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CORSO DI DOTTORATO DI RICERCA IN FISICA XXXII CICLO

Investigating new physics through cosmology

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There isn't time, so brief is life, for bickerings, apologies, heartburnings,
callings to account. There is only time for loving,
and but an instant, so to speak, for that.

(Mark Twain)

Abstract

Throughout the years, the cosmic microwave background (CMB) has given a crucial contribution to build the standard cosmological model as we know today, becoming a privileged observational tool of modern cosmology. In particular, in the so-called era of high-precision cosmology, CMB data can put stringent constraints about the physics of the primordial Universe, when, according to the standard paradigm, the first cosmological perturbations formed.

The measurements of anisotropies in the temperature of CMB and of CMB polarization are perfectly compatible with a Universe that in the past has experienced an inflationary phase of accelerated expansion (simply known as *inflation*), which is assumed to be driven by a scalar field, the so-called *inflaton* field. During inflation quantum perturbations of the inflaton field were stretched on very large (super-horizon) cosmological scales, where they got frozen. These are thought to be the seeds which later on evolved into the large scale structures that we observe today. In particular, from the precise analysis of the CMB maps, we can constrain a large variety of theoretical models aiming to describe the inflationary epoch. In fact, the CMB is for cosmologists what the colliders are for particle-physicists, allowing to test modified gravity theories, symmetries breaking, and the particle content during inflation. The CMB observations currently seem to favor the simplest inflationary models, the so-called slow-roll models, where a single slowly-rolling scalar field drives the expansion of the Universe under the effects of Einstein gravity, sourcing both scalar and tensor perturbations, which are the so-called primordial curvature perturbation and primordial gravitational waves.

However, our understanding of inflation is far away to be complete both for theoretical and observational aspects. For instance, we still do not know the precise mechanism realizing inflation. In fact, primordial gravitational waves as predicted by slow-roll models have not been detected yet. Therefore, we do not have detailed information on their (power spectrum) statistics, which would be crucial to determine the exact inflationary model. Another example is the recent confirmation from the *Planck* satellite of the presence of some anomalies in the CMB map, suggesting a possible violation of some symmetries (e.g. the parity symmetry) at some point in the evolution of the Universe. Moreover, we are still facing the so-called trans-Planckian problem, i.e. the fact that the CMB physical observational scales could have been inside the Planck scale at the beginning of inflationary epoch, thus requiring an ultra-violet completion of the theory. All these aspects force us to push our research efforts further on about the physics of

inflation.

The physics of the CMB can be used to provide an insight into all these different issues related to the inflationary epoch. On the other hand, it also provides an observational tool to test other aspects of fundamental physics. For example, in this Thesis, we will show how modifications of the photon-fermion interactions as we know from the Standard Model of particle physics may lead to alternative theoretical predictions about the expected level of CMB polarization anisotropies generated during and after the recombination epoch. For example, they can generate non-zero circular polarization which is not allowed in the standard lore, but not completely excluded by the CMB experiments.

Motivated by all these considerations, in this Thesis we will review some fundamental aspects of the standard cosmological model (regarding in particular the inflationary epoch) and investigate new scenarios that go beyond the standard paradigm, leading to predictions that could be tested with current and future CMB experiments which will focus on CMB polarization. The Thesis is organized as follows:

- In part I, we will review different aspects of the standard cosmological model, focusing on the inflationary epoch and the connection between primordial perturbations and CMB anisotropies.

- In part II, we will study parity breaking signatures in the gravity sector during the primordial Universe. In particular, we will consider the modifications to the statistics of primordial perturbations in presence of Chern-Simons gravity, which is the first order parity breaking modified gravity theory arising from an effective field theory modification of Einstein gravity. We will show that the expected amount of chirality in the power spectrum statistics of gravitational waves is expected to be very small, in a way that is difficult to have a detection with current and futuristic CMB experiments.

  Thus, we will make an analysis of the parity breaking signatures induced to the higher order (bispectra) statistics of primordial perturbations, showing interesting prospects for the 2 gravitons-1 scalar bispectrum. We will perform a Fisher-matrix forecast on the possibility to detect primordial signatures of Chern-Simons gravity from this bispectrum, with next decade CMB experiments focusing on CMB B modes.

  In particular, we will show that, in general, an improvement in the angular resolution of CMB experiments, together with lensing subtraction, can significantly enhance the sensitivity of CMB bispectra to parity breaking signatures in the primordial Universe. On the contrary, we will show that no significant improvements in constraining primordial parity breaking signatures are expected from the power spectra statistics of futuristic CMB experiments, even with very high angular resolution and perfect lensing subtraction. This result suggests that CMB bispectra statistics could be, in the future, a crucial observable to constrain parity breaking signatures from inflation.

- In part III, we will consider how CMB can be possibly exploited to probe new physics beyond Standard Model of particle physics. We will study the generation of CMB circular polarization from the forward scattering between CMB photons and other particles. This is the same physical mechanism that also generates the neutrino flavor mixings in the Standard Model of fundamental interactions, causing the oscillation of neutrino flavors.

In particular, firstly we will study the forward scattering between CMB photons and gravitons, showing a non-zero generation of V modes in case of anisotropies in the statistics of primordial gravitons. However, we will show that the final amount of V modes expected today is too low to be detected by both current and futuristic CMB experiments. Nevertheless, we will derive general equations that can be applied to a general photon-graviton interaction in contexts different from the CMB, e.g. for searching of gravitational wave events of astrophysical origin.

We will then study the polarization mixing due to the forward scattering of CMB photons and generic fermions. In particular, we will provide a general parametrization of the photon-fermion forward-scattering amplitude (assuming only gauge invariance and CPT symmetry) and compute the corresponding mixing terms between the different CMB polarization states. We will consider different general extensions of Standard Model interactions which violate discrete symmetries, while preserving the combination of charge conjugation, parity and time reversal. We will show that it is possible to source CMB circular polarization by violating parity and charge conjugation symmetries. Instead, a B-mode generation (in absence of primordial gravitational waves) is associated to the violation of symmetry for time-reversal. Our final results will be expressed in terms of some free parameters characterizing different kinds of forward scattering interactions, thus offering in the future a viable and general tool to put constraints on fundamental physics properties beyond the standard paradigms using CMB data.

