

## 2.

### **Individual level diffusion models: the Cellular Automata approach**

#### 2.1 Introduction

In the review proposed by Mahajan, Muller and Bass (1990) it is pointed out that a crucial aspect of the Bass model is that it focuses on markets in the aggregate: in fact, the typical observed variable is the cumulative number of adopters who have bought the product at a given time. The aggregate approach allows a direct understanding of the structure and evolution of an entire context of diffusion, a level in which managers are obviously interested, just estimating three parameters, and is particularly suitable for sales' forecasting, since aggregate models can be estimated with market level data, relatively easy to obtain. Such an attractive parsimony may be a sufficient reason for explaining the huge body of literature on aggregate diffusion models, emerged after the publication of the Bass model: actually part of this literature was aimed at extending the structure of it in order to include marketing mix effects (Bass, Krishnan and Jain, 1994) and other phenomena related to consumer behaviour, such as repeated purchases, stages of awareness in new product adoption (Sharif and Ramanathan, 1982; Mahajan, Muller and Kerin, 1984; Kalish, 1985) and non uniform word-of-mouth effects (Easingwood, Mahajan and Muller, 1983). The aggregate approach to diffusion represents the earliest and most influential for studying the dynamics of new product adoption: however, many attempts were made beginning from the 1970s for explaining adoption behaviour also at the individual level. Indeed, the decision to adopt is individual specific and all the

potential adopters probably do not have the same probability to adopt (see Roberts and Urban, 1988): a possible approach for dealing with this aspect may consider classical utility assumptions taken from economic theory and then transform them into probabilities of purchase. Models that in the past tried to describe diffusion by specifying adoption decisions at individual level were proposed by Hiebert (1974), Stoneman (1981), Feder and O'Mara (1982), Jensen (1982), Oren and Schwartz (1988), Roberts and Urban (1988), Chatterjee and Eliashberg (1989). Despite these attempts, Mahajan, Muller and Bass (1990) still indicated the linkage between diffusion at aggregate level and adoption decisions at individual one as an open issue that needed to be addressed by future research. Roberts and Lattin (2000) have provided a notable review on disaggregate level diffusion models, synthesising the contributions in modeling terms given on this topic. Interestingly, they notice that describing individual behaviour may require various parameters and even more if this behaviour changes over time: in particular, they say, if there is heterogeneity on all dimension of consumer behaviour, then the number of involved parameters could be multiplied by up to the size of the population considered. As a consequence, individual level models do not present the desired characteristic of parsimony. Moreover, the availability of data may seriously limit their empirical application. In general they are composed of three elements, a utility function, an updating process like bayesian learning and a choice decision based on that utility: all this information is typically not available, which renders such models essentially intractable in empirical terms.

In their recent review on new product growth, Muller, Peres and Mahajan (2007) maintain that individual level diffusion is still in its early phases of development: though aggregate diffusion models describe the collective result of many individual choices, the

combination of these two dimensions is not so easy to obtain. The attempts made in the 1970s and 1980s often proposed too rich models that finally ended in rather cumbersome descriptions of a single agent behaviour. Reminding the crucial aspect of empirical tractability, Muller, Peres and Mahajan suggest that individual adoption should be studied from a perspective that can be later generalised into an aggregate model, since most of the data available come in aggregate form. As a consequence, the interface between the two levels should be explored, making individual level models a standard tool for research in diffusion. Interestingly, we may notice in this review a new perspective on how individual level models should be used: though heterogeneity among agents is still the main driver for building individual models, its formal specification chooses to abandon particular characterizations of single agent behaviour, like learning procedures and utility functions, that ultimately are not empirically testable. On the contrary, *heterogeneity among agents should be explored for the consequences it has on the aggregate level*. A better connotation of individual behaviour should serve the purpose of detecting phenomena not explicitly discussed in the Bass model, like disadoptions, resistance to innovation, social network formation, network externalities, spatial connectivity among agents, that indeed have effect on new product growth. In other words, according to the point of view expressed by Muller, Peres and Mahajan (2007), *individual level models should be employed for explaining phenomena that affect diffusion and may depend on individual agents' choices*.

One of the most recent proposals for dealing with individual level diffusion suggests the use of the Complex System Analysis, traditionally employed in natural sciences, like biology, ecology, physics and mathematics (see for example Goldenberg, Libai and Muller, 2001a).

Defining a complex system is not a simple matter. However, there is a certain agreement about the properties a system must have to be considered complex. Following Boccara (2004) we say that a system qualified as complex is composed of a large number of interacting agents that dispose only of *local information*, exhibits the property of *emergence*, i.e. the result of these interactions is visible only at the macro level of analysis and this emergent behaviour does not result from the existence of a social controller (self-organizing behaviour). The term complex system is generally used to indicate a very broad and interdisciplinary field of research including neuroscience, physics, mathematics, biology, computer science, social sciences and economics. Although the interest and application of Complex System Analysis in economics is quite recent, the number of issues addressed in literature is expanding rapidly.

In quantitative marketing research, the perceived complexity of organizations and markets in which many agents interact between them has suggested to ground on Complex System Analysis and in particular on Agent-Based Models like Cellular Automata Models for improving the understanding of the collective behaviour we normally inspect in aggregate data. Cellular Automata models have been recently employed to investigate various diffusion related phenomena: for example Goldenberg, Libai and Muller (2001a) have treated the problem of consumers' heterogeneity, in terms of their individual probability of adoption, Goldenberg, Libai and Muller (2001b) have tried to analyze the word-of-mouth effect as a complex adaptive system, Moldovan and Goldenberg (2004) the problem of resistance to innovations, Goldenberg and Efroni (2001) the emergence of innovations. Garber, Goldenberg, Libai and Muller (2004) have proposed the use of Cellular Automata for the early prediction of new product

success, while Goldenberg, Libai and Muller (2005) have used these models to analyze the issue of network externalities.

