He could consider the intensity of light proportional to the width of the slit and, for each compared position of the spectrum, ten setting were made. In this way, he could obtain four experimental series, two using lithium light (670 nm) and two using white light for comparison. The measurements were conducted taken into account the effect due to the Purkinje phenomena (see footnote 32). The following tables show the obtained intensity values, in the first case using lithium light, and in the second case using white light as reference.



Beobachter: E. Br. (grbl.)

λ	I	λ	I	λ	I
670 μμ	2,04	560 μμ	5,08	475 μμ	0,0725
640 „	6,40	545 „	3,39	465 „	0,0409
620 „	8,73	530 „	2,12	450 „	0,0214
605 „	9,15	515 „	0,933	438 „	0,0136
590 „	8,47	500 „	0,330		
575 „	6,95	487 „	0,136		

Table 4.2
Brodhun's obtained intensity values for lithium light dispersion spectrum (Brodhun 1887, p. 33)



Beobachter: E. Br. (grbl.)

λ	I	λ	I	λ	I
670 μμ	0,49	560 μμ	9,00	475 μμ	1,11
640 „	2,00	545 „	8,28	465 „	0,817
620 „	4,98	530 „	7,29	450 „	0,644
605 „	6,04	515 „	4,84	438 „	0,575
590 „	8,29	500 „	2,62		
575 „	8,97	487 „	1,51		

Table 4.3
Brodhun's obtained intensity values for sunlight dispersion spectrum (Brodhun 1887, p. 33)

Brodhun, then, reported the values of the second table (Table 4.3) in a coordinate system, where on the x-axis wavelengths are shown, as we can see in Fig. 4.29.

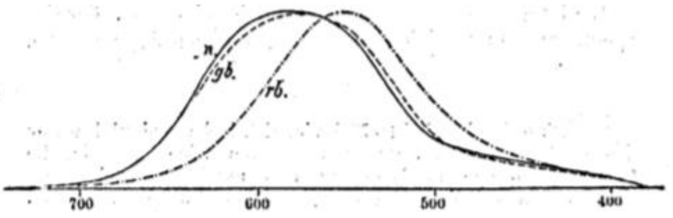


Fig. 4.29
Brodhun's intensity curves for a normal trichromat (denoted by *n*), a red-blind subject (denoted by *rb*) and a green-blind subject (denoted by *gb*) (Brodhun 1887, p. 33)

Comparing his curve (*gb*) with his elementary sensation functions, he noticed a similarity between the former and his *W*-curve. He obtained, in fact, the following ratios (*I/W*), considering white light as reference:

λ (nm)	670	640	620	605	590	575	560	545
$\frac{I}{W}$	1.02	1.08	0.97	0.99	1.03	1.03	1.05	1.04

Table 4.4
Ratio I/W calculated by Brodhun in the wavelength rage 670-545 nm (Brodhun 1887, p. 34)

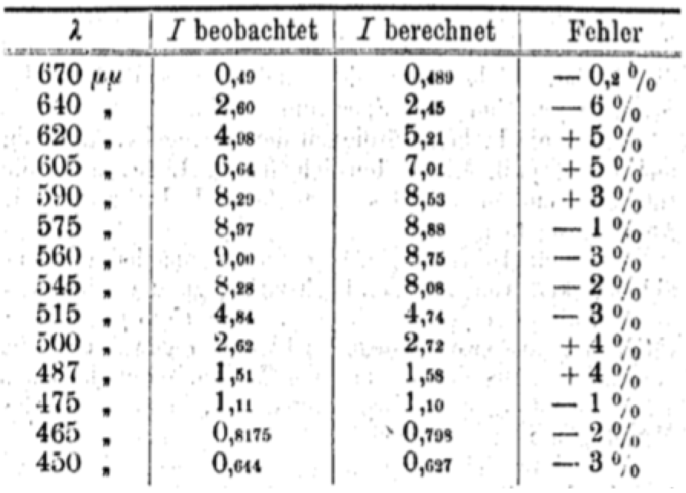
He proceeded, them, writing a simple relationship between *K*, *W*, and *I*, assuming the intensity of a mixture as equal to the sum of the intensities of the components:

$$I = aW + bK \tag{4.17}$$

The intensity curve should appear, in fact, as a superposition of the two elementary sensation functions. For what concerns constants a and b, Brodhun could find, using all the equations he had obtained, the most probable values of them:

$$a = 1.018;\ b = 0.03945$$

Table 4.5 shows the values of *I* obtained by calculation (*berechnet*) and those of *I* obtained by observation (*beobachtet*), in order to offer a direct comparison:



λ	*I* beobachtet	*I* berechnet	Fehler
670 μμ	0,49	0,489	— 0,2 %
640 „	2,60	2,45	— 6 %
620 „	4,98	5,21	+ 5 %
605 „	6,64	7,01	+ 5 %
590 „	8,29	8,53	+ 3 %
575 „	8,97	8,88	— 1 %
560 „	9,00	8,75	— 3 %
545 „	8,28	8,08	— 2 %
515 „	4,84	4,74	— 3 %
500 „	2,62	2,72	+ 4 %
487 „	1,51	1,58	+ 4 %
475 „	1,11	1,10	— 1 %
465 „	0,8175	0,798	— 2 %
450 „	0,644	0,627	— 3 %

Table 4.5
Intensity values *I* obtained by calculation (*berechnet*) and intensity values *I* obtained by observation (*beobachtet*) (Brodhun 1887, p. 35)

From the following equation:

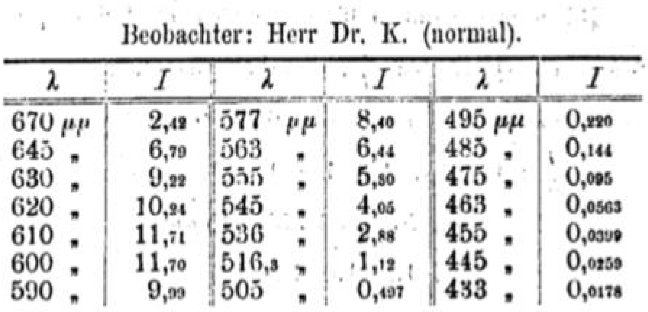
$$\int I d\lambda = a \int W d\lambda + b \int K d\lambda \tag{4.18}$$

where $\int I d\lambda$ corresponds to the intensity of white light and $\int W d\lambda = \int K d\lambda$, Brodhun could obtain the intensity ratio between components *K* and *W*:

$$\frac{b}{a} = 0.0388$$

This ratio has a very small size compared to the significant qualitative difference between extreme red and white. These results allowed Brodhun to suggest the following hypothesis: the *W*-elementary sensation should be involved for every light sensation, whereas the *K*-elementary sensation should have only a qualitative influence.

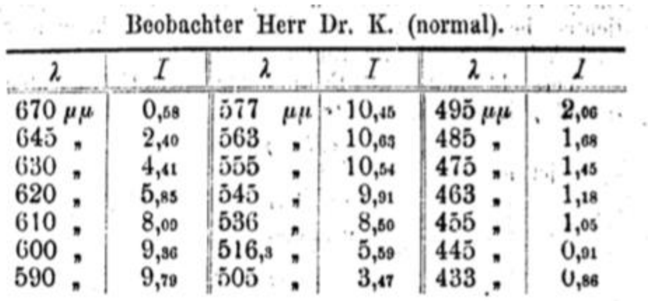
Moreover, Brodhun reported results obtained from the observations of König (with normal colour vision) and those of another observer with a red-blind colour vision, Mr. Ritter.
In the following two tables we can see the data recorded by König, for the dispersion spectrum of gas light and for the interference spectrum of the sun.



Beobachter: Herr Dr. K. (normal).

λ	I	λ	I	λ	I
670 $\mu\mu$	2,42	577 $\mu\mu$	8,40	495 $\mu\mu$	0,220
645 „	6,79	563 „	6,44	485 „	0,144
630 „	9,22	555 „	5,30	475 „	0,095
620 „	10,24	545 „	4,05	463 „	0,0563
610 „	11,71	536 „	2,88	455 „	0,0399
600 „	11,70	516,3 „	1,12	445 „	0,0259
590 „	9,99	505 „	0,497	433 „	0,0178

Table 4.6
König's intensity values related to the dispersion spectrum of gas light (Brodhun 1887, p. 37)



Beobachter Herr Dr. K. (normal).

λ	I	λ	I	λ	I
670 $\mu\mu$	0,58	577 $\mu\mu$	10,45	495 $\mu\mu$	2,06
645 „	2,40	563 „	10,63	485 „	1,68
630 „	4,41	555 „	10,54	475 „	1,45
620 „	5,85	545 „	9,91	463 „	1,18
610 „	8,00	536 „	8,50	455 „	1,05
600 „	9,36	516,3 „	5,59	445 „	0,91
590 „	9,79	505 „	3,47	433 „	0,86

Table 4.7
König's intensity values related to the intensity spectrum of sunlight (Brodhun 1887, p. 37)

For a normal eye, a direct comparison between the intensity curve with the three elementary sensation functions reveals a great similarity between the former and the R-curve. At this point, Brodhun tried to represent the intensity curve as a superposition of the R-, G-, and V-curves:

$$I = \alpha R + \beta G + \gamma V \tag{4.19}$$

where coefficients β and γ had very small values. The best calculation was obtained by assuming that the G excitation had no influence at all on the brightness. The ratio I/R was then calculated; we report the obtained values in the table below, related to the first part of the spectrum:

λ (nm)	670	645	630	620	610	600	590	577	563	555	545
$\frac{I}{R}$	1.12	1.11	1.11	1.12	1.21	1.26	1.18	1.17	1.12	1.11	1.13

Table 4.8
Ratio I/R calculated by Brodhun in the wavelength rage 670-545 nm (Brodhun 1887, p. 38)

He also calculated the values of the coefficients α and γ from the equation:

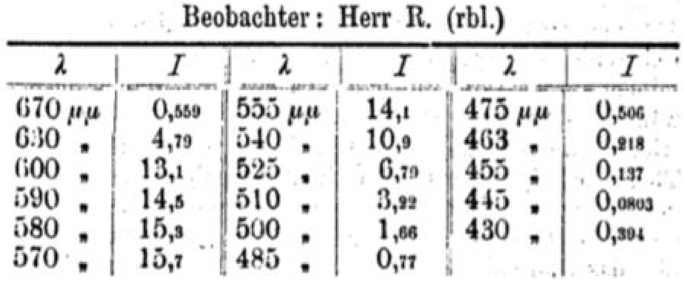
$$I = \alpha R + \gamma V \tag{4.20}$$

obtaining the following results:

$$\alpha = 1.14;\ \gamma = 0.083;\ \frac{\gamma}{\alpha} = 0.073$$

In this case, as Brodhun recognized, the intensity curve obtained by calculation was not in full agreement with the observed one.

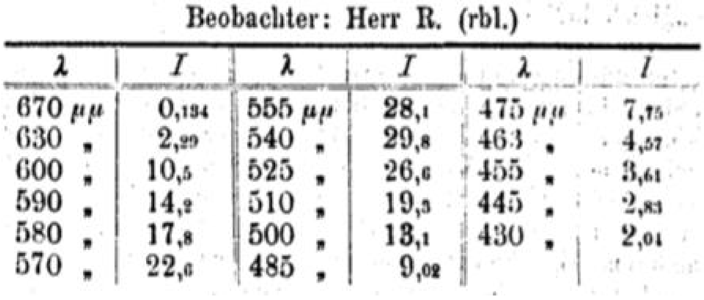
Tables 4.9 and 4.10 show the data recorded from Ritter's observations, also in this case, for both the dispersion spectrum of gas light and the interference spectrum of the sun.



Beobachter: Herr R. (rbl.)

λ	I	λ	I	λ	I
670 $\mu\mu$	0,559	555 $\mu\mu$	14,1	475 $\mu\mu$	0,506
630 „	4,79	540 „	10,9	463 „	0,218
600 „	13,1	525 „	6,79	455 „	0,137
590 „	14,5	510 „	3,22	445 „	0,0803
580 „	15,3	500 „	1,66	430 „	0,394
570 „	15,7	485 „	0,77		

Table 4.9
Ritter’s intensity values related to the dispersion spectrum of gas light (Brodhun 1887, p. 39)



Beobachter: Herr R. (rbl.)

λ	I	λ	I	λ	I
670 $\mu\mu$	0,134	555 $\mu\mu$	28,1	475 $\mu\mu$	7,75
630 „	2,29	540 „	29,8	463 „	4,57
600 „	10,5	525 „	26,6	455 „	3,61
590 „	14,2	510 „	19,3	445 „	2,83
580 „	17,8	500 „	13,1	430 „	2,01
570 „	22,6	485 „	9,02		

Table 4.10
Ritter’s intensity values related to the intensity spectrum of sunlight (Brodhun 1887, p. 39)

The curve obtained from Ritter’s observations is shown in Fig. 4.29 and denoted by *rb*. It presents a distinctly different shape (the maximum is located in a different position) and to a great extent it is highly consistent with the red-blind *W*-curve. He could, thus, conclude that the relation subsisting between the *W*-sensation and the intensity could approximately be considered as the same as in the case of green-blindness. In this last investigation, Brodhun was not able to build the *I*-curve by superposition because Ritter’s elementary sensation functions were not determined.

Before concluding his work, Brodhun offered a brief review of the obtained results. Analysing the three intensity curves *rb*, *gb* and *n*, depicted in Fig. 4.29, he stated that the intensity curves related to normal and green-blind colour vision shared almost the same shape, whereas the shape of the curve related to red-blind colour vision deviated considerably from the other two. I would like, at this point, report Brodhun’s original words of thanks addressed to Helmholtz and his assistant König:

Am Schlusse meiner Arbeit darf ich nicht unterlassen, meinem hochverehrten Lehrer, Herrn Geheimrath von Helmholtz, für die mir in den fünf Semestern, während welcher ich im Berliner Physikalischen Institut tähtig war, gewährte Belehrung und Antheilnahme, sowie Herrn Doctor A. König für die speciell bei dieser Arbeit geleistete eingehende Unterstützung und Anregung meinen tiefgefühlten Dank anzusprechen (At the end of my work, I must not refrain from expressing my profound gratitude to my honoured teacher, Herr Geheimrath von Helmholtz, for the instruction and *the assistance given to me in the five semesters during which I was working at the Berliner Physikalischen Institut and to Herr Doctor A. König, for the support and encouragement given specifically for this work)* (Brodhun 1887, p. 40).

4.8 Brodhun's investigation on the validity of Newton's law of colour mixing

My intention is to report here a work published by König in 1887 with the title *Über Newton's Gesetz der Farbenmischung und darauf bezügliche Versuche des Hrn. Eugen Brodhun* (König and Engelmann 1903, pp. 108-115) containing Brodhun's investigation on the validity of Newton's law of colour mixing. König began his paper summarizing with these words the assumptions lying at the basis of Newton's law, whose validity was to be tested for all colour vision system: *Bei genauer Betrachtung ergiebt sich, daß dem von NEWTON in Verbindung mit seiner Anordnung der Farben zu einer Farbentafel aufgestellten Gesetz der Farbenmischung mehrere Annahmen zu Grunde liegen, deren Bestätigung durch die Erfahrung erst dem Gesetze seine Gültigkeit verleiht. Diese Annahmen sind zuerst von H. GRASSMANN und dann durch Hrn. H. von Helmholtz in folgender Fassung ausgesprochen worden (The law of colour mixing established by Newton, in connection with his arrangement of colours in a colour chart, is based on several assumptions, the confirmation of which gives validity to the law. These assumptions were first expressed by H. Grassmann and then by Helmholtz)* (König and Engelmann 1903, p. 108). The three assumptions are reported below.

1. Any compound colour must look the same as the mixture of a specific saturated colour with white.
2. When one of the two lights involved in a mixture changes constantly, the appearance of the compound colour also changes constantly.
3. Same-looking mixed colours give the same-looking compounds.

König examined whether these three conditions could still be regarded as fulfilling at his time. He took into account only trichromatic and dichromatic colour systems.

For what concerns the first assumption, König reformulated the sentence in the following way, in relation to trichromatic systems:

1. Any compound colour is either white or equal to one of the spectral colours supplemented by the addition of the purple to a self-contained series or equal to a mixture of these colours with white.

And, considering dichromatic colour systems, where instead of a colour chart, a colour line appears:

1. Any arbitrary compound colour is equal to a spectral colour.

This change does not affect Newton's law of colour mixing, but rather allows a small expansion of its realm, as it renders it possible to have colour equations, which were formerly considered not producible.

The second sentence, in its entirety, was still in accordance with the observations.

König then concentrated on the third sentence, which contains the requirement that each colour equation must be independent of the intensity. He proceeded writing the colour equation for any intensity in the following form:

$$\alpha \cdot L_1 + \beta \cdot L_2 = a \cdot L_3 + b \cdot L_4 \tag{4.21}$$

where L stands for *Licht (light)* and α, β, a and b are coefficients.

This colour equation must remain unchanged by varying the intensity n times on both sides:

$$n \cdot \alpha \cdot L_1 + n \cdot \beta \cdot L_2 = n \cdot a \cdot L_3 + n \cdot b \cdot L_4 \tag{4.22}$$

In the production of a colour equation, the coefficients a and b should not depend on n, according to the third sentence. König then introduced his investigation on dichromatic systems, in which a certain homogeneous light appears the same as the white undecomposed sunlight: *Bei dichromatischen Farbensystemen hat ein gewisses homogenes Licht dieselbe Farbe, wie das weiße unzerlegte Sonnelicht* (König and Engelmann 1903, p. 108). In this case, he introduced a colour equation containing, on one side, a simple spectral colour (the *neutral point* of the colour blind, that point in the spectrum of a dichromat which he can match with white light), and, on the other side, the sum of all the *constituents of sunlight*, as he wrote.

König reported Brodhun's experiments, which played a central role in the investigation and in which he compared a mixture of lights of wavelengths 615 nm and 460 nm with light of wavelength included between them. At the lowest intensity, he could write the colour equation in the following form:

$$n \cdot L_{\lambda} = n \cdot a \cdot L_{615} + n \cdot b \cdot L_{460} \tag{4.23}$$

He found out that only in the case of wavelength 480 nm the coefficients a and b were independent from n. Coefficient a increased and b decreased as the intensity increased, in contradiction, therefore, with Newton's third assumption. König, reporting this crucial result, concluded the paper with the following words: *Überhaupt ist durch die Erschütterung der dritten Vorausetzung des NEWTON'schen Farbenmischgesetzes und durch durch die damit verbundene Einschränkung der Gültigkeit des letzteren dem physiologisch-optischen Versuche ein neues, noch völlig unbegrenzten Gebiet übergeben, auf dem wahrscheinlich die Schlüssel für viele bisher noch unerklärte Tatsachen zu finden sind (In general, the upheaval of the third assumption of NEWTON's colour mixing law and the consequent restriction on its validity give to the physiological-optical experiments a new, still completely unlimited, field, in which, probably, the keys to many as yet unexplained facts can be found)* (König and Engelmann, p. 115).

Brodhun's results and considerations on the subject were then published, in complete form, in his paper of 1893 with the title *Die Gültigkeit des NEWTON'schen Farbenmischungsgesetzes bei dem sog. Grünblinden Farbensystem (The validity of the NEWTON colour mixing law in the so-called green blind colour system)* (Brodhun 1893). The pivotal conclusions obtained by Brodhun are discussed below, after offering a brief overview of König's study of 1884, that provided the background for Brodhun's investigation.

In 1884, in fact, König wrote a paper with the title *Zur Kenntniss dichromatischer Farbensysteme (On the knowledge of dichromatic colour systems)* (König and Engelmann 1903, pp. 11-22), which contains a review of the state of knowledge at that time of dichromic colour system. His aim was the determination of the neutral point for both *Rothblinden* (red-blind subjects) and *Gründblinden* (green-blind subjects), belonging to the more general class of *Roth-grünverwechsler* (subjects that confuse red and green, also called *red-green blindness* by the adherents of Müller theory of vision), at fixed intensity at first, and then under different intensity conditions. He also analysed its position in the spectrum for both cases. In describing the experimental set-up, König reported all the possible source of errors that could occur in conducting experiments of this kind and underlined that: *Die Umgehung aller dieser Fehlerquellen geschah, indem ich ein MAXWELL zuerst ausgeführtes und von Herrn v. HELMHOLTZ bei der Construction seines Apparates zur subjectiven Mischung zweier Spectralfarben befolgtes Princip meiner Untersuchungsmethode zu Grunde legte (In order to circumvent of all these sources of error, I base my investigation method on a principle first adopted by MAXWELL and by Mr. HELMHOLTZ, in the construction of his apparatus for the subjective mixture of two spectral colours)* (König and Engelmann 1903, p. 14). Here König referred to

Helmholtz colour mixer and Maxwell's famous colour box, used by him, as we have seen in chapter 3 (sections 3.10 and 3.14), to obtain the intensity curves for a normal and for a red-blind eye. I would like to remark the importance of Maxwell's ideas lying behind his colour box. This simple principle was adopted, in fact, to conduct several investigations in the field of colour vision over the centuries, from Helmholtz, König and Dieterici's colour mixer up to the colour-matching experiments of the 20th century.

My intention is to focus on König's results and considerations, without entering into the detail of the experimental set-up. First, he experimented with one eye at the time for every observer, in order to ascertain if deviations from one eye to the other should be taken into consideration. He could conclude that the wavelength value of the neutral point calculated separately for each eye deviated very little from one to the other; the difference could, therefore, be neglected.

For what concerns the determination of the neutral point at fixed intensity, he found out, analysing the results obtained by six red-blind and seven green-blind observers, that the wavelength of the neutral point was nearly the same for both classes: *Eine scharfe Trennung dieser beiden Classen ist also aus meinen Untersuchungen nicht zu folgern, vielmehr das Gegentheil (A sharp separation of these two classes cannot, therefore, be inferred from my investigations, rather the opposite)* (König and Engelmann 1903, p. 18). I report the obtained measurements with measurement errors in Table 4.11:

λ_n **for red-blind observers** (nm)	λ_n **for green-blind observers** (nm)
Hr. Dr. W.= $491{,}70 \pm 0{,}09$	Hr. Dr. K. = $492{,}04 \pm 0{,}09$
Hr. Dr. B.= $492{,}25 \pm 0{,}19$	Hr. Dr. Lu. = $495{,}92 \pm 0{,}36$
Hr. Dr. S.= $493{,}08 \pm 0{,}13$	Hr. Dr. F. = $496{,}01 \pm 0{,}23$
Hr. C. = $493{,}80 \pm 0{,}36$	Hr. Le. = $496{,}08 \pm 0{,}40$
Hr. Schw.= $497{,}37 \pm 0{,}48$; $497{,}68 \pm 0{,}34$[55]	Hr. E. W. = $499{,}44 \pm 0{,}20$
Hr. R. H. = $497{,}66 \pm 0{,}14$	Hr. W. H. = $499{,}71 \pm 0{,}16$
	Hr. J. P. = $504{,}75 \pm 0{,}15$

Table 4.11
König's wavelength measurements of the neutral point for red-blind and green-blind subject at fixed intensity (König and Engelmann 1903, p. 18)

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[55] The second measurement was taken several days after the first one.

From König original data, we have calculated the weighted mean and weighted average uncertainty for both cases obtaining the following wavelength values: $(493,45 \pm 0.06)\,nm$ and $(496,49 \pm 0.06)\,nm$ for red-blind and green-blind subjects respectively. I would like to show that, although the difference is small, the neutral point for the green-blind subjects under investigation underwent a just perceptible shift toward the green region of the spectrum.

König investigated, then, the neutral point's dependence on light intensity. He selected two red-blind subjects and only one green-blind observer. He considered as a unitary intensity the one kept fixed in the previous investigation. Fig. 4.30 shows the obtained curves.

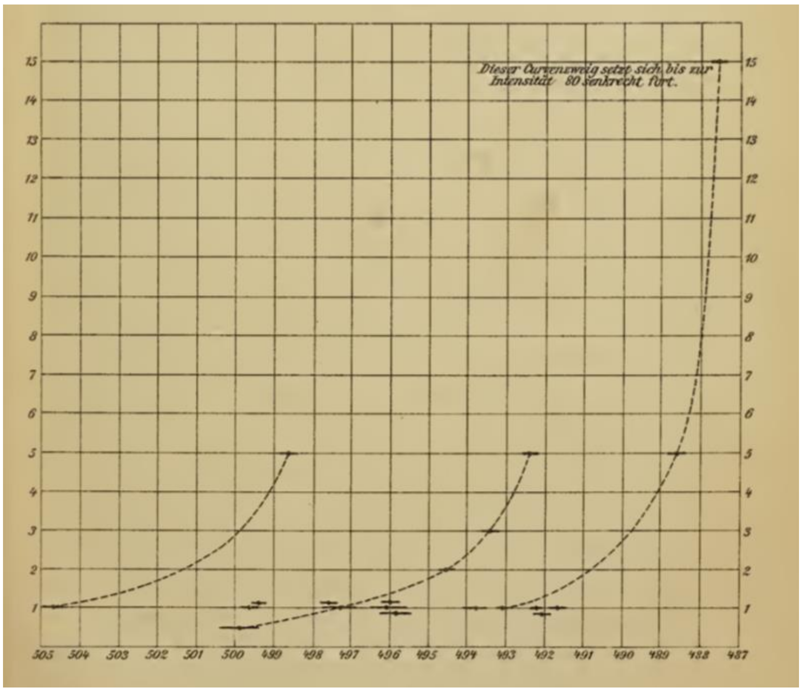

Fig. 4.30

The position of the neutral point in function of light intensity (y-axis). The first curve on the left refers to the green-blind subject's observations. On the x-axis is reported the wavelength range 487-505 nm. All three curves show a similar behaviour in the range of light intensity between 1 and 5 (König and Engelmann 1903)

From Fig. 4.30 it is evident that at lower intensity, the neutral point shifted toward the red region of the spectrum, whereas, for what concerns the red-blind subjects, it remained unchanged at higher intensities. For both cases, weighted mean and weighted average uncertainty were calculated from König's original data.

Class of colour blindness	$\boldsymbol{\lambda_n\ (nm)}$ **Intensity range 1-5**	$\boldsymbol{\lambda_n\ (nm)}$ **Intensity range 5-80**
Green-blindness	$502{,}2 \pm 0{,}1$	-
Red-blindness	$493{,}29 \pm 0{,}08$	$487{,}5 \pm 0{,}1$

Table 4.12

Calculated weighted mean and weighted average uncertainty of λ_n in two intensity ranges, for both protanopes (red-blind subjects) and deuteranopes (green-blind subject)

I would like to remark that Maxwell reported in a letter of 1860 addressed to Stokes the obtained intensity curves for a red-blind subject, whose intersection point, i.e. neutral point, was located near Fraunhofer line F (486 nm), just shifted toward the red region of the spectrum. This result seems to be in agreement with König's observations, although offering a quantitative comparison is beyond our ability[56]. Before introducing Brodhun's paper on the subject, I would like to report König's observations and conclusions with his original words: *Diese graphische Aufzeichnung lehrt [...] dass die Form der Curven, welche bei demselben Individuum die Abhängigkeit zwischen der Wellenlänge des neutralen Punktes und der Intensität darstellt, bei allen drei beobachteten Personen dieselbe ist (This graphic representation teaches that the shape of the curves, which for every subject represents the dependence between the wavelength of the neutral point and the intensity, is the same for all three persons under examination)* (König and Engelmann 1903, p. 22).