- In part IV, we will provide our conclusions and final considerations about the research developed in this Thesis.
- Finally, in part V, we will provide an Appendix, where some formulas and formalisms employed in this work are reviewed.

List of papers

- Bartolo N., Hoseinpur A., Matarrese S., Orlando G., Zarei M. (2019), *CMB circular and B-mode polarization from new interactions*, Phys. Rev. D, 100, 043516, (arXiv), (PRD).
- Bartolo N., Orlando G., Shiraishi M. (2018), *Measuring chiral gravitational waves in Chern-Simons gravity with CMB bispectra*, Journ. of Cosmol. and Astrop. Phys., 01, 050 (arXiv), (JCAP).
- Bartolo, N., Hoseinpur, A., Orlando G., Matarrese S., Zarei M. (2018), *Photon-graviton scattering: A new way to detect anisotropic gravitational waves?*, Phys. Rev. D, 98, 023518, (arXiv), (PRD).
- Bartolo, N., Orlando G. (2017), *Parity breaking signatures from a Chern-Simons coupling during inflation: the case of non-Gaussian gravitational waves*, Journ. of Cosmol. and Astrop. Phys., 7, 034, (arXiv), (JCAP).

Contents

Part I
Introduction

In this first part, we will review the basic elements of the standard cosmological model, introducing the inflationary epoch (Chap. 1), and showing the slow-roll models predictions both about the power spectrum and bispectrum statistics of primordial (scalar and tensor) perturbations (Chaps. 2 and 3). Moreover, we will qualitatively review how we can link the early-time statistics of these perturbations to the CMB anisotropies we observe now, showing the experimental constraints on inflationary models from the last *Planck* data (Chap. 4). Finally, we will review the effective field theory formulation of inflation, allowing to study in a model independent way all the possible deviations from slow-roll models of inflation (Chap. 5).

The main aim of this part is to clarify the formalism and conventions adopted in this work, in order to help the reader in understanding the original scientific results presented in parts II and III. Throughout this Thesis, if not alternatively specified, we will use natural units, thus $c = \hbar = 1$, together with the reduced mass Planck $M_{\rm Pl} = (8\pi G)^{-1/2}$, and the metric signature $(-,+,+,+)$.

Chapter 1

Standard cosmological model

1.1 General description

The current cosmological model relies on a fundamental feature of the Universe, i.e. its large scale homogeneity and isotropy. To date, there have been surprisingly only few tests of these two symmetries. Most of these have looked for a redshift dependence of cosmological parameters or have used galaxy density projections on the sky (see e.g. [1–4]). The most reliable test is the isotropy of the cosmic microwave background (CMB) made by the *Planck* mission (see the *Planck* paper [5]). All these observations naturally support the hypothesis that we do not occupy a peculiar space in the Universe, and this is the basics of the well known *cosmological principle*, defining the role of the so-called *comoving observer*. In just one sentence, we can resume this principle as follows:

"Each comoving observer sees the Universe around him, at a fixed time, as homogeneous and isotropic, on sufficiently large scales". When we say "large scales", we refer to distances bigger than about 100 Mpc (Megaparsec) or, better to say, distances that are much bigger than the characteristic dimension of a galaxy.

With the mathematical instruments of general relativity, we can reformulate this fundamental principle in the following way: the Universe is a four dimensional manifold with three space-like coordinates ($x_i, i = 1, 2, 3$) and one time-like coordinate (t) where the three dimensional submanifolds at fixed time Σ_t are maximally symmetric manifolds. This automatically implies that the metric of the Universe, apart for diffeomorphisms, has to take the following form[1]

$$ds^2 = -dt^2 + a(t)^2 \left[d\mathbf{x}^2 + k \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{1 - k\mathbf{x}^2} \right] , \tag{1.2}$$

where (t, $\mathbf{x}$) are a set of coordinates of the comoving observer, in particular t being the

[1]Or an equivalent form

$$ds^2 = a(t)^2 \left[-d\eta^2 + d\mathbf{x}^2 + k \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{1 - k\mathbf{x}^2} \right] \tag{1.1}$$

in terms of the *conformal time* η defined as $dt = ad\eta$. In this work, we will use both η or τ to denote the conformal time depending by the context in order to avoid conflicts of notation with other parameters.

so-called *cosmological time*. In polar coordinates the metric (1.2) reads

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] . \tag{1.3}$$

The quantity $a(t)$ is the so-called *scale factor* and is a positive parameter which characterizes the time evolution of the physical dimensions of space-like hypersurfaces. Usually, it is normalized to be equal to 1 today, thus $a_o = 1$.[2] The parameter k, called *spatial curvature*, represents the curvature of these space-like hypersurfaces. We have three different spatial geometries of the Universe depending on the value of k:

- $k < 0$, the hyper-surfaces Σ_t have negative curvature, it follows an open Universe.
- $k = 0$, the hyper-surfaces Σ_t are euclidean, it follows a flat Universe.
- $k > 0$, the hyper-surfaces Σ_t have positive curvature, it follows a closed Universe.

What we have just described is the so-called Friedmann-Robertson-Walker (FRW) metric. Normally, one has to find the metric of a given space solving the so-called Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \, , \tag{1.4}$$

where $R_{\mu\nu}$ is the Ricci tensor built on the metric $g_{\mu\nu}$, R is the scalar curvature, G is the gravitational constant and $T_{\mu\nu}$ in the energy-momentum tensor of the matter field. However, this procedure would require a precise knowledge of the energy-momentum tensor of the Universe. Since it is not realistically possible to precisely map all the matter contained in any point of the Universe, then cosmologists can only assume the metric to be FRW (according to the symmetries imposed by the cosmological principle), and investigate about the time evolution of the energy-momentum tensor of the Universe.

