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RESULTS ON CONTROLLABILITY AND NUMERICAL APPROXIMATION OF THE MINIMUM TIME FUNCTION

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To my parents

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Abstract

This thesis focuses on the *unconstrained* and *constrained* minimum time problems, in particular on regularity, numerical approximation, feedback and synthesis aspects.

We first consider the problem of *small-time local controllability* for nonlinear finite-dimensional time-continuous control systems in presence of state constraints. More precisely, given a nonlinear control system subject to state constraints and a closed set S, we provide sufficient conditions to steer to S every point of a suitable neighborhood of S along admissible trajectories of the system, respecting the constraints, and giving also an upper estimate of the minimum time needed for each point to reach the target.

Then in framework of control affine nonlinear systems, sufficient conditions to reach a target for a suitable discretization of a given dynamics are provided. We make use of an approach based on Hamilton-Jacobi theory to prove the convergence of the solution of a fully discrete scheme to the (true) minimum time function, together with error estimates. We also design an approximate suboptimal discrete feedback and provide an error estimate for the time to reach the target through the discrete dynamics generated by this feedback.

We next propose a new formulation of the minimum time problem in which we employ the *signed* minimum time function positive outside of the target, negative in its interior and zero on its boundary. Under some standard assumptions, we prove the so called *Bridge Dynamic Programming Principle* (BDPP) which is a relation between the value functions defined on the complement of the target and in its interior. Then owing to BDPP, we obtain the error estimates of a semi-Lagrangian discretization of the resulting Hamilton-Jacobi-Bellman equation.

The remainder of this thesis is devoted to introducing an approach to compute the approximate minimum time function of control problems which is based on reachable set approximation. In particular, the theoretical justification of the proposed approach is restricted to a class of linear control systems and uses arithmetic operations for convex sets. The error estimate of the fully discrete reachable set is provided by employing Hausdorff distance. The detailed procedure solving the corresponding discrete problem is described. Under standard assumptions, by means of convex analysis and knowledge of regularity of the true minimum time function, we estimate the error of its approximation. Finally, we reconstruct discrete suboptimal trajectories which reach a set of supporting points from a given target for a class of linear control problems and also proving the convergence of discrete optimal controls by the use of nonsmooth and variational analysis.

Riassunto

La tesi è dedicata a problemi di tempo minimo finito dimensionali, sia con vincoli di stato che senza, con particolare riguardo alla regolarità all'approssimazione numerica e ad aspetti collegati di sintesi.

Si considera in primo luogo il problema della *controllabilità locale per tempi piccoli* con vincoli di stato: si forniscono condizioni sufficienti per portare ad un bersaglio in tempo finito una traiettoria del sistema dato, senza violare i vincoli, e si dà una stima del tempo necessario.

Nell'ambito di problemi affini rispetto al controllo, si danno condizioni sufficienti per la controllabilità a rispetto ad una particolare discretizzazione della dinamica. Tale risultato è motivato da un approccio all'approssimazione del tempo minimo T basato sulla sua caratterizzazione mediante un'equazione di Hamilton-Jacobi. Il contributo di questa parte della tesi consiste in un risultato teorico che estende la teoria esistente al caso in cui T non sia Lipschitz (cioè sotto ipotesi deboli di controllabilità) e nella costruzione di un *feedback* approssimato con la relativa stima dell'errore.

Si propone inoltre una nuova formulazione del problema del tempo minimo, nella quale si fa uso di un tempo *negativo* quando la traiettoria è penetrata all'interno del bersaglio, allo scopo di ridurre l'errore di approssimazione vicino alla frontiera. Si dimostra una nuova versione del principio della programmazione dinamica (il "Principio Ponte"), che stabilisce una relazione tra il tempo minimo all'interno e all'esterno del bersaglio. Si studia poi una discretizzazione della corrispondente equazione di Hamilton-Jacobi e si forniscono stime dell'errore.

La parte finale della tesi è dedicata all'introduzione di un nuovo approccio per il calcolo approssimato di T basato sull'approssimazione degli insiemi raggiungibili mediante l'aritmetica degli insiemi convessi, valido per sistemi lineari. Si fornisce una stima dell'errore mediante la distanza di Hausdorff per gli insiemi raggiungibili e per il tempo minimo. Si costruiscono inoltre traiettorie subottimali discrete e si prova la convergenza dei corrispondenti controlli al controllo ottimo.

Contents

Basic notations

	$\mathcal{C}(\mathbb{R}^n)$	the set of convex, compact, nonempty subsets of $\mathbb{R}^n$;
$\|x\|_1$	$= \sum_1^n \|x_i\|$	the l^1-norm of $x \in \mathbb{R}^n$;
$\langle x, y\rangle$	$:= \sum_{i=1}^n x_i y_i$	the *scalar product* in $\mathbb{R}^n$;
$\|x\|$	$:= \sqrt{\langle x, x\rangle}$	the *Euclidean norm* in $\mathbb{R}^n$;
$\|M\|$	$:= \sup_{\substack{x\in\mathbb{R}^n \\ x\neq 0}} \frac{\|Mx\|}{\|x\|}$	the *lub-norm* of M with respect to $\|\cdot\|$;
	∂S, $\operatorname{int} S$, $\overline{S}$	the *topological boundary*, *interior* and *closure* of S;
$\operatorname{diam}(S)$	$:= \sup\{\|z_1 - z_2\| : z_1, z_2 \in S\}$	the *diameter* of S;
S_{n-1}	$:= \{w \in \mathbb{R}^n : \|w\| = 1\}$	the *unit sphere* (centered at the origin);
$B(y, r)$	$= B_r(y) := \{z \in \mathbb{R}^n : \|z - y\| < r\}$	the *open ball* centered at y of radius r;
$\mathrm{d}_K(y)$	$= \mathrm{d}(y, K) := \min\{\|z - y\| : z \in K\}$	the *distance* of y from K;
	$\mathrm{d}_H(S, K)$	the *Hausdorff distance* between S and K;
$\pi_K(y)$	$:= \{z \in K : \|z - y\| = d_K(y)\}$	the *set of projections* of y onto K;
$\operatorname{co}(S)$	$:= \bigcap_{\substack{C \text{ convex} \\ C\supseteq S}} C$	the *convex hull* of S;
S^c	$:= \mathbb{R}^n \setminus S$	the *complement* of S;
S_δ	$:= B(S, \delta) := \{y \in \mathbb{R}^n : d_S(y) \leq \delta\}$	the δ*-neighborhood* of S;
$S_{-\delta}$	$:= \{x \in \mathbb{R}^n : d_{\overline{S^c}}(x) \geq \delta\}$	the δ*-shrinking* of S;
	$N_K^P(x)$	the *proximal normal cone to K at x*;
$\operatorname{dom} f$	$:= \{x \in X : f(x) < +\infty\}$	the *domain* of f;
$\operatorname{epi} f$	$:= \{(x, \beta) \in X \times \mathbb{R} : x \in \operatorname{dom} f,\ \beta \geq f(x)\}$	the *epigraph* of f;
$\operatorname{hypo} f$	$:= \{(x, \alpha) \in X \times \mathbb{R} : x \in \operatorname{dom} f,\ \alpha \leq f(x)\}$	the *hypograph* of f;
	$\partial_P f(x)$, $\partial^P f(x)$	the *proximal subdifferential,* the *proximal superdifferential* of f at x (see Definition 2.2.4);

	$C^{k,\alpha}_{\mathrm{loc}}(U)$	see Definition 2.2.2;
	$\mathrm{Id}_{\mathbb{R}^n}$	the identity function in $\mathbb{R}^n$;
	$[\cdot,\cdot]$	Lie bracket;
$\chi_S(y)$	$:= \begin{cases} 1, & \text{if } y \in S, \\ 0, & \text{otherwise.} \end{cases}$	the *characteristic function of* S;
$I_A(x)$	$= \begin{cases} 0 \text{ if } x \in A \\ +\infty \text{ otherwise} \end{cases}$	the *indicator function* of A;
$]a,b[$	$= (a,b) = \{x\colon a < x < b\}$	an open interval in $\mathbb{R}$;
$[a,b[$	$= [a,b) = \{x\colon a \leq x < b\}$	an haft-closed interval in $\mathbb{R}$.

Chapter 1

Introduction

Consider the control system

$$\dot{y}(t) = f(y(t), u(t)) \tag{1.1}$$

with $u(t) \in U$ for a.e. t, $U \subset \mathbb{R}^m$ a compact set, together with the initial condition

$$y(0) = \xi. \tag{1.2}$$

Under standard assumptions, for any $u(\cdot)$ measurable and any ξ, the solution $y(\cdot, \xi, u)$ of (1.1) and (1.2) is unique and globally defined. By associating (1.1), (1.2) with different cost functions, one can formulate a variety of optimal control problems, e.g., infinite horizon problem, Bolza problem, minimum time problem, etc.

Let $S \subset \mathbb{R}^n$ be a nonempty closed set, the target and $x \in \mathbb{R}^n$. The *unconstrained* minimum time problem consists in finding a measurable control $u \in U$ to reach S by following the dynamics (1.1), (1.2) in the smallest amount of time, $T_S(\xi)$. Here $T_S(\xi)$ is called the *unconstrained* minimum time function. Furthermore, if $y(\cdot, \xi, u)$ is restricted to stay in $\overline{\Omega}$, where $\Omega \in \mathbb{R}^n$ is a given open subset, we obtain the so-called *constrained* minimum time problem. Then the minimum time function complying with the restriction of state space is referred to as the *constrained* minimum time function, $T_{S,\Omega}(\xi)$. It is obvious if Ω is the whole space $\mathbb{R}^n$, the problem turns back to the unconstrained case. From now on, for the sake of shortness, we will omit the adjective *unconstrained* when applicable.

This thesis is focused on the *unconstrained* and *constrained* minimum time problems, which are studied by several authors in different aspects such as: regularity, numerical approximation, feedback and synthesis, sensitivity analysis, necessary conditions... This thesis contains contributions on the three first topics. The scope of this work is devoted to the following main issues:

1) *small-time controllability* on S (STCS), i.e. studying sufficient conditions on f, U, S and Ω ensuring that all the points sufficiently near to S can be steered to S by *admissible* trajectories of the system in finite time T_S, ($T_{S,\Omega}$ for the constrained case), and establishing suitable continuity properties of T_S.

2) providing efficient approaches to compute $T_S(x)$ approximately in different contexts and designing discrete, both open and closed loop, suboptimal controls together with trajectories.

1.1 Controllability

In the absence of state constraints, one of the most common assumptions ensuring controllability is Petrov's condition, which can be stated as follows in the case of compact target S: there exist $\delta, \mu > 0$, such that for every $x \in \mathbb{R}^n \setminus S$ whose distance $d_S(x)$ from S is less than δ there exist $u \in U$ and a point $\bar{x} \in S$ with $\|x - \bar{x}\| = d_S(x)$ and

$$\langle x - \bar{x}, f(x, u)\rangle \leq -\mu d_S(x).$$

When the function $d_S(\cdot)$ is differentiable at $x \notin S$, the above condition can be written as $\langle \nabla d_S(x), f(x, u)\rangle \leq -\mu$. In a smooth setting, we can interpret Petrov condition in the following way: for each point sufficiently near to x there exists an admissible trajectory which points *sufficiently well towards* to the target. Petrov condition later referred to as *the first order one* is very strong, and it is well known that it is equivalent to the local Lipschitz continuity of the minimum time function $T_S(\cdot)$ in a neighborhood of the target (see, e.g., [16, Section 4.1], [23, Section 8.2]). Moreover, it is equivalent for any given x near to the target to the existence of an admissible trajectory starting from x and reaching the target in time less than $Cd_S(x)$ for a suitable $C > 0$ depending on μ but not on x.

Chapter 3: Continuous small-time controllability. Roughly speaking, Petrov condition guarantees the existence of an admissible trajectory along which the distance decreases in a neighborhood of S. An idea generalizing Petrov condition could be the following one: given x as above we look for a curve $t \mapsto y_x(t)$ such that $y_x(t)$ can be reached from x in time t and such that

$$d_S(y_x(t)) - d_S(x) \leq -r \text{ with } r > 0, \tag{1.3}$$

without paying any attention to the behavior of the distance along the admissible trajectory joining x and $y_x(t)$. Such the map $t \mapsto y_x(t)$ is called an $\mathscr{A}$-trajectory. The first step in this direction was taken by Krastanov-Quincampoix in [53]. The later papers, see, e.g., [54, 61, 63], treat the case under various degrees of requirements on the smoothness of the terms appearing in the expression of an $\mathscr{A}$-trajectory and the regularity of the target set S.

One of difficulties in this approach is to describe the set of $\mathscr{A}$-trajectories on which the conditions for STCS must be checked. Fortunately, if we confine to *control-affine systems*, i.e., the dynamics is given by $\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{N} u_i(t) f_i(x(t))$, $\|u_i\|_{L^\infty} \leq 1$, additional information can be obtained by studying the Lie algebra generated by the vector fields $\{f_0 \pm f_i\}_{i=1,\dots,N}$. Observe that the presence of a nontrivial drift term f_0 breaks the time-reversal symmetry (which would hold if $f_0 = 0$), which is an essential ingredient for many results obtained by means of Lie algebraic methods in the 60's and in the 70's by Kalman, Hermes, Sussmann, Hörmander and many other authors. We refer to [53] for a brief history of these results.

Most of the results in this direction, see, e.g, [53,54,61,63], are obtained for systems in which the state space is the whole of $\mathbb{R}^n$. The first paper investigating the problem of STCS in the presence of restriction on the state space was [22]. The authors extended first-order Petrov's condition to such kind of systems by assuming on the boundary of the constraint an *inward pointing condition*. In its simplest form (i.e., the system is autonomous and the constraint is given by $\overline{\Omega}$, where Ω is an open bounded subset of $\mathbb{R}^n$), this condition amounts to ask that at every point of $\partial\Omega$ there are admissible velocities belonging to the interior of the tangent cone to Ω (see Remark 3.3 in [22]). It turns out that even in its full general version this condition implies that $\partial\Omega$ is locally Lipschitz continuous (see Remark 3.2 in [22]).

Our first aim is to weaken the smoothness assumptions on the terms appearing in the expression of the $\mathscr{A}$-trajectory and also the regularity hypothesis on S. Moreover, the presence of the additional state constraint $y_x(t) \in \overline{\Omega}$, where Ω is an open subset of $\mathbb{R}^n$ with $\bar{\Omega} \setminus S \neq \emptyset$, is taken into account. The first result, contained in Theorem 3.1.6, relies on the assumption of the existence of certain $\mathscr{A}$-trajectories fulfilling suitable properties and respecting the state constraints. The presence of state constraints would affect only the existence of such curves, which is assumed to be granted. So, in particular, state constraints do not play explicitly any role in the proof of this first result. Furthermore, we also show that several main results from [53,54,61,63] are covered by Theorem 3.1.6.

After proving such general STCS results, we turn our attention to control-affine systems in presence of state constraints. By providing an approximate representation formula for the elements of the set of $\mathscr{A}$-trajectory, we are able to prove a better sufficient condition for STCS in this case. The presence of state constraints is taken into account by means of a condition different from the inward pointing condition of [22]. Here we assume some smoothness of the distance function near to the boundary of the constraint, which in particular allows us to treat systems with a class of constraints whose boundaries are not necessarily Lipschitz continuous. The main ingredient of this part is a suitable approximate representation formula for the expansion of the distance function along the trajectories of control-affine systems with nontrivial drift. The comparison between the inward pointing condition of [22] and ours is postponed to the end of Section 3.3.

Chapter 4: Regularity of the minimum time function under weak controllability assumptions. Besides Petrov condition discussed above, finer controllability conditions are well known in the literature, in particular when S is an equilibrium point of the dynamics and $f(x,u) = f_0(x) + \sum_{i=1}^{M} f_i(x)u_i$, $M \in \mathbb{N} \setminus \{0\}$. They are called *higher order* conditions, in the sense that for every x close enough to S there exists a Lie bracket of the vector fields which points towards S. If this Lie bracket can be approximated by admissible trajectories of the controlled dynamics, by switching between suitable controls $\pm u$, then one can prove that it is possible to reach S in finite time from a neighborhood, and T_S is Hölder continuous with a suitable exponent depending on the maximal order of the Lie brackets. This is the case, for example, if S is the origin and the dynamics is linear and satisfies the classical Kalman rank condition. More in general, in [61,63], see also references therein, higher order controllability conditions were established for a rather general target and some classes of nonlinear, control affine, dynamics, and several examples were presented. In this chapter we will recall some sufficient conditions appearing in [63] such that the reachable set $\mathcal{R}^S$ be a neighborhood of S and T_S is Hölder continuous in $\mathcal{R}^S$. Moreover, we will also provide sufficient conditions guaranteeing *robustness of the controllability condition* with respect to a suitable shrinking $S_{-\sigma}$ of S. This result will be used for the discrete controllability and approximate feedback control construction in Chapter 5.

1.2 Approximation of the minimum time function

To our knowledge, there are two main approaches to compute the minimum time function T_S numerically. The first one is based on direct methods and the approximate solutions constructed in this way in general depend strongly on an initial guess. Therefore such methods in most of cases are able to provide only local results, see, e.g., [30] for more details. The second one employs tools from PDE solvers which in contrary can give global results. This thesis is focused on the latter approach.

Chapter 5: A Hamilton-Jacobi-Bellman approach under weak controllability assumptions. The work [15] opened the door to the approximation of the minimum time function T_S through numerical schemes for a suitable boundary value problem of Hamilton-Jacobi type. The first paper on this subject was [17], where a semidiscrete scheme was developed under the assumption of Lipschitz continuity of T_S in a neighborhood of S, or equivalently the Petrov controllability condition.

However, as discussed in Section 1.1, $T_S(x)$ is merely Hölder continuous in many situations under *some higher order controllability condition.* Therefore, it is natural generalizing to the case of higher order controllability numerical methods which were established under the first order condition [17]. The idea is first considering a one step discretization method for the dynamics, namely a discrete controlled dynamical system which approximates the given continuous time system. If one can reach the target S, subject to this approximate discrete dynamics, within a time which is bounded by a fractional power of the distance to S of the initial point, then the approximate time converges to the true one as the time discretization step tends to zero. What remains to do, then, is transferring to a suitable discrete approximate dynamics the controllability which holds for the continuous time system. In the k-th ($k \geq 1$) order case, at each step the gain in the distance to the target is a k-th power of the time length. Thus, the order of the numerical scheme must be at least $k+1$, in order not to destroy this gain. This is exactly what is done here: we prove that some controllability conditions on the original dynamics are also sufficient for a suitable one step discretization to reach the target and prove the desired estimate on the time (see Sections 5.1, 5.2, and 5.3). That given, a fully discrete approximation together with error estimates follows from well established arguments (see Sections 5.4 and 5.5). The remainder part of Chapter 5 is devoted to the design of an approximate feedback. It is well known that the steepest descent feedback (i.e., the feedback $u(x)$ suggested by the dynamic programming equation, see, e.g., [25]) is - in general - discontinuous, and so the O.D.E. $\dot{x} = f(x, u(x))$ may not admit solutions. Moreover, it is well known that generalized solutions (of Krasovskiĭ or Filippov type) are not always satisfactory, as they even may not reach the target (see, e.g., [67]). Following a well established method (see, e.g., [34, 37, 38]), the idea is substituting the continuous time dynamical system with a discrete one: this way, the problem of existence of solutions is bypassed. The approximate feedback is obtained, as one can expect, by choosing a control which minimizes a discretized Hamiltonian. Of course the point is proving that this strategy is suboptimal. To this aim, in order to be sure to reach the desired target S, one needs to consider the problem of reaching a suitable shrinking of S. In Chapter 4 we show that higher order sufficient conditions for both discrete and continuous time controllability are indeed robust with respect to a shrinking of S, provided the target is regular enough. Essentially we allow S to be nonsmooth but rule out outward angles and inward cusps; technically, we require S to be wedged and to satisfy a uniform internal sphere condition. To illustrate the approach numerical examples are presented.

Chapter 6: *Bridge* dynamic programming and a new Hamilton-Jacobi-Bellman approach. The global solution of the minimum time problem – after a transformation – is efficiently obtained via the solution of the associated Hamilton-Jacobi-Bellman equation. Indeed, the unique viscosity solution of this equation is the optimal value function of the problem, whose knowledge can in a subsequent step be used in order to synthesize the optimal control functions. For the numerical solution of this Hamilton-Jacobi-Bellman equation, semi-Lagrangian schemes – which consist of a semi-discretization in time followed by a finite element discretization in space – are particularly attractive because they are unconditionally stable and allow to combine different discretization methods in space and time [37]. Most importantly, however, the semi-discretization is directly linked to a discrete time approximation to the original minimum time problem, which facilitates both the interpretation of the numerical results and the synthesis of approximately optimal feedback laws from the numerical approximation.

Nevertheless, one of the main disadvantages of the semi-Lagrangian approach is the fact that the semi-discretization of the standard minimum time problem leads to a piecewise constant optimal value function whose discontinuities pose problems, e.g., for the subsequent spatial discretization. The discontinuities stem from the fact that the optimal value function is fixed to be 0 on the target set of the minimum time problem. In order to improve the approximation, it does hence appear to be a good idea to use a formulation of the minimum time problem which avoids setting the optimal value function to 0 by extending the problem inside the target in a meaningful way. This is what is done in Chapter 6, which presents the theoretical foundations of a new formulation of the minimum time problem as well as its numerical discretization including an error analysis of the resulting semi-Lagrangian scheme. Numerical examples show that under suitable conditions the new formulation is indeed able to significantly reduce the numerical error compared to the classical approach.

Chapter 7: Reconstruction of the minimum time function through the approximation of reachable sets for linear control systems. Reachable sets have attracted several mathematicians for a longer time both in theoretical and in numerical analysis. One common definition collects end points of feasible solutions of a control problem starting from a common inital set and reaching a point *up to* a given end time, the other definition is similar but prescribes a *fixed* end time in which the point is reached.

Reachable sets with and without control constraints appear in control theory (e.g., in stability results), in optimal control (e.g., in analysis for robustness) and in set-valued analysis. For reachable sets at a given end time of linear or nonlinear control problems, properties like convexity for linear control problems at a given end time (due to Aumann and his study of Aumann's integral for set-valued maps in [6]), closedness and connectedness under weak assumptions for nonlinear systems (see, e.g., [5, 29]), ... are well-known. The Lipschitz continuity of reachable sets with respect to the initial value is also established and is a result of the Filippov theorem which proves the existence of neighboring solutions for Lipschitz systems.

The approaches for the numerical computation of reachable sets mainly split into two classes, those for reachable sets up to a given time and the other ones for reachable sets at a given end time. We will list here only some of them, since the literature is very rich. There are methods based on overestimation and underestimation of reachable sets based on ellipsoids [55], zonotopes [3, 44] or on approximating the reachable set with support functions resp. supporting points [14, 52, 58]. Other popular and well-studied approaches involve level-set methods, semi-Lagrangian schemes and the computation of an associated Hamilton-Jacobi-Bellman equation, see, e.g., [16, 21, 34, 37, 45] . Further methods [8, 13, 14] are set-valued generalizations of quadrature methods and Runge-Kutta methods are initiated by [31–33, 73, 76]...

Here, we will focus on set-valued quadrature methods and set-valued Runge-Kutta methods with the help of support functions or supporting points, since they do not suffer on the wrapping effect or on an exploding number of vertices and the error of restricting computations only for finitely many directions can be easily estimated. Furthermore, they belong to the most efficient and fast methods (see [3, Sec. 3.1], [57, Chap. 9, p. 128]) for linear control problems to which we restrict the computation of the minimum time function $T_S(x)$.

In the HJB approach, the minimal requirement on the regularity of $T_S(x)$ is the continuity, see, e.g., [17, 26, 47]. The solution of a HJB equation with suitable boundary conditions gives immediately – after a transformation – the minimum time function and its level sets provide a description of the reachable sets. A natural question occurring is whether it is also possible to do the other way around, i.e. reconstruct the minimum time function $T_S(x)$ if knowing the reachable sets. One of the attempts was done in [20, 21], where the approach is based on PDE solvers and on the reconstruction of the optimal control and solution via the value function. On the other hand, our approach in this work is completely different. It is based on very efficient quadrature methods for convex reachable sets as described in Section 7.2.

In the last chapter we present a novel approach for calculating the minimum time function. The basic idea is to use set-valued methods for approximating reachable sets at a given end time with computations based on support functions resp. supporting points. By reversing the time and start from the convex target as an initial set we compute the reachable sets for times on a (coarser) time grid. Due to the strictly expanding condition for reachable sets, the corresponding end time is assigned to all boundary points of the computed reachable sets. Since we discretize in time and in space (by choosing a finite number of outer normals for the computation of supporting points), the vertices of the polytopes forming the fully discrete reachable sets are considered as data points of an irregular triangulated domain. On this simplicial triangulation, a piecewise linear approximation yields a fully discrete approximation of the minimum time function. The well-known interpolation error and the convergence results for the set-valued method can be applied to yield an easy-to-prove error estimate by taking into account the regularity of the minimum time function. It requires at least the continuity and involves the maximal diameter of the simplices in the used triangulation. A second error estimate is proved without explicitely assuming the continuity of the minimum time function and depends only on the time interval between the computed (backward) reachable sets. The computation does not need the nonempty interior of the target set in contrary to the Hamilton-Jacobi-Bellman approach, for singletons the error estimate even improves. It is also able to compute discontinuous minimum time functions, since the underlying set-valued method can also compute lower-dimensional reachable sets. There is no explicit dependence of the algorithm and the error estimates on the smoothness of optimal solutions or controls. These results are devoted to reconstructing discrete optimal trajectories which reach a set of supporting points from a given target for a class of linear control problems and also proving the convergence of discrete optimal controls by the use of nonsmooth and variational analysis. The main tool is Attouch's theorem that allows to benefit from the convergence of the discrete reachable sets to the time-continuous one. To illustrate the error estimates and to demonstrate differences to other numerical approaches we consider a series of numerical examples which either allow higher order of convergence or where the regularity cannot be sufficiently granted.

1.3 Structure of the thesis

This thesis consists of seven chapters. The first two chapters are devoted to the introduction and the background of this work. All of the main results are described in Chapters 3 – 7:

- Chapter 3 is based on [56].
- Chapters 4 and 5 are taken from [26].
- Chapter 6 is presented in [47].
- Chapter 7 is derived from [11, 12].

Chapter 2

Preliminaries

This chapter is devoted to introducing some definitions, notations and basic knowledge which are used through this work. Let $S \subset \mathbb{R}^n$ be a closed set, $\|\cdot\|$ be the Euclidean norm and $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^n$ and $\delta > 0$ be given. We set, for $x \in \mathbb{R}^n$,

$$d(x, S) = d_S(x) := \min\{\|y - x\| : y \in S\}$$

and

$$S_\delta = \{x \in \mathbb{R}^n : d_S(x) \leq \delta\}.$$

Denoting S^c as the complement of S, $S^c = \mathbb{R}^n \setminus S$, we also define

$$S_{-\delta} = \{x \in \mathbb{R}^n : d_{\overline{S^c}}(x) \geq \delta\}.$$

Of course $S_{-\delta}$ may be empty, but we will consider mainly cases where this behavior does not occur. Let $\mathcal{C}(\mathbb{R}^n)$ be the set of convex, compact, nonempty subsets of $\mathbb{R}^n$, $B(x_0, r)$ or $B_r(x_0)$ be the (Euclidean) ball with radius $r > 0$ centered at x_0 and S_{n-1} be the unique sphere in $\mathbb{R}^n$. Let A be a subset of $\mathbb{R}^n$, M be an $n \times n$ real matrix, then $B(A, r) := \bigcup_{x \in A} B(x, r)$, $\|M\|$ denotes the *lub-norm* of M with respect to $\|\cdot\|$, i.e. the spectral norm. The *convex hull*, the *boundary* and the *interior* of a set A are signified by $\mathrm{co}(A)$, ∂A, $\mathrm{int}(A)$ respectively. The characteristic function of S is defined as

$$\chi_S(y) = \begin{cases} 1, & \text{if } y \in S, \\ 0, & \text{otherwise.} \end{cases} .$$

The following recalls the definition of multi-index notation.

Definition 2.0.1 (Multi-index)**.** *Given $n \in \mathbb{N} \setminus \{0\}$, an* n-dimensional multi-index *is a n-tuple $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with $\alpha_i \neq 0$ for all $i = 1, \ldots, n$. For* n-dimensional multi-indices *α, β and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we define:*

1. *Componentwise sum and difference: $\alpha \pm \beta = (\alpha_1 \pm \beta_1,\, \alpha_2 \pm \beta_2, \ldots,\, \alpha_n \pm \beta_n)$;*
2. *Partial order: $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i \quad \forall\, i \in \{1, \ldots, n\}$;*

3. *Length:* $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$;

4. *Factorial:* $\alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_n!$;

5. *Binomial coefficient:* $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\cdots\binom{\alpha_n}{\beta_n} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$;

6. *Multinomial coefficient:* $\binom{k}{\alpha} = \frac{k!}{\alpha_1!\alpha_2!\cdots\alpha_n!} = \frac{k!}{\alpha!}$, *where* $k := |\alpha|$;

7. *Power:* $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$;

8. *Higher-order derivative* $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\dots\partial_n^{\alpha_n}$, *where* $\partial_i^{\alpha_i} := \partial^{\alpha_i}/\partial x_i^{\alpha_i}$.

2.1 Control theory

We now are going to recall some basic notations of control theory. Consider the controlled dynamics and its inversed one in $\mathbb{R}^n$

$$\begin{cases} \dot{y}(t) &= f(y(t), u(t)) \\ y(t_0) &= y_0 \end{cases}, \quad (2.1) \qquad \begin{cases} \dot{y}^-(t) &= -f(y^-(t), u(t)) \\ y^-(t_0) &= y_0^- \end{cases}, \quad (2.2)$$

where $u(t) \in U$ for a. e. $t > t_0 \geq 0$, $U \subset \mathbb{R}^m$ a nonempty compact set, the control set. Under standard assumptions, the existence and uniqueness of (2.1) as well as (2.2) are guaranteed for any $u(\cdot)$ measurable and any $y_0, y_0^- \in \mathbb{R}^n$. Let $S \subset \mathbb{R}^n$, a nonempty compact set, be the target and the set of admissible controls

$$\mathcal{U} := \{u \colon [t_0, +\infty) \to U, \ \textit{measurable}\}.$$

Let $y(\cdot, y_0, u)$, $y^-(\cdot, y_0^-, v)$ be the solution of (2.1), (2.2) initiating from y_0, y_0^- and following the controls u, $v \in U$ respectively. We define the minimum time to reach S and to $\overline{S^c}$ by following some $u(\cdot) \in \mathcal{U}$ from $y_0 \notin S$ and $y_0^- \notin \overline{S^c}$ respectively

$$\begin{aligned} t_S(y_0, u) &= \min\{t \geq t_0 : y(t, y_0, u) \in S\} \leq +\infty, \\ t_{S^c}(y_0^-, u) &= \min\{t \geq t_0 : y^-(t, y_0^-, u) \in \overline{S^c}\} \leq +\infty. \end{aligned} \tag{2.3}$$

Then the minimum time functions to reach S and to $\overline{S^c}$ from y_0 and y_0^- are defined as

$$T_S(y_0) = \inf_{u \in \mathcal{U}} \{t_S(y_0, u)\},$$

$$T_{S^c}(y_0^-) = \inf_{u \in \mathcal{U}} \{t_{S^c}(y_0^-, u)\}.$$

Under standard assumptions, the infimum is attained, provided it is not $+\infty$. We also define the *reachable sets for fixed end time* $t \geq t_0$, *up to time* t, *up to a finite time* respectively as follows:

$$
\begin{aligned}
\mathcal{R}^S(t) &:= \{y_0 \in \mathbb{R}^n : \textit{ there exists } u \in \mathcal{U},\ y(t, y_0, u) \in S\},\\
\mathcal{R}^S_{\leq}(t) &:= \bigcup_{s\in[t_0,t]} \mathcal{R}^S(s) = \{y_0 \in \mathbb{R}^n : \textit{ there exists } u \in \mathcal{U},\ y(s, y_0, u) \in S \text{ for some } s \in [t_0, t]\},\\
\mathcal{R}^S &:= \{y_0 \in \mathbb{R}^n : T_S(y_0) < +\infty\} = \bigcup_{t\in[t_0,\infty)} \mathcal{R}^S(t),\\
\mathcal{R}^{S^c} &:= \{y_0^- \in \mathbb{R}^n : T_{S^c}(y_0^-) < +\infty\}.
\end{aligned}
$$

By definition

$$\mathcal{R}^S_{\leq}(t) = \{y_0 \in \mathbb{R}^n \colon T_S(y_0) \leq t\} \tag{2.4}$$

is a sublevel set of the minimum time function. We define level sets in a neighborhood of ∂S by setting, given $\tau > 0$,

$$
\begin{aligned}
S^+_\tau &= \{x \notin S,\ T_S(x) < \tau\},\\
S^-_\tau &= \{x \notin \overline{S^c},\ T_{S^c}(x) < \tau\}.
\end{aligned}
\tag{2.5}
$$

Given an open subset $\Omega \subseteq \mathbb{R}^n$. In presence of state constraint Ω, i.e.

$$
\begin{cases}
\dot{y}(t) = f(y(t), u(t)), \ \text{ for a.e. } t > t_0,\\
y(t_0) = y_0 \in \overline{\Omega},\\
y(t) \in \overline{\Omega},\ t \geq t_0.
\end{cases}
\tag{2.6}
$$

We introduce the following additional definition which used in Chapter 3 as follows. Given the system (2.6), we define the *state constrained reachable set from* $y_0 \in \overline{\Omega}$ *at time* $\tau \geq t_0$:

$$\mathcal{R}^{S,\Omega}_{y_0}(\tau) := \Big\{y(\tau) : y(\cdot) \text{ is a solution of (2.6) defined on } [t_0, \tau]\Big\}.$$

The *state constrained minimum time function* from $y_0 \in \overline{\Omega}$ is

$$
T_{S,\Omega}(y_0) :=
\begin{cases}
+\infty, & \text{if } \mathcal{R}^{S,\Omega}_{y_0}(\tau) \cap S = \emptyset \text{ for all } \tau \geq t_0,\\
\inf\{\tau \geq t_0 : \mathcal{R}^{S,\Omega}_{y_0}(\tau) \cap S \neq \emptyset\}, & \text{otherwise.}
\end{cases}
$$

Throughout this work, we reserve the letters $U, \mathcal{U}, S$ for the control set, the admissible control set and the target set respectively. We will let $t_0 = 0$ except for Chapter 7.

2.2 Nonsmooth and variational analysis

The followings are some concepts of nonsmooth analysis which will be needed in this sequel. Among many reference books on the subject we choose to quote here [24], which also contains an introduction to control problems.

Let $S \subset \mathbb{R}^n$ be a closed set. We say that $v \in \mathbb{R}^n$ is a proximal normal to S at $x \in S$ if there exists $\sigma = \sigma(v,x) \geq 0$ such that

$$\langle v, y - x\rangle \leq \sigma \left\|y - x\right\|^2, \quad \forall y \in S.$$

The set of such vectors is the proximal normal cone to S at x, $N_S^P(x)$. The cone of limiting normals is denoted by $N_S^L(x)$, and consists of those $v \in \mathbb{R}^n$ for which there exist sequences $\{x_i\} \subset S$, and $\{v_i\}$, with $v_i \in N_S^P(x_i)$, such that $x_i \to x$ and $v_i \to v$. This cone never trivializes if $x \in \partial S$, the boundary of S. If S is convex, then $N_S^P = N_S^L = N_S$, the normal cone of Convex Analysis. The Clarke normal cone $N_S^C(x)$ equals the closed convex hull of $N_S^L(x)$. Let A be a subset in $\mathbb{R}^n$ and $f : A \to \mathbb{R} \cup \{+\infty\}$ be a function. The *epigraph* of f be defined as

$$\operatorname{epi} f = \{(x,r) \in \mathbb{R}^n \times \mathbb{R} \colon x \in A,\ r \geq f(x)\}. \tag{2.7}$$

Definition 2.2.1. *If X is a topological vector space, we say that $f : X \to \mathbb{R} \cup \{+\infty\}$ is* lower semicontinuous *(l.s.c.) if* epi f *is closed in $X \times \mathbb{R}$ with respect to the product topology on $X \times \mathbb{R}$, i.e.,*

$$\liminf_{y \to x} f(y) \geq f(x).$$

A function $g : X \to \mathbb{R} \cup \{-\infty\}$ is called upper semicontinuous *(u.s.c.) if $-g$ is l.s.c.*

Definition 2.2.2. *Given the Banach spaces X and Y, $U \subseteq X$, $V \subseteq Y$ be open, $0 \leq \alpha \leq 1$, a function $f : U \to V$ is said to be a* Hölder continuous function of exponent α *or* α-Hölder continuous *($f \in C^{0,\alpha}(U)$) if there exists $C > 0$, such that for every $x_1, x_2 \in U$*

$$\|f(x_1) - f(x_2)\|_Y \leq C\|x_1 - x_2\|_X^\alpha.$$

A function $f : U \to V$ is called locally Hölder continuous of exponent α *or* locally α-Hölder continuous *($f \in C^{0,\alpha}_{\text{loc}}(U)$) if it is Hölder continuous with exponent α on every compact set of U. We will call* $\text{Lip}(U) := C^{0,1}(U)$ *(resp.* $\text{Lip}_{\text{loc}}(U) := C^{0,1}_{\text{loc}}(U)$*) the set of* Lipschitz continuous *functions (resp.* locally Lipschitz continuous *functions). If $f \in \text{Lip}(U)$ or $f \in \text{Lip}_{\text{loc}}(U)$, the constant $C > 0$ appearing in the definition is called a* Lipschitz constant *for f.*

Given $k \geq 1$, we define the sets

$$\begin{aligned} C^{k,\alpha}(U) :=& \{g : U \to Y : g \text{ is } k \text{ times continuously differentiable with } \alpha\text{-Hölder continuous} \\ & k\text{-th differential}\}, \\ C^{k,\alpha}_{\text{loc}}(U) :=& \{g : U \to Y : g \text{ is } k \text{ times continuously differentiable with locally } \alpha\text{-Hölder} \\ & \text{continuous } k\text{-th differential}\}. \end{aligned}$$

We recall the following classical result on regularity of Lipschitz functions in finite-dimensional spaces (the proof can be found e.g., in Corollary 4.19 p.148 of [24]):

Theorem 2.2.3 (Rademacher's Theorem)**.** *Let X be a finite-dimensional Banach space, $V \subseteq X$ be open, and $f : V \to \mathbb{R}$ be a locally Lipschitz function. Then f is differentiable almost everywhere.*

Definition 2.2.4 (Proximal subdifferential)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be given and $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous map, $x \in \mathrm{dom}(f)$, and $v \in \mathbb{R}^n$. We say that v is a* proximal subgradient *of f at x if $(v, -1) \in N^P_{\mathrm{epi}(f)}(x, f(x))$. The possibly empty set of all proximal subgradients of f at x will be denoted by $\partial_P f(x)$ and called the* proximal subdifferential *of f at x. The following* proximal inequality formula *gives another characterization of $\partial_P f(x)$:*

$$v \in \partial_P f(x) \textit{ iff there exist } \sigma, \eta > 0 \textit{ s.t. } f(y) \geq f(x) + \langle v, y - x\rangle - \sigma\|y - x\|^2,$$

for all $y \in B(x, \eta)$. Symmetrically, it is possible to define the proximal superdifferential *$\partial^P g(x)$ of an upper semicontinuous function $g : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ by taking $\partial^P g(x) = \{v \in \mathbb{R}^n : (-v, 1) \in N^P_{\mathrm{hypo}\,(g)}(x)\}$. In this case the proximal inequality formula becomes*

$$v \in \partial^P g(x) \textit{ iff there exist } \sigma, \eta > 0 \textit{ s.t. } g(y) \leq g(x) + \langle v, y - x\rangle + \sigma\|y - x\|^2,$$

for all $y \in B(x, \eta)$.

Our controllability results will be largely based on some properties of the *distance function*, in connection with suitable regularity assumptions on the target. We now recall such assumptions.

Definition 2.2.5. *(1) Let $S \in \mathbb{R}^n$ be closed and let $\rho > 0$. We say that S satisfies a ρ–internal sphere condition if S is the union of closed spheres of radius ρ, i.e., for any $x \in S$ there exists y such that $x \in \overline{B_\rho(y)} \subset S$.*
(2) Let $K \subseteq \mathbb{R}^n$ be closed, $a \in K$, C be a nonempty compact subset of $\mathbb{R}^n$ with $K \cap C \neq \emptyset$. We set

$$\begin{aligned} \mathrm{reach}(K, a) &:= \sup\{r \geq 0 : \pi_K(y) \textit{ is a singleton for every } y \in B(a, r)\}, \\ \mathrm{reach}_C(K) &:= \inf_{a \in K \cap C} \mathrm{reach}(K, a), \\ \mathrm{reach}(K) &:= \inf_{a \in K} \mathrm{reach}(K, a). \end{aligned}$$

We say that K has locally positive reach *if for every compact C with $K \cap C \neq \emptyset$ we have* $\mathrm{reach}_C(K) > 0$*, equivalently there exists an open set $U \supseteq K$ such that for every $y \in U$ there exists a unique $x \in K$ with $d_K(y) = \|y - x\|$. If* $\mathrm{reach}(K) > 0$ *we say that K has positive reach. If* $\mathrm{reach}(K, a) = \rho > 0$ *for every $x \in S$, we say that S has reach ρ, it can be proved that there exists a neighborhood V of a such that $d_S(\cdot)$ is of class $C^{1,1}(V \setminus S)$. This class of sets was introduced in [39], and has been extensively studied by many authors both in finite and infinite dimensions. We refer the reader to [27] and [62] for further details and extension of such kind of results.*

Relations between the above concepts were studied in detail in [64]. We recall, in particular, that if S has reach ρ, then the closure of its complement, $\overline{S^c}$, satisfies a ρ–internal sphere condition, but the converse is not true in general. The main property of the distance we are going to use is its *semiconcavity*. We say that a function $f : \Omega \to \mathbb{R}$

is locally semiconcave if for every $x \in \Omega$ there exists a ball $B_r(x)$ and a positive constant C such that

$$\lambda f(y) + (1-\lambda) f(y') \leq f(\lambda y + (1-\lambda) y') + C \left\| y - y' \right\|^2 \tag{2.8}$$

for all $y, y' \in B_r(x)$ and all $\lambda \in [0,1]$. Global semiconcavity means that the above inequality is satisfied by every $y, y' \in \Omega$ such that the segment $[y, y'] \subset \Omega$ with the same constant C. The constant C appearing in (2.8) is labeled as semiconcavity constant. We say that f is *locally semiconvex* if $-f$ is locally semiconcave.

Semiconcave functions enjoy some remarkable properties, summarized in the following.

Proposition 2.2.6. *Let $f : \Omega \to \mathbb{R}$ be a function. Then*

1. *f is semiconcave if and only if there exists $c > 0$ such that $f(x) - \dfrac{c\|x\|^2}{2}$ is concave in every convex subset of Ω.*

2. *If $f : \Omega \to \mathbb{R}$ is both semiconcave and semiconvex, then $f \in C^{1,1}(\Omega)$.*

3. *Let $f : \Omega \to \mathbb{R}$ be semiconvex. Then f is locally Lipschitz in Ω and $\partial_P f(x) = \partial f(x)$ at every $x \in \Omega$. In particular, the proximal subdifferential $\partial_P f(x) \neq \emptyset$ at each point. If f is semiconcave, the same results hold, with the proximal superdifferential instead of the proximal subdifferential.*

4. *If f is semiconcave, then it is twice differentiable a.e. in the domain.*

Alternative characterizations of semiconcave functions can be given (see [23]).

Proposition 2.2.7. *Let Ω be an open subset of $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}$ be a function, and $c \geq 0$. The following are equivalent:*

1. *f is semiconcave with semiconcavity constant c;*

2. *for every $p \in \partial^P f(x)$ we have*

$$f(y) - f(x) \leq \langle p, y - x \rangle + c\|y - x\|^2;$$

3. *for any $w \in \mathbb{R}^n$ such that $\|w\| = 1$ we have $\dfrac{\partial^2 v}{\partial w^2} \leq c$ in the sense of distributions in Ω.*

The following results are well known (see, e.g., Proposition 2.2.2 in [23] and Section 4 in [39]).

Proposition 2.2.8. *Let $S \subset \mathbb{R}^n$ be closed. Then the distance function d_S satisfies the following properties:*

(i) d_S *is locally semiconcave in* $\mathbb{R}^n \setminus S$. *More precisely, given a set* $K \subset \mathbb{R}^n \setminus S$ *such that* $\inf_{x \in K} d_S(x) = \delta > 0$, d_S *is semiconcave in* K *with semiconcavity constant equal to* $\frac{1}{\delta}$.

(ii) If S *satisfies a* ρ*- internal sphere condition, then* d_S *is semiconcave in* $\overline{S^c}$ *with semiconcavity constant* $\frac{1}{\rho}$.

(iii) If S *has reach* $\rho > 0$, *then* d_S *is differentiable, and* ∇d_S *is locally Lipschitz, in* $S_\rho \setminus S$.

(iv) d_S^2 *is semiconcave on the whole of* $\mathbb{R}^n$ *with semiconcavity constant equal to* 2.

Using the metric projection, i.e., the set

$$\pi_S(x) = \{y \in S : \|y - x\| = d_S(x)\},$$

it is possible to characterize the (super-)differential of the distance function (see, e.g, Corollary 3.4.5 in [23] and Section 4 in [39]). We have, for all $x \in \mathbb{R}^n \setminus S$,

$$\partial^P d_S(x) = \frac{x - \text{co } \pi_S(x)}{d_S(x)},$$

where "co" denotes the convex hull. Moreover, if S has reach $\rho > 0$ and $x \in S_\rho \setminus S$, then $\pi_S(x)$ is a singleton and

$$\nabla d_S(x) = \frac{x - \pi_S(x)}{d_S(x)},$$

and $\nabla d_S(x) \in N_S^P(\pi_S(x))$.

Some basic notions of variational analysis which are needed in constructing and proving the convergence of controls are now introduced. The main references for this part are [24, 69]. Let $A \subset \mathbb{R}^n$, then the *indicator function* of A be defined as

$$I_A(x) = \begin{cases} 0 \text{ if } x \in A \\ +\infty \text{ otherwise} \end{cases}.$$

Proposition 2.2.9. *Let* A *be a closed, convex and nonempty set. Then* I_A *is a lower semicontinuous, convex function and* $\operatorname{epi} I_A$ *is a closed, convex set.*

Proof. see e.g. [24, Exercise 2.1]. ☐

Definition 2.2.10 ((Painlevé-Kuratowski) convergence of sets in [69, Sec. 4.A–4.B])**.** *For a sequence* $\{A^i\}_{i \in \mathbb{N}}$ *of subsets of* $\mathbb{R}^n$, *the* outer limit *is the set*

$$\limsup_{i \to \infty} A^i = \{x \colon \limsup_{i \to \infty} \mathrm{d}(x, A^i) = 0\},$$

and the inner limit *is the set*

$$\liminf_{i\to\infty} A^i = \{x\colon \liminf_{i\to\infty} \mathrm{d}(x, A^i) = 0\},$$

The limit *of the sequence exists if the outer and inner limit sets are equal:*

$$\lim_{i\to\infty} A^i := \liminf_{i\to\infty} A^i = \limsup_{i\to\infty} A^i.$$

We also need two more convergence terms for set-valued maps and functions.

Definition 2.2.11 (graphical and epi-graphical convergence)**.** *Consider $A \subset \mathbb{R}^n$ and the set-valued map $F : A \rightrightarrows \mathbb{R}^n$. Then the* graph *of F is defined as*

$$\operatorname{gph} F := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \colon y \in F(x),\ x \in A\}.$$

A sequence of functions $f^i : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, $i \in \mathbb{N}$, converges epi-graphically*, if the outer and the inner limit of their epigraphs* $(\operatorname{epi} f^i)_{i\in\mathbb{N}}$ *coincide. The* epi-limit *is the function whose epigraph* $\operatorname{epi} f$ *coincides with the set limit of the epigraphs in the sense of Painlevé-Kuratowski (see [69, Definition 7.1]).*

We say that the sequence of set-valued maps $(F^i)_{i\in\mathbb{N}}$ with $F^i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ converges graphically to a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ if and only if its graphs, i.e. the sets $(\operatorname{gph} F^i)_{i\in\mathbb{N}}$*, converge to* $\operatorname{gph} F$ *in the sense of Definition 2.2.10 (see [69, Definition 5.32]).*

We cite here Attouch's theorem in a reduced version which plays an important role for convergence results of discrete optimal controls and solutions.

Theorem 2.2.12 (see [69, Theorem 12.35])**.** *Let $(f^i)_i$ and f be lower semicontinuous, convex, proper functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{\infty\}$.*
Then the epi-convergence of $(f^i)_{i\in\mathbb{N}}$ to f is equivalent to the graphical convergence of the subdifferential maps $(\partial_P f^i)_{i\in\mathbb{N}}$ to $\partial_P f$.

2.3 Operations on sets and Aumann integral

In this section we will recall definitions as well as basic knowledge of convex analysis for later use especially in Chapter 7. We define the support function, the supporting points in a given direction and the set arithmetic operations as follows.

Definition 2.3.1. *Let $A \in \mathcal{C}(\mathbb{R}^n)$, $l \in \mathbb{R}^n$. The* support function *and the* supporting face *of A in the direction l are defined as, respectively,*

$$\delta^*(l, A) := \max_{x\in A} \langle l, x\rangle,$$
$$\mathrm{Y}(l, A) := \{x \in A \colon \langle l, x\rangle = \delta^*(l, A)\}.$$

An element of the supporting face is called supporting point*.*

Definition 2.3.2. *Let* $A, B \in \mathcal{C}(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$, $M \in \mathbb{R}^{m\times n}$. *Then the* scalar multiplication*, the* image of a set under a linear map *and the* Minkowski sum *are defined as follows:*

$$\begin{aligned}
\lambda A &:= \{\lambda a\colon a \in A\},\\
MA &:= \{Ma\colon a \in A\},\\
A + B &:= \{a + b\colon a \in A,\, b \in B\}.
\end{aligned}$$

In the following propositions we will recall known properties of the convex hull, the support function and the supporting points when applied to the set operations introduced above (see e.g. [5, Chap. 0], [4, Sec. 4.6, 18.2], [3, 7, 57]). Especially, the convexity of the arithmetic set operations becomes obvious.