This stream of research in marketing shares several purposes and modelling tools with another in economics called Agent-Based computational economics, ACE. The purpose of ACE is to study and model economies as evolving systems of interacting agents (see Tesfatsion, 2001): thus it may be fairly considered a specific field of economics based on the complex systems paradigm (Holland, 1992). Following a tradition opened by Smith (1937) and Hayek (1948), ACE focuses on economies as self-organized systems and aims to study processes in which global regularities and social order can emerge and persist in decentralized market economies despite the absence of top-down planning and control (Tesfatsion, 2001). The challenge of ACE is to demonstrate *constructively* how these global patterns might arise from the *bottom-up*, through repeated local interactions of autonomous agents.

Combining the perspectives developed by these recent fields of research in economics and marketing, this chapter analyses Cellular Automata models as a valid tool for a better investigation of diffusion related phenomena, whose effect may be explored and detected with a bottom-up analysis. Section 2.2 presents the characteristics of Cellular Automata models and illustrates an example of a basic diffusion framework represented with them. Some aspects on the implementation of these models are discussed, arguing that the typical techniques based on computer simulations may be a powerful tool to be completed with statistical analysis, rather than a full substitute of it. Section 2.3 proposes a particular class of Cellular Automata models, that allows to represent in a constructive way the existing relationship between Agent-based models and aggregate models for innovation diffusion. Section 2.4 grounds on these results for

designing a new aggregate diffusion model that, through an explicit use of individual level rules of behaviour, takes into account disadoption dynamics. Section 2.5 is left for some concluding remarks, summarizing the results obtained and the perspectives of research.

## 2.2 Cellular Automata models

Proposed for the first time by Ulam and Von Neumann in the 1950s and developed during the 1980s by Wolfram, that dedicated to them the famous book “A New Kind of Science” (2002), Cellular Automata models, CA, have fascinated many researchers from various disciplines, looking for a formal structure able to describe *global situations arising from local interactions* between the members of a certain population (see Wolfram, 1983). These members may be biologic organisms in ecosystems, automobiles in traffic, units in a physical system, individual agents in a particular social context. In 1970 the mathematician Conway proposed his famous *Game of Life*, receiving widespread interest among researchers. Since then, the increasing popularity of CA has been certainly related to their simplicity on the one hand and to the powerful ability to represent complex systems on the other.

These models are normally composed of an environment in which individuals locally interact between them, given a procedural rule that defines the way these interactions can occur. The implementation of such models consists in tracking the characteristics of each individual through time. A CA is composed of a finite grid of cells, that can be placed in a long line (a one-dimensional CA), in a rectangular array or even occasionally in a three dimensional cube. Cells represent individual units and each of them is in a specific state, given a finite number of  $k$  possible states like  $k = 2$ , adopter

or neutral, alive or dead, on or off. The state of each single cell can change in discrete time according to a given *transition rule* or procedural rule, which specifies how that state depends on the previous state of that cell and the states of the cell's immediate neighbours. The same rule is used to update the state of every cell in the grid, so that the model is homogeneous with respect to rules (Gilbert and Troitzsch, 1999). If we define, following Boccara and Fuks (1999), with  $s(i,t)$  the state of cell  $i$  at time  $t$  we will have

$$s(i,t+1) = f(s(i-r,t), s(i-r+1,t) \dots s(i,t), s(i+1,t) \dots s(i+r,t)) \quad (1)$$

This equation, called Cellular Automaton of ray  $r$ , in which function  $f$  represents a quite general transition rule, tells that the state of cell  $i$  at time  $t+1$ , depends on the state at time  $t$  of the cells being within the ray  $r$ . In the simplest version of the model the number of possible states is  $k=2$ , so that in an innovation diffusion context there are two possible types of individuals, adopters,  $s(i,t)=1$  and neutrals,  $s(i,t)=0$ . Cellular Automata are suitable for modeling situations in which interactions are local (individual), because the rules make reference to the state of other cells in a cell's neighbourhood. For example, analysing the codes proposed by Wolfram (1986), we can find a rule (254) stating that an individual adopts if at least, one of his neighbours is an adopter. Consider this simple one-dimensional CA in which the clusters of zeros (neutrals) are separated by clusters of ones (adopters).

1	<u>0</u>	0	<u>0</u>	1	1	<u>0</u>	0	0	<u>0</u>	1
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Only neutral cells adjacent to a cluster of ones change their state (underlined), while the others remain neutral. Wolfram (1986) has proposed many other elementary rules governing the changing state of cells in one-dimensional CA, showing that even these

simple rules can reproduce complex behaviour. The common element of these rules is that the system evolves and neutral cells become adopters if there exists at least one cell already active at the beginning of the process. In the case of two-dimensional grids, cells' neighbourhoods generally take two possible configurations, termed *Moore* and *Von Neumann* neighbourhood respectively. The first one considers the eight cells surrounding a specific cell, while the second just takes the cells directly above, below, to the left and to the right of the cell.

