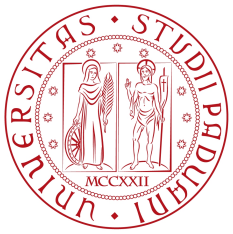

UNIVERSITÀ DEGLI STUDI DI PADOVA

Sede Amministrativa: Università degli studi di Padova

Dipartimento di: Matematica Pura ed Applicata

Scuola di dottorato di ricerca in: Scienze Matematiche

Indirizzo: di Matematica Computazionale - XXII ciclo

Interest rate derivatives pricing when the short rate is a continuous time finite state Markov process

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Sommario

Nell' articolo [6] Filipovic' e Zabczyk presentano un approccio per ottenere l'analogo in tempo discreto della struttura a termine affine a tempo continuo, considerando il tasso spot $r(t)$ come una catena di Markov (MC) con uno spazio degli stati discreto. Quando il tasso spot è una MC a tempo discreto, il numero dei salti in un qualsiasi intervallo temporale fissato è deterministico.

Tuttavia, nel mercato reale, il tasso spot non cambia generalmente ad istanti fissi ma "salta" ad istanti aleatori.

Questo suggerisce di considerare il tasso spot come una catena di Markov a tempo continuo (CTMC) con uno spazio degli stati finito $E = \{r^1, r^2, \dots, r^N\}$, $N \in \mathbb{N}$, $r^i \in \mathbb{R}$, $i = 1, \dots, N$. Seguendo la procedura conosciuta come *martingale modeling*, assumiamo che sotto una misura di martingala $\tilde{\mathbb{P}}$ la matrice delle intensità di transizione sia data da $Q = \{q_{i,j}\}_{i,j=1,\cdots,N}$. Fissata una maturità T e una data di valutazione t, il numero di salti del tasso spot tra t e T (indicato con $\nu_{t,T}$), ovvero il numero delle transizioni della MC, è aleatorio e può assumere un valore arbitrariamente grande.

L'obiettivo del nostro lavoro è di ottenere, nel caso in cui il tasso spot evolve come una CTMC, delle formule esplicite (facilmente implementabili per ottenere risultati numerici) per il prezzaggio dei bond e dei derivati come i cap, le swaption e le bond option. La difficoltà maggiore nel nostro studio è data dall'aleatorietà dei tempi di salto. Nel caso in cui si considera un modello diffusivo con salti per il tasso spot, l'articolo di Björk-Kabanov-Runggaldier [1] illustra come, assumendo una struttura a termine affine, il prezzo di un bond può essere espresso in termini della soluzione di un sistema di equazioni differenziali ordinarie. Questo approccio è interessante dal punto di vista teorico, tuttavia non considera i prezzi dei derivati e risulta difficile da implementare nella pratica. Nell'articolo di Eberlein and Kluge [5], dove si considera un modello generale di Levy, sono prezzati anche i cap e le swaption; gli autori ottengono inoltre delle formule analitiche esplicite che richiedono degli strumenti matematici abbastanza sofisticati. La maggiore differenza con il nostro lavoro consiste nel fatto che noi riusciamo a prezzare con un approccio unificato sia i bond che i derivati sui tassi d'interesse, pur considerando dei processi stocastici meno generici.

Nella tesi verrà mostrato come i prezzi dei bond e dei derivati sui tassi d'interesse, come anche quello delle opzioni sui bond, sono particolari casi del prezzo di uno strumento finanziario fittizio detto "Prodotto Prototipo", legato concettualmente al prezzaggio dei titoli Arrow-Debreu. Il prezzaggio del prodotto Prototipo rappresenta un approccio che prezza in modo unificato diversi prodotti finanziari sui tassi d'interesse e che permette una facile im-

plementazione. Considereremo una approssimazione superiore del prezzo la cui differenza col prezzo reale del prodotto in questione è trascurabile.

Un **Prodotto Prototipo** è un prodotto finanziario che garantisce la consegna alla scadenza T di un certo payoff $\vartheta_0(r(T))$ che dipende dal valore assunto dal tasso spot nella data di scadenza T:

$$\vartheta_0(\cdot) = \sum_{i=1}^{N} w_i \mathbf{I}_{\{\cdot = r^i\}} \text{ with } r^i \in E \text{ and } w_i \in \{0\} \cup \mathbb{R}_+.$$

Per un determinato stato attuale $r^i \in E$ $(1 \leq i \leq N)$, indicato genericamente con v, ricordiamo che

- il tempo di interarrivo è una variabile aleatoria esponenzialmente distribuita con parametro (intensità) $q(v) \triangleq q_i = \sum_{j=1, j\neq i}^{N} q_{i,j}$;
- le probabilità di transizione dallo stato v allo stato $u = r^j$ è
  $p_{v,u} \triangleq p_{i,j} = \begin{cases} p_{i,j} = \frac{q_{i,j}}{q_i} & \text{if } i \neq j \\ p_{i,j} = 0 & \text{if } i = j. \end{cases}$

Inoltre, indicando con ν_t il numero di salti della CTMC fino al tempo t ed $\mathfrak{M} \triangleq \left\{\vartheta : E \to \{0\} \cup \mathbb{R}_+ \mid \vartheta(v) = \sum_{i=1}^{N} w_i \mathbf{I}_{\{v=r^i\}},\ w_i \in \{0\} \cup \mathbb{R}_+,\ \forall i = 1, \ldots, N\right\}$ dotato della norma del sup, otteniamo i seguenti risultati:

Proposition 0.1 *Sotto una misura di martingala $\tilde{\mathbb{P}}$, il prezzo del prodotto Prototipo all'istante $t < T$ può essere rappresentato come*

$$V_{\vartheta_0,t,T}(r_{\nu_t}) = \sum_{k=0}^{+\infty} \vartheta_k(r_{\nu_t}) \tilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t}) \tag{1}$$

dove le funzioni ϑ_k sono ottenute ricorsivamente dopo k passi iterando l'operatore $\mathcal{T}$:

$$\mathcal{T}\vartheta(v) \triangleq \int_{\mathbb{R}} q(v) e^{-(v+q(v))s} \Big(\sum_{u \in E} p_{v,u} \vartheta(u) \Big) ds, \quad \vartheta \in \mathfrak{M}.$$

Proposition 0.2 *L'operatore $\mathcal{T} : \mathfrak{M} \to \mathfrak{M}$ è un operatore contraente con punto fisso $\vartheta^* = 0$ e costante di contrazione $\gamma \triangleq \sup_{v \in E} \frac{q(v)}{v+q(v)}$.*

Per quanto riguarda la distribuzione di $\nu_{t,T}$ otteniamo la seguente

Proposition 0.3 *Valgono le seguenti formule*

$$\begin{cases} \tilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r^m) = \displaystyle\sum_{\substack{i_1, \ldots, i_k = 1 \\ i_1 \neq m, i_2 \neq i_1, \ldots, i_k \neq i_{k-1}}}^{N} e^{q_m t - q_{i_k} T} \varphi_k(Q) \cdot \Psi_k(t, T, Q) \\ \tilde{\mathbb{P}}(\nu_{t,T} = 0 | r_{\nu_t} = r^m) = e^{-q_m (T-t)} \end{cases} \tag{2}$$

dove m è un intero in $\{1,\dots,N\}$, Ψ_k è l'integrale multiplo

$$\Psi_k(t,T,Q) \triangleq \int_t^T e^{(q_{i_1}-q_m)t_1} \int_{t_1}^T e^{(q_{i_2}-q_{i_1})t_2} \dots \int_{t_{k-1}}^T e^{(q_{i_k}-q_{i_{k-1}})t_k} \mathrm{dt_k} \dots \mathrm{dt_2 dt_1} \tag{3}$$

e

$$\varphi_k(Q) \triangleq q_{m,i_1} \cdot \dots \cdot q_{i_{k-1},i_k}.$$

La formula (1) contiene un numero infinito di termini da calcolare: tuttavia, poiché le funzioni ϑ_k sono ottenute applicando k volte l'operatore contraente alla funzione ϑ_0, esiste -per un ϵ piccolo a piacere- un numero naturale n_ϵ tale che $V^\epsilon_{\vartheta_0,t,T}(r_{\nu_t}) \triangleq \sum_{k=0}^{n_\epsilon} \vartheta_k(r_{\nu_t}) \tilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t})$ è una buona approssimazione del prezzo reale $V_{\vartheta_0,t,T}$ nel senso che

$$|V^\epsilon_{\vartheta_0,t,T}(r_{\nu_t}) - V_{\vartheta_0,t,T}(r_{\nu_t})| < \epsilon \text{ uniformemente a } (t,T,r_{\nu_t}). \tag{4}$$

Il prezzaggio dei bond e degli altri derivati sui tassi d'interesse può essere ottenuto come segue:

P1 : un bond che matura al tempo T può essere visto come un prodotto Prototipo con payoff $\varphi_0(\cdot) = \sum_{i=1}^N w_i \mathbf{I}_{\{\cdot=r^i\}}$ dove $w_i \equiv 1$: il prezzo di un T-bond valutato all'istante t è uguale a $V_{\varphi_0,t,T}$;

P2 : i prezzi dei caplet e delle swaption possono essere rappresentati come combinazioni lineari di diversi N prodotti Prototipo $V_{\psi_0^n,t,T}$ con payoff ψ_0^n definiti, per ogni $n \in \{1,\dots,N\}$, da

$$\begin{cases} \psi_0^n(\cdot) = \sum_{i_0=1}^N w_{i_0}(n) \mathbf{I}_{\{\cdot=r^{i_0}\}} \\ w_{i_0}(n) = \begin{cases} 0, & i_0 \neq n \\ 1, & i_0 = n \end{cases} \end{cases}$$

Un risultato analogo si ottiene per il prezzo delle opzioni sui bond.

Inoltre, sulla base del risultato in (4), riusciamo ad ottenere delle formule esplicitamente calcolabili per i prodotti finanziari sopra indicati:

P1$_\epsilon$: una "buona" approssimazione del prezzo del bond è ottenuta considerando $V^\epsilon_{\varphi_0,t,T}$ invece della quantità $V_{\varphi_0,t,T}$ indicata in **P1**;

P2$_\epsilon$: "buone" approssimazioni dei prezzi di caplet, swaption e opzioni sui bond sono ottenute considerando $V^\epsilon_{\psi_0^n,t,T}$ invece delle quantità $V_{\psi_0^n,t,T}$, per ogni $n \in \{1,\dots,N\}$, enunciate in **P2**.

Generalizziamo inoltre il modello unifattoriale precedentemente presentato considerando un modello multifattoriale in cui il tasso spot dipende da diverse CTMC correlate tra loro. Considerando un particolare modello multifattoriale, i bond, i caplet, le swaption e le opzioni sui bond possono essere visti come particolari casi di un prodotto Prototipo il cui prezzo è rappresentabile da una formula esplicita e calcolabile anche quando il tasso spot dipende da più fattori. Il modello multifattoriale può essere inoltre applicato al prezzaggio dei defaultable bond (ovvero bond soggetti al rischio che l'emittente sia incapace di far fronte al pagamento) dove la formula di prezzaggio dipende dal tasso spot e dal tasso di fallimento, fornendo una rappresentazione esplicita del prezzo di questi contratti.

Infine deriviamo dei risultati numerici per mostrare la performance del nostro approccio. In particolare consideriamo un altro approccio numerico per prezzare i bond e i derivati sui tassi d'interesse nel caso unifattoriale: il metodo basato sugli alberi ricombinanti. Paragoniamo la bontà del nostro metodo con quella del metodo ad albero ricombinante quando entrambi sono usati per il prezzaggio dei bond in una struttura a termine affine in tempo continuo. Evidenziamo il fatto che, mentre i metodi ad albero funzionano bene per modelli scalari senza la presenza dei salti, il nostro approccio è applicabile senza difficoltà aggiuntive sostanziali anche al caso multivariato e può essere usato per approssimare i prezzi in modelli a tempo continuo con salti. Inoltre il nostro approccio è specificamente ideato per modelli in cui il tasso spot è una CTMC, che sembra essere più realistico di un modello diffusivo con salti o di un modello a tempo discreto con istanti di salto deterministici.

Summary

In their article [6] Filipovic' and Zabczyk present an approach to obtain in discrete time the analog of the affine term structure model in continuous time. They consider the spot rate $r(t)$ a Markov chain (MC) with a finite state space. Since the short rate is a MC in discrete time, the number of jumps in a fixed time interval is deterministic.

However, in real markets the spot rate does generally not change at deterministic times but it rather "jumps" at random times.

This suggests to model the spot rate as a continuous time Markov chain (CTMC) with a finite state space $E = \{r^1, r^2, \ldots, r^N\}$, $N \in \mathbb{N}$, $r^i \in \mathbb{R}$, $i = 1, \ldots, N$. Following the procedure known as *martingale modeling*, we assume that under a martingale measure $\mathbb{P}$ the transitions intensity matrix of the chain is given by $Q = \{q_{i,j}\}_{i,j=1,\cdots,N}$. For a maturity T and an evaluation time t, the number of jumps of the spot rate between t and T (denoted by $\nu_{t,T}$), namely the number of transition of the MC, is random and can take arbitrarily large values.

The purpose of our study is now to obtain, in a setup where the short rate evolves as a continuous time Markov process, explicit formulae for bond prices and derivatives such as caps, swaptions and bond options that can be easily implemented to obtain numerical results. The major difficulty to this effect is given by the randomness of the jump times. For the case of jump diffusions the article by Björk-Kabanov-Runggaldier [1] illustrates how, by assuming an affine structure, the bond price can be expressed in terms of solutions of a system of ODE's. This approach is theoretically interesting but does not consider derivative prices and it turns out to be difficult to implement in practice. For a more general Levy driven model, the article by Eberlein and Kluge [5] considers also Caps and Swaptions. Here the authors obtain explicit analytic solution formulae that however require rather sophisticated mathematical tools. The main difference with our setup is that, while ours is less general, it leads however to a unified approach for the pricing of bonds and interest rate derivatives.

The pricing of bonds and interest derivatives as well as bond options will be shown to be particular cases of the pricing of a fictitious financial product, namely the "Prototype product"that is related to Arrow-Debreu prices. The pricing of the Prototype product represents a unified approach to the pricing of interest rate related products and which can be implemented by rather elementary calculations. We shall actually consider un upper approximation price such that the difference with the true price of this product is negligible.

We call **Prototype product** a financial product which guarantees to deliver at maturity T a certain payoff $\vartheta_0(r(T))$ which depends on the value taken by the spot rate at the date of

maturity T:

$$\vartheta_0(\cdot) = \sum_{i=1}^{N} w_i \mathbf{I}_{\{\cdot = r^i\}} \text{ with } r^i \in E \text{ and } w_i \in \{0\} \cup \mathbb{R}_+.$$

Recall that for a given current state $r^i \in E$ $(1 \leq i \leq N)$ which below we generically denote by v:

- the time to the next jump is an exponential random variable with parameter (intensity) $q(v) \triangleq q_i = \sum_{j=1, j\neq i}^{N} q_{i,j}$;
- the transition probability to the next state $u = r^j$ is $p_{v,u} \triangleq p_{i,j} = \begin{cases} p_{i,j} = \frac{q_{i,j}}{q_i} & \text{if } i \neq j \\ p_{i,j} = 0 & \text{if } i = j. \end{cases}$

Furthermore, let ν_t denote the number of jumps of the CTMC up to time t and $\mathfrak{M} \triangleq \{\vartheta : E \to \{0\} \cup \mathbb{R}_+ \mid \vartheta(v) = \sum_{i=1}^{N} w_i \mathbf{I}_{\{v=r^i\}},\ w_i \in \{0\} \cup \mathbb{R}_+,\ \forall i = 1, \ldots, N\}$ endowed with the sup norm. We have now the following results:

Proposition 0.1 *For a martingale measure $\tilde{\mathbb{P}}$, the price of the Prototype product at time $t < T$ can be represented as*

$$V_{\vartheta_0,t,T}(r_{\nu_t}) = \sum_{k=0}^{+\infty} \vartheta_k(r_{\nu_t}) \tilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t}) \tag{1}$$

where the functions ϑ_k are obtained recursively, after k steps, by iterating the operator $\mathcal{T}$:

$$\mathcal{T}\vartheta(v) \triangleq \int_{\mathbb{R}} q(v) e^{-(v+q(v))s} \Big(\sum_{u \in E} p_{v,u} \vartheta(u) \Big) ds, \quad \vartheta \in \mathfrak{M}.$$

Proposition 0.2 *The operator $\mathcal{T} : \mathfrak{M} \to \mathfrak{M}$ is a contraction operator with fixed point $\vartheta^* = 0$ and contraction constant $\gamma \triangleq \sup_{v \in E} \frac{q(v)}{v + q(v)}$.*

For the distribution of $\nu_{t,T}$ we have:

Proposition 0.3 *The following holds*

$$\begin{cases} \tilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r^m) = \displaystyle\sum_{\substack{i_1, \ldots, i_k = 1 \\ i_1 \neq m, i_2 \neq i_1, \ldots, i_k \neq i_{k-1}}}^{N} e^{q_m t - q_{i_k} T} \varphi_k(Q) \cdot \Psi_k(t, T, Q) \\ \tilde{\mathbb{P}}(\nu_{t,T} = 0 | r_{\nu_t} = r^m) = e^{-q_m (T-t)} \end{cases} \tag{2}$$

where m is a fixed index in $\{1, \ldots, N\}$, Ψ_k is the multiple integral

$$\Psi_k(t, T, Q) \triangleq \int_t^T e^{(q_{i_1} - q_m) t_1} \int_{t_1}^T e^{(q_{i_2} - q_{i_1}) t_2} \ldots \int_{t_{k-1}}^T e^{(q_{i_k} - q_{i_{k-1}}) t_k} \mathrm{dt_k} \ldots \mathrm{dt_2 dt_1} \tag{3}$$

and

$$\varphi_k(Q) \triangleq q_{m,i_1} \cdot \ldots \cdot q_{i_{k-1}, i_k}.$$

Formula (1) involves an infinite number of terms to compute; however, since the ϑ_ks are obtained by applying k times a contraction operator on ϑ_0, there exists -for an arbitrarily small ϵ- a natural number n_ϵ such that $V^\epsilon_{\vartheta_0,t,T}(r_{\nu_t}) \triangleq \sum_{k=0}^{n_\epsilon} \vartheta_k(r_{\nu_t})\tilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t})$ approximates arbitrarily well the real price $V_{\vartheta_0,t,T}$ in the sense that

$$|V^\epsilon_{\vartheta_0,t,T}(r_{\nu_t}) - V_{\vartheta_0,t,T}(r_{\nu_t})| < \epsilon \text{ uniformily in } (t, T, r_{\nu_t}). \tag{4}$$

The pricing of bonds and other interest rate derivatives can now be obtained as follows

P1 : a bond which matures at time T can be viewed as a Prototype product with payoff $\varphi_0(\cdot) = \sum_{i=1}^N w_i \mathbf{I}_{\{\cdot = r^i\}}$ where $w_i \equiv 1$: the price of a T-bond evaluated at time t is equal to $V_{\varphi_0,t,T}$;

P2 : the prices of both caplets and swaptions can be represented as linear combinations of the prices of N Prototype products $V_{\psi_0^n,t,T}$ with payoffs ψ_0^n defined as follows for each $n \in \{1, \ldots, N\}$:

$$\begin{cases} \psi_0^n(\cdot) = \sum_{i_0=1}^N w_{i_0}(n)\mathbf{I}_{\{\cdot = r^{i_0}\}} \\ w_{i_0}(n) = \begin{cases} 0, & i_0 \neq n \\ 1, & i_0 = n \end{cases} \end{cases}$$

An analogous result is obtained for the price of bond options.

Moreover, on the basis of the result in (4), we are able to obtain computable expressions for the financial products already mentioned above:

P1$_\epsilon$: a "good" approximation of the bond price is obtained by considering $V^\epsilon_{\varphi_0,t,T}$ instead of $V_{\varphi_0,t,T}$ in the expression for **P1**;

P2$_\epsilon$: "good" approximations of the prices of caplets, swaptions and bond options are obtained by considering $V^\epsilon_{\psi_0^n,t,T}$ instead of $V_{\psi_0^n,t,T}$ for each $n \in \{1, \ldots, N\}$ in the expression for **P2**.

We furthermore generalize the one-factor short rate model discussed above by considering a multi-factor short rate model in which the spot rate depends on several correlated CTMCs. Under a particular multi-factor short rate model, bonds, caps, swaptions and bond options can be viewed as particular cases of a Prototype product whose price admits a computable explicit formula also when the short rate is driven by more factors. The multi-factor short rate model can also be applied to the pricing of defaultable bonds, where the pricing formula depends on the short rate and default intensity processes, and we give an explicit representation of the prices of these contracts in the scalar case.

Finally we derive numerical results to illustrate the performance of our approach. In particular we consider also another numerical approach to compute prices of bonds and interest rate derivatives, namely the approach based on a recombining tree. We compare the performance of our method with that of a tree-based method when both are considered for the price of a bond in a continuous time affine term structure model. We point out that, while tree methods work well for scalar models without jumps, our approach is applicable without substantial additional difficulties also to the multivariate case and it can be used to approximate

prices in continuous time models involving jumps. We also would like to stress the fact that our approach is specifically designed for CTMC models for the short rate which appears to be more realistic than diffusion-type models or discrete time models with fixed time instants.

Chapter 1

Introduction

In their article [6] Filipovic' and Zabczyk present an approach to obtain in discrete time the analog of the affine term structure model in continuous time. They consider the spot rate $r(t)$ a Markov chain (MC) with a finite state space. Since the short rate is a MC in discrete time, the number of jumps in a fixed time interval is deterministic.

In real markets the spot rate does not generally change at deterministic times but it rather "jumps" at random times. This suggests to model the spot rate as a continuous time Markov chain (CTMC) with a finite state space $E = \{r^1, r^2, \dots, r^N\}$, $N \in \mathbb{N}$, $r^i \in \mathbb{R}$, $i = 1, \dots, N$. Under a martingale measure $\tilde{\mathbb{P}}$ equivalent to the physical measure $\mathbb{P}$, the transition intensity matrix of the chain is given by $Q = \{q_{i,j}\}_{i,j=1,\cdots,N}$. Such a modeling approach appears also to be more realistic with respect to the traditional diffusion-type models for the short rate. The innovation introduced by this model with respect to [6] is represented by the fact that the number of jumps of the spot rate between an evaluation time t and a maturity T (denoted by $\nu_{t,T}$), namely the number of transition of the MC, is random and can take arbitrarily large values.

Continuous-time term structure models that allow also for jumps have already been considered in the literature. We limit ourselves to mention here just a couple of them. For the case of jump diffusions the article by Björk-Kabanov-Runggaldier [1] illustrates how, by assuming an affine structure, the bond price can be expressed in terms of solutions of a system of ODE's. This approach is theoretically interesting but does not consider derivative prices and it turns out to be difficult to implement in practice. For a more general Levy driven model, the article by Eberlein and Kluge [5] considers also Caps and Swaptions. Here the authors obtain explicit analytic solution formulae in the scalar case that require however rather sophisticated mathematical tools; moreover their numerical results do not concern the prices as such, which is our main goal.

On the other hand, in our setup where the short rate evolves as a continuous time Markov process, we are able to obtain explicit formulae for bond prices and derivatives such as caps, swaptions and bond options that can actually be implemented to obtain numerical results. In fact, the pricing of bonds and interest derivatives as well as bond options will be shown to be particular cases of the pricing of a fictitious financial product, namely the "Prototype product". The pricing of the Prototype product represents a unified approach to the pricing

of interest rate related products and which can be implemented by rather elementary calculations. We shall actually consider un upper approximation price such that the difference with the true price of this product is negligible. We obtain a computable expression of the price of the Prototype product by using a technique based on a contracting operator and on the distribution of $\nu_{t,T}$. Even though by our approach we face a difficulty represented by the randomness of the jump times of the spot rate, we are able to give an explicitly computable formula for the distribution of $\nu_{t,T}$.

We furthermore generalize the one-factor short rate model discussed above by considering a multi-factor short rate model in which the spot rate depends on several correlated CTMCs. Under a particular multi-factor short rate model, bonds, caps, swaptions and bond options can be viewed as particular cases of the Prototype product whose price admits a computable explicit formula also when the short rate is driven by more factors. The multi-factor short rate model can also be applied to the pricing of defaultable bonds, where the pricing formula depends on the short rate and default intensity processes, and we give an explicit representation of the prices of these contracts. Finally we derive numerical results to illustrate the performance of our approach. In particular we consider also another numerical approach to compute prices of bonds and interest rate derivatives in the scalar case, namely the approach based on a recombining tree. We compare the performance of our method with that of a tree-based method when both are considered for the price of a bond in a continuous time affine term structure model. We point out that, while tree methods work well for scalar models without jumps, our approach is applicable without substantial additional difficulties also to the multivariate case and it can be used to approximate prices in continuous time models involving jumps. We also would like to stress the fact that our approach is specifically designed for CTMC models for the short rate which appears to be more realistic than diffusion-type models or discrete time models with fixed time instants.

In Chapter 2 we discuss the pricing approach based on the Prototype product when the short rate is a CTMC; in Chapter 3 we derive the prices of bonds, caps, swaptions and bond options by using the results obtained for the Prototype product pricing. In Chapter 4, by assuming that the short rate depends on several CTMCs, we discuss the pricing of both bonds and other interest rate derivatives, also in a defaultable setup, by starting from the pricing of the Prototype product under a particular multi-factor short rate model. To conclude, in Chapter 5, the numerical results are presented in support of the theory developed in the previous chapters.

Chapter 2

Pricing of interest rate derivatives with a Markov short rate: the Prototype product

2.1 Market model

Let a filtered probability space be given by $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in\mathbb{R}}, \mathbb{P})$ where $\mathbb{P}$ is the physical measure.

Consider the price $p(t,T)$ at time t of a zero coupon bond that matures in $T > t$. In a general setting the bond price has the following representation

$$p(t,T) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\int_t^T r(u)du\Big)|\mathcal{F}_t\Big]$$

where, in order to avoid arbitrage, $\widetilde{\mathbb{P}}$ is a martingale measure equivalent to $\mathbb{P}$. If we assume a Markov short rate, $p(t,T)$ can be expressed by means of specific quantities that we are going to introduce. In particular, we consider the spot rate $r(t)$ a continuous time Markov chain (CTMC) with a state space $E = \{r^1, r^2, \ldots, r^N\}$, $N \in \mathbb{N}$ and $r^i \in \mathbb{R}^+,\ i = 1, \ldots, N$. Denote

- $Q = (q_{i,j})_{1\le i,j\le N}$ the transition intensity kernel independent with respect to the time;
- $q_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} q_{i,j},\ i = 1, \ldots, N$ the intensities associated with the state r^i;
- the transition probabilities from the state r^i to r^j

$$\begin{cases} p_{i,j} = \frac{q_{i,j}}{q_i} & \text{if } i \neq j \\ p_{i,j} = 0 & \text{if } i = j. \end{cases} \tag{2.1}$$

Hence $r(t)$ is a stochastic process with right-continuous piecewise constant trajectories where the jump times T_i $(i = 1, 2, \ldots)$ are random variables and, conditionally on a generic value r^h $(h = 1, \ldots, N)$ of the process at time T_i, the interarrival times $\mathrm{T}_{i+1} - \mathrm{T}_i$ are exponentially distributed, namely

$$(\mathrm{T}_{i+1} - \mathrm{T}_i | r(\mathrm{T}_i) = r^h) \sim \mathcal{E}xp(q_h) \tag{2.2}$$

Therefore

$$p(t,T) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp(r_{\nu_t}(t - \mathrm{T}_{\nu_t}))exp\Big(-\sum_{i=\nu_t}^{\nu_T-1} r_i(\mathrm{T}_{i+1} - \mathrm{T}_i) - r_{\nu_T}(T - \mathrm{T}_{\nu_T})\Big)|\mathcal{F}_t^r\Big] \tag{2.3}$$

where, for a generic time s, ν_s denotes the number of jumps of the Markov process until s, $r(s) = r_i$ for $s \in [\mathrm{T}_i, \mathrm{T}_{i+1})$ and, since $\{\mathrm{T}_i \leq s\} \in \mathcal{F}_s^r$ where T_i is an $\mathcal{F}^r$-stopping time, we let

$$\mathcal{F}_{\mathrm{T}_i}^r = \{A \in \mathcal{F}_T^r | A \cap \{\mathrm{T}_i \leq s\} \in \mathcal{F}_s^r, \forall s \leq T\}.$$

For simplicity of notation we denote $\mathcal{F}_i^r$ the $\sigma-$algebra $\mathcal{F}_{\mathrm{T}_i}^r$ and, since the factor $exp(r_{\nu_t}(t - \mathrm{T}_{\nu_t})$ in formula (2.3) can be explicitly computed on the basis of the information at time t, we assume without lost of generality that $t = \mathrm{T}_{\nu_t}$; consequently the bond price $p(t,T)$ can be written as

$$p(t,T) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_T-1} r_i(\mathrm{T}_{i+1} - \mathrm{T}_i) - r_{\nu_T}(T - \mathrm{T}_{\nu_T})\Big)|\mathcal{F}_{\nu_t}^r\Big] \tag{2.4}$$

where $\mathcal{F}_{\nu_t}^r \equiv \mathcal{F}_t^r$. More generally, given two generic times $\underline{s}$ and $\overline{s}$ such that $\underline{s} < \overline{s}$, denote by $\nu_{\underline{s},\overline{s}}$ the number of jumps in the interval $[\underline{s}, \overline{s})$.

The expression of the bond price in (2.4) leads to the idea of introducing a fictitious financial product which we call *Prototype product* and that is related to Arrow-Debreu prices: we are able to obtain an expression of the price of both bond and interest rate derivatives as linear combinations of Prototype Products. In the following section we consider at the pricing of the *Prototype product*.

2.2 Prototype product pricing: $V_{\vartheta_0,t,T}(\cdot) = \sum_{k=0}^{+\infty} \vartheta_k(\cdot)\widetilde{\mathbb{P}}(\nu_{t,T} = k|\cdot)$

The representation of the bond price given by (2.4) suggests introducing a new financial product of which the zero-coupon bond is a particular case: the *Prototype product*.

Definition 2.1 *A **Prototype product** is a financial product which guarantees to deliver a certain payoff $\vartheta_0(r_{\nu_T})$ at maturity T: this payoff depends on the value taken by the spot rate at the date of maturity T.*
Its price at time $t < T$ is represented by

$$V_{\vartheta_0,t,T}(r_{\nu_t}) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_T-1} r_i(\mathrm{T}_{i+1} - \mathrm{T}_i) - r_{\nu_T}(T - \mathrm{T}_{\nu_T})\Big)\vartheta_0(r_{\nu_T})|\mathcal{F}_{\nu_t}^r\Big] \tag{2.5}$$

*and the **Prototype payoff** $\vartheta_0(\cdot)$ is supposed to have the following form*

$$\vartheta_0(\cdot) = \sum_{i=1}^{N} w_i \mathbf{I}_{\{\cdot = r^i\}}, r^i \in E,\ w_i \in \{0\} \cup \mathbb{R}_+ \tag{2.6}$$

Let us now introduce $\overline{V}_{\vartheta_0,t,T}$ and $\underline{V}_{\vartheta_0,t,T}$ as un upper and a lower approximation respectively of the price of the Prototype Product. They are defined as

$$\overline{V}_{\vartheta_0,t,T} \triangleq \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_T-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\nu_T})|\mathcal{F}^r_{\nu_t}\Big], \tag{2.7}$$

by dropping the last term in the sum in (2.5), and

$$\underline{V}_{\vartheta_0,t,T} \triangleq \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_T} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\nu_T})|\mathcal{F}^r_{\nu_t}\Big] \tag{2.8}$$

where the last term in the sum in (2.5) is extended up to T_{ν_T}. We shall give sufficient conditions to have the difference $\pi \triangleq \overline{V}_{\vartheta_0,t,T} - \underline{V}_{\vartheta_0,t,T}$ a negligible quantity (see Remark 2.5 and Lemma 2.13 below) and we shall see that $\pi \simeq 0$ in many cases such as in the numerical results described in Chapter 5. For this reason, from now on we shall concentrate on the upper approximation $\overline{V}_{\vartheta_0,t,T}$ instead of the true price $V_{\vartheta_0,t,T}$, which is easier to compute and, for simplicity of notation, we will denote $\overline{V}_{\vartheta_0,t,T}$ by $V_{\vartheta_0,t,T}$.