Brodhun conducted a more detail investigation on his own deficiency, i.e. green-blindness, in order to show his neutral point's shift at low intensity.

He started his work of 1893 with the definition of neutral point, remarking its crucial role. In fact, its observation in function of the intensity could verify or invalidate Newton's third law of colour mixture, in the form expressed by König. Brodhun reported and analysed König's results, obtained with Helmholtz's apparatus for colour mixing. He found out that at high intensities the position of the neutral point remained unchanged, whereas at low intensities it shifted toward the red end of the spectrum, as we have seen above. In order to determine the position of the neutral point, it was sufficient to build a colour equation with a spectral light, on one side, and white light, on the other. The observations conducted on a large number of colour blind subjects yielded, as a result, to

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[56] Too many experimental uncertainties, e.g., related to the precision and accuracy of the experimental set-up and to the intensity determination do not allow us to offer a significant quantitative comparison.

deviations to Newton's law. Dieterici and König considered, however, these deviations as restricted to a small portion of the spectrum.

Brodhun, conducting experiments by using the Helmholtz colour mixer to test the inconstancy of the neutral point for his own colour system, was able to build colour equations containing a homogeneous colour, on one side, and a mixture of only two homogeneous colours, on the other, instead of white light. Two wavelengths were selected, i.e. 615 and 460 nm, which mixed together, for a given intensity, appeared equal to the wavelength of 496 nm. Subsequently, he decided to diminish the intensity of the mixture with the aim of finding the wavelength which appeared equal to that mixture. The obtained results are reported in Fig. 4.31, that shows intensity on the y-axis and wavelength on the x-axis.

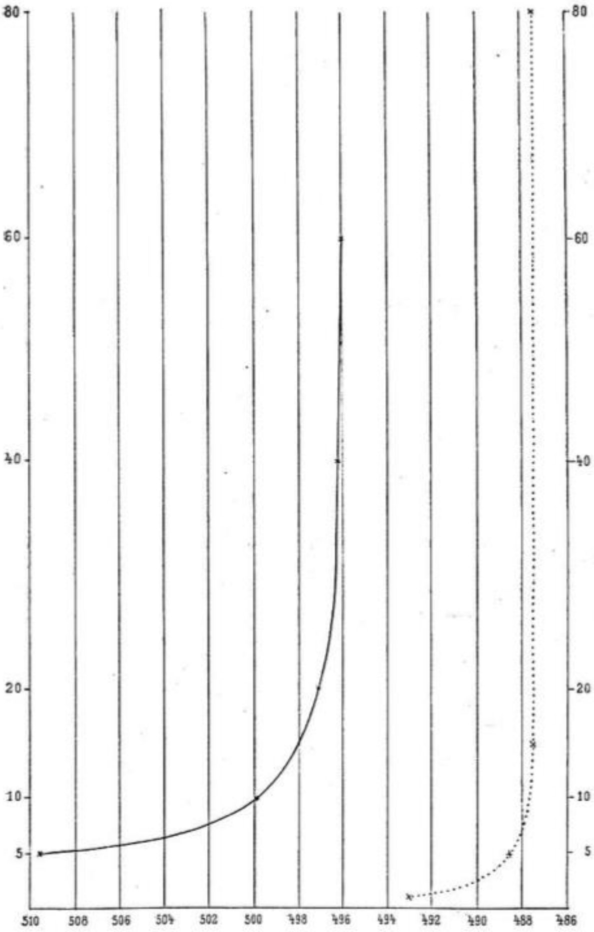

Fig. 4.31
The position of the neutral point in function of the intensity for Brodhun (full curve) and König (dashed curve) (Brodhun 1893, p. 327)

The dotted curve represents König's results, obtained under different intensity conditions, and shares the same shape of Brodhun's full curve. From Fig. 4.31, it is evident that, for lower intensities, the colour equation changed much faster than for the higher ones.

He decided to investigate in which portion of the spectrum and under which intensity conditions deviations occurred from Newton's law. The next step was, therefore, that of building the elementary sensation curves at different intensities, in order to better visualize the shift of the neutral point. He described the method: two components were chosen, the same as indicated above, of wavelengths 615 and 460 nm, that were mixed in order to appear equal to a certain homogenous light of wavelength lying in between, specifically 480, 500, 520, 540, 560 and 580 nm. The investigation was carried out under different intensity conditions. Although the selected components did not belong to the end of the spectrum, where only differences in intensity and no saturation differences were to be seen, they were close to it. Thus, it was possible to produce very nearly the elementary curves *W* and *K* for the gaslight dispersion spectrum used for different brightness. The colour corresponding to wavelength 480 nm appeared the same as that of the mixture for all intensities under examination. For the wavelength of 500 nm, the colour change, at low intensity, was already significant. This change was also noticeable for what concerns the wavelength of 600 nm.

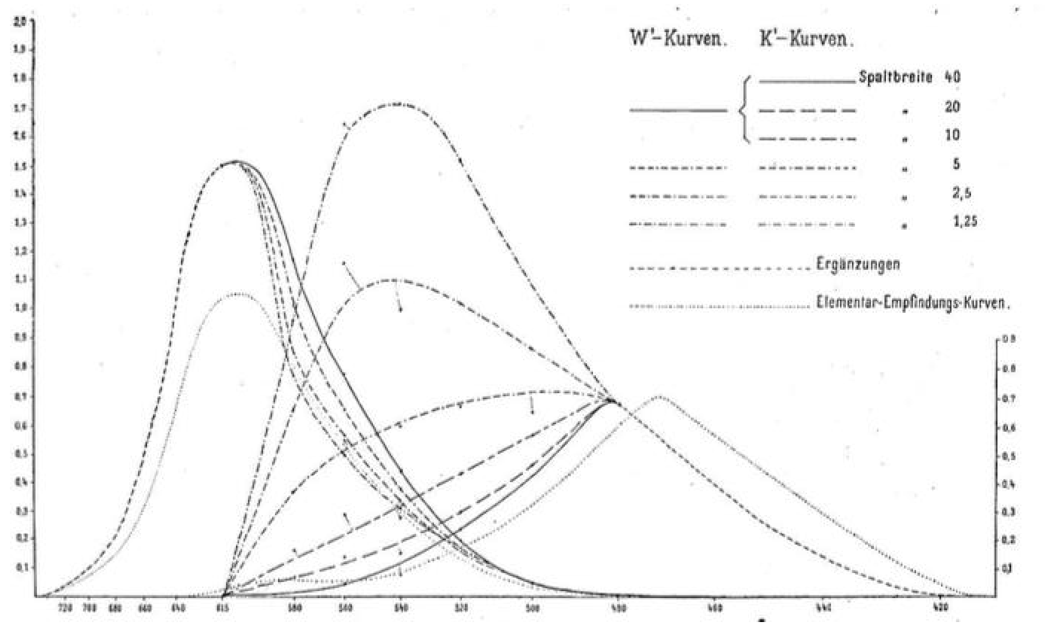

Fig. 4.32

Brodhun's elementary curves K' and W' for different intensities of light; in the legend, *Spaltbreite* is the width of the slit through which light was allowed to pass (the wider the opening the higher the intensity). Brodhun reported the intensity on the y-axis and the wavelength (720-420 nm) on the x-axis (Brodhun 1893, p. 332)

Brodhun indicated his own curves by K' and W' to differentiate them from those of König and Dieterici, which are denoted by *K* and *W*. It is worth reporting here Brodhun's original considerations in order to better understand the graph in Fig. 4.32: *Um meine* W'*- und* K'*-Kurven mit den auf mein Farbensystem bezüglichen W- und K-Kurven, wie sie in der mehrfach citierten Arbeit der Hrn. A.*

KÖNIG und C. DIETERICI angegeben sind, bequem vergleichen zu können, habe ich diese ebenfalls punktiert eingezeichnet. Da die Hrn. KÖNIG und DIETERICI nur bei hohen Intensitäten arbeiten, so muß natürlich meine auf die Spaltbreite 60 bezügliche Kurve zum Vergleich herangezogen werden. Es zeigt sich, daß die Abweichung zwischen W' *und W sehr gering, zwischen* K' *und K aber auch nicht größer ist, als sie aus dem Umstand, daß meine kurzwelligen Mischungskomponente sich noch sehr merkbar von meinem gesättigsten Blau unterscheidet, vorher zu erwarten war (In order to be able to compare my curves* W' *and* K' *with the W- and K-curves, which are given in the most-cited work of Mr. KÖNIG and C. DIETERICI, I have also drawn them dotted* [Elementar-Empfindungs-Kurven in the legend]. *Since Mr. KÖNIG and DIETERICI work only at high intensities, it is obvious that my curve relating to the slit width 60*[57] *must be used for comparison. It turns out that the difference between* W' *and W is very small, but also not larger between* K' *and K than it was previously to be expected from the fact that my short-wavelength component of the mixture differs very noticeably from my most saturated blue)* (Brodhun 1893, p. 333).

From Fig. 4.32, it is clear that the position of the neutral point, identified by the intersection point of the two curves K' and W', moves toward the red end of the spectrum with decreasing intensity. It can be seen that the middle region of the spectrum the inconstancy of the colour equation is highly significant. Furthermore, Brodhun stated that for every wavelength in the range between 520 and 580 nm, he could find an intensity at which the wavelength under consideration appeared to him equal to one, and always the same, mixture of 615 and 460 nm. The most pronounced deviations from Newton's law occurred at 540 and 560 nm; they, however, decreased everywhere with increasing intensity. For what concerns the highest intensities, a broad range could be detected, in which colour equations are independent of the intensity. Here, mixtures were obtained and the size of the components was determined without indicating the illumination conditions. The elementary sensation curves built by König and Dieterici should belong, therefore, to this range.

Brodhun concluded this essay with some considerations about trichromatic colour systems: *Somit scheint es erwiesen, daß auch bei Trichromaten Abweichungen von dem NEWTONschen Gesetz vorhanden sind. Sie sind aber offenbar weniger hervortretende, als bei den Farbenblinden, und lassen sich infolgedessen auch erheblich schwerer quantitativ feststellen (Also in this case, deviations from Newton's law could be noticed but they are less prominent than for colour blind and, as a result, they are much more difficult to quantify)* (Brodhun 1893, p. 334).

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[57] In the graph legend of Fig. 4.32, the indication of *Spaltbreite* (slit-width) 60 is missing. This could be due to the fact that Brodhun could obtain only two results for that intensity (see Brodhun 1893, p. 331); we can, therefore, assume that the points he detected lie along the curves obtained by König and Dieterici (indicated by *Elementar-Empfindungs-Kurven* in the graph legend).

All these crucial considerations and material, collected by König, Dieterici and Brodhun, served as a starting point for Helmholtz's work on the geometry of colour space. In the next chapter, in fact, we will see how Helmholtz came to the first definition of line element in colour space, extending the psychophysical law of Weber-Fechner to a complex of two and three dimensions and exploiting his assistants' precious results.

Chapter 5

Helmholtz and the geometry of colour space

He had picked up Young's ideas on a three-receptor mechanism of color vision, imbibed Fechner's ideas on the relationship between physical input and sensory output, and seen Riemann develop his non-euclidean geometry. In his extraordinary universal mind these ingredients fused into that mathematical expression which, until the present day, formed the basis for any line element developed (J. J. Vos, Vos 1979, p. 208)

In this chapter I will focus on Helmholtz's attempt to define a line element in colour space (*kürzeste Linien im Farbensystem*), a three-dimensional mathematical model used to describe colour differences in terms of colour distances. Line elements can be defined as *mathematical formulations that describe subjective colour differences in terms of model characteristics* (Vos 1979, p. 208). In other words, line elements express the relationship subsisting between fundamental colour responses, e.g. cone responses, and just noticeable difference (see chapter 4) in colour space. This could be counted among one of Helmholtz's great achievements and considered as the first step toward a metrically significant model of colour space. He succeeded, in fact, to produce the first accurate determination of linear sequences through colour space. I would like to remember that Helmholtz measured, by using his colour mixing apparatus, the relative distance from white to the perimeter of the colour diagram as a function of wavelength and found out an *asymmetry*. Mixing complementary colours, he recognized that the quantity of light required to produce white varied systematically with position around the Newton colour circle. Adopting Newton's analogy of the centre of gravity to predict the outcome of optical mixture of light, therefore, the shape of the colour diagram should be neither a circle nor a triangle, as proposed by Maxwell, but rather a truncated hyperbola, as we have reported in chapter 3. Helmholtz realized at this point that colour space could not be uniform, i.e. equal distances did not correspond to equal perceptual difference. This consideration made him turn toward the one-dimensional Weber-Fechner law of the new-born field of psychophysics, which related physical stimuli and psychological experience, to build the first line element in colour space. His intention was that of extending the law to three dimensions, because three is the number of independent variables in colour space in the case of normal vision. Helmholtz's mathematical treatment and related considerations are contained in three papers, published between 1891 and 1892, with the titles *Versuch einer erweiterten Anwendung des Fechnerschen Gesetzes im Farbensystem (An attempt to extend the application of Fechner's law in the colour system)*, *Versuch, das psychophysische Gesetz auf die Farbenunterschiede trichromatischer Augen anzuwenden (An attempt to apply the psychophysical law to colour differences in trichromatic eyes)* and *Kürzeste Linien im Farbensystem (The shortest lines in the colour system)*, which have not been yet translated into English.

In the first sections, I will briefly present the content of Riemann's inaugural dissertation, *On the Hypotheses which Lie at the Bases of Geometry*, held in 1854 and of Helmholtz's essay of 1868, *On the Facts Underlying Geometry*, which contain not only their general reflections upon the foundations of geometry, as we will discuss in the following paragraph, but also crucial considerations relating the geometry of colour space. Riemann and Helmholtz were the first, in fact, to interpret the geometry of colour space as non-Euclidean. My intention, then, is to retrace the path followed by Helmholtz to obtain his line element model to describe the mechanism of human colour vision. In sections 5.5 and 5.6 an analysis of Helmholtz's papers of 1891 and 1892 will be provided.

5.1 Geometry: axiomatic or empirical approach?

Bernhard Riemann's inaugural dissertation *On the Hypotheses which Lie at the Bases of Geometry* of 1854 and Hermann von Helmholtz's essay *On the Facts Underlying Geometry*, published in 1868, opened the door to a new view of the nature and character of geometrical knowledge. Their works are characterized by an empirical approach to the study of geometry and its foundations, as we will see in the following lines, as an alternative to Euclid's ideas. According to Riemann and Helmholtz, geometry should be founded on our perception and construction of space (which is not necessarily the one constituting the basis for Euclidean geometry). When we are dealing with geometry, they though, we are not simply dealing with immediately evident axioms, with given truths, but rather with hypothetical truths that depend on the validity of certain premises and whose truth depends on empirical choices. In fact, Riemann opened this dissertation with the following words:

It is known that geometry assumes, as things given, both the notion of space and the first principles of constructions in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor, a priori, *whether it is possible. From Euclid to Legendre (to name the most famous of modern reforming geometers) this darkness was cleared up neither by mathematicians nor by such philosophers as concerned themselves with it* (Jost 2016, p. 31).

One of the most debate topic among mathematicians at that time concerned the foundations of geometry. This gave a significant boost to the development of non-Euclidean geometries, from the 1820s onwards[58]. After the development of non-Euclidean geometries and Riemann and Helmholtz's above-mentioned works, mathematicians were free to construct infinitely many geometries and to see which one better applied to physical space; it had become clear that Euclidean geometry was not the only possible option. In Riemann and Helmholtz's view (unlike, for example, Poincaré) this had to be decided by recourse to experience.

I would like to offer a brief overview of the content of Riemann and Helmholtz's famous works, in which colour space was cited by both authors as an example of non-Euclidean geometry realized in nature. Starting from this consideration, in fact, Helmholtz defined the first line element in colour space, combining his knowledge of non-Euclidean geometry and psychophysics.

5.2 Riemann: *Über die Hypothesen, welche der Geometrie zu Grunde liegen*

Begriffe, deren Bestimmungsweisen eine discrete Mannigfaltigkeit bilden, sind so häufig, dass sich für beliebig gegebene Dinge wenigstens in den gebildeteren Sprachen immer ein Begriff auffinden lässt, unter welchem sie enthalten [...], dagegen sind die Veranlassungen zur Bildung von Begriffen, deren Bestimmungsweisen eine stetige Mannigfaltigkeit bilden, im gemeinen Leben so selten, dass die Orte der Sinnengegenstände und ***die Farben*** *wohl die einzigen einfachen Begriffe sind, deren Bestimmungsweisen eine mehrfach ausgedehnte Mannigfaltigkeit bilden.*

(Notions whose specialisations form a discrete manifoldness are so common that at least in the cultivated languages any things being given it is always possible to find a notion in which they are

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[58] One of the most debate topic in the history of mathematic was the proof of Euclid's V postulate, also known as the Parallel Postulate, which had raised a great deal of controversy within the mathematical field over the centuries. Starting from the first decade of the nineteenth century, in attempting to demonstrate the V postulate, the idea of a geometry different from the Euclidean one was taking root. For a complete overview of the history of non-Euclidean geometry the reader is referred to *Il flauto di Hilbert: storia della matematica moderna e contemporanea* written by Bottazzini (Bottazzini 1990, pp. 167-190), in which the works of Gauss, Bolyai and Lobacevskij are reported in detail, and to the contribution of Tazzioli, *L'Ottocento: matematica. La geometria non euclidea*, which contains a brief but precise account of the founders of non-Euclidean geometry and its diffusion (Tazzioli 2003).

included […], on the other hand, so few and far between are the occasions for forming notions whose specialisations make up a continuous manifoldness, that the only simple notions whose specialisations form a multiply extended manifoldness are the positions of perceived objects and ***colours****.)* (Jost 2016, p. 32)

On June 10, 1854, Riemann gave a public lecture, *Über die Hypothesen, welche der Geometrie zu Grunde liegen (On the Hypotheses which Lie at the Bases of Geometry),* on the occasion of his *Habilitation* as a *Privatdozent* in Göttingen, in which he proposed a general notion of manifold (*Mannigfaltigkeit*). The work was published posthumously by Dedekind in 1867, one year after Riemann's death, and it can be considered one of the most important contributions to geometry. My intention is to offer a short account of the content of Riemann's *Habilitation*, which contains a crucial consideration relating the geometry of colour space.

I would like to underline that Riemann's work was deeply influenced by the contribution his predecessors, Herbart and Gauss. As Riemann wrote in paragraph I of the *Habilitationsvortrag*: *In proceeding to attempt the solution of the first of these problems, the development of the notion of a multiply extended magnitude, I think I may the more claim indulgent criticism in that I am not practised in such undertakings of a philosophical nature where the difficulty lies more in the notions themselves than in the construction; and that besides some very short hints on the matter given by Privy Councillor Gauss in his second memoir on Biquadratic Residues, in the Göttingen Gelehrte Anzeige, and in his Jubilee-book, and some philosophical researches of Herbart, I could make use of no previous labors* (Jost 2016, p. 32).

As Ferreoirós pointed out, the concept of manifold may have been suggested by Herbart[59]. In the 1850s Riemann studied closely the work of the German philosopher Johann Friedrich Herbart, who was Professor in Göttingen until 1841. From a manuscript note of Riemann, we can read: *The author is a Herbartian in psychology and the theory of knowledge (methodology and eidology), but for the most part he cannot embrace Herbart's philosophy of nature and the metaphysical disciplines that are related to it (ontology and synechology)* (Scholz 1982, p. 414*)*. Here we find the peculiar names given by Herbart to the different parts of his doctrine. As pointed out by Ferreirós, Riemann's

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[59] Herbart introduced *serial forms (continuerliche Reihenformen)* of concepts. As Erhald Scholz wrote, Herbart considerations on space and time were the starting point for generating these general continuous serial forms. But, as Scholz reported reading Riemann's manuscript notes: *Riemann's excerpts suggest that he did not bother about specific procedures to generate "serial forms", although he was interested in how all of this related to Herbart's geometrical thinking, because the very general idea made it possible to transfer spatial concept into a nongeometric contest* (Scholz 1982, p. 422).

philosophical writings extended Herbart's considerations on psychology and epistemology (Ferreirós 2007 p. 45).

From Riemann's work a deep interrelation of mathematical, physical and philosophical thought emerges. The *Habilitationsvortrag*, in fact, contains a generalization of Gauss' differential geometry of surfaces together with a deep contribution to the question of physical space. Riemann's aim was to present a deeper analysis of the concept of space and its presupposition, trying to liberate physical explanation from conceptual limitations. He reported a series of conditions that limited the properties of space, which he called *hypothesis*, not axioms. In fact, he used the words *hypothesis* in order to refer to empirical facts, which are not known with certainty[60].

Riemann reworked in a very original way the precise structures which characterized Herbart's thought in order to give body to a new philosophical system that allowed to adequately respond to the problems posed by the sciences, in line with Helmholtz and Fechner's intention. In the philosophy of Herbart, Riemann found the possibility of keeping the realism of the sciences on one side and, on the other side, the possibility of releasing science from an immediate correspondence with reality. Science should be thought, in fact, as an increasingly precise approximation of reality, which remains unattainable. This renders it possible to build different models to describe it, because none of them could pretend to be the only one to faithfully reflect reality. From this fact, Riemann was convinced that, when we are dealing with geometry, it is absurd to assume that we are dealing simply with immediately obvious *axioms*, given truths, but instead we have to do with *hypothetical truths*, which depend on the validity of certain premises and whose truth depends only on precise *empirical* choices. For Riemann, the concept of manifold was the fundamental concept of mathematics, which allowed to give a precise mathematical formulation to scientific problems.

The failure of traditional geometry was due, according to Riemann, to the lack of a general concept of multiply extended magnitudes: *I have in the first place, therefore, set myself the task of constructing the notion of a multiply extended magnitude out of general notions of magnitude. It will follow from this that a multiply extended magnitude is capable of different measure-relations, and consequently that space is only a particular case of a triply extended magnitude* (Jost 2016, p. 31). He started from the concept of metrics, a multiply extended quantity, and proposed to construct it starting from general concepts of quantity. The aim was to show that a n-dimensional quantity could admit different metric relationships. In paragraph I, Riemann introduced the notion of an n-ply

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[60] Riemann wrote in fact: *These matters of fact are – like all matters of fact – not necessary, but only of empirical certainty; they are hypotheses. We may therefore investigate their probability, which within the limits of observation is of course very great, and inquire about the justice of their extension beyond the limits of observation, on the side both of the infinitely great and of the infinitely small* (Jost 2016, pp. 31-32).

extended magnitude. Manifolds can be continuous or discrete, as he stated: *Magnitude-notions are only possible where there is an antecedent general notion which admits of different specialisations. According as there exists among these specialisations a continuous path from one to another or not, they form a continuous or discrete manifoldness: the individual specialisations are called in the first case points, in the second case elements, of the manifoldness. Notions whose specialisations form a discrete manifoldness are so common that at least in the cultivated languages any things being given it is always possible to find a notion in which they are included. (Hence mathematicians might unhesitatingly found the theory of discrete magnitudes upon the postulate that certain given things are to be regarded as equivalent.) On the other hand, so few and far between are the occasions for forming notions whose specialisations make up a continuous manifoldness, that the only simple notions whose specialisations form a multiply extended manifoldness are the positions of perceived objects and colours. More frequent occasions for the creation and development of these notions occur first in the higher mathematic* (Jost 2016, p. 32). Riemann made here a comparison between the position of perceived objects and the space of possible colours. In both cases, as Isaac recognized (Isaac 2013), there are different determinations which may obtain, and in both cases the transition through these points is continuous, idea which was already present in Herbart's work[61]. From this excerpt, we can understand that Riemann considered the geometry of colour space as non-Euclidean. Helmholtz, as we will see in the following lines, shared Riemann's idea and succeeded in the determination of linear sequences through colour space.

After mentioning the possibility of studying the continuous varieties independently of the metric, Riemann presented the concept of one-dimensional manifold, such as a line, from which one could generate two, three, ... n dimensions. A point of a n-dimensional manifold is identified by n variables $x_1, x_2, \ldots, x_n$. Riemann faced the problem of determining the possible metric relations in a manifold, assuming that the line has a length independently of its position on the variety. Inspired by Gauss'

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[61] I would like to report Scholz's words in this regard: *Space did not exist for Herbart; instead there was a collection of spaces for which the modes of existence were completely different. His two main examples were the "line of sound" (Tonlinie) and the colour triangle with blue, red, and yellow at the corners and the mixing colors in the two-dimensional continuum in between* (Scholz 1982, p. 422). In fact, Herbart wrote in his *Psychologie als Wissenschaft*: "*Die Farben haben ebenfalls zum wenigsten zwei Dimensionen, in dem schon Roth, Blau und Gelb, paarweise genommen, eine Folge von Nüancen in gerader Linie zwischen sich einschliessen, und alle drei in der Tat ein gleichseitiges Dreieck zu bilden scheinen, in welchem jedoch weder Weiss noch Schwarz, noch selbst, wie es scheint, das reine Braun mit eingeschlossen liegt" (Colours have at least two dimensions, in which red, blue and yellow, taken in pairs, enclose a series of shades, in straight line between them, and all the three seem to form an equilateral triangle in which neither black nor white nor brown are included)* (Herbart 1824, p. 186). Herbart also made a distinction between line of tones and line of colours with the following words: *Hiezu kommt nun allerdings noch der eigenthümliche Unterschied der Tonlinie, die nach zwey Seiten ins Unendliche geht, und der Farben, die nur ein begränztes, obwohl flächenförmiges, und in so fern grösseres Continuum bilden (To this, however, comes the peculiar difference of the tone-line, which is infinite, and of the colours, which form only a limited, though planar, and in so far greater, continuum)* (Herbart 1825, p. 91).

work on the subject, Riemann found an expression for the infinitesimal element ds, given by the square root of a positive second-degree homogeneous function of the dx_i. The manifolds are called *flat* if the element ds can be expressed as[62]:

$$ds = \sqrt{\sum dx_i^{\,2}} \tag{5.1}$$

This is the case of ordinary plane and ordinary space, once introduced a system of orthogonal Cartesian coordinates.

From the definition of the infinitesimal arc element, Riemann could introduce the concept of geodesics on a manifold, using the notion of curvature defined by Gauss for surfaces. Riemann stated that the measure of curvature of a flat n-dimensional manifold is equal to zero at all points in every direction and manifolds whose curvature is constantly zero may be treated as a special case of those whose curvature is constant (Jost 2016, p. 37). In paragraph III, *Application to Space*, Riemann went on making a distinction between mere extensive relations and measure relations: *In this respect there is a real distinction between mere extensive relations, and measure-relations; in so far as in the former, where the possible cases form a discrete manifoldness, the declarations of experience are indeed not quite certain, but still not inaccurate; while in the latter, where the possible cases form a continuous manifoldness, every determination from experience remains always inaccurate: be the probability ever so great that it is nearly exact. This consideration becomes important in the extensions of these empirical determinations beyond the limits of observation to the infinitely great and infinitely small* (Jost 2016, p. 39). For what concerns the infinitely great, Riemann made a distinction between unboundedness and infinite extent, *the former belongs to the extent relations, the latter to the measure relations* (Jost 2016, p. 39). As Bottazzini pointed out, the distinction between extensive and measure relations was of fundamental importance because carried with it the possibility of a geometry different from the Euclidean one, a geometry in which not only the Euclid's fifth postulate, the parallel postulate, but also the second, i.e. "A finite straight line may be extended continuously in a straight line", have to be abandoned.