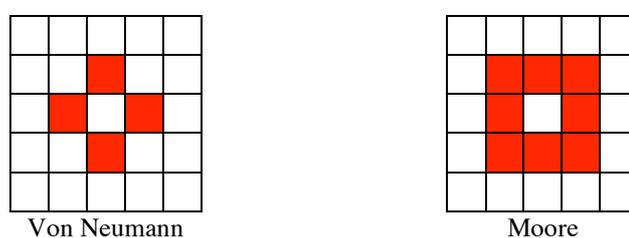


Figure 1. Von Neumann and Moore neighbourhoods: cells in colour are “active”

One of the most famous applications of CA is the *Game of Life*, proposed by Conway. In the Game of Life a cell survives if two or three of its neighbours are alive. If there are more or fewer neighbours alive then the cell dies, from either overcrowding or loneliness. Moreover, a dead cell is brought back to life if it has three living neighbours (see Gilbert and Troitzsch, 1999). From these simple rules, many patterns of global behaviour can emerge. Indeed, despite the relatively simple construction of Cellular Automata models, they are able to depict complex situations. For most of them, a general method for describing the global behaviour arising from the individual evolution of each cell is to run computer simulations with various initial configurations. Thus, a major direction for research has been to study the so-called *Local to Global Mapping*, i.e. the evolving dynamics of a CA at different time steps and with different

initial configurations. For a detailed review on this topic see Ganguly, Sikdar, Deutsch, Canright, Chaudhuri (2003). Once the purpose of characterizing global behaviour from local interactions has been achieved, one should be able to reconstruct the inverse path, that is the *Global to Local Mapping*. The so-called “inverse problem” is aimed at finding the local rules of a CA from a given global behaviour and is considered an extremely difficult task. One of the methodologies currently employed to address this aspect relies on evolutionary computation techniques, like genetic algorithms.

### 2.2.1 An explanatory example: innovation diffusion in a simple CA

Cellular Automata models are increasingly being used in economics and social science to study specific processes by which social order can emerge from self-interested micro behaviour. Pioneering works were proposed by Schelling (1971), who studied a model of racial segregation, and Axelrod (1978), who first analysed dynamics of cooperation within a CA framework. Typical applications of CA concern urban and spatial economics, to model, for instance, traffic flow and to design patterns of urban development. In general, Cellular Automata models can be applied to any situation involving local interactions between agents, such as the diffusion of information or the adoption of new technologies. This section provides a simple example of the diffusion of an innovation described with a CA (see Gilbert and Troitzsch, 1999). We consider a 10 x 10 grid, so that the number of cells is  $n=100$ . Each cell represents an individual, who may be in one of two states: adopter (coloured cell) or neutral (white cell). The rules governing the process are defined as follows: an individual cell becomes adopter if one of its Von Neumann neighbours has already adopted and once the individual has adopted, it cannot reverse its state (irreversibility of choice). Simulating through several

iterations the collective behaviour of the system given that at time  $t=1$  only one cell has adopted, yields the results presented in Figure 2.