Proposition 2.3.3. *Let* $A, B \in \mathbb{R}^n$, $M \in \mathbb{R}^{m\times n}$ *and* $\lambda \in \mathbb{R}$. *Then,*

$$\begin{aligned}
\operatorname{co}(A + B) &= \operatorname{co}(A) + \operatorname{co}(B),\\
\operatorname{co}(\lambda A) &= \lambda \operatorname{co}(A),\\
\operatorname{co}(MA) &= M \operatorname{co}(A).
\end{aligned}$$

Proposition 2.3.4. *Let* $A, B \in \mathcal{C}(\mathbb{R}^n)$, $\lambda \geq 0$, $M \in \mathbb{R}^{m\times n}$ *and* $l \in \mathbb{R}^n$.
Then λA, $A + B \in \mathcal{C}(\mathbb{R}^n)$ *and* $MA \in \mathcal{C}(\mathbb{R}^m)$. *Moreover,*

$$\begin{aligned}
\delta^*(l, \lambda A) &= \lambda \delta^*(l, A), & \mathrm{Y}(l, \lambda A) &= \lambda\, \mathrm{Y}(l, A),\\
\delta^*(l, A + B) &= \delta^*(l, A) + \delta^*(l, B), & \mathrm{Y}(l, A + B) &= \mathrm{Y}(l, A) + \mathrm{Y}(l, B),\\
\delta^*(l, MA) &= \delta^*(M^T l, A), & \mathrm{Y}(l, MA) &= M\, \mathrm{Y}(M^T l, A).
\end{aligned}$$

By means of the support function or the supporting points, one can fully represent a convex compact set, either as intersection of halfspaces by the Minkowski duality or as convex hull of supporting points.

Proposition 2.3.5. *Let* $A \in \mathcal{C}(\mathbb{R}^n)$. *Then*

$$\begin{aligned}
A &= \bigcap_{l \in S_{n-1}} \big\{x \in \mathbb{R}^n\colon \langle l, x\rangle \leq \delta^*(l, A)\big\},\\
A &= \operatorname{co}\Big(\bigcup_{l \in S_{n-1}} \{y(l, A)\}\Big),\\
\partial A &= \bigcup_{l \in S_{n-1}} \{\mathrm{Y}(l, A)\},
\end{aligned}$$

where $y(l, A)$ *is an arbitrary selection of* $\mathrm{Y}(l, A)$.

We also recall the definition of Hausdorff distance which is the main tool to measure the error of reachable set approximation.

Definition 2.3.6. *Let $C, D \in \mathcal{C}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$. Then the Hausdorff distance between C and D is defined as*

$$\mathrm{d}_H(C, D) := \max\{\max_{x\in C} \mathrm{d}_D(x), \max_{y\in D} \mathrm{d}_C(y)\},$$

or equivalently

$$\mathrm{d}_H(C, D) := \min\{r \geq 0\colon C \subset B(D, r) \text{ and } D \subset B(C, r)\}.$$

The next proposition will be used for a special form of the space discretization of convex sets via the convex hull of finitely many supporting points.

Proposition 2.3.7 ([9, Proposition 3.4]). *Let $A \in \mathcal{C}(\mathbb{R}^n)$, choose $\varepsilon > 0$ with a finite set of normed directions*

$$S_{n-1}^{\Delta} := \bigcup_{k=1,\dots,N_n} \{l^k\} \subset S_{n-1}$$

with $N_n \in \mathbb{N}$, $\mathrm{d}_H(S_{n-1}, S_{n-1}^{\Delta}) \leq \varepsilon$ and consider the approximating polytope

$$A_\Delta = \mathrm{co}\{\bigcup_{k=1,\dots,N_n} \{y(l^k, A)\}\} \subset A,$$

where $y(l^k, A)$ is an arbitrary selection of $\mathrm{Y}(l^k, A)$, $k = 1, \dots, N_n$. Then

$$\mathrm{d}_H(A, A_\Delta) \leq 2\,\mathrm{diam}(A) \cdot \varepsilon,$$

where $\mathrm{diam}(A)$ *stands for the diameter of the set A.*

We also recall the notion of Aumann's integral [6] of a set-valued mapping defined as follows.

Definition 2.3.8. *Consider $t_f \in [t_0, \infty)$ and the set-valued map $F : [t_0, t_f] \rightrightarrows \mathbb{R}^n$ with nonempty images. With the help of the set of integrable selections*

$$\mathcal{F} := \{f\colon [t_0, t_f] \to \mathbb{R}^n\colon f \text{ is integrable over } [t_0, t_f] \text{ and } f(t) \in F(t) \text{ for a.e. } t \in [t_0, t_f]\}$$

the Aumann's integral *of $F(\cdot)$ is defined as*

$$\int_{t_0}^{t_f} F(s)ds := \{\int_{t_0}^{t_f} f(s)ds\colon f \in \mathcal{F}\}.$$

Part I

Controllability

Chapter 3

Continuous small-time controllability

In this chapter, we deal with the problem of *small-time controllability on S* (STCS) for nonlinear finite-dimensional time-continuous control systems in presence of state constraints. More precisely, given a nonlinear control system subjected to state constraints and a closed set S, our aim is to provide sufficient conditions to steer to S every point of a suitable neighborhood of S through admissible trajectories of the system, respecting the constraints, and giving also an upper estimate of the minimum time needed for each point to reach the target.

The chapter is structured as follows: in Section 3.1 we state and prove our general results on STCS with state constraints. Section 3.2 is devoted to a detailed comparison between the results of [53, 54, 61, 63] and ours. In Section 3.3 we turn our attention to control-affine systems, providing some explicit sufficient conditions for STCS. Finally, in Section 3.4 we give an example illustrating our approach.

3.1 A general result for small-time controllability

Given an open subset $\Omega \subseteq \mathbb{R}^n$ and a compact set $U \subseteq \mathbb{R}^m$, we will consider the following *state constrained control system*:

$$\begin{cases} \dot{y}(t) & = f(y(t), u(t)), \text{ for a.e. } t > 0, \\ y(0) & = x_0 \in \overline{\Omega}, \\ y(t) & \in \overline{\Omega},\ t \geq 0. \end{cases} \tag{3.1}$$

where for every compact K there exists $L = L_K > 0$ such that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies

$$\|f(x, u) - f(y, u)\| \leq L_K \|x - y\|, \text{ for all } x, y \in K,\ u \in U,$$

$u(\cdot) \in \mathcal{U} = \{v \colon [0, +\infty[\to U : v \text{ is measurable}\}$, and a closed subset $S \subseteq \mathbb{R}^n$ (called the *target set*) with $S \cap \overline{\Omega} \neq \emptyset$ is given.

A special case of the above system is given by the following *state constrained control-affine system*:

$$\begin{cases} \dot{y}(t) & = f_0(y(t)) + \displaystyle\sum_{i=1}^{N} u_i(t) f_i(y(t)), \quad \text{for a.e. } t > 0, \\ y(0) & = x_0 \in \overline{\Omega}, \\ y(t) & \in \overline{\Omega},\ t \geq 0, \end{cases} \tag{3.2}$$

where $f_0, f_i \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, $u_i \in \mathcal{U}$, with $i = 1, \dots, N$ and

$$\mathcal{U} = \{v \colon [0, +\infty[\to [-1, 1] : \ v \text{ is measurable}\}.$$

If $\Omega = \mathbb{R}^n$, in both cases we will omit the adjective *constrained* and we will write simply $\mathcal{R}^S_x(t)$ and $T_S(x)$.

Remark 3.1.1 (Estimates on trajectories)**.** *Consider the system 3.1. Let $\bar{x} \in \mathbb{R}^n$, $\delta_{\bar{x}} > 0$. Choose*

$$M_{\bar{x}} \geq \max\{\|f(z, u)\| : \ z \in \overline{B(\bar{x}, \delta_{\bar{x}})},\ u \in U\}.$$

Then for any $0 < \delta' < \delta_{\bar{x}}$ we have $\mathcal{R}^S_x(t) \subseteq \overline{B(\bar{x}, \delta_{\bar{x}})}$ for all $x \in B(\bar{x}, \delta')$ and $0 \leq t \leq \dfrac{\delta_{\bar{x}} - \delta'}{M_{\bar{x}}}$. The proof is classical, and is based on Schauder's fixed point theorem and Gronwall's inequality. See e.g. Section 5 in Chapter III of [16].

The property in which we are interested is the following (see also [53], [54], [63]).

Definition 3.1.2 (STCS)**.** *We say that S is* small-time controllable *for the system* (3.1) *if for any $T > 0$ there exists an open neighborhood $U_T \subseteq \mathbb{R}^n$ of S such that $T_{S,\Omega}(x) \leq T$ for all $x \in U_T \cap \overline{\Omega}$.*

Remark 3.1.3. *A sufficient condition for STCS is to be able to bound from above the minimum time function $T_{S,\Omega}(\cdot)$ in a relative neighborhood of the target by a continuous increasing function of the distance from the target itself, and vanishing exactly on the target.*

Lemma 3.1.4 (Localization)**.** *Consider system* (3.1) *with closed target $S \subseteq \mathbb{R}^n$. Assume that for every $\bar{x} \in \partial S \cap \overline{\Omega}$ there exists a continuous increasing function $\omega_{\bar{x}}$: $[0, +\infty[\to [0, +\infty[$ and $0 < \delta_{\bar{x}} < +\infty$ such that*

1. *$\omega_{\bar{x}}(p) = 0$ if and only if $p = 0$,*
2. *$T_{S,\Omega}(x) \leq \omega_{\bar{x}}(d_S(x))$ for all $x \in B(\bar{x}, \delta_{\bar{x}}) \cap \overline{\Omega}$.*

Then STCS holds. If moreover $\partial S \cap \overline{\Omega}$ is compact, then $\omega_{\bar{x}}(\cdot)$ and $\delta_{\bar{x}}$ can be chosen independently on $\bar{x}$.

Proof. Let $T > 0$ be fixed. For any $\bar{x} \in \partial S \cap \overline{\Omega}$ there exists $r_{\bar{x}} > 0$ such that $\omega_{\bar{x}}(s) \leq T$ for all $s \in [0, r_{\bar{x}}]$. Set $U_{\bar{x}} := B(\bar{x}, \delta_{\bar{x}}) \cap S_{r_{\bar{x}}}$, and notice that, in particular, if $x \in U_{\bar{x}} \cap \overline{\Omega}$ we have $T_{S,\Omega}(x) \leq \omega_{\bar{x}}(d_S(\bar{x})) \leq T$. Moreover, trivially, we have $T_{S,\Omega}(x) \leq T$ for all $x \in S$. Defined

$$U_T := \left(\bigcup_{\bar{x} \in \partial S \cap \overline{\Omega}} U_{\bar{x}} \cup S \right),$$

we have that $U_T \cap \overline{\Omega}$ is an open neighborhood of S in the topology of $\overline{\Omega}$ and, by construction, we have $T_{S,\Omega}(x) \leq T$ for all $x \in U_T \cap \overline{\Omega}$. So STC$S$ holds.

In the compact case, we can cover $\partial S \cap \overline{\Omega}$ by finitely many balls $\{B(\bar{x}_i, \delta_{\bar{x}_i}) : i = 1 \ldots, N\}$, where $\bar{x}_i \in \partial S \cap \overline{\Omega}$ and $\delta_{\bar{x}_i} > 0$, thus we can set $\omega(p) = \max_{i=1,\ldots,N} \omega_{\bar{x}_i}(p)$ and $\delta = \min_{i=1,\ldots,N} \delta_{\bar{x}_i}$. □

The following simple result will be extensively used in our analysis.

Lemma 3.1.5. *Let $\delta > 0$ be a constant, $\lambda : [0, +\infty[\times [0, +\infty[\to \mathbb{R}$, $\theta : [0, +\infty[\to [0, +\infty[$ be continuous functions such that*

1. $r \mapsto \dfrac{\theta(r)}{\lambda(\theta(r), r)}$ *is bounded from above by a nonincreasing nonnegative function $\beta(\cdot) \in L^1(]0, \delta[)$;*
2. $\lambda(\theta(r), r) > 0$ *for* $0 < r < \delta$, *and* $\lambda(0, r) = 0$ *for* $r > 0$.

Consider any sequence $\{r_i\}_{i \in \mathbb{N}}$ in $[0, \delta]$ satisfying for all $i \in \mathbb{N}$:

(S_1) $r_{i+1} - r_i \leq -\lambda(\theta(r_i), r_i)$.

(S_2) $\theta(r_i) \neq 0$ *implies* $r_i \neq 0$.

Then we have

a.) $r_i \to 0$,

b.) $\displaystyle\sum_{i=0}^{\infty} \theta(r_i) \leq \int_0^{r_0} \beta(r)\, dr.$

Proof. According to (S_1), the sequence $\{r_i\}_{i \in \mathbb{N}}$ is monotonically decreasing and bounded from below, thus it admits a limit. Let $r_\infty = \lim_{i \to +\infty} r_i$, it is evident that $0 \leq r_\infty < \delta$. Now we would like to show that $r_\infty = 0$. Assume, by contradiction, $r_\infty > 0$. By passing to the limit for $i \to +\infty$ in (S_1) and recalling that $\lambda(\cdot, \cdot)$ is a continuous function and $\lambda(\theta(r_i), r_i) \geq 0$, we obtain that $0 = \lambda(\theta(r_\infty), r_\infty)$ and this contradicts the fact that

$\lambda(\theta(r), r) > 0$ for any $0 < r < \delta$, thus $r_\infty = 0$. Since if $\theta(r_i) \neq 0$ we have $r_i \neq 0$ and $\dfrac{r_i - r_{i+1}}{\lambda(\theta(r_i), r_i)} \geq 1$, we obtain

$$\begin{aligned}\sum_{i=0}^{\infty} \theta(r_i) = \sum_{\substack{i=0\\ \theta(r_i)\neq 0}}^{\infty} \theta(r_i) &\leq \sum_{\substack{i=0\\ \theta(r_i)\neq 0}}^{\infty} \frac{\theta(r_i)}{\lambda(\theta(r_i), r_i)}(r_i - r_{i+1}) \\ &\leq \sum_{\substack{i=0\\ \theta(r_i)\neq 0}}^{\infty} \beta(r_i)(r_i - r_{i+1}) \leq \int_0^{r_0} \beta(r)dr,\end{aligned}$$

recalling the monotonicity property of $r \mapsto \beta(r)$. □

We will apply this Lemma to the following situation: take a sequence of points $\{x_i\}_{i\in\mathbb{N}}$, define $r_i = d_S^2(x_i)$ for all $i \in \mathbb{N}$, and assume that the time needed to reach x_{i+1} from x_i is $\theta(r_i)$. Then we can bound from above $\displaystyle\sum_{i=0}^{\infty} \theta(r_i)$, and thus the time needed to steer x_0 to S, provided that we are able to construct $\lambda(\cdot)$ fulfilling the assumptions of Lemma 3.1.5. If the bound is locally uniform in a neighborhood of S, STCS follows from Remark 3.1.3.

The map $\lambda(t, r)$ will measure the *decrease of the squared distance from the target* starting from a point that is at distance r from the target after having followed a particular admissible trajectory for time t. Roughly speaking, if the decrease of the distance is *too slow*, we will not be able to reach the target in finite time, so we need a quantitive bound of the ratio between the time passed and the amount of the decrease of the distance.

It is clear that, in the above discussion, we can replace the squared distance $d_S^2(\cdot)$ with a general nonnegative locally Lipschitz continuous function $\Phi_S(\cdot)$ (satisfying some extra regularity assumptions) and such that $S = \{x \in \mathbb{R}^n : \Phi_S(x) \leq 0\}$.

We state and prove now a first general result on STCS in the spirit of Remark 3.1.3.

Theorem 3.1.6 (General controllability)**.** *Consider the system* (3.1)*. Let $\bar{x} \in \partial S \cap \overline{\Omega}$, $\delta_{\bar{x}} > 0$ and assume that*

$$S \cap \overline{B(\bar{x}, \delta_{\bar{x}})} := \{x \in \overline{B(\bar{x}, \delta_{\bar{x}})} : \Phi_S(x) \leq 0\},$$

where $\Phi_S : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function. Set $\Phi_{\bar{x}} = \max\limits_{x\in\overline{B(\bar{x},\delta_{\bar{x}})}} \{\Phi_S(x)\}$, denote by $L(r) > 0$ a Lipschitz constant of $\Phi_S(\cdot)$ on $\overline{B(\bar{x}, \delta_{\bar{x}})} \cap \{x : \Phi_S(x) \leq r\}$, and let $M_{\bar{x}} = \max\limits_{\substack{v\in U\\ z\in\overline{B(\bar{x},\delta_{\bar{x}})}}} \{\|f(z, v)\|\}$. Let $\sigma, \delta, \mu, \chi : [0, +\infty[\times[0, +\infty[\to [0, +\infty[$, and $\tau, \theta : [0, +\infty[\to [0, +\infty[$ be continuous function. We assume that:

1) $\tau(r) = 0$ iff $r = 0$, $0 < \theta(r) \leq \tau(r)$ for every $0 < r < \Phi_{\bar{x}}$;

2) for any $x \in (\overline{B(\bar{x}, \delta_{\bar{x}})} \cap \overline{\Omega}) \setminus S$ *and* $0 < t \leq \tau(\Phi_S(x))$ *the following holds*

2.a) $[\mathcal{R}_x^{S,\Omega}(t)]_{\delta(t,\Phi_S(x))} \cap S_{2\delta_{\bar{x}}} \neq \{x\}$,

2.b) if $\mathcal{R}_x^{S,\Omega}(t) \cap S = \emptyset$*, there exists* $y_{t,x} \in [\mathcal{R}_x^{S,\Omega}(t)]_{\delta(t,\Phi_S(x))} \cap \overline{B(x, \chi(t,r))}$ *with*

$$\min_{\zeta \in \partial^P \Phi_S(x)} \langle \zeta, y_{t,x} - x \rangle + K \|y_{t,x} - x\|^2 \leq -\mu(t, \Phi_S(x)) + \sigma(t, \Phi_S(x));$$

2.c) $\Phi_S(\cdot)$ *is semiconcave on* $\overline{B(\bar{x}, \delta_{\bar{x}})}$ *with semiconcavity constant* $K = K_{\bar{x}} > 0$.

3) the continuous function $\lambda : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$*, defined as*

$$\lambda(t, r) := \mu(t, r) - \sigma(t, r) - (L(r) + K\delta(t, r) + 2K\chi(t, r))\, \delta(t, r),$$

satisfies the following properties:

(3.a) $0 < \lambda(\theta(r), r) < r$, $\lambda(0, r) = 0$ *for all* $0 < r < \Phi_{\bar{x}}$;

(3.b) $r \mapsto \dfrac{\theta(r)}{\lambda(\theta(r), r)}$ *is bounded from above by a nonincreasing nonnegative function* $\beta(\cdot) \in L^1(]0, \Phi_{\bar{x}}[)$.

Then, if we set

$$\omega(r_0) := \int_0^{r_0} \beta(r)\, dr,$$

we have that there exists $\delta'_{\bar{x}} > 0$ *such that* $T_{S,\Omega}(x) \leq \omega(\Phi_S(x))$ *for any* $x \in B(\bar{x}, \delta'_{\bar{x}}) \cap \overline{\Omega} \setminus S$.

Before the proof of Theorem 3.1.6, we make some comment on the assumptions.

i. Assumption (1) is just technical, and fix an upper bound $\tau(\Phi_S(x))$ on time sampling, depending only on the level set of $\Phi_S(\cdot)$ to which the considered starting point x belongs.

ii. Assumption (2.a) states that sufficiently near to the target there are no points where the unique admissible trajectory is the constant one. This is quite reasonble, since if $\bar{x}$ would be one of such points, we would have $T_S(\bar{x}) = +\infty$, so STCS could not hold.

iii. Assumption (2.b) provides a quantitative estimate of the variation of the Φ_S between two sampling times in the case that we are not able to reach the target in the sampling time $\tau(\Phi_S(x))$.

iv. Assumption (2.c) gives the technical assumptions on $\Phi_S(\cdot)$ (see also Remark 3.1.7).

v. Assumption (3) ensures that between two sampling times the function Φ_S actually decreases with a decreasing rate *fast enough* to reach the target in finite time, thanks to Lemma 3.1.5.

Proof. If $\Phi_{\bar{x}} = 0$ then $B(\bar{x}, \delta_{\bar{x}}) \subseteq S$, and so $T_{S,\Omega}(x) = 0$ for all $x \in B(\bar{x}, \delta_{\bar{x}}) \cap \overline{\Omega}$, and there is nothing to prove. We suppose now $\Phi_{\bar{x}} > 0$. Since f is bounded on $\overline{B(\bar{x}, \delta_{\bar{x}})}$, by Remark 3.1.1 we can choose $0 < \delta'_{\bar{x}} < \frac{\delta_{\bar{x}}}{2}$ such that, if we set

$$T_{\delta'_{\bar{x}}} = \max_{x \in \overline{B(\bar{x}, \delta'_{\bar{x}}) \setminus S}} \int_0^{\Phi_S(x)} \beta(s)\, ds,$$

we have $\mathcal{R}^S_x(t) \subseteq \overline{B(\bar{x}, \delta_{\bar{x}})}$ for all $0 < t \leq T_{\delta'_{\bar{x}}}$ and $x \in \overline{B(\bar{x}, \delta'_{\bar{x}})}$, recalling that, by continuity of $\Phi_S(\cdot)$, by the definition of S, and by the fact that $\beta \in L^1$, we have $T_{\delta'_{\bar{x}}} \to 0^+$ as $\delta'_{\bar{x}} \to 0^+$. Moreover, we have also $\mathcal{R}^S_x(t) \subseteq \overline{B(x, M_{\bar{x}} t)}$ for all $0 < t \leq T_{\delta'_{\bar{x}}}$.

We define a sequence of points and times $\{(x_i, t_i, r_i)\}_{i \in \mathbb{N}}$ by induction as follows. We choose $x_0 \in (\overline{B(\bar{x}, \delta'_{\bar{x}})} \cap \overline{\Omega}) \setminus S$, and set $r_0 = \Phi_S(x_0)$, $t_0 = \theta(r_0)$. Suppose to have defined x_i, t_i, r_i. We distinguish the following cases:

1. if $x_i \in S$, we define $x_{i+1} = x_i$, $t_{i+1} = 0$, $r_{i+1} = 0$.

2. if $x_i \notin S$

   (a) if $\mathcal{R}^{S,\Omega}_{x_i}(t_i) \cap S \neq \emptyset$, take $x_{i+1} \in \mathcal{R}^{S,\Omega}_{x_i}(t_i) \cap S$ and define $r_{i+1} = 0$, $t_{i+1} = 0$.

   (b) if $\mathcal{R}^{S,\Omega}_{x_i}(t_i) \cap S = \emptyset$, we choose $y_i \in [\mathcal{R}^{S,\Omega}_{x_i}(t_i)]_{\delta(t_i, r_i)} \cap \overline{B(x_i, \chi(x_i, r_i))}$ such that

   $$\min_{\zeta_{x_i} \in \partial^P \Phi_S(x_i)} \langle \zeta_{x_i}, y_i - x_i \rangle + \|y_i - x_i\|^2 \leq -\mu(t_i, r_i) + \sigma(t_i, r_i).$$

   We select $x_{i+1} \in \mathcal{R}^{S,\Omega}_{x_i}(t_i)$ such that $\|y_i - x_{i+1}\| \leq \delta(t_i, r_i)$, and define $r_{i+1} = \Phi_S(x_{i+1})$, $t_{i+1} = \theta(r_{i+1})$. According to the semiconcavity of $\Phi_S(\cdot)$ (with semiconcavity constant K), we have that there exists $\zeta_{x_i} \in \partial \Phi_S(x_i)$ such that

   $$\begin{aligned} r_{i+1} - r_i &\leq \langle \zeta_{x_i}, x_{i+1} - x_i \rangle + K\|x_{i+1} - x_i\|^2 \\ &\leq \langle \zeta_{x_i}, x_{i+1} - y_i + y_i - x_i \rangle + K\|(x_{i+1} - y_i) + (y_i - x_i)\|^2 \\ &\leq \langle \zeta_{x_i}, x_{i+1} - y_i + y_i - x_i \rangle + K\|x_{i+1} - y_i\|^2 \\ &+ 2K\|x_{i+1} - y_i\|\|y_i - x_i\| + K\|y_i - x_i\|^2. \end{aligned}$$

   By recalling that by assumption 2b) and the selection of x_{i+1}, we have $\|y_i - x_i\| \leq \chi(t_i, x_i)$ and $\|x_{i+1} - y_i\| \leq \delta(t_i, x_i)$. Therefore

   $$\begin{aligned} r_{i+1} - r_i &\leq \langle \zeta_{x_i}, x_{i+1} - y_i \rangle + \langle \zeta_{x_i}, y_i - x_i \rangle + K\delta^2(t_i, x_i) \\ &+ 2K\delta(t_i, x_i)\chi(t_i, x_i) + K\|y_i - x_i\|^2 \\ &\leq L(r_i)\delta(t_i, r_i) + \big(\langle \zeta_{x_i}, y_i - x_i \rangle + K\|y_i - x_i\|^2\big) + K\delta^2(t_i, x_i) + 2K\delta(t_i, x_i)\chi(t_i, x_i) \\ &\leq L(r_i)\delta(t_i, r_i) - \mu(t_i, r_i) + \sigma(t_i, r_i) + K\delta^2(t_i, x_i) + 2K\delta(t_i, x_i)\chi(t_i, x_i) \\ &\leq \big(L(r_i) + K\delta(t_i, r_i) + 2K\chi(t_i, r_i)\big)\delta(t_i, r_i) - \mu(t_i, r_i) + \sigma(t_i, r_i) = -\lambda(t_i, r_i), \end{aligned} \tag{3.3}$$

   recalling that $\|\zeta_{x_i}\| \leq L(r_i)$ by definition of $L(\cdot)$. We notice that in this case $x_{i+1} \notin S$ since $x_{i+1} \in \mathcal{R}^{S,\Omega}_{x_i}(t_i)$ and $\mathcal{R}^{S,\Omega}_{x_i}(t_i) \cap S = \emptyset$, thus $t_{i+1} > 0$ and $r_{i+1} > 0$.

The assumptions of Lemma 3.1.5 are satisfied:

1. $r_{i+1} - r_i \leq -\lambda(\theta(r_i), r_i)$,
2. it is obvious that $\theta(r_i) \neq 0$ implies $r_i \neq 0$. Indeed, assume that $r_i = 0$. Since $0 \leq \theta(r) \leq \tau(r)$, and $\tau(r) = 0$ iff $r = 0$, we have $\theta(0) = 0$.
3. by assumption, there exists $\beta \in L^1(]0, \delta_0[)$ such that $\dfrac{\theta(s)}{\lambda(\theta(s), s)} \leq \beta(r)$.

Applying Lemma 3.1.5, we have that

a.) $r_i \to 0$,

b.) $\displaystyle\sum_{i=0}^{\infty} \theta(r_i) \leq \int_0^{r_0} \beta(r)\, dr$.

Since $\displaystyle\sum_{i=0}^{\infty} t_i \leq \sum_{i=0}^{\infty} \theta(s_i)$, we have $\displaystyle\sum_{i=0}^{\infty} t_i \leq \int_0^{r_0} \beta(r)\, dr \leq T_{\delta'_x}$.

Noticing that $\{x_i\}_{i\in\mathbb{N}} \subseteq \mathcal{R}_x^S\left(\displaystyle\sum_{i=0}^{\infty} t_i\right)$ and $\displaystyle\sum_{i=0}^{\infty} t_i \leq T_{\delta'}$, we have that $\{x_i\}_{i\in\mathbb{N}} \subseteq \overline{B(\bar{x}, \delta_{\bar{x}})}$, thus is bounded. Up to subsequence, still denoted by $\{x_i\}_{i\in\mathbb{N}}$, we have that there exists $x_\infty \in \mathbb{R}^n$ such that $x_i \to x_\infty$. Since $\Phi_S(x_i) \to 0$, we have $x_\infty \in S$ and so $T_{S,\Omega}(x_0) \leq \displaystyle\sum_{i=0}^{\infty} t_i \leq \omega(\Phi_S(x_0))$. □

Remark 3.1.7. *Recalling Proposition 2.2.8, for every general closed set S we can take $\Phi_S(\cdot) = d_S^2(\cdot)$ with $K = 2$ and $L(r) = 2\sqrt{r}$. In this case, given $x \notin S$, we have $\zeta \in \partial^P d_S^2(x)$ if and only if $\zeta = 2d_S(x)\xi$ with $\xi \in \partial^P d_S(x)$. If S satisfies the ρ-internal sphere condition, we can take $\Phi_S(\cdot) = d_S(\cdot)$ with $K = 1/\rho$, $L(r) \equiv 1$.*

We will show now that the following well-known controllability condition can be fitted in the framework of the above result.

Corollary 3.1.8. *[Petrov's condition: the free case] Consider the system (3.1) with $\Omega = \mathbb{R}^n$, U compact, S closed, and $f \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n \times U; \mathbb{R}^n)$. Assume that for any $\bar{x} \in \partial S$ there exist $\eta_{\bar{x}} > 0$ and $\delta_{\bar{x}} > 0$ with*

$$\min_{\substack{u\in U\\ \zeta_x \in \partial^P d_S(x)}} \langle \zeta_x, f(x,u)\rangle < -\eta_{\bar{x}}, \qquad \forall x \in B(\bar{x}, \delta_{\bar{x}}) \setminus S,$$

then STCS holds. Moreover, if S is compact, there exist $\delta', C' > 0$ such that $T_S(x) \leq C' d_S(x)$ for all $x \in S_{\delta'}$.

Proof. Take $\Phi_S(\cdot) = d_S^2(\cdot)$, $K = K_{\bar{x}} = 2$, $L(r) = 2\sqrt{r}$. Let $M_{\bar{x}} > 0$ be such that $M_{\bar{x}} > \|f(z,v)\| + \|\nabla f(z,v)\|$ for all $v \in U$, $z \in \overline{B(\bar{x},\delta_{\bar{x}})}$.

Fix $\bar{x} \in \partial S$ and define the continuous functions:

$$\delta(t,r) := \frac{M_{\bar{x}}^2}{2}t^2, \quad \mu(t,r) := 2t\sqrt{r}\eta_{\bar{x}}, \quad \sigma(t,r) = 2M_{\bar{x}}^2 t^2, \quad \chi(t,r) = M_{\bar{x}}t.$$

and set $\beta(r) = \dfrac{1}{\sqrt{r}\eta_x} \in L^1(]0,\Phi_{\bar{x}}[)$. Then, $\lambda(\cdot,\cdot)$ is defined as in Theorem 3.1.6 (3), we have

$$\lambda(t,r) = 2t\sqrt{r}\eta_{\bar{x}} - 2M_{\bar{x}}^2 t^2 - \left(2\sqrt{r} + 2\frac{M_{\bar{x}}^2}{2}t^2 + 4M_{\bar{x}}t\right)\frac{M_{\bar{x}}^2}{2}t^2.$$

We notice that there exists a constant $C > 0$, independent of r, such that $t\sqrt{r}\eta_x \leq \lambda(t,r) < r$ for all $0 \leq t \leq C\sqrt{r} =: \tau(r)$ and $r \geq 0$. Furthermore, we choose $0 < \delta'_{\bar{x}} < \frac{\delta_{\bar{x}}}{2}$ such that, if we set $\Phi_{\bar{x}} := \sup\limits_{z \in \overline{B(\bar{x},\delta'_{\bar{x}})}} \Phi_S(x) = (\delta'_{\bar{x}})^2$, for all $0 < t < \tau(\Phi_{\bar{x}})$ and $x \in B(\bar{x},\delta'_{\bar{x}})$ we have $M_{\bar{x}}t \leq \dfrac{\delta_{\bar{x}}}{2}$, and so $\mathcal{R}_x^S(t) \subseteq B(\bar{x},\delta_{\bar{x}})$.

Take $x \in \overline{B(\bar{x},\delta'_{\bar{x}})} \setminus S$, and choose $u_x \in U$ and $\xi_x \in \partial^P d_S(x)$ such that

$$\langle \xi_x, f(x,u_x)\rangle < -\eta_{\bar{x}}. \tag{3.4}$$

Consider the solution $y_x(\cdot)$ of (3.1) starting from x at time $t = 0$ and generated by any constant control $u(t) \equiv u_x \in U$, and set $y_{t,x} = x + tf(x,u_x)$.

We have

$$\|y_x(t) - y_{t,x}\| \leq \int_0^t \|f(y_x(s),u_x) - f(x,u_x)\|\,ds \leq M_{\bar{x}}\int_0^t \|y_x(s) - x\|\,ds = \frac{M_{\bar{x}}^2}{2}t^2 = \delta(t,r),$$

thus $y_{t,x} \in [\mathcal{R}_x^S(t)]_{\delta(t,\Phi_S(x))}$.

If $x \notin S$, choose $\zeta_x \in \partial^P d_S^2(x)$ such that $\zeta_x = 2d_S(x)\xi_x$ with $\xi_x \in \partial^P d_S(x)$ realizing (3.4). Thus

$$\begin{aligned}\langle \zeta_x, y_{t,x} - x\rangle + 2\|y_{t,x} - x\|^2 &\leq 2d_S(x)t\langle \xi_x, f(x,u_x)\rangle + 2M_{\bar{x}}^2 t^2\\ &\leq -2d_S(x)t\eta_x + 2M_{\bar{x}}^2 t^2 = -\mu(x,\Phi_S(x)) + \sigma(x,\Phi_S(x))\end{aligned}$$

Recalling that for all $x \in B(\bar{x},\delta'_{\bar{x}}) \setminus S$ and $0 < t \leq \tau(r)$ we have $0 < \lambda(\theta(r),r) < r$, $\lambda(0,r) = 0$ for all $0 < r < \Phi_{\bar{x}}$, we can take $\theta(r) = \tau(r)$. Clearly, (1)-(2) in Theorem 3.1.6 are satisfied.

We notice that

$$\frac{\theta(r)}{\lambda(\theta(r),r)} \leq \frac{1}{\sqrt{r}\eta_{\bar{x}}} = \beta(r),$$

thus also (3) in Theorem 3.1.6 holds, and so, applying Theorem 3.1.6,

$$T_S(x) \leq \int_0^{\Phi_S(x)} \frac{1}{\sqrt{r}\eta_{\bar{x}}}\,dr = \frac{2}{\eta_{\bar{x}}}d_S(x).$$

The last part of the statement comes from Lemma 3.1.4. □

3.2 Comparison with previous results

In order to apply Theorem 3.1.6 we need to prove some estimate on the set $\mathcal{R}_x^{S,\Omega}(t)$ for x sufficiently near to S and t sufficiently small. A natural choice is to assume the existence of a *selection* $(t,x) \mapsto y_x(t)$ of the set-valued map $(t,x) \mapsto \mathcal{R}_x^{S,\Omega}(t)$ satisfying $y_x(0) = x$ along which the distance (squared) from S is strictly decreasing. Further assumptions ensuring that the selection satisfies certain smoothness property are requested in order to estimate decreasing rate of the distance. Finally, if the aforementioned rate is sufficiently high, we obtain STCS.

In the *free* case (i.e., without state constraints), this strategy was employed in several papers, among which we mention [53, 54, 61, 63], under different degrees of generality and geometric assumptions on the target S. The aim of this section is to show that all these results can be seen as consequences of Theorem 3.1.6.

We recall this definition.

Definition 3.2.1 ($\mathscr{A}^\Omega$-trajectory)**.** *Let $\bar{x} \in \mathbb{R}^n$, $T > 0$. We say that a continuous curve $y_{\bar{x}} : [0,T] \to \mathbb{R}^n$ is an* $\mathscr{A}^\Omega$-trajectory *starting from $\bar{x}$ if we have $y_{\bar{x}}(0) = \bar{x}$ and $y_{\bar{x}}(t) \in \mathcal{R}_{\bar{x}}^{S,\Omega}(t)$ for any $t \in [0,T]$ (see also Section 3.1 in [53]). If $\Omega = \mathbb{R}^n$ we will omit it.*

Remark 3.2.2. *The small-time controllability studied in this chapter is sometimes referred to as* small-time local attainability*, see, e.g., [53, 54].*

The following is the main result of [53], and was proved for the first time in Theorem 3.1 of that paper.

Corollary 3.2.3 (Krastanov-Quincampoix)**.** *Let S be a closed subset of $\mathbb{R}^n$. Let α, $s > 0$, r_0 and $T_0 > 0$ be given. Let $\bar{x} \in \partial S$. We assume the following conditions:*

(A1) starting from any x from $\overline{B(\bar{x}, r_0) \setminus S}$, there exists a $\mathscr{A}$-trajectory $x(\cdot)$ such that for every $t \in [0, T_0]$ it holds

$$x(t) = x + a(t;x) + t^\alpha A(x) + o(t^\alpha; x) \in \mathcal{R}_x^S(t); \tag{3.5}$$

(A2) there exist positive constants N and β such that

$$\max_{x \in \overline{B(\bar{x},r_0) \setminus S}} \|o(t^\alpha; x)\| \leq N\ t^{\alpha+\beta};$$

(A3) there exists some Lipschitz continuous negative function $b(\cdot)$ with a Lipschitz constant L_b on $\overline{B(\bar{x}, r_0) \setminus S}$ such that

$$\max_{\substack{x \in \overline{B(\bar{x},r_0) \setminus S} \\ \pi_x \in \pi(x)}} \left\langle \frac{x - \pi_x}{\|x - \pi_x\|}, A(x) \right\rangle \leq b(x) < 0;$$

(A4) *there exists* $L_0 > 0$ *such that for all* $x, y \in \overline{B(\bar{x}, r_0) \setminus S}$

$$\|A(x) - A(y)\| \leq L_0 \|x - y\|;$$

(A5) *there exists a Lipschitz continuous nonnegative function* $c(\cdot)$ *such that*

$$\max_{x \in \overline{B(\bar{x}, r_0) \setminus S}} \|a(t; x)\| \leq t^s c(x), \text{ and } \lim_{\substack{d_S(x) \to 0 \\ x \in \overline{B(\bar{x}, r_0) \setminus S}}} c(x) = 0.$$

Then for every sufficiently small $T > 0$ *there exists a neighborhood* $B(\bar{x}, \theta)$ *of* $\bar{x}$*, with* $\theta > 0$*, such that for every point* $x \in B(\bar{x}, \theta) \setminus S$ *there exists* $t \in [0, T_0]$ *such that* $\mathcal{R}_x^S(t) \cap S \neq \emptyset$*.*

Proof. The proof will follow the argument of Corollary 3.1.8. We set $\delta_{\bar{x}} = r_0$.

Fix $\bar{x} \in \partial S$, and take $M_{\bar{x}} = \max\{\|f(z, v)\| : v \in U, z \in \overline{B(\bar{x}, \delta_{\bar{x}})}\}$, $\ell = 1 + \max_{x \in \overline{B(\bar{x}, \delta_{\bar{x}})}} \|A(x)\|$, $\mu = -\max_{x \in \overline{B(\bar{x}, \delta_{\bar{x}})}} b(x) > 0$.

Take $\Phi_S(\cdot) = d_S^2(\cdot)$, thus, according to the notations of Theorem 3.1.6, we have $K_{\bar{x}} = 2$, $L(r) = 2\sqrt{r}$, $\Phi_{\bar{x}} = \delta_{\bar{x}}^2$, and (2.c) in Theorem 3.1.6 is satisfied.

Define

$$\begin{aligned} \delta(t, r) &:= N t^{\alpha+\beta}, \quad \mu(t, r) := 2\mu\sqrt{r} t^\alpha, \\ \chi(t, r) &:= k t^s \sqrt{r} + \ell t^\alpha, \quad \sigma(t, r) = 2krt^s + 2\chi^2(t, r), \end{aligned}$$

and set as in Theorem 3.1.6 (3)

$$\lambda(t, r) := \mu(t, r) - \sigma(t, r) - (L(r) + K\delta(t, r) + 2K\chi(t, r))\, \delta(t, r).$$

We notice that there exist $C > 0$, independent on r, and $\bar{\rho} > 0$ such that if we define $\tau(r) = (Cr)^{1/2\alpha}$ we have $0 \leq \mu\sqrt{C}r \leq \lambda(\tau(r), r) < r$ for all $0 < r < \bar{\rho}$.

Clearly, (1) and (3) in Theorem 3.1.6 are satisfied by taking $\theta(r) = \tau(r)$ and $\beta(r) = \dfrac{\theta(r)}{\mu\sqrt{C}r}$. Moreover, there exists $C'' > 0$ such that

$$\omega(p) = \int_0^p \beta(r)\, dr = C'' p^{1/2\alpha}.$$

We check now (2.a) and (2.b) in Theorem 3.1.6. Choose $0 < \delta'_{\bar{x}} < \min\{\bar{\rho}, \delta_{\bar{x}}/2\}$ such that $M_{\bar{x}}\omega(\delta'^2_{\bar{x}}) < \delta_{\bar{x}}/2$ and $\omega(\delta'^2_{\bar{x}}) \leq T_0$. This implies that $\mathcal{R}_x^S(t) \subseteq \overline{B(\bar{x}, \delta_{\bar{x}})}$ for all $0 < t \leq \omega(\Phi_S(x))$ and $x \in \overline{B(\bar{x}, \delta'_{\bar{x}})}$. Given $x \in B(\bar{x}, \delta'_{\bar{x}}) \setminus S$, according to $(A1) - (A2)$ we have

$$y_{t,x} := x + a(t; x) + t^\alpha A(x) \in [\mathcal{R}_x^S(t)]_{\delta(t, \Phi_S(x))}.$$

Since the assumption $(A5)$ implies that there exists $k > 0$ such that $|c(x)| \leq k d_S(x)$, where $k > 1$ is a suitable constant greater than the Lipschitz constant of $c(\cdot)$, we have

$$\|y_{t,x} - x\| = \|a(t; x) + t^\alpha A(x)\| \leq t^s |c(x)| + \ell t^\alpha \leq k t^s d_S(x) + \ell t^\alpha,$$

Choosing $\zeta_x \in \partial^P d_S^2(x)$, if $x \notin S$ we have $\zeta_x = 2d_S(x)\xi_x$ with $\xi_x \in \partial^P d_S(x)$, thus

$$\begin{aligned}\langle \zeta_x, y_{t,x} - x\rangle + 2\|y_{t,x} - x\|^2 &\leq 2d_S(x)\langle \xi_x, a(t;x) + t^\alpha A(x)\rangle + 2\|y_{t,x} - x\|^2 \\ &\leq 2d_S(x)\|a(t;x)\| + 2d_S(x)t^\alpha\langle \xi_x, A(x)\rangle + 2\|y_{t,x} - x\|^2 \\ &\leq 2d_S(x)t^s|c(x)| + 2d_S(x)t^\alpha\langle \xi_x, A(x)\rangle + 2\|y_{t,x} - x\|^2 \\ &\leq 2kd_S^2(x)t^s + 2d_S(x)t^\alpha\langle \xi_x, A(x)\rangle + 2(kt^s d_S(x) + \ell t^\alpha)^2.\end{aligned}$$

So, according to $(A3)$ and $(A5)$, we have

$$\begin{aligned}\min_{\zeta_x \in \partial^P d_S(x)} \langle \zeta_x, y_{t,x} - x\rangle + 2\|y_{t,x} - x\|^2 &\leq -2\mu d_S(x)t^\alpha + 2kd_S^2(x)t^s + 2(kt^s d_S(x) + \ell t^\alpha)^2 \\ &= -\mu(t, \Phi_S(x)) + \sigma(t, \Phi_S(x)),\end{aligned}$$

fulfilling (2.b) in Theorem 3.1.6. So by Theorem 3.1.6 we obtain

$$T_S(x) \leq \omega(\Phi_S^2(x)) = C'' d_S^{1/\alpha}(x)$$

for all $x \in B(\bar{x}, \theta)$ with $\theta = \delta'_{\bar{x}}$. □

In our framework we can prove also the following refinement of Corollary 3.2.3, which was proved firstly by Krastanov in Theorem 3.1 of [54] improving also the results of [61], which allowed also the scalar product between the proximal normal ζ_x and the *leading term* of the $\mathscr{A}$-trajectory, to vanish *sufficiently slowly* as $d_S(x) \to 0$, while in (A3) of Corollary 3.2.3 the same quantity was strictly negative and bounded away from zero.

Corollary 3.2.4 (Krastanov). *In the same assumptions of Corollary 3.2.3, assume $\alpha \geq 1$, $\beta > 0$, and $0 < s \leq \alpha$ and define*

$$\sigma_1(\alpha, s, \beta) = \begin{cases} \alpha + \beta, & \text{if } \alpha = s; \\ \dfrac{\alpha}{\alpha - s}, & \text{if } \alpha > s; \end{cases} \qquad \sigma_2(\alpha, \beta) = \begin{cases} \alpha + \beta, & \text{if } \beta \geq \alpha; \\ \dfrac{\alpha}{\alpha - \beta}, & \text{if } \alpha > \beta; \end{cases}$$

$$\sigma_3(\alpha, s, \beta) = \begin{cases} \alpha + \beta, & \text{if } \beta + s \geq \alpha; \\ \dfrac{\alpha}{\alpha - \beta - s}, & \text{if } \beta + s < \alpha. \end{cases}$$

$$\tilde{\sigma}(\alpha, s, \beta) = \min\left\{\sigma_1(\alpha, s, \beta), \sigma_2(\alpha, \beta), \sigma_3(\alpha, s, \beta), 1 + \frac{\min\{s, 1\}}{2\alpha - \min\{s, 1\}}\right\}.$$

Choose $1 \leq \lambda < \tilde{\sigma}(\alpha, s, \beta)$, and replace (A3) with the following condition

(A3') there exists $\delta > 0$ such that

$$\max_{\substack{x\in B(x_0,r_0)\setminus S\\ \pi_x\in\pi(x)}} \langle\, x-\pi_x, A(x)\rangle \le -\delta d_S^\lambda(x) < 0.$$

Then the same conclusion of Corollary 3.2.3 holds true.

Proof. We notice that $1 \le \lambda < 2$. Without loss of generality, we can assume $0 < \delta_{\bar{x}} < 1$. The proof follows exactly the same line of the corresponding proof of Corollary 3.2.3, replacing the constant μ by the function $\tilde{\mu}(r) = \delta r^{\frac{\lambda-1}{2}}$ in the definition of $\mu(t,r)$ and setting $\tau(r) = \theta(r) = Cr^{\lambda/2\alpha}$, where $C > 0$ is a suitable positive constant to be determined. So we have $\mu(\tau(r),r) := 2\delta C^\alpha r^\lambda$, $\delta(\tau(r),r) := NC^{\alpha+\beta} r^{\lambda\frac{\alpha+\beta}{2\alpha}}$, $\chi(t,r) = kC^s r^{\frac{1}{2}+\frac{\lambda s}{2\alpha}} + \ell C^\alpha r^{\frac{\lambda}{2}}$, and

$$\sigma(\tau(r),r) = 2kC^s r^{\frac{\lambda s}{2\alpha}+1} + 2(kC^s r^{\frac{1}{2}+\frac{\lambda s}{2\alpha}} + \ell C^\alpha r^{\frac{\lambda}{2}})^2 = \ell^2 C^{2\alpha} r^\lambda + o(r^\lambda),$$

recalling the choice of λ. We observe that, still by the choice of λ, we have also $r^{\frac{1}{2}+\lambda\frac{\alpha+\beta}{2\alpha}} = o(r^\lambda)$, $\delta^2(\tau(r),r) = o(r^\lambda)$, and $\chi(\tau(r),r)\delta(\tau(r),r) = o(r^\lambda)$. Thus, recalling the definition of $\lambda(t,r)$ given in Theorem 3.1.6 (3), we have

$$\begin{aligned}\lambda(\tau(r),r) &:= \mu(\tau(r),r) - \sigma(\tau(r),r) - (L(r) + K\delta(\tau(r),r) + 2K\chi(\tau(r),r))\,\delta(t,r)\\ &= C^\alpha(2\delta - \ell^2 C^\alpha) r^\lambda + o(r^\lambda),\end{aligned}$$

so there exists $\bar{\rho} > 0$ such that for all $0 < r < \bar{\rho}$ we have $\lambda(\tau(r),r) \le 2C^\alpha\delta r^\lambda$. We choose $C > 0$ such that $2C^\alpha\delta < 1$ and $\ell^2 C^\alpha \le \delta$, thus, recalling that $\lambda \ge 1$ and $0 < r \le 1$, we have $C^\alpha \delta r^\lambda \le \lambda(\tau(r),r) < r^\lambda \le r$, so we can take $\theta(r) = \tau(r)$. We have

$$\frac{\theta(r)}{\lambda(\theta(r),r)} \le \frac{Cr^{\frac{\lambda}{2\alpha}}}{C^\alpha\delta r^\lambda} = \frac{C^{1-\alpha}}{\delta} r^{\lambda(\frac{1}{2\alpha}-1)} =: \beta(r),$$

and so there exists $C' > 0$ such that

$$T_S(x) \le \int_0^{d_S^2(x)} \beta(r)\,dr = C'[d_S(x)]^{2+\frac{\lambda}{\alpha}-2\lambda},$$

and STCS holds. $\square$

In Example 5.21 of [63] is presented a situation where Corollary 3.2.4 cannot be used, since the requirement (A4) of Lipschitz continuity of the function $A(\cdot)$ prevents the choice of λ. This requirement was essential in the proof of Theorem 3.1 in [54]. The first main result of [63], i.e., Theorem 5.10 in [63], used a different argument which do not require that Lipschitz condition, but it works only under the additional assumption of the internal sphere condition of the target S. This was the case of Example 5.21 in [63], to which that result can be applied

In our proof of Corollary 3.2.4 and 3.2.3, based on Theorem 3.1.6, we never used (A4) and we do not impose any internal sphere condition on the target, thus generalizing also Theorem 5.10 of [63].

However, the second main results of [63] which exploits also generalized curvature of the target S – which is supposed sufficiently smooth to have d_S at least of class $C^{1,1}$ in a neighborhood of it – is still not covered by Theorem 3.1.6, as shown by Example 5.22 in [63], where Theorem 5.10 of [63] fails even if the target S is smooth, since to have STCS is essential to exploit also its curvature properties.

3.3 Controllability conditions for control-affine systems

We turn now our attention to *control-affine systems* described in (3.2). For such kind of systems it turns out that it is possible to construct *explicit approximations* of $\mathcal{R}_x^{S,\Omega}(t)$, on which we are going to check the conditions of Theorem 3.1.6.

Our aim is to provide for these system conditions on the data of problem (i.e., on vector fields f_j, on S and on Ω as appear in (3.2)) ensuring the applicability of Theorem 3.1.6 for a *given* system.

The problem can be split in two parts:

1. construct suitable approximated $\mathscr{A}$-trajectories of the systems approaching the target sufficiently fast;

2. among the previous trajectories, select the ones along which it is possible to provide a suitable lower bound of the distance from Ω^c, thus granting the fulfillment of the state constraints.

The first issue is strictly related to the possibility of providing a description at least of some suitable subsets of $\mathcal{R}_x^S(t)$ for any $x \in \mathbb{R}^n$ near to the target and $t > 0$ sufficiently small.