In particular, it can be shown (see e.g. [6]) that the energy-momentum tensor of the Universe necessarily takes the same form as for a perfect fluid

$$T_{\mu\nu} = (\rho(t) + p(t)) u_\mu u_\nu + p(t) g_{\mu\nu} \, , \tag{1.5}$$

where u_μ is the 4-velocity of the cosmic fluid, ρ and p the energy density and the isotropic pressure respectively. Inserting expression (1.5) into Einstein's field equations we find the well-known Friedmann equations

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \, , \tag{1.6}$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3p) \, , \tag{1.7}$$

where $H = \dot{a}/a$ is the so-called Hubble parameter labeling the expansion rate of the Universe. The latest experimental value of H today is provided by the *Planck* mission as [7]

$$H_o = (67.4 \pm 0.5) \ \mathrm{km\,Mpc^{-1} s^{-1}}. \tag{1.8}$$

[2] In the following, we will use the suffix "o" to indicate quantities evaluated today.

However, this is not the only experimental value of H_o available in literature.[3]

Returning to our general description, after specifying an equation of state, i.e. $p = p(\rho)$, Eqs. (1.6) and (1.7) form a complete set which can be used to determine the two unknown functions $a(t)$ and $\rho(t)$. Typically, the Universe is well-approximated by a baryotropic fluid which follows an equation of state like [6]

$$p = \omega\rho\,, \tag{1.10}$$

where ω take different values depending on which kind of fluid well describes the Universe composition at any stage. In the following, we list all the different types of fluids that may compose the Universe.

- Matter fluid (M): it can consist of non relativistic matter and/or *dark matter* (DM). Non-relativistic matter is typically baryonic matter with an energy mass $E_m = m$ greater than the thermal energy $E_T \sim T$ of the Universe at a certain time (i.e. in natural units $m > T$). The DM component represents matter that interacts only gravitationally in the Universe and does not have any electromagnetic interaction and for this reason we cannot observe it directly through fotons emission. Moreover, we know from observations that in order to be a dominant component today in the matter fluid the DM component must be non-relativistic today (also known as *cold dark matter*). For such fluid, we can take $p \simeq 0$, $w \simeq 0$.

- Radiation fluid (R): it consists of relativistic matter and radiation. The relativistic matter is baryonic matter with an energy mass much smaller than the thermal energy of the Universe ($m \ll T$). Radiation is associated to cosmological photons (like the CMB). For such fluid, $w = 1/3$.

- Vacuum energy fluid (Λ): this kind of fluid is currently associated to regions of vacuum in the Universe. For this fluid, $w = -1$, i.e. it has negative pressure. Physically, this means that it makes an opposition towards the gravitational pressure, like the radiation pressure when we try to indefinitely compress a fluid. In this last case, the energy momentum tensor becomes of the kind $T_{\mu\nu} = \Lambda g_{\mu\nu}$, with Λ a constant. In literature, this fluid is also called *cosmological constant*, because it originally appeared in the left-hand side of the Einstein equations (1.4) as the cosmological constant firstly introduced by Einstein to find a solution for a static Universe [10]. It is also known as *dark energy* (DE), because it consists of an unknown form of energy that seems to contrast the attractive gravitational force.

[3] For instance, it is worth to mention a recent local measurement through infrared observations of 70 long-period Cepheids in the Large Magellanic Cloud, which gave the value [8]

$$H_o^{\text{local}} = (74.03 \pm 1.42)\ \text{km}\,\text{Mpc}^{-1}\text{s}^{-1}. \tag{1.9}$$

This measurement results in a 4.4σ tension with respect to the latest result from *Planck* satellite. This is the so-called *Hubble tension*. For more details on this aspect and how to relax the tension, see e.g. [9] and references therein. In the rest of this work we will not go trough this point since it is not related to the PhD research project.

Our Universe is composed by all of these three fluids at the same time. Therefore, the dynamics of the Universe is driven by the fluid which is dominant in term of the energy density at a given epoch. In particular, we can define the *density parameter* Ω_i of the i-th species as

$$\Omega_i = \frac{\rho_i}{\rho_c}\,, \tag{1.11}$$

where $\rho_c = 3H^2/8\pi G$ is the so-called critical energy density, being the density of a spatially flat Universe. The total density parameter is obtained by the sum over all the three species

$$\Omega = \Omega_M + \Omega_r + \Omega_\Lambda\,, \qquad \Omega_M = \Omega_b + \Omega_c\,, \tag{1.12}$$

where b and c stand for baryons and cold dark matter respectively. The last experimental values of the parameters Ω_i^o measured by the *Planck* space mission are tabulated in [7].[4] These confirm that today the predominant component of the Universe is an energy vacuum like fluid with a relative abundance $\Omega_\Lambda^o \simeq 0.69$. The cold dark matter component have a relative abundance of $\Omega_c^o \simeq 0.26$, while non relativistic baryons and radiation represent only a minor part of the cosmic fluid ($\Omega_b^o \simeq 0.05$ and $\Omega_r^o \simeq 10^{-5}$). The current observations indicate also that today the Universe is fully consistent with a spatially flat Universe ($k = 0$). If we assume $k \simeq 0$ during all the Universe history (this assumption works well because in the past the spatially flat solution is an attractive solution of Eq. (1.7)), then we can solve exactly Eqs. (1.6) and (1.7), finding

$$\rho(t) = \begin{cases} \rho(t_0)\, t^{-3(1+w)}, & \text{if } w \neq -1\,, \\ \rho(t_0), & \text{if } w = -1\,, \end{cases} \tag{1.13}$$

$$a(t) = \begin{cases} a(t_0)\, t^{\frac{2}{3(1+w)}}, & \text{if } w \neq -1\,, \\ a(t_0) e^{Ht}, & \text{if } w = -1\,, \end{cases} \tag{1.14}$$

where t_0 denotes some initial time. In Tab. (1.1) these results are summarized for each of the three fluids.

