Università degli Studi di Padova

Sede Amministrativa: Università degli Studi di Padova
Dipartimento di Matematica "Tullio Levi-Civita"
CORSO DI DOTTORATO DI RICERCA IN SCIENZE MATEMATICHE
CURRICOLO MATEMATICA
CICLO XXXII

A topological approach to the Fourier transform of an elementary D-module

Coordinatore: Ch.mo Prof. Martino Bardi
Supervisore: Ch.mo Prof. Andrea D'Agnolo

Dottorando: Davide Barco

Riassunto

Prendiamo in considerazione $\mathcal{M}$ $\mathcal{D}_{\mathbb{V}}$-modulo olonomo sulla retta complessa $\mathbb{V}$. La trasformata di Fourier-Laplace $\mathcal{M}^{\wedge}$ di $\mathcal{M}$ è un $\mathcal{D}$-modulo olonomo sulla retta duale $\mathbb{V}^*$. In generale, anche se $\mathcal{M}$ è regolare, $\mathcal{M}^{\wedge}$ è irregolare. É naturale chiedersi quali siano i legami tra la struttura di Stokes di $\mathcal{M}^{\wedge}$ e quella di $\mathcal{M}$ e ricavare dati sulla prima dalla seconda. La letteratura al riguardo è , citiamo in particolare i lavori di Malgrange [3], Mochizuki [4], Hien-Sabbah [2], D'Agnolo-Hien-Morando-Sabbah [1].

Malgrange ha dato una trattazione esauriente dell'argomento, ma il suo lavoro risulta alle volte di difficile lettura. Mochizuki ha prodotto una descrizione completa della trasformata di Fourier-Laplace di $\mathcal{M}$ utilizzando l'omologia a decrescita rapida introdotta da Bloch-Esnault. Per un tipo particolare di $\mathcal{D}_{\mathbb{V}}$-module, i cosiddetti elementari, Hien-Sabbah hanno dato una descrizione più esplicita. Utilizzando la corrispondenza di Riemann-Hilbert come enunciata da Deligne e Malgrange, hanno introdotto una trasformazione topologica locale di Laplace per i sistemi locali Stokes-filtrati, e l'hanno calcolata in termini della coomologia di Cech.

Un prospettiva differente sullo studio del fenomeno di Stokes è data nella versione della corrispondenza di Riemann-Hilbert come descritta da D'Agnolo-Kashiwara. Questa associa ad un $\mathcal{D}$-modulo olonomo l'ind-fascio arricchito delle sue soluzioni arricchite. In più, per funtorialità, questa corrispondenza sostituisce trasformata di Fourier-Laplace per $\mathcal{D}$-moduli olonomi con la trasformata di Fourier-Sato per ind-fasci arricchiti. In questo contesto, D'Agnolo-Hien-Morando-Sabbah hanno calcolato esplicitamente la struttura di Stokes di $\mathcal{M}^{\wedge}$, nel caso di $\mathcal{M}$ olonomo regolare.

In questa tesi, utilizzando questo ultimo punto di vista, il nostro scopo è quello di ottenere una descrizione della trasformata di Fourier-Laplace di un $\mathcal{D}_{\mathbb{V}}$-modulo elementare. Al contrario di quanto fatto da Hien-Sabbah, il nostro approccio è puramente topologico. Come nel caso di D'Agnolo-Hien-Morando-Sabbah, è basato su calcoli fatti in termini di classi di omologia di Borel-Moore. In particolare, scegliamo le classi naturalmente associate a questo contesto, quelle provenienti dai cicli a decrescita rapida.

Abstract

Let $\mathbb{V}$ be a complex affine line, and $\mathcal{M}$ a holonomic $\mathcal{D}_{\mathbb{V}}$-module. The Fourier-Laplace transform $\mathcal{M}^{\wedge}$ of $\mathcal{M}$ is a holonomic $\mathcal{D}$-module on the dual affine line $\mathbb{V}^*$. Even if $\mathcal{M}$ is regular, $\mathcal{M}^{\wedge}$ is irregular in general. It is natural and important to try to describe the Stokes structure of $\mathcal{M}^{\wedge}$ in terms of the Stokes structure of $\mathcal{M}$. In the literature dealing with this problem, let us mention in particular the papers by Malgrange [3], Mochizuki [4], Hien-Sabbah [2], D'Agnolo-Hien-Morando-Sabbah [1].

Malgrange gave a comprehensive treatment, but his work is notoriously difficult. Mochizuki has given a recipe for a complete description of the Fourier-Laplace transform of a general $\mathcal{M}$ using the rapid decay homology theory introduced by Bloch-Esnault. For a particular kind of $\mathcal{D}_{\mathbb{V}}$-module, so-called elementary, Hien-Sabbah gave a more explit description. Using the Riemann-Hilbert correspondence of Deligne-Malgrange, they introduced a topological local Laplace transformation at the level of Stokes-filtered local systems, and computed it in terms of Cech cohomology.

A different point of view to the study of the Stokes phenomena is given by the Riemann-Hilbert correspondence, as stated by D'Agnolo-Kashiwara. This associates to a holonomic $\mathcal{D}$-module the enhanced ind-sheaf of its enhanced solutions. Moreover, by functoriality, such correspondence interchanges Fourier-Laplace transform for holonomic $\mathcal{D}$-modules with Fourier-Sato transform for enhanced ind-sheaves. Using this point of view, D'Agnolo-Hien-Morando-Sabbah explicitly computed the Stokes structure of $\mathcal{M}^{\wedge}$, for $\mathcal{M}$ regular holonomic.

In this thesis, using this same point of view, our aim is to get a description of the Fourier-Laplace transform of an elementary $\mathcal{D}_{\mathbb{V}}$-module. Unlike Hien-Sabbah, our approach is purely topological. Like D'Agnolo-Hien-Morando-Sabbah, it is based on computations in terms of Borel-Moore homology classes. For that, we choose the most natural classes, namely those attached to steepest descent cycles.

References

[1] Andrea D'Agnolo, Marco Hien, Giovanni Morando and Claude Sabbah. (2017). *Topological computation of some Stokes phenomena on the affine line*, `arXiv:1705.07610`

[2] Marco Hien and Claude Sabbah. (2014). *The local Laplace transform of an elementary irregular meromorphic connection.* Rend. Semin. Mat. Univ. Padova **134**, 133–196.

[3] Bernard Malgrange. (1991). Équations différentielles à coefficients polynomiaux, Birkhauser, Advances in Mathematics, 232 pp.

[4] Takuro Mochizuki. (2010). *Note on the Stokes structure of Fourier transform.* Acta Math. Vietnam. **35**, no. 1, 107–158

A TOPOLOGICAL APPROACH TO THE FOURIER TRANSFORM OF AN ELEMENTARY $\mathcal{D}$-MODULE

DAVIDE BARCO

Contents

Date: December 2, 2019.

1. Introduction

1.1. **Fourier-Laplace transform.** Consider the affine line $\mathbb{V} = \mathbb{C}_z$, with coordinate z, and denote by $\mathbb{P} := \mathbb{V} \cup \{\infty\}$ its projective compactification. Denote by $D_{\mathbb{V}} = \mathbb{C}[z]\langle\partial_z\rangle$ the Weyl algebra, and by $\mathcal{D}_{\mathbb{V}}^{al}$ the sheaf of rings of algebraic differential operators. Recall that $D_{\mathbb{V}} = \Gamma(\mathbb{V}; \mathcal{D}_{\mathbb{V}}^{al})$, and that this induces an equivalence between holonomic $D_{\mathbb{V}}$-modules and holonomic $\mathcal{D}_{\mathbb{V}}^{al}$-modules.

Let $\mathbb{V}^* = \mathbb{C}_w$ be the dual affine line, so that the pairing is given by $(z, w) \mapsto zw$, and consider the projections

$$\mathbb{V} \xleftarrow{p} \mathbb{V} \times \mathbb{V}^* \xrightarrow{q} \mathbb{V}^*.$$

The Fourier-Laplace transform of a $D_{\mathbb{V}}$-module M is the $D_{\mathbb{V}^*}$-module $M^\wedge$ which coincides with M as a $\mathbb{C}$-vector space, and with the actions of w and ∂_w given by

$$w \cdot m = -\partial_z \cdot m,$$
$$\partial_w \cdot m = z \cdot m,$$

respectively. By [12], this transform reads as follows in terms of holonomic $\mathcal{D}^{al}_{\mathbb{V}}$-modules:

$$\mathcal{M}^\wedge_{al} := Dq_*(\mathcal{E}^{-zw}_{al}[1] \otimes^D Dp^*\mathcal{M}_{al}) \in D^b(\mathcal{D}^{al}_{\mathbb{V}^*}). \tag{1.1}$$

Here, $D^b(\mathcal{D}^{al}_{\mathbb{V}^*})$ denotes the bounded derived category of $\mathcal{D}^{al}_{\mathbb{V}^*}$-modules, Dq_*, $\otimes^D$ and Dp^* denote the operations for $\mathcal{D}^{al}$-modules, and $\mathcal{E}^{-zw}_{al}$ is the $\mathcal{D}^{al}_{\mathbb{V}\times\mathbb{V}^*}$-module associated with the connection $(\mathcal{O}^{al}_{\mathbb{V}\times\mathbb{V}^*}, d + d(zw))$. Note that $\mathcal{M}^\wedge_{al}$ is concentrated in degree zero.

Even if $\mathcal{M}_{al}$ is regular, $\mathcal{M}^\wedge_{al}$ is irregular in general, giving rise to Stokes phenomena. It is natural and important to try to describe the Stokes structure of $\mathcal{M}^\wedge_{al}$ in terms of the Stokes structure of $\mathcal{M}_{al}$. In the literature dealing with this problem, let us mention in particular the papers by Malgrange [13], Mochizuki [15], Hien-Sabbah [9], D'Agnolo-Hien-Morando-Sabbah [6].

Malgrange gave a comprehensive treatment in [13]. Mochizuki has given a recipe for a complete description of the Fourier-Laplace transform of a general $\mathcal{M}_{al}$ using the rapid decay homology theory introduced by Bloch-Esnault. For a particular kind of $\mathcal{D}^{al}_{\mathbb{V}}$-module, so-called elementary, Hien-Sabbah gave a more explicit description. Using the Riemann-Hilbert correspondence of Deligne-Malgrange, they introduced a topological local Laplace transformation at the level of Stokes-filtered local systems, and computed it in terms of Cech cohomology.

A different point of view to the study of the Stokes phenomena is given by the Riemann-Hilbert correspondence, as stated by D'Agnolo-Kashiwara [4]. This associates to a (analytic) holonomic $\mathcal{D}$-module the enhanced ind-sheaf of its enhanced solutions. Moreover, by functoriality, such correspondence interchanges Fourier-Laplace transform for holonomic $\mathcal{D}$-modules with Fourier-Sato transform for enhanced ind-sheaves. Using this point of view, D'Agnolo-Hien-Morando-Sabbah explicitly computed the Stokes structure of $\mathcal{M}^\wedge$, for $\mathcal{M}$ regular holonomic.

In this thesis, using this same point of view, our aim is to get a description of the Fourier-Laplace transform of an elementary $\mathcal{D}_{\mathbb{V}}$-module. Unlike Hien-Sabbah, our approach is purely topological. Like D'Agnolo-Hien-Morando-Sabbah, it is based on computations in terms of Borel-Moore homology classes. For that, we choose the most natural classes, namely those attached to steepest descent cycles.

Let us give more details.

There are several ways of encoding the Stokes structure of a holonomic $\mathcal{D}^{al}_{\mathbb{V}}$-module. We will later do it in the framework of enhanced ind-sheaves.

For now, recall that another way, obtained using the classical Riemann-Hilbert correspondence as stated by Deligne and Malgrange, is by assigning: a local system on $\mathbb{P}\setminus S$, for $S\subset\mathbb{P}$ the finite set of singular points (including ∞), a Stokes-filtered local system (refer e.g. to [20]) on the circle of directions at each element of S, and some additional data on S corresponding to the vanishing cycles. From these data one recovers the so-called exponential factors as the degrees where the Stokes filtration jumps. Let us briefly recall how they can be obtained, and how they transform by Fourier-Laplace.

1.2. **Elementary modules and stationary phase formula.** Let $\rho : \mathbb{C}_u \to \mathbb{V}$ be the ramification of order p given by $z=\rho(u)=u^p$. Let $\varphi\in u^{-1}\mathbb{C}[u^{-1}]$, and denote by $\mathcal{E}^{\varphi}_{al}$ the $\mathcal{D}^{al}_{\mathbb{C}_u}$-module associated with the meromorphic connection $(\mathcal{O}^{al}_{\mathbb{C}_u}(*0), d-d\varphi)$. Let $\mathcal{R}^{al}$ be a regular holonomic $\mathcal{D}^{al}_{\mathbb{C}_u}$-module with 0 as its only singularity at finite distance.

Definition 1.1. *One considers the holonomic $\mathcal{D}^{al}_{\mathbb{V}}$-module*

$$El(\rho,\varphi,\mathcal{R}^{al}) := D\rho_*(\mathcal{E}^{\varphi}_{al}\otimes^D \mathcal{R}^{al}),$$

and calls it an elementary $\mathcal{D}^{al}_{\mathbb{V}}$-module at 0. An elementary $\mathcal{D}^{al}_{\mathbb{V}}$-module at $a\in\mathbb{P}=\mathbb{V}\cup\{\infty\}$ is obtained by replacing the local coordinate z at 0 with the local coordinate z_a at a given by

$$z_a=\begin{cases} z-a & \text{for } a\in\mathbb{V},\\ z^{-1} & \text{for } a=\infty.\end{cases}$$

Let $\mathcal{M}$ be a holonomic $\mathcal{D}^{al}_{\mathbb{V}}$-module with a singularity at $a\in\mathbb{P}$, and denote by $\mathcal{M}(*a)\hat{\rceil}_a$ its formalization. At the formal level, the Hukuhara-Levelt-Turrittin theorem states that there is a finite number of elementary $\mathcal{D}^{al}_{\mathbb{V}}$-modules at a such that

$$\mathcal{M}(*a)\hat{\rceil}_a \simeq \bigoplus_j El(\rho_j,\varphi_j,\mathcal{R}^{al}_j)\hat{\rceil}_a,$$

and that this isomorphism also holds at the asymptotic level on small sectors with vertex at a. The $\mathcal{D}^{al}_{\mathbb{V}}$-module $\bigoplus_j El(\rho_j,\varphi_j,\mathcal{R}^{al}_j)$ is called the formal type of $\mathcal{M}$ at a. Each function φ_j can be considered as ramified function of z_a, and each of its determinations is called an exponential factor of $\mathcal{M}$ at a.

Remark 1.2. *In this thesis we will discuss the Fourier-Laplace transform of elementary $\mathcal{D}$-modules. By the above result, these may be considered as some kind of building blocks of general holonomic $\mathcal{D}$-modules.*

One knows that the Fourier-Laplace transform of an elementary $\mathcal{D}$-module at 0 has singularities only at 0 and ∞, with ∞ the only irregular singularity. The first information one can obtain about such a Fourier-Laplace transform is its formal type at ∞. This is a classical result, called stationary phase formula. An explicit such formula is

Theorem 1.3. *(see [19] or [8]) The formal type of $El(\rho,\varphi,\mathcal{R}^{al})^\wedge$ is that of $El(\hat\rho,\hat\varphi,\hat{\mathcal{R}}^{al})$, where*

(1) $\hat\rho:\mathbb{C}_\zeta\to(\mathbb{V}^*\setminus 0)$, $w^{-1}=\hat\rho(\zeta)=-\frac{\rho'(\zeta)}{\varphi'(\zeta)}$ is a ramification of order $n+p$, where n is the pole order of φ at 0,

(2) $\hat{\varphi}(\eta) = \varphi(\eta) - \frac{\rho(\eta)}{\rho'(\eta)}\varphi'(\eta)$, *which has pole order* n,

(3) $\hat{\mathcal{R}}^{al} = \mathcal{R}^{al} \otimes^{D} L_n$ *where* L_n *is the* $\mathcal{D}$*-module associated with the connection* $(\mathcal{O}_{\mathbb{C}_\eta}, d - \frac{n}{2}\frac{d\eta}{\eta})$

A similar statement holds when considering the Fourier-Laplace transform of an elementary $\mathcal{D}$*-module at* $a \in \mathbb{P} \setminus \{0\}$*.*

1.3. **Enhanced ind-sheaves.** After finding the formal structure of

$$El(\rho, \varphi, \mathcal{R}^{al})^{\wedge},$$

the next step is to describe its Stokes structure at ∞. We will do it in the framework of the Riemann-Hilbert correspondence introduced by D'Agnolo-Kashiwara [4], using their theory of enhanced ind-sheaves. Let us briefly recall that framework and theory, referring to the the next sections for more details.

Denote by $\mathcal{D}_{\mathbb{V}}$ the sheaf of rings of analytic differential operators on the complex manifold underlying $\mathbb{V}$. Consider the bordered space $\mathbb{V}_\infty = (\mathbb{V}, \mathbb{P})$, and denote by $Mod_{hol}(\mathcal{D}_{\mathbb{V}_\infty})$ the full subcategory of the category of holonomic $\mathcal{D}_{\mathbb{P}}$-modules, whose objects $\mathcal{M}$ are localized at ∞, i.e. are such that $\mathcal{M} \simeq \mathcal{M}(*\infty)$. Then $Mod_{hol}(\mathcal{D}_{\mathbb{V}_\infty})$ is equivalent to the category of holonomic $\mathcal{D}^{al}_{\mathbb{V}}$-modules, and a similar equivalence holds at the level of the derived category $D^b_{hol}(\mathcal{D}_{\mathbb{V}_\infty})$.

For $\mathbf{k}$ a field, denote by $D^b(\mathbf{k}_{\mathbb{V}})$ the bounded derived category of sheaves of $\mathbf{k}$-vector spaces on $\mathbb{V}$. Consider an additional real variable $t \in \mathbb{R}$, and let $P \supset \mathbb{R}_t$ be the real projective line. Denote by $\tilde{E}^b_{\mathbb{R}-c}(\mathbf{k}_{\mathbb{V}_\infty})$ the full subcategory of $D^b(\mathbf{k}_{\mathbb{V}\times\mathbb{R}})$ whose objects $\mathcal{K}$ extend as $\mathbb{R}$-constructible objects on $\mathbb{P} \times P$, and satisfy $\mathbf{k}_{\{t\geq 0\}} \overset{+}{\otimes} \mathcal{K} \xrightarrow{\sim} \mathcal{K}$ for $\overset{+}{\otimes}$ the convolution in the t variable. Recall that any object of the triangulated category $E^b_{\mathbb{R}-c}(\mathbf{I}\mathbf{k}_{\mathbb{V}_\infty})$ of $\mathbb{R}$-constructible enhanced ind-sheaves can be expressed as the stabilization $\mathbf{k}^E \overset{+}{\otimes} \mathcal{K}$ of some $\mathcal{K} \in \tilde{E}^b_{\mathbb{R}-c}(\mathbf{k}_{\mathbb{V}_\infty})$.

For $\mathbf{k} = \mathbb{C}$, the Riemann-Hilbert correspondence provides a fully faithful functor

$$\mathcal{S}ol^E \colon D^b_{hol}(\mathcal{D}_{\mathbb{V}_\infty}) \to E^b_{\mathbb{R}-c}(\mathbf{I}\mathbb{C}_{\mathbb{V}_\infty}),$$

so that the Stokes structure of a holonomic $\mathcal{D}_{\mathbb{V}_\infty}$-module $\mathcal{M}$ is encoded in $\mathcal{S}ol^E(\mathcal{M})$. For $\varphi(u)$ a meromorphic functions with poles at 0 and ∞, let $\mathcal{E}^\varphi$ be the $\mathcal{D}_{(\mathbb{C}_u)_\infty}$-module associated with the meromorphic connection $(\mathcal{O}_{\mathbb{P}}(*\{0,\infty\}), d - d\varphi)$. Recall that one has $\mathcal{S}ol^E(\mathcal{E}^\varphi) \simeq \mathbf{k}^E \overset{+}{\otimes} E^\varphi$, where

$$E^\varphi := \mathbf{k}_{\{t+\operatorname{Re}\varphi(z)\geq 0\}}.$$

Let a be a singularity of a holonomic $\mathcal{D}_{\mathbb{V}_\infty}$-module $\mathcal{M}$. By the Hukuhara-Levelt-Turrittin theorem, one knows that on any small sector with vertex at a, the object $\mathcal{S}ol^E(\mathcal{M})$ decomposes as a finite direct sum of objects of the form $\mathbf{k}^E \overset{+}{\otimes} R\tilde{\rho}_{j!}(E^{\varphi_j} \otimes \pi^{-1}F_j)$, for φ_j the exponential factors of $\mathcal{M}$ at a and $F_j = \mathcal{S}ol(\mathcal{R}_j)$ is the classical solution complex of $\mathcal{R}_j$.

More in details, for each $\theta \in S_a\mathbb{V}$ circle of directions at a, there exists an open sector $D_\theta \subset \mathbb{V} \setminus \{a\}$ such that

$$\pi^{-1}k_{D_\theta} \otimes \mathcal{S}ol^E(\mathcal{M}) \simeq \pi^{-1}k_{D_\theta} \otimes \bigoplus_j \mathbf{k}^E \overset{+}{\otimes} R\tilde{\rho}_{j!}(E^{\varphi_j} \otimes \pi^{-1}F_j)$$

One can use such a covering to encode the Stokes structure of $\mathcal{M}$.

Indeed, for any $\theta \neq \theta' \in S_a\mathbb{V}$ with $D_\theta \cap D_{\theta'} \neq \emptyset$ we can consider

$$\begin{array}{ccc} & & \pi^{-1}k_{D_\theta \cap D_{\theta'}} \otimes \bigoplus_j \mathbf{k}^E \overset{+}{\otimes} R\tilde{\rho}_{j!}(E^{\varphi_j} \otimes \pi^{-1}F_j) \\ & \overset{\sim}{\nearrow} & \\ \pi^{-1}k_{D_\theta \cap D_{\theta'}} \otimes \mathcal{S}ol^E(\mathcal{M}) & & \alpha_{\theta,\theta'} \Big\downarrow \wr \\ & \underset{\sim}{\searrow} & \\ & & \pi^{-1}k_{D_\theta \cap D_{\theta'}} \otimes \bigoplus_j \mathbf{k}^E \overset{+}{\otimes} R\tilde{\rho}_{j!}(E^{\varphi_j} \otimes \pi^{-1}F_j) \end{array}$$

obtained by restricting the decomposition for $D_\theta, D_{\theta'}$ to $D_\theta \cap D_{\theta'}$.

The Stokes matrices are then the family of morphisms

$$\{\alpha_{\theta,\theta'} \in End_{E^b_{\mathbb{R}-c}(k_{\mathbb{V}_\infty})}(\pi^{-1}k_{D_\theta \cap D_{\theta'}} \otimes \bigoplus_j \mathbf{k}^E \overset{+}{\otimes} R\tilde{\rho}_{j!}(E^{\varphi_j} \otimes \pi^{-1}F_j))\}$$

arising from the diagram.

Remark 1.4. *Recall that there are two sets of particular directions associated with two distinct exponential factors $\phi_j, \phi_{j'}$, they are*

$$\mathcal{S}t(\{\phi_j, \phi_{j'}\}) = \{\theta \in S_a\mathbb{V} : \mathrm{Re}(\phi_j - \phi_{j'})(re^{i\theta}) = 0, r \to 0\}$$
$$\mathcal{AS}t(\{\phi_j, \phi_{j'}\}) = \{\theta \in S_a\mathbb{V} : \mathrm{Im}(\phi_j - \phi_{j'})(re^{i\theta}) = 0, r \to 0\}$$

By taking the union over pairs of distinct exponential factors of these sets, we obtain the classical definition of Stokes and anti-Stokes directions.

The Stokes matrices associated with a cover by sectors such that each sector contains only one Stokes direction for each pair of distinct exponential factors are called Stokes multipliers.

1.4. **Fourier transform and Fourier-Sato transform.** Consider the projections

$$\mathbb{V}_\infty \xleftarrow{p} \mathbb{V}_\infty \times \mathbb{V}^*_\infty \xrightarrow{q} \mathbb{V}^*_\infty.$$

The Fourier-Laplace transform at the level of $\mathcal{D}_{\mathbb{V}_\infty}$-modules has the same expression as (1.1)

As first noticed in [11], the Riemann-Hilbert correspondence of [4] interchanges the Fourier transform for holonomic $\mathcal{D}$-modules with the Fourier-Sato transform for enhanced ind-sheaves. Such an operation can also be defined at the level of enhanced ind-sheaves. More precisely, let $\mathcal{M}$ be a holonomic $\mathcal{D}_{\mathbb{V}_\infty}$-module and let $\mathcal{K} \in \tilde{E}^b_{\mathbb{R}-c}(\mathbf{k}_{\mathbb{V}_\infty})$ be such that $\mathcal{S}ol^E(\mathcal{M}) \simeq \mathbf{k}^E \overset{+}{\otimes} \mathcal{K}$. Then one has

$$\mathcal{S}ol^E(\mathcal{M}^\wedge) \simeq \mathbf{k}^E \overset{+}{\otimes} \mathcal{K}^\curlywedge,$$

where we set

$$\mathcal{K}^{\curlywedge} = R\tilde{q}_{!}(E^{-zw}[1] \overset{+}{\otimes} \tilde{p}^{-1}\mathcal{K}),$$

with $\tilde{q} = q \times id_{\mathbb{R}}$ and $\tilde{p} = p \times id_{\mathbb{R}}$.

1.5. **Fourier transform and Stokes phenomenon.** The aim of this work is to give another different description of the Fourier transform of an elementary $\mathcal{D}$-module. This will be achieved in the framework and with the language of enhanced ind-sheaves as explained at the end of section 1.3. The idea comes from the already cited articles [15],[16]. Following [1], the transform is described via the rapid decay homology theory introduced by Bloch and Esnault in [2]. This approach relies on the presence of peculiar families of integration cycles: in [15], Mochizuki explicitly construct a family of such cycles for the elementary case.

Instead of using the explicit family constructed by Mochizuki, we choose to rely on a family of cycles coming from differential geometry. This family is well known for its properties related to Fourier transform: it is the family of steepest descent paths. Following [21], we are able to relate such paths to the study of critical level sets of the imaginary part of the Legendre transform.

The study of these cycles and their interactions provides the complete description of the Fourier transform we are looking for in term of decompositions on a suitable covering and associated Stokes matrices.

We have already recalled that, in their work [9], Hien and Sabbah performed the same task: in particular, they provided a full set of Stokes multipliers of the Fourier-Laplace transform of an elementary $\mathcal{D}$-module.

We want to stress out that we are not computing Stokes multipliers: each of the sectors constituting our covering only contains one Stokes line for a single pair of exponential factors.

Theoretically, we can recover the Stokes multipliers through our approach, by computing the product of the right number of Stokes matrices obtained in the right order.

We decided to maintain the description in term of Stokes matrices.

Apart from the difficulties coming from keeping track of the combinatorics involved in the definition of these matrices, our choice is due to the work made by Mochizuki.

Indeed, he also provided Stokes matrices in the elementary case as the stepping stone from which he proceeds towards the complete description of the Fourier transform of a general $\mathcal{D}$-module.

Our hope is that this work provides a possible first step towards the same goal in the framework of D'Agnolo-Kashiwara Riemann-Hilbert correspondence.

1.6. **The results.** Let us describe in details our results: the notation and definition are the same as in section 1.2.

Denote by F the solution complex of $\mathcal{R}$.

The hypothesis made on $\mathcal{R}$ implies that F gives rise to a local system L when restricted to $\mathbb{C}_u \setminus \{0\}$: we will denote by V its stalk and by T its monodromy.

We will assume in this work that n, p are coprime, and our primary focus will be on $\varphi(u) = -\alpha u^{-n}$ with $\alpha = \beta e^{i\mu} \in \mathbb{C}^\times$.

Consider $El(\rho, \varphi, \mathcal{R})^\wedge$: as recalled, it is ramified with ramification given by

$$w^{-1} = \hat{\rho}(\zeta) = \frac{p}{n\alpha}\zeta^{n+p}.$$

Our aim is to study the Stokes structure of the $\mathcal{D}$-module

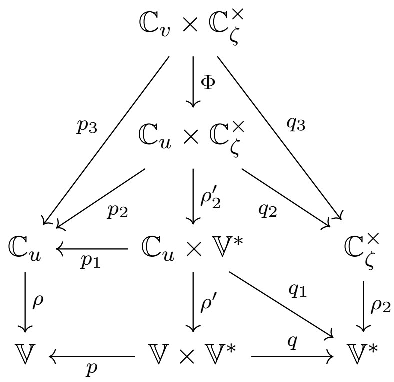
$$\mathcal{M} := D\rho_2^*(El(\rho, \varphi, \mathcal{R})^\wedge)$$

where $\rho_2 = \frac{1}{\hat{\rho}}$.

By D'Agnolo-Kashiwara Riemann-Hilbert correspondence, this is equivalent to study the pullback via ρ_2 of the enhanced Fourier-Sato transform of the enhanced sheaf $R\rho_!(E^{-\alpha u^{-n}} \otimes \pi^{-1}F)$, i.e.,

$$K := \tilde{\rho}_2^{-1}(R\tilde{\rho}_!(E^{\alpha u^{-n}} \otimes \pi^{-1}F))^\curlywedge =$$
$$= \tilde{\rho}_2^{-1} R\tilde{q}_!(E^{-zw}[1] \overset{+}{\otimes} \tilde{p}^{-1}(R\tilde{\rho}_!(E^{\alpha u^{-n}} \otimes \pi^{-1}F)))$$

Consider the diagram with Cartesian squares



with $\Phi(v, \zeta) = (v\zeta, \zeta)$ the blow-up.

Via this diagram, our object of study reduces to (see §8)

$$K \simeq R\tilde{q}_{3!}(E^{-\tilde{\Psi}} \otimes \tilde{p}_3^{-1}(\pi^{-1}F))[1].$$

with $\tilde{\Psi}(v, \zeta) = n\alpha\zeta^{-n}(\frac{v^{-n}}{n} + \frac{v^p}{p})$.

Let us denote with v_m the $n+p$-th root of unity $e^{i\frac{2m\pi}{n+p}}$ and with θ_m its argument.

Define $\tilde{\Psi}_m$ as the fiber of $\tilde{\Psi}$ at v_m, i.e.,

$$\tilde{\Psi}_m(\zeta) = \tilde{\Psi}(v_m, \zeta).$$

The first goal in our study is to reach a complete knowledge of

$$K_{(\zeta_0, t_0)} = R\Gamma(\mathbb{C}_v, k_{\{\mathrm{Re}\,\tilde{\Psi}_{\zeta_0} \leq t_0\}} \otimes \Phi_{\zeta_0}^{-1}L)[1]$$

where $\tilde{\Psi}_{\zeta_0}(v) = \tilde{\Psi}(v, \zeta_0)$.

Our approach makes use of the steepest descent paths coming from Morse-Witten theory.

Notice that the critical points of $\tilde{\Psi}_{\zeta_0}$ are $\{v_m : m = 0, ..., n+p-1\}$: the steepest descent paths are cycles associated with each v_m and representing the most natural integration cycles for the transform.

Namely, they are constructed as solutions to the so-called downward gradient equation

$$\frac{dv(t)}{dt} = -\nabla \operatorname{Re} \tilde{\Psi}_{\zeta_0}(v(t))$$

with limit condition

$$\lim_{t\to-\infty} v(t) = v_m.$$

Since we are dealing with the real part of a meromorphic function, these curves have an additional property: they are contained in the critical level set $\{\operatorname{Im} \tilde{\Psi}_{\zeta_0}(v) = \operatorname{Im} \tilde{\Psi}_{\zeta_0}(v_m)\}$.

In particular, the steepest descent path $\Gamma^{\Psi}_{m,\zeta_0}$ associated to v_m is the analytic curve component of such critical level set abutting from v_m along the negative eignespace of $Hess \operatorname{Re} \tilde{\Psi}_{\zeta_0}(v_m)$.

We will see that they contain all needed information concerning the cohomology with compact support in exam; indeed, if we denote by $\mathcal{S}\Gamma$ the support of a curve Γ, we have

Proposition 1.5. *For all $t_0 \in \mathbb{R}$, we have*

$$R\Gamma_c(\mathbb{C}_v, \boldsymbol{k}_{\{\operatorname{Re}\Psi_{\zeta_0}\le t_0\}\setminus \bigcup_{m:\operatorname{Re}\lambda_m\le t_0} \mathcal{S}\Gamma^{\Psi}_{m,\zeta_0}}) = 0$$

In particular this implies that, if t_0 is big enough

$$R\Gamma_c(\mathbb{C}_v, \mathbf{k}_{\{\operatorname{Re}\Psi_{\zeta_0}\le t_0\}}) \simeq R\Gamma_c(\mathbb{C}_v, \mathbf{k}_{\mathcal{S}\Gamma^{\Psi}_{\zeta_0}})$$

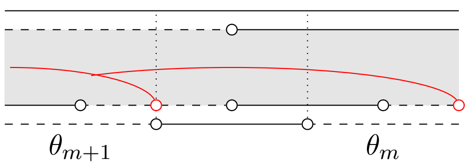
where we denote by $\mathcal{S}\Gamma^{\Psi}_{\zeta_0}$ the union $\bigcup_{m=0}^{n+p-1} \mathcal{S}\Gamma^{\Psi}_{m,\zeta_0}$.

Such cycles are disjoint for all ζ_0 but a finite set of lines, whose directions are given by

$$\mathcal{AS}t_{\mathcal{T}rp}(\Psi) := \{\Theta \in S^1_\zeta | \exists m \in [0, ..., n+p-1] : \operatorname{Im}(\tilde{\Psi}_m - \tilde{\Psi}_{m+1})(e^{i\Theta}) = 0\}.$$

In particular, to each argument in $\mathcal{AS}t_{\mathcal{T}rp}(\Psi)$ we can associate the pairs $(m, m+1)$ satisfying the defining condition.

At these directions, pairs of steepest descent paths associated to the consecutive critical points v_m, v_{m+1} degenerate: they are no more disjoint, but have a critical point in common.



We will call the configuration arising tripod.

This phenomenon is the source of the obstruction to a global trivialization of K.

We can provide a complete description of $\mathcal{AS}t_{\mathcal{T}rp}(\Psi)$.

Recall that, since n, p are coprime, also $p, n+p$ are: let $ap + b(n+p) = 1$ be one of the associated Bezout identities.

Proposition 1.6. *We will distinguish three cases, according to the class of $n+p$ modulo 4.*

(i) *Assume that* $n+p$ *is odd. Then* $\mathcal{AS}t_{\mathcal{T}rp}(\Psi) = \{\Theta_h \colon h \in \mathbb{Z}\}$, *with*

$$\Theta_h = \frac{\mu}{n} + \frac{2h+1}{2n(n+p)}\pi.$$

The defining condition is satisfied for a single pair (m_h, m_h+1), *with*

$$m_h = a\frac{n+p+1}{2}(\frac{n-p+1}{2}+h) \pmod{n+p}.$$

(ii) *Assume that* $n+p = 0 \pmod 4$. *Then* $\mathcal{AS}t_{\mathcal{T}rp}(\Psi) = \{\Theta_h \colon h \in \mathbb{Z}\}$, *with*

$$\Theta_h = \frac{\mu}{n} + \frac{2h+1}{n(n+p)}\pi.$$

The defining condition is satisfied for two pairs (m_h^1, m_h^1+1) *and* (m_h^2, m_h^2+1), *with*

$$m_h^1 = a(\frac{n-p+2}{4}+h) \pmod{n+p},$$
$$m_h^2 = a(\frac{n-p+2}{4}+h) + \frac{n+p}{2} \pmod{n+p}.$$

(iii) *Assume that* $n+p = 2 \pmod 4$. *Then* $\mathcal{AS}t_{\mathcal{T}rp}(\Psi) = \{\Theta_h \colon h \in \mathbb{Z}\}$, *with*

$$\Theta_h = \frac{\mu}{n} + \frac{2h}{n(n+p)}\pi.$$

The defining condition is satisfied for two pairs (m_h^1, m_h^1+1) *and* (m_h^2, m_h^2+1), *with*

$$m_h^1 = a(\frac{n-p+4}{4}+h) \pmod{n+p},$$
$$m_h^2 = a(\frac{n-p+4}{4}+h) + \frac{n+p}{2} \pmod{n+p}.$$

As explained at the end of section 1.3, the description of K is obtained by covering a punctured neighbourhood of $0 \in \mathbb{C}_\zeta$ with open sectors over which K admits a trivialization and then study how different decompositions behave at overlaps.

In the same way as in [6], we can perform this description with closed sectors instead of open ones, and simply study the behaviour on overlapping sides.

By all considerations made so far, the covering we use is given by the closed sectors delimited by consecutive elements in $\mathcal{AS}t_{\mathcal{T}rp}(\Psi)$.

We will denote by Σ_h the open sector delimited by Θ_{h-1}, Θ_h.

Using the family of steepest descent paths, in Σ_h we have a well defined decomposition given by the fact that all cycles are disjoint for $arg(\zeta_0) \notin \mathcal{AS}t_{\mathcal{T}rp}(\Psi)$.

We will extend the behavior of this family of disjoint paths to the boundary of Σ_h by considering the limit cycles $\Gamma_{m,\Theta_{h-1}+}, \Gamma_{m,\Theta_h-}$.

They are obtained by taking the limit in the construction of $\Gamma_{m,\zeta_0 e^{i\delta}}$ with $\arg(\zeta_0) = \Theta_{h-1}, \Theta_h$, for $\delta \to 0^+$ for Θ_{h-1} and $\delta \to 0^-$ for Θ_h.

Such limit cycles are closed components of the tripods arising at the boundary: thanks to them, we are able to overcome the degeneration and state the first result about representation of K in sectors.