The second issue amounts to provide a quantitative estimate of the variation of the (squared) distance function from Ω^c along the $\mathscr{A}$-trajectories, in a very similar way as it was done with the (squared) distance function from the target S, or, more generally, of $\Phi_S(\cdot)$. While in the latter case we provided an *upper estimate* by means of a *semiconcavity inequality* satisfied by Φ_S, in the first case we will need the *reverse inequality*, i.e., a *semiconvexity inequality* to bound the (squared) distance function from below. Without any extra smoothness hypothesis, the (squared) distance from a set is not semiconvex, thus, while for the upper bound we do not put any smoothness assumption on S, for the lower bound we will need some regularity hypothesis on Ω^c.

Definition 3.3.1 (Characters, alphabet and words)**.** *We consider a nonempty finite set of symbols* $\boldsymbol{X} := \{x_0, \dots, x_N\}$*, called the* alphabet*. The elements of* $\boldsymbol{X}$ *will be called* characters*. A* word *on* $\boldsymbol{X}$ *is any* finite *sequence of characters* $w = x_{i_1} x_{i_2} \dots x_{i_M}$*, where* $i_j \in \{0, \dots, N\}$ *for all* $j \in \{0, \dots, M\}$*. In this case,* $|w| = M$ *is called the* length *of the word. The* empty word *is the unique word of length* 0 *and will be denoted by* Λ*. If* $w = x_{i_1} x_{i_2} \dots x_{i_M} \neq \Lambda$*, we will define* $I = (i_1, \dots, i_M) \in \mathbb{N}^M$ *and write* $w = x_I$*. Given two words* $x_I = x_{i_1} \dots x_{i_M}$ *and* $x_J = x_{j_1} \dots x_{j_H}$*, we define their* concatenation $x_{IJ} = x_I x_J = x_{i_1} \dots x_{i_M} x_{j_1} \dots x_{j_H}$*. We have clearly* $x_I \Lambda = \Lambda x_I = x_I$ *for all words* x_I*. The set* $\Sigma(\boldsymbol{X})$ *of all words on* $\boldsymbol{X}$ *together with the operation of concatenation is a monoid, since this operation is associative (but in general noncommutative) with* Λ *as the identity element. Given* $k \in \mathbb{N}$*, we will denote by* $\Sigma_k(\boldsymbol{X})$ *the subset of* $\Sigma(\boldsymbol{X})$ *made of words of length less or equal than* k*.*

Definition 3.3.2 (Free Lie algebras)**.** *Given an alphabet* $\boldsymbol{X}$ *and the set of words* $\Sigma(\boldsymbol{X})$*, we can consider the free module on* $\mathbb{R}$ *generated by* $\Sigma(\boldsymbol{X})$*, i.e., the set of all formal finite linear combinations of words* $\sum_{h=1}^{P} c_h w_h$*, where* $w_h \in \Sigma(\boldsymbol{X})$ *and* $c_h \in \mathbb{R}$*, with the usual identifications: i.e., if* $c = 1$ *then* $cw = w$ *for all* $w \in \Sigma(\boldsymbol{X})$*, and for all* $w_1, \dots, w_P \in \Sigma(\boldsymbol{X})$*,* $c_1, \dots, c_P \in \mathbb{R}$*, we have* $\sum_{h=1}^{P} c_h w_h = \sum_{\substack{h=1 \\ h \neq j}}^{P} c_h w_h$ *if* $c_j = 0$*. The free module on* $\mathbb{R}$ *generated by* $\Sigma(\boldsymbol{X})$ *together with the operation of concatenation is the free algebra* $A(\boldsymbol{X})$ *generated by* $\Sigma(\boldsymbol{X})$*, namely, the product of two words is given by* $(c_1 x_I)(c_2 x_J) = c_1 c_2 x_{IJ}$ *for all* $c_1, c_2 \in \mathbb{R}$ *and* $x_I, x_J \in \Sigma(\boldsymbol{X})$*. On* $A(\boldsymbol{X}) \times A(\boldsymbol{X})$*, we define the* Lie bracket *(or* commutator*) by setting* $[w, z] = wz - zw \in A(\boldsymbol{X})$ *for every* $w, z \in \Sigma(\boldsymbol{X})$*, where* wz *is the concatenation of* w *and* z*; and then extending it on the whole of* $A(\boldsymbol{X})$ *by linearity. The Lie bracket operation gives to* $A(\boldsymbol{X})$ *the structure of a Lie algebra. Given* $k \in \mathbb{N}$*, we will denote by* $A_k(\boldsymbol{X})$ *the subset of* $A(\boldsymbol{X})$ *made of all finite linear combinations of words in* $\Sigma_k(\boldsymbol{X})$ *with real coefficients.*

Definition 3.3.3 (Chen-Fliess series)**.** *Given the alphabet* $\boldsymbol{X} := \{x_0, \dots, x_N\}$*, consider now the following Cauchy problem in* $A(\boldsymbol{X})$

$$\begin{cases} \dot{S}(t) = S(t) \cdot \left(\displaystyle\sum_{j=0}^{N} u_j(t) x_j \right), \quad t > 0, \\[2ex] S(0) = \Lambda, \end{cases} \tag{3.6}$$

where the maps $u_j \in \mathcal{U}$ *for any* $j = 0, \dots, N$ *and* $\cdot$ *denotes here the concatenation* *operation. Given* $u(\cdot) = (u_0(\cdot), \dots, u_N(\cdot)) \in \mathcal{U}^{N+1}$*,* $t > 0$*, we set*

$$\begin{cases} \Upsilon_\Lambda(t,u) = 1, \\ \Upsilon_{x_j}(t,u) = \displaystyle\int_0^t u_j(s)\,ds, & \textit{for } j = 0,\dots,N, \\ \Upsilon_{wx_j}(t,u) = \displaystyle\int_0^t \Upsilon_w(s,u)u_j(s)\,ds, & \textit{for } w \in \Sigma(\boldsymbol{X}),\ j = 0,\dots,N. \end{cases}$$

This defines by recurrence a map $\Upsilon : \Sigma(\boldsymbol{X}) \times [0,+\infty[\times\mathcal{U}^{N+1} \to \mathbb{R}$*, which can be extended by linearity to a map* $\Upsilon : A(\boldsymbol{X}) \times [0,+\infty[\times\mathcal{U}^{N+1} \to \mathbb{R}$*. With this definition, the explicit solution of* (3.6) *is given by* Chen-Fliess *series*

$$S(t) = \sum_{n\in\mathbb{N}} \sum_{\substack{w\in\Sigma(\boldsymbol{X}) \\ |w|=n}} \Upsilon_w(t,u)w.$$

Remark 3.3.4. *The number of terms appearing in* $\displaystyle\sum_{\substack{w\in\Sigma(\boldsymbol{X}) \\ |w|=n}} \Upsilon_w(t,u)w$ *increases very rapidly with* n*. Since many terms turn out to be repeated or can collected into terms involving possibly nested commutators of lower-length words* w', w''*, this motivates the need of finding* alternative description *for the Chen-Fliess series, exploiting as much as possible the symmetries in the iterated products and factorizing the words* w *appearing in the sum with respect to suitable Lie algebra basis (e.g. Hall-Viennot basis) for which the terms can be computed efficiently.*

We want to link now the above abstract setting to the original control-affine system (3.2). We will give an idea of this connection, referring the reader to [50] for the details.

Definition 3.3.5. *Consider the system* (3.2)*, and let* $\boldsymbol{X} = \{x_0, \dots, x_N\}$*. Assume that* $f_0, \dots, f_N$ *are of class* $C^{k,1}$ *for some* $k \geq 1$*. Define a map* ψ *on* $\Sigma_k(\boldsymbol{X})$ *by setting for all* $j = 0, \dots, N$*,* $w \in \Sigma_{k-1}(\boldsymbol{X})$ *and* $\varphi \in C^\infty(M)$

$$\begin{cases} \psi(\Lambda)\varphi = \varphi, \\ \psi(x_j)\varphi = f_j\varphi, \\ \psi(wx_j)\varphi = \psi(w)(f_j\varphi), \end{cases}$$

where $f_j\varphi$ *is the usual action of the vector field* f_j *on* M *as a differential operator on the function* φ*. By linearity, we can extend* ψ *by linearity on the whole of* $A(\boldsymbol{X})$*.*

Remark 3.3.6. *Fixed a coordinate system around* x_0 *on the* d*-dimensional manifold* M*, and chosen* φ_h *as the* h*-th coordinate function, it has been proved by Sussmann that, when all the vector fields are analytic, the vector-valued series*

$$(\psi(S(t))\varphi)(x_0) := ((\psi(S(t))\varphi_1)(x_0), \dots, (\psi(S(t))\varphi_d)(x_0))$$

converges exactly to the solution of (3.2) *(we set* $u_0 \equiv 1$*). When the vector fields are not analytic, we cannot expect convergence of this series in any sense, not even if they are* C^∞*, however its truncation yields an approximation of the solution.*

When we consider $M = \mathbb{R}^n$, we can take the coordinate functions $\varphi = (\varphi_1, \dots, \varphi_d)$ to be the identity function. In this case, we can identify a differential operator acting on φ with a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. We will use systematically this identification for systems in $\mathbb{R}^n$.

Lemma 3.3.7. *Consider the system* (3.2) *in* $\mathbb{R}^n$*, and let* $\boldsymbol{X} = \{x_0, \dots, x_N\}$*. Assume that* $f_0, \dots, f_N$ *are of class* $C^{k,1}_{\mathrm{loc}}(\mathbb{R}^n)$ *for some* $k \geq 1$*. Consider the* m*-th partial sum with* $0 \leq m \leq k$

$$S_m(t) = \sum_{w \in \Sigma_m(\boldsymbol{X})} \Upsilon_w(t, u) w.$$

Then for each compact neighborhood K *of* x*, there exists* $t_K > 0$ *and* $C_K > 0$ *such that*

$$\|\psi(S(t))(x) - \psi(S_m(t))(x)\| \leq C_K t^{m+1}, \text{ for any } 0 < t < t_K,$$

thus, in particular, we have that

$$\left\{ \sum_{w \in \Sigma_m(\boldsymbol{X})} \Upsilon_w(t, u)\psi(w)(x) : t \in]0, t_K[, u(\cdot) = (1, u_1(\cdot), \dots, u_N(\cdot)) \in \mathcal{U}^{N+1} \right\} \subseteq [\mathcal{R}^S_x(t)]_\delta$$

with $\delta = C_K t^{m+1}$*.*

Proof. The result is a special case in $\mathbb{R}^n$ of equation (2.13) of Section 2.4.4 in [1] obtained by choosing $\varphi = \mathrm{Id}_{\mathbb{R}^n}$ and the identification for any $w \in \Sigma_m(\boldsymbol{X})$ of the differential operator $\psi(w)$ acting on φ with a map $\psi(w)(\cdot)$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. $\square$

Remark 3.3.8. *Of course, in Lemma 3.3.7 we can choose convenient subsets* $\mathscr{C}$ *of* $\mathcal{U}^{N+1}$ *to obtain similar inclusions. For instance, we can restrict ourselves to use only piecewise constant controls in order to simplify the computation of the iterated integrals appearing in* $\Upsilon_w(t, u)$*. This will give a tool to check in practice condition (2.b) of Theorem 3.1.6.*

Lemma 3.3.9. *Fix* $t > 0$*, a partition* $0 = t_0 < \dots < t_N = T$ *of* $[0, t]$*,* $\{\lambda_i\}_{i=1,\dots,N} \in \mathbb{R}$*. Define* $I_i = [t_{i-1}, t_i]$*,* $\ell_i = t_i - t_{i-1}$ *for* $i = 1, \dots, N$ *and* $u(\cdot) = (u_1(\cdot), \dots, u_N(\cdot))$ *with* $u_i(\tau) = \lambda_i \chi_{I_i}(\tau)$ *for all* $\tau \in [0, t]$*,* $i = 1, \dots, N$*. Let* $\boldsymbol{X} = \{x_1, \dots, x_N\}$ *and consider* $w \in \Sigma(\boldsymbol{X}) \setminus \{\Lambda\}$*. Then*

1. *if* $\Upsilon_w(\tau, u) \neq 0$ *with* $w = x_{j_1} \dots x_{j_n}$*, the sequence* $\{j_h\}_{h=1}^n$ *must be nondecreasing;*
2. *if* $w = x_{h_1}^{\alpha_1} x_{h_2}^{\alpha_2} \dots x_{h_m}^{\alpha_m}$*, with* $\alpha_i \in \mathbb{N} \setminus \{0\}$ *and* $h_1 < \dots < h_m$*, then*

$$\Upsilon_w(t, u) = \prod_{h=1}^{m} \frac{(\lambda_{j_h} \ell_{j_h})^{\alpha_h}}{\alpha_h!}.$$

Proof. With the above choice of $u(\cdot)$, given $w = x_{j_1} \dots x_{j_n} \in \Sigma(\boldsymbol{X})$, and according to the definition of $\Upsilon_w(\tau, u)$, we have

$$\Upsilon_w(\tau,u) = \lambda_{j_1} \cdots \lambda_{j_n} \int \dots \int_{0 \le s_n \le \dots s_1 \le t} \chi_{I_{j_1}}(s_n) \dots \chi_{I_{j_n}}(s_1)\, ds_n \dots ds_1$$

$$\frac{d}{d\tau}\Upsilon_{wx_j}(\tau,u) = \lambda_j \chi_{I_j}(\tau)\Upsilon_w(\tau,u), \text{ for all } \tau \in [0,t] \setminus \{t_{j-1}, t_j\}.$$

In particular, $\Upsilon_{wx_j}(\tau,u) = \Upsilon_{wx_j}(t_j,u)$ for all $t_j \le \tau \le t$. Similarly, if $0 \le \tau \le t_{j_n-1}$ we have $\Upsilon_w(\tau,u) = \Upsilon_w(0,u) = 0$.

1. consider $\Upsilon_{vx_ix_j}(\tau,u)$ with a given $v \in \Sigma(\boldsymbol{X})$. If $j < i$, $\Upsilon_{vx_i}(\tau,u) = 0$ for $\tau \in]t_{j-1}, t_j[$ since $t_j \le t_{i-1}$. Then

$$\frac{d}{d\tau}\Upsilon_{vx_ix_j}(\tau,u) = \lambda_j \chi_{I_j}(\tau)\Upsilon_{vx_i}(\tau,u) = 0, \qquad \text{for all } \tau \in]t_{j-1}, t_j[.$$

   Due to absolute continuity of $\Upsilon_{vx_ix_j}(\cdot,u)$, $\Upsilon_{vx_ix_j}(\cdot,u)$ is constant in $]t_{j-1},t_j[$ and hence on the whole of $[0,t]$. Thus $\Upsilon_{vx_ix_j}(\tau,u) = \Upsilon_{vx_ix_j}(0,u) = 0$. By induction, this implies that if $j < i$ we have $\Upsilon_{vx_ix_jv'}(\tau,u) = 0$ for all $\tau \in [0,t]$ and $v' \in \Sigma(\boldsymbol{X})$. Therefore if $\Upsilon_w(\tau,u) \ne 0$, the sequence $\{j_h\}_{h=1}^n$ must be nondecreasing.

2. assume now $w = x_{h_1}^{\alpha_1} x_{h_2}^{\alpha_2} \dots x_{h_m}^{\alpha_m}$, with $\alpha_i \in \mathbb{N} \setminus \{0\}$ and $0 < h_1 < \dots < h_m$. Recall $\Upsilon_w(\tau,u) = 0$ for $0 \le \tau \le t_{h_m-1}$ and $\Upsilon_w(\tau,u) = \Upsilon_w(t_{h_m},u)$ for $t_{h_m} \le \tau \le t$. Given $\tau \in]t_{h_m-1}, t_{h_m}[$, we have for all $0 < \alpha \le \alpha_m$.

$$\frac{d^\alpha}{d\tau^\alpha}\Upsilon_w(\tau,u) = \lambda_j^\alpha \Upsilon_{w'x_{h_m}^{\alpha_m-\alpha}}(\tau,u),$$

   where $w' = x_{h_1}^{\alpha_1} x_{h_2}^{\alpha_2} \dots x_{h_{m-1}}^{\alpha_{m-1}}$. In particular, recalling the smoothness of $\Upsilon_w(\cdot,u)$, we have

$$\lim_{\tau \to t_{h_{m-1}}^+} \frac{d^\alpha}{d\tau^\alpha}\Upsilon_w(\tau,u) = \begin{cases} 0, \text{ for } 0 < \alpha < \alpha_m, \\ \lambda_j^{\alpha_m}\Upsilon_{w'}(t_{h_{m-1}},u), \text{ for } \alpha = \alpha_m, \\ 0, \text{ for } \alpha > \alpha_m. \end{cases}$$

   The third case is obtained since $\tau \in]t_{h_m-1}, t_{h_m}[$, which has an empty intersection with $I_{h_{m-1}}$ by the assumption $h_m > h_{m-1}$, and $\Upsilon_{w'}(\tau,u)$ is constant if $\tau \notin I_{h_{m-1}}$. This implies

$$\Upsilon_w(\tau,u) = \begin{cases} 0, & \text{if } 0 \le \tau \le t_{j_m-1}, \\ \dfrac{(\lambda(\tau - t_{j_m-1}))^{\alpha_m}}{\alpha_m!}\Upsilon_{w'}(\tau,u), & \text{if } t_{j_m-1} \le \tau \le t_{j_m}, \\ \dfrac{(\lambda_{j_m}\ell_{j_m})^{\alpha_m}}{\alpha_m!}\Upsilon_{w'}(\tau,u), & \text{if } t_{j_m} \le \tau \le t. \end{cases}$$

   The assertion now follows by repeating the argument on w' and choosing $\tau = t$.

□

Example 3.3.10. *Consider the system in* $\mathbb{R}^n$

$$\dot{x}(t) = u_0(t)f_0(x(t)) + u_1(t)f_1(x(t)) + u_2(t)f_2(x(t)). \tag{3.7}$$

Define

$$\dot{x}(t) = \bar{u}_1(t)g_1(x(t)) + \bar{u}_2(t)g_2(x(t)) + \bar{u}_3(t)g_3(x(t)) + \bar{u}_4(t)g_4(x(t)), \tag{3.8}$$

where $g_1(x) = f_0(x) + f_1(x)$, $g_2(x) = f_0(x) + f_2(x)$, $g_3(x) = f_0(x) - f_1(x)$, $g_4(x) = f_0(x) - f_2(x)$. *Consider the partition* $\{t_0 = 0, t_1 = t/4, t_2 = t/2, t_3 = 3t/4, t_4 = t\}$, *set* $\lambda_i = 1$, $\ell_i = t/4$, *and define* $\bar{u}_i(s) = \chi_{[t_{i-1},t_i]}(s)$, *for all* $i = 1, \ldots, 4$, $s \in [0, t]$. *We consider the alphabet* $\boldsymbol{X} = \{x_1, x_2, x_3, x_4\}$. *In this case,*

$$\begin{aligned}\Sigma_2(\boldsymbol{X}) = \{&\Lambda, x_1, x_2, x_3, x_4, x_1x_1, x_1x_2, x_1x_3, x_1x_4, x_2x_1, x_2x_2, x_2x_3, x_2x_4, x_3x_1,\\ &x_3x_2, x_3x_3, x_3x_4, x_4x_1, x_4x_2, x_4x_3, x_4x_4\}.\end{aligned}$$

According to Lemma 3.3.9, the contribution to the Chen-Fliess series comes only from those words whose sequence of letters is nondecreasing, i.e.

$$\{\Lambda, x_1, x_2, x_3, x_4, x_1x_1, x_1x_2, x_2x_2, x_1x_3, x_2x_3, x_3x_3, x_1x_4, x_2x_4, x_3x_4, x_4x_4\}.$$

Thus in this case we have

$$\begin{aligned}\psi \circ S(t)(x) =& x + \frac{t}{4}(g_1 + g_2 + g_3 + g_4)(x) + \frac{t^2}{16}(g_1g_2 + g_1g_3 + g_1g_4 + g_2g_3 + g_2g_4 +\\ &+ g_3g_4)(x) + \frac{t^2}{32}(g_1g_1 + g_2g_2 + g_3g_3 + g_4g_4)(x) + o(t^2).\end{aligned}$$

Notice that for each choice of $\{\bar{u}_i\}$, *there exists a corresponding one of* $\{u_i\}$ *such that the right hand side of* (3.7) *and* (3.8) *coincide. Thus if we substitute the expression of* g_i, *we obtain again*

$$\psi \circ S(t)(x) = x + tf_0(x) + \frac{t^2}{16}\left(8f_0f_0 + 2[f_1 + f_2, f_0] + [f_1, f_2]\right)(x) + o(t^2),$$

as before. The drifless case can be deduced from the above formula by taking $f_0 = 0$ *or noticing that all the words containing* x_0 *give no contribution, since the coefficient* $\Upsilon_w(t, u)$ *vanishes because of the presence of* $u_0(s) \equiv 0$. *Thus we have the well-known formula for the commutator of the flows of* f_1 *and* f_2, *i.e.,*

$$\psi \circ S(t)(x) = x + \frac{t^2}{16}[f_1, f_2](x) + o(t^2).$$

Motivated by the above examples, the following result allows us to construct the desired approximation of $\mathcal{R}^S_x(t)$ using Lemma 3.3.9.

Lemma 3.3.11. *Consider the system (3.2) in $\mathbb{R}^n$ with $f_0, f_i \in C^{k,1}_{\mathrm{loc}}(\mathbb{R}^n)$, $i = 1, \dots, N$, $0 < m \le k$.*

Let $M > 0$, $\boldsymbol{X} := \{x_1, \dots, x_M\}$, $\sigma = (\sigma_1, \dots, \sigma_M) \in \{1, \dots, N\}^M$, $\ell = (\ell_1, \dots, \ell_M) \in]0, 1]^M$ such that $\sum_{j=1}^M \ell_j = 1$, $\lambda = (\lambda_1, \dots, \lambda_M) \in [-1, 1]^M$. Define $g = (g_1, \dots, g_M)$ by setting $g_i = f_0 + \lambda_i f_{\sigma_i}$, $i = 1, \dots, M$.

Given a word $v \in \Sigma(\boldsymbol{X})$, define

$$\tilde{\psi}_v(t) = \begin{cases} t^{|\alpha|}\dfrac{(\ell_i)^\alpha}{\alpha!}(g_i)^\alpha, & \text{if } i_1 < i_2 < \dots < i_h, \\ \mathrm{Id}_{\mathbb{R}^n}, & \text{if } v = \Lambda, \\ 0, & \text{otherwise}, \end{cases}$$

where if $v \neq \Lambda$ we wrote it in a unique way as $v = x_{i_1}^{\alpha_1} \dots x_{i_h}^{\alpha_h}$ with $i_j \in \{1, \dots, M\}$, $\alpha_j \in \mathbb{N} \setminus \{0\}$ and $i_j \neq i_{j+1}$ for all $j = 1, \dots, h-1$, and we have denoted $(\ell_i)^\alpha = {\ell_{i_1}}^{\alpha_1} \dots {\ell_{i_h}}^{\alpha_h}$, and $(g_i)^\alpha = g_{i_1}^{\alpha_1} \cdot \dots \cdot g_{i_h}^{\alpha_h}$.

Then, if we set

$$P_x^{m,M,\ell,\lambda,\sigma}(t) := x + \sum_{w \in \Sigma_m \setminus \{\Lambda\}} \tilde{\psi}_w(t)(x),$$

for any compact neighborhood K of x there exists $t_K > 0$ and $C_K > 0$ such that $x + P_x^{M,\ell,\lambda,\sigma}(t) \in [\mathcal{R}_x^S(t)]_{C_K t^{m+1}}$ for all $0 < t < t_K$.

Proof. Indeed, for each $t > 0$ we consider the partition $t_0 = 0 < t_1 < \dots < t_M = t$, where $t_i - t_{i-1} = \ell_i t$. We set $I_i = [t_{i-1}, t_i]$, and define $u_{\sigma_i}(s) = \lambda_i \chi_{I_i}(s)$ for $i = 1, \dots, M$. This is equivalent to consider the system $\dot{x}(s) = \sum_{i=1}^M \mu_i(s) g_i(x(s))$, where $\mu_{\sigma_i}(s) = \chi_{I_i}(s)$ for $i = 1, \dots, M$. According to Lemma 3.3.9, Chen-Fliess series in this case truncated at order m is $P_x^{m,M,\ell,\lambda,\sigma}(t)$ and, together with the error estimate of Lemma 3.3.7, this concludes the proof. □

Remark 3.3.12. *$x + P_x^{m,M,\ell,\lambda,\sigma}(t)$ is the approximation at m-th order of the point reached at time t by an admissible trajectory using piecewise constant controls activating at each time only one controlled vector field and with total amount of switchings equal to M. Some variants are possible, for example we may consider λ, σ are $M \times N$ constant matrices with $\lambda_{ij} \in [-1, 1]$, $\sigma_{ij} \in \{1, ..., N\}$, $i = 1, ..., M$, $j = 1, ..., N$ and define accordingly $g_i = f + \sum_{j=1}^N \lambda_{ij} f_{\sigma_{ij}}$, $i = 1, \dots, M$, keeping inalterate the definitions of $\tilde{\psi}_v$ and of $P_x^{m,M,\ell,\lambda,\sigma}(t)$. The result still holds exactly with the same proof, and in this way we drop the restriction to use only one controlled vector field at each time.*

Definition 3.3.13. *Consider the system (3.2) in $\mathbb{R}^n$ with $f_0, f_i \in C^{k,1}_{\mathrm{loc}}(\mathbb{R}^n)$, $i = 1, \dots, N$, $0 < m \le k$. Define*

$$\mathscr{T}_x^m := \left\{ P_x^{m,M,\ell,\lambda,\sigma}(\cdot) : M > 0, \sigma \in \{1, \dots, N\}^M, \ell \in]0, 1]^M, \sum_{j=1}^M \ell_j = 1, \lambda \in [-1, 1]^M \right\}.$$

According to Lemma 3.3.11, given $P_x(\cdot) \in \mathscr{T}_x^m$, for any compact neighborhood K of x there exists $t_K > 0$ and $C_K > 0$ such that $x + P_x(t) \in [\mathcal{R}_x^S(t)]_{C_K t^{m+1}}$ for all $0 < t < t_K$.

Now we state the second main result of the paper, concerning sufficient conditions for STCS in the control-affine case (3.2).

Theorem 3.3.14 (Local STCS for constrained control-affine systems)**.** *Consider the system* (3.2) *with* $f_0, f_1, \dots, f_N \in C^{k,1}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$. *Fix* $\bar{x} \in \partial S \cap \overline{\Omega}$, $\delta_{\bar{x}} > 0$, *and assume that*

$$S \cap \overline{B(\bar{x},\delta_{\bar{x}})} \cap \overline{\Omega} := \{x:\ \Phi_S(x) \le 0\},$$
$$\overline{B(\bar{x},\delta_{\bar{x}})} \cap \Omega^c := \{x:\ \Psi_{\Omega^c}(x) \le 0\},$$

for suitable locally Lipschitz functions $\Phi_S, \Psi_{\Omega^c} : \mathbb{R}^n \to \mathbb{R}$. *Let* $C_{\bar{x}} > 0$, $\tau_{\bar{x}} > 0$ *be the constants appearing in Definition 3.3.13 by taking* $\overline{B(\bar{x},\delta_{\bar{x}})}$ *as a compact neighborhood of* $\bar{x}$.

Assume that $\Phi_S(\cdot)$ *is semiconcave on* $\overline{B(\bar{x},\delta_{\bar{x}})}$ *with semiconcavity constant* $K_{\bar{x}} > 0$. *Define* $\Phi_{\bar{x}}$, $L(\cdot) > 0$, $M_{\bar{x}}$, $K = K_{\bar{x}}$, $\sigma, \mu, \chi, \tau, \theta, \beta$ *as in Theorem 3.1.6. If* $\bar{x} \in \partial S \cap \partial\Omega$, *suppose that* Ψ_{Ω^c} *is semiconvex on* $\overline{B(\bar{x},\delta_{\bar{x}})}$.

Let $\varepsilon :]0,+\infty[\times]0,+\infty[\to \mathbb{R}$ *be a continuous function such that* $\displaystyle\lim_{t\to 0^+} \frac{\varepsilon(t,r)}{t^2} = +\infty$, *uniformly w.r.t.* $r \in \left]0, \displaystyle\max_{z\in\overline{B(\bar{x},\delta_{\bar{x}})}} \{\Psi_{\Omega^c}(z)\}\right]$.

Assume that for every $x \in \overline{\Omega} \cap B(\bar{x},\delta_{\bar{x}}) \setminus S$ *there exist* $0 < k_x \le k$, $P_x(\cdot) \in \mathscr{T}^{k_x}_x$, *and* $\zeta_x \in \partial^P \Phi_S(x)$, $\theta_x \in \partial_P \Psi_{\Omega^c}$ *satisfying for all* $0 \le t \le \tau(\Phi_S(x))$:

(App) *approaching condition:*

$$\langle \zeta_x, P_x(t)\rangle + K\|P_x(t)\|^2 \le -\mu(t, d_S(x)) + \sigma(t, d_S(x)),$$

(Con) *constraint condition: if* $\bar{x} \in \partial\Omega$ *we require also that*

$$\langle \theta_x, P_x(t)\rangle > \varepsilon(t, \Psi_{\Omega^c}(x)),\ \textit{for all } x \in \Omega \cap \overline{B(\bar{x},\delta_{\bar{x}})} \setminus S.$$

Moreover, set $\delta(t,r) = C_{\bar{x}} t^k$ *and suppose that (1), (3) in Theorem 3.1.6 are satisfied. Then there exists* $0 < \delta''_{\bar{x}} < \frac{\delta_{\bar{x}}}{2}$ *and a continuous increasing function* $\omega_{\bar{x}} : [0,+\infty[\to [0,+\infty[$ *such that* $\omega_{\bar{x}}(0) = 0$ *and* $T_{S,\Omega}(x) \le \omega_{\bar{x}}(\Phi_S(x))$ *for every* $x \in B(\bar{x},\delta''_{\bar{x}}) \cap \overline{\Omega}$.

Before proving the theorem, we make some comments on the assumptions.

1. We are considering the approximation of $\mathcal{R}^S_x(t)$ provided by all the truncation of Chen-Fliess series obtained by using piecewise constant controls, this gives a family of $\mathscr{A}$-trajectories, among which we assume to be able to apply Theorem 3.1.6, ignoring for the moment any state constraint.

2. The semiconvexity assumption on Ψ_{Ω^c} together with Assumption **(Con)** yields a quantitative estimate of the variation of $\Psi_c(\cdot)$ along the $\mathscr{A}$-trajectory.

3. The assumptions on $\varepsilon(\cdot)$ will prevent the vanishing of $\Psi_c(\cdot)$, thus implying that the unconstrained $\mathscr{A}$-trajectory will satisfy also the state constraint, and so that it is actually an $\mathscr{A}^\Omega$-trajectory.

Proof. Without loss of generality, we may assume that if $\bar{x} \in \partial S \cap \Omega$ we have $\overline{B(\bar{x}, \delta_{\bar{x}})} \cap \Omega^c = \emptyset$ and $\displaystyle\int_0^{\Phi_{\bar{x}}} \beta(r)\,dr < 1$. As in the proof of Theorem 3.1.6, we can choose $0 < \delta'_{\bar{x}} < \dfrac{\delta_{\bar{x}}}{2}$ such that $\mathcal{R}^S_x(t) \subseteq \overline{B(\bar{x}, \delta_{\bar{x}})}$ for all $x \in B(\bar{x}, \delta'_{\bar{x}})$ and $0 < t \le \displaystyle\max_{x \in \overline{B(\bar{x}, \delta'_{\bar{x}}) \setminus S}} \int_0^{\Phi_S(x)} \beta(r)\,dr$, where $\beta(\cdot) \in L^1$ is a function as in Theorem 3.1.6.

Given $P_x(\cdot) \in \mathscr{T}^{k_x}_x$ as in the statement, we can find $P'_x(\cdot) \in \mathscr{T}^k_x$ such that $\|P'_x(t) - P_x(t)\| = o(t^{k_x})$. We set $y_{t,x} = x + P'_x(t)$. Then we apply Theorem 3.1.6 ignoring the state constaint to obtain the upper bound $T_S(x) \le \displaystyle\int_0^{\Phi_S(x)} \beta(r)\,dr$.

If $\bar{x} \in \partial S \cap \Omega$ the proof is concluded, recalling that in this case $\mathcal{R}^S_x(t) = \mathcal{R}^{S,\Omega}_x(t)$ for all $0 \le t \le T_S(x)$, and so $T_S(x) = T_{S,\Omega}(x)$.

Assume that $\bar{x} \in \partial\Omega \cap \partial S$, and take $x \in B(\bar{x}, \delta'_{\bar{x}}) \cap \Omega \setminus S$. Let $C,\ \tau_C > 0$ and $y_x(\cdot)$ be an admissible trajectory for the unconstrained system such that $\|y_x(t) - (x + P_x(t))\| \le Ct^2$ for $0 < t \le \tau_C$. By taking $0 < \delta''_{\bar{x}} < \delta'_{\bar{x}}$ such that $\displaystyle\int_0^{\Phi_S(x)} \beta(r)\,dr < \tau_C$ for all $x \in \overline{B(\bar{x}, \delta''_{\bar{x}})} \cap \Omega$, we will show that $y_x(t) \in \overline{\Omega}$ for all $0 \le t \le T_S(x)$.

We denote by $\Psi_{\bar{x}}$ the semiconvexity constant of Ψ_{Ω^c} on $\overline{B(\bar{x}, \delta_{\bar{x}})}$. For $0 < t < T_S(x)$, there exists $\theta_x \in \partial_P \Psi_{\Omega^c}(x)$ such that

$$\begin{aligned}\Psi_{\Omega^c}(y_x(t)) &\ge \Psi_{\Omega^c}(x) + \langle \theta_x, y_x(t) - x\rangle - \Psi_{\bar{x}}\|y_x(t) - x\|^2 \\ &\ge \Psi_{\Omega^c}(x) + \langle \theta_x, P_x(t)\rangle - \|\theta_x\| Ct^2 - \Psi_{\bar{x}}(\|P_x(t)\|^2 + 2Ct^2\|P_x(t)\| + C^2t^4).\end{aligned}$$

Since $\Psi_{\Omega^c}(\cdot)$ is locally Lipschitz continuous, we have that $\|\theta_{\bar{x}}\|$ is uniformly bounded in $\overline{B(\bar{x}, \delta''_{\bar{x}})}$, furthermore, by the smoothness of the vector fields, we have that $\dfrac{\|P_x(t)\|}{t}$ is uniformly bounded for all $x \in \overline{B(\bar{x}, \delta''_{\bar{x}})}$ and $t \in]0, T_S(x)]$. In particular, there exists $D > 0$ and $\tau_D > 0$ such that for all $0 < t < \tau_D$ we have

$$\Psi_{\Omega^c}(y_x(t)) \ge \Psi_{\Omega^c}(x) + \varepsilon(t, \Psi_{\Omega^c}(x)) - Dt^2 \ge \Psi_{\Omega^c}(x) > 0,$$

for all $x \in B(\bar{x}, \delta''_{\bar{x}})$ due to the assumptions of $\varepsilon(\cdot)$. Thus we take $0 < \delta''_{\bar{x}} < \delta'_{\bar{x}}$ such that $\displaystyle\int_0^{\Phi_S(x)} \beta(r)\,dr < \min\{\tau_C,\ \tau_D\}$ for all $x \in \overline{B(\bar{x}, \delta''_{\bar{x}})} \cap \overline{\Omega}$.

This implies that x_i and x_{i+1}, constructed as in the proof of Theorem 3.1.6 for the unconstrained system starting from $x \in \overline{B(\bar{x}, \delta''_{\bar{x}})} \cap \Omega$, are actually connected by an admissible trajectory also for the constrained system for every $i \in \mathbb{N}$, since Ψ_{Ω^c} is nondecreasing, and so in particular it remains strictly positive. Thus also in this case we have

$$T_{S,\Omega}(x) \le \int_0^{\Phi_S(x)} \beta(r)\,dr,$$

for all $x \in \overline{B(\bar{x}, \delta''_{\bar{x}})} \cap \Omega$.

Finally, assume that $\bar{x} \in \partial\Omega \cap \partial S$, and take $x \in B(\bar{x}, \delta''_{\bar{x}}) \cap \partial\Omega \setminus S$. We can find a sequence of points $\{z_i\}_{i\in\mathbb{N}} \subseteq B(\bar{x}, \delta''_{\bar{x}}) \cap \Omega \setminus S$, a sequence of admissible trajectories $\{y_{z_i}(\cdot)\}_{i\in\mathbb{N}}$ and a sequence of times$\{T_i\}_{i\in\mathbb{N}}$, such that $y_{z_i}(0) = z_i$, $y_{z_i}(t) \in \Omega \cap \overline{B(\bar{x}, \delta_{\bar{x}})}$ for all $0 \leq t \leq T_i$, $y_{z_i}(T_i) \in S$, and $T_i \leq \displaystyle\int_0^{\Phi_\Omega(z_i)} \beta(s)\,ds$.

It is well known (see e.g. Theorem 1.11 in Chapter 4 of [24]) that up to passing to a subsequence, we have that $\{y_{z_i}(\cdot)\}_{i\in\mathbb{N}}$ uniformly converges to an admissible trajectory $y_x(\cdot)$ satisfying $y_x(0) = x$, and $T_i \to T_\infty$. Since the constraint and the target set are closed, we have also $y_x(t) \in \overline{\Omega} \cap \overline{B(\bar{x}, \delta''_{\bar{x}})}$ for all $0 \leq t \leq T_\infty$ and $y_x(T_\infty) \in S$. Thus

$$T_{S,\Omega}(x) \leq T_\infty \leq \lim_{i\to\infty} \int_0^{\Phi_S(z_i)} \beta(r)\,dr = \int_0^{\Phi_S(x)} \beta(r)\,dr.$$

□

Applying Lemma 3.1.4, we can give a global STCS estimate.

Corollary 3.3.15 (Global STCS for constrained control-affine systems)**.** *Consider the system* (3.2)*. Assume that at every $\bar{x} \in \partial S \cap \overline{\Omega}$ the assumptions of Theorem 3.3.14 are satisfied, then STCS holds. Moreover, if $\partial S \cap \overline{\Omega}$ is compact we have that there exists $\delta_S > 0$ and a continuous function $\omega : [0, +\infty[\to [0, +\infty[$ such that $\omega(0) = 0$ and $T_{S,\Omega}(x) \leq \omega(d_S(x))$ for every $x \in S_{\delta_S} \cap \overline{\Omega}$.*

Proof. It is a straightforward application of Lemma 3.1.4. □

As already said, if we want to take $\Psi_{\Omega^c} = d_{\Omega^c}$, we have to assume some smoothness property on the constraint $\overline{\Omega}$. To this end we give the following remarks.

Remark 3.3.16. Given $r \geq 0$, we have $0 \leq r < \operatorname{reach}(K, a)$ if and only if every $y \in B(a, r)$ admits an unique closest element in K. For instance, if K is closed and convex it is known that every point of $\mathbb{R}^n$ has an unique projection on K, hence in this case r can be chosen arbitrary large, so $\operatorname{reach}(K, a) = \infty$ for all $a \in K$, thus $\operatorname{reach}(K) = +\infty$ (the class of sets with positive reach contains all convex sets). More generally, even if K is smooth, it may no longer possible to choose arbitrary large r. If $K := \{(x, y) : y \leq x^2\}$, we have that the point $(0, z)$ projects uniquely on $(0, 0)$ if and only if $z < 1/2$, so $\operatorname{reach}(K, (0, 0)) \leq 1/2$, (actually it can be proved that equality holds). On the other hand, if $K = \{(x, y) : y \leq |x|\}$, we have that the points $z_n := (0, y)$ have two projections on K for every $y > 0$, so $\operatorname{reach}(K, (0, 0)) = 0$. Positive reach property at $a \in K$ is strictly related to local smoothness of the distance function in $B(a, r) \setminus K$, as shown in [39].

Remark 3.3.17 (Comparison of the inward pointing condition of [22] and ours)**.** In Section 2 of [22], given a closed subset $S \subseteq \mathbb{R}^n$ and $x \in S$ the following definition of *tangent cone* is given:

$$\operatorname{Tan}_S(x) := \left\{ v \in \mathbb{R}^n : \limsup_{\substack{y\to x\\ t\to 0^+}} \frac{d_S(y + tv) - d_S(y)}{t} = 0 \right\}.$$

According to Remark 3.2 of [22], assumption (3.1) of [22] implies that $\mathrm{int}(\mathrm{Tan}_{\overline{\Omega}}(x)) \neq \emptyset$ at all $x \in \partial\Omega$. The classes of wedged sets and of sets whose complement has locally positive reach are distinct (even if smooth $C^{1,1}$ sets belong to both of them). In particular, if in $\mathbb{R}^2$ we take $\Omega_1 =]-1,1[\times]-1,1[$, we have that Ω_1^c can not have positive reach at $(\pm 1, \pm 1)$ because the sequence of points $\left\{\left(1-\dfrac{1}{n}, 1-\dfrac{1}{n}\right)\right\}_{n\in\mathbb{N}} \subseteq \Omega_1$ converges to $(1,1)$ but each element of the sequence has two projections on Ω_1^c. The other vertices can be treated similarly. However, the set $\overline{\Omega}_1$ is convex, so by [24] we have that $\mathrm{Tan}_{\overline{\Omega}_1}(x) = \{v : \langle v, w\rangle \leq 0 \text{ for all } w \in N^P_{\overline{\Omega}_1}(x)\}$. We deduce that $\mathrm{Tan}_{\overline{\Omega}_1}(x)$ coincides with an half space at every point $x \in \partial\overline{\Omega}_1$ different from the vertices, and with the intersection of two half spaces at the vertices with nonparallel boundaries. In both cases, it has nonempty interior, so $\overline{\Omega}_1$ is wedged. Conversely, if we take $f(x) = \sqrt{|x|}$ and $\Omega_2 = \mathbb{R}^2 \setminus \mathrm{epi} f$, we have that $\Omega_2^c = \mathrm{epi}\, f$ has positive reach, while $\overline{\Omega}_2 = \mathrm{hypo}\, f$ is not wedged, since $\partial\Omega_2$ is not Lipschitz continuous. We conclude that the class of admissible constraints of our work and [22] are different, thus the results are not directly comparable. For other relevant properties of tangents and normal to sets with positive reach, we refer the reader to [39] or [28].

3.4 An example

In this section we present an example illustrating our approach.

Example 3.4.1. In $\mathbb{R}^3$ we consider the control-affine system (3.2) with $N = 2$, and set

$$\begin{aligned}
S &:= \overline{B(0,1/2)},\\
f_0(x_1,x_2,x_3) &= \frac{1}{8}(-x_2, x_1, 0),\\
f_1(x_1,x_2,x_3) &= (x_1x_3, x_2x_3, 0),\\
f_2(x_1,x_2,x_3) &= (0,0,1),\\
\Omega &:= \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 > \frac{1}{16}\right\}.
\end{aligned}$$

We take $\Phi_S(x) = \|x\| - 1/2$. This map agrees with $d_S(\cdot)$ on $\overline{\mathbb{R}^n \setminus S}$ and is smooth on $\mathbb{R}^n \setminus \{0\}$. We have $\nabla\Phi_S(x) = \dfrac{x}{\|x\|}$ for all $x \neq 0$. Moreover, $\partial^P\Phi_S(x) = \left\{\dfrac{x}{\|x\|}\right\}$ for all $x \neq 0$, and $\Phi_S(\cdot)$ is semiconcave of constant $K = K_{\bar{x}} = 2$ in every ball $\overline{B(\bar{x}, \delta_{\bar{x}})}$ with $\bar{x} \in \partial S$ and $\delta_{\bar{x}} \leq 1/4$ according to Proposition 2.2.8. Finally, we have $L(r) = 1$ by 1-Lipschitz continuity of $\Phi_S(\cdot)$. Notice that the constraint has positive reach, thus we take $\Psi_{\Omega^c}(x) = d_{\Omega^c}(x) = \sqrt{x_1^2 + x_2^2} - \frac{1}{4}$. We consider now the unconstrained problem. We notice that at every point $(x_1,x_2,x_3) \in \partial S$ we have

$$\langle \nabla\Phi_S(x), (f_0(x) + u_1 f_1(x) + u_2 f_2(x)\rangle = 2x_3((x_1^2 + x_2^2)u_1 + u_2) \tag{3.9}$$

For any $\bar{x} \in D := \partial S \cap \Omega \setminus \{x_3 = 0\}$, there exists $\delta_{\bar{x}} > 0$ such that Petrov's condition for any $x \in B(\bar{x}, \delta_{\bar{x}})$: indeed, by choosing $u_2(s) = -\text{sign}(x_3)$, $-1 \leq u_1 \leq 1$, we have that 3.9 is continuous and strictly negative on D thus every point of D possesses a neighborhood where the above expression remains bounded away from 0. Given $t > 0$, we consider the following choice of controls:

$$u_1(s) = \begin{cases} 1, & \text{if } 8s/t \in [0,1] \cup [6,7], \\ -1, & \text{if } 8s/t \in [2,3] \cup [4,5], \\ 0, & \text{elsewhere}, \end{cases} \qquad u_2(s) = u_1(s - t/8).$$

Given $x \in \mathbb{R}^3$, in this case Chen-Fliess series yields an $\mathscr{A}$-trajectory of the form

$$\tilde{y}_x(t) = x + tf_0(x) + \frac{t^2}{32}(16f_0f_0 + [f_1, f_2])(x) + o(t^2)$$

and, by the smoothness of the vector fields, there exists $L > 0$ such that $\|o(t^2)\| \leq Lt^3$ for every $x \in B(0,1) \supset S$. Set $P_x(t) = tf_0(x) + \frac{t^2}{32}(16f_0f_0 + [f_1, f_2])(x)$. We notice that, by the smoothness of the vector fields, the map $x \mapsto P_x(t)$ is continuous. Given $x = (x_1, x_2, 0) \in \partial S$, we have

$$\langle \nabla\Phi_S(x), P_x(t)\rangle + 2\|P_x(t)\|^2 = \frac{t^2(-128 + 64(x_1 + x_2))}{16384} + o(t^2) \leq \frac{-t^2}{256} + o(t^2).$$

In particular, there exist $\tau, C > 0$ such that for $0 \leq t \leq \tau$ and every $x = (x_1, x_2, 0) \in \partial S$

$$\langle \nabla\Phi_S((x), P_x(t)\rangle + 2\|P_x(t)\|^2 < -2Ct^2.$$

Thus every point $\bar{x} \in \partial S \cap \{x_3 = 0\}$ possesses a neighborhood $V_{\bar{x}}$ such that

$$\langle \nabla\Phi_S((x), P_x(t)\rangle + 2\|P_x(t)\|^2 < -Ct^2.$$

Thus we can define $k_x = 2$, $\mu(t, r) = Ct^2$, $\delta(t, r) = Lt^3$, $\sigma(t, r) = 0$, $\chi(t, r) = 1$, $\tau(r) = \theta(r) = \min\left\{\tau, \frac{1}{L}, \frac{1}{2C}\sqrt{r}\right\}$. Condition **(App)** thus holds at these points, and by Theorem 3.3.14 and Corollary 3.3.15, we obtain that there exists $C' > 0$ such that $T_S(x) \leq C' d_S^{1/2}(x)$ on a suitable neighborhood of S. Now we pass to consider the constraints. Since $\partial S \cap \overline{\Omega} \cap \{x_3 = 0\} = \emptyset$, for any $\bar{x} \in \partial S \cap \overline{\Omega}$ there exists $\delta_{\bar{x}} > 0$ such that Petrov's condition holds at every $x \in \overline{B(\bar{x}, \delta_{\bar{x}})} \setminus S$ by taking $u_2 = -\text{sign}(x_3)$ and $-1 \leq u_1 \leq 1$. Moreover, for all $x \in \Omega \cap \overline{B(\bar{x}, \delta_{\bar{x}})} \setminus S$ we have

$$\begin{aligned}\langle \nabla\Psi_{\Omega^c}(x_1, x_2, x_3), &f_0(x_1, x_2, x_3) + u_1 f_1(x_1, x_2, x_3) - \text{sign}(x_3) f_2(x_1, x_2, x_3)\rangle = \\ &= u_1\sqrt{x_1^2 + x_2^2}\, x_3 \geq \frac{u_1 x_3}{4}.\end{aligned}$$

By taking $u_1 = \text{sign}(x_3)$ we have that the above expression is strictly positive at any point of $\Omega \cap \overline{B(\bar{x}, \delta_{\bar{x}})} \setminus S$, thus both (App) and the constraint condition are fulfilled, so $T_{S,\Omega}(x) \leq C' d_S^{1/2}(x)$ on a suitable neighborhood of S in $\overline{\Omega}$ by Theorem 3.3.14 and Corollary 3.3.15.

Chapter 4

Regularity of the minimum time function under weak controllability assumptions

The first section is devoted to giving some estimates on bang-bang trajectories which are needed in Section 4.2. The remainder part is reserved for recalling some controllability conditions appearing in [63], as well as providing sufficient conditions for robustness of controllability with respect to a suitable shrinking of S. Throughout this chapter we consider the affine control system in $\mathbb{R}^n$

$$\dot{x} = f_0(x) + \sum_{i=1}^{M} f_i(x)u_i, \tag{4.1}$$

where $f_0, f_i \colon \mathbb{R}^n \to \mathbb{R}^n$ are C^∞-maps and $u = (u_1, ..., u_M) \in [-1, 1]^M$, together with the initial condition

$$x(0) = \xi. \tag{4.2}$$

For the sake of simplicity , we set $f(x, u) := f_0(x) + \sum_{i=1}^{M} f_i(x)u_i$. The standard assumptions on $f_0, ..., f_M$ and the target set S we need are the following:

Assumptions 4.0.2. *(1) f_0, f_i are C^∞ and all partial derivatives are Lipschitz with Lipschitz constant $L > 0$, $i = 1, ..., M$; moreover,*

$$\|f_0(y)\|, \|f_i(y)\| \leq K_0(1 + \|y\|)$$

for all $y \in \mathbb{R}^n$, where K_0 is a positive constant.
(2) S is compact.

Such assumptions will be always supposed to be satisfied in this chapter and also in Chapter 5 and we label them as *standard assumptions* on the dynamics.