In the last section of his *Habilitationsvortrag*, Riemann went on to treat measure relations in the

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[62] In Riemann's original words: *Manifoldnesses in which, as in the Plane and in Space, the line-element may be reduced to the form* $\sqrt{\sum dx^2}$*, are therefore only a particular case of the manifoldnesses to be here investigated; they require a special name, and therefore these manifoldnesses in which the square of the line-element may be expressed as the sum of the squares of complete differentials I will call flat* (Jost 2016, p. 35).

infinitely small[63], when the independence of the bodies from the position occupied by them ceased to exist. Riemann then offered a reflection on the validity of the hypothesis of geometry in the infinitely small. He conjectured here that either the reality which underlies space form a discrete manifold, which contains within itself the foundation of a metric, or the ground of metric relations must be sought outside it, in the binding forces which act upon it.

5.3 Helmholtz: *Über die Tatsachen, die der Geometrie zugrunde liegen*

Es lagen mir gerade in der physiologischen Optik zwei Beispiele von anderen, räumlich darstellbaren und in mehrfachem Sinne veränderlichen Mannigfaltigkeiten vor, nämlich das ***System der Farben****, welches auch Riemann zitiert, und die Ausmessung des Gesichtsfeldes durch das Augenmaß. Beide zeigen gewisse fundamentale Unterschiede von dem Messungssysteme der Geometrie und regten zu einer Vergleichung an.*
(Precisely in physical optics, two examples were available to me of other manifolds which can be portrayed spatially and are variable in several respects. Namely the ***colour system****, which Riemann also cites, and the measuring out of the visual field by visual estimation. Both show certain fundamental differences from the metrical system of geometry, and stimulated comparison.)* (von Helmholtz 1977, p. 40)

In 1868, at the time he was Professor of Physiology at Heidelberg University, Helmholtz published the essay *Über die Tatsachen, die der Geometrie zugrunde liegen (On the Facts Underlying Geometry)*[64], which was sent to the Göttingen Scientific Society. This paper, together with his *Über den Ursprung und die Bedeutung der geometrischen Axiome, On the Origin and Significance of Geometrical Axioms* of 1870 and Riemann's above-mentioned work were, citing Königsberger, *epoch-making for the development of the mathematico-philosophical conceptions of the second half of the last century* (Königsberger 1906, p. 254).
In 1868 his friend Ernst Schering, Professor of mathematics at Göttingen and editor of Gauss' works, informed Helmholtz that Riemann had written his *On the Hypotheses which Lie at the Bases of*

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[63] Riemann focused on the infinitely small because, as he wrote, *The questions about the infinitely great are for the interpretation of nature useless questions. But this is not the case with the questions about the infinitely small* (Jost 2016, p. 39).

[64] Helmholtz also reflected on the foundations of geometry, which he called *facts*. Riemann and Helmholtz, called with different names the same concept. For Riemann, in fact, as we have shown above, *hypotheses* were facts, i.e. notions founded on an empirical certainty. The two works, therefore, in spite of the titles, correspond.

Geometry and that it had been published posthumously. Helmholtz, who was unaware of this, wrote to Schering: *In thanking you for sending me the two little notes about Riemann, there is one question I should like to ask. In your notice of his life I find it stated that he gave a Habilitationsvorlesung on the Hypotheses of Geometry. I have myself been occupied with this subject for the last two years in connexion with my work in physiological optics, but have not yet completed or published the work, because I hoped to make certain points more general* (Königsberger 1906, p. 254).

And, after having received a copy of Riemann's work, he wrote: *I am much obliged for the copy of Riemann's Habilitationsschrift. Herewith I send you a short account of the part of my own studies of this subject which is not covered by Riemann's work, begging you to lay it before the Royal Society to be published in the Gottinger Anzeigen (Proceedings of the Society)* (Königsberger 1906, p. 255).

Although the publication of Riemann's work had cancelled the priority of his own essay, Helmholtz found here a valid indication that the path he was following was the right one: *For the rest I must observe, that even if the publication of Riemann's work has cancelled the priority of a whole series of my own results, it is of no little importance to me, in regard to such a recondite and hitherto discredited subject, to find that so distinguished a mathematician should have though these questions worthy of his attention, and it has been to me a certain guarantee of the validity of the way, when I found him upon it as my companion* (Königsberger 1906, p. 261).

Helmholtz had independently reached the same conclusion that Riemann did by mathematical reasoning, i.e. Euclidean geometry is not the only possible description of space. The question, which can be considered as the starting point for both Riemann and Helmholtz's investigation, was not under what conditions geometrical axioms might be valid, but under what hitherto not explained conditions the knowledge of them can be achieved.

For a more complete analysis of Helmholtz's work the reader is referred to the volume *Epistemological Writings: The Paul Hertz/Moritz Schlick centenary edition of 1921, with notes and commentary by the editors* (von Helmholtz 1977, pp. 39-71), which contains notes and comments of Paul Hertz. I will limit myself to a brief description of the content, without entering into the details.

Helmholtz's starting point for his investigation were his physiological experiences, i.e. his research on spatial intuitions in the visual field, which had given him the opportunity to carry out investigations of the origin and the essence of our spatial intuitions. After his research on colour, Helmholtz realized that colour space, as defined by the colour mixing law, could not be uniform. He cited, in fact, such as Riemann did, the system of colour as an example of non-Euclidean geometry. As we will see in the next sections, Helmholtz went further; he was the first to perform empirical measurements on the psychological distances between colour experiences and succeeded in demonstrating that the geometry of colour is non-Euclidean.

In *On the Facts Underlying Geometry* Helmholtz's main aim was to find out to what extent geometric propositions have an objectively validity[65]. As Königsberger underlined, Helmholtz's work, although it can be considered as implicit in that as the work of Riemann, contains an original particular. Indeed, Helmholtz tried to define the condition under which the Pythagorean Law as generalized by Riemann would be valid and made this condition, that Riemann reported at the end of his paper, the basis of his treatment. This condition was that spatial figures should have the degree of mobility postulated in geometry (Königsberger 1906, p. 261). Helmholtz started with the consideration that all primitive measures of space are based on observation of congruence, as he declared: *My starting point was that the primary measurement of space is entirely based upon the observation of congruence; [...]. However, one cannot at all speak of congruence unless fixed bodies or point systems can be moved up to one another without changing form, and unless the congruence of two spatial magnitudes is a fact whose existence is independent of all motions. So I presupposed from the outset that the measurement of space through ascertaining congruence was possible, and set myself the task of looking for the most general analytical form of a severally extended manifold in which motions of the kind thus demanded are possible* (von Helmholtz 1977, p. 41).

It follows from these words that he was especially interested in the application to the physical space of the general theory of the n-dimensional manifolds outlined by Riemann. The geometry of a three-dimensional variety, such as the space, according to Helmholtz could be characterized by some principles, the facts underlying the geometry: he considered, such as Riemann did, space of n-dimensions as an n-extended manifold; he assumed the existence of bodies, and respectively points

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[65] I would like to report here Helmholtz's original words with which he started the work: *My investigations on spatial intuitions in the field of vision induced me also to start investigations on the question of the origin and essential nature of our general intuitions of space. The question which then forced itself upon me, and one which also obviously belongs to the domain of the exact sciences, was at first only the following: how much of the propositions of geometry has an objectively valid sense?* (von Helmholtz 1977, p. 39).

systems, which are mobile but rigid for themselves[66]; completely free mobility of fixed bodies was presupposed; he also assumed that two congruent bodies are still congruent after one of them has undergone a complete rotation about any axis of rotation.

The conclusions drawn by Helmholtz coincided with the starting point assumed by Riemann, since he came to show that *there exists a homogeneous second degree expression of the differentials, which remains unchanged during any motion of two points which are fixedly connected to each other and whose separation is vanishingly small* (von Helmholtz 1977, p. 56).

Helmholtz, however, incurred in an error, corrected afterwards by Eugenio Beltrami. In his work, in fact, Helmholtz stated that if infinite extension of space was required, no geometry other than the Euclidean was possible, whereas Beltrami showed that the geometry of Lobachevsky was also admissible, where in a space extended infinitely in all directions we can build in every part congruent figures but only one shortest line can be detected between any two points. Moreover, in such a geometry, the axiom of parallel lines no longer holds. Such a space becomes Euclidean only when the measure of spatial curvature is everywhere zero. If the measure of curvature is constant and positive, we have a spherical space, in which there are no parallels. If, on the other hand, the measure of curvature is constant and negative, we have pseudo-spherical surfaces in which the straightest lines proceed to infinity and the number of parallels is infinite.

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[66] Helmholtz here made some considerations about the colour system*: I will note that the postulate laid down above, according to which in space there exists an equation between every pair of fixedly connected points, distinguishes space from the colour system. In the latter there exists in general, on account of the law of mixtures, an equation only between five points and not less. Or in the special case in which a colour can be mixed together from two others, between these three. The case corresponding to this in space would be if all fixed body could be expanded arbitrarily in the direction of three principal axes* (Helmholtz 1977, p. 43). The five points Helmholtz was probably talking about were the origin of the coordinate axes, the three chosen primary colours and the colour resulting from the sum of the three primaries. If we associate any point with a vector of three components, we obtain, as Paul Hertz noted, 15 coordinates of the point quintuples arising from a given one by affine transformation, given by 15 equations with 12 arbitrary constants. Hertz, in fact, in order to explain Helmholtz's passage, considered an affine transformation to which is subjected the system of point corresponding to one and the same set of colours when the choice of basic colours is changed. If we consider a colour as the sum of the three primaries, we obtain three equations between four points, i.e. one point corresponds to the colour under examination and the other three correspond to the three primaries. Hertz proceeded stating that: *we should probably understand the preceding passage, that Helmholtz further wants to give all possible situations to the coordinate system of the notation space in which the points are set. The affine transformations which then enter into consideration are characterised by 12 coefficients* (von Helmholtz 1977, p. 62). Helmholtz's intention here was to underline that in the colour system we have to deal with constraints that are different from the Euclidean ones related to space. In fact, the facts underlying the geometry of colour space are facts of colour perception and Helmholtz recognized that, in order to explain them, a geometry different from the Euclidean one was required.

After a lively correspondence with Beltrami[67] and after having read Beltrami's essay on the interpretation of the geometry of Lobachevsky [68], Helmholtz correct his erroneous statement in a brief note that appeared the following year.

To conclude, I would like to report an interesting passage contained in his paper *Über den Ursprung und die Bedeutung der geometrischen Axiome (On the origin and significance of the axioms of geometry)* published in 1870, after a lecture given to the *Docentenverein* at Heidelberg, in which Helmholtz described how we can picture the appearance of a pseudo-spherical world extending in all directions: *Think of the image of the world in a convex mirror. [...] The image of a man measuring, with a measuring rod, a straight line stretching away from the mirror, would progressively shrivel up as its original moved away. But the man in the image would count up, with his equally shrivelled measuring rod, exactly the same number of centimetres as the man in actuality. [...] All congruences would match in the images, if the bodies concerned were actually laid against each other, just as in the outer world. All lines of sight in the outer world would be replaced by straight lines of sight in the mirror. In short, I do not see how the men in the mirror could bring it out that their bodies were not fixed bodies and their experience [not] good examples of the correctness of Euclid's axioms. But if they could look out into our world, as we look into theirs, without being able to cross the boundary, then they would have to pronounce our world to be the image of a convex mirror, and speak of us just as we of them. [...] Now Beltrami's mapping of pseudospherical space onto a whole sphere of Euclidean space is of a quite similar kind, except that the background surface is not a plane but a sphere, and that the proportion in which the images contract, as they approach the surface of the sphere, has a different mathematical expression. [...]*

We can deduce how, the objects of a pseudospherical world would appear to an observer, whose visual estimation and spatial experiences had, exactly as ours, been developed in flat space, if he could enter such a world. Such an observer would continue to see the lines of light rays, or the lines of sight of his eyes, as straight lines like those existing in flat space, and like they actually are in the spherical image of pseudospherical space. The visual image of the objects in pseudospherical space would therefore give him the same impression as if he were at the centre of Beltrami's spherical image (von Helmholtz 1977, pp. 20-21).

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[67] Five letters written between 1869 and 1878 by Beltrami to Helmholtz are preserved at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften* (Nachlass Hermann von Helmholtz, Briefe 36 Eugenio Beltrami).

[68] In 1868, at the time he was Professor at the University of Bologna, Beltrami wrote a paper with the title *Saggio di interpretazione della geometria non-euclidea (Essay on the interpretation of non-Euclidean geometry),* in which he presented a model of the hyperbolic geometry of Bolyai and Lobachevsky.

5.4 Toward the definition of the line element of colour space: Helmholtz's papers of 1891 and 1892

One of Helmholtz's major contribution to the field of colour was the first determination of a line element in dichromatic and trichromatic colour spaces. In fact, the attempt to extend the Weber-Fechner psychophysical law to complex of more than one dimension, had led Helmholtz to the quantitative determination of the nature of a colour sensation in dichromatic and trichromatic eyes by two and three independent variables respectively. These works date back to 1891 and 1892[69]; at that time, indeed, he was harvesting the fruits of a life-long interest in mathematics and physiological optics and collecting the pivotal results obtained in collaboration with his assistants König, Dieterici and Brodhun. I would like to present the content of the three papers containing this precious formulation, trying to follow the same path that Helmholtz trodden and to underline also the key role played by his collaborators' investigations.

5.5 *Versuch einer erweiterten Anwendung des Fechnerschen Gesetzes im Farbensystem*

Helmholtz had prepared the study by previously analysing the validity of the Weber-Fechner law. In this first paper on the subject, *Versuch einer erweiterten Anwendung des Fechnerschen Gesetzes im Farbensystem (An attempt to extend the application of Fechner's law in the colour system)*, published in 1891, Helmholtz tried at first to generalize Weber-Fechner law to two-dimensional colour space of dichromats, basing the all treatment on Brodhun's results (see chapter 4). He wrote indeed: *Für die Verwerthung der Beobachtungen über die kleinsten erkennbaren Unterschiede der Farbentöne des Spectrum liegen bisher ausreichende Beobachtungen nur für das dichromatische Auge des Herrn Brodhun vor. [...] Denn die Erweiterung des FECHNERschen Gesetzes auf ein Gebiet von mehreren Dimensionen ja leichter vollziehen lassen, als für drei [...], dass sowohl die Farbengleichung als auch namentlich die Vergleiche der Helligkeit von Dichromaten sicherer und schärfer vollzogen werden, als von Trichromaten (So far, in order to investigate the smallest perceptible differences of colour tones of the spectrum, the only available observations are those related to the dichromatic eye of Mr. Brodhun. [...] In this case, it is easier to accomplish the extension of the FECHNER's law to*

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[69] The original manuscript version of these papers is preserved at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften*, Berlin.

a complex of several dimensions than for three dimensions, because both colour matching and especially brightness comparisons for dichromats are performed in a safer and sharper way than for trichromats) (von Helmholtz 1891, p. 11).

Brodhun, as reported in chapter 4, was green-blind and starting from 1887, when he submitted his doctoral dissertation, devoted himself to the study of colour perception, collecting a series of precious results, which Helmholtz could exploit to pursue his aim.

At the beginning of the paper, Helmholtz explained the reasons that led him to explore this new field: *die bisher von E. H. WEBER, FECHNER und ihren Nachfolgern angestellten Messungen beziehen sich, so viel ich weiß, alle nur auf Veränderungen, die ausschießlich in einer einzigen Richtung vorgingen. Das Gebiet der Farbenempfindungen bietet die Gelegenheit solche Studien auch für eine nach drei Dimensionen sich erstreckende Mannigfaltigkeit zu machen (All the measurements hitherto carried out by E. H. WEBER, FECHNER and their successors, as far as I know, refer only to changes which proceeded exclusively in a single direction. The field of colour sensations offers the opportunity to carry on such studies also for a manifold extending in three dimensions)* (von Helmholtz 1891, p. 2). Helmholtz was aware that the available empirical material to define laws in this unexplored field was uncomplete at that time and based his work on researches undertaken by his collaborators König, Dieterici and Brodhun. He designated his communications as hypotheses, which required a more precise test, but he was convinced that such an attempt must be made in order to obtain the preliminary orientations in this new field.

The first investigation was conducted by means of the Maxwell's colour top and Helmholtz summarized the obtain results in paragraph *Eigene Versuche mit Farbenscheiben (Own experiments with colour discs)*. If the brightness of two differently coloured lights has to be compared, a point could be detected at which the perceptible difference of colour reaches a minimum of clearness. Helmholtz's aim was that of ascertaining this point of least recognizable difference for a series of mixed colours, obtained from the same colour elements. In order to achieve this goal, he made use of colour discs to compare the luminosity of two different colours, one darker than the other. He adjusted the sectors in order to let the darker colour unchanged while the other one was subjected to small changes in luminosity and composition by introducing narrow black sections, which rendered it a little darker until the boundaries of the rings, where these colours appeared mostly at the same luminosity. The required ratios were then recorded. I would like to offer a brief description of the method used by Helmholtz. The colour mixtures filled five concentric rings on the disc. The discs were cut from papers painted of colours, as saturated as possible, and they were split along a radius according to Maxwell's method. At the top of Fig. 5.1, each slice is reproduced with a slightly smaller radius to show how they lied between each other (as Helmholtz pointed out, for the experiment, the discs were bounded by congruent circle lines of equal radius). The disc of slightly lighter colour R and the black one had a simple radial incision, the former denoted as bb and the latter as dd. On the other hand, the disc of darker colour B had a boundary line with crenelated protrusions between aa and cc. The radius was denoted by aa, whereas cc indicated a parallel line to this radius, so that the angular values of the arcs between aa and cc became larger for the inner circles than for the outer one. The black disc was also set in such a way that its boundary line *dd* was not exactly in the position of a radius, but it was located parallel to the radius aa lying next to it; thus the black stripe, which looked like the crenelated cut-outs, maintained the same width everywhere and the height of the battlements decreased everywhere by the same fraction.

The position of the radius b could be arbitrarily shifted to almost the entire circumference, with the exception of the strip taken by the pinnacles, so that the two colours R and B could appear as pure as possible. When a small arc of the colour B was added to the colour mixture determined by the two sectors lying between the radii bb and aa in the protrusions of field B reaching cc, an equal arc of colour R had to be removed. On the other hand, in the cut-out of the battlements, only a narrow strip *aa*-*dd* of *R* was taken away by black. If we consider a colour B a little darker, a luminosity lost would occur between aa and cc, where it would appear instead of R. In order to compensate this lost, part of the colour R should be taken away.

Once detected a fixed brightness ratio for the two colours under examination, also a fixed relationship between the two small arcs *ac* and *ad* could be found, which produced, in the rings of different colour tone, always the same brightness, independently from the ratio between the two major sectors R and B. Helmholtz, then, supposed that lights of the same luminosity but of different colour tone could be expressed by the following formula, mixing two primary colours:

$$H = A \cdot x + B \cdot y \tag{5.2}$$

where A and B are constants, H is a function of the luminosity, x and y are quanta of two chosen elementary colours. Thus, for an adjacent colour in the series of equal bright mixture, he could write:

$$0 = A \cdot dx + B \cdot dy \tag{5.3}$$

The relationship between dx and dy was independent from the value of x and y, and the luminosity was fixed. Helmholtz reported then the obtained results considering mixtures of green and red, blue and red and blue and green, founding the following relation for the brightness: green > red > blue (von Helmholtz 1891, p. 7).

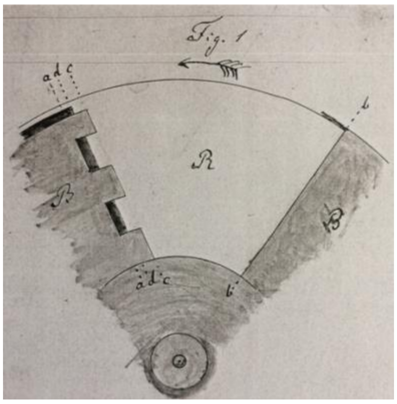

Fig. 5.1
Helmholtz's original sketch of the colour top used for luminosity comparison preserved at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften* (Nachlass Hermann von Helmholtz, 568 Versuch einer erweiterten Anwendung des Fechnerschen Gesetzes in Farbensystem)

From this experiment, Helmholtz found out that the effect of an increment of any colour upon the brightness was attenuated by the amount of the same colour already present in a determined mixture. Then, if a series of different colours of equal brightness were determined, starting from a highly saturated colour, the total light involved in these mixtures could not remain unchanged. Taken as starting point the highly saturated red, the brightness should be less diminished by subtracting a small quantity of red than strengthen by addition of an equal quantity of blue. The comparison in brightness between two approximately equal and highly saturated colours appeared to be different from that between very differently coloured fields. From a long series of experiments using the colour top, Helmholtz came to the conclusion that the recognisability of low gradation of the intensity of coloured light was less affected by the simultaneous presence of a second and dissimilar colour in the field than it was by the presence of an equally bright quantity of the same colour.

For the next investigation, Helmholtz based his considerations on Brodhun's results obtained by means of the top. The subject of study was the comparison of the brightness of very different colours (*Vergleichung der Helligkeit sehr verschiedener Farben*). Helmholtz stated that for his normal eye, it was difficult to pursue the aim but he was aware that subjects affected by dichromacy could be able to obtain a certain degree of accuracy. Brodhun, indeed, was able to obtain brightness comparisons directly by using the colour top. He made use of coloured paper discs, two painted red (dark red and light red) and two painted blue (dark blue and light blue), one painted grey and another one painted black in order to build colour equations[70]. Comparing the brightness of the two varieties of red and blue with that of grey, he obtained the following results:

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[70] Brodhun's manuscript colour equations contained in Helmholtz's notebook are preserved at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften* (see Appendix C).

$$Light\ red = \frac{160}{360} Grey$$

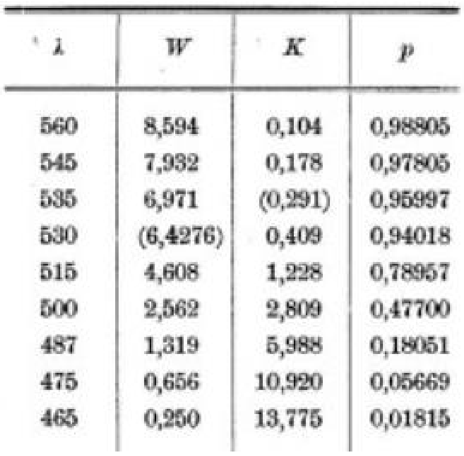
$$Dark\ red = \frac{110}{360} Grey$$

$$Light\ blue = \frac{60}{360} Grey$$

$$Dark\ blue = \frac{25}{360} Grey$$

Helmholtz proceeded reporting Brodhun's linear relation for the intensity obtained from his measurements of the intensity curves and contained in Brodhun's dissertation of 1887 (see chapter 4, section 4.7): $J = 1{,}018\, W + 0{,}03915\, K$. Helmholtz defined W and K as the quanta of the chosen primaries, calculated according to a unit, which set them equal in saturation and baptized these two quantities as *Einheiten von gleichem Farbenwerth* (units of the same colour); he calculated that *W* was about 26 times brighter than *K*. Helmholtz then continued stating that: *Wenn wir dagegen mit w und k Einheiten bezeichnen, die bei gleicher Anzahl größeren Lichtstärken gleiche Helligkeit geben [...]; so ergiebt sich, daß das Violett k 26 mal größere Farbenintensität, als die wahrscheinliche gelbliche Farbe w besitz (If, on the other hand, we denote by w and k units which give equal brightness with the same amount of intensity of light, we find that the violet k has 26 times greater colour intensity than the probably yellowish colour w)* (von Helmholtz 1891, p. 10).

The next step was that of analysing colour differences of spectral colours (*Farbenunterschiede der Spetcralfarben*) for subjects affected by colour blindness, in order to extend the Weber-Fechner law to a complex of two dimensions, such as the colour system of dichromats. He examined colours of the same brightness but of different wavelengths, exposing at first the results obtained by Brodhun between 1885 and 1886 and published in the *Verhandlungen der physiologischen Gesellschaft zu Berlin*, on the sensitivity of the eye to wavelength differences. The method used by Brodhun was based on the same principle applied in the experiment with the top; the error of each setting was read on the colour mixing apparatus. The average error of the settings was calculated as a measure of the uncertainty of the comparison. I would like to report in Table 5.1 the obtained results for the wavelength range 560-465 nm.

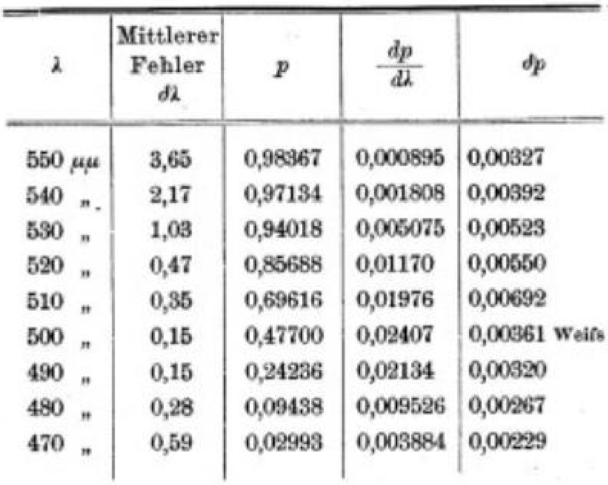


λ	W	K	p
560	8,594	0,104	0,98805
545	7,932	0,178	0,97805
535	6,971	(0,291)	0,95997
530	(6,4276)	0,409	0,94018
515	4,608	1,228	0,78957
500	2,562	2,809	0,47700
487	1,319	5,988	0,18051
475	0,656	10,920	0,05669
465	0,250	13,775	0,01815

Table 5.1
W and K values obtained by Brodhun in the wavelength range 560-465 nm (von Helmholtz 1891, p. 12)

Table 5.1 shows the wavelength range for which he specified the mixing ratio of the warm and cold colours (W and K) in units of equal colouring power (*nach Einheiten von gleicher färbender Kraft*) and $p = \frac{W}{W+K}$.

In the following table are reported also average error, $\delta\lambda$, and the corresponding error in the value of p, $\delta p = \frac{dp}{d\lambda}\delta\lambda$.



λ	Mittlerer Fehler $d\lambda$	p	$\frac{dp}{d\lambda}$	dp
550 $\mu\mu$	3,65	0,98367	0,000895	0,00327
540 "	2,17	0,97134	0,001808	0,00392
530 "	1,03	0,94018	0,005075	0,00523
520 "	0,47	0,85688	0,01170	0,00550
510 "	0,35	0,69616	0,01976	0,00692
500 "	0,15	0,47700	0,02407	0,00361 Weifs
490 "	0,15	0,24236	0,02134	0,00320
480 "	0,28	0,09438	0,009526	0,00267
470 "	0,59	0,02993	0,003884	0,00229

Table 5.2
Brodhun's results in the wavelength range 550-470 nm with indication of $\delta\lambda$ and the corresponding error δp (von Helmholtz 1891, p. 13)

The numbers in column p refer to light units of the same colouring power. Helmholtz recognized at this point that, in order to render them useful for the comparison of the difference of sensitivity, the numbers had to be converted in units of equal brightness. Therefore, maintaining unchanged the value of W, the value of K should be replaced by:

$$\frac{1}{n} \cdot K; where\ n = \frac{a}{b} = 26$$

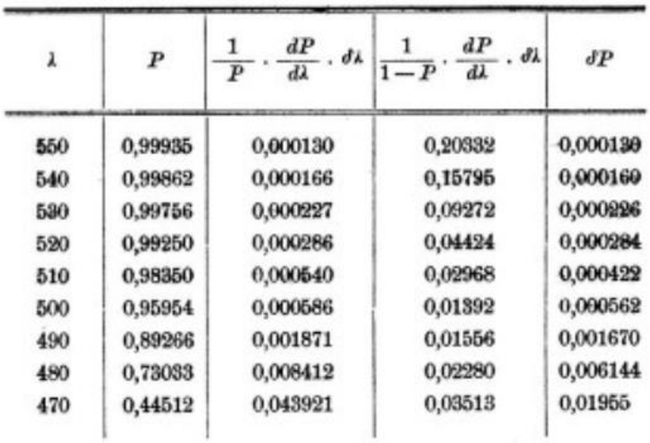
where a and b are constants introduced first by Brodhun (the reader is referred to Brodhun's dissertation, chapter 4, section 4.7).