Theorem 1.7. *For each $h \in \mathbb{Z}$, we have an isomorphism in $E^b_{\mathbb{R}-c}(\boldsymbol{k}_{\mathbb{C}^\times_{\zeta,\infty}})$*

$$\pi^{-1}\boldsymbol{k}_{\bar{\Sigma}_h} \otimes K \xrightarrow[\sim]{\sigma_h} \pi^{-1}\boldsymbol{k}_{\bar{\Sigma}_h} \otimes \bigoplus_{m=0}^{n+p-1} V \otimes E^{-\tilde{\Psi}_m}$$

Now, at each line $l_h := \Sigma_h \cap \Sigma_{h+1}$ with direction Θ_h, we have two decomposition provided by σ_h, σ_{h+1}:

$$\begin{array}{ccc} & & \pi^{-1}k_{l_h} \otimes \bigoplus_{m=0}^{n+p-1} (E^{-\tilde{\Psi}_m} \otimes V) \\ & \overset{\sigma_h}{\nearrow} & \\ \pi^{-1}k_{l_h} \otimes K & & \Big\downarrow \alpha_h \\ & \underset{\sigma_{h+1}}{\searrow} & \\ & & \pi^{-1}k_{l_h} \otimes \bigoplus_{m=0}^{n+p-1} (E^{-\tilde{\Psi}_m} \otimes V) \end{array}$$

Since

$$End_{E^b_{\mathbb{R}-c}(k_{\mathbb{C}^*_{\zeta,\infty}})}\Big(\bigoplus_{m=0}^{n+p-1} (E^{-\tilde{\Psi}_m} \otimes V)\Big) \subset End_k\Big(\bigoplus_{m=0}^{n+p-1} V\Big)$$

the left vertical line of the above diagram is determined by the linear map it induces on a stalk at a point (ζ_0, t_0) with $\zeta_0 \in l_h$ and $t_0 >> 0$ big enough. In particular, at the level of stalk, we have

$$R\Gamma_c(\mathbb{C}_v, \mathbf{k}_{\mathcal{S}\Gamma^\Psi_{\zeta_0}} \otimes \Phi^{-1}_{\zeta_0} L)[1] \xrightarrow[\sim]{\sigma_h} \bigoplus_{m=0}^{n+p-1} V$$

$$R\Gamma_c(\mathbb{C}_v, \mathbf{k}_{\mathcal{S}\Gamma^\Psi_{\zeta_0}} \otimes \Phi^{-1}_{\zeta_0} L)[1] \xrightarrow[\sim]{\sigma_{h+1}} \bigoplus_{m=0}^{n+p-1} V.$$

By the construction of σ_h, σ_{h+1}, the morphism $\alpha_h = \sigma_{h+1} \circ \sigma_h^{-1}$ arises then from the linear map obtained by the comparison at l_h of the limit cycles $\Gamma_{m,\Theta_h+}, \Gamma_{m,\Theta_h+}$.

Computations of σ_h, σ_{h+1} involves the use of cohomology with compact support.

However, since we want to compare cycles, it is more convenient to pass to the dual side and use Borel-Moore homology in order to compute more easily the multipliers.

The dual of the morphisms above is then

$$H_1^{BM}(\mathcal{S}\Gamma^\Psi_{\zeta_0}, (\Phi^{-1}_{\zeta_0} L)^*) \xleftarrow[\sim]{\sigma_h^*} \bigoplus_{m=0}^{n+p-1} V^*$$

$$H_1^{BM}(\mathcal{S}\Gamma^\Psi_{\zeta_0}, (\Phi^{-1}_{\zeta_0} L)^*) \xleftarrow[\sim]{\sigma_{h+1}^*} \bigoplus_{m=0}^{n+p-1} V^*.$$

and the dual morphism σ_h^* (σ_{h+1}^*) are defined by taking as a basis for $H_1^{BM}(\mathcal{S}\Gamma_{\zeta_0}^{\Psi})$ the limit cycles Γ_{m,Θ_h+} (Γ_{m,Θ_h-}).

The dual morphism α_h^* then arises by the change of bases arising.

The algebraic framework in which we will express the result is the following.

Denote by $\mathbb{1}_m$ the complex vector space of rank one with chosen basis element 1_m. We will keep track of the indices in $\bigoplus_{m=0}^{n+p-1} V$ by writing

$$\bigoplus_{m=0}^{n+p-1} V = \bigoplus_{m=1}^{n+p-1} (\mathbb{1}_m \otimes V)$$

Thus we will describe α_h^* as the morphisms arising from elements in $End_k(\bigoplus_{m=1}^{n+p-1} (\mathbb{1}_m^* \otimes V^*))$.

We will use the following convention

$$1_{n+p}^* \otimes \mathrm{v}^* := 1_0^* \otimes (T^*)^{-1}\mathrm{v}^*.$$

Theorem 1.8. *(i) Let $n+p$ odd, the morphism α_h^* comes from the element in $End_k(\bigoplus_{m=1}^{n+p-1} (\mathbb{1}_m^* \otimes V^*))$ given, $\forall h = 0, ..., 2n(n+p)-2$, by the assignments*

$$\begin{cases} 1_m^* \otimes \mathrm{v}^* \mapsto 1_m^* \otimes \mathrm{v}^* & \text{for } m \neq m_h, m_{h+1}, \\ 1_{m_h}^* \otimes \mathrm{v}^* \mapsto 1_{m_h}^* \otimes \mathrm{v}^* + 1_{m_h+1}^* \otimes \mathrm{v}^*, \\ 1_{m_h+1}^* \otimes \mathrm{v}^* \mapsto 1_{m_h+1}^* \otimes \mathrm{v}^*. \end{cases}$$

if h is even and $\sin(p\theta_{m_h} - n\Theta_h) = 1$ or if h odd and $\sin(p\theta_{m_h} - n\Theta_h) = -1$ and

$$\begin{cases} 1_m^* \otimes \mathrm{v}^* \to 1_m^* \otimes \mathrm{v}^* & \text{for } m \neq m_h, m_{h+1}, \\ 1_{m_h}^* \otimes \mathrm{v}^* \to 1_{m_h}^* \otimes \mathrm{v}^* \\ 1_{m_h+1}^* \otimes \mathrm{v}^* \to 1_{m_h+1}^* \otimes \mathrm{v}^* - 1_{m_h}^* \otimes \mathrm{v}^* \end{cases}$$

if h is odd and $\sin(p\theta_{m_h} - n\Theta_h) = 1$ or if h even and $\sin(p\theta_{m_h} - n\Theta_h) = -1$.

The morphism $\alpha_{2n(n+p)-1}$ is defined by composing the morphism obtained by applying the rule above with $(-1)^n diag((T^*)^{-1})$.

(ii) Let $n+p$ even, the morphism α_h^* comes from the element in

$$End_k(\bigoplus_{m=1}^{n+p-1} (\mathbb{1}_m^* \otimes V^*))$$

defined, $\forall h = 0, ..., n(n+p)-2$, by the assignment

$$\begin{cases} 1_m^* \otimes \mathrm{v}^* \mapsto 1_m^* \otimes \mathrm{v}^* & \text{if } m \notin \{m_h^i, m_h^i+1 : i = 1,2\} \\ 1_{m_h^1}^* \otimes \mathrm{v}^* \mapsto 1_{m_h^1}^* \otimes \mathrm{v}^* + 1_{m_h^1+1}^* \otimes \mathrm{v}^* \\ 1_{m_h^1+1}^* \otimes \mathrm{v}^* \mapsto 1_{m_h^1+1}^* \otimes \mathrm{v}^* \\ 1_{m_h^2}^* \otimes \mathrm{v}^* \mapsto 1_{m_h^2}^* \otimes \mathrm{v}^* \\ 1_{m_h^2+1}^* \otimes \mathrm{v}^* \mapsto 1_{m_h^2+1}^* \otimes \mathrm{v}^* - 1_{m_h^2}^* \otimes \mathrm{v}^*. \end{cases}$$

if $\sin\Big(p\theta_{m_h^1} - n\Theta_h\Big) = +1$ *or by the assignment*

$$\begin{cases} 1^*_m \otimes \mathrm{v}^* \mapsto 1^*_m \otimes \mathrm{v}^* & \textit{if } m \notin \{m_h^i, m_h^i + 1 : i = 1,2\} \\ 1^*_{m_h^1} \otimes \mathrm{v}^* \mapsto 1^*_{m_h^1} \otimes \mathrm{v}^* \\ 1^*_{m_h^1+1} \otimes \mathrm{v}^* \mapsto 1^*_{m_h^1+1} - 1^*_{m_h^1} \otimes \mathrm{v}^* \\ 1^*_{m_h^2} \otimes \mathrm{v}^* \mapsto 1^*_{m_h^2,+} \otimes \mathrm{v}^* + 1^*_{m_h^2+1} \otimes \mathrm{v}^* \\ 1^*_{m_h^2+1} \otimes \mathrm{v}^* \mapsto 1^*_{m_h^2+1} \otimes \mathrm{v}^*. \end{cases}$$

if $\sin\Big(p\theta_{m_h^1} - n\Theta_h\Big) = -1.$

The morphism $\alpha_{n(n+p)-1}$ *is defined by composing the morphism obtained by applying the rule above with* $(-1)^n diag((T^*)^{-1})$*.*

In the case of a generic

$$\varphi(u) = \alpha u^{-n} + \sum_{j=1}^{n-1} \alpha_j u^{-j} \in u^{-1}\mathbb{C}[u^{-1}],$$

we are reduced to study the same object K where the exponent is a perturbation $\tilde{\Psi}^{prt}$ of the function $\tilde{\Psi}$ obtained by αu^{-n}.

We will then display results concerning the estension of the description of K in this case.

1.7. **Structure of the work.** Sections 3,4,5 deals with preliminaries about the theory of enhanced (ind-)sheaves, the formulation of Riemann-Hilbert correspondence in that framework and the theory of Fourier transform for $\mathcal{D}$-module and enhanced sheaves.

Section 6 provides the tools and definitions from Morse-Witten theory needed, while Section 7.1 introduces Borel-Moore homology in the subanalytic framework.

In Section 8 we compute the Fourier-Sato transform for the object in exam and provide the setting for making its study as easy as possible.

Section 9 deals with the description of level sets related with the geometric description of the transform: from this we will deduce, in Section 10, how steepest descent cycles degenerate.

In Sections 11,12 we give the proofs to the main results.

Section 13 shows how the main results stated and their proofs are sufficient for the result to extend to the general case.

1.8. **Acknowledgments.** I would like to express my deepest appreciation to my supervisor, Andrea D'Agnolo, for the constant support, patience and nurturing he gave me in my Ph.D. studies and in the drawing of this thesis.

I would like to thank the Ph.D Program in Mathematical Sciences for the financial support provided from their funds in all these three years' travels.

I would like to express my gratitude to Marco Hien and Claude Sabbah for guesting me in Augsburg and Paris and for their warm hospitality: their help was precious in understanding Stokes phenomenon and Fourier transform towards the completion of this work.

Moreover, since they have been my referees, I would also like to thank them for their good evaluation of my work and the useful advices.

In addition, a thank you to Takuro Mochizuki for answering my questions about his article, which was the stepping stone for this work.

A particular thank to my family and friends for the continuous support they have given me during this journey, in particular during the period in which I was drawing up this work.

2. An example

Before dealing with the main matter, let us give an example of the methods used to prove our results.

2.1. The Fourier transform of $\mathcal{E}^{-u^{-1}}$. We will discuss here the Fourier transform of the elementary $\mathcal{D}^{al}$-module

$$\mathcal{E}_{al}^{-z^{-1}} = D(id)_* \mathcal{E}_{al}^{-u^{-1}} = El(id, -u^{-1}, \mathcal{O}^{al}_{\mathbb{C}_u}(*0))$$

associated with the trivial ramification $id(u) = z$ and the trivial regular part $\mathcal{O}^{al}_{\mathbb{C}_u}(*0)$.

The stationary phase formula associates with such elementary $\mathcal{D}^{al}$-module the ramification $w^{-1} = \hat{id}(\zeta) = \zeta^{-2}$. Consider the diagram

$$\begin{array}{ccccc}
 & & \mathbb{C}_v \times \mathbb{C}_\zeta^\times & & \\
 & & \downarrow{\scriptstyle \Phi} & \overset{q_2}{\searrow} & \\
\mathbb{C}_u & \xleftarrow{p_1} & \mathbb{C}_u \times \mathbb{C}_\zeta^\times & \xrightarrow{q_1} & \mathbb{C}_\zeta^\times \\
{\scriptstyle id}\downarrow & & \downarrow & & \downarrow{\scriptstyle \rho_2} \\
\mathbb{V} & \xleftarrow{p} & \mathbb{V} \times (\mathbb{V}^* \setminus \{0\}) & \xrightarrow{q} & \mathbb{V}^* \setminus \{0\},
\end{array}$$

where $\Phi(v, \zeta) = (v\zeta, \zeta)$ is the blow-up and $\rho_2 = (\hat{id})^{-1}$.

If we denote by $j : \mathbb{C}_u^\times \to \mathbb{C}_u$ the open inclusion, we notice that

$$\mathcal{S}ol^E(\mathcal{O}^{al}_{\mathbb{C}_u}(*0)) = \mathbf{k}^E_{\mathbb{C}_u} \overset{+}{\otimes} \pi^{-1}\mathbf{k}_{\mathbb{C}_u}.$$

At the level of enhanced sheaves, we are then interested in computing

$$\begin{aligned}
K = \tilde{\rho}_2^{-1}((\tilde{id}_!(E^{-u^{-1}}))^\curlywedge) &\simeq R\tilde{q}_{1!}(E^{-u\zeta^{-2}} \overset{+}{\otimes} \tilde{p}_1^{-1} E^{-u^{-1}})[1] \\
&\simeq R\tilde{q}_{1!} E^{-u\zeta^{-2} - u^{-1}}[1] \\
&\simeq R\tilde{q}_{2!} E^{-\zeta^{-1}(v^{-1}+v)}[1] \\
&\simeq R\tilde{q}_{2!} \mathbf{k}_{\{\mathrm{Re}\,\tilde{\Psi}(v,\zeta) \leq t\}}[1],
\end{aligned}$$

where we set $\tilde{\Psi}(v, \zeta) = \zeta^{-1}(v^{-1} + v)$ (see §8 for more details).

We then need to study the property of the sublevel set $\{\mathrm{Re}\,\tilde{\Psi} \leq t\}$ with respect to the direct image with proper support via $\tilde{q}_2$. We will achieve this by first studying its stalk at $(\zeta_0, t_0) \in \mathbb{C}_\zeta^\times \times \mathbb{R}$:

$$K_{(\zeta_0,t_0)} \simeq (R\tilde{q}_{2!}\mathbf{k}_{\{\tilde{\Psi}(v,\zeta) \leq t\}}[1])_{(\zeta_0,t_0)} \simeq R\Gamma_c(\{\mathrm{Re}\,\tilde{\Psi}(v, \zeta_0) \leq t_0\})[1],$$

where we write for brevity $R\Gamma_c(A) := R\Gamma_c(\mathbb{C}_v, \mathbf{k}_A)$ for $A \subset \mathbb{C}_v$.

The geometric objects we will use in order to achieve our goal are the steepest descent paths. Our aim is to show that they provide a decomposition for the stalk which dually corresponds to a set of generators for the Borel-Moore homology of $\{\mathrm{Re}\,\tilde{\Psi}(v,\zeta_0) \leq t_0\}$.

Let us briefly recall here the definition and properties of these paths, see §6 for a complete presentation.

The steepest descent cycles are oriented cycles attached to any critical point p of a Morse function f and defined as closures of solutions to limit problems of the downward gradient equation

$$\begin{cases} \frac{dx(t)}{dt} = -g\nabla f(x(t)) \\ \lim_{t\to-\infty} v(t) = p \end{cases}$$

with $t \in]-\infty, 0]$ and g a Riemannian metric on the domain of f. It is noticeable that f is non-increasing on flow lines solutions to this problem. Denote by $\tilde{\Psi}_{\zeta_0}(v) := \tilde{\Psi}(v,\zeta_0)$ the fiber of $\tilde{\Psi}$ at $\zeta_0 \in \mathbb{C}^\times_\zeta$: its critical points are ±1 and are non-degenerate.

Thus we can associate steepest descent paths for $\mathrm{Re}\,\tilde{\Psi}_{\zeta_0}$ to its critical points ±1.

Since our framework is complex analytic, we also know that such paths are contained in critical level sets (i.e. level set at critical values) for $\mathrm{Im}\,\tilde{\Psi}_{\zeta_0}$.

Our idea is to study such critical level sets and single out the steepest descent paths as the branch where $\mathrm{Re}\,\tilde{\Psi}_{\zeta_0}$ is decreasing on the two sides of the critical point.

It is easy to notice that the critical level sets for $\mathrm{Im}\,\tilde{\Psi}_{\zeta_0}$ does not depend on $\epsilon_0 := |\zeta_0|$, but only on $\Theta_0 = \arg(\zeta_0)$. It is hence sufficient to study

$$\Psi_{\Theta_0}(v) := e^{-i\Theta_0}(v^{-1} + v)$$

If (r,θ) is the set of polar coordinates for $\mathbb{C}_v$, we have

$$f_{\Theta_0}(r,\theta) := \mathrm{Im}\,\Psi_{\Theta_0}(r,\theta) = -r^{-1}\sin(\theta+\Theta_0) + r\sin(\theta-\Theta_0).$$

The critical points are $(1,0),(1,\pi)$ with critical values

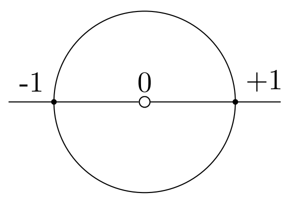
$$f_{\Theta_0}(1,0) = -2\sin(\Theta_0)$$
$$f_{\Theta_0}(1,\pi) = 2\sin(\Theta_0).$$

Before studying in details these level sets for Θ_0 varying, we have to notice that, for $\Theta_0 = 0,\pi$, the critical values are not distinct: the steepest descent for ±1 are then contained in the same critical level set.

Let us deal with this case first: the equation for the level set becomes:

$$\{\sin(\theta)(r^2-1) = 0\} = \{\theta = 0\} \cup \{\theta = \pi\} \cup \{r = 1\}.$$

The critical level set is then made of two analytic curves orthogonally intersecting at ±1.



It is easy to determine the steepest descent paths: it suffices to restrict $\mathrm{Re}\,\Psi_{\Theta_0}$ to the single components of the level set and pick the ones along which such function is decreasing.

Since

$$g_{\Theta_0}(r,\theta) := \mathrm{Re}\,\Psi_{\Theta_0}(r,\theta) = -r^{-1}\cos(\theta+\Theta_0) - r\cos(\theta-\Theta_0)$$

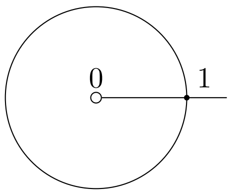
we have

- for $r=1$, $g_{\Theta_0}(1,\theta) = -2\cos(\theta)\cos(\Theta_0)$,
- if $\theta = 0$, $g_{\Theta_0}(r,0) = -g_{\Theta_0}(r,\pi) = -\cos(\Theta_0)(r^{-1}+r)$
- if $\theta = \pi$, $g_{\Theta_0}(r,0) = -g_{\Theta_0}(r,\pi) = \cos(\Theta_0)(r^{-1}+r)$

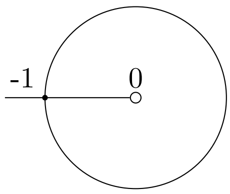
Since $\lim_{r\to 0}(r^{-1}+r) = +\infty$ and the same holds for the limit $r \to +\infty$, the sign of $\cos(\Theta_0)$ determines which line between $\{\theta = 0\}$ and $\{\theta = \pi\}$ is of steepest descent.

If $\Theta_0 = 0$, $\{\theta = 0\}$ is the steepest descent path associated with 1, while $\{\theta = \pi\}$ is the one associated with -1 for $\Theta_0 = \pi$.

Let us focus on $\{r = 1\}$. If $\Theta_0 = 0$, notice that g_{Θ_0} is decreasing moving on both sides of the circle from -1 to $+1$, while the opposite happens for $\Theta_0 = \pi$: this means that $\{r = 1\}$ is of steepest descent for both $\mathrm{Re}\,\Psi_0$ and $\mathrm{Re}\,\Psi_\pi$.



(A) Steepest descent paths for $\Theta_0 = 0$



(B) Steepest descent paths for $\Theta_0 = \pi$

Let us now study the case $\Theta_0 \neq 0, \pi$ and focus on the steepest descent path associated to $(1, 0)$.

We have then to study the equation

$$r^2\sin(\theta-\Theta_0) - 2r\sin(\Theta_0) - \sin(\theta+\Theta_0) = 0. \tag{2.1}$$

First of all, consider $\theta = \Theta_0$: the defining equation becomes

$$-2r\sin(\Theta_0) - \sin(2\Theta_0) = 0 \Rightarrow r = -\cos(\Theta_0)$$

and hence $(-\cos(\Theta_0), \Theta_0)$ is a point of the level set if $\cos(\Theta_0) < 0$. For generic θ, the discriminant of (2.1) is

$$\begin{aligned}\frac{\Delta}{4} &= \sin^2(\Theta_0) + \sin(\theta-\Theta_0)\sin(\theta+\Theta_0) = \\ &\sin^2(\Theta_0) + \frac{1}{2}[\cos(2\Theta_0) - \cos(2\theta)] = \\ &\sin^2(\Theta_0) + \frac{1}{2}[\cos^2(\Theta_0) - \sin^2(\Theta_0) - \cos(2\theta)] = \\ &\frac{1-\cos(2\theta)}{2} = \sin^2(\theta)\end{aligned}$$

Hence we have two solutions for the variable r:

$$r_1(\theta) = \frac{\sin(\Theta_0) - \sin(\theta)}{\sin(\theta - \Theta_0)}$$

$$r_2(\theta) = \frac{\sin(\Theta_0) + \sin(\Theta_0)}{\sin(\theta - \Theta_0)}$$

The condition $r > 0$ implies that r_1 above gives well defined radial parametrization for a branch of the considered level set if

$$\begin{cases} \pi - \Theta_0 < \theta < \Theta_0 + \pi & \text{if } 0 < \Theta_0 < \pi \\ \Theta_0 - \pi < \theta < 3\pi - \Theta_0 & \text{if } \pi < \Theta_0 < 2\pi \end{cases}$$

while r_2 is a radial parametrization in

$$\begin{cases} \Theta_0 < \theta < 2\pi - \Theta_0 & \text{if } 0 < \Theta_0 < \pi \\ 2\pi - \Theta_0 < \theta < \Theta_0 & \text{if } \pi < \Theta_0 < 2\pi \end{cases}$$

Furthermore, it is easy to notice the following facts concerning the behaviour of r_1, r_2 at the boundary of their domain of definition.

- if $0 < \Theta_0 < \pi$

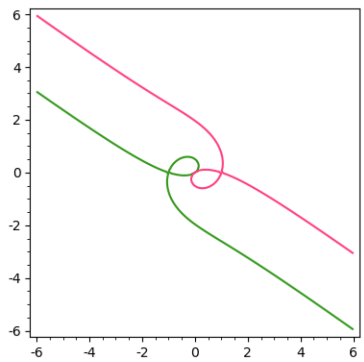
$$\begin{aligned} \lim_{\theta \to (\Theta_0+\pi)^-} r_1(\theta) &= \lim_{\theta \to \Theta_0^+} r_2(\theta) = +\infty \\ \lim_{\theta \to (\pi-\Theta_0)^+} r_1(\theta) &= \lim_{\theta \to (2\pi-\Theta_0)^-} r_2(\theta) = 0 \end{aligned} \tag{2.2}$$

- if $-\pi < \Theta_0 < 2\pi$

$$\begin{aligned} \lim_{\theta \to (\pi+\Theta_0)^+} r_1(\theta) &= \lim_{\theta \to \Theta_0^-} r_2(\theta) = +\infty \\ \lim_{\theta \to (3\pi-\Theta_0)^-} r_1(\theta) &= \lim_{\theta \to (2\pi-\Theta_0)^+} r_2(\theta) = 0 \end{aligned} \tag{2.3}$$

Hence the critical level set is made of two analytic curves connecting 0 and ∞ and orthogonally intersecting at 1.

The following picture represents the critical level sets for a Θ_0 with $\frac{\pi}{2} < \Theta_0 < \pi$.



Since the endpoints of the level set are the singular points $0, \infty$ of g_{Θ_0}, it suffices to study what happens when the two branches come nearer these points in order to determine the steepest descent path.

From (2.2),(2.3), we know for which argument in the parametrization the curves go to ∞, hence:

- if $0 < \Theta_0 < 2\pi$

$$\lim_{\theta\to(\Theta_0+\pi)^-} g_{\Theta_0}(r_1(\theta),\theta) = \lim_{\theta\to(\Theta_0+\pi)^-} r_1(\theta)\cos(\theta-\Theta_0) = +\infty,$$
$$\lim_{\theta\to\Theta_0^+} g_{\Theta_0}(r_2(\theta),\theta) = \lim_{\theta\to\Theta_0^+} -r_2(\theta)\cos(\theta-\Theta_0) = -\infty$$

- similarly, if $\pi < \Theta_0 < 2\pi$

$$\lim_{\theta\to(\Theta_0+\pi)^+} g_{\Theta_0}(r_1(\theta),\theta) = +\infty$$
$$\lim_{\theta\to\Theta_0^-} g_{\Theta_0}(r_1(\theta),\theta) = -\infty$$

Same computations and result holds when dealing with 0.

We get then that the steepest descent path associated to 1 has radial parametrization given by r_2.

Similarly we can compute for the steepest descent path at -1 and find a radial parametrization given, this time, by

$$r(\theta) = \frac{-\sin(\Theta_0)+\sin(\theta)}{\sin(\theta-\Theta_0)}.$$

We can give an orientation to the paths constructed so far with $\Theta_0 \neq 0,\pi$ in the following way.

The theory of flow lines for the downward gradient equation states that, near the critical point p chosen as limit condition, the solution curves are abutting from p with tangents given by eigenvectors of the Hessian matrix of f computed at p and relative to the negative eigenvalue.

It is sufficient to pick one of these eigenvector: the direction chosen will give the orientation.

In our case

$$\begin{aligned} Hess(\operatorname{Re}\Psi_{\Theta_0})(1,0) =& 2\begin{pmatrix} -\cos(\Theta_0) & -\sin(\Theta_0) \\ -\sin(\Theta_0) & \cos(\Theta_0) \end{pmatrix} \\ Hess(\operatorname{Re}\Psi_{\Theta_0})(1,\pi) =& 2\begin{pmatrix} \cos(\Theta_0) & \sin(\Theta_0) \\ \sin(\Theta_0) & -\cos(\Theta_0) \end{pmatrix} \end{aligned}$$

Both have determinant equals to -4 and ± 2 as eigenvalues.

The eigenspace for -2 is generated by

$$\begin{pmatrix} \cos\left(\frac{\Theta}{2}\right) \\ \sin\left(\frac{\Theta}{2}\right) \end{pmatrix}$$

if the critical point is $(1,0)$ and by

$$\begin{pmatrix} -\sin\left(\frac{\Theta}{2}\right) \\ \cos\left(\frac{\Theta}{2}\right) \end{pmatrix}$$

if the critical point is $(1,\pi)$.

We will use these eigenvectors to orientate our cycles. It is noticeable that, since all steepest descent are parametrized by θ, it suffices to consider only the second entry of the eigenvalues, the one relative to the θ component.

The resulting oriented cycles associated with $+1$ and -1 will be denoted by $\Gamma^{\Psi}_{0,\Theta_0}$ and $\Gamma^{\Psi}_{1,\Theta_0}$, respectively.

In particular, $\Gamma^{\Psi}_{0,\Theta_0}$ will be oriented counterclockwise if $\sin\left(\frac{\Theta}{2}\right) > 0$, clockwise if $\sin\left(\frac{\Theta}{2}\right) < 0$. The same holds by changing $\Gamma^{\Psi}_{0,\Theta_0}$ with $\Gamma^{\Psi}_{1,\Theta_0}$ and $\sin\left(\frac{\Theta}{2}\right)$ with $\cos\left(\frac{\Theta}{2}\right)$.

The importance of the steepest descent cycles is due to the following fact (see §section 6.5 for a proof): if we denote by $\mathcal{S}C$ the support of a curve C, we have

$$R\Gamma_c(\{\operatorname{Re}\Psi \leq t_0\} \setminus (\mathcal{S}\Gamma^{\Psi}_{-1,\Theta_0} \cup \mathcal{S}\Gamma^{\Psi}_{+1,\Theta_0})) = 0$$

equivalent to

$$R\Gamma_c(\{\operatorname{Re}\Psi \leq t_0\}) \simeq R\Gamma_c(\mathcal{S}\Gamma^{\Psi}_{-1,\Theta_0} \cup \mathcal{S}\Gamma^{\Psi}_{+1,\Theta_0})) \tag{2.4}$$

Since $\forall\Theta_0 \neq 0, \pi$ the steepest descent paths are disjoint, (2.4) gives a decomposition for the cohomology with compact support we are dealing with, i.e.

$$H^1_c(\{\operatorname{Re}\Psi \leq t_0\})[1] \simeq H^1_c(\mathcal{S}\Gamma^{\Psi}_{-1,\Theta_0})[1] \oplus H^1_c(\mathcal{S}\Gamma^{\Psi}_{+1,\Theta_0})[1] \tag{2.5}$$

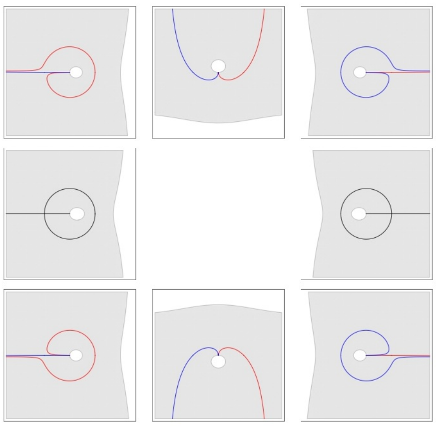
At $\Theta_0 = 0, \pi$, the degeneration does not allow us to state the same: indeed, as shown, they intersect at $+1$ if $\Theta_0 = 0$ and at -1 if $\Theta_0 = \pi$. We will refer to the degenerated configurations at $\Theta_0 = 0, \pi$ as

$$\begin{aligned} T_0 &:= \mathcal{S}\Gamma^{\Psi}_{-1,0} \cup \mathcal{S}\Gamma^{\Psi}_{+1,0} \\ T_\pi &:= \mathcal{S}\Gamma^{\Psi}_{-1,\pi} \cup \mathcal{S}\Gamma^{\Psi}_{+1,\pi}. \end{aligned} \tag{2.6}$$

This degeneration is the obstruction for a global trivialization of the enhanced sheaf K and the source for the Stokes phenomenon underlying the transform.

The following picture represents the steepest descent for some value of Θ_0.

FIGURE 2. Steepest descent paths



Let us focus on the left and right side of the picture: they are the steepest descent paths for $0 \pm \delta$ (on the right) and $\Theta_0 = \pi \pm \delta$ (on the left).

It is noticeable that in the limit $\delta \to 0$ they define two closed subsets of T_0, T_π.

We will denote them by $\Gamma^{\Psi}_{0,\Theta_0\pm}$ and $\Gamma_{1,\Theta_0\pm}$ for $\Theta_0 = 0, \pi$: we can extend the orientation accordingly to the limit.

We will show that this construction provides two decompositions for $\Theta_0 = 0, \pi$ (see §10):

$$\begin{array}{c} H^1_c(\mathcal{S}\Gamma^{\Psi}_{0,\Theta_0-})[1] \oplus H^1_c(\mathcal{S}\Gamma^{\Psi}_{1,\Theta_0-})[1] \\ \sigma_{\Theta_0-}\Big\uparrow\sim \\ H^1_c(T_{\Theta_0})[1] \\ \sigma_{\Theta_0+}\Big\downarrow\sim \\ H^1_c(\mathcal{S}\Gamma^{\Psi}_{0,\Theta_0+})[1] \oplus H^1_c(\mathcal{S}\Gamma^{\Psi}_{1,\Theta_0+})[1] \end{array} \tag{2.7}$$

We will use the families of curves constructed so far to determine a trivialization for the enhanced sheaf on the sectors $\Sigma_1 = \{\zeta : 0 \leq \Theta \leq \pi\}$ and $\Sigma_2 = \{\zeta : \pi \leq \Theta \leq 2\pi\}$ (see §11).

$$\begin{aligned} \pi_2^{-1}\mathbf{k}_{\Sigma_1} \otimes \tilde{\rho}_2^{-1}(E^{-\frac{1}{u}})^{\curlywedge} &\xrightarrow[\sim]{\sigma_1} \mathbf{k}_{\Sigma_1} \otimes (E^{2\zeta^{-1}} \oplus E^{-2\zeta^{-1}})[1] \\ \pi_2^{-1}\mathbf{k}_{\Sigma_2} \otimes \tilde{\rho}_2^{-1}(E^{-\frac{1}{u}})^{\curlywedge} &\xrightarrow[\sim]{\sigma_2} \pi_2^{-1}\mathbf{k}_{\Sigma_2} \otimes (E^{2\zeta^{-1}} \oplus E^{-2\zeta^{-1}})[1] \end{aligned} \tag{2.8}$$

In particular the families $\Gamma_{0,\Theta_0\pm}$ and $\Gamma_{1,\Theta_0\pm}$ for $\Theta_0 = 0, \pi$ will be used to extend the well defined decomposition (2.5) valid for all $\Theta \neq 0, \pi$ to the boundary of the sectors Σ_1 Σ_2.

The isomorphisms (2.8) induce the following diagrams for rays l_h with $h = 1, 2$

$$\begin{array}{ccc} & & \pi_2^{-1}\mathbf{k}_{l_1} \otimes (E^{2\zeta^{-1}} \oplus E^{-2\zeta^{-1}})[1] \\ & \overset{\sigma_2}{\nearrow} & \Big\downarrow \alpha_1 \\ \pi_2^{-1}\mathbf{k}_{l_1} \otimes K & & \\ & \underset{\sigma_1}{\searrow} & \\ & & \pi_2^{-1}\mathbf{k}_{l_1} \otimes (E^{2\zeta^{-1}} \oplus E^{-2\zeta^{-1}})[1] \end{array}$$

$$\begin{array}{ccc} & & \pi_2^{-1}\mathbf{k}_{l_2} \otimes (E^{2\zeta^{-1}} \oplus E^{-2\zeta^{-1}})[1] \\ & \overset{\sigma_1}{\nearrow} & \Big\downarrow \alpha_\pi \\ \pi_2^{-1}\mathbf{k}_{l_2} \otimes K & & \\ & \underset{\sigma_2}{\searrow} & \\ & & \pi_2^{-1}\mathbf{k}_{l_2} \otimes (E^{2\zeta^{-1}} \oplus E^{-2\zeta^{-1}})[1] \end{array}$$

We can read the Stokes multipliers from the stalk at (ζ_0, t_0) (with t_0 big enough) of the morphism at the right.

In particular the stalk of the left part of the diagram corresponds, by construction of σ_1, σ_2, to 2.7.

Hence the Stokes multiplier α_h can be computed as $\sigma_{\Theta_0+} \circ \sigma^{-1}_{\Theta_0-}$.

The best way to express this morphism is to pass to the dual side using Borel-Moore homology based on subanalytic chains (see §7.1).

We recall that, if X is a real analytic manifold, Z is a subanalytic locally closed subset of X and L is a local system on X, we have

$$(H_c^j(X, \mathbf{k}_Z \otimes L))^* \simeq H_j^{BM}(Z, L^*).$$

An element of this homology is represented by

$$\Delta \otimes s$$

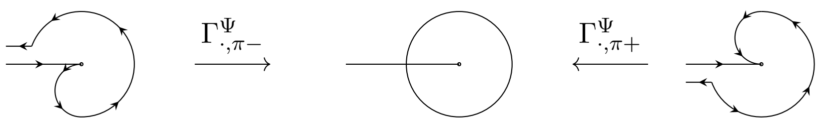
where Δ is an oriented subanalytic j-chain in Z and s a section of L^* along it.

By passing then to the dual side with Borel-Moore homology and denoting by $H_1^{BM}(A) = H_1^{BM}(A, \mathbf{k})$, we can rephrase (2.7) as the fact that $\{\Gamma_{0,\Theta_0+}, \Gamma_{1,\Theta_0+}\}$ and $\{\Gamma^\Psi_{0,\Theta_0-}, \Gamma^\Psi_{1,\Theta_0-}\}$ provides two pairs of basis for

$$H_1^{BM}(T_{\Theta_0})$$

with $\Theta_0 = 0, \pi$. The change of bases arising can be deduced by comparing the behaviour of the cycles in Fig.2.

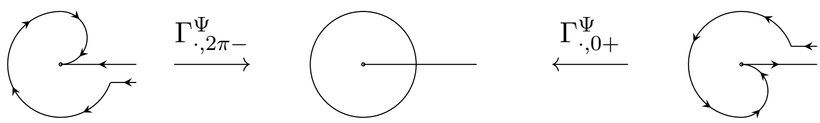
If $\Theta_0 = \pi$, the embeddings of the curves $\Gamma_{0,\pi\pm}$, $\Gamma_{1,\pi\pm}$ can be schematically reproduced as in the picture



It is then easy to deduce that α_1^* is given by the assignment

$$\Gamma^\Psi_{0,\pi-} \mapsto \Gamma^\Psi_{0,\pi+} - 2\Gamma^\Psi_{1,\pi+}$$
$$\Gamma^\Psi_{1,\pi-} \mapsto \Gamma^\Psi_{1,0+}$$

At $\Theta_0 = 0$, we have to compare the cycles $\Gamma^\Psi_{0,2\pi-}, \Gamma^\Psi_{1,2\pi-}$ with $\Gamma^\Psi_{0,0+}, \Gamma^\Psi_{1,0+}$



We have that α_2^* is then defined by

$$\Gamma^\Psi_{0,0-} \mapsto -\Gamma^\Psi_{0,0+}$$
$$\Gamma^\Psi_{1,0-} \mapsto -\Gamma^\psi_{1,0+} - 2\Gamma^\Psi_{0,0+}$$

3. Enhanced sheaves and ind-sheaves

For the content of this section, we make use of [6, Section 1], which we reproduce here for the reader's convenience.

Let us briefly recall the theory of enhanced (ind-)sheaves.

Let M be a real analytic manifold and $\mathbf{k}$ be a field.

3.1. **Sheaves.** Denote by $D^b(\mathbf{k}_M)$ the bounded derived category of sheaves of $\mathbf{k}$-vector spaces on M. For $S \subset M$ a locally closed subset, denote by $\mathbf{k}_S$ the zero extension to M of the constant sheaf on S with stalk $\mathbf{k}$.

For $f : M \to N$ a morphism of real analytic manifolds, denote by $\otimes, f^{-1}, Rf_!, R\mathcal{H}om, Rf_*, f^!$ the six Grothendieck operations for sheaves.

3.2. **Convolution.** Consider the maps

$$\mu, q_1, q_2 : M \times \mathbb{R} \times \mathbb{R} \to M \times \mathbb{R}$$

given by $q_1(x, t_1, t_2) = (x, t_1)$, $q_2(x, t_1, t_2) = (x, t_2)$ and $\mu(x, t_1, t_2) = (x, t_1 + t_2)$. For $F_1, F_2 \in D^b(\mathbf{k}_{M\times\mathbb{R}})$ the functors of convolution in the variable t are defined by

$$\begin{aligned} F_1 \overset{*}{\otimes} F_2 &= R\mu_!(q_1^{-1}F_1 \otimes q_2^{-1}F_2) \\ R\mathcal{H}om^*(F_1, F_2) &= Rq_{1*}R\mathcal{H}om(q_2^{-1}F_1, \mu^! F_2) \end{aligned} \tag{3.1}$$

The convolution product $\overset{*}{\otimes}$ makes $D^b(\mathbf{k}_{M\times\mathbb{R}})$ into a commutative tensor category with $\mathbf{k}_{M\times\{0\}}$ as unit object. We will often write $\mathbf{k}_{\{t=0\}}$ instead of $\mathbf{k}_{M\times\{0\}}$, and similarly for $\mathbf{k}_{\{t\geq 0\}}$, $\mathbf{k}_{\{t\leq 0\}}$, etc.