4.1 Lie brackets and estimates on bang-bang trajectories

Given the target $S \subset \mathbb{R}^n$, we will state some sufficient conditions in order to reach S from every ξ in a neighborhood in finite time and give an upper bound for the minimum time $T_S(\xi)$. Such conditions involve *Lie brackets* of the vector fields $f_0, ..., f_M$. We recall their definition for general C^1 vector fields X, Y. We set

$$[X, Y](x) = \nabla Y(x)X(x) - \nabla X(x)Y(x),$$

and higher order brackets are defined recursively, provided X, Y are smooth enough. Let now Φ_t^X and Φ_t^Y, $t \geq 0$, be the flows generated by the vector fields X and Y, namely $\Phi_t^X(\xi)$, respectively $\Phi_t^Y(\xi)$, are the solution at time t of the Cauchy problems

$$\dot{x} = X(x), \quad x(0) = \xi; \qquad \dot{x} = Y(x), \quad x(0) = \xi.$$

It is well known that $\Phi_t^X(\cdot)$ and $\Phi_t^Y(\cdot)$ are diffeomorphisms for all $t \geq 0$ small enough. The *formal Lie bracket* between Φ_t^X and Φ_t^Y is defined by setting

$$[\Phi^X, \Phi^Y]_t(\xi) = (\Phi_t^Y)^{-1} \circ (\Phi_t^X)^{-1} \circ (\Phi_t^Y) \circ (\Phi_t^X)(\xi)$$

The procedure may be iterated and the order of such iterations can defined by induction. If Φ is either $\Phi_t^X(\cdot)$ or $\Phi_t^Y(\cdot)$, then $ord(\Phi) = 1$; otherwise, if A and B are nested Lie brackets of $\Phi_t^X(\cdot)$ and $\Phi_t^Y(\cdot)$, we set $ord([A, B]) = ord(A) + ord(B)$. The power of a Lie bracket B, $pw(B)$, is set to 1 if B consists of a single diffeomorphism, while $pw([A, B]) = 2 \times pw(A) + 2 \times pw(B)$. The following classical result establishes a relation between the two types of Lie brackets.

Theorem 4.1.1. *Let $\{X_i\}_{i\in\mathbb{N}}$ be smooth vector fields and let B be a nested formal Lie bracket of order $\bar{k} \in \mathbb{N}$ of the corresponding flows $\{\Phi_t^{X_i}\}_{i\in\mathbb{N}}$, for $t > 0$ small enough, $B = B(\Phi^{X_{i_1}}, ..., \Phi^{X_{i_k}})$, $k \leq \bar{k}$. Then*

$$\frac{\partial^j}{\partial t^j} B(\Phi^{X_{i_1}}, ..., \Phi^{X_{i_k}}) \mid_{t=0} = 0, \quad \forall 1 \leq j < \bar{k},$$
$$\frac{1}{\bar{k}!}\frac{\partial^{\bar{k}}}{\partial t^{\bar{k}}} B(\Phi^{X_{i_1}}, ..., \Phi^{X_{i_k}}) \mid_{t=0} = B(X_{i_1}, ..., X_{i_k}).$$

In what follows we will consider iterated Lie brackets of the vector fields $f_0 \pm f_i$, $i = 1, ..., M$, where f_0, f_i appear in (4.1), possibly with $f_0 \equiv 0$. We denote by $\mathcal{L}$ the set of all iterated Lie brackets of the above vector fields.

Let B be such a non-vanishing Lie bracket with order k, $ord(B) = k$, and power $pw(B)$. Let $x_\xi^B(\cdot)$ be the trajectory of (4.1) and (4.2) corresponding to B, namely the trajectory which uses bang-bang controls ± 1, according to the vector fields appearing in B. We obtain immediately from Theorem 4.1.1 the following expansion:

$$x_\xi^B(pw(B)t) = \xi + B(\xi)t^k + o(t^k), \qquad t \to 0^+, \tag{4.3}$$

where for each compact C containing ξ, there exists $K_C > 0$ such that

$$\left\|o(t^k)\right\| \le K_C t^{k+1} \qquad \text{for all } t \text{ small enough.} \tag{4.4}$$

Now we proceed by applying the above approximation (4.3) to an estimate of the distance from the target of suitable trajectories of (4.1).

Proposition 4.1.2. *Let S be a closed set and let $\xi \notin S$. Let B be a non-vanishing Lie bracket of order k of the vector fields $f_0 \pm f_i$, $i = 1, ..., M$. Let $x^B_\xi(\cdot)$ be the corresponding trajectory of (4.1) and (4.2). Let $t > 0$ and assume that $x^B_\xi(s) \notin S$ for all $s \in [0,t]$. Let $\zeta \in \partial^P d_S(\xi)$. Then we have, for every compact set C containing ξ,*

$$d_S\Big(x^B_\xi(pw(B)t)\Big) \le d_S(\xi) + \langle \zeta, B(\xi)\rangle t^k + K\left(t^{k+1} + \frac{t^{2k}}{d_S(\xi)}\right), \tag{4.5}$$

where K depends only on the constant K_C appearing in (4.4).

Moreover, if S satisfies a ρ-internal sphere condition, then $\frac{1}{d_S(\xi)}$ can be substituted by $\frac{1}{\rho}$, and $x^B_\xi(\cdot)$ may touch S.

Proof. Set $x^B_\xi(pw(B)t) = x_t$. By putting together Proposition 2.2.8 (i), (ii) and (4.3), (4.4) we obtain

$$\begin{aligned} d_S(x_t) &\le d_S(\xi) + \langle \zeta, x_t - \xi\rangle + \frac{K}{d_S(\xi)}\left\|x_t - \xi\right\|^2 \\ &\le d_S(\xi) + \langle \zeta, B(\xi)\rangle t^k + K' t^{k+1} + \frac{\bar{K} t^{2k}}{d_S(\xi)}, \end{aligned}$$

for suitable constants $K', \bar{K}$ satisfying the desired properties. □

Remark 4.1.3. *The regularity requirements on f_0 and f_i can be weakened if Lie brackets only up to a fixed order k are considered. Actually, in most of our results we need only that, for a given $k \in \mathbb{N}$, f_0 and f_i, $i = 1, \ldots, M$, are of class C^k and all partial derivatives up to the order k are Lipschitz with the same constant.*

4.2 Hölder continuity of the minimum time function

We state here two controllability results, proved in [63], which are at the basis of our results. We treat separately the case where the target S satisfies an internal and an external sphere condition.

We say that a Lie bracket B is *compatible* with the controlled dynamics (4.1) if the (direct and reversed) flows appearing in the formal Lie bracket of Theorem 4.1.1 are flows of (4.1). A simple sufficient condition ensuring compatibility is, of course, the drift f_0 to be zero. More in general, compatibility can be seen as a time reversibility of the dynamics. In Section 4.3, controllability conditions which do not require time reversibility will be given for the case of second order Lie brackets.

Theorem 4.2.1 (see Corollaries 5.9 and 5.11 in [63]). *Let S be compact and let one of the two following assumptions be valid. Either,*

(IS) *let S be satisfying a ρ–internal sphere condition and assume there exist $\delta > 0$, $\mu > 0$, and $k \in \mathbb{N}$ such that for every $\xi \in S_{2\delta} \setminus S$ there exist $\zeta_\xi \in \partial^P d_S(\xi)$ and a compatible $B_\xi \in \mathcal{L}$, with $ord(B_\xi) \leq k$, enjoying the following property:*

$$\langle \zeta_\xi, B_\xi(\xi) \rangle \leq -\mu. \tag{4.6}$$

Or, alternatively,

(ES) *let S have reach $\rho > 0$ and assume there exist $0 < \delta < \frac{\rho}{2}$, $\mu > 0$, and $k \in \mathbb{N}$ such that for every $\xi \in S_\delta \setminus S$ there exists a compatible $B_\xi \in \mathcal{L}$, with $ord(B_\xi) \leq k$, enjoying the following property:*

$$\langle \nabla d_S(\xi), B_\xi(\xi) \rangle \leq -\mu. \tag{4.7}$$

Then the minimum time function to reach S from ξ subject to the dynamics (4.1), $T_S(\xi)$, is (finite and) Hölder continuous with exponent $\frac{1}{k}$ on S_δ. More precisely, there exists a constant Λ, depending only on ρ, δ, μ and on the vector fields $f_0, f_i, i = 1, ..., M$, such that for all $\xi_1, \xi_2 \in S_\delta$ it holds

$$|T_S(\xi_1) - T_S(\xi_2)| \leq \Lambda \, \|\xi_1 - \xi_2\|^{1/k}. \tag{4.8}$$

Remark 4.2.2. *Observe that assumptions (IS) and (ES) are of a different nature, because an external sphere condition is assumed either on the closure of the complement of S (case (IS)), or on S (case (ES)). If S satisfies both an internal and an external sphere condition, then its boundary is of class $C^{1,1}$.*

In Chapter 5 we will need to ensure that small time controllability holds not only with respect to S, but also to a suitable shrinking or enlargement of S. The following are the relevant statements. The first one requires, in addition to the internal sphere condition, a uniform external cone condition (see (4.10) in Theorem 4.2.3 below). Of course, such additional requirement is satisfied if the boundary of the target is of class $C^{1,1}$.

Theorem 4.2.3. *Let the assumption (IS) hold and let k, $\rho, \delta, \mu > 0$ be as in (IS). Let $L_{\mathcal{L}}$ be the Lipschitz constant of all $B \in \mathcal{L}, ord(B) \leq k$, on S_δ and set*

$$C_B := \max\{\|B(x)\| : x \in S_\delta, B \in \mathcal{L}, ord(B) \leq k\}. \tag{4.9}$$

Assume furthermore that there exists $0 < \mu' < \frac{\mu}{2}$ such that

$$\max\left\{ \|\zeta' - \zeta\| : \|\zeta\| = \|\zeta'\| = 1, \;\; \zeta, \zeta' \in N_S^C(x), x \in S \right\} < \frac{\mu'}{C_B}. \tag{4.10}$$

Then there exists $0 < \bar{\sigma} < \rho$, depending only on μ', C_B and $L_{\mathcal{L}}$, such that for all $0 < \sigma < \bar{\sigma}$ assumption (IS) holds for $S_{-\sigma}$. More precisely, for every $\xi \in S_\delta \setminus S_{-\sigma}$ there exist $\zeta_\xi \in \partial^P d_{S_{-\sigma}}(\xi)$ and $B_\xi \in \mathcal{L}$, with $ord(B_\xi) \leq k$, enjoying (4.6) with $\min\{\mu, \mu'\}$ in place of μ. Consequently, if the Lie brackets appearing in (IS) are compatible, the minimum time function to reach $S_{-\sigma}$ is finite and Hölder continuous with exponent $\frac{1}{k}$ on S_δ. Moreover, the constant Λ appearing in (4.8) can be chosen independently of σ.

Proof. We claim first that if $0 < \bar{\sigma} < \rho$, then $S_{-\bar{\sigma}}$ satisfies a uniform $(\rho - \bar{\sigma})$-internal sphere condition. Indeed, recalling Corollaries 16 and 19 in [64], the external cone condition implies that $(\operatorname{int} S)^c$ has ρ-positive reach. Therefore, if $0 < \bar{\sigma} < \rho$, then $(\operatorname{int} S_{-\bar{\sigma}})^c$ has $(\rho - \bar{\sigma})$-positive reach. Invoking again Corollaries 16 and 19 in [64] we obtain that $S_{-\bar{\sigma}}$ satisfies a uniform $(\rho - \bar{\sigma})$-internal sphere condition.

Consequently, in order to prove that the minimum time function to reach $S_{-\sigma}$ ($0 < \sigma < \bar{\sigma}$) is Hölder on S_δ it is enough to establish the analogue of (4.6) for $S_{-\sigma}$ as claimed in the statement of the theorem.

To this aim, fix first $\xi \in S_\delta \setminus S$. We claim that

$$\partial^P d_S(\xi) = \partial^P d_{S_{-\sigma}}(\xi).$$

Indeed, set $\bar{\xi} = \pi_S(\xi)$. Since $\overline{S^c}$ has positive reach, S has at $\bar{\xi}$ both an internal and an external nontrivial proximal normal. Thus $\bar{\zeta} := \dfrac{\bar{\xi} - \xi}{d_S(\xi)}$ is the unique unit normal to $\overline{S^c}$ at $\bar{\xi}$. Define, for $0 \leq t < \rho$, $\xi_t = \bar{\xi} + t\bar{\zeta}$. Observe that, by the internal ρ-positive reach condition, $\bar{\xi}$ is the unique projection of ξ_t onto $\overline{S^c}$, so that, in particular, $d_{\overline{S^c}}(\xi_t) = t$. Therefore, $\xi_\sigma \in S_{-\sigma}$, and so $d_{S_{-\sigma}}(\xi) \leq d_S(\xi) + \sigma$. On the other hand, for all $\xi' \in S_{-\sigma}$ one has obviously $\|\xi' - \xi\| \geq d_S(\xi) + \sigma$, whence

$$d_{S_{-\sigma}}(\xi) = d_S(\xi) + \sigma, \tag{4.11}$$

and the claim follows. By (IS), there exist $B_\xi \in \mathcal{L}, ord(B_\xi) \leq k$, and $\zeta_\xi \in \partial^P d_S(\xi)$, such that (4.6) holds. Since $\partial^P d_S(\xi) = \partial^P d_{S_{-\sigma}}(\xi)$ the proof is completed for the case $\xi \in S_\delta \setminus S$. Observe that in this case one can choose $\mu' = \mu$.

Fix now $\xi \in S \setminus S_{-\sigma}$. Let x be the unique projection of ξ onto $\overline{S^c}$. Assume first that $N_S^P(x) \neq \{0\}$ and let ζ be the (unique) unit vector in $N_S^P(x)$. Let $x_n = x + \frac{\zeta}{n}, n \in \mathbb{N}$. Then, since $\nabla d_S(x_n) = \zeta$ for all n large enough, the assumption (IS) yields that there exist $B_n \in \mathcal{L}, ord(B_n) \leq k$, such that

$$\langle \zeta, B_n(x_n) \rangle \leq -\mu, \quad \forall n \text{ large enough.}$$

Since the order of the B_n's is bounded, up to a subsequence we may assume that $B_n(x) = B(x)$ is independent of n. Therefore, by passing to the limit we obtain

$$\langle \zeta, B(x) \rangle \leq -\mu. \tag{4.12}$$

Let now $N_S^P(x) = \{0\}$ and let $\zeta \in N_S^L(x), \|\zeta\| = 1$. By definition of limiting normal, there exist sequences $\{x_n\} \subset S, \{\zeta_n\} \subset \mathbb{R}^n$ such that $\zeta_n \in N_S^P(x_n)$, $\|\zeta_n\| = 1$, $x_n \to x$, and $\zeta_n \to \zeta$ as $n \to \infty$. Then, for every n there exists $B_n \in \mathcal{L}, ord(B_n) \leq k$, such that

$$\langle \zeta_n, B_n(x_n) \rangle \leq -\mu.$$

By passing to the limit as above, we obtain (4.12).

Now we wish to prove that an inequality of the type (4.12) holds at ξ. Recalling that $x = \pi_{\overline{S^c}}(\xi)$. As before, we assume first that $N^P_S(x) \neq \{0\}$. Then, since S has both an inner and an outer nonvanishing proximal normal at x, we have that $N^P_S(x) = \zeta\mathbb{R}^+ = -N^P_{\overline{S^c}}(x)$ for a suitable unit vector ζ, and

$$d_{S_{-\sigma}}(\xi) = \sigma - d_{\overline{S^c}}(\xi). \tag{4.13}$$

Thus $d_{S_{-\sigma}}$ is differentiable at ξ and moreover

$$\nabla d_{S_{-\sigma}}(\xi) = -\nabla d_{\overline{S^c}}(\xi) = \zeta.$$

By the uniform Lipschitz continuity of Lie brackets of order $\leq k$, we obtain from (4.12) that

$$\langle \zeta, B(\xi) \rangle \leq \langle \zeta, B(x) \rangle + L_{\mathcal{L}} \|\xi - x\| \leq -\mu + L_{\mathcal{L}}\sigma.$$

Therefore, if $\sigma < \bar{\sigma} := \frac{\mu-\mu'}{L_{\mathcal{L}}}$ we obtain

$$\langle \zeta, B(\xi) \rangle \leq -\mu',$$

which was to be proved.

Assume now again that $N^P_S(x) = \{0\}$. Recalling Lemma 5 in [72][1], we have that $N^C_S(x) = -N^P_{\overline{S^c}}(x)$. Therefore, by (4.13), $\zeta' := \nabla d_{S_{-\sigma}}(\xi) = \frac{x-\xi}{\|x-\xi\|} \in N^C_S(x)$. By the assumption (IS), there exist a unit vector $\zeta \in N^L_S(x)$ and a Lie bracket $B \in \mathcal{L}$, with $ord(B) \leq k$, such that $\langle \zeta, B(x) \rangle \leq -\mu$. Recalling (4.10) we obtain $\|\zeta - \zeta'\| < \frac{\mu'}{C_B}$. By putting the above inequalities together, we finally have

$$\langle \zeta', B(\xi) \rangle = \langle \zeta' - \zeta, B(\xi) \rangle + \langle \zeta, B(x) \rangle + \langle \zeta, B(\xi) - B(x) \rangle < \mu' - \mu + L_{\mathcal{L}}\sigma.$$

Therefore, if $\sigma \leq \bar{\sigma} := \frac{\mu-2\mu'}{L_{\mathcal{L}}}$ we finally reach $\langle \zeta', B(\xi) \rangle < -\mu'$. The proof is concluded. □

The second perturbation result is concerned with the case where the target S has positive reach.

Proposition 4.2.4. *Let the assumption (ES) of Theorem 4.2.1 hold and let* $0 < \sigma < \delta$. *Then the minimum time to reach* S_σ *from* $S_\delta \setminus S_\sigma$ *(is finite and) satisfies (4.8), where the constant* Λ *is independent of* σ.

Proof. It is enough to observe that if $\xi \in S_\delta \setminus S_\sigma$, then $d_{S_\sigma}(\xi) = d_S(\xi) - \sigma$. □

Remark 4.2.5. *Observe that, under the assumptions of Proposition 4.2.4, the enlargement of* S_σ *satisfies an internal sphere condition, and so, as far as it is enough to consider an approximation of the target, one can concentrate only on the (IS) case.*

[1]The statement of Lemma 5 in [72] actually requires S to be convex, but this is used only to provide wedgedness, which indeed we assume.

Part II

Approximation of minimum time function

Chapter 5

A Hamilton–Jacobi–Bellman approach under weak controllability assumptions

This chapter is devoted to designing a suitable fully discrete scheme for the approximation of T_S. We follow the well established method based on dynamic programming, which was first designed by Bardi and Falcone [17] (see also [18], [16], and [36] and references therein). We apply to T_S the Kružkov transform and then, through discrete dynamic programming, we approximate the viscosity solution of a suitable boundary value problem. Since, due to controllability assumptions which are based on higher order Lie brackets, T_S is not locally Lipschitz, we need to use a scheme which is of a suitably high order in time and of first order in space.

This chapter is divided into a number of sections. First we present a higher order one step semidiscrete scheme for our dynamics (4.1), taking controls subject to suitable switchings. Given a step size h, for every initial condition ξ we construct a discrete trajectory which converges as $h \to 0$ to a suitable trajectory of (4.1). Moreover, the time needed to reach the target is bounded by a fractional power of $d_S(\xi)$ (discrete controllability). Next we apply Kružkov transform to T_S, and relying on a discrete dynamic programming principle and a convergence results due to [17] we prove that a discrete value function v_h converges to the transformation v_S of T_S, also providing an error estimate. Finally, we introduce a fully discrete scheme and prove its convergence and a related error estimate. The standard assumptions 4.0.2 is assumed to be fulfilled in this whole chapter.

5.1 Time discretization

Given the control system (4.1), (4.2), we write

$$\dot{x} = f(x, u), \quad x(0) = \xi, \tag{5.1}$$

where $f(x,u) = f_0(x) + \sum_{i=1}^M f_i(x)u_i$, $u = (u_1, ..., u_M) \in [-1,1]^M$. Given a fixed step $h > 0$ small enough, we approximate (5.1) by a one step $(q+1)$-th order scheme which has the form

$$\begin{cases} y_{n+1} &= y_n + h\Phi(y_n, A_n, h) \\ y_0 &= \xi \end{cases} \tag{5.2}$$

where A_n is a $M \times l$ matrix, $A_n = (u_n^1, ..., u_n^l)$ with $u_n^i \in [-1,1]^M$. Here $l > 0$ depends on the specific method. We make the following assumptions on the method:

$$\begin{cases} \lim_{h\to 0} \Phi(\xi, (\bar{u}, ..., \bar{u}), h) = f(\xi, \bar{u}) \quad (l \text{ copies of } \bar{u}), \\ \|\Phi(\xi_1, A, h) - \Phi(\xi_2, A, h)\| \le L_\Phi \|\xi_1 - \xi_2\|. \end{cases} \tag{5.3}$$

In order to prove the discrete controllability, we now consider the following Cauchy problem, instead of (5.1)

$$\dot{x} = f(x, \upsilon) = f_0(x) + \sum_{i=1}^M f_i(x)u_i, \quad x(0) = \xi \tag{5.4}$$

where $\upsilon = (u_1, ..., u_M)$, $u_i \in \{-1, 1\}$, is supposed to be constant in an interval $[0, \tau]$, $0 < \tau \le 1$. Let $0 < h < \tau$ and $k \in \mathbb{N}$, $k \ge 1$, be given. Here k will play the role of the order of a suitable Lie bracket which will be identified later. We consider the one step order scheme (5.2) for (5.4). In this case, the control matrix A is generated by l copies of υ, therefore the conditions (5.3) can be rewritten, by an abuse of notation, in the following way:

$$\begin{cases} \lim_{h\to 0} \Phi(\xi, \upsilon, h) = f(\xi, \upsilon), \\ \|\Phi(\xi_1, \upsilon, h) - \Phi(\xi_2, \upsilon, h)\| \le L_\Phi \|\xi_1 - \xi_2\|. \end{cases}$$

Furthermore, we require a suitably high order of approximation, namely

$$\|x_\upsilon(h, \xi) - (\xi + h\Phi(\xi, \upsilon, h))\| \le C_\Phi h^{q+2}, \tag{5.5}$$

where $q \ge k$ and $x_\upsilon(\cdot, \xi)$ is the exact solution of (5.4). The classical Runge-Kutta method, for example, enjoys the above properties (see [48]). Set

$$\begin{cases} \xi_0 &= \xi, \\ \xi_{n+1} &= \xi_n + h\Phi(\xi_n, \upsilon, h), \end{cases} \tag{5.6}$$

and, for $N \in \mathbb{N}$, $N \ge 1$,

$$h = \frac{\tau}{N}. \tag{5.7}$$

Then there exists C_Φ such that

$$\|x_\upsilon(\tau, \xi) - \xi_N(\tau, \xi, \upsilon)\| \le C_\Phi h^{q+1}, \tag{5.8}$$

for all h small enough (see, e.g., Theorem 3.6 in [48]). Observe that the point $\xi_N(\tau,\xi,\upsilon)$ defined through (5.6) and (5.7) depends on the M-tuple υ. We denote this point by $y(\upsilon,\tau,h,\xi)$, i.e.,

$$y(\upsilon,\tau,h,\xi) := \xi_N(\tau,\xi,\upsilon).$$

Let $p \in \mathbb{N}, p \geq 1$. We consider now a p-tuple $\underline{u}$ of M-tuples of controls $u_i \in \{-1,1\}$, namely $\underline{u} = (\upsilon^1,...,\upsilon^p)$, where $\upsilon^j = (u_1^j,...,u_M^j) \in \{-1,1\}^M$ and subsequently apply the process (5.6), with υ^j in place of υ, N times for each j. More precisely, we set

$$\begin{cases} y^1 = \; y(\upsilon^1,\tau,h,\xi), \\ \;\vdots \\ y^j = \; y(\upsilon^j,\tau,h,y^{j-1}), \end{cases} \tag{5.9}$$

where $j = 2,...,p$. We denote the point y^p constructed above by $y^p(\underline{u},\tau,h,\xi)$.

Let $x_{\underline{u}}(\tau,\xi)$ be the final point of the exact solution of (5.4) corresponding to the p-tuples of controls $\underline{u}$. More precisely, we set

$$\begin{cases} x^1 = x_{\upsilon^1}(\tau,\xi), \\ \;\vdots \\ x^m = x_{\upsilon^m}(\tau,x^{m-1}), \qquad m = 2,\dots,p. \end{cases}$$

By applying (5.8) subsequently on p intervals of length τ, we obtain

$$\|x^p - y^p\| \leq C_p h^{q+1}, \tag{5.10}$$

where C_p is a suitable constant depending only on p,Φ.

5.2 Discrete controllability

The following result falls in the framework of (approximate) discrete controllability: under assumptions including either (IS) or (ES), given $0 < \eta < \delta$, for all $\xi \in S_\delta \setminus S_\eta$ we construct a finite sequence of points of the types y^p described just above, say $y_1,...,y_{n(\xi)}$, and of increasing times t_i, $i = 0,...,n(\xi)-1$, such that $d_S(y_{n(\xi)}) < \eta$ and the time to reach $y_{n(\xi)}$, namely $\sum_{i=0}^{n(\xi)-1} p_{i+1}(t_{i+1}-t_i)$, is bounded from above by $d_S(\xi)^{1/k}$. The number of discretization steps, namely $[(t_{i+1}-t_i)/h]$ where $h > 0$ is fixed, will be labeled here for simplicity as N.

Theorem 5.2.1. *Let $S \subset \mathbb{R}^n$ be closed and let $\delta,\rho,\mu > 0$, $k \in \mathbb{N}, k \geq 1$ be given, with $\delta < 1$. Assume that for every $\xi \in S_{2\delta} \setminus S$ there exist $\zeta_\xi \in \partial^P d_S(\xi)$ and a compatible $B_\xi \in \mathcal{L}$, with ord$(B_\xi) \leq k$, such that*

$$\langle \zeta_\xi, B_\xi(\xi) \rangle \leq -\mu. \tag{5.11}$$

Let $0 < \eta < \delta$ be given and consider a one step $(q+1)$-th order scheme with $q \geq k$. Then for every $\xi \in S_\delta \setminus S_\eta$ there exist a number of steps N, independent of ξ, and finite sequences of natural numbers p_{i+1}, of p_{i+1}-tuples $\{\underline{u}_{i+1}\}$ of M-tuples of ± 1, of points $\{y_i\}$, and of times $\{t_i\}$, $t_{i+1} > t_i$, $i = 0, ..., n(\xi) - 1$, satisfying the properties

$$\begin{cases} t_0 = 0, \ y_0 = \xi, \\ y_{i+1} = y^{p_i}\Big(\underline{u}_{i+1}, p_{i+1}(t_{i+1} - t_i), \dfrac{t_{i+1} - t_i}{N}, y_i\Big), \\ i = 0, ..., n(\xi) - 1, \end{cases}$$

$$y_{n(\xi)} \in S_\eta,$$

$$\sum_{i=0}^{n(\xi)-1} p_{i+1}(t_{i+1} - t_i) \leq C\big(d_S(\xi)\big)^{1/k}, \tag{5.12}$$

for a suitable constant C independent of ξ. Here y^{p_i} is defined according to (5.9).

Proof. Fix $\xi \in S_\delta \setminus S_\eta$. By our assumptions, there exist $\zeta_\xi \in \partial^P d_S(\xi)$ and a Lie bracket $B_\xi \in \mathcal{L}$ with $ord(B_\xi) \leq k$ such that (5.11) holds. Now we are going to prove that there exist a time $0 < t_1 \leq 1$, a number $p_1 \geq 1$, a (finite) sequence of M-tuples of ± 1, say $\underline{u}_1 = (\upsilon_1^1, ..., \upsilon_1^{p_1})$, $\upsilon_1^j \in \{-1, 1\}^M$, corresponding to B_ξ through Theorem 4.1.1, such that the trajectory $x_\xi^{B_\xi}(\cdot)$ of (4.1), (4.2) associated to this sequence of controls satisfies the following properties for all $t \in [0, t_1]$:

$$\begin{cases} d_S\Big(x_\xi^{B_\xi}(p_1 t)\Big) > \dfrac{d_S(\xi)}{2}, \\ d_S\Big(x_\xi^{B_\xi}(p_1 t)\Big) \leq d_S(\xi) - \mu t^k + K\Big(t^{k+1} + \dfrac{2t^{2k}}{d_S(\xi)}\Big), \end{cases} \tag{5.13}$$

where K is the constant appearing in (4.5). Indeed, in order to obtain the first inequality in (5.13), recalling (4.3), (4.4), and (4.9) it is enough to choose $0 < t_1 \leq 1$ such that

$$(C_B + K_{S_\delta})t_1^k \leq \frac{d_S(\xi)}{2}, \tag{5.14}$$

where K_{S_δ} is the constant appearing in (4.4) with S_δ in place of C, while the second one follows from (4.5) in Proposition 4.1.2 together with (5.11). In particular, we obtain

$$0 < d_S\Big(x_\xi^{B_\xi}(p_1 t_1)\Big) < 2\delta.$$

Observe furthermore that there exists a constant p_k (the maximal power of a Lie bracket of order $\leq k$ in $\mathbb{R}^n$) depending only on k, such that

$$p_1 \leq p_k.$$

Let $N \in \mathbb{N}, N \geq 1$, and set $h_1 = \frac{t_1}{N}$. We assume N to be so large that the discretization error corresponding to the step size h_1 satisfies (5.8). Let y_1 be the point $y^{p_1}(\underline{u}_1, p_1 t_1, h_1, \xi)$ constructed according to (5.6), (5.9). By (5.10) we have

$$\left\| x_\xi^{B_\xi}(p_1 t_1) - y_1 \right\| \leq C_{p_k} h_1^{q+1}. \tag{5.15}$$

Remembering that $q \geq k$ and putting together the above inequality and (5.13), we receive

$$d_S(y_1) \leq d_S(\xi) - \mu t_1^k + K\Big(t_1^{k+1} + \frac{2t_1^{2k}}{d_S(\xi)}\Big) + C_{p_k}\Big(\frac{t_1}{N}\Big)^{k+1}.$$

We rewrite the above estimate as

$$\begin{aligned} d_S(y_1) &\leq d_S(\xi) - \mu t_1^k + \Big(K + \frac{C_{p_k}}{N^{k+1}}\Big)t_1^{k+1} + \frac{2K}{d_S(\xi)}t_1^{2k} \\ &=: d_S(\xi) - \mu t_1^k + K_1 t_1^{k+1} + \frac{K_2}{d_S(\xi)}t_1^{2k}. \end{aligned} \tag{5.16}$$

By imposing the supplementary conditions

$$t_1 K_1 + \frac{K_2}{d_S(\xi)}t_1^k \leq \frac{\mu}{2} \qquad \text{and} \qquad C_{p_k} t_1^{k+1} \leq \frac{N^{k+1} d_S(\xi)}{4}, \tag{5.17}$$

we obtain from (5.13), (5.15), and (5.16)

$$\frac{d_S(\xi)}{4} \leq d_S(y_1) \leq d_S(\xi) - \frac{\mu}{2}t_1^k. \tag{5.18}$$

Observe that all conditions previously imposed on t_1 (in particular (5.14) and (5.17)) are satisfied if

$$0 < t_1 = \min\left\{1, \sqrt[k]{\frac{N^{k+1}d_S(\xi)}{4C_{p_k}}}, \sqrt[k]{\frac{\mu d_S(\xi)}{4K_2}}, \frac{\mu}{4K_1}, \sqrt[k]{\frac{d_S(\xi)}{2(C_B + K_{S_\delta})}}\right\}. \tag{5.19}$$

Assume now that we have constructed recursively times t_i, numbers p_i, controls $\underline{u}_i = (\upsilon_i^1, ..., \upsilon_i^{p_i})$, $\upsilon_i^j \in \{-1, 1\}^M$ and points y_i up to $i = \bar{\imath}$, such that

$$t_{i-1} < t_i,$$

$$t_i - t_{i-1} = \min\{t_1, \sqrt[k]{\frac{N^{k+1}d_S(y_{i-1})}{4C_{p_k}}}, \sqrt[k]{\frac{\mu d_S(y_{i-1})}{4K_2}}, \sqrt[k]{\frac{d_S(y_{i-1})}{2(C_B + K_{S_\delta})}}\},$$

and

$$\frac{d_S(y_{i-1})}{4} \leq d_S(y_i) \leq d_S(y_{i-1}) - \frac{\mu}{2}(t_i - t_{i-1})^k. \tag{5.20}$$

We are now going to construct the next step. By the assumptions, there exist a Lie bracket $B_{y_{\bar{\imath}}}$ and $\zeta_{y_{\bar{\imath}}} \in \partial^P d_S(y_{\bar{\imath}})$ such that $\langle \zeta_{y_{\bar{\imath}}}, B_{y_{\bar{\imath}}}(y_{\bar{\imath}})\rangle \leq -\mu$. By applying again

Proposition 4.1.2 and the argument designed for t_1 we find a time $t_{\bar{i}+1}$, a number $p_{\bar{i}+1} \leq p_k$, a control $\underline{u}_{\bar{i}+1} \in \{-1,1\}^{M\times p_{\bar{i}+1}}$, and a point $y_{\bar{i}+1}$ satisfying the properties

$$0 < t_{\bar{i}+1} - t_{\bar{i}} < t_1, \tag{5.21}$$

$$t_{\bar{i}+1} := t_{\bar{i}} + \min\{\sqrt[k]{\frac{N^{k+1}d_S(y_{\bar{i}})}{4C_{p_k}}}, \sqrt[k]{\frac{\mu d_S(y_{\bar{i}})}{4K_2}}, \sqrt[k]{\frac{d_S(y_{\bar{i}})}{2(C_B + K_{S_\delta})}}\}, \tag{5.22}$$

from which, taking into account (5.21) and (5.22), we obtain finally

$$\frac{d_S(y_{\bar{i}})}{4} \leq d_S(y_{\bar{i}+1}) \leq d_S(y_{\bar{i}}) - \frac{\mu}{2}(t_{\bar{i}+1} - t_{\bar{i}})^k,$$

which concludes our construction. Now we are going to show that we can reach S_η after finitely many iterations $n(\xi)$ and that (5.12) holds. To this aim, set

$$\alpha = \min\{1, \frac{\mu}{4K_1}\}, \qquad \beta = \min\{\frac{1}{\sqrt[k]{2(C_B + K_{S_\delta})}}, \sqrt[k]{\frac{N^{k+1}}{4C_{p_k}}}, \sqrt[k]{\frac{\mu}{4K_2}}\}.$$

Then, for every $i \in \mathbb{N}$, we have $t_{i+1} - t_i = \min\{\alpha, \beta\sqrt[k]{d_S(y_i)}\}$. Observe that (5.20) implies that the sequence $\{d_S(y_i)\}$ is strictly decreasing. Therefore, there exists an index $\bar{i}$ such that for all $i \geq \bar{i}$ we have

$$t_{i+1} - t_i = \beta\sqrt[k]{d_S(y_i)}. \tag{5.23}$$

Let $d = \lim_{i\to\infty} d_S(y_i)$. From (5.20) and (5.23) we obtain, for all $i \geq \bar{i}$,

$$d_S(y_{i+1}) - d_S(y_i) \leq -\frac{\mu\beta^k}{2}d_S(y_i),$$

from which necessarily $d = 0$. Therefore, there exists some index $n(\xi)$ such that $d_S(y_{n(\xi)}) \leq \eta$. Finally, we deal with (5.12). Owing again to (5.20) and (5.23), we have for all $i \leq n(\xi) - 1$

$$d_S(y_{i+1}) - d_S(y_i) \leq -\frac{\mu}{2}(t_{i+1} - t_i)^k = -\frac{\mu}{2}\min\left\{\alpha^{k-1}, \beta^{k-1}d_S(y_i)^{\frac{k-1}{k}}\right\}(t_{i+1} - t_i).$$

Thus,

$$t_{i+1} - t_i \leq \frac{2}{\mu}\left(\frac{d_S(y_i) - d_S(y_{i+1})}{\beta^{k-1}d_S(y_i)^{\frac{k-1}{k}}} + \frac{d_S(y_i) - d_S(y_{i+1})}{\alpha^{k-1}}\right).$$

By summing the above inequalities and recalling that $p_i \leq p_k$ for each i, we obtain

$$\begin{aligned}\sum_{i=0}^{n(\xi)-1} p_{i+1}(t_{i+1} - t_i) &\leq \frac{2p_k}{\mu}\sum_{i=0}^{n(\xi)-1}\left(\frac{d_S(y_i) - d_S(y_{i+1})}{\beta^{k-1}\,d_S(y_i)^{\frac{k-1}{k}}} + \frac{d_S(y_i) - d_S(y_{i+1})}{\alpha^{k-1}}\right)\\ &\leq \frac{2p_k}{\mu}\int_0^{d_S(\xi)}\left(\frac{1}{\beta^{k-1}r^{\frac{k-1}{k}}} + \frac{1}{\alpha^{k-1}}\right)dr\\ &\leq \frac{2p_k}{\mu}\left(\frac{k}{\beta^{k-1}}\sqrt[k]{d_S(\xi)} + \frac{1}{\alpha^{k-1}}d_S(\xi)\right),\end{aligned}$$

which, recalling that $d_S(\xi) < \delta < 1$, implies (5.12) and concludes our proof. $\square$

The second result of this section requires some regularity on the target S. Under suitable conditions we prove that the discretized trajectory reaches the target (*not* a neighborhood) after finitely many steps of constant length, and establish an estimate of the type (5.12). Such upper bound will be used in the proof of the convergence of a suitable discretized value function to the viscosity solution of a Hamilton-Jacobi equation.

To this aim, we define the discrete minimum time as follows. Given a step size $h > 0$ and a sequence of control matrices $\{A_i\} \subset [-1, 1]^{Ml}$, we recall the discrete dynamics defined in the previous section for the control system (5.1)

$$\begin{cases} y_{n+1} &= y_n + h\Phi(y_n, A_n, h) \\ y_0 &= \xi. \end{cases} \tag{5.24}$$

We define the function

$$n_h(\{A_i\}, \xi) = \min\{n \in \mathbb{N} : y_n \in S\} \leq +\infty, \tag{5.25}$$

where $n_h = \infty$ if y_n never reaches S. Let $N_h(\xi)$ be the minimum number of steps to reach S, namely,

$$N_h(\xi) = \min_{\{A_i\}\in[-1,1]^{Ml}} \{n_h(\{A_i\}, \xi)\}. \tag{5.26}$$

The discrete minimum time function is now defined by setting

$$T_h(\xi) = hN_h(\xi). \tag{5.27}$$

We define also the discrete reachable set $\mathcal{R}_h^S$ by $\mathcal{R}_h^S = \{\xi \in \mathbb{R}^n : N_h(\xi) < +\infty\}$.

Theorem 5.2.2. *Let the assumptions of Theorem 4.2.3 hold and let the target S, ρ, σ, k be as S, ρ, $\bar{\sigma}$, k in Theorem 4.2.3. Consider the discrete dynamics (5.24), generated by a one step scheme Φ which satisfies (5.3) and (5.5) for some $q \geq k$, and the discrete minimum time function (5.27).*

Then there exist $\bar{\delta}$, $\bar{h}$, $C > 0$ such that for every $0 < \delta < \bar{\delta}$, $h \leq \bar{h}$, $\xi \in S_\delta \setminus S$, we have

$$T_h(\xi) \leq C \sqrt[k]{d_S(\xi)}. \tag{5.28}$$

Proof. Fix $x_0 \in \partial S$ and consider $\xi \in B_{\sigma/2}(x_0)\setminus S$. Recalling the proof of Theorem 4.2.3, we obtain that $S_{-\sigma}$ satisfies a $(\rho - \sigma)$-internal sphere condition, and so the distance function to $S_{-\sigma}$, $d_{S_{-\sigma}}(\cdot)$, is semiconcave with constant $(\rho - \sigma)^{-1}$. According to (4.11), $\partial^P d_{S_{-\sigma}}(\xi) = \partial^P d_S(\xi)$. Then, for each $\zeta_\xi \in \partial^P d_S(\xi)$ we obtain

$$d_{S_{-\sigma}}(y) \leq d_{S_{-\sigma}}(\xi) + \langle \zeta_\xi, y - \xi \rangle + \frac{1}{\rho - \sigma} \|y - x\|^2$$

for every $y \in B_\sigma(x_0)$.

Recalling (4.3) and (4.4), we can find $p \geq 1$, a sequence of bang-bang controls $\{\pm 1\}$, say $\underline{u} = (v^1, ..., v^p) \in \{-1, 1\}^{M \times p}$ and $t \in (0, 1]$, such that the corresponding trajectory $x_\xi^{B_\xi}(\cdot)$ of (4.1), (4.2) has the form (4.3), i.e.,

$$x_\xi^{B_\xi}(pt) = \xi + B_\xi t^k + o(t^k), \tag{5.29}$$

where $\|o(t^k)\| \leq K_{B_\sigma} t^{k+1}$. If furthermore $(C_B + K_{B_\sigma})t^k < \frac{\sigma}{2}$, where we recall that C_B was defined in (4.9), then putting together (5.29), (4.5) and the estimate on the semiconcavity constant of $d_{S_{-\sigma}}$ we have also

$$d_{S_{-\sigma}}\left(x_\xi^{B_\xi}(pt)\right) \leq d_{S_{-\sigma}}(\xi) - \mu t^k + K_{B_\sigma} t^{k+1} + \frac{\left(C_B + K_{B_\sigma}\right)^2}{\rho - \sigma} t^{2k}. \tag{5.30}$$

Set $t = Nh$, $N \in \mathbb{N}$, and let y be the point $y^p(\underline{u}, pNh, h, \xi)$ constructed according to (5.6), (5.9). From (5.10) we receive

$$\left\|x_\xi^B(pNh) - y\right\| \leq C_p h^{q+1}. \tag{5.31}$$

By putting together (5.30) and (5.31), we obtain now

$$d_{S_{-\sigma}}(y) \leq d_{S_{-\sigma}}(\xi) - \mu(Nh)^k + \left(K_{B_\sigma} + C_p\right)(Nh)^{k+1} + \frac{\left(K_{B_\sigma} + C_B\right)^2}{\rho - \sigma}(Nh)^{2k}.$$

Then

$$d_{S_{-\sigma}}(y) \leq d_{S_{-\sigma}}(\xi) - \frac{\mu}{2}(Nh)^k,$$

provided

$$Nh < \min\{1, \sqrt[k]{\frac{\sigma}{2(C_B + K_{B_\sigma})}}, \frac{\mu}{4(K_{B_\sigma} + C_p)}, \sqrt[k]{\frac{\mu(\rho - \sigma)}{4\left(K_{B_\sigma} + C_B\right)^2}}\} =: \alpha. \tag{5.32}$$

On the other hand, we want to impose the condition $d_{S_{-\sigma}}(\xi) - \frac{\mu}{2}(Nh)^k \leq \sigma$, which yields $d_{S_{-\sigma}}(y) \leq \sigma$ and so $y \in S$. This condition, in view of (4.11), is equivalent to

$$Nh \geq \sqrt[k]{\frac{2d_S(\xi)}{\mu}}. \tag{5.33}$$

Now, in order to make (5.32) and (5.33) compatible, we impose a condition on $d_S(\xi)$, namely

$$2\frac{d_S(\xi)}{\mu} \leq 2\frac{\delta}{\mu} < \alpha^k.$$

Then, to reach S it is enough to choose $N^\star \in \mathbb{N}$ and $h^\star$ so that

$$N^\star h^\star = \sqrt[k]{\frac{2d_S(\xi)}{\mu}}.$$

Due to the compactness of S, we finally obtain

$$T_h(\xi) \le pN^\star h^\star = p\sqrt[k]{\frac{2d_S(\xi)}{\mu}} \le C\sqrt[k]{d_S(\xi)},$$

for a suitable constant C, which is the desired estimate. $\square$

Remark 5.2.3. *A result similar to Theorem 5.2.2 can be proved without restrictions on* $\delta > 0$ *(except* $\delta < 1$*).*

Indeed, the following statement can be proved.

Under the assumptions of Theorem 5.2.2, let $0 < \delta < 1$*,* $k \in \mathbb{N}$ *be such that for every* $\xi \in S_\delta \setminus S$ *there exist* $\zeta_\xi \in \partial^P d_S(\xi)$ *and* $B_\xi \in \mathcal{L}$*, with* $ord(B_\xi) \le k$*, such that (5.11) holds. Let* $0 < \sigma < \delta$*. Then for every step size* h *small enough, there exists a number* $N^\star$ *such that for every* $\xi \in S_\delta \setminus S$ *we can find controls* $\upsilon_1, ..., \upsilon_{N^\star}$ *for which we can reach* S_δ *by* $N^\star$ *iterations of (5.24).*

The proof is a combination of arguments of the proofs of Theorems 5.2.1 and (5.2.2).

5.3 A further result on discrete controllability

We consider now the case where the approximation of trajectories of (4.1), (4.2) with Lie brackets contains also lower order terms. This case occurs in general when the drift term f_0 does not vanish or when the system is not necessarily time reversible. Our reference is the second order controllability result proved by [61]. For simplicity we treat only one of the sufficient conditions proved in [61, Proposition 4], but an entirely similar result can be obtained with the other one.

Proposition 5.3.1 (Proposition 4 in [61])**.** *Consider the controlled system (4.1), (4.2) with* $M = 1$ *and let the target* S *satisfy the* ρ*-internal sphere condition. Let* $\delta, \mu > 0$ *be given and assume that for all* $x \in S_\delta \setminus S$ *there exists a control* $u \in [-1, 1]$*, and* $\zeta_x \in \partial^P d_S(x)$ *such that the following inequalities hold:*
Either
(IS.0) $\langle f_0(x) + f_1(x)u, \zeta_x\rangle \le -\mu,$
or
(IS.1) $\langle f_0(x), \zeta_x\rangle \le 0,$
(IS.2) $\langle 2\nabla f_0(x) f_0(x) + u[f_1, f_0](x), \zeta_x\rangle + \frac{4}{\rho}\|f_0(x)\|^2 \le -\mu.$
Then $\mathcal{R}^S$ *contains* S *in its interior and* T_S *is Hölder continuous with exponent* $1/2$ *in* $\mathcal{R}^S$*.*

Remark 5.3.2. *Robustness of the controllability condition of Proposition 5.3.1 with respect to a shrinking* $S_{-\sigma}$ *of the target.*

Let $\rho > \sigma > 0$ and S satisfy the same properties as in Theorem 4.2.3, namely ρ-internal sphere condition and wedgedness. By the same arguments as in the proof

of Theorem 4.2.3, for every $\xi \in S_\delta \setminus S_{-\sigma}$ the inequality (IS.2) still holds with some $\zeta_\xi \in \partial^P d_{S_{-\sigma}}(\xi)$ and a suitable $\mu' \leq \mu$ in place of μ. In order to preserve the discrete controllability under a shrinking of the target, the condition (IS.1), instead, needs to be strengthened as follows:
(IS'.1) *for all $\xi \in S_\delta \setminus S_{-\sigma}$ there exists $\zeta_\xi \in \partial^P d_{S_{-\sigma}}$ such that $\langle f_0(x), \zeta_x \rangle \leq 0$.*

The following result contains our second order discrete controllability condition in the case where the drift term cannot be neglected.

Theorem 5.3.3. *Let the target S, and ρ, δ, σ be as $S, \rho, \delta, \bar{\sigma}$ in Theorem 4.2.3 and let the assumptions (IS.0), (IS'.1) and (IS.2) hold true for $x \in S_\delta \setminus S_{-\sigma}$. Consider the discrete dynamics (5.24) generated by the one step scheme Φ which satisfies (5.3) and (5.5) for some $q \geq 2$, and the discrete minimum time function (5.27).*

Then there exist $\bar{\delta}$, $\bar{h}$, $C > 0$ such that for every $0 < \delta_1 < \bar{\delta}$, $h \leq \bar{h}$, $\xi \in S_{\delta_1} \setminus S$, we have

$$T_h(\xi) \leq C\sqrt{d_S(\xi)}.$$

Proof. Fix $x_0 \in \partial S$ and consider $\xi \in B_{\sigma/2}(x_0) \setminus S$. By the same argument as at the beginning of the proof of Theorem 5.2.2, for each $\zeta_\xi \in \partial^P d_S(\xi)$ we obtain

$$d_{S_{-\sigma}}(y) \leq d_{S_{-\sigma}}(\xi) + \langle \zeta_\xi, y - \xi \rangle + \frac{1}{\rho - \sigma} \|y - x\|^2 \tag{5.34}$$

for every $y \in B_\sigma(x_0)$. Assume first that (IS'.1) and (IS.2) hold at ξ. Recalling Lemma 1 in [61], for every $\bar{u} \in [-1, 1]$ and $t \in (0, 1]$, if we follow first the flow of $f_0 + \bar{u} f_1$ and then of $f_0 - \bar{u} f_1$, each one for a time t, the corresponding trajectory $x_\xi(\cdot)$ of (4.1), (4.2) has the form

$$x_\xi(2t) = \xi + 2t f_0(\xi) + t^2\big(2Df_0(\xi)f_0(\xi) + u[f_1, f_0](\xi)\big) + o(t^2), \tag{5.35}$$

where $\|o(t^2)\| \leq K_{B_\sigma} t^3$, for a suitable constant K_{B_σ}. Set now

$$\begin{aligned} C_f &:= \max\{\|f_0(x)\| : x \in S_\delta \setminus S_{-\sigma}\}, \\ C_{ff} &:= \max\{\|Df_0(x)f_0(x)\| : x \in S_\delta \setminus S_{-\sigma}\}, \\ C_{fg} &:= \max\{\|[f_1, f_0](x)\| : x \in S_\delta \setminus S_{-\sigma}\}, \end{aligned}$$

and $M_1 = (2C_f + 2C_{ff} + C_{fg} + K_{B_\sigma})$ and assume that $M_1 t < \frac{\sigma}{2}$. Then, by putting together (5.35), (5.34), (IS'.1), and (IS.2) we have also

$$d_{S_{-\sigma}}\Big(x_\xi(2t)\Big) \leq d_{S_{-\sigma}}(\xi) - \mu t^2 + K_{B_\sigma} t^3 + \frac{M_2^2}{\rho - \sigma} t^4, \tag{5.36}$$

where $M_2 := 2C_{ff} + C_{fg} + K_{B_\sigma}$.
Consider now the one step method (5.2) with $q = 2$ and set $t = Nh$ and $\underline{u} := \{\bar{u}, -\bar{u}\}$. Then let $y := y(\underline{u}, 2Nh, h, \xi)$ be the final point of the discrete dynamical system (5.9) after choosing $\bar{u}$ for the first N iterations and $-\bar{u}$ for other N. From (5.10) we receive

$$\|x_\xi(2Nh) - y\| \leq C_2 h^3. \tag{5.37}$$

By putting together (5.36) and (5.37), we obtain now

$$d_{S_{-\sigma}}(y) \le d_{S_{-\sigma}}(\xi) - \mu(Nh)^2 + \big(K_{B_\sigma} + C_2\big)(Nh)^3 + \frac{M_2^2}{\rho-\sigma}(Nh)^4,$$

so that

$$d_{S_{-\sigma}}(y) \le d_{S_{-\sigma}}(\xi) - \frac{\mu}{2}(Nh)^2,$$

provided

$$Nh < \min\{1, \frac{\sigma}{2M_1}, \frac{\mu}{4(K_{B_\sigma}+C_2)}, \sqrt{\frac{\mu(\rho-\sigma)}{4M_2^2}}\} =: \alpha. \tag{5.38}$$

On the other hand, we want to impose the condition $d_{S_{-\sigma}}(\xi) - \frac{\mu}{2}(Nh)^2 < \sigma$, which yields $d_{S_{-\sigma}}(y) \le \sigma$ and so $y \in S$. This condition, in view of (4.11), is equivalent to

$$Nh \ge \sqrt{\frac{2d_S(\xi)}{\mu}}. \tag{5.39}$$

Now, in order to make (5.38) and (5.39) compatible, we impose

$$2\frac{d_S(\xi)}{\mu} \le 2\frac{\delta}{\mu} < \alpha^2.$$

Then, to reach S it is enough to choose $N^\star \in \mathbb{N}$ and $h^\star$ so that

$$N^\star h^\star = \sqrt{\frac{2d_S(\xi)}{\mu}}.$$

where $h^\star < \bar{h}$, $\delta_1 < \bar{\delta} := \min\{1, \sigma, \frac{\mu\alpha^2}{2}\}$. Due to the compactness of S, we finally obtain

$$T_h(\xi) \le 2N^\star h^\star = 2\sqrt{\frac{2d_S(\xi)}{\mu}},$$

which is the desired estimate.