In this way, the new value of P could be obtained:

$$P = \frac{W}{W + \frac{1}{n}K} \tag{5.4}$$

from which also the value of δP, that represents the average error in the discrimination of the two coloured quanta (*den mittleren Fehler in der Bestimmung der beiden Farbenquanta*), could be calculated, and the proportion to the total amount of light of both colours was set equal to:

$$P + (1 - P) = 1 \tag{5.5}$$



λ	P	$\frac{1}{P} \cdot \frac{dP}{d\lambda} \cdot \delta\lambda$	$\frac{1}{1-P} \cdot \frac{dP}{d\lambda} \cdot \delta\lambda$	δP
550	0,99935	0,000130	0,20332	0,000130
540	0,99862	0,000166	0,15795	0,000166
530	0,99756	0,000227	0,09272	0,000226
520	0,99250	0,000286	0,04424	0,000284
510	0,98350	0,000540	0,02968	0,000422
500	0,95954	0,000586	0,01392	0,000562
490	0,89266	0,001871	0,01556	0,001670
480	0,73033	0,008412	0,02280	0,006144
470	0,44512	0,043921	0,03513	0,01955

Table 5.3

Brodhun's results obtained in the wavelength range 550-470 nm with indication of P, $\frac{\delta P}{P}$, $\frac{\delta P}{1-P}$, and the corresponding error δP (von Helmholtz 1891, p. 14)

From the values of P and δP, Helmholtz calculated the ratio $\frac{\delta P}{P}$, i.e. the least perceptual difference of the element of the warm colour for the same brightness and colouring power, and $\frac{\delta P}{1-P}$, i.e. the least perceptual difference of the element of the cold colour for the same brightness and colouring power for the considered wavelength range.

It is worth to specify the luminosity condition, at which all the measurements conducted by Brodhun, and reported by Helmholtz, were obtained. Referring to the paper of König and Brodhun of 1888 (see chapter 4), Helmholtz stated that the chosen intensity value could correspond to a value of 100 in König's investigation. It was always sought to achieve, in fact, a light intensity which was above the limit at which the Purkinje's phenomenon[71] occurred.

Helmholtz then made a comparison between the values of δP and those of $\frac{\delta r}{r}$, when $r = 1$, obtained by König and Brodhun for the intensity under consideration and could state that the obtained values of δP were much smaller than the errors aroused in an intensity comparison. I would like to remark that this was in agreement with Maxwell's prediction. Comparing quantitatively the tone and the intensity discrimination, in fact, Helmholtz verified that the human eye is more sensitive to colour tone differences than to variations in intensity.

Analysing the results, Helmholtz could conclude that, with the simultaneous presence of a strongly different colour in the field, the discrimination of small level gradations of coloured light intensity was affected much less than it was in the presence of an equally luminous colour of the same colour (*Durch das gleichzeitige Vorhandensein einer zweiten stark abweichenden Farben im Felde, wird die Erkennbarkeit kleiner Abstufungen von Intensitätsstufen farbigen Lichtes viel weniger beeinträchtigt, als durch das Vorhandensein eines gleich hellen Quantum derselben Farbe*). In other words, it is more difficult to recognize differences in colour intensity in the presence of a coloured light very different to the colour of the field, compared to the presence of an equally luminous colour quantum of the same colour of the field.

Equal values of δP should correspond with equal gradations of the yellow-blue elementary effect of the colour series. As he noticed, the values of the average error δP showed a high sensitivity for gradations of what he called an *intense yellow elementary effect* (*Elementarwirkung*) and a lower sensitivity for gradations of an *intense blue elementary effect*. On the other hand, considering the sensations of a dichromic colour series of two primary colours corresponding to the end colours of the spectrum, Helmholtz referred to the ratios $\frac{\delta P}{P}$ and $\frac{\delta P}{1-P}$ for making a comparison with the Weber-Fechner's law, whose values are reported in columns III and IV of Table 5.3. It is noteworthy to report Helmholtz's consideration on this point: *die Abstufungen des Violett sind in der That alle groß genug, um wahrgenommen zu werden, oder kommen der wahrscheinlichen Grenze wenigstens ganz nahe, während die Abstufungen der Roth, soweit die Beobachtungen reichen, zu kleine Verhältnisse des* $\frac{\delta P}{P}$ *bieten, mit Ausnahme der letzten Zahl. Leider ist diese die einzige, wobei P< ½, d. h. der violette Antheil der Farbe heller ist als der gelbe (beziehlich rothe). Für den Rest des Spectrum fehlen*

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[71] See footnote 32.

die Beobachtungen der Mischungsverhältnisse der beiden Farben, da die Mengen des eingemischten Roth hier wohl zu schwach für eine sichere Bestimmung waren (Gradations of the violet are, in fact, all big enough to be perceived, or at least quite close to the limit, while gradations of the red, as far as the observations are concerned, offer too small ratios, except the last number. Unfortunately, this is the only one where P <½, i.e. the violet component of the colour is lighter than the yellow (red). For the rest of spectrum, the observations of the mixing ratios of the two colours are lacking, since the quantities of the mixed red were probably too weak for a reliable determination) (von Helmholtz 1891, p. 17). From Table 5.3, we can see that the lowest value of the least perceptual difference of the element of the *warm* colour was reached at wavelength 550 nm and it started increasing toward the smaller wavelengths; whereas, the value of the least perceptual difference of the element of the *cold* colour reached the highest value at wavelength 550 nm, assuming the lowest values for the wavelength range 520-470 nm.

Helmholtz recognized that the values of $\frac{\delta P}{1-P}$ for the linear scale of just noticeable difference steps of the violet were not constant, as they should be, at least approximately, according to Fechner's law for the just noticeable differences in intensity of an isolated colour. This could be due to the fact, suggested Helmholtz, that, at the violet end of the spectrum, whitish fluorescent light of the retina came into play and has to be added to the incident violet light, as König and Dieterici had quantitatively estimated for both trichromatic and dichromatic eyes. Moreover, also the contributions should be considered, which the internal excitation of the retina, according to Fechner, add to the excitement of the external light, and which must be taken into account for the calculation of the linear scale of just noticeable difference steps (*Empfindungsstufen*) in very small external quantities of light. Indeed, in the more saturated colours of the spectrum, traces of other mixed colours are small enough that their perceptibility can be appreciably affected by the intrinsic light. Thus, it could be explained that where the mixtures of violet in the warm background colour are very small, the sensitivity to colour tone changes is not quite as great as expected according to the original form of Weber-Fechner law.

At this point, Helmholtz wrote, it did not seem hopeless to use the Fechner law as a model and to try to apply it everywhere in the spectrum; the size of the linear scale of just noticeable difference step for each primary colour depends only on the quantity of similar colours, but can be considered as independent of the quantities of other primary colours covering the field at the same time. Before proceeding in the determination of the form of the *Empfindungsunterschiede*[72] (differences in

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[72] It is to be noticed that Helmholtz used the word *Empfindungsunterschied* (difference in sensation) almost interchangeably with the word *Unterschiedsempfindung* (sensation of difference).

sensation) dE, i.e. the line element in colour space, Helmholtz made another key assumption. In fact, in their experiments, at the border of two compared fields under examination there were always gradations of two primary colours, and indeed, on one hand, an increase in the determination of the just noticeable difference steps in brightness of a constant colour, and on the other hand an increase in the determination of the just noticeable difference steps in colour tone for the same brightness. In order to define the *Empfindungsunterschied* dE, Helmholtz stated that: *Wenn man die mögliche Form eines solchen Gesetzes überlegt, so ist klar, dass die Empfindungsunterschied an der Grenze zweier Felder nur dann ganz verschwinden kann, wenn keine von den drei Grundempfindungen daselbst eine Abstufung zeigt (When considering the possible form of such a law, it is clear that the difference in sensation at the boundary of two fields can disappear only if none of the three basic sensations shows a gradation.)* (von Helmholtz 1891, p. 18). Therefore, he recognized that, denoting by dE_1, dE_2 and dE_3 the differences in sensations, then dE must be a function of the latter such that $dE = 0$ could occur only if $dE_1 = dE_2 = dE_3 = 0$ at the same time. Helmholtz considered here the case of normal colour vision, in which three independent variables had to be taken into consideration. The valued of dE can be zero only if dE_1, dE_2 and dE_3 are all zero because, as Stiles underlined (Stiles 1972, p. 3), a sensation difference in one fundamental colour cannot cancel out a sensation difference in another colour.

After an analysis of the conditions at which the above-mentioned equality could be reached, he proposed an expression for the differences in sensation, starting from the one-dimensional equation, which he introduced first in his *Handbuch*:

$$dE = \frac{k \cdot dx}{(a\ + x) \cdot F} \tag{5.6}$$

for an increment dx of colour denoted by x, where F is a function of the brightness. The constant denoted by a should assume a smaller value for the violet and must be taken into account only in the case of low intensities, whereas for higher intensity it could be neglected. The constant indicated by k corresponds to the one appearing in the Weber-Fechner law. Helmholtz, then, considering the three dimensional colour space, defined F as a homogenous function of x, y, z, the three fundamental colours, and defined the differences in sensation in the following way:

$$dE_1 = \frac{k \cdot dx}{(a\ + x) \cdot [1 + lx + my + nz]} \tag{5.7}$$

$$dE_2 = \frac{k \cdot dy}{(\beta \ + y) \cdot [1 + lx + my + nz]} \tag{5.8}$$

$$dE_3 = \frac{k \cdot dz}{(\gamma \ + z) \cdot [1 + lx + my + nz]} \tag{5.9}$$

where *lx, my* and *nz* represents equal bright quanta of the three fundamental colours and *l*, *m* and *n* indicate the *Blendungscoefficienten*, i.e. glare coefficients. For each fundamental colour, for the same brightness, equal amounts of dE must correspond to equal ratios $\frac{dx}{x}$, $\frac{dy}{y}$ and $\frac{dz}{z}$, that is the case in which equal amounts of *lx*, *my* and *nz* correspond to equal brightness, at sufficiently high levels of brightness[73]. At this point, Helmholtz gave a definition of what he called *Helligkeit* (brightness), in terms of *Unterschiedsempfindlichkeit* (difference in sensitivity). It is worth reporting Helmholtz's original words, in order to better understand his future considerations: *Gleich hell sind differente Farben, welche gleiche Blendung und gleiche Unterschiedsempfindlichkeit haben. Letztre entscheidet namentlich bei niederen Lichtstärken, wo die Blendung aufhört. Auf Feldern von gleicher Unterschiedsempfindlichkeit kann man zarte Schatten, von Modulierung der Oberfläche herrührend, und kleine Objecte gleich gut unterscheiden (Colours have the same brightness when they have the same glare and possess the same differences in sensitivity. The latter decides, especially at low light intensities, where the glare stops. On fields of equal differences in sensitivity, one can distinguish delicate shadows, originating from modulation of the surface, and equally small objects.)* (von Helmholtz 1891, p. 21).

I would like to report here the first application of the line element made by Helmholtz, which is highly significant. In paragraph *Ähnlichste Farben (Most similar colours)*, he determined pairs of neighbouring colours of the greatest similarity. His purpose was that of extending the Weber-Fechner law to a complex of two dimensions considering two fundamental colours, whose *quanta* were denoted by x and y. He could write, according to the Weber-Fechner law:

$$dE_1 = k \cdot \frac{dx}{x} \tag{5.10}$$

$$dE_2 = k \cdot \frac{dy}{y} \tag{5.11}$$

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[73] At such high levels, the element dE predicts that, for pure brightness discrimination, when $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$, the Weber fraction will have the same value for all colours, according to König and Brodhun's results (see chapter 4).

and, therefore:

$$dE = k \cdot \sqrt{\left(\frac{dx}{x}\right)^2 + \left(\frac{dy}{y}\right)^2} \tag{5.12}$$

He envisaged a comparison of one colour, having fixed colour tone and fixed intensity with another having a slightly different colour tone and variable intensity. He denoted colour tone and intensity by *p* and *r* respectively. For what concerns the first colour, he could write:

$$x = r \cdot p \tag{5.13}$$

$$y = r \cdot (1 - p) \tag{5.14}$$

and for the second colour we have:

$$x = (r + dr) \cdot (p + dp) \tag{5.15}$$

$$y = (r + dr) \cdot (1 - p - dp) \tag{5.16}$$

The line element allowed him to calculate the intensity of the second colour at which the quantity dE^2 is minimal, and the two colours then are said to have the *greatest similarity*. Maintaining, in fact, the colour tone unchanged, Helmholtz could find the value of *dr* that minimized dE[74], obtaining an equation of an equilateral hyperbola: *Wenn wir in einem rechtwinkeligen Coordinatensystem die Quanta der Farbe x als Ordinaten und die Quanta von y als Abscissen auftragen, so stellt die Gleichung [*$log(x \cdot y) = Const.$*] eine Curve dar, in denen die Farben kleinsten Unterschiedes neben einander liegen. Diese Curve ist eine gleichseitige Hyperbel, deren Asymptoten in der Entfernung sich den Coordinataxen anschließen (If we plot the quanta of the colour x along the ordinate and the quanta of y along the abscissa in a rectangular coordinate system, then the equation [*$log(x \cdot y) = Const.$*] represents a curve in which the colours of the smallest difference lie next to each other. This curve is an equilateral hyperbola whose asymptotes, in the distance, follow the coordinate axes.)* (von Helmholtz 1891, p. 24). Colours of the greatest similarity, therefore, are located along a curve, instead

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[74] As Stiles pointed out, this procedure corresponds with a colorimetric method, called step-by step method, used to determine a relative luminous efficiency curve, and neighbouring colours of the greatest similarity could also be described as colours of the same brightness, or of the same small-step brightness (Stiles 1972, p. 4).

of a straight line; this was a valuable clue for the non-Euclidean interpretation of the geometry of colour space. On the other hand, explained Helmholtz, if we consider very different colours, as in the case of Brodhun's experiment with the colour top, the mixed colours lie in a straight line.

Before concluding the paper, Helmholtz developed the general bases of the calculation of the line element for a dichromatic eye, on the basis of König and Bodhun's results contained in their paper of 1888. He started defying:

$$dE_x = \frac{dx}{x} \cdot X \tag{5.17}$$

$$dE_y = \frac{dy}{y} \cdot Y \tag{5.18}$$

where x and y denote the intensities of two fundamental colours; X and Y represent function of x and y respectively and, within the normal range of Weber-Fechner law, are almost constant, but they rise for very low and very high light intensities. He went further expressing x and y in terms of Brodhun's colour elements, what he called *Quanta der Endfarben des Spectrum*, quanta of the end colours of the spectrum, which in turn were expressed in terms of two variables, *P*, which indicates the mixing ratio, and *R*, which denotes the light intensity. Helmholtz, then, came back to the notation used by Brodhun and König for the determination of the just noticeable difference in intensity for different wavelengths, to determine the value of the constant k of the Weber-Fechner law (k should be > 1.8238) and was able to calculate the differential threshold for every wavelength in the range 550-470 nm[75].

I would like at this point to introduce Helmholtz's remarks and conclusions. First, analysing the formula for the just noticeable difference, he recognized that colour differences disappear at very low and very high (if we consider the glare factor) light intensity. He then concluded his work expressing a key result, i.e. lines expressing the smallest colour differences (*kürzeste Linien im Farbenfelde*) do not follow the lines of the same colour mixtures and, therefore, between colours of on the one hand higher, and on the other hand lower brightness the same mixing ratios will not always be detected. These conclusions were of fundamental importance because they opened the door to the successive works, which contained the first determination of line element in trichromatic colour space.

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[75] The all mathematical treatment can be found from page 24, to page 28 of Helmholtz's paper.

5.6 The papers of 1892: the definition of the line element

Helmholtz's papers of 1892, which contain the definition of the line element in trichromatic colour space, were both published in *Zeitschrift für Psychologie and Physiologie der Sinnesorgane* with the titles *Versuch, das psychophysische Gesetz auf die Farbenunterschiede trichromatischer Augen anzuwenden (Attempt to apply the psychophysical law to colour differences of trichromatic eyes)* and *Kürzeste Linien im Farbensystem (The shortest lines in the colour system)* and were communicated to the Academy on December 17, 1891. Helmholtz's considerations and definitions contained in this two papers are of crucial importance because they opened the doors to all subsequent investigations on the subject.

5.6.1 *Versuch, das psychophysische Gesetz auf die Farbenunterschiede trichromatischer Augen anzuwenden*

I would like to start with an analysis of the content of the first of the two papers. Here, Helmholtz returned to the conclusions laid down by himself and Riemann, i.e. that all the characteristics of colour space can be derived from the fact that the value of the distances between two neighbouring points could be expressed by the corresponding increments of the coordinates in analogy with the rigid body treatment, which requires that the interval between any two points of the body are completely defined by their terminal points, and remains the same in all possible rotations or displacements. Starting from the fact that each colour may be represented by the combination of the corresponding measured quantity of the three chosen fundamental colours, he could write the formula for the *Empfindungsunterschied* dE, expressed in terms of the corresponding degrees of distinctness (*Deutlichkeitsgrade*) of the individual sensations dE_1, dE_2 and dE_3, maintaining the same notation of the previous work, in the following form:

$$dE^2 = dE_1^2 + dE_2^2 + dE_3^2 \tag{5.19}$$

Therefore, he could write, according to the Weber-Fechner law and repeating the procedure described in the paper of 1891:

$$dE_1 = k \cdot \frac{dx}{x} \tag{5.20}$$

$$dE_2 = k \cdot \frac{dy}{y} \tag{5.21}$$

$$dE_3 = k \cdot \frac{dz}{z} \tag{5.22}$$

$$dE = k \cdot \sqrt{\left(\frac{dx}{x}\right)^2 + \left(\frac{dy}{y}\right)^2 + \left(\frac{dz}{z}\right)^2} \tag{5.23}$$

In order to switch from colour to brightness discrimination, he first applied the equation above to the case in which only the intensity of two colours of the same quality (Helmholtz used the German word *Qualität* to indicate colour tone) had to be compared, i.e. he considered the case in which the intensities of the three fundamental colours exceed that of the others by an equal fraction, that he denoted by ε. This is what Stiles defined condition of small-step brightness (Stiles 1972). Then, he could write: $dx = \varepsilon \cdot x$; $dy = \varepsilon \cdot y$; $dz = \varepsilon \cdot z$, obtaining the following expression for dE:

$$dE = k \cdot \varepsilon \cdot \sqrt{3} \tag{5.24}$$

The first step was that of applying the formula to determine what he called *ähnlichster Farbenpaare* (the most similar colour pairs). Helmholtz referred, here, to a pair of composed colours, one of which contains the quanta of the fundamental colours x, y, z, the other is composed of $(x + dx)$, $(y + dy)$ and $(z + dz)$, and the light intensity of the first colour can be increased in the ratio $1{:}\,(1 + \varepsilon)$, so that its components become: $x(1 + \varepsilon)$, $y(1 + \varepsilon)$, $z(1 + \varepsilon)$. The equation of the difference in sensation could be written in the following form:

$$\frac{dE^2}{k^2} = \left(\frac{dx - \varepsilon x}{x}\right)^2 + \left(\frac{dy - \varepsilon y}{y}\right)^2 + \left(\frac{dz - \varepsilon z}{z}\right)^2 \tag{5.25}$$

He found the value of ε which minimized the equation, and once found it, the minimum value of dE^2 could be reached, which satisfied the condition of small-step brightness:

$$\frac{dE^2}{k^2} = \frac{1}{3}\left[\left(\frac{dx}{x} - \frac{dy}{y}\right)^2 + \left(\frac{dy}{y} - \frac{dz}{z}\right)^2 + \left(\frac{dz}{z} - \frac{dx}{x}\right)^2\right] \tag{5.26}$$

These results were needed to Helmholtz to explain König and Dieterici's measurements of the smallest perceptible wavelength differences reported in their paper of 1884 (see chapter 4).
The observed wavelength differences, $\delta\lambda$, was given by the above expression for dE^2 on inserting the values appropriate to the spectral colours for x, y and z and their variations, and assuming for dE a small fixed value:

$$\frac{dE}{k} = \frac{\delta\lambda}{\sqrt{3}}\sqrt{\left[\left(\frac{1}{x}\cdot\frac{dx}{d\lambda} - \frac{1}{y}\cdot\frac{dy}{d\lambda}\right)^2 + \left(\frac{1}{y}\cdot\frac{dy}{d\lambda} - \frac{1}{2}\cdot\frac{dz}{d\lambda}\right)^2 + \left(\frac{1}{z}\cdot\frac{dz}{d\lambda} - \frac{1}{x}\cdot\frac{dx}{d\lambda}\right)^2\right]} \tag{5.27}$$

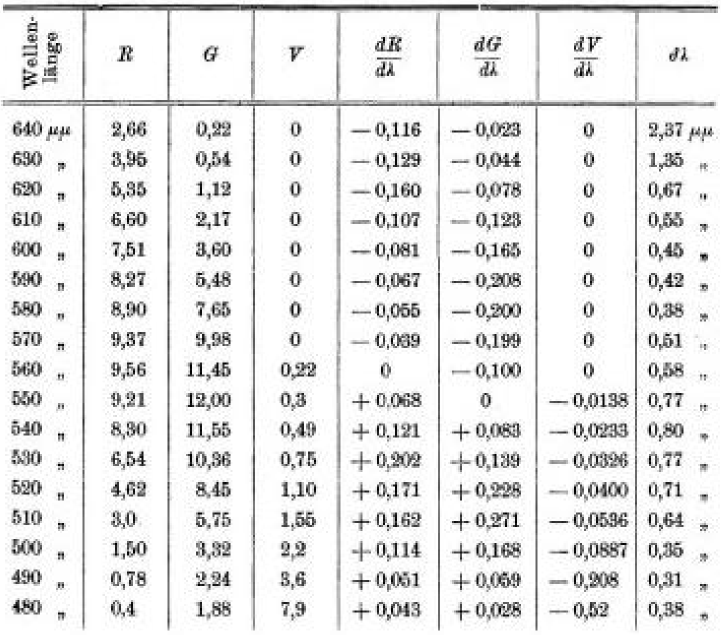
The next step was that of expressing x, y and z in terms of the *Elementarfarben*, the elementary colours used by König and Dieterici:

$$x = a_1 \cdot R + b_1 \cdot G + c_1 \cdot V \tag{5.28}$$

$$y = a_2 \cdot R + b_2 \cdot G + c_2 \cdot V \tag{5.29}$$

$$z = a_3 \cdot R + b_3 \cdot G + c_3 \cdot V \tag{5.30}$$

The choice of the coefficients, according to Young's theory, was limited only to the fact that the values of R, G, V belonging to the spectral colours must not yield negative values of x, y and z. Once determined the values of the coefficients, Helmholtz reported in Table 5.4 the data for the calculation, in which $\delta\lambda$ indicate the average errors obtained by König, calculated for ten attempts to set the Helmholtz colour mixer to the same colours.

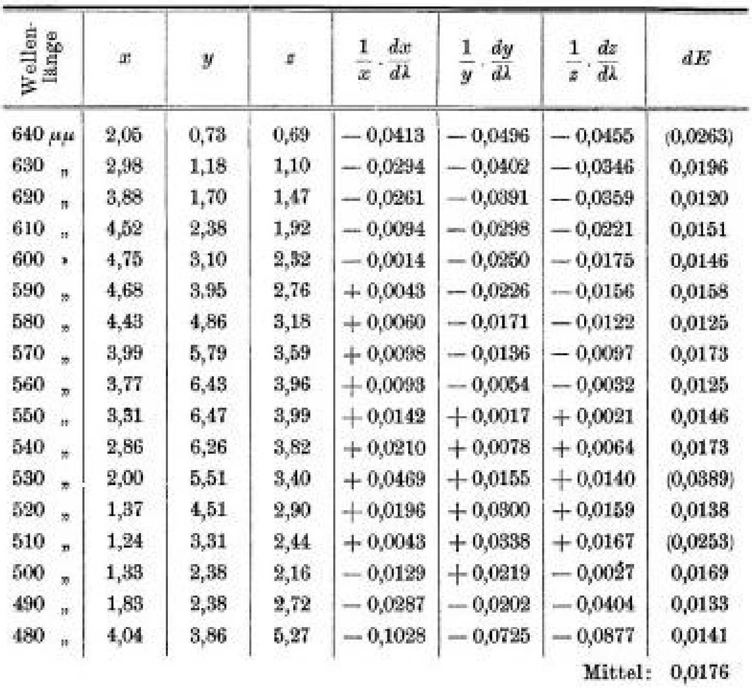


Wellenlänge	R	G	V	$\frac{dR}{d\lambda}$	$\frac{dG}{d\lambda}$	$\frac{dV}{d\lambda}$	$d\lambda$
640 $\mu\mu$	2,66	0,22	0	− 0,116	− 0,023	0	2,37 $\mu\mu$
630 „	3,95	0,54	0	− 0,129	− 0,044	0	1,35 „
620 „	5,35	1,12	0	− 0,160	− 0,078	0	0,67 „
610 „	6,60	2,17	0	− 0,107	− 0,123	0	0,55 „
600 „	7,51	3,60	0	− 0,081	− 0,165	0	0,45 „
590 „	8,27	5,48	0	− 0,067	− 0,208	0	0,42 „
580 „	8,90	7,65	0	− 0,055	− 0,200	0	0,38 „
570 „	9,37	9,98	0	− 0,039	− 0,199	0	0,51 „
560 „	9,56	11,45	0,22	0	− 0,100	0	0,58 „
550 „	9,21	12,00	0,3	+ 0,068	0	− 0,0138	0,77 „
540 „	8,30	11,55	0,49	+ 0,121	+ 0,083	− 0,0233	0,80 „
530 „	6,54	10,36	0,75	+ 0,202	+ 0,139	− 0,0326	0,77 „
520 „	4,62	8,45	1,10	+ 0,171	+ 0,228	− 0,0400	0,71 „
510 „	3,0	5,75	1,55	+ 0,162	+ 0,271	− 0,0536	0,64 „
500 „	1,50	3,32	2,2	+ 0,114	+ 0,168	− 0,0887	0,35 „
490 „	0,78	2,24	3,6	+ 0,051	+ 0,059	− 0,208	0,31 „
480 „	0,4	1,88	7,9	+ 0,043	+ 0,028	− 0,52	0,38 „

Table 5.4

Helmholtz's table containing the data, obtained by König in 1884, which Helmholtz used for the calculation of x, y and z and of the line element dE. The first column contains the wavelength (Wellenlänge) values (von Helmholtz 1892a, p. 9)

We can find the results of the calculation in the table below.