3.3. **Enhanced sheaves.** Consider the projection

$$M \times \mathbb{R} \xrightarrow{\pi} M \tag{3.2}$$

The triangulated category $E^b(\mathbf{k}_M)$ of enhanced sheaves is then the quotient of $D^b(\mathbf{k}_{M\times\mathbb{R}})$ by the stable subcategory $\pi^{-1}D^b(\mathbf{k}_M)$. It splits as $E^b(\mathbf{k}_M) \simeq E^b_+(\mathbf{k}_M) \oplus E^b_-(\mathbf{k}_M)$ where $E^b_\pm(\mathbf{k}_M)$ is the quotient of $D^b(\mathbf{k}_{M\times\mathbb{R}})$ by the stable subcategory of objects K satisfying $\mathbf{k}_{\{\mp t\geq 0\}} \overset{*}{\otimes} K \simeq 0$.

The quotient functor

$$Q : D^b(\mathbf{k}_{M\times\mathbb{R}}) \to E^b(\mathbf{k}_M) \tag{3.3}$$

has a left and a right adjoint which are fully faithful.

Let us denote by $\tilde{E}^b_+(\mathbf{k}_M) \subset D^b(\mathbf{k}_{M\times\mathbb{R}})$ the essential image of $E^b_+(\mathbf{k}_M)$ by the left adjoint, that is, the full subcategory whose objects F satisfy $\mathbf{k}_{\{t\geq 0\}} \overset{*}{\otimes} F \simeq F$. Thus, one has an equivalence

$$Q : \tilde{E}^b_+(\mathbf{k}_M) \xrightarrow{\sim} E^b_+(\mathbf{k}_M). \tag{3.4}$$

The functor

$$\epsilon : D^b(\mathbf{k}_M) \to \tilde{E}^b_+(\mathbf{k}_M), \quad G \to \mathbf{k}_{\{t\geq 0\}} \otimes \pi^{-1}G. \tag{3.5}$$

is fully faithful. For $f : M \to N$ a morphism of real analytic manifolds, it interchanges the operations $\otimes$, f^{-1} and $Rf_!$ with $\overset{*}{\otimes}$, $\tilde{f}^{-1}$ and $R\tilde{f}_!$, respectively. Here, we set

$$\tilde{f} := f \times id_{\mathbb{R}} : M \times \mathbb{R} \to N \times \mathbb{R}. \tag{3.6}$$

3.4. **Ind-sheaves.** An ind-sheaf is an ind-object in the category of sheaves with compact support. There is a natural embedding of sheaves into ind-sheaves, and it has an exact left adjoint α given by $\alpha(``\varinjlim" F_i) = \varinjlim F_i$. The functor α has an exact fully faithful left adjoint, denoted by β.

Denote by $D^b(\mathrm{I}\mathbf{k}_M)$ the bounded derived category of ind-sheaves. Denote by $\otimes$, $R\mathcal{IH}om$, f^{-1}, Rf_*, $Rf_{!!}$ and $f^!$ the six Grothendieck operations for ind-sheaves.

3.5. **Enhanced ind-sheaves.** Consider the morphisms

$$M \xrightarrow{i_\infty} M \times \bar{\mathbb{R}} \xrightarrow{\pi} M, \quad M \times \mathbb{R} \xrightarrow{j} M \times \bar{\mathbb{R}} \tag{3.7}$$

where $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the real projective line , $i_\infty(x) = (x, \infty)$, π is the projection and j the embedding.

The triangulated category $E^b(\mathrm{I}\mathbf{k}_M)$ of enhanced ind-sheaves is defined by two consecutive quotients of $D^b(\mathrm{I}\mathbf{k}_{M\times\bar{\mathbb{R}}})$: first by the subcategory of objects of the form $Ri_{\infty*}F$ and then by the subcategory of objects of the form $\pi^{-1}F$. As for enhanced sheaves, the quotient functor has a left and a right adjoint which are fully faithful. It follows that there are two realizations of $E^b(\mathrm{I}\mathbf{k}_M)$ as a full subcategory of $D^b(\mathrm{I}\mathbf{k}_{M\times\bar{\mathbb{R}}})$. Enhanced ind-sheaves are endowed with an analogue of the convolutions functors, denoted $\overset{+}{\otimes}$ and $R\mathcal{I}hom^+$. For $f : M \to N$ a morphism of real analytic manifolds, one also has external operations Ef^{-1}, Ef_*, $Ef_{!!}$ and $Ef^!$. Here Ef^{-1} is the functor induced by $\tilde{f}^{-1}$ at the level of ind-sheaves, and similarly for the other operations.

There is a natural embedding $E^b(\mathbf{k}_M) \subset E^b(\mathrm{I}\mathbf{k}_M)$ induced by $Rj_!$ or, equivalently, by Rj_*. Set

$$\mathbf{k}_M^E := ``\varinjlim_{a\to\infty}" \mathbf{k}_{\{t\geq a\}} \in E^b(\mathbf{Ik}_M) \tag{3.8}$$

We will need the following two lemmas on compatibility for enhanced ind-sheaves and for usual sheaves with an additional variable.

One has:

Lemma 3.1. *Let $F, F_1, F_2 \in D^b(\boldsymbol{k}_{M\times\mathbb{R}})$ and $G \in D^b(\boldsymbol{k}_{N\times\mathbb{R}})$. One has:*

$$\begin{aligned}(\boldsymbol{k}_M^E \overset{+}{\otimes} QF_1) \overset{+}{\otimes} (\boldsymbol{k}_M^E \overset{+}{\otimes} QF_2) &\simeq \boldsymbol{k}_M^E \overset{+}{\otimes} (QF_1 \overset{+}{\otimes} QF_2)\\ &\simeq \boldsymbol{k}_M^E \overset{+}{\otimes} Q(F_1 \overset{*}{\otimes} F_2)\\ Ef^{-1}Q(\boldsymbol{k}_N^E \overset{+}{\otimes} G) &\simeq \boldsymbol{k}_M^E \overset{+}{\otimes} Q\tilde{f}^{-1}G.\end{aligned} \tag{3.9}$$

If moreover f is proper, then

$$Ef_{!!}Q(\boldsymbol{k}_M^E \overset{+}{\otimes} F) \simeq \boldsymbol{k}_N^E \overset{+}{\otimes} QR\tilde{f}_!F. \tag{3.10}$$

Moreover, we recall in the following lemma properties of the convolution needed later (see [4, Prop. 4.1.5][4, Lemma 4.3.1],[4, Prop 4.5.10])

Lemma 3.2. *Let $K_1, K_2 \in E^b(\mathrm{I}\boldsymbol{k}_M)$, $F' \in D^b(\boldsymbol{k}_M)$ $L_1, L_2 \in E^b(\mathrm{I}\boldsymbol{k}_N)$, $f : M \to N$ morphism of real analytic manifolds. Then*

-
$$(K_1 \overset{+}{\otimes} K_2) \overset{+}{\otimes} K_3 \simeq (K_1 \overset{+}{\otimes} K_2) \overset{+}{\otimes} K_3,$$

- $$\pi^{-1}F' \otimes (K_1 \overset{+}{\otimes} K_2) \simeq (\pi^{-1}F' \otimes K_1) \overset{+}{\otimes} K_2,$$
- *(Projection formula)*
$$Ef_{!!}(Ef^{-1}L_1 \overset{+}{\otimes} K_1) \simeq L_1 \overset{+}{\otimes} Ef_{!!}K_1,$$
- $$Ef^{-1}(L_1 \overset{+}{\otimes} L_2) \simeq Ef^{-1}L_1 \overset{+}{\otimes} Ef^{-1}L_2.$$

Moreover also the following result holds (see [4, Prop. 4.5.11])

Proposition 3.3. *Consider a Cartesian diagram of real analytic manifold as below*

$$\begin{array}{ccc} M' & \xrightarrow{f'} & N' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{f} & N \end{array}$$

There are then isomorphisms of functors $E^b(\mathrm{I}\boldsymbol{k}_M) \to E^b(I\boldsymbol{k}_{N'})$

$$Eg^{-1}Ef_{!!} \simeq Ef'_{!!}Eg'^{-1} \qquad Eg^! Ef_* \simeq Ef'_* Eg'^!$$

Thanks to the preceding compatibility lemma, we can rewrite the above result also in term of enhanced sheaves.

The category $E^b(\mathrm{I}\mathbf{k}_M)$ has a natural hom functor $R\mathcal{H}om^E$ with values in $D^b(\mathbf{k}_M)$.

One has

Lemma 3.4. *For* $F \in D^b(\boldsymbol{k}_{M\times\mathbb{R}})$ *one has*

$$R\mathcal{H}om^E(\boldsymbol{k}_{\{t\geq 0\}}, \boldsymbol{k}^E_M \overset{+}{\otimes} QF) \simeq R\pi_*(\boldsymbol{k}_{\{t\geq 0\}} \overset{*}{\otimes} F). \tag{3.11}$$

3.6. $\mathbb{R}$**-constructibility.** Denote by $D^b_{\mathbb{R}-c}(\mathbf{k}_M)$ be the full subcategory of objects with $\mathbb{R}$-constructible cohomologies. Using notations introduced above, denote $D^b_{\mathbb{R}-c}(\mathbf{k}_{M\times\mathbb{R}_\infty})$ the full subcategory of $D^b_{\mathbb{R}-c}(\mathbf{k}_{M\times\mathbb{R}})$ whose objects F are such that $Rj_!F$ (or, equivalently, Rj_*F) is $\mathbb{R}$-constructible in $M\times\bar{\mathbb{R}}$. Since $Rj_!$ is fully faithful, we will consider $D^b_{\mathbb{R}-c}(\mathbf{k}_{M\times\mathbb{R}_\infty})$ as a full subcategory of $D^b_{\mathbb{R}-c}(\mathbf{k}_{M\times\bar{\mathbb{R}}})$.

The triangulated category $\tilde{E}^b_{\mathbb{R}-c}(\mathbf{k}_M)$ of $\mathbb{R}$-constructible enhanced sheaves is the full subcategory of $D^b_{\mathbb{R}-c}(\mathbf{k}_{M\times\mathbb{R}_\infty})$ whose objects F satisfy $F \simeq \mathbf{k}_{\{t\geq 0\}} \overset{*}{\otimes} F$. It is a full subcategory of $\tilde{E}^b_+(\mathbf{k}_M)$.

The category $E^b_{\mathbb{R}-c}(\mathrm{I}\mathbf{k}_M)$ of $\mathbb{R}$-constructible enhanced ind-sheaves is defined as the full subcategory of $E^b(\mathrm{I}\mathbf{k}_M)$ whose objects K satisfy the following property: for any relatively compact open subset $U \subset M$ there exists $F \in \tilde{E}^b_{\mathbb{R}-c}(\mathbf{k}_M)$ such that

$$\pi^{-1}\mathbf{k}_U \otimes K \simeq \mathbf{k}^E_M \overset{+}{\otimes} QF. \tag{3.12}$$

The object $\mathbf{k}^E_M$ plays the role of the constant sheaf in $E^b_{\mathbb{R}-c}(\mathbf{Ik}_M)$

4. Riemann-Hilbert correspondence

For the content of this section, we make use of [6, Section 2], which we reproduce here for the reader's convenience.

We recall here the Riemann-Hilbert correspondence for (not necessarily regular) holonomic $\mathcal{D}$-modules. This is based on the theory of enhanced ind-sheaves. One of the key ingredients of the proof is the description of the structure of flat meromorphic connections by Kedlaya and Mochizuki.

Let X be a complex manifold. We set for short $d_X = dim X$.

4.1. **$\mathcal{D}$-modules.** Denote by $\mathcal{O}_X$ and $\mathcal{D}_X$ the rings of holomorphic functions and of differential operators on X, respectively.

Denote by $D^b(\mathcal{D}_X)$ the bounded derived category of left $\mathcal{D}_X$-modules. For $f : X \to Y$ a morphism of complex manifolds, denote by $\otimes^D$, Df^*, Df_* the operations for $\mathcal{D}$-modules.

Denote $D^b_{hol}(\mathcal{D}_X)$ the full subcategory of $D^b(\mathcal{D}_X)$ of objects with holonomic cohomologies, and by $D^b_{g-hol}(\mathcal{D}_X)$ the full subcategory of objects with good and holonomic cohomologies.

Let $D \subset X$ a complex analytic hypersurface and denote by $\mathcal{O}_X(*D)$ the sheaf of meromorphic functions with poles along D. Set $U = X \setminus D$.

For $\varphi \in \mathcal{O}_X(*D)$, set

$$\begin{aligned} \mathcal{D}_X e^{\varphi} &= \mathcal{D}_X / \{P : Pe^{\varphi} = 0 \text{ on } U\} \\ \mathcal{E}^{\varphi}_{U|X} &= \mathcal{D}_X e^{\varphi} \otimes^D \mathcal{O}_X(*D). \end{aligned} \tag{4.1}$$

4.2. **Tempered solutions.** By the functor β, there is a natural notion of $\mathcal{D}_X$-module in the category of ind-sheaves. We denote $D^b(\mathrm{I}\mathcal{D}_X)$ the corresponding derived category.

Denote by $\mathcal{O}^t_X$ the complex of tempered holomorphic functions. It is related to the functor $\mathcal{T}hom$ by the relation

$$\alpha R\mathcal{I}hom(F, \mathcal{O}^t_X) \simeq \mathcal{T}hom(F, \mathcal{O}_X) \tag{4.2}$$

for any $F \in D^b_{\mathbb{R}-c}(\mathbf{k}_X)$.

Consider the tempered solution functor

$$\mathcal{S}ol^t_X : D^b(\mathcal{D}_X)^{op} \to D^b(\mathrm{I}\mathbf{k}_X), \quad \mathcal{M} \to R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^t_X). \tag{4.3}$$

4.3. **Enhanced solutions.** There is a natural notion of $\mathcal{D}_X$-module in the category of enhanced ind-sheaves, and we denote by $E^b(\mathrm{I}\mathcal{D}_X)$ the corresponding triangulated category.

Let $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ be the complex projective line, and let

$$i : X \times \bar{\mathbb{R}} \to X \times \mathbb{P} \tag{4.4}$$

be the closed embedding. Denote by $\tau \in \mathbb{P}$ the affine coordinate, so that $\tau \in \mathcal{O}_{\mathbb{P}}(*\infty)$. Consider the exponential $\mathcal{D}_{\mathbb{P}}$-module $\mathcal{E}^{\tau}_{\mathbb{C}|\mathbb{P}}$.

The enhanced solution functor is given by

$$\mathcal{S}ol^E_X : D^b(\mathcal{D}_X)^{op} \to E^b(\mathrm{I}\mathbf{k}_X), \quad \mathcal{M} \to i^{-1}\mathcal{S}ol^t_{X\times\mathbb{P}}(\mathcal{M} \boxtimes^D \mathcal{E}^{\tau}_{\mathbb{C}|\mathbb{P}})[2] \tag{4.5}$$

The functorial properties of $\mathcal{S}ol^E$ are summarized in the next theorem. The statements on direct and inverse images are easy consequences of the corresponding results for tempered holomorphic solutions. The statement on the tensor product is specific to enhanced solutions.

Theorem 4.1. *Let $f : X \to Y$ be a complex analytic map. Let $\mathcal{M} \in D^b_{g-hol}(\mathcal{D}_X)$, $\mathcal{M}_1, \mathcal{M}_2 \in D^b_{hol}(\mathcal{D}_X)$ and $\mathcal{N} \in D^b_{hol}(\mathcal{D}_Y)$. Assume supp$\mathcal{M}$ is proper over Y. Then one has*

$$\begin{aligned} \mathcal{S}ol^E_X(Df^*\mathcal{N}) &\simeq Ef^{-1}\mathcal{S}ol^E_Y(\mathcal{N}), \\ \mathcal{S}ol^E_Y(Df_*\mathcal{M})[d_Y] &\simeq Ef_{!!}\mathcal{S}ol^E_X(\mathcal{M})[d_X], \\ \mathcal{S}ol^E_X(\mathcal{M}_1) \overset{+}{\otimes} \mathcal{S}ol^E_X(\mathcal{M}_2) &\simeq \mathcal{S}ol^E_X(\mathcal{M}_1 \otimes^D \mathcal{M}_2). \end{aligned} \tag{4.6}$$

Notation: let $D \subset X$ be a closed hypersurface and set $U = X \setminus D$. For $\varphi \in \mathcal{O}_X(*D)$, we set

$$\begin{aligned} E^\varphi &:= \mathbf{k}_{\{t+Re\varphi(x)\geq 0\}} \in \tilde{E}^b_{\mathbb{R}-c}(\mathbf{k}_X) \\ \mathbb{E}^\varphi &:= \mathbf{k}^E_X \overset{+}{\otimes} QE^\varphi \in E^b_{\mathbb{R}-c}(\mathbf{Ik}_X) \end{aligned} \tag{4.7}$$

where we set for short

$$\{t + Re\varphi \geq 0\} = \{(x,t) \in X \times \mathbb{R} : x \in U, t + Re\varphi(x) \geq 0\} \tag{4.8}$$

We will also need the following computation.

Theorem 4.2. *With the above notations, one has*

$$\mathcal{S}ol^E_X(\mathcal{E}^\varphi_{U|X}) \simeq \mathbb{E}^\varphi. \tag{4.9}$$

4.4. **Riemann-Hilbert correspondence.** Let us state the Riemann-Hilbert correspondence for holonomic $\mathcal{D}$-modules.

Theorem 4.3. *The enhanced solution functor induces a fully faithful functor*

$$\mathcal{S}ol^E_X : D^b_{hol}(\mathcal{D}_X)^{op} \to E^b_{\mathbb{R}-c}(\mathrm{I}\boldsymbol{k}_X) \tag{4.10}$$

Moreover, there is a functorial way of reconstructing $\mathcal{M} \in D^b_{hol}(\mathcal{D}_X)$ from $\mathcal{S}ol^E_X(\mathcal{M})$.

This implies in particular that the Stokes structure of a flat meromorphic connection $\mathcal{M}$ is encoded in $\mathcal{S}ol^E(\mathcal{M})$ (see [4, §9.8]).

4.5. **A lemma.** We will use the following remark. Let $D \subset X$ be a closed hypersurface, set $U = X \setminus D$ and denote by $j : U \to X$ the embedding.

Lemma 4.4. *With the above notations, let $\mathcal{M} \in D^b_{hol}(\mathcal{D}_X)$ be such that $\mathcal{M} \simeq \mathcal{M}(*D)$. Assume that X is compact. Then there exists $F \in \tilde{E}^b_{\mathbb{R}-c}(\boldsymbol{k}_U)$ such that $R\tilde{j}_!F \in \tilde{E}^b_{\mathbb{R}-c}(\boldsymbol{k}_X)$ and*

$$\mathcal{S}ol^E_X(\mathcal{M}) \simeq \boldsymbol{k}^E_X \overset{+}{\otimes} QR\tilde{j}_!F. \tag{4.11}$$

5. Fourier transform

For the content of this section, we make use of [6, Section 3], which we reproduce here for the reader's convenience.

By functoriality, the enhanced solution functor interchanges integral transforms at the level of holonomic $\mathcal{D}$-modules with integral transforms at the level of enhanced ind-sheaves. (This was observed in [11], where the non-holonomic case is also discussed.) We recall here some consequences of this fact, dealing in particular with the Fourier transform.

5.1. **Integral transforms.** Consider a diagram of complex manifolds

$$\begin{array}{ccc} & S & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & & Y \end{array}$$

At the level of $\mathcal{D}$-modules, the integral transform with kernel $\mathcal{L} \in D^b(\mathcal{D}_S)$ is the functor

$$* \overset{D}{\circ} \mathcal{L} : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y), \quad \mathcal{M} \overset{D}{\circ} \mathcal{L} := Dq_*(\mathcal{L} \otimes^D Dp^* \mathcal{M}). \tag{5.1}$$

At the level of enhanced ind-sheaves, the integral transform with kernel $H \in E^b(\mathrm{I}\mathbf{k}_S)$ is the functor

$$* \overset{+}{\circ} H : E^b(\mathrm{I}\mathbf{k}_X) \to E^b(\mathrm{I}\mathbf{k}_Y), \quad K \overset{+}{\circ} H = Eq_{!!}(H \overset{+}{\otimes} Ep^{-1} K) \tag{5.2}$$

Using Theorem 4.1, we get the following

Corollary 5.1. *Let $\mathcal{M} \in D^b_{g-hol}(\mathcal{D}_X)$ and $\mathcal{L} \in D^b_{g-hol}(\mathcal{D}_S)$. Assume that $p^{-1} supp(\mathcal{M}) \cap supp(\mathcal{L})$ is proper over Y. Set $K = \mathcal{S}ol^E_X(\mathcal{M})$ and $H = \mathcal{S}ol^E_S(\mathcal{L})$. Then there is a natural isomorphism in $E^b_{\mathbb{R}-c}(\mathrm{I}\boldsymbol{k}_Y)$*

$$\mathcal{S}ol^E_Y(\mathcal{M} \overset{D}{\circ} \mathcal{L}) \simeq K \overset{+}{\circ} H[d_S - d_Y]. \tag{5.3}$$

Remark 5.2. *There is a similar statement with the solution functor replaced by the de Rham functor. This has been extended to the case where $\mathcal{M}$ is good but not necessarily holonomic.*

5.2. **Globally $\mathbb{R}$-constructible enhanced ind-sheaves.** Consider the diagram of real analytic manifolds induced by 5.1

$$\begin{array}{ccc} & S \times \mathbb{R} & \\ {}^{\tilde{p}}\swarrow & & \searrow^{\tilde{q}} \\ X \times \mathbb{R} & & Y \times \mathbb{R} \end{array}$$

where $\tilde{p} = p \times id_{\mathbb{R}}$ and $\tilde{q} = q \times id_{\mathbb{R}}$.

The natural integral transform for $\mathbb{R}$-constructible enhanced sheaves with kernel $L \in \tilde{E}^b_{\mathbb{R}-c}(\mathrm{I}\mathbf{k}_S)$ is the functor

$$* \overset{*}{\circ} L : \tilde{E}^b_{\mathbb{R}-c}(\mathrm{I}\mathbf{k}_X) \to \tilde{E}^b_{\mathbb{R}-c}(\mathrm{I}\mathbf{k}_Y), \quad F \overset{*}{\circ} L = R\tilde{q}_!(L \overset{*}{\otimes} \tilde{p}^{-1} F). \tag{5.4}$$

By corollary 5.1 and Lemma 4.4 we obtain

Proposition 5.3. *Let $\mathcal{M} \in D^b_{g-hol}(\mathcal{D}_X)$, $\mathcal{L} \in D^b_{g-hol}(\mathcal{D}_S)$, and assume that $p^{-1} supp(\mathcal{M}) \cap supp(\mathcal{L})$ is proper over Y. Let $F \in \tilde{E}^b_{\mathbb{R}-c}(\boldsymbol{k}_X)$, $L \in \tilde{E}^b_{\mathbb{R}-c}(\boldsymbol{k}_S)$ and assume that there are isomorphisms*

$$(5.5) \qquad \mathcal{S}ol^E_X(\mathcal{M}) \simeq \boldsymbol{k}^E_X \overset{+}{\otimes} QF, \quad \mathcal{S}ol^E_S(\mathcal{L}) \simeq \boldsymbol{k}^E_S \overset{+}{\otimes} QL$$

Then there is a natural isomorphism in $E^b_{\mathbb{R}-c}(\mathrm{I}\boldsymbol{k}_Y)$

$$(5.6) \qquad \mathcal{S}ol^E_Y(\mathcal{M} \overset{D}{\circ} \mathcal{L}) \simeq \boldsymbol{k}^E_Y \overset{+}{\otimes} Q(F \overset{*}{\circ} L)[d_S - d_Y].$$

Note that if X and S are compact, then for any $\mathcal{M} \in D^b_{hol}(\mathcal{D}_X)$ and $\mathcal{L} \in D^b_{hol}(\mathcal{D}_S)$ there exists $F \in \tilde{E}^b_{\mathbb{R}-c}(\mathbf{k}_X)$ and $L \in \tilde{E}^b_{\mathbb{R}-c}(\mathbf{k}_S)$ satisfying the hypothesis of Proposition 5.3.

5.3. **Fourier-Laplace transform.** Let $\mathbb{V}$ be a finite dimensional complex vector space, denote by $\mathbb{P}$ its projective compactification and set $\mathbb{H} = \mathbb{P} \setminus \mathbb{V}$.

Definition 5.4. *Let $D^b_{hol}(\mathcal{D}_{\mathbb{V}_\infty})$ be the full triangulated subcategory of $D^b_{hol}(\mathcal{D}_{\mathbb{P}})$ whose objects $\mathcal{M}$ satisfy $\mathcal{M} \simeq \mathcal{M}(*\mathbb{H})$.*

Let $\mathbb{V}^*$ the dual vector space of $\mathbb{V}$, denote by $\mathbb{P}^*$ its projective compactification, and set $\mathbb{H}^* = \mathbb{P}^* \setminus \mathbb{V}^*$. The pairing

$$(5.7) \qquad \mathbb{V} \times \mathbb{V}^* \to \mathbb{C}, \quad (z, w) \to \langle z, w \rangle$$

defines a meromorphic function on $\mathbb{P} \times \mathbb{P}^*$ with poles along $(\mathbb{P} \times \mathbb{H}^*) \cup (\mathbb{H} \times \mathbb{P}^*) = (\mathbb{P} \times \mathbb{P}^*) \setminus (\mathbb{V} \times \mathbb{V}^*)$. Consider the projections:

$$\begin{array}{ccc} & \mathbb{P} \times \mathbb{P}^* & \\ \swarrow & & \searrow \\ \mathbb{P} & & \mathbb{P}^* \end{array}$$

Definition 5.5. *Set*

$$(5.8) \qquad \mathcal{L} = \mathcal{E}^{-\langle z,w \rangle}_{\mathbb{V} \times \mathbb{V}^* | \mathbb{P} \times \mathbb{P}^*}, \quad \mathcal{L}^a = \mathcal{E}^{\langle w,z \rangle}_{\mathbb{V}^* \times \mathbb{V} | \mathbb{P}^* \times \mathbb{P}}$$

The Fourier-Laplace transform of $\mathcal{M} \in D^b_{hol}(\mathcal{D}_{\mathbb{V}_\infty})$ is given by

$$(5.9) \qquad \mathcal{M}^\wedge = \mathcal{M} \overset{D}{\circ} \mathcal{L} \in D^b_{hol}(\mathcal{D}_{\mathbb{V}^*_\infty}).$$

The inverse Fourier-Laplace transform of $\mathcal{N} \in D^b_{hol}(\mathcal{D}_{\mathbb{V}^*_\infty})$ is given by

$$(5.10) \qquad \mathcal{N}^\vee = \mathcal{N} \overset{D}{\circ} \mathcal{L}^a \in D^b_{hol}(\mathcal{D}_{\mathbb{V}_\infty}).$$

Theorem 5.6. *The Fourier-Laplace transform gives an equivalence of categories*

$$(5.11) \qquad {}^\wedge : D^b_{hol}(\mathcal{D}_{\mathbb{V}_\infty}) \to D^b_{hol}(\mathcal{D}_{\mathbb{V}^*_\infty})$$

A quasi-inverse is given by $\mathcal{N} \to \mathcal{N}^\vee$.

This result is classical. The idea of the proof is as follows. Denoting by $D_{\mathbb{V}}$ the Weyl algebra, there is an equivalence of categories $D^b(D_{\mathbb{V}}) \simeq D^b(\mathcal{D}_{\mathbb{V}_\infty})$.

Under this equivalence, the Fourier-Laplace transform is induced by the algebra isomorphism $D_{\mathbb{V}} \simeq D_{\mathbb{V}^*}$ given by $z_i \to -\partial_{w_i}, \partial_{z_i} \to w_i$.

Using the Riemann-Hilbert correspondence and a result from , we give an alternate topological proof of the above theorem in the remark below.

5.4. **Enhanced Fourier-Sato transform.** Recall that $j : \mathbb{V} \to \mathbb{P}$ denotes the embedding.

Definition 5.7. *Let $\tilde{E}^b_{\mathbb{R}-c}(\boldsymbol{k}_{\mathbb{V}_\infty})$ be the full triangulated subcategory of $\tilde{E}^b_{\mathbb{R}-c}(\boldsymbol{k}_{\mathbb{V}})$ whose objects F satisfy $R\tilde{j}_!F \in \tilde{E}^b_{\mathbb{R}-c}(\boldsymbol{k}_{\mathbb{P}})$.*

Consider the projections

$$\begin{array}{ccccc} & & \mathbb{V} \times \mathbb{V}^* \times \mathbb{R} & & \\ & \swarrow^{\tilde{p}} & \downarrow{\scriptstyle \bar{p}} & \searrow^{\tilde{q}} & \\ \mathbb{V} \times \mathbb{R} & & \mathbb{V} & & \mathbb{V}^* \times \mathbb{R} \end{array}$$

Definition 5.8. *Recall the notation introduced for exponential enhanced (ind-)sheaves and set*

$$L = E^{-\langle z,w\rangle}[1], \quad L^a = E^{\langle w,z\rangle}[1]. \tag{5.12}$$

The enhanced Fourier-Sato transform of $F \in \tilde{E}^b_{\mathbb{R}-c}(\boldsymbol{k}_{\mathbb{V}_\infty})$ is given by

$$F^{\curlywedge} = F \overset{*}{\circ} L \in \tilde{E}^b_{\mathbb{R}-c}(\boldsymbol{k}_{\mathbb{V}^*_\infty}) \tag{5.13}$$

The enhanced inverse Fourier-Sato transform of $G \in \tilde{E}^b_{\mathbb{R}-c}(\boldsymbol{k}_{\mathbb{V}^*_\infty})$ is given by

$$G^{\curlyvee} = G \overset{*}{\circ} L^a \in \tilde{E}^b_{\mathbb{R}-c}(\boldsymbol{k}_{\mathbb{V}_\infty}) \tag{5.14}$$

The transform $F \to F^{\curlywedge}$ has been investigated by Tamarkin (in the more generale case of vector spaces over $\mathbb{R}$). He proved in particular the following result.

Proposition 5.9. *The enhanced Fourier-Sato transform gives an equivalence of categories*

$$\curlywedge : \tilde{E}^b_{\mathbb{R}-c}(\boldsymbol{k}_{\mathbb{V}_\infty}) \xrightarrow{\simeq} \tilde{E}^b_{\mathbb{R}-c}(\boldsymbol{k}_{\mathbb{V}^*_\infty}) \tag{5.15}$$

A quasi-inverse is given by $G \to G^{\curlyvee}$.

Denoting $u : \mathbb{V} \times \mathbb{V}^* \to \mathbb{P} \times \mathbb{P}^*$ the embedding, one has

$$\mathcal{S}ol^E_{\mathbb{P}\times\mathbb{P}^*}(\mathcal{L}) = \mathbb{E}^{\langle z,w\rangle}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{P}\times\mathbb{P}^*} \simeq \mathbf{k}^E_{\mathbb{P}} \overset{+}{\otimes} QR\tilde{u}_!L \tag{5.16}$$

The next result is easily checked

Lemma 5.10. *Denote by $k : \mathbb{V}^* \to \mathbb{P}^*$. Let $\mathcal{M} \in D^b_{hol}(\mathcal{D}_{\mathbb{V}_\infty})$ and $F \in \tilde{E}^b_{\mathbb{R}-c}(\boldsymbol{k}_{\mathbb{V}_\infty})$ satisfy*

$$\mathcal{S}ol^E_{\mathbb{P}} \simeq \boldsymbol{k}^E_{\mathbb{P}} \overset{+}{\otimes} QR\tilde{j}_! F. \tag{5.17}$$

Then, there is an isomorphism

$$\mathcal{S}ol^E_{\mathbb{P}^*}(\mathcal{M}^\wedge) \simeq \boldsymbol{k}^E_{\mathbb{P}^*} \overset{+}{\otimes} QR\tilde{k}_! F^\curlywedge[dim \mathbb{V}]. \tag{5.18}$$

Note that, in view of Lemma, for any $\mathcal{M} \in D^b_{hol}(\mathcal{D}_{\mathbb{V}_\infty})$ there is an $F \in \tilde{E}^b_{\mathbf{k}_{\mathbb{V}_\infty}}$ satisfying (5.17).

6. Morse-Witten theory

By the definition of Fourier-Sato transform introduced in the last section, we need to compute direct image with proper support of exponential sheaves.

In particular, in order to proceed with our computations, we will need a way to get information about the cohomology with compact support of subsets of $\{\mathrm{Re}\,\varphi \leq t_0\}$, where φ is a holomorphic function defined in some 1-dimensional domain.

The right framework is provided by Morse-Witten theory.

Since we will be dealing with meromorphic functions, the analysis provided from such theory should not be needed.

However, in order to provide the right geometric framework where the computation are made and from where the ideas came, we want to give a sketch of the fundamental principles of such theory.

6.1. Morse-Witten theory in the compact case.

The explanation below is based on [21]. Let X be a smooth compact real manifold of dimension n, $f : X \to \mathbb{R}$ a smooth function.

Let $p \in X$, denote by TX_p the tangent space of X at p. We say that p is a critical point for f if the morphism induced at the level of tangent spaces $Tf_p : TX_p \to T\mathbb{R}_{f(p)}$ is zero. If $x = (x_1, ..., x_n)$ is a local coordinate system given by a chart centered at p, this means that

$$\frac{\partial f}{\partial x_i}(0) = 0, \forall i = 1, ..., n. \tag{6.1}$$

A critical point is called non degenerate if and only if the Hessian matrix

$$(\frac{\partial^2 f}{\partial x_i \partial x_j}(0))_{i,j=1,\dots,n} \tag{6.2}$$

is non-singular (i.e. its determinant in non-zero). We call f a Morse function if all its critical points are non-degenerate. In this case, the number of negative eigenvalues of the Hessian of f at p is called the Morse index of p.

In general, any Morse function f can be used to compute upper bounds on the rank of real homology (or cohomology) groups of X in terms of its critical points. More in detail, the rank of the q-dimensional homology of X (also known as the q-Betti number) is at most the number of critical points of f of index q. The inequalities arising are called Morse inequalities.

If the differences between the indexes of distinct critical points are never equal to ± 1, then f is called perfect Morse function and the inequality above becomes an equality. In fact, in this case even the integral homology can be described via the critical points.

The classical approach (see [14] for an exposition) is to show that X is a CW complex with a q-cell for each critical point of index q. This is achieved by studying the change in the homology of the sublevel sets of f when crossing critical values of f.

Another approach, more categorical, is given by the introduction of the Morse-Witten complex and the homology associated with it.

In [22], Witten described a geometric realization for the boundary operators of such complex, showing that more information about the homology of the manifold X under exam could be extracted from peculiar paths associated to f and passing by each critical point of f.

In particular, he gave a recipe for constructing a k-cycle $\mathcal{J}_p$ attached to each critical point p with index k, and representing an element of the k-th homology group: these cycles generate the homology with integer coefficients.

Let us briefly recall how the construction of $\mathcal{J}_p$ is made.

In general, on any manifold X with real coordinates x, pick a Riemannian metric g on it and consider the downward flow or gradient flow equation

$$\frac{dx(t)}{dt} = -g\nabla f(x(t)) \tag{6.3}$$

The reason why this equation is called downward flow is given in the following lemma.

Lemma 6.1. *Let f and g as above. Then, except for stationary solutions which sits at a critical point for all t, f is always non-increasing along a flow for* (6.3)

Proof. It suffices to compute

$$\frac{df(x(t))}{dt} = -(\nabla f(x(t)))^T \frac{dx(t)}{dt} = -||\nabla f(x(t))||_g^2 < 0 \tag{6.4}$$

□

An important property of the flow equation is that if $x(t)$ equals a critical point at some t, then the flow equation implies that $x(t)$ is constant for all t. So a nonconstant flow can only reach a critical point at $t = \pm\infty$.

Let p be a nondegenerate critical point of f, and consider the downward flow equations on the half-line $]-\infty, 0]$ with the boundary condition $\lim_{t\to-\infty} x(t) = p$.

If p has index k, the moduli space of such solutions is a k-dimensional manifold, since there are k independent directions of downward flow from the critical point p.

We can think of such moduli space as a submanifold of X, by mapping a downward flow line $x(t)$ to the corresponding point $x(0) \in X$.

This gives an embedding in X since, as the downward flow equation is first order in time, a flow is uniquely determined by its value at $t = 0$.

We can also define it as the submanifold of X consisting of points that can be reached at any t by a flow that starts at p at $t = -\infty$.

Given a flow line $x(t)$ that reaches a point $x_* \in X$ at $t = t_*$, the flow line $x(t-t_*)$ arrives at x_* at $t = 0$. A flow line defined on the full line $]-\infty, +\infty[$ will be called a complete flow line.

A similar discussion can be made by considering the gradient equation with limit condition $\lim_{t\to+\infty} x(t) = p$: we will denote by $\mathcal{I}_p$ the submanifold arising from the same construction above.

6.2. **Morse-Witten theory in the non-compact case.** If X is a compact manifold, f is automatically bounded above and below, and the critical points of a perfect Morse function determine the ordinary homology of X.

If X is not compact, we can possibly deal with a Morse function f that is unbounded above and below. According to [21], in such a case, the critical points of f will no longer determine ordinary homology groups, but relative homology groups.

They will be the homology groups $H_k(X, X_{-T})$, where $X_{-T} := \{x \in X : f(x) \leq -T\}$ and T is a large constant.

The construction of $\mathcal{J}_p$ is the same as above. It is noticeable that, since X is no more compact, the gradient flow lines are no more necessarily defined for all t. Moreover, since f is no more bounded and non-increasing along flow lines, flow lines can possibly approach regions where $f \to -\infty$ and have endpoints at poles of f.

Once we pick an orientation of $\mathcal{J}_p$, $\mathcal{J}_p$ will define a cycle in the relative homology $H_k(X, X_{-T})$ if it is closed, i.e., any sequence of points in $\mathcal{J}_p$ has a subsequence on which f either converges or tends to $-\infty$. This fails precisely if there is a complete flow line l that starts at p at $t = -\infty$ and ends at another critical point q at $t = +\infty$. In that case, $\mathcal{J}_p$ is not closed, since l is contained in $\mathcal{J}_p$, but a sequence of points in l can converge to q, which is not contained in $\mathcal{J}_p$.

It is in general not easy to describe when such degeneration occurs.

In our case, it will be easy to determine a necessary condition for a flow line to connect two distinct critical points, we will moreover refine such condition to a sufficient one.

6.3. **Morse-Witten theory in the complex analytic setting.** We want to apply the theory explained in the preceding subsections to the real part of a meromorphic function φ on the projective line $\mathbb{P}^1_v$ with effective poles at S with $\infty \in S$. Set $U = \mathbb{P}^1_v \setminus S \subset \mathbb{C}_v$.