Assume now that (IS.0) holds at x. In this case (5.34) yields, for a suitable constant M_3

$$d_{S_{-\sigma}}\Big(x_\xi(t)\Big) \le d_{S_{-\sigma}}(\xi) - \mu t + \frac{M_3}{\rho-\sigma}t^2.$$

Then the argument is analogous and simpler than the previous one. Note that in this case the estimate on the discrete time is

$$T_h(\xi) \le C d_S(\xi).$$

□

5.4 The discrete dynamic programming approach and convergence

Following the well established literature on the dynamic programming approach (see [17], [37] and references therein) we consider the Kružkov transformation, namely we define

$$v_S(x) = \begin{cases} 1 - e^{-T_S(x)} & x \in \mathcal{R}^S, \\ 1 & x \notin \mathcal{R}^S, \end{cases} \tag{5.40}$$

and recall that v_S is the unique bounded viscosity solution of the boundary value problem

$$\begin{cases} v_S(x) + \sup_{u\in[-1,1]^M}\{\langle -f(x,u), \nabla v_S(x)\rangle\} = 1 & \text{in } \mathbb{R}^n \setminus S, \\ v_S(x) = 0 & \text{on } S \end{cases} \tag{5.41}$$

where $f(x,u) = f_0(x) + \sum_{i=1}^{M} f_i(x)u_i$ (see Theorem IV.2.6 and Proposition II.2.5 in [16]).

We define also, for a given step size $h > 0$,

$$v_h(x) = 1 - e^{-T_h(x)}, \tag{5.42}$$

where $T_h(x)$ is the discretized minimum time function which was defined in (5.27). Observe that $v_h(x)$ is the value function of a discrete optimal control problem, namely,

$$v_h(x) = \begin{cases} \min_{\{A_i\}\subset[-1,1]^{Ml}} J_x^h(\{A_i\}) & \text{for } x \in \mathcal{R}_h^S \\ 1 & \text{for } x \notin \mathcal{R}_h^S, \end{cases} \tag{5.43}$$

where

$$J_x^h(\{A_i\}) = 1 - e^{-hn_h(\{A_i\},x)} = \Big(\sum_{j=0}^{n_h(\{A_i\},x)-1} e^{-jh}\Big)(1-e^{-h})\chi_{S^c}(x), \tag{5.44}$$

and $\chi_{S^c}(x) = 1$ if $x \notin S$ and 0 otherwise. Following Theorem 2.3 in [17], we observe that v_h is the unique bounded solution of the following problem:

$$\begin{cases} V(x) = A(V(x)) & \forall x \in \mathbb{R}^n \setminus S \\ V(x) = 0 & \forall x \in S \end{cases} \tag{5.45}$$

where $A(V(x)) = \inf_{A\in[-1,1]^{Ml}}\{e^{-h}V(x + h\Phi(x,A,h))\} + 1 - e^{-h}$.

Furthermore, owing to (5.28) and Remark 5.2.3, there exists a constant C such that

$$T_h(x) \le C\sqrt[k]{d_S(x)}, \forall x \in \mathcal{R}^S.$$

Therefore, by Theorem 3.3 in [17], we obtain the following

Theorem 5.4.1. *Let the assumptions of Theorem 5.2.2 hold and let v_S, v_h be defined according to (5.40), (5.42), respectively. Then $v_h \to v_S$ locally uniformly in $\mathbb{R}^n$ and $hN_h \to T_S$ locally uniformly in $\mathcal{R}^S$.*

5.5 Fully discrete scheme and error estimates

Let $S \subset \mathbb{R}^n$ be a compact nonempty set and let the assumptions of Theorem 5.2.2 or of Theorem 5.3.3 hold on some compact neighborhood of S, S_δ. Before continuing, we observe that it is enough to consider any one step method which has at least $(k+1)$-th order of convergence. To make a slightly more general approach, in the sequel we always consider a method with order higher or equal to $k+1$. We will describe our results only for the case of Theorem 5.2.2, since the other one requires only small modifications. We recall that error estimates for pursuit evasion differential games, under Hölder continuity assumptions but with a first order time discretization, were obtained in [70].

We recall that, according to Theorem 4.2.1, under our assumptions the minimum time T_S is Hölder continuous on S_δ and there exists a constant C such that

$$T_S(x) \le C \sqrt[k]{d_S(x)}, \quad \forall x \in S_\delta \setminus S. \tag{5.46}$$

This inequality implies that $v_S(x) \in C^{0,1/k}(S_\delta)$ (see, e.g., [16, Remark 1.7, p. 230]). Moreover, the discrete minimum time function T_h is finite on S_δ and satisfies

$$T_h(x) \le C \sqrt[k]{d_S(x)}, \quad \forall x \in S_\delta \setminus S, \tag{5.47}$$

provided $h > 0$ is small enough (see Theorem 5.2.2).

We consider the dynamical system (5.1) and its corresponding one step $(q+1)$-th order scheme (5.2). We make the following assumptions on the scheme to preserve the order of the method:

(A.1) For any $x \in \mathbb{R}^n$ and any measurable $u\colon [0,h) \to [-1,1]^M$ there exists a $M \times l$ (where l depends on the chosen method) matrix $A \in [-1,1]^{Ml}$ such that

$$\|y(h,x,u) - y_h(h,x,A)\| \le C h^{q+2}, \tag{5.48}$$

where C is a constant, $q \ge k$, and $y(h,x,u)$ stands for the exact solution of (5.1) following the control u and $y_h(h,x,A) = x + h\Phi(x,A,h)$.

Conversely,

(A.2) for any matrix $A \in [-1,1]^{Ml}$, there exists a measurable control $u\colon [0,h) \to [-1,1]^M$ such that (5.48) holds.

Such assumptions are used, for example, in [36, 40]. Higher order one step methods satisfying (5.48) for control systems of the type considered here are constructed in [46]. The assumption (A.2) is satisfied by taking u to be piecewise constant (with entries of A) on subsequent intervals of length h/l.

We now deal with space discretization. For convenience we recall that $v_h(x) = 1 - e^{-T_h(x)}$ is the unique bounded solution of the problem

$$\begin{cases} v_h(x) = \inf_{A \in [-1,1]^{Ml}} \{e^{-h} v_h(x + h\Phi(x,A,h))\} + 1 - e^{-h} & \text{on } \mathbb{R}^n \setminus S \\ v_h(x) = 0 & \text{on } S \end{cases} \tag{5.49}$$

provided that $h > 0$ is small enough.

Let $\Gamma = \{x_i : i = 1, ..., I\}$ be a space grid for the domain $\Omega \subset \mathcal{R}^S$, with $\overline{\Omega} = \cup_j S_j$, such that the diameter of each cell S_j corresponding to Γ is less than or equal to Δx. Let

$$W^k = \{\omega : \Omega \to \mathbb{R} : \omega(\cdot) \in C(\Omega), D\omega(x) = a_j, \forall x \in S_j, \forall j\}$$

be the class of piecewise linear functions on Ω. We look for an approximate solution of (5.49) belonging to W^k. For any $\phi(\cdot)$ defined on Ω, let $I^1_\Gamma[\phi](x) = \sum_i^I \lambda_i(A)\phi(x_i)$, where $x = \sum_i^I \lambda_i(A)x_i$, $\lambda_i(A) \in [0, 1]$, $\sum_{i=1}^I \lambda_i(A) = 1$ for any $A \in [-1, 1]^{Ml}$, see [34] for more information.

Now we are going to replace (5.49) with its fully discrete version by substituting $v_h(x_i + h\Phi(x_i, A, h))$ with

$$I^1_\Gamma[v_h](x_i + h\Phi(x_i, A, h)).$$

More precisely, in order to construct a fully discretized minimum time function we set $\Gamma^\star := \{x \in \Gamma$: there exists a control matrix A such that $x + h\Phi(x, A, h) \in \Omega\}$ and consider the problem

$$\begin{cases} v_h^{\Delta x}(x) = \min_{A\in[-1,1]^{Ml}}\{e^{-h} I^1_\Gamma[v_h^{\Delta x}](x + h\Phi(x, A, h))\} + 1 - e^{-h} & \text{if } x \in \Gamma^\star \setminus S, \\ v_h^{\Delta x}(x) = 0 & \text{if } x \in \Gamma^\star \cap S, \\ v_h^{\Delta x}(x) = 1 & \text{if } x \in \Gamma \setminus \Gamma^\star. \end{cases} \tag{5.50}$$

Let V be a function on the grid Γ and define the operator $A_h^{\Delta x}[V](x)$ by setting, for all $x \in \Gamma \setminus S$,

$$A_x^{\Delta x}[V](x) = \min_{A\in[-1,1]^{Ml}} \{e^{-h} I^1_\Gamma[V](x + h\Phi(x, A, h))\} + 1 - e^{-h}.$$

By using the same arguments of Section 5.2 in [37], it not difficult to prove that $A_h^{\Delta x}$ is monotone, namely if $V_1(x) \le V_2(x)$ for all $x \in \Gamma$, then

$$A_h^{\Delta x}[V_1](x) \le A_h^{\Delta x}[V_2](x).$$

Moreover, $A_h^{\Delta x}[\cdot]$ considered componentwise is a contraction from $\mathbb{R}^I$ to $\mathbb{R}^I$ with contraction coefficient e^{-h}. Therefore the fixed point problem (5.50) has indeed a unique solution for all $0 < h < 1$ and $\Delta x > 0$, which we label $v_h^{\Delta x}$. Notice that $v_h^{\Delta x}$ is computed only at the grid nodes, but it can be extended by interpolation over the whole of Ω. More precisely, from now on, for every $x \in \Omega$ $v_h^{\Delta x}(x)$ means that

$$\begin{cases} v_h^{\Delta x}(x) \text{ is the solution of (5.50)} & \text{if } x \in \Gamma, \\ v_h^{\Delta x}(x) = I^1_\Gamma[v_h^{\Delta x}](x) & \text{if } x \in \Omega \setminus \Gamma. \end{cases} \tag{5.51}$$

The next results are devoted to error estimates. The first lemmas deal with the (semi)discrete minimum time function. More precisely we will prove that $\|v_S - v_h\|_{\infty,\Omega} \le Ch^{\frac{q+1}{k}}$. We denote by $\|\cdot\|_{\infty,\Omega}$ the usual supremum norm taken on Ω and recall that the functions $n(\{A_i\}, x)$ and $N_h(x)$ were defined in (5.25) and (5.26), respectively.

Lemma 5.5.1. *Assume that (5.46) holds in a neighborhood S_δ of the target S (in particular this happens under the assumptions of Theorem 5.2.2), together with (A.2). Then there exist two positive constants $\bar{h}$ and C such that*

$$T_S(x) - hN_h(x) \le Ch^{\frac{q+1}{k}}, \text{ for any } x \in \Omega,\ h \le \bar{h}.$$

Proof. For any fixed $x \in \Omega$, we choose a sequence of control matrices $\{A_i\} \subset [-1,1]^{Ml}$ such that $n(\{A_i\}, x) = N_h(x)$. According to (A.2) and to the fact that x belongs to the compact set Ω, there exists a measurable control u, with $u_i(t) \in [-1,1]^M$ a.e., such that

$$\|y(hN_h(x), x, u) - y_h(hN_h(x), x, \{A_i\})\| \le C_\Phi h^{q+1}.$$

By choosing $h \le \sqrt[q+1]{\frac{\delta}{C_\Phi}}$, we obtain $y(hN_h(x), x, \{u_i\}) \in S_\delta$. Then due to (5.46) we obtain the inequality

$$T_S(x) \le hN_h(x) + C(C_\Phi h^{q+1})^{1/k}.$$

Equivalently, $T_S(x) - hN_h(x) \le Ch^{\frac{q+1}{k}}$, for a suitable constant C. □

The analogous estimate for $hN_h(x) - T_S(x)$ can be obtained by using (A.1) in place of (A.2).

Lemma 5.5.2. *Assume that (5.47) and (A.1) hold in a neighborhood S_δ of the target S. Then there exist $\bar{h}$ and $C > 0$ such that*

$$hN_h(x) - T_S(x) \le Ch^{\frac{q+1}{k}}, \text{ for any } x \in \Omega,\ h \le \bar{h}.$$

Proof. Let u be an optimal control steering x to S and fix a discretization step $h > 0$ small enough. By (A.1), there exists a sequence of control matrices $\{A_n\}$, $n = 0, \dots, N < +\infty$, with entries in $[-1,1]$ such that

$$\|y(T_S(x), x, u) - y_h(T_S(x), x, \{A_n\})\| \le C_\Phi h^{q+1}.$$

Then by choosing $h \le \sqrt[q+1]{\frac{\delta}{C_\Phi}}$, we obtain $y_h(hN_h(x), x, \{A_n\}) \in S_\delta$. Thus by (5.47), we receive

$$hN_h(x) \le T_S(x) + Ch^{\frac{q+1}{k}}$$

and the proof is concluded. □

In the sequel, for the sake of simplicity, we will sometimes use the same letter for different constants. Combining Lemma 5.5.1 and (5.5.2), we obtain

$$|hN_h(x) - T_S(x)| \le Ch^{\frac{q+1}{k}} \tag{5.52}$$

and applying the mean value theorem, from (5.52) we obtain

$$|v_S(x) - v_h(x)| \le Ch^{\frac{q+1}{k}}. \tag{5.53}$$

Remembering that C may depend on $|x|$, we can choose a global constant C such that (5.53) holds for every $x \in \Omega$. Thus we obtain a uniform estimate for v_S, namely

$$\|v_S - v_h\|_{\infty,\Omega} \leq C h^{\frac{q+1}{k}}. \tag{5.54}$$

The following result is devoted to establishing an error estimate for the fully discrete value function, namely an upper bound for $\left\|v_S - v_h^{\Delta x}\right\|_{\infty,\Omega}$.

Theorem 5.5.3. *Assume that the assumptions of Lemmas 5.5.2 and 5.5.1 hold. Then there exist suitable constants C_1, C_2, $\bar{h}$ such that for every $h \in (0, \bar{h}]$*

$$\left\|v_S - v_h^{\Delta x}\right\|_{\infty,\Omega} \leq C_1 h^{\frac{q+1}{k}-1} + C_2 \frac{(\Delta x)^{1/k}}{h}.$$

Proof. Recalling the semidiscrete and the fully discrete dynamic programming principle, for any $x \in \Gamma \setminus S$ we have

$$v_h(x) = \inf_{A \in [-1,1]^{Ml}} \{e^{-h} v_h(x + h\Phi(x, A, h))\} + 1 - e^{-h}, \tag{5.55}$$

$$v_h^{\Delta x}(x) = \inf_{A \in [-1,1]^{Ml}} \{e^{-h} I_\Gamma^1[v_h^{\Delta x}](x + h\Phi(x, A, h))\} + 1 - e^{-h}. \tag{5.56}$$

Let $A^\star$ be an optimal control matrix in (5.55). Then for any $x \in \Gamma$ we obtain

$$\begin{aligned}
v_h^{\Delta x}(x) - v_h(x) &\leq e^{-h} I_\Gamma^1[v_h^{\Delta x}](x + h\Phi(x, A^\star, h)) - e^{-h} v_h(x + h\Phi(x, A^\star, h)) \\
&\leq e^{-h} \Big(\left| I_\Gamma^1[v_h^{\Delta x}](x + h\Phi(x, A^\star, h)) - I_\Gamma^1[v_h](x + h\Phi(x, A^\star, h)) \right| \\
&\quad + \left| I_\Gamma^1[v_h](x + h\Phi(x, A^\star, h)) - I_\Gamma^1[v_S](x + h\Phi(x, A^\star, h)) \right| \\
&\quad + \left| I_\Gamma^1[v_S](x + h\Phi(x, A^\star, h)) - v_S(x + h\Phi(x, A^\star, h)) \right| \\
&\quad + \left| v_S(x + h\Phi(x, A^\star, h)) - v_h(x + h\Phi(x, A^\star, h)) \right| \Big) \\
&\leq e^{-h} \left\|v_h^{\Delta x} - v_h\right\|_{\infty,\Gamma} + C_2 (\Delta x)^{1/k} + C_1 h^{\frac{q+1}{k}}
\end{aligned}$$

where in the last inequality we used the monotonicity of $I_\Gamma^1[\cdot]$, the Hölder continuity of $v_S(\cdot)$, and (5.54). In an entirely similar way, we also obtain

$$v_h(x) - v_h^{\Delta x}(x) \leq e^{-h} \left\|v_h^{\Delta x} - v_h\right\|_{\infty,\Gamma} + C_1 h^{\frac{q+1}{k}} + C_2 (\Delta x)^{1/k}.$$

Thus $(1 - e^{-h}) \left\|v_h - v_h^{\Delta x}\right\|_{\infty,\Gamma} \leq C_1 h^{\frac{q+1}{k}} + C_2 (\Delta x)^{1/k}$. Since $1 - e^{-h} = h + O(h^2)$, by possibly modifying C_1 and C_2 we receive, for all $x \in \Gamma$,

$$\left\|v_h - v_h^{\Delta x}\right\|_{\infty,\Gamma} \leq C_1 h^{\frac{q+1}{k}-1} + C_2 \frac{(\Delta x)^{1/k}}{h}.$$

Therefore, for every $x \in \Omega$,

$$\begin{aligned} v_h^{\Delta x}(x) - v_h(x) &\le |I_\Gamma^1[v_h^{\Delta x}](x) - I_\Gamma^1[v_h](x)| + |I_\Gamma^1[v_h](x) - I_\Gamma^1[v_S](x)| \\ &\quad + |I_\Gamma^1[v_S](x) - v_S(x)| + |v_S(x) - v_h(x)| \\ &\le \left\| v_h^{\Delta x} - v_h \right\|_{\infty,\Gamma} + \|v_h - v_S\|_{\infty,\Omega} + |I_\Gamma^1[v_S](x) - v_S(x)| \\ &\quad + \|v_S - v_h\|_{\infty,\Omega} \\ &\le C_1 h^{\frac{q+1}{h}-1} + C_2 \frac{(\Delta x)^{1/k}}{h}. \end{aligned}$$

Analogously, we receive the same estimate for the reversed direction, i.e.,

$$v_h(x) - v_h^{\Delta x}(x) \le C_1 h^{\frac{q+1}{h}-1} + C_2 \frac{(\Delta x)^{1/k}}{h}.$$

Thus, we have

$$\left\| v_h^{\Delta x} - v_h \right\|_{\infty,\Omega} \le C_1 h^{\frac{q+1}{k}-1} + C_2 \frac{(\Delta x)^{1/k}}{h}, \tag{5.57}$$

for every $x \in \Omega$. Putting together (5.54) and (5.57), we obtain the error estimate of the fully discrete value function

$$\left\| v_S - v_h^{\Delta} \right\|_{\infty,\Omega} \le C_1 h^{\frac{q+1}{h}-1} + C_2 \frac{(\Delta x)^{1/k}}{h}$$

The proof is complete. □

5.6 Approximate feedback controls and suboptimal trajectories

This section is devoted to constructing (approximate) suboptimal feedback controls, together with obtaining an error estimate for the related cost function.

Recall that the semidiscrete dynamic programming principle (SDDPP) was stated in (5.49). The *semidiscrete feedback* is defined, for a given time discretization step h, by picking any control matrix $A_h(x)$ such that

$$A_h(x) \in \operatorname{argmin}_{A \in [-1,1]^{Ml}} \{e^{-h} v_h(x + h\Phi(x, A, h))\}.$$

We define also a sequence of control matrices $A_h(y_m)$, where y_m is the solution of the discrete dynamical system

$$\begin{cases} y_{m+1} &= y_m + h\Phi(y_m, A_h(y_m), h) \\ y_0 &= x. \end{cases}$$

According to (A.2), there exists a measurable control $u_h(y_m)$, corresponding to each $A_h(y_m)$, such that

$$\|y(h, y_m, u_h(y_m)) - y_h(h, y_m, A_h(y_m))\| \le C h^{q+2}.$$

Let $S_{-\sigma}$ be a shrinking of the target S. Consider the (SDDPP) for $S_{-\sigma}$, namely

$$v_{h,\sigma}(x) = \inf_{A\in[-1,1]^{Ml}} \{e^{-h} v_{h,\sigma}(x + h\Phi(x, A, h))\} + 1 - e^{-h}, \quad v_{h,\sigma}(x) = 0 \text{ on } S_{-\sigma}. \tag{5.58}$$

Let $A_{h,\sigma}(y_m)$, $u_{h,\sigma}(y_m)$ be defined as $A_h(y_m)$, $u_h(y_m)$ above and set

$$A^{\star,m}_{h,\sigma} := A_{h,\sigma}(y_m) \quad \text{and} \quad u^\star_{h,\sigma}(s) := u_{h,\sigma}(y_m), \tag{5.59}$$

for $s \in [mh, (m+1)h)$, $m = 1, \ldots$. Set also

$$J(u, x) = 1 - e^{-t_S(u,x)}, \qquad J_{h,\sigma}(\{A_i\}, x) = 1 - e^{-hn_{h,\sigma}(\{A_i\},x)},$$

where $t_S(u, x)$ was defined in (2.3) and $n_{h,\sigma}$ is the smallest integer n (if any) such that $y_h(nh, x, \{A_i\})$ belongs to $S_{-\sigma}$. The first result of this section is concerned with an error estimate for the cost function $J(u^\star_{h,\sigma}(\cdot), x)$ compared with $\inf_{u(\cdot)\in U} J(u(\cdot), x)$, under suitable assumptions.

Proposition 5.6.1. *Assume that there exists $\bar{h} > 0$ such that, for $0 < h < \bar{h}$, (A.1), (A.2), and the assumptions of Theorem 5.2.2 hold, where σ is chosen sufficiently small. Then $J(u^\star_{h,\sigma}(\cdot), x) \le \inf_{u(\cdot)\in U} J(u(\cdot), x) + \epsilon(\sigma, h)$ for every $x \in \Omega$, where $\epsilon(\sigma, h) \to 0$, as $\sigma, h \to 0$.*

Proof. Recall that, according to Theorem 5.2.2, for all $\bar{x} \in S \setminus S_{-\sigma}$ we have

$$T_{h,\sigma}(\bar{x}) \le C\sqrt[k]{\sigma} =: \omega(\sigma), \tag{5.60}$$

where k is the maximal order of Lie brackets appearing in Theorem 5.2.2. Let $x \in \Omega \setminus S$ and assume there exists $N \in \mathbb{N}$ and a sequence of control matrices $\{A^{\star,m}_{h,\sigma}\}$, $m = 1, \ldots, N$, constructed according to (5.59) such that $y_h(Nh, x, \{A^{\star,m}_{h,\sigma}\}) \in S_{-\sigma}$. By the assumption (A.2), there exists a corresponding control $u^\star_{h,\sigma}(\cdot) \in U$ such that $y(Nh, x, u^\star_{h,\sigma}) \in S$ with $0 < h < \bar{h}$, whence we obtain

$$J(u^\star_{h,\sigma}(\cdot), x) \le J_{h,\sigma}(\{A^{\star,m}_{h,\sigma}\}, x) = v_{h,\sigma}(x).$$

Thus

$$\begin{aligned} J(u^\star_{h,\sigma}(\cdot), x) - \inf_{u(\cdot)\in U} J(u(\cdot), x) &\le J_{h,\sigma}(\{A^{\star,m}_{h,\sigma}\}, x) - v_S(x) \\ &= v_{h,\sigma}(x) - v_h(x) + v_h(x) - v_S(x) \le \epsilon(\sigma, h), \end{aligned}$$

where we used the mean value theorem and (5.60), together with (5.54). Note that the desired estimate is trivial for any $x \in \Omega$ where there does not exist any sequence of control matrices $\{A^{\star,m}_{h,\sigma}\}$ which steers x to $S_{-\sigma}$ or for any $x \in S \cap \Omega$. The proof is complete. □

We consider now the fully discrete version of (5.58) and use it to define our approximate feedback.

Definition 5.6.2. *Let the space mesh Γ, with cell diameter Δx, for the domain Ω and $\sigma > 0$ and $h > 0$ be fixed. For each $x \in \Omega \setminus S_{-\sigma}$ we define the* approximate (fully discrete) feedback $A^{\sigma}_{\Delta x,h}(x)$*, relative to Δx, h and $S_{-\sigma}$, by picking any*

$$A^{\sigma}_{\Delta x,h}(x) \in \operatorname{argmin}_{A\in[-1,1]^{Ml}}\{e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A, h))\}. \tag{5.61}$$

As we did for the semidiscrete case, we consider the sequence of control matrices $A^{\sigma}_{\Delta x,h}(y_m)$, where y_m is computed by

$$\begin{cases} y_{m+1} &= y_m + h\Phi(y_m, A^{\sigma}_{\Delta x,h}(y_m), h) \\ y_0 &= x. \end{cases}$$

Again, according to (A.2), there exists a measurable control $u^{\sigma}_{\Delta x,h}(y_m)$ corresponding to $A^{\sigma}_{\Delta x,h}(y_m)$ such that

$$\left\| y(h, y_m, u^{\sigma}_{\Delta x,h}(y_m)) - y_h(h, y_m, A^{\sigma}_{\Delta x,h}(y_m)) \right\| \leq Ch^{q+2}. \tag{5.62}$$

Let

$$u^{\star,\sigma}_{\Delta x,h}(s) := u^{\sigma}_{\Delta x,h}(y_m), \qquad A^{\star,\sigma,m}_{\Delta x,h} := A^{\sigma}_{\Delta x,h}(y_m) \tag{5.63}$$

for $s \in [mh, (m + 1)h)$, $m = 1, \ldots, N$.

We are interested in estimating the difference between the cost $J(u^{\star,\sigma}_{\Delta x,h}(\cdot), x)$, resp. $J_h(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x)$, and the value function $v_S(x)$. We prove first a preliminary lemma, similar to Theorem 1.7 in [34].

Lemma 5.6.3. *Let $v^{\Delta x}_{h,\sigma}(\cdot)$ and $A^{\star,\sigma,m}_{\Delta x,h}$ be defined, respectively, by (5.51) and (5.63). Then, for every $x \in \Omega$, $J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) \leq v^{\Delta x}_{h,\sigma}(x) + \frac{\varepsilon(h,\Delta x)}{1-e^{-h}}$, where*

$$\varepsilon(h, \Delta x) := C_1 h^{\frac{q+1}{k}-1} + C_2 \frac{(\Delta x)^{1/k}}{h}.$$

Proof. Recall that for all $x \in \Gamma \setminus S$ the equality

$$v^{\Delta x}_{h,\sigma}(x) = e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A^{\sigma}_{\Delta x,h}(x), h)) + 1 - e^{-h}$$

holds. We are now interested in estimating the difference between $v^{\Delta x}_{h,\sigma}(x)$ and

$$e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}]\big(x + h\Phi(x, A^{\sigma}_{\Delta x,h}(x), h)\big) + 1 - e^{-h}$$

for every $x \in \Omega \setminus S$. Recalling that the dynamic programming principle for $S_{-\sigma}$ reads as

$$v_{S,\sigma}(x) = \min_{u\in U}\{e^{-h} v_{S,\sigma}(y(h, x, u)) + 1 - e^{-h}\},$$

let

$$u^\star_\sigma(x) \in \operatorname{argmin}_{u\in U}\{e^{-h} v_{S,\sigma}(y(h, x, u)) + 1 - e^{-h}\}$$

and $A^\star_\sigma \in [-1,1]^{Ml}$ be such that

$$\|y(h,x,u^\star_\sigma(x)) - y_h(h,x,A^\star_\sigma(x))\| \le Ch^{q+2}.$$

Then we have

$$\begin{aligned}
&e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A^\sigma_{\Delta x,h}(x), h)) + 1 - e^{-h} - v^{\Delta x}_{h,\sigma}(x)\\
&\le e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A^\sigma_{\Delta x,h}(x), h)) + 1 - e^{-h} - v_{S,\sigma}(x) + |v_{S,\sigma}(x) - v^{\Delta x}_{h,\sigma}(x)|\\
&\le e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A^\star_\sigma(x), h)) + 1 - e^{-h} - (e^{-h} v_{S,\sigma}(y(h, u^\star_\sigma, x)) + 1 - e^{-h})\\
&\qquad + |v_{S,\sigma}(x) - v^{\Delta x}_{h,\sigma}(x)|\\
&\quad \le e^{-h} |I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A^\star_\sigma(x), h)) - I^1_\Gamma[v_{S,\sigma}](x + h\Phi(x, A^\star_\sigma(x), h))|\\
&\qquad + |I^1_\Gamma[v_{S,\sigma}](x + h\Phi(x, A^\star_\sigma(x), h)) - v_{S,\sigma}(x + h\Phi(x, A^\star_\sigma(x), h))|\\
&\qquad + |v_{S,\sigma}(x + h\Phi(x, A^\star_\sigma(x), h)) - v_{S,\sigma}(y(h, u^\star_\sigma, x))| + |v_{S,\sigma}(x) - v^{\Delta x}_{h,\sigma}(x)|\\
&\quad \le C_1 h^{\frac{q+1}{k}-1} + C_2 \frac{\Delta x^{1/k}}{h}.
\end{aligned}$$

Therefore, for every $x \in \Omega \setminus S$ we obtain

$$e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A^\sigma_{\Delta x,h}(x), h)) + 1 - e^{-h} - v^{\Delta x}_{h,\sigma}(x) \le C_1 h^{\frac{q+1}{k}-1} + C_2 \frac{\Delta x^{1/k}}{h},$$

or equivalently

$$1 - e^{-h} \le v^{\Delta x}_{h,\sigma}(x) - e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A^\sigma_{\Delta x,h}(x), h)) + C_1 h^{\frac{q+1}{k}-1} + C_2 \frac{\Delta x^{1/k}}{h}. \quad (5.64)$$

By multiplying both sides of (5.64) by e^{-mh} and taking $x = y_m$, we obtain

$$e^{-mh}(1 - e^{-h}) \le e^{-mh}\Big(v^{\Delta x}_{h,\sigma}(y_m) - e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](y_m + h\Phi(y_m, A^{\star,\sigma,m}_{\Delta x,h})\Big) + e^{-mh}\varepsilon(h, \Delta x).$$

Let N be the minimum number of steps to reach $S_{-\sigma(h)}$ by $\{y_m\}$. Then, by summing over m, we obtain

$$\begin{aligned}
\sum_{m=0}^{N-1} e^{-mh}(1 - e^{-h}) \le \sum_{m=0}^{N-1} e^{-mh}\Big(v^{\Delta x}_{h,\sigma}(y_m) - e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](y_m + h\Phi(y_m, A^{\star,\sigma,m}_{\Delta x,h}, h))\Big)\\
+ \varepsilon(h, \Delta x) \sum_{m=0}^{N-1} e^{-mh}.
\end{aligned}$$

After simplifying, the proof is complete. □

Now we are ready to state and prove the main result of this section. It shows that the feedback defined by (5.61) through numerical approximation is suboptimal. For the sake of clarity, we choose $\Delta x = h^{q+1}$ and set $\gamma := \frac{q+1}{k} - 1 \,(> 0)$.

Theorem 5.6.4. *Let the assumptions of Proposition 5.6.1 hold. Then, for every $x \in \Omega$,*

$$J(u^{\star,\sigma}_{\Delta x,h}(\cdot), x) \leq \inf_{u(\cdot)\in U} J(u(\cdot), x) + R(\sigma, h),$$

moreover,

$$J_h(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) \leq \inf_{u(\cdot)\in U} J(u(\cdot), x) + R(\sigma, h),$$

where $R(\sigma, h) = C\Big(\frac{h^\gamma}{1-e^{-h}} + \omega(\sigma)\Big)$, C being a suitable constant, and $u^{\star,\sigma}_{\Delta x,h}(\cdot)$, $\{A^{\star,\sigma,m}_{\Delta x,h}\}$ and $\omega(\sigma)$ are defined according to (5.63), (5.60), respectively.

Proof. From (5.57) we obtain

$$\left\| v^{\Delta x}_{h,\sigma} - v_{h,\sigma} \right\|_{\infty,\Omega} \leq C' h^\gamma,$$

whence, recalling Lemma 5.6.3 we have

$$J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) - v_{h,\sigma}(x) \leq v^{\Delta x}_{h,\sigma}(x) + \frac{\varepsilon(h, \Delta x)}{1 - e^{-h}} - v_{h,\sigma}(x) \leq \frac{C h^\gamma}{1 - e^{-h}}. \tag{5.65}$$

If $\{A^{\star,\sigma,m}_{\Delta x,h}\}$ does not steer x to $S_{-\sigma}$ through $\{y_m\}$, then $J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) = 1$. Thus $J(u^{\star,\sigma}_{\Delta x,h}(\cdot), x) \leq J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x)$. Otherwise, let $N^\star$ be the minimum number of steps to reach $S_{-\sigma}$ by $\{y_m\}$. By the assumption (A.2), there exists a control $u^{\star,\sigma}_{\Delta x,h}(\cdot) \in U$ such that $y(hN^\star, x, u^{\star,\sigma}_{\Delta x,h}) \in S$ for $0 < h < \bar{h}$ small enough, and so $J(u^{\star,\sigma}_{\Delta x,h}(\cdot), x) \leq J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x)$. Therefore

$$\begin{aligned} J(u^{\star,\sigma}_{\Delta x,h}(\cdot), x) - \inf_{u(\cdot)\in U} J(u(\cdot), x) &\leq J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) - v_S(x) \\ &= J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) - v_{h,\sigma}(x) + v_{h,\sigma}(x) - v_h(x) \\ &\qquad + v_h(x) - v_S(x) \\ &\leq R(\sigma, h), \end{aligned} \tag{5.66}$$

where the last inequality is due to (5.65) and the mean value theorem, together with (5.60), and (5.54) as in the proof of Proposition 5.6.1. To prove $J_h(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) \leq \inf_{u(\cdot)\in U} J(u(\cdot), x) + R(\sigma, h)$, we just remark that $J_h(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) \leq J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x)$, then by following the same procedure as (5.66) the proof is concluded. □

Remark 5.6.5. *The idea of using the argmin of the discretized Hamiltonian in order to construct approximate feedbacks is classical and was used [34,35,37,41] in the framework of infinite horizon problems. Adapting this idea to minimum time problems requires to handle the discontinuity of the semidiscrete value function and to make use of a shrinking of the target. To this aim, we used the robust discrete controllability result of Theorem 4.2.3.*

5.7 Numerical tests

This section is devoted to showing the output of an implementation of our scheme to two examples where the minimum time function is not Lipschitz. The papers [61, 63] contain several cases – including the two ones we are going to describe – where the assumptions of Theorem 4.2.1 are satisfied. Since in our examples the target is smooth, the assumptions of Theorem 4.2.3, on which all results of the Sections 5.5 and 5.6 devoted to algorithms are based, are satisfied as well.

The simplest example where our method applies is the well known double integrator, $\ddot{x} = u$, $|u| \le 1$. It is well known that the minimum time to reach the origin subject to this dynamics is Hölder continuous with exponent $1/2$ on the whole of $\mathbb{R}^2$. Our method applies when the target satisfies the assumptions of Theorem 5.3.3. In the second example contained in [61, Section 6], it is shown that such assumptions are satisfied (with $k = 2$) if the target is a ball centered at the origin with any radius r small enough. Figure 5.1 shows the discrete trajectories obtained via the numerical feedback (left) and the graph of the value function (right, after Kružkov transformation). The computed trajectories agree with the theoretical computations which can be made through Pontryagin's Maximum Principle. In particular, the two optimal trajectories which reach the target tangentially are correct. For this example, we refer to Section 6.3 for the error table, which is presented in comparison with a new approach.

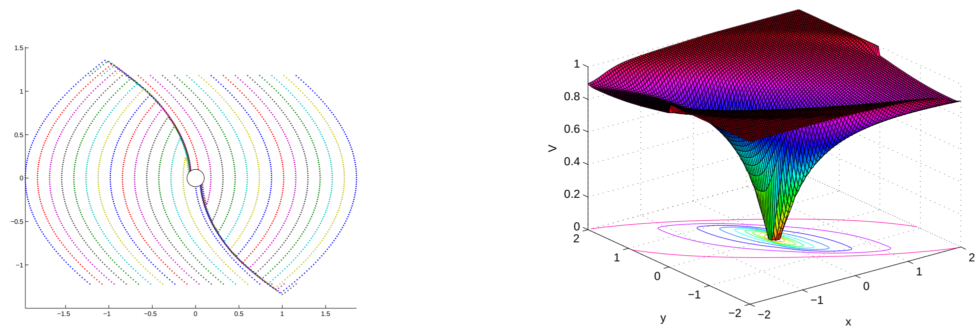

Figure 5.1: Double integrator: computed optimal trajectories (only trajectories issuing from the two horizontal segments are shown) and graph of the value function with radius of the target $r = 0.1$, $h = 0.025$, $\Delta x = 0.02$, 3rd order Runge-Kutta scheme

The second example is bilinear and is taken from [63, Example 5.19], up to the factor $1/8$ in place of 10^{-3}. The dynamics is

$$\begin{cases} \dot{x}_1 &= -\frac{x_2}{8} - x_2 u \\ \dot{x}_2 &= \frac{x_1}{8} + 2x_1 u, \end{cases} \tag{5.67}$$

where $|u| \le 1$, and the target is the unit ball. In [63] the authors prove that the assumptions of Theorem 5.3.3 are satisfied with $k = 2$. To be more precise, it is not difficult to prove that the first order condition (IS.0) is satisfied in the complement of the union of two strips centered at the axes, while in the two strips the second order conditions (IS'.1) and (IS.2) hold. Figure 5.2 is the analogue of Figure 5.1 for the dynamics (5.67). Observe that if the initial point is close enough to the target, the estimated optimal trajectory is a "bang" one, while otherwise the estimated trajectory has some switchings. The plot of the value function reveals that the time to reach the target is rapidly decreasing. This is not surprising, since the minimum time function, due to the second order controllability condition, is majorized only by the square root of the distance to the target. Equivalently, approaching to the target is very slow (like a sailor which has to beat to windward and therefore proceeds slowly in the desired direction). Since the true solution is not available, we obtain the following error table by using the solution on a fine grid as a reference. The table shows that the error behavior of the solution is the same as that of the first example (see Table 6.3) due to switchings, the Hölder continuity of the (true) solution.

Δx	h	**Error**
0.015	0.05	0.1046
0.015	0.025	0.0578
0.015	0.0125	0.0550
0.015	0.00625	0.0853

Table 5.1: Error estimates for Example (5.67) ($r = 1$, 3rd order Runge-Kutta scheme)

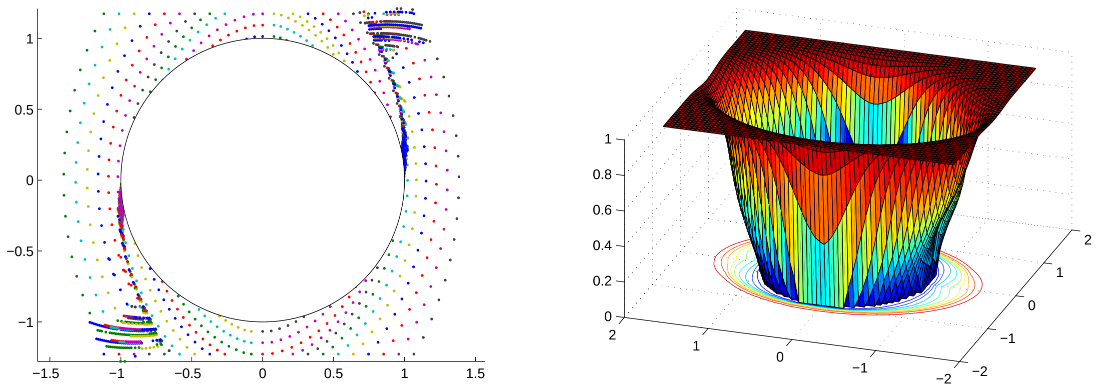

Figure 5.2: Dynamics (5.67): computed optimal trajectories (only trajectories issuing from the two horizontal segments are shown) and graph of the value function resp. with $h = 0.05$, $\Delta x = 0.027$, 3rd order Runge-Kutta scheme. .

Chapter 6

Bridge dynamic programming and a new Hamilton–Jacobi–Bellman approach

The chapter is organized as follows. Section 6.1 introduces the new formulation of the minimum time problem from a theoretical point of view and proves the bridge dynamic programming principle (BDPP) as the main technical tool for the subsequent analysis. In Section 6.2 the discretization is introduced and the numerical error is analyzed. The performance of the new approach is finally illustrated by several numerical examples in Section 6.3.

In this chapter, we consider the following controlled dynamics and its reversed one in $\mathbb{R}^n$

$$\begin{cases} \dot{y}(t) & = f(y(t), u(t)) \\ y(0) & = \xi \end{cases}, \qquad (6.1) \qquad \begin{cases} \dot{y}^-(t) & = -f(y^-(t), u(t)) \\ y(0) & = \xi \end{cases}. \qquad (6.2)$$

The following assumptions are supposed to be satisfied throughout this chapter.

Assumptions 6.0.1.

a) $f(x,u)$ is globally Lipschitz continuous in x, uniformly in u and satisfies $\|f(x,u)\| \leq K(1+\|x\|)$, for all $x \in \mathbb{R}^n$, $u \in U \in \mathbb{R}^m$, where K is a positive constant,

b) S is a compact set with C^2 boundary,

c) (f,U), $(-f,U)$ are small time controllable on S, $\overline{S^c}$ respectively. Moreover, assume $T_S(\cdot)$, $T_{S^c}(\cdot)$ are locally Hölder continuous with exponent $\frac{1}{k}$ in $\mathcal{R}^S$, $\mathcal{R}^{S^c}$, $k \in \mathbb{N}\setminus\{0\}$, i.e., for all compact subsets $K^S \subset \mathcal{R}^S$ and $K^{S^c} \subset \mathcal{R}^{S^c}$ there exists a constant $L > 0$ such that

$$\begin{aligned} |T_S(x) - T_S(y)| &\leq L\,\|x-y\|^{\frac{1}{k}} \quad \textit{for all } x,y \in K^S \\ |T_{S^c}(x) - T_{S^c}(y)| &\leq L\,\|x-y\|^{\frac{1}{k}} \quad \textit{for all } x,y \in K^{S^c}. \end{aligned} \qquad (6.3)$$

Remark 6.0.2. *Readers can find the definition of small-time controllability in [16], and sufficient conditions which guarantee the Hölder continuity of the minimum time function in, for instance, [16, 59, 61, 63]. If $k = 1$, T_S, T_{S^c} are Lipschitz continuous.*

6.1 A new formulation of the minimum time problem

Our aim in this section is to introduce a new approach to the minimum time problem, so that it is possible to design a numerical scheme to solve its corresponding discrete problem more efficiently in some cases. For $x \in \mathcal{R}^S$, consider the control system (6.1) and the target set S. Let $y^+(\cdot, \xi, u)$ with some $u \in U$ be the solution (6.1) for the sake of clarity in this chapter. Under our assumptions, recall that for all $T_S(x) \geq t > 0$ the function $T_S(x)$ satisfies the dynamic programming principle

$$T_S(x) = \inf_{\alpha \in \mathcal{U}} \{t + T_S(y^+(t, x, \alpha))\} \tag{6.4}$$

and is the unique viscosity solution of the following boundary value problem (see [16])

$$\begin{cases} \sup_{u \in U}\{-f(x,u)\nabla T_S(x)\} - 1 = 0 & \textit{in } \mathcal{R}^S \setminus S \\ T_S(x) = 0 & \textit{on } \partial S \\ T_S(x) = +\infty & \textit{as } x \to x_0 \in \partial \mathcal{R}^S. \end{cases} \tag{6.5}$$

In the *classical* approach described in Chapter 5 (see also, e.g., [17, 34]), one does care only what happens in $\overline{S^c}$, and, by definition, $T_S(x)$ is set to be zero whenever $x \in \operatorname{int} S$, where $\operatorname{int} S$ is the interior of S. Then, after the Kruzkov transform (5.40) $v_S(x)$ satisfies the dynamic programming principle

$$v_S(x) = \inf_{\alpha \in \mathcal{U}} \{\int_0^t e^{-s} ds + e^{-t} v_S(y^+(t, x, \alpha))\} \tag{6.6}$$

for all $T_S(x) \geq t > 0$ and is the unique bounded viscosity solution of (5.41), i.e.

$$\begin{cases} v_S(x) + \sup_{u \in U}\{-f(x,u)\nabla v_S(x)\} - 1 = 0 & \textit{in } \mathbb{R}^n \setminus S \\ v_S(x) = 0 & \textit{on } S. \end{cases} \tag{6.7}$$

The full discretization of (6.7) can be constructed by a semi-Lagrangian approach [37]. In this approach, first proposed for the minimum time problem in [17, 18], the problem is first discretized in time and then in space. More specifically, here we follow the approach in [26] which uses a high order one step numerical approximation in time and a first order interpolation in space. We refer to Chapter 5 for more details. As already observed in [17], when applied to this procedure (6.7), the semi-discretization in time is a piecewise constant function with jumps of size $\approx h$, a fact which may deteriorate the convergence properties of the subsequent spatial discretization. From the interpretation of the semi-discrete problem as a discrete time optimal control problem (see, e.g., [17, Section 2]) it is easily seen that this piecewise constant behavior stems from the fact that the approximate solution of (6.7) is equal to 0 on $\operatorname{int} S$. Our goal is thus to reformulate the problem so that this issue is reduced, at least, for some classes of the control systems. To this aim, instead of letting $T_S(x)$ be zero in $\operatorname{int} S$, we do as in what follows. For $x \in \mathcal{R}^{S^c}$, consider the reversed dynamics (6.2) and the new target set as the closure of the complement of the original target set S, $\overline{S^c}$. Due to the same arguments as above, $T_{S^c}(x)$ is the unique viscosity solution of

$$\begin{cases} \sup_{u\in U}\{f(x,u)\nabla T_{S^c}(x)\} - 1 = 0 & \text{in } \mathcal{R}^{S^c} \setminus \overline{S^c} \\ T_{S^c}(x) = 0 & \text{on } \partial S^c \\ T_{S^c}(x) = +\infty & \text{as } x \to x_0 \in \partial\mathcal{R}^{S^c}. \end{cases} \tag{6.8}$$

Now we are going to redefine the minimum time function as

$$T(x) = \begin{cases} T_S(x) & \text{if } x \in \mathcal{R}^S \\ 0 & \text{if } x \in \partial S \\ -T_{S^c}(x) & \text{if } x \in \mathcal{R}^{S^c} \\ +\infty & \text{if } x \to x_0 \in \partial\mathcal{R}^S, \\ -\infty & \text{if } x \to x_0 \in \partial\mathcal{R}^{S^c} \end{cases} \tag{6.9}$$

and the value function as

$$v(x) = \begin{cases} 1 - e^{-T(x)} & \text{if } x \in \mathbb{R}^n \setminus S \\ 0 & \text{if } x \in \partial S \\ e^{T(x)} - 1 & \text{if } x \in \mathbb{R}^n \setminus \overline{S^c}. \end{cases} \tag{6.10}$$

Then the minimum time problem is reformulated as $T(x)$ is the unique viscosity solution of

$$\begin{cases} \sup_{u\in U}\{-f(x,u)\nabla T(x)\} - 1 = 0 & \text{in } \mathcal{R}^S \setminus S \text{ or } \mathcal{R}^{S^c} \setminus \overline{S^c} \\ T(x) = 0 & \text{on } \partial S \\ T(x) = +\infty & \text{as } x \to x_0 \in \partial\mathcal{R}^S \text{ or } \partial\mathcal{R}^{S^c}. \end{cases} \tag{6.11}$$

It is easy to check that the transformation (6.10) satisfies the required properties of Proposition 2.5 in [16], thus $v(x)$ is the unique bounded viscosity solution of

$$\begin{cases} v(x) + \sup_{u\in U}\{-f(x,u)\nabla v(x)\} - 1 = 0 & \text{in } S^c \\ -v(x) + \sup_{u\in U}\{-f(x,u)\nabla v(x)\} - 1 = 0 & \text{in } \operatorname{int} S \\ v(x) = 0 & \text{on } \partial S, \end{cases} \tag{6.12}$$

The remaining part of this section is devoted to proving some results necessary for error estimates later on. It is easy to see that T and v satisfy the dynamic programming principles (6.4) and (6.6) whenever the optimal trajectories y^+ on the right hand side of these principles stay in S^c. Likewise, it is straightforward to see that (6.4) and (6.6) with y^- in place of y^+ hold for $-T$ and $-v$, respectively, whenever y^- stays in S. However, it remains to be clarified how these principles change for trajectories crossing ∂S. To this end, observe that under the assumptions on S, there exists $\rho > 0$, such that S satisfies both *the ρ-internal and the ρ-external sphere condition* and $S_\rho \in \mathcal{R}^S$, $S_{-\rho} \in \mathcal{R}^{S^c}$. Let $\tau_1 = \min_{x \in \partial S_\rho} T_S(x)$, $\tau_2 = \min_{x \in \partial S_{-\rho}} T_{S^c}(x)$ and $\tau = \min\{\tau_1, \tau_2\}$.