Wellenlänge	x	y	z	$\frac{1}{x}\cdot\frac{dx}{d\lambda}$	$\frac{1}{y}\cdot\frac{dy}{d\lambda}$	$\frac{1}{z}\cdot\frac{dz}{d\lambda}$	dE
640 $\mu\mu$	2,05	0,73	0,69	− 0,0413	− 0,0496	− 0,0455	(0,0263)
630 „	2,98	1,18	1,10	− 0,0294	− 0,0402	− 0,0346	0,0196
620 „	3,88	1,70	1,47	− 0,0261	− 0,0391	− 0,0359	0,0120
610 „	4,52	2,38	1,92	− 0,0094	− 0,0298	− 0,0221	0,0151
600 „	4,75	3,10	2,32	− 0,0014	− 0,0250	− 0,0175	0,0146
590 „	4,68	3,95	2,76	+ 0,0043	− 0,0226	− 0,0156	0,0158
580 „	4,43	4,86	3,18	+ 0,0060	− 0,0171	− 0,0122	0,0125
570 „	3,99	5,79	3,59	+ 0,0098	− 0,0136	− 0,0097	0,0173
560 „	3,77	6,43	3,96	+ 0,0093	− 0,0054	− 0,0032	0,0125
550 „	3,31	6,47	3,99	+ 0,0142	+ 0,0017	+ 0,0021	0,0146
540 „	2,86	6,26	3,82	+ 0,0210	+ 0,0078	+ 0,0064	0,0173
530 „	2,00	5,51	3,40	+ 0,0469	+ 0,0155	+ 0,0140	(0,0389)
520 „	1,37	4,51	2,90	+ 0,0196	+ 0,0300	+ 0,0159	0,0138
510 „	1,24	3,31	2,44	+ 0,0043	+ 0,0338	+ 0,0167	(0,0253)
500 „	1,33	2,38	2,16	− 0,0129	+ 0,0219	− 0,0027	0,0169
490 „	1,83	2,38	2,72	− 0,0287	− 0,0202	− 0,0404	0,0133
480 „	4,04	3,86	5,27	− 0,1028	− 0,0725	− 0,0877	0,0141
						Mittel:	0,0176

Table 5.5

Helmholtz's calculated values of x, y, z, $\frac{1}{x}\cdot\frac{dx}{d\lambda}, \frac{1}{y}\cdot\frac{dy}{d\lambda}, \frac{1}{z}\cdot\frac{dz}{d\lambda}$ and dE, with indication also of its mean value (*Mittel*) (von Helmholtz 1892a, p. 10)

Helmholtz, at this point, plotted on a graph both the wavelength discrimination curves obtained by his calculation and by König's direct measurements. It is worth to report the graph below in order to offer a direct comparison between the two curves.

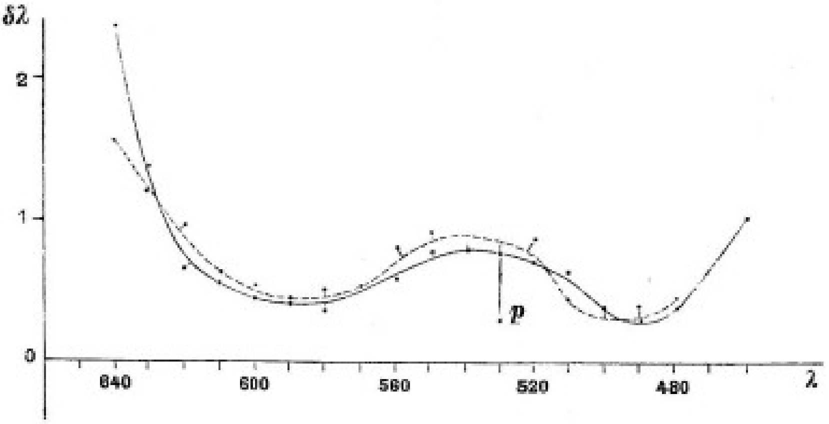


Fig. 5.2
König's wavelength discrimination curves (straight line) and Helmholtz's calculated curve (dotted line) obtained requiring a constant value for dE (von Helmholtz 1892a, p. 10)

He compared his theoretical calculation with König's measured data with the following words: *Man sieht daß eine ziemlich ähnlich verlaufende Curve, wie die der beobachteten Werthe, durch die gegebene Theorie erreicht werden kann. Auch würden weitere Verbesserungen der Constanten* a, b, c *wohl noch merklich bessere Übereinstimmung haben erreichen lassen, als es bisher gelungen ist. Die auffallendste Abweichung ist bei* $\lambda = 530\ \mu\mu$, *wo ein einzelner ganz kleiner Werth von* $\delta\lambda$, *[...] mit* p *bezeichnet [...], mitten zwischen solchen erscheint, die dem dort bestehenden Maximum von* $\delta\lambda$ *entsprechen. Es liegt diese Stelle im Grün nahe bei der Linie E und dort mußte ein besonders weites Intervall (von* $\lambda = 536\ \mu\mu$ *bis* $\lambda = 516{,}5\ \mu\mu$*) durch Interpolation ausgefüllt werden, wodurch die Werthe der Differentialquotienten an jener Stelle erheblich unsicher werden (It can be seen that a fairly similar curve, like that of the observed values, can be achieved by the given theory. Also, further improvements of the constants* a, b, c *would probably have achieved noticeably better agreement than was previously achieved. The most conspicuous deviation is for* $\lambda = 530\ nm$ *in the case where a single, very small value of* $\delta\lambda$ *[...], denoted by* p *[...], appears in the middle of those corresponding to the maximum of* $\delta\lambda$*. This point lies in the green region near the line E and, there, a very wide interval (from* $\lambda = 536\ nm$ *to* $\lambda = 516{,}5\ nm$*) had to be filled in by interpolation, whereby the values of the differential ratios at that point become considerably uncertain.)* (von Helmholtz 1892a, p. 11). As we can notice from Fig. 5.2, in fact, the curves show a similar behaviour except for a significant deviation that occurred at 530 nm. This could be due to the fact, as Helmholtz explained, that in the green region there is a wide interval at which the differential ratios become uncertain. He recognized that this deviation could not be eliminated by any combination of the constants a, b, c and it was due basically to the value of the term $\frac{1}{x} \cdot \frac{dx}{d\lambda}$. Helmholtz stated also that one of the curves of the spectral colours could present a sudden change in the differential quotient and in such a place, his interpolation calculations, which proceeded from the assumption of a continuous curvature of the curves, must be misleading. Moreover, it should be noted that everywhere, where the light intensity of one of the three colours recedes from the others, the diminished sensitivity to the just noticeable difference of weak light prevails. There, if the differential ratios in λ do not become very small everywhere at the same time, the sensitivity to colour differences could be smaller, than it should be expected according to the theory. This is in fact so, except for the interval already indicated by Helmholtz, between 530 and 510 nm, where the mixed red is very weak; this is also the case of the green elementary colour at the red end of the spectrum. This position, in fact, corresponds to another deviation from König's curve, as we can see in Fig. 5.2, which occurs at 640 nm.

I would like to proceed presenting Helmholtz's key considerations contained in paragraph entitled *Die gefundene Grundfarben (The found fundamental colours).* Here, Helmholtz illustrated in a colour triangle the relationship subsisting between the new fundamental colours, x, y and z, which he found, at least provisionally, by calculation, and the spectral colours, *R*, *G* and *V*.

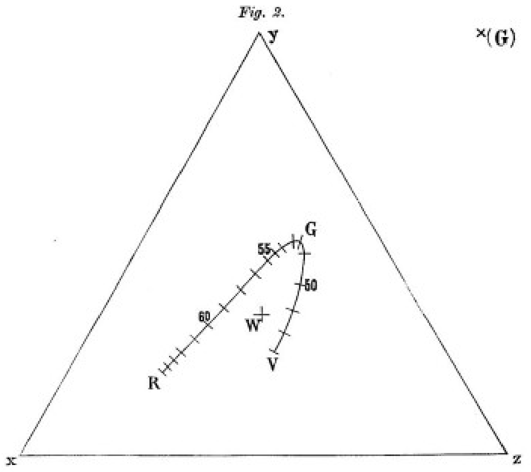


Fig. 5.3
The position of the spectral locus in Helmholtz's colour triangle, which has the fundamentals x, y and z at the corners with indication of point (G) which shows the position of the green elementary colour in König's investigation (von Helmholtz 1892a, p. 12)

Fig. 5.3 depicts the vertices x, y, z of the colour triangle, at the centre of which pure white, *W*, can be found. The curve *RGV* represents the spectral colour series and, as we can notice, the spectral colours lie quite far from the vertices of the triangle; they are indeed strongly mixed, even the red and the violet. From these results we can understand, as Königsberger pointed out, that Helmholtz left once more undecided which colour sensations correspond with the three elementary excitations of the nervous apparatus of the eye (Königsberger 1906, p. 397). His investigation yielded the following results for the three fundamental colours: spectral red should be a whitish, slightly yellow modification of the fundamental colour x, which has to be a highly saturated carmine-red; for what concerns the violet, it should be a pale red modification of the third fundamental colour z, which may therefore be compared with ultramarine in its tone; the fundamental colour y would correspond, in colour tone, to the position between 540 and 560 nm, where $x = z$, and may be compared more or less with the green of vegetation.

The strong curvature of the spectral curve at point *G* corresponds with the spectral green at Fraunhofer line E. We can see that the irregularity which Helmholtz detected in the discrimination curve at 530 nm falls straight into this strong curvature in the green, which may explain the uncertainty of the measurements and interpolations made there. Helmholtz's curve indicates that all simple colours excite all the light-sensitive nerve elements of the trichromatic eye simultaneously and with only moderate differences in intensity.

Letter *G* outside the triangle and enclosed by brackets refers to the green proposed, originally, as elementary colour by König and Dieterici for their colour mixing experiments. It is important to underline that, in König's subsequent investigation, this colour was already located outside the colour triangle, which presented König's fundamental colours $\mathfrak{R}$, $\mathfrak{G}$ and $\mathfrak{B}$ at the vertices[76]. König and Dietrici's paper published in 1892 with the title *Die Grundempfindungen in normalen und anomalen Farbensystem und ihre Intensitätsvertheilung im Spectrum (The fundamental sensation functions in normal and anomalous colour systems and their intensity distribution in the spectrum)*, already mentioned in chapter 4, contains their colour triangle with the fundamental $\mathfrak{R}$, $\mathfrak{G}$ and $\mathfrak{B}$ at the vertices (König and Engelmann 1903, pp. 214-321), which I would like to report below in order to enable a direct comparison.

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[76] For greater clarity, I would like to remember that König made a distinction between elementary colours, which he denoted by R, G and B, and fundamental colours, $\mathfrak{R}$, $\mathfrak{G}$ and $\mathfrak{B}$, obtained from linear combination of the first ones (see chapter 4).

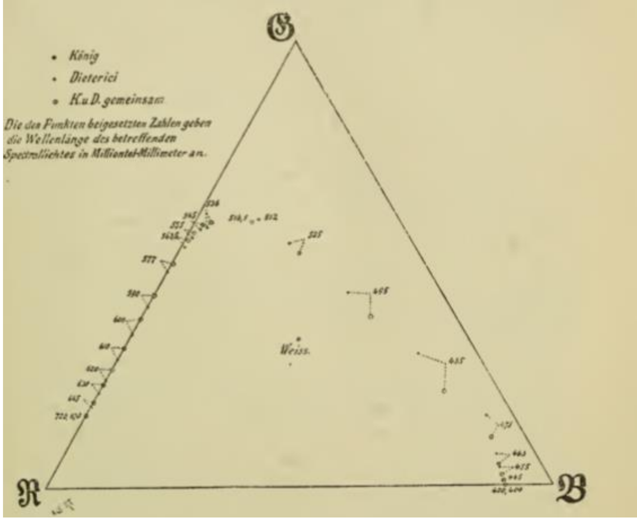

Fig. 5.4
The position of the spectral locus in König and Dieterici's colour triangle, which has the fundamentals, $\mathfrak{R}$, $\mathfrak{G}$ and $\mathfrak{B}$, at the vertices (König and Engelmann 1903, p. 317)

From the König's colour diagram arouse the following nuances corresponding to the fundamental colours: $\mathfrak{R}$ corresponded to a red which deviates slightly from the red of the long-wave final range in the spectrum to the purple; $\mathfrak{G}$ corresponded to a green of wavelength 505 nm and $\mathfrak{B}$ to a wavelength of 470 nm.

Helmholtz proceeded offering a significant comparison between trichromatic and dichromatic eyes, basing his investigation on König and Dieterici's results contained in their paper of 1892. He could express the fundamental colours $\mathfrak{R}$ and $\mathfrak{G}$, identified as individually missing in the two kinds of dichromatic eyes by his collaborators, in terms of his fundamentals colours x, y and z in order to depict them inside his own colour triangle (see Fig. 5.3). König and Dieterici expressed the missing fundamental colour for a red-blind subject and for a green-blind subject, $\mathfrak{R}$ and $\mathfrak{G}$ respectively, in terms of the elementary colours (König and Engelmann 1903, pp. 307-308):

$$\mathfrak{R} = \frac{20\,R - 3\,G + 2\,V}{19} \tag{5.31}$$

$$\mathfrak{G} = \frac{1}{5}R + \frac{4}{5}G \tag{5.32}$$

The next step was, therefore, that of expressing them in terms of his fundamentals in the following way[77]:

$$\mathfrak{R} = 1{,}434 \cdot x + 1{,}797 \cdot y - 2{,}132 \cdot z \tag{5.33}$$

$$\mathfrak{G} = -0{,}1442 \cdot x + 2{,}715 \cdot y - 1{,}483 \cdot z \tag{5.34}$$

He could conclude this analysis stating that negative coefficients indicate that the defined colours have to lie outside of the colour triangle, for both the missing colours of the two classes of dichromates. And, using Helmholtz's words: *Die fehlende Farbe der Grünblinden würde zwischen den verlängerten Seiten des Farbendreiecks liegen, die sich im Grün schneiden, näher dem vom Rot kommenden Schenkel, die der Rotblinden ausserhalb der Rot-Grün-Linie, deren Mitte etwa gegenüber, aber ziemlich entfernt (The missing colour of the green-blind would lie between the extended sides of the colour triangle, which intersect in the green, and that of the red-blind would lie closer to the side coming from the red, but quite far outside the red-green line, in front of its midpoint but rather far from it* (von Helmholtz 1892a, p. 19).

The end part of the paper is dedicated to a comparison of sensitivity to differences of brightness and to differences of colour, *Vergleichung der Empfindlichkeit für Helligkeitsunterschiede mit der für Farbenunterschiede*. Helmholtz started the paragraph introducing the value of the ratio of the smallest perceptual difference in brightness found by König into his equation of the line element. In fact, König's second paper of 1888 entitled *Experimentelle Untersuchungen über die psycho-physische Fundamentalformel in Bezug auf den Gesichtssinn (Experimental investigation on the psycho-physical fundamental formula in relation to the sense of vision)* contains the values of the differential threshold in brightness for white light with several intensities found by the author, in collaboration with Brodhun (see chapter 4). Here, Helmholtz used the value $\frac{\delta r}{r} = 0{,}0173$, which corresponds to an input intensity of 50000 units (as can be derived from the table contained in König and Dieterici's paper of 1889 (König, and Engelmann 1903, p. 138)). Therefore, replacing ε with 0,0173 in equation 5.24:

$$dE = k \cdot 0{,}0173 \cdot \sqrt{3} \tag{5.35}$$

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[77] The complete mathematical treatment can be found from page 14, to page 19 of the paper under examination.

The values of k, furthermore, as Helmholtz found out from his colour comparison experiments, must be taken 1,8238 times larger than in the brightness comparisons (see section 5.5). Hence, the expression for the smallest perceptual difference in brightness could take the following form:

$$dE = \frac{1}{1{,}8238} \cdot 0{,}0173 \cdot \sqrt{3} = 0{,}01643 \tag{5.36}$$

whereas, for what concerns the sensitivity to colour difference, he reported the average value of dE reported in Table 5.5:

$$dE = 0{,}0176$$

He commented this correspondence stating that this was in line with his assumption; Helmholtz had assumed that the perception differences in colour originated with the perception of differences in brightness. Both differences in brightness and differences in colour, indeed, contribute to a geometric representation of colour.

As concluding remark, Helmholtz stated that a further examination of the law laid down here could be carried out by direct mixture of two spectral colours of different mixing ratios, which could be read directly on the apparatus, and different comparison could be made between neighbouring spectral colours.

5.6.2 *Kürzeste Linien im Farbensystem*

The paper *Kürzeste Linien im Farbensystem (The shortest lines in the colour system)* contains Helmholtz's complete definition of the line element in trichromatic colour space. My intention is to briefly present this main achievements, which constitute the basis for all future studies on colour metrics. At the beginning of the work, Helmholtz declared the starting point of his investigation: *Wir wollen im Folgenden von einer geometrischen Darstellung des Farbensystems ausgehen, welche Lambert's Farbenpyramide entspricht, indem wir jede besondere Farbe als hergestellt durch die Vereinigung der passend abgemessenen Quanta dreier passend gewählter Grundfarben ansehen, und die Werthe dieser Quanta gleich setzten den drei positiven rechtwinkeligen Coordinaten x, y, z. Dann ist jede Farbe durch einen Punct innerhalb der dreikantigen Ecke vertreten, welche die drei (In what* *follows, we shall proceed from a geometrical representation of the colour system which corresponds to Lambert's colour pyramid by considering each particular colour as made by the union of the appropriately measured quanta of three suitably chosen fundamental colours, and the values of that quanta equals the three positive right-angled coordinates x, y, z. Then each colour is represented by a point within the triangular corners enclosed between the positive coordinate axes)* (von Helmholtz 1892b, p. 108). He went further with a remark of his and Riemann's interpretation of the geometry of colour space and returned to the question concerning the determination of the sensitivity to differences (*Deutlichkeit des Unterschiedes*) reporting the possible way to definite it; it could be determined equally well in relation to the brightness of qualitatively equal colour as well as in relation to the colour tone of equally bright lights, and a direct comparison between the two could be made. Helmholtz was aware of the priority of his work; no one, in fact, had ever attempt to extend the Weber-Fechner law to a complex of more than one dimension[78]. Referring to his previous work of 1892, however, Helmholtz recognized that the obtained results were not yet completely reliable. In fact, it turned out that all the spectral colours, including the final colours at the red and violet end of the spectrum, receive rather strong quanta from all three primary colours, which correspond in colour tone to crimson, ultramarine and leaf green but must be considerably more saturated than these.

At this point, he derived the famous expression for the smallest perceptual difference, (*Deutlichkeit des Unterschieds*) between two colours, i.e. the line element, one of which is composed of quanta of primary colours x, y and z, which, as he pointed out, represented the *physiologische Urfarben* (physiological primary colours), and the other one is composed of $(x + dx)$, $(y + dy)$ and $(z + dz)$, namely:

$$dE^2 = \left(\frac{dx}{x+a}\right)^2 + \left(\frac{dy}{y+b}\right)^2 + \left(\frac{dz}{z+c}\right)^2 \tag{5.37}$$

where a, b and c are constants which refer to the self-light constant (*Farbencomponenten des Eigenlichts*). It is to be noted that x, y, z cannot be replaced by any linear function of the primary

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[78] As it follows from his following words: *Ich habe darin versucht, den Grad der Deutlichkeit zweier Farben anzugeben, die sich gleichzeitig in den Quanten aller drei Grundfarben von einander unterschieden, welche in ihre Zusammensetzung eingehen, also gleichzeitig sich in Helligkeit und in der Qualität unterscheiden können, während bisher nur diejenige Seite des Gesetzes durchgearbeitet war, welche sich auf Helligkeitsunterschiede allein, bei unveränderter Qualität, bezieht (I tried to indicate the degree of sensitivity to differences of two colours, which at the same time differ from one another in the quanta of all three primary colours, which can be included in their composition, and thus differ in brightness and quality at the same time; whereas so far only the law was applied to brightness differences alone, maintaining the quality unchanged)* (von Helmholtz 1892b, p. 110).

colours, as it occurs for the colour mixing law. Helmholtz called this smallest perceptual difference *Kürzeste Linie*, shortest line, i.e. the expression describing infinitesimal distances in a three-dimensional non-Euclidean space. Lines of smallest colour difference were taken as geodesic, in other words as shortest lines between colours as points in the colour space.

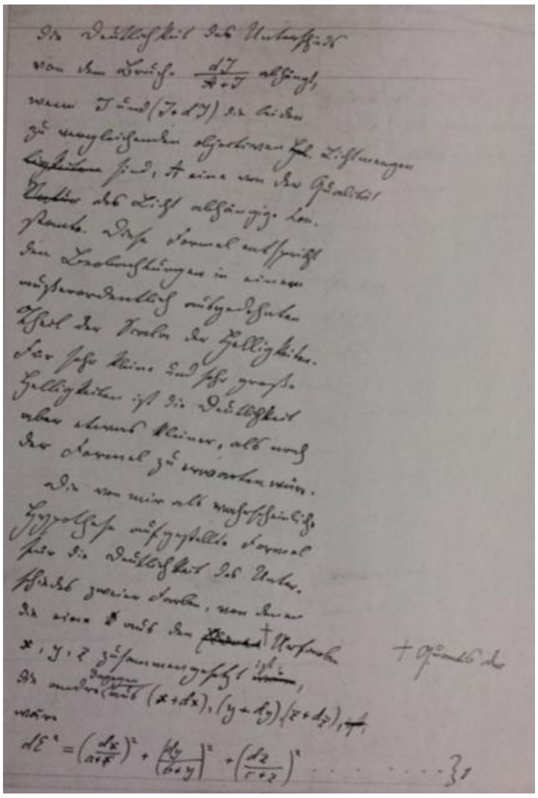

Fig. 5.5
Helmholtz's manuscript formula of the line element (*Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften* (Nachlass Hermann von Helmholtz, 576 Kürzeste Linien im Farbensystem)

Helmholtz moved to a transformed space in which coordinates are the logarithms of the three primaries plus the self-light constant. As Stiles remarked, in this space the geodesic between two points is simply the straight line joining them (Stiles 1972, p. 5). My intention is to present here one significant conclusion which Helmholtz drew from his analysis. He investigated the colour differences for just one colour tone, and specifically the one in which the tristimulus values, i.e. the amount of the three fundamentals required to match a given stimulus, are in the same proportion as the corresponding self-lights a, b and c, for different light intensity. Helmholtz found out in this case that a change in intensity could produce no change in apparent hue. Yellowish-white and all other lights tending to this, as their intensities are made very high, have this property, as Helmholtz recognized experimentally. In his discussion of colour appearances at lower intensity levels, on the other hand, the self-light constants again play an important role and a change in hue becomes perceptible as light intensity changes[79].

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[79] Helmholtz's complete mathematical treatment can be found from page 115, to page 119 of the paper under examination.

Chapter 6

The Helmholtz and Maxwell’s legacy

Colour science has excellent credentials as it has claimed the interest of some of the greatest scientists of all time, with Thomas Young, Helmholtz and Maxwell being outstanding in the 19th century, following on the foundations laid by Isaac Newton some 100 years or more before (W. D. Wright, Schanda 2007, pp. 9-10)

This last chapter is dedicated to a brief account of the development of colorimetry, after Maxwell and Helmholtz’s precious work. My intention is, therefore, to highlight the reasons why Maxwell and Helmholtz are considered the founders of modern colorimetry.

I will start reporting some implementations of Helmholtz’s line element model, which represents a turning point in the application of psychophysical laws to colour experience. I will focus on the significant attempts conducted by Schrödinger in 1920 and by Stiles in 1946.

Then, I will present Guild and Wright’s pivotal works, conducted between the 1920s and 1930s, that allowed them to determine colour matching functions exploiting the method used by Maxwell to obtain his first colour equations. It is worth to highlight that all the subsequent colour matching experiments, which were undertaken over the course of the years, were based on the principle underlying Maxwell’s colour box, i.e. the realization of a bipartite field in order to allow a direct comparison, in colour and intensity, between coloured lights.

We will also understand why Wright and Guild’s experimental results were combined to give birth to the first CIE colour space, RGB, the direct evolution of Maxwell’s equilateral triangle and from which the CIE XYZ colour space was derived.

I will also spend few words presenting the pioneering attempts conducted by Wright and MacAdam in 1941 and 1942, to measure just noticeable differences across the CIE (x, y) colour diagram. The famous MacAdam’s ellipses will be then presented, which have become over the course of the years a key set of data used as test data for line elements and colour difference formulae. The last section is dedicated to a short overview of the meeting of *the International Color Association* held in 1971 in Driebergen on the topic of small colour differences. Here important aspects of colour metrics were discussed and many open problems were illustrated, some of which have remained unsolved until the present time.

My aim, far from giving a complete and detailed review of all subsequent contributions to the field up to 1971, will be to underline the pivotal role played by Helmholtz and Maxwell's studies in the development of colour science, whose main mathematical language has remained to the present day the Riemannian geometry based on differential thresholds measured along arbitrary directions in colour space.

6.1 Following Helmholtz: Schrödinger and Stiles' line elements

Helmholtz's pioneering work on the line element in colour space can be considered the first attempt to define a metrically significant model of colour space. This work on the subject, however, had remained almost forgotten, until Schrödinger and other scientists and colour theorists, such as Stiles, after him rediscovered its precious value. In this section, I will underline the fundamental role played by Helmholtz's investigation in the development of more complex line elements over the years, of which I will briefly present some examples. It is worth to highlight that, for what concerns colour differences and line elements, there are, at the present time, many different formulae and colour space models to predict visual difference between two colours but it is still challenging because a perfect uniform colour space does not exist.

6.1.1 Schrödinger's line element

Helmholtz's works of 1891 and 1892 attracted the attention of Erwin Schrödinger, who, in the 1920s published his articles on colour theory, containing his new and more complex form of the line element. Schrödinger detected some shortcomings of Helmholtz's model. He recognized that Helmholtz's form of the line element did not have the same luminous efficiency curve as obtained from the observed curve, which is nowadays indicated by $V(\lambda)$. It was at that time, in fact, that the modern concept of luminance saw birth, thanks to elaborate photometric studies (Vos 1979). Schrödinger also underlined another key point, which he expressed with the following words: *Es ist in letzter Zeit [...] bewiesen worden, daß die Helligkeit, jedenfalls sehr angenähert, eine additive Eigenschaft der Farben ist, d.h. daß gleichhelle Lichter gemischt, gleichhelle Lichter ergeben. Diese Additivität der Helligkeit folgt keineswegs aus der Additivität der Farbgleichungen, sie wurde sogar* *von Helmholtz [...] bezweifelt, und zwar auf Grund einiger sehr interessanter Versuche am Farbenkreisel (It has been proven lately [...] that the brightness, at least very close, is an additive property of the colours, i.e. equally bright lights, when mixed, will give, as a result, equally bright lights. This additivity of brightness follows by no means from the additivity of the equations of colour; it was doubted even by Helmholtz [...] on the basis of some very interesting experiments on the colour top)* (Schrödinger 1920, pp. 493-494). From this excerpt, it is clear that Schrödinger recognized that the additivity of brightness was essential to formulate line element, something that had escaped Helmholtz's attention.