Notice that, with this set-up, φ is proper and has a finite number of critical points $\{v_m\} \subset U$, we will require that they are non-degenerate. We further require that φ has distinct critical values $\{\lambda_m := \varphi(v_m)\} =: Crit(\varphi)$, i.e. $\lambda_m \neq \lambda_{m'}$ $\forall v_m \neq v_{m'}$

By considering the real coordinates given by $(v, \bar{v})$ and the Kahler metric $ds^2 = |dv|^2$, the downward flow (or gradient flow) equation for $\operatorname{Re}\varphi$ rewrites as

$$\frac{dv}{dt} = -\partial_{\bar{v}}\bar{\varphi}, \frac{d\bar{v}}{dt} = -\partial_v \varphi \tag{6.5}$$

The downward flow lines have another property in this case.

Lemma 6.2. $\operatorname{Im}\varphi$ *is preserved along the flow lines of the downward gradient equation.*

Proof. Indeed

$$\frac{d\operatorname{Im}\varphi(v(t))}{dt} = \frac{1}{2i}\frac{d(\varphi-\bar{\varphi})(v(t))}{dt} = -\frac{1}{2i}(\partial_v\varphi\frac{dv(t)}{dt} - \partial_{\bar{v}}\bar{\varphi}\frac{d\bar{v}(t)}{dt}) = 0$$

□

We will denote the closure of the cycles $\mathcal{J}_{v_m}$ arising from the construction described above (operation which amounts to add the critical point v_m) by Γ_m^{φ} and call it the steepest descent path associated to v_m.

By Lemma 6.2, they are contained in the critical level set $\{\operatorname{Im}\varphi = \operatorname{Im}\lambda_m\}$.

It is noticeable that $\operatorname{Re}\varphi$ restricted to such paths has a maximum at v_m.

The non-degeneracy condition and the fact that $\operatorname{Re}\varphi$ is harmonic implies that v_m are saddles for $\operatorname{Re}\varphi$.

Remark 6.3. *By considering the cycles $\mathcal{I}_{v_m}$, one obtain another family of paths, called steepest ascent paths.*

Contrary to the steepest descent case, φ restricted to such subsets has a minimum in v_m.

Since the downward gradient equation can be linearized near a critical point v_m by the Hessian associated with $\operatorname{Re}\varphi$, we have the following result

Lemma 6.4. *The submanifold $\mathcal{J}_{v_m}$ consist, in a small punctured neighbourhood of v_m, of two curves abutting from v_m with tangents given by eigenvectors of the Hessian of* $\operatorname{Re}\varphi$ *associated with the negative eigenvalue.*

The same statement holds by considering $\mathcal{I}_{v_m}$ and positive eigenvalues of the Hessian.

Proof. This is a particular case of a general result, the Stable Manifold Theorem (see [17, Section 2.7, pp. 107]). □

Remark 6.5. *By Lemma 6.2, this last result is analogous to the description of the critical level set* $\{\operatorname{Im}\varphi = \operatorname{Im}\lambda_m\}$ *in a neighboruhood of v_m given by Morse theory (see* [14]*).*

The following result shows the relation between the level set $\{\operatorname{Im}\varphi = \operatorname{Im}\bar{\lambda}\}$ with $\bar{\lambda} \in Crit(\varphi)$ and solutions to the downward gradient equation.

Lemma 6.6. *The connected components of* $\{\operatorname{Im}\varphi = \operatorname{Im}\bar{\lambda}\} \setminus \{v_m\}$ *are solutions to the downward gradient equation. More specifically, a connected component C satisfies only one of the following:*

(i) C is an analytic curve with both endpoints in S, in particular $\operatorname{Re}\varphi_{|C}$ *is a non-increasing diffeomorphism from C to $]-\infty,+\infty[$,*

(ii) at least one of the endpoints of C is a critical point $\bar{v}$ with $\varphi(\bar{v}) = \bar{\lambda}$ and C is one of the curves abutting from $\bar{v}$ described in Lemma 6.4.

Proof. By the implicit function theorem, we know that $\{\operatorname{Im}\varphi = \operatorname{Im}\bar{\lambda}\}\setminus\{v_m\}$ is the union of analytic curves. Moreover Morse lemma assures the presence of quadruples of them orthogonally intersecting at critical points contained in $\varphi^{-1}(\lambda_m)$.

Since $\operatorname{Im}\varphi$ is harmonic, $\{\operatorname{Im}\varphi = \operatorname{Im}\bar{\lambda}\}$ can not be closed in U, henceforth its boundary has to be in S. Hence the endpoints of the connected components are in $S \cup \{\bar{v} : \varphi(\bar{v}) = \bar{\lambda}\}$.

Suppose C has both endpoints in S: C is then the complete flow line for the downward gradient problem with initial condition given by one of its points.

This, together with the fact that endpoints of C are poles of φ, implies that $\operatorname{Re}\varphi_{|C}$ is a non-increasing diffeomorphism from C to $]-\infty, +\infty[$.

The statement concerning connected components with at least a critical point as endpoint follows from remark 6.5.

□

6.4. **Orientation.** As stated in Lemma 6.4, steepest descent paths are approximated by curves abutting from the critical point with tangents given by eigenvectors of the Hessian matrix of $\operatorname{Re}\varphi$ relative to the negative eigenvalue.

In order to give an orientation to Γ^{φ}_m, it suffices to choose one such eigenvector.

6.5. **A vanishing result.** Our first aim in this thesis is to provide a decomposition in sectors for the Fourier-Sato transform as sum of exponential factors.

The first step will be proving this fact at the level of stalk. In the next result, we will show how steepest descent paths are the suited objects for the purpose.

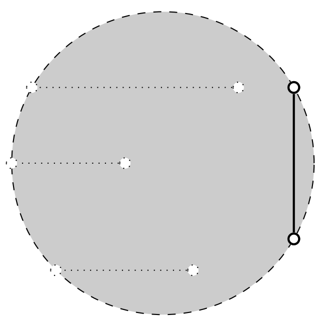
Proposition 6.7. *Let φ, v_m and Γ^{φ}_m as in the previous subsection. Then, $\forall t_0 \in \mathbb{R}$, we have*

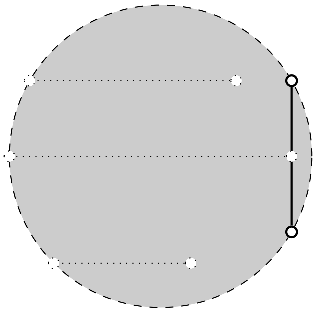
$$R\Gamma_c(\mathbb{C}_v, \boldsymbol{k}_{\{\operatorname{Re}\varphi \le t_0\} \setminus \bigcup_{m:\operatorname{Re}\lambda_m \le t_0} \Gamma^{\varphi}_m}) = 0 \tag{6.6}$$

Proof. Let τ be the affine coordinate on the target space of φ and consider, for all $t_0 \in \mathbb{R}$, the set

$$N^{\varphi}_{t_0} := \{\operatorname{Re}\tau \le t_0\} \setminus \bigcup_{m=1}^{d} (\lambda_m + \mathbb{R}_{\le 0}) \subset \mathbb{C}_\tau.$$

Notice that, since $N^{\varphi}_{t_0}$ is a closed half plane with semiclosed intervals removed, $R\Gamma_c(\mathbb{C}_\tau, N^{\varphi}_{t_0}) = 0$.





We will denote by $\bar{\varphi}_{t_0}$ the restriction

$$\bar{\varphi}_{t_0} : \varphi^{-1}(N^{\varphi}_{t_0}) \to N^{\varphi}_{t_0}$$

Let us prove the following

Lemma 6.8. *$R\Gamma_c(\mathbb{C}_v, \varphi^{-1}(N^{\varphi}_{t_0})) = 0$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\varphi^{-1}(N^{\varphi}_{t_0}) & \xrightarrow{j_u} & U & \xrightarrow{a_u} & \{pt\} \\
\downarrow{\scriptstyle \bar{\varphi}_{t_0}} & & \downarrow{\scriptstyle \varphi} & \nearrow{\scriptstyle a_\tau} & \\
N^{\varphi}_{t_0} & \xrightarrow{j_\tau} & \mathbb{C}_\tau & &
\end{array}$$

Notice that, since φ is a finite ramified covering of the projective line to itself, $\bar{\varphi}_{t_0}$ is a finite covering, in particular it satisfies

$$R\bar{\varphi}_{t_0 !}\bar{\varphi}^{-1}_{t_0} G \simeq \bigoplus_{i=1}^{d} G$$

for every $G \in D^b(\mathbf{k}_{\mathbb{C}_\tau})$.

Then

$$\begin{aligned}
R\Gamma_c(\mathbb{C}_v, \varphi^{-1}(N^{\varphi}_{t_0})) \simeq & Ra_{u!}Rj_{u!}j_u^{-1}a_u^{-1}\mathbf{k} \simeq \\
\simeq Ra_{\tau!}R\varphi_! Rj_{u!}j_u^{-1}\varphi^{-1}a_u^{-1}\mathbf{k} \simeq & Ra_{\tau!}R(\varphi \circ j_u)_!(\varphi \circ j_u)^{-1}a_\tau^{-1} \simeq \\
\simeq Ra_{\tau!}R(j_\tau \circ \bar{\varphi}_{t_0})_!(j_\tau \circ \bar{\varphi}_{t_0})^{-1}a_\tau^{-1} \simeq & Ra_{\tau!}Rj_{\tau!}R\bar{\varphi}_{t_0!}\bar{\varphi}_{t_0}^{-1}j_\tau^{-1}a_\tau^{-1} \simeq \\
\simeq \bigoplus_{i=1}^{d} Ra_{\tau!}Rj_{\tau!}j_\tau^{-1}a_\tau^{-1}\mathbf{k} \simeq & \bigoplus_{i=1}^{d} R\Gamma_c(\mathbb{C}_\tau, N^{\varphi}_{t_0}) = 0.
\end{aligned} \tag{6.7}$$

□

We have

$$\varphi^{-1}(N^{\varphi}_{t_0}) = \{\operatorname{Re}\varphi \leq t_0\} \setminus \bigcup_m (\{\operatorname{Im}\varphi = \operatorname{Im}\lambda_m\} \cap \{\operatorname{Re}\varphi \leq \operatorname{Re}\lambda_m\}) \tag{6.8}$$

Since $\Gamma^{\varphi}_m \subset \{\operatorname{Im}\varphi = \operatorname{Im}\lambda_m\} \cap \{\operatorname{Re}\varphi \leq \operatorname{Re}\lambda_m\}$, we obtain

$$\varphi^{-1}(N^{\varphi}_{t_0}) \subset \{\operatorname{Re}\varphi \leq t_0\} \setminus \bigcup_m \Gamma^{\varphi}_m$$

In order to complete the proof, it is then sufficient to show that these two sets have the same cohomology with compact support or, equivalently, that

$$(\{\operatorname{Re}\varphi \leq t_0\} \setminus \bigcup_m \Gamma_m^{\varphi}) \setminus \varphi^{-1}(N_{t_0}^{\varphi})$$

is cohomologically trivial.

This last set can be rewritten as

$$\{\operatorname{Re}\varphi \leq t_0\} \cap \bigcup_m [(\{\operatorname{Im}\varphi = \operatorname{Im}\lambda_m\} \setminus \Gamma_m^{\varphi}) \cap \{\operatorname{Re}\varphi \leq \operatorname{Re}\lambda_m\}]$$

which is union of diffeomorphic images of semi-closed intervals.

Indeed, notice that

$$(\{\operatorname{Im}\varphi = \operatorname{Im}\lambda_m\} \setminus \Gamma_m^{\varphi}) \cap \{\operatorname{Re}\varphi \leq \operatorname{Re}\lambda_m\} \tag{6.9}$$

is obtained by removing from the critical level set $\{\operatorname{Im}\varphi = \operatorname{Im}\lambda_m\}$ all solutions to the downward gradient equation having a critical point as endpoint.

By Lemma 6.6, the above set then only consists of complete flow lines γ connecting two points of $S = \partial U$.

Again by Lemma 6.6, $\operatorname{Re}\varphi_{|_\gamma}$ is a diffeomorphism, therefore, $\forall t_0 \in \mathbb{R}$, we can find a point v_γ on each γ such that $\operatorname{Re}\varphi$ has value t_0.

This means that (6.9) is made of the semi-closed part of the γ's going from v_γ to the endpoint of γ where $\operatorname{Re}\varphi_{|_\gamma} \to -\infty$.

□

7. Borel-Moore homology

While Morse-Witten theory will provide the tools in order to find trivialization for the Fourier transform in sectors, Borel-Moore homology will provide the right framework in which we will compute the Stokes multipliers.

It is a classical tool for extending the singular homology theory to the case of general locally compact spaces.

There are several way to define the integral Borel-Moore homology

$$H_*^{BM}(X, \mathbb{Z})$$

for a locally compact space X:

- by introducing locally finite chains and taking the homology of the complex obtained
- as the relative homology $H_*(Y, Y \setminus X)$ with respect to a compactification Y of X.
- as

$$H_j^{BM}(X, \mathbb{Z}) = H^{-j} R\Gamma(X, \omega_X) \tag{7.1}$$

  where ω_X is the dualizing sheaf $a_X^!\mathbb{Z}$ with $a_X : X \to \{pt\}$.

All definitions can be extended to define the Borel-Moore homology

$$H_*^{BM}(X, L)$$

with coefficients in a local system L.

For a definition that suits our work, we will refer to the nice description made in Appendix A in [6].

7.1. **Borel-Moore homology for subanalytic spaces.** Let X be a subanalytic space, i.e., an $\mathbb{R}$-ringed space locally modeled on closed subanalytic subsets of real analytic manifolds.

In [10, 9.2], the sheaf $\mathcal{CS}_p^X$ is introduced and defined as the sheaf associated to the following presheaf.

For $V \subset X$ open subset, define $CS_p^X(V)$ as the **k**-vector space spanned by the symbols $[S]$, where S ranges through the family of subanalytic p dimensional oriented submanifolds of X, with relations

1. $[S_1 \cup S_2] = S_1 + S_2$ if $S_1 \cap S_2 = \emptyset$,
2. $[S] = [S']$ if $S \subset S'$ is open, subanalytic and endowed with the induced orientation,
3. $[S^a] = -[S]$, where S^a is the submanifold S endowed with the opposite orientation.

There is a boundary map $\partial : \mathcal{CS}_p^X \to \mathcal{CS}_{p-1}^X$ explicitly constructed in [10] and inducing a complex of sheaves.

This map is the same as the one defined in the usual singular homology, i.e., $\partial[S] = [\partial S]$ in the case of interest to us: the oriented p-dimensional submanifold S and the embedding $S \subset \bar{S}$ are locally modeled on $\{x_1 > 0\}$ and $\{x_1 > 0\} \subset \{x_1 \geq 0\}$ for $\mathbb{R}^p \ni (x_1, ..., x_p)$.

For $G \in Mod_{\mathbb{R}-c}(\mathbf{k}_X)$, we define the space of subanalytic Borel-Moore p-chains relative to G as the subspace

$$BM_p^X(G) \subset Hom(G, \mathcal{CS}_p^X) \tag{7.2}$$

of morphisms $\phi \in Hom(G, \mathcal{CS}_p^X)$ such that for any relatively compact subset U of X, and $s \in G(U)$, there exists $\sigma \in \mathcal{C}_p^X(X)$ with $\sigma_{|U} = \phi(s)$. This requirement is equivalent to ask supp $\phi(s)$, which is closed subanalytic in U, is subanalytic in X.

The boundary map ∂ induces a complex $BM_\bullet^X(G)$.

Using language and result from the theory of subanalytic sheaves, it is possible to show the following

Lemma 7.1. *There is a functorial isomorphism*

$$BM_\bullet^X(G) \simeq RHom(G, \omega_X) \simeq DR\Gamma_c(X, G) \tag{7.3}$$

where ω_X is the dualizing sheaf and $D = RHom(\cdot, \boldsymbol{k})$ is the dual in $D^b(\boldsymbol{k})$.

For the proof, see [6].

In particular, if we apply the lemma to the case of L local system of finite rank and $Z \subset X$ locally closed subanalytic, we obtain

$$\begin{aligned} BM_\bullet^X(\mathbf{k}_Z \otimes L) \simeq RHom(\mathbf{k}_Z \otimes L\omega_X) &\simeq RHom(L, R\Gamma_Z\omega_X) \\ &\simeq R\Gamma(X, R\Gamma_Z\omega_X \times L *^{)} . \end{aligned} \tag{7.4}$$

It is then natural to define

Definition 7.2. *Let L be a local system of finite rank, and $Z \subset X$. For $j \in \mathbb{Z}$, the j-th Borel-Moore homology of Z, with coefficients in L^* is given*

$$H_j^{BM}(Z, L^*) := H_j BM_\bullet^X(\boldsymbol{k}_Z \otimes L). \tag{7.5}$$

We immediately obtain from Lemma 7.1

$$H_j^{BM}(Z, L^*) := (H_c^j(X, \mathbf{k}_Z \otimes L))^*. \tag{7.6}$$

An element in $H_j^{BM}(Z, L^*)$ will be represented as

$$\sigma \otimes s \tag{7.7}$$

with σ and s respecting the condition as in the definition of $BM_\bullet^X(G)$ given above, for $G = \mathbf{k}_Z \otimes L$.

8. The Fourier transform of an elementary module

8.1. **The Fourier transform of elementary $\mathcal{D}$-module.** Let $\rho : \mathbb{C}_u \to \mathbb{V}$ be a ramification of order p of the variable z, i.e. $\rho(u) = u^p$,$\mathcal{R} \in D^b_{reg-hol}(k_{\mathbb{C}_u})$ and $\varphi \in u^{-1}\mathbb{C}[u^{-1}]$.
Let us give the following definition:

Definition 8.1. *Define the elementary $\mathcal{D}$-module $El(\rho, \varphi, R)$ as*

$$El(\rho, \varphi, R) = D\rho_*(\mathcal{E}^\varphi \overset{D}{\otimes} \mathcal{R}). \tag{8.1}$$

We will from now on focus on $\varphi(u) = -\alpha u^{-n}$ with $\alpha \in \mathbb{C}^\times$. We will deal with $\varphi \in u^{-1}\mathbb{C}[u^{-1}]$ in section 13 and show that the results obtained for the simple case extends to the general case.

Since $\mathcal{E}^\varphi(*0) \simeq E^\varphi$, we can suppose $R \simeq R(*0)$. This, in turn, implies that $F \simeq j_! L$ for some local system L on $\mathbb{C}_u^\times$ with $j : \mathbb{C}_u^\times \to \mathbb{C}_u$ open inclusion.

As recalled in section 1.2, the stationary phase formula provides then the formal type of the Fourier transform of this $\mathcal{D}$-module, in particular we know that it is ramified with ramification given by

$$w^{-1} = \hat{\rho}(\zeta) := -\frac{\rho'(\zeta)}{\varphi'(\zeta)} = \frac{p}{n\alpha}\zeta^{n+p}$$

In order for the exponential factors and the ramification to be defined in the same space $\mathbb{C}_\zeta^\times$, we will consider

$$\mathcal{M} = D\rho_2^*(D\rho_*(\mathcal{E}^\varphi \overset{D}{\otimes} \mathcal{R}))^\wedge$$

with $\rho_2(\zeta) := \frac{1}{\hat{\rho}(\zeta)} = \frac{n\alpha}{p}\zeta^{-(n+p)}$.

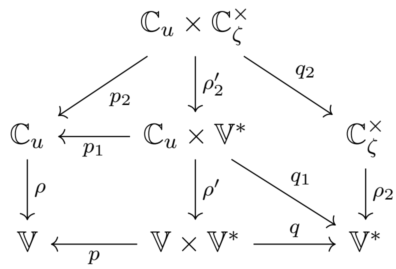
Our aim is to study $\mathcal{M}$ and the Stokes phenomenon underlying it. We will achieve this in the framework provided by the theory of enhanced sheaves.

As recalled in section 5, the Fourier trasform for $\mathcal{D}$-modules is interchanged with the enhanced Fourier-Sato transform for enhanced ind-sheaves.

Since the enhanced solution functor also commutes with the inverse image,

$$\mathcal{K} := E\rho_2^{-1}(E\rho_{!!}(\mathbb{E}^\varphi \overset{+}{\otimes} e(F)))^\wedge = E\rho_2^{-1}Eq_{!!}(\mathbb{E}^{-zw}[1] \overset{+}{\otimes} E\rho_{!!}(\mathbb{E}^\varphi \overset{+}{\otimes} e(F)))$$

Consider the diagram with Cartesian squares



where p_1, p_2, q_1, q_2 are standard projections, $\rho' = \rho \times id_{\mathbb{V}^*}$, $\rho'_2 = id_{\mathbb{C}_u} \times \rho_2$. The enhanced ind-sheaf $\mathcal{K}$ can hence be computed as:

$$
\begin{aligned}
& E\rho_2^{-1}(E\rho_!(\mathbb{E}^{-\alpha u^{-n}} \overset{+}{\otimes} e(F)))^{\curlywedge} = \\
&= E\rho_2^{-1}Eq_{!!}(\mathbb{E}^{-zw} \overset{+}{\otimes} Ep^{-1}E\rho_{!!}(\mathbb{E}^{-\alpha u^{-n}} \overset{+}{\otimes} e(F)))[1] \\
&\simeq E\rho_2^{-1}Eq_{!!}(\mathbb{E}^{-zw} \overset{+}{\otimes} E\rho'_{!!}Ep_1^{-1}(\mathbb{E}^{-\alpha u^{-n}} \overset{+}{\otimes} e(F)))[1] \simeq \\
&\simeq E\rho_2^{-1}Eq_{!!}\rho'_{!!}[E\rho'^{-1}\mathbb{E}^{-zw} \overset{+}{\otimes} Ep_1^{-1}(\mathbb{E}^{-\alpha u^{-n}} \overset{+}{\otimes} e(F))][1] \\
&\simeq E\rho_2^{-1}Eq_{1!!}[\mathbb{E}^{-u^p w} \overset{+}{\otimes} Ep_1^{-1}(\mathbb{E}^{-\alpha u^{-n}} \overset{+}{\otimes} e(F))][1] \\
&\simeq Eq_{2!!}E\rho_2'^{-1}[\mathbb{E}^{-u^p w} \overset{+}{\otimes} Ep_1^{-1}(\mathbb{E}^{-\alpha u^{-n}} \overset{+}{\otimes} e(F))][1] \\
&\simeq Eq_{2!!}(\mathbb{E}^{-\frac{n\alpha}{p}\frac{u^p}{\zeta^{n+p}}} \overset{+}{\otimes} Ep_2^{-1}\mathbb{E}^{-\alpha u^{-n}} \overset{+}{\otimes} Ep_2^{-1}e(F))[1] \\
&\simeq Eq_{2!!}(\mathbb{E}^{-\alpha(\frac{n}{p}\frac{u^p}{\zeta^{n+p}}+u^{-n})} \overset{+}{\otimes} Ep_2^{-1}e(F))[1]
\end{aligned}
\tag{8.2}
$$

Our study will then focus on the enhanced sheaf (recall the functor ϵ defined in (3.5))

$$
K := R\widetilde{q}_{2!}(E^{-\frac{n\alpha}{p}\frac{u^p}{\zeta^{n+p}}-\alpha u^{-n}} \overset{+}{\otimes} \widetilde{p}_2^{-1}\epsilon(F))[1] \tag{8.3}
$$

which satisfy $\mathcal{K} \simeq \mathbf{k}^E_{\mathbb{C}^\times_\zeta} \overset{+}{\otimes} K$.

Recalling Lemma 3.2, we also have

$$
\begin{aligned}
K &= R\widetilde{q}_{2!}(E^{-\frac{n\alpha}{p}\frac{u^p}{\zeta^{n+p}}-\alpha u^{-n}} \overset{+}{\otimes} \widetilde{p}_2^{-1}(\mathbf{k}_{\{t\geq 0\}} \otimes \pi^{-1}F))[1] \simeq \\
&\simeq R\widetilde{q}_{2!}(E^{-\frac{n\alpha}{p}\frac{u^p}{\zeta^{n+p}}-\alpha u^{-n}} \overset{+}{\otimes} (\mathbf{k}_{\{t\geq 0\}} \otimes \widetilde{p}_2^{-1}\pi^{-1}F))[1] \simeq \\
&\simeq R\widetilde{q}_{2!}((E^{-\frac{n\alpha}{p}\frac{u^p}{\zeta^{n+p}}-\alpha u^{-n}} \overset{+}{\otimes} \mathbf{k}_{\{t\geq 0\}}) \otimes \widetilde{p}_2^{-1}\pi^{-1}F)[1] \simeq \\
&\simeq R\widetilde{q}_{2!}(E^{-\frac{n\alpha}{p}\frac{u^p}{\zeta^{n+p}}-\alpha u^{-n}} \otimes \widetilde{p}_2^{-1}\pi^{-1}F)[1]
\end{aligned}
$$

For computational purposes, we will make a further addition.

Consider the blow-up $\Phi : \mathbb{C}_v \times \mathbb{C}^\times_\zeta \to \mathbb{C}_u \times \mathbb{C}^\times_\zeta$ defined by

$$
\Phi(v, \zeta) = (v\zeta, \zeta)
$$

Since $R\tilde{\Phi}_!\tilde{\Phi}^{-1}H \simeq H$, by considering its insertion in 8.2, we get

$$\begin{aligned} K \simeq & R\tilde{q_2}_! R\tilde{\Phi}_!\tilde{\Phi}^{-1}(E^{-\alpha(\frac{n}{p}\frac{u^p}{\zeta^{n+p}}+u^{-n})} \otimes \tilde{p_2}^{-1}\pi^{-1}F)[1] \simeq \\ \simeq & R\tilde{q_3}_!(E^{-\Phi^*(\alpha(\frac{n}{p}\frac{u^p}{\zeta^{n+p}}+u^{-n}))} \otimes \tilde{p_3}^{-1}\pi^{-1}F)[1] \simeq \\ \simeq & R\tilde{q_3}_!(E^{-\tilde{\Psi}} \otimes \tilde{p_3}^{-1}\pi^{-1}F)[1] \end{aligned}$$

where $q_2 = q_1 \circ \Phi$, $p_3 = p_2 \circ \Phi$ and

$$\begin{aligned} \tilde{\Psi}(v,\zeta) := & \Phi^*(\alpha(\frac{n}{p}\frac{u^p}{\zeta^{n+p}} + u^{-n})) = \alpha(\frac{n}{p}\frac{(v\zeta)^p}{\zeta^{n+p}} + (v\zeta)^{-n}) = \\ = & \alpha\zeta^{-n}(v^{-n} + \frac{n}{p}v^p) = n\alpha\zeta^{-n}(\frac{v^{-n}}{n} + \frac{v^p}{p}) \end{aligned} \tag{8.4}$$

We will show that $\tilde{\Psi}$ contains all needed information about $\mathcal{M}$, from its exponential factors to the Stokes multipliers.

Remark 8.2. *Notice that we can already obtain the exponential factors, via the stationary phase formula. Since*

$$\begin{aligned} \hat{\varphi}(\zeta) = & \varphi(\zeta) - \frac{\rho(\zeta)}{\rho'(\zeta)}\varphi'(\zeta) = -\alpha u^{-n} - \frac{\zeta^p}{p\zeta^{p-1}}(n\alpha u^{-n-1}) = \\ = & -\alpha(1+\frac{n}{p})\zeta^{-n}, \end{aligned}$$

the exponential factors for $\mathcal{M}$ are

$$\hat{\varphi}_m(\zeta) = \hat{\varphi}(\zeta e^{i\frac{2\pi m}{n+p}}) = -\alpha(1+\frac{n}{p})\zeta^{-n}e^{-i(n\frac{2m\pi}{n+p})} \quad m = 0, ..., n+p-1$$

Now, since $e^{-in\theta_m} = e^{ip\theta_m}$

$$\tilde{\Psi}_m(\zeta) := \tilde{\Psi}(e^{i\theta_m},\zeta) = \alpha\zeta^{-n}(e^{-in\theta_m} + \frac{n}{p}e^{ip\theta_m}) = \alpha(1+\frac{n}{p})\zeta^{-n}e^{-i(n\frac{2m\pi}{n+p})} \tag{8.5}$$

and hence the exponential factors are encoded in $\tilde{\Psi}$.

This is nothing new: the fact that $\tilde{\Psi}$ can recover the exponential factors in this way comes from the classical well known studies about Legendre transform. For further reference, one can see [5].

9. Morse-theoretic properties for level sets of $\tilde{\Psi}$

In this section, we want to focus on

$$K_{(\zeta_0,t_0)} = R\Gamma_c(\mathbb{C}_v, k_{\{\mathrm{Re}\,\tilde{\Psi}_{\zeta_0}\le t_0\}} \otimes \tilde{\Phi}_{\zeta_0}^{-1}\pi^{-1}F)[1]$$

where $\tilde{\Psi}_{\zeta_0}(v) = \tilde{\Psi}(v,\zeta_0)$ and $\Phi_{\zeta_0}(v) = \Phi(v,\zeta_0)$.

Thanks to the result in section 6, we already know that all information about this cohomology is contained in the family of steepest descent paths for $\mathrm{Re}\,\tilde{\Psi}_{\zeta_0}$. We also know that such paths are contained in the critical level sets for $\mathrm{Im}\,\tilde{\Psi}_{\zeta_0}$.

Our aim in this section is to investigate precisely the level sets of $\mathrm{Im}\,\tilde{\Psi}_{\zeta_0}$, more specifically we will prove the following

Proposition 9.1. *There exists a covering of $\mathbb{C}_v^\times$ made of open sectors with vertices in 0 and local radial parametrizations of the branches of $\{\operatorname{Im}\tilde{\Psi}_{\zeta_0} = t_0\}$ in these sectors such that*

(i) if the sector contains a critical point of $\operatorname{Im}\tilde{\Psi}_{\zeta_0}$, there are two branches extending the local quadratic behaviour of the level set around that critical point,

(ii) if the sector does not contain a critical point, the level set in this sector consists of a single curve.

Proof. See Proposition 9.6, Proposition 9.7, Proposition 9.8, Proposition 9.9. □

In particular ,the local radial parametrizations provided and their properties will allow us to study the change on the behaviour of steepest descent paths with respect to ζ_0 (see section 10).

9.1. **Framework and reductions.** Let (ϵ, Θ) be the polar coordinates for ζ, i.e. $\zeta = \epsilon e^{i\Theta}$, write $\beta e^{i\mu}$ for the coefficient α.

Denote by $\tilde{\Psi}_{\zeta_0}$ the fiber of $\tilde{\Psi}$ at $\zeta_0 \in \mathbb{C}_\zeta^\times$, i.e.,

$$\tilde{\Psi}_{\zeta_0}(v) := \tilde{\Psi}(v, \zeta_0) = n\beta\epsilon_0^{-n} e^{\mu - n\Theta_0}\left(\frac{v^{-n}}{n} + \frac{v^p}{p}\right). \tag{9.1}$$

We would like to apply Morse-Witten theory as explained in section 6.3 to $\tilde{\Psi}_{\zeta_0}$.

It is easy to notice that critical level sets of $\operatorname{Im}\tilde{\Psi}_{\zeta_0}$ do not depend on positive constant, hence on β, n and ϵ_0. We then define

$$\Psi(v, \zeta) = e^{i(\mu - n\Theta)}\left(\frac{v^{-n}}{n} + \frac{v^p}{p}\right).$$

Introduce the variable $\kappa \in \mathbb{R}$ and set $\nu(\Theta) = \mu - n\Theta =: \kappa$.

$$\Psi_\kappa(v) = e^{i\kappa}\left(\frac{v^{-n}}{n} + \frac{v^p}{p}\right).$$

The following result summarize the main property of Ψ_κ.

Lemma 9.2. $\forall \zeta_0 \in \mathbb{C}_\zeta^\times$, *the function Ψ_{κ_0} with $\kappa_0 = \nu(\Theta_0)$ has non-degenerate critical points $\{v_m = e^{i\frac{2\pi m}{n+p}} : m = 0, ..., n+p-1\}$ with distinct critical values $\{\lambda_m = (\frac{1}{n} + \frac{1}{p})e^{ip\frac{2\pi m}{n+p}}\}$.*

Proof. The first two complex derivatives of Ψ_{κ_0} are

$$\partial_v \Psi_{\kappa_0}(v) = e^{i\kappa_0}(-v^{-n-1} + v^{p-1})$$
$$\partial_v^2 \Psi_{\kappa_0}(v) = e^{i\kappa_0}((n+1)v^{-n-1} + (p-1)v^{p-1})$$

The critical points of the functions are then solution of $v^{n+p} = 1$, i.e., the $n+p$-th roots of unity $e^{i\frac{2\pi m}{n+p}}$ for $m = 0, ..., n+p-1$.

The critical points are non-degenerate: indeed, since $e^{-in\theta_m} = e^{ip\theta_m}$, we have

$$\partial_v^2 \Psi_{\kappa_0}(v_m) = e^{i\kappa_0}(n+p)e^{ip\theta_m} \neq 0.$$

The critical values of Ψ_{κ_0} are

$$\Psi_{\kappa_0}(e^{i\theta_m}) = (\frac{1}{n} + \frac{1}{p})e^{i(p\theta_m+\kappa_0)*}.$$

They are distinct, indeed

$$e^{ip\theta_m+\kappa_0} = e^{ip\theta_{m'}+\kappa_0} \Leftrightarrow p(\theta_m - \theta_{m'}) = 0 \pmod{2\pi} \Leftrightarrow$$
$$\Leftrightarrow p(m - m') = 0 \pmod{n+p}$$

and, since $(n,p) = 1 \Rightarrow (p, n+p) = 1$, the system above is equivalent to $m = m' \pmod{n+p}$. □

In the following, we will denote by Ξ the set of arguments of the $n+p$-root of unity v_m, i.e. $\Xi = \{\theta_m := \frac{2m}{n+p}\pi : m = 0, ..., n+p-1\}$.

This lemma implies that we can apply Morse-Witten theory to

$$g_{\kappa_0}(r,\theta) := \operatorname{Re}\Psi_{\kappa_0}(r,\theta) = \frac{r^{-n}}{n}\cos(n\theta - \kappa_0) + \frac{r^p}{p}\cos(p\theta + \kappa_0)$$

and construct the steepest descent paths associated to each critical point v_m: we will denote them by $\Gamma^{\Psi}_{m,\kappa_0}$.

9.2. **Level sets of harmonic functions.** First of all, some generalities about the level sets of the imaginary part of a holomorphic function $\varphi : U \to \mathbb{C}$.

It is known that $\operatorname{Im}\varphi$ is harmonic. This implies that its level sets can not be made of closed curve in the domain of definition of our function: if so, by the maximum principle, f would be constant.

This means then that all level curves have extremes at the boundary of U.

Moreover, suppose a is a pole of φ of order n. Then, the local structure theorem for meromorphic functions tells us that there are exactly $2n$ lines of the level set $\{\operatorname{Im}\varphi = t_0\}$ $\forall t_0 \in \mathbb{R}$, with a as an endpoint and crossing the singularity at equal angles.

Let us apply what said so far to $\operatorname{Im}\Psi_{\kappa_0}$ which, written in polar coordinates, is:

$$f_{\kappa_0}(r,\theta) = \operatorname{Im}\Psi_{\kappa_0}(r,\theta) = -\frac{r^{-n}}{n}\sin(n\theta - \kappa_0) + \frac{r^p}{p}\sin(p\theta_m + \kappa_0)$$

It is harmonic and well-defined in $]0, +\infty[\times S^1$ and there are exactly $2p$ lines of the level sets from ∞ and $2n$ lines from 0.

Moreover the angle at which each of these lines cross their pole can be read from the coefficients of $\frac{r^p}{n}$ and $\frac{r^{-n}}{n}$.

For example, for f_{κ_0}, they are the arguments on which $\sin(p\theta + \kappa_0) = 0$ at ∞ and the ones for which $\sin(n\theta + \kappa_0)$ vanishes at 0.

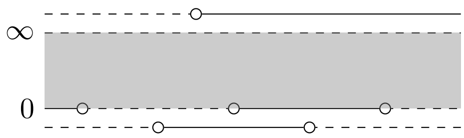
These facts allows us to say that the level sets for f_{κ_0} are made of $n+p$ disjoint analytic curves for each t_0 which is not a critical value.

Indeed, in this case, we have analytic curves thanks to the implicit function theorem and each curve must join two germs of curves at 0 or ∞, and two only: since we have $2(n+p)$ such germs, we have exactly $n+p$ curves arising.

Concerning critical level sets, we have no more regular disjoint curves, however Morse theory still assures us the presence of $n+p$ analytic curves, some of which cross orthogonally at critical points.

9.3. **A model.** In order to proceed into a more visual description, we need to introduce a model where to picture our level set.

It is similar to the model used by Mochizuki in [15] to introduce his rapid decay cycles, with some minor changes and additions. It can be thought of as the plane $\mathbb{C}_v$ of which we have taken the real blow-up at 0 and ∞ (the light gray part in the below figure).



In this model, the argument is considered to be growing when moving from the right to the left.

There are four lines depicted, each of them with its own meaning.

(1) the upper exterior line keeps track of the sign of $\sin(p\theta+\kappa_0)$
(2) the upper interior line keeps track of the sign of $\cos(p\theta+\kappa_0)$
(3) the lower exterior line keeps track of the sign of $\sin(n\theta-\kappa_0)$
(4) the lower interior line keeps track of the sign of $\cos(n\theta-\kappa_0)$

Dashed and straight segments characterize the sign of the function associated to the line (negative, positive). The dots represent zeroes of such functions.

The dots of a exterior segment is the midpoint of an interior segment on the same height: this follows from well known properties of cos and sin.

With upper and lower lines we aim to represent the signs of f_{κ_0} and g_{κ_0}: we recall that, near the poles $0,\infty$ they display the behaviour of the term with the highest pole order.

The two upper/lower lines can be hence thought as picturing the sign of the coefficients of the corresponding pole leading term on $S^1_{r=+\infty}/S^1_{r=0}$.

We will use the exterior lines to picture the level set of f_{κ_0} and the interior lines to keep track of the behaviour of g_{κ_0} along level sets of f_{κ_0}.

In particular, this will allow us to determine, between all lines of the critical level sets, which ones are steepest descent paths.

Indeed, they will be the analytic curves contained in the critical level set connecting dashed interior lines, which represent regions where $\Psi_{\kappa_0}\to-\infty$ or semi-closed curves connecting a dashed line and a critical point (in the case there degeneration).

9.4. **Subset of radial critical points.** In order to describe what happens in sectors, it is crucial to study some properties of the level set of the first partial derivative of f_κ with respect to r, denoted by C and called subset of radial critical points.

C is then defined by $\{\partial_r f_{\kappa_0}=+r^{-n-1}\sin(n\theta+\kappa_0)+r^{p-1}\sin(p\theta+\kappa_0)=0\}$: since $r\neq 0$ we can rewrite it as $r^{n+p}\sin(p\theta+\kappa_0)+\sin(n\theta-\kappa_0)=0$.