Proposition 6.1.1 (Bridge Dynamic Programming Principle (BDPP) for T)**.** *Under Assumptions 6.0.1,*

$$T(x) = \inf_{\alpha \in \mathcal{U}} \{t + T(y^+(t,x,\alpha))\} \quad \textit{for } x \in S_\tau^+,\ T_S(x) < t \le \tau, \tag{6.13}$$

$$T(x) = \sup_{\alpha \in \mathcal{U}} \{-t + T(y^-(t,x,\alpha))\} \quad \textit{for } x \in S_\tau^-,\ T_{S^c}(x) < t \le \tau. \tag{6.14}$$

Proof. By means of (6.9), (6.13) can be rewritten as

$$T_S(x) = \inf_{\alpha \in \mathcal{U}} \{t - T_{S^c}(y^+(t,x,\alpha))\}.$$

Fix $\alpha \in \mathcal{U}$ such that $t = t_S(x,\alpha) + t_{S^c}(z,\bar{\alpha})$, where $\bar{\alpha}(s) = \alpha(t-s)$, $z = y^+(t,x,\alpha)$. Then

$$t_S(x,\alpha) = t - t_{S^c}(z,\bar{\alpha}) \le t - T_{S^c}(z),$$

by taking the infimum over $\mathcal{U}$,

$$T_S(x) \le \inf_{\alpha \in \mathcal{U}} \{t - T_{S^c}(y^+(t,x,\alpha))\}.$$

Now let α, $\alpha_1 \in \mathcal{U}$, such that $t = t_S(x,\alpha) + t_{S^c}(y,\alpha_1)$ and $T_{S^c}(y) \ge t_{S^c}(y,\alpha_1) - \varepsilon$ for any fixed $\varepsilon > 0$, where $y = y^+(t,x,\bar{\alpha})$,

$$\bar{\alpha}(s) = \begin{cases} \alpha(s) & s \le t_S(x,\alpha) \\ \alpha_1(t-s) & s > t_S(x,\alpha), \end{cases}$$

then $t_S(x,\bar{\alpha}) = t - t_{S^c}(y,\alpha_1) \ge t - T_{S^c}(y) - \varepsilon$. By letting $\varepsilon \to 0^+$ and taking the infimum over $\mathcal{U}$,

$$T_S(x) \ge \inf_{\alpha \in \mathcal{U}} \{t - T_{S^c}(y^+(t,x,\alpha))\}$$

which completes the proof (6.13).

For (6.14), by exchanging the roles of S and $\overline{S^c}$, we obtain, from (6.13),

$$T_{S^c}(x) = \inf_{\alpha \in \mathcal{U}} \{t - T_S(y^-(t,x,\alpha))\},$$

or, equivalently, $-T_{S^c}(x) = \sup_{\alpha \in \mathcal{U}} \{-t + T_S(y^-(t,x,\alpha))\}$, i.e (6.14). $\square$

Proposition 6.1.2 (BDPP for v).
Under Assumptions 6.0.1,

$$v(x) = \inf_{\alpha\in\mathcal{U}}\{\int_0^t e^{-s}ds + e^{-T(x)}v(y^+(t,x,\alpha))\} \quad \text{for } x \in S_\tau^+,\ T_S(x) < t \le \tau, \tag{6.15}$$

$$v(x) = \sup_{\alpha\in\mathcal{U}}\{-\int_0^t e^{-s}ds + e^{-T(x)}v(y^-(t,x,\alpha))\} \quad \text{for } x \in S_\tau^-,\ T_{S^c}(x) < t \le \tau. \tag{6.16}$$

Proof. To start with, we prove (6.15). Let $J(x,\alpha) = \int_0^{t_S(x,\alpha)} e^{-s}ds$, by definition, $v(x) = \inf_{\alpha\in\mathcal{U}} J(x,\alpha)$. Let $\alpha \in \mathcal{U}$ such that $t = t_S(x,\alpha)+t_{S^c}(z,\bar{\alpha})$ and $t_S(x,\alpha) \le T_S(x)+\varepsilon\ (< t)$ for any fixed $\varepsilon > 0$ small enough, where $z = y^+(t,x,\alpha)$, $\bar{\alpha}(s) = \alpha(t-s)$. We have

$$\begin{aligned} J(x,\alpha) &= \int_0^t e^{-s}ds - \int_{t_S(x,\alpha)}^t e^{-s}ds = \int_0^t e^{-s}ds + e^{-t_S(x,\alpha)}(e^{-(t-t_S(x,\alpha))}-1) \\ &= \int_0^t e^{-s}ds + e^{-t_S(x,\alpha)}(e^{-t_{S^c}(z,\bar{\alpha})}-1) \\ &\le \int_0^t e^{-s}ds + e^{-t_S(x,\alpha)}v(y^+(t,x,\alpha)) \\ &\le \int_0^t e^{-s}ds + e^{-T_S(x)-\varepsilon}v(y^+(t,x,\alpha)). \end{aligned} \tag{6.17}$$

From (6.17), by letting $\varepsilon \to 0^+$ and taking the infimum over $\mathcal{U}$, we obtain

$$v(x) \le \int_0^t e^{-s}ds + e^{-T_S(x)}v(y^+(t,x,\alpha)).$$

Let α, $\alpha_1 \in \mathcal{U}$, such that $t = t_S(x,\alpha)+t_{S^c}(z,\alpha_1)$ and $t_{S^c}(z,\alpha_1) \le T_{S^c}(z)+\varepsilon$ for any fixed $\varepsilon > 0$, where $z = y^+(t,x,\bar{\alpha})$,

$$\bar{\alpha}(s) = \begin{cases} \alpha(s) & s \le t_S(x,\alpha) \\ \alpha_1(t-s) & s > t_S(x,\alpha) \end{cases}.$$

Then

$$\begin{aligned} J(x,\bar{\alpha}) &= \int_0^t e^{-s}ds + e^{-t_S(x,\alpha)}(e^{-t_{S^c}(z,\alpha_1)}-1) \\ &\ge \int_0^t e^{-s}ds + e^{-t_S(x,\alpha)}(e^{-T_{S^c}(z)-\varepsilon}-1) \\ &\ge \int_0^t e^{-s}ds + e^{-T_S(x)}(e^{-T_{S^c}(z)-\varepsilon}-1). \end{aligned} \tag{6.18}$$

By taking the infimum over $\mathcal{U}$ and letting $\varepsilon \to 0^+$ in (6.18), we receive

$$v(x) \ge \int_0^t e^{-s}ds + e^{-T_S(x)}v(y^+(t,x,\alpha)).$$

Thus (6.15) is proved. Analogously for (6.16), the proof is completed. □

Remark 6.1.3. *We remark that the factor* $e^{-T(x)}$ *in front of* v *in* (6.15) *and* (6.16) *is different from the factor* e^{-t} *in the "usual" dynamic programming principle* (6.6). *This is due to the fact that the Kruzkov transform, i.e., the exponential transformation in* (6.10), *acts with different signs inside and outside* S *and the different factor is needed as a correction term when the optimal trajectories are crossing* ∂S.

6.2 Discretization and error estimates

In this section, we are going to introduce the semi- and fully discrete version of (6.12) constructed by means of a high order one step numerical approximation in time and a first order interpolation in space. We will also provide error estimates of the value function $\|v_h - v\|_{\infty,\Omega}$, $\|v_\Delta - v\|_{\infty,\Omega}$, where $\|\cdot\|_{\infty,\Omega}$ is a usual supremum norm taken on Ω and v_h, v_Δ, Ω will be specified later, carried out under the continuity property of v, the local error of the numerical scheme, in conjunction with Propositions 6.1.1 and 6.1.2.

6.2.1 Discretization

Fix $h > 0$ sufficiently small and let $q > 0$ be a given integer number. We approximate (6.1),(6.2) by means of a one step $q-$th order numerical scheme which is fully described in Chapter 5. The numerical approximations of (6.1),(6.2) can be written in the following classical forms, respectively,

$$\begin{cases} y^+_{n+1} &= y^+_n + h\Phi^+(y^+_n, A_n, h) \\ y^+_0 &= \xi \end{cases}, \quad (6.19) \qquad \begin{cases} y^-_{n+1} &= y^-_n + h\Phi^-(y^-_n, A_n, h) \\ y^-_0 &= \xi \end{cases}, \quad (6.20)$$

where A_n is an $m \times l$ control matrix, $A_n \in U^l$, and $l \in \mathbb{N} \setminus \{0\}$ depends on the specific method. The increment functions Φ^+, Φ^- satisfy the properties as those of Φ in Section 5.1. We also need the assumptions (A.1) and (A.2) (see Section 5.5) on the schemes (6.19), (6.19) to preserve the order of the method for proving error estimate later. Let recall them for convenience.

Assumptions 6.2.1.

(O.1) For any $x \in \mathbb{R}^n$ *and any measurable* $u, v\colon [0, h) \to U$ *there exists* $m \times l$ *matrices* A, $\bar{A} \in U^l$ *such that*

$$\begin{aligned} \left\|y^+(h, x, u) - y^+_h(h, x, A)\right\| &\le Ch^{q+1}, \\ \left\|y^-(h, x, v) - y^-_h(h, x, \bar{A})\right\| &\le Ch^{q+1}, \end{aligned} \tag{6.21}$$

where C *is a constant.*

Conversely,

(O.2) for any two matrices A, $\bar{A} \in U^l$, there exists measurable controls $u, \upsilon\colon [0,h) \to U$ such that (6.21) holds.

For the construction of higher order one step methods satisfying Assumptions 6.2.1 for some classes of control systems we refer to in [46]. We only remark that the simplest scheme satisfying Assumptions 6.2.1 (with $q = 1$) is the Euler scheme with $A = u$ and $\Phi^+(y,u,h) = f(y,u)$, $\Phi^-(y,u,h) = -f(y,u)$.

Consider the following problem as the discrete version of (6.12)

$$\begin{cases} \upsilon_h(x) = \inf_{A\in U^l}\{e^{-h}\upsilon_h(x + h\Phi^+(x,A,h))\} + 1 - e^{-h} & \textit{for } x \in S^c \\ \upsilon_h(x) = \sup_{A\in U^l}\{e^{-h}\upsilon_h(x + h\Phi^-(x,A,h))\} + e^{-h} - 1 & \textit{for } x \in \operatorname{int} S \\ \upsilon_h(x) = 0 & \textit{for } x \in \partial S. \end{cases} \tag{6.22}$$

Now we are going to show that (6.22) has a unique bounded solution υ_h by a contraction mapping argument, and υ_h can be considered as an approximation of υ by proving that an upper bound of the error estimate $\|\upsilon_h - \upsilon\|_{\infty,\Omega}$ on some compact domain is bounded by some function depending on h continuously.

Theorem 6.2.2. *The problem* (6.22) *admits a unique bounded solution $\upsilon \in L^\infty(\mathbb{R}^n)$. Moreover, $\|\upsilon\|_\infty \le 1$.*

Proof. Denoting

$$\begin{aligned} B_{L^\infty}(0,r) &= \{\upsilon \in L^\infty(\mathbb{R}^n)\colon\ \|\upsilon\|_\infty \le r\}, \\ (S_{out}\upsilon)(x) &= \inf_{A\in U^l}\{e^{-h}\upsilon(x + h\Phi^+(x,A,h))\} + 1 - e^{-h}, \\ (S_{int}\upsilon)(x) &= \sup_{A\in U^l}\{e^{-h}\upsilon(x + h\Phi^-(x,A,h))\} + e^{-h} - 1, \\ (S\upsilon)(x) &= \begin{cases} (S_{out}\upsilon)(x) & \textit{for } x \in S^c \\ (S_{int}\upsilon)(x) & \textit{for } x \in \operatorname{int} S, \end{cases} \end{aligned}$$

(6.22) can be rewritten as

$$\begin{cases} \upsilon(x) = (S_{out}\upsilon)(x) & \textit{for } x \in S^c \\ \upsilon(x) = (S_{int}\upsilon)(x) & \textit{for } x \in \operatorname{int} S \\ \upsilon(x) = 0 & \textit{for } x \in \partial S \end{cases}$$

or equivalently

$$\begin{cases} \upsilon(x) = (S\upsilon)(x) & \textit{for } x \in S^c \cup \operatorname{int} S \\ \upsilon(x) = 0 & \textit{for } x \in \partial S. \end{cases}$$

We will divide the proof into two main steps. We first prove that S maps $B_{L^\infty}(0,1)$ into $B_{L^\infty}(0,1)$ and then that S is a contraction mapping. Then the existence of a unique solution follows from Banach's fixed point theorem.

Step 1: $S\colon B_{L^\infty}(0,1) \to B_{L^\infty}(0,1)$. For $\|v\|_\infty \leq 1$, we observe that

$$(S_{out}v)(x) \leq e^{-h}\|v\|_\infty + 1 - e^{-h} \leq 1,$$
$$(S_{out}v)(x) \geq -e^{-h}\|v\|_\infty + 1 - e^{-h} \geq 1 - 2e^{-h} > -1.$$

Similarly, we also have

$$(S_{int}v)(x) \leq e^{-h}\|v\|_\infty + e^{-h} - 1 \leq 2e^{-h} - 1 < 1,$$
$$(S_{int}v)(x) \geq -e^{-h}\|v\|_\infty + e^{-h} - 1 \geq -1.$$

Then, $\|S_{out}v\|_\infty \leq 1$, $\|S_{int}v\|_\infty \leq 1$ for $v \in B_{L^\infty}(0,1)$ and therefore $\|Sv\|_\infty \leq 1$ and thus $Sv \in B_{L^\infty}(0,1)$.

Step 2: S is a contraction mapping. Let u, v be bounded and $u = v = 0$ on ∂S. Let $x \notin S$, A be a control matrix such that

$$(S_{out}v)(x) \geq e^{-h}v(x + h\Phi^+(x,A,h)) + 1 - e^{-h} - \varepsilon,$$

for any fixed $\varepsilon > 0$. Then

$$\begin{aligned}(S_{out}u)(x) - (S_{out}v)(x) &\leq e^{-h}(u(x + h\Phi^+(x,A,h)) \\ &\qquad - v(x + h\Phi^+(x,A,h))) + \varepsilon \\ &\leq e^{-h}\|u - v\|_\infty + \varepsilon,\end{aligned}$$

which implies

$$\|S_{out}u - S_{out}v\|_{\infty,S^c} \leq e^{-h}\|u - v\|_\infty + \varepsilon. \tag{6.23}$$

Analogously,

$$\|S_{int}u - S_{int}v\|_{\infty,\operatorname{int} S} \leq e^{-h}\|u - v\|_\infty + \varepsilon^\star. \tag{6.24}$$

Putting together (6.23), (6.24) and owing to $u = v = 0$ on ∂S, we obtain, for ε, $\varepsilon^\star \to 0^+$,

$$\|Su - Sv\|_\infty \leq \alpha\|u - v\|_\infty, \text{ where } \alpha = e^{-h} < 1, \tag{6.25}$$

i.e., S is a contraction mapping. Consequently, there exists a unique bounded solution of (6.22).

□

Let us turn to the discretization in space. To this end, as in Section 5.5, let Ω containing S be a compact subset of $\mathcal{R}^S$. We construct a grid over Ω with space step Δx. Set $\Gamma = \{x_1, x_2, ..., x_I\}$, where I is a number of the grid points. In order to construct a numerical algorithm for (6.12), (6.22) has to be discretized in state variables as well. To do this, we use, for simplicity, the first order interpolation, see Section 5.5 or [34] for the details. The fully discrete problem of (6.12) reads as

$$\begin{cases} v_\Delta(x) = \inf_{A\in U^l}\{e^{-h} I^1_\Gamma[v_\Delta](x + h\Phi^+(x, A, h))\} + 1 - e^{-h} & \textit{for } x \in S^c \\ v_\Delta(x) = \sup_{A\in U^l}\{e^{-h} I^1_\Gamma[v_\Delta](x + h\Phi^-(x, A, h))\} + e^{-h} - 1 & \textit{for } x \in \operatorname{int} S \\ v_\Delta(x) = 0 & \textit{for } x \in \partial S, \end{cases} \tag{6.26}$$

where $I^1_\Gamma[v_\Delta](x) = \sum\limits_1^I \beta_i v_\Delta(x_i)$ if provided $x = \sum\limits_1^I \beta_i x_i$ and $\sum\limits_1^I \beta_i = 1$, $\beta_i \geq 0$.

After solving (6.26), we obtain the numerical solution over the whole domain Ω. However, if one is interested in the solution defined on only $\overline{S^c}$, one can recover it by taking all of the values of $v_\Delta(x)$ with $x \in \Gamma \cap \overline{S^c}$ the due to the construction.

6.2.2 Error estimate

To begin with, notice that Hölder continuity of T implies Hölder continuity of v with the same exponent. We will employ this property of v in the next theorems. Under Assumptions 6.2.1, the construction of the discrete problem, together with the results proved in Section 6.1, we are now able to provide error estimates as follows.

Theorem 6.2.3. *Under Assumptions 6.0.1 and 6.2.1,*

$$\|v_h - v\|_{\infty,\Omega} \leq C_1 h^{\frac{q+1}{k}-1} + C_2 h,$$

where v, v_h are solutions of (6.12), (6.22) *respectively and C_1, C_2 are positive constants.*

Proof. In this proof, we consider two cases as follows

case 1: let $x \notin S$. If $T(x) \geq h$, let $u \in \mathcal{U}$ be the minimizer of

$$v(x) = \inf_\alpha\{e^{-h} v(y^+(h, x, \alpha)) + 1 - e^{-h}\},$$

and A be an $m \times l$ control matrix such that $|y^+(h, x, \alpha) - y^+_h(h, x, A)| \leq Ch^{q+1}$. Then

$$\begin{aligned} v_h(x) - v(x) &\leq e^{-h}(v_h(x + h\Phi^+(x, A, h)) - v(y^+(h, x, u))) \\ &\leq e^{-h}\Big(v_h(x + h\Phi^+(x, A, h)) - v(x + h\Phi^+(x, A, h)) \\ &\qquad + v(x + h\Phi^+(x, A, h)) - v(y^+(h, x, u))\Big) \\ &\leq e^{-h}\|v_h - v\|_{\infty,\Omega} + C_1 h^{\frac{q+1}{k}}. \end{aligned} \tag{6.27}$$

If $T(x) < h$, let $\bar{u}$ be the minimizer of

$$v(x) = 1 - e^{-h} + \inf_\alpha\{e^{-T(x)} v(y^+(h, x, \alpha))\},$$

and $\bar{A}$ be an $m \times l$ control matrix such that

$$|y^+(h,x,\alpha) - y_h^+(h,x,\bar{A})| \leq Ch^{q+1}.$$

We receive

$$\begin{aligned}
v_h(x) - v(x) &\leq e^{-h} v_h(x + h\Phi^+(x,\bar{A},h)) - e^{-T(x)} v(y^+(h,x,\bar{u})) \\
&\leq e^{-h}\Big(v_h(x + h\Phi^+(x,\bar{A},h)) - v(x + h\Phi^+(x,\bar{A},h)) \\
&\qquad + v(x + h\Phi^+(x,\bar{A},h)) - v(y^+(h,x,\bar{u}))\Big) \\
&\quad + (e^{-h} - e^{-T(x)}) v(y^+(h,x,\bar{u})) \\
&\leq e^{-h} \|v_h - v\|_{\infty,\Omega} + C_1 h^{\frac{q+1}{k}} + C_2(h - T(x))^2 \\
&\leq e^{-h} \|v_h - v\|_{\infty,\Omega} + C_1 h^{\frac{q+1}{k}} + C_2 h^2.
\end{aligned} \tag{6.28}$$

From (6.27), (6.28),

$$v_h(x) - v(x) \leq e^{-h} \|v_h - v\|_{\infty,\Omega} + C_1 h^{\frac{q+1}{k}} + C_2 h^2.$$

Similarly, we can prove the reverse inequality, i.e.

$$v(x) - v_h(x) \leq e^{-h} \|v_h - v\|_{\infty,\Omega} + C_1 h^{\frac{q+1}{k}} + C_2 h^2.$$

Therefore,

$$\|v_h - v\|_{\infty,S^c} \leq e^{-h} \|v_h - v\|_{\infty,\Omega} + C_1 h^{\frac{q+1}{k}} + C_2 h^2 \text{ for } x \notin S. \tag{6.29}$$

case 2: let $x \in \operatorname{int} S$. By applying the corresponding formulas of $v(x)$, $v_h(x)$ with respect to $x \in \operatorname{int} S$ and follow the similar technique as that of Case 1, we have

$$\|v_h - v\|_{\infty,\operatorname{int} S} \leq e^{-h} \|v_h - v\|_{\infty,\Omega} + C_1 h^{\frac{q+1}{k}} + C_2 h^2 \tag{6.30}$$

By putting (6.29), (6.30) together, we derive

$$\|v_h - v\|_{\infty,\Omega} \leq e^{-h} \|v_h - v\|_{\infty,\Omega} + C_1 h^{\frac{q+1}{k}} + C_2 h^2, \tag{6.31}$$

Consequently,

$$\|v_h - v\|_{\infty,\Omega} \leq C_1 h^{\frac{q+1}{k} - 1} + C_2 h.$$

□

Theorem 6.2.4. *Under Assumptions 6.0.1 and 6.2.1,*

$$\|v_\Delta - v\|_{\infty,\Omega} \leq C_1 \frac{\Delta x^{\frac{1}{k}}}{h} + C_2 h^{\frac{q+1}{k} - 1} + C_3 h,$$

where v, v_Δ are solutions of (6.12), (6.26) *respectively and C_1, C_2, C_3 are positive constants.*

Proof. We divide the proof into two cases as follows

case 1: consider $x \notin S \cap \Gamma$. If $T(x) \geq h$, let $u \in \mathcal{U}$ be the minimizer of

$$v(x) = 1 - e^{-h} + \inf_{\alpha}\{e^{-h} v(y^+(h, x, \alpha))\},$$

and A be an $m \times l$ control matrix such that $|y^+(h, x, \alpha) - y_h^+(h, x, A)| \leq Ch^{q+1}$. Then

$$\begin{aligned}
&v_\Delta(x) - v(x) \leq e^{-h}\big(I_\Gamma^1[v_\Delta](x + h\Phi^+(x, A, h)) - v(y^+(h, x, u))\big) \\
&\leq e^{-h}\Bigg(I_\Gamma^1[v_\Delta](x + h\Phi^+(x, A, h)) - I_\Gamma^1[v](x + h\Phi^+(x, A, h)) + I_\Gamma^1[v](x + h\Phi^+(x, A, h)) \\
&- v(x + h\Phi^+(x, A, h)) + v(x + h\Phi^+(x, A, h)) - v(y^+(h, x, u))\Bigg) \\
&\leq e^{-h} \left\| v_\Delta - v \right\|_{\infty,\Gamma} + C_1 \Delta x^{\frac{1}{k}} + C_2 h^{\frac{q+1}{k}}.
\end{aligned} \tag{6.32}$$

If $T(x) < h$, let $\bar{u}$ be the minimizer of

$$v(x) = 1 - e^{-h} + \inf_{\alpha}\{e^{-T(x)} v(y^+(h, x, \alpha))\},$$

and $\bar{A}$ be an $m \times l$ control matrix such that

$$|y^+(h, x, \alpha) - y_h^+(h, x, \bar{A})| \leq Ch^{q+1}.$$

We have

$$\begin{aligned}
v_\Delta(x) - v(x) &\leq e^{-h} I_\Gamma^1[v_\Delta](x + h\Phi^+(x, \bar{A}, h)) - e^{-T(x)} v(y^+(h, x, u)) \\
&\leq e^{-h}\Bigg(I_\Gamma^1[v_\Delta](x + h\Phi^+(x, \bar{A}, h)) - I_\Gamma^1[v](x + h\Phi^+(x, \bar{A}, h)) \\
&\qquad + I_\Gamma^1[v](x + h\Phi^+(x, \bar{A}, h)) - v(x + h\Phi^+(x, \bar{A}, h)) \\
&\qquad + v(x + h\Phi^+(x, \bar{A}, h)) - v(y^+(h, x, u))\Bigg) \\
&\quad + (e^{-h} - e^{-T(x)}) v(y^+(h, x, \bar{u})) \leq e^{-h} \left\| v_\Delta - v \right\|_{\infty,\Gamma} \\
&\quad + C_1 \Delta x^{\frac{1}{k}} + C_2 h^{\frac{q+1}{k}} + C_3 h^2.
\end{aligned} \tag{6.33}$$

From (6.32),(6.28), we obtain

$$v_\Delta(x) - v(x) \leq e^{-h} \left\| v_\Delta - v \right\|_{\infty,\Gamma} + C_1 \Delta x^{\frac{1}{k}} + C_2 h^{\frac{q+1}{k}} + C_3 h^2.$$

The proof of the reverse inequality is similar, so we dismiss it. Conclusively,

$$\left\| v_\Delta - v \right\|_{\infty, S^c \cup \Gamma} \leq e^{-h} \left\| v_\Delta - v \right\|_{\infty,\Gamma} + C_1 \Delta x^{\frac{1}{k}} + C_2 h^{\frac{q+1}{k}} + C_3 h^2. \tag{6.34}$$

case 2: consider $x \notin S^c \cap \Gamma$. In the same way, we derive

$$\|v_\Delta - v\|_{\infty, \mathrm{int}\, S \cup \Gamma} \leq e^{-h} \|v_\Delta - v\|_{\infty,\Gamma} + C_1 \Delta x^{\frac{1}{k}} + C_2 h^{\frac{q+1}{k}} + C_3 h^2. \tag{6.35}$$

(6.34),(6.35) imply

$$\|v_\Delta - v\|_{\infty,\Gamma} \leq C_1 \frac{\Delta x^{\frac{1}{k}}}{h} + C_2 h^{\frac{q+1}{k}-1} + C_3 h \text{ for } x \in \Gamma.$$

Let $x \in \Omega$, we have

$$v_\Delta(x) - v(x) \leq |I^1_\Gamma[v_\Delta](x) - I^1_\Gamma[v](x)| + |I^1_\Gamma[v](x) - v(x)| \leq C_1 \frac{\Delta x^{\frac{1}{k}}}{h} + C_2 h^{\frac{q+1}{k}-1} + C_3 h.$$

and, analogously,

$$v(x) - v_\Delta(x) \leq C_1 \frac{\Delta x^{\frac{1}{k}}}{h} + C_2 h^{\frac{q+1}{k}-1} + C_3 h.$$

Finally,

$$\|v_\Delta - v\|_{\infty,\Omega} \leq C_1 \frac{\Delta x^{\frac{1}{k}}}{h} + C_2 h^{\frac{q+1}{k}-1} + C_3 h.$$

□

6.3 Numerical tests

In this section we illustrate the performance of our proposed scheme compared to the classical approach described in [26, 34]. We first consider a one-dimensional example which illustrates that the size of the jumps in the semi-discretization v_h is indeed reduced by our new approach. Afterwards we evaluate the numerical error of the schemes for three two-dimensional examples for which the exact solutions are known.

6.3.1 A test in 1d

The first test we perform uses the simple one dimensional dynamics

$$\dot{x} = u, \;\; u \in [-1, 1]. \tag{6.36}$$

We consider this example on $\Omega = [-1, 1]$ with target $S = [-0.25, 0.25]$. In one space dimension, it is possible to make the spatial step size Δx so small that the resulting approximation very accurately represents the semi-discretization v_h. The resulting graphs in Figure 6.1 indicate that the size of the jumps in the solution in the new approach (right) is only half as large as in the classical approach (left). This observation can also be explained analytically if we look at the jump at ∂S. For the classical approach, it is easily seen that the smallest cost $v_h(x)$ for $x \notin S$ is $1 - e^{-h} = h + \mathcal{O}(h^2)$. On the other hand, for the new approach, we can use the fact that the system moves fastest towards ∂S for $u \equiv 1$ in case $x < 0$ and for $u \equiv -1$ for $x > 0$ and the observation that $y_h^+(h, x, u) = x + hu$ and $y_h^-(h, x, u) = x - hu$ holds for any Runge-Kutta scheme. Thus, for each $x \notin S$ sufficiently close to S, the bridge dynamic programming principles (6.15) applied to $x \notin S$ and (6.16) applied to $y^+(h, x, u) \in S$ yield

$$v_h(x) = e^{-h}(e^{-h}v_h(x) + e^{-h} - 1) + 1 - e^{-h} \quad \Rightarrow \quad v_h(x) = \frac{(1 - e^{-h})^2}{1 - e^{-2h}} = \frac{h}{2} + \mathcal{O}(h^2).$$

While for this example with the chosen parameters the improvement in accuracy is

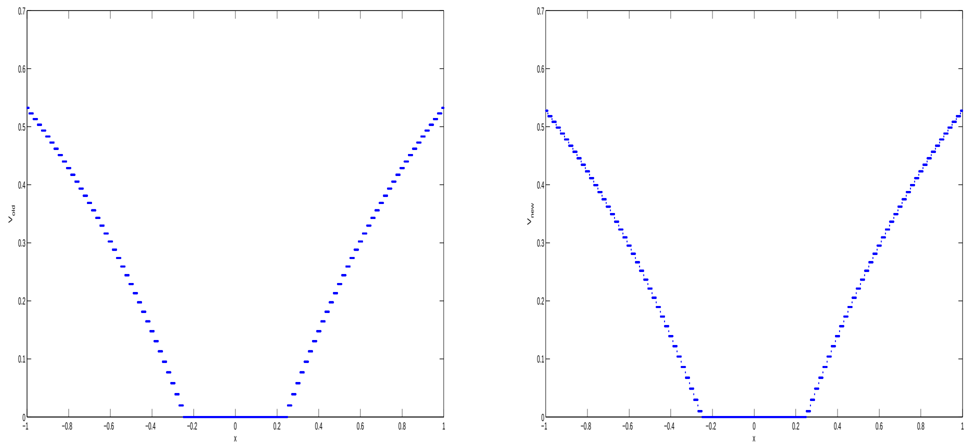

Figure 6.1: Example (6.36): positive part of the value function obtained by the classical approach (left) and by the proposed new approach (right) ($\Omega = [-1, 1]$, $S = [-0.25.0.25]$, $\Delta x = 0.0002$, $h = 0.02$).

almost precisely equal to $h/2$, in the next section we will see that for more reasonable (i.e., larger) choices of Δx the improvement can be substantially larger, which is due to the fact that smaller jumps in v_h do not only improve the accuracy of the semi-discretization v_h itself (which is what is visible here) but also have a significant positive effect on the accuracy of the subsequent spatial discretization.

6.3.2 Tests in 2d

Next we consider three numerical examples to see the error behavior of the solutions obtained by the new and the classical approaches. In all examples, we take $\Omega = [-2, 2]^2$, S as a ball with radius r centered at the origin and a third order Runge-Kutta method. For each example, the table shows the L^∞ numerical errors of the recovered numerical solutions of both approaches. Moreover, we provide two plots for the value function computed by the proposed new approach for each example. The first figure shows the graph of the value function on the whole set Ω, while the second shows the graph on

$\Omega \setminus S$. The first example uses the dynamics

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ \ (u_1,\ u_2) \in [-1, 1]^2. \tag{6.37}$$

The dynamics of the second example is

Δx	h	**new approach**	**classical approach**
0.02	0.1	0.0495	0.0948
0.02	0.05	0.0126	0.0484
0.02	0.025	0.0076	0.0243
0.02	0.0125	0.0045	0.0168
0.01	0.1	0.0498	0.0951
0.01	0.05	0.0250	0.0487
0.01	0.025	0.0059	0.0246
0.01	0.0125	0.0036	0.0123

Table 6.1: Comparison of error estimates for Example (6.37) ($r = 0.5$)

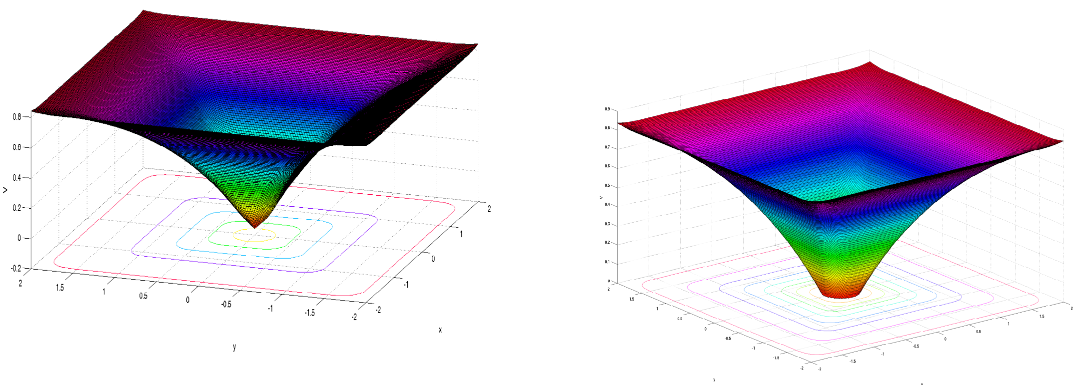

Figure 6.2: Example (6.37): value function on Ω and on $\Omega \setminus S$ resp. obtained by the new approach (radius of the target $r = 0.25$, $h = 0.025$, $\Delta x = 0.01$, 3rd order Runge-Kutta scheme).

$$\dot{x}_1 = -x_2 + x_1\, u,\ \dot{x}_2 = x_1 + x_2\, u,\ \ u \in [-1, 1]. \tag{6.38}$$

It is easy to check that the Petrov condition holds for the first and second examples, thus T_S is Lipschitz continuous.

The last example is the classical double integrator, i.e.

Δx	h	**new approach**	**classical approach**
0.02	0.1	0.0212	0.0944
0.02	0.05	0.0113	0.0480
0.02	0.025	0.0059	0.0364
0.02	0.0125	0.0084	0.0358
0.01	0.1	0.0253	0.0950
0.01	0.05	0.0117	0.0486
0.01	0.025	0.0051	0.0245
0.01	0.0125	0.0029	0.0190

Table 6.2: Comparison of error estimates for Example (6.38) ($r = 0.5$)

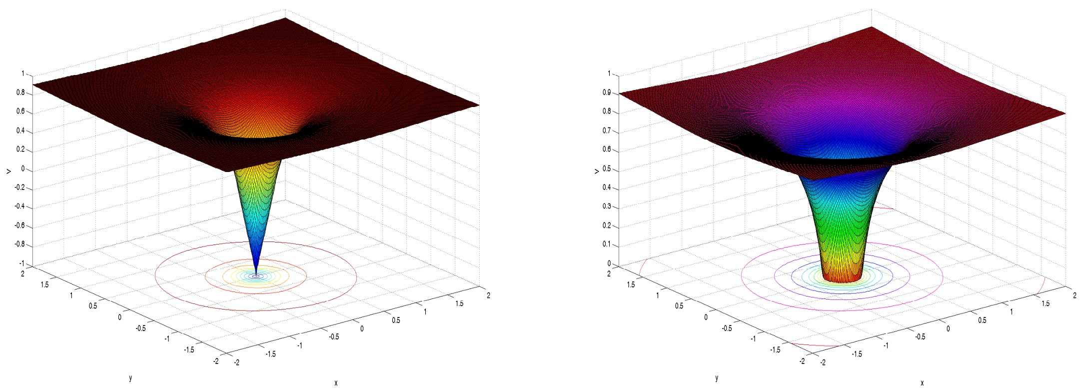

Figure 6.3: Example (6.38): value function on Ω and on $\Omega \setminus S$ resp. obtained by the new approach (radius of the target $r = 0.25$, $h = 0.01$, $\Delta x = 0.01$, 3rd order Runge-Kutta scheme).

$$\ddot{x} = u, \;\; u \in [-1, 1]. \tag{6.39}$$

T_S is Hölder continuous (see [61]). Therefore, Assumptions 6.0.1 are satisfied and the new approach works for all the examples. In conclusion, we observe that in the first two examples the numerical errors obtained by means of the new approach are reduced significantly in comparison with the classical one. This is due to the fact that in this examples the Petrov condition is fulfilled, hence the optimal value functions are Lipschitz. This implies that the discretization error is of the order $\mathcal{O}(h)$ which is exactly the order of the size of the jumps. Consequently the jumps in v_h are a dominant error source and their reduction has a visible effect on the error. In the last example the situation is different since v is only Hölder continuous along a curve extending from the target S to the boundary of Ω, which is also the place where the maximal errors are located. Due to the non-Lipschitzness, in this example, the order of the error is larger

Δx	h	**new approach**	**classical approach**
0.016	0.05	0.0786	0.0818
0.016	0.025	0.0760	0.0795
0.016	0.0125	0.0778	0.0783
0.016	0.00625	0.0815	0.0819
0.008	0.05	0.0858	0.0887
0.008	0.025	0.0716	0.0741
0.008	0.0125	0.0702	0.0718
0.008	0.00625	0.0705	0.0709

Table 6.3: Comparison of error estimates for Example (6.39) ($r = 0.1$)

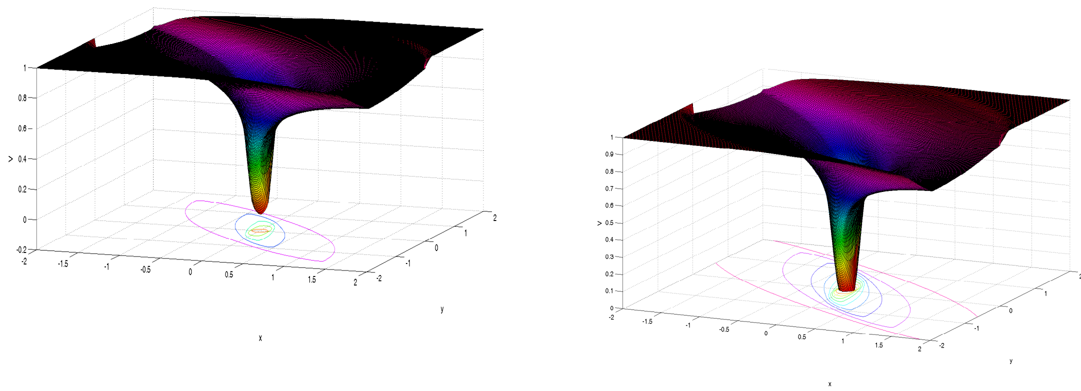

Figure 6.4: Example (6.39): value function on Ω and on $\Omega \setminus S$ resp. obtained by the new approach (radius of the target $r = 0.1$, $\Delta x = 0.016$, $h = 0.01$, 3rd order Runge-Kutta scheme).

than $\mathcal{O}(h)$. Hence, the reduction of the jumps which yields a reduction of the error of order $\mathcal{O}(h)$ is hardly visible here because it is dominated by the larger error along the curve where the solution is merely Hölder continuous.

Chapter 7

Reconstruction of the minimum time function through the approximation of reachable sets for linear control systems

The plan of this chapter is as follows. In Section 7.1 the convexity of the reachable set for linear control problems and the characterization of its boundary via the level-set of the minimum time function is the basis for the algorithm formulated. Section 7.2 is devoted to constructing the approximate minimum time function. First we briefly introduce the reader to set-valued quadrature methods and Runge-Kutta methods and their implementation and discuss the convergence order for the fully discrete approximation of reachable sets at a given time both in time and in space. Then we present the error estimate for the fully discrete minimum time function which depends on the regularity of the continuous minimum time function and on the convergence order of the underlying set-valued method. Another error estimate expresses the error only on the time period between the calculated reachable sets. Section 7.3 discusses the construction of discrete optimal trajectories and convergence of discrete optimal controls. The remainder of this chapter presents some numerical examples to illustrate this approach.

In this chapter, we will consider a special class of control systems (2.1), the linear time-variant controlled dynamics in $\mathbb{R}^n$,

$$\begin{cases} \dot{y}(t) = A(t)y(t) + B(t)u(t) & \text{for a.e. } t \in [t_0, \infty), \\ u(t) \in U & \text{for a.e. } t \in [t_0, \infty), \\ y(t_0) = y_0. \end{cases} \tag{7.1}$$

The coefficients $A(t), B(t)$ are $n \times n$ and $n \times m$ matrices respectively, $y_0 \in \mathbb{R}^n$ is the initial value, $U \in \mathcal{C}(\mathbb{R}^m)$ is the set of control values.

For a given maximal time $t_f > t_0$ and some $t \in I = [t_0, t_f]$, $\mathcal{R}^S(t)$ is the set of points *reachable from the target in time t* by the *time-reversed system*

$$\dot{y}(t) = \bar{A}(t)y(t) + \bar{B}(t)u(t), \tag{7.2}$$
$$y(t_0) \in S, \tag{7.3}$$

where $\bar{A}(t) := -A(t_0 + t_f - t)$, $\bar{B}(t) := -B(t_0 + t_f - t)$ for shortening notations. In other words, $\mathcal{R}^S(t)$ equals the set of starting points from which the system can reach the target in time t. Sometimes $\mathcal{R}^S(t)$ is called the *backward reachable set* which is also considered in [20] for computing the minimum time function by solving a Hamilton-Jacobi-Bellman equation.

The following standing hypotheses are assumed to be fulfilled in the sequel.

Assumptions 7.0.1.

(i) *$A(t)$, $B(t)$ are $n \times n$, $n \times m$ real-valued matrices defining integrable functions on any compact interval of $[t_0, \infty)$.*

(ii) *The control set $U \subset \mathbb{R}^m$ is convex, compact and nonempty, i.e. $U \in \mathcal{C}(\mathbb{R}^m)$.*

(iii) *The target set $S \in \mathbb{R}^n$ is convex, compact and nonempty, i.e. $S \in \mathcal{C}(\mathbb{R}^n)$. Especially, the target set can be a singleton.*

(iv) *$\mathcal{R}^S(t)$ is* strictly expanding *on the compact interval $[t_0, t_f]$, i.e. $\mathcal{R}^S(t_1) \subset \operatorname{int} \mathcal{R}^S(t_2)$ for all $t_0 \le t_1 < t_2 \le t_f$*

Remark 7.0.2. *The reader can find sufficient conditions for Assumption 7.0.1(iv) for $S = \{0\}$ in [49, Chap. 17], [59, Sec. 2.2–2.3]. Under this assumption, it is obvious that*

$$\mathcal{R}^S(t) = \mathcal{R}^S_{\le}(t).$$

7.1 Description and main properties of reachable sets

Under our standard hypotheses, the control problem (7.2) can equivalently be replaced by the following linear differential inclusion

$$\dot{y}(t) \in \bar{A}(t)y(t) + \bar{B}(t)U \quad \text{for a.e. } t \in [t_0, \infty) \tag{7.4}$$

with absolutely continuous solutions $y(\cdot)$ (see [71, Appendix A.4].

All the solutions of (7.3)–(7.4) are represented as

$$y(t) = \Phi(t, t_0)y_0 + \int_{t_0}^{t} \Phi(t, s)\bar{B}(s)u(s)ds$$

for all $y_0 \in S$, $u \in \mathcal{U}$, and $t_0 \le t < \infty$, where $\Phi(t, s)$ is the *fundamental solution matrix* of the homogeneous system

$$\dot{y}(t) = \bar{A}(t)y(t), \tag{7.5}$$

with $\Phi(s,s) = I_n$, the $n \times n$ identity matrix. Using the Minkowski addition and the Aumann's integral, the reachable set can be described by means of Aumann's integral as follows

$$\mathcal{R}^S(t) = \Phi(t,t_0)S + \int_{t_0}^{t} \Phi(t,s)\bar{B}(s)U ds. \tag{7.6}$$

For time-invariant systems, i.e. $\bar{A}(t) = \bar{A}$, we have $\Phi(t,t_0) = e^{\bar{A}(t-t_0)}$.

For the linear control system (7.1) the reachable set at a fixed end time is convex which allows to apply support functions or supporting points for its approximation. Furthermore, the reachable sets change continuously with respect to the end time. The following theorem will summarize the needed properties.

Theorem 7.1.1. *Let the Assumptions 7.0.1(i)–(iii) be fulfilled and consider the linear control process* (7.1) *in* $\mathbb{R}^n$*. Then* $\mathcal{R}^S(t)$ *is convex, compact and nonempty. Moreover,* $\mathcal{R}^S(t)$ *varies continuously with* $t_0 \le t < \infty$*.*

Proof. Recall that for $t \ge t_0$

$$\mathcal{R}^S(t) = \Phi(t,t_0)S + \int_{t_0}^{t} \Phi(t,s)\bar{B}(s)U ds. \tag{7.7}$$

Observe that the integral term $\int_{t_0}^{t} \Phi(t,s)\bar{B}(s)U ds$ is actually the reachable set at time $t \ge t_0$ initiating from the origin, it is compact and convex due to the convexity of Aumann's integral, see e.g. [6]. The same properties hold for the reachable set $\mathcal{R}^S(t)$ due to Proposition 2.3.4 (see also [59, Sec. 2.2, Theorem 1]) and the assumptions on S. This reference also states the continuity of the set-valued map $t \mapsto \mathcal{R}^S(t)$. The proof is completed. □

Lemma 7.1.2. *Under the same assumptions as in Theorem 7.1.1, the map* $t \mapsto \mathcal{R}^S_{\le}(t)$ *has nonempty compact images and varies continuously with respect to* $t \in I = [t_0, t_f]$*.*

Proof. The integrable linear growth condition holds due to Assumptions 7.0.1(i) so that the Filippov-Gronwall theorem in [42, Theorem 2.3] applies yielding the compactness of the closure of the set of solutions in the maximum norm on I. As a consequence the compactness of $\mathcal{R}^S_{\le}(t)$ follows easily.

Let $s, t \in [t_0, t_f]$ and consider $x \in \mathcal{R}^S_{\le}(s)$. Then, there exists $\widetilde{s} \in [t_0, s]$ with $x \in \mathcal{R}^S(\widetilde{s})$. We distinguish two cases.

case (i): $\widetilde{s} \le t$

$$\mathrm{d}(x, \mathcal{R}^S_{\le}(t)) \le \mathrm{d}(x, \mathcal{R}^S(\widetilde{s})) = 0$$

case (ii): $\widetilde{s} > t$

$$\mathrm{d}(x, \mathcal{R}^S_{\le}(t)) \le \mathrm{d}(x, \mathcal{R}^S(t)) \le \sup_{x \in \mathcal{R}^S(\widetilde{s})} \mathrm{d}(x, \mathcal{R}^S(t)) \le \mathrm{d}_H(\mathcal{R}^S(\widetilde{s}), \mathcal{R}^S(t))$$

For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $s \in [t-\delta, t+\delta] \cap [t_0, t_f]$ we have

$$\mathrm{d}_H(\mathcal{R}^S(s), \mathcal{R}^S(t)) \leq \varepsilon$$

which also holds for $\widetilde{s}$ instead of s, since $0 \leq \widetilde{s} - t \leq s - t$.

This proves the continuity of $\mathcal{R}^S_{\leq}(\cdot)$. $\square$

The following proposition is to provide the connection between $\mathcal{R}^S(t)$ and the level set of $T_S(\cdot)$ at time t which is essential for this approach. We will benefit from the sublevel representation in (2.4). The result is related to [20, Theorem 2.3], where the minimum time function at x is the minimum for which x lies on a zero-level set bounding the backward reachable set.

Proposition 7.1.3. *Let Assumption 7.0.1 be fulfilled and $t \in (t_0, t_f]$. Then*

$$\partial \mathcal{R}^S(t) = \{y_0 \in \mathbb{R}^n \colon T_S(y_0) = t\}. \tag{7.8}$$

Proof. "$\subset$": Assume that there exists $x \in \partial \mathcal{R}^S(t)$ with $x \notin \{y_0 \in \mathbb{R}^n \colon T_S(y_0) = t\}$. Clearly, $x \in \mathcal{R}^S_{\leq}(t)$ and (2.4) shows that $T_S(x) \leq t$. By definition there exists $s \in [t_0, t]$ with $x \in \mathcal{R}^S(s)$. Assuming $s < t$ we get the contradiction $x \in \mathcal{R}^S(s) \subset \operatorname{int} \mathcal{R}^S(t)$ from Assumption 7.0.1(iv).

"$\supset$": Assume that there exists $x \in \{y_0 \in \mathbb{R}^n \colon T_S(y_0) = t\}$ (i.e. $T_S(x) = t$) be such that $x \notin \partial \mathcal{R}^S(t)$. Since $x \in \mathcal{R}^S(t)$ by (2.4) and we assume that $x \notin \partial \mathcal{R}^S(t)$, then $x \in \operatorname{int}(\mathcal{R}^S(t))$.

Hence, there exists $\varepsilon > 0$ with $x + \varepsilon B_1(0) \subset \mathcal{R}^S(t)$. The continuity of $\mathcal{R}^S(\cdot)$ ensures for $t_1 \in [t-\delta, t+\delta] \cap I$ that

$$\mathrm{d}_H(\mathcal{R}^S(t), \mathcal{R}^S(t_1)) \leq \frac{\varepsilon}{2}.$$

Hence,

$$x + \varepsilon B_1(0) \subset \mathcal{R}^S(t) \subset \mathcal{R}^S(t_1) + \frac{\varepsilon}{2} B_1(0).$$

The order cancellation law in [65, Theorem 3.2.1] can be applied, since $\mathcal{R}^S(t_1)$ is convex and all sets are compact. Therefore,

$$\begin{aligned}(x + \frac{\varepsilon}{2} B_1(0)) + \frac{\varepsilon}{2} B_1(0) &\subset \mathcal{R}^S(t_1) + \frac{\varepsilon}{2} B_1(0) \\ \Rightarrow\ x + \frac{\varepsilon}{2} B_1(0) &\subset \mathcal{R}^S(t_1)\end{aligned}$$

Hence, $x \in \operatorname{int}(\mathcal{R}^S(t_1))$ with $t_1 < t$ so that $T_S(x) \leq t_1 < t$ which is again a contradiction. Therefore, $\{y_0 \in \mathbb{R}^n \colon T_S(y_0) = t\} \subset \partial \mathcal{R}^S(t)$. The proof is completed. $\square$

In the previous characterization of the boundary of the reachable set at fixed end time the assumption of monotonicity of the reachable sets played a crucial role. As stated in Remark 7.0.2, Assumption 7.0.1(iv) also guarantees that the union of reachable sets coincides with the reachable set at the largest end time and is trivially convex. If we drop this assumption, we can only characterize the boundary of the *union* of reachable sets up to a time under relaxing the expanding property (iv) while demanding convexity as can be seen in the following proposition.

Proposition 7.1.4. *Let $t > t_0$, Assumptions 7.0.1(i)–(iii) and Assumption*

> *(iv)' $\mathcal{R}^S_{\leq}(t)$ has convex images and is strictly expanding on the compact interval $[t_0, t_f]$, i.e.*
>
> $$\mathcal{R}^S_{\leq}(t_1) \subset \operatorname{int} \mathcal{R}^S_{\leq}(t_2) \quad \textit{for all } t_0 \leq t_1 < t_2 \leq t_f.$$

Then

$$\partial \mathcal{R}^S_{\leq}(t) = \{x \in \mathbb{R}^n : T_S(x) = t\} \tag{7.9}$$

Proof. For the first inclusion "$\supset$" consider $x \in \mathbb{R}^n$ with $T_S(x) = t$. Assume by contradiction that $x \notin \partial \mathcal{R}^S_{\leq}(t)$, so that $x \in \operatorname{int} \mathcal{R}^S_{\leq}(t)$ and there exists $\varepsilon > 0$ with

$$x + \varepsilon B_1(0) \subset \mathcal{R}^S_{\leq}(t).$$

The continuity of $\mathcal{R}^S_{\leq}(\cdot)$ due to Lemma 7.1.2 ensures for $t_1 \in [t - \delta, t + \delta] \cap I$ that

$$\mathrm{d}_H(\mathcal{R}^S_{\leq}(t), \mathcal{R}^S_{\leq}(t_1)) \leq \frac{\varepsilon}{2}.$$

Hence,

$$x + \varepsilon B_1(0) \subset \mathcal{R}^S_{\leq}(t) \subset \mathcal{R}^S_{\leq}(t_1) + \frac{\varepsilon}{2} B_1(0).$$

Again, the order cancellation law

$$\begin{aligned} (x + \frac{\varepsilon}{2} B_1(0)) + \frac{\varepsilon}{2} B_1(0) &\subset \mathcal{R}^S_{\leq}(t_1) + \frac{\varepsilon}{2} B_1(0) \\ \Rightarrow \; x + \frac{\varepsilon}{2} B_1(0) &\subset \mathcal{R}^S_{\leq}(t_1) \end{aligned}$$

in [65, Theorem 3.2.1] can be applied, since $\mathcal{R}^S_{\leq}(t_1)$ is assumed to be convex and all sets are compact. Hence, $x \in \operatorname{int}(\mathcal{R}^S_{\leq}(t_1))$ with $t_1 < t$ and $T_S(x) \leq t_1 < t$ which is not possible so that x must lie in $\partial \mathcal{R}^S_{\leq}(t)$.