The representation of the upper approximation of the Prototype product price given by (2.7) motivates us to consider particular functions ϑ_h defined as in the following Lemma:

Lemma 2.2 *Let $r(t)$ be a CTMC with state space E and $\vartheta_0(\cdot)$ the* Prototype payoff *as in (2.6). Let the sequence of functions $\vartheta_h(\cdot)$ be defined recursively as follows: for fixed $k, \eta \in \mathbb{N}$, put*

$$\begin{cases} \vartheta_h(r_{\eta+k-h}) \triangleq \mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_{\eta+k-h}(\mathrm{T}_{\eta+k-h+1}-\mathrm{T}_{\eta+k-h})}\vartheta_{h-1}(r_{\eta+k-h+1})\big|\mathcal{F}^r_{\eta+k-h}\big] \\ \forall h = 1,\dots,k \end{cases} \tag{2.9}$$

then

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\eta+k})|\mathcal{F}^r_{\eta}\Big] = \vartheta_k(r_\eta) \tag{2.10}$$

Proof We denote $\mathcal{F}^{\mathrm{r}}_k \triangleq \mathcal{F}^{\mathrm{r}}_{\mathrm{T}_k} = \{A \in \mathcal{F}^r_T | A \cap \{\mathrm{T}_i \le s\} \in \mathcal{F}^r_s, \forall s \le T\}$. Let us show that for $h < k$, namely $\mathrm{T}_h < \mathrm{T}_k$, one has $\mathcal{F}^{\mathrm{r}}_h \subset \mathcal{F}^{\mathrm{r}}_k$. More generically, for two stopping times τ and σ with $\tau \le \sigma$ it follows that $\mathcal{F}_\tau \subset \mathcal{F}_\sigma$. In fact, for any $t \le T$ one has that $\{\sigma \le t\} \subset \{\tau \le t\}$. By taking $A \in \mathcal{F}_\tau$ it follows that $A \in \mathcal{F}_\sigma$ because

$$A \cap \{\sigma \le t\} = A \cap \{\sigma \le t\} \cap \{\tau \le t\} = (A \cap \{\tau \le t\}) \cap \{\sigma \le t\} \in \mathcal{F}_t$$

where in the last passage we use the fact that $A \cap \{\tau \le t\} \in \mathcal{F}_t$ and $\{\sigma \le t\} \in \mathcal{F}_t$ by hypothesis. The statement is proved.

Inspired by Filipović-Zabczyk [6], we can divide the proof into k steps:

1st STEP

$$\begin{aligned}
&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\eta+k})|\mathcal{F}_\eta^r\Big]\\
=\;&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\big[exp\Big(-\sum_{i=\eta}^{\eta+k-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\eta+k})\big|\mathcal{F}_{\eta+k-1}^r\vee\mathcal{F}_{\eta+k-1}^{\mathrm{T}}\big]\Big|\mathcal{F}_\eta^r\Big]\\
=\;&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_{\eta+k-1}(\mathrm{T}_{\eta+k}-\mathrm{T}_{\eta+k-1})}exp\Big(-\sum_{i=\eta}^{\eta+k-2} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\eta+k})\big|\mathcal{F}_{\eta+k-1}^r\vee\mathcal{F}_{\eta+k-1}^{\mathrm{T}}\big]\Big|\mathcal{F}_\eta^r\Big]\\
=\;&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-2} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_{\eta+k-1}(\mathrm{T}_{\eta+k}-\mathrm{T}_{\eta+k-1})}\vartheta_0(r_{\eta+k})\big|\mathcal{F}_{\eta+k-1}^r\vee\mathcal{F}_{\eta+k-1}^{\mathrm{T}}\big]\Big|\mathcal{F}_\eta^r\Big]\\
=\;&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-2} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_{\eta+k-1}(\mathrm{T}_{\eta+k}-\mathrm{T}_{\eta+k-1})}\vartheta_0(r_{\eta+k})\big|r_{\eta+k-1}\big]\Big|\mathcal{F}_\eta^r\Big] \qquad (2.11)
\end{aligned}$$

where the last passage is due to the fact that, for a generic $i\in\mathbb{N}$, conditionally on $\mathcal{F}_i^r\vee\mathcal{F}_i^{\mathrm{T}}$, both the distributions of the interarrival time $\mathrm{T}_{i+1}-\mathrm{T}_i$ and of the visited state r_{i+1} depend only on the initial state r_i by the properties of the CTMC's;
hence

$$(2.11)\overset{(2.9)}{=}\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-2} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_1(r_{\eta+k-1})\Big|\mathcal{F}_\eta^r\Big]. \qquad (2.12)$$

2nd STEP

$$\begin{aligned}
(2.11)\;&=\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-2} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_1(r_{\eta+k-1})\Big|\mathcal{F}_\eta^r\Big]\\
=\;&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\big[exp\Big(-\sum_{i=\eta}^{\eta+k-2} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_1(r_{\eta+k-1})\big|\mathcal{F}_{\eta+k-2}^r\vee\mathcal{F}_{\eta+k-2}^{\mathrm{T}}\big]\Big|\mathcal{F}_\eta^r\Big]\\
=\;&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_{\eta+k-2}(\mathrm{T}_{\eta+k-1}-\mathrm{T}_{\eta+k-2})}\vartheta_1(r_{\eta+k-1})exp\Big(-\sum_{i=\eta}^{\eta+k-3} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\big|\mathcal{F}_{\eta+k-2}^r\vee\mathcal{F}_{\eta+k-2}^{\mathrm{T}}\big]\Big|\mathcal{F}_\eta^r\Big]\\
=\;&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-3} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_{\eta+k-2}(\mathrm{T}_{\eta+k-1}-\mathrm{T}_{\eta+k-2})}\vartheta_1(r_{\eta+k-1})\big|\mathcal{F}_{\eta+k-2}^r\vee\mathcal{F}_{\eta+k-2}^{\mathrm{T}}\big]\Big|\mathcal{F}_\eta^r\Big]\\
=\;&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-3} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_{\eta+k-2}(\mathrm{T}_{\eta+k-1}-\mathrm{T}_{\eta+k-2})}\vartheta_1(r_{\eta+k-1})\big|r_{\eta+k-2}\big]\Big|\mathcal{F}_\eta^r\Big]\\
=\;&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-3} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_2(r_{\eta+k-2})\Big|\mathcal{F}_\eta^r\Big] \qquad (2.13)
\end{aligned}$$

where the last passage is again justified by the properties of the CTMC's recalled in the first step;
hence, recursively until the last step, we obtain

kth STEP

$$\begin{aligned}
&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\eta+k})|\mathcal{F}_\eta^r\Big]=\cdots=\\
&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-k} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_{k-1}(r_{\eta+k-(k-1)})\Big|\mathcal{F}_\eta^r\Big]=\\
&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{r_\eta(\mathrm{T}_{\eta+1}-\mathrm{T}_\eta)}\vartheta_{k-1}(r_{\eta+1})\Big|\mathcal{F}_\eta^r\Big]\overset{(2.9)}{=}\vartheta_k(r_\eta).
\end{aligned}$$

□

The introduction of functions of the type ϑ_k in Lemma 2.2 gives the possibility to represent $V_{\vartheta_0,t,T}$ as an expectation of $\vartheta_{\nu_{t,T}}$.

Lemma 2.3 *The price at time t of the Prototype product with maturity T admits the following representation*

$$V_{\vartheta_0,t,T}(r_{\nu_t}) = \sum_{k=0}^{+\infty} \vartheta_k(r_{\nu_t}) \widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t}) \tag{2.14}$$

where $\nu_{t,T} = \nu_T - \nu_t$ represents the number of jumps occurring in the interval $[t,T]$ and the quantity $\vartheta_k(r_{\nu_t})$, for each $k \in \mathbb{N}$, is recursively defined by relation (2.9) with $\eta = \nu_t$.

Proof The upper approximation $V_{\vartheta_0,t,T}$ of the price of Prototype product can be represented as follows

$$\begin{aligned}
V_{\vartheta_0,t,T}(r_{\nu_t}) &\overset{(2.7)}{=} \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_T-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\nu_T})|\mathcal{F}^r_{\nu_t}\Big] \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_{t,T}+\nu_t-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\nu_T})|\mathcal{F}^r_{\nu_t}\vee\sigma\{\nu_{t,T}\}\Big]|\mathcal{F}^r_{\nu_t}\Big] \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_{t,T}+\nu_t-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\nu_T})|\mathcal{F}^r_{\nu_t}\vee\sigma\{\nu_{t,T}\}\Big]|r_{\nu_t}\Big] \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}[\vartheta_{\nu_{t,T}}(r_{\nu_t})|r_{\nu_t}]
\end{aligned} \tag{2.15}$$

where in the last passage we have considered the result of Lemma 2.2 with $\eta = \nu_t$ and $k = \nu_{t,T}$; since $\nu_{t,T}$ is a discrete random variable, we can write the expectation as the following sum

$$V_{\vartheta_0,t,T}(r_{\nu_t}) = \sum_{k=0}^{+\infty} \vartheta_k(r_{\nu_t}) \widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t})$$

□

In order to evaluate the difference between the upper and lower approximations of the Prototype product in Remark 2.5, in the next Remark we suggest a useful representation of the lower approximation $\underline{V}_{\vartheta_0,t,T}$:

Remark 2.4 *The lower approximation $\underline{V}_{\vartheta_0,t,T}$ of the price of the Prototype product defined by (2.8) admits the following representation*

$$\underline{V}_{\vartheta_0,t,T} = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_T-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_1(r_{\nu_T})|\mathcal{F}^r_{\nu_t}\Big] \tag{2.16}$$

where $\vartheta_1(\cdot)$ is given by formula (2.9) with $h = 1$, $\eta = \nu_t$ and $k = \nu_{t,T}$.

Remark 2.5 *Denoting by π the difference between $\overline{V}_{\vartheta_0,t,T} - \underline{V}_{\vartheta_0,t,T}$, we have that*

$$\pi = \widetilde{\mathbb{E}}^{\mathbb{P}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_T-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)(\vartheta_0(r_{\nu_T})-\vartheta_1(r_{\nu_T}))|\mathcal{F}^r_{\nu_t}\Big] \tag{2.17}$$

in accordance with the representations in (2.7) and (2.16). We will give a sufficient condition to have a negligible π in Subsections 2.2.2 where we will suggest explicit formulae for the functions $\vartheta_0(\cdot)$ and $\vartheta_1(\cdot)$.

As regards the quantity $V_{\vartheta_0,t,T}$, the representation (2.14) stresses the fact that we need to know both the distribution of $\nu_{t,T}$ and an explicit expression of the functions ϑ_k, which we shall study later on, and that to obtain a computable expression for $V_{\vartheta_0,t,T}$ one has to truncate the infinite sum. For this purpose we are going to introduce a contracting operator which gives the possibility to approximate the price of the Prototype product by a truncated series.

2.2.1 A computable expression of $V_{\vartheta_0,t,T}$ based on a contracting operator

Inspired by formula (2.9) for the functions ϑ_h, we consider a function space $\mathfrak{M}$ defined by

$$\mathfrak{M} \triangleq \{\vartheta : E \to \{0\} \cup \mathbb{R}_+\} \tag{2.18}$$

so that $\vartheta(v) = \sum_{i=1}^N w_i \mathbf{I}_{\{v=r^i\}}$ where $w_i \in \{0\}\cup\mathbb{R}_+$, $\forall i = 1,\dots,N$. We introduce the operator $\mathcal{T}$ on $\mathfrak{M}$:

$$\begin{cases} \mathcal{T}\vartheta(v) \triangleq \mathbb{E}_v[e^{-v\mathfrak{T}}\vartheta(u)] \\ \mathfrak{T} \sim \mathcal{E}xp(q(v)), \quad \vartheta \in \mathfrak{M} \end{cases} \tag{2.19}$$

where, by considering a generic $i \in \mathbb{N}$, the quantities introduced in the above definition can be interpreted as follows:

- v and u are the spot rate r_i at jump time T_i and r_{i+1} at jump time T_{i+1} respectively,
- $q(v)$ is the intensity associated with each v of state space E.

We show how the functions $(\vartheta_h)_{h\in\mathbb{N}}$ defined as in Lemma 2.2 can be expressed in terms of the operator $\mathcal{T}$:

Corollary 2.6 *Let $r(t)$ be a CTMC with state space E such that the interarrival times $\mathrm{T}_{i+1} - \mathrm{T}_i$ are exponentially distributed as in (2.2) for $i \in \mathbb{N}$.*
If ϑ_h are as defined in Lemma 2.2 and ϑ_0 is the Prototype payoff as in (2.6) , then also the ϑ_h's are elements of $\mathfrak{M}$ and they can be obtained by iterating the operator $\mathcal{T}$ defined by (2.19):

$$\vartheta_h = \mathcal{T}\vartheta_{h-1} \in \mathfrak{M}$$

Proof Comparing (2.9) and (2.19) it is clear that the two representations are equivalent if we assume that $v = r_i$, $u = r_{i+1}$ and $\left(\mathrm{T}_{i+1} - \mathrm{T}_i\right) \sim \mathcal{E}xp(q(r_i))$ for every natural number i. It remains to prove that $\vartheta_h \in \mathfrak{M}$, $\forall k \in \mathbb{N}$ by induction.
Base Case ($h = 0$): $\vartheta_0 \in \mathfrak{M}$ by definition (2.6) in accordance with the hypothesis.
Inductive Step: let

$$\vartheta_{h-1}(u) = \sum_{m=1}^{N} w_{h-1}^m \mathbf{I}_{\{u=r^m\}} \in \mathfrak{M}, w_{h-1}^m \in \{0\} \cup \mathbb{R}_+ \, \forall m = 1, \ldots, N \quad , \tag{2.20}$$

then

$$\begin{aligned} \vartheta_h(v) = \mathcal{T}\vartheta_{h-1}(v) &\overset{(2.19)}{=} \mathbb{E}_v[e^{-v\mathfrak{T}}\vartheta_{h-1}(u)] \\ &\overset{(2.20)}{=} \mathbb{E}_v[e^{-v\mathfrak{T}} \cdot \sum_{i=1}^{N} w_i^{h-1}\mathbf{I}_{\{u=r^i\}}] \\ &= \sum_{m=1}^{N}\sum_{i=1}^{N} w_i^{h-1}\mathbb{E}[e^{-v\mathfrak{T}} \cdot \mathbf{I}_{\{u=r^i\}}|v = r^m]\mathbf{I}_{\{v=r^m\}} \\ &= \sum_{m=1}^{N} w_m^h \mathbf{I}_{\{v=r^m\}} \end{aligned} \tag{2.21}$$

with $w_m^h \triangleq \sum_{i=1}^{N} w_i^{h-1}\mathbb{E}[e^{-v\mathfrak{T}} \cdot \mathbf{I}_{\{u=r^i\}}|v = r^m] \in \mathbb{R}$ for every $m = 1, \ldots, N$. □

In the following Propositions we shall show that the contraction property of $\mathcal{T}$ allows to obtain a computable approximation of the real Prototype product price $V_{\vartheta_0,t,T}$ that can be made arbitrarily close:

Proposition 2.7

a) The function space $\mathfrak{M}$ is closed w.r.t. $\mathcal{T}$, *namely* $\mathcal{T} : \mathfrak{M} \to \mathfrak{M}$

b) $\mathcal{T}$ is a contracting operator: *For ϑ, $\vartheta' \in \mathfrak{M}$ we have*

$$||\mathcal{T}\vartheta - \mathcal{T}\vartheta'|| \leq \gamma||\vartheta - \vartheta'|| \tag{2.22}$$

with the norm $||\cdot||$ defined by $||f|| \triangleq \sup_{v\in E}|f(v)|$, E the finite state space and $\gamma \triangleq \sup_{v\in E}\frac{q(v)}{v+q(v)} < 1$

c) The fixed point of $\mathcal{T}$ is identically equal to zero: *$\mathcal{T}\vartheta^* = \vartheta^*$ where $\vartheta^* \equiv 0$.*

Proof
a) Being $\vartheta(u) \in \mathfrak{M}$ it can be represented as follows:

$$\vartheta(u) = \sum_{m=1}^{N} b_m \mathbf{I}_{\{u=r^m\}}$$

Analogously to (2.21) we obtain that

$$\begin{aligned}\mathcal{T}\vartheta(v) &\overset{(2.19)}{=} \mathbb{E}_v[e^{-v\mathfrak{T}}\cdot\vartheta(u)]\\ &= \sum_{m=1}^{N}\tilde{w}_m\mathbf{I}_{\{v=r^m\}}\end{aligned}$$

with $\tilde{w}_m \triangleq \sum_{i=1}^{N} w_i\mathbb{E}[e^{-v\mathfrak{T}}\cdot\mathbf{I}_{\{u=r^i\}}|v=r^m]\in\mathbb{R}$ for every $m=1,\dots,N$.

b)

$$\begin{aligned}|\mathcal{T}\vartheta(v)-\mathcal{T}\vartheta'(v)| &\leq \mathbb{E}_v[|e^{-v\mathfrak{T}}|\cdot|\vartheta(u)-\vartheta'(u)|]\\ &\leq \mathbb{E}_v[|e^{-v\mathfrak{T}}|\sup_{u\in E}|\vartheta(u)-\vartheta'(u)|]\\ &= \mathbb{E}_v[e^{-v\mathfrak{T}}]\cdot||\vartheta(u)-\vartheta'(u)||\\ &= \int_0^{+\infty}e^{-vs}q(v)e^{-q(v)s}\mathrm{ds}\cdot||\vartheta(u)-\vartheta'(u)||\\ &= \frac{q(v)}{v+q(v)}\cdot||\vartheta(u)-\vartheta'(u)||;\end{aligned}$$

taking now the supremum over $v\in E$ we obtain

$$||\mathcal{T}\vartheta(v)-\mathcal{T}\vartheta'(v)||\leq\gamma||\vartheta(u)-\vartheta'(u)||$$

with $\gamma\triangleq\sup_{v\in E}\frac{q(v)}{v+q(v)}$.
It is easy to observe that the quantity $\frac{q(v)}{v+q(v)}<1$ for every v in E because $q(v)$ and v are always positive quantities: γ is strictly lower than 1.

c) By considering $\vartheta(\cdot)=0$ in definition (2.19) we obtain $\mathcal{T}\vartheta(v)=\mathbb{E}_v[e^{-v\mathfrak{T}}\cdot 0]=0$. By its unicity the fixed point of the operator $\mathcal{T}$ is thus equal to zero. □

Proposition 2.8 *Let the functions ϑ_k be defined as in Lemma 2.2 for a given ϑ_0. For an arbitrarily small ϵ, for γ as in b) of Proposition 2.7 and for $n_\epsilon\in\mathbb{N}$ such that*

$$n_\epsilon\geq\left\lceil\frac{\log(\epsilon(1-\gamma))}{\log(\gamma)}-\frac{\sup_{v\in E}|\vartheta_1(v)-\vartheta_0(v)|}{\log(\gamma)}\right\rceil, \tag{2.23}$$

we have that

$$V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t})\triangleq\sum_{k=0}^{n_\epsilon}\vartheta_k(r_{\nu_t})\widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t}) \tag{2.24}$$

approximates the real price of the Prototype product defined as in (2.14) in the sense that

$$|V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t})-V_{\vartheta_0,t,T}(r_{\nu_t})|<\epsilon \text{ uniformily in } (t,T,r_{\nu_t}) \tag{2.25}$$

Proof Denote the norm $||f(v)|| = \sup_{v\in E}|f(v)|$ for every $f \in \mathfrak{M}$. We consider the contracting map $\mathcal{T}$ on the functions ϑ_h defined as in Lemma 2.2. Hence we use a classical result of the contracting map theory, that is

$$||\vartheta_n - \vartheta^*|| \leq \frac{\gamma^n}{1-\gamma}||\vartheta_1 - \vartheta_0||$$

where γ and ϑ^* are defined as in b) and c) of Proposition 2.7 respectively. By taking $n = n_\epsilon \in \mathbb{N}$ such that $\frac{\gamma^n}{1-\gamma}||\vartheta_1 - \vartheta_0|| < \epsilon$, namely

$$n_\epsilon > \frac{\log(\epsilon(1-\gamma))}{\log(\gamma)} - \frac{||\vartheta_1 - \vartheta_0||}{\log(\gamma)}$$

which is a condition equivalent to (2.23), we assure that the sup-norm distance between ϑ_n and $\vartheta^* = 0$ is smaller than ϵ for all $n \in \{n_\epsilon + 1, n_\epsilon + 2, \ldots\}$:

$$||\vartheta_n|| < \epsilon \quad \forall n \in (n_\epsilon, +\infty) \cap \mathbb{N}. \tag{2.26}$$

Furthermore

$$(2.26) \Rightarrow \sup_{v\in E}|\vartheta_n(v)| < \epsilon \Rightarrow |\vartheta_n(v)| < \epsilon \quad \forall v \in E,$$

so we have

$$-\epsilon < \vartheta_n(v) < \epsilon, \ \forall n \in \mathbb{N}\cap]n_\epsilon, +\infty[, \ \forall v \in E \tag{2.27}$$

Now we consider the price of the Prototype product at time t:

$$\begin{aligned} V_{\vartheta_0,t,T}(r_{\nu_t}) &\overset{(2.14)}{=} \sum_{k=0}^{+\infty} \vartheta_k(r_{\nu_t})\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t}) \\ &= \sum_{k=0}^{n_\epsilon} \vartheta_k(r_{\nu_t})\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t}) + \sum_{k=n_\epsilon+1}^{+\infty} \vartheta_k(r_{\nu_t})\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t}) \end{aligned}$$

and we define $V^\epsilon_{\vartheta_0,t,T}(r_{\nu_t}) \triangleq \sum_{k=0}^{n_\epsilon} \vartheta_k(r_{\nu_t})\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t}) = V_{\vartheta_0,t,T}(r_{\nu_t}) - \sum_{k=n_\epsilon+1}^{+\infty} \vartheta_k(r_{\nu_t})$ $\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t})$; by observing that, by (2.27),

$$\sum_{k=n_\epsilon+1}^{+\infty} -\epsilon\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t}) < \sum_{k=n_\epsilon+1}^{+\infty} \vartheta_k(r_{\nu_t})\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t}) < \sum_{k=n_\epsilon+1}^{+\infty} \epsilon\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t})$$

$$\Downarrow$$

$$-\epsilon\Big(1 - \sum_{k=0}^{n_\epsilon} \widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t})\Big) < -\sum_{k=n_\epsilon+1}^{+\infty} \vartheta_k(r_{\nu_t})\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t}) < \epsilon\Big(1 - \sum_{k=0}^{n_\epsilon} \widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t})\Big)$$

and that $\sum_{k=0}^{n_\epsilon} \widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t}) < 1$, we obtain the following confidence interval for $V_{\vartheta_0,t,T}(r_{\nu_t})$:

$$V_{\vartheta_0,t,T}(r_{\nu_t}) \in]V^\epsilon_{\vartheta_0,t,T}(r_{\nu_t}) - \epsilon, V^\epsilon_{\vartheta_0,t,T}(r_{\nu_t}) + \epsilon[.$$

Since n_ϵ does not depend on (t, T, r_{ν_t}, ν_t), we finally conclude that

$$|V^\epsilon_{\vartheta_0,t,T}(r_{\nu_t}) - V_{\vartheta_0,t,T}(r_{\nu_t})| < \epsilon \text{ uniformily in } (t, T, r_{\nu_t}).$$

□

Remark 2.9 *In the proof above we have observed that, in practice, it is not possible to compute the price of the (upper approximation of the) Prototype Product $V_{\vartheta_0,t,T}(r_{\nu_t})$ but only its approximation $V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t})$. Moreover the explicit representation of $V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t})$ uses the fact that the operator $\mathcal{T}$ defined in (2.18) is contracting.*
In the Chapter "Numerical results" we will discuss a "full simulation" approach which computes directly the price $V_{\vartheta_0,t,T}(r_{\nu_t})$ and does not require the contraction property of the operator $\mathcal{T}$.

2.2.2 The functions ϑ_k, explicit formula

We are going to give a computable representation for the ϑ_k's that are necessary to obtain a closed formula of $V_{\vartheta_0,t,T}(r_{\nu_t})$.

Lemma 2.10 *Let $r(t)$ be a CTMC with state space E and transition kernel $Q = (q_{i,j})_{1\le i,j\le N}$. Given the Prototype payoff defined by $\vartheta_0(\cdot) = \sum_{i_0=1}^{N} w_{i_0}\mathbf{I}_{\{\cdot = r^{i_0}\}}$ with $w_1,\dots,w_N \in \{0\}\cup\mathbb{R}_+$ N non negative real values, the functions ϑ_k defined as in Lemma 2.2 satisfy the following formula*

$$\vartheta_k(\cdot) = \sum_{i_k=1}^{N} w^k_{i_k}\mathbf{I}_{\{\cdot = r^{i_k}\}},\ k\ge 1 \tag{2.28}$$

where, for $m = 1,\dots,N$,

$$\begin{cases} w^1_m \triangleq \displaystyle\sum_{\substack{i_0=1\\ i_0\neq m}}^{N} w_{i_0}\frac{q_{m,i_0}}{r^m + q_m}, & k=1 \\ w^k_m \triangleq \displaystyle\sum_{\substack{i_0,\dots,i_{k-1}=1\\ i_0\neq i_1,\dots,i_{k-1}\neq m}}^{N} w_{i_0}\frac{q_{m,i_{k-1}}}{r^m+q_m}\left[\prod_{h=1}^{k-1}\frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}}\right], & k>1 \end{cases} \tag{2.29}$$

Proof We consider w.l.o.g the functions ϑ_k evaluated in r_i, the state of the process at a generic transition time T_i. Let us prove the statement by an induction method.

Base Case $(k=1)$

$$\begin{aligned} \vartheta_1(r_i) &\overset{(2.9)}{=} \mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\vartheta_0(r_{i+1})\big|r_i\big] \\ &\overset{(2.6)}{=} \mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\sum_{i_0=1}^{N} w_{i_0}\mathbf{I}_{\{r_{i+1}=r^{i_0}\}}\big|r_i\big] \\ &= \sum_{i_0,i_1=1}^{N} w_{i_0}\mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\mathbf{I}_{\{r_{i+1}=r^{i_0}\}}\big|r_i=r^{i_1}\big]\mathbf{I}_{\{r_i=r^{i_1}\}} \end{aligned} \tag{2.30}$$

Now

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\mathbf{I}_{\{r_{i+1}=r^{i_0}\}}\big|r_i=r^{i_1}\Big] = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\big|r_i=r^{i_1}\Big]\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbf{I}_{\{r_{i+1}=r^{i_0}\}}\big|r_i=r^{i_1}\Big] \tag{2.31}$$

because, conditionally on r_i, the interarrival time $\mathrm{T}_{i+1} - \mathrm{T}_i$ and r_{i+1} (the value of the process at the transition time T_{i+1}) are independent by the properties of the CTMCs.
Moreover we have that

- $\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\Big|r_i = r^{i_1}\Big] = \int_0^\infty e^{-r^{i_1}u} q_{i_1} e^{-q_{i_1}u} du = \frac{q_{i_1}}{r^{i_1}+q_{i_1}}$
because $\mathrm{T}_{i+1} - \mathrm{T}_i$ is exponentially distributed in accordance with (2.2),
- $\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbf{I}_{\{r_{i+1}=r^{i_0}\}}\Big|r_i = r^{i_1}\Big] = \widetilde{\mathbb{P}}(r_{i+1} = r^{i_0}|r_i = r^{i_1}) = p_{i_1,i_0}$ the transition probability from state r^{i_1} to r^{i_0};

hence

$$\begin{aligned}\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\mathbf{I}_{\{r_{i+1}=r^{i_0}\}}\Big|r_i = r^{i_1}\Big] &\overset{(2.31)}{=} p_{i_1,i_0}\frac{q_{i_1}}{r^{i_1}+q_{i_1}} \\ &\overset{(2.1)}{=} \frac{q_{i_1,i_0}}{r^{i_1}+q_{i_1}}, \quad \forall i_0 \neq i_1 \end{aligned} \tag{2.32}$$

We obtain that

$$\vartheta_1(r_i) \overset{(2.30)}{=} \sum_{\substack{i_0,i_1=1 \\ i_0\neq i_1}}^{N} w_{i_0}\frac{q_{i_1,i_0}}{r^{i_1}+q_{i_1}}\mathbf{I}_{\{r_i=r^{i_1}\}} = \sum_{i_1=1}^{N} w_{i_1}^1 \mathbf{I}_{\{r_i=r^{i_1}\}} \tag{2.33}$$

where $w_{i_1}^1 = \sum_{\substack{i_0=1 \\ i_0\neq i_1}}^{N} w_{i_0}\frac{q_{i_1,i_0}}{r^{i_1}+q_{i_1}}$ defined as in (2.29).

Inductive step
By Lemma 2.2

$$\vartheta_k(r_i) = \mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\vartheta_{k-1}(r_{i+1})\big|r_i\big] = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\sum_{i_{k-1}=1}^{N} w_{i_{k-1}}^{k-1}\mathbf{I}_{\{r_{i+1}=r^{i_{k-1}}\}}\right]. \tag{2.34}$$

By the induction hypothesis

$$
\begin{aligned}
(2.34) \quad &= \mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)} \sum_{\substack{i_0,\ldots,i_{k-1}=1 \\ i_0\neq i_1,\ldots,i_{k-2}\neq i_{k-1}}}^{N} w_{i_0}\left[\prod_{h=1}^{k-1}\frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}}\right]\mathbf{I}_{\{r_{i+1}=r^{i_{k-1}}\}}\big|r_i\right] \\
&= \sum_{\substack{i_0,\ldots,i_{k-1}=1 \\ i_0\neq i_1,\ldots,i_{k-2}\neq i_{k-1}}}^{N} w_{i_0}\left[\prod_{h=1}^{k-1}\frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}}\right]\mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\mathbf{I}_{\{r_{i+1}=r^{i_{k-1}}\}}\big|r_i\right] \\
&= \sum_{i_k=1}^{N}\left\{\sum_{\substack{i_0,\ldots,i_{k-1}=1 \\ i_0\neq i_1,\ldots,i_{k-2}\neq i_{k-1}}}^{N} w_{i_0}\left[\prod_{h=1}^{k-1}\frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}}\right]\right. \\
&\qquad \left.\mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\mathbf{I}_{\{r_{i+1}=r^{i_{k-1}}\}}\big|r_i=r^{i_k}\right]\mathbf{I}_{\{r_i=r^{i_k}\}}\right\} \\
&\overset{(2.32)}{=} \sum_{i_k=1}^{N}\left\{\sum_{\substack{i_0,\ldots,i_{k-1}=1 \\ i_0\neq i_1,\ldots,i_{k-1}\neq i_k}}^{N} w_{i_0}\frac{q_{i_k,i_{k-1}}}{r^{i_k}+q_{i_k}}\left[\prod_{h=1}^{k-1}\frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}}\right]\right\}\mathbf{I}_{\{r_i=r^{i_k}\}} \\
&= \sum_{\substack{i_0,\ldots,i_k=1 \\ i_0\neq i_1,\ldots,i_{k-1}\neq i_k}}^{N} w_{i_0}\left[\prod_{h=1}^{k}\frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}}\right]\mathbf{I}_{\{r_i=r^{i_k}\}} = \sum_{i_k=1}^{N} w_{i_k}^{k}\mathbf{I}_{\{r_i=r^{i_k}\}}
\end{aligned}
$$

where $w_{i_k}^k = \sum_{\substack{i_0,\ldots,i_{k-1}=1 \\ i_0\neq i_1,\ldots,i_{k-1}\neq i_k}}^{N} w_{i_0}\frac{q_{i_k,i_{k-1}}}{r^{i_k}+q_{i_k}}\left[\prod_{h=1}^{k-1}\frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}}\right]$ defined as in (2.29). □

By Corollary 2.6 we know that the $\vartheta_k(\cdot)$ belong to $\mathfrak{M}$, $\forall k \in \mathbb{N}$ because the Prototype payoff $\vartheta_0(\cdot) \in \mathfrak{M}$. In the previous Lemma we have proved that $\vartheta_k(\cdot) = \sum_{i_k=1}^{N} w_{i_k}^k \mathbf{I}_{\{\cdot=r^{i_k}\}}$ with coefficients $w_{i_k}^k$ known when the initial values $w_1,\ldots,w_N$ are given. We present now a simpler representation for $\vartheta_k(\cdot)$ by introducing vector notation:

Definition 2.11 *Let $\underline{r} = [r^1,\ldots,r^N]'$ be the N-dimensional vector with components the values of state space E and define*

- $\theta_0(\underline{r}) \triangleq [w_1,\ldots,w_N]'$ *where the components correspond to the Prototype payoff $\vartheta_0(\cdot)$*
- $\theta_k(\underline{r}) = [w_1^k,\ldots,w_N^k]'$, *with w_m^k defined in Lemma 2.10.*

In other terms, for $k \in \mathbb{N}$, $\{\vartheta_k(r^i)\}_{\{r^i\in E\}}$ is the collection of all possible values assumed by the function ϑ_k and, for a fixed $r^i \in E$, $\vartheta_k(r^i)$ is the i-th component of vector $\theta_k(\underline{r})$.