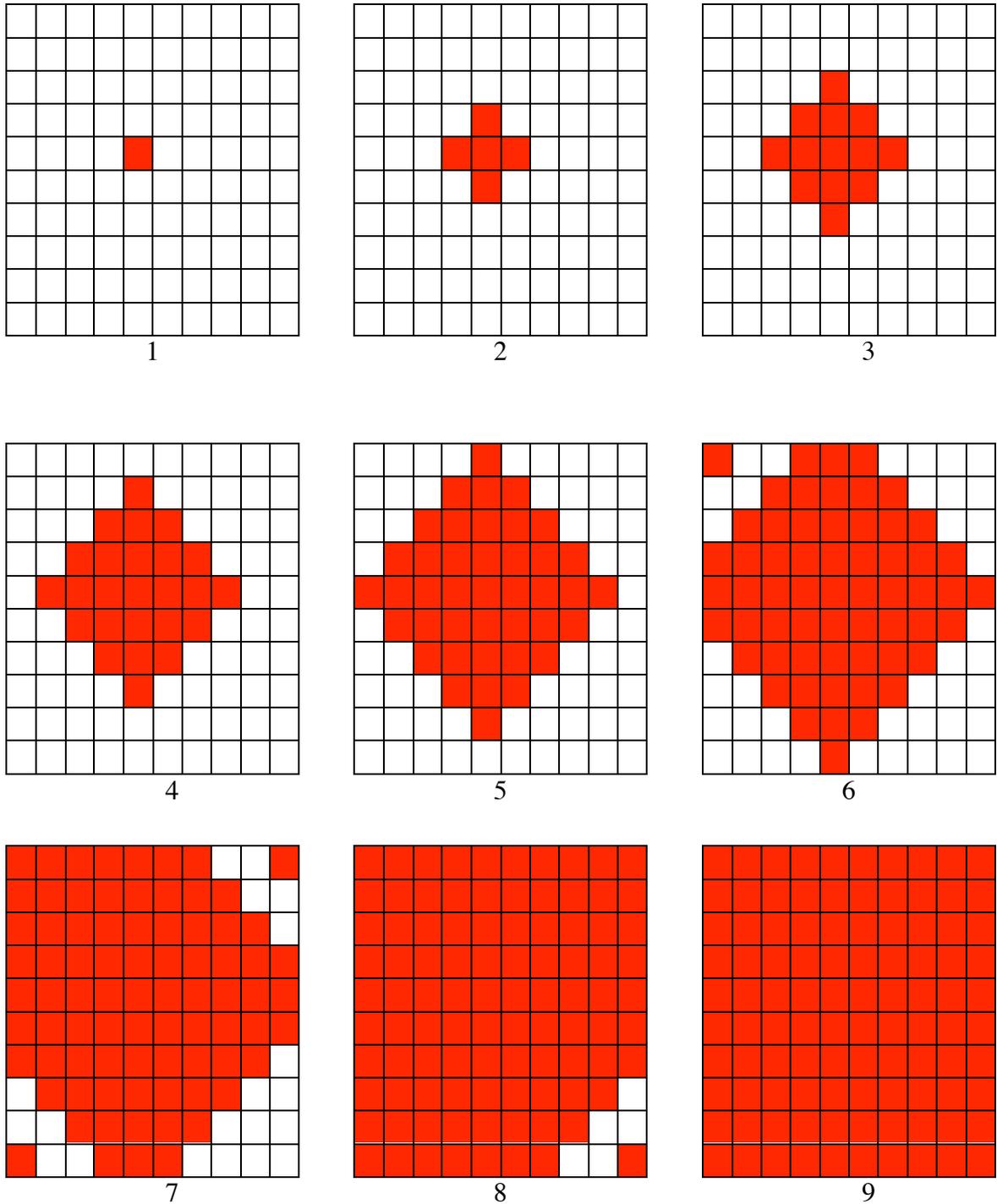


Figure 2. The diffusion of an innovation in a simple CA: the process ends after nine iterations.

In this example the innovation has been adopted by all individuals after nine iterations. This is a rather simplified representation of reality, as once an agent has bought the new product, all the others will do so as well. Extensions of the model provided in Gilbert and Troitzsch (1999) incorporate stochastic elements into the way a cell is affected by its neighbours. However, this example is sufficient for our purposes: we just need to take the cumulative number of adoptions realised at each period of time.

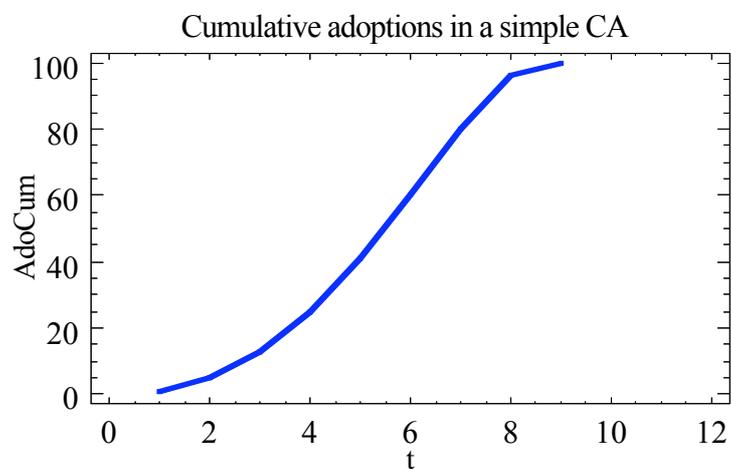


Figure 3. Plot of cumulative adoptions simulated with a CA.

The interesting result of this application is that the the plot of cumulative adoptions derived from the CA simulation produces a Bass-like curve, suggesting that the Cellular Automata approach leads to results coherent with the aggregate one. This intuition is confirmed by performing a non-linear regression (NLS), which tests if a standard Bass model fits the data obtained with the CA simulation. The results of the NLS procedure are summarised in Table 1.

Cumulative adoptions data from a CA simulation			
m	p	q	R <sup>2</sup>
110, 689	0,01155	0,718	99,8863 percent

Table 1. Parameter estimates with a standard Bass model.

Parameter estimates show that a standard Bass model fits quite well the data produced with the CA simulation ( $R^2= 99,8863$ ), though we may observe a slight overestimation of the market potential. Clearly this application does not have any predictive value, as the model has been implemented with the complete data series. Indeed, the primary purpose of this example was to demonstrate empirically that there exists an interesting relationship between aggregate models and Cellular Automata models in representing innovation diffusion processes. In particular, a focused inquiry on this relationship may suggest to employ Cellular Automata models with statistical methods rather than with computer simulations. Though the implementation of CA with simulations may allow to explore how different initial configurations can lead to different final results, it is true that the reliability of simulations is related to the availability of information about the context analysed, so that “the better is a simulation for its own purposes, by the inclusion of all relevant details, the more difficult it is to generalize its conclusions” (Maynard Smith, 1974). As Maynard Smith (1974) pointed out whereas a good simulation should include as much detail as possible, a good model should include as little as possible. Since information on a diffusion process normally comes in form of aggregate data of adoption, CA should be treated in order to be finally incorporated in aggregate models.

## 2.3 A probabilistic Cellular Automata for diffusion of innovations

This section presents a method for representing a global behaviour starting from a formal specification of a Cellular Automata model without performing simulations, based on a probabilistic Cellular Automata model for innovation diffusion first developed by Boccara and Fuks (1999) and subsequently extended by Guseo and Guidolin (2007a). This probabilistic CA considers a usual two-state situation, i.e. adopter = 1, neutral = 0.

The specification of the transition rule,  $f$ , considers a structure by which the state of each cell at time  $t + 1$ ,  $s(i, t + 1)$  depends on its previous state,  $s(i, t)$ , and on the state of its neighbouring cells within a local neighbourhood of ray  $n$ . This neighbouring pressure is defined as a *mean value*

$$\sigma(i, t) = \sum_{n=-\infty}^{\infty} s(i + n, t)p(n) \quad \sum_{n=-\infty}^{\infty} p(n) = 1, p(n) \geq 0 \quad (2)$$

where  $n$  is the ray of interaction of the generic cell  $i$  and  $p(n)$  is a probability distribution, and represents a social pressure on cell  $i$  to adopt.