Moreover, as Vos reported, one of the consequences of Helmholtz's construction was that lines of equal colours and lines of equal brightness were not mutually complementary, i.e. orthogonal (Vos 2006). To meet all his criticism of the Helmholtz's line element, Schrödinger proposed the following form:

$$(ds)^2 = \frac{1}{l_R R + l_G G + l_B B}\left[\frac{l_R (dR)^2}{R} + \frac{l_G (dG)^2}{G} + \frac{l_B (dB)^2}{B}\right] \tag{6.1}$$

where l_R, l_G and l_B are constants corresponding approximately to the luminous efficiency functions of the three fundamental colour vision processes and R, G and B are the fundamental processes, as experimentally determined by König and Dieterici. The line element obeys the Weber-Fechner law while being additive. In this way, Schrödinger circumvented the orthogonality problem and produced an additive luminous efficiency function $V(\lambda)$.

However, as Vos underlined, Schrödinger's line element has drawn even less attention than Helmholtz's construction. This could be due to the difficulties of calculating relative values of constants.

6.1.2 Stiles' line element

Another example of line element which I would like to briefly present here is that developed by the physicist and mathematician Walter S. Stiles, who spent most of his professional life in the Light Division of the National Physical Laboratory at Teddington. Stiles proposed a version of a line element based on his experimental investigations (Stiles 1946). He recognized that different Weber fractions had to be applied to the three cone systems rather than the common fraction used by

Helmholtz and Schrödinger[80]. Stiles found, in fact, that the Weber fractions for the three visual processes were far from equal, as Helmholtz assumed. He proposed his line element in the following form:

$$(ds)^2 = \left[\frac{\zeta(R)}{\rho}(dR)\right]^2 + \left[\frac{\zeta(G)}{\gamma}(dG)\right]^2 + \left[\frac{\zeta(B)}{\beta}(dB)\right]^2 \tag{6.2}$$

where $\zeta(...)$ are empirical functions which define the shape of the curve showing the increase of threshold with field intensity and ρ, γ and β are proportional to the Weber fractions:

$$\rho{:}\gamma{:}\beta = 0.78{:}1{:}4.46$$

Introducing different Weber fractions for the three-receptor systems, Stiles could reconcile the Helmholtz's concept with the orthogonality requirement. He obtained Weber fractions with values of 1,28 for ρ, 1,65 for γ and 7,35 for β. This should suggest that the *blue* process is much less sensitive than the *red* and the *green*.

Stiles could obtain rather satisfactory results. In fact, over the course of the years, Stiles' line element was tested extensively and was found to provide good approximations to several sets of visual data.

One of the most advanced attempts to define a line element in colour space was the line element proposed by Vos and Walraven, who were working together at the Institute for Perception TNO in Soesterberg in 1972 (Vos and Walraven 1972). For a complete review of the contribution of the line-element approach to the understanding of colour metrics, the reader is referred to the works of Vos of 1979, *Line elements and Physiological Models of Color Vision* (Vos 1979) and of 2006, *From lower to higher colour metrics: a historical account* (Vos 2006).

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[80] Stiles, at the beginning of his paper of 1946, speaking about the Helmholtz's line element, which he reported, stated that: *He [Helmholtz] was able to find a set of primaries which satisfied this test [that the values of the hue limen through the spectrum should agree with the measurements of this quantity made by König and Dieterici] fairly well. But the corresponding fundamental response curves [...] each have two pronounced maxima and are quite unlike the* Grundempfindungen *proposed by König from consideration of the properties of colour-blinds. Moreover, as Schrödinger (1920) pointed out, they lead to an impossible double peak form for the step-by-step visibility curve (the curve showing the relative energies of neighbouring spectrum colours which appear equally bright)* (Stiles 1946 pp. 42-23).

6.2 Following Maxwell: Guild and Wright and the CIE 1931 colour systems

I think Mr Wright is to be congratulated on an exceedingly valuable contribution to the subject of colour vision and colorimetrics. His conclusion that the evaluation of the primaries from two colour matches with monochromatic light instead of, as is usual, from a single match with white gives results in which the differences encountered between different observers are very much diminished, although not completely removed, is in accordance with our experience at the National Physical Laboratory (J. Guild, Wright 1928-29, p. 160)

Between 1920 and 1930 two English physicists J. Guild and W. D. Wright, working independently, carried out a series of experiments, exploiting Maxwell's ideas to obtain his colorimetric equations, that allowed them to determine colour matching functions, which took the place of König and Dieterici's colour fundamentals (see chapter 4), that until that time had remained the accepted ones.

My intention here is to briefly present the method used by Wright to determined his colour matching functions, without entering into the details, in order to show its simplicity and effectiveness and also to underline the influence of Maxwell's colour mixing experiments[81] on the field of colorimetry.

In his paper *A re-determination of the trichromatic coefficients of the spectral colours*, Wright reported a series of colour matches through the spectrum made by ten observes (Wright 1928-29). He could determine the trichromatic coefficients of the spectral colours, i.e. the proportion of the three selected primaries, red, green and blue lights, which were required to match monochromatic coloured lights of the spectrum. The obtained data, when combined with luminosity data, were used to compute the spectral mixtures and the sensation curves. The apparatus used to conduct these colour matching experiments was a trichromatic colorimeter, which Wright described in detail in his previous paper of 1928, *A trichromatic colorimeter with spectral primaries* (Wright 1928), and colour matching equations were obtained exploiting the basic principle of Maxwell's colour box. Wright described the method with the following words: *A spectrometer system is used in which two spectra are formed from the same source. From one of these, three portions to act as primaries are reflected back through a lower part of the dispersing system, so that the mixing of the three radiations is effected by neutralising the prismatic dispersion by which the colours were first separated. From the other*

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[81] In this regard, Wright acknowledged Maxwell's precious work with the following words: *I think I should first make a brief reference to the pioneer work done by Maxwell. Among the various papers I have managed to collect, I am fortunate to have a reprint of Maxwell's 1860 paper in which he describes his colour box and colour mixture curves with he produced* (Boynton 1979, p. 397).

spectrum the test colour and a desaturating colour are selected and mixed in a similar manner, and the composite beams are then brought into the two halves of a simple bipartite field (Wright 1928, p. p. 141). In this way, Wright could obtain a bipartite field in order to allow a direct comparison, in colour and intensity, of a given coloured light, instead of white light used by Maxwell (see chapter 3, section 3.10), with a mixture of the three chosen primary spectral lights, in the desired proportions. The intensity of each of these lights could be independently adjusted, until the two sides of the field appeared perceptually identical. Matches could be obtained, in this way, with any given test light.

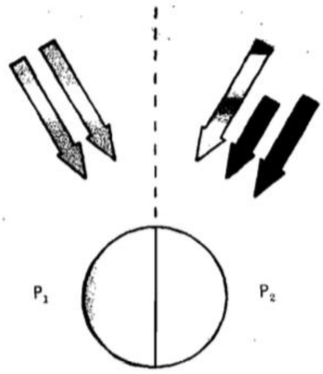

Fig. 6.1
The two halves of a colorimeter field in which the test field, indicated by P_1, is matched in colour and brightness by a suitable mixture of red, green and blue lights in the comparison field, indicated by P_2 (Wright 1944, p. 54)

Wright, like Maxwell and Helmholtz, was aware of the subjective aspect of colour perception. Two persons were required, one for observing and the other one for noting down the readings and for setting new colours to be matched. In order to avoid errors due to subjective defects, he wrote: *The observer is thus solely concerned with colour matching, a fact that tends to create the right mental atmosphere, an important consideration in work such as this, in which the temperament of the observer plays a big part in the accuracy and reliability of this observations. Whilst no indication was obtained that the state of adaptation of the observer affected any match, it was thought desirable to allow the observer five or ten minutes' dark adaptation before commencing a series of observations* (Wright 1928-29, p. 145).

To obtained colorimetric functions Wright used, as we have seen above, monochromatic primaries, whereas Guild used broadband primaries (see Guild 1931[82]). As Fairchild reported, since the primaries from one experiment could be expressed in terms of tristimulus values to match them using the other system, it was possible, by means of a linear transformation, to convert tristimulus values

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[82] Although Guild did not publish his results until 1931, it is worth to underline that he began his experiments much earlier, toward the end of 1926 (Guild 1931, p. 151).

from one set of primaries to another. A transformation was, thus, derived to obtain a common set of primaries (Fairchild 2005, p. 72).

Once this was done, Guild and Wright found a very good agreement between their results, despite the fact that they conducted their experiments in different laboratories with different apparatus and using different primaries. Their works have remained an accepted and standard part of colorimetry. In fact, this surprisingly good agreement increased confidence that their data were good enough for constructing a standard set of colour matching functions, based on the mean results of Guild and Wright. The *International Commission on Illumination*, CIE, could, at this point, create the CIE 1931 RGB colour space. The mean functions were transformed to RGB primaries: 700,0 nm (red), 546,1 nm (green) and 435, 8 nm (blue)[83]. The spectral tristimulus values for the CIE 1931 RGB system are illustrated in Fig. 6.2.

In the CIE 1931 RGB system, the tristimulus values R, G and B, for a given stimulus are obtained by multiplying the colour matching functions, indicated by $\bar{r}(\lambda)$, $\bar{g}(\lambda)$ and $\bar{b}(\lambda)$, by the amount of energy in the stimulus at each wavelength and integrating across the spectrum, from 380 to 780 nm. The equations for calculating the tristimulus values of a stimulus with spectral power distribution, denoted by $\Phi(\lambda)$, are reported below:

$$R = \int_{380}^{780} \Phi(\lambda)\, \bar{r}(\lambda)\, d\lambda \tag{6.3}$$

$$G = \int_{380}^{780} \Phi(\lambda)\, \bar{g}(\lambda)\, d\lambda \tag{6.4}$$

$$B = \int_{380}^{780} \Phi(\lambda)\, \bar{b}(\lambda)\, d\lambda \tag{6.5}$$

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[83] At the beginning of his paper of 1931, Guild reported the values of Maxwell's primaries, in μm, in order to offer a direct comparison: *CLERK MAXWELL first determined the spectrum locus on a colour triangle constructed in accordance with NEWTON'S law of colour mixture, and also deduced mixture curves of the spectrum for himself and his assistant [his wife Katherine], using as primaries spectral radiations of wave-lengths 0,6307 μ, 0,5286 μ and 0,4573 μ, approximately* (Guild 1931, p. 149).

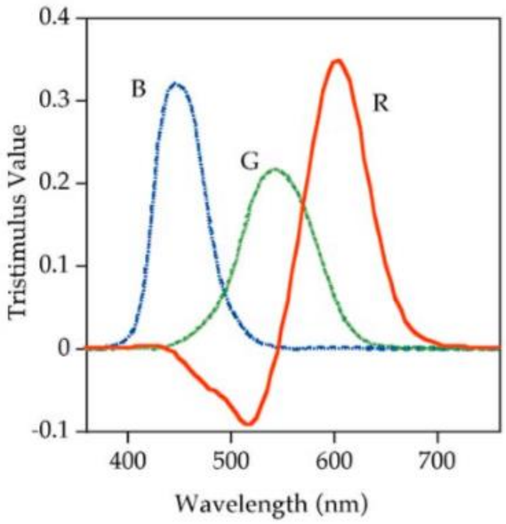

Fig. 6.2
Spectral tristimulus values for the CIE 1931 RGB system with monochromatic primaries at 435,8, 546,1 and 700,0 nm (Fairchild 2005, p. 72)

As we can see in Fig. 6.2, some of the tristimulus values plotted (in particular for the red one) are negative, just as for Maxwell's intensity curves (see chapter 3, section 3.10).

Before proceeding, I would like to report Wright's significant considerations on Maxwell's colour diagram. The RGB system was the direct evolution of Maxwell's equilateral triangle. However, as Wright recognized, in a unit trichromatic equation, *once any two of the three coefficients are known, the third can be found as the difference between unity and the sum of the other two*. Therefore, if we denote by *C* the outcome of a mixture of the three primaries, red, green and blue, *the expression for C is then a function of only two variables instead of three, and C can consequently be represented graphically on a plane diagram* (Wright 1944, pp. 60-61).

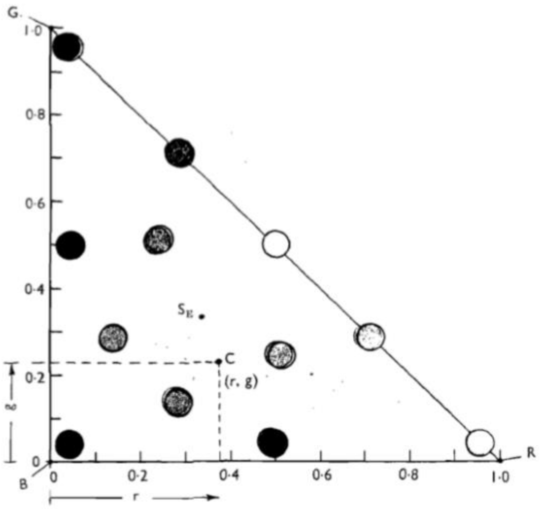

Fig. 6.3
The chromaticity diagram in which the quality of a stimulus is defined in terms of two coordinates. The circles, originally coloured, were intended to give an indication of the distribution of the colours when the diagram related to a set of reference stimuli, R, G and B (Wright 1944, p. 60)

In Fig. 6.3, the horizontal axis represents the red coefficient, r, and the vertical axis represents the green one, g. A colour C can be identified by the coordinates (r, g) and its corresponding value for b can be derived from the equation: $b = 1 - (r + g)$. White, denoted by S_E, is given by the point (0,333, 0,333), the red, R, by (1,000, 0,000), the green, G, by (0,000, 1,000), and the blue, B, by (0,000, 0,000). The line connecting R and G is the locus of all points in which $r + g = 1$ and for which the value of b is zero. From the geometry of the diagram, Wright could conclude that *RG is the locus of saturated orange, yellow and yellow-green colours. Similarly, BG is the locus of colours in which r is zero, and includes blue-greens of varying predominance of blue or green, while RG is the locus of purples in which g is zero* (Wright 1944, p. 61).

In order to remove the negative tristimulus values of the RGB system, the CIE derived from the CIE RGB colour space another colour space, called CIE XYZ. New primaries were chosen to match all physically realizable colour stimuli via purely positive linear combinations[84]. However, this could be achieved only with imaginary primaries, i.e. not physically realizable, more saturated than monochromatic lights. The colour matching functions for the XYZ primaries are indicated by $\bar{x}(\lambda)$, $\bar{y}(\lambda)$ and $\bar{z}(\lambda)$ and are known as the colour matching functions of the CIE 1931 standard observer.

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[84] The precise mathematical treatment can be found in the text of Wright, *The Measurement of Colour*, of 1944 (Wright 1944, pp. 63-73).

The values of the XYZ tristimulus can be obtained in the same fashion as the RGB tristimulus values[85]. The equations are reported below:

$$X = k \int_{380}^{780} \Phi(\lambda)\, \bar{x}(\lambda)\, d\lambda \tag{6.6}$$

$$Y = k \int_{380}^{780} \Phi(\lambda)\, \bar{y}(\lambda)\, d\lambda \tag{6.7}$$

$$Z = k \int_{380}^{780} \Phi(\lambda)\, \bar{z}(\lambda)\, d\lambda \tag{6.8}$$

where k is a normalizing constant, $\bar{x}(\lambda)$, $\bar{y}(\lambda)$ and $\bar{z}(\lambda)$ are the colour matching functions and $\Phi(\lambda)$ is the spectral power distribution.

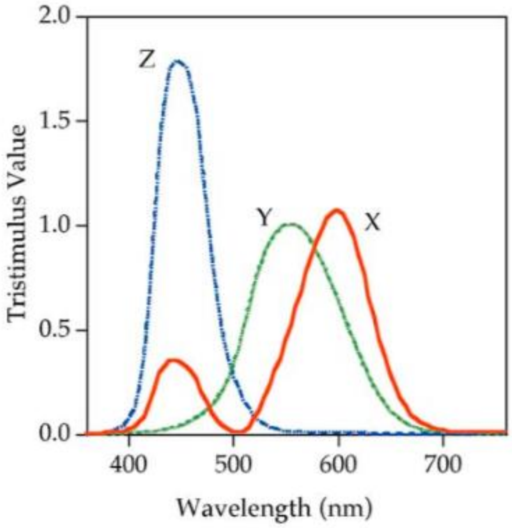

Fig. 6.4
Spectral tristimulus values of the CIE 1931 XYZ (Fairchild 2005, p. 73)

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[85] The two imaginary primaries X and Z were chosen such that they produced no luminance response, all the luminance response is left to the third primary Y. In this way, forcing one of the colorimetric functions to equal the photopic luminous efficiency function $V(\lambda)$, the CIE system of photometry, established in 1924, could be incorporated into the CIE system of colorimetry, established in 1931, as we have seen above (Fairchild 2005).

The primaries XYZ are the basis vectors of the CIE 1931 XYZ colour space with coordinates x, y and z, and $x + y + z = 1$. Hence, if x and y are known, z can always be deduced from $1 - x - y$. In fact, it is possible to build two-dimensional chromaticity diagrams, usually represented in the (x, y) plane, where $x = \frac{X}{X+Y+Z}$ and $y = \frac{Y}{X+Y+Z}$. An example is reported in Fig. 6.5, where the equi-energy stimulus is denoted by S_E and its coordinates are $x = 1/3$, $y = 1/3$ and $z = 1/3$.

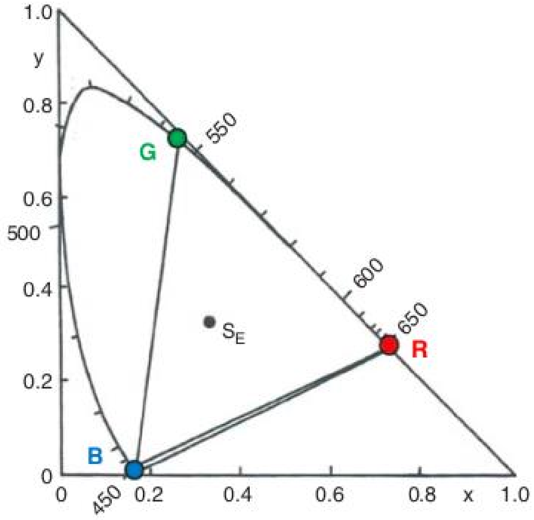

Fig. 6.5
The CIE (x, y) chromaticity diagram, showing the spectrum locus and equi-energy stimulus S_E, containing the CIE red, green and blue matching stimuli, R, G, B (Hunt and Pointer 2011, p. 43)

In the CIE (x, y) colour space depicted in Fig. 6.5 the colour representing an additive mixture of two colour stimuli is represented in the diagram by a point always lying on the line joining the two points representing the two original colours.

Although the CIE (x, y) chromaticity diagram has been widely used, I would like to point out that it presents a serious disadvantage, i.e. the distribution of the colour on it is non-uniform. Ideally, in fact, equal colour differences should be represented by lines of equal length, but it is far from being the case, as we have seen in the previous sections. Wright, who was aware of this problem, made the first serious attempt to measure just noticeable differences across this colour space. Wright's attempt was followed by the work carried out by the American physicist MacAdam, who derived the famous MacAdam ellipses in colour space, as I will briefly present in the following paragraph.

6.3 Wright and MacAdam's attempts to measure just noticeable differences

In 1941 Wright published a paper with the title *The sensitivity of the eye to small colour differences*, (Wright 1941) in which he presented his attempt to measure just noticeable differences across the CIE (x, y) colour diagram. Wright determined, by direct observations, just noticeable differences along straight lines in a plane of constant brightness. He made use of his colorimeter to realize a bipartite field. This time, however, each side of the field was illuminated by the same monochromatic lights, $\lambda_1$and λ_2, as he stated: *For convenience, the two halves of the field were first equated in colour and intensity. [...] The amount of, say,* λ_1 *was then altered until a difference in colour between the two halves of the field was detected, the intensity of the other half of the field being simultaneously adjusted to remove any brightness difference that might otherwise have been interpreted as a colour difference* (Wright 1941, p. 95). Then, the intensity of $\lambda_1$was returned to its starting point and the subject had to adjust it but in the opposite direction, until a difference was detected. The all procedure was repeated three times and the average values were taken in order to determine the step size. Wright used the same method applied by König, Dieterici and Brodhun, described in chapter 4, for their experiments on colour perception by using Helmholtz colour mixer.

Wright allowed the subject to adjust only one of the lights at a time in order to control the directionality along the line of interest through the CIE 1931 (x, y) colour diagram. The data obtained by Wright can be seen plotted in Fig. 6.6.

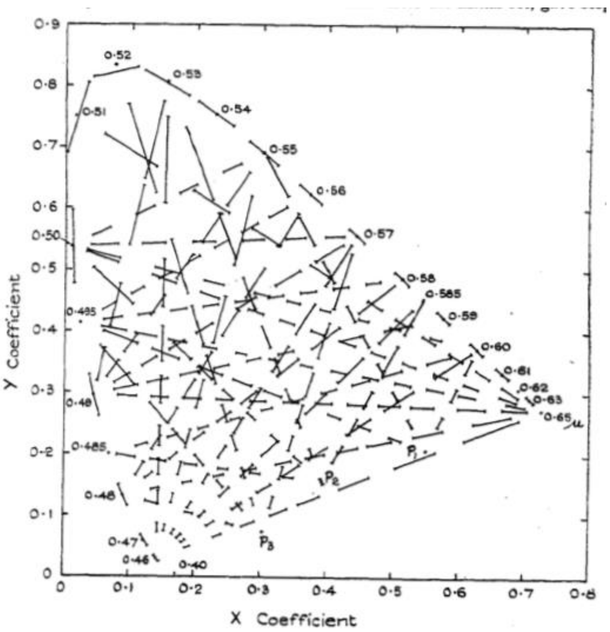

Fig. 6.6
Equal perceptual colour differences shown in the CIE (x, y) chromaticity diagram. Each bar indicates two noticeable differences, as measured on both sides of a mean colour stimulus by Wright (Wright 1941, p. 99)

Noticeable differences were assessed to be equal for each measurement. However, as Helmholtz first realized, equal perceptual differences did not correspond to equal distances in the CIE (x, y) chromaticity diagram, as clearly depicted in Fig. 6.6.

In 1942, MacAdam published his paper entitled *Visual sensitivities to color differences in daylight* (MacAdam 1942), which contains his experimental determination of the famous ellipses in CIE (x, y) chromaticity diagram. His aim was that of deriving just noticeable difference thresholds for chromaticity discrimination of 25 colour samples. He used a bipartite comparison field of view; one colour was a test and the other colour was subjected to small variation in chromaticity by the observer, until it matched the test colour. MacAdam worked on a plane of constant CIE luminance, as he wrote: *the luminance of the test field was maintained constant at 15 millilamberts* (MacAdam 1942, p. 247), which correspond to a luminance of about $48\ cd/m^2$. The results were then plotted in the CIE (x, y) diagram. He found out that colours, within a certain small region in the diagram, appeared identical to a human eye. As we can see in Fig. 6.7, all the matches fell into an ellipse on the diagram and each ellipse contains all the colours that are indistinguishable to the human eye from the colour at the centre of the ellipse. The perimeter of the ellipses represents the just noticeable difference of chromaticity; therefore, the smaller the perimeter, the better the discrimination of chromaticity. In Fig. 6.7 the ellipses vary so much in size, shape and orientation; the reason of this is contained in Helmholtz's papers of 1891 and 1892.

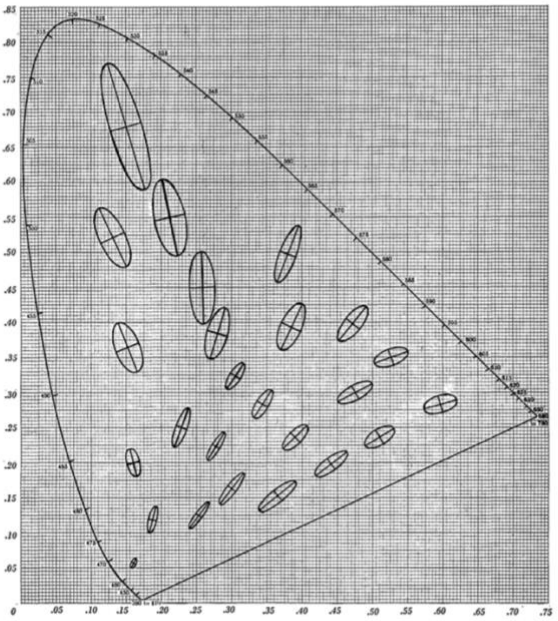

Fig. 6.7
MacAdam's ellipses, represented ten times their true size, on the CIE (x, y) chromaticity diagram, as determined by MacAdam (MacAdam 1942, p. 271)

In order to offer a better comparison of the discrimination of chromaticity for the 25 test colours, I report here a coloured version of the original plot of MacAdam.

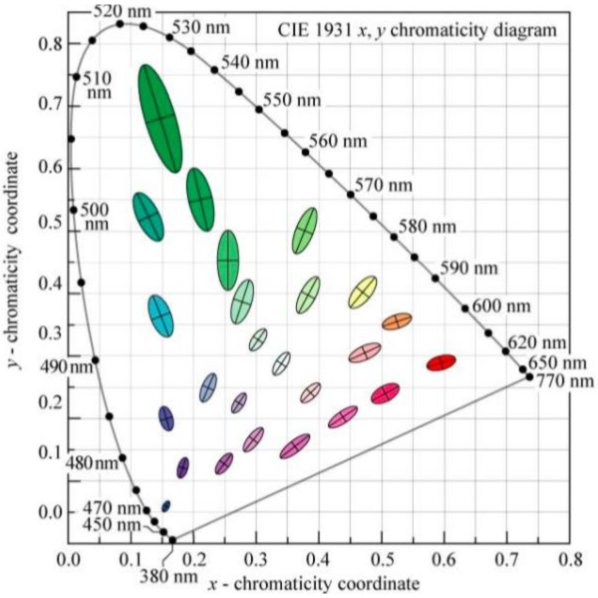

Fig. 6.8
Coloured version of MacAdam's ellipses represented on the CIE (x, y) chromaticity diagram (Schubert 2006, p. 297)

From Fig. 6.8, it is evident that the ellipses in the blue region of the spectrum are much smaller than those in the green region; this should indicate that the human eye is more discriminating in blue hues than green. However, this is true if we restrict our considerations to the luminance value under examination and to the particular colour space in which the results are plotted. In this respect, I would like to report MacAdam's original words: *The noticeabilities of colour differences involving simultaneous luminance and chromaticity variations are expected to define ellipsoids around the points representing every color in the color solid*[86]*. Such ellipsoids will be required for the complete representation of color differences. [...] The development of a more elegant manner of specifying the noticeabilities of colour differences than the system of ellipses [...] requires an application of the principle of differential geometry* (MacAdam 1942, p. 272).