Lemma 2.12 *Let $r(t)$ be a CTMC with state space E and transition kernel $Q = (q_{i,j})_{1\le i,j\le N}$. The vectors $\theta_k(\underline{r})$ in Definition 2.11 admit, for $k \in \mathbb{N}$, the representation*

$$\theta_k(\underline{r}) = \widetilde{Q}^k \cdot \theta_0(\underline{r}), k \in \mathbb{N} \tag{2.35}$$

with $\widetilde{Q}^0 \triangleq I_N$ *the identity matrix and*

$$\widetilde{Q} = \begin{bmatrix} 0 & \frac{q_{1,2}}{r^1+q_1} & \frac{q_{1,3}}{r^1+q_1} & \cdots & \frac{q_{1,N}}{r^1+q_1} \\ \frac{q_{2,1}}{r^2+q_2} & 0 & \frac{q_{2,3}}{r^2+q_2} & \cdots & \frac{q_{2,N}}{r^2+q_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{q_{N-1,1}}{r^{N-1}+q_{N-1}} & \frac{q_{N-1,2}}{r^{N-1}+q_{N-1}} & \cdots & 0 & \frac{q_{N-1,N}}{r^{N-1}+q_{N-1}} \\ \frac{q_{N,1}}{r^N+q_N} & \frac{q_{N,2}}{r^N+q_N} & \cdots & \frac{q_{N,N-1}}{r^N+q_N} & 0 \end{bmatrix}. \tag{2.36}$$

Proof
We prove the statement by an induction method.

Base Case $k = 0$
By Definition 2.11 we have

$$\theta_0(\underline{r}) = [w_1, \ldots, w_N]' = \widetilde{Q}^0 \cdot \theta_0(\underline{r})$$

Inductive step
Observing that relation (2.35) is equivalent to

$$\theta_k(\underline{r}) = \widetilde{Q} \cdot \theta_{k-1}(\underline{r}),\ k > 0 \tag{2.37}$$

it is sufficient to prove (2.37) by using the induction hypothesis $\theta_{k-1}(\underline{r}) = \widetilde{Q}^{k-1} \cdot \theta_0(\underline{r})$. Letting $D(\xi) = \{i_1 \neq i_2, \ldots, i_{k-2} \neq \xi\}$ we have in fact

$$\theta_k(\underline{r}) = \begin{bmatrix} w_1^k \\ w_2^k \\ \vdots \\ w_N^k \end{bmatrix} \overset{(2.29)}{=} \begin{bmatrix} \sum\limits_{\substack{i_0,\ldots,i_{k-1}=1 \\ i_0 \neq i_1,\ldots,i_{k-1} \neq 1}}^{N} w_{i_0} \frac{q_{1,i_{k-1}}}{r^1+q_1} \prod\limits_{h=1}^{k-1} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}} \\ \sum\limits_{\substack{i_0,\ldots,i_{k-1}=1 \\ i_0 \neq i_1,\ldots,i_{k-1} \neq 2}}^{N} w_{i_0} \frac{q_{2,i_{k-1}}}{r^2+q_2} \prod\limits_{h=1}^{k-1} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}} \\ \vdots \\ \sum\limits_{\substack{i_0,\ldots,i_{k-1}=1 \\ i_0 \neq i_1,\ldots,i_{k-1} \neq N}}^{N} w_{i_0} \frac{q_{N,i_{k-1}}}{r^N+q_N} \prod\limits_{h=1}^{k-1} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}} \end{bmatrix}$$

$$
= \begin{bmatrix}
\displaystyle\sum_{\substack{i_{k-1}=1\\ i_{k-1}\neq 1}}^{N} \frac{q_{1,i_{k-1}}}{r^1+q_1} \Bigg\{ \sum_{\substack{i_0,\dots,i_{k-2}=1\\ i_0\neq i_1,\dots,i_{k-2}\neq i_{k-1}}}^{N} w_{i_0} \frac{q_{i_{k-1},i_{k-2}}}{r^{i_{k-1}}+q_{i_{k-1}}} \prod_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}} \Bigg\} \\
\displaystyle\sum_{\substack{i_{k-1}=1\\ i_{k-1}\neq 2}}^{N} \frac{q_{2,i_{k-1}}}{r^2+q_2} \Bigg\{ \sum_{\substack{i_0,\dots,i_{k-2}=1\\ i_0\neq i_1,\dots,i_{k-2}\neq i_{k-1}}}^{N} w_{i_0} \frac{q_{i_{k-1},i_{k-2}}}{r^{i_{k-1}}+q_{i_{k-1}}} \prod_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}} \Bigg\} \\
\vdots \\
\displaystyle\sum_{\substack{i_{k-1}=1\\ i_{k-1}\neq N}}^{N} \frac{q_{N,i_{k-1}}}{r^N+q_N} \Bigg\{ \sum_{\substack{i_0,\dots,i_{k-2}=1\\ i_0\neq i_1,\dots,i_{k-2}\neq i_{k-1}}}^{N} w_{i_0} \frac{q_{i_{k-1},i_{k-2}}}{r^{i_{k-1}}+q_{i_{k-1}}} \prod_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}} \Bigg\}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\frac{q_{1,2}}{r^1+q_1} \Big\{ \displaystyle\sum_{\substack{i_0,\dots,i_{k-2}=1\\ D(2)}}^{N} w_{i_0} \frac{q_{2,i_{k-2}}}{r^2+q_2} \prod_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}} \Big\} + \cdots + \frac{q_{1,N}}{r^1+q_1} \Big\{ \displaystyle\sum_{\substack{i_0,\dots,i_{k-2}=1\\ D(N)}}^{N} w_{i_0} \frac{q_{N,i_{k-2}}}{r^N+q_N} \prod_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}} \Big\} \\
\frac{q_{2,1}}{r^2+q_2} \Big\{ \displaystyle\sum_{\substack{i_0,\dots,i_{k-2}=1\\ D(1)}}^{N} w_{i_0} \frac{q_{1,i_{k-2}}}{r^1+q_1} \prod_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}} \Big\} + \cdots + \frac{q_{2,N}}{r^2+q_2} \Big\{ \displaystyle\sum_{\substack{i_0,\dots,i_{k-2}=1\\ D(N)}}^{N} w_{i_0} \frac{q_{N,i_{k-2}}}{r^N+q_N} \prod_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}} \Big\} \\
\vdots \\
\frac{q_{N,1}}{r^N+q_N} \Big\{ \displaystyle\sum_{\substack{i_0,\dots,i_{k-2}=1\\ D(1)}}^{N} w_{i_0} \frac{q_{1,i_{k-2}}}{r^1+q_1} \prod_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}} \Big\} + \cdots + \frac{q_{N-1,N}}{r^{N-1}+q_{N-1}} \Big\{ \displaystyle\sum_{\substack{i_0,\dots,i_{k-2}=1\\ D(N-1)}}^{N} w_{i_0} \frac{q_{N-1,i_{k-2}}}{r^N+q_N} \prod_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}} \Big\}
\end{bmatrix}
$$

$$
= \widetilde{Q} \cdot \begin{bmatrix}
\displaystyle\sum_{\substack{i_0,\dots,i_{k-2}\\ D(1)}}^{N} w_{i_0} \frac{q_{1,i_{k-2}}}{r^1+q_1} \prod_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}} \\
\displaystyle\sum_{\substack{i_0,\dots,i_{k-2}\\ D(2)}}^{N} w_{i_0} \frac{q_{2,i_{k-2}}}{r^2+q_2} \prod_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}} \\
\vdots \\
\displaystyle\sum_{\substack{i_0,\dots,i_{k-2}\\ D(N)}}^{N} w_{i_0} \frac{q_{N,i_{k-2}}}{r^N+q_N} \prod_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r^{i_h}+q_{i_h}}
\end{bmatrix} = \widetilde{Q} \cdot \theta_{k-1}(\underline{r})
$$

□

The explicit representation of the functions ϑ_k in Lemma 2.10 allows us to evaluate the quantity π introduced in Remark 2.5. We are able to give a sufficient condition to have $\pi \simeq 0$ in the cases in which we are interested, that is when the Prototype payoff $\vartheta_0(\cdot) = \sum_{i=1}^{N} w_i \mathbf{I}_{\{\cdot = r^i\}}$ has coefficients $w_i \equiv 1$ for $i = 1, \dots, N$.

Lemma 2.13 *Let $r(t)$ be a CTMC with state space $E \in \mathbb{R}^N$ and let us suppose that the Prototype payoff $\vartheta_0(\cdot) = \sum_{i=1}^{N} w_i \mathbf{I}_{\{\cdot = r^i\}}$ with $w_i \equiv 1$, $i = 1, \dots, N$, then the quantity π in (2.17) is negligible if*

$$
\frac{r^i}{q_i} \simeq 0 \qquad \forall i = 1, \dots, N \tag{2.38}
$$

where $r^i \in E$ and q_i represents the intensity associated to the state r^i.

Proof By Remark 2.5 we have that

$$\pi = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_T-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)(\vartheta_0(r_{\nu_T})-\vartheta_1(r_{\nu_T}))|\mathcal{F}^r_{\nu_t}\Big].$$

A sufficient condition to obtain $\pi \simeq 0$ is then given by the following relation

$$\vartheta_0(r) - \vartheta_1(r) \simeq 0, \qquad r \in E. \tag{2.39}$$

By the explicit formulae of the functions ϑ_0 and ϑ_1 given in Lemma 2.10 we obtain

$$\begin{aligned}
\vartheta_0(\cdot)-\vartheta_1(\cdot) &= \sum_{i=1}^{N}\Big\{w_i - \sum_{\substack{i_0=1\\ i_0\neq i}}^{N} w_{i_0}\frac{q_{i,i_0}}{r^i+q_i}\Big\}\mathbb{I}_{\{\cdot=r^i\}} \\
&= \sum_{i=1}^{N}\Big\{1 - \sum_{\substack{i_0=1\\ i_0\neq i}}^{N} \frac{q_{i,i_0}}{r^i+q_i}\Big\}\mathbb{I}_{\{\cdot=r^i\}} \\
&= \sum_{i=1}^{N}\Big\{1 - \frac{q_i}{r^i+q_i}\Big\}\mathbb{I}_{\{\cdot=r^i\}} = \sum_{i=1}^{N}\Big\{1 - \frac{1}{\frac{r^i}{q_i}+1}\Big\}\mathbb{I}_{\{\cdot=r^i\}}
\end{aligned}$$

where in the second passage we have used the hypotheses $w_i = 1$, $i = 1,\dots,N$. Hence, $\vartheta_0(\cdot)-\vartheta_1(\cdot) \simeq 0$ if and only if $1 - \frac{1}{\frac{r^i}{q_i}+1} \simeq 0$ for each $i = 1,\dots,N$, which is equivalent to (2.38). □

Remark 2.14 *The condition (2.38) can be seen to be satisfied when the short rate takes values that are relatively small with respect to the intensities, which is e.g. the case when there are frequent jumps. In the numerical results this condition is satisfied and in fact the short rate frequently jumps.*

2.2.3 Distribution of $\nu_{t,T}$

The discrete random variable $\nu_{t,T}$, as we have introduced it at the beginning of the chapter, represents the number of jumps of the process $r(\cdot)$ between t and T. We now compute $\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t} = r^m)$, namely the probability of k jumps occurring in the interval $[t,T]$ when the process $r(\cdot)$ at time t is equal to r^m, for all $k \in \mathbb{N}$ and $r^m \in E$, $m \in \{1,\dots,N\}$.

Lemma 2.15 *Let $r(t)$ be a CTMC with state space E and transition kernel $Q = (q_{i,j})_{1\leq i,j\leq N}$, then, for every positive $k \in \mathbb{N}$ and $r^m \in E$ with $m \in \{1,\dots,N\}$, we have*

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t} = r^m) = \sum_{\substack{i=1\\ i\neq m}}^{N} q_{m,i}\int_t^T e^{-q_m(s-t)}\widetilde{\mathbb{P}}(\nu_{s,T} = k-1|r_{\nu_s} = r^i)ds, \ \textit{with } t < T. \tag{2.40}$$

Proof Let us denote the first jump time after t as the random variable $\widehat{\tau}$, that is $\widehat{\tau} = \inf\{u > 0 : \nu_{t+u} > \nu_{(t+u)^-}\}$. By properties of CTMC's we observe that $(\widehat{\tau}|r_{\nu_t} = r^m) \sim \mathcal{E}xp(q_m)$. Consider now the random variable τ defined as follows:

$$\tau \triangleq \widehat{\tau} + t;$$

since the density function of $\widehat{\tau}$ is $\widetilde{\mathbb{P}}(\widehat{\tau} \in ds|r_{\nu_t} = r^m) = q_m e^{-q_m s} ds$, the density function of τ is given by

$$\widetilde{\mathbb{P}}(\tau \in ds|r_{\nu_t} = r^m) = \widetilde{\mathbb{P}}(\widehat{\tau} + t \in ds|r_{\nu_t} = r^m) = q_m e^{-q_m(s-t)} ds, \ s > t. \tag{2.41}$$

We can now proceed to prove the statement. In fact, by the law of total probability, we have that

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t} = r^m) = \int_t^T \sum_{i=1}^N \widetilde{\mathbb{P}}(\tau \in ds, r_{\nu_\tau} = r^i|r_{\nu_t} = r^m)\widetilde{\mathbb{P}}(\nu_{s,T} = k-1|r_{\nu_s} = r^i) \tag{2.42}$$

where r^i $(i = 1, \ldots, N)$ are all possible states reachable at the jump time τ. By properties of CTMC's, the random variables τ and r_{ν_τ} are, conditionally on r_{ν_t}, independent and so

$$\begin{aligned} \widetilde{\mathbb{P}}(\tau \in ds, r_{\nu_\tau} = r^i|r_{\nu_t} = r^m) &= \widetilde{\mathbb{P}}(\tau \in ds|r_{\nu_t} = r^m)\widetilde{\mathbb{P}}(r_{\nu_\tau} = r^i|r_{\nu_t} = r^m) \\ &\overset{(2.41)}{=} q_m e^{-q_m(s-t)} p_{m,i} ds. \end{aligned}$$

Hence, by (2.42), we obtain

$$\begin{aligned} \widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t} = r^m) &= \int_t^T \sum_{i=1}^N p_{m,i} q_m e^{-q_m(s-t)} \widetilde{\mathbb{P}}(\nu_{s,T} = k-1|r_{\nu_s} = r^i) ds \\ &\overset{(2.1)}{=} \sum_{\substack{i=1 \\ i \neq m}}^N q_{m,i} \int_t^T e^{-q_m(s-t)} \widetilde{\mathbb{P}}(\nu_{s,T} = k-1|r_{\nu_s} = r^i) ds. \end{aligned}$$

□

We derive now an explicit expression for the probabilities defined in (2.40).

Lemma 2.16 *Under the same hypotheses of Lemma 2.15, assuming w.l.o.g that $\nu_t = h \in \mathbb{N}$ and denoting $m = i_h$ for simplicity of notations, it follows that*

$$\begin{cases} \widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t} = r^{i_h}) = \displaystyle\sum_{\substack{i_{h+1}, \ldots, i_{h+k} = 1 \\ i_{h+1} \neq i_h, i_{h+2} \neq i_{h+1}, \ldots, i_{h+k} \neq i_{h+k-1}}}^N e^{q_{i_h} t - q_{i_{h+k}} T} \varphi_{h,k}(Q) \cdot \Psi_{h,k}(t, T, Q) \\ \widetilde{\mathbb{P}}(\nu_{t,T} = 0|r_{\nu_t} = r^{i_h}) = e^{-q_{i_h}(T-t)} \end{cases} \tag{2.43}$$

with i_h a fixed index in $\{1, \ldots, N\}$, $\Psi_{h,k}$ the following multiple integral

$$\begin{aligned} \Psi_{h,k}(t, T, Q) \triangleq & \int_t^T e^{(q_{i_{h+1}} - q_{i_h}) t_{h+1}} \int_{t_{h+1}}^T e^{(q_{i_{h+2}} - q_{i_{h+1}}) t_{h+2}} \ldots \\ & \ldots \int_{t_{h+k-1}}^T e^{(q_{i_{h+k}} - q_{i_{h+k-1}}) t_{h+k}} dt_{h+k} \ldots dt_{h+2} dt_{h+1} \end{aligned}$$

and

$$\varphi_{h,k}(Q) \triangleq q_{i_h,i_{h+1}} \cdot \ldots \cdot q_{i_{h+k-1},i_{h+k}}.$$

Proof The case $k = 0$ is proved directly by properties of CTMC's. Let us prove by an induction method the general case for $k \geq 1$.

Base Case $(k = 1)$

$$\begin{aligned}
\widetilde{\mathbb{P}}(\nu_{t,T} = 1 | r_{\nu_t} = r^{i_h}) &\overset{(2.40)}{=} \sum_{\substack{i_{h+1}=1 \\ i_{h+1}\neq i_h}}^{N} q_{i_h,i_{h+1}} \int_t^T e^{-q_{i_h}(t_{h+1}-t)} \widetilde{\mathbb{P}}(\nu_{t_{h+1},T} = 0 | r_{\nu_{t_{h+1}}} = r^{i_{h+1}}) dt_{h+1} \\
&= \sum_{\substack{i_{h+1}=1 \\ i_{h+1}\neq i_h}}^{N} q_{i_h,i_{h+1}} \int_t^T e^{-q_{i_h}(t_{h+1}-t)} e^{-q_{i_{h+1}}(T-t_{h+1})} dt_{h+1} \\
&= \sum_{\substack{i_{h+1}=1 \\ i_{h+1}\neq i_h}}^{N} q_{i_h,i_{h+1}} e^{(q_{i_h}t - q_{i_{h+1}}T)} \int_t^T e^{(q_{i_{h+1}} - q_{i_h})t_{h+1}} dt_{h+1}
\end{aligned}$$

Inductive step:

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r^{i_h}) \overset{(2.40)}{=} \sum_{\substack{i_{h+1}=1 \\ i_{h+1}\neq i_h}}^{N} q_{i_h,i_{h+1}} \int_t^T e^{-q_{i_h}(t_{h+1}-t)} \widetilde{\mathbb{P}}(\nu_{t_{h+1},T} = k-1 | r_{\nu_{t_{h+1}}} = r^{i_{h+1}}) dt_{h+1}$$

$$= \sum_{\substack{i_{h+1}=1 \\ i_{h+1}\neq i_h}}^{N} q_{i_h,i_{h+1}} \int_t^T e^{-q_{i_h}(t_{h+1}-t)} \cdot$$

$$\cdot \Big\{ \sum_{\substack{i_{h+2},\ldots,i_{h+k}=1 \\ i_{h+2}\neq i_{h+1}, i_{h+3}\neq i_{h+2}, \ldots, i_{h+k}\neq i_{h+k-1}}}^{N} e^{q_{i_{h+1}}t_{h+1} - q_{i_{h+k}}T} \varphi_{h+1,k-1}(Q) \cdot \Psi_{h+1,k-1}(t_{h+1}, T, Q) \Big\} dt_{h+1}$$

$$\begin{aligned}
&= \sum_{\substack{i_{h+1},\ldots,i_{h+k}=1 \\ i_{h+1}\neq i_h, i_{h+2}\neq i_{h+1}, \ldots, i_{h+k}\neq i_{h+k-1}}}^{N} q_{i_h,i_{h+1}} e^{q_{i_h}t - q_{i_{h+k}}T} \varphi_{h+1,k-1}(Q) \\
&\qquad \int_t^T e^{(q_{i_{h+1}} - q_{i_h})t_{h+1}} \Psi_{h+1,k-1}(t_{h+1}, T, Q) dt_{h+1} \\
&= \sum_{\substack{i_{h+1},\ldots,i_{h+k}=1 \\ i_{h+1}\neq i_h, i_{h+2}\neq i_{h+1}, \ldots, i_{h+k}\neq i_{h+k-1}}}^{N} e^{q_{i_h}t - q_{i_{h+k}}T} \varphi_{h,k}(Q) \Psi_{h,k}(t, T, Q) dt_{h+1}
\end{aligned}$$

where in the last passage we have used the fact that

$$\varphi_{h+1,k-1} = q_{i_{h+1},i_{(h+1)+1}} \cdot \ldots \cdot q_{i_{(h+1)+(k-1)-1},i_{(h+1)+(k-1)}} = \frac{\varphi_{h,k}}{q_{i_h,i_{h+1}}}$$

and

$$\Psi_{h+1,k-1}(t_{h+1},T,Q) = \int_{t_{h+1}}^{T} e^{(q_{i_{h+2}}-q_{i_{h+1}})t_{h+2}} \int_{t_{h+2}}^{T} e^{(q_{i_{h+3}}-q_{i_{h+2}})t_{h+3}} \dots \\ \dots \int_{t_{h+k-1}}^{T} e^{(q_{i_{h+k}}-q_{i_{h+k-1}})t_{h+k}} dt_{h+k} \dots dt_{h+3} dt_{h+2}$$

□

Remark 2.17 *In the previous Lemma, the dependence of the probability* $\widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t}=r^{i_0})$ *on* $h=\nu_t$ *(the number of jumps occurred until the time* t*) is due only to technical reasons during the proof. In fact the following representation of these probabilites is equal to the one given by (2.43):*

$$\widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t}=r^{i_0}) = \sum_{\substack{i_1,\dots,i_k=1 \\ i_1\neq i_0, i_2\neq i_1,\dots,i_k\neq i_{k-1}}}^{N} e^{q_{i_0}t-q_{i_k}T}\varphi_k(Q)\cdot\Psi_k(t,T,Q) \tag{2.44}$$

with i_0 *a fixed index in* $\{1,\dots,N\}$*,* Ψ_k *the following multiple integral*

$$\Psi_k(t,T,Q) \triangleq \int_t^T e^{(q_{i_1}-q_{i_0})t_1} \int_{t_1}^T e^{(q_{i_2}-q_{i_1})t_2} \dots \int_{t_{k-1}}^T e^{(q_{i_k}-q_{i_{k-1}})t_k} dt_k \dots dt_2 dt_1 \tag{2.45}$$

and

$$\varphi_k(Q) \triangleq q_{i_0,i_1}\cdot\ldots\cdot q_{i_{k-1},i_k}.$$

We recall now an interesting property of an homogeneous time CTMC -namely a CTMC with transition kernel $Q=(q_{i,j})_{1\leq i,j\leq N}$ independent with respect the time- which is very useful in our context:

Remark 2.18 *Let* ν_τ *be the discrete random variable which represents the number of jumps until* τ *such that* $\tau=T-t$ *with* $t,T\in\mathbb{R}_+$*. When the process* $r(\cdot)$ *is an homogeneous time CTMC -as in our case- the random variable* $\nu_{t,T}$ *is equivalent in law to* $\nu_\tau\equiv\nu_{0,\tau}$*. Hence, by Remark 2.17,*

$$\widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t}=r^{i_0}) = \widetilde{\mathbb{P}}(\nu_{T-t}=k|r(0)=r^{i_0}) \tag{2.46}$$
$$= \sum_{\substack{i_1,\dots,i_k=1 \\ i_1\neq m, i_2\neq i_1,\dots,i_k\neq i_{k-1}}}^{N} e^{-q_{i_k}(T-t)}\varphi_k(Q)\cdot\Psi_k(0,T-t,Q)$$

where the quantities involved are defined as in the previous Remark.

2.2.4 Price of Prototype product $V_{\vartheta_0,t,T}$: closed formula

We are now able to give an explicit representation of the price of the Prototype product represented as in Lemma 2.3 and its approximation $V^{\epsilon}_{\vartheta_0,t,T}$ introduced in Proposition 2.8.

Proposition 2.19 *Under the same hypotheses and notations of Proposition 2.8 and Lemma 2.10 and assuming that at the evaluation time t the spot rate is equal to a fixed $r^i \in E$, the Prototype product price $V_{\vartheta_0,t,T}(r_{\nu_t})$ and its approximation $V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t})$ admit respectively the following representations*

$$V_{\vartheta_0,t,T}(r_{\nu_t})|_{r_{\nu_t}=r^i} = \sum_{k=0}^{+\infty} [\widetilde{Q}^k \cdot \theta_0(\underline{r})]_i \widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r^i), \tag{2.47}$$

and

$$V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t})|_{r_{\nu_t}=r^i} = \sum_{k=0}^{n_\epsilon} [\widetilde{Q}^k \cdot \theta_0(\underline{r})]_i \widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r^i), \tag{2.48}$$

where $\widetilde{Q}$ is defined as in (2.36), $\theta_0(\underline{r})$ is as in Definition 2.11 and $[v]_i$ denotes, for a generic vector v, its i-th element.

Proof By formulas (2.14) and (2.35) we have that

$$V_{\vartheta_0,t,T}(r_{\nu_t})|_{r_{\nu_t}=r^i} = \mathbb{E}^{\widetilde{\mathbb{P}}}[[\widetilde{Q}^{\nu_{t,T}} \cdot \theta_0(\underline{r})]_i | r_{\nu_t} = r^i];$$

it follows that

$$V_{\vartheta_0,t,T}(r_{\nu_t})|_{(r_{\nu_t})=r^i} = \sum_{k=0}^{+\infty} [\widetilde{Q}^k \cdot \theta_0(\underline{r})]_i \widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r^i). \tag{2.49}$$

We obtain a similar expression for $V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t})$, that is

$$V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t})|_{r_{\nu_t}=r^i} = \mathbb{E}^{\widetilde{\mathbb{P}}}_{\epsilon}[[\widetilde{Q}^{\nu_{t,T}} \cdot \theta_0(\underline{r})]_i | r_{\nu_t} = r^i] = \sum_{k=0}^{n_\epsilon} [\widetilde{Q}^k \cdot \theta_0(\underline{r})]_i \widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r^i)$$

□

We are able to give a simpler explicit formula of the price of the Prototype Product in the particular case when the matrix $\widetilde{Q}$ in (2.36) is diagonalizable:

Proposition 2.20 *Let us suppose that $\widetilde{Q}$ defined by (2.36) is diagonalizable. Under the same hypotheses and notations of Proposition 2.8 and Lemma 2.10, $V_{\vartheta_0,t,T}(r_{\nu_t})$ defined as in (2.14) and $V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t})$ admit respectively the following representations*

$$V_{\vartheta_0,t,T}(r_{\nu_t})|_{r_{\nu_t}=r^i} = e_i' \cdot S \cdot \begin{bmatrix} \mathbb{E}^{\widetilde{\mathbb{P}}}[d_1^{\nu_{t,T}} | r_{\nu_t} = r^i] & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbb{E}^{\widetilde{\mathbb{P}}}[d_N^{\nu_{t,T}} | r_{\nu_t} = r^i] \end{bmatrix} \cdot S^{-1} \cdot \theta_0(\underline{r}), \tag{2.50}$$

and

$$V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu t})|_{r_{\nu_t}=r^i} = e_i' \cdot S \cdot \begin{bmatrix} \mathbb{E}^{\widetilde{\mathbb{P}}}_{\epsilon}[d_1^{\nu_{t,T}}|r_{\nu_t}=r^i] & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbb{E}^{\widetilde{\mathbb{P}}}_{\epsilon}[d_N^{\nu_{t,T}}|r_{\nu_t}=r^i] \end{bmatrix} \cdot S^{-1} \cdot \theta_0(\underline{r}), \quad (2.51)$$

where S is an $N \times N$ matrix the columns of which are the eigenvectors of $\widetilde{Q}$, $(d_j)_{j=1,\ldots,N}$ are the eigenvalues of $\widetilde{Q}$, e_i is the i^{th} unit vector, $\theta_0(\underline{r})$ is as in Definition 2.11 and $\mathbb{E}^{\widetilde{\mathbb{P}}}_{\epsilon}[d^{\nu_{t,T}}|r_{\nu_t}=r^i] \triangleq \sum_{k=0}^{n_\epsilon} d^k \widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t}=r^i)$ for a real number d and n_ϵ given by (2.23).

Proof By (2.35) we have $\theta_{\nu_{t,T}}(\underline{r}) = \widetilde{Q}^{\nu_{t,T}} \cdot \theta_0(\underline{r})$ and it follows that $w_i^{\nu_{t,T}}(Q)$, the i^{th} element of $\theta_{\nu_{t,T}}(\underline{r})$, is equal to $[\widetilde{Q}^{\nu_{t,T}} \cdot \theta_0(\underline{r})]_i$: hence

$$\vartheta_{\nu_{t,T}}(r^i) = [\widetilde{Q}^{\nu_{t,T}} \cdot \theta_0(\underline{r})]_i. \quad (2.52)$$

By formula (2.15) we have

$$\begin{aligned} V_{\vartheta_0,t,T}(r_{\nu t})|_{r_{\nu_t}=r^i} &= \mathbb{E}^{\widetilde{\mathbb{P}}}[\vartheta_{\nu_{t,T}}(r_{\nu_t})|r_{\nu_t}=r^i] \\ &\overset{(2.52)}{=} \mathbb{E}^{\widetilde{\mathbb{P}}}[[\widetilde{Q}^{\nu_{t,T}} \cdot \theta_0(\underline{r})]_i|r_{\nu_t}=r^i] \\ &= \mathbb{E}^{\widetilde{\mathbb{P}}}[[S \cdot D^{\nu_{t,T}} \cdot S^{-1}\theta_0(\underline{r})]_i|r_{\nu_t}=r^i] \\ &= [S \cdot \mathbb{E}^{\widetilde{\mathbb{P}}}[D^{\nu_{t,T}}|r_{\nu_t}=r^i] \cdot S^{-1} \cdot \theta_0(\underline{r})]_i \\ &= e_i' \cdot S \cdot \mathbb{E}^{\widetilde{\mathbb{P}}}[D^{\nu_{t,T}}|r_*=r^i] \cdot S^{-1} \cdot \theta_0(\underline{r}) \\ &= e_i' \cdot S \cdot \begin{bmatrix} \mathbb{E}^{\widetilde{\mathbb{P}}}[d_1^{\nu_{t,T}}|r_{\nu_t}=r^i] & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbb{E}^{\widetilde{\mathbb{P}}}[d_N^{\nu_{t,T}}|r_{\nu_t}=r^i] \end{bmatrix} \cdot S^{-1} \cdot \theta_0(\underline{r}) \end{aligned}$$

where in the second passage we have diagonalized the matrix $\widetilde{Q} = SDS^{-1}$ with S the matrix of eigenvectors and D the diagonal matrix with diagonal elements $(d_j)_{j=1,\ldots,N}$. Analogously we obtain an explicit matrix representation for $V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu t}) = \sum_{k=0}^{n_\epsilon} \vartheta_k(r_{\nu t})\widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t})$. □

2.2.5 The Prototype product pricing under a more general market model

So far we have considered a market model under which the spot rate is assumed to be a CTMC. It is possible to generalize this framework by considering the above interest rate as a renewal process. In fact, by definition, a renewal process is a point process characterized by the fact that the successive interarrival times are independent identically distributed (the CTMC is a renewal process where the interarrival times are exponentially distributed in accordance with (2.2)).

The pricing formula for the Prototype Product $V_{\vartheta_0,t,T}(r_{\nu t}) = \sum_{k=0}^{+\infty} \vartheta_k(r_{\nu t})\widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t})$ given by (2.14) holds also when the spot rate $r(t)$ is a general renewal process which takes

values in a finite set $E = \{r^1, \dots, r^N\}$. In this case we are going to show that the approach to price the Prototype Product does not change but the quantities $\vartheta_k(r_{\nu_t})$ and $\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t})$ have a different representation with respect to the previous sections.

Functions $\vartheta_{\mathbf{k}}(\mathbf{r}_{\nu_\mathbf{t}})$:

for a fixed k, $\vartheta_k(r_{\nu_t})$ can be represented recursively, as in Corollary 2.6, by using an operator $\mathcal{T}$ defined similarly to the one introduced in (2.19), namely

$$\begin{cases} \mathcal{T}\vartheta(v) \triangleq \mathbb{E}_v^{\widetilde{\mathbb{P}}}[e^{-v\mathfrak{T}}\vartheta(u)] \\ \mathfrak{T} \sim F(q(v)), \quad \vartheta \in \mathfrak{M} \end{cases} \tag{2.53}$$

where $\mathfrak{T}$ represents the interarrival time with a general distribution F which depends on the parameter $q(v)$ and $\mathfrak{M}$ is defined by (2.18) (see Subsection 2.2.1 for the other notations used in formula (2.53)).