Given these assumptions, a simplified CA model describing an innovation diffusion process is defined as follows: a neutral cell may become an adopter with probability equal to  $\sigma(i, t)$  and its decision to adopt is not reversible. Note that  $\sigma(i, t)$  is itself a probability distribution. The transition rule  $f$  is indirectly specified through the probabilities of change, depending on the state of cell  $i$  and on the neighbourhood  $\sigma(i, t)$

$$P(s(i, t + 1) = 0) = (1 - s(i, t))(1 - \sigma(i, t))$$

$$P(s(i, t + 1) = 1) = 1 - (1 - s(i, t))(1 - \sigma(i, t)) \quad (3)$$

The probability of change, called *transition probability*, is defined as

$$P_{b \leftarrow a} = P(s(i, t+1) = b | s(i, t) = a) \quad (4)$$

and transition probabilities may be collected in a unique matrix, called *transition probability matrix*

$$\mathbf{P} = \begin{pmatrix} P_{0 \leftarrow 0} & P_{0 \leftarrow 1} \\ P_{1 \leftarrow 0} & P_{1 \leftarrow 1} \end{pmatrix} = \begin{pmatrix} 1 - \sigma(i, t) & 0 \\ \sigma(i, t) & 1 \end{pmatrix}. \quad (5)$$

Transition probability matrix (5) may be used for calculating the average global behaviour, defining with  $\rho(t)$  the average density (proportion) of adopters at time  $t$  and with  $\eta(t) = 1 - \rho(t)$  the average density of neutrals. The values of  $\rho(t+1)$  and  $\eta(t+1)$  can be calculated by introducing in the transition probability matrix an operator  $\langle \rangle$  that denotes a *spatial average*, such that

$$\begin{pmatrix} \eta(t+1) \\ \rho(t+1) \end{pmatrix} = \begin{pmatrix} \langle P_{0 \leftarrow 0} \rangle & \langle P_{0 \leftarrow 1} \rangle \\ \langle P_{1 \leftarrow 0} \rangle & \langle P_{1 \leftarrow 1} \rangle \end{pmatrix} \begin{pmatrix} \eta(t) \\ \rho(t) \end{pmatrix}. \quad (6)$$

Observe that average dynamics of diffusion are obtained through a probability matrix in which individual probabilities of change have been substituted by an appropriate spatial average excluding the effect of individual probability distribution  $p(n)$ . In particular, the average density of adopters  $\rho(t+1)$  will be defined as

$$\rho(t+1) = \rho(t) \langle P_{1 \leftarrow 1} \rangle + \eta(t) \langle P_{1 \leftarrow 0} \rangle \quad (7)$$

that is

$$\rho(t+1) = \rho(t) \langle P_{1 \leftarrow 1} \rangle + (1 - \rho(t)) \langle P_{1 \leftarrow 0} \rangle \quad (7a)$$

or

$$\rho(t+1) = \rho(t) + (1 - \rho(t))\langle\sigma(i,t)\rangle. \quad (7b)$$

In this simple model the probability of becoming an adopter for a neutral cell equals the value of the neighbouring pressure,  $\sigma(i,t)$ . In other words a cell becomes an adopter exclusively depending on its neighbours' status. This is clearly a rather simplifying assumption, so that, under a more realistic hypothesis, the probability of adoption should not be equal, but only proportional  $\sigma(i,t)$ , namely

$$P_{1\leftarrow 0} = q\sigma(i,t) \quad q \in (0,1) \quad (8)$$

Parameter  $q$  may be considered as an individual aptitude for imitative adoption, combined with the neighbouring pressure,  $\sigma(i,t)$ . Both parameters  $q$  and  $\sigma(i,t)$  refer to stochastic independent events. Updating equation (6b) with this new assumption yields

$$\rho(t+1) = \rho(t) + (1 - \rho(t))q\langle\sigma(i,t)\rangle \quad (9)$$

The average density of adopters  $\rho(t)$  may be considered as a *mean field approximation*, that is a *limiting behaviour* of  $\langle\sigma(i,t)\rangle$ , when the range of interactions tends to infinity, for a spreading distribution  $p(n)$ , equation (9) may be rewritten as a finite difference equation,

$$\rho(t+1) - \rho(t) = q\rho(t)(1 - \rho(t)) \quad (10)$$

One may easily see that the continuous version of equation (10) reproduces the structure of a standard logistic equation, that is

$$y'(t) = qy(t)(1 - y(t)) \quad (11)$$

This fact allows to propose a reasonable extension of equation (11), in which the probability of becoming an adopter has no longer a logistic pattern

$\langle P_{1 \leftarrow 0} \rangle = q \langle \sigma(i, t) \rangle$  and takes a Bass-like form, so that adoption depends on the sum of two different forces, the external and the internal. Note that this extension is not present in Boccara and Fuks (1999) and is due to Guseo and Guidolin (2007a).

Through this change the individual probability of adoption is

$$\langle P_{1 \leftarrow 0} \rangle = p + q \langle \sigma(i, t) \rangle \quad (12)$$

where  $p$  represents the external pressure due to institutional communication. Consistently with the Bass approach, equation (12) expresses the fact that the conditional probability of becoming an adopter refers to two mutually exclusive events, either an innovative behaviour or an imitative one.