From our solutions it follows that, independently by which fluid composes the Universe, the curve $a(t)$ versus t must be concave downward and must have reached $a(t) = 0$ at some finite time in the past. This is the initial singularity universally known as *Big Bang.*

In particular, according to the current *Lambda-Cold Dark Matter* (ΛCDM) model,[5] the Universe was initially an high energy plasma composed merely of radiation fluid in thermodynamic equilibrium. This is the so-called *radiation dominated* epoch. Then,

[4]It is important to underline that the constraints on cosmological parameters derived by the experiments are typically sensitive to the details of the underlying cosmological model, i.e. a modification of the standard paradigm may give different experimental outcomes. In this work, we will not give all the details of the standard cosmological model, but we will introduce only those elements which may help the reader to understand the research goals fulfilled. For more details about the full ΛCDM cosmological model, we refer the reader to [7].

[5]It is called in this way because it predicts that the cosmological constant and cold dark matter dominate the composition of the Universe today.

Fluid	w	$\rho(a)$	$a(t)$
M	0	a^{-3}	$t^{2/3}$
R	1/3	a^{-4}	$t^{1/2}$
Λ	-1	a^{0}	e^{Ht}

Table 1.1. The time dependence of density and scale factor for a spatially flat Universe with a dominant component of matter (M), radiation (R) and cosmological constant (Λ) fluid.

the Universe expanded according to the FRW solution, becoming colder and colder, and favouring the formation of hydrogen atoms from free electrons and protons. When the fraction of hydrogen atoms became higher than the relative abundance of free electrons and protons, we had the so-called *recombination* epoch. The consequent reduction in the number of free electrons determined soon after also the decoupling of photons from the baryonic matter. In this moment, radiation and matter ceased to be part of the same plasma, starting to evolve in time with a separate thermodynamical equilibrium. Since the energy density of the matter fluid decreased slower in time than the radiation fluid (see Tab. 1.1), then at a certain time the energy density of the radiation fluid became the same as the matter fluid in an epoch denominated *equivalence epoch*. Then, a new epoch started, the so-called *matter dominated* epoch. During this epoch, all the large scale structures that we observe in the Universe, such as galaxies and cluster of galaxies, started to form through gravitational instability guided by the *dark matter*. It is during this epoch that we had the so-called *reionization* of hydrogen atoms, when the Universe reverted from being neutral to once again being an ionized plasma. After this epoch, at very recent times, the cosmological constant fluid became dominant with respect to both matter and radiation, because of the growing of vacuum regions in the Universe, leading to the today acceleration in the Universe expansion. In particular, as showed above, nowadays radiation represents only a negligible component of the Universe, while (cold) dark matter and cosmological constant are the main components. This is a very brief description of the standard cosmological model just to give a general idea to the reader. We refer to e.g. Ref. [6] for more details.

An important quantity to date a particular event or source of gamma rays in the Universe history is the so-called cosmological redshift z. This redshift is the result of the cosmological expansion of the Universe. It is defined as [6]

$$z = \frac{\lambda_{\rm obs} - \lambda_{\rm emis}}{\lambda_{\rm obs}}\,, \tag{1.15}$$

where λ_{emis} is the wavelength emitted by an electromagnetic source at a certain time $t_{\rm emis}$ and $\lambda_{\rm obs}$ is instead the wavelength observed today, which results stretched by the cosmological expansion. We can link this observable to the scale factor at the emission time $a(t_{\rm emis})$ as

$$1 + z(t_{\rm emis}) = \frac{a_{\rm o}}{a(t_{\rm emis})} \,. \tag{1.16}$$

Using Eq. (1.16), one can adopt the variable $z(t)$ instead of $a(t)$ to refer to a particular cosmological epoch in the Universe history.

A fundamental prediction of the standard cosmological model is the existence of a background blackbody radiation emitted at the recombination epoch and with the temperature of $T \simeq 2.7\,\mathrm{K}$ today, which has been observed and is the well known CMB mentioned above. One further outstanding success is the prediction of light-element abundances produced during cosmological nucleosynthesis, which agree with current observations (see Ref. [11] for a recent review).

However, this picture suffers from at least one major unresolved problem (related to the cosmological constant) and some other fine-tuning problems (related to initial conditions). Moreover, it lacks the answer to the fundamental question about the origin of the first primordial inhomogeneities that gave rise to the cosmic structures that we observe today. In the next sections, we will see these problems in detail and how we can solve the initial conditions problems and explain the rising of primordial inhomogeneities by introducing an *inflationary period.*

1.2 The cosmological constant problem

There are several issues regarding the vacuum energy density fluid (or the so-called cosmological constant) that we now observe in the Universe. In this section, we will briefly summarize the main ones.

First of all, there is an anthropological problem: from the last *Planck* satellite measurements we got that today the Universe is dominated by the matter and the cosmological constant fluids with relative abundances $\Omega_M^o \simeq 0.31$ and $\Omega_\Lambda^o \simeq 0.69$. We immediately see that, apart for a factor of 2, the two fluids are now present in the Universe with the same order of magnitude in abundance. The problem relies in the following question: why this happens just in our epoch? In particular, this fact was most likely crucial for our galaxy and our solar system to be as we now observe, thus for us to exist. This is a sort of coincidence problem. This question is still nowadays without a convincing answer.

The other important issue is the famous disagreement between the theoretical and experimental value of the cosmological constant energy density. This problem was firstly pointed out by Weinberg [12] as follows. In natural units, the observed energy density of Λ is given by

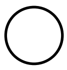
$$\rho_\Lambda^{\rm obs} = \Omega_\Lambda^o \, \rho_c \approx 10^{-46}\,\mathrm{GeV}^4 \,. \tag{1.17}$$

In quantum mechanics to each normal mode k of frequency ω_k we can attribute a zero point energy of $\hbar\omega_k/2$. Thus, the computation of the total vacuum energy density depends on the frequency interval of the normal modes and is given by the following summation

$$\rho_\Lambda = \frac{\sum_k \frac{1}{2}\hbar\omega_k}{V} \simeq \frac{\hbar}{2\pi^2 c^3} \int_0^{\omega_{\rm max}} \omega^3 \, d\omega = \frac{\hbar}{8\pi^2 c^3} \omega_{\rm max}^4 \,, \tag{1.18}$$

where $\omega_{\max}$ is associated to a cut-off scale, i.e. the energy at which we believe that our current understanding of the physics fails in describing the laws of nature. For instance, if we take the electroweak scale as the cut-off scale, then we find $\rho_{\rm vac}^{\rm EW} \sim 10^8\,{\rm GeV}^4$ which is already $\sim 10^{55}$ times larger than the observed value. If we consider the Planck scale as the cut-off scale, then we find $\rho_{\rm vac}^{\rm EW} \sim 10^{76}\,{\rm GeV}^4$ with a disagreement respect to the experimental value of 123 orders of magnitude. When this problem was originally pointed out, it was labeled as one of the greatest disagreements between the theory and the observation.