The contraction constant γ of $\mathcal{T}$, depending on the distribution F, could also be greater than 1 (contrary to the case in which the random variable $\mathfrak{T}$ is exponentially distributed and, according to point **b)** of Proposition 2.7, $\gamma \triangleq \sup_{v \in E} \frac{q(v)}{v+q(v)} < 1$). Nevertheless, recalling Remark 2.9, we are able to compute numerically the price of the Prototype Product even if $\mathcal{T}$ is not a contraction map.

Probabilities $\widetilde{\mathbb{P}}(\nu_{\mathbf{t},\mathbf{T}} = \mathbf{k}|\mathbf{r}_{\nu_\mathbf{t}})$:

the probabilities $\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t})$ can be obtained by a formula similar to (2.40). More precisely, while in the proof of Lemma 2.15 the time τ is exponentially distributed (see formula (2.41)), in this case τ is distributed according to F.

Hence the price of the Prototype Product $V_{\vartheta_0,t,T}$ can be explicitly represented also when the spot rate is assumed to be a more general point process, namely a renewal process, with a finite state space E.

Chapter 3

Bond, Cap, Swaption and Bond Option pricing with a Markov short rate

Once we have presented the Prototype product pricing under the assumption that the spot rate is a CTMC, we are able to give, for specific contracts (bond, cap, swaption and bond option), a representation of the price as a linear combination of Prototype products.

3.1 Bond pricing

3.1.1 The pricing formula

The bond price $p(t,T)$ is simply a Prototype product with a particular payoff ϑ_0. We shall denote $p(t,T)$ always by $p(t,T;r(t))$ because a T-bond price evaluated at time t depends on the value of the spot rate at time t.

Proposition 3.1 *Let $r(t)$ be a CTMC with state space E and transition kernel $Q=(q_{i,j})_{1\leq i,j\leq N}$; let its value, at the initial time t when ν_t jumps have occurred, be $r(t)=r^i$ for a fixed $i=1,\dots,N$. A zero-coupon bond which matures at time T can be viewed as a Prototype product characterized by the following payoff:*

$$\begin{cases} \vartheta_0(\cdot)=\phi_0(\cdot)\triangleq\sum_{i_0=1}^{N} w_{i_0}\mathbf{I}_{\{\cdot=r^{i_0}\}} \\ w_{i_0}=1,\ i_0=1,\dots,N \end{cases} \tag{3.1}$$

Hence the bond price admits the following representation

$$p(t,T;r(t))|_{r(t)=r^i}=\sum_{k=0}^{+\infty}[\widetilde{Q}^k\cdot\theta_0(\underline{r})]_i\widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t}=r^i) \tag{3.2}$$

where $\widetilde{Q}$ is defined as in (2.36), $\theta_0(\underline{r})=[1,\dots,1]'\in\mathbb{R}^N$, the distribution of $\nu_{t,T}$ conditionally on $r(t)$ is given by (2.44) and $[v]_i$ denotes, for a generic vector v, the i-th element of v.

Moreover, under the assumption that $\widetilde{Q}$ is diagonalizable, we have an alternative representation:

$$p(t,T;r(t))|_{r(t)=r^i} = e_i' \cdot S \cdot \begin{bmatrix} \mathbb{E}^{\widetilde{\mathbb{P}}}[d_1^{\nu_{t,T}}|r(t)=r^i] & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbb{E}^{\widetilde{\mathbb{P}}}[d_N^{\nu_{t,T}}|r(t)=r^i] \end{bmatrix} \cdot S^{-1} \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad (3.3)$$

where the columns of S are the eigenvector of $\widetilde{Q}$, $(d_j)_{j=1,\dots,N}$ are the eigenvalues of $\widetilde{Q}$ and e_i is the i^{th} unit vector.

Proof Assuming $r(t) = r^i$ and the payoff ϑ_0 given by ϕ_0 as in (3.1), by (2.4) we have

$$\begin{aligned} p(t,T;r(t)) &= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_T-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)|\mathcal{F}^r_{\nu_t}\Big] \\ &= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_t+k-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\big|\mathcal{F}^r_{\nu_t}\vee\sigma\{\nu_{t,T}\}\Big]\Big|\mathcal{F}^r_{\nu_t}\Big] \\ &= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_t+k-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\phi_0(r_{\nu_T})\big|\mathcal{F}^r_{\nu_t}\vee\sigma\{\nu_{t,T}\}\Big]\Big|\mathcal{F}^r_{\nu_t}\Big] \\ &= \mathbb{E}^{\widetilde{\mathbb{P}}}[\phi_{\nu_{t,T}}(r_{\nu_t})|\mathcal{F}^r_{\nu_t}] = \mathbb{E}^{\widetilde{\mathbb{P}}}[\phi_{\nu_{t,T}}(r(t))|r(t)] \end{aligned} \quad (3.4)$$

where the last two passages follow by Lemma 2.2 when we set $\eta = \nu_t$, $k = \nu_T - \nu_t = \nu_{t,T}$ and we denote the functions ϑ_k by ϕ_k. Looking at formula (2.15) of $V_{\vartheta_0,t,T}(r_{\nu_t})$, we have that $p(t,T;r(t)) = V_{\phi_0,t,T}(r(t))$. Hence, under the assumptions of Proposition 2.19, by formula (2.47) the bond price can be written as follows

$$p(t,T;r(t))|_{r(t)=r^i} = \sum_{k=0}^{+\infty} [\widetilde{Q}^k \cdot \theta_0(\underline{r})]_i \widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t}=r^i)$$

where

- the components of $\theta_0(\underline{r})$, according to Definition 2.11, are given by the coefficients w_{i_0} of $\vartheta_0(\cdot)$ that in our case are equal to 1 by (3.1);
- $\nu_{t,T}$, conditionally on $r(t)$, is distributed as in (2.44).

In the case of $\widetilde{Q}$ diagonalizable, by using the results and the notations of Proposition 2.20, the bond price has the following explicit expression

$$p(t,T;r(t))|_{r(t)=r^i} = e_i' \; S \begin{bmatrix} \mathbb{E}^{\widetilde{\mathbb{P}}}[d_1^{\nu_{t,T}}|r(t)=r^i] & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbb{E}^{\widetilde{\mathbb{P}}}[d_N^{\nu_{t,T}}|r(t)=r^i] \end{bmatrix} S^{-1}\theta_0(\underline{r})$$

□

3.1.2 A computable bond pricing formula

An analogous result can be achieved when we look for a computable approximation of the bond price:

Proposition 3.2 *Under the same hypotheses and notations of Proposition 3.1, for an arbitrarily small ϵ and $n_\epsilon \in \mathbb{N}$ such that*

$$n_\epsilon \geq \left\lceil \frac{\log(\epsilon(1-\gamma))}{\log(\gamma)} - \frac{\sup_{v\in E} |\sum_{i=1}^{N} \left(\frac{q_i}{r^i+q_i} - 1\right)\mathbf{I}_{\{v=r^i\}}|}{\log(\gamma)} \right\rceil, \tag{3.5}$$

it follows that, if we denote by ϕ_k the functions given by relation (2.9) with the starting point equal to ϕ_0, the quantity $p_\epsilon(t,T;r(t)) \triangleq \sum_{k=0}^{n_\epsilon} \phi_k(r(t))\widetilde{\mathbb{P}}(\nu_{t,T}=k|r(t)=r^i)$ approximates the real bond price $p(t,T;r(t))$ represented by (3.3) in the sense that

$$|p_\epsilon(t,T;r(t)) - p(t,T;r(t))| < \epsilon \text{ uniformly in } (t,T,r(t)). \tag{3.6}$$

Moreover, the approximating price $p_\epsilon(t,T;r(t))$ can be written as follows

$$p_\epsilon(t,T;r(t))|_{r(t)=r^i} = \sum_{k=0}^{n_\epsilon} [\widetilde{Q}^k \cdot \theta_0(\underline{r})]_i \widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t}=r^i) \tag{3.7}$$

and, assuming the diagonalizability of $\widetilde{Q}$, it admits the representation below

$$p_\epsilon(t,T;r(t))|_{r(t)=r^i} = e_i' \; S \begin{bmatrix} \mathbb{E}_\epsilon^{\widetilde{\mathbb{P}}}[d_1^{\nu_{t,T}}|r(t)=r^i] & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbb{E}_\epsilon^{\widetilde{\mathbb{P}}}[d_N^{\nu_{t,T}}|r(t)=r^i] \end{bmatrix} S^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \tag{3.8}$$

with $\mathbb{E}_\epsilon^{\widetilde{\mathbb{P}}}[d_j^{\nu_{t,T}}|r(t)=r^i] \triangleq \sum_{k=0}^{n_\epsilon} d_j^k \widetilde{\mathbb{P}}(\nu_{t,T}=k|r(t)=r^i),\ j=1,\ldots,N.$

Proof By hypothesis let us denote the ϑ_ks in Lemma 2.2 by the ϕ_ks with the starting point defined by the Prototype payoff as in (3.1): then we have that

$$\vartheta_0(\cdot) = \phi_0(\cdot) = \sum_{i=1}^{N} w_i \mathbf{I}_{\{\cdot=r^i\}} \overset{(3.1)}{=} \sum_{i=1}^{N} \mathbf{I}_{\{\cdot=r^i\}}$$

and

$$\begin{aligned} \vartheta_1(\cdot) &= \phi_1(\cdot) \overset{(2.28),(2.29)}{=} \sum_{\substack{i_0,i_1=1 \\ i_0\neq i_1}}^{N} w_{i_0} \frac{q_{i_1,i_0}}{r^{i_1}+q_{i_1}} \mathbf{I}_{\{\cdot=r^{i_1}\}} \\ &\overset{(3.1)}{=} \sum_{\substack{i_0,i_1=1 \\ i_0\neq i_1}}^{N} \frac{q_{i_1,i_0}}{r^{i_1}+q_{i_1}} \mathbf{I}_{\{\cdot=r^{i_1}\}} = \sum_{i_1=1}^{N} \frac{1}{r^{i_1}+q_{i_1}} \Big(\sum_{\substack{i_0=1 \\ i_0\neq i_1}}^{N} q_{i_1,i_0} \Big) \mathbf{I}_{\{\cdot=r^{i_1}\}} \\ &= \sum_{i=1}^{N} \frac{q_i}{r^i+q_i} \mathbf{I}_{\{\cdot=r^i\}} \end{aligned}$$

Recalling Proposition 2.8, the condition (2.23) on n_ϵ then becomes

$$n_\epsilon \geq \left\lceil \frac{\log(\epsilon(1-\gamma))}{\log(\gamma)} - \frac{\sup_{v\in E} |\sum_{i=1}^{N} \left(\frac{q_i}{r^i+q_i} - 1\right)\mathbf{I}_{\{v=r^i\}}|}{\log(\gamma)} \right\rceil,$$

and we obtain that

$$|p_\epsilon(t,T;r(t)) - p(t,T;r(t))| < \epsilon \text{ uniformly in } (t,T,r(t))$$

where the uniformly convergence with respect to $(t,T,r(t))$ is due to the independence of n_ϵ on the triple $(t,T,r(t))$.

In addition, an explicit expression for $p_\epsilon(t,T;r(t))$ is obtained directly by the closed formula of $V^\epsilon_{\vartheta_0,t,T}(r(t))$ in (2.48) (or by the explicit formula in (2.51) when $\widetilde{Q}$ is diagonalizable) analogously to the previous Proposition. □

To conclude this section, we observe how, by considering the spot rate $r(\cdot)$ as an homogeneous time CTMC, the price of a T-bond evaluated at the time t does not depend separately on t and T but only on the length $T-t$: this result will be very useful in the pricing of interest rate derivatives.

Proposition 3.3 *Let $r(t)$ be a CTMC with state space E and stationary transition kernel $Q = (q_{i,j})_{1\leq i,j\leq N}$, then the price of a T-bond evaluated at the time t with the initial value of the spot rate equal to $r^m \in E$ admits the following representation*

$$p(t,T;r(t))|_{r(t)=r^m} = p(r^m, T-t) \tag{3.9}$$

where $p(r^m,T-t) \triangleq p(0,T-t;r(0)|_{r(0)=r^m}$. The same property is valid for $p_\epsilon(t,T;r(t))$, defined as in Proposition 3.2, which approximates the bond price:

$$p_\epsilon(t,T;r(t))|_{r(t)=r^m} = p_\epsilon(r^m, T-t) \tag{3.10}$$

with $p_\epsilon(r^m,T-t) \triangleq p_\epsilon(0,T-t;r(0))|_{r(0)=r^m}$.

Proof Looking at (3.4) in the proof of Proposition 3.1, we have that

$$p(t,T;r(t))|_{r(t)=r^m} = \mathbb{E}^{\widetilde{\mathbb{P}}}[\phi_{\nu_{t,T}}(r(t))|r(t)=r^m] = \sum_{k=0}^{+\infty} \phi_k(r^m)\widetilde{\mathbb{P}}(\nu_{t,T}=k|r(t)=r^m).$$

Taking into account (2.46) in Remark 2.18, it follows that

$$p(t,T;r(t))|_{r(t)=r^m} = \sum_{k=0}^{+\infty} \phi_k(r^m)\widetilde{\mathbb{P}}(\nu_{T-t}=k|r(0)=r^m) = p(0,T-t;r(0))|_{r(0)=r^m}.$$

The quantity $p(0,T-t;r(0))|_{r(0)=r^m}$ is indicated by $p(r^m,T-t)$ because we want to emphasize that the representation of the probability $\widetilde{\mathbb{P}}(\nu_{T-t}=k|r(0)=r^m)$ in (2.46) highlights the independence of the bond price on the time of evaluation. In an analogous way (3.10) is proved. □

3.1.3 An approach based on the affine term structure representation

The affine term structure models have been successful in the interest rate theory because they allow computations to be carried out more easily. We now use the result in [1] (Proposition below) to calculate the bond price in our context.

Definition 3.4 *An interest rate model is said to have an affine term structure if bond prices can be described as*

$$p(t,T;r(t)) = exp[A(t,T) - B(t,T)r(t)] \tag{3.11}$$

where A, B are deterministic functions and $r(t)$ is the spot rate.

Assumption 3.5 *The short rate $r(t)$ is solution of the following stochastic differential equation under a martingale measure Q*

$$dr(t) = a(t,r(t))dt + b(t,r(t))d\widetilde{W}_t + \int_D q(t,r(t),x)\mu(dt,dx) \tag{3.12}$$

where $\mu(dt,dx)$ is a point process measure on the mark space D and $a(t,r)$, $b(t,r)$, $q(t,r,x)$ are given deterministic functions. The process $\widetilde{W}_t$ is Wiener under Q and μ has a predictable Q-intensity

$$\lambda(\omega,t,dx) = \lambda(t,r(t^-),dx) \tag{3.13}$$

where $\lambda(t,r,dx)$ is a deterministic measure for each t and r.

Proposition 3.6 *Suppose that the r-dynamics under Q are given by (3.12) and the parameters a, b, q and λ have the following structure*

$$\begin{aligned} a(t,r) &= \alpha_1(t) + \alpha_2(t)r & (3.14)\\ b(t,r) &= \sqrt{\beta_1(t) + \beta_2(t)r} & (3.15)\\ q(t,r,x) &= q(t,x) & (3.16)\\ \lambda(t,r,dx) &= l_1(t,dx) + l_2(t,dx)r. & (3.17) \end{aligned}$$

Then the model has an affine term structure of the form (3.11) where the functions $A(\cdot,T)$ and $B(\cdot,T)$ solve the following system of ODE's on $[0,T]$

$$\begin{aligned} \frac{\partial B(t,T)}{\partial t} + \alpha_2(t)B(t,T) - \frac{1}{2}\beta_2(t)B^2(t,T) + \psi_2(t,B(t,T)) &= -1\\ B(T,T) &= 0 \end{aligned} \tag{3.18}$$

$$\begin{aligned} \frac{\partial A(t,T)}{\partial t} + \alpha_1(t)B(t,T) + \frac{1}{2}\beta_1(t)B^2(t,T) + \psi_1(t,B(t,T)) &= 0\\ A(T,T) &= 0 \end{aligned} \tag{3.19}$$

and

$$\psi_i(t,y) = \int_D \{1 - e^{-yq(t,x)}\} l_i(t,dx) \ , i = 1,2. \tag{3.20}$$

We show now that this result can be used in our context where $r(t)$ is assumed to be a CTMC with finite state space E. We consider the simplest case, namely $r(t)$ is a CTMC with $E = \{r^1, r^2\}$ ($r^1 = 1$, $r^2 = 2$) and transition kernel $Q = (q_{i,j})_{1 \leq i,j \leq 2}$ such that $q_{1,2} = -q_{1,1} = q_1 = \lambda$ and $q_{2,1} = -q_{2,2} = q_2 = \mu$ where $\lambda, \mu > 0$.

Lemma 3.7 *Let $r(t)$ be a CTMC with a state space $E = \{1, 2\}$ and transition kernel $Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$, then the spot rate is solution of the following SDE*

$$dr(t) = q(t, r(t), 1)dN_t^1 - q(t, r(t), -1)dN_t^2 \tag{3.21}$$

where $q(t, r(t), 1) = 1,\ q(t, r(t), -1) = -1$ and $N_t^i \sim Poiss(\lambda_t^i(r(t_-)))$, $i = 1, 2$ such that

$$\lambda_t(t, r(t_-), 1) \triangleq \lambda_t^1(r(t_-)) = \begin{cases} \lambda & \text{if } r(t_-) = 1 \\ 0 & \text{if } r(t_-) = 2 \end{cases}$$

and

$$\lambda_t(t, r(t_-), -1) \triangleq \lambda_t^2(r(t_-)) = \begin{cases} 0 & \text{if } r(t_-) = 1 \\ \mu & \text{if } r(t_-) = 2 \end{cases}$$

Proof The SDE (3.21) is a particular case of (3.12) if $r(t)$ is considered a continuous Markov chain. In fact a continuous time Markov chain is a "pure jump" process: it can be represented by a SDE of form (3.12) without the diffusive part ($a = b = 0$). In our case the state space $E = \{1, 2\}$ and the mark space $D = \{1, -1\}$, which contains all possible jump sizes, are discrete sets, so that the dynamics of $r(t)$ are given by

$$dr(t) = \sum_{k \in D} q(t, r(t), k)\mu(dt, k)$$

with

$$q(t, r, 1) = 1,\ q(t, r, -1) = -1$$

and

$$\mu(dt, k) = \begin{cases} dN_t^1 \sim Poiss(\lambda_t^1(r(t_-))) & \text{if } k = 1 \\ dN_t^2 \sim Poiss(\lambda_t^2(r(t_-))) & \text{if } k = -1 \end{cases}$$

such that

$$\lambda_t^1(r(t_-)) = \begin{cases} \lambda & \text{if } r(t_-) = 1 \\ 0 & \text{if } r(t_-) = 2 \end{cases}$$

$$\lambda_t^2(r(t_-)) = \begin{cases} 0 & \text{if } r(t_-) = 1 \\ \mu & \text{if } r(t_-) = 2 \end{cases}$$

The choice of these parameters is motivated by the following argument: at time t_- the process r can take two values

- if $r(t_-) = 1$ then the jump occurs according to the Poisson process N_t^1 with intensity λ and the jump size is equal to 1 ($q(t, r, 1) = 1$) to reach $r(t) = 2$;
- if $r(t_-) = 2$ then the jump occurs according to the Poisson process N_t^2 with intensity μ and the jumps size is equal to -1 ($q(t, r, -1) = -1$) to reach $r(t) = 1$. □

Corollary 3.8 *Suppose that the r-dynamics under Q is given by (3.21), then we are in the presence of an affine term structure where $A(\cdot,T)$ and $B(\cdot,T)$ solve (3.18)-(3.19) with*

$$\begin{aligned}
&\alpha_1(t) = \alpha_2(t) = \beta_1(t) = \beta_2(t) = 0 \\
&\psi_1(t,y) = 2\lambda(1-e^{-y}) - \mu\lambda(1-e^{y}) \\
&\psi_2(t,y) = -\lambda(1-e^{-y}) + \mu\lambda(1-e^{y}).
\end{aligned} \tag{3.22}$$

i.e.

$$\begin{aligned}
\frac{\partial B(t,T)}{\partial t} - \lambda(1-e^{-B(t,T)}) + \mu(1-e^{B(t,T)}) &= -1 \\
B(T,T) &= 0,
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\frac{\partial A(t,T)}{\partial t} + 2\lambda(1-e^{-B(t,T)}) - \mu(1-e^{B(t,T)}) &= 0 \\
A(T,T) &= 0.
\end{aligned} \tag{3.24}$$

Proof Assuming $r(t)$ solution of (3.21), by Lemma 3.7 we obtain

$$q(t,1) \triangleq q(t,r,1) = 1,\ q(t,-1) \triangleq q(t,r,-1) = -1$$

$$\lambda_t^1(r(t_-)) = \begin{cases} \lambda & \text{if } r(t_-)=1 \\ 0 & \text{if } r(t_-)=2 \end{cases} \Rightarrow \lambda_t^1(r(t_-)) = 2\lambda - \lambda r(t_-) \tag{3.25}$$

$$\lambda_t^2(r(t_-)) = \begin{cases} 0 & \text{if } r(t_-)=1 \\ \mu & \text{if } r(t_-)=2 \end{cases} \Rightarrow \lambda_t^2(r(t_-)) = -\mu + \mu r(t_-). \tag{3.26}$$

The Lemma is proved by verifying the hypotheses of Proposition 3.6 and showing that with r-dynamics as given by (3.21), the results of that Proposition lead to (3.23)-(3.24):

- $\alpha_1(t) = \alpha_2(t) = \beta_1(t) = \beta_2(t) = 0$ because there is no diffusive part in (3.21);
- the components of the intensities λ_t^1 and λ_t^2 are

$$\begin{aligned}
&l_1(t,1) = 2\lambda,\ l_2(t,1) = -\lambda \\
&l_1(t,2) = -\mu,\ l_2(t,2) = \mu;
\end{aligned}$$

  by (3.25) and (3.26);

- the ψ's in (3.20), with a discrete mark space D , become

$$\begin{aligned}
\psi_1(t,y) &= \sum_{k\in D}(1-e^{q(t,k)y})l_1(t,k) \\
&= 2\lambda(1-e^{-y}) - \mu(1-e^{y}),
\end{aligned}$$

$$\begin{aligned}
\psi_2(t,y) &= \sum_{k\in D}(1-e^{q(t,k)y})l_2(t,k) \\
&= -\lambda(1-e^{-y}) + \mu(1-e^{y}).
\end{aligned}$$

Equations (3.18) and (3.19) now become (3.23) and (3.24). □

Remark 3.9 *The functions $A(\cdot,T)$ and $B(\cdot,T)$, solutions of (3.23)-(3.24), can be expressed by explicit but very complicated formulas: for this reason it is practically not possible to develop an interest rate derivatives theory based on the affine term structure (3.11) when the spot rate is a CTMC.*

3.2 Cap pricing

Following the notations in Brigo-Mercurio [2], let us consider a set of payment dates $S_{\alpha,\beta} = \{S_{\alpha+1}, \ldots, S_\beta\}$, $\alpha < \beta \in \mathbb{N}$ such that, for a fixed date $t > 0$, $t < S_\alpha < S_{\alpha+1} < \ldots < S_\beta$ and this implies a set of tenors $\{s_i \triangleq S_i - S_{i-1}; i = \alpha+1, \ldots, \beta\}$. For the cap pricing we limit ourselves to the caplets because the cap is viewed as a sum of caplets.

3.2.1 Caplet pricing

For a fixed $i \in \{\alpha+1, \ldots, \beta\}$, the i-th caplet is a call option on the Libor rate $L_i(t) \triangleq L(t, S_{i-1}, S_i) = \frac{1}{s_i}\big(\frac{p(t,S_{i-1})}{p(t,S_i)} - 1\big)$. Assuming a unitary nominal capital, we have on the given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in\mathbb{R}}, \widetilde{\mathbb{P}})$ with $\widetilde{\mathbb{P}}$ a martingale measure

$$Cpl(t,S_i) = s_i \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\int_t^{S_i} r(u)du\Big)(L_i(S_{i-1}) - K)^+ \Big| \mathcal{F}_t\Big] \tag{3.27}$$

where K is the strike price. We derive now an alternative representation for the caplet price as a bond put option:

$$\begin{aligned}
Cpl(t,S_i) &\overset{(3.27)}{=} s_i \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-\int_t^{S_i} r(u)du}(L_i(S_{i-1}) - K)^+ \Big| \mathcal{F}_t\Big] \\
&= s_i \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}[e^{-\int_t^{S_i} r(u)du}(L_i(S_{i-1}) - K)^+ | \mathcal{F}_{S_{i-1}}] \Big| \mathcal{F}_t\Big] \\
&= s_i \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-\int_t^{S_{i-1}} r(u)du} e^{-\int_{S_{i-1}}^{S_i} r(u)du}\Big(\frac{1 - p(S_{i-1},S_i)(1+Ks_i)}{s_i p(S_{i-1},S_i)}\Big)^+ \Big| \mathcal{F}_{S_{i-1}}\Big] \Big| \mathcal{F}_t\Big] \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-\int_t^{S_{i-1}} r(u)du}\Big(\frac{1 - p(S_{i-1},S_i)(1+Ks_i)}{p(S_{i-1},S_i)}\Big)^+ \mathbb{E}^{\widetilde{\mathbb{P}}}[e^{-\int_{S_{i-1}}^{S_i} r(u)du} | \mathcal{F}_{S_{i-1}}] \Big| \mathcal{F}_t\Big] \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-\int_t^{S_{i-1}} r(u)du} p(S_{i-1},S_i)\Big(\frac{1 - p(S_{i-1},S_i)(1+Ks_i)}{p(S_{i-1},S_i)}\Big)^+ \Big| \mathcal{F}_t\Big] \\
&= (1+Ks_i)\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-\int_t^{S_{i-1}} r(u)du}\Big(\frac{1}{1+Ks_i} - p(S_{i-1},S_i)\Big)^+ \Big| \mathcal{F}_t\Big]
\end{aligned} \tag{3.28}$$

We shall denote $Cpl(t,S_i)$ always by $Cpl(t,S_i;r(t))$ because the price of the i-th caplet evaluated at time t depends on the value of the spot rate at time t. Considering the spot rate as a CTMC, we can particularize the formula (3.28) as we have done for the bond pricing

in (2.4):

$$Cpl(t,S_i;r(t)) = (1+Ks_i)\mathbb{E}^{\widetilde{\mathbb{P}}}\left[\exp\Big(-\sum_{j=\nu_t}^{\nu_{S_{i-1}}-1} r_j(\mathrm{T}_{j+1}-\mathrm{T}_j)\Big)\Big(\frac{1}{1+Ks_i}-p(S_{i-1},S_i;r(S_{i-1}))\Big)^+\Big|\mathcal{F}^r_{\nu_t}\right] \tag{3.29}$$

where $p(S_{i-1},S_i;r(S_{i-1}))$ denotes the bond price $p(S_{i-1},S_i)$ which depends on the value of the spot rate at time S_{i-1}.

We have shown in the previous chapter that the bond price can be viewed as a particular Prototype product. We are now going to show that the caplet price can be written as a linear combination of the prices of N different Prototype products.

Proposition 3.10 *Let $r(t)$ be a CTMC with state space E and transition kernel $Q=(q_{i,j})_{1\le i,j\le N}$; let its value, at the initial time t when ν_t jumps have occurred, be $r(t)=r^l$ for a fixed $l=1,\dots,N$. The price $Cpl(t,S_i;r(t))$ of the i-th caplet can be represented as follows*

$$Cpl(t,S_i;r(t))|_{r(t)=r^l} = (1+Ks_i)\sum_{m=1}^{N}\left(\frac{1}{1+Ks_i}-p(r^m,s_i)\right)^+ V_{\psi_0^m,t,S_{i-1}}(r^l) \tag{3.30}$$

where, for each $m\in\{1,\dots,N\}$, $p(r^m,s_i)$ is defined by (3.9) and $V_{\psi_0^m,t,S_{i-1}}$ represents the price of a Prototype product with payoff ψ_0^m defined by

$$\begin{cases} \psi_0^m(\cdot)=\vartheta_0(\cdot)\triangleq\sum_{i_0=1}^N w_{i_0}(m)\mathbf{I}_{\{\cdot=r^{i_0}\}} \\ w_{i_0}(m)=\begin{cases} 0, & i_0\neq m \\ 1, & i_0=m \end{cases} \end{cases} \tag{3.31}$$

Proof By (3.29) we have

$$\begin{aligned}\frac{Cpl(t,S_i;r(t))}{1+Ks_i} &= \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\exp\Big(-\sum_{j=\nu_t}^{\nu_{S_{i-1}}-1} r_j(\mathrm{T}_{j+1}-\mathrm{T}_j)\Big)\Big(\frac{1}{1+Ks_i}-p(S_{i-1},S_i;r(S_{i-1}))\Big)^+\Big|\mathcal{F}^r_{\nu_t}\right] \\ &= \mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[\sum_{m=1}^{N}\exp\Big(-\sum_{j=\nu_t}^{\nu_{S_{i-1}}-1} r_j(\mathrm{T}_{j+1}-\mathrm{T}_j)\Big) \\ &\qquad \Big(\frac{1}{1+Ks_i}-p(S_{i-1},S_i;r^m)\Big)^+\mathbf{I}_{\{r_{\nu_{S_{i-1}}}=r^m\}}\Big|\mathcal{F}^r_{\nu_t}\Bigg]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{N} \Bigg\{ \Big(\frac{1}{1+Ks_i} - p(S_{i-1}, S_i; r^m) \Big)^+ \\
&\qquad \widetilde{\mathbb{E}}^{\widetilde{\mathbb{P}}} \Bigg[\exp\Big(- \sum_{j=\nu_t}^{\nu_{S_{i-1}}-1} r_j(\mathrm{T}_{j+1} - \mathrm{T}_j) \Big) \mathbf{I}_{\{r_{\nu_{S_{i-1}}} = r^m\}} \Big| \mathcal{F}^r_{\nu_t} \Bigg] \Bigg\} \\
&\overset{(3.9)}{=} \sum_{m=1}^{N} \Bigg\{ \Big(\frac{1}{1+Ks_i} - p(r^m, s_i) \Big)^+ \\
&\qquad \widetilde{\mathbb{E}}^{\widetilde{\mathbb{P}}} \Bigg[\exp\Big(- \sum_{j=\nu_t}^{\nu_{S_{i-1}}-1} r_j(\mathrm{T}_{j+1} - \mathrm{T}_j) \Big) \mathbf{I}_{\{r_{\nu_{S_{i-1}}} = r^m\}} \Big| \mathcal{F}^r_{\nu_t} \Bigg] \Bigg\}
\end{aligned}
$$

where, by setting in Lemma 2.2 $\eta = \nu_t$, $k = \nu_{S_{i-1}} - \nu_t = \nu_{t,S_{i-1}}$ and choosing $\vartheta_0(r_{\eta+k}) = \psi_0(r_{\eta+k})$ as defined in (3.31), we have that

$$
\begin{aligned}
&\widetilde{\mathbb{E}}^{\widetilde{\mathbb{P}}} \Bigg[\exp\Big(- \sum_{j=\nu_t}^{\nu_{S_{i-1}}-1} r_j(\mathrm{T}_{j+1} - \mathrm{T}_j) \Big) \mathbf{I}_{\{r_{\nu_{S_{i-1}}} = r^m\}} \Big| \mathcal{F}^r_{\nu_t} \Bigg] \\
&\overset{(3.31)}{=} \widetilde{\mathbb{E}}^{\widetilde{\mathbb{P}}} \Bigg[\exp\Big(- \sum_{j=\nu_t}^{\nu_{S_{i-1}}-1} r_j(\mathrm{T}_{j+1} - \mathrm{T}_j) \Big) \psi_0^m(r_{\nu_{S_{i-1}}}) \Big| \mathcal{F}^r_{\nu_t} \Bigg] \\
&= \widetilde{\mathbb{E}}^{\widetilde{\mathbb{P}}} \Bigg[\widetilde{\mathbb{E}}^{\widetilde{\mathbb{P}}} \Big[\exp\Big(- \sum_{j=\nu_t}^{\nu_{S_{i-1}}-1} r_j(\mathrm{T}_{j+1} - \mathrm{T}_j) \Big) \psi_0^m(r_{\nu_{S_{i-1}}}) | \mathcal{F}^r_{\nu_t} \vee \sigma\{\nu_{t,S_{i-1}}\} \Big] \Big| \mathcal{F}^r_{\nu_t} \Bigg] \\
&\overset{(2.10)}{=} \widetilde{\mathbb{E}}^{\widetilde{\mathbb{P}}} \Big[\psi^m_{\nu_{t,S_{i-1}}}(r_{\nu_t}) | \mathcal{F}^r_{\nu_t} \Big] \overset{(2.15)}{=} V_{\psi_0^m, t, S_{i-1}}(r_{\nu_t})
\end{aligned}
$$

Hence, when $r(t) = r^l$, it follows that

$$
Cpl(t, S_i; r(t))|_{r(t)=r^l} = (1+Ks_i) \sum_{m=1}^{N} \left(\frac{1}{1+Ks_i} - p(r^m, s_i) \right)^+ V_{\psi_0^m, t, S_{i-1}}(r^l).
$$

□

3.2.2 A computable caplet pricing formula

Since $p(r^m, s_i)$ and $V_{\psi_0^m, t, S_{i-1}}$ have to be computed as infinite sums, similarly to the bond pricing, we are interested in a computable approximation of the caplet price.