For the opposite inclusion "$\subset$" let $x \in \partial \mathcal{R}^S_{\leq}(t)$ and assume by contradiction that $T_S(x) < t$. Observe that $T_S(x)$ is continuous in $\mathcal{R}^S$ due to Assumption (iv)', see [16, Chap. IV, Proposition 1.6]. Then by the continuity of $T_S(x)$ there exist $\delta > 0$ and a neighborhood $U(x)$ of x such that $T_S(y) \leq t - \delta < t$ for all $y \in U(x)$. The neighborhood exists, since $\mathcal{R}^S$ has nonempty interior by Assumption (iv)'. By (2.4) $x \in U(x) \subset \mathcal{R}^S_{\leq}(t - \delta) \subset \operatorname{int} \mathcal{R}^S_{\leq}(t)$ which contradicts $x \in \partial \mathcal{R}^S_{\leq}(t)$. $\square$

Remark 7.1.5. *Assumption (iv)' implies that the considered system is small-time controllable, see [16, Chap. IV, Definition 1.1]. Moreover, under the assumption of small-time controllability the nonemptiness of the interior of $\mathcal{R}^S$ and the continuity of the minimum time function in $\mathcal{R}^S$ are consequences, see [16, Chap. IV, Propositions 1.2, 1.6]. Assumption (iv)' is essentially weaker than (iv), since the convexity of $\mathcal{R}^S_{\leq}(t)$ and the strict expandedness of $\mathcal{R}^S_{\leq}(\cdot)$ follows by Remark 7.0.2.*

In the previous proposition we can allow that $\mathcal{R}^S_{\leq}(t)$ is lower-dimensional and are still able to prove the inclusion "$\supset$" in (7.9), since the interior of $\mathcal{R}^S_{\leq}(t)$ would be empty and x cannot lie in the interior which also creates the (wanted) contradiction.

For the other inclusion "$\subset$" the nonemptiness of the interior of $\mathcal{R}^S(t)$ in Proposition 7.1.3 resp. the one of $\mathcal{R}^S_{\leq}(t)$ in Proposition 7.1.4 is essential. Therefore, the expanding property in Assumptions (iv) resp. (iv)' cannot be relaxed by assuming only monotonicity in the sense

$$\mathcal{R}^S(s) \subset \mathcal{R}^S(t) \quad \textit{or} \quad \mathcal{R}^S_{\leq}(s) \subset \mathcal{R}^S_{\leq}(t) \tag{7.10}$$

for $s < t$.

7.2 Computation of the numerical minimum time function

7.2.1 Set-valued discretization methods

Consider the linear controlled dynamics (7.1). For a given $x \in \mathbb{R}^n$, the problem of computing approximately the minimum time $T_S(x)$ to reach S by following the dynamics (7.1) is deeply investigated in literature. It was usually obtained by solving the associated discrete Hamilton-Jacobi-Bellman equation (HJB), see Chapter 5, 6 and references therein. Neglecting the space discretization we obtain an approximation of $T_S(x)$. In this paper, we will introduce another approach to treat this problem based on approximation of the reachable set of the corresponding linear differential inclusion. The approximate minimum time function is not derived from the PDE solver, but from iterative set-valued methods or direct discretization of control problems.

Our aim now is to compute $\mathcal{R}^S(t)$ numerically up to a maximal time t_f based on the representation (7.6) by means of set-valued methods to approximate Aumann's integral. There are many approaches to achieving this goal. We will describe three known options for discretizing the reachable set which are used in the following.

Consider for simplicity of notations an equidistant grid over the interval $I = [t_0, t_f]$ with N subintervals, step size $h = \frac{t_f - t_0}{N}$ and grid points $t_i = t_0 + ih$, $i = 0, \ldots, N$.

(I) Set-valued quadrature methods with the exact knowledge of the fundamental solution matrix of (7.5) (see e.g. [14, 32, 73], [7, Sec. 2.2]):
As in the pointwise case, we replace the integral $\int_{t_0}^{t} \Phi(t,s)\bar{B}(s)U ds$ by some

quadrature scheme of order p with non-negative weights. Therefore, (7.6) is approximated by

$$\mathcal{R}_h^S(t_N) = \Phi(t_N, t_0)S + h\sum_{i=0}^{N} c_i\Phi(t_N, t_i)\bar{B}(t_i)U \tag{7.11}$$

with weights $c_i \geq 0$, $i = 0, \ldots, N$. Moreover, the error estimate

$$\mathrm{d}_H(\int_{t_0}^{t_N} \Phi(t_N, s)\bar{B}(s)Uds, h\sum_{i=0}^{N} c_i\Phi(t_N, t_i)\bar{B}(t_i)U) \leq Ch^p$$

holds. Obviously, the following recursive formula is valid for $i = 0, \ldots, N-1$

$$\mathcal{R}_h^S(t_{i+1}) = \Phi(t_{i+1}, t_i)\mathcal{R}_h^S(t_i) + h\sum_{j=0}^{1} \widetilde{c}_{ij}\Phi(t_{i+1}, t_{i+j})\bar{B}(t_{i+j})U, \tag{7.12}$$

$$\mathcal{R}_h^S(t_0) = S \tag{7.13}$$

with suitable weights $\widetilde{c}_{ij} \geq 0$ due to the semigroup property of the fundamental solution matrix, i.e.

$$\Phi(t+s, t_0) = \Phi(t+s, s)\Phi(s, t_0) \quad \text{for all } t \in I,\ s \geq t_0 \text{ with } s+t \in I.$$

For example, the set-valued trapezoidal rule uses the settings

$$c_0 = c_N = \frac{1}{2}, \quad c_i = 1 \ \ (i = 1, 2, \ldots, N-1)$$
$$\widetilde{c}_{ij} = \frac{1}{2} \ \ (i = 0, 1, \ldots, N-1,\ j = 0, 1).$$

(II) Set-valued combination methods (see e.g. [14], [7, Sec. 2.3]):
We replace $\Phi(t_N, t_i)$ in method (I) by its approximation (e.g. via ODE solvers of the corresponding matrix equation) such that

a) $\Phi_h(t_{m+n}, t_0) = \Phi_h(t_{m+n}, t_m)\Phi_h(t_m, t_0)$ for all $m \in \{0, \ldots, N\}$, $n \in \{0, \ldots, N-m\}$
The use of e.g. Euler's method or Heun's method yields

$$\Phi_h(t_{i+1}, t_i) = I_n + hA(t_i), \tag{7.14}$$

$$\Phi_h(t_{i+1}, t_i) = I_n + \frac{h}{2}(A(t_i) + A(t_{i+1})) + \frac{h^2}{2}A(t_{i+1})A(t_i), \tag{7.15}$$

respectively for $i = 0, 1, \ldots, N-1$.

b) $\sup_{0 \leq i \leq N} \|\Phi(t_N, t_i) - \Phi_h(t_N, t_i)\| \leq Ch^p$.

The following global resp. local recursive approximation together with (7.13) holds for the discrete reachable sets:

$$\mathcal{R}_h^S(t_N) = \Phi_h(t_N, t_0)S + h\sum_{i=0}^{N} c_i \Phi_h(t_N, t_i)\bar{B}(t_i)U, \tag{7.16}$$

$$\mathcal{R}_h^S(t_{i+1}) = \Phi_h(t_{i+1}, t_i)\mathcal{R}_h^S(t_i) + h\sum_{j=0}^{1} \widetilde{c}_{ij} \Phi_h(t_{i+1}, t_{i+j})\bar{B}(t_{i+j})U. \tag{7.17}$$

(III) Set-valued Runge-Kutta methods (see e.g. [8, 9, 33, 74, 76]):
We can approximate (7.4) by set-valued analogues of Runge-Kutta schemes. The discrete reachable set is computed recursively with the starting condition in (7.13) for the set-valued Euler scheme (see e.g. [33]) as

$$\mathcal{R}_h^S(t_{i+1}) = \Phi_h(t_{i+1}, t_i)\mathcal{R}_h^S(t_i) + hB(t_i)U \tag{7.18}$$

with (7.14) or with (7.15) for the set-valued Heun's scheme with piecewise constant selections (see e.g. [74]) as

$$\mathcal{R}_h^S(t_{i+1}) = \Phi_h(t_{i+1}, t_i)\mathcal{R}_h^S(t_i) + \frac{h}{2}\Big((I + hA(t_{i+1}))B(t_i) + B(t_{i+1})\Big)U. \tag{7.19}$$

For linear differential inclusions these methods can be regarded as perturbed set-valued combination methods (see [8]).

Further options are possible, for instance, methods based on Fliess expansion and Volterra series [40, 46, 51, 60, 68].

The purpose of this paper is not to focus on the set-valued numerical schemes themselves, but on the approximative construction of $T_S(\cdot)$. Thus we just choose the scheme described in (II) and (III) to present our idea from now on. In practice, there are several strategies in control problems to discretize the set of controls $\mathcal{U}$, see e.g. [9]. Here we choose a *piecewise constant* approximation $\mathcal{U}_h$ for the sake of simplicity which corresponds to use only one selection on the subinterval $[t_i, t_{i+1}]$ in the corresponding set-valued quadrature method. This choice is obvious in the approaches (7.18), (7.19) and e.g. for set-valued Riemann sums in (7.11) or (7.16). In the recursive formulas for the set-valued Riemann sum, this means that $\widetilde{c}_{i1} = 0$ for $i = 0, 1, \ldots, N-1$. Recall that from (II) the discrete reachable set reads as follows.

$$\begin{aligned}\mathcal{R}_h^S(t_N) = \{y \in \mathbb{R}^n \colon\ &\text{there exists a piecewise constant control } u_h \in \mathcal{U}_h \text{ and } y_0 \in S\\ &\text{such that } y = \Phi_h(t_N, t_0)y_0 + h\sum_{i=0}^{N} c_i \Phi_h(t_N, t_i)\bar{B}(t_i)u_h(t_i)\}\end{aligned}$$

or equivalently

$$\mathcal{R}_h^S(t_N) = \Phi_h(t_N, t_0)S + h\sum_{i=0}^{N} c_i \Phi_h(t_N, t_i)\bar{B}(t_i)U.$$

We set

$$t_h(y_0, y, u_h) = \min\{t_n \colon n \in \mathbb{N},\ y = \Phi_h(t_n, t_0)y_0 + h\sum_{i=0}^{n} c_i \Phi_h(t_n, t_i)\bar{B}(t_i)u_h(t_i)\}$$

for some $y \in \mathbb{R}^n$, $y_0 \in S$ and a piecewise constant grid function u_h with $u_h(t_i) = u_i \in U$, $i = 0, \dots, n$. If there does not exist such a grid control u_h which reaches y from y_0 by the corresponding discrete trajectory, $t_h(y_0, y, u_h) = \infty$. Then the discrete minimum time function $T_h(\cdot)$ is defined as

$$T_h(y) = \min_{\substack{u_h \in \mathcal{U}_h \\ y_0 \in S}} t_h(y_0, y, u_h).$$

Proposition 7.2.1. *In all of the constructions (I)–(III) described above, $\mathcal{R}_h^S(t_N)$ is a convex, compact and nonempty set.*

Proof. The key idea of the proof of this proposition is to employ the linearity of (7.4), in conjunction with the convexity of S, U and Proposition 2.3.4. In particular, it follows analogously to the proof of [9, Proposition 3.3]. □

Theorem 7.2.2. *Consider the linear control problem* (7.3)–(7.4)*. Assume that the set-valued quadrature method and the ODE solver have the same order* p*. Furthermore, assume that $\bar{A}(\cdot)$ and $\delta^*(l, \Phi(t_f, \cdot)\bar{B}(\cdot)U)$ have absolutely continuous* $(p-2)$*-nd derivative, the* $(p-1)$*-st derivative is of bounded variation uniformly with respect to all $l \in S_{n-1}$ and $\sum_{i=0}^{N} c_i \left\| B(t_i)U \right\|$ is uniform bounded for $N \in \mathbb{N}$. Then*

$$\mathrm{d}_H(\mathcal{R}^S(t_N), \mathcal{R}_h^S(t_N)) \le Ch^p, \tag{7.20}$$

where C is a non-negative constant.

Proof. See [14, Theorem 3.2]. □

Remark 7.2.3. *For $p = 2$ the requirements of Theorem 7.2.2 are fulfilled if $A(\cdot)$, $B(\cdot)$ are absolutely continuous and $A'(\cdot)$, $B'(\cdot)$ are bounded variation (see [31], [7, Secs. 1.6, 2.3]).*

The next subsection is devoted to the full discretization of the reachable set, i.e. we consider the space discretization as well. Since we will work with supporting points, we do this implicitly by discretizing the set S_{n-1} of normed directions. This error will be adapted to the error of the set-valued numerical scheme caused by the time discretization to preserve its order of convergence with respect to time step size as stated in Theorem 7.2.2. Then we will describe in detail the procedure to construct the graph of the minimum time function based on the approximation of the reachable sets. We will also provide the corresponding overall error estimate.

7.2.2 Implementation and error estimate of the reachable set approximation

For a particular problem, according to its smoothness in an appropriate sense we are first able to choose a difference method with a suitable order, say $O(h^p)$ for some $p > 0$, to solve (7.5) numerically effectively, for instance Euler scheme, Heun's scheme or Runge-Kutta scheme etc. Then we approximate Aumann's integral in (7.6) by a quadrature formula with the same order, for instance Riemann sum, trapezoid rule, or Simpson's rule etc. to obtain the discrete scheme of the global order $O(h^p)$.

We implement the set arithmetic operations in (7.17) only approximately as indicated in Proposition 2.3.7 and work with finitely many normed directions

$$\begin{aligned} S_{\mathcal{R}}^{\Delta} &:= \{\, l^k \,:\, k = 1, \ldots, N_{\mathcal{R}} \,\} \subset S_{n-1}, \\ S_{U}^{\Delta} &:= \{\, \eta^r \,:\, r = 1, \ldots, N_U \,\} \subset S_{m-1} \end{aligned} \tag{7.21}$$

satisfying

$$\begin{aligned} \mathrm{d}_H(S_{n-1}, S_{\mathcal{R}}^{\Delta}) &\le Ch^p, \\ \mathrm{d}_H(S_{m-1}, S_{U}^{\Delta}) &\le Ch^p \end{aligned}$$

to preserve the order of the considered scheme approximating the reachable set.

With this approximation we generate a finite set of supporting points of $\mathcal{R}_h^S(\cdot)$ and with its convex hull the fully discrete reachable set $\mathcal{R}_{h\Delta}^S(\cdot)$. To reach this target, we discretize the target set S and the control set U appearing in (7.13) and (7.17), e.g. along the line of Proposition 2.3.7:

$$\begin{aligned} \widetilde{S}_{\Delta} &:= \textstyle\bigcup_{l^k \in S_{\mathcal{R}}^{\Delta}} \{y(l^k, S)\}, \quad S_{\Delta} := \mathrm{co}(\widetilde{S}_{\Delta}) \\ \widetilde{U}_{\Delta} &:= \textstyle\bigcup_{\eta^r \in S_{U}^{\Delta}} \{y(\eta^r, U)\}, \quad U_{\Delta} := \mathrm{co}(\widetilde{U}_{\Delta}) \end{aligned} \tag{7.22}$$

Hence, S_Δ, U_Δ are polytopes approximating S resp. U.

Let $T_{h\Delta}(\cdot)$ be the fully discrete version of $T_S(\cdot)$ (it will be defined later in details). Our aim is to construct the graph of $T_{h\Delta}(\cdot)$ up to a given time t_f based on the knowledge of the reachable set approximation. We divide $[t_0, t_f]$ into K subintervals each of length Δt. Setting

$$\Delta t = \frac{t_f - t_0}{K}, \; h = \frac{\Delta t}{N},$$

we have $t_f - t_0 = KNh$ and compute subsequently the sets of supporting points $Y_{h\Delta}(\Delta t)$,..., $Y_{h\Delta}(t_f)$ by the algorithm described below yielding fully discrete reachable sets $\mathcal{R}_{h\Delta}^S(i\Delta t)$, $i = 1, \ldots, K$. Here K decides how many sublevel sets of the graph of $T_{h\Delta}(\cdot)$ we would like to have and h is the step size of the numerical scheme computing $Y_{h\Delta}(i\Delta t)$ starting from $Y_{h\Delta}((i-1)\Delta t)$.

Due to (7.7) and (7.8), the description of each sublevel set of $T_S(\cdot)$ can be formulated only with its boundary points, i.e. the supporting points of the reachable sets at the corresponding time. For the discrete setting, at each step, we will determine the value of $T_{h\Delta}(x)$ for $x \in Y_{h\Delta}(\cdot)$. Therefore, we only store this information for constructing the graph of $T_{h\Delta}(\cdot)$ on the subset $[t_0, t_f]$ of its range.

Algorithm 7.2.4.

step 1: Set $Y_{h\Delta}(t_0) = \tilde{S}_\Delta$, $\mathcal{R}^S_{h\Delta}(t_0) := S_\Delta$ *as in* (7.22), $i = 0$.

step 2: Compute $\widetilde{Y}_{h\Delta}(t_{i+1})$ *as follows*

$$\widetilde{Y}_{h\Delta}(t_{i+1}) = \Phi_h\big(t_{i+1}, t_i\big)Y_{h\Delta}\big(t_i\big) + h\sum_{j=0}^{N} c_j \Phi_h(t_{i+1}, t_{ij})\bar{B}(t_{ij})\tilde{U}_\Delta,$$

$$\widetilde{\mathcal{R}^S}_{h\Delta}(t_{i+1}) = \mathrm{co}\,\big(\widetilde{Y}_{h\Delta}(t_{i+1})\big),$$

where

$$t_i = t_0 + i\Delta t,\ t_{ij} = t_i + jh \quad (j = 0, 1, \ldots, N). \tag{7.23}$$

step 3: Compute the set of the supporting points $\bigcup_{l^k \in S^\Delta_{\mathcal{R}}} \{y(l^k, \widetilde{\mathcal{R}^S}_{h\Delta}(t_{i+1}))\}$ *and set*

$$Y_{h\Delta}(t_{i+1}) = \bigcup_{l^k \in S^\Delta_{\mathcal{R}}} \big\{y\big(l^k, \widetilde{\mathcal{R}^S}_{h\Delta}(t_{i+1})\big)\big)\big\} \tag{7.24}$$

where $y(l^k, \widetilde{\mathcal{R}^S}_{h\Delta}(t_{i+1}))$ *is an arbitrary element of* $\mathrm{Y}(l^k, \widetilde{\mathcal{R}^S}_{h\Delta}(t_{i+1}))$ *and set*

$$\mathcal{R}^S_{h\Delta}(t_{i+1}) := \mathrm{co}(Y_{h\Delta}(t_{i+1})).$$

step 4: If $i < K - 1$, *set* $i = i + 1$ *and go back to step 2. Otherwise, go to step 5.*

step 5: Construct the graph of $T_{h\Delta}(\cdot)$ *by the (piecewise) linear interpolation based on the values* t_i *at the points* $Y_{h\Delta}(t_i)$, $i = 0, \ldots, K$.

The algorithm computes the set of vertices $Y_{h\Delta}(t_i)$ of the polygon $\mathcal{R}^S_{h\Delta}(t_i)$ which are supporting points in the directions $l^k \in S^\Delta_{\mathcal{R}}$. The following proposition is the error estimate between the fully discrete reachable set $\mathcal{R}^S_{h\Delta}(\cdot)$ and $\mathcal{R}^S(\cdot)$.

Proposition 7.2.5. *Let Assumptions 7.0.1(i)–(iii), together with*

$$\mathrm{d}_H\left(\mathcal{R}^S_h(t_i), \mathcal{R}^S(t_i)\right) \leq C_s h^p \tag{7.25}$$

for the set-valued combination method (7.16) *in (II), be valid. Furthermore, finitely many directions* S^Δ_U, $S^\Delta_{\mathcal{R}} \subset S_{n-1}$ *are chosen with*

$$\max(\mathrm{d}_H(S_{n-1}, S^\Delta_U), \mathrm{d}_H(S_{n-1}, S^\Delta_{\mathcal{R}})) \leq C_\Delta h^p.$$

Then, for h small enough,

$$\begin{aligned} \mathrm{d}_H\left(\mathcal{R}^S_{h\Delta}(t_i), \mathcal{R}^S_h(t_i)\right) &\le C_f h^p, \\ \mathrm{d}_H\left(\mathcal{R}^S_{h\Delta}(t_i), \mathcal{R}^S(t_i)\right) &\le C_f h^p, \end{aligned} \tag{7.26}$$

where C_s, C_Δ, C_f are some positive constants and $t_i = t_0 + i\Delta t$, $i = 0, \dots, K$.

Proof. Recall that by construction we have

$$\widetilde{Y}_{h\Delta}(t_{i+1}) = \Phi_h\big(t_{i+1}, t_i\big) Y_{h\Delta}\big(t_i\big) + h \sum_{j=0}^{N} c_j \Phi_h(t_{i+1}, t_{ij}) \bar{B}(t_{ij}) \tilde{U}_\Delta,$$

and by definition together with the semigroup property of $\mathcal{R}^S_h(\cdot)$ in (7.17)

$$\mathcal{R}^S_h(t_{i+1}) = \Phi_h(t_{i+1}, t_i) \mathcal{R}^S_h(t_i) + h \sum_{j=0}^{N} c_j \Phi_h(t_{i+1}, t_{ij}) \bar{B}(t_{ij}) U,$$

where we used the notations (7.23) in Algorithm 7.2.4.

Setting $C_S := 2C_\Delta \operatorname{diam}(S)$, $C_U := 2C_\Delta \operatorname{diam}(U)$ and $C_{li} := 2C_\Delta \operatorname{diam}(\mathcal{R}^S_{h\Delta}(t_i))$, $i = 1, \dots, K$.

The reachable sets $\mathcal{R}^S(t_i)$ are all bounded by some constant C_R (see e.g. [10]) so that

$$\|\mathcal{R}^S_{h\Delta}(t_i)\| \le \|\mathcal{R}^S_h(t_i)\| \le \mathrm{d}_H(\mathcal{R}^S_h(t_i), \mathcal{R}^S(t_i)) + \|\mathcal{R}^S(t_i)\| \le C_s h^p + C_R.$$

Therefore, the doubled diameters of the fully discrete reachable sets $\mathcal{R}^S_{h\Delta}(t_i)$ are bounded by some constant C_l. We first observe that

$$\mathrm{d}_H(S_\Delta, S) \le C_S h^p, \ \mathrm{d}_H(U_\Delta, U) \le C_U h^p, \tag{7.27}$$

$$\mathrm{d}_H(\widetilde{\mathcal{R}^S}_{h\Delta}(t_i), \mathcal{R}^S_{h\Delta}(t_i)) \le C_{li} h^p \le C_l h^p, \ i = 1, \dots, K, \tag{7.28}$$

since $\mathcal{R}^S_{h\Delta}(t_i) := co\Big(\bigcup_{l^k \in S^\Delta_{\mathbb{R}}} \{y(l^k, \widetilde{\mathcal{R}^S}_{h\Delta}(t_i)\}\Big)$ by definition and Proposition 2.3.7 holds. Let

$$\begin{aligned} &C_A := \max_{t \in [t_0, t_f]} \left\|A(t)^T\right\|, \, C_B = \max_{t \in [t_0, t_f]} \left\|B(t)^T\right\|, \\ &C_0 := \max\{C_l, C_S\}, \\ &C_i := e^{\Delta t \cdot C_A} (1 + Ch)\Big(C_{i-1} + \Delta t \cdot C_B C_U\Big), \ i = 0, 1, \dots, K-1. \end{aligned} \tag{7.29}$$

We will prove the following by induction

$$\mathrm{d}_H\left(\mathcal{R}^S_{h\Delta}(t_i), \mathcal{R}^S_h(t_i)\right) \le C_i h^p, \ i = 0, \dots, K. \tag{7.30}$$

Since $\mathcal{R}_h^S(t_0) = S$, $\mathcal{R}_{h\Delta}^S = S_\Delta$, $C_S \leq C_0$ and (7.27) hold, the inequality (7.30) trivially follows for $i = 0$. Assume that (7.30) holds up to the index $i \geq 0$ with $i \leq K-1$. Since $\Phi(t,s)$ is the fundamental solution matrix of (7.5), we have

$$\Phi(\tau, s) = \Phi(s,s) + \int_s^\tau \bar{A}(v)\Phi(v,s)dv.$$

We have

$$\left\|\Phi(\tau,s)^T\right\| \leq \left\|\Phi(s,s)^T\right\| + \int_s^\tau \left\|\Phi(v,s)^T\right\| \left\|A(t_0+t_f-v)^T\right\| dv,$$

therefore, Gronwall's inequality yields

$$\left\|\Phi(\tau,s)^T\right\| \leq e^{\int_s^\tau \left\|A(t_0+t_f-v)^T\right\|dv} \leq e^{(\tau-s)C_A},$$

Since $\widetilde{\mathcal{R}^S}_{h\Delta}(t_i)$, $\mathcal{R}_h^S(t_i) \in \mathcal{C}(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\mathrm{d}_H(\widetilde{\mathcal{R}^S}_{h\Delta}(t_{i+1}), \mathcal{R}_h^S(t_{i+1})) = \max_{l\in S_{n-1}} |\delta^*(l, \widetilde{\mathcal{R}^S}_{h\Delta}(t_{i+1})) - \delta^*(l, \mathcal{R}_h^S(t_{i+1}))| \\
&= \max_{l\in S_{n-1}} \Bigg|\delta^*\Big(l, co\{\Phi_h\big(t_{i+1},t_i\big)Y_{h\Delta}(t_i) + h\sum_{j=0}^N c_j\Phi_h(t_{i+1},t_{ij})\bar{B}(t_{ij})\widetilde{U}_\Delta\}\Big) \\
&\quad - \delta^*\Big(l, \Phi_h\big(t_{i+1},t_i\big)\mathcal{R}_h^S(t_i) + h\sum_{j=0}^N c_j\Phi_h(t_{i+1},t_{ij})\bar{B}(t_{ij})U\Big)\Bigg| \\
&= \max_{l\in S_{n-1}} \Bigg|\delta^*\Big(\Phi_h\big(t_{i+1},t_i\big)^T l, \mathrm{co}(Y_{h\Delta}(t_i))\Big) + h\sum_{j=0}^N c_j\delta^*\Big((\Phi_h(t_{i+1},t_{ij})\bar{B}(t_{ij}))^T l, \mathrm{co}(\widetilde{U}_\Delta)\Big) \\
&= -\delta^*\Big(\Phi_h\big(t_{i+1},t_i\big)^T l, \mathcal{R}_h^S(t_i)\Big) - h\sum_{j=0}^N c_j\delta^*\Big((\Phi_h(t_{i+1},t_{ij})\bar{B}(t_{ij}))^T l, U\Big) \\
&= \max_{l\in S_{n-1}} \Bigg|\delta^*\Big(\Phi_h\big(t_{i+1},t_i\big)^T l, \mathcal{R}_{h\Delta}^S(t_i)\Big) - \delta^*\Big(\Phi_h\big(t_{i+1},t_i\big)^T l, \mathcal{R}_h^S(t_i)\Big)\Bigg| \\
&\quad + h\sum_{j=0}^N c_j \max_{l\in S_{n-1}} \Bigg|\delta^*\Big((\Phi_h(t_{i+1},t_{ij})\bar{B}(t_{ij}))^T l, U_\Delta\Big) - \delta^*\Big((\Phi_h(t_{i+1},t_{ij})\bar{B}(t_{ij}))^T l, U\Big)\Bigg| \\
&\leq \max_{l\in S_{n-1}} \left\|\Phi_h\big(t_{i+1},t_i\big)^T l\right\| \max_\xi \Bigg|\delta^*\Big(\xi, \mathcal{R}_{h\Delta}^S(t_i)\Big) - \delta^*\Big(\xi, \mathcal{R}_h^S(t_i)\Big)\Bigg| \\
&\quad + h\sum_{j=0}^N c_j \max_{l\in S_{n-1}} \left\|(\Phi_h(t_{i+1},t_{ij})\bar{B}(t_{ij}))^T l\right\| \max_\xi \Bigg|\delta^*\Big(\xi, U_\Delta\Big) - \delta^*\Big(\xi, U\Big)\Bigg| \\
&\leq \left\|\Phi\big(t_{i+1},t_i\big)^T + I_n Ch^p\right\| \max_\xi \Bigg|\delta^*\Big(\xi, \mathcal{R}_{h\Delta}^S(t_i)\Big) - \delta^*\Big(\xi, \mathcal{R}_h^S(t_i)\Big)\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&+ h\sum_{j=0}^{N} c_j \left\|((\Phi(t_{i+1},t_{ij}) + I_n Ch^p)\bar{B}(t_{ij}))^T\right\| \max_{\xi} \left| \delta^*\Big(\xi, U_\Delta\Big) - \delta^*\Big(\xi, U)\Big) \right| \\
\le{}& \left\| \Phi\big(t_{i+1},t_i\big)^T + I_n Ch^p \right\| \max_{\xi} \left| \delta^*\Big(\xi, \mathcal{R}^S_{h\Delta}(t_i)\Big) - \delta^*\Big(\xi, \mathcal{R}^S_h(t_i)\Big) \right| \\
&+ h\sum_{j=0}^{N} c_j \left\|\Phi(t_{i+1},t_{ij})^T + I_n Ch^p\right\| \left\|\bar{B}(t_{ij})^T\right\| \max_{\xi} \left| \delta^*\Big(\xi, U_\Delta\Big) - \delta^*\Big(\xi, U)\Big) \right| \\
\le{}& (e^{\Delta t C_A} + Ch^p) \max_{\xi} \left| \delta^*\Big(\xi, \mathcal{R}^S_{h\Delta}(t_i)\Big) - \delta^*\Big(\xi, \mathcal{R}^S_h(t_i)\Big) \right| \\
&+ h\sum_{j=0}^{N} c_j (e^{\Delta t C_A} + Ch^p) C_B \max_{\xi} \left| \delta^*\Big(\xi, U_\Delta\Big) - \delta^*\Big(\xi, U)\Big) \right| \\
\le{}& (e^{\Delta t C_A} + Ch^p)\Big(\mathrm{d}_H(\mathcal{R}^S_{h\Delta}(t_i), \mathcal{R}^S_h(t_i)) + h\sum_{j=0}^{N} c_j C_B \,\mathrm{d}_H(U_\Delta, U)\Big) \\
\le{}& e^{\Delta t C_A}(1+Ch)\Big(C_i + \Delta t \cdot C_B C_U\Big) h^p,
\end{aligned}
$$

since the order of convergence of the quadrature method is at least 1 so that $h\sum_{j=0}^{N} c_j = \Delta t$. Hence,

$$
\mathrm{d}_H(\widetilde{\mathcal{R}^S}_{h\Delta}(t_{i+1}), \mathcal{R}^S_h(t_{i+1})) \le C_{i+1} h^p. \tag{7.31}
$$

It remains to estimate the constant C_i uniformly for all $i = 0, 1, \ldots, K$ as

$$
C_i \le e^{i\Delta t \cdot C_A}(1+Ch)^i (C_0 + i\Delta t \cdot C_B C_U) \quad (i = 0, 1, \ldots, K)
$$

which follows per induction.
For $i = 0$ this estimate is obvious. Assume that it holds up to some $i \ge 0$ with $i < K$. Then,

$$
\begin{aligned}
C_{i+1} &= e^{\Delta t \cdot C_A}(1+Ch)\Big(C_i + \Delta t \cdot C_B C_U\Big) \\
&\le e^{\Delta t \cdot C_A}(1+Ch)\Big(e^{i\Delta t \cdot C_A}(1+Ch)^i\big(C_0 + i\Delta t \cdot C_B C_U\big) + \Delta t \cdot C_B C_U\Big) \\
&\le e^{\Delta t \cdot C_A}(1+Ch)e^{i\Delta t \cdot C_A}(1+Ch)^i\Big(C_0 + i\Delta t \cdot C_B C_U + \Delta t \cdot C_B C_U\Big) \\
&= e^{(i+1)\Delta t \cdot C_A}(1+Ch)^{i+1}(C_0 + (i+1)\Delta t \cdot C_B C_U)
\end{aligned}
$$

so that

$$
\begin{aligned}
C_i &\le e^{C_A(t_f - t_0)}\Big(1 + C\frac{\Delta t}{N}\Big)^i (C_0 + C_B C_U(t_f - t_0)) \\
&\le e^{C_A(t_f - t_0)} e^{CK\frac{\Delta t}{N}} (C_0 + C_B C_U(t_f - t_0)), \\
C_i &\le \overline{C} := e^{(C_A + C)(t_f - t_0)}(C_0 + C_B C_U(t_f - t_0)) \quad (i = 0, 1, \ldots, K).
\end{aligned} \tag{7.32}
$$

Putting (7.31), (7.28) together and noticing that Hausdorff distance satisfies triangle inequality, we receive

$$\mathrm{d}_H(\mathcal{R}^S_{h\Delta}(t_{i+1}), \mathcal{R}^S_h(t_{i+1})) \le \max\{C_l, C_{i+1}\}h^p \le C_{i+1}h^p, \tag{7.33}$$

therefore, (7.30) holds for $i = 0, \dots, K$. Moreover, taking into account the estimate (7.32), (7.33) becomes

$$\mathrm{d}_H\left(\mathcal{R}^S_{h\Delta}(t_i), \mathcal{R}^S_h(t_i)\right) \le \overline{C}h^p \quad \text{for } i = 0, \dots, K. \tag{7.34}$$

In conclusion, denoting $C_f = \max\{\overline{C}, C_s\}$ and combining (7.34) with (7.25) we have

$$\mathrm{d}_H(\mathcal{R}^S_{h\Delta}(t_i), \mathcal{R}^S(t_i)) \le C_f h^p.$$

The proof is completed. □

Remark 7.2.6. *If S is a singleton, we do not need to discretize the target set. The overall error estimate in* (7.26) *even improves in this case, since*

$$\mathrm{d}_H\left(\widetilde{\mathcal{R}^S}_{h\Delta}(t_0), \mathcal{R}^S_h(t_0)\right) = 0.$$

As we can see in this subsection the convexity of the reachable set plays a vital role. Therefore, this approach can only be extended to special nonlinear control systems with convex reachable sets.

In the following subsection, we provide the error estimation of $T_{h\Delta}(\cdot)$ obtained by the indicated approach under Assumptions 7.0.1, the regularity of $T_S(\cdot)$ and the properties of the numerical approximation.

7.2.3 Error estimate of the minimum time function

After computing the fully discrete reachable sets in Subsection 7.2.2, we obtain the values of $T_{h\Delta}(x)$ for all $x \in \bigcup_{i=0,\dots,K} Y_{h\Delta}(t_i)$, $t_i = t_0 + i\Delta t$. For all boundary points $x \in \partial\mathcal{R}^S_{h\Delta}(t_i)$ and some $i = 1, \dots, K$, we define

$$T_{h\Delta}(x) = t_i \text{ for } x \in \partial\mathcal{R}^S_{h\Delta}(t_i), \tag{7.35}$$

together with the initial condition

$$T_{h\Delta}(x) = t_0 \quad \text{for } x \in S_\Delta.$$

The task is now to define a suitable value of $T_{h\Delta}(x)$ in the computational domain

$$\Omega := \bigcup_{i=0,\dots,K} \mathcal{R}^S_{h\Delta}(t_i),$$

if x is neither a boundary point of reachable sets nor lies inside the target set. First we construct a simplicial triangulation $\{\Gamma_j\}_{j=1,\dots,M}$ over the set $\Omega \setminus \mathrm{int}(S)$ of points with grid nodes in $\bigcup_{i=0,\dots,K} Y_{h\Delta}(t_i)$. Hence,

- $\Gamma_j \subset \mathbb{R}^n$ is a simplex for $j = 1, \ldots, M$,
- $\Omega \setminus \operatorname{int}(S) = \bigcup_{j=1,\ldots,M} \Gamma_j$,
- the intersection of two different simplices is either empty or a common face
- all supporting points in the sets $\{Y_{h\Delta}(t_i)\}_{i=0,\ldots,K}$ are vertices of some simplex,
- all the vertices of each simplex have to belong either to the fully discrete reachable set $\mathcal{R}^S_{h\Delta}(t_i)$ or to $\mathcal{R}^S_{h\Delta}(t_{i+1})$ for some $i = 0, 1, \ldots, K-1$.

For the triangulation as in Figure 7.1, we introduce the maximal diameter of simplices as

$$\Delta_\Gamma := \max_{j=1,\ldots,M} \operatorname{diam}(\Gamma_j).$$

Assume that x is neither a boundary point of one of the computed discrete reachable

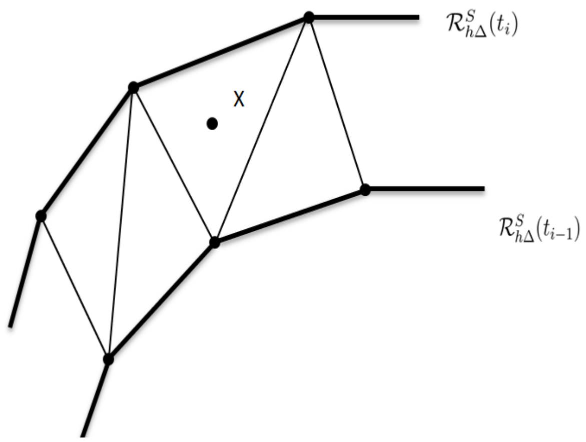

Figure 7.1: part of the triangulation

sets $\{\mathcal{R}^S_{h\Delta}(t_i)\}_{i=0,\ldots,K}$ nor an element of the target set S and let Γ_j be the simplex containing x. Then

$$T_{h\Delta}(x) = \sum_{\nu=1}^{n+1} \lambda_\nu T_{h\Delta}(x_\nu), \tag{7.36}$$

where $x = \sum_{\nu=1}^{n+1} \lambda_\nu x_\nu$, $\sum_{\nu=1}^{n+1} \lambda_\nu = 1$ with $\lambda_\nu \geq 0$ and $\{x_\nu\}_{\nu=1,\ldots,n+1}$ being the vertices of Γ_j.

If x lies in the interior of Γ_j, the index j of this simplex is unique. Otherwise, x lies on the common face of two or more simplices due to our assumptions on the simplicial triangulation and (7.36) is well-defined. Let i be the index such that $\Gamma_j \in \mathcal{R}^S_{h\Delta}(t_i) \setminus \operatorname{int}(\mathcal{R}^S_{h\Delta}(t_{i-1}))$.

Since $T_{h\Delta}(x_\nu)$ is either t_i or t_{i-1} due to (7.35), we have

$$T_{h\Delta}(x) = \sum_{\nu=1}^{n+1} \lambda_\nu T_{h\Delta}(x_\nu) \le t_i,$$
$$\partial\mathcal{R}^S_{h\Delta}(t_i) = \{y \in \mathbb{R}^n : T_{h\Delta}(y) = t_i\}.$$

The latter holds, since the convex combination is bounded by t_i and equality to t_i only holds, if all vertices with positive coefficient λ_ν lie on the boundary of the reachable set $\mathcal{R}^S_{h\Delta}(t_i)$.

The following theorem is about the error estimate of the minimum time function obtained by this approach.

Theorem 7.2.7. *Assume that $T_S(\cdot)$ is continuous with a non-decreasing modulus $\omega(\cdot)$ in $\mathcal{R}^S$, i.e.*

$$|T_S(x) - T_S(y)| \le \omega(\|x-y\|) \quad \textit{for all } x, y \in \mathcal{R}^S. \tag{7.37}$$

Let Assumptions 7.0.1 be fulfilled, furthermore assume that

$$\mathrm{d}_H(\mathcal{R}^S_{h\Delta}(t), \mathcal{R}^S(t)) \le Ch^p \tag{7.38}$$

holds. Then

$$\|T_S - T_{h\Delta}\|_{\infty,\Omega} \le \omega(\Delta_\Gamma) + \omega(Ch^p). \tag{7.39}$$

where $\|\cdot\|_{\infty,\Omega}$ is the supremum norm taken over Ω.

Proof. We divide the proof into two cases.

case 1: $x \in \partial\mathcal{R}^S_{h\Delta}(t_i)$ for some $i = 1, \ldots, K$.
Let us choose a best approximation $\bar{x} \in \partial\mathcal{R}^S(t_i)$ of x so that

$$\|x - \bar{x}\| = \mathrm{d}(x, \partial\mathcal{R}^S(t_i)) \le \mathrm{d}_H(\partial\mathcal{R}^S_{h\Delta}(t_i), \partial\mathcal{R}^S(t_i)) = \mathrm{d}_H(\mathcal{R}^S_{h\Delta}(t_i), \mathcal{R}^S(t_i)),$$

where we used [75] in the latter equality. Clearly, (7.8), (7.36) show that $T_{h\Delta}(x) = T_S(\bar{x}) = t_i$. Then

$$\begin{aligned} |T_S(x) - T_{h\Delta}(x)| &\le |T_S(x) - T_S(\bar{x})| + |T_S(\bar{x}) - T_{h\Delta}(x)| \\ &\le \omega(\|x-\bar{x}\|) \le \omega\big(\mathrm{d}_H(\mathcal{R}^S_{h\Delta}(t_i), \mathcal{R}^S(t_i))\big) \le \omega(Ch^p) \end{aligned} \tag{7.40}$$

due to (7.38).

case 2: $x \in \operatorname{int}\big(\mathcal{R}^S_{h\Delta}(t_i)\big) \setminus \mathcal{R}^S_{h\Delta}(t_{i-1})$ for some $i = 1, \ldots, K$.
Let Γ_j be a simplex containing x with the set of vertices $\{x_j\}_{j=1,\ldots,n+1}$. Then

$$T_{h\Delta}(x) = \sum_{j=1}^{n+1} \lambda_j T_{h\Delta}(x_j),$$

where $x = \sum_{j=1}^{n+1} \lambda_j x_j$, $\sum_{j=1}^{n+1} \lambda_j = 1, \lambda_j \geq 0$. We obtain

$$
\begin{aligned}
&|T_S(x) - T_{h\Delta}(x)| = |T_S(x) - \sum_{j=1}^{n+1} \lambda_j T_{h\Delta}(x_j)| \\
&\leq |T_S(x) - \sum_{j=1}^{n+1} \lambda_j T_S(x_j)| + |\sum_{j=1}^{n+1} \lambda_j T_S(x_j) - \sum_{j=1}^{n+1} \lambda_j T_{h\Delta}(x_j)| \\
&\leq \sum_{j=1}^{n+1} \lambda_j \Big(|T_S(x) - T_S(x_j)| + |T_S(x_j) - T_{h\Delta}(x_j)| \Big) \leq \omega(\Delta_\Gamma) + \omega(Ch^p),
\end{aligned}
$$

where we applied the continuity of $T_S(\cdot)$ for the first term and the error estimate (7.40) of case 1 for the other.

Combining two cases and noticing that $T_S(x) = T_{h\Delta}(x) = t_0$ if $x \in S_\Delta$, we get

$$\|T_S - T_{h\Delta}\|_{\infty,\Omega} := \max_{x\in\Omega} |T_S(x) - T_{h\Delta}(x)| \leq \omega(\Delta_\Gamma) + \omega(Ch^p). \tag{7.41}$$

The proof is completed. □

Remark 7.2.8. *Theorem 7.2.2 provides sufficient conditions for set-valued combination methods such that* (7.38) *holds. See also e.g. [33] for set-valued Euler's method resp. [74] for Heun's method. If the minimum time function is Hölder continuous on* Ω, (7.39) *becomes*

$$\|T_S - T_{h\Delta}\|_{\infty,\Omega} \leq C\Big((\Delta_\Gamma)^{\frac{1}{k}} + h^{\frac{p}{k}} \Big) \tag{7.42}$$

for some positive constant C. The inequality (7.42) *shows that the error estimate is improved in comparison with the one obtained in [26] and does not assume explicitly the regularity of optimal solutions as in [18]. One possibility to define the modulus of continuity satisfying the required property of non-decrease in Theorem 7.2.7 is as follows:*

$$\omega(\delta) = \sup\{|T_S(x) - T_S(y)| : \|x - y\| \leq \delta\}$$

An advantage of the methods of Volterra type studied in [26] which benefit from non-standard selection strategies is that the discrete reachable sets converge with higher order than 2. The order 2 is an order barrier for set-valued Runge-Kutta methods with piecewise constant controls or independent choices of controls, since many linear control problems with intervals or boxes for the control values are not regular enough for higher order approximations (see [74]).

Remark 7.2.9. *There are many different triangulations based on the same data. Among them, we can always choose the one with a smaller diameter close to the Hausdorff distance of the two sets by applying standard grid generators. For example, from the same set of data we can build the two following grids and it is easy to see in Figure 7.2 that the left one (for which only three edges are emerging from the corner of the bigger reachable set) gives a better approximation, since the maximal diameter in the triangulation at the right is much bigger.*

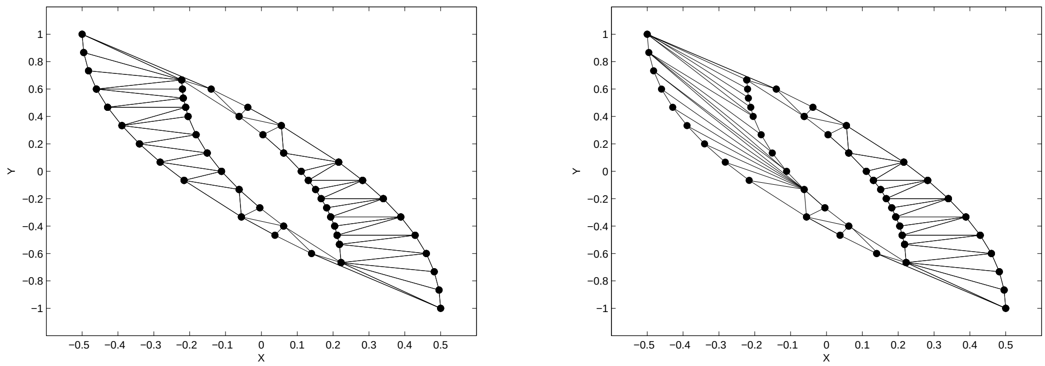

Figure 7.2: two triangulations for the linear interpolation of the minimum time function

Proposition 7.2.10. *Let the conditions of Theorem 7.2.7 be fulfilled. Furthermore assume that the step size h is so small such that Ch^p in* (7.38) *is smaller than $\frac{\varepsilon}{3}$, where*

$$\mathcal{R}^S(t_i) + \varepsilon B_1(0) \subset \operatorname{int} \mathcal{R}^S(t_{i+1}) \quad \text{for all } i = 0, \dots, K-1. \tag{7.43}$$

Then

$$\mathcal{R}^S_{h\Delta}(t_i) + \frac{\varepsilon}{3} B_1(0) \subset \operatorname{int} \mathcal{R}^S_{h\Delta}(t_{i+1}) \tag{7.44}$$

and

$$\|T_S - T_{h\Delta}\|_{\infty,\Omega} \le 2\Delta t. \tag{7.45}$$

where $\|\cdot\|_{\infty,\Omega}$ is the supremum norm taken over Ω.

Proof. For some $i = 0, \dots, K-1$ we choose a constant $M_{i+1} > 0$ such that $\mathcal{R}^S(t_{i+1}) \subset M_{i+1} B_1(0)$. Since $\mathcal{R}^S(t_i)$ does not intersect the complement of $\operatorname{int} \mathcal{R}^S(t_{i+1})$ bounded with $M_{i+1} B_1(0)$ and both are compact sets, there exists $\varepsilon > 0$ such that

$$\mathcal{R}^S(t_i) + \varepsilon B_1(0) \subset \operatorname{int} \mathcal{R}^S(t_{i+1}) \subset M_{i+1} B_1(0). \tag{7.46}$$

We will show that a similar inclusion as (7.46) holds for the discrete reachable sets for small step sizes. If the step size h is so small that Ch^p in (7.38) is smaller than $\frac{\varepsilon}{3}$, then we have the following inclusions:

$$\begin{aligned} \operatorname{int} \mathcal{R}^S(t_{i+1}) &\subset \operatorname{int} \big(\mathcal{R}^S_{h\Delta}(t_{i+1}) + Ch^p B_1(0)\big) = \operatorname{int} \mathcal{R}^S_{h\Delta}(t_{i+1}) + Ch^p \operatorname{int} B_1(0), \\ \mathcal{R}^S(t_i) + \varepsilon B_1(0) &\subset \operatorname{int} \mathcal{R}^S(t_{i+1}) \subset \operatorname{int} \mathcal{R}^S_{h\Delta}(t_{i+1}) + \frac{\varepsilon}{3} B_1(0). \end{aligned}$$

By the order cancellation law of convex compact sets in [65, Theorem 3.2.1]

$$\mathcal{R}^S(t_i) + \frac{2}{3}\varepsilon B_1(0) \subset \operatorname{int} \mathcal{R}^S_{h\Delta}(t_{i+1})$$

and

$$\mathcal{R}^S_{h\Delta}(t_i) + \frac{\varepsilon}{3}B_1(0) \subset \big(\mathcal{R}^S(t_i) + \frac{\varepsilon}{3}B_1(0)\big) + \frac{\varepsilon}{3}B_1(0) \subset \operatorname{int}\mathcal{R}^S_{h\Delta}(t_{i+1}). \tag{7.47}$$

We have

$$|T_S(x) - T_{h\Delta}(x)| = \sum_{j=1}^{n+1} \lambda_j |T_S(x) - T_{h\Delta}(x_j)|, \tag{7.48}$$

in order to obtain the estimate, we observe that

1) $x_j \in \partial\mathcal{R}^S_{h\Delta}(t_i)$, then $t_\nu \le T_S(x_j) \le t_{i+1}$ with $\nu = \max\{0, i-1\}$.
2) $x \in \operatorname{int}(\mathcal{R}^S_{h\Delta}(t_i)) \setminus \mathcal{R}^S_{h\Delta}(t_{i-1})$, then $t_\nu < T_S(x) \le t_{i+1}$ with $\nu = \max\{0, i-2\}$.