However, nowadays this problem has been reformulated under a different perspective which helps to alleviate it. The modern grown up way to calculate the vacuum energy density is to compute the vacuum loop diagrams for each particle species in the Standard Model of the particle physics. For example, the one loop self-energy diagram for a free canonical scalar field, ϕ, of mass m, can be computed using the techniques of dimensional regularization, yielding [13]



$$\begin{aligned} &= -\frac{m^4}{(8\pi)^2}\left[-\frac{2}{\epsilon}+\log\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma-\frac{3}{2}\right]\int d^dx \\ &= -\rho_{\rm vac}\int d^dx\,, \end{aligned} \tag{1.19}$$

where γ is the Euler-Mascheroni constant, μ is an arbitrary mass scale, and the $1/\epsilon$ term is a divergent term ($\epsilon \to 0$ corresponds to the $d \to 4$ limit). From this result, we have that the 1 loop contribution to the vacuum energy from the scalar field under consideration yields

$$\rho_{\rm vac} = \frac{m^4}{(8\pi)^2}\left[-\frac{2}{\epsilon}+\log\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma-\frac{3}{2}\right]. \tag{1.20}$$

The divergence requires us to add the following counterterm which depends on an arbitrary subtraction scale, M,

$$\rho_{\rm count} \sim \frac{m^4}{(8\pi)^2}\left[\frac{2}{\epsilon}+\log\left(\frac{4\pi\mu^2}{M^2}\right)\right]. \tag{1.21}$$

So, the renormalised vacuum energy (at one loop) is given by

$$\rho_{\rm vac,ren} = \frac{m^4}{(8\pi)^2}\left[\log\left(\frac{m^2}{M^2}\right)+\text{finite terms}\right]. \tag{1.22}$$

Because this depends explicitly on the arbitrary scale M, we do not have a concrete theoretical prediction for the renormalised vacuum energy. If we extend the same 1 loop computation for all the fields of the Standard Model and we sum up all the contributions, then the problem would have been solved since the final value of the energy density is not predicted by the theory, but can be given only by the observations. It seems that, using the experiment, we can determine the exact value of M so that the theoretical and experimental values coincide, treating M like a sort of new fundamental constant of the physics. Unfortunately, the problem arises again when we consider an higher loops computation. In such a case, the finite terms in Eq. (1.22) slightly change since, after reabsorbing the new divergences with other counterterms, we have additional subdominant contributions to $\rho_{\rm vac,ren}$. As a result, we have that the 2 loops value of M which solves the cosmological constant problem is slightly different from the 1 loop value. But then the same thing happens when we go to three loops, then four, and so on. At each successive order in perturbation theory we are required to retune the value of M to extreme accuracy. This leads to a radiative instability. The only way to avoid it is to require the higher loop corrections to be very highly suppressed, which would require a fine-tuning of the parameters of the underlying quantum field theory (and this is not the case of Standard Model of particle physics). This new way to express the cosmological constant problem suggests that it is probably related to our ignorance about a complete theory of gravity at high energies. You can find out more about this point in [13]. Moreover, in literature many other ways to explain the nature of the cosmological constant fluid have been proposed, with the common purpose to outflank the vacuum energy problem. These usually rely on modified gravity scenarios whose purpose is to "mimic" the role of this fluid (see e.g. [14–17]), without requiring a contribution from the vacuum energy.

1.3 The horizon problem

We start enunciating the problem by defining the *comoving particle horizon* χ_p as the maximum distance a light ray can travel from time 0 to time t. In a FRW Universe it can be written as

$$\chi_p = \int_0^t \frac{dt'}{a(t')} = \int_{a(0)}^{a(t)} d\ln a \left(\frac{1}{aH}\right) , \tag{1.23}$$

where the quantity $(aH)^{-1}$ is instead the so-called *comoving Hubble radius* (r_H), which quantifies the comoving distance covered by a light ray during the characteristic Universe time $\tau = H^{-1}$. The comoving Hubble radius represents an extimation of the distance under which two points in the comoving Universe are causally connected at a fixed time t.

The physical size of the particle horizon is simply

$$d(t) = a(t)\chi_p \, . \tag{1.24}$$

Using solutions (1.14), it can be shown (see Refs. [6, 18, 19]) that the comoving Hubble radius increases in time like

$$r_H \propto a^{(1+3w)/2} \, , \tag{1.25}$$

where for a matter fluid $w = 0$ and for a radiation fluid $w = 1/3$. This means that the fraction of the Universe in causal contact grows in time, or, in other words, that the causally connected Universe was much smaller in the past. In particular, going backward until the time of recombination, one concludes that the last scattering surface is made of several independent patches that had never causally connected in the past, but incredibly have the same degree of isotropy nowadays. Since particles belonging to different patches could not interact in the past, the currently situation with the CMB being a bath of photons with the same properties everywhere in the sky is extremely improbable. The lack of a microphysical explanation to this unlikely fine tuning is known as the *horizon problem*.