Proposition 3.11 *Under the same hypotheses and notations of Proposition 3.10 and recalling the definition of γ in point **b)** of Proposition 2.7, let us consider an arbitrarily small ϵ and*

$(n_\epsilon^m)_{m=1,\dots,N} \in \mathbb{N}$ *such that*

$$\begin{cases} n_\epsilon^m \geq \left\lceil \dfrac{\log(\epsilon(1-\gamma))}{\log(\gamma)} - \dfrac{\sup_{v\in E}\left|\psi_1^m(v)-\psi_0^m(v)\right|}{\log(\gamma)} \right\rceil, \text{ with} \\ \psi_1^m(v) \triangleq \displaystyle\sum_{\substack{i_0,i_1=1 \\ i_0\neq i_1}}^{N} w_{i_0}(m)\frac{q_{i_1,i_0}}{r^{i_1}+q_{i_1}}\mathbf{I}_{\{v=r^{i_1}\}},\ \psi_0^m(v) \triangleq \displaystyle\sum_{i_0=1}^{N} w_{i_0}(m)\mathbf{I}_{\{v=r^{i_0}\}} \end{cases} \quad m=1,\dots,N, \tag{3.32}$$

then, letting $p_\epsilon(r^m, s_i)$ *be as in (3.10) and* $V^\epsilon_{\psi_0^m,t,S_{i-1}}$ *as in (2.51) for* $\vartheta_0 = \psi_0^m, T = S_{i-1}$,

$$Cpl_\epsilon(t,S_i;r(t))|_{r(t)=r^l} \triangleq (1+Ks_i)\sum_{m=1}^{N}\left(\frac{1}{1+Ks_i} - p_\epsilon(r^m,s_i)\right)^+ V^\epsilon_{\psi_0^m,t,S_{i-1}}(r^l) \tag{3.33}$$

is a good approximation of the caplet price defined as in (3.30) in the sense that

$$Cpl_\epsilon(t,S_i;r(t)) \overset{\epsilon\to 0}{\longrightarrow} Cpl(t,S_i;r(t)) \text{ uniformly in } (t,S_{i-1},S_i,r(t)). \tag{3.34}$$

Proof Recalling the representation of the caplet price as in (3.30) of Proposition 3.10 and the definition of $Cpl_\epsilon(t,S_i;r(t))$ in (3.33),to prove the convergence in (3.34) it is sufficient to prove that

1. $p_\epsilon(r^m, S_i - S_{i-1}) \overset{\epsilon\to 0}{\longrightarrow} p(r^m, S_i - S_{i-1})$ uniformly in (S_{i-1}, S_i, r^m)
2. $V^\epsilon_{\psi_0^m,t,S_{i-1}}(r^l) \overset{\epsilon\to 0}{\longrightarrow} V_{\psi_0^m,t,S_{i-1}}(r^l)$ uniformly in (t, S_{i-1}, r^l), $\forall m = 1,\dots,N$

The first statement follows directly from Proposition 3.2 taking into account that, according to Proposition 3.3, $p_\epsilon(t,T;r(t))|_{r(t)=r^m} = p_\epsilon(r^m, T-t)$ and $p(t,T;r(t))|_{r(t)=r^m} = p(r^m, T-t)$. The second statement is proved by considering Proposition 2.8 : for an arbitrarily small ϵ and $n_\epsilon^m \in \mathbb{N}$ such that the condition (3.32) is satisfied, we obtain that $V^\epsilon_{\psi_0^m,t,S_{i-1}}$, represented by the closed formula in (2.51) for $\vartheta_0 = \psi_0$ and $T = S_{i-1}$, converges to $V_{\psi_0^m,t,S_{i-1}}$ in the sense of (2.25). Hence it follows that

$$V^\epsilon_{\psi_0^m,t,S_{i-1}}(r^l) \overset{\epsilon\to 0}{\longrightarrow} V_{\psi_0^m,t,S_{i-1}}(r^l) \text{ uniformly in } (t,S_{i-1},r^l).$$

□

3.3 Swaption pricing

Using the same notations as in Section 3.2, a swaption is the option to enter into an interest rate swap at a specified future date S_α. The payoff can be written not only in terms of the *forward swap rate* but also in terms of an IRS (Interest Rate Swap), a contract which exchanges payments between two differently indexed legs. In the Payer IRS (i.e. PFS) the contract owner pays the "fixed leg" and receives the "floating leg" defined as follows:

fixed leg: $\sum_{i=\alpha+1}^{\beta} K s_i p(t, S_i)$, that is the value at time t of the total amount to be paid with a *fixed* interest rate K along the set of payments dates $S_{\alpha,\beta}$

floating leg: $\sum_{i=\alpha+1}^{\beta} L_i(t) s_i p(t, S_i)$, that is the value at time t of the total amount to be received at a *floating* rate (namely the Libor rate $L_i(t)$ for $i = \alpha+1, \ldots, \beta$) along $S_{\alpha,\beta}$.

It can be easily seen that the value of the PFS at time t can be expressed as

$$PFS_{\alpha}^{\beta}(t, K) = p(t, S_\alpha) - p(t, S_\beta) - K \sum_{h=\alpha+1}^{\beta} s_h p(t, S_h). \tag{3.35}$$

3.3.1 The swaption pricing formula

The swaption can now be viewed also as the option of entering the PFS, so its price can be represented, similarly to the caplet price in (3.29), by

$$Swopt_t(S_\alpha, S_{\alpha,\beta}; r(t)) = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\exp\Big(- \sum_{j=\nu_t}^{\nu_{S_\alpha}-1} r_j(\mathrm{T}_{j+1} - \mathrm{T}_j)\Big) (PFS_{\alpha}^{\beta}(S_\alpha, K))^+ \Big| \mathcal{F}_{\nu_t}^r \right] \tag{3.36}$$

More precisely, the swaption price can be expressed as a linear combinations of the prices of N Prototype products with maturity S_α and payoff ψ_0^m $(m = 1, \ldots, N)$ defined as in (3.31), namely we have

Proposition 3.12 *Let $r(t)$ be a CTMC with state space E and transition kernel $Q = (q_{i,j})_{1\le i,j\le N}$; let its value, at the initial time t when ν_t jumps have occurred, be $r(t) = r^l$ for a fixed $l = 1, \ldots, N$. The price $Swopt_t(S_\alpha, S_{\alpha,\beta}; r(t))$ can be represented as follows*

$$Swopt_t(S_\alpha, S_{\alpha,\beta}; r(t))|_{r(t)=r^l} = \sum_{m=1}^{N} \left(1 - p(r^m, S_\beta - S_\alpha) - K \sum_{h=1+\alpha}^{\beta} s_h p(r^m, S_h - S_\alpha) \right)^+ V_{\psi_0^m, t, S_\alpha}(r^l) \tag{3.37}$$

where, for each $m \in \{1, \ldots, N\}$, $p(r^m, S_h - S_\alpha)$ is defined by (3.9) while $V_{\psi_0^m, t, S_\alpha}$ represents the price of a Prototype product with payoff ψ_0^m defined by (3.31).

Proof Let us denote $p(t, T)$ by $p(t, T; r(t))$ because a T-bond price evaluated at time t depends on the value of the spot rate at time t.

By (3.36) and (3.35) we have

$$\begin{aligned}
Swopt_t(S_\alpha, S_{\alpha,\beta}; r(t)) &= \mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[\exp\Big(-\sum_{j=\nu_t}^{\nu_{S_\alpha}-1} r_j(\mathrm{T}_{j+1}-\mathrm{T}_j)\Big)\cdot \\
&\qquad \cdot\Big(1-p(S_\alpha,S_\beta;r(S_\alpha))-K\sum_{h=\alpha+1}^{\beta} s_h p(S_\alpha,S_h;r(S_\alpha))\Big)^+\Big|\mathcal{F}^r_{\nu_t}\Bigg] \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[\sum_{m=1}^{N}\exp\Big(-\sum_{j=\nu_t}^{\nu_{S_\alpha}-1} r_j(\mathrm{T}_{j+1}-\mathrm{T}_j)\Big)\cdot \\
&\qquad \cdot\Big(1-p(S_\alpha,S_\beta;r^m)-K\sum_{h=\alpha+1}^{\beta} s_h p(S_\alpha,S_h;r^m)\Big)^+\mathbf{I}_{\{r_{\nu_{S_\alpha}}=r^m\}}\Big|\mathcal{F}^r_{\nu_t}\Bigg] \\
&= \sum_{m=1}^{N}\Bigg\{\Big(1-p(S_\alpha,S_\beta;r^m)-K\sum_{h=\alpha+1}^{\beta} s_h p(S_\alpha,S_h;r^m)\Big)^+ \\
&\qquad \mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[\exp\Big(-\sum_{j=\nu_t}^{\nu_{S_\alpha}-1} r_j(\mathrm{T}_{j+1}-\mathrm{T}_j)\Big)\mathbf{I}_{\{r_{\nu_{S_\alpha}}=r^m\}}\Big|\mathcal{F}^r_{\nu_t}\Bigg]\Bigg\} \\
&\overset{(3.9)}{=} \sum_{m=1}^{N}\Bigg\{\Big(1-p(r^m,S_\beta-S_\alpha)-K\sum_{h=\alpha+1}^{\beta} s_h p(r^m,S_h-S_\alpha)\Big)^+ \\
&\qquad \mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[\exp\Big(-\sum_{j=\nu_t}^{\nu_{S_\alpha}-1} r_j(\mathrm{T}_{j+1}-\mathrm{T}_j)\Big)\mathbf{I}_{\{r_{\nu_{S_\alpha}}=r^m\}}\Big|\mathcal{F}^r_{\nu_t}\Bigg]\Bigg\}
\end{aligned}$$

where, by setting in Lemma 2.2 $\eta=\nu_t$, $k=\nu_{S_\alpha}-\nu_t=\nu_{t,S_\alpha}$ and choosing $\vartheta_0(r_{\eta+k})$ as defined in (3.31), we have that

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[\exp\Big(-\sum_{j=\nu_t}^{\nu_{S_\alpha}-1} r_j(\mathrm{T}_{j+1}-\mathrm{T}_j)\Big)\mathbf{I}_{\{r_{\nu_{S_\alpha}}=r^m\}}\Big|\mathcal{F}^r_{\nu_t}\Bigg] = V_{\psi_0^m,t,S_\alpha}(r_{\nu_t})$$

similarly to the proof of Proposition 3.10. By considering $r(t)=r^l$ we obtain

$$Swopt_t(S_\alpha,S_{\alpha,\beta};r(t))|_{r(t)=r^l} = \sum_{m=1}^{N}\Big(1-p(r^m,S_\beta-S_\alpha)-K\sum_{h=1+\alpha}^{\beta} s_h p(r^m,S_h-S_\alpha)\Big)^+ V_{\psi_0^m,t,S_\alpha}(r^l).$$

□

3.3.2 A computable swaption pricing formula

Again, since the exact expressions for $p(r^m,\cdot)$ and $V_{\psi_0^m,t,S_\alpha}$ are based on an infinite sum, we present also for the swaptions a computable approximation of the price represented by (3.37).

Proposition 3.13 *Under the same hypotheses and notations of Proposition 3.12 and recalling the definition of γ in point **b)** of Proposition 2.7, let us consider an arbitrarily small ϵ and $(n_\epsilon^m)_{m=1,\dots,N} \in \mathbb{N}$ such that*

$$\begin{cases} n_\epsilon^m \geq \left\lceil \dfrac{\log(\epsilon(1-\gamma))}{\log(\gamma)} - \dfrac{\sup_{v\in E}\left|\psi_1^m(v) - \psi_0^m(v)\right|}{\log(\gamma)} \right\rceil, \text{ with} \\ \psi_1^m(v) \triangleq \displaystyle\sum_{\substack{i_0,i_1=1 \\ i_0\neq i_1}}^{N} w_{i_0}(m)\frac{q_{i_1,i_0}}{r^{i_1}+q_{i_1}}\mathbf{I}_{\{v=r^{i_1}\}},\ \psi_0^m(v) \triangleq \displaystyle\sum_{i_0=1}^{N} w_{i_0}(m)\mathbf{I}_{\{v=r^{i_0}\}} \end{cases} \quad m=1,\dots,N,$$

then, by letting $p_\epsilon(r^m, S_h - S_\alpha)$ be as in (3.10) and $V^\epsilon_{\psi_0^m,t,S_\alpha}$ as in (2.51),

$$\begin{aligned} Swopt_t^\epsilon(S_\alpha, S_{\alpha,\beta}; r(t))|_{r(t)=r^l} &= \sum_{m=1}^{N}\left(1 - p_\epsilon(r^m, S_\beta - S_\alpha) - K\sum_{h=1+\alpha}^{\beta} s_h p_\epsilon(r^m, S_h - S_\alpha)\right)^+ \cdot \\ &\quad \cdot V^\epsilon_{\psi_0^m,t,S_\alpha}(r^l) \end{aligned} \tag{3.38}$$

is a good approximation of the swaption price defined as in (3.37) in the sense that

$$Swopt_t^\epsilon(S_\alpha, S_{\alpha,\beta}; r(t)) \xrightarrow{\epsilon\to 0} Swopt_t(S_\alpha, S_{\alpha,\beta}; r(t)) \text{ uniformly in } (t, S_\alpha, S_{\alpha,\beta}, r(t)). \tag{3.39}$$

Proof Completely analogous to the proof of Proposition 3.11. □

3.4 Bond option pricing

The approach used to price caps and swaptions in the previous Sections is valid also for generic bond options. Consider an option, written on a S-maturity bond, which matures at time $T < S$: its price is given by

$$OptBond_t(T,S) = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{-\int_t^T r(s)ds} h(p(T,S))|\mathcal{F}_t\right] \tag{3.40}$$

where $h(\cdot)$ is a generic payoff. In our situation it can be equivalently represented as

$$OptBond_t(T,S;r(t)) = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\exp\Big(-\sum_{i=\nu_t}^{\nu_T-1} r_i(\mathrm{T}_{i+1} - \mathrm{T}_i)\Big) h(p(T,S;r(T)))\Big|\mathcal{F}^r_{\nu_t}\right]$$

where we have denoted $OptBond_t(T,S)$ by $OptBond_t(T,S;r(t))$ because the price of a bond option evaluated at time t depends on the value of the spot rate at time t.

In fact, we can compare the above expression with formulas (3.29) and (3.36) obtained respectively for caplets and swaptions (which can be viewed as particular cases of bond options). Hence we achieve similar results as were obtained for caplets and swaptions:

Proposition 3.14 *Let $r(t)$ be a CTMC with state space E and transition kernel $Q = (q_{i,j})_{1\leq i,j\leq N}$; let its value, at the initial time t when ν_t jumps have occurred, be $r(t) = r^l$ for a fixed $l = 1, \ldots, N$. The price $OptBond_t(T, S; r(t))$ can be represented as follows*

$$OptBond_t(T, S; r(t))|_{r(t)=r^l} = \sum_{m=1}^{N} h(p(r^m, S-T))V_{\psi_0^m,t,T}(r^l) \tag{3.41}$$

with $p(r^m, S-T)$ defined as in (3.9) and $V_{\psi_0^m,t,T}$ as in (2.50) for $\vartheta_0 = \phi_0^m$ with ψ_0^m as in (3.31).

Proof It is sufficient to choose in Proposition 3.10 $S_{i-1} = T$, $S_i = S$ and to consider $h(p(T, S))$ instead of the particular function $(1 + Ks_i)(\frac{1}{1+Ks_i} - p(S_{i-1}, S_i))^+$: in fact, looking at the proof, the representation obtained for the caplet price is independent of the payoff that is being considered. □

Proposition 3.15 *Under the same hypotheses and notations of Proposition 3.14 and with γ as defined in point **b)** of Proposition 2.7, let us consider an arbitrarily small ϵ and $(n_\epsilon^m)_{m=1,\ldots,N} \in \mathbb{N}$ such that*

$$\begin{cases} n_\epsilon^m \geq \left\lceil \dfrac{\log(\epsilon(1-\gamma))}{\log(\gamma)} - \dfrac{\sup_{v\in E}\left|\psi_1^m(v) - \psi_0^m(v)\right|}{\log(\gamma)} \right\rceil, \text{ with} \\ \psi_1^m(v) \triangleq \displaystyle\sum_{\substack{i_0,i_1=1 \\ i_0\neq i_1}}^{N} w_{i_0}(m)\frac{q_{i_1,i_0}}{r^{i_1}+q_{i_1}}\mathbf{I}_{\{v=r^{i_1}\}},\ \psi_0^m(v) \triangleq \displaystyle\sum_{i_0=1}^{N} w_{i_0}(m)\mathbf{I}_{\{v=r^{i_0}\}} \end{cases} \quad m = 1, \ldots, N,$$

then, letting $p_\epsilon(r^m, S-T)$ be as in (3.10) and $V^\epsilon_{\psi_0^m,t,T}$ as in (2.51),

$$OptBond_t^\epsilon(T, S; r(t))|_{r(t)=r^l} \triangleq \sum_{m=1}^{N} h(p_\epsilon(r^m, S-T))V^\epsilon_{\psi_0^m,t,T}(r^l) \tag{3.42}$$

is a good approximation of the bond option price defined in (3.41) in the sense that

$$OptBond_t^\epsilon(T, S; r(t)) \overset{\epsilon\to 0}{\longrightarrow} OptBond_t(T, S; r(t)) \text{ uniformly in } (t, T, S, r(t)). \tag{3.43}$$

Proof Analogous to the proof of Proposition 3.11. □

Chapter 4

Pricing of interest rate derivatives when the short rate depends on several correlated CTMCs: a multi-factor approach

Until now we have considered the pricing of interest rate derivatives when the short rate is given by a single CTMC. However one can suppose that the value of the short rate depends on several factors (such as e.g. the credit spread, inflation rate, etc.) to obtain a more flexible model for the evolution of the spot rate. In the following we will present a two-factor model in which the short rate can be represented by a linear combination of two correlated CTMC's and we shall see how the approach developed in the Chapter 2 can be generalized when more factors are considered.

4.1 Market model

As in the previous chapters we assume that the stochastic processes which we are going to introduce evolve under a martingale measure $\widetilde{\mathbb{P}}$ in accordance with the theory of the *"martingale modeling"*.

As a means to introduce correlation we consider two CTMCs X and Y with the respective transition kernels dependent on a discrete random variable Z taking values in $\mathcal{Z} = \{z_1, \ldots, z_M\}$ with distributions $\pi = \{\pi_1, \ldots, \pi_M\} = \{\widetilde{\mathbb{P}}(Z = z_1), \ldots, \widetilde{\mathbb{P}}(Z = z_M)\}$. We make the following assumptions

Assumption 4.1 *$X(t; Z)$ denotes a CTMC with state space $E^X = \{x^1, \ldots, x^N\}$ and transition intensity matrix $Q^X(Z) = (q(Z)^X_{i,j})_{1 \leq i,j \leq N}$ in the following sense: given a fixed value $\bar{z} \in \mathcal{Z}$, the process $X(t) \triangleq (X(t; Z)|Z = \bar{z})$ is a CTMC with state space $E^X = \{x^1, \ldots, x^N\}$ ($N \in \mathbb{N}$ and $x^i \in \mathbb{R}^+$ for each $i = 1, \ldots, N$) where*

- *$Q^X(\bar{z}) = (q(\bar{z})^X_{i,j})_{1 \leq i,j \leq N}$ is the transition kernel homogeneous with respect the time*

(with $q(\bar{z})^X_{i,j} \in \mathbb{R}$),

- $q(\bar{z})^X_i = \sum_{\substack{j=1 \\ j\neq i}}^N q(\bar{z})^X_{i,j}$, $i = 1, \ldots, N$ *is the intensity associated with the state* x^i.

Moreover, being T^X_i *the random time at which the i-th jump of* $X(t;Z)$ *occurs, we have that*

- *given a generic value* x^h $(h = 1, \ldots, N)$ *of the process* $X(t;\bar{z})$ *at time* T^X_i, *the interarrival time* $\mathrm{T}^X_{i+1} - \mathrm{T}^X_i$ *is exponentially distributed with parameter* $q(\bar{z})^X_h$ *under the measure* $\widetilde{\mathbb{P}}$ *(namely* $(\mathrm{T}^X_{i+1} - \mathrm{T}^X_i | X(\mathrm{T}_i) = x^h) \sim \mathcal{E}xp(q(\bar{z})^X_h)$*);*
- *for a generic time* $s \geq 0$, $\nu(\bar{z})^X_s$ *denotes the number of jumps of* $X(t;Z)$ *until* s*;*
- *for each* $s \in [\mathrm{T}^X_i, \mathrm{T}^X_{i+1})$, *the filtration generated by the process* X *at the stopping time* T^X_i *is denoted by* $\mathcal{H}_{\mathrm{T}^X_i} = \{A \in \mathcal{F}^X_T | A \cap \{\mathrm{T}^X_i \leq s\} \in \mathcal{F}^X_s, \forall s \leq T\}$ *and* $X_i \triangleq X(s,Z)$ *(similarly* $X_i(\bar{z}) \triangleq X(s,\bar{z})$ *if we consider a realization* $\bar{z}$ *of the r.v.* Z*);*
- *for two generic times* $\underline{s}$ *and* $\overline{s}$ *such that* $\underline{s} < \overline{s}$, $\nu(\bar{z})^X_{\underline{s},\overline{s}}$ *denotes the number of jumps in the interval* $[\underline{s}, \overline{s})$.

Assumption 4.2 *$Y(t;Z)$ (analogously to the definition of $X(t;Z)$) denotes a CTMC with state space $E^Y = \{y^1, \ldots, y^{\widetilde{N}}\}$ ($\widetilde{N} \in \mathbb{N}$) and transition intensity matrix $Q^Y(Z) = (q(Z)^Y_{i,j})_{1\leq i,j\leq \widetilde{N}}$. The notations introduced for the stochastic process X are also valid for $Y(t;Z)$, but now the filtration generated by the process Y at a generic time stopping time $\mathrm{T}^Y_i \geq s$ is denoted by $\mathcal{G}_{\mathrm{T}^Y_i} = \{A \in \mathcal{G}^Y_T | A \cap \{\mathrm{T}^Y_i \leq s\} \in \mathcal{G}^Y_s, \forall s \leq T\}$ and $Y_i \triangleq Y(s,Z)$ (similarly $Y_i(\bar{z}) \triangleq Y(s,\bar{z})$ if we consider a realization $\bar{z}$ of the r.v. Z).*

Remark 4.3 *From the Assumptions 4.1-4.2 it follows implicitly that, conditionally on Z, the CTMCs $X(t;Z)$ and $Y(t;Z)$ are mutually independent.*

Furthermore we consider the short rate as given by

$$r(t) = aX(t;Z) + bY(t;Z) \;\; a, b \in \mathbb{R}, \quad t \geq 0. \tag{4.1}$$

This particular representation for the short rate suggests the following remark:

Remark 4.4 *In the real market it is reasonable that the short rate depends on two factors which are correlated: according to the formula above the factors X and Y both depend on the random variable Z. In general, it could become complicated to price interest rate derivatives when one chooses a short-rate model depending on two correlated factors, but in our situation this is not the case because of the conditional independence of $Y(t;Z)$ and $X(t;Z)$.*

We are now able to give a representation for the bond price in this market model. Let us assume without lost of generality that $t = T^X_{\nu^X_t} = T^Y_{\nu^Y_t}$ as we have done for the one-factor case in Chapter 2. Given the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in\mathbb{R}}, \tilde{\mathbb{P}})$ where $\mathcal{F}_t \triangleq \mathcal{H}_t \vee \mathcal{G}_t \equiv$

$\mathcal{H}_{T^X_{\nu_t^X}} \vee \mathcal{G}_{T^Y_{\nu_t^Y}}$, the price of a zero coupon bond at time t that matures in $T > t$ can be represented by

$$p(t,T;X_{\nu_t^X},Y_{\nu_t^Y}) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(- a \sum_{i=\nu_t^X}^{\nu(Z)_T^X-1} X_i(\mathrm{T}^X_{i+1} - \mathrm{T}^X_i) - aX_{\nu_T^X}(T - \mathrm{T}_{\nu_T^X}) \quad (4.2)$$
$$-b \sum_{j=\nu_t^Y}^{\nu(Z)_T^Y-1} Y_j(\mathrm{T}^Y_{j+1} - \mathrm{T}^Y_j) - bY_{\nu_T^Y}(T - \mathrm{T}_{\nu_T^Y})\Big)|\mathcal{F}_t\Big]$$

In this Chapter we will consider upper approximations of the price $p(t,T;X_{\nu_t^X},Y_{\nu_t^Y})$ and of the Prototype product price analogously to what we had done in the scalar case (see beginning of Section 2.2). We essentially consider the following discount factor

$$exp\Big(- a \sum_{i=\nu_t^X}^{\nu(Z)_T^X-1} X_i(\mathrm{T}^X_{i+1} - \mathrm{T}^X_i) - b \sum_{j=\nu_t^Y}^{\nu(Z)_T^Y-1} Y_j(\mathrm{T}^Y_{j+1} - \mathrm{T}^Y_j)\Big)$$

instead of the true one

$$exp\Big(-a \sum_{i=\nu_t^X}^{\nu(Z)_T^X-1} X_i(\mathrm{T}^X_{i+1}-\mathrm{T}^X_i)-aX_{\nu_T^X}(T-\mathrm{T}_{\nu_T^X})-b \sum_{j=\nu_t^Y}^{\nu(Z)_T^Y-1} Y_j(\mathrm{T}^Y_{j+1}-\mathrm{T}^Y_j)-bY_{\nu_T^Y}(T-\mathrm{T}_{\nu_T^Y})\Big)$$

and we justify this choice with the same motivations given for the one-factor case in Chapter 2. By considering a two-factor model where the short rate evolves according to (4.1), the price of the Prototype Product introduced in Section 2.2 will be represented differently because now the Prototype payoff and the discount factor in formula (2.7) both depend on X and Y. In the following Section we shall see how, by choosing a two-factor short-rate model, one can generalize the definition of the Prototype Product and obtain a closed formula for its price.

4.2 The Prototype product and an explicit representation for the pricing formula

In order to generalize the price of the Prototype Product as in (2.7), we present a result which will be useful to reduce the problem of the Prototype Product pricing to one that is simpler to treat:

Lemma 4.5 *Let $X(Z) \doteq (X(s,Z),\mathcal{H}_s)_{s\in[t,T]}$ and $Y(Z) \doteq (Y(s,Z),\mathcal{G}_s)_{s\in[t,T]}$ be two stochastic processes of which the dynamics depend on a random variable Z taking values in $\mathcal{Z} = \{z_1,\dots,z_M\}$ with distribution $\pi = \{\pi_1,\dots,\pi_M\}$. By assuming that, conditionally on Z, the processes X and Y are independent, it follows that*

$$\mathbb{E}^{\widetilde{\mathbb{P}}}[f(X(Z))g(Y(Z))|\mathcal{H}_t \vee \mathcal{G}_t] = \sum_{h=1}^{M} \mathbb{E}^{\widetilde{\mathbb{P}}}[f(X(z_h))|\mathcal{H}_t]\mathbb{E}^{\widetilde{\mathbb{P}}}[g(Y(z_h))|\mathcal{G}_t]\pi_h, \ \forall T > t \geq 0 \quad (4.3)$$

where $f,g:\mathbb{R}\to\mathbb{R}$ are two generic functions.