To prove 1) the inequality $T_S(x_j) >= t_0$ is clear. Assume that $T_S(x_j) < t_{i-1}$ for some $i > 1$. Then $x_j \in \mathcal{R}^S(t_{i-1})$. By the estimates (7.38), (7.47) and $Ch^p < \frac{\varepsilon}{3}$, it follows that

$$x_j \in \mathcal{R}^S_{h\Delta}(t_{i-1}) + Ch^p B_1(0) \subset \operatorname{int}\mathcal{R}^S_{h\Delta}(t_i)$$

which is a contradiction to the assumption $x_j \in \partial\mathcal{R}^S_{h\Delta}(t_i)$. Hence, $T_S(x_j) \ge t_{i-1}$. Assume that $T_S(x_j) > t_{i+1}$. Then, $x_j \notin \mathcal{R}^S(t_{i+1})$. Furthermore, x_j cannot be an element of $\mathcal{R}^S_{h\Delta}(t_i)$, since otherwise

$$x_j \in \mathcal{R}^S_{h\Delta}(t_i) \subset \mathcal{R}^S(t_i) + Ch^p B_1(0) \subset \operatorname{int}\mathcal{R}^S(t_{i+1})$$

which is a contradiction to $x_j \notin \mathcal{R}^S(t_{i+1})$.
Therefore, $x_j \notin \mathcal{R}^S_{h\Delta}(t_i)$ which contradicts $x_j \in \partial\mathcal{R}^S_{h\Delta}(t_i)$. Hence, the starting assumption $T_S(x_j) > t_{i+1}$ must be wrong which proves $T_S(x_j) \le t_{i+1}$.
To prove 2) if we assume $T_S(x) \le t_{i-2}$ for some $i \ge 2$, then $x \in \mathcal{R}^S(t_{i-2})$ and

$$x \in \mathcal{R}^S_{h\Delta}(t_{i-2}) + Ch^p B_1(0) \subset \operatorname{int}\mathcal{R}^S_{h\Delta}(t_{i-1})$$

by estimate (7.38). But this contradicts $x \notin \mathcal{R}^S_{h\Delta}(t_{i-1})$. Therefore, $T_S(x) > t_{i-2}$.

Assuming $T_S(x) > t_{i+1}$ for some $i < K-1$, then $x \notin \mathcal{R}^S(t_{i+1})$. Furthermore, if x is an element of $\mathcal{R}^S_{h\Delta}(t_i)$,

$$x \in \mathcal{R}^S_{h\Delta}(t_i) \subset \mathcal{R}^S(t_i) + Ch^p B_1(0) \subset \operatorname{int}\mathcal{R}^S(t_{i+1})$$

which is a contradiction to $x \notin \mathcal{R}^S(t_{i+1})$.
Therefore, $x \notin \mathcal{R}^S_{h\Delta}(t_i)$ which contradicts $x \in \operatorname{int}(\mathcal{R}^S_{h\Delta}(t_i)) \setminus \mathcal{R}^S_{h\Delta}(t_{i-1})$. Hence, the starting assumption $T_S(x) > t_{i+1}$ must be wrong which proves $T_S(x) \le t_{i+1}$. Consequently, 1) and 2) are proved.
Notice that

a) the case 1) means

$$T_S(x_j) \in [t_{i-1}, t_{i+1}] \quad (i \ge 1),$$
$$T_S(x_j) = t_0 \quad (i = 0),$$

and $|T_S(x_j) - T_{h\Delta}(x_j)| \le \Delta t$ due to $T_{h\Delta}(x_j) = t_i$, $i = 0, \ldots, K$.

b) from the case 2), we obtain

$$\begin{aligned}
T_S(x) &\in (t_{i-2}, t_{i+1}] \quad (i \geq 2),\\
T_{h\Delta}(x_j) - T_S(x) &< t_i - t_{i-2} = 2\Delta t,\\
T_{h\Delta}(x_j) - T_S(x) &> t_{i-1} - t_{i+1} = -2\Delta t.
\end{aligned}$$

Therefore, $|T_S(x) - T_{h\Delta}(x_j)| \leq 2\Delta t$ for $i \geq 2$ (similarly with estimates for $i = 0, 1$). Altogether, (7.45) is proved. □

7.3 Convergence of open loop controls and reconstruction of discrete suboptimal trajectories

In this section we first prove the convergence of the normal cones of $\mathcal{R}^S_{h\Delta}(\cdot)$ to the ones of the continuous-time reachable set $\mathcal{R}^S(\cdot)$ in an appropriate sense. Using this result we will be able to reconstruct discrete optimal trajectories to reach the target from a set of given points and also derive the proof of L^1-convergence of discrete optimal controls. In this section only convergence under weaker assumptions and no convergence order 1 as in [2] are proved (see more references therein for the classical field of direct discretization methods). We also restrict to linear minimum time problems.

The following theorem plays an important role in this reconstruction and will deal with the convergence of the normal cones. If the normal vectors of $\mathcal{R}^S_{h\Delta}(\cdot)$ converge to the corresponding ones of $\mathcal{R}^S(\cdot)$, the discrete optimal controls can be computed with the discrete Pontryagin Maximum Principle under suitable assumptions.

For the remaining part of this section let us consider a fixed index $i \in \{1, 2 \ldots, K\}$. We choose a space discretization $\Delta = \Delta(h)$ with $\mathcal{O}(\Delta) = \mathcal{O}(h^p)$ (compare with [7, Sec. 3.1]) and often suppress the index Δ for the approximate solutions and controls.

Theorem 7.3.1. *Consider a discrete approximation of reachable sets of type (I)–(III) with*

$$\lim_{h \downarrow 0} \mathrm{d}_H(\mathcal{R}^S_{h\Delta}(t_i), \mathcal{R}^S(t_i)) = 0. \tag{7.49}$$

Under Assumptions 6.0.1, the set-valued maps $x \mapsto N_{\mathcal{R}^S_{h\Delta}(t_i)}(x)$ converge graphically to the set-valued map $x \mapsto N_{\mathcal{R}^S(t_i)}(x)$ for $i = 1, \ldots, K$.

Proof. Let us recall that, under Assumptions 6.0.1 and by the construction in Subsec. 7.2.1, $\mathcal{R}^S_{h\Delta}(t_i)$, $\mathcal{R}^S(t_i)$ are convex, compact and nonempty sets. Moreover, we also have that the indicator functions $I_{\mathcal{R}^S_{h\Delta}(t_i)}(\cdot), I_{\mathcal{R}^S(t_i)}(\cdot)$ are lower semicontinuous convex functions (see Proposition 2.2.9). By [69, Example 4.13] the convergence in (7.49) with respect to the Hausdorff set also implies the set convergence in the sense of Definition 2.2.10. Hence, [69, Proposition 7.4(f)] applies and shows that the corresponding indicator functions converge epi-graphically. Since the subdifferential of the (convex) indicator functions coincides with the normal cone by [69, Exercise 8.14], Attouch's Theorem 2.2.12 yields the graphical convergence of the corresponding normal cones. □

The remainder deals with the reconstruction of discrete optimal trajectories and the proof of convergence of optimal controls in the L^1-*norm*, i.e. $\int_0^{t_i} \|\hat{u}(t) - \hat{u}_h(t)\|_1 dt \to 0$ as $h \downarrow 0$ for $\hat{u}(\cdot)$, $\hat{u}_h(\cdot)$ being defined later, where the ℓ_1-*norm* is defined for $x \in \mathbb{R}^n$ as $\|x\|_1 = \sum_{i=1}^n |x_i|$. To illustrate the idea, we confine to a special form of the target and control set, i.e. $S = \{0\}$, $U = [-1,1]^m$, $t \in [0, t_i]$ and the time invariant time-reversed linear system

$$\begin{cases} \dot{y}(t) &= \bar{A}y(t) + \bar{B}u(t), \;\; u(t) \in [-1,1]^m, \\ y(0) &= 0. \end{cases} \tag{7.50}$$

Algorithm 7.2.4 can be interpreted pointwisely in this context as follows. For any $y_{(i-1)N} \in Y_{h\Delta}(t_i)$ there exists a sequence of controls $\{u_{kj}\}_{j=0,\dots,N}^{k=1,\dots,i-1}$ such that

$$\begin{cases} y_{(k-1)N} &= \Phi_h\big(t_k, t_{k-1}\big)y_{(k-1)0} + h\sum_{j=0}^N c_{kj}\Phi_h(t_k, t_{(k-1)j})\bar{B}u_{(k-1)j}, \\ y_{00} &= 0 \end{cases} \tag{7.51}$$

for $k = 1, \dots, i$. Thus

$$y_{(i-1)N} = h\sum_{k=1}^{i}\sum_{j=0}^{N} c_{kj}\Phi_h(t_i, t_{(k-1)j})\bar{B}u_{(k-1)j}.$$

The continuous-time adjoint equation of (7.50) written for n-row vectors reads as

$$\begin{cases} \dot{\eta}(t) &= -\eta(t)\bar{A} \\ \eta(t_i) &= \zeta, \end{cases} \tag{7.52}$$

and its discrete version, approximated by the same method (see [43, Chap. 5]) as the one used to discretize (7.50), i.e. (7.51), can be written as follows. For $k = i-1, i-2, \dots, 0$ and $j = N, N-1, \dots, 1$,

$$\begin{cases} \eta_{k(j-1)} &= \eta_{kj}\Phi_h\big(t_{kj}, t_{k(j-1)}\big) \\ \eta_{(i-1)N} &= \zeta_h, \end{cases} \tag{7.53}$$

where ζ, ζ_h will be clarified later. By the definition of t_{kj} (see Algorithm 7.2.4) the index $k0$ can be replaced by $(k-1)N$, the solution of (7.53) in backward time is therefore possible. Here, the end condition will be chosen subject to certain transversality conditions, see the latter reference for more details.

Due to well-known arguments (see e.g. [59, Sec. 2.2]) the end point of the time-optimal solution lies on the boundary of the reachable set and the adjoint solution $\eta(\cdot)$ is an outer normal at this end point. Similarly, this also holds in the discrete case. The following proposition formulates this fact by a discrete version of [59, Sec. 2.2, Theorem 2]. The proof is just a translation of the one of the cited theorem in [59] to the discrete language. For the sake of clarity, we will formulate and prove it in detail.

Proposition 7.3.2. *Consider the system* (7.50) *in* $\mathbb{R}^n$ *with its adjoint problem* (7.52) *as well as their discrete pendants* (7.51), (7.53) *respectively. Let* $\{u_{kj}\}$ *be a sequence of controls,* $\{y_{kj}\}$ *be its corresponding discrete solution. Then under Assumptions 6.0.1, for* h *small enough,* $y_{(i-1)N} \in Y_{h\Delta}(t_i)$ *if and only if there exists nontrivial solution* $\{\eta_{kj}\}$ *of* (7.53) *such that*

$$\eta_{kj}\bar{B}u_{kj} = \max_{u\in U}\{\eta_{kj}\bar{B}u\}$$

for $k = 0, ..., i-1,\ j = 0, ..., N$*, where* $Y_{h\Delta}(t_i)$ *is defined as in Algorithm 7.2.4.*

Proof. Assume that $\{u_{jk}\}$ is such that $y_{(i-1)N}$ by the response

$$y_{(i-1)N} = h\sum_{k=1}^{i}\sum_{j=0}^{N} c_{kj}\Phi_h(t_i, t_{(k-1)j})\bar{B}u_{(k-1)j}.$$

Since $\mathcal{R}^S_{h\Delta}(t_i)$ is a compact and convex set by construction, there exists a supporting hyperplane γ to $\mathcal{R}^S_{h\Delta}(t_i)$ at $y_{(i-1)N}$. Let ζ_h be the outer normal vector of $\mathcal{R}^S_{h\Delta}(t_i)$ at $y_{(i-1)N}$. Define the nontrivial discrete adjoint response (7.53), i.e.

$$\begin{cases}\eta_{k(j-1)} &= \eta_{kj}\Phi_h(t_{kj}, t_{k(j-1)})\\ \eta_{(i-1)N} &= \zeta_h,\end{cases}$$

Then $\eta_0 = \eta_{(i-1)N}\Phi_h(t_i, 0) = \zeta_h\,\Phi_h(t_i, 0)$. Noticing that $\Phi_h(t_{kj}, t_{k(j-1)})$ is a perturbation of the identity matrix I_n, there exists $\bar{h}$ such that $\Phi_h(t_{kj}, t_{k(j-1)})$ is invertible for $h \in [0, \bar{h}]$ and so is $\Phi_h(t_i, 0)$. Therefore, $\eta_{(i-1)N} = \eta_0\Phi_h^{-1}(t_i, 0)$. Now we compute the inner product of $\eta_{(i-1)N}$, $y_{(i-1)N}$:

$$\begin{aligned}\eta_{(i-1)N}\,y_{(i-1)N} &= \eta_0\Phi_h^{-1}(t_i,0)\Big(h\sum_{k=1}^{i}\sum_{j=0}^{N} c_{kj}\Phi_h(t_i, t_{(k-1)j})\bar{B}u_{(k-1)j}\Big)\\ &= h\sum_{k=1}^{i}\sum_{j=0}^{N} c_{kj}\eta_0\Phi_h^{-1}(t_i,0)\Phi_h(t_i, t_{(k-1)j})\bar{B}u_{(k-1)j}\\ &= h\sum_{k=1}^{i}\sum_{j=0}^{N} c_{kj}\eta_0\Phi_h^{-1}(t_{(k-1)j},0)\Phi_h^{-1}(t_i, t_{(k-1)j})\Phi_h(t_i, t_{(k-1)j})\bar{B}u_{(k-1)j}\\ &= h\sum_{k=1}^{i}\sum_{j=0}^{N} c_{kj}\eta_0\Phi_h^{-1}(t_{(k-1)j},0)\bar{B}u_{(k-1)j} = h\sum_{k=1}^{i}\sum_{j=0}^{N} c_{kj}\eta_{(k-1)j}\bar{B}u_{(k-1)j}\end{aligned}$$

Now assume that $\eta_{kj}\bar{B}u_{kj} < \max_{u\in U}\{\eta_{kj}\bar{B}u\}$ for some indices k, j. Then define another sequence of controls as follows

$$\tilde{u}_{kj} = \begin{cases}u_{kj} & \text{if } \eta_{kj}\bar{B}u_{kj} = \max_{u\in U}\{\eta_{kj}\bar{B}u\}\\ \max_{u\in U}\{\eta_{kj}\bar{B}u\} & \text{otherwise.}\end{cases}$$

Let $\tilde{y}_{(i-1)N}$ be the end point of the discrete trajectory following $\{\tilde{u}_{kj}\}$. We have

$$\eta_{(i-1)N}\,\tilde{y}_{(i-1)N} = h\sum_{k=1}^{i}\sum_{j=0}^{N} c_{kj}\eta_{(k-1)j}\bar{B}\tilde{u}_{(k-1)j}$$

which implies that $\eta_{(i-1)N}\,y_{(i-1)N} < \eta_{(i-1)N}\,\tilde{y}_{(i-1)N}$ or $\eta_{(i-1)N}(\tilde{y}_{(i-1)N} - y_{(i-1)N}) > 0$ which contradicts the construction of $\eta_{(i-1)N} = \zeta_h$, an outer normal vector of $\mathcal{R}^S_{h\Delta}(t_i)$ at $y_{(i-1)N}$. Therefore, $\eta_{kj}\bar{B}u_{kj} = \max_{u\in U}\{\eta_{kj}\bar{B}u\}$.
Conversely, assume that for some nontrivial discrete adjoint response $\eta_{(i-1)N} = \eta_0\Phi_h^{-1}(t_i, 0)$, the controls satisfies

$$\eta_{kj}\bar{B}u_{kj} = \max_{u\in U}\{\eta_{kj}\bar{B}u\} \tag{7.54}$$

for every indices $k = 0, ..., i-1$, $j = 0, ..., N$. We will show that the end point $y_{(i-1)N}$ of the corresponding trajectory $\{y_{kj}\}$ will lie at the boundary of $\mathcal{R}^S_{h\Delta}(t_i)$, not at any point belonging to its interior. Suppose, by contradiction, $y_{(i-1)N}$ lies in the interior of $\mathcal{R}^S_{h\Delta}(t_i)$. Let $\tilde{y}_{(i-1)N}$ be a point reached by a sequence of controls $\{\tilde{u}_{kj}\}$ in $\mathcal{R}^S_{h\Delta}(t_i)$ in such that

$$\eta_{(i-1)N}y_{(i-1)N} < \eta_{(i-1)N}\tilde{y}_{(i-1)N}. \tag{7.55}$$

Our assumption (7.54) implies that

$$\eta_{kj}\bar{B}\tilde{u}_{kj} \le \eta_{kj}\bar{B}u_{kj} \tag{7.56}$$

for all k, j. As above, due to (7.56), we show that $\eta_{(i-1)N}\tilde{y}_{(i-1)N} \le \eta_{(i-1)N}y_{(i-1)N}$ which is a contradiction to (7.55). Consequently, $y_{(i-1)N} \in \partial\mathcal{R}^S_{h\Delta}(t_i) = Y_{h\Delta}(t_i)$. $\square$

Motivated by the outer normality of the adjoints in continuous resp. discrete time and the maximum conditions, we define the optimal controls $\hat{u}(t)$, $\hat{u}_h(t)$ as follows

$$\begin{cases} \hat{u}(t) = \operatorname{sign}(\eta(t)\bar{B})^\top & \text{for } (t \in [0, t_i]), \\ \hat{u}_h(t) = \hat{u}_{kj} & \text{if } t \in [t_{kj}, t_{k(j+1)}),\ k = 0, ..., i-1,\ j = 0, ..., N-1, \\ \hat{u}_h(t) = \hat{u}_{(i-1)(N-1)} & \text{for } t = t_{(i-1)N}, \end{cases} \tag{7.57}$$

where $\hat{u}_{kj} = \operatorname{sign}(\eta_{kj}\bar{B})^\top$, $k = 0, ..., i-1$, $j = 0, ..., N$ and

$$w := \operatorname{sign}(v) \text{ with } w_\mu = \begin{cases} 1 & \text{if } v_\mu > 0, \\ 0 & \text{if } v_\mu = 0, \\ -1 & \text{if } v_\mu < 0 \end{cases}$$

is the *signum function* and $v, w \in \mathbb{R}^m$, $\mu = 1, \ldots, m$.

Owing to Theorem 7.3.1, we have that the set-valued maps $(N_{\mathcal{R}^S_{h\Delta}(t_i)}(\cdot))_h$ converge graphically to $N_{\mathcal{R}^S(t_i)}(\cdot)$ which implies that for every sequence $(y_{(i-1)N}, \eta_{(i-1)N})_N$ in the graphs there exists an element $(y(t_i), \eta(t_i))$ of the graph such that

$$(y_{(i-1)N}, \eta_{(i-1)N}) \to (y(t_i), \eta(t_i)) \text{ as } h \downarrow 0, \tag{7.58}$$

where $\eta_{(i-1)N} \in N_{\mathcal{R}^S_{h\Delta}(t_i)}(y_{(i-1)N})$, $\eta(t_i) \in N_{\mathcal{R}^S(t_i)}(y(t_i))$. Thus ζ, ζ_h are chosen such that (7.58) is realized. Then it is obvious that $\eta_{kj} \to \eta(t_{kj})$ as $h \downarrow 0$ with $k = 0, ..., i-1$ uniformly in $j = 0, ..., N$.

For a function $g\colon I \to \mathbb{R}^m$, we denote the total variation $V(g, I) := \sum_1^m V(g_i, I)$, where $V(g_i, I)$ is a usual total variation of the i-th components of g over a bounded interval $I \in \mathbb{R}$. Now if we assume furthermore that if the system (7.50) is normal, $\hat{u}_h(t)$ converges to $\hat{u}(t)$ in the L^1-norm.

Proposition 7.3.3. *Consider that the minimum time problem with the dynamics* (7.50) *in* $\mathbb{R}^n$. *Assume that the normality condition holds, i.e.*

$$\operatorname{rank}\{B\omega, AB\omega, \dots, A^{n-1}B\omega\} = n \tag{7.59}$$

for each (nonzero) vector ω *along an edge of* $U = [-1, 1]^m$ *or along the two end points of the interval* $U = [-1, 1]$ *if* $m = 1$. *Then, under Assumptions 6.0.1,* $\int_0^{t_i} \|\hat{u}(t) - \hat{u}_h(t)\|_1 dt \to 0$ *as* $h \to 0$ *for any* $i \in \{1, \dots, K\}$.

Proof. Due to (7.59) $\hat{u}(t)$ defined as in (7.57) on $t_0 \le t \le t_i$ is the optimal control to reach the state $\hat{y}(t_i)$ of the corresponding optimal solution from the origin. Moreover, it has a finite number of switchings see [59, Sec. 2.5, Corollary 2]. Therefore, the total variation, $V(\hat{u}(t), [t_0, t_i])$, is bounded. Let $I_{kj} = [t_{kj}, t_{k(j+1)})$, for $k = 0, \dots, i-1$, $j = 0, \dots, N-1$, and except for $I_{(i-1)(N-1)} = [t_{(i-1)(N-1)}, t_{(i-1)N}]$. Then

$$\begin{aligned} &\int_{I_{kj}} \|\hat{u}(t) - \hat{u}_h(t)\|_1 dt \le \int_{I_{kj}} (\|\hat{u}(t) - \hat{u}(t_{kj})\|_1 + \|\hat{u}(t_{kj}) - \hat{u}_h(t_{kj})\|_1) dt \\ &\le hV(\hat{u}(t), I_{kj}) + h\| \operatorname{sign}(\eta(t_{kj})\bar{B})^\top - \operatorname{sign}(\eta_{kj}\bar{B}))^\top \|_1 \end{aligned} \tag{7.60}$$

Taking a sum over $k = 0, \dots, i-1$, $j = 0, \dots, N-1$ we obtain

$$\int_{t_0}^{t_i} \|\hat{u}(t) - \hat{u}_h(t)\|_1 dt \le hV(\hat{u}(t), [t_0, t_i]) + h \sum_{k=0}^{i-1} \sum_{j=0}^{N-1} \| \operatorname{sign}(\eta(t_{kj})\bar{B}))^\top - \operatorname{sign}(\eta_{kj}\bar{B}))^\top \|_1.$$

Since $\hat{u}(t)$ has a finite number of switchings and $\eta_{kj}, \eta(t_{kj})$ are non-trivial with the convergence $\eta_{kj} \to \eta(t_{kj})$ as $h \to 0$ for $k = 0, \dots, i$, $j = 0, \dots, N$, the variation $V(\hat{u}(t), [t_0, t_i])$ and $\sum_{k=0}^{i} \sum_{j=0}^{N-1} \| \operatorname{sign}(\eta(t_{kj})\bar{B}))^\top - \operatorname{sign}(\eta_{kj}\bar{B}))^\top \|_1$ are bounded. Therefore,

$$\int_{t_0}^{t_i} \|\hat{u}(t) - \hat{u}_h(t)\|_1 dt \to 0 \text{ as } h \to 0.$$

The proof is completed. □

7.4 Numerical tests

This section is devoted to illustrating the performance of the error behavior of our proposed approach. We consider several linear examples with various target and control sets and study different levels of regularity of the corresponding minimum time function. The control sets are either one- or two-dimensional polytopes (a segment or a square) or balls and are varied to study different regularity allowing high or low order of convergence for the underlying set-valued quadrature method. In all linear examples, we apply a set-valued combination method of order 1 and 2 (the set-valued Riemann sum combined with Euler's method resp. the set-valued trapezoidal rule with Heun's method). For the last nonlinear example, we would like to approximate the reversed dynamics of (7.66) directly by Euler's and Heun's method. This example demonstrates that this approach is not restricted to the class of linear control systems. The space discretization follows the presented approach in Subsection 7.2.2 and uses supporting points in directions

$$
\begin{aligned}
l^k &:= \left(\cos\left(2\pi \frac{k-1}{N_{\mathcal{R}}-1}\right),\ \sin\left(2\pi \frac{k-1}{N_{\mathcal{R}}-1}\right)\right)^\top,\ k = 1, \ldots, N_{\mathcal{R}} \\
\eta^r &:= \begin{cases} -1 + 2(r-1) & \text{if } U = [-1,1],\ r = 1, \ldots, N_U, \\ l^r & \text{if } U \subset \mathbb{R}^2,\ r = 1, \ldots, N_U \end{cases}
\end{aligned}
$$

and normally choose either $N_U = 2$ for one-dimensional control sets or $N_U = N_{\mathcal{R}}$ for $U \subset \mathbb{R}^2$ in the discretizations of the unit sphere (7.21).

The comparison of the two applied methods is done by computing the error with respect to the L^∞-norm of the difference between the approximate and the true minimum time function evaluated at test points. The true minimum time function is delivered analytically by tools from control theory. The test grid points are distributed uniformly over the domain $\mathcal{G} = [-1,1]^2$ with step size $\Delta x = 0.02$.

7.4.1 Linear examples

In the linear, two-dimensional, time-invariant Examples 7.4.1–7.4.4 we can check Assumption 7.0.1(iv)

> $\mathcal{R}^S(t)$ is *strictly expanding* on the compact interval $[t_0, t_f]$, i.e. $\mathcal{R}^S(t_1) \subset \operatorname{int} \mathcal{R}^S(t_2)$ for all $t_0 \le t_1 < t_2 \le t_f$

in several ways. From the numerical calculations we can observe this property in the shown figures for the fully discrete reachable sets. Secondly, we can use the available analytical formula for the minimum time function resp. the reachable sets or check the Kalman rank condition

$$\operatorname{rank} \Big[B, AB \Big] = 2$$

for time-invariant systems if the target is the origin (see [49, Theorems 17.2 and 17.3]).

We start with an example having a Lipschitz continuous minimum time function and verify the error estimate in Theorem 7.2.7. Observe that the numerical error here is only contributed by the spatial discretization of the target set or control set.

Example 7.4.1. Consider the control dynamics , see [19, 47],

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ (u_1, u_2)^\top \in U \text{ with } U := B_1(0) \text{ or } U := [-1, 1]^2\ . \tag{7.61}$$

We consider either the small ball $B_{0.25}(0)$ or the origin as target set S. This is a simple time-invariant example with $\bar{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Its fundamental solution matrix is the identity matrix, therefore

$$\mathcal{R}^S(t) = \Phi(t, t_0)S + \int_{t_0}^{t} \Phi(t, s)\bar{B}(s)U = S + (t_0 - t)U,$$

and any method from (I)–(III) gives the exact solution, i.e.

$$\mathcal{R}_h^S(t) = \mathcal{R}^S(t) = S + (t - t_0)U$$

due to the symmetry of U. For instance, the set-valued Euler scheme with $h = \frac{t_{j+1}-t_j}{N}$ yields

$$\begin{cases} \mathcal{R}_h^S(t_{j+1}) = \mathcal{R}_h^S(t_j) + h(\bar{A}\mathcal{R}_h^S(t_j) + \bar{B}U) = \mathcal{R}_h^S(t_j) - hU, \\ \mathcal{R}_h^S(t_0) = S, \end{cases}$$

therefore, $\mathcal{R}_h^S(t_N) = S - NhU = S + (t_N - t_0)U$ and the error is only due to the space discretizations $\mathcal{S}_\Delta \approx S$, $U_\Delta \approx U$ and does not depend on h (see Table 7.1). The error would be the same for finer step size h and Δt in time or if a higher-order method is applied. Note that the error for the origin as target set (no space discretization error) is in the magnitude of the rounding errors of floating point numbers.

We choose $t_f = 1$, $K = 10$ and $N = 2$ for the computations. The set-valued Riemann sum combined with Euler's method is used.

It is easy to check that the minimum time function is Lipschitz continuous, since one of the equivalent Petrov conditions in [66], [16, Chap. IV, Theorem 1.12] with $U = B_1(0)$ or $[-1, 1]^2$ hold:

$$\begin{aligned} &0 > \min_{(u_1,u_2)^\top \in U} \langle \nabla d(x, S), (u_1, u_2)^\top \rangle, \\ &0 \in \operatorname{int}\left(\bigcup_{u \in U} f(0, u) \right) \quad \text{with } f(x, u) = Ax + Bu. \end{aligned}$$

Moreover, the support function with respect to the time-reversed dynamics (7.61)

$$\delta^*(l, \Phi(t, \tau)\bar{B}(\tau)U) = \begin{cases} \|l\| & \text{if } U = B_1(0), \\ |l_1| + |l_2| & \text{if } U = [-1, 1]^2 \end{cases}$$

is constant with respect to the time t, so it is trivially arbitrarily continuously differentiable with respect to t with bounded derivatives uniformly for all $l \in S_{n-1}$. In Figure

$N_{\mathcal{R}} = N_U$	$U = B_1(0)$, $S = B_{0.25}(0)$	$U = [-1,1]^2$, $S = B_{0.25}(0)$	$U = [-1,1]^2$, $S = \{0\}$
100	6.14×10^{-4}	4.9×10^{-4}	8.9×10^{-16}
50	24×10^{-4}	19×10^{-4}	8.9×10^{-16}
25	0.0258	0.0073	8.9×10^{-16}

Table 7.1: error estimates for Example 7.4.1 with different control and target sets

7.3 the minimum time functions are plotted for Example 7.4.1 for two different control sets $U = B_1(0)$ (left) and $U = [-1,1]^2$ (right) with the same two-dimensional target set $S = B_{0.25}(0)$. The minimum time function is in general not differentiable everywhere. Since it is zero in the interior of the target, one has at most Lipschitz continuity at the boundary of S. In Figure 7.4 the minimum time function is plotted for the same control set as in Figure 7.3 (right), but this time the target set is the origin and not a small ball.

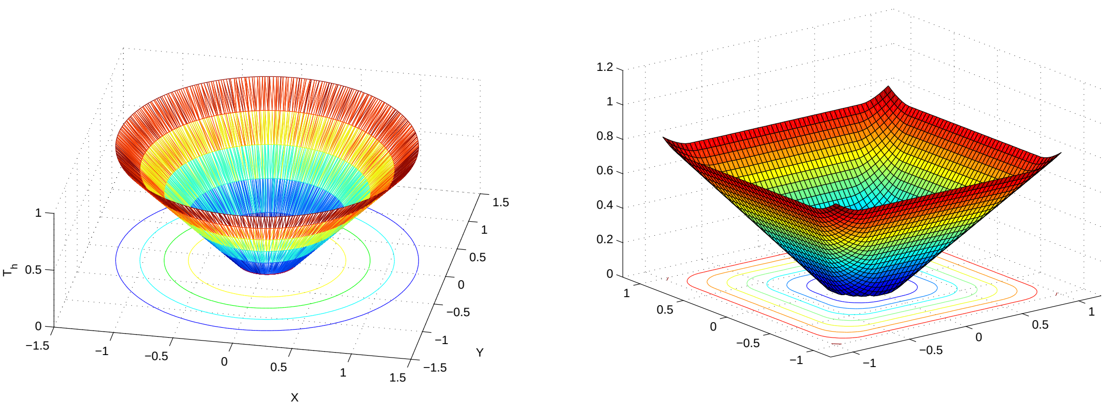

Figure 7.3: minimum time functions for Example 7.4.1 with different control sets

We now study well-known dynamics as the double integrator and the harmonic oscillator in which the control set is one-dimensional. The classical rocket car example with Hölder-continuous minimum time function was already computed by the Hamilton-Jacobi-Bellman approach in [34, Test 1] and [26, 47], where numerical calculations are carried out by enlarging the target (the origin) by a small ball.

Example 7.4.2. a) The following dynamics is the *double integrator*, see e.g. [26].

$$\dot{x}_1 = x_2,\ \dot{x}_2 = u,\ u \in U := [-1,1]. \tag{7.62}$$

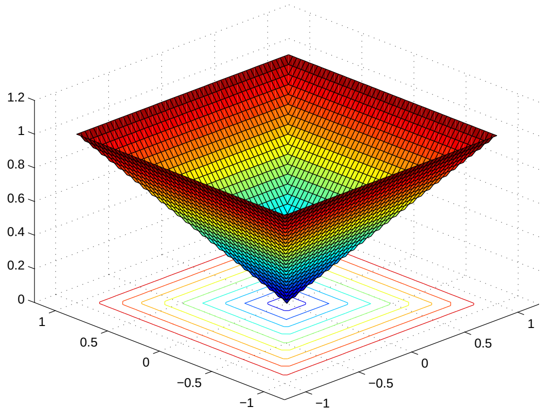

Figure 7.4: minimum time function for Example 7.4.1 with $U = [-1, 1]^2$, $S = \{0\}$

We consider either the small ball $B_{0.05}(0)$ or the origin as target set S. Then the minimum time function is $\frac{1}{2}$–Hölder continuous for the first choice of S see [26, 61] and the support function for the time-reversed dynamics (7.62)

$$\delta^*(l, \Phi(t,\tau)\bar{B}(\tau)[-1,1]) = \delta\left(l, \begin{bmatrix} 1 & -(t-\tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} [-1,1]\right) = \left|(t-\tau, -1) \cdot l\right|$$

is only absolutely continuous with respect to τ for some directions $l \in S_1$ with $l \neq 0$. Hence, we can expect that the convergence order for the set-valued quadrature method is at most 2. We fix $t_f = 1$ as maximal computed value for the minimum time function and $N = 5$.

In Table 7.2 the error estimates for two set-valued combination methods are compared (order 1 versus order 2). Since the minimum time function is only $\frac{1}{2}$–Hölder continuous we expect as overall convergence order $\frac{1}{2}$ resp. 1. A least squares approximation of the function Ch^p for the error term reveals $C = 1.37606$, $p = 0.4940$ for Euler scheme combined with set-valued Riemann sum resp. $C = 22.18877$, $p = 1.4633$ (if $p = 1$ is fixed, then $C = 2.62796$) for Heun's method combined with set-valued trapezoidal rule. Hence, the approximated error term is close to the expected one by Theorem 7.2.7 resp. Remark 7.2.8. Very similar results are obtained with the Runge-Kutta methods of order 1 and 2 in Table 7.3 in which the set-valued Euler method is slightly better than the combination method of order 1 in Table 7.2, and the set-valued Heun's method coincides with the combination method of order 2, since both methods use the same approximations of the given dynamics.

Here we have chosen to double the number of directions $N_{\mathcal{R}}$ each time the step size is halfened which is suitable for a first order method. For a second order method we should have multiplied $N_{\mathcal{R}}$ by 4 instead. From this point it is not surprising that there is no improvement of the error in the fifth row for step size $h = 0.0025$. As in

h	$N_{\mathcal{R}}$	**Euler scheme & Riemann sum**	**Heun's scheme & trapezoid rule**
0.04	50	0.2951	0.2265
0.02	100	0.1862	0.1180
0.01	200	0.1332	0.0122
0.005	400	0.1132	0.0062
0.0025	800	0.0683	0.0062

Table 7.2: error estimates for Ex. 7.4.2 a) for combination methods of order 1 and 2

h	$N_{\mathcal{R}}$	**set-valued Euler method**	**set-valued Heun method**
0.04	50	0.2330	0.2265
0.02	100	0.1681	0.1180
0.01	200	0.1149	0.0122
0.005	400	0.0753	0.0062
0.0025	800	0.0318	0.0062

Table 7.3: error estimates for Ex. 7.4.2 a) for Runge-Kutta meth. of order 1 and 2

Example 7.4.1 we can consider the dynamics (7.62) with the origin as a target (see the minimum time function in Figure 7.6 (left). In this case, the numerical computation by PDE approaches, i.e. the solution of the associated Hamilton-Jacobi-Bellman equation (see e.g. [34]) requires the replacement of the target point 0 by a small ball $B_\varepsilon(0)$ for suitable $\varepsilon > 0$. This replacement surely increases the error of the calculation (compare the minimum time function in Figure 7.5 for $\varepsilon = 0.05$). However, the proposed approach works perfectly regardless of the fact whether S is a two-dimensional set or a singleton.

b) harmonic oscillator dynamics (see [59, Chap. 1, Section 1.1, Example 3])

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ \ u \in U := [-1, 1]. \tag{7.63}$$

Since the Kalman rank condition

$$\operatorname{rank}\Big[B, AB\Big] = 2,$$

the minimum time function $T_S(\cdot)$ is also continuous. The plot for $T_S(x)$ for the harmonic oscillator with the origin as target, $t_f = 6$, $N_{\mathcal{R}} = 100$, $N = 5$ and $K = 40$ is shown in Figure 7.6 (right).

According to Section 7.3 we construct open-loop time-optimal controls for the discrete problem with target set $S = \{0\}$ by Euler's method. In Fig. 7.7 the corresponding discrete open-loop time-optimal trajectories for Examples 7.4.2a) (left) and b) (right) are depicted.

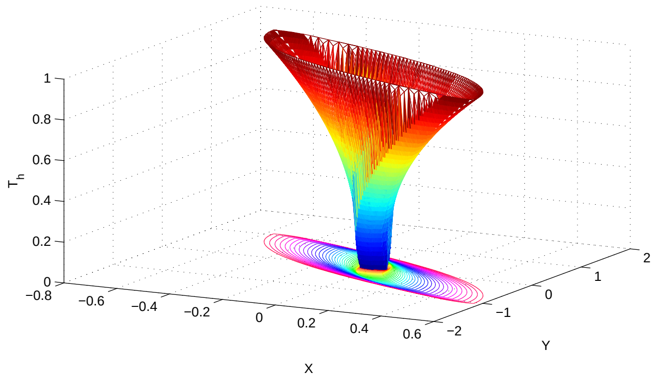

Figure 7.5: minimum time function for Example 7.4.2a) with target set $B_{0.05}(0)$

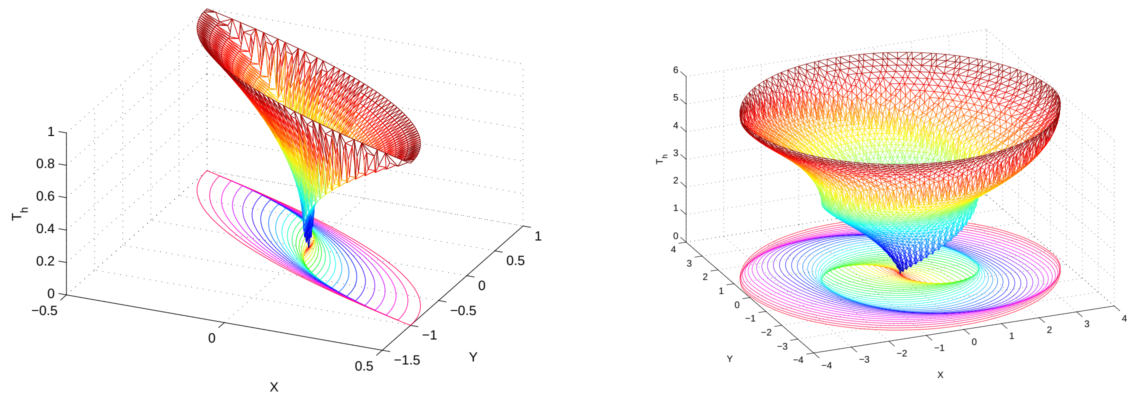

Figure 7.6: minimum time functions for Example 7.4.2a) resp. b)

The following two examples exhibit smoothness of the support functions and would even allow for methods with order higher than two with respect to time discretization. The first example has a special linear dynamics and is smooth, although the control set is a unit square.

Example 7.4.3. In the third linear two-dimensional example the reachable set for various end times t is always a polytope with four vertices and coinciding outer normals at its faces. Therefore, it is a smooth example which would even justify the use of methods with higher order than 2 to compute the reachable sets (see [13, 14]). It is a variant of [13, Example 2].

Again, we fix $t_f = 1$ as maximal time value and compute the result with $N = 2$. We choose $N_{\mathcal{R}} = 50$ normed directions, since the reachable set has only four different

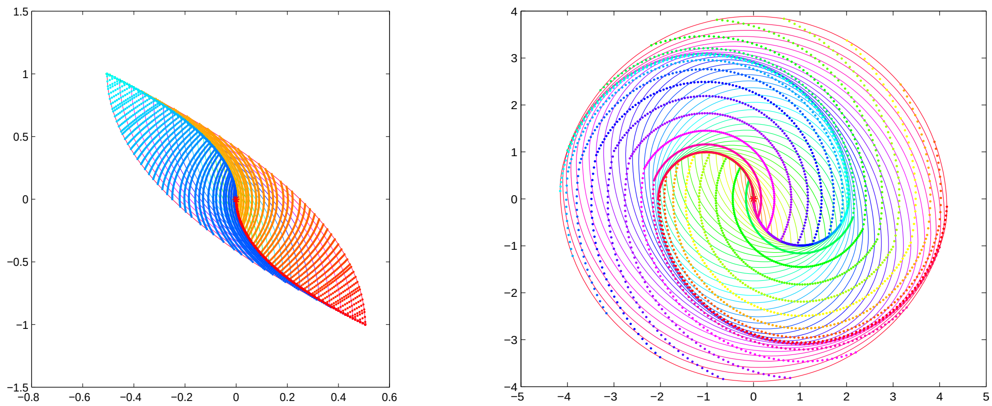

Figure 7.7: approximate optimal trajectories for Example 7.4.2a) resp. b)

vertices.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{7.64}$$

where $(u_1,\, u_2)^\top \in [-1,1]^2$. Let the origin be the target set S. The fundamental solution matrix of the time-reversed dynamics of (7.64) is given by

$$\Phi(t,\tau) = \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix}.$$

This is a smooth example in the sense that the support function for the time-reversed set-valued dynamics of (7.64),

$$\delta^*(l, \Phi(t,\tau)\bar{B}(\tau)[-1,1]^2) = e^{-(t-\tau)}|l_1 - l_2| + e^{-2(t-\tau)}|l_1 - 2l_2|,$$

is smooth with respect to τ uniformly in $l \in S_1$.

The analytical formula for the (time-continuous) minimum time function is as follows:

$$T_S((x_1,x_2)^\top) = \max\{t\colon\ t \geq 0 \text{ is the solution of one of the equations}$$
$$x_2 = -2x_1 \pm (e^{-t} - 1),\, x_2 = -x_1 \pm 1/2(1 - e^{-2t})\}.$$

A least squares approximation of the function Ch^p for the error term reveals $C = 2.14475$, $p = 0.8395$ for the set-valued combination method of order 1 and $C = 23.9210$, $p = 1.7335$ (if $p = 2$ is fixed, then $C = 70.1265$) for the one of order 2. The values are similar to the expected ones from by Remark 7.2.8, since the minimum time function (see Figure 7.8 (left)) is Lipschitz (see [16, Sec. IV.1, Theorem 1.9]).

Similarly, another variant of this example with a one-dimensional control can be constructed by deleting the second column in matrix B. The resulting (discrete and continuous-time) reachable sets would be line segments. Thus, the algorithm would compute the fully discrete minimum time function on this one-dimensional subspace. The absence of interior points in the reachable sets is not problematic for this approach in contrary to common approaches based on the Hamilton-Jacobi-Bellman equation, see [12].

h	**Euler scheme & Riemann sum**	**Heun's scheme & trapezoid rule**
0.05	0.170	0.1153
0.025	0.095	0.0470
0.0125	0.0599	0.0133
0.00625	0.0285	0.0032

Table 7.4: error estimates for Example 7.4.3 for methods of order 1 and 2

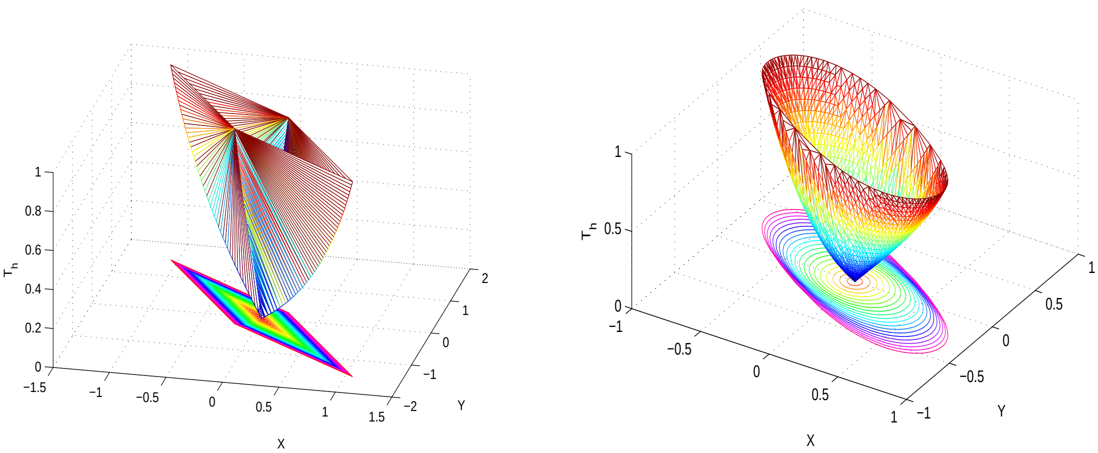

Figure 7.8: minimum time functions for Examples 7.4.3 and 7.4.4

The next example involves a ball as control set and leads naturally to a smooth support function.

Example 7.4.4. *The following smooth example is very similar to the previous example. It is given in [9, Example 4.2], [14, Example 4.4]*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B_1(0) \tag{7.65}$$

and uses a ball as control set. This is a less academic example than Example 7.4.3 (in which the matrix $B(t)$ was carefully chosen), since a ball as control set often allows the use of higher order methods for the computation of reachable sets (see [8, 14]). Here, no analytic formula for the minimum time function is available so that we can study only numerically the minimum time function (see Figure 7.8 (right)). Obviously, the support function is again smooth with respect to τ uniformly in all normed directions l, since

$$\delta^*(l, \Phi(t,\tau)B_1(0)) = \|\Phi(t,\tau)^\top l\|.$$

7.4.2 A nonlinear example

The following special bilinear example with convex reachable sets may provide the hope to extend our approach to some class of nonlinear dynamics.

Example 7.4.5. The nonlinear dynamics is one of the examples in [47].

$$\dot{x}_1 = -x_2 + x_1 u,\ \dot{x}_2 = x_1 + x_2 u,\ u \in [-1, 1]. \tag{7.66}$$

With this dynamics, after computing the true minimum time function we observe that $T_S(\cdot)$ is Lipschitz continuous and its sublevel set, which is exactly the reachable set at the corresponding time, satisfies the required properties. The target set S is $B_{0.25}(0)$.

We fix $t_f = 1$ as maximal computed value for the minimum time function and $N = 2$. Estimating the error term Ch^p in Table 7.5 by least squares approximation yields the values $C = 0.3293133$, $p = 1.8091$ for the set-valued Euler method and $C = 0.5815318$, $p = 1.9117$ for the Heun method.

The unexpected good behavior of Euler's method stems from the specific behavior of trajectories. Although the distance of the end point of the Euler iterates for halfened step size to the true end point is halfened, but the distance of the Euler iterates to the boundary of the true reachable set is almost shrinking by the factor 4 due to the specific tangential approximation. In Figure 7.9 the Euler iterates are marked with $*$ in red color, while Heun's iterates are shown with $\circ$ marks in blue color. The symbol $\bullet$ marks the end point of the corresponding true solution.

Observe that the dynamics originates from the following system in polar coordinates

$$\dot{r} = ru,\ \dot{\varphi} = 1,\ u \in [-1, 1].$$

Hence, the reachable set will grow exponentially with increasing time. The minimum time function for this example is shown in Figure 7.10.

7.5 Outline

Although the underlying set-valued method approximating reachable sets in linear control problems is very efficient, the numerical implementation is a first realization only

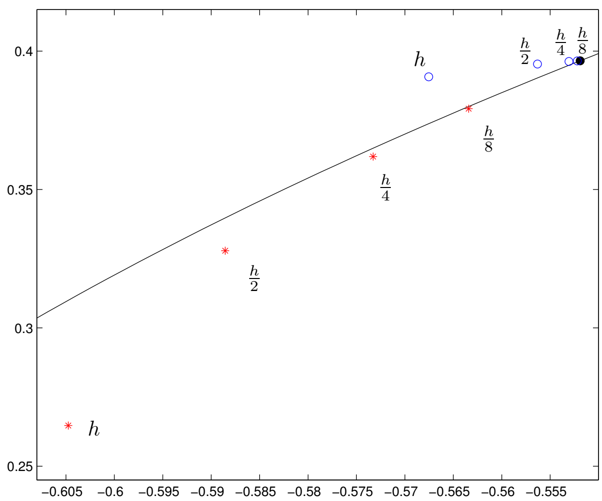

Figure 7.9: Euler and Heun's iterates for Example 7.4.5

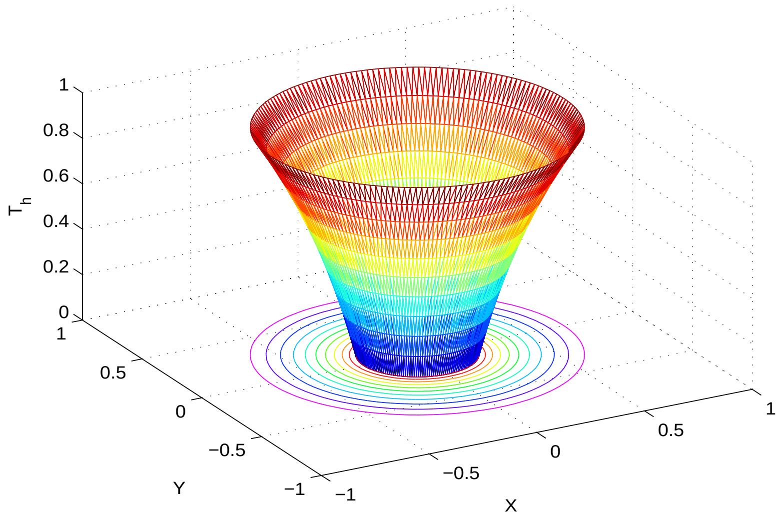

Figure 7.10: minimum time function for Example 7.4.5

h	$N_{\mathcal{R}}$	**set-valued Euler scheme**	**set-valued Heun's scheme**
0.5	50	0.0848	0.1461
0.1	100	0.0060	0.0076
0.05	200	0.0015	0.0020
0.025	400	0.00042	0.000502
0.0125	800	0.000108	0.000126

Table 7.5: error estimates for Example 7.4.5 with set-valued methods of order 1 and 2

and can still be considerably improved. Especially, step 3 in Algorithm 7.2.4 can be computed more efficiently as in our test implementation. Furthermore, higher order methods like the set-valued Simpson's rule combined with the Runge-Kutta(4) method are an interesting option in examples where the underlying reachable sets can be computed with higher order of convergence than 2, especially if the minimum time function is Lipschitz. But even if it is merely Hölder-continuous with $\frac{1}{2}$, the higher order in the set-valued quadrature method can balance the missing regularity of the minimum time function and improves the error estimate.

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