We can obtain a parametrization:

$$\theta \to (r_C(\theta), \theta) \tag{9.2}$$

where $r_C(\theta) := [-\frac{\sin(n\theta-\kappa_0)}{\sin(p\theta+\kappa_0)}]^{\frac{1}{n+p}}$

It describes a curve in each of the sectors composing the subset

$$\{\sin(p\theta + \kappa_0)\sin(n\theta - \kappa_0) < 0\}$$

domain of definition of the parametrization.

This parametrization completely describes all of C as far as we avoid κ_0 at which $\sin(p\theta + \kappa_0) = \sin(n\theta - \kappa_0) = 0$ for some θ.

Let us suppose this does not happen for the time being.

Then, it is easy to notice the following:

$$\lim_{\theta\to\bar{\theta}} r_C(\theta) = \begin{cases} 0 & \text{for } \sin\big(n\bar{\theta} - \kappa_0\big) = 0, \sin\big(p\bar{\theta} + \kappa_0\big) \neq 0 \\ +\infty & \text{for } \sin\big(p\bar{\theta} + \kappa_0\big) = 0, \sin\big(n\bar{\theta} - \kappa_0\big) \neq 0 \end{cases} \tag{9.3}$$

where the limit has to be intended on the domain of definition of the parametrization.

Moreover, the critical points of f_{κ_0} clearly belongs to C; not only, such points are also critical for $f_{\kappa_0|_C}$.

$$\begin{aligned} f_{\kappa_0|_C}(\theta) =& r_C(\theta)^{-n}(-\frac{1}{n}\sin(n\theta-\kappa_0) + \frac{r_C(\theta)^{n+p}}{p}\sin(p\theta_m+\kappa_0)) = \\ =& -(\frac{1}{n}+\frac{1}{p})r_C(\theta)^{-n}\sin(n\theta-\kappa_0) \\ \frac{d}{d\theta}r_C(\theta) =& r_C(\theta)\frac{n\cos(n\theta-\kappa_0)\sin(p\theta+\kappa_0) - p\sin(n\theta-\kappa_0)\cos(p\theta+\kappa_0)}{(n+p)\sin(n\theta-\kappa_0)\sin(p\theta+\kappa_0)} \\ \frac{d}{d\theta}f_{\kappa_0|_C}(\theta) =& -r_C(\theta)^{-n}\frac{\sin((n+p)\theta)}{\sin(p\theta+\kappa_0)} \end{aligned}$$

Equaling the last formula to zero gives us the possible critical points of $f_{\kappa_0}|_C$: they are

$$\sin((n+p)\theta) = 0 \Rightarrow \theta = \frac{h}{n+p}\pi.$$

We will distinguish two components for this set: the first is the already introduced Ξ, the second is

$$\tilde{\Xi} := \{\xi_m := \frac{2m+1}{n+p}\pi : m = 0, ..., n+p-1\}$$

Since $\sin(p\theta_m + \kappa_0)\sin(n\theta_m - \kappa_0) > 0$, the family Ξ does not give contribution to the critical locus.

In the end, by studying the sign of $\frac{d}{d\theta}f \restriction_C (\theta)$, we have that ξ_m is a maximum if $\sin(p\xi_m + \kappa_0) > 0$ and a minimum if $\sin(p\xi_m + \kappa_0) < 0$.

Let us now consider what happens for value of κ_0 where the parametrization is ill defined at some direction θ.

It is clear that, in this case, the line connecting 0 and ∞ with direction θ is a component of C.

Let us solve then $\sin(p\theta + \kappa_0) = \sin(n\theta - \kappa_0) = 0$.

The condition

$$\sin(p\theta + \kappa_0) = \sin(n\theta - \kappa_0)$$

is equivalent to $\cos\big(\frac{n+p}{2}\theta\big)\sin\big(\frac{p-n}{2}\theta+\kappa_0\big)=0$.

We have then two possible scenarios

Case 1: let $\cos\big(\frac{n+p}{2}\theta\big)=0$.

This means that $\theta=\frac{2m+1}{n+p}\pi\in\tilde{\Xi}$. Since we also require $\sin(p\theta+\kappa_0)=0$, this happens for the set of directions

$$\kappa_0=h\pi-p\frac{2m+1}{n+p}\pi \text{ with } h\in\mathbb{Z}, m=0,...,n+p-1$$

Since $\sin(p\xi_m+\kappa_0)=\sin(n\xi+\kappa_0)$ and $\cos(p\xi_m+\kappa_0)=-\cos(n\theta+\kappa_0)$, ξ_m divides two sectors where $\sin(p\xi_m+\kappa_0)\sin(n\theta+\kappa_0)<0$ for such directions.

Indeed

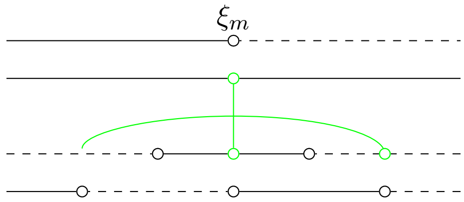
$$\begin{aligned}\sin(p(\xi_m\pm\epsilon)+\kappa_0)\sin(n(\xi_m\pm\epsilon)-\kappa_0)&=\\ \sin(p\epsilon)\sin(n\epsilon)\cos(p\xi_m+\kappa_0)\cos(n\xi_m-\kappa_0)&=\\ \sin(p\epsilon)\sin(n\epsilon)\cos^2(p\xi_m+\kappa_0)&<0\end{aligned}$$

Hence there is another branch of C defined in the sectors nearby ξ_m and intersecting the line with direction ξ_m at

$$\lim_{\theta\to\xi_m} r_C(\theta)=\lim_{\theta\to\xi_m}(-\frac{\sin(n\theta-\kappa_0)}{\sin(p\theta+\kappa_0)})^{\frac{1}{n+p}}=$$
$$=\lim_{\theta\to\xi_m}(-\frac{n\cos(n\theta+\kappa_0)}{p\cos(p\theta+\kappa_0)})^{\frac{1}{n+p}}=(\frac{n}{p})^{\frac{1}{n+p}}$$

where we used L'Hôpital's rule in the second passage.

We can picture the situation in our model once we choose a sign for $\cos(p\xi_m+\kappa_0)$: for instance, if it is equal to 1



Case 2: let $\sin\big(\frac{p-n}{2}\theta+\kappa_0\big)=0$.

We get $\kappa_0=h\pi+\frac{n-p}{2}\theta$, by applying $\sin(p\theta+\kappa_0)=0$ we get $\sin\big(\frac{n+p}{2}\theta\big)=0$.

Hence we obtain that $\theta=\theta_m\in\Xi$ for some m and $\kappa_0=h\pi+\frac{n-p}{n+p}m\pi$.

Since in this case $\cos(p\theta_m+\kappa_0)=\cos(n\theta_m-\kappa_0)$ and $\sin(p\theta_m+\kappa_0)=-\sin(n\theta_m-\kappa_0)$ we have, with computations similar to the ones above, that

$$\sin(p(\theta_m\pm\epsilon)+\kappa_0)\sin(n(\theta_m\pm\epsilon)-\kappa_0)>0$$

so that there is no other subset of C in the sectors abiding θ_m.

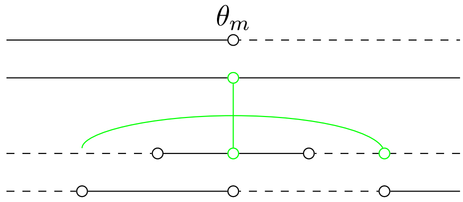
9.5. **The level set $\{f_{\kappa_0}=0\}$.** It is noticeable that we can study in the same way as in the last paragraph the level set $Z:=\{f_{\kappa_0}=0\}$.

The only difference comes from the change of sign in the parametrization:

$$r_Z(\theta):=[\frac{\sin(n\theta-\kappa_0)}{\sin(p\theta+\kappa_0)}]$$

This forces the domain of definition and the role of Ξ and $\tilde{\Xi}$ to reverse with respect to C.

As an example,take κ_0 at which the parametrization is ill-defined and $\theta_m \in \Xi$ with $\cos(p\theta_m + \kappa_0) > 0$: the picture is



9.6. **Level set of f_{κ_0} at θ_0 fixed.** Let $\theta_0 \in S^1$, we will denote by f_{κ_0,θ_0} the function f_{κ_0} restricted to the line $\{\theta = \theta_0\}$, $f_{\kappa_0,\theta_0}(r) = -\frac{r^{-n}}{n}\sin(n\theta_0 - \kappa_0) + \frac{r^p}{p}\sin(p\theta_0 + \kappa_0)$.

We are interested in the level sets of such function, $\{f_{\kappa_0,\theta_0} = t_0\} = \{f_{\kappa_0} = t_0\} \cap \{\theta = \theta_0\}$ $\forall t_0 \in \mathbb{R}$.

The following results provide the first step towards the local description of the level set.

Lemma 9.3.
- *Suppose* $\sin(p\theta_0 + \kappa_0) > 0$*,* $\sin(n\theta_0 - \kappa_0) < 0$*. Then*

$$\{f_{\kappa_0,\theta_0} = t_0\} = \begin{cases} \{r_1(\theta_0, t_0), r_2(\theta_0, t_0)\} & \text{if } t_0 > f_{\kappa_0,\theta_0}(r_C(\theta_0)) \\ \{r_C(\theta_0)\} & \text{if } t_0 = f_{\kappa_0,\theta_0}(r_C(\theta_0)) \\ \emptyset & \text{if } t_0 < f_{\kappa_0,\theta_0}(r_C(\theta_0)) \end{cases}$$

where $r_1(\theta_0, t_0) < r_C(\theta_0) < r_2(\theta_0, t_0)$ $\forall \theta_0, t_0$ *for which they are defined.*
- *Suppose* $\sin(p\theta_0 + \kappa_0) < 0$*,* $\sin(n\theta_0 - \kappa_0) > 0$*. Then*

$$\{f_{\kappa_0,\theta_0} = t_0\} = \begin{cases} \emptyset & \text{if } t_0 > f_{\kappa_0,\theta_0}(r_C(\theta_0)) \\ \{r_C(\theta_0)\} & \text{if } t_0 = f_{\kappa_0,\theta_0}(r_C(\theta_0)) \\ \{r_1(\theta_0, t_0), r_2(\theta_0, t_0)\} & \text{if } t_0 < f_{\kappa_0,\theta_0}(r_C(\theta_0)) \end{cases}$$

where $r_1(\theta_0, t_0) < r_C(\theta_0) < r_2(\theta_0, t_0)$ $\forall \theta_0, t_0$ *for which they are defined.*

Proof. The possible critical points of $f_{\kappa_0,\theta_0}(r)$ are given by solving:

$$\frac{d}{dr} f_{\kappa_0,\theta_0}(r) = r^{-n-1}\sin(n\theta_0 - \kappa_0) + r^{p-1}\sin(p\theta_0 + \kappa_0) = 0$$

hence, a point in C.

Since $\sin(p\theta_0 + \kappa_0) \neq 0$ we can rewrite the above as:

$$r^{n+p} = -\frac{\sin(n\theta_0 - \kappa_0)}{\sin(p\theta_0 + \kappa_0)} \tag{9.4}$$

Concerning the first statement, we notice that there is exactly one critical point $r_C(\theta_0)$.

It is then straightforward to notice the following properties
- $\frac{d}{dr} f_{\kappa_0,\theta_0}(r) > 0 \quad \forall r > r_C(\theta_0)$, $\frac{d}{dr} f_{\kappa_0,\theta_0}(r) < 0 \quad \forall 0 < r < r_C(\theta_0)$

- $$\lim_{r\to+\infty} f_{\theta_0}(r) = +\infty$$
- $$\lim_{r\to 0^+} f_{\theta_0}(r) = +\infty$$

from which we obtain the desired result.

The second statement can be proved similarly. □

Lemma 9.4. *Suppose* $\sin(p\theta_0+\kappa_0)\sin(n\theta_0-\kappa_0) > 0$. *Then* $\forall t_0 \in \mathbb{R}$

$$\{f_{\kappa_0,\theta_0} = t_0\} = \{r_\sim(\theta_0, t_0)\}$$

where, if $t_0 \lesseqgtr 0$

$$\begin{cases} r_\sim(\theta_0,t_0) \lesseqgtr r_Z(\theta_0) & \text{if } \sin(p\theta_0+\kappa_0) > 0,\ \sin(n\theta_0-\kappa_0) > 0 \\ r_\sim(\theta_0,t_0) \gtreqless r_Z(\theta_0) & \text{if } \sin(p\theta_0+\kappa_0) < 0,\ \sin(n\theta_0-\kappa_0) < 0 \end{cases}$$

Proof. Concerning the first statement, (9.4) has no solution, hence there is no critical point.

If $\sin(p\theta_0+\kappa_0) > 0$, $\sin(n\theta_0-\kappa_0) > 0$, we immediately notice the following properties:

- $\frac{d}{dr} f_{\kappa_0,\theta_0}(r) > 0 \quad \forall r > 0$
- $$\lim_{r\to+\infty} f_{\kappa_0,\theta_0}(r) = +\infty$$
- $$\lim_{r\to 0^+} f_{\kappa_0,\theta_0}(r) = -\infty$$

Therefore f_{κ_0,θ_0} is a diffeomorphism and its level set for each t_0 is always a point, let us denote it by $r_\sim(\theta_0,t_0)$

Moreover, since f_{κ_0,θ_0} is increasing, we moreover get $r_\sim(\theta_0,t_0) \lesseqgtr r_Z(\theta_0)$ if $t_0 \lesseqgtr 0$.

The proof in the case $\sin(p\theta_0+\kappa_0) < 0$, $\sin(n\theta_0-\kappa_0) < 0$ is similar. □

Let us now deal with the case of one or more sines vanishing.

Lemma 9.5. *(1) Suppose* $\sin(p\theta_0+\kappa_0) = 0$. *Then*

- *if* $\sin(n\theta_0-\kappa_0) < 0$

$$\{f_{\kappa_0,\theta_0} = t_0\} = \begin{cases} \left(\frac{-\sin(n\theta_0-\kappa_0)}{nt_0}\right)^{\frac{1}{n}} & \text{if } t_0 > 0 \\ \emptyset & \text{if } t_0 \le 0 \end{cases}$$

- *if* $\sin(n\theta_0-\kappa_0) > 0$

$$\{f_{\kappa_0,\theta_0} = t_0\} = \begin{cases} \left(\frac{\sin(n\theta_0-\kappa_0)}{nt_0}\right)^{\frac{1}{n}} & \text{if } t_0 < 0 \\ \emptyset & \text{if } t_0 \ge 0 \end{cases}$$

(2) Suppose $\sin(n\theta_0-\kappa_0) = 0$. *Then*

- *if* $\sin(p\theta_0+\kappa_0) < 0$

$$\{f_{\kappa_0,\theta_0} = t_0\} = \begin{cases} \emptyset & \text{if } t_0 > 0 \\ \left(\frac{-\sin(p\theta_0+\kappa_0)}{nt_0}\right)^{\frac{1}{n}} & \text{if } t_0 \le 0 \end{cases}$$

- *if* $\sin(p\theta_0+\kappa_0)>0$

$$\{f_{\kappa_0,\theta_0}=t_0\}=\begin{cases}\emptyset & \text{if } t_0<0\\ (\frac{-\sin(p\theta_0+\kappa_0)}{nt_0})^{\frac{1}{n}} & \text{if } t_0\geq 0\end{cases}$$

(3) If $\sin(p\theta_0+\kappa_0)=\sin(n\theta_0-\kappa_0)=0$, *we have*

$$\{f_{\kappa_0,\theta_0}=t_0\}=\begin{cases}\mathbb{R}_{>0} & \text{if } t_0=0\\ \emptyset & \text{if } t_0\neq 0\end{cases}$$

Proof. If $\sin(p\theta_0+\kappa_0)0\ \sin(n\theta_0-\kappa_0)\neq 0$, the function we are considering is $f_{\kappa_0,\theta_0}(r)=-\frac{r^{-n}}{n}\sin(n\theta_0-\kappa_0)$, the first statement is then trivial. Similar considerations hold for the second.

The third statement is trivial since the function f_{κ_0,θ_0} is identically 0. □

9.7. **Level set of f_{κ_0} in sectors with constant coefficient signs.** In this subsection, we will describe the geometry of the level set $\{f_{\kappa_0}=t_0\}$ in sectors where the sign of the dominant coefficients is unchanged, using Lemma 9.3 and Lemma 9.4.

Notice that $v_m\in\{v\in\mathbb{C}:\sin(p\arg(v)+\kappa_0)\sin(n\arg(v)-\kappa_0)<0\}$.

We will consider a sector $S=\theta^{<}<\arg(v)<\theta^{>}$ contained in a subset where such dominant coefficients have constant sign for all the arguments of its elements and maximal with respect to this requirement and call it elementary.

Proposition 9.6. *Let* κ_0, f_{κ_0} *and* $S\subset\{\sin(p\theta+\kappa_0)>0,\sin(n\theta-\kappa_0)<0\}$ *elementary with* $v_m\in S$. *Then* $\{f_{\kappa_0}=t_0\}$ *is empty if* $t_0\leq 0$, *it is the union of two analytic curves* E_1^m, E_2^m *defined as follows in all other cases.*

- *If* $t_0>f_{\kappa_0}(1,\theta_m)$*:*

$$E_i^m :]\theta^{<},\theta^{>}[\to\mathbb{C}_v$$
$$\theta\longrightarrow r_i(\theta,t_0)e^{i\theta}$$

for $i=1,2$, $r_1(\theta),r_2(\theta)$ *as in Lemma 9.3 and*

$$\lim_{\theta\to\theta^{\lessgtr}} r_1(\theta)=\begin{cases}0 & \text{if } \sin\big(n\theta^{\lessgtr}-\kappa_0\big)=0\\ (\frac{-\sin(n\theta^{\lessgtr}-\kappa_0)}{nt_0})^{\frac{1}{n}} & \text{if } \sin\big(\theta^{\lessgtr}+\kappa_0\big)=0\end{cases}\tag{9.5}$$

$$\lim_{\theta\to\theta^{\lessgtr}} r_2(\theta)=\begin{cases}(\frac{\sin(p\theta^{\lessgtr}+\kappa_0)}{nt_0})^{\frac{1}{n}} & \text{if } \sin\big(n\theta^{\lessgtr}-\kappa_0\big)=0\\ +\infty & \text{if } \sin\big(p\theta^{\lessgtr}+\kappa_0\big)=0.\end{cases}\tag{9.6}$$

- *If* $t_0=f(1,\theta_m)$*:*

$$E_1^m(\theta)=\begin{cases}r_1(\theta,t_0)e^{i\theta} & \text{if } \theta\leq\theta_m\\ r_2(\theta,t_0)e^{i\theta} & \text{if } \theta>\theta_m\end{cases}$$

$$E_2^m(\theta)=\begin{cases}r_2(\theta,t_0)e^{i\theta} & \text{if } \theta\leq\theta_m\\ r_1(\theta,t_0)e^{i\theta} & \text{if } \theta>\theta_m\end{cases}$$

with same limit conditions (9.5),(9.6) *as above.*

- *If* $f(1,\theta_m) > t_0 > 0$*, then* $\exists \theta_1(\kappa_0,t_0), \theta_2(\kappa_0,t_0)$ *with*
$$\theta^< < \theta_1(\kappa_0,t_0) < \theta_m < \theta_2(\kappa_0,t_0) < \theta^>$$
such that r_1, r_2 *are defined on* $]\theta^<, \theta_1(\kappa_0,t_0)[\cup]\theta_2(\kappa_0,t_0), \theta^>[$*. Then* E_1^m *is defined by the union of the two curves parametrized by*
$$]\theta^<, \theta_1(\kappa_0,t_0)[\ni \theta \to r_1(\theta,t_0)e^{i\theta}$$
$$]\theta^<, \theta_1(\kappa_0,t_0)[\ni \theta \longrightarrow r_2(\theta,t_0)e^{i\theta}$$
with $r_1(\theta_1(\kappa_0,t_0),t_0) = r_2(\theta_1(\kappa_0,t_0),t_0)$ *, while* E_2^m *arises in the same fashion from*
$$]\theta_2(\kappa_0,t_0), \theta^>[\ni \theta \to r_1(\theta,t_0)e^{i\theta}$$
$$]\theta_2(\kappa_0,t_0), \theta^>[\ni \theta \longrightarrow r_2(\theta,t_0)e^{i\theta}$$

Proof. If $t_0 \neq f(1,\theta_m)$, by implicit function theorem, $\{f_{\kappa_0} = t_0\}$ is union of analytic curves in the sector $]\theta^<, \theta^>[$. Moreover the values $r_1(\theta,t_0), r_2(\theta,t_0)$ defined in Lemma 9.3 describes the level set for θ varying and give hence rise locally to radial parametrization of such analytic curves.

In the case $t_0 > f(1,\theta_m)$, r_1 and r_2 are defined everywhere in $]\theta^<, \theta^>[$ and divided by the component of the subset of radial critical points in this sector (see first part of Lemma 9.3).

Therefore they are global radial parametrizations for the level set.

If $0 < t_0 < f(1,\theta_m)$, recall that $f_{\kappa_0}|_C$ attains a maximum at θ_m with value $f_{\kappa_0}(1,\theta_m)$ and goes to 0 moving towards $\theta^{\lessgtr}$. We can argue from this the existence of $\theta_1(\kappa_0,t_0), \theta_2(\kappa_0,t_0)$ such that $f_{\kappa_0}(r_C(\theta_1(\kappa_0,t_0)), \theta_1(\kappa_0,t_0)) = f_{\kappa_0}(r_C(\theta_2(\kappa_0,t_0)), \theta_2(\kappa_0,t_0)) = t_0$.

It is then clear that r_1 and r_2 are well defined on $]\theta^<, \theta_1(\kappa_0,t_0)[\cup]\theta_2(\kappa_0,t_0), \theta^>[$ and that on each of the two components they glue to give rise to a parametrization of a branch of the level set.

If $t_0 = f(1,\theta_m)$, by Morse theory the level set is made by two analytic curves intersecting orthogonally at $(1,\theta_m)$.

The same argument as above tells us that r_1 and r_2 are well defined and analytic in $]\theta^<, \theta^>[\setminus\{\theta_m\}$. Since the two curves intersect transversally, a radial parametrization is given by switching r_1 and r_2 when crossing θ_m.

Since $r_1(\theta,t_0) < r_C(\theta) < r_2(\theta,t_0)$:

$$\lim_{\theta\to\theta^{\lessgtr}} r_1(\theta,t_0) < \lim_{\theta\to\theta^{\lessgtr}} r_C(\theta) = 0 \text{ if } \sin\big(n\theta^{\lessgtr} - \kappa_0\big) = 0 \tag{9.7}$$

$$\lim_{\theta\to\theta^{\lessgtr}} r_2(\theta,t_0) > \lim_{\theta\to\theta^{\lessgtr}} r_C(\theta) = +\infty \text{ if } \sin\big(p\theta^{\lessgtr} + \kappa_0\big) = 0 \tag{9.8}$$

The remaining limits are obtained by Lemma 9.4 and its opposite counterpart.

Indeed they are equivalent to studying the fiber of the level set at a direction where one of the leading cosines annihilates.

At last, since $f_{\kappa_0} > 0$ in the sector, the level set for $t_0 < 0$ is empty. □

Similarly one can describe the situation in the case $\sin(p\theta + \kappa_0) < 0$ and $\sin(n\theta - \kappa_0) > 0$.

Proposition 9.7. *Let κ_0, f_{κ_0} and $S \subset \{\sin(p\theta + \kappa_0) < 0, \sin(n\theta - \kappa_0) > 0\}$ elementary with $v_m \in S$. Then $\{f_{\kappa_0} = t_0\}$ is empty if $t_0 \geq 0$, it is the union of two analytic curves E_1^m, E_2^m defined as follows in all other cases.*

- *If $t_0 < f_{\kappa_0}(1, \theta_m)$:*

$$E_i^m :]\theta^<, \theta^>[\to \mathbb{C}_v$$
$$\theta \longrightarrow r_i(\theta, t_0)e^{i\theta}$$

  for $i = 1, 2$, $r_1(\theta), r_2(\theta)$ as in Lemma 9.3 and

$$\lim_{\theta \to \theta^{\lessgtr}} r_1(\theta) = \begin{cases} 0 & \text{if } \sin\big(n\theta^{\lessgtr} - \kappa_0\big) = 0 \\ \big(\frac{-\sin(n\theta^{\lessgtr} - \kappa_0)}{nt_0}\big)^{\frac{1}{n}} & \text{if } \sin\big(\theta^{\lessgtr} + \kappa_0\big) = 0 \end{cases} \tag{9.9}$$

$$\lim_{\theta \to \theta^{\lessgtr}} r_2(\theta) = \begin{cases} \big(\frac{\sin(p\theta^{\lessgtr} + \kappa_0)}{nt_0}\big)^{\frac{1}{n}} & \text{if } \sin\big(n\theta^{\lessgtr} - \kappa_0\big) = 0 \\ +\infty & \text{if } \sin\big(p\theta^{\lessgtr} + \kappa_0\big) = 0. \end{cases} \tag{9.10}$$

- *If $t_0 = f(1, \theta_m)$:*

$$E_1^m(\theta) = \begin{cases} r_1(\theta, t_0)e^{i\theta} & \text{if } \theta \leq \theta_m \\ r_2(\theta, t_0)e^{i\theta} & \text{if } \theta > \theta_m \end{cases}$$

$$E_2^m(\theta) = \begin{cases} r_2(\theta, t_0)e^{i\theta} & \text{if } \theta \leq \theta_m \\ r_1(\theta, t_0)e^{i\theta} & \text{if } \theta > \theta_m \end{cases}$$

  with same limit conditions (9.9),(9.10).
- *If $f(1, \theta_m) < t_0 < 0$, then $\exists \theta_1(\kappa_0, t_0), \theta_2(\kappa_0, t_0)$ with*

$$\theta^< < \theta_1(\kappa_0, t_0) < \theta_m < \theta_2(\kappa_0, t_0) < \theta^>$$

  such that r_1, r_2 are defined on $]\theta^<, \theta_1(\kappa_0, t_0)[\cup]\theta_2(\kappa_0, t_0), \theta^>[$. Then E_1^m is defined by the union of the two curves parametrized by

$$]\theta^<, \theta_1(\kappa_0, t_0)[\ni \theta \to r_1(\theta, t_0)e^{i\theta}$$
$$]\theta^<, \theta_1(\kappa_0, t_0)[\ni \theta \longrightarrow r_2(\theta, t_0)e^{i\theta}$$

  with $r_1(\theta_1(\kappa_0, t_0), t_0) = r_2(\theta_1(\kappa_0, t_0), t_0)$, while E_2^m arises in the same fashion from

$$]\theta_2(\kappa_0, t_0), \theta^>[\ni \theta \to r_1(\theta, t_0)e^{i\theta}$$
$$]\theta_2(\kappa_0, t_0), \theta^>[\ni \theta \longrightarrow r_2(\theta, t_0)e^{i\theta}$$

We will now deal with dominant coefficients having the same sign. Notice that, in this case, no critical point v_m is in $\{\sin(p\theta + \kappa_0)\sin(n\theta - \kappa_0) > 0\}$, however, every elementary sector for it is surrounded by two elementary sectors for $\{\sin(p\theta + \kappa_0)\sin(n\theta - \kappa_0) > 0\}$. Let us say that these neighbouring elementary sectors contains two consecutive critical points v_m, v_{m+1}.

Proposition 9.8. *Let $S \subset \{\sin(p\theta + \kappa_0) > 0, \sin(n\theta + \kappa_0) > 0\}$ elementary.*

Then $\{f_{\kappa_0} = t_0\}$ in this sector is given by an analytic curve

$$E^{m,m+1} :]\theta^<, \theta^>[\to \mathbb{C}_v$$
$$\theta \longrightarrow r_{E^{m,m+1}}(\theta, t_0)e^{i\theta}$$

In particular:

- *if* $t_0 > 0$, $r_{E^{m,m+1}} > r_Z$ *and*

$$\lim_{\theta\to\theta^{\lessgtr}} r_{E^{m,m+1}}(\theta) = \begin{cases} +\infty & \text{if } \sin\bigl(p\theta^{\lessgtr}+\kappa_0\bigr)=0 \\ \bigl(\frac{\sin(p\theta^{\lessgtr}+\kappa_0)}{nt_0}\bigr)^{\frac{1}{n}} & \text{if } \sin\bigl(\theta^{\lessgtr}-\kappa_0\bigr)=0 \end{cases}$$

- *if* $t_0 < 0$, $r_{E^{m,m+1}} < r_Z$ *and*

$$\lim_{\theta\to\theta^{\lessgtr}} r_{E^{m,m+1}}(\theta) = \begin{cases} \bigl(\frac{-\sin(n\theta^{\lessgtr}-\kappa_0)}{nt_0}\bigr)^{\frac{1}{n}} & \text{if } \sin\bigl(p\theta^{\lessgtr}+\kappa_0\bigr)=0 \\ 0 & \text{if } \sin\bigl(n\theta^{\lessgtr}-\kappa_0\bigr)=0 \end{cases}$$

Proof. It is clear from Lemma 9.4 that $r_\sim(\theta,t_0) =: r_{E^{m,m+1}}(\theta,t_0)$ is a radial parametrization for the level set in the sector.

The conditions at the extremes are determined in the same way as in the last Proposition and by recalling the properties of r_Z. □

Similarly one can prove

Proposition 9.9. *Let* $S \subset \{\sin(p\theta+\kappa_0) < 0, \sin(n\theta+\kappa_0) < 0\}$ *elementary.*

Then $\{f_{\kappa_0} = t_0\}$ *in this sector is given by an analytic curve*

$$E^{m,m+1} :]\theta^{<},\theta^{>}[\to \mathbb{C}_v$$
$$\theta \longrightarrow r_{E^{m,m+1}}(\theta,t_0)e^{i\theta}$$

In particular:

- *if* $t_0 < 0$, $r_{E^{m,m+1}} > r_Z$ *and*

$$\lim_{\theta\to\theta^{\lessgtr}} r_{E^{m,m+1}}(\theta) = \begin{cases} +\infty & \text{if } \sin\bigl(p\theta^{\lessgtr}+\kappa_0\bigr)=0 \\ \bigl(\frac{\sin(p\theta^{\lessgtr}+\kappa_0)}{nt_0}\bigr)^{\frac{1}{n}} & \text{if } \sin\bigl(\theta^{\lessgtr}-\kappa_0\bigr)=0 \end{cases}$$

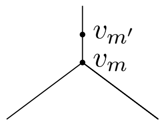
- *if* $t_0 > 0$, $r_{E^{m,m+1}} < r_Z$ *and*

$$\lim_{\theta\to\theta^{\lessgtr}} r_{E^{m,m+1}}(\theta) = \begin{cases} \bigl(\frac{-\sin(n\theta^{\lessgtr}-\kappa_0)}{nt_0}\bigr)^{\frac{1}{n}} & \text{if } \sin\bigl(p\theta^{\lessgtr}+\kappa_0\bigr)=0 \\ 0 & \text{if } \sin\bigl(n\theta^{\lessgtr}-\kappa_0\bigr)=0 \end{cases}$$

10. Degeneration of steepest descent cycles

As explained in 6, the steepest descent path Γ^Ψ_{m,κ_0} relative to v_m begins and ends in $S = \{0,\infty\}$ as far as there is no complete flow line for the downward flow equation connecting v_m with another critical point. In this section, we will determine a necessary and sufficient condition showing when such degeneration arises.

10.1. **A necessary and sufficient condition for degeneration.** Assume that there is a complete flow line for the downward gradient field associated to $\operatorname{Re}\Psi_{\kappa_0}$ and connecting two critical points v_m and $v_{m'}$. In this case, and only in this case, the closures of Γ^Ψ_{m,κ_0} and $\Gamma^\Psi_{m',\kappa_0}$ intersect. We call tripod the configuration given by their union, and denote it by $T_{m,m'} := \Gamma^\Psi_{m,\kappa_0} \cup \Gamma^\Psi_{m',\kappa_0}$. Assuming for example that $\operatorname{Re}\Psi_{\kappa_0}(v_{m'}) > \operatorname{Re}\Psi_{\kappa_0}(v_m)$, the tripod $T_{m,m'}$ is schematically pictured as



We will use the following notations

Definition 10.1. *Recall that we introduced the map $\nu : \Theta \to \kappa = \mu - n\Theta$. Let us set*

$$\mathcal{T}rp(\Psi_\kappa) := \{(m, m') : \operatorname{Im}\Psi_\kappa(v_m) = \operatorname{Im}\Psi_\kappa(v_{m'}) \text{ and } |m' - m| = 1\},$$
$$\mathcal{AS}t^{\nu}_{\mathcal{T}rp}(\Psi) := \{\kappa : \operatorname{Im}\Psi_\kappa(v_m) = \operatorname{Im}\Psi_\kappa(v_{m+1}) \text{ for some } m\},$$
$$\mathcal{AS}t_{\mathcal{T}rp}(\Psi) := \nu^{-1}(\mathcal{AS}t^{\nu}_{\mathcal{T}rp}(\Psi)) \subset \mathbb{R}_\Theta.$$

We write $m \in \mathcal{T}rp(\Psi_\kappa)$ to mean that $(m, m') \in \mathcal{T}rp(\Psi_\kappa)$ for some m'. In that case, we set $T_m = T_{m,m'}$.

Our aim is to prove the following

Proposition 10.2. *There is a complete flow line for the downward gradient field associated to $\operatorname{Re}\Psi_{\kappa_0}$ connecting two critical points v_m and $v_{m'}$ if and only if $\kappa_0 \in \mathcal{AS}t^{\nu}_{\mathcal{T}rp}(\Psi)$ and $(m, m') \in \mathcal{T}rp(\Psi_{\kappa_0})$.*

In order to prove this result, more information about the level sets of $\operatorname{Im}\Psi_{\kappa_0}$ is needed, and we will deal with it in the following subsections.

10.2. **Orientation.** Since in this section we will exhibit explicit steepest descent paths, we also need to provide them with an orientation.

As explained in section 6.4, we can give an orientation to the steepest descent cycles $\Gamma^{\Psi}_{m,\kappa_0}$ by choosing an eigenvector of the Hessian matrix of $g_{\kappa_0} = \operatorname{Re}\psi_{\kappa_0}$ at the critical point $(1, \theta_m)$ and associated with its negative eigenvalue.

Now we have

$$Hess(g_{\kappa_0})(1, \theta_m) = (n+p)\begin{pmatrix} \cos(p\theta_m + \kappa_0) & -\sin(p\theta_m + \kappa_0) \\ -\sin(p\theta_m + \kappa_0) & \cos(p\theta_m + \kappa_0) \end{pmatrix} \tag{10.1}$$

The determinant of this matrix is $-(n+p)^2$ with eigenvalues $\pm(n+p)$. We will use the eigenvector

$$\begin{pmatrix} \sin\left(\frac{p\theta_m + \kappa_0}{2}\right) \\ \cos\left(\frac{p\theta_m + \kappa_0}{2}\right) \end{pmatrix} \tag{10.2}$$

relative to $-(n+p)$ to give an orientation to $\Gamma^{\Psi}_{m,\kappa_0}$.

Notice that, since Γ_{m,κ_0} can be parametrized by θ for almost all κ_0, such orientation only depends on the sign of $\cos(p\theta_m + \kappa_0)$.

10.3. **The fundamental case: the tripod.** In this subsection we will show that steepest descent paths degenerate in the case prescribed by Proposition 10.2. We will assume that $n > p$: remark about the case $n < p$ will be done later. Suppose that $\kappa_0 \in \mathcal{AS}t^{\nu}_{\mathcal{T}rp}(\Psi)$ and $(m, m+1) \in \mathcal{T}rp(\Psi_{\kappa_0})$. Then one has $\sin(p\theta_m + \kappa_0) = \sin(p\theta_{m+1} + \kappa_0)$. Recalling that we set $\xi_m = (\theta_m + \theta_{m+1})/2$, this implies $\cos(p\theta_m + \kappa_0) = -\cos(p\theta_{m+1} + \kappa_0)$ and $\cos(p\xi_m + \kappa_0) = 0$.

There are two possible configurations for this situation, depending on the sign of $\sin(p\xi_m + \kappa_0)$. We depict them in Figure 3.

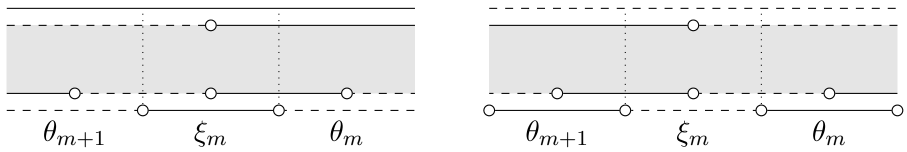
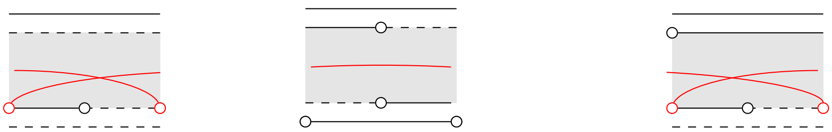


FIGURE 3. The case $\sin(p\xi_m + \kappa_0) = 1$ and $\sin(p\xi_m + \kappa_0) = -1$, respectively

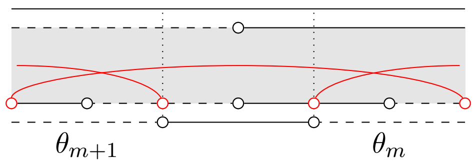
Since they are clearly symmetric, it suffices to deal with one of them.

Suppose then that $\sin(p\theta_m + \kappa_0) = 1$. This implies that $\cos(p\theta_m + \kappa_0) > 0$. As explained above, it suffices to study what happens in the three distinct sectors composing the picture above, where the sign of the sines is constant.

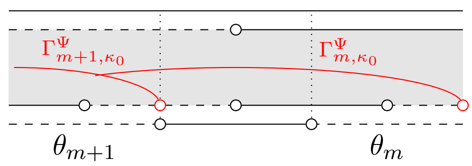
Using Propositions 9.6,9.8, we obtain the following pictures for the level set $\{\operatorname{Im}\Psi_{\kappa_0} = \lambda\}$ with $\lambda = \operatorname{Im}\Psi_{\kappa_0}(v_m) = \operatorname{Im}\Psi_{\kappa_0}(v_{m+1})$.



Notice that the analytic curve in the central sector has to connect the two branches coming from the first and the third sector by continuity, the resulting picture is

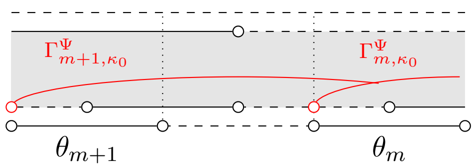


Recalling that $\operatorname{Re}\Psi_{\kappa_0}(v_m) > \operatorname{Re}\Psi_{\kappa_0}(v_{m+1})$, one has that $\Gamma^{\Psi}_{m,\kappa_0}$ and $\Gamma^{\Psi}_{m+1,\kappa_0}$ are



and the complete flow line is the part of the red region exiting from v_m and entering v_{m+1}.