Proof By the *tower property* of conditional expectations it follows that

$$\mathbb{E}^{\widetilde{\mathbb{P}}}[f(X(Z))g(Y(Z))|\mathcal{H}_t \vee \mathcal{G}_t] = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}[f(X(Z))g(Y(Z))|\sigma\{Z\} \vee \mathcal{H}_t \vee \mathcal{G}_t]\Big|\mathcal{H}_t \vee \mathcal{G}_t\Big] \tag{4.4}$$

Hence

$$\begin{aligned}
(4.4) &= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}[f(X(Z))|\sigma\{Z\} \vee \mathcal{H}_t \vee \mathcal{G}_t]\mathbb{E}^{\widetilde{\mathbb{P}}}[g(Y(Z))|\sigma\{Z\} \vee \mathcal{H}_t \vee \mathcal{G}_t]\Big|\mathcal{H}_t \vee \mathcal{G}_t\Big] \\
&= \sum_{h=1}^{M} \mathbb{E}^{\widetilde{\mathbb{P}}}[f(X(Z))|Z = z_h, \mathcal{H}_t \vee \mathcal{G}_t]\mathbb{E}^{\widetilde{\mathbb{P}}}[g(Y(Z))|Z = z_h, \mathcal{H}_t \vee \mathcal{G}_t]\widetilde{\mathbb{P}}(Z = z_h|\mathcal{H}_t \vee \mathcal{G}_t) \\
&= \sum_{h=1}^{M} \mathbb{E}^{\widetilde{\mathbb{P}}}[f(X(z_h))|\mathcal{H}_t \vee \mathcal{G}_t]\mathbb{E}^{\widetilde{\mathbb{P}}}[g(Y(z_h))|\mathcal{H}_t \vee \mathcal{G}_t]\widetilde{\mathbb{P}}(Z = z_h) \qquad (4.5)
\end{aligned}$$

where in the first passage we use the fact that, conditioned on $\sigma\{Z\} \vee \mathcal{H}_t \vee \mathcal{G}_t$, $f(X(Z))$ and $g(Y(Z))$ are independent and the last passage is motivated by the independence of Z on the σ-algebra $\mathcal{H}_t \vee \mathcal{G}_t$. By the hypothesis, $f(X(z_h))$ is independent by $\mathcal{G}_t$ and $g(Y(z_h))$ is independent by $\mathcal{H}_t$ because we have fixed $Z = z_h$, so

$$(4.5) = \sum_{h=1}^{M} \mathbb{E}^{\widetilde{\mathbb{P}}}[f(X(z_h))|\mathcal{H}_t]\mathbb{E}^{\widetilde{\mathbb{P}}}[g(Y(z_h))|\mathcal{G}_t]\pi_h \tag{4.6}$$

where π_h denotes the probability $\widetilde{\mathbb{P}}(Z = z_h)$. □

From now on we will use the following notations:

Notation 4.6 *We represent the CTMCs defined as in Assumptions 4.1 and 4.2 on the time interval $[t, T]$ with the following notation: $X(Z) \doteq (X(s, Z), \mathcal{H}_s)_{s\in[t,T]}$ and $Y(Z) \doteq (Y(s, Z), \mathcal{G}_s)_{s\in[t,T]}$. In fact we are interested in the dynamics of both processes X and Y only for times included between t and T because our purpose is to price interest rate derivatives which mature at time T and that are evaluated at the date t.*

We introduce now the Prototype Product when the short rate is given by (4.1) and $X(Z)$ and $Y(Z)$ are CTMCs as in Notation 4.6:

Definition 4.7 *A **Prototype product** is a financial product which guarantees to deliver a certain payoff Θ_0 at maturity T. This payoff depends on the value taken by the spot rate at the date of maturity T. Under the two-factor short-rate model (4.1) with the factors X and Y defined as in Assumptions 4.1 and 4.2, the price of the Prototype product at time $t < T$, analogously to (2.7), is represented by*

$$V_{\Theta_0,t,T}(X_{\nu_t^X}, Y_{\nu_t^Y}) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[DF(t, T; r) \cdot \Theta_0(X_{\nu(Z)_T^X}, Y_{\nu(Z)_T^Y})|\mathcal{H}_t \vee \mathcal{G}_t\Big] \tag{4.7}$$

where

- $DF(t, T; r) \triangleq exp\left(- a \sum_{i=\nu_t^X}^{\nu(Z)_T^X - 1} X_i(\mathrm{T}_{i+1}^X - \mathrm{T}_i^X) - b \sum_{j=\nu_t^Y}^{\nu(Z)_T^Y - 1} Y_j(\mathrm{T}_{j+1}^Y - \mathrm{T}_j^Y) \right)$ *is the discount*

factor;

- Θ_0 *is the* ***Prototype payoff*** *supposed to have the following form*

$$\Theta_0(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{\widetilde{N}} w_i\widetilde{w}_j \mathbf{I}_{\{x=x^i\}}\mathbf{I}_{\{y=y^j\}},\ x^i \in E^X,\ y^j \in E^Y,\ w_i, \widetilde{w}_j \in \{0\}\cup\mathbb{R}_+ \tag{4.8}$$

The price of the Prototype Product defined above can be represented, by using the results of Lemma 4.5, by an expression similar to the pricing formula of the Prototype Product in the one-factor short-rate model (see formula (2.7)):

Lemma 4.8 *Let us suppose the dynamics of the short rate to be given by (4.1) and the factors X and Y to be defined as in Assumptions 4.1 and 4.2 respectively. Then the price $V_{\Theta_0,t,T}(X_{\nu_t^X}, Y_{\nu_t^Y})$ at time t of the Prototype Product with maturity T and Prototype payoff Θ_0 defined by (4.8) admits the following representation:*

$$V_{\Theta_0,t,T}(X_{\nu_t^X}, Y_{\nu_t^Y}) = \sum_{h=1}^{M} U_{\vartheta_0,t,T}(X_{\nu_t^X}, z_h) U_{\widetilde{\vartheta}_0,t,T}(Y_{\nu_t^Y}, z_h)\pi_h \tag{4.9}$$

where

$$\begin{cases} U_{\vartheta_0,t,T}(X_{\nu_t^X}, z_h) \doteq \widetilde{\mathbb{E}^{\mathbb{P}}}\Big[exp\Big(-a\sum\limits_{i=\nu_t^X}^{\nu(z_h)_T^X-1} X_i(z_h)\cdot(\mathrm{T}_{i+1}^X - \mathrm{T}_i^X)\Big)\vartheta_0(X_{\nu(z_h)_T^X})|\mathcal{H}_t\Big] \\ \text{with } \vartheta_0(\cdot) \triangleq \sum_{i=1}^{N} w_i \mathbf{I}_{\{\cdot=x^i\}},\ x^i \in E^X, \end{cases} \tag{4.10}$$

and

$$\begin{cases} U_{\widetilde{\vartheta}_0,t,T}(Y_{\nu_t^Y}, z_h) \doteq \widetilde{\mathbb{E}^{\mathbb{P}}}\Big[exp\Big(-b\sum\limits_{j=\nu_t^Y}^{\nu(z_h)_T^Y-1} Y_j(z_h)\cdot(\mathrm{T}_{j+1}^Y - \mathrm{T}_j^Y)\Big)\widetilde{\vartheta}_0(Y_{\nu(z_h)_T^Y})|\mathcal{G}_t\Big] \\ \text{with } \widetilde{\vartheta}_0(\cdot) \triangleq \sum_{j=1}^{\widetilde{N}} \widetilde{w}_j \mathbf{I}_{\{\cdot=y^j\}},\ y^j \in E^Y. \end{cases} \tag{4.11}$$

Moreover, the Prototype payoff Θ_0 can be expressed in terms of the functions ϑ_0 and $\widetilde{\vartheta}_0$ as

$$\Theta_0(\cdot,*) = \vartheta_0(\cdot)\widetilde{\vartheta}_0(*) \tag{4.12}$$

Proof Formula (4.12) follows directly from the definitions of Θ_0, ϑ_0 and $\widetilde{\vartheta}_0$.
Hence we proceed to prove that the price of the Prototype product can be represented as in

(4.9): by Definition 4.7 and using $\vartheta_0(\cdot)$ and $\widetilde{\vartheta}_0(\cdot)$ as in (4.10) and (4.11)

$$
\begin{aligned}
V_{\Theta_0,t,T}(X_{\nu_t^X}, Y_{\nu_t^Y}) &= \mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[exp\Bigg(-a\sum_{i=\nu_t^X}^{\nu(Z)_T^X-1} X_i(\mathrm{T}_{i+1}^X-\mathrm{T}_i^X) - b\sum_{j=\nu_t^Y}^{\nu(Z)_T^Y-1} Y_j(\mathrm{T}_{j+1}^Y-\mathrm{T}_j^Y)\Bigg)\cdot \\
&\qquad \cdot\Bigg\{\sum_{i=1}^{N}\sum_{j=1}^{\widetilde{N}} w_i\widetilde{w}_j \mathbf{I}_{\{X_{\nu(Z)_T^X}=x^i\}}\mathbf{I}_{\{Y_{\nu(Z)_T^Y}=y^j\}}\Bigg\}|\mathcal{H}_t\vee\mathcal{G}_t\Bigg] \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[\Bigg\{exp\Bigg(-a\sum_{i=\nu_t^X}^{\nu(Z)_T^X-1} X_i(\mathrm{T}_{i+1}^X-\mathrm{T}_i^X)\Bigg)\sum_{i=1}^{N} w_i\mathbf{I}_{\{X_{\nu(Z)_T^X}=x^i\}}\Bigg\}\cdot \\
&\qquad \Bigg\{exp\Bigg(-b\sum_{j=\nu_t^Y}^{\nu(Z)_T^Y-1} Y_j(\mathrm{T}_{j+1}^Y-\mathrm{T}_j^Y)\Bigg)\sum_{j=1}^{\widetilde{N}}\widetilde{w}_j\mathbf{I}_{\{Y_{\nu(Z)_T^Y}=y^j\}}\Bigg\}|\mathcal{H}_t\vee\mathcal{G}_t\Bigg] \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[\Bigg\{exp\Bigg(-a\sum_{i=\nu_t^X}^{\nu(Z)_T^X-1} X_i(\mathrm{T}_{i+1}^X-\mathrm{T}_i^X)\Bigg)\vartheta_0(X_{\nu(Z)_T^X})\Bigg\}\cdot \\
&\qquad \Bigg\{exp\Bigg(-b\sum_{j=\nu_t^Y}^{\nu(Z)_T^Y-1} Y_j(\mathrm{T}_{j+1}^Y-\mathrm{T}_j^Y)\Bigg)\widetilde{\vartheta}_0(Y_{\nu(Z)_T^Y})\Bigg\}|\mathcal{H}_t\vee\mathcal{G}_t\Bigg] \qquad (4.13)
\end{aligned}
$$

Hence, by considering Lemma 4.5 when the processes $X(Z) \doteq (X(s,Z),\mathcal{H}_s)_{s\in[t,T]}$ and $Y(Z) \doteq (Y(s,Z),\mathcal{G}_s)_{s\in[t,T]}$ denote the two CTMCs specified as in Assumptions 4.1 and 4.2 respectively and given two real functions f and g

$$
f(X(Z)) = exp\Bigg(-a\sum_{i=\nu_t^X}^{\nu(Z)_T^X-1} X_i(\mathrm{T}_{i+1}^X-\mathrm{T}_i^X)\Bigg)\vartheta_0(X_{\nu(Z)_T^X})\Bigg)
$$

$$
g(Y(Z)) = exp\Bigg(-b\sum_{j=\nu_t^Y}^{\nu(Z)_T^Y-1} Y_j(\mathrm{T}_{j+1}^Y-\mathrm{T}_j^Y)\Bigg)\widetilde{\vartheta}_0(Y_{\nu(Z)_T^Y})\Bigg)
$$

we obtain that

$$
\begin{aligned}
(4.13) &= \sum_{h=1}^{M}\mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[exp\Big(-a\sum_{i=\nu_t^X}^{\nu(z_h)_T^X-1} X_i(z_h)\cdot(\mathrm{T}_{i+1}^X-\mathrm{T}_i^X)\Big)\vartheta_0(X_{\nu(z_h)_T^X})|\mathcal{H}_t\Bigg]\cdot \\
&\qquad \cdot\mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[exp\Big(-b\sum_{j=\nu_t^Y}^{\nu(z_h)_T^Y-1} Y_j(z_h)\cdot(\mathrm{T}_{j+1}^Y-\mathrm{T}_j^Y)\Big)\widetilde{\vartheta}_0(Y_{\nu(z_h)_T^Y})|\mathcal{G}_t\Bigg]\pi_h
\end{aligned}
$$

□

Remark 4.9 *We can compare formula (4.10) (or equivalently (4.11)) with the representation of the Prototype Product in the one-factor short-rate model (formulas (2.6)-(2.7)):*

- *assuming the short rate is given by the CTMC denoted by $X(z_h)$, the quantity $U_{\vartheta_0,t,T}(X_{\nu_t^X}, z_h)$ can be viewed as the price of the Prototype product with the function ϑ_0 defined by (4.10) as the Protoype payoff;*
- *similarly, by considering the CTMC denoted by $Y(z_h)$ as the short rate, the quantity $U_{\widetilde{\vartheta}_0,t,T}(Y_{\nu_t^Y}, z_h)$ can be viewed as the price of the Prototype product with the function $\widetilde{\vartheta}_0$ defined by (4.11) as the Protoype payoff;*

It follows that all the results on the Prototype Product pricing under a one-factor short-rate model (Chapter 2) can be easily carried over to give an explicit representation of the expectations $U_{\vartheta_0,t,T}(X_{\nu_t^X}, z_h)$ and $U_{\widetilde{\vartheta}_0,t,T}(Y_{\nu_t^Y}, z_h)$ for each z_h with $h = \{1, \dots, M\}$.

Hereafter we present all the results -derived by the Lemmas proved in Chapter 2- useful to give an explicit expression to formula (4.10) (and similarly to (4.11)):

Lemma 4.10 (analog of **Lemma 2.2**)
Let $X(Z)$ be a CTMC defined as in Assumption 4.1 and a a real number. For fixed $k, \eta \in \mathbb{N}$ and $Z = z_l$, let

$$\begin{cases} \vartheta_h(X_{\eta+k-h}, z_l) \triangleq \mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-aX_{\eta+k-h}(\mathrm{T}^X_{\eta+k-h+1} - \mathrm{T}^X_{\eta+k-h})}\vartheta_{h-1}(X_{\eta+k-h+1}, Z)\big|Z = z_l, \mathcal{H}^X_{\eta+k-h}\big] \\ \forall h = 1, \dots, k \end{cases} \tag{4.14}$$

with starting point independent of Z and given by the function $\vartheta_0(\cdot)$ defined as in (4.10), then

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-a\sum_{i=\eta}^{\eta+k-1} X_i(\mathrm{T}^X_{i+1} - \mathrm{T}^X_i)\Big)\vartheta_0(X_{\eta+k})|Z = z_l, \mathcal{H}^X_\eta\Big] = \vartheta_k(X_\eta, z_l) \tag{4.15}$$

Proof The proof is similar to the proof of Lemma 2.2: instead of the filtrations $\mathcal{F}^r_i$ (for each $i \in \{\eta, \dots, \eta+k-1\}$) we consider the σ-algebra $\{Z = z_l, \mathcal{H}^X_i\}$ where we have denoted $\mathcal{H}^X_i$ the σ-algebra $\mathcal{H}_{\mathrm{T}^X_i}$ for simplicity of notation (we choose $\{Z = z_l, \mathcal{H}^X_i\}$ because the functions ϑ_h do not depend only on the value of X at the step $\eta+k-h$, but also on the value z_l assumed by the random variable Z). □

Proposition 4.11 (analog of **Proposition 2.8**)
Consider the same hypotheses of Lemma 4.10, given an arbitrarily small ϵ, by defining for $l = 1, \dots, M$ the constant $\gamma(z_l) \triangleq \sup_{i\in\{1,\dots,N\}} \dfrac{q(z_l)^X_i}{ax^i + q(z_l)^X_i}$ and letting $n^X_\epsilon(z_l) \in \mathbb{N}$ be such that

$$n^X_\epsilon(z_l) \geq \left\lceil \frac{\log(\epsilon(1-\gamma(z_l)))}{\log(\gamma(z_l))} - \frac{\sup_{i\in\{1,\dots,N\}}|\vartheta_1(x^i, z_l) - \vartheta_0(x^i)|}{\log(\gamma(z_l))} \right\rceil \tag{4.16}$$

we have that $U^{\epsilon}_{\vartheta_0,t,T}(X_{\nu_t^X}, z_l) = \sum_{k=0}^{n_\epsilon^X(z_l)} \vartheta_k(X_{\nu_t^X}, z_l)\widetilde{\mathbb{P}}(\nu(z_l)^X_{t,T} = k|X_{\nu_t^X})$ *approximates the real value of* $U_{\vartheta_0,t,T}(X_{\nu_t^X}, z_l)$ *defined in (4.10), in the sense that*

$$|U^{\epsilon}_{\vartheta_0,t,T}(X_{\nu_t^X}, z_l) - U_{\vartheta_0,t,T}(X_{\nu_t^X}, z_l)| < \epsilon \text{ uniformily in } (t, T, X_{\nu_t^X}),\ \forall z_l \in \mathcal{Z} \tag{4.17}$$

Proof See the proof of Proposition 2.8. □

Lemma 4.12 (analog of **Lemma 2.10**)
Consider the same hypotheses of Lemma 4.10. Recalling that the function $\vartheta_0(\cdot) = \sum_{i=1}^N w_i \mathbf{I}_{\{\cdot = x^i\}}$ *defined in (4.10) has the coefficients* w_i *independent of* Z*, the functions* ϑ_k *satisfy the following formula*

$$\vartheta_k(\cdot, z_l) = \sum_{i_k=1}^N w^k_{i_k}(z_l)\mathbf{I}_{\{\cdot = x^{i_k}\}},\ k \geq 1,\ \text{for every fixed } z_l \in \mathcal{Z}, \tag{4.18}$$

where, for $m = 1, \ldots, N$,

$$\begin{cases} w^1_m(z_l) \triangleq \displaystyle\sum_{\substack{i_0=1 \\ i_0 \neq m}}^N w_{i_0} \frac{q(z_l)^X_{m,i_0}}{ax^m + q(z_l)^X_m}, & k = 1 \\ w^k_m(z_l) \triangleq \displaystyle\sum_{\substack{i_0,\ldots,i_{k-1}=1 \\ i_0 \neq i_1,\ldots,i_{k-1} \neq m}}^N w_{i_0} \frac{q(z_l)^X_{m,i_{k-1}}}{ax^m + q(z_l)^X_m}\left[\prod_{h=1}^{k-1} \frac{q(z_l)^X_{i_h,i_{h-1}}}{ax^{i_h} + q(z_l)^X_{i_h}}\right], & k > 1 \end{cases} \tag{4.19}$$

Proof See the proof of Lemma 2.10. □

Lemma 4.13 (analog of **Lemma 2.16**)
Under the same hypotheses of Lemma 4.10, assuming w.l.o.g that $\nu_t^X = h \in \mathbb{N}$ *and considering* $X_{\nu_t^X} = x^{i_h}$ *for a fixed index* $i_h \in \{1, \ldots, N\}$*, it follows that*

$$\begin{cases} \widetilde{\mathbb{P}}(\nu(z_l)^X_{t,T} = k|X_{\nu_t^X} = x^{i_h}) = \displaystyle\sum_{\substack{i_{h+1},\ldots,i_{h+k}=1 \\ i_{h+1}\neq i_h, i_{h+2}\neq i_{h+1},\ldots,i_{h+k}\neq i_{h+k-1}}}^N \exp\left(q(z_l)^X_{i_h} t - q(z_l)^X_{i_{h+k}} T\right) \cdot \\ \qquad \cdot \varphi_{h,k}(Q^X(z_l))\Psi_{h,k}(t, T, Q^X(z_l)); \\ \widetilde{\mathbb{P}}(\nu(z_l)^X_{t,T} = 0|X_{\nu_t^X} = x^{i_h}) = e^{-q(z_l)^X_{i_h}(T-t)}, \end{cases} \tag{4.20}$$

with $\Psi_{h,k}$ *the following multiple integral*

$$\begin{aligned} \Psi_{h,k}(t, T, Q^X(z_l)) \triangleq & \int_t^T e^{\left(q(z_l)^X_{i_{h+1}} - q(z_l)^X_{i_h}\right)t_{h+1}} \int_{t_{h+1}}^T e^{\left(q(z_l)^X_{i_{h+2}} - q(z_l)^X_{i_{h+1}}\right)t_{h+2}} \cdots \\ & \cdots \int_{t_{h+k-1}}^T e^{\left(q(z_l)^X_{i_{h+k}} - q(z_l)^X_{i_{h+k-1}}\right)t_{h+k}} dt_{h+k} \ldots dt_{h+2} dt_{h+1} \end{aligned}$$

and

$$\varphi_{h,k}(Q^X(z_l)) \triangleq q(z_l)^X_{i_h,i_{h+1}} \cdot \ldots \cdot q(z_l)^X_{i_{h+k-1},i_{h+k}}.$$

Proof See the proof of Lemma 2.16. □

Proposition 4.14 (analog of **Propositions 2.19-2.20**)
Consider the same hypotheses and notations of Proposition 4.11 and Lemma 4.12. Assuming that, at the evaluation time t, the process X is equal to a fixed $x^i \in E^X$, then the quantities $U_{\vartheta_0,t,T}(X_{\nu_t^X}, z_l)$ and their approximations $U^\epsilon_{\vartheta_0,t,T}(X_{\nu_t^X}, z_l)$ admit respectively the following representations

$$U_{\vartheta_0,t,T}(X_{\nu_t^X}, z_l) = \sum_{k=0}^{+\infty} [\widetilde{Q}^X(z_l)^k \cdot \theta_0(\underline{X})]_i \widetilde{\mathbb{P}}(\nu(z_l)^X_{t,T} = k | X_{\nu_t^X} = x^i), \tag{4.21}$$

and

$$U^\epsilon_{\vartheta_0,t,T}(X_{\nu_t^X}, z_l) = \sum_{k=0}^{n^X_\epsilon(z_l)} [\widetilde{Q}^X(z_l)^k \cdot \theta_0(\underline{X})]_i \widetilde{\mathbb{P}}(\nu(z_l)^X_{t,T} = k | X_{\nu_t^X} = x^i), \tag{4.22}$$

where

- $\widetilde{Q}^X(z_l) = (\widetilde{q}(z_l)^X_{i,j})_{1\le i,j,\le N} \triangleq \begin{cases} \frac{q(z_l)^X_{i,j}}{ax^i + q(z_l)^X_i}, & i \neq j \\ 0, & i = j \end{cases}, \ \forall l = 1, \ldots, M;$
- $\theta_0(\underline{X}) \triangleq [w_1, \ldots, w_N]'$ *whose components* w_i *are the coefficients of the function* ϑ_0 *defined by (4.10);*
- *the probabilities* $\widetilde{\mathbb{P}}(\nu(z_l)^X_{t,T} = k | X_{\nu_t^X} = x^i)$ *are given by (4.20);*
- $n^X_\epsilon(z_l)$ *is given by (4.16).*

Assuming that $\widetilde{Q}^X(z_l)$ *is diagonalizable for each* $l \in \{1, \ldots, M\}$, *we obtain*

$$U_{\vartheta_0,t,T}(x^i, z_l) = e_i' \cdot S \cdot \begin{bmatrix} \mathbb{E}^{\widetilde{\mathbb{P}}}[d(z_l)_1^{\nu(z_l)^X_{t,T}} | X_{\nu_t^X} = x^i] & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbb{E}^{\widetilde{\mathbb{P}}}[d(z_l)_N^{\nu(z_l)^X_{t,T}} | X_{\nu_t^X} = x^i] \end{bmatrix} \cdot S^{-1} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \tag{4.23}$$

and

$$U^\epsilon_{\vartheta_0,t,T}(x^i, z_l) = e_i' \cdot S \cdot \begin{bmatrix} \mathbb{E}^{\widetilde{\mathbb{P}}}_\epsilon[d(z_l)_1^{\nu(z_l)^X_{t,T}} | X_{\nu_t^X} = x^i] & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbb{E}^{\widetilde{\mathbb{P}}}_\epsilon[d(z_l)_N^{\nu(z_l)^X_{t,T}} | X_{\nu_t^X} = x^i] \end{bmatrix} \cdot S^{-1} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \tag{4.24}$$

where S *is a* $N \times N$ *matrix the columns of which are the eigenvectors of* $\widetilde{Q}^X(z_l)$, $(d(z_l)_j)_{j=1,\ldots,N}$ *are the eigenvalues of* $\widetilde{Q}^X(z_l)$, e_i *is the* i^{th} *unit vector and* $\mathbb{E}^{\widetilde{\mathbb{P}}}_\epsilon[d^{\nu(z_l)^X_{t,T}} | r_{\nu_t^X} = x^i] \triangleq \sum_{k=0}^{n^X_\epsilon(z_l)} d^k \widetilde{\mathbb{P}}(\nu(z_l)^X_{t,T} = k | X_{\nu_t^X} = x^i)$ *for a real number d.*

Proof See the proof of Propositions 2.19 and 2.20. □

Remark 4.15 *From now on we will give the representation of the price of the Prototype Product in terms of $U_{\vartheta_0,t,T}$ and $U^{\epsilon}_{\vartheta_0,t,T}$ defined as in (4.21) and (4.22) and of the analogous quantities $U_{\widetilde{\vartheta}_0,t,T}$ and $U^{\epsilon}_{\widetilde{\vartheta}_0,t,T}$ by implying that, in the case when $\widetilde{Q}^X(z_l)$ is diagonalizable for each $l \in \{1,\dots,M\}$, we can consider also the representations as in (4.23) and (4.24).*

We are now able to give an explicit representation of the price of the Prototype product:

Proposition 4.16 *Under the same hypotheses of Lemma 4.8, let us assume that the CTMCs X and Y both satisfy the assumptions of Proposition 4.14. Assuming that $X_{\nu_t^X} = x^n$ and $Y_{\nu_t^Y} = y^m$, the price of the Prototype product $V_{\Theta_0,t,T}(X_{\nu_t^X}, Y_{\nu_t^Y})$ as in (4.7) with Prototype payoff $\Theta_0(X_{\nu(Z)_T^X}, Y_{\nu(Z)_T^Y}) \overset{(4.8)}{=} \sum_{i=1}^{N}\sum_{j=1}^{\widetilde{N}} w_i\widetilde{w}_j \mathbf{I}_{\{X_{\nu(Z)_T^X}=x^i\}}\mathbf{I}_{\{Y_{\nu(Z)_T^Y}=y^j\}}$ can be represented as*

$$V_{\Theta_0,t,T}(X_{\nu_t^X}, Y_{\nu_t^Y})|_{X_{\nu_t^X}=x^n, Y_{\nu_t^Y}=y^m} = \sum_{h=1}^{M} U_{\vartheta_0,t,T}(x^n, z_h) U_{\widetilde{\vartheta}_0,t,T}(y^m, z_h)\pi_h \tag{4.25}$$

where, $\forall z_h \in \mathcal{Z}$,

$$U_{\vartheta_0,t,T}(x^n, z_h) = \sum_{k=0}^{+\infty} [\widetilde{Q}^X(z_h)^k \cdot \theta_0(\underline{X})]_n \widetilde{\mathbb{P}}(\nu(z_h)_{t,T}^X = k | X_{\nu_t^X} = x^n) \tag{4.26}$$

and

$$U_{\widetilde{\vartheta}_0,t,T}(y^m, z_h) = \sum_{k=0}^{+\infty} [\widetilde{Q}^Y(z_h)^k \cdot \tilde{\theta}_0(\underline{Y})]_m \widetilde{\mathbb{P}}(\nu(z_h)_{t,T}^Y = k | Y_{\nu_t^Y} = y^m) \tag{4.27}$$

with the following notations

- $\widetilde{Q}^X(z_h) = (\widetilde{q}(z_h)_{i,j}^X)_{1\le i,j,\le N} \triangleq \begin{cases} \frac{q(z_l)_{i,j}^X}{ax^i + q(z_h)_i^X}, & i \neq j \\ 0, & i = j \end{cases} \quad \forall h = 1,\dots,M;$
- $\widetilde{Q}^Y(z_h) = (\widetilde{q}(z_h)_{i,j}^Y)_{1\le i,j,\le \tilde{N}} \triangleq \begin{cases} \frac{q(z_l)_{i,j}^Y}{by^i + q(z_h)_i^Y}, & i \neq j \\ 0, & i = j \end{cases} \quad \forall h = 1,\dots,M;$
- *$\theta_0(\underline{X}) \triangleq [w_1,\dots,w_N]'$ and $\tilde{\theta}_0(\underline{Y}) \triangleq [\tilde{w}_1,\dots,\tilde{w}_{\tilde{N}}]'$ whose components w_i and $\tilde{w}_i$ are the coefficients of the function ϑ_0 defined by (4.10) and of $\tilde{\vartheta}_0$ defined by (4.11) respectively;*
- *the probabilities $\widetilde{\mathbb{P}}(\nu(z_h)_{t,T}^X = k | X_{\nu_t^X} = x^n)$ and $\widetilde{\mathbb{P}}(\nu(z_h)_{t,T}^Y = k | Y_{\nu_t^Y} = y^m)$ relative to the processes $X(z_h)$ and $Y(z_h)$ respectively in formula (4.20), are given by Lemma 4.13;*

Proof The proof follows directly from Lemma 4.8 and Proposition 4.14:

- according to Lemma 4.8 the price of the Prototype product $V_{\Theta_0,t,T}$ can be represented as in (4.9) with $U_{\vartheta_0,t,T}$ defined by (4.10) and $U_{\widetilde{\vartheta}_0,t,T}$ defined by (4.11);

- by result (4.21) in Proposition 4.14, $U_{\vartheta_0,t,T}$ and, analogously, $U_{\widetilde{\vartheta}_0,t,T}$ admit the representations (4.26) and (4.27) respectively.

□

As we have done in Chapter 2, we finally present a computable expression of the price of the Prototype product suggested by the result of Lemma 4.11:

Proposition 4.17 *Consider the same hypotheses and notations of the previous Proposition. For an arbitrarily small ϵ, we define the approximation of the real price of the Prototype product as*

$$V^{\epsilon}_{\Theta_0,t,T}(X_{\nu_t^X}, Y_{\nu_t^Y})|_{X_{\nu_t^X}=x^n, Y_{\nu_t^Y}=y^m} \triangleq \sum_{h=1}^{M} U^{\epsilon}_{\vartheta_0,t,T}(x^n, z_h) U^{\epsilon}_{\widetilde{\vartheta}_0,t,T}(y^m, z_h)\pi_h \tag{4.28}$$

where, $\forall z_h \in \mathcal{Z}$,

$$U^{\epsilon}_{\vartheta_0,t,T}(x^n, z_h) = \sum_{k=0}^{n^X_{\epsilon}(z_h)} [\widetilde{Q}^X(z_h)^k \cdot \theta_0(\underline{X})]_n \widetilde{\mathbb{P}}(\nu(z_h)^X_{t,T} = k | X_{\nu_t^X} = x^n) \tag{4.29}$$

and

$$U^{\epsilon}_{\widetilde{\vartheta}_0,t,T}(y^m, z_h) = \sum_{k=0}^{n^Y_{\epsilon}(z_h)} [\widetilde{Q}^Y(z_h)^k \cdot \tilde{\theta}_0(\underline{Y})]_m \widetilde{\mathbb{P}}(\nu(z_h)^Y_{t,T} = k | Y_{\nu_t^Y} = y^m) \tag{4.30}$$

with

$$\begin{cases} n^X_{\epsilon}(z_h) \geq \left\lceil \frac{\log(\epsilon(1-\gamma(z_h)))}{\log(\gamma(z_h))} - \frac{\sup_{i\in\{1,\dots,N\}} |\vartheta_1(x^i,z_h)-\vartheta_0(x^i)|}{\log(\gamma(z_h))} \right\rceil, \\ \gamma(z_h) \triangleq \sup\limits_{i\in\{1,\dots,N\}} \dfrac{q(z_h)^X_i}{ax^i + q(z_h)^X_i}, \end{cases}$$

and

$$\begin{cases} n^Y_{\epsilon}(z_h) \geq \left\lceil \frac{\log(\epsilon(1-\gamma(z_h)))}{\log(\gamma(z_h))} - \frac{\sup_{j\in\{1,\dots,\widetilde{N}\}} |\widetilde{\vartheta}_1(y^j,z_h)-\widetilde{\vartheta}_0(y^j)|}{\log(\gamma(z_h))} \right\rceil, \\ \gamma(z_h) \triangleq \sup\limits_{j\in\{1,\dots,\widetilde{N}\}} \dfrac{q(z_h)^Y_j}{by^j + q(z_h)^Y_j}. \end{cases}$$

Then $V^{\epsilon}_{\Theta_0,t,T}(X_{\nu_t^X}, Y_{\nu_t^Y})$ is a good approximation of $V_{\Theta_0,t,T}(X_{\nu_t^X}, Y_{\nu_t^Y})$ in the sense that

$$V^{\epsilon}_{\Theta_0,t,T}(X_{\nu_t^X}, Y_{\nu_t^Y}) \xrightarrow{\epsilon\to 0} V_{\Theta_0,t,T}(X_{\nu_t^X}, Y_{\nu_t^Y}) \text{ uniformily in } (t, T, X_{\nu_t^X}, Y_{\nu_t^Y}). \tag{4.31}$$

Proof By the definition of $V^{\epsilon}_{\Theta_0,t,T}$ as in (4.28), to prove the convergence in (4.31) it is sufficient to prove that, $\forall z_h \in \mathcal{Z}$,

1. $U^{\epsilon}_{\vartheta_0,t,T}(X_{\nu_t^X}, z_h) \xrightarrow{\epsilon\to 0} U_{\vartheta_0,t,T}(X_{\nu_t^X}, z_h)$ uniformily in $(t, T, X_{\nu_t^X})$,

2. $U^{\epsilon}_{\widetilde{\vartheta}_0,t,T}(Y_{\nu_t^Y}, z_h) \xrightarrow{\epsilon\to 0} U_{\widetilde{\vartheta}_0,t,T}(Y_{\nu_t^Y}, z_h)$ uniformily in $(t, T, Y_{\nu_t^Y})$.

The result now follows immediately from Proposition 4.11. □

4.3 Bond, Cap, Swaption and Bond Option pricing under a two-factor short-rate model

We can now proceed to price bonds and the other interest rate derivatives in the presence of a two-factor short-rate model (by considering upper approximation prices as we have done in the scalar case by using Remark 2.5 and the results in Lemma 2.13).

We will see that -as we have shown in Chapter 3 when we have considered the short rate as a CTMC- bonds, caps, swaptions and bond options can be represented as linear combinations of Prototype products. We will omit the bond option pricing for sake of similarity with the caplet and swaption ones, but we additionally will give a representation for defaultable bonds.

4.3.1 Bond pricing

Since the bond price depends on the spot rate at the date of evaluation t, that is on the values assumed by $X(t)$ and $Y(t)$, from now on we denote $p(t,T)$ by $p(t,T;X(t),Y(t))$. Let us see the representation of the bond price in Proposition 4.18 and a computable bond pricing formula in Proposition 4.19:

Proposition 4.18 *Let us suppose that the dynamics of the short rate are given by (4.1) and the factors X and Y are defined as in Assumptions 4.1-4.2 respectively. Assuming that, at the evaluation time t, $X(t) = x^n$ for a fixed $n \in \{1,\dots,N\}$ and $Y(t) = y^m$ for a fixed $m \in \{1,\dots,\widetilde{N}\}$, a zero-coupon bond which matures at time T can be viewed as a Prototype product characterized by the following payoff:*

$$\begin{cases} \Theta_0(\cdot,*) = \Phi_0(\cdot,*) \triangleq \sum_{i=1}^{N}\sum_{j=1}^{\widetilde{N}} w_i\widetilde{w}_j \mathbf{I}_{\{\cdot=x^i\}}\mathbf{I}_{\{*=y^j\}} \\ w_i = 1,\ i = 1,\dots,N \\ \widetilde{w}_j = 1,\ j = 1,\dots,\widetilde{N} \end{cases} \tag{4.32}$$

In other terms the price of a T-bond admits the following representation

$$p(t,T;X_{\nu_t^X},Y_{\nu_t^Y})|_{X_{\nu_t^X}=x^n, Y_{\nu_t^Y}=y^m} = \sum_{h=1}^{M} U_{\vartheta_0,t,T}(x^n,z_h)U_{\widetilde{\vartheta}_0,t,T}(y^m,z_h)\pi_h \tag{4.33}$$

where, by using the same notations as in Proposition 4.16, for each $z_h \in \mathcal{Z}$ we have that

$$\begin{cases} U_{\vartheta_0,t,T}(x^n,z_h) = \sum_{k=0}^{+\infty}[\widetilde{Q}^X(z_h)^k \cdot \theta_0(\underline{X})]_n \widetilde{\mathbb{P}}(\nu(z_h)^X_{t,T} = k | X_{\nu_t^X} = x^n) \\ U_{\widetilde{\vartheta}_0,t,T}(y^m,z_h) = \sum_{k=0}^{+\infty}[\widetilde{Q}^Y(z_h)^k \cdot \tilde{\theta}_0(\underline{Y})]_m \widetilde{\mathbb{P}}(\nu(z_h)^Y_{t,T} = k | Y_{\nu_t^Y} = y^m) \end{cases} \tag{4.34}$$

with $\theta_0(\underline{X}) = [1,\dots,1]' \in \mathbb{R}^N$ and $\tilde{\theta}_0(\underline{Y}) = [1,\dots,1]' \in \mathbb{R}^{\tilde{N}}$.