The MacAdam's ellipses rapidly became a key set of data used as test data for line elements and colour difference formulae. W. R. J. Brown (1957), G. Wyszecki and G. H. Fielder (1971) and many other researchers and colour theorists derived chromaticity difference data sets based on MacAdam's ellipses[87]. In the 1950s and 1960s attempts were made to convert the ellipses to circle of equal size, modifying locally the chromaticity diagram. However, from the results of such calculation, it was clear that the linear transformation of the CIE chromaticity diagram does not yield a uniform chromaticity diagram in which the MacAdam ellipses form circles of equal size. The first serious attempt to define an approximately uniform colour space was the model proposed in 1960, the CIE Luv diagram, which was obtained via a linear transformation of the 1931 CIE diagram. Here, colours are represented by specifying *L*, the luminance and two normalized proportions, *u* and *v*. However, this space presents a problem, i.e. yellow, brown, orange and red colours, which are colours of great importance for what concerns the textile and printing industries, appeared compressed into a small area. To face this problem, another model was derived via a non-linear transformation of the 1960 CIE Luv diagram, i.e. the 1976 CIE Lu'v' system, which presents a further improvement in the correlation of Euclidean distances between colours and perceived colour differences. Another model was derived in the same year via another non-linear transformation, this time directly from the 1931 CIE colour space, the CIE Lab colour space, in order to achieve a truly three-dimensional representation of colours. Here, colours are organized around three perpendicular axes: a vertical axis is denoted by *L*, which ranges from white to black, and two horizontal axes, *a* and *b*, which range from red to green and from yellow to blue respectively (Hasting and Rubin 2012, pp. 136-137). Over the course of the years, many different colour differences formulae and colour space models have been developed to meet the needs of more and more sophisticated printing, textiles and computer

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[86] Ellipsoids in a three-dimensional space which included differences in luminance were, in fact, first obtained by Brown and MacAdam (Brown and MacAdam 1949).

[87] For further details, the reader is referred to chapter 8 of the text written by Kuehni and Schwarz (Kuehni and Schwarz 2008, pp. 311-335), which offers a more complete view of the topic with related bibliography, and to the text written by Fairchild, *Color appearance models* (Fairchild 2005).

graphics industries[88]. However, as already mentioned, to predict in a precise manner visual difference between two colours is still challenging because of the non-existence of a perfect uniform colour space.

6.4 Helmholtz Memorial Symposium on Color Metrics

In this last section, I would like to say a few words on the meeting of the *International Color Association*, AIC, held in 1971 in Driebergen on the topic of small colour differences, in order to underline, once again, the impact of Helmholtz's work on the field of colour metrics. The title of the symposium was *"Helmholtz Memorial Symposium on Color Metrics",* not only in occasion of the 150th anniversary of the birth of Helmholtz, but also because he was considered as the undisputed founder of colour metrics. The symposium *was meant to be a meeting between workers in the applied fields, confronted daily with problems of industrial color tolerances, and psychophysics, primarily concerned with understanding basic mechanisms*, as declared by the Organizing Committee. In fact, a whole range of different colour difference formulae passed the review. Among the 64 participants, I would like to cite MacAdam, Stiles, Vos and Walraven, whose contributions were published in the conference proceedings[89]. Stiles opened the conference presenting a detailed review of the history of the line element in colour theory, from Helmholtz's pioneering work up to the models proposed in the 1960s (Stiles 1972), anticipating also the contribution of Vos and Walraven with the following words: *As Drs Vos and Walraven are presenting their line-element in a paper at this Symposium, I need not attempt any summary in a few words. I would add however that this work seems to me an example, par excellence, of the inductive approach to the line element* (Stiles 1972, p. 22).

It is in this occasion, in fact, that Vos and Walraven presented their line element in a contribution entitled *A zone-fluctuation line element describing colour discrimination* (Vos and Walraven 1972). Without entering into details, I would just mention that their line element was based on photon noise methodology in place of the empirical Weber-Fechner law. They assumed that, before a difference signal can reach the brain, the output signals of the individual cone mechanism are integrated by combining two chromatic and one luminance signal. With their line element, they could easily calculate the MacAdam's ellipses for different luminance levels and for some of the MacAdam's points, seen in the previous section, by computer program. I would like to remark that in the late

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[88] One important task of modern colorimetry, for example, is to develop a colour appearance model capable of predicting appearance across a wide range of viewing conditions, in order to satisfy the recent demands on cross-media reproduction to match the appearance of a colour on a display to that on hard copy paper.

[89] The proceedings, edited by Vos, Friele and Walraven, were published in 1972 (Vos et al. 1972).

1960s, a wide number of controversies and open questions abounded, and the question of change in ellipse’s size as a function of luminance can be counted among them. Vos and Walraven found out that *the influence of luminance remains restricted to smaller effects on shape and orientation* (Vos and Walraven 1972, p. 76). I report here in Figs. 6.9, 6.10 and 6.11 the JND ellipses calculated for three different luminance levels.

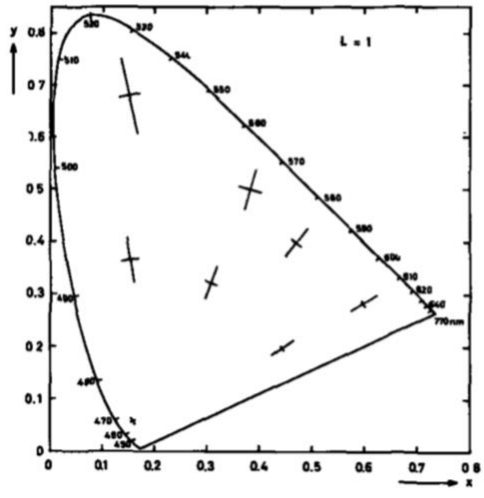

Fig. 6.9
JND ellipses calculated by Vos and Walraven for low luminance levels (Vos and Walraven 1972, p. 77)

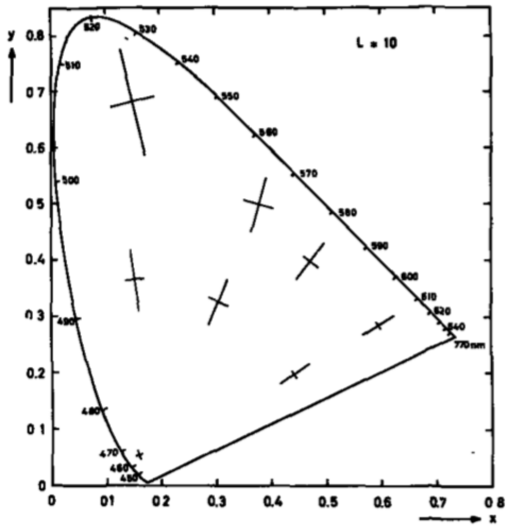

Fig. 6.10
JND ellipses calculated by Vos and Walraven for medium luminance levels (Vos and Walraven 1972, p. 77)

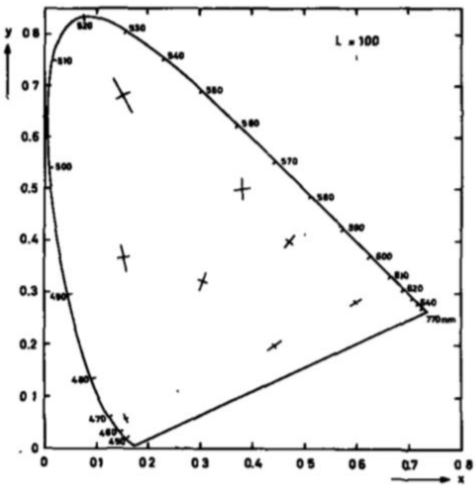

Fig. 6.11
JND ellipses calculated by Vos and Walraven for high luminance levels (Vos and Walraven 1972, p. 77)

Vos and Walraven offered also a direct comparison between their theoretical results and the experimental ellipses obtained by MacAdam. For the lowest brightness levels, they obtained an equally good agreement between theoretical and experimental results, whereas it began to decrease with increasing brightness.

MacAdam's contribution was related to the above mentioned topic. He made some observations, in fact, on the role of luminance increments in small colour differences (MacAdam 1972). He analysed the four colour difference formulae recommended for trial by the *International Commission on Illumination* and reported the findings of two colour theorists, Schulze and Gall, who demonstrated that all of them, except one, define equal-colour-difference ellipsoids with tops tipped toward white in the CIE colour space.

During the general discussions of the conference, key questions arose and open problems were discussed. It is worth reporting some of the debate topics because they provided valuable insights into the future of colour metrics.

The discussion leader of the first general discussion, the well-known colorimetry expert R. W. G. Hunt, formulated the following question:

What is a line-element? How does it differ from a colour difference formula?

I report the answers provided by some of the participants below.

***Stiles**:*

I might try to say what I think is the essence of a line element. But I would hesitate to say in what way a colour difference formula is essentially different. A line element, in my opinion, is an expression which relates the perceptibility of differences between neighbouring colours in a form which should *show how these differences become perceptible through different mechanisms of the eye. This is the inductive view, of course. The parameters should, if possible, have some physiological or physical meaning and the element should also be applicable, if possible, to extreme as well as the common conditions met with in practice. It should also be capable of modification in the light of new additions to our ideas of the visual system.*

MacAdam:

The concept of a line element, in my opinion, introduced notions foreign to color and might better be avoided in discussions of color-difference formulas. It prejudices most people against formulas associated with noneuclidean geometry, which formulas are not essentially more difficult and may be more appropriate than the euclidean forms. Positive definite quadratic forms, in general are appropriate for color differences because, at threshold, the effect of a positive increment is equal to the effect of a negative increment of the same kind and amount, from the same starting color. Such behavior can be represented, in general, only by a positive definite quadratic form. It is only a coincidence, and a misleading one, that the same form is also used in Riemannian geometry. Only physiological and colorimetric considerations should govern the formulation of color-difference formulas. Geometry is foreign to them, and has the effect of limiting the options.

Wright:

The term "line element" does not seem to me to have any visual meaning at all and to have no obvious connection with the visual processes responsible for colour discrimination. I also find it objectionable because it encourages the discussion of colour discrimination in purely mathematical terms. For example, in Dr. Stiles' very impressive lecture this morning, I do not think he once referred to the colour sensations themselves, and yet without the generation of redness, greenness, etc. we would have no colour discrimination at all.

Stiles:

Just some 10-20 years before Helmholtz published his line element concept, Riemann had been developing analytical geometry and used this expression for the distance measurement. I had not essentially to do with colour, of course. Helmholtz, himself a great mathematical physicist as well, referred to Riemann in his treatise and took the term from it. It has a drawback, of course, because it tends to make us think of colour differences in terms of a sum of squares or at least as a quadratic form- which is not experimentally certain. I do not think, however that a more elaborate form is likely to help much.

Wyszecki:

It satisfies our desire to put some mathematical formalism into our world of color.

Wright:

I think this is a disadvantage. It tends to conceal the physiological and perceptual aspects of the problem.

MacAdam:

And yet each maker of systems—Oswald, Munsell etc.—has thought of colour in terms of a mathematical model. It is almost inherent to it. Riemann (1854) himself wrote: "The position that the objects of sense and the colors are probably the only familiar things whose specifications constitute a multiply extended manifold". So this association of colors with points in space is as old as Riemannian geometry —and I think it is inevitable. Now unfortunately, to truly simulate the relations of colors, this geometrical model is going to be awfully complicated. It is not as simple as euclidean space. I have often thought that we might better grow out of thinking in terms of this geometrical model. We have to stay close to the data. We want a formula for color difference and may have to stop cluttering up our minds and the literature with line elements. That's why I don't speak, or write or draw graphs of the metric coefficients, g_{ik}, *any more.*

Stiles:

I disagree on Dr. MacAdam's last remark. At the moment I don't see any prospect of getting away from the quadratic summing concept of the line element as a workable theoretical tool.

Wyszecki:

It is just a mathematical formula, I don't see why we have to talk about it in terms of geometrical concepts.

Stiles:

No, but I am talking about it as an expression with structure whose terms can be associated with different mechanisms.

Wright:

This illustrates my point. All this mathematics has nothing to do with the colour sensations themselves.

Nimeroff:

The problem we have been discussing in this session is a philosophical one in which there is the real world of stimuli and responses on one hand and the abstract mathematical world on the other. In order to use the abstract mathematics to help understand the real world we need to establish the interfacing between these two worlds (Vos et al. 1972, pp. 114-123).

From these statements, it emerges clearly Wright's purpose of leaving mathematics separated from colour sensations. Also MacAdam declared his intention to stay closer to the data and to distance himself from the notion of line element. Nimeroff, on the contrary, recognized that the beauty of the line element treatment lies precisely in the intertwining of psychophysics and mathematics. This was, in fact, Helmholtz's original idea.

Maxwell acknowledged the work carried out by Helmholtz to create connections between isolated fields of knowledge. In a *Rede Lecture* with the title *The telephone*, held in Cambridge on 14 May 1878, a year before his death, Maxwell expressed a few considerations on the new-born field of psychophysics, which he recognized as an important result of what he called *cross-fertilization of the sciences* (see also chapter 3, section 3.16). It is worth reporting Maxwell's original words:

Among all the recent steps in the progress of science, I know none of which the truly scientific or science-producing consequences are likely to be so influential as the rise of school of physiologists, who investigate the conditions of our sensations by producing on the external senses impressions, the physical conditions of which can be measured with precision, and then recording the verdict of consciousness as to the similarity or difference of the resulting sensations. [...] No man has done more than Helmholtz to open up paths of communication between isolated departments of human knowledge; and one of these, lying in a more attractive region than that of elementary psychology, might be explored under exceptionally favourable conditions, by some of the fresh minds now coming up to Cambridge (Niven 1890, vol. II, pp. 753-754).

In view of the above reported discussion between the conference participants, I would like to add a short remark on the role played by mathematics in the natural sciences. In the inaugural lecture held before the British Association in Liverpool on 15 September 1870, as the President of the Mathematical and Physical Sections, Maxwell reflected on the way scientific knowledge grows. He stated: *It was a great step in science when men became convinced that, in order to understand the nature of things, they must begin by asking, not whether a thing is good or bad, noxious or beneficial, but of what kind is it? and how much is there of it? Quality and Quantity were then first recognized as the primary features to be observed in scientific inquiry. [...] It is this scientific method of directing our attention to those features of phenomena which may be regarded as quantities which brings physical research under the influence of mathematical reasoning. [...] The student who wishes to master any particular science must make himself familiar with the various kinds of quantities which belong to that science. [...] But when the student has become acquainted with several different sciences, he finds that mathematical processes and trains of reasoning in one science resemble those in another so much that his knowledge of the one science may be made a most useful help in the study of the other. When he examines into the reason of this, he finds that in the two sciences he has been dealing with systems of quantities, in which the mathematical forms of the relations of the quantities are the same in both systems, though the physical nature of the quantities may be utterly different* (Niven 1890, vol. II, pp. 217-218).

With these words Maxwell expressed in a very clear manner his conception that the same mathematical reasoning could be applied to different fields of knowledge in order to enhance our understanding of both sciences[90]. From this fruitful exchange between mathematics and physics, according to Maxwell, scientific knowledge can grow. Wright was afraid that mathematics could *conceal the physiological and perceptual aspects of the problem*, instead of recognizing the precious role that mathematical processes could play, as a tool, to shed light on human colour perception, improving, therefore, our understanding of colour vision. To conclude, I would like to report some considerations of Eugene Wigner, who offered an analysis of the role of mathematics in the natural sciences in his paper *The unreasonable effectiveness of mathematics in the natural sciences* (Wigner 1960). In order to answer to the question as to why mathematics works so well in describing some parts of the natural world, Wigner provided some examples of mathematical concepts, developed in one context, and then, turned out to have an unexpected but effective application also in another context, confirming, thus, Maxwell's idea. He cited, for example, the complex Hilbert space which revealed to be invaluable in the formulation of quantum mechanics. In relation to this, Wigner wrote: *It is difficult to avoid the impression that a miracle confronts us here* (Wigner 1960, p. 7). He described the role of mathematics in the natural sciences as *bordering on the mysterious* and *there is no rational explanation of it* (Wigner 1960, p. 2). And he went further stating that: *it is important to point out that the mathematical formulation of the physicist's often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena. This shows that the mathematical language has more to commend it than being the only language which we can speak; it shows that it is, in a very real sense, the correct language* (Wigner 1960, p. 8). Therefore, to Wright's statement that *All this mathematics has nothing to do with the colour sensations themselves,* we can reply using Wigner's words: *The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning* (Wigner 1960, p. 14).

During the last general discussion, some significant considerations emerged on the representation of colour in terms of vectors and on the need to develop a unified colour model, which I would like to report and briefly comment.

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[90] Maxwell exploited these formal similarities between different fields of knowledge in what he defined the method of *physical analogy* (see also chapter 3, section 3.2). For a complete account of Maxwell's use of analogies and metaphors, the reader is referred to the text of Peruzzi, chapter I (Peruzzi 2010, pp. 11-30).

Nimeroff:
I have missed in our discussions the term vector. We talk about color-differences as if they were only scalar, but they have both size and direction! We have here emphasized only the magnitude aspects and neglected the important directional aspects.
Vos:
I don't understand your problem. Did not we talk about ellipses, with size and direction —so did not we discuss vectors, be it under another name?
MacAdam:
A vector that depends on both location and direction—that is what we call a tensor. The ellipses, or in general ellipsoids, portray the perceptibility of color difference (or, operationally, the variance of color matching) as a tensor. But, I don't think we need get involved with such semantic niceties or with the mathematics of vectors or tensors. The minimum requirement that is not now satisfied and does not seem likely to be satisfied in the near future, is a generally accepted formula or method for calculating a single number to specify any color difference. If we had that, its usefulness would not be destroyed if someone called it a scalar. If we had it, we could then consider how to resolve it, i.e., break it down into its components—hue, saturation and brightness, or any other triad you prefer. But, first of all, we need to agree on a single formula for color difference that we will all use, and not fly off in all directions deriving, or using, others (Vos et al. 1972, pp. 330-331).

Nimeroff, as we can read, tried to draw attention back to the vector nature of colour, topic that was first introduced by Grassmann (see chapter 3, section 3.2), which was, in his view, neglected during the course of the conference. MacAdam, however, considering the mathematics of vectors or tensor as a marginal aspect in their discussion, underlined another important task, which had to be discussed, i.e. the requirement of a general accepted formula or method to specify any colour difference.
Since the CIE established the first colour systems, which represent the basis for modern colorimetry, researchers and colour theorists have developed over the course of the years many different colour difference formulae and colour spaces in the hope of finding a unified colour model. I would like to report Fairchild's point of view on this subject with his original words: *Will there ever be a single color appearance model that is universally accepted and used for all applications? Absolutely not! The problem is too complex to be solved for all imaginable applications by a single model* (Fairchild 2005, p. 357). Different specific applications require a wide varieties of different colour models; therefore, although considerable progress has been made in the last years, a unified model is still missing at the present time.

Chapter 7

Concluding Remarks

Colour vision has always attracted the attention of philosophers, painters and scientists throughout the ages but we have to wait until the 1850s before assisting to the birth of the modern science of colour, inaugurated by Maxwell and Helmholtz's fundamental contributions. The present study offers an historical account of the development of Maxwell and Helmholtz's theories of colour. Especially, particular attention has been devoted to the role played by Newton and Young's pivotal ideas on the evolution of their own work, ideas which they exploited, refined and elaborated, both theoretically and experimentally, and which therefore constitute a fundamental starting point for their research. Although the topic has already been explored by many historians of physics, historians of science and philosophers of science, whose major contributions can be found in the bibliography, our intention was to provide an original contribution to the historical studies on the rise of the modern theory of colour, offering an overall picture of Maxwell and Helmholtz's works, enriched with unpublished material, which was still missing to date, as far as we know.

Moreover, from a deep analysis of Maxwell and Helmholtz's major and minor contributions to colour science, an important aspect has emerged, which both scientists recognized, i.e. in the science of colour vision the quantitative dimension is intimately linked with the subjective one. In this regards, as we have shown in the text, Helmholtz expressed pivotal considerations on the geometry of colour space, starting with an analysis of his colour diagram, in which equal distances did not correspond to equal perceptual differences. Helmholtz, having at his disposal rich material provided by his collaborators and assistants and drawing on the Weber-Fechner law of the new-born field of psychophysics, was able to produce a metrically significant model of colour space. This model constitutes the first serious attempt to "quantify the subjective" (Isaac 2013) in the field of colour. This has led us to carry out a study of the original works of Helmholtz's collaborators, as well as a deep analysis of Helmholtz's papers of 1891 and 1892, which contain the definition of his line element in colour space. The goal was to retrace the path followed by Helmholtz to obtain his line element model to describe the mechanism of human colour vision. The translations in English of German primary sources, which we have provided in the text, will be useful instruments for historians of physics and more generally for historians of science, who will be interested in further developing the presented subject. It is worth to highlight, in fact, that the breadth and complexity of the subject may unveil other aspects and insights for future investigations. Especially, in light of the outlined importance of Helmholtz's contributions to the geometry of colour space, a more complete and detailed review of all subsequent contributions to the field of colour could be of historical relevance, to better understand the development of modern colour spaces and colour different formulae, as well as the modern colorimetric techniques in use today, and also to shed light on some problems which have remained unsolved until the present time.

I would like to underline another fascinating aspect which has emerged, namely that the present study, as we have tried to emphasize in the body of the text, involves different fields of knowledge, providing a precious link between history of physics, mathematics, psychophysics and experimental psychology, link first recognized in the mid-19th century by Maxwell and Helmholtz. This has offered the possibility to create fruitful contacts with various research groups, who provided valuable support to the research activity. Firstly, I would like to acknowledge the work of Sara Magrin and Roberto Temporin, responsible of the Didactic Laboratories of the Department of Physics and Astronomy of the University of Padova, who provided, in addition to the technical equipment, complete assistance and precious advices for the realization of a preliminary replica of König's experiment described in his paper of 1888. Moreover, of pivotal importance for the research process was the interactions with librarians working at the *Biblioteca di Storia della Scienza* and at the *Biblioteca di Fisica ed Astronomia* of the University of Padova, who kindly supplied textbooks and articles, and with archivists, who provided access to rich unpublished material, such as Helmholtz and Maxwell's manuscript collections, which underwent only a partial analysis, circumscribed to the present work, constituting therefore the object of future studies. It is worth, thus, to express our gratitude to all personnel working at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften,* BBAW, at the *Handschriftenabteilung of the Staatsbibliothek zu Berlin* in Berlin and at the archive of the *Deutsches Museum* in Munich.

As a concluding remark, I would like to highlight that, from a deep analysis of both primary and secondary sources, aimed to retrace the path followed by Maxwell and Helmholtz to elaborate their theory of colour, the interdisciplinary nature of the topic under examination has clearly emerged, which thus represent a perfect example of *cross-fertilization of the sciences* and could open the doors to future collaborations between the different research fields mentioned above.

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Appendix A

Maxwell's colour photograph

A hundred years ago the great physicist projected a photograph in full color. How this was done has been something of a puzzle (R. M. Evans, Evans 1961, p. 118)

Among Maxwell's great achievements, it is worth mentioning that he produced the world's first colour photographic image. In this appendix, I will give a brief account of the method used by Maxwell to obtain the first photograph, an accidental invention, as we will see in the following lines. In his paper of 1855 *Experiments on Colour, as perceived by the Eye, with Remarks on Colour-Blindness*, Maxwell proposed a *supposed case*[91], citing his original words, to support Young's three-receptor theory. It is worth reporting Maxwell's words to describe the method to produce a full colour image: *This theory [The Young's three-receptor theory] of colour may be illustrated by a supposed case taken from the art of photography. Let it be required to ascertain the colours of a land- scape, by means of impressions taken on a preparation equally sensitive to rays of every colour. Let a plate of red glass be placed before the camera, and an impression taken. The positive of this will be transparent wherever the red light has been abundant in the landscape, and opaque where it has been wanting. Let it now be put in a magic lantern, along with the red glass, and a red picture will be thrown on the screen. Let this operation be repeated with a green and a violet glass, and, by means of three magic lanterns, let the three images be superimposed on the screen. The colour of any point on the screen will then depend on that of the corresponding point of the landscape; and, by properly adjusting the intensities of the lights, & c., a complete copy of the landscape, as far as visible colour is concerned, will be thrown on the screen. The only apparent difference will be, that the copy will be more subdued, or less pure in tint, than the original. Here, however, we have the process performed twice- first on the screen, and then on the retina* (Maxwell 1855, pp. 283-284).

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[91] This expression used by Maxwell corresponds to what we now call *though experiment*, which is the English translation of the German word *Gedankenexperiment*, term coined by the Danish physicist Hans Christian Ørsted within the context of German *Naturphilosophie* (Witt-Hansen 1976).

Six years later, on 17 May 1861, Maxwell was able to apply this recipe exhibiting the trichromatic colour photograph of a tartan ribbon during a lecture before the Royal Institution in London. For the photographic details, Maxwell could rely on the technical assistance of Thomas Sutton, a teacher and lecturer on photography. By using separate magic lanterns, three images were superposed, obtained by photographing the ribbon separately through red, green and blue filters. Maxwell suggested that the images on his photographs represented the red, the green and the blue parts of the ribbon separately: *Three photographs of a coloured ribbon taken through the three coloured solution respectively, were introduced into the camera, giving images representing the red, the green and the blue parts separately, as they would be seen by each of Young's three sets of nerves separately. When these were superposed, a coloured image was seen, which, if the red and green images had been as fully photographed as the blue, would have been a truly-coloured image of the ribbon* (Niven 1890, vol. I, p. 449). For a complete account of the technique used by Maxwell, the reader is referred to the paper *Maxwell's color photograph* written by Evans, who worked at the *Color Technology Division of Eastman Kodak*, in occasion of the centenary of the birth of colour photography (Evans 1961). As Evans remarked, there was one surprising thing about Maxwell's demonstration: it should not have worked. For filters, Sutton used ammoniated cupric sulfate for blue, cupric chloride for green and ferric thiocyanate for red. It is certain that these photographic emulsions were sensitive only to the extreme blue end of the spectrum and not to green and red. Maxwell was aware of the limitations of the materials used to product photography at his time, as testified by the following words: *By finding photographic materials more sensitive to the less refrangible rays, the representation of the colours of objects might be greatly improved* (Niven 1890, vol. I, p. 449). Nevertheless, the demonstration was successful and impressed the audience. Many doubts have been raised over the course of the years on how Maxwell actually managed to obtain a trichromatic colour photograph.

Fig. A.1
Reproduction of the first colour photographic image of a tartan ribbon obtained by Evans and collaborators on Ektacolor Print Film from copies of Maxwell's original transparencies (Evans 1961, p. 119)

Fig. A.2
The look of Maxwell's separation positives. Here the positives are printed in blue, green and red ink to simulate light of the colour in which they were projected (Evans 1961, p. 119)

Fig. A.3
Solutions of metallic salts in glass cells used by Sutton as filters. We can see the blue filter of ammoniated cupric sulfate (left), the red filter of ferric thiocyanate (second from the left) and the green filter of cupric chloride (remaining samples). The solutions contained in the three cells represent the concentration used by Sutton, whereas the two flasks show us how cupric chloride shifts to bluish-green if diluted (Evans 1961, p. 120)

The mystery was unveiled by Evans and his collaborators at the *Color Technology Division of Eastman Kodak* in 1960.

The best way to solve the mystery was that of reproducing the original procedure of Sutton. The group, in fact, used a photographic material with the same sensitivity as that implied by Sutton and recreated the chemical solutions that he used as filters. This gave rise to a colourful reproduction of the original image obtained by Maxwell (see Fig. A.1). At this point, Evans and collaborators were able to explain the surprising result. Following the same procedure of Sutton, they obtained a film sensitive only to extreme blue and ultraviolet. As Evans explained, in fact, the emulsion was sensitive to ultraviolet light, which was reflected by the red pigments of the ribbon and transmitted by the filter and camera lens, which were transparent in correspondence to that specific frequency band. *As a result*, reporting Evans' words, *a red object can produce a strong image on the "red" plate not because it is red but because it is more ultraviolet than objects that to our eyes look blue and green* (Evans 1961 p. 125). Hence, it was thanks to a series of favourable coincidences that Maxwell could obtain the desired result, impressing the audience.