In the same way one can deal with the second configuration above. The picture this time is



In the case $n < p$, we can argue in the same way that tripods arises. The only difference is that, instead of two endpoints being 0 and one ∞, the tripod will have two endpoints at ∞ and one at 0.

10.4. **Around the tripod.** For future reference, it is useful to describe what happens for $\kappa_0 \pm \epsilon$ for small $\epsilon > 0$ and $\kappa_0 \in \mathcal{AS}t^{\nu}_{\mathcal{T}rp}(\Psi)$ as above. Let us start with the case $\sin(p\xi_m + \kappa_0) = 1$.

Notice that

(1) for $\kappa_0 - \epsilon$ one has $\sin(p\theta_m + \kappa_0 - \epsilon) < \sin(p\theta_{m+1} + \kappa_0 - \epsilon)$ and $\cos(p\xi_m + \kappa_0 - \epsilon) = -\sin(p\xi_m + \kappa_0)\sin(\epsilon) < 0$
(2) for $\kappa_0 + \epsilon$ one has $\sin(p\theta_{m+1} + \kappa_0 + \epsilon) < \sin(p\theta_m + \kappa_0 + \epsilon)$ and $\cos(p\xi_m + \kappa_0 + \epsilon) = \sin(p\xi_m + \kappa_0)\sin(\epsilon) > 0$

The two configurations arising from $\kappa_0 \pm \epsilon$ are then as in Figure 5.

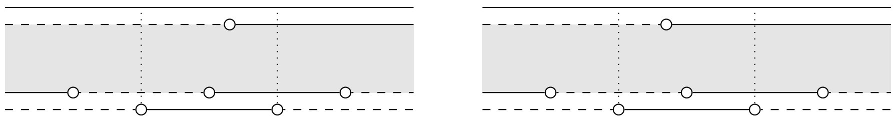


FIGURE 5. The situation for $\kappa_0 - \epsilon$ and $\kappa_0 + \epsilon$, respectively

We have still to deal with the level set in three sectors: while the situation in the second and the third is the same as in the presence of a tripod, the analytic curve in the second sector does not degenerate in the critical point v_{m+1} anymore.

Indeed, this is due to $\sin(p\theta_m + \kappa_0 \pm \epsilon) \neq \sin(p\theta_{m+1} + \kappa_0 \pm \epsilon)$.

Recalling Propositions Proposition 9.6,Proposition 9.8

(1) for $\kappa_0 - \epsilon$, since $\sin(p\theta_m + \kappa_0 - \epsilon) < \sin(p\theta_{m+1} + \kappa_0 - \epsilon)$, the steepest descent cycles Γ_{m,κ_0} ends in 0 without intersecting $\Gamma_{m+1,\kappa_0-\epsilon}$
(2) for $\kappa_0+\epsilon$, since $\sin(p\theta_{m+1} + \kappa_0 + \epsilon) < \sin(p\theta_m + \kappa_0 + \epsilon)$,the steepest descent cycles Γ_{m,κ_0} ends in ∞ without intersecting $\Gamma_{m+1,\kappa_0+\epsilon}$

(A) $\kappa_0 - \epsilon$: first sector (B) $\kappa_0 + \epsilon$: first sector

By singling out the steepest descent paths, the complete pictures become

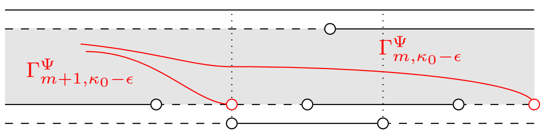


FIGURE 7. $\kappa_0 - \epsilon$: Steepest descent paths

Symmetrically, we obtain the description for $\sin(p\xi_m + \kappa_0) = -1$.

In the case $n < p$, the situation can be dealt with in a similar way. However, the result is symmetric with respect to $\kappa_0 \pm \epsilon$.

This follows from the fact that the local situation for $n < p$ at $0, \infty$ comes from reversing in our picture model the role of upper and lower lines.

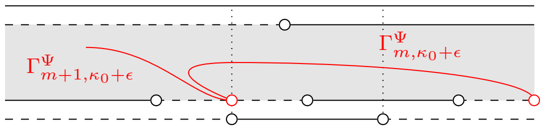


FIGURE 8. $\kappa_0+\epsilon$: Steepest descent paths

10.5. **Orientation issues.** Once we try to assign an orientation to the steepest descent paths just constructed, we discover that the behaviour of this choice depends on $n \lessgtr p$.

We can prove without problems that the orientation of the steepest descent cycles does not change when crossing a tripodal direction, take for example $\Gamma^{\Psi}_{m,\kappa_0\pm\epsilon}$.

By section 10.2, orientation depends on the sign of $\cos(p\theta_m+\kappa_0\pm\epsilon)$, hence it suffices to consider

$$\cos(p\theta_m+\kappa_0+\epsilon)\cos(p\theta_m+\kappa_0-\epsilon)=\frac{1}{2}(\cos(p\theta_m+\kappa_0)+\cos(\epsilon)) \tag{10.3}$$

Notice now that $\cos(p\theta_m+\kappa_0)\neq -1$: indeed

$$\cos(p\theta_m+\kappa_0)=\cos\left(p(\xi_m-\frac{\pi}{n+p})+\kappa_0\right)=$$
$$\cos(p\xi_m+\kappa_0)\cos\left(p\frac{\pi}{n+p}\right)+\sin(p\xi_m+\kappa_0)\sin\left(p\frac{\pi}{n+p}\right)=$$
$$-\sin\left(p\frac{\pi}{n+p}\right).$$

and the last term is different from -1 as far as $n\neq p$.

Then, for ϵ small enough, (10.3) is bigger than 0 and the orientation is preserved. Same computations hold for $\Gamma^{\Psi}_{m+1,\kappa_0\pm\epsilon}$.

On the other hand, when we try to perform the same computation for $\Gamma^{\Psi}_{m+1,\kappa_0+\epsilon}$ and $\Gamma^{\Psi}_{m,\kappa_0+\epsilon}$ (in order to see how they are oriented with respect to each other on the same side of the tripodal direction κ_0), we have

$$\cos(p\theta_m+\kappa_0+\epsilon)\cos(p\theta_{m+1}+\kappa_0+\epsilon)=$$
$$\frac{1}{2}[\cos\left(\frac{p(\theta_m+\theta_{m+1})}{2}+\kappa_0+\epsilon\right)+\cos\left(\frac{p(\theta_m-\theta_{m+1})}{2}\right)]=$$
$$\frac{1}{2}[\cos(p\xi_m+\kappa_0+\epsilon)+\cos\left(p\frac{\pi}{n+p}\right)]$$

Notice that $\cos(p\xi_m+\kappa_0+\epsilon)\to 0$ as $\epsilon\to 0$ by property of ξ_m, hence the sign of the last term depends uniquely on $\cos\left(p\frac{\pi}{n+p}\right)$. Since $\frac{p}{n+p}<1$, we only have to check the following cases

$$\cos\left(p\frac{\pi}{n+p}\right)>0\Leftrightarrow 0<p\frac{\pi}{n+p}<\frac{\pi}{2}\Leftrightarrow 0<2p<n+p\Leftrightarrow n>p$$
$$\cos\left(p\frac{\pi}{n+p}\right)<0\Leftrightarrow \frac{\pi}{2}<p\frac{\pi}{n+p}<\pi\Leftrightarrow n+p<2p<2(n+p)\Leftrightarrow n<p$$

This means that

- if $n > p$, the cycles are oriented in the same way with respect to the variable θ.
- if $n < p$, they are oriented in a opposite way with respect with the the variable θ.

10.6. **Closed components of the tripod.** Consider the families spanned by the steepest descent paths $\Gamma^{\Psi}_{m,\kappa}$, given by

$$C_{\pm} := \bigcup_{0<\epsilon\ll 1} \left(\Gamma^{\Psi}_{m,\kappa_0\pm\epsilon} \times \{\kappa_0 \pm \epsilon\}\right) \subset \mathbb{C}_u \times \mathbb{R}_{\kappa}.$$

Denote by $\Gamma^{\Psi}_{m,\kappa_0\pm}$ the fiber at κ_0 of the closure of $C_{\pm}$. This amounts to consider the limit of the paths $\Gamma^{\Psi}_{m,\kappa}$ for $\kappa \to \kappa_0\pm$.

For $m \notin \mathcal{T}rp^{\nu}_{\kappa_0}(\Psi)$, we have $\Gamma^{\Psi}_{m,\kappa_0\pm} = \Gamma^{\Psi}_{m,\kappa_0}$.

For $m \in \mathcal{T}rp^{\nu}_{\kappa_0}(\Psi)$, the set $\Gamma^{\Psi}_{m,\kappa_0\pm}$ is a closed subset of the tripod T_m. Assume $(m, m+1) \in \mathcal{T}rp^{\nu}_{\kappa_0}(\Psi)$, so that $T_m = T_{m,m+1}$. (The case $(m, m-1) \in \mathcal{T}rp^{\nu}_{\kappa_0}(\Psi)$ is similarly treated.)

The work done in the preceding subsection shows that, if $n > p$

(1) if $\sin(p\xi_m + \kappa_0) = 1$, we have

$$\Gamma^{\Psi}_{m+1,\kappa_0\pm} = \Gamma^{\Psi}_{m+1,\kappa_0}$$

while $\Gamma^{\Psi}_{m,\kappa_0\pm}$ define two closed subsets of $T_{m,m+1}$ as pictured below

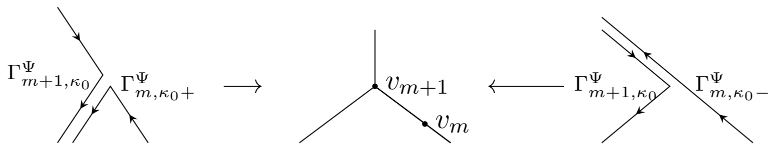


FIGURE 9. $\sin(p\xi_m + \kappa_0) = 1$

(2) if $\sin(p\xi_m + \kappa_0) = -1$, we have

$$\Gamma^{\Psi}_{m,\kappa_0\pm} = \Gamma^{\Psi}_{m,\kappa_0}$$

while $\Gamma^{\Psi}_{m+1,\kappa_0\pm}$ define two closed subsets of $T_{m,m+1}$. We can schematically picture the situation as

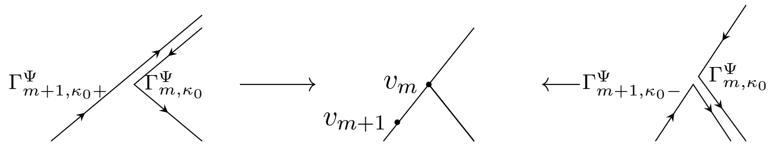


FIGURE 10. $\sin(p\xi_m + \kappa_0) = -1$

For the case $n < p$, the resulting pictures are symmetric to the ones above with respect to the central tripod , moreover, the orientation of one of the cycles is reversed.

10.7. **Borel-Moore homology for the tripod.** We want to show that the construction above provides a pair of basis for the Borel-Moore homology of tripod.

The change of basis arising is of the utmost importance for the description of the Stokes multipliers of the Fourier transform and will be described in 7.1.

Proposition 10.3. *Let* $\kappa_0 \in \mathcal{ASt}^{\nu}_{\mathcal{T}rp}(\Psi)$, $m \in \mathcal{T}rp(\Psi_{\kappa_0})$ *and* $T_{m,m+1}$ *the associated tripod. Then the paths* $\Gamma^{\Psi}_{m,\kappa_0\pm}, \Gamma^{\Psi}_{m+1,\kappa_0\pm}$ *are generators for the Borel-Moore homology of* $T_{m,m+1}$, *i.e. the morphisms*

$$H_1^{BM}(\tilde{\Gamma}^{\Psi}_{m,\kappa_0\pm}) \oplus H_1^{BM}(\tilde{\Gamma}^{\Psi}_{m+1,\kappa_0\pm}) \xrightarrow{\sim} H_1^{BM}(T_{m,m+1})$$

induced by the embeddings of $\Gamma^{\Psi}_{m,\kappa_0\pm}$ *and* $\Gamma^{\Psi}_{m+1,\kappa_0\pm}$ *in* $T_{m,m+1}$ *are isomorphisms. Moreover, the associated change of basis is*

(1)

$$\begin{cases} \Gamma^{\Psi}_{m,\kappa_0+} \mapsto \Gamma^{\Psi}_{m,\kappa_0-} - \Gamma^{\Psi}_{m+1,\kappa_0-} \\ \Gamma^{\Psi}_{m+1,\kappa_0+} \mapsto \Gamma^{\Psi}_{m+1,\kappa_0-} \end{cases}$$

if $\sin(p\xi_m + \kappa_0) = 1$

(2)

$$\begin{cases} \Gamma^{\Psi}_{m,\kappa_0+} \mapsto \Gamma^{\Psi}_{m,\kappa_0-} \\ \Gamma^{\Psi}_{m+1,\kappa_0+} \mapsto \Gamma^{\Psi}_{m+1,\kappa_0-} + \Gamma^{\Psi}_{m,\kappa_0-} \end{cases}$$

if $\sin(p\xi_m + \kappa_0) = 1$

Proof. The embeddings give rise to the following distinguished triangles

$$R\Gamma_c(T_{m,m+1}) \to R\Gamma_c(\tilde{\Gamma}^{\Psi}_{m,\kappa_0\pm}) \oplus R\Gamma_c(\tilde{\Gamma}^{\Psi}_{m+1,\kappa_0\pm}) \to R\Gamma_c(\tilde{\Gamma}^{\Psi}_{m,\kappa_0\pm} \cap \tilde{\Gamma}^{\Psi}_{m+1,\kappa_0\pm}) \xrightarrow{+1}$$

Observe that $\Gamma^{\Psi}_{m,\kappa_0\pm} \cap \Gamma^{\Psi}_{m,\kappa_0\pm}$ is diffeomorphic to a semi-closed interval, hence the last term of such triangles is 0. We obtain two isomorphisms

$$R\Gamma_c(T_{m,m+1}) \xrightarrow{\sim} R\Gamma_c(\tilde{\Gamma}^{\Psi}_{m,\kappa_0-}) \oplus R\Gamma_c(\tilde{\Gamma}^{\Psi}_{m+1,\kappa_0-}) \tag{10.4}$$

Passing to the dual, the first statement follows.

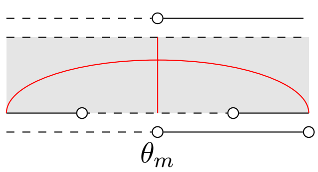
We have already summarized the situation for $\sin(p\xi_m + \kappa_0) = \pm 1$ in Fig. 9, 10: one obtains the changes of bases stated by comparing the paths on the left to the ones on the right. □

10.8. **Level set of** $\operatorname{Im}\Psi_{\kappa_0}$ **for a critical value with** $\operatorname{Re}\Psi_{\kappa_0}$ **negative.** Our goal in this subsection is to prove the following result.

Lemma 10.4. *The steepest descent path* $\Gamma^{\Psi}_{m,\kappa_0}$ *associated to a critical point* v_m *with* $\cos(p\theta_m + \kappa_0) \le 0$ *has as endpoints* 0 *and* ∞.

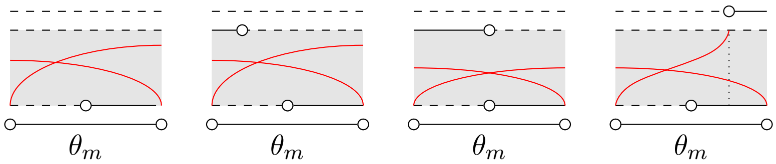
Proof. (i) First, suppose $\operatorname{Re}\Psi_{\kappa_0}(v_m) = -(\frac{1}{n} + \frac{1}{p})$, i.e., $\cos(p\theta_m + \kappa_0) = -1$.

We are interested in the zero level set of $\operatorname{Im}\Psi_{\kappa_0}$, which is



Since steepest descent paths connects regions where $\operatorname{Re}\Psi_{\kappa_0}$ tends to $-\infty$ and these region are represented here as inner dashed segments, we argue that $\Gamma^{\Psi}_{m,\kappa_0}$ is the straight vertical line connecting 0 and ∞ along the direction θ_m. The result then holds.

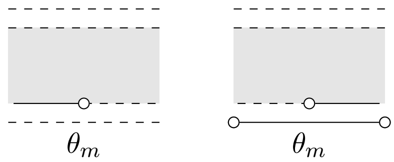
(ii) Suppose now that $\sin(p\theta_m + \kappa_0) < 0$. It is easy to see that one starting point for $\Gamma^{\Psi}_{m,\kappa_0}$ is 0 by checking the possible configurations in a sector around θ_m:



In particular, the last picture shows the result for θ_m nearest to α such that $\sin(p\alpha + \kappa_0) = 0$.

In general, however, the sector between θ_m and such an α is decomposed in the union of several elementary sectors.

For the ones which have not α as right side, we have two possible configurations:



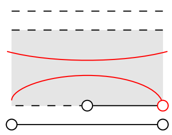
It is follows from (9.8) that there is only one branch of the level set in the first case.

Concerning the second case, notice the following: $\forall\theta_{m'}$ satisfying $\alpha < \theta_m < \theta_{m'}$, we have

$$\sin(p\theta_{m'} + \kappa_0) > \sin(p\theta_m + \kappa_0),$$

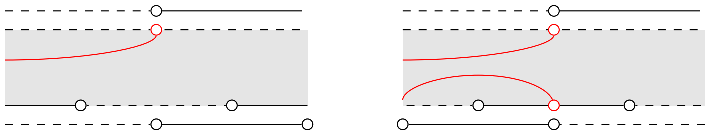
which is a consequence of the fact that $\sin(p\theta + \kappa_0) < 0$ is increasing in the model going from left to right.

This means that the resulting picture for the level set around such $\theta_{m'}$ is



These last pictures show that $\Gamma^{\Psi}_{m,\kappa_0}$ starting at 0 in the elementary sector containing θ_m never stops in one of the intermediate elementary sectors before α.

Concerning the sectors with right side α, we have the following configurations and branches of the level set.



The steepest descent cycle being the upper one, it always has an endpoint at ∞ tangent to the direction α.

The proof is symmetric for $\sin(p\theta_m + \kappa_0) > 0$.

$\square$

10.9. **Proof of Proposition 10.2.**

Proof. The study of the tripod perfomed in section 10.3 show the sufficiency.

Concerning the necessity, assume $\sin(p\theta_m + \kappa_0) = \sin(p\theta_{m'} + \kappa_0)$ with $m' \neq m+1$. As already remarked, this means that $\cos\left(p\frac{\theta_m+\theta_{m'}}{2} + \kappa_0\right) = \cos\left(p(\frac{\theta_m+\theta_{m'}}{2} + \pi) + \kappa_0\right) = 0$ and $\cos(p\theta_m + \kappa_0) > 0$, $\cos(p\theta_{m+1} + \kappa_0) < 0$.

We have two possibilities for $\frac{\theta_m+\theta_{m'}}{2}$ (the same holds for $\frac{\theta_m+\theta_{m'}}{2} + \pi$):

(1) it is equal to some $\theta_{\bar{m}} \in \Xi$,
(2) it is equal to some $\xi_{\bar{m}} \in \tilde{\Xi}$

In the first case, $\Gamma^{\Psi}_{\bar{m},\kappa_0}$ is an analytic curve connecting 0 and ∞ as a consequence of Lemma 10.4: since $\cos(p\theta_m + \kappa_0), \cos(p\theta_{m+1} + \kappa_0) \neq 0$, $\bar{m} \neq m, m'$.

In the second case, there is the tripod $T_{\bar{m},\bar{m}+1}$: notice that under the assumption $m' \neq m+1$, $m \neq \bar{m}$ and $m' \neq \bar{m}+1$.

In both cases, it is not possible for $\{f_{\kappa_0} = (\frac{1}{n} + \frac{1}{p}) \sin(p\theta_m + \kappa_0) =: \lambda\}$ to intersect such paths, since they are contained in a critical level set relative to critical values different from λ.

This prevent the steepest descent paths relative to v_m and $v_{m'}$ to degenerate.

$\square$

11. Sectorial representation

Recall that our object of study is

$$K = R\tilde{q_3}_!(E^{-\tilde{\Psi}} \otimes \tilde{p_3}^{-1}\pi^{-1}F)[1]$$

with $q_3 : \mathbb{C}_v \times \mathbb{C}^{\times}_{\zeta} \to \mathbb{C}^{\times}_{\zeta}$ standard projection, $p_3 : \mathbb{C}_v \times \mathbb{C}_{\zeta} \to \mathbb{C}_u$ defined by $p_3(v, \zeta) = (v\zeta, \zeta)$ and

$$\tilde{\Psi}(v, \zeta) = \alpha\zeta^{-n}(v^{-n} + \frac{n}{p}v^p).$$

with $|\alpha| = \beta$. We can now prove that K admits a trivialization in sectors delimited by directions in $\mathcal{AS}t_{\mathcal{T}rp}(\Psi)$ with the exponential factors prescribed from stationary phase formula.

We recall that the perverse sheaf F is of the form $j_!L$ for some local system L on $\mathbb{C}^{\times}_u$ and $j : \mathbb{C}^{\times}_u \to \mathbb{C}_u$ the open inclusion. We will denote by V the $\mathbf{k}$-vector space generic stalk of L.

Recall that we defined

$$\tilde{\Psi}_m(\zeta) := \tilde{\Psi}(v_m, \zeta) = \alpha(1 + \frac{1}{p})\zeta^{-n} e^{-in\theta_m}$$

in remark 8.2.

Theorem 11.1. *Let* $\Sigma \subset \mathbb{C}^\times_\zeta$ *be an open sector centered in* 0 *delimited by two consecutive elements* $\Theta_1, \Theta_2 \in \mathcal{AS}t_{\mathcal{T}rp}(\Psi)$*. Then we have an isomorphism in* $E^b_{\mathbb{R}-c}(\boldsymbol{k}_{\mathbb{C}^\times_{\zeta,\infty}})$

$$\pi^{-1}\boldsymbol{k}_{\overline{\Sigma}} \otimes K \xrightarrow[\sim]{\sigma} \pi^{-1}\boldsymbol{k}_{\overline{\Sigma}} \otimes \bigoplus_{m=0}^{n+p-1} (V \otimes E^{-\tilde{\Psi}_m})$$

Proof. We will use the following terminology: if $f : X \to Y$ is a function and $A \subset X$, $y \in f(A)$, we will call fiber of A at y the set $f^{-1}(y) \cap A$ and denote it by A_y. We will be mainly concerned with fibers of the function $\tilde{q}_3$.

Denote by $\mathcal{Z}$ the closed subset $\{\operatorname{Re}\tilde{\Psi} \leq t\} \cap \tilde{q}_3^{-1}(\tilde{\Sigma} \times \mathbb{R}) \subset \mathbb{C}^\times_v \times \mathbb{C}^\times_\zeta \times \mathbb{R}$ and let $\kappa_i = \nu(\Theta_i)$ for $i = 1, 2$.

For $\zeta_0 = \epsilon_0 e^{i\Theta_0} \in \mathbb{C}^\times_\zeta$, consider the flow lines of the downward gradient equation (6.5) in $\mathbb{C}^\times_v \setminus \{v_m : m = 0, ..., n + p - 1\}$ associated to $\operatorname{Re}\tilde{\Psi}_{\zeta_0}$ with the limit condition

$$\lim_{t \to -\infty} v(t) = v_m.$$

We have already noticed that such cycles for $\operatorname{Re}\tilde{\Psi}_{\zeta_0}$ are the same for the ones for $\operatorname{Re}\Psi_{\kappa_0}$, where $\kappa_0 = \nu(\Theta_0)$.

Consider

$$A := \bigcup_{\zeta \in \Sigma} \Gamma^{\Psi}_{m,\nu(\Theta)} \times \{\zeta\}$$

the family spanned by the flow lines as a subset of $\mathbb{C}^\times_v \times \mathbb{C}^\times_\zeta$.

A is a subset of the connected component of the level set

$$\operatorname{Im}(\tilde{\Psi} - \tilde{\Psi}_m)(v, \zeta) = 0$$

in $\mathbb{C}^\times_v \times \Sigma$ containing $\{v_m\} \times \Sigma$.

This closed subset is the family spanned by the components of the level sets $\operatorname{Im}(\Psi_\zeta(v) - \Psi_\zeta(v_m)) = 0$ containing v_m, intersections of two analytic curves at v_m (see Lemma 6.4).

If we remove $\{v_m\} \times \Sigma$, we then obtain 4 connected components of dimension 3, the families spanned by the 4 branches of analytic curves departing from v_m.

In this framework, A is obtained by taking the two connected components relative to the negative eigenspace of the Hessian of

$$\operatorname{Re}\tilde{\Psi}(v, \zeta)$$

at (v_m, ζ) with $\zeta \in \Sigma$ and rejoining $\{v_m\} \times \Sigma$.

Consider now $\bar{A}$ closure of A.

Since there are no limit points for A in $q_3(\Sigma)$ by the above description, we can argue that $q_3(\Sigma) \cap \bar{A} = A$, i.e. that the fiber of $\bar{A}$ at $\zeta_0 \in \Sigma$ is then exactly the steepest descent path $\Gamma^{\Psi}_{m,\nu(\Theta_0)}$.

From the construction in section 10.6, it is clear that the fiber of $\bar{A}$ at ζ with $\arg(\zeta) = \Theta_1, \Theta_2$ consists of the limit cycles $\mathcal{S}\Gamma_{m,\kappa_1,+}, \mathcal{S}\Gamma_{m,\kappa_2,-}$.

Denote by Γ_m the closed subset of $\mathbb{C}^\times_v \times \mathbb{C}^\times_\zeta \times \mathbb{R}$ given by $\bar{A} \times \mathbb{R}$ and define

$$\mathcal{W}_m = \Gamma_m \cap \mathcal{Z}$$

From the considerations made about A, it is easy to notice that its fiber at (ζ_0, t_0) is

$$(\mathcal{W}_m)_{(\zeta_0,t_0)} = \begin{cases} \mathcal{S}\Gamma^\Psi_{m,\nu(\Theta_0)} & \Psi_{\nu(\Theta_0)}(v_m) \leq \frac{t_0\epsilon_0^n}{n\beta} \text{ and } \Theta_1 < \Theta_0 < \Theta_2 \\ \mathcal{S}\Gamma^\Psi_{m,\kappa_1,+} & \Psi_{\nu(\Theta_0)}(v_m) \leq \frac{t_0\epsilon_0^n}{n\beta} \text{ and } \Theta_0 = \Theta_1 \\ \mathcal{S}\Gamma^\Psi_{m,\kappa_2,-} & \Psi_{\nu(\Theta_0)}(v_m) \leq \frac{t_0\epsilon_0^n}{n\beta} \text{ and } \Theta_0 = \Theta_2 \\ \emptyset & \text{otherwise.} \end{cases} \tag{11.1}$$

Denote by $\mathcal{W}$ the union $\bigcup_{m=0}^{n+p-1} \mathcal{W}_m$.

Notice that, for $i = 1, 2$ and $m \in \mathcal{T}rp_{\kappa_i}$, we have

$$\mathcal{S}\Gamma^\Psi_{m,\kappa_i,\pm} \cup \mathcal{S}\Gamma^\Psi_{m+1,\kappa_i,\pm} = T_{m,m+1} = \mathcal{S}\Gamma^\Psi_{m,\kappa_i} \cup \mathcal{S}\Gamma^\Psi_{m+1,\kappa_i}$$

Consider the following exact sequence:

$$0 \to \mathbf{k}_{\mathcal{Z}\setminus\mathcal{W}} \to \mathbf{k}_{\mathcal{Z}} \to \mathbf{k}_{\mathcal{W}} \to 0$$

which induces the distinguished triangle

$$R\tilde{q}_{3!}(\mathbf{k}_{\mathcal{Z}\setminus\mathcal{W}} \overset{+}{\otimes} \tilde{p}_3{}^{-1}\pi^{-1}F)[1] \to R\tilde{q}_{3!}(\mathbf{k}_{\mathcal{Z}} \overset{+}{\otimes} \tilde{p}_3{}^{-1}\pi^{-1}F)[1] \to$$
$$\xrightarrow{\sigma} R\tilde{q}_{3!}(\mathbf{k}_{\mathcal{W}} \overset{+}{\otimes} \tilde{p}_3{}^{-1}\pi^{-1}F)[1] \xrightarrow{+1}$$

Since $\forall(\zeta_0, t_0) \in \bar{\Sigma}^\times \times \mathbb{R}$

$$\begin{aligned}(\mathcal{Z} \setminus \mathcal{W})_{(\zeta_0,t_0)} =& \mathcal{Z}_{(\zeta_0,t_0)} \setminus \mathcal{W}_{(\zeta_0,t_0)} = \\ =& \{\operatorname{Re} \tilde{\Psi}(v, \zeta_0) \leq t_0\} \setminus \bigcup_{\substack{m: \\ \operatorname{Re} \tilde{\Psi}(v_m,\zeta_0) \leq t_0}} \mathcal{S}\Gamma^\Psi_{m,\nu(\Theta_0)} \\ =& \{\operatorname{Re} \Psi_{\nu(\Theta_0)} \leq \frac{t_0\epsilon_0^n}{n\beta}\} \setminus \bigcup_{\substack{m: \\ \operatorname{Re} \Psi_{\nu(\Theta_0)}(v_m) \leq \frac{t_0\epsilon_0^n}{n\beta}}} \mathcal{S}\Gamma^\Psi_{m,\nu(\Theta_0)}\end{aligned}$$

is cohomologically trivial by section 6.5, we obtain

$$(R\tilde{q}_{3!}\mathbf{k}_{\mathcal{Z}\setminus\mathcal{W}})_{(\zeta_0,t_0)} = 0$$

Thus

$$R\tilde{q}_{3!}(\mathbf{k}_{\mathcal{Z}\setminus\mathcal{W}} \otimes \tilde{p}_3^{-1}\pi^{-1}F)[1] \simeq 0$$

and σ is a isomorphism.

We moreover have the following exact sequence

$$0 \to \mathbf{k}_{\mathcal{W}} \to \bigoplus_m \mathbf{k}_{\mathcal{W}_m} \to \bigoplus_{m,m'} \mathbf{k}_{\mathcal{W}_m \cap \mathcal{W}_{m'}} \to 0$$

inducing the following distinguished triangle

$$R\tilde{q}_{3!}(\mathbf{k}_{\mathcal{W}} \otimes \tilde{p}_3^{-1}\pi^{-1}F)[1] \xrightarrow{\eta} \bigoplus_m R\tilde{q}_{3!}(\mathbf{k}_{\mathcal{W}_m} \otimes \tilde{p}_3^{-1}\pi^{-1}F)[1] \to$$
$$\to \bigoplus_{m,m'} R\tilde{q}_{3!}(\mathbf{k}_{\mathcal{W}_m \cap \mathcal{W}_{m'}} \otimes \tilde{p}_3^{-1}\pi^{-1}F)[1] \xrightarrow{+1}$$

Notice that

$$(\mathcal{W}_m \cap \mathcal{W}_{m'})_{(\zeta_0,t_0)} = \begin{cases} \bigcup\limits_{m\in \mathcal{T}rp_{\kappa_1}} \mathcal{S}\Gamma^{\Psi}_{m,\kappa_1-} \cap \mathcal{S}\Gamma^{\Psi}_{m+1,\kappa_1-} \\ \bigcup\limits_{m\in \mathcal{T}rp_{\kappa_2}} \mathcal{S}\Gamma^{\Psi}_{m,\kappa_2+} \cap \mathcal{S}\Gamma^{\Psi}_{m+1,\kappa_2+} \\ \emptyset & \text{otherwise} \end{cases}$$

Since this set is cohomologically trivial (see section 10.6), we have

$$(R\tilde{q}_{3!}\mathbf{k}_{\mathcal{W}_m\cap\mathcal{W}_{m'}})_{(\zeta_0,t_0)} = 0,$$

thus

$$R\tilde{q}_{3!}(\mathbf{k}_{\mathcal{W}_m\cap\mathcal{W}_{m'}} \otimes \tilde{p}_3^{-1}\pi^{-1}F)[1] \simeq 0$$

and η is an isomorphism.

Then, η and σ induces an isomorphism at the level of enhanced sheaves

$$\pi^{-1}\mathbf{k}_{\bar{\Sigma}} \otimes K \to \pi^{-1}\mathbf{k}_{\bar{\Sigma}} \otimes \bigoplus_m R\tilde{q}_{3!}(\mathbf{k}_{\mathcal{W}_m} \otimes \tilde{p}_3^{-1}\pi^{-1}F)[1]$$

In order to conclude, notice that

$$\tilde{q}_3(\mathcal{W}_m) = \begin{cases} (\zeta_0,t_0) \in \bar{\Sigma}\times\mathbb{R} & \text{if } \operatorname{Re}\tilde{\Psi}(v_m,\zeta_0) \leq t_0 = \\ \emptyset & \textit{otherwise}, \end{cases}$$

so

$$\begin{aligned} \tilde{q}_3(\mathcal{W}_m) =& \{(\zeta,t)\in\bar{\Sigma}\times\mathbb{R} : \operatorname{Re}\tilde{\Psi}(v_m,\zeta)\leq t\} = \\ =& \{\operatorname{Re}\tilde{\Psi}_m \leq t\} \cap \pi^{-1}\bar{\Sigma}. \end{aligned}$$

We get then by (11.1) that $\tilde{q}_3$ restricted to $\mathcal{W}_m$ is a trivial fiber bundle on $\{\operatorname{Re}\tilde{\Psi}_m \leq t\} \cap \bar{\Sigma}$ with fiber $\mathbb{R}$.

Indeed we can construct an isomorphism

$$B : \mathbb{R} \times (\{\operatorname{Re}\tilde{\Psi}_m \leq t\} \cap \pi^{-1}(\bar{\Sigma})) \xrightarrow{\sim} \mathcal{W}_m$$

via the parametrization of $\Gamma^{\Psi}_{m,\Theta_0}$ at each $(\zeta_0,t_0) \in \{\operatorname{Re}\tilde{\Psi}_m \leq t\} \cap \pi^{-1}(\bar{\Sigma})$.

Consider the commutative diagram

$$\begin{array}{ccccc} & & \mathbb{R}\times(\{\operatorname{Re}\tilde{\Psi}_m \leq t\}\cap\pi^{-1}(\bar{\Sigma})) & & \\ & \swarrow{\scriptstyle\sim} & \downarrow{\scriptstyle B} & \searrow{\scriptstyle\bar{\pi}} & \\ \tilde{\Phi}(\mathcal{W}_m) & \xleftarrow{\sim} & \mathcal{W}_m & \xrightarrow{\tilde{q}_{3|}} & \{\operatorname{Re}\tilde{\Psi}_m \leq t\}\cap\pi^{-1}(\bar{\Sigma}) \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_1} \\ \mathbb{C}_u\times\mathbb{C}^{\times}_{\zeta}\times\mathbb{R} & \xleftarrow{\tilde{\Phi}} & \mathbb{C}_v\times\mathbb{C}^{\times}_{\zeta}\times\mathbb{R} & \xrightarrow{\tilde{q}_3} & \mathbb{C}^{\times}_{\zeta}\times\mathbb{R} \end{array}$$

where the isomorphisms of the upper left triangle are induced by B and $\tilde{\Phi}$, i, i_1, i_2 denote the closed embeddings and $\bar{\pi}$ the standard projection.

Then

$$\begin{aligned}&R\tilde{q}_{3!}(\mathbf{k}_{\mathcal{W}_m}\otimes\tilde{p}_3^{-1}\pi^{-1}F)[1]\simeq\\&R\tilde{q}_{3!}R(\tilde{\Phi}^{-1})_!(\Phi^{-1})^{-1}(\mathbf{k}_{\mathcal{W}_m}\otimes\tilde{p}_3^{-1}\pi^{-1}L)[1]\simeq\\&R\tilde{q}_{2!}(\mathbf{k}_{\Phi(\mathcal{W}_m)}\otimes\tilde{p}_2^{-1}\pi^{-1}L)[1]\simeq\\&R\tilde{q}_{2!}i_{1!}((\tilde{p}_2^{-1}\pi^{-1}L)_{|\Phi(\mathcal{W}_m)})[1].\end{aligned}$$

where $\tilde{q}_2=\tilde{\Phi}^{-1}\circ\tilde{q}_3$ and $\tilde{p}_2=\tilde{\Phi}^{-1}\circ\tilde{p}_3$ are the standard projections (see 8). Now $p_2^{-1}\pi^{-1}L$ is the local system on $\mathbb{C}^\times_u\times\mathbb{C}^\times_\zeta\times\mathbb{R}$ with stalk V. Since the fiber of $\Phi(\mathcal{W}_m)$ with respect to the projection to $\{\operatorname{Re}\tilde{\Psi}_m\leq t\}\cap\pi^{-1}(\bar{\Sigma})$ is isomorphic to $\mathbb{R}$ via B, we can trivialize L on each such fiber and hence on $\Phi(\mathcal{W}_m)$.

$$\begin{aligned}R\tilde{q}_{2!}i_{1!}(\tilde{p}_2^{-1}\pi^{-1}L)_{|\Phi(\mathcal{W}_m)}[1]\simeq R\tilde{q}_{2!}i_{1!}(V_{|\Phi(\mathcal{W}_m)})[1]\simeq\\\simeq R\tilde{q}_{2!}(\mathbf{k}_{\Phi(\mathcal{W}_m)}\otimes V)[1]\simeq R\tilde{q}_{2!}\mathbf{k}_{\Phi(\mathcal{W}_m)}[1]\otimes V\end{aligned}$$

It remains to study $R\tilde{q}_{2!}\mathbf{k}_{\Phi(\mathcal{W}_m)}[1]$.

$$\begin{aligned}&R\tilde{q}_{2!}\mathbf{k}_{\Phi(\mathcal{W}_m)}[1]\simeq R\tilde{q}_{2!}R\tilde{\Phi}_!\tilde{\Phi}^{-1}\mathbf{k}_{\Phi(\mathcal{W}_m)}[1]\simeq\\&R\tilde{q}_{3!}\mathbf{k}_{\mathcal{W}_m}[1]\simeq R\tilde{q}_{3!}i_!(\mathbf{k}_{|\mathcal{W}_m})[1]\simeq\\&R\tilde{q}_{3!}i_!RB_!B^{-1}(\mathbf{k}_{|\mathcal{W}_m})[1]\simeq R(\tilde{q}_3\circ i\circ B)_!\mathbf{k}_{|\mathbb{R}\times(\{\operatorname{Re}\tilde{\Psi}_m\leq t\}\cap\pi^{-1}(\bar{\Sigma}))}[1]\simeq\\&R(i_1\circ\pi)_!\mathbf{k}_{|\mathbb{R}\times(\{\operatorname{Re}\tilde{\Psi}_m\leq t\}\cap\pi^{-1}(\bar{\Sigma}))}[1]\simeq i_{1!}R\pi_!\mathbf{k}_{|\mathbb{R}\times(\{\operatorname{Re}\tilde{\Psi}_m\leq t\}\cap\pi^{-1}(\bar{\Sigma}))}[1]\simeq\\&\ i_{1!}\mathbf{k}_{|\{\operatorname{Re}\tilde{\Psi}_m\leq t\}\cap\pi^{-1}(\bar{\Sigma})}\simeq\mathbf{k}_{\{\operatorname{Re}\tilde{\Psi}_m\leq t\}\cap\pi^{-1}(\bar{\Sigma})}\simeq\pi^{-1}\mathbf{k}_{\bar{\Sigma}}\otimes E^{-\tilde{\Psi}_m}.\end{aligned}$$

We then obtain the desired isomorphism $\sigma_{\Theta_1}^{\Theta_2}$ at the level of enhanced sheaves

$$\pi^{-1}\mathbf{k}_{\bar{\Sigma}}\otimes K\xrightarrow[\sim]{\sigma}\pi^{-1}\mathbf{k}_{\bar{\Sigma}}\otimes\bigoplus_m(V\otimes E^{-\tilde{\Psi}_m})$$

□

By convolution with $\mathbf{k}^E_{\mathbb{C}^\times_{\zeta,\infty}}$ we deduce the isomorphisms:

$$\sigma^E:\pi^{-1}\mathbf{k}_{\bar{\Sigma}}\otimes\mathcal{K}\xrightarrow{\sim}\pi^{-1}\mathbf{k}_{\bar{\Sigma}}\otimes\bigoplus_{m=0}^{n+p-1}(\mathbb{E}^{-\tilde{\Psi}_m}\otimes V)[1]\tag{11.2}$$

12. Behaviour at an anti-Stokes direction

In this section we will describe the Stokes multipliers associated to the sectorial representation of Theorem 11.1.