Proof Using the same notations of Definition 4.7 of the Prototype Product, we can write the price of a bond which matures at time T as the price of a Prototype Product with the same maturity:

$$\begin{aligned} p(t,T;X_{\nu_t^X},Y_{\nu_t^Y}) &\overset{(4.2)}{=} \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[DF(t,T;r)|\mathcal{H}_t \vee \mathcal{G}_t\Big] \\ &= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[DF(t,T;r)\cdot\Theta_0(X_{\nu(Z)_T^X},Y_{\nu(Z)_T^Y})|\mathcal{H}_t \vee \mathcal{G}_t\Big] \\ &= V_{\Theta_0,t,T}(X_{\nu_t^X},Y_{\nu_t^Y}) \end{aligned} \tag{4.35}$$

where the second passage is due to the fact that, by definition (4.32), $\Theta_0(x^i,y^j)=1$ for each pair (i,j) such that $i\in\{1,\dots,N\}$ and $j\in\{1,\dots,\tilde{N}\}$. Hence the statement is proved directly by the results of Proposition 4.16. □

Proposition 4.19 *Consider the same hypotheses and notations of the previous Proposition. For an arbitrarily small ϵ, we define the approximation of the price of a T-bond at the date of evaluation t as*

$$p_\epsilon(t,T;X_{\nu_t^X},Y_{\nu_t^Y})|_{X_{\nu_t^X}=x^n,Y_{\nu_t^Y}=y^m} \triangleq \sum_{h=1}^{M} U^\epsilon_{\vartheta_0,t,T}(x^n,z_h)U^\epsilon_{\tilde{\vartheta}_0,t,T}(y^m,z_h)\pi_h \tag{4.36}$$

where, $\forall z_h \in \mathcal{Z}$, $U^\epsilon_{\vartheta_0,t,T}(x^n,z_h)$ and $U^\epsilon_{\tilde{\vartheta}_0,t,T}(y^m,z_h)$ are given by (4.29) and (4.30) with $\theta_0(\underline{X})=[1,\dots,1]'\in\mathbb{R}^N$ and $\tilde{\theta}_0(\underline{Y})=[1,\dots,1]'\in\mathbb{R}^{\tilde{N}}$ respectively.

It follows that $p_\epsilon(t,T;X_{\nu_t^X},Y_{\nu_t^Y})$ is a good approximation of the actual bond price $p(t,T;X_{\nu_t^X},Y_{\nu_t^Y})$ in the sense that

$$p_\epsilon(t,T;X_{\nu_t^X},Y_{\nu_t^Y}) \overset{\epsilon\to 0}{\longrightarrow} p(t,T;X_{\nu_t^X},Y_{\nu_t^Y}) \text{ uniformily in } (t,T,X_{\nu_t^X},Y_{\nu_t^Y}). \tag{4.37}$$

Proof See proof of Proposition 4.17. □

We conclude by giving a result due to the time homogeneity property of the factors $X(t,Z)$ and $Y(t,Z)$ on which the spot rate depends:

Proposition 4.20 *Under the same hypotheses of Proposition 4.18, a T-bond price and its approximation can be written as follows*

$$\begin{cases} p(t,T;X_{\nu_t^X},Y_{\nu_t^Y})|_{X_{\nu_t^X}=x^n,Y_{\nu_t^Y}=y^m} = p(T-t;x^n,y^m) \\ p_\epsilon(t,T;X_{\nu_t^X},Y_{\nu_t^Y})|_{X_{\nu_t^X}=x^n,Y_{\nu_t^Y}=y^m} = p_\epsilon(T-t;x^n,y^m) \end{cases} \tag{4.38}$$

where $p(T-t;x^n,y^m) \triangleq p(0,T-t;X(0),Y(0))|_{X(0)=x^n,Y(0)=y^m}$ and $p_\epsilon(T-t;x^n,y^m) \triangleq p_\epsilon(0,T-t;X(0),Y(0))|_{X(0)=x^n,Y(0)=y^m}$.

Proof Remark 2.18 holds also for the probabilities in formula (4.34) because X and Y are both CTMC homogeneous with respect to the time. Therefore, similarly to the proof of Proposition 3.3, we are able to prove (4.38). □

4.3.2 Defaultable bonds: an application of the Prototype Product pricing approach

Until now we have solved the bond pricing problem, under the assumption of a *"default free market"*, when the short rate is considered a single CTMC (see Section 3.1) or when it is taken as a linear combination of two correlated CTMCs (see Subsection 4.3.1). We are however also able to give a representation of the price of defaultable bonds by using the Prototype Product pricing approach introduced in Section 4.2.

In a general setting of the reduced form approach to credit risk, the price of a defaultable bond at time of today t and maturity T can be written as

$$\Pi(t,T) = \mathbf{I}_{\{\tau>t\}}\widetilde{\mathbb{E}}^{\mathbb{P}}[e^{\int_t^T r(s)+\lambda(s)ds}|\mathcal{F}_t] \tag{4.39}$$

where the processes r and λ represent the spot rate and the default intensity respectively, τ is the time of default and $\mathcal{F}_t$ is the filtration generated by the two-dimensional process (r,λ).

Let us suppose that $r(t) \equiv X(t,Z)$ as defined in Assumption 4.1 and the default intensity $\lambda(t) \equiv Y(t,Z)$ as defined in Assumption 4.2: under these assumptions, the price of a defaultable bond $\Pi(t,T)$ admits a representation as in (4.2) with $a = b = 1$ which is the pricing formula of a default free bond when the spot rate depends on two correlated CTMCs $X(t,Z)$ and $Y(t,Z)$. In this way we can adapt all the results presented in Subsection 4.3.1 to the pricing of defaultable bonds.

4.3.3 Cap pricing

As we have done in Section 3.2, we bring back the cap pricing to the that of a caplet and we are going to use the same notations of Subsection 3.2.1.

To this purpose we generalize the pricing formula of the i-th caplet given by (3.28) when we consider that the spot is given by (4.1) under the Assumptions 4.1-4.2:

$$Cpl(t,S_i;X_{\nu_t^X},Y_{\nu_t^Y}) = (1+Ks_i)\widetilde{\mathbb{E}}^{\widetilde{\mathbb{P}}}\Big[DF(t,S_{i-1};r)\Big(\frac{1}{1+Ks_i} - p(S_{i-1},S_i;X_{\nu^X_{S_{i-1}}},Y_{\nu^Y_{S_{i-1}}})\Big)^+\Big|\mathcal{H}_t\vee\mathcal{G}_t\Big] \tag{4.40}$$

where

$$DF(t,S_{i-1};r) \triangleq exp\Bigg(-a\sum_{i=\nu_t^X}^{\nu(Z)^X_{S_{i-1}}-1} X_i(\mathrm{T}^X_{i+1}-\mathrm{T}^X_i) - b\sum_{j=\nu_t^Y}^{\nu(Z)^Y_{S_{i-1}}-1} Y_j(\mathrm{T}^Y_{j+1}-\mathrm{T}^Y_j)\Bigg)$$

is the discount factor previously introduced in Definition 4.7.

We are able to give the representation of the caplet price as a linear combination of the prices of $N\times\tilde{N}$ Prototype Products:

Proposition 4.21 *Let us consider the spot rate given by relation (4.1) and that, at the date of evaluation t, $X_{\nu_t^X} = x^n$ and $Y_{\nu_t^Y} = y^m$ for fixed $n\in\{1,\dots,N\}$ and $m\in\{1,\dots,\tilde{N}\}$, then*

the price of the i-th caplet can be written as

$$\frac{Cpl(t,S_i;X_{\nu_t^X},Y_{\nu_t^Y})|_{X_{\nu_t^X}=x^n,Y_{\nu_t^Y}=y^m}}{1+Ks_i}=\sum_{l=1}^{N}\sum_{j=1}^{\tilde{N}}\Big(\frac{1}{1+Ks_i}-p(s_i;x^l,y^j)\Big)^+V_{\Psi_0^{l,j},t,S_{i-1}}(x^n,y^m) \tag{4.41}$$

where we use the notation introduced in Proposition 4.20 for the bond prices $p(s_i;x^l,y^j)$ and the quantity $\Psi_0^{l,j}$ is as follows

$$\begin{cases}\Psi_0^{l,j}(\cdot,*)=\sum_{i_0=1}^{N}\sum_{i_1=1}^{\tilde{N}}w_{i_0}(l)\tilde{w}_{i_1}(j)\mathbf{I}_{\{\cdot=x^{i_0}\}}\mathbf{I}_{\{*=y^{i_1}\}},\\ w_{i_0}(l)=\begin{cases}0,\ i_0\neq l\\ 1,\ i_0=l\end{cases}\quad and\ \tilde{w}_{i_1}(j)=\begin{cases}0,\ i_1\neq j\\ 1,\ i_1=j.\end{cases}\end{cases} \tag{4.42}$$

Proof By (4.40) we have

$$\begin{aligned}\frac{Cpl(t,S_i;X_{\nu_t^X},Y_{\nu_t^Y})}{1+Ks_i}&=\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[DF(t,S_{i-1};r)\Big(\frac{1}{1+Ks_i}-p(S_{i-1},S_i;X_{\nu^X_{S_{i-1}}},Y_{\nu^Y_{S_{i-1}}})\Big)^+\Big|\mathcal{H}_t\vee\mathcal{G}_t\Big]\\
&=\mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[\sum_{l=1}^{N}\sum_{j=1}^{\tilde{N}}DF(t,S_{i-1};r)\Big(\frac{1}{1+Ks_i}-p(S_{i-1},S_i;x^l,y^j)\Big)^+\cdot\\
&\qquad\cdot\mathbf{I}_{\{X_{\nu^X_{S_{i-1}}}=x^l\}}\mathbf{I}_{\{Y_{\nu^Y_{S_{i-1}}}=y^j\}}\Big|\mathcal{H}_t\vee\mathcal{G}_t\Bigg]\\
&=\sum_{l=1}^{N}\sum_{j=1}^{\tilde{N}}\Bigg\{\Big(\frac{1}{1+Ks_i}-p(S_{i-1},S_i;x^l,y^j)\Big)^+\cdot\\
&\qquad\cdot\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[DF(t,S_{i-1};r)\mathbf{I}_{\{X_{\nu^X_{S_{i-1}}}=x^l\}}\mathbf{I}_{\{Y_{\nu^Y_{S_{i-1}}}=y^j\}}\Big|\mathcal{H}_t\vee\mathcal{G}_t\Big]\Bigg\}\\
&\overset{(4.38)}{=}\sum_{l=1}^{N}\sum_{j=1}^{\tilde{N}}\Bigg\{\Big(\frac{1}{1+Ks_i}-p(s_i;x^l,y^j)\Big)^+\cdot\\
&\qquad\cdot\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[DF(t,S_{i-1};r)\mathbf{I}_{\{X_{\nu^X_{S_{i-1}}}=x^l\}}\mathbf{I}_{\{Y_{\nu^Y_{S_{i-1}}}=y^j\}}\Big|\mathcal{H}_t\vee\mathcal{G}_t\Big]\Bigg\}\\
&\overset{(4.42)}{=}\sum_{l=1}^{N}\sum_{j=1}^{\tilde{N}}\Bigg\{\Big(\frac{1}{1+Ks_i}-p(s_i;x^l,y^j)\Big)^+\cdot\\
&\qquad\cdot\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[DF(t,S_{i-1};r)\Psi_0^{l,j}(X_{\nu^X_{S_{i-1}}},Y_{\nu^Y_{S_{i-1}}})\Big|\mathcal{H}_t\vee\mathcal{G}_t\Big]\Bigg\}\\
&=\sum_{l=1}^{N}\sum_{j=1}^{\tilde{N}}\Big(\frac{1}{1+Ks_i}-p(s_i;x^l,y^j)\Big)^+V_{\Psi_0^{l,j},t,S_{i-1}}(X_{\nu_t^X},Y_{\nu_t^Y})\end{aligned}$$

where the last passage is due to the Definition of the price of the Prototype Product at time

t with maturity S_{i-1} and payoff $\Psi_0^{l,j}$. Hence, when $X_{\nu_t^X} = x^n$ and $Y_{\nu_t^Y} = y^m$, it follows that

$$\frac{Cpl(t, S_i; X_{\nu_t^X}, Y_{\nu_t^Y})|_{X_{\nu_t^X}=x^n, Y_{\nu_t^Y}=y^m}}{1+Ks_i} = \sum_{l=1}^{N}\sum_{j=1}^{\tilde{N}} \left(\frac{1}{1+Ks_i} - p(s_i; x^l, y^j)\right)^+ V_{\Psi_0^{l,j},t,S_{i-1}}(x^n, y^m)$$

□

To obtain a computable price of the already mentioned i-th caplet, it suffices to apply the results obtained in Proposition 4.17 to the Prototype Product prices $V_{\Psi_0^{l,j},t,S_{i-1}}$ with payoffs $\Psi_0^{l,j}$ defined by (4.42) for each $l \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, \tilde{N}\}$:

Proposition 4.22 *Under the assumptions of the previous Proposition we define, for an arbitrarily small ϵ, the approximation of the price of the i-th caplet $Cpl(t, S_i; X_{\nu_t^X}, Y_{\nu_t^Y})$ as*

$$\frac{Cpl_\epsilon(t, S_i; X_{\nu_t^X}, Y_{\nu_t^Y})|_{X_{\nu_t^X}=x^n, Y_{\nu_t^Y}=y^m}}{1+Ks_i} \triangleq \sum_{l=1}^{N}\sum_{j=1}^{\tilde{N}} \left(\frac{1}{1+Ks_i} - p_\epsilon(s_i; x^l, y^j)\right)^+ V^\epsilon_{\Psi_0^{l,j},t,S_{i-1}}(x^n, y^m)$$

where, for each couple of indexes (l, j), $V^\epsilon_{\Psi_0^{l,j},t,S_{i-1}}$ is defined by (4.28) with $\Theta_0 = \Psi_0^{l,j}$ and $T = S_{i-1}$. It follows that $Cpl_\epsilon(t, S_i; X_{\nu_t^X}, Y_{\nu_t^Y})$ is a good approximation of the real price $Cpl(t, S_i; X_{\nu_t^X}, Y_{\nu_t^Y})$ in the sense that

$$Cpl_\epsilon(t, S_i; X_{\nu_t^X}, Y_{\nu_t^Y}) \xrightarrow{\epsilon\to 0} Cpl(t, S_i; X_{\nu_t^X}, Y_{\nu_t^Y}) \text{ uniformly in } (t, S_{i-1}, S_i, X_{\nu_t^X}, Y_{\nu_t^Y}). \quad (4.43)$$

Proof The statement is proved directly by the results of Proposition 4.17. □

4.3.4 Swaption pricing

Hereafter we give an explicit formula for the price of a swaption under the market model introduced at the beginning of this Chapter. By using the same notations of Section 3.3, we apply the general formula of a swaption given by (3.36), namely

$$Swopt_t(S_\alpha, S_{\alpha,\beta}; X_{\nu_t^X}, Y_{\nu_t^Y}) = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[DF(t, S_\alpha; r)(PFS_\alpha^\beta(S_\alpha, K))^+ \middle| \mathcal{H}_t \vee \mathcal{G}_t\right] \quad (4.44)$$

where

- $DF(t, S_\alpha; r) \triangleq exp\left(- a \sum_{i=\nu_t^X}^{\nu(Z)^X_{S_\alpha}-1} X_i(\mathrm{T}^X_{i+1} - \mathrm{T}^X_i) - b \sum_{j=\nu_t^Y}^{\nu(Z)^Y_{S_\alpha}-1} Y_j(\mathrm{T}^Y_{j+1} - \mathrm{T}^Y_j)\right)$ is the discount factor (analogously to (4.40));
- $PFS_\alpha^\beta(t, K) \triangleq p(t, S_\alpha; X_{\nu_t^X}, Y_{\nu_t^Y}) - p(t, S_\beta; X_{\nu_t^X}, Y_{\nu_t^Y}) - K\sum_{h=\alpha+1}^{\beta} s_h p(t, S_h; X_{\nu_t^X}, Y_{\nu_t^Y})$ which has the same meaning as that introduced in Section 3.3 (see (3.35)).

Similarly to the caplet pricing, we get the representation of the price of a swaption and its approximation:

Proposition 4.23 *Let us consider the spot rate given by relation (4.1) and that, at the date of evaluation t, $X_{\nu_t^X} = x^n$ and $Y_{\nu_t^Y} = y^m$ for fixed $n \in \{1, \dots, N\}$ and $m \in \{1, \dots, \tilde{N}\}$, then the price of the swaption can be written as*

$$Swopt_t(S_\alpha, S_{\alpha,\beta}; X_{\nu_t^X}, Y_{\nu_t^Y})|_{X_{\nu_t^X}=x^n, Y_{\nu_t^Y}=y^m} = \sum_{l=1}^{N}\sum_{j=1}^{\tilde{N}}(g(S_\alpha, S_{\alpha,\beta}; x^l, y^j))^+ V_{\Psi_0^{l,j},t,S_\alpha}(x^n, y^m) \tag{4.45}$$

where, for each $l \in \{1, \dots, N\}$ and $j \in \{1, \dots, \tilde{N}\}$,

- $g(S_\alpha, S_{\alpha,\beta}; x^l, y^j) = 1 - p(S_\beta - S_\alpha; x^l, y^j) - K\sum_{h=\alpha+1}^{\beta} s_h p(S_h - S_\alpha; x^l, y^j)$,
- $p(S_h - S_\alpha; x^l, y^j) = p(S_\alpha, S_h; X_{\nu_{S_\alpha}^X}, Y_{\nu_{S_\alpha}^Y})|_{X_{\nu_{S_\alpha}^X}=x^l, Y_{\nu_{S_\alpha}^Y}=y^j}$, $\forall h = \alpha+1, \dots, \beta$ *by using the same notations introduced in Proposition 4.20,*
- $\Psi_0^{l,j}$ *is defined by (4.42).*

Proposition 4.24 *Under the assumptions of the previous Proposition we define, for an arbitrarily small ϵ, the approximation of the price of the swaption $Swopt_t(S_\alpha, S_{\alpha,\beta}; X_{\nu_t^X}, Y_{\nu_t^Y})$ as*

$$Swopt_t^\epsilon(S_\alpha, S_{\alpha,\beta}; X_{\nu_t^X}, Y_{\nu_t^Y})|_{X_{\nu_t^X}=x^n, Y_{\nu_t^Y}=y^m} = \sum_{l=1}^{N}\sum_{j=1}^{\tilde{N}}(g^\epsilon(S_\alpha, S_{\alpha,\beta}; x^l, y^j))^+ V^\epsilon_{\Psi_0^{l,j},t,S_{i-1}}(x^n, y^m)$$

where $g^\epsilon(S_\alpha, S_{\alpha,\beta}; x^l, y^j) = 1 - p_\epsilon(S_\beta - S_\alpha; x^l, y^j) - K\sum_{h=\alpha+1}^{\beta} s_h p_\epsilon(S_h - S_\alpha; x^l, y^j)$ and, for each pair of indexes (l, j), $V^\epsilon_{\Psi_0^{l,j},t,S_\alpha}$ is defined by (4.28) with $\Theta_0 = \Psi_0^{l,j}$ and $T = S_\alpha$. It follows that $Swopt_t^\epsilon(S_\alpha, S_{\alpha,\beta}; X_{\nu_t^X}, Y_{\nu_t^Y})$ is a good approximation of the actual price $Swopt_t(S_\alpha, S_{\alpha,\beta}; X_{\nu_t^X}, Y_{\nu_t^Y})$ in the sense that

$$Swopt_t^\epsilon(S_\alpha, S_{\alpha,\beta}; X_{\nu_t^X}, Y_{\nu_t^Y}) \overset{\epsilon\to 0}{\longrightarrow} Swopt_t^\epsilon(S_\alpha, S_{\alpha,\beta}; X_{\nu_t^X}, Y_{\nu_t^Y}) \text{ uniformly in } (t, S_\alpha, S_{\alpha,\beta}, X_{\nu_t^X}, Y_{\nu_t^Y}). \tag{4.46}$$

We omit the proof of the Propositions above for sake of similarity to Propositions 4.21 and 4.22.

Chapter 5

Numerical results

The aim of this chapter is to test numerically the pricing approach proposed up to now for bonds and other interest rate derivatives under the assumption that the spot rate is a CTMC (or more generally the spot rate is a linear combination of several correlated CTMCs). In what follows we call it ***"Prototype Product Approach"***. In the following sections we present numerical results both for the one-factor short rate model and the two-factor one.

5.1 Numerical results when the short rate is a single-factor CTMC

For the numerical tests we shall treat only the pricing of zero-coupon bonds because, as seen in the previous chapters, in our approach the prices of caps and swaptions can be written as functions of prices of bonds and other Prototype Products (this holds also for bond options). We shall test numerically the validity of our approach by proceeding as follows: consider a continuous time short-rate model for which the bond price admits an explicit closed formula and compare this exact price obtained with the one given by (3.7) after approximating the short rate by a CTMC so as to apply the Prototype Product Approach.

Let us choose the following continuous-time affine model, known as the Cox-Ingersoll-Ross (**CIR**) or square-root model, for the short rate:

$$\begin{cases} dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW_t \\ r(0) = \widetilde{r} \end{cases} \tag{5.1}$$

where W_t is a Wiener process under an equivalent martingale measure $\tilde{\mathbb{P}}$ as introduced in Chapter 2 and the long-run mean θ, the rate of mean reversion k, the volatility σ and the initial spot rate $\tilde{r}$ are positive constants. Moreover, to ensure that the process remains positive, the following condition has to be satisfied

$$2k\theta > \sigma^2. \tag{5.2}$$

The CIR model is largely used in finance to model the short rate: it guarantees the positivity of the square-root process in (5.1) and the bond price admits a closed formula.

First approximate the square-root process by a CTMC by using a suitable approximation that we call *"Kushner approximation"* (which we are going to introduce next), so that we can apply the Prototype Product Approach to compute the bond price and compare it with the exact price. We shall price zero-coupon bonds with the Prototype Product Approach both by computing explicit formulae and by a full simulation approach based on Monte Carlo techniques. Moreover, in order to have a further possibility of comparison, we shall also consider a widely used approach, namely the *lattice method*, to compute approximations of the bond price starting from the continuous-affine term short rate model. We shall thus compute prices in the following four ways:

a) ***Explicit Closed formula***

b) ***Lattice Method***

c) ***Prototype Product Approach*** (after approximating the diffusion by a CTMC with the ***Kushner approximation***):

- **c.1)** ***Explicit Formulae***
- **c.2)** ***Monte Carlo simulations***

We are now going to describe in more detail each of the just mentioned alternatives:

5.1.1 Explicit Closed formula

Under the CIR affine term structure model, the price at time t of a zero-coupon bond with maturity T is given by

$$p(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{5.3}$$

with

$$\begin{cases} A(t,T) = \left(\frac{2he^{\frac{(k+h)(T-t)}{2}}}{2h+(k+h)(e^{h(T-t)}-1)}\right)^{\frac{2k\theta}{\sigma^2}} \\ B(t,T) = \frac{2e^{(e^{h(T-t)}-1)}}{2h+(k+h)(e^{h(T-t)}-1)} \\ h = \sqrt{k^2+2\sigma^2}. \end{cases} \tag{5.4}$$

For more details see Brigo-Mercurio [2].

5.1.2 Lattice Method

The lattice method is widely used in finance and it consists in building a recombining tree which approximates the evolution of a diffusion process (in this case the short rate as given by the CIR model).

Here we consider the lattice algorithm suggested in Costabile-Leccadito-Massabò [3]: they propose an approach based on a direct discretization of the process $r(t)$ by means of a recombining binomial tree with a number of nodes that grows linearly with the number of steps; then, by an argument based on absence of arbitrage, they compute the bond price by working backwards along the tree. To solve a frequent problem in lattice methods, namely that the transition probabilities have to belong in $[0,1]$, the authors introduce *multiple upward* and *downward jumps* that satisfy an appropriate matching condition.

5.1.3 Prototype Product Approach

If we consider a CIR affine term structure model as in (5.1) for the short rate, in order to apply the Prototype Product Approach (namely all the theory developed in Chapter 3) to the bond pricing, we have first to approximate the short rate, which is a diffusion process, by a CTMC: we propose the Kushner approximation as described e.g. in Di Masi-Runggaldier [4] which is appropriate for this purpose. The name derives from the fact that this approximation was developed by H. Kushner and coworkers (for a reference see [8]).

Kushner approximation: let us consider the square-root process as in (5.1) and denote by $r^{h,n}(t)$ the CTMC obtained by first discretizing with respect to the space variable (with spatial step length h) the infinitesimal generator of the diffusion $r(t)$, thus obtaining a denumerable CTMC $\{r^h(t)\}$, and then stopping $r^h(t)$ at the boundary of the interval $I = (0, N)$ (with $N \triangleq hn$ suitably chosen where n represents the number of subintervals in which I is divided).

We have that the state space of $r^{h,n}(t)$ is given by

$$E^{h,n} = \{r^0, \dots, r^N\} = \{0, h, \dots, h(n-1), hn\} \in \mathbb{R}^{N+1} \tag{5.5}$$

and the transition intensity kernel is represented by the matrix $Q^{h,n} = (q_{i,j}^{h,n})_{\{1 \le i,j \le N+1\}}$ with the first and last rows identically equal to zero (absorption at the boundary) and with the i-th row given by

$$[0, \dots, 0, q_-^{h,n}(r^i), q^{h,n}(r^i), q_+^{h,n}(r^i), 0, \dots, 0], \tag{5.6}$$

where $q^{h,n}(r^i)$ is in the diagonal and

$$\begin{cases} q_-^{h,n}(r^i) = \frac{(k(\theta - r^i))^-}{h} + \frac{\sigma^2 r^i}{2h^2} \\ q^{h,n}(r^i) = -\frac{|k(\theta - r^i)|}{h} - \frac{\sigma^2 r^i}{h^2} \\ q_+^{h,n}(r^i) = \frac{(k(\theta - r^i))^+}{h} - \frac{\sigma^2 r^i}{2h^2} \end{cases} \tag{5.7}$$

with $(\cdot)^+$ and $(\cdot)^-$ denoting the positive and negative parts respectively. Moreover, the intensity associated with a generic state $r^i \in E^{h,n}$ can be represented by

$$q_i^{h,n} = \sum_{\substack{j=1 \\ j \neq i}}^{N+1} q_{i,j}^{h,n} = -q^{h,n}(r^i). \tag{5.8}$$

The CTMC $r^{h,n}(t)$ converges to $r(t)$ as $n \to +\infty$ and $h \to 0$ in the sense of weak convergence of the induced probability measures.

Once discretized, the short rate becomes a CTMC and so we can compute the bond price with the Prototype Product Approach by using either the explicit formulae given in Chapter 3 or by using an approach based on the simulation of the trajectories of the process $r^{h,n}(t)$. Below we refer to the first alternative as ***"Explicit Formulae"*** and the second as ***"Monte Carlo simulations"***.

Remark 5.1 *If we have the spot rate as the CTMC $r^{h,n}(t)$ given by the Kushner approximation, the upper approximation price of the zero-coupon bond is a good approximation of its true price: we can apply Lemma 2.13 because the sufficient condition to have a negligible difference between the upper and lower approximations of the true price holds. In fact in this case condition (2.38) is equivalent to have*

$$\frac{r^i}{q^{h,n}(r^i)} = \frac{h(i-1)}{\frac{|k(\theta-h(i-1))|}{h} + \frac{\sigma^2}{h^2}} \qquad i = 1, \ldots, N+1. \tag{5.9}$$

By considering h at most of the order of 10^{-2}, $h(i-1)$ at most equal to 0.03 and suitable CIR parameters as in the following numerical results, the expression in (5.9) is close to zero.

Prototype Product Approach (Explicit Formulae)

According to (3.7), for an arbitrarily small ϵ, a computable price of a T-bond at time t (under the market model where $r^{h,n}(t)$ is supposed to be the short rate) is given by

$$p_\epsilon(t, T; r^{h,n}(t))|_{r^{h,n}(t)=r^i} = \sum_{k=0}^{n_\epsilon} [\widetilde{Q^{h,n}}^k \cdot \theta_0(\underline{r}^{h,n})]_i \widetilde{\mathbb{P}}(\nu_{t,T} = k | r^{h,n}_{\nu_t} = r^i) \tag{5.10}$$

where

- $r^i \in E^{h,n}$ is the value assumed by the spot rate at time t;
- the matrix $\widetilde{Q^{h,n}}$ takes a simpler form with respect to its general definition (2.36) because of the tridiagonal structure of the transition kernel $Q^{h,n}$: the first and the last rows are identically equal to zero and the j-th row is given by

  $$\left[0, \ldots, 0, \frac{q^{h,n}_-(r^j)}{r^j - q^{h,n}(r^j)}, 0, \frac{q^{h,n}_+(r^j)}{r^j - q^{h,n}(r^j)}, 0, \ldots, 0\right]$$

  where the $(j+1)$-th and the $(j-1)$-th terms are different from zero;
- $\theta_0(\underline{r}^{h,n}) \triangleq [1, \ldots, 1]' \in \mathbb{R}^{N+1}$;
- for a fixed $k \in \{0\} \cup \mathbb{N}_+$, the probability $\widetilde{\mathbb{P}}(\nu_{t,T} = k | r^{h,n}_{\nu_t} = r^i)$ is given by (2.44) when we set $q_{i,j} = q^{h,n}_{i,j}$ as the general element of the transition kernel and $q_i = q^{h,n}_i$ as the intensity associated with the state r^i for each $i, j \in \{1, \ldots, N+1\}$;
- n_ϵ is a natural number such that

  $$n_\epsilon \geq \left\lceil \frac{\log(\epsilon(1-\gamma))}{\log(\gamma)} - \frac{\sup_{v \in E^{h,n}} |\sum_{i=1}^{N+1} \left(\frac{q^{h,n}_i}{r^i + q^{h,n}_i} - 1\right) \mathbf{I}_{\{v=r^i\}}|}{\log(\gamma)} \right\rceil,$$

  with $\gamma \triangleq \sup_{1 \leq j \leq N+1} \frac{q^{h,n}_j}{r^j + q^{h,n}_j}$.

We are going now to see how the quantities involved in (5.10) can be computed in practice:

- the i-th element of the vector $\widetilde{Q^{h,n}}^k \cdot \theta_0(\underline{r}^{h,n})$ is easy to compute for each k: in fact the following recurrence relation holds

$$\begin{cases} a_0 = \theta_0(\underline{r}^{h,n}) \\ a_k = \widetilde{Q^{h,n}} \cdot a_{k-1} \quad k = 1, 2, \dots \end{cases}$$

  where $a_0, a_k \in \mathbb{R}^{N+1}$ and $a_k = \widetilde{Q^{h,n}}^k \cdot \theta_0(\underline{r}^{h,n})$;

- the probabilities $\widetilde{\mathbb{P}}(\nu_{t,T} = k | r^{h,n}_{\nu_t} = r^i)$ can be computed exactly only for not too large values of k: the computational complexity relative to these probabilities, for h which goes to zero and n which grows, increases because it requires the solution of the $k(N+1)$ multiple integrals $\Psi_k(t, T, Q^{h,n})$ in (2.45).
  For each choice of the pair (h, n), there exists a certain natural number $\overline{k}$ such that
  - up to $\overline{k}$ it is numerically feasible to compute exactly the probabilities $\widetilde{\mathbb{P}}(\nu_{t,T} = k | r^{h,n}_{\nu_t} = r^i)$ by the recursion as in Lemma 2.15,
  - beyond $\overline{k}$ the probabilities $\widetilde{\mathbb{P}}(\nu_{t,T} = k | r^{h,n}_{\nu_t} = r^i)$ are more conveniently computed as in (2.44) where we use a multidimensional Monte Carlo integration for the quantities $\Psi_k(t, T, Q^{h,n})$.