To mark the centenary of Maxwell's demonstration, the Colour Group, in collaboration with The Institute of Physics and The Physical Society and the Inter-Society Colour Council of America, organized a conference on 16, 17 and 18 May 1961. The extended abstracts of the 18 papers on the subjects Trichromatic Principles, Colour Reproduction and Colour Appearance were published in the proceedings of the conference.

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Appendix B

Colour vision deficiencies

This appendix provides a short overview of the main types of colour deficiency, in order to better understand König, Dieterici and Brodhun’s pioneering work on the subject, which can be considered as a milestone in the evolution of colour vision science. We will basically refers to the works of Parry[92], *Color vision deficiencies*, published in 2015, that of Malacara[93], *Color vision and colorimetry: theory and applications*, published in 2011 and to the contribution of the famous American physicist, Dean B. Judd, who published in 1943 *Facts of Color-Blindness,* an important contribution to the field of colorimetry.

Before introducing deviations to the normal human colour vision, I would like to offer a brief outline of the basic mechanism of colour vision.

In the human retina two types of light-sensitive elements are present, i.e. *rods* and *cones*. The rods and the cones are responsible for night (scotopic) vision and daylight (photopic) colour vision, respectively. The colour-sensitive elements are of three different kinds, *L-cones*, more sensitive to long wavelengths, *M-cones*, more sensitive to medium wavelengths and the *S-cones*, for short wavelengths. For what concerns the location of the cones, the majority of them is located in the central part of the retina, the fovea, where the sharpest vision occurs and where no rods are present. There are fewer S-cones than L- and M-cones, and they possess spatial properties that make them different from the other two types. I will cite only some of them. They are sparsely distributed in the retina and they are almost absent in the fovea; they are not able to detect the border of an imagine; although they do not contribute to the luminance, they highly contribute to the hue and chroma (mainly yellow-blue discrimination). The rods are the most prevalent in the retina, constituting more than 95 % of all photoreceptor cells; although they do not contribute to colour vision, they activate under limited twilight conditions.

A precise knowledge of the cone spectral sensitivities is crucial to understand the human normal colour vision, and, therefore, the reduced forms of colour vision, in which one or more of the cone

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[92] Clinical Scientist at the Manchester Royal Eye Hospital.

[93] Professor at the Centro de Investigaciones en Optica, Leon, Gto, Mexico.

types is missing. The study of cone spectral sensitivities nowadays embraces many fields of inquiry, including physics, psychophysics, anatomy, biophysics and physiology. As we have shown in chapter 4, the first estimates of the three cone spectral sensitivities, called *Elementarempfindungskurven* (elementary sensation functions) and *Grundempfindungskurven* (fundamental sensation functions), was obtained by König and Dieterici. Their derivation depended on the hypothesis that what they called *green-blind* and *red-blind* subjects lack one of the three cone types but their two curves were assumed to be identical to their counterparts in normal colour system. Since 1886, several estimates of the three cone sensitivities were obtained, e.g. by Guild and Wright (1931), Judd (1945), Wyszecki and Stiles (1967) and Vos, Estévez and Walraven (1990). Several methods to determine the sensitivities have been used over the course of the years. One of this method, as reported by Malacara, consists in the isolation of one type of cone in order to measure it by using as many different procedures as possible. In a work published by Stockman and Sharpe (Stockman and Sharpe, 1999) with the title *Cone spectral sensitivities and color matching* we can find a good review of the subject. For a general overview of the different measuring techniques, the reader is referred to the text of Malacara (Malacara 2011, pp. 51-58). Malacara also reported that the latest numerical values of the cones' sensitivities have been reported by Stockman and Sharpe in 2000 (Malacara 2011, p. 52). In Fig. B.1 the relative spectral sensitivities for the three types of cones are reported.

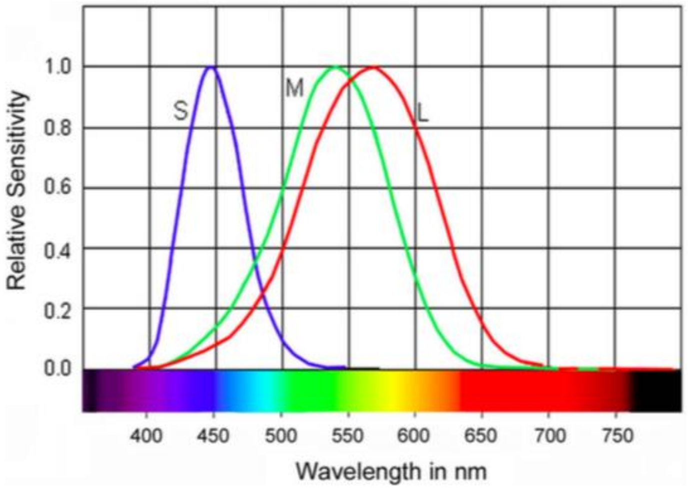

Fig. B.1
The relative cone spectral sensitivities, taken from *Color vision and colorimetry: theory and applications* (Malacara 2011, p. 53)

B.1 Trichromacy

Trichromatic, or colour-normal, individuals possess all three types of cones S, M and L, spectrally tuned to around 420, 530 and 560 nm, respectively. As mentioned by Parry, the term trichromacy comes from colour-matching studies, in which the observer should compare a mixture of three spectral lights with a given hue. In order to match any colour, trichromats need to use a combination of three lights. It exists a case of anomalous trichromacy, first investigated by König and Dieterici. Individuals affected by anomalous trichromacy have a defect in the photopigment coding for either L- or M-cones. This defect is called *protanomaly* when the L-cone is affected, whereas *deuteranomaly* indicates a defect of the M-cone. It is still not clear today whether *tritanomaly* exists[94], i.e. a defect affecting the S-cone. An observer possessing anomalous trichromatic vision requires a combination of three lights to produce all colours which he is capable of experiencing, as a normal trichromatic individual does. What changes, in this case, is the proportions required to produce a given colour. If an individual is extremely affected by anomalous trichromacy, the equal-energy spectrum will scarcely differ from that of a dichromat. However, as Judd reported, the anomalous observer can make with difficulty distinctions between colours, that a dichromat cannot make at all.

B.2 Dichromacy

An individual possessing a dichromatic visual system requires a combination of only two lights to produce all colours which he is capable of experiencing. He is able to make two visual discriminations, one achromatic and the other one chromatic, either yellow-blue or red-green (Judd 1943, p. 298). Commonly, individuals lack either L-cones, and in this case they are called *protanopes*[95] and *protanopia* (or red-blindness) their deficiency, or M-cones, leading to *deuteranopia* (or green-blindness) and the affected individuals are called *deuteranopes*. For both types of dicrhomats, the spectrum appears to them in two hues only, i.e. the short-wave end of the spectrum appears blue and the long-wave end appears yellow. These two regions are separated by the so called *neutral point*, which has for them no hue at all. Judd reported that this region is located at 495 nm for both *protanopes* and *deuteranopes* (Judd 1943), whereas an investigation conducted by Hecht and

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[94] Such a case was discovered by Engelking in 1925 (Judd 1943).

[95] Johannes Adolf von Kries proposed the terms *protanopia* and *deuteranopia* in 1897 (Judd 1943, p. 298).

Schlaer in 1935 on twenty-one *protanopes* and twenty-five *deuteranopes* furnished the values of 496,5 and 510,2 nm for the position of the neutral points of *protanopes* and *deuteranopes*, respectively (Hecht and Schlaer 1936).

Another form of dichromacy is the yellow-blue blindness, a rare form associated chiefly with diseases of the eye. The first type of yellow-blue blindness is called *tritanopia*. An individual affected by tritanopia sees only one neutral point, which divided a red from a bluish green region. This neutral point is located at about 570 nm, as reported by Judd. For what concerns the second type of yellow-blue blindness is called *tetartanopia*. Individuals affected by this deficiency see two neutral points, one at about 580 nm and the other at about 470 nm (Judd 1943, p. 299). To a tetartanope, Judd wrote, the spectrum appears red from the long-wave end down to the first neutral point, green between the two neutral points and red again at the short-wave end.

B.3 Monochromatism

Individual affected by *monochromaticy*, also known as *achromatopsia*, are the real colour blind; in fact, they only need one light to match any hue. They can match any spectral light only on the basis of luminance, or, in other words, they are capable of light-dark discrimination but not of chromatic discrimination. In this case, there is no need to speak of neutral points because their spectrum is all neutral, as we can see in Fig. B.2. There are two mechanisms of monochromacy. Individuals called *rod monochromats* have no functioning cones in the retina, therefore, all their vision is due to the presence of the rods. *S-cone monochromats*, instead, possess rods and S-cones, although with poor colour vision and poor acuity. There are two less common forms of monochromatism, *L-cone and M-cone monochromacy*, although little is known about these two forms. They could be due to disease of the eye and the optical nerve. One plausible mechanism, as reported by Parry, is the possession of independent mutations of the S-cone and L/M-cone gene arrays. The individuals possess only rods and either L-cone or M-cone; they would be expected to have virtually absent colour vision but normal visual acuity as well as good foveal vision (Parry 2015).

I would like to remember that in his book, *The Island of the Colorblind,* first published in 1996, Oliver Sacks, a renowned British neurologist, naturalist and author, gives a fascinating account of a Micronesian island community in which all inhabitants are affected by rod monochromacy, offering a detailed report of this peculiar condition.

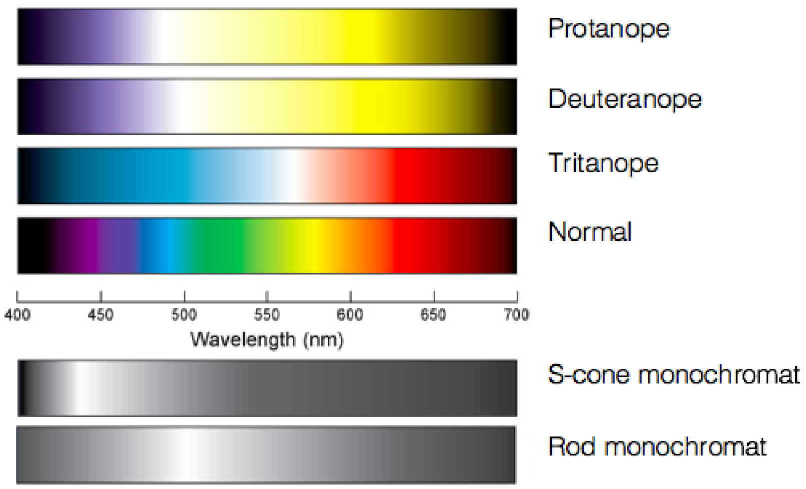

Fig. B.2
The appearance of the visible spectrum for different types of colour blindness: *protanopia*, *deuteranopia*, *tritanopia*, *S-cone monochromacy* and *rod monochromacy*. Here neutral points for dichromats are shown (corresponding to the white regions in the spectra) (Sharpe et al. 1999, p. 31)

To conclude, I would like to introduce the concept of *dichromatic confusion line*. The dichromatic confusion lines are constant-dichromatic-chromaticity lines depicted on the plane of the CIE (International Commission on Illumination) chromaticity diagram of normal trichromats (Oleari 1996) and they converge to *dichromatic confusion points*. Maxwell first attempted to draw these lines analysing red-blind subjects, i.e. protanopes, on his triangular colour diagram (see chapter 3, section 3.12). Fig. B.3 shows dichromatic confusion lines on the CIE 1931 chromaticity diagram for tritanopes, deuteranopes and protanopes (Parry 2015).

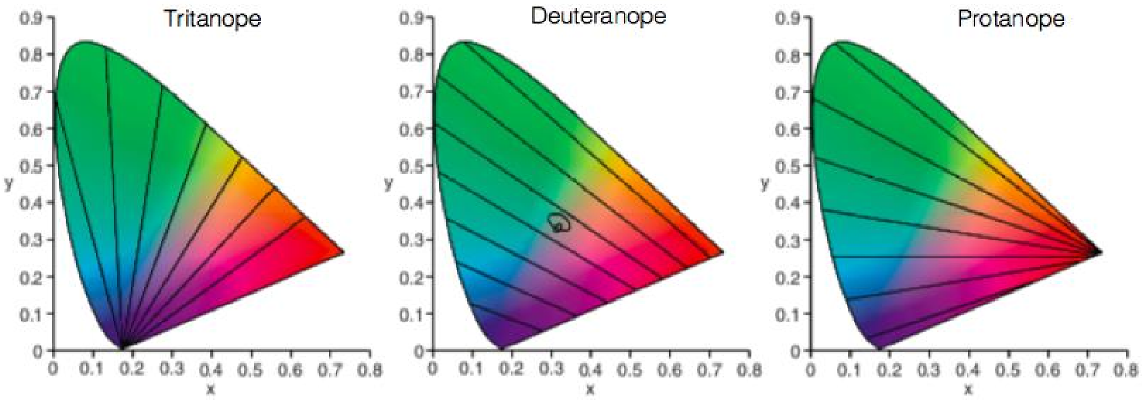

Fig. B.3
Dichromatic confusion lines, which radiate from the confusion points, or co-punctual points; for deuteranopes, this point lies outside the diagram. Any two points that lie on a confusion line originating from the co-punctual point are considered as confusion points for that specific defect (Parry 2015, p. 224)

In the central panel we can see two ellipses showing the size and orientation of regions of the colour space in which colours appear the same for trichromats (small ellipse) and for deuteranopes (larger ellipses).

After the work of the American physicist and colour scientist David Lewis MacAdam of 1942 (MacAdam 1942), it was possible to characterize the discrimination ellipses in colour space, i.e. regions containing all colours, which are indistinguishable from the colour of the centre of the ellipses. The perimeter of each ellipse, therefore, represents the *just noticeable difference* of chromaticity (see chapter 6): the smaller the perimeter of the ellipse, the better the hue discrimination.

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Appendix C

Helmholtz's *Rechnungen zur Anwendung des phsychophisischen Gesetzes im Farbensystem*

In this appendix, I would like to report some precious manuscript material, which is preserved at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften*. Notes on experiments on colour and on colour perception, undertaken by Helmholtz and his collaborators, are collected under the title *Rechnungen zur Anwendung des phsychophisischen Gesetzes im Farbensystem (Calculations on the application of the psychophysical law to the colour system)*. This manuscript collection contains, in fact, some sketches of intensity curves and colour diagrams, which may be attributed to his assistants König and Dieterici, and a notebook with the title *Farbenunterschiede* (*colour differences*), in which we can find notes and calculations on colour differences taken in December 1887, four years before Helmholtz's paper of 1891 was published. The notebook encloses Brodhun's manuscript colour equations obtained by using Maxwell's colour top, which we have reported in chapter 5. Brodhun's experiments with the colour top were inserted by Helmholtz in the paper of 1891, in paragraph *Vergleichung der Helligkeit sehr verschiedener Farben (Comparison of the brightness of very different colours)* and his results were exploited by him to extend the Weber-Fechner law to two-dimensional colour space for dichromats (see chapter 5). It is worth to remark the crucial role played by the top, which, after about 30 years from Maxwell's refinement, was still considered as a fundamental starting point for investigations in the field of colour perception.

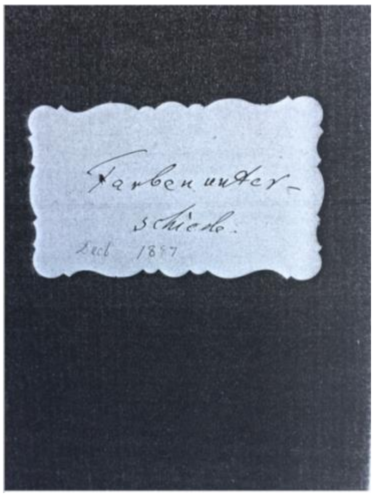
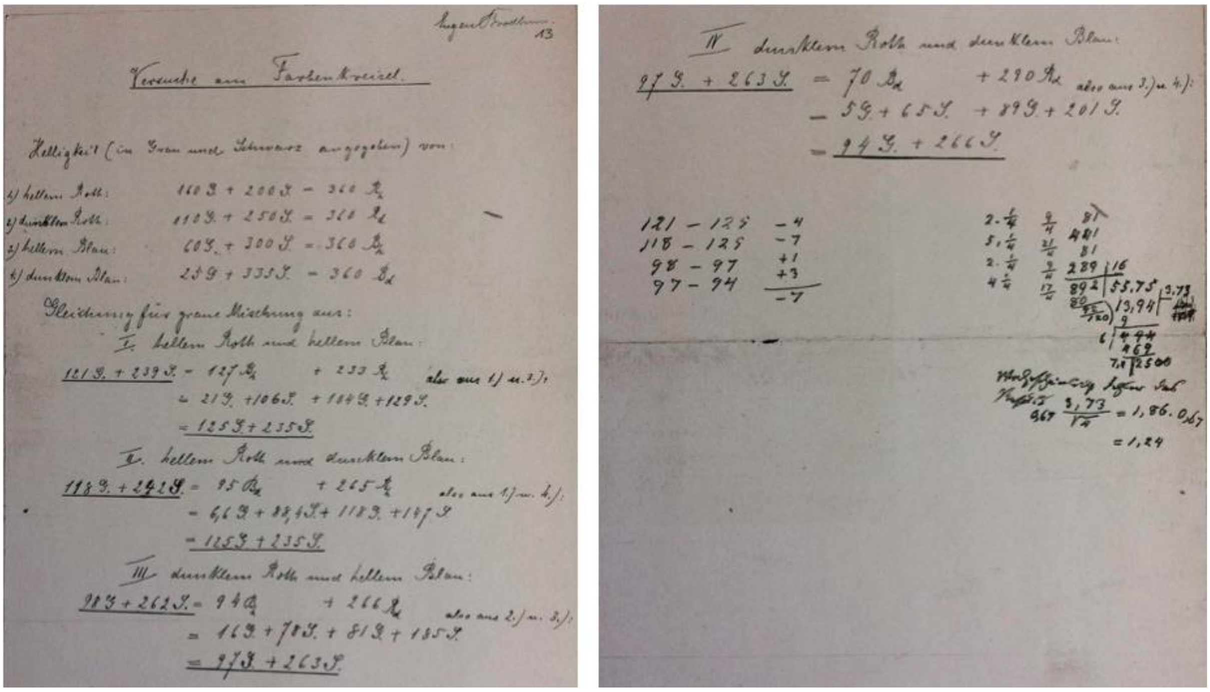


Fig. C.1
Cover of Helmholtz's notebook entitled *Farbenunterschiede. Dez. 1887 (Colour differences. December 1887)* preserved at the at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften* (Nachlass Hermann von Helmholtz, 569 Rechnungen zur Anwendung des psychophysischen Gesetzes im Farbensystem, Bl. 1-15)

C.1 Brodhun's manuscript colour equations



Eugen Brodhun. 13

Versuche am Farbenkreisel.

Helligkeit (in Grau und Schwarz angegeben) von:

1) hellem Roth: 160 G. + 200 S. = 360 R_h
2) dunklem Roth: 110 G. + 250 S. = 360 R_d
3) hellem Blau: 60 G. + 300 S. = 360 B_h
4) dunklem Blau: 25 G. + 335 S. = 360 B_d

Gleichung für graue Mischung aus:

I. hellem Roth und hellem Blau:
121 G. + 239 S. = 127 B_h + 233 R_h also aus 1) u. 3).
= 21 G. + 106 S. + 104 G. + 129 S.
= 125 G. + 235 S.

II. hellem Roth und dunklem Blau:
118 G. + 242 G. = 95 B_d + 265 R_h also aus 1) u. 4).
= 6,6 G. + 88,4 S. + 118 G. + 147 S.
= 125 G. + 235 S.

III. dunklem Roth und hellem Blau:
98 G. + 262 S. = 94 B_h + 266 R_d also aus 2.) u. 3.).
= 16 G. + 78 S. + 81 G. + 185 S.
= 97 G. + 263 S.

IV dunklem Roth und dunklem Blau:
97 G. + 263 S. = 70 B_d + 290 R_d also aus 3.) u. 4.):
= 5 G. + 65 S. + 89 G. + 201 S.
= 94 G. + 266 S.

121 − 125 −4
118 − 125 −7
98 − 97 +1
97 − 94 +3
−7

2 · ¼ 9/4 81
5 · ¼ 21/4 441
2 · ¼ 9/4 81
4 ¼ 17/4 289 | 16
892 | 55,75 | 3,73
13,94

Wahrscheinlicher Fehler des Resultats
0,67 $\frac{3,73}{\sqrt{4}}$ = 1,86 · 0,67
= 1,24

Fig. C.2
Brodhun's manuscript colour equations contained in Helmholtz's notebook (Nachlass Hermann von Helmholtz, 569 Rechnungen zur Anwendung des psychophysischen Gesetzes im Farbensystem, Bl. 13, Versuche am Farbenkreisel (experiments with the colour top))

We report the text with English translation below.

Versuche am Farbenkreisel (Experiment with the colour top)

Helligkeit (im Grau G und Schwarz S angegeben) von: [Brightness (given in gray G and black S) of:]

1. *Hellem Roth* (bright red): $160G + 200S = 360\,R_h$
2. *Dunkelm Roth* (dark red): $110G + 250S = 360\,R_d$
3. *Hellem Blau* (bright blue): $60G + 300S = 360\,B_h$
4. *Dunklem Blau* (dark blue): $25G + 335S = 360\,B_d$

Gleichung für graue Mischung aus (Equation for gray mixture of):

1. *Hellem Rot und hellem Blau: (also aus 1 und 3)* [bright red and bright blue: (thus, from 1 and 3)]

$$121G + 239S = 127B_h + 233R_h$$

$$121G + 239S = 21G + 106S + 104G + 129S$$

$$121G + 239S = 125G + 235S$$

2. *Hellem Rot und dunklem Blau: (also aus 1 und 4)* [bright red and dark blue: (thus, from 1 and 4)]

$$118G + 242S = 95B_d + 265R_h$$

$$118G + 242S = 6.6G + 88.4S + 118G + 147S$$

$$118G + 242S = 125G + 235S$$

3. *Dunklem Roth und hellem Blau: (also aus 2 und 3)* [dark red and bright blue: (thus, from 2 and 3)]

$$98G + 262S = 94B_h + 266R_d$$

$$98G + 262S = 16G + 78S + 81G + 185S$$

$$98G + 262S = 97G + 263S$$

4. *Dunklem Roth und dunklem Blau: (also aus 3 und 4)* [dark red and dark blue: (thus, from 3 and 4)]

$$27G + 263S = 70B_d + 290R_d$$

$$27G + 263S = 5G + 65S + 89G + 201S$$

$$27G + 263S = 94G + 266S$$

C.2 Sketch of the fundamental curves for normal trichromats

Fig. C.3 shows a first sketch of the fundamental curves ℜ, 𝔊 and 𝔅 obtained, presumably, by König and Dieterici. For a direct comparison between these curves and the final version of the fundamental curves, published in the papers of 1886 and 1892, the reader is referred to chapter 4, section 4.3.

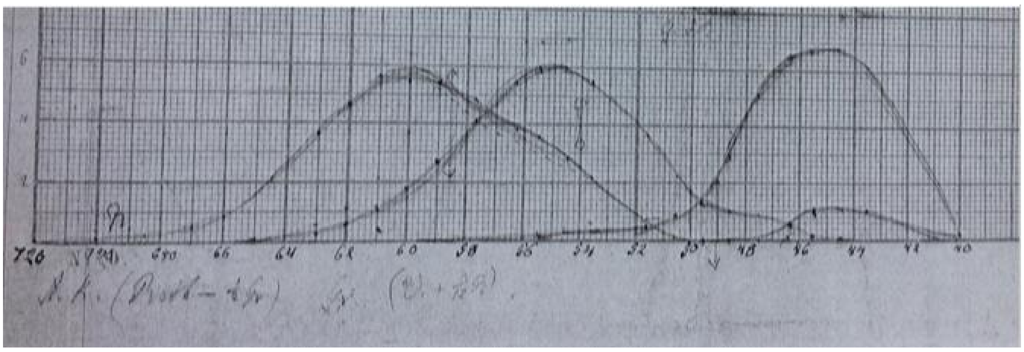
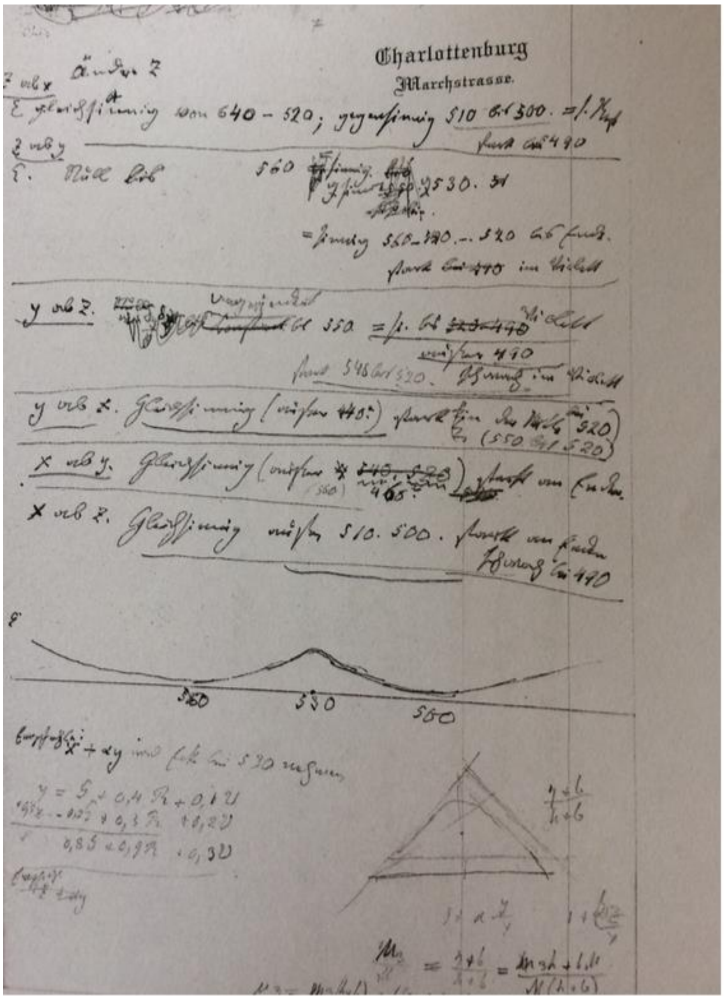


Fig. C.3
Sketch of the three fundamental curves for people with normal colour vision (Nachlass Hermann von Helmholtz, 569 Rechnungen zur Anwendung des psychophysischen Gesetzes im Farbensystem)

C.3 Drawing of Helmholtz’s colour diagram



Charlottenburg
Marchstrasse.

Fig. C.4
Helmholtz’s calculations to obtain a sketch of his colour diagram inside Maxwell’s colour triangle (Nachlass Hermann von Helmholtz, 569 Rechnungen zur Anwendung des psychophysischen Gesetzes im Farbensystem)

Fig. C.4 shows Helmholtz’s calculations to obtained a draft of his colour diagram inside Maxwell’s equilateral triangle, whose first version was already printed in the second volume of his *Handbuch* (see chapter 4, Fig. 4.10). From the stamp on the upper right corner, we can guess that he carried out these calculations after 1889, when he moved to the residence assigned to the *President of the Physikalisch-Technische Reichsanstalt* in the “Marchstrasse”, Charlottenburg.

Ringraziamenti

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