12.1. **When do the tripods occur?** Our goal in this subsection is to achieve a better understanding of the conditions $\kappa\in\mathcal{ASt}^\nu_{\mathcal{T}rp}(\Psi)$ and $m\in\mathcal{T}rp(\Psi_\kappa)$, identifying the presence of tripods.

Recall that we assume $(n,p)=1$. This implies $(p,n+p)=1$ and hence, by Bezout identity, $ap+b(n+p)=1$ for some $a,b\in\mathbb{Z}$. Note that such an a is unique modulo $n+p$, and its class is the inverse of p in $\mathbb{Z}/(n+p)\mathbb{Z}$.

Proposition 12.1. *We will distinguish three cases, according to the class of* $n+p$ *modulo* 4. *Let* a *be as above.*

(i) Assume that $n+p$ *is odd. Then* $\mathcal{AS}t^{\nu}_{\mathcal{T}rp}(\Psi) = \{\kappa_q \colon q \in \mathbb{Z}\}$, *with*

$$\kappa_q = \frac{2q+1}{2(n+p)}\pi.$$

The set $\mathcal{T}rp(\Psi_{\kappa_q})$ *has a single element* $(m^{\nu}_q, m^{\nu}_q + 1)$, *with*

$$m^{\nu}_q = a\frac{n+p+1}{2}(\frac{n-p-1}{2} - q) \pmod{n+p}.$$

(ii) Assume that $n+p = 0 \pmod 4$. *Then* $\mathcal{AS}t^{\nu}_{\mathcal{T}rp}(\Psi) = \{\kappa_q \colon q \in \mathbb{Z}\}$, *with*

$$\kappa_q = \frac{2q+1}{n+p}\pi.$$

The set $\mathcal{T}rp(\Psi_{\kappa_q})$ *is the set with two elements* $(m^{\nu,1}_q, m^{\nu,1}_q + 1)$ *and* $(m^{\nu,2}_q, m^{\nu,2}_q + 1)$, *with*

$$m^{\nu,1}_q = a(\frac{n-p-2}{4} - q) \pmod{n+p},$$
$$m^{\nu,2}_q = a(\frac{n-p-2}{4} - q) + \frac{n+p}{2} \pmod{n+p}.$$

(iii) Assume that $n+p = 2 \pmod 4$. *Then* $\mathcal{AS}t^{\nu}_{\mathcal{T}rp}(\Psi) = \{\kappa_q \colon q \in \mathbb{Z}\}$, *with*

$$\kappa_q = \frac{2q}{n+p}\pi.$$

The set $\mathcal{T}rp(\Psi_{\kappa_q})$ *is the set with two elements* (m^1_q, m^1_q+1) *and* (m^2_q, m^2_q+1), *with*

$$m^1_q = a(\frac{n-p}{4} - q) \pmod{n+p},$$
$$m^2_q = a(\frac{n-p}{4} - q) + \frac{n+p}{2} \pmod{n+p}.$$

Proof. Recall that $\kappa \in \mathcal{AS}t^{\nu}_{\mathcal{T}rp}(\Psi)$ means that $\operatorname{Im}\Psi_\kappa(v_m) = \operatorname{Im}\Psi_\kappa(v_{m+1})$ for some m. This is equivalent to

$$\sin(p\theta_m + \kappa) = \sin(p\theta_{m+1} + \kappa),$$

which in turn is equivalent to

$$\sin\left(p\frac{\theta_m - \theta_{m+1}}{2}\right)\cos\left(p\frac{\theta_m + \theta_{m+1}}{2} + \kappa\right) = 0.$$

Since $(\theta_m - \theta_{m+1})/2 = \pi/(n+p)$, the condition $\sin\left(p\frac{\theta_m-\theta_{m+1}}{2}\right) = 0$ is equivalent to $p = 0 \pmod{n+p}$. This is never satisfied since $(p, n+p) = 1$. We are thus left with the condition

$$\cos\left(p\frac{\theta_m + \theta_{m+1}}{2} + \kappa\right) = 0.$$

Since $(\theta_m + \theta_{m+1})/2 = (2m+1)\pi/(n+p)$, this is equivalent to

$$\kappa = -\frac{p(2m+1)}{n+p}\pi + \frac{\pi}{2} + h\pi = [n+p-2p(2m+1)+2(n+p)h]\frac{\pi}{2(n+p)}$$

for some $h \in \mathbb{Z}$. In other words, $\kappa = d\frac{\pi}{2(n+p)}$ for $d \in \mathbb{Z}$ satisfying

$$2p(2m+1) = n+p-d \pmod{2(n+p)}$$

for some m or, equivalently

$$4pm = n-p-d \pmod{2(n+p)} \tag{12.1}$$

(i) If $n+p$ is odd, also $n-p$ is odd. A necessary condition for (12.1) is that $d = 2q+1$ is also odd. Let us show that this condition is also sufficient by providing an $m = m_q$ such that

$$2pm_q = \frac{n-p-1}{2} - q \pmod{n+p}.$$

Recall that a is an inverse of p modulo $n+p$, and note that $2\frac{n+p+1}{2} = 1 \pmod{n+p}$. Then we have

$$m_q = a\frac{n+p+1}{2}(\frac{n-p-1}{2} - q) \pmod{n+p}.$$

(ii) We assume now that $n+p = 0 \pmod 4$. In particular $n+p$ is even, and since $(n,p) = 1$ this implies that both n and p are odd. Since $n+p$ is even, (12.1) implies that $n-p-d = 0 \pmod 4$. Since p is odd, $2p = 2 \pmod 4$ and hence $n+p = 0 \pmod 4$ implies $n-p = 2 \pmod 4$. Thus, we have $d = 2 \pmod 4$, or equivalently $d = 2(2q+1)$.

In order to show that this condition is also sufficient, it is enough to provide a solution $m = m_q$ of

$$pm_q = \frac{n-p-2}{4} - q \pmod{\frac{n+p}{2}}.$$

Since a is an inverse of p also modulo $(n+p)/2$, the solution is

$$m_q = a(\frac{n-p-2}{4} - q) \pmod{\frac{n+p}{2}}.$$

From this, one gets the solutions $m = m_q^1$ and $m = m_q^2$ of

$$2pm_q^i = \frac{n-p-2}{2} - 2q \pmod{n+p}$$

given by

$$m_q^1 = a(\frac{n-p-2}{4} - q) \pmod{n+p},$$
$$m_q^2 = a(\frac{n-p-2}{4} - q) + \frac{n+p}{2} \pmod{n+p}.$$

(iii) The proof for the case $n+p = 2 \pmod 4$ is similar. □

Corollary 12.2. *With the same notation of 12.1, we have*

(1) if $n+p$ odd

$$m_{q+1} - m_q = -a(\frac{n+p+1}{2}),$$

(2) if $n+p$ even

$$m_{q+1}^i - m_q^i = -a$$

for $i = 1, 2$.

Proposition 12.1 extends to the following results by using the correspondence ν.

Proposition 12.3. *We will distinguish three cases, according to the class of $n+p$ modulo 4. Let a be as above.*

(i) Assume that $n+p$ is odd. Then $\mathcal{AS}t_{\mathcal{T}rp}(\Psi) = \{\Theta_h : h \in \mathbb{Z}\}$, with

$$\Theta_h = \frac{\mu}{n} + \frac{2h+1}{2n(n+p)}\pi.$$

The set $\mathcal{T}rp(\Psi_{\nu(\Theta_h)})$ has a single element (m_h, m_h+1), with

$$m_h = a\frac{n+p+1}{2}(\frac{n-p+1}{2}+h) \pmod{n+p}.$$

(ii) Assume that $n+p = 0 \pmod 4$. Then $\mathcal{AS}t_{\mathcal{T}rp}(\Psi) = \{\Theta_h : h \in \mathbb{Z}\}$, with

$$\Theta_h = \frac{\mu}{n} + \frac{2h+1}{n(n+p)}\pi.$$

The set $\mathcal{T}rp(\Psi_{\nu(\Theta_h)})$ is the set with two elements (m_h^1, m_h^1+1) and (m_h^2, m_h^2+1), with

$$m_h^1 = a(\frac{n-p+2}{4}+h) \pmod{n+p},$$
$$m_h^2 = a(\frac{n-p+2}{4}+h) + \frac{n+p}{2} \pmod{n+p}.$$

(iii) Assume that $n+p = 2 \pmod 4$. Then $\mathcal{AS}t_{\mathcal{T}rp}(\Psi) = \{\Theta_h : h \in \mathbb{Z}\}$, with

$$\Theta_h = \frac{\mu}{n} + \frac{2h}{n(n+p)}\pi.$$

The set $\mathcal{T}rp(\Psi_{\nu(\Theta_h)})$ is the set with two elements (m_h^1, m_h^1+1) and (m_h^2, m_h^2+1), with

$$m_h^1 = a(\frac{n-p+4}{4}+h) \pmod{n+p},$$
$$m_h^2 = a(\frac{n-p+4}{4}+h) + \frac{n+p}{2} \pmod{n+p}.$$

Proof. Clearly the given sets are the inverse images via ν of $\mathcal{AS}t^{\nu}_{\mathcal{T}rp}$. For the description of $\mathcal{T}rp(\Psi_{\nu(\Theta_h)})$, it suffices to notice the following

$$\nu(\frac{\mu}{n} + \frac{2h+1}{2n(n+p)}\pi) = -\frac{2h+1}{2n(n+p)}\pi$$
$$\nu(\frac{\mu}{n} + \frac{2h+1}{n(n+p)}\pi) = -\frac{2h+1}{n(n+p)}\pi$$
$$\nu(\frac{\mu}{n} + \frac{2h}{n(n+p)}\pi) = -\frac{2h}{n(n+p)}\pi$$

In particular, we can pass from the parameter q for $\mathcal{AS}t^{\nu}_{\mathcal{T}rp}$ to h by $q = -1-h$. The results then follows by applying this substitution in Proposition 12.1. □

The following corollary is immediate.

Corollary 12.4. *With the same notation of 12.3, we have*

(1) if $n+p$ odd

$$m_{h+1} - m_h = a(\frac{n+p+1}{2}),$$

(2) if $n+p$ even

$$m^i_{h+1} - m^i_h = a$$

for $i = 1, 2$.

12.2. **Tripodal directions and anti-Stokes directions.** Recall that, with the set $\{\tilde{\Psi}_m\}$ of exponential factors of K, we have the classical sets of Stokes and anti-Stokes directions.

The following result holds

Proposition 12.5. *We have $\mathcal{AS}t_{\mathcal{T}rp}(\tilde{\Psi}) \subset \mathcal{AS}t(\{\tilde{\Psi}_m\})$, with equality when $n+p$ is odd.*

Proof. Recall that

$$\mathcal{AS}t(\{\tilde{\Psi}_m\}) = \{\Theta\colon \operatorname{Im}(\tilde{\Psi}_m - \tilde{\Psi}_{m'})(\epsilon e^{i\Theta}) = 0, \epsilon \to 0\}$$

and hence it all revolves about computing

$$\begin{aligned} \operatorname{Im}(\epsilon^{-n} e^{-in\Theta}(e^{ip\theta_m} - e^{ip\theta_{m'}})) &= 0 \\ \operatorname{Im} e^{i(p\theta_m + \mu - n\Theta)} &= \operatorname{Im} e^{i(p\theta_{m'} + \mu - n\Theta)} \\ \sin(p\theta_m + \kappa) &= \sin(p\theta_{m'} + \kappa) \end{aligned} \tag{12.2}$$

Recalling the definition of $\mathcal{AS}t_{\mathcal{T}rp}(\tilde{\Psi})$, the first result follows.

Notice that the last equation in (12.2) is equivalent to

$$\sin\left(p\frac{\theta_m - \theta_{m'}}{2}\right)\cos\left(p\frac{\theta_m + \theta_{m'}}{2} + \kappa\right) = 0$$

It is straightforward to notice that the solutions κ only depends on $m + m'$ $\pmod{n+p}$.

Let now $n+p$ be odd, notice that the equation

$$m + m' = 2\bar{m} + 1 \pmod{n+p}$$

always has one and only one solution $\bar{m} \pmod{n+p}$. □

12.3. **Stokes matrices.** We are now ready to show how the sectorial representations behave at a tripodal direction.

Consider then $\Theta_h, \Theta_{h-1}, \Theta_{h+1} \in \mathcal{AS}t_\nu(\Psi)$.

Denote by Σ_h (respectively Σ_{h+1}) the sector delimited by Θ_{h-1}, Θ_h (respectively Θ_h, Θ_{h+1}) and by σ_h, σ_{h+1} the isomorphisms coming from Theorem 11.1 over Σ_h, Σ_{h+1} respectively.

If we denote by l_h the ray with argument Θ_h, **??** gives us the following diagram

$$\begin{array}{ccc} & & \pi_2^{-1}\mathbf{k}_{l_h} \otimes \bigoplus_{m=0}^{n+p-1} (E^{-\Psi_m} \otimes V) \\ & \overset{\sigma_h}{\nearrow} & \\ \pi_2^{-1}\mathbf{k}_{l_h} \otimes K & & \Big\downarrow \alpha_h \\ & \underset{\sigma_{h+1}}{\searrow} & \\ & & \pi_2^{-1}\mathbf{k}_{l_h} \otimes \bigoplus_{m=0}^{n+p-1} (E^{-\Psi_m} \otimes V) \end{array}$$

Due to natural isomorphism

$$End_{E^b(\mathbf{k}_{\mathbb{C}^{\times}_{\zeta,\infty}})}(\pi_2^{-1}\mathbf{k}_{l_{\Theta_h}} \otimes \bigoplus_{m=0}^{n+p-1} (E^{-\Psi_m} \otimes V)) \subset End_{\mathbf{k}}(\bigoplus_{m=0}^{n+p-1} V)$$

the morphism α_h is determined by the linear map it induces on a stalk at a point

$$(\zeta_0, t_0) \in \pi_2^{-1}(l_h) \cap \bigcap_{m=0}^{n+p-1} \{t - \operatorname{Re}\Psi_m(\zeta) \geq 0\}.$$

In particular, we will choose $(e^{i\Theta_h}, t_0)$, which satisfies the requirement above for t_0 big enough.

Moreover, in order to keep track of the different copies of V constituting V^{n+p}, we will use the following notation. Denote by $\mathbb{1}_m$ the complex vector space of rank one with chosen basis element 1_m, we will write

$$V^{n+p} = \bigoplus_{m=0}^{n+p-1} (\mathbb{1}_m \otimes V)$$

Recall that, by section 6.5

$$K_{(\zeta_0,t_0)} = R\Gamma_c(\mathbb{C}_v, \mathbf{k}_{\mathcal{Z}_{(\zeta_0,t_0)}} \otimes \Phi_{\zeta_0}^{-1}L)[1] \simeq R\Gamma_c(\mathbb{C}_v, \mathbf{k}_{\mathcal{S}\Gamma^{\Psi}_{\nu(\Theta_h)}} \otimes \Phi_{\zeta_0}^{-1}L)[1]$$

where we denoted by $\mathcal{S}\Gamma^{\Psi}_{\nu(\Theta_h)}$ the set $\bigcup_{m=1}^{n+p-1} \mathcal{S}\Gamma^{\Psi}_{m,\nu(\Theta_h)}$.

On the level of stalks σ_h, σ_{h+1} give rise to isomorphisms σ_h, σ_{h+1}

$$R\Gamma_c(\mathbb{C}_v, \mathbf{k}_{\mathcal{S}\Gamma^{\Psi}_{\nu(\Theta_h)}} \otimes \Phi_{\zeta_0}^{-1}L)[1] \underset{\sim}{\xrightarrow{\sigma_h}} \bigoplus_{m=0}^{n+p-1} (\mathbb{1}_m \otimes V)$$

$$R\Gamma_c(\mathbb{C}_v, \mathbf{k}_{\mathcal{S}\Gamma^{\Psi}_{\nu(\Theta_h)}} \otimes \Phi_{\zeta_0}^{-1}L)[1] \underset{\sim}{\xrightarrow{\sigma_{h+1}}} \bigoplus_{m=0}^{n+p-1} (\mathbb{1}_m \otimes V).$$

Recall that such morphisms are induced by the embedding of the curves $\mathcal{S}\Gamma^{\Psi}_{m,\nu(\Theta_h)\pm}$ in $\mathcal{S}\Gamma^{\Psi}_{\Theta_h}$.

The morphism α_h can then be recovered as $\sigma_{h+1} \circ \sigma_h^{-1}$.

The easiest way to represent the morphism α_h is to pass to the dual side and consider the Borel-Moore homology.

By applying what described in Lemma 7.1, we can consider the dual of the morphisms above:

$$H_1^{BM}(\mathcal{S}\Gamma^{\Psi}_{\nu(\Theta_h)}, (\Phi_{\zeta_0}^{-1}L)^*) \xleftarrow[\sim]{\sigma_h^*} \bigoplus_{m=0}^{n+p-1} (\mathbb{1}_m^* \otimes V^*)$$

$$H_1^{BM}(\mathcal{S}\Gamma^{\Psi}_{\nu(\Theta_h)}, (\Phi_{\zeta_0}^{-1}L)^*) \xleftarrow[\sim]{\sigma_{h+1}^*} \bigoplus_{m=0}^{n+p-1} (\mathbb{1}_m^* \otimes V^*).$$

Our aim is then to describe, using definitions and results from section 12.1, the family of morphisms $\{\alpha_h^* = (\sigma_{h+1}^*)^{-1} \circ \sigma_h^* : \Theta_h \in \mathcal{AS}t_{\mathcal{T}rp}(\Psi)\}$

12.4. **Stokes matrices for $L = \mathbf{k}_{\mathbb{C}_u^\times}$.** Before computing the Stokes matrices in the case of a generic local system L, let us show how they look like when considering the trivial local system $\mathbf{k}_{\mathbb{C}_u^\times}$.

Notice that, in this particular case, $\Phi_{\zeta_0}^{-1}\mathbf{k}_{\mathbb{C}_u^\times} = \mathbf{k}_{\mathbb{C}_v^\times}$ and the morphism σ_h, σ_{h+1} are associated with the two bases $\{\Gamma^{\Psi}_{m,\Theta_h\pm} : m = 0, ..., n+p-1\}$ of $H_1^{BM}(\mathcal{S}\Gamma^{\Psi}_{\nu(\Theta_h)}$.

Proposition 12.6. *(i) Let $n+p$ odd, the morphism α_h^* comes from the element in $End_k(\bigoplus_{m=1}^{n+p-1}(\mathbb{1}_m^* \otimes V^*))$ given, $\forall h = 0, ..., 2n(n+p)-2$, by the assignments*

$$\begin{cases} 1_m^* \mapsto 1_m^* & \text{for } m \neq m_h, m_{h+1}, \\ 1_{m_h}^* \mapsto 1_{m_h}^* + 1_{m_h+1}^*, \\ 1_{m_h+1}^* \mapsto 1_{m_h+1}^*. \end{cases}$$

if h is even and $\sin(p\theta_{m_h} - n\Theta_h) = 1$ or if h odd and $\sin(p\theta_{m_h} - n\Theta_h) = -1$ and

$$\begin{cases} 1_m^* \to 1_m^* & \text{for } m \neq m_h, m_{h+1}, \\ 1_{m_h}^* \to 1_{m_h}^* \\ 1_{m_h+1}^* \to 1_{m_h+1}^* - 1_{m_h}^* \end{cases}$$

if h is odd and $\sin(p\theta_{m_h} - n\Theta_h) = 1$ or if h even and $\sin(p\theta_{m_h} - n\Theta_h) = -1$.

The morphism $\alpha_{2n(n+p)-1}$ is defined by composing the morphism obtained by applying the rule above with $(-1)^n diag((T^*)^{-1})$.

(ii) Let n+p even, the morphism α_h^* comes from the element in $End_k(\bigoplus_{m=1}^{n+p-1}(V^* \otimes \mathbb{1}_m^*))$ defined, $\forall h = 0, ..., n(n+p)-2$, by the assignment

$$\begin{cases} 1_m^* \mapsto 1_m^* & \text{if } m \notin \{m_h^i, m_h^i + 1 : i = 1, 2\} \\ 1_{m_h^1}^* \mapsto 1_{m_h^1}^* + 1_{m_h^1+1}^* \\ 1_{m_h^1+1}^* \mapsto 1_{m_h^1+1}^* \\ 1_{m_h^2}^* \mapsto 1_{m_h^2}^* \\ 1_{m_h^2+1}^* \mapsto 1_{m_h^2+1}^* - 1_{m_h^2}^*. \end{cases}$$

if $\sin\Big(p\theta_{m_h^1} - n\Theta_h\Big) = +1$ *or by the assignment*

$$\begin{cases} 1^*_m \mapsto 1^*_m & \textit{if } m \notin \{m_h^i, m_h^i+1 : i = 1,2\} \\ 1^*_{m_h^1} \mapsto 1^*_{m_h^1} \\ 1^*_{m_h^1+1} \mapsto 1^*_{m_h^1+1} - 1^*_{m_h^1} \\ 1^*_{m_h^2} \mapsto 1^*_{m_h^2,+} \otimes \mathrm{v}^* + 1^*_{m_h^2+1} \\ 1^*_{m_h^2+1} \mapsto 1^*_{m_h^2+1}. \end{cases}$$

if $\sin\Big(p\theta_{m_h^1} - n\Theta_h\Big) = -1.$

The morphism $\alpha_{n(n+p)-1}$ *is defined by composing the morphism obtained by applying the rule above with* $(-1)^n diag((T^*)^{-1})$.

Proof. For all $m \neq m_h, m_h+1$, we have $\Gamma^\Psi_{m,\nu(\Theta_h)-} = \Gamma^\Psi_{m,\nu(\Theta_h)+}$ and no interaction with other steepest descent paths, since they are all disjoint.

Hence the morphism α_h^* is the identity on $\bigoplus_{m\neq m_h, m_h+1} \mathbb{1}_m$.

Given the construction made in Theorem 11.1, the morphisms $\sigma_h^*, \sigma_{h+1}^*$ can be read, at the level of tripod T_{m_h,m_h+1}, as

$$\begin{array}{c} H_1^{BM}(\mathcal{S}\Gamma^\Psi_{m_h,\Theta_h-}, \mathbf{k}_{\mathbb{C}_v^\times}) \oplus H_1^{BM}(\mathcal{S}\Gamma^\Psi_{m_h+1,\Theta_h-}, \mathbf{k}_{\mathbb{C}_v^\times}) \\ \big\downarrow \sigma_h^* \\ H_1^{BM}(T_{m_h,m_h+1}, \mathbf{k}_{\mathbb{C}_v^\times}) \\ \sigma_{h+1}^* \big\uparrow \\ H_1^{BM}(\mathcal{S}\Gamma^\Psi_{m_h,\Theta_h+}, \mathbf{k}_{\mathbb{C}_v^\times}) \oplus H_1^{BM}(\mathcal{S}\Gamma^\Psi_{m_h+1,\Theta_h+}, \mathbf{k}_{\mathbb{C}_v^\times}) \end{array}$$

From 10.3 that we already know that the arising change of basis (which is α_h) depends on the sign of $\sin(p\theta_{m_h} + \nu(\Theta_h))$.

Let us use the explicit description given in Proposition 12.3 to achieve a better knowledge of the sign condition and compute $\sin(p\theta_{m_h} + \nu(\Theta_h))$ in term of $\sin(p\theta_{m_0} + \nu(\Theta_0))$

Recall the Bezout identity for $p, n+p$, $ap + b(n+p) = 1$. Suppose at first that $n+p$ is odd, we have

$$\begin{aligned} p\theta_{m_0+hac} + \nu(\Theta_h) &= p\frac{2(m_0 + ha(\frac{n+p+1}{2}))}{n+p}\pi + \nu(\Theta_0) - \frac{h}{n+p}\pi = \\ &= p\theta_{m_0} + \nu(\Theta_0) + h\frac{ap(n+p+1)-1}{n+p}\pi = \\ &= p\theta_{m_0} + \nu(\Theta_0) + h\frac{(n+p+1)(1-b(n+p))-1}{n+p}\pi \\ & p\theta_{m_0} + \nu(\Theta_0) + h\frac{n+p+1+b(n+p)(n+p+1)-1}{n+p}\pi = \\ &= p\theta_{m_0} + \nu(\Theta_0) + h\frac{(n+p)(1-b(n+p+1))}{n+p}\pi \\ p\theta_{m_0} + \nu(\Theta_0) + h(1-b(n+p)-b)\pi &= p\theta_{m_0} + \nu(\Theta_0) + h(ap-b)\pi. \end{aligned}$$

Thus

$$\sin(p\theta_{m_0+hac} + \nu(\Theta_h)) = (-1)^{q(ap-b)} \sin(p\theta_{m_0} + \nu(\Theta_0))$$

If we reduce the Bezout identity $ap + b(n+p) = 1$ modulo 2, we obtain $ap + b(n+p) = ap + b = 1 \pmod 2$ and hence $ap - b = 1 \pmod 2$. The value of the term then only depends on the parity of h and on the value of $\sin(p\theta_{m_0} + \nu(\Theta_0))$.

The last statement about $h = 2n(n+p)$ is due to the fact that, after a complete turn around 0 with the argument Θ, we have performed n turns around 0 in the variable κ.

It is then sufficient to recall that after a turn in the variable κ, the steepest descent paths reverses orientation.

If $n+p$ is even, we have to perform the following computations

$$p\theta_{m_h^i} + \nu(\Theta_h) = p\frac{2(m_0^i + ha)}{n+p} + \nu(\Theta_0) - \frac{2h}{n+p}\pi =$$
$$= p\theta_{m_0^i} + \kappa_0 + 2q\frac{ap-1}{n+p}\pi = p\theta_{m_0^i} + \kappa_0 - 2qb\pi.$$

Hence we have

$$\sin\Big(p\theta_{m_h^i} + \nu(\Theta_h)\Big) = \sin\Big(p\theta_{m_0^i} + \nu(\Theta_0)\Big).$$

Since $m_0^2 - m_0^1 = \frac{n+p}{2} \pmod{n+p}$ and p is odd in this case

$$\sin\Big(p\theta_{m_0^2} + \nu(\Theta_0)\Big) = \sin\Big(p\theta_{m_0^1} + \nu(\Theta_0) + p\pi\Big) = -\sin\Big(p\theta_{m_0^1} + \nu(\Theta_0)\Big).$$

and the form of the morphism depends only on the sign of $\sin\Big(p\theta_{m_0^1} + \nu(\Theta_0)\Big)$.

The last statement about $\alpha^*_{n(n+p)-1}$ follows in the same way as in the odd case.

□

12.5. **Stokes multipliers for a generic local system L.** Recall that it is classical to represent the local system L as the pair (V,T) of its local sections at a point different from 0 and the monodromy action T. Since we need to deal with the Borel-Moore homology with coefficient in the dual system L^* of L, we recall that the pair associated with L^* is given by

(1) the generic fiber of L^*, V^*
(2) the monodromy T^* which is characterized by the condition

$$\langle T^* w, Tv\rangle = \langle w, v\rangle$$

for all $v \in V$ and $w \in V^*$

In particular, one can argue from the last condition that the transpose of the monodromy isomorphism T is $(T^*)^{-1}$.

In order to deal with this case, we need to make some adjustments. Recall that we introduced the diffeomorphism

$$\Phi : \mathbb{C}_v \times \mathbb{C}_\zeta^\times \to \mathbb{C}_u \times \mathbb{C}_\zeta^\times$$
$$(v,\zeta) \to (v\zeta,\zeta)$$

in order to ease the computations involved with the descriptions of the steepest descent paths. When dealing with the contribution to the Stokes multipliers given by the local system L and its monodromy, it is however easier to perform computations in $\mathbb{C}_u$.

Therefore, we will consider all objects introduced so far on $\mathbb{C}_u$ via the blow up Φ. In particular we can use the ζ-fiber of $\Phi_\zeta(v) = v\zeta$ to define $\tilde{\Gamma}^{\Psi}_{m,\Theta}$ as $\Phi_\zeta(\Gamma^{\Psi}_{m,\Theta})$.

Note that the $\tilde{\Gamma}^{\Psi}_{m,\Theta}$ are then of steepest descent for $\operatorname{Re}\Phi_*(\Psi_\zeta)$ and are attached to the critical point $e^{i(\theta_m+\Theta)}$.

All results given up to now can be rephrased in this framework without changes.

Under this change, the stalk we need to study becomes

$$K_{\zeta_0,t_0} = R\Gamma_c(\mathbb{C}_v, \mathbf{k}_{\mathcal{S}\Gamma^{\Psi}_{m,\Theta}} \otimes \Phi^{-1}_{\zeta_0} L)[1] \simeq R\Gamma_c(\mathbb{C}_u, \mathbf{k}_{\mathcal{S}\tilde{\Gamma}^{\Psi}_{m,\Theta}} \otimes L)[1]$$

and the decomposition given by the limit cycles can be rewritten accordingly.

In particular, we can then study the isomorphisms $\sigma^*_h, \sigma^*_{h+1}$, which can be read as

$$\begin{array}{c} \bigoplus H_1^{BM}(\mathcal{S}\tilde{\Gamma}^{\Psi}_{m,\nu(\Theta_h)-}, L^*) \\ \Big\downarrow \sigma^*_h \\ H_1^{BM}(\mathcal{S}\Gamma^{\Psi}_{m,\Theta}, L^*) \\ \sigma^*_{h+1} \Big\uparrow \\ \bigoplus H_1^{BM}(\mathcal{S}\tilde{\Gamma}^{\Psi}_{m,\nu(\Theta_h)+}, L^*) \end{array}$$

In order to rewrite such morphisms in term of the dual V^* of the general fiber of L, we need an isomorphism for each $\Theta \in S^1_\zeta$

$$H_1^{BM}(\mathcal{S}\tilde{\Gamma}^{\Psi}_{m,\nu(\Theta)}, L^*) \simeq H_1^{BM}(\mathcal{S}\tilde{\Gamma}^{\Psi}_{m,\nu(\Theta)}) \otimes V^* \tag{12.3}$$

i.e., we need to fix a determination for the section of L^* along $\mathcal{S}\tilde{\Gamma}^{\Psi}_{m,\Theta}$.

Notice that it suffices to give an element of $(L^*)_{e^{i(\theta_m+\Theta_0)}}$ in order to have a local section of L^* along $\mathcal{S}\tilde{\Gamma}^*_{m,\Theta}$ by extension.

Such isomorphism is constructed as follows. Consider a determination for L^* at $1 \in \mathbb{C}_u$, i.e., an isomorphism $V^* \xrightarrow{\sim} (L^*)_1$ which associates to each $\mathrm{v}^* \in V^*$ an element $(s^*)_1$, stalk of a local section s^* at 1.

Consider the analytic continuation $\tilde{s}$ of s along the path

$$\begin{aligned} \gamma : [0, \theta_m + \Theta] &\to S^1_u \\ \tilde{\theta} &\to e^{i\tilde{\theta}} \end{aligned}$$

This gives rise to an isomorphism $V^* \xrightarrow{\sim} (L^*)_{e^{i(\theta_+\Theta)}}$ given by $\mathrm{v}^* \mapsto (\tilde{s})_{e^{i\theta_m+\Theta}}$ which provides the isomorphism required.

If $\bar{\Theta} \in \mathcal{AS}t_{\mathcal{T}rp}(\Psi)$, we can define $s^{\Psi}_{m,\bar{\Theta}\pm}$ sections of L^* on $\tilde{\Gamma}^{\Psi}_{m,\bar{\Theta}\pm}$ by taking the limit in the construction described in 10.6.

The above definition and construction adapts to $\Gamma^{\Psi}_{m,\nu(\Theta)\pm}$.

We set

$$\tilde{\Gamma}^{\Psi}_{n+p,\nu(\Theta)\pm} \otimes \mathrm{v}^* = \tilde{\Gamma}^{\Psi}_{0,\nu(\Theta)\pm} \otimes (T^*)^{-1}\mathrm{v}^*. \tag{12.4}$$

Theorem 12.7. *(i) Let $n+p$ odd, the morphism α^*_h comes from the element in $End_k(\bigoplus_{m=1}^{n+p-1}(\mathbb{1}^*_m \otimes V^*))$ given, $\forall h = 0, ..., 2n(n+p)-2$, by the assignments*

$$\begin{cases} 1^*_m \otimes \mathrm{v}^* \mapsto 1^*_m \otimes \mathrm{v}^* & \text{for } m \neq m_h, m_{h+1}, \\ 1^*_{m_h} \otimes \mathrm{v}^* \mapsto 1^*_{m_h} \otimes \mathrm{v}^* + 1^*_{m_h+1} \otimes \mathrm{v}^*, \\ 1^*_{m_h+1} \otimes \mathrm{v}^* \mapsto 1^*_{m_h+1} \otimes \mathrm{v}^*. \end{cases}$$

if h is even and $\sin(p\theta_{m_h} - n\Theta_h) = 1$ or if h odd and $\sin(p\theta_{m_h} - n\Theta_h) = -1$ and

$$\begin{cases} 1^*_m \otimes \mathrm{v}^* \to 1^*_m \otimes \mathrm{v}^* & \text{for } m \neq m_h, m_{h+1}, \\ 1^*_{m_h} \otimes \mathrm{v}^* \to 1^*_{m_h} \otimes \mathrm{v}^* \\ 1^*_{m_h+1} \otimes \mathrm{v}^* \to 1^*_{m_h+1} \otimes \mathrm{v}^* - 1^*_{m_h} \otimes \mathrm{v}^* \end{cases}$$

if h is odd and $\sin(p\theta_{m_h} - n\Theta_h) = 1$ or if h even and $\sin(p\theta_{m_h} - n\Theta_h) = -1$.

The morphism $\alpha_{2n(n+p)-1}$ is defined by composing the morphism obtained by applying the rule above with $(-1)^n diag((T^*)^{-1})$.

(ii) Let $n+p$ even, the morphism α^*_h comes from the element in

$$End_k(\bigoplus_{m=1}^{n+p-1}(\mathbb{1}^*_m \otimes V^*))$$

defined, $\forall h = 0, ..., n(n+p)-2$, by the assignment

$$\begin{cases} 1^*_m \otimes \mathrm{v}^* \mapsto 1^*_m \otimes \mathrm{v}^* & \text{if } m \notin \{m^i_h, m^i_h+1 : i=1,2\} \\ 1^*_{m^1_h} \otimes \mathrm{v}^* \mapsto 1^*_{m^1_h} \otimes \mathrm{v}^* + 1^*_{m^1_h+1} \otimes \mathrm{v}^* \\ 1^*_{m^1_h+1} \otimes \mathrm{v}^* \mapsto 1^*_{m^1_h+1} \otimes \mathrm{v}^* \\ 1^*_{m^2_h} \otimes \mathrm{v}^* \mapsto 1^*_{m^2_h} \otimes \mathrm{v}^* \\ 1^*_{m^2_h+1} \otimes \mathrm{v}^* \mapsto 1^*_{m^2_h+1} \otimes \mathrm{v}^* - 1^*_{m^2_h} \otimes \mathrm{v}^*. \end{cases}$$

if $\sin\Big(p\theta_{m^1_h} - n\Theta_h\Big) = +1$ or by the assignment

$$\begin{cases} 1^*_m \otimes \mathrm{v}^* \mapsto 1^*_m \otimes \mathrm{v}^* & \text{if } m \notin \{m^i_h, m^i_h+1 : i=1,2\} \\ 1^*_{m^1_h} \otimes \mathrm{v}^* \mapsto 1^*_{m^1_h} \otimes \mathrm{v}^* \\ 1^*_{m^1_h+1} \otimes \mathrm{v}^* \mapsto 1^*_{m^1_h+1} - 1^*_{m^1_h} \otimes \mathrm{v}^* \\ 1^*_{m^2_h} \otimes \mathrm{v}^* \mapsto 1^*_{m^2_h,+} \otimes \mathrm{v}^* + 1^*_{m^2_h+1} \otimes \mathrm{v}^* \\ 1^*_{m^2_h+1} \otimes \mathrm{v}^* \mapsto 1^*_{m^2_h+1} \otimes \mathrm{v}^*. \end{cases}$$

if $\sin\Big(p\theta_{m^1_h} - n\Theta_h\Big) = -1$.

The morphism $\alpha_{n(n+p)-1}$ is defined by composing the morphism obtained by applying the rule above with $(-1)^n diag((T^*)^{-1})$.

Proof. Let $\Theta_h \in \mathcal{ASt}_{\mathcal{T}rp}(\Psi)$. We already know from Proposition 12.6 how α_h behaves at the level of the Borel-Moore homology with coefficients in $\mathbf{k}$.

Clearly, with the exception of tripod pairs, the definition of the morphism extends unchanged when adding a regular part.

It only remains to examine what happens generically at tripods.

Let (m_h, m_{h+1}) the coordinate of tripod, consider $\sin(p\xi_{m_h} + \nu(\Theta_h)) = +1$.

Since $\tilde{\Gamma}^{\Psi}_{m_h+1,\nu(\Theta_h)-} = \tilde{\Gamma}^{\Psi}_{m_h+1,\nu(\Theta_h)+}$, the definition of the morphism $\alpha^*_{\Theta_h}$ extends unchanged.