Incorporating this assumption in equation (9) yields

$$\rho(t+1) - \rho(t) = p + q\rho(t)(1 - \rho(t)) \quad (13)$$

and if we approximate the finite difference  $\rho(t+1) - \rho(t)$  with the prime derivative  $\rho'(t)$  the continuous time version of equation (13) is

$$\rho'(t) = (p + q\rho(t))(1 - \rho(t)) \quad (14)$$

or

$$y' = (p + qy)(1 - y) \quad (14a)$$

using the Bass model notation.

In this way it is possible to recover the initializing aspects of diffusion due to parameter  $p$ , whose presence is a prerequisite for the whole process. Notice that also the approximation in equation (13) is not present in Boccara and Fuks (1999) and is proposed in Guseo and Guidolin (2007a).

From a theoretical point of view, the transition from the individual (CA) to the aggregate level (Bass model) may represent a trial to face the so-called inverse problem, from global to local mapping: “find a Cellular Automata rule that will have some preselected global properties”. Indeed, the parameters of the aggregate model,  $m$ ,  $p$ ,  $q$ , may be considered as global evolutive patterns of the entire process, allowing to determine the aggregate dynamics  $y(t)$  (or  $\rho(t)$ ).

The average density of adopters  $\rho(t)$  may be seen as a limiting behaviour of  $\sigma(i,t)$ , whose value is necessary for the transition rule that defines the CA evolution. Clearly, the optimal solution would require the knowledge of the real  $\sigma(i,t)$ , but this would mean to know the individual distribution  $p(n)$ . Since this information is generally unavailable, the CA transition rule may be constructed using a limiting value of  $\sigma(i,t)$ ,  $\rho(t)$ , inferred from a process historically observed, for which data are available. This procedure seems more reliable than one using direct simulations based on a transition rule whose values are defined probably in an arbitrary manner.

Such a use of Cellular Automata in statistical terms seems suitable for building individual-level diffusion models, that are models able to capture the evolution of the market for a new product over time, through explicit use of individual rules of behaviour (see Muller, Peres and Mahajan, 2007).

## 2.4 Modelling the diffusion of services

Since its publication in Management Science in 1969, the Bass model was meant to describe the life cycle of durable goods: however, its application to non durable goods, like services, has provided notable results, confirming the versatility of this model. The Bass has been criticized for not considering repeated purchases, that are typical of most

services: probably this is an unfounded concern, since the model is estimated with aggregate data on *adoptions* and not on *adopters*. A one-to-one correspondence between adopters and adoptions is not required by the model. At the same time we must notice that new services, representing a large share of current innovations, are contributing to highlight the particularly phenomenon of disadoption. Indeed, most services require a subscription and the possibility that customers, after the first purchase, decide to unsubscribe is obviously a critical aspect for firms providing them. Disadoption may clearly occur for dissatisfaction reasons or availability of competitive alternatives. When a customer decides to abandon a service provider choosing a competitor, we talk about *churn*, which is particularly evident in very competitive industries like cellular telephony. Besides, a customer may decide to disadopt completely the service, regardless of possible alternatives: as noticed by Hogan, Lemon and Libai (2003) this may be typical of markets using very innovative technologies, in which the risk associated to innovation adoption is particularly high.

Actually, one of the main concerns of the service sector is today the retention of gained customers, through the establishment of long-term relationships: the field of marketing called Customer Relationship Marketing (CRM) is focused on “attracting, developing and retaining customer relationships” (Berry and Parasuraman, 1991). As pointed out by Groenroos (2004) marketing from a relational perspective can be seen as the process of identifying and establishing, maintaining, enhancing and, when necessary, terminating relationships with customers and other stakeholders.

Libai, Muller and Peres (2006) correctly observe that not all services are characterised by a potential long-term relationship, because there exist many situations

ending with a single transaction and it is therefore important to focus on services that imply a registration, such as phones, Internet provision or financial services.

Although the strategy of CRM tends to a one-to-one management of relationships, using techniques like data mining, that permit to maintain information about each single customer and, as much as possible, *customize* the provision of the service, the evolutionary perspective of aggregate diffusion models still plays a crucial role in facing issues of customer retention and disadoption along time.

In summarizing some contributions in the diffusion literature dealing with long-range perspectives Libai, Muller and Peres (2006) recall, among others, the work of Norton and Bass (1987, 1992) and Mahajan and Muller (1996) on successive generations of products, and the work of Lilien, Rao and Kalish (1981) on trial and repeat purchase models, observing that these models, in spite of their long term approach, do not take into account disadoption aspects. Interestingly, the issue of disadoption may be carefully evaluated in the case of successive generations, since a correct analysis of disadoption dynamics may give useful indications to manage the timing of the new generation of product.

From a diffusion perspective, one of the most important consequences of not considering the loss of customers, clearly relates to a biased estimation of the potential market  $m$ . Indeed, the need of a simultaneous account of inward and outward flows of adopters is less perceived in the case of durable goods, where the possibility of repeated adoptions is more difficult, rather it seems of strategic importance for those goods characterised by multiple purchases, like services.

### 2.4.1 An innovation diffusion model with exit rules: the BFG model

This section presents a new diffusion model, able to take into account inward and outward flows of adopters, developed by Guseo and Guidolin (2007a) with Cellular Automata models, starting from the specification of the individual probability of disadoption. This procedure proves to be effective for a step-by-step building of the final model.