1.4 The flatness problem

We can rewrite the Friedmann equation (1.6) as

$$\Omega(t) - 1 = \frac{k}{(aH)^2}\,, \tag{1.26}$$

where

$$\Omega(t) = \frac{\rho(t)}{\rho_c} = \frac{\rho(t)}{3M_{Pl}^2 H^2}\,, \tag{1.27}$$

where $M_{\rm Pl}$ is the reduced Planck mass. As we have seen in the previous section, the comoving Hubble radius is growing in time, thus the quantity $\Omega - 1$ decreases going backward in time. The measured value of $\Omega_{\rm o} - 1$ is very close to zero, $|\Omega_{\rm o} - 1| = 0.001 \pm 0.002$ (68 % CL) [7], implying that in the past it should have been even smaller. Doing a precise computation, we can show that at the Planck scale $|\Omega_{\rm Pl} - 1| < \mathcal{O}(10^{-64})$ [20]. Nedless to say, this points again to an extreme fine tuning of the initial conditions, which is known as the *flatness problem*.

1.5 The cosmological relics problem

According to several extensions of the Standard Model of particles (e.g. Grand Unification Theories (GUT) or string theories), if the primordial Universe had very high energies, in the early Universe at very high energies various cosmological defects, such as magnetic monopoles, could have been produced, which would still be present in the Universe, with an energy density that would overclose the critical density by many orders of magnitudes. These are called *cosmological relicts*. The magnetic monopoles are an example of these cosmological relics. See Refs. [21, 22] for more details about other types of cosmological relics.

1.6 The inflationary solution to initial conditions problem

Horizon, flatness and cosmological relics problems are known to be *initial conditions* problems. In the previous section, we saw how crucial is the role of the comoving Hubble radius in the formulation of the horizon and flatness problems: both of them appear since $(aH)^{-1}$ is strictly increasing. However, this also suggests that the horizon and flatness problems can be solved by the same mechanism: making the comoving Hubble radius to decrease in time in the very early Universe. In this way, the flatness problem is trivially solved as $\Omega - 1$ would naturally converge to zero at early times (from Eq. (1.26)), before the standard FRW evolution begins. The horizon problem is also solved, as the region that will become the observable Universe today actually becomes smaller during this period, so that what appear now as causally disconnected regions in the sky were in causal contact in the past. Thus, this mechanism requires a period of accelerated expansion since by definition

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0 \;\Leftrightarrow\; \ddot{a} > 0\,. \tag{1.28}$$

This period is called *inflation*. The search for the solution to the problems of horizon and flatness was the historical motivation for inflation. Its ability to motivate one or more of the initial conditions of the standard cosmological model was noticed by several authors (see e.g. Refs. [23–26]) and acquired universal appreciation with the papers [27–29]. However, nowadays inflation has become a fundamental part of the modern cosmology for another reason. In fact, inflation provides us with a powerful mechanism to generate the energy density perturbations of the Universe, necessary for the formation of large scale structures. Before the advent of inflationary solution, the initial fluctuations were only postulated and taken as initial conditions designed to explain observational data. Today, inflation explains the origin of the first primordial inhomogeneities as small quantum fluctuations of the inflaton field that are stretched on very large scales by the Universe expansion. Nedless to say, this offers a concrete way to make predictions for the spectrum of these primordial perturbations, that are confirmed by the analysis of the CMB anisotropies (see Chap. 4).

Since the evolution of the Universe obeys Friedmann equations (1.6) and (1.7), it is clear that in order to have a period of accelerated expansion (i.e. $\ddot{a} > 0$) we need to satisfy the condition

$$\rho + 3p < 0. \tag{1.29}$$

From this last equation, we get that for an accelerated expansion it is necessary that the pressure of the Universe is negative, i.e. $p < -\rho/3$. Neither a radiation-dominated or a matter-dominated phase (for which respectively $p = \rho/3$ and $p = 0$) satisfies this condition. In the following, we will see a simple field-theoretical model realizing inflation which is also the current standard paradigm of the inflationary epoch. Later on, we will also discuss its theoretical implications.

We start by writing down the well-known action of one scalar field ϕ, which we call the *inflaton*, auto-interacting through a generic potential $V(\phi)$:

$$S = \int d^4x\sqrt{g}\left[\frac{1}{2}\partial_\mu\phi\partial_\nu\phi\, g^{\mu\nu} + V(\phi)\right]\,, \tag{1.30}$$

where $g_{\mu\nu}$ is the FRW metric (1.2) and $g = -det[g_{\mu\nu}] = a^6$. Writing the energy-momentum tensor of the scalar field,

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}\,, \tag{1.31}$$

we can find out the corresponding energy and pressure densities as

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{(\nabla\phi)^2}{2a^2}\,, \tag{1.32}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) - \frac{(\nabla\phi)^2}{2a^2}\,. \tag{1.33}$$

Now, we shift the inflaton field into an homogeneous and isotropic background and a quantum perturbation around the background as

$$\phi = \phi_0(t) + \delta\phi(t,x)\,. \tag{1.34}$$

For this moment, we consider only the dynamical evolution of the vacuum expectation value of the field, i.e. ϕ_0. The background part behaves like a perfect fluid with

$$\rho_0 = \frac{1}{2}\dot{\phi}_0^2 + V(\phi_0)\,, \tag{1.35}$$

$$p_0 = \frac{1}{2}\dot{\phi}_0^2 - V(\phi_0)\,. \tag{1.36}$$

Under the hypothesis that the potential energy is larger than the kinetic energy, i.e. $\dot{\phi}_0^2 \ll V(\phi_0)$, we obtain

$$w = \frac{p_0}{\rho_0} \simeq -1 < -\frac{1}{3}\,, \tag{1.37}$$

that gives an accelerated expansion of the Universe.