Prototype Product Approach (Monte Carlo simulations)

Instead of considering the relation (3.7) which gives a computable pricing formula for a zero-coupon bond, we can consider the full pricing formula as in (3.2), that is

$$p(t, T; r^{h,n}(t))|_{r^{h,n}(t)=r^i} = \sum_{k=0}^{+\infty} [\widetilde{Q^{h,n}}^k \cdot \theta_0(\underline{r}^{h,n})]_i \widetilde{\mathbb{P}}(\nu_{t,T} = k | r^{h,n}_{\nu_t} = r^i) \tag{5.11}$$

or equivalently

$$p(t, T; r^{h,n}(t))|_{r^{h,n}(t)=r^i} = \mathbb{E}^{\widetilde{\mathbb{P}}}[[\widetilde{Q^{h,n}}^{\nu_{t,T}} \cdot \theta_0(\underline{r}^{h,n})]_i | r^{h,n}_{\nu_t} = r^i]. \tag{5.12}$$

An approach to compute the price in the above expression can then also be obtained by using the *MonteCarlo* technique, that is based on

$$\frac{1}{M} \sum_{l=1}^{M} [\widetilde{Q^{h,n}}^{\nu^l_{t,T}} \cdot \theta_0(\underline{r}^{h,n})]_i \overset{M \to \infty}{\longrightarrow} p(t, T; r^{h,n}(t))|_{r^{h,n}(t)=r^i} \quad \widetilde{\mathbb{P}} - \text{a.s.} \tag{5.13}$$

where $\nu^l_{t,T}$ is the l-th simulation outcome of the random variable $\nu_{t,T}$. Hence, by recalling that the probability transition matrix of the imbedded Markov chain $r^{h,n}$ (see (2.1)) is given by

$$P^{h,n} = (p^{h,n}_{i,j})_{\{1 \le i,j \le N+1\}} \triangleq \begin{cases} \frac{q^{h,n}_{i,j}}{q^{h,n}_i} & \text{if } i \neq j \\ 0 & \text{if } i = j, \end{cases}$$

we are going to describe the algorithm used to simulate $\nu_{t,T}$ when $r^{h,n}(t) = r^m$.

Algorithm to simulate $\nu_{\mathbf{t},\mathbf{T}}$

- initialize $number\_of\_jumps = 0$
- generate the value $interarrival\_time$ by simulating a random variable with the distribution $\mathcal{E}xp(q_m^{h,n})$
- initialize $current\_index = m$ (index of the currently visited state)
- initialize $arrival\_time = t + interarrival\_time$
- if $(arrival\_time > T)$

  return $number\_of\_jumps$
- while $(arrival\_time < T)$
  - initialize the vector $[\pi_1, \ldots, \pi_{N+1}] = current\_index$-th row of the transition probability matrix $P^{h,n}$
  - generate the value r^j by simulating a discrete random variable taking values in the state space $E^{h,n}$ with distributions $[\pi_1, \ldots, \pi_{N+1}]$
  - generate the value $interarrival\_time$ by simulating a random variable with the distribution $\mathcal{E}xp(q_j^{h,n})$
  - set $current\_index = j$
  - set $arrival\_time = arrival\_time + interarrival\_time$
  - set $number\_of\_jumps = number\_of\_jumps + 1$
- return $number\_of\_jumps$.

We present now some tables where the bond prices, for several maturities and several values of the CIR parameters in (5.1), are obtained as follows:

CF: the exact closed formula;

RBT: the lattice method, namely the recombining binomial tree according to [3], where we have chosen a number of steps ***"stepsRBT"*** always equal to 500;

PPA(EF)+K-A: the Prototype Product Approach after discretizing the short rate with the Kushner approximation (**K-A**) and by using the explicit formulae (**EF**) discussed in our study;

PPA(MC)+K-A: the Prototype Product Approach after discretizing the short rate with the Kushner approximation (**K-A**) and by using a full simulation approach based on the Monte Carlo technique (**MC**). We have chosen the number of steps for the Monte Carlo simulations ***"stepsRBT"***, namely M in formula (5.13), always equal to 500.

As regards the other parameters, we have considered the date of today as $t = 0$ years, three different times of maturity T (namely 0.5, 2 and 5 years) and the parameters of the CIR model such that the condition (5.2) is verified. In Table 1 and Table 2 the numerical results relative to **CF**, **RBT** and **PPA(MC)+K-A** are presented when the values of the initial spot rate $\widetilde{r}$ and the mean-reversion constant θ in formula (5.1) are of the order of one hundredth; in Table 3 we present also some results relative to **PPA(EF)+K-A** when $\widetilde{r}$ and θ are of the order of one tenth.

Remark 5.2 *In Tables 1-2, the bond prices computed with **PPA(EF)+K-A** are not considered because of a theoretical reason due to the Kushner approximation. When we consider $\widetilde{r}$ and θ of the order of one hundredth, a more accurate **K-A** is required to well discretize the diffusion process: the number n of the steps in which the state space $E^{h,n}$ is divided (denoted by* **"n(K-A)"***) has then to be taken of the order of one thousand and the spatial step length h (denoted by* **"h(K-A)"***) of the order of 10^{-4} (10^{-5} in some cases). It follows that, in accordance with relation (5.7), the elements of the transition kernel $Q^{h,n}$ and the intensities associated with each state of $E^{h,n}$ are of the order of 10^9. Consequently, if we use the approach **PPA(EF)+K-A**, when we have to evaluate the probability $\widetilde{\mathbb{P}}(\nu_{t,T} = 0 | r^{h,n}_{\nu_t} = r^i)$ as in (2.43) and the probabilities $\widetilde{\mathbb{P}}(\nu_{t,T} = k | r^{h,n}_{\nu_t} = r^i)$ for $k \geq 1$ as in (2.40), we face an error of* "Overflow" *because the negative exponential functions in formulas (2.43) and (2.40) respectively cannot take numbers of the order of 10^9 as inputs. The above problem is avoided when we consider $\widetilde{r}$ and θ of the order of one tenth: in fact a less accurate **K-A** is then required to well approximate the square-root process by a CTMC.*

We have performed the numerical simulations on a single core Intel x86 Linux machine equipped with 2GB of RAM and we have implemented a C/C++ framework by using the well-known GNU Scientific Library to handle the data structure.

Table 1: bond prices with **CF**, **RBT** and **PPA(MC)+K-A** (***stepsMC*=*stepsRBT***=500) (for ***n(K-A)*** and ***h(K-A)*** see Remark 5.2)

T(years)	0.5	2	5	0.5	2	5
$\widetilde{r}$	0.01	0.01	0.01	0.02	0.02	0.02
θ	0.01	0.01	0.01	0.02	0.02	0.02
k	0.8	0.8	0.8	0.5	0.5	0.5
σ	0.1	0.1	0.1	0.05	0.05	0.05
n(K-A)	600	600	700	600	600	700
h(K-A)	0.00005	0.00005	0.00005	0.0001	0.0001	0.0001
CF	0.995014	0.980245	0.951463	0.990051	0.960822	0.905047
RBT	0.995042	0.980302	0.951556	0.99007	0.960898	0.905226
PPA(MC)+K-A	**0.995024**	**0.980276**	0.951621	0.990143	0.960734	0.905318

Table 2: bond prices with **CF**, **RBT** and **PPA(MC)+K-A** (***stepsMC***=***stepsRBT***=500)

T(years)	0.5	2	5	0.5	2	5
$\widetilde{r}$	0.03	0.03	0.03	0.02	0.02	0.02
θ	0.03	0.03	0.03	0.02	0.02	0.02
k	1.1	1.1	1.1	1.2	1.2	1.2
σ	0.1	0.1	0.1	0.1	0.1	0.1
n(K-A)	600	600	700	600	600	700
h(K-A)	0.00015	0.00015	0.00015	0.0001	0.0001	0.0001
CF	0.985116	0.941861	0.861095	0.990053	0.960849	0.905072
RBT	0.985146	0.941974	0.86135	0.990072	0.960926	0.905251
PPA(MC)+K-A	**0.985128**	**0.941968**	**0.861319**	**0.990059**	0.95647	0.90193

Table 3: bond prices with **CF**, **RBT**,**PPA(MC)+K-A** and **PPA(EF)+K-A** (***stepsMC***=***stepsRBT***=500)

T(years)	0.5	0.5	0.5	0.5
$\widetilde{r}$	0.1	0.1	0.2	0.3
θ	0.1	0.1	0.2	0.3
k	0.1	0.4	0.2	0.3
σ	0.1	0.05	0.2	0.3
n(K-A)	300	300	300	300
h(K-A)	0.01	0.01	0.02	0.03
CF	0.951249	0.951234	0.904977	0.86114
RBT	0.951343	0.951329	0.905157	0.861394
PPA(MC)+K-A	0.951022	0.950859	0.905229	0.861104
PPA(EF)+K-A	**0.951324**	0.951723	**0.905012**	0.861756

Remark 5.3 *Both results with* ***PPA(EF)+K-A*** *and* ***PPA(MC)+K-A*** *are competitive with the* ***RBT*** *method and they generally differ only at the fourth decimal digit. In fact, in spite of the Kushner approximation which is necessary for the comparison with the prices of the continuous affine term structure model given in (5.3)-(5.4), our methods based on the Prototype Product pricing work roughly as the lattice method which does not require previously any approximation to be applied and consequently does not feel the effect of the error due to* ***K-A****. Furthermore,* ***PPA(EF)+K-A*** *and* ***PPA(MC)+K-A*** *work sometimes better than the lattice methods (see results in bold). In any case our approach is designed for r given directly by a CTMC and the Kushner approximation was introduced only for comparison purposes.*

Moreover it is known that lattice methods work well under a one-factor short rate model, but it becomes more difficult to implement them if the short rate depends on several correlated processes. On the contrary, the Prototype Product Approach applies well also to a particular multi-factor short rate model (see Chapter 4) and in this case it can be easily implemented. In the following section we are going to show an example of a short rate model depending on two correlated factors and we will discuss the numerical results obtained for this case.

5.2 Numerical results under a two-factor short rate model

We present now a simple example of numerical result for the bond pricing under a multi-factor short rate model. This choice is motivated by the fact that, by considering the market model proposed in Section 4.1, there exist no closed formula nor other lattice approaches to be compared with the Prototype Product approach when the short rate is defined as in (4.1).

We consider a *"bull and bear"* market described by a random variable Z which takes two values $\{z_1, z_2\} = \{0, 1\}$ with probabilities $\{\pi_1, \pi_2\} = \{0.4, 0.6\}$; then we define the spot rate as a linear combination of two correlated processes:

$$r(t) = X(t; Z) + Y(t; Z)$$

where $X(t; Z)$ and $Y(t; Z)$ are CTMCs depending on Z in the following sense:

- the state space of $X(t; Z)$ is $E^X = \{x^1, x^2, x^3, x^4\} = \{0.005, 0.01, 0.015, 0.02\}$ and the transition kernel $Q^X(Z)$ is such that

$$Q^X(Z) = \begin{cases} \begin{bmatrix} -0.5 & 0.5 & 0 & 0 \\ 0.1 & -0.4 & 0.3 & 0 \\ 0 & 0.5 & -0.8 & 0.3 \\ 0 & 0 & 0.8 & -0.8 \end{bmatrix} & \text{if } Z = z_1 \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0.5 & -0.6 & 0.1 & 0 \\ 0 & 2 & -2.5 & 0.5 \\ 0 & 0 & 0.2 & -0.2 \end{bmatrix} & \text{if } Z = z_2 \end{cases}$$

- the state space of $Y(t; Z)$ is $E^Y = \{y^1, y^2, y^3, y^4\} = \{0.005, 0.01, 0.012, 0.016\}$ and the transition kernel $Q^Y(Z)$ is such that

$$Q^Y(Z) = \begin{cases} \begin{bmatrix} -0.7 & 0.7 & 0 & 0 \\ 0.7 & -0.8 & 0.1 & 0 \\ 0 & 0.2 & -0.5 & 0.3 \\ 0 & 0 & 0.3 & -0.3 \end{bmatrix} & \text{if } Z = z_1 \\ \begin{bmatrix} -0.4 & 0.4 & 0 & 0 \\ 0.8 & -1.3 & 0.5 & 0 \\ 0 & 3 & -4 & 1 \\ 0 & 0 & 0.1 & -0.1 \end{bmatrix} & \text{if } Z = z_2 \end{cases}$$

We present below the prices of zero-coupon bonds, at the date of evaluation $t = 0$ years and maturities 0.5, 2 and 5 years, obtained by applying the Prototype Product approach with both the Explicit Formulae and the Monte Carlo simulations. We use the same notations of Section 5.1.

Table 4 (***stepsMC***=100000)

T (years)	0.5	2	5
PPA (MC)	0.990106	0.961139	0.905623
PPA (EF)	0.990086	0.961048	0.905008

Remark 5.4 *In Table* 4 *we have chosen a number of* ***stepsMC*** *(*100000*) greater than the number of* ***stepsMC*** *considered in the previous subsections (that is* 500*). In Tables* 1-2-3 *we have presented the prices of bonds obtained with the Prototype Product Approach after the diffusion process had been discretized by* $r^{h,n}$ *with the* ***K-A*** *approach. By construction, the transition intensities associated with each state of* $E^{h,n}$ *(namely the quantities* $q_i^{h,n}$ *defined in (5.8)) turn out to be of the order of* 10^4*: since the interarrival times between two successive jumps of* $r^{h,n}$ *are exponentially distributed with parameter* $q_i^{h,n}$*, the average time at which a jump occurs is* $\frac{1}{q_i^{h,n}}$ *(namely the process* $r^{h,n}$ *jumps frequently). It follows that, by having a large number of jumps for each simulation* $\nu_{t,T}^i$ *of the random variable* $\nu_{t,T}$*, a number of Monte Carlo simulation* ***stepsMC***$=$ 500 *is sufficient to obtain convergence in formula (5.13). On the contrary, in Table* 4 *we consider CTMCs with transition intensities of the order of* 10^{-1}*: we need* 100000 ***stepsMC*** *to obtain an acceptable convergence for the bond price.*

5.3 Conclusions

We briefly sum up the results obtained in this chapter: if we consider a one-factor short rate model, the Prototype Product Approach, by using either the explicit formulae or the Monte Carlo simulations, is competitive with the lattice method which is widely used to compute the price of zero-coupon bonds. To allow for such a comparison we had to start from a continuous time diffusion model which required a preliminary discretization to obtain a CTMC for which our methods are designed. Moreover we are able to obtain numerical results for prices of caps, swaptions and bond options with the same complexity as required for the computation of bond prices (considered as a particular case of Prototype Product) because all the prices of these interest rate derivatives can be viewed as linear combinations of Prototype Product prices (see Sections 3.2-3.3-3.4).

Under the two-factor short rate model specified in Chapter 4, we are able to price zero-coupon bonds without additional computational complexity with respect to the one-factor case and the results obtained with the Explicit Formulae **PPA(EF)** or the Monte Carlo simulations **PPA(MC)** coincide up to the third decimal digit (see Table 4). Furthermore, also when the short rate depends on two correlated factors, we are able to compute prices of caps, swaptions and bond options as linear combinations of Prototype Product prices (see Section 4.3).

Appendix A

Particular cases

The purpose of this section is to consider two particular cases:

- the case when the number of elements of the state space E is $N = 2$;
- the case when the CTMC $r(t)$ reduces to a Poisson process, namely when instead of a transition intensity matrix Q we have just a jump intensity λ and the random variable $\nu_{t,T}$ -studied in Section 2.2.3- becomes a Poisson random variable distributed with parameter $\lambda(T-t)$.

In the first statement, namely in Corollary A.1 we present the expression of the distribution of $\nu_{t,T}$ when $r(\cdot)$ is a CTMC with two states and transition intensities λ and μ; in Corollary A.3, we show that this CTMC becomes a Poisson process with parameter λ when μ converges to λ.

Corollary A.1 *Let $r(t)$ be a CTMC with state space $E = \{1, 2\}$ and $Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$, then assuming w.l.o.g. that $\nu_t = h \in \mathbb{N}$, it follows that*

$$\begin{cases} \widetilde{\mathbb{P}}(\nu_{t,T} = k | r_h = 1) = \mu^{\frac{k}{2}} \lambda^{\frac{k}{2}} e^{-\lambda(T-t)} \Psi_{h,k}(t, T, \mu, \lambda) & , k \text{ even} \\ \widetilde{\mathbb{P}}(\nu_{t,T} = k | r_h = 1) = \mu^{\frac{k-1}{2}} \lambda^{\frac{k+1}{2}} e^{-\mu T + \lambda t} \Psi_{h,k}(t, T, \mu, \lambda) & , k \text{ odd} \end{cases}$$

$$\begin{cases} \widetilde{\mathbb{P}}(\nu_{t,T} = k | r_h = 2) = \lambda^{\frac{k}{2}} \mu^{\frac{k}{2}} e^{-\mu(T-t)} \Psi_{h,k}(t, T, \mu, \lambda) & , k \text{ even} \\ \widetilde{\mathbb{P}}(\nu_{t,T} = k | r_h = 2) = \lambda^{\frac{k-1}{2}} \mu^{\frac{k+1}{2}} e^{-\lambda T + \mu t} \Psi_{h,k}(t, T, \mu, \lambda) & , k \text{ odd} \end{cases}$$

where $\Psi_{h,k}$ denotes the following multiple integral

$$\Psi_{h,k}(t, T, \mu, \lambda) \triangleq \int_t^T \int_{t_{h+1}}^T \int_{t_{h+2}}^T \dots \int_{t_{h+k-1}}^T exp\Big(\sum_{i=h+1}^{h+k} (-1)^i (\lambda - \mu) t_i \Big) dt_{h+k} \dots dt_{h+3} dt_{h+2} dt_{h+1} \quad \text{(A.1)}$$

Proof At first we recall that the notations for a CTMC with two states $r^1 = 1$ and $r^2 = 2$ are the following

- the transition kernel elements are $q_{1,2} = \lambda$, $q_{2,1} = \mu$, $q_{1,1} \triangleq -\sum_{\substack{j=1 \\ j\neq 1}}^{2} q_{1,j} = -\lambda$ and $q_{2,2} \triangleq -\sum_{\substack{j=1 \\ j\neq 2}}^{2} q_{2,j} = -\mu$
- the intensities defined by $q_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} q_{i,j},\ i = 1, \ldots, N$ become $q_1 = -q_{1,1} = \lambda$ and $q_2 = -q_{2,2} = \mu$.

Hence we are going to use the results of Lemma 2.16 applied to a CTMC with two states: let us divide the proof into four cases ($r_{\nu_t} = r^1$ *and* k *even*, $r_{\nu_t} = r^2$ *and* k *even*, $r_{\nu_t} = r^1$ *and* k *odd*, $r_{\nu_t} = r^2$ *and* k *odd*).

1) *Case* $r_{\nu_t} = r^{i_h} = r^1$ *and* k *even*

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_h = r^1) = \sum_{\substack{i_{h+1},\ldots,i_{h+k}=1 \\ i_{h+1}\neq 1, i_{h+2}\neq i_{h+1},\ldots,i_{h+k}\neq i_{h+k-1}}}^{2} e^{q_1 t - q_{i_{h+k}} T} \varphi_{h,k}(Q) \cdot \Psi_{h,k}(t,T,Q)$$

Hence, by observing that every index can take only one specific value, (i.e $i_{h+1} = 2, i_{h+2} = 1, i_{h+3} = 2, \ldots, i_{h+k} = 1$ because $i_{h+1} \neq 1, i_{h+2} \neq i_{h+1}, \ldots, i_{h+k} \neq i_{h+k-1}$ and $i_{h+1}, \ldots, i_{h+k} \in \{1, 2\}$), the sum has only one term,

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_h = r^1) = e^{-q_1(T-t)} \varphi_{h,k}(Q) \cdot \Psi_{h,k}(t,T,Q) \tag{A.2}$$

where $\Psi_{h,k}(t,T,Q)$ and $\varphi_{h,k}(Q)$ are the following quantities

$$\begin{aligned}
\Psi_{h,k}(t,T,Q) &= \int_t^T e^{(q_2-q_1)t_{h+1}} \int_{t_{h+1}}^T e^{(q_1-q_2)t_{h+2}} \ldots \int_{t_{h+k-1}}^T e^{(q_1-q_2)t_{h+k}} dt_{h+k} \ldots dt_{h+2} dt_{h+1} \\
&= \int_t^T \int_{t_{h+1}}^T \int_{t_{h+2}}^T \ldots \int_{t_{h+k-1}}^T exp\Big(\sum_{i=h+1}^{h+k} (-1)^i (q_1 - q_2) t_i \Big) dt_{h+k} \ldots dt_{h+3} dt_{h+2} dt_{h+1},
\end{aligned}$$

$$\begin{aligned}
\varphi_{h,k}(Q) &= q_{1,i_{h+1}} q_{i_{h+1},i_{h+2}} \cdot \ldots \cdot q_{i_{h+k-1},i_{h+k}} \\
&= q_{1,2} \cdot q_{2,1} \cdot \ldots \cdot q_{2,1} = q_{1,2}^{\frac{k}{2}} q_{2,1}^{\frac{k}{2}};
\end{aligned}$$

so we have

$$\begin{aligned}
(A.2) &= q_{2,1}^{\frac{k}{2}} q_{1,2}^{\frac{k}{2}} e^{-q_1(T-t)} \\
&\quad \int_t^T \int_{t_{h+1}}^T \int_{t_{h+2}}^T \ldots \int_{t_{h+k-1}}^T exp\Big(\sum_{i=h+1}^{h+k} (-1)^i (q_1 - q_2) t_i \Big) dt_{h+k} \ldots dt_{h+3} dt_{h+2} dt_{h+1} \\
&= \mu^{\frac{k}{2}} \lambda^{\frac{k}{2}} e^{-\lambda(T-t)} \Psi_{h,k}(t,T,\mu,\lambda)
\end{aligned}$$

where $\Psi_{h,k}(t,T,\mu,\lambda)$ is as defined in (A.1).

2) *Case* $r_{\nu_t} = r_{i_h} = r^2$ *and* k *even*

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_h = r^2) \quad = \sum_{\substack{i_{h+1},\dots,i_{h+k}=1 \\ i_{h+1}\neq 2, i_{h+2}\neq i_{h+1},\dots,i_{h+k}\neq i_{h+k-1}}}^{2} e^{q_2 t - q_{i_{h+k}} T} \varphi_{h,k}(Q) \cdot \Psi_{h,k}(t,T,Q)$$

Hence, by observing again that every index takes only one specific value (i.e. $i_{h+1} = 1$, $i_{h+2} = 2, i_{h+3} = 1, \dots, i_{h+k} = 2$), the sum has only one term,

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_h = r^2) = e^{-q_2(T-t)} \varphi_{h,k}(Q) \cdot \Psi_{h,k}(t,T,Q) \tag{A.3}$$

with

$$\begin{aligned}\Psi_{h,k}(t,T,Q) &= \int_t^T e^{(q_1-q_2)t_{h+1}} \int_{t_{h+1}}^T e^{(q_2-q_1)t_{h+2}} \dots \int_{t_{h+k-1}}^T e^{(q_2-q_1)t_{h+k}} dt_{h+k} \dots dt_{h+2} dt_{h+1} \\ &= \int_t^T \int_{t_{h+1}}^T \int_{t_{h+2}}^T \dots \int_{t_{h+k-1}}^T exp\Big(\sum_{i=h+1}^{h+k} (-1)^i (q_2 - q_1) t_i \Big) dt_{h+k} \dots dt_{h+3} dt_{h+2} dt_{h+1},\end{aligned}$$

$$\begin{aligned}\varphi_{h,k}(Q) &= q_{2,i_{h+1}} q_{i_{h+1},i_{h+2}} \cdot \dots \cdot q_{i_{h+k-1},i_{h+k}} \\ &= q_{2,1} \cdot q_{1,2} \cdot \dots \cdot q_{1,2} = q_{2,1}^{\frac{k}{2}} q_{1,2}^{\frac{k}{2}};\end{aligned}$$

so we have

$$\begin{aligned}(A.3) \quad &= \quad q_{1,2}^{\frac{k}{2}} q_{2,1}^{\frac{k}{2}} e^{-q_2(T-t)} \\ &\quad \int_t^T \int_{t_{h+1}}^T \int_{t_{h+2}}^T \dots \int_{t_{h+k-1}}^T exp\Big(\sum_{i=h+1}^{h+k} (-1)^i (q_2 - q_1) t_i \Big) dt_{h+k} \dots dt_{h+3} dt_{h+2} dt_{h+1} \\ &= \quad \lambda^{\frac{k}{2}} \mu^{\frac{k}{2}} e^{-\mu(T-t)} \Psi_{h,k}(t,T,\lambda,\mu)\end{aligned}$$

where $\Psi_{h,k}(t,T,\lambda,\mu)$ is again defined as in (A.1).

3) *Case* $r_{\nu_t} = r_{i_h} = r^1$ *and* k *odd*

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_h = r^1) \quad = \sum_{\substack{i_{h+1},\dots,i_{h+k}=1 \\ i_{h+1}\neq 1, i_{h+2}\neq i_{h+1},\dots,i_{h+k}\neq i_{h+k-1}}}^{2} e^{q_1 t - q_{i_{h+k}} T} \varphi_{h,k}(Q) \cdot \Psi_{h,k}(t,T,Q)$$

where now $i_{h+1} = 2, i_{h+2} = 1, i_{h+3} = 2, \dots, i_{h+k} = 2$; therefore

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_h = r^1) = e^{-q_2 T + q_1 t} \varphi_{h,k}(Q) \cdot \Psi_{h,k}(t,T,Q) \tag{A.4}$$

with $\Psi_{h,k}(t,T,Q)$ the same as in the first case and $\varphi_{h,k}(Q)$ is defined as follows

$$\begin{aligned}\varphi_{h,k}(Q) &= q_{1,i_{h+1}} \cdot q_{i_{h+1},i_{h+2}} \cdot \dots \cdot q_{i_{h+k-1},i_{h+k}} \\ &= q_{1,2} \cdot q_{2,1} \cdot \dots \cdot q_{1,2} = q_{1,2}^{\frac{k+1}{2}} q_{2,1}^{\frac{k-1}{2}};\end{aligned}$$

so we have

$$\begin{aligned}(A.4) &= q_{2,1}^{\frac{k+1}{2}} q_{1,2}^{\frac{k-1}{2}} e^{-q_2 T + q_1 t} \\ &\int_t^T \int_{t_{h+1}}^T \int_{t_{h+2}}^T \dots \int_{t_{h+k-1}}^T exp\Big(\sum_{i=h+1}^{h+k} (-1)^i (q_1 - q_2) t_i \Big) dt_{h+k} \dots dt_{h+3} dt_{h+2} dt_{h+1} \\ &= \mu^{\frac{k-1}{2}} \lambda^{\frac{k+1}{2}} e^{-\mu T + \lambda t} \Psi_{h,k}(t, T, \mu, \lambda)\end{aligned}$$

where $\Psi_{h,k}(t, T, \mu, \lambda)$ is as defined in (A.1).

4) *Case $r_{\nu_t} = r_{i_h} = r^2$ and k odd*

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_h = r^2) = \sum_{\substack{i_{h+1}, \dots, i_{h+k} = 1 \\ i_{h+1} \neq 2, i_{h+2} \neq i_{h+1}, \dots, i_{h+k} \neq i_{h+k-1}}}^{2} e^{q_2 t - q_{i_{h+k}} T} \varphi_{h,k}(Q) \cdot \Psi_{h,k}(t, T, Q)$$

where now $i_{h+1} = 1, i_{h+2} = 2, i_{h+3} = 1, \dots, i_{h+k} = 1$; therefore

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_h = r^2) = e^{-q_1 T + q_2 t} \varphi_{h,k}(Q) \cdot \Psi_{h,k}(t, T, Q) \tag{A.5}$$

with $\Psi_{h,k}(t, T, Q)$ the same as in the second case and $\varphi_{h,k}(Q)$ defined as follows

$$\begin{aligned}\varphi_{h,k}(Q) &= q_{2,i_{h+1}} \cdot q_{i_{h+1}, i_{h+2}} \cdot \dots \cdot q_{i_{h+k-1}, i_{h+k}} \\ &= q_{2,1} \cdot q_{1,2} \cdot \dots \cdot q_{2,1} = q_{2,1}^{\frac{k+1}{2}} q_{1,2}^{\frac{k-1}{2}};\end{aligned}$$

so we have

$$\begin{aligned}(A.5) &= q_{1,2}^{\frac{k-1}{2}} q_{2,1}^{\frac{k+1}{2}} e^{-q_1 T + q_2 t)} \\ &\int_t^T \int_{t_{h+1}}^T \int_{t_{h+2}}^T \dots \int_{t_{h+k-1}}^T exp\Big(\sum_{i=h+1}^{h+k} (-1)^i (q_2 - q_1) t_i \Big) dt_{h+k} \dots dt_{h+3} dt_{h+2} dt_{h+1} \\ &= \lambda^{\frac{k-1}{2}} \mu^{\frac{k+1}{2}} e^{-\lambda T + \mu t} \Psi_{h,k}(t, T, \lambda, \mu)\end{aligned}$$

where $\Psi_{h,k}(t, T, \lambda, \mu)$ is again defined as in (A.1). □

Remark A.2 *A CTMC with a state space $E = \{1, 2\}$ and relative intensities λ and μ, for μ which tends to λ, becomes a Poisson process with parameter λ.*

In this case we have the following result that allows us to check the correctness of the formulas in the previous Corollary by showing that this result coincides with the known result for the Poisson case:

Corollary A.3 *For a Poisson process with parameter λ, the probabilities of Lemma 2.16 have the following representation*

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = 1) = \widetilde{\mathbb{P}}(\nu_{t,T}) = k | r_{\nu_t} = 2) = \frac{[\lambda(T-t)]^k}{k!} e^{-\lambda(T-t)}.$$

Proof By Remark 2.17 the multiple integral $\Psi_{h,k}$ does not really depend on h: $\Psi_{h,k} \equiv \Psi_k$ where Ψ_k is defined as in (2.45). Analogously, in the case when $r(\cdot)$ is a CTMC with only two states, the quantity $\Psi_{h,k}$ defined as in (A.1) does not depend on h: if we denote $\Psi_{h,k}|_{h=0} = \Psi_k$, we show that

$$\Psi_k(t,T,\mu,\lambda) \overset{\mu\mapsto\lambda}{\mapsto} \frac{(T-t)^k}{k!}, \tag{A.6}$$

then the Corollary is proved.
By the definition of Ψ and the *dominated convergence theorem* (let $\lambda > \mu$ w.l.o.g., then there always exists $C \in \mathbb{R}_+$ such that $\left|e^{(\lambda-\mu)\sum_{i=h+1}^{h+k}(-1)^i t_i}\right| \leq e^{CT(k-1)}$ which is a constant), we observe that

$$\Psi_k(t,T,\mu,\lambda) \overset{\mu\mapsto\lambda}{\mapsto} V_k(t,T) \triangleq \int_t^T \int_{t_1}^T \int_{t_2}^T \dots \int_{t_{k-1}}^T \mathrm{dt_k} \dots \mathrm{dt_3 dt_2 dt_1},$$

so we show that

$$V_k(t,T) = \frac{(T-t)^k}{k!} \tag{A.7}$$

by an induction method.

Proof by induction of (A.7)

Base Case $k = 1$

$$V_1(t,T) = \int_t^T \mathrm{dt_1} = T - t$$

Inductive step

$$\begin{aligned} V_k(t,T) &\triangleq \int_t^T \int_{t_1}^T \int_{t_2}^T \dots \int_{t_{k-1}}^T \mathrm{dt_k} \dots \mathrm{dt_3 dt_2 dt_1} \\ &= \int_t^T V_{k-1}(t_1,T)\mathrm{dt_1} \\ &= \int_t^T \frac{(T-t_1)^{(k-1)}}{(k-1)!}\mathrm{dt_1} = \frac{(T-t)^k}{k!} \end{aligned}$$

by using the induction hypothesis in the third passage.

□

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