Starting from the CA model presented in section 2.3 and considering the possibility of disadoption, the probability of adoption reversibility has no longer a null value and may be defined as follows

$$P_{0 \leftarrow 1} = r \quad 0 < r < 1 \quad (15)$$

and clearly the complementary probability is

$$P_{1 \leftarrow 0} = 1 - r. \quad (16)$$

Updating the transition probability matrix with this new assumption implies

$$\begin{pmatrix} P_{0 \leftarrow 0} & P_{0 \leftarrow 1} \\ P_{1 \leftarrow 0} & P_{1 \leftarrow 1} \end{pmatrix} = \begin{pmatrix} 1 - p + q\sigma(i, t) & r \\ p + q\sigma(i, t) & 1 - r \end{pmatrix}. \quad (17)$$

The average collective dynamics are calculated using the updated matrix

$$\begin{pmatrix} \eta(t+1) \\ \rho(t+1) \end{pmatrix} = \begin{pmatrix} \langle P_{0 \leftarrow 0} \rangle & \langle P_{0 \leftarrow 1} \rangle \\ \langle P_{1 \leftarrow 0} \rangle & \langle P_{1 \leftarrow 1} \rangle \end{pmatrix} \begin{pmatrix} \eta(t) \\ \rho(t) \end{pmatrix} \quad (18)$$

that is

$$\begin{pmatrix} \eta(t+1) \\ \rho(t+1) \end{pmatrix} = \begin{pmatrix} 1 - p + q\langle \sigma(i, t) \rangle & r \\ p + q\langle \sigma(i, t) \rangle & 1 - r \end{pmatrix} \begin{pmatrix} \eta(t) \\ \rho(t) \end{pmatrix}. \quad (19)$$

Recalling that  $\rho(t)$  may be considered a limiting value of  $\langle \sigma(i,t) \rangle$ , equation (19) can be rewritten as

$$\begin{pmatrix} \eta(t+1) \\ \rho(t+1) \end{pmatrix} = \begin{pmatrix} 1-p+q\rho(t)r & \\ p+q\rho(t) & 1-r \end{pmatrix} \begin{pmatrix} \eta(t) \\ \rho(t) \end{pmatrix}. \quad (20)$$

Focusing on the density of adopters, we obtain

$$\rho(t+1) = (1-r)\rho(t) + (p+q\rho(t))(1-\rho(t)) \quad (21)$$

$$\rho(t+1) - \rho(t) = -r\rho(t) + (p+q\rho(t))(1-\rho(t)) \quad (21a)$$

and approximating the finite difference  $\rho(t+1) - \rho(t)$  with the prime derivative  $\rho'(t)$  yields

$$\rho'(t) = -r\rho(t) + (p+q\rho(t))(1-\rho(t)) \quad (22)$$

with hazard rate

$$h(t) = p + q\rho(t) - r \frac{\rho(t)}{1-\rho(t)}. \quad (23)$$

Using a notation consistent with that of the Bass model, equation (22) may be written as

$$y' = -ry + (p+qy)(1-y) \quad (24)$$

that is the BFG model proposed in Guseo and Guidolin (2007a).

This is a generalised version of the Bass model considering the variation over time of adoptions as the result of two processes, *adoption* and *disadoption*. Notice that parameter  $r$  is correctly multiplied by  $y(t)$ , i.e. the number of adopters at time  $t$ . A more general version of the BFG model takes into account the presence of possible

interventions, like marketing mix interventions, managerial strategies, environmental constraints acting on the diffusion process, incorporating function  $x(t)$

$$y' = \{-ry + (p + qy)(1 - y)\}x(t). \quad (25)$$

Therefore, the standard BFG model is a special case under  $x(t) = 1$ . The Generalized Bass model is obtained for  $r = 0$ . Finally the standard Bass model is attained under both conditions,  $x(t) = 1$  and  $r = 0$ . It should be noticed that a very similar version of the standard BFG model has been proposed in Libai, Muller and Peres (2006), although without an explicit derivation of the final equation describing the model.

#### 2.4.2 Solution of a perturbed BFG

The equation describing a perturbed BFG model is

$$y' = \{-ry + (p + qy)(1 - y)\}x(t). \quad (26)$$

The factorization of this equation yields

$$y' + q(y - r_1)(y - r_2)x(t) = 0 \quad (27)$$

with real roots, for  $D = \sqrt{(r + p - q)^2 + 4pq} > 0$ , equal to

$$r_i = \frac{-(r + p - q) \pm D}{2q}, \quad i=1, 2 \quad (28)$$

so that we have  $a(r_2 - r_1) = D$  and the closed form solution for  $y(0)=0$  is therefore

$$y(t) = \frac{1 - e^{-D \int_0^t x(\tau) d\tau}}{\frac{1}{r_2} - \frac{1}{r_1} e^{-D \int_0^t x(\tau) d\tau}}. \quad (29)$$

Notice that  $\lim_{t \rightarrow +\infty} y(t) = r_2$  and that the absolute scale representation of the diffusion process may be obtained in a direct way  $z(t) = My(t)$ , where  $M$  is a scale parameter defining the absolute size of the process. Consequently, the asymptotic behaviour is

$$m = \lim_{t \rightarrow +\infty} z(t) = Mr_2. \quad (30)$$

Recall that in the Bass model (BM, GBM), the exit parameter has a null value,  $r = 0$  and  $r_2 = 1$  so that the asymptotic behaviour of diffusion and the potential market (carrying capacity) coincide, i.e.  $m = M$ .

It is easy to see that in the case of the BFG model the asymptote is lower than in the case of the Bass model, because of disadoption dynamics. Evidently the *effective maximum number* of adopters, taking into account disadoptions, must be smaller than in the case of a standard Bass model. Interestingly, the shape of the curve of the BFG may depend on the specification of parameter  $r$ , that has been modeled as constant along the whole process, but may have a variable structure, in order to represent, for example, an increasing speed in disadoptions (see for example Guseo and Guidolin, 2007a). The difference between the two models (Bass and BFG) may be inspected also in graphic terms in Figure 4.