Consider then $\tilde{\Gamma}^{\Psi}_{m_h,\nu(\Theta_h)-} \otimes \mathrm{v}$: by (12.3), this corresponds to $\tilde{\Gamma}^{\Psi}_{m_h,\nu(\Theta_h)-} \otimes (\tilde{s}^*)_{e^{i(\theta_{m_h}+\Theta_h)}}$.

The embedding $\tilde{\Gamma}^{\Psi}_{m_h,\nu(\Theta_h)-} \to T_{m_h,m_h+1}$ gives then a well defined element

$$\tilde{\Gamma}^{\Psi}_{m_h,\nu(\Theta_h)-} \otimes (\tilde{s}^*)_{e^{i(\theta_{m_h})}} = \tilde{\Gamma}^{\Psi}_{m_h,\nu(\Theta_h)-} \otimes (\tilde{s}^*)_{e^{i(\theta_{m_h}+\Theta_h)}} \in H_1^{BM}(T_{m_h,m_h+1}) \otimes V^*.$$

Since $\tilde{\Gamma}^{\Psi}_{m_h,\nu(\Theta_h)-} = \tilde{\Gamma}^{\Psi}_{m_h,\nu(\Theta_h)+} + \tilde{\Gamma}^{\Psi}_{m_h+1,\nu(\Theta_h)+}$, we have

$$\begin{aligned} &\tilde{\Gamma}^{\Psi}_{m_h,\nu(\Theta_h)-} \otimes (\tilde{s}^*)_{e^{i(\theta_{m_h}+\Theta_h)}} = \\ &= (\tilde{\Gamma}^{\Psi}_{m_h,\nu(\Theta_h)+} + \tilde{\Gamma}^{\Psi}_{m_h+1,\nu(\Theta_h)+}) \otimes (\tilde{s}^*)_{e^{i(\theta_{m_h}+\Theta_h)}} = \\ &= \tilde{\Gamma}^{\Psi}_{m_h,\nu(\Theta_h)+}) \otimes (\tilde{s}^*)_{e^{i(\theta_{m_h})}} + \tilde{\Gamma}^{\Psi}_{m_h+1,\nu(\Theta_h)+}) \otimes (\tilde{s}^*)_{e^{i(\theta_{m_h}+\Theta_h)}} \end{aligned}$$

The choice of the determination made above implies that

$$\tilde{\Gamma}^{\Psi}_{m_h,\nu(\Theta_h)+}) \otimes (\tilde{s}^*)_{e^{i(\theta_{m_h}+\Theta_h)}} = \tilde{\Gamma}^{\Psi}_{m_h,\nu(\Theta_h)+}) \otimes \mathrm{v}^*.$$

Notice that the analytic continuation along the path connecting $e^{i(\theta_{m_h}+\Theta_h)}$ and $e^{i(\theta_{m_h+1}+\Theta_h)}$ in the tripod is the same as the analytic continuation along the piece of S^1_u connecting the same two points, as the path are homotopic.

Hence $\tilde{\Gamma}^{\Psi}_{m_h+1,\nu(\Theta_h)+}) \otimes (\tilde{s}^*)_{e^{i(\theta_{m_h}+\Theta_h)}} = \tilde{\Gamma}^{\Psi}_{m_h+1,\nu(\Theta_h)+}) \otimes (\tilde{s}^*)_{e^{i(\theta_{m_h+1}+\Theta_h)}}$.

We then need only to study what happens to $(\tilde{s}^*)_{e^{i(\theta_{m_h+1}+\Theta_h)}}$.

(1) If $m_h \neq n+p-1$, $(\theta_{m_h+1} + \Theta_h) - (\theta_{m_h} + \Theta_h) = \frac{2\pi}{n+p} < 2\pi$: there is no monodromy involved since we are not moving more than 2π, and hence
$$(\tilde{s}^*)_{e^{i(\theta_{m_h+1}+\Theta_h)}} = \mathrm{v},$$
(2) if $m_h = n+p-1$, the section considered is
$$(\tilde{s}^*)_{e^{i(\theta_{n+p}+\Theta_h)}} = (\tilde{s}^*)_{e^{i(\theta_0+2\pi+\Theta_h)}} = (T^*)^{-1}(\tilde{s}^*)_{e^{i(\theta_0+\Theta_h)}}$$
and hence
$$\tilde{\Gamma}^{\Psi}_{n+p,\nu(\Theta_h)+}) \otimes (\tilde{s}^*)_{e^{i(\theta_{n+p}+\Theta_h)}} = \tilde{\Gamma}^{\Psi}_{0,\nu(\Theta_h)+}) \otimes (T^*)^{-1}(\tilde{s}^*)_{e^{i(\theta_0+\Theta_h)}}$$

We can similarly deal with the case $\sin(p\xi_{m_h} + \nu(\Theta_h)) = -1$.

□

13. The perturbated case

Let us consider the general case, where the exponential factor of the elementary $\mathcal{D}$-module is given by by $\varphi(u) = -\alpha u^{-n} - \sum_{j=1}^{n-1} \alpha_j u^{-j}$.

If so, stationary phase formula prescribes a ramification

$$\begin{aligned} w =&\hat{\rho}^{-1}(\zeta) := \frac{n\alpha}{p}\zeta^{-(n+p)} + \sum_{j=1}^{n-1}\frac{j\alpha_j}{p}\zeta^{-j-p} = \\ =&\frac{n\alpha}{p}\zeta^{-(n+p)}(1+\sum_{j=1}^{n-1}\frac{j\alpha_j}{n\alpha}\zeta^{n-j}) = \frac{n\alpha}{p}\zeta^{-(n+p)}(1+o(\zeta)). \end{aligned} \tag{13.1}$$

Notice that, since $1+\sum_{j=1}^{n-1}\frac{j\alpha_j}{n\alpha}\zeta^{n-j}$ is different from zero in $0 \in \mathbb{C}_\zeta$, we can consider one of its $n+p$-th roots h defined in a sufficiently small ball around 0.

Thanks to the change of coordinate $\zeta \to \tilde{\zeta} = \zeta h(\zeta)$, the ramification takes the form $\hat{\rho}^{-1}(\zeta) = \frac{n\alpha}{p}\zeta^{-(n+p)}$ around 0.

We can then perform the same computations as in section 8, by considering a small neighbourhood of 0 instead of the whole $\mathbb{C}_\zeta$ and with the same ρ_2: we obtain

$$K \simeq R\tilde{q}_{3!}(E^{-\tilde{\Psi}^{prt}} \otimes \tilde{p}_3^{-1}\pi^{-1}F)[1] \tag{13.2}$$

with

$$\tilde{\Psi}^{prt}(v,\zeta) = \alpha n\zeta^{-n}(\frac{v^{-n}}{n} + \frac{v^p}{p} + \sum_{j=1}^{n-1}\frac{\alpha_j}{\alpha n}v^{-j}\zeta^{n-j}). \tag{13.3}$$

Our original aim was to prove that the same feature and description of the Fourier transform given in the preceding subsections for the non-perturbated case extends to this generic case.

However, this goal still needs some work in order to be perfected.

In the following we will then describe the point reached with the computations.

In particular, we will show that we can obtain from the non-perturbated case (the one with $\alpha_j = 0\ \forall j$ described so far) results concerning the asymptotic behaviour of the generic transform on particular open subsets related to $\mathcal{AS}t_{\mathcal{T}rp}(\tilde{\Psi})$.

Moreover, we will show that the degeneration and consequent arise of tripods happens in the same way in this case along a particular subvariety related to $\mathcal{AS}t_{\mathcal{T}rp}(\tilde{\Psi})$.

We will proceed as follows: after proving interesting estimates concerning $\tilde{\Psi}^{prt}$ and $\tilde{\Psi}$, we will use them to provide a result about general behaviour for steepest descent paths of $\tilde{\Psi}^{prt}$. We will then focus on degeneration, and how the presence of tripods arises also in this case.

13.1. **Notations and observations for $\tilde{\Psi}^{prt}$.** As in the non perturbated case, it suffices to consider the study of the critical level sets of

$$\Psi^{prt}(v,\zeta) := e^{i(\mu-n\Theta)}(\frac{v^{-n}}{n} + \frac{v^p}{p} + \sum_{j=1}^{n-1}\frac{\alpha_j}{\alpha n}v^{-j}\zeta^{n-j})$$

Denote by $\{v_m(\zeta)\}$ the set of critical points of Ψ_ζ^{prt} and notice that, in a small neighbourhood of $0 \in \mathbb{C}_\zeta$, they can be approximated as

$$v_m(\zeta) = v_m + o(\zeta)$$

since $v_m(0) = v_m$. Denote by $\{\lambda_m(\zeta)\}$ the corresponding set of critical values, again, we have

$$\lambda_m(\zeta) = \lambda_m + o(\zeta).$$

We will more precise about the properties of the critical points and values in the following subsection. Notice moreover that, by similar considerations, the critical points of Ψ_ζ^{prt} are non-degenerate and the critical values are distinct for each ζ in a small neighbourhood of 0. Denote by $\mathcal{AS}t^{prt}_{\mathcal{T}rp}$ the set

$$\mathcal{AS}t^{prt}_{\mathcal{T}rp} := \{\zeta | \exists m \colon \operatorname{Im}\lambda_m(\zeta) = \operatorname{Im}\lambda_{m+1}(\zeta)\} \tag{13.4}$$

Notice that, since Ψ and Ψ^{prt} have the same most polar part with respect to ζ, $\mathcal{AS}t^{prt}_{\mathcal{T}rp}$ consists of the union of analytic curves abutting from 0 with tangent in $\mathcal{AS}t_{\mathcal{T}rp}$.

13.2. **Some estimates.**

Proposition 13.1. *Let* $0 < r_0 << 1, 0 << R_0$, $A = \max_{j=1,\dots,n-1}\{|\alpha_j|\}$ *and notation as above. Then*

(1) $\forall C > 0$ *we have*

$$|\Psi^{prt} - \Psi| \le C \text{ on } \{r_0 < |v| < R_0\} \times \{|\zeta| \le \frac{Cr_0^{n-1}}{A(n-1)}\} \tag{13.5}$$

(2) $\forall 0 < \bar{r} < \min\{\{|v_m - v_{m'}| : m \neq m' \pmod{n+p}\}, 1\}$ *we have*

$$\{\partial_v \Psi^{prt} = 0\} \subset (\bigcup B_{\bar{r}}(v_m)) \times \{|\zeta| \le M(\bar{r})\} \tag{13.6}$$

with $M(\bar{r}) \to 0$ *for* $\bar{r} \to 0$

(3) $\forall C > 0$ *there exists* $\bar{r}_0 > 0$ *such that*

$$|\lambda_m(\zeta) - \lambda_m| < C \text{ for } \zeta : |\zeta| \le M(\bar{r}_0) \tag{13.7}$$

with M *as in* (13.6).

Proof. Let K be a compact in $\mathbb{C}_v$ and suppose

$$\begin{aligned} K_{max} &:= \max_{v \in K} |v| > 1 \\ 0 < K_{min} &:= \min_{v \in K} |v| < 1 \end{aligned} \tag{13.8}$$

Notice that conditions on K_{max}, K_{min} imply

$$\begin{aligned} K_{min}^{-1} >> 1 &\Rightarrow K_{min}^{-j} < K_{min}^{-k} \text{ for } j < k \\ 0 < K_{max}^{-1} << 1 &\Rightarrow K_{max}^{-j} > K_{max}^{-k} \text{ for } j < k \end{aligned} \tag{13.9}$$

Since

$$\begin{aligned} |\Psi_\kappa(v)| &= |e^{i\kappa}(\frac{v^{-n}}{n} + \frac{v^p}{p})| = |v|^{-n}|\frac{1}{n} + \frac{v^{n+p}}{p}| \\ |\partial_v \Psi_\kappa(v)| &= |v^{p-1} - v^{-n-1}| = |v|^{-n-1}|v^{n+p} - 1| \end{aligned} \tag{13.10}$$

the absolute values do not depend on κ: we will hence simply write $|\Psi|$. We have the following estimates on K

$$
\begin{aligned}
|\Psi| \leq |v|^{-n}(\frac{1}{n}+\frac{|v|^{n+p}}{p}) < K_{min}^{-n}(\frac{1}{n}+\frac{K_{max}^{n+p}}{p}) \\
|\Psi| \geq K_{max}^{-n}\min\{|\frac{K_{max}^{n+p}}{p}-\frac{1}{n}|,|\frac{K_{min}^{n+p}}{p}-\frac{1}{n}|\} \\
|\partial_v\Psi| \leq K_{min}^{-n-1}(K_{max}^{n+p}+1) \\
|\partial_v\Psi| \geq K_{max}^{-n-1}\min\{|K_{max}^{n+p}-1|,|K_{min}^{n+p}-1|\}
\end{aligned}
\tag{13.11}
$$

We have moreover the following estimate again on K for $|\zeta|<1$

$$
\begin{aligned}
|\Psi^{\underline{a}}-\Psi| = |\sum_{j=1}^{n-1}\frac{\alpha_j}{n}v^{-j}| \leq \sum_{j=1}^{n-1}|\frac{\alpha_j}{n}||v|^{-j}|\zeta|^{n-j} < \\
< AK_{min}^{-n+1}\sum_{j=1}^{n-1}|\zeta|^{n-j} < AK_{min}^{-n+1}(n-1)|\zeta|
\end{aligned}
\tag{13.12}
$$

where A is the real number $\max_{j=1,\dots,n-1}\{|\frac{\alpha_j}{\alpha n}|\}$

If we apply this last estimate to $K=\{r_0 \leq |v| \leq R_0\}$ for $0<r_0<<1$ and $R_0>>1$ we obtain

$$
|\Psi^{\underline{a}}-\Psi| < C \text{ if } |\zeta| < \frac{Cr_0^{n-1}}{A(n-1)}
\tag{13.13}
$$

for $C>0$.

Notice that the conditions on $\bar{r}, r_0, R_0$ imply

$$
\begin{aligned}
0<(1-\bar{r})<1 \Rightarrow (1-\bar{r})^{-1}>1, \\
1<(1+\bar{r})<2 \Rightarrow \frac{1}{2}<(1+\bar{r})^{-1}<1,
\end{aligned}
\tag{13.14}
$$

and that

$$
\max_{v\in\partial B_{\bar{r}}}|v| = 1+\bar{r}, \ \min_{v\in\partial B_{\bar{r}}}|v| = 1-\bar{r}.
\tag{13.15}
$$

Now we want to show that the critical point $v_m(\zeta)$ is as near as we want to $e^{i\theta_m}$ for $|\zeta|$ small enough. We will achieve this via the following classical theorem

Theorem 13.2 (Rouche)**.** *Let $U \subset \mathbb{C}_v$, $f,g: U \to \mathbb{C}$ be two holomorphic function, suppose that ∂U is a simple contour (i.e., without self-intersections). If $|g(v)|<|f(v)|$ on ∂U, then f and $f+g$ have the same number of zeroes inside U, where each zero is counted as many times as its multiplicity.*

More in details, consider the functions

$$
f = \partial_v\Psi = e^{i\kappa}(-v^{-(n+1)}+v^{p-1})
\tag{13.16}
$$

and

$$
g = \partial_v(\Psi^{\underline{a}}-\Psi) = -\sum_{j=1}^{n-1} j\frac{\alpha_j}{n}v^{-j-1}\zeta^{n-j}.
\tag{13.17}
$$

on $U = B_{\bar{r}} := e^{i\theta_m} + B(0, \bar{r})$. We want to find suitable conditions on $|\zeta|$ such that the hypotesis of the theorem holds. Choose $\bar{r}$ satisfying

$$0 < \bar{r} < \min\{\{|e^{i\theta_m} - e^{\theta_{m'}}| : m \neq m' \pmod{n+p}\}, 1\} \tag{13.18}$$

and notice that

$$\max_{v \in \partial B_{\bar{r}}} |v| = 1 + \bar{r} > 1, \ \min_{v \in \partial B_{\bar{r}}} |v| = 1 - \bar{r} < 1 \tag{13.19}$$

hence the estimates above hold. The condition about $\bar{r}$ also implies that

$$\begin{gathered} 1 < 1 + \bar{r} < 2 \Rightarrow 2^{-1} < (1+\bar{r})^{-1} < 1 \\ 0 < 1 - \bar{r} < 1 \Rightarrow (1-\bar{r})^{-1} > 1 \end{gathered} \tag{13.20}$$

By (13.12) and the Cauchy estimate on the first derivative we get

$$|\partial_v(\Psi^{\underline{a}} - \Psi)| < \frac{A(1-\bar{r})^{-n+1}(n-1)}{\bar{r}}|\zeta| \tag{13.21}$$

Consider now the second estimate in (13.11). We need to compute the term $\min\{|K_{max}^{n+p} - 1|, |K_{min}^{n+p} - 1|\}$. In this case

$$\begin{gathered} |K_{max}^{n+p} - 1| = (1+\bar{r})^{n+p} - 1 \\ |K_{min}^{n+p} - 1| = 1 - (1 - (1-\bar{r})^{n+p}) \end{gathered} \tag{13.22}$$

Since

$$\begin{gathered} (1+\bar{r})^{n+p} - 1 = \bar{r} \sum_{k=0}^{n+p-1} (1+\bar{r})^k \\ 1 - (1 - (1-\bar{r})^{n+p}) = \bar{r} \sum_{k=0}^{n+p-1} (1-\bar{r})^k \end{gathered} \tag{13.23}$$

we have $(1+\bar{r})^{n+p} - 1 > 1 - (1-\bar{r})^{n+p}$ and the minimum we are looking for is the latter.

The estimate we get is then

$$|\partial_v \Psi| \geq (1+\bar{r})^{-n}(1 - (1-\bar{r})^{n+p}) > 2^{-n}(1 - (1-\bar{r})^{n+p}) \tag{13.24}$$

The condition for Rouche's theorem then rewrites as

$$\frac{A(1-\bar{r})^{-n+1}(n-1)}{\bar{r}}|\zeta| < 2^{-n}(1 - (1-\bar{r})^{n+p}) \tag{13.25}$$

Hence, for

$$|\zeta| < \frac{\bar{r}(1-\bar{r})^{n-1}(1 - (1-\bar{r})^{n+p})}{(n-1)2^n A} \tag{13.26}$$

Rouche theorem applies to $\partial_v(\Psi^{\underline{a}} - \Psi)$ and $\partial_v \Psi$. The two functions have then the same number of zeroes in $B_{\bar{r}}$: since there is only one zero with multiplicity 1 for the first, the same holds for the second, proving what claimed.

Lastly, let us show the following

$$|\Psi(e^i\theta_m) - \Psi^{\underline{a}}(v_m(\theta))| < C \tag{13.27}$$

for $C > 0$ and $|\zeta|$ satisfying (13.26) for $\bar{r} > 0$ small enough. Notice that, under these conditions, $v_m(\theta) \in B_{\bar{r}}$ and hence its absolute value satisfies all inequalities from belonging to that ball.

The inequality can be rewritten as

(13.28)
$$\begin{aligned}
&|F_\kappa(e^i\theta_m) - \tilde{F}_\kappa(v_m(\theta))| = \\
&|F_\kappa(e^i\theta_m) - F_\kappa(v_m(\theta)) + F_\kappa(v_m(\theta)) - \tilde{F}_\kappa(v_m(\theta))| \leq \\
&\leq |F_\kappa(e^i\theta_m) - F_\kappa(v_m(\theta))| + |F_\kappa(v_m(\theta)) - \tilde{F}_\kappa(v_m(\theta))| \leq \\
&\leq \max_{\partial B_{\bar{r}}} |F'_\kappa| \cdot \bar{r} + \max_{\partial B_{\bar{r}}} |\tilde{F}_\kappa - F_\kappa| \leq \\
&\leq \bar{r}[(1+\bar{r})^{n+p} + 1] + A(1-\bar{r})^{-n+1}(n-1)|\zeta| \leq \\
&\leq \bar{r}[(1+\bar{r})^{n+p} + 1] + A(1-\bar{r})^{-n+1}(n-1)\frac{\bar{r}(1-\bar{r})^{n-1}(1-(1-\bar{r})^{n+p})}{(n-1)2^n A} \leq \\
&\leq \frac{\bar{r}}{(1-\bar{r})^n}[1 + (1+\bar{r})^{n+p} + 2^{-n}(1-\bar{r})^n(1-(1-\bar{r})^{n+p})]
\end{aligned}$$

Let us denote this last term by $M(\bar{r})$, it is then easy to notice that

$$M(0) = 0 \text{ and } M(\bar{r}) > 0,\ \forall 0 < \bar{r} < 1$$
$$\lim_{\bar{r}\to 1} M(r) = +\infty$$
$$\frac{dM}{d\bar{r}}(0) > 0$$

These conditions imply that there exists a right neighbourhood of 0 (i.e. of the kind $]0, \bar{r}_0]$) in which M is positive and increasing, in particular a diffeomorphism on such interval. This means the inequality holds

(1) for any $C > M(\bar{r}_0)$ on $B_{\bar{r}}$ for any $\bar{r} < \bar{r}_0$,
(2) for any $M(\bar{r}_0) > C > 0$ on $B_{\bar{r}}$ for any $\bar{r} < M^{-1}(C)$

i.e. for any $C > 0$ and sufficiently near each non-perturbated critical point $e^{i\theta_m}$. □

13.3. **Consequences of the estimates.** In this subsection, we will derive, as consequence of the estimates just obtained, that the behaviour of level sets for the perturbated case is indeed the same as in the absence of perturbation.

In particular, we want to show what happens away from directions in $\mathcal{AS}t_{\mathcal{T}rp}(\Psi)$, denote by S^c_Θ the closed sector in $\mathbb{C}^\times_\zeta$ delimited by $\Theta \pm c$

Proposition 13.3. *There exists $d_0 > 0$ such that, $\forall d_0 > d > 0\ \exists c = c(d) > 0$ with $c \to 0$ as $d \to 0$ such that*

(13.29)
$$\forall \zeta \in \{|\zeta| < d\} \setminus \bigcup_{\Theta \in \mathcal{AS}t_{\mathcal{T}rp}(\Psi)} S^c_\Theta$$

the steepest descent path for $\operatorname{Re}\Psi^{prt}_\zeta$ *are deformations of the steepest descent paths for* $\operatorname{Re}\Psi_\zeta$ *satisfying the following properties:*

- *they have the same endpoints and tangents,*
- *they are contained in tubular neighbourhood of steepest descent paths for* $\operatorname{Re}\Psi$*,*
- *the critical point $v_m(\zeta)$ is contained in a small ball around v_m.*

Proof. First of all, notice that, since $\operatorname{Im}\Psi^{prt}_\zeta$ and $\operatorname{Im}\Psi_\zeta$ have the same polar structure at the singular points $0, \infty$, the two functions have the same level set structure near such points.

Let us fix a radius $d_0 > 0$ which satisfies both the requirement for h to be well defined and for $\mathcal{AS}t^{prt}_{\mathcal{T}rp}(\Psi^{prt})$ to be written as the disjoint union of analytic curves.

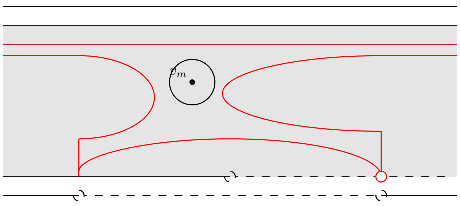
We will restrict our focus at a singular critical point $v_m(\zeta)$. As a consequence of Proposition 13.1, we have the following properties

- $\exists C > 0$ such that

$$\{\operatorname{Im}\Psi^{prt}_\zeta = \operatorname{Im}\lambda_m(\zeta)\} \subset \{\operatorname{Im}\lambda_m(\zeta) - C < \operatorname{Im}\Psi < \operatorname{Im}\lambda_m(\zeta) + C\}$$

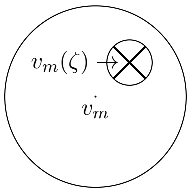
- $\exists \bar{r} > 0$ such that $v_m(\zeta) \in B_{\bar{r}}(v_m)$
- for C, r as above $B_{\bar{r}}(v_m) \subset \{\operatorname{Im}\lambda_m(\zeta) - C < \operatorname{Im}\Psi < \operatorname{Im}\lambda_m(\zeta) + C\}$

By restricting $\bar{r}$ if necessary, we can always suppose that $B_{\bar{r}}$ is a subset of the elementary sector containing v_m. In the picture above we have an example of the situation in the case $\lambda_m(\zeta) > 0$.



Since $\operatorname{Im}\lambda_m(\zeta) - C < \operatorname{Im}\lambda_m < \operatorname{Im}\lambda_m(\zeta) + C$, $\{\operatorname{Im}\lambda_m(\zeta) - C < \operatorname{Im}\Psi_\zeta < \operatorname{Im}\lambda_m(\zeta) + C\}$ is a tubular neighbourhood of the critical level set $\{\operatorname{Im}\Psi_\zeta = \lambda\}$.

Furthermore, we know the structure of the level set $\{\operatorname{Im}\Psi^{prt}_\zeta = \operatorname{Im}\lambda_m(\zeta)\}$ in a small neighbourhood U_m around $v_m(\zeta)$: it is the union of orthogonally intersecting analytic curves (see Lemma 6.4).



The resulting four branches exiting from U_m have to connect with the germ of the level set at the singular points without self-intersecting and remain in the tubular neighbourhood defined above.

The statement about the steepest descent then follows from noticing that the behavior of $\operatorname{Re}\Psi$ and $\operatorname{Re}\Psi^{prt}$ near the singular points $0, \infty$ is the same, hence

□

It remains to show what happens at $\mathcal{AS}t^{prt}_{\mathcal{T}rp}(\Psi^{prt})$. We have already noticed that this variety is contained in sectors around directions in $\mathcal{AS}t_{\mathcal{T}rp}(\Psi)$.

Proposition 13.4. *For $\zeta \in \mathcal{AS}t^{prt}_{\mathcal{T}rp}(\Psi^{prt})$, the steepest descent paths for Ψ^{prt}_ζ degenerate in a tripod $T^{prt}_{m,m+1}$ for steepest descent associated with $v_m(\zeta), v_{m+1}(\zeta)$ and $m \in \mathcal{T}rp_{\nu(\Theta)}$.*

Proof. Since $\operatorname{Im}\lambda_m(\zeta) = \operatorname{Im}\lambda_{m+1}(\zeta) =: \lambda$, we have

(13.30)

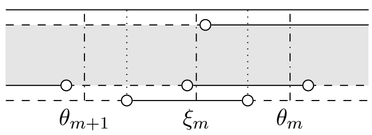
$$\{\operatorname{Im}\Psi_\zeta^{prt} = \operatorname{Im}\lambda_m(\zeta)\}, \{\operatorname{Im}\Psi_\zeta^{prt} = \operatorname{Im}\lambda_{m+1}(\zeta)\} \subset \{\operatorname{Im}\lambda - C < \operatorname{Im}\Psi < \operatorname{Im}\lambda + C\}$$

Moreover, by above consideration, we know that $\mathcal{AS}t^{prt}_{\mathcal{T}rp}(\Psi^{prt}) \subset S^h_{\Theta_0}$. Hence we have 6 possibilities for Θ:

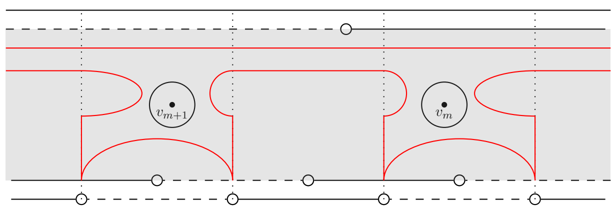
- $\Theta < \Theta_0$ and $\sin(p\theta_m + \nu(\Theta_0)) = \pm 1$,
- $\Theta = \Theta_0$ and $\sin(p\theta_m + \nu(\Theta_0)) = \pm 1$,
- $\Theta > \Theta_0$ and $\sin(p\theta_m + \nu(\Theta_0)) = \pm 1$.

It is sufficient to focus on one of them, the others can be dealt in a similar way.

Suppose then $\Theta < \Theta_0$ with $\sin(p\theta_m + \nu(\Theta_0)) = 1$, the configuration in the model is then as pictured below



The situation in the two extremal sectors have already been dealt in the previous Proposition, we have only to notice that the two local situations are linked by a tunnel passing in the intermediate sector.



Both the critical level sets have to enter the tunnel and reach the opposite critical point: the branch is then forcibly unique by the requirement and a tripod $T^{prt}_{m,m+1}$ arises in the same fashion as $T_{m,m+1}$. □

13.4. **An observation.** We want to stress out the following fact: we are able to deduce properties of the steepest descent paths for the perturbated case in term of the non-perturbated one only away from $\mathcal{AS}t^p_{\mathcal{T}rp}rt$ and $\mathcal{AS}t_{\mathcal{T}rp}$.

Indeed, our estimates do not allow the same study made above for points neighbouring such varieties.

As an example, just notice that tripods arise in different points depending on perturbated/non-perturbated case.

Theoretically, for points contained between a branch of $\mathcal{AS}t^p_{\mathcal{T}rp}rt$ and the line in $\mathcal{AS}t_{\mathcal{T}rp}$ having the same tangent at 0, the steepest descent cycles are no more deformation of the other.

Since an endpoint of the steepest descent path changes at a tripodal direction and we have the same behaviour on one side of the tripodal variety, there must be in general a different behaviour for steepest descent for points in the middle of the two subvarieties by continuity.

13.5. **Real blow-up.** In order to express what we can achieve from Proposition 13.3, we will the language and framework provided by the real blow-up of $\mathbb{C}_\zeta$ at 0.

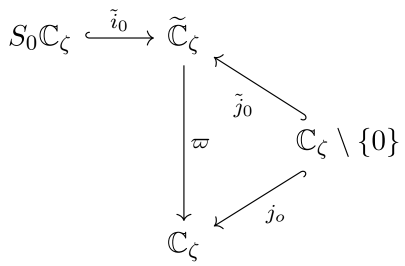
The total real blow-up

$$\varpi^{tot} : \widetilde{\mathbb{C}}^{tot}_\zeta \to \mathbb{C}_\zeta$$

of $\mathbb{C}_\zeta$ at 0 is the map of smooth manifolds locally defined as follows

$$\begin{aligned} \widetilde{\mathbb{C}}^{tot}_\zeta &= \{(\epsilon, \Theta) \in \mathbb{R} \times S^1\} \\ \varpi^{tot} : \widetilde{\mathbb{C}}^{tot}_\zeta &\to \mathbb{C}_\zeta, \quad (\epsilon, \Theta) \to \epsilon e^{i\Theta} \end{aligned} \tag{13.31}$$

The real blow up $\widetilde{\mathbb{C}}_\zeta$ of $\mathbb{C}_\zeta$ at 0, is the closed subset $\{\epsilon \geq 0\}$ of $\widetilde{\mathbb{C}}^{tot}_\zeta$. Setting $\varpi = \varpi^{tot}_{|\widetilde{\mathbb{C}}_\zeta}$, consider the commutative diagram



where $S_0\mathbb{C}_\zeta = \varpi^{-1}(0) \simeq S^1$ is the sphere of tangent directions at 0. Let $\Theta \in S_a\mathbb{C}_\zeta$ and $V \subset \mathbb{C}_\zeta$. One says that V is a sectorial neighborhood of Θ if $V \subset \mathbb{C}_\zeta \setminus \{0\}$ and $S_0\mathbb{C}_\zeta \cup \varpi^{-1}(V)$ is a neighborhood of Θ in $\widetilde{\mathbb{C}}_\zeta$. This is equivalent to saying that $V = \varpi^{-1}(0)$ for some neighborhood U of Θ in $\widetilde{\mathbb{C}}_\zeta$. For $\Theta \in S_0\mathbb{C}_\zeta$, we will:

(1) write for short $x \to \Theta$ instead of $\tilde{j}_0(x) \to \Theta$,
(2) write $\Theta \,\dot{\in}\, V$ to indicate that V is a sectorial neighborhood of Θ.

One says that $U \subset \mathbb{C}_\zeta \setminus \{0\}$ is a sectorial neighborhood of $I \subset S_a\mathbb{C}_\zeta$ if $U \,\dot{\ni}\, \Theta$ for any $\Theta \in I$.

13.6. **Conclusions.**

Proposition 13.5. *For all* $\Theta_h, \Theta_{h+1} \in \mathcal{AS}t_{\mathcal{T}rp}(\tilde{\Psi})$ *there exists a sectorial neighbourhood* V_h *of* $]\Theta_h, \Theta_{h+1}[$ *at each point of which steepest descent paths for* $\tilde{\Psi}^{prt}_\zeta$ *satisfy the same properties described in Proposition 13.3.*

Proof. For all $d < d_0$, the space $\{|\zeta| < d\} \setminus \bigcup_{\Theta \in \mathcal{AS}t_{\mathcal{T}rp}(\Psi)} S^c_\Theta$ consists of the union over h of open sectors

$$\{\zeta \colon |\zeta| < d, \Theta_h + c < \Theta < \Theta_{h+1} - c\}$$

. The stated neighbourhood V_h is then the union over $d < d_0$ of these sectors for a fixed h. Notice that, as $c \to 0$ for $d \to 0$, the arising V_h is a sectorial neighbourhood of $]\Theta_h, \Theta_{h+1}[$ as stated. ☐

In this sense, with our estimates we are able to deduce the behaviour asymptotic to consecutive components of $\mathcal{AS}t^{prt}_{\mathcal{T}rp}(\tilde{\Psi}^{prt})$ at 0 of the Fourier transform in the perturbated case.

We can then clearly extend Theorem 11.1 to the perturbated case using the sectorial neighbourhoods provided by the above Proposition.

Let $\tilde{\Psi}^{prt}_m(\zeta) = \tilde{\Psi}(v_m(\zeta), \zeta)$, we have the following

Theorem 13.6. *Let $V_h \subset \mathbb{C}^\times_\zeta$ be one of the sectorial neighbourhood described in Proposition 13.5. Then we have an isomorphism in $E^b_{\mathbb{R}-c}(\boldsymbol{k}_{\mathbb{C}^\times_{\zeta,\infty}})$*

$$\pi^{-1}\boldsymbol{k}_{V_h} \otimes K \xrightarrow[\sim]{\sigma} \pi^{-1}\boldsymbol{k}_{V_h} \otimes \bigoplus_{m=0}^{n+p-1} (V \otimes E^{-\tilde{\Psi}^{prt}_m})$$

Notice that, in order to compute the Stokes matrices, we need the elements of a covering of 0 where the enhanced sheaf K has a trivialization to have non-trivial intersection.

In the non-perturbated case, this was achieved by closed sectors with common sides in $\mathcal{AS}t_{\mathcal{T}rp}(\tilde{\Psi})$; in this case, our covering with sectorial neighbourhoods has no intersection where to compare the jumps in the steepest descent paths.

Following the spirit of this work and Proposition 13.4, the comparison should happen at the curves constituting $\mathcal{AS}t^{prt}_{\mathcal{T}rp}(\tilde{\Psi}^{prt})$.

The missing point is then how to extend the different trivializations coming from Theorem 13.6 to be comparable on $\mathcal{AS}t^{prt}_{\mathcal{T}rp}(\tilde{\Psi}^{prt})$ where.

Once this issue is solved, it will be clear that the Stokes matrices for K are exactly the ones described in **??**.

References

[1] Alexander Beilinson, Spencer Bloch, Pierre Deligne and Hélène Esnault. (2005). *Periods for irregular connections on curves*, preprint

[2] Spencer Bloch and Hélène Esnault. (2004). *Homology for irregular connections*, J. Theor. Nombres Bordeaux **16**, 357–371

[3] Armand Borel and John Coleman Moore. (1960). *Homology theory for locally compact spaces*. Michigan Math. J., Volume 7, Issue 2, 137-159

[4] Andrea D'Agnolo and Masaki Kashiwara. (2016). *Riemann-Hilbert correspondence for holonomic D-modules*. Publ. Math. Inst. Hautes Études Sci. **123**, 69–197

[5] Andrea D'Agnolo and Masaki Kashiwara. (2017). *A microlocal approach to the enhanced Fourier-Sato transform in dimension one*. Advances in Mathematics **339**, 1–59.

[6] Andrea D'Agnolo, Marco Hien, Giovanni Morando and Claude Sabbah. (2017). *Topological computation of some Stokes phenomena on the affine line*, `arXiv:1705.07610`

[7] Charles Ehresmann. (1950). Les connexions infinitésimales dans un espace fibré différentiable, Colloque de Topologie, Bruxelles, pp. 29–55

[8] Jiangxue Fang. (2009). *Calculation of local Fourier transforms for formal connections*, Sci. China Ser. A **52**, no. 10, 2195–2206

[9] Marco Hien and Claude Sabbah. (2014). *The local Laplace transform of an elementary irregular meromorphic connection*. Rend. Semin. Mat. Univ. Padova **134**, 133–196.

[10] Masaki Kashiwara and Pierre Schapira. (1990). Sheaves on manifolds, Springer, Grundlehren der Mathematischen Wissenschaften **292**.

[11] Masaki Kashiwara and Pierre Schapira. (2014). *Irregular holonomic kernels and Laplace transform*. Selecta Mathematica **22**, no. 1, 55–109.

[12] Nicholas M. Katz and Gérard Laumon. (1985). *Transformation de Fourier et majoration de sommes exponentielles*, Inst. Hautes Études Sci. Publ. Math., **62**, 361–418.

[13] Bernard Malgrange. (1991). Équations différentielles à coefficients polynomiaux, Birkhauser, Advances in Mathematics, 232 pp.

[14] John Milnor. (1963). Morse Theory. Annals of Mathematical Studies **51**, Princeton University Press.

[15] Takuro Mochizuki. (2010). *Note on the Stokes structure of Fourier transform.* Acta Math. Vietnam. **35**, no. 1, 107–158

[16] Takuro Mochizuki. (2018). *Stokes shells and Fourier transforms.* `arXiv:1808.01037`

[17] Laurence Perko. (2001). *Differential equations and dynamical systems.* Springer-Verlag New York, Texts in Applied Mathematics **7**.

[18] Frédéric Pham. (1985). *La descente des cols par les onglets de Lefschetz avec vue sur Gauss-Manin.* in Differential systems and singularities (Luminy, 1983), Astérisque **130**, 11–47.

[19] Claude Sabbah. (2007). *An explicit stationary phase formula for the local formal Fourier-Laplace transform.* in Singularities I, Contemp. Math.

[20] Claude Sabbah. (2013). *Introduction to Stokes structures.* Lecture Notes in Mathematics **2060**, Springer, xiv+249 pp.

[21] Edward Witten. (2011). *Analytic Continuation Of Chern-Simons Theory.* In Chern-Simons gauge theory: 20 years after, AMS/IP Stud. Adv. Math. **50**, AMS, 347–446

[22] Edward Witten. (1982). *Supersimmetry and Morse theory.* J. Differential Geom., Volume 17, Number 4, 661-692