In order to express better this condition, let us write down the equation of motion for the inflaton ϕ_0 derived from action (1.30):

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + V'(\phi_0) = 0\,. \tag{1.38}$$

Physically, if $\dot{\phi}_0^2 \ll V(\phi_0)$, the field is slowly rolling down its potential, hence the name of *slow-roll inflation*. If we want that this condition is maintained in time, then we should also expect $\ddot{\phi}_0$ to be very small, i.e. $\ddot{\phi}_0 \ll 3H\dot{\phi}_0$. Under these assumptions the Friedmann equation (1.6) becomes

$$3M_{\rm Pl}^2 H^2 \simeq V(\phi_0)\,, \tag{1.39}$$

while Eq. (1.38) becomes

$$3H\dot{\phi}_0 + V'(\phi_0) \simeq 0\,. \tag{1.40}$$

Differentiating with respect to the time Eq. (1.39) and using Eq. (1.40), we find

$$\dot{H} \simeq -\frac{1}{2}\frac{\dot{\phi}_0^2}{M_{\rm Pl}^2} \simeq -\frac{1}{6}\frac{V'(\phi_0)^2}{3M_{\rm Pl}^2 H^2}\,. \tag{1.41}$$

This last equation tell us that, for the potential energy to dominate the energy density of the Universe, the potential of the inflaton should be very flat:

$$\dot{\phi}_0^2 \ll V(\phi_0) \quad \Longrightarrow \quad \frac{V'(\phi_0)^2}{V(\phi_0)} \ll H^2\,. \tag{1.42}$$

This is the first slow-roll condition. Now, differentiating with respect to the time Eq. (1.40) we get

$$\ddot{\phi}_0 \simeq \frac{V''(\phi_0)\,\dot{\phi}_0}{3H}\,. \tag{1.43}$$

Thus, it follows

$$\ddot{\phi}_0 \ll 3H\dot{\phi}_0 \quad \Longrightarrow \quad V''(\phi_0) \ll H^2\,, \tag{1.44}$$

which is the second slow-roll condition.

Both conditions (1.42) and (1.44) can be expressed in more generality using only the Hubble parameter H. In fact, since in the current model $w \simeq -1$ (Eq. (1.37)), slow-roll inflation is a quasi-de Sitter stage, with H almost constant. To achieve this, it is enough to require that the fractional change of the Hubble parameter during one Hubble time H^{-1} is much less than unity, i.e.

$$\epsilon = -\frac{\dot{H}}{H^2} \ll 1\,. \tag{1.45}$$

This is the definition of the *first slow-roll parameter* and corresponds to condition (1.42) if inflation is driven by a scalar field with action (1.30). At the same time, also the time variation of $\dot{H}$ must be small during inflation, i.e.

$$\frac{\ddot{H}}{\dot{H}H} \ll 1\,. \tag{1.46}$$

This leads to the definition of a *second slow-roll parameter*, η, which corresponds to the condition (1.44) and is defined in terms of ϵ through

$$\eta = 2\epsilon - \frac{1}{2}\frac{\dot{\epsilon}}{\epsilon H}\,. \tag{1.47}$$

Slow roll parameters can be expressed in terms of the first and second order derivatives of the slow-roll potential $V(\phi)$ as

$$\epsilon = \frac{1}{2}\left(\frac{M_{\rm Pl}V'(\phi_0)}{V(\phi_0)}\right)^2 \simeq \frac{1}{2}\frac{\dot{\phi}_0^2}{H^2 M_{\rm Pl}^2}\,, \tag{1.48}$$

$$\eta = \frac{M_{\rm Pl}^2 V''(\phi_0)}{V(\phi_0)} \simeq -\frac{\ddot{\phi}_0}{H\dot{\phi}_0} + \frac{1}{2}\frac{\dot{\phi}_0^2}{H^2 M_{\rm Pl}^2}\,. \tag{1.49}$$

Notice that we can write

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = (1-\epsilon)H^2\,. \tag{1.50}$$

Therefore, the condition $\epsilon < 1$ is fundamental to achieve $\ddot{a} > 0$: as soon as it is violated, inflation comes to an end.

The total amount of inflation is measured by the number of e-folds of accelerated expansion as

$$N^{\rm e-f} = \int_{a_i}^{a_f} d\ln a = \int_{t_i}^{t_f} H(t)dt\,, \tag{1.51}$$

where the subscripts i and f denotes respectively the beginning and the end of inflation. In the case of slow-roll inflation driven by a scalar field, we have

$$H dt = \frac{H}{\dot{\phi}} d\phi \simeq \frac{1}{\sqrt{2\epsilon} M_{\rm Pl}} d\phi \,, \tag{1.52}$$

so that Eq. (1.51) can be rewritten as an integral in the field space

$$N^{\rm e-f} = \int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{2\epsilon} M_{\rm Pl}} d\phi \simeq \frac{1}{\sqrt{2\epsilon} M_{\rm Pl}} \Delta\phi \,. \tag{1.53}$$

Inflation can successfully solve the horizon and flatness problems if $N^{\rm e-f} \geqslant 50 - 60$ (see e.g. Ref. [20]).

Notice that, since any physical process occurring in the Universe is irreversible (time-translation is not an isometry of the FRW metric (1.2)), then inflation, at its inset, erases any information regarding both the geometry and the composition of the Universe, giving "new" initial conditions that are in accordance with the today observations and the subsequent FRW evolution. For this reason, we automatically solve also the problem regarding the initial singularity that was given by the FRW solutions (1.14) (it is enough to assume that these solutions are valid only after the time t_f when inflation ends). Notice also that, if the theory of inflation is correct, then we are no more interested on what happened at the times $t < t_i$ before the beginning of inflation. In fact, for what we said, we can not access any physical information about the Universe before this time.

To conclude this section, we very briefly mention the inflationary solution to the problem of the cosmological relics. The basic idea is very simple: the density of the cosmological relics can be strongly diluted by the accelerated expansion taking place during inflation, to such low levels that justify the fact that they are not observed today. This is just the description of the qualitative solution to the problem. For more details see e.g. Refs. [21, 22].