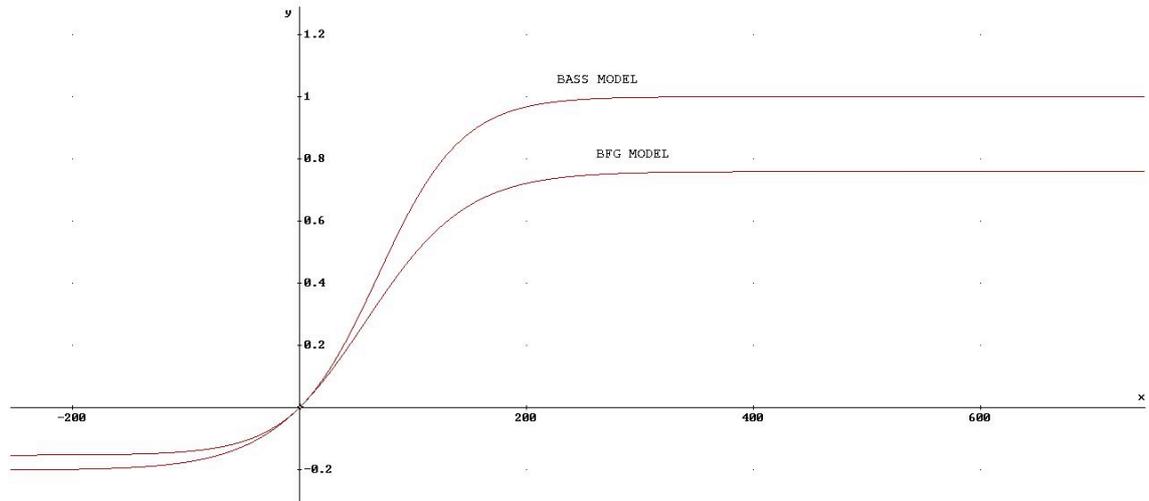


Figure 4: Bass model versus BFG model: disadoption dynamics lower the level of the carrying capacity.

## 2.5 Final remarks and discussion

This chapter has analysed the issue of modelling innovation diffusion at the individual level. A considerable body of literature has been produced in the 1970s and 1980s in order to characterise adoption decisions of the single agent. However, the model proposed were too rich, thus essentially intractable in empirical terms.

More recently, research in quantitative marketing has suggested to use Complex Systems models, especially Cellular Automata models, that permit to reconstruct aggregate dynamics of adoption, starting from local rules of behaviour of single agents. The advantage provided by this class of models is the possibility to test how a collective structure may change as individual conditions do and therefore to provide a representation of how single choices, once aggregated, may produce results not obvious *a priori*. In this sense, Cellular Automata models represent a very interesting opportunity to research on diffusion, offering a new way to study the effect of individuals' heterogeneity on diffusion, which is hardly understandable just from an

aggregate point of view. The implementation of these models generally relies on computer simulations: however, a formal connection between individual and aggregate level models is necessary for studying individual adoption in a way that can be later generalised in aggregate form. In this sense, simulations may be better employed for analysing particular artificial cases or as an exploratory tool to be completed with statistical analyses and historical data.

This chapter has proposed a method for connecting a class of Cellular Automata with the Bass model, based on a *mean field approximation*. Such an approximation surely introduces some simplifications, as it neglects local interactions and spatial correlations among agents, but it allows to point out conclusively that there exists a analytical connection between CA and the aggregate approach to diffusion. In particular, the step-by-step building of the final aggregate model has suggested some important insights. For instance, it has been possible to explain that individual imitative behaviour,  $q\sigma(i,t)$  results from the conjoint existence of two independent probabilities: an individual aptitude towards imitation,  $q$ , combined with a local evidence, pressure  $\sigma(i,t)$ .

The Cellular Automata approach allows to study internal dynamics of adoption related to individual choices and therefore to build generalizations of the Bass model that account for other phenomena. In this sense, it is interesting to notice that while the Generalized Bass model, GBM, presented in chapter 1 has proven its importance for modelling all those diffusion processes strongly affected by external factors, modelling internal factors has required a new methodology, to which CA models have offered an important chance.

In this chapter, the structure of a new aggregate model for innovation diffusion with simultaneous dynamics of adoption and disadoption, based on a Cellular Automata model, has been developed by Guseo and Guidolin (2007a) and has been presented as a constructive example of how individual-level models may be designed in order to take into account phenomena not considered in standard diffusion models, that indeed have a considerable impact on new-product growth and depend on agents' behaviour: the model proposed has analyzed the effect of disadoption on diffusion, highlighting that not giving the right importance to these dynamics may introduce strong bias on models and subsequent forecasts especially on market potential estimates. An interesting consequence of this first model is the possibility to account for a market potential (or carrying capacity) whose structure may not be stable over the entire life cycle of the product or service. Further effort is surely due in this direction of research in order to produce concrete applications of this model, whose empirical testing is hindered by the difficulty to find databases that simultaneously consider arrivals and departures of adopters and by some instability problems in parameters' estimation.

However, this first model represents a strong incentive for further developing new aggregate models based on individual rules of behaviour, with a particular attention to the structure of the market potential, whose size may be time-dependent. Some proposals in this direction will be presented in the following chapters.