Chapter 2

Quantum perturbations during inflation

If during inflationary epoch we do not consider the presence of any primordial perturbation, none of the cosmic structures we see today would have never formed. Our current understanding of the large scale structures of the Universe is that they had their origin from tiny perturbations in the energy density at a very early epoch. In absence of a physical mechanism able to produce them, they must be put by hand as initial conditions. Fortunately, an explanation is offered by the same mechanism that solves the horizon and flatness problems: during inflation, small quantum fluctuations are generated and, while the Hubble radius remains almost constant, their wavelength soon exceeds the Hubble radius itself. At this point, microscopic physics does not affect their evolution anymore, and their amplitude "gets frozen" at a non-zero value, which remains almost unchanged until the end of inflation. Then, since in the subsequent radiation and matter dominated epochs the Hubble radius increases faster than the scale factor, wavelengths that have gone outside the horizon during inflation eventually re-enter. The fluctuations that went outside the Hubble horizon around 60 e-foldings before the end of inflation re-enter with physical wavelength in the range accessible to cosmological observations like the CMB, and provide us with distinctive signatures of the high-energy physics of the early Universe (see Chap. 4). In this chapter, we will review the slow-roll models prediction about the power spectrum statistics of primordial perturbations and see which are the fundamental parameters that we use to characterize it.

2.1 Perturbation theory

Let us recall Eq. (1.34) where we considered a fluctuation of the inflaton field around the background

$$\phi = \phi_0(t) + \delta\phi(t, x)\,. \tag{2.1}$$

It is evident that, when we perturb the inflaton filed, we are automatically perturbing its energy momentum tensor $T_{\mu\nu}$ (1.31), which, in turn, will perturb the metric tensor though the Einstein equations (1.4) in the following way

$$g_{\mu\nu} = g^0_{\mu\nu}(t) + \delta g_{\mu\nu}(t,x)\,. \tag{2.2}$$

where the background metric is the FRW metric (1.2) with $k = 0$. The metric perturbations are usually decompose into objects with definite transformation properties with respect to the underlying three-dimensional manifold, since they appear to be uncoupled at first order [30, 31]. Following this reasoning, the metric tensor can be decomposed as (see e.g. [32])

$$g_{00} = -\Big(1 + 2\sum_{s=1}^{\infty}\frac{1}{s!}\Psi^{(s)}\Big) \tag{2.3}$$

$$g_{0i} = a(t)^2\sum_{s=1}^{\infty}\frac{1}{s!}\omega_i^{(s)} \tag{2.4}$$

$$g_{ij} = a(t)^2\left[\Big(1 + 2\sum_{s=1}^{\infty}\frac{1}{s!}\Phi^{(s)}\Big)\delta_{ij} + \sum_{s=1}^{\infty}\frac{1}{s!}h_{ij}^{(s)}\right] \tag{2.5}$$

where the functions $\Psi^{(s)}$, $\Phi^{(s)}$, $\omega_i^{(s)}$ and $h_{ij}^{(s)}$ represent the s-th order perturbations of the metric. We can decompose these in a way that we have objects with a well-defined transformation under spatial rotations. We can split ω_i into a transverse and longitudinal part (that will be called respectively transverse vector and scalar) as

$$\omega_i = \partial_i\omega^{||} + \omega_i^{\perp}\,. \tag{2.6}$$

Moreover, we can decompose h_{ij} into a scalar function, a transverse vector field, and a transverse and trace-less (TT) tensor as

$$h_{ij} = D_{ij}h^{||} + \partial_i h_j^{\perp} + \partial_j h_i^{\perp} + h_{ij}^{TT}\,, \tag{2.7}$$

where $D_{ij} = \partial_i\partial_j - \delta_{ij}\nabla^2/3$ is a traceless operator and we omitted the suffix (s) in the perturbations for simplicity of notation.

Because of the invariance of general relativity under general coordinate transformations

$$x^\mu \to x^\mu + \xi^\mu(t,\mathbf{x})\,, \tag{2.8}$$

not all the perturbations introduced are physical, but there will be some gauge modes that can be reabsorbed performing appropriate coordinate transformations. Once we have removed all the gauge modes, we have completely fixed one of many possible gauges. In particular, the perturbations can be different among the various gauges. Thus, gauge invariance implies that there are many different ways in which we can choose a map between the FRW background space and its associated perturbed Universe. Each of these descriptions is physically equivalent. Thus, in order to remove the gauge redundancy, we need to fix one of these maps before making a perturbative expansion of our physical objects. This procedure takes the name of *gauge fixing*. The issue of gauge invariance in cosmological perturbations is well-known in literature (see e.g. Refs. [30, 31, 33–36]) and we will not discuss it further in this section. Usually, the most clever way to deal with the gauge invariance is to define gauge invariant perturbations, i.e. perturbations which are independent by the choice of the gauge, giving an uniform description. In the following, two particular convenient gauge choices will be presented.

Moreover, in literature, we can find out an alternative formal way to express the perturbed metric tensor when dealing with cosmological perturbations. This is made trough the so-called *ADM (Arnowitt-Deser-Misner) formalism* of the metric, firstly introduced in [37, 38]. This is an Hamiltonian reformulation of the theory of gravity in which we foliate the total 4-dimensional space-time into 3-dimensional space-like hypersufaces at fixed time. The 3-metric $h_{ij} = g_{ij}$ on these hyper-surfaces can be splitted in the same way as in Eq. (2.5). The remaining components of the 4-metric can be rewritten in terms of the so-called lapse function $N = (-g^{00})^{-1/2}$ and the shift vector $N_i = g_{0i}$ in a way in which the metric in the ADM form reads

$$\begin{aligned} g_{00} &= -(N^2 - N^i N_i)\,, \\ g_{0i} &= N_i\,, \\ g_{ij} &= h_{ij}\,, \end{aligned} \tag{2.9}$$

and the inverse metric components read

$$\begin{aligned} g^{00} &= -\frac{1}{N^2}\,, \\ g^{0i} &= -\frac{N_i}{N^2}\,, \\ g^{ij} &= h^{ij} - \frac{N^i N^j}{N^2}\,. \end{aligned} \tag{2.10}$$

We can rewrite in an "ADM form" all the fundamental tensors of general relativity, in a way in which all the contractions are done only with the 3-metric h_{ij} (see Appendix A). This formalism is useful since N and N_i will play the role of Lagrange multipliers in the action (1.30) (at least in standard gravity and in effective field theory extensions of Einstein gravity) and their algebraic equations of motion can be solved in terms of dynamical perturbations and substituted back in the action. Another interesting feature of working in this formalism is the following: if we are interested in an expansion of the action (1.30) until cubic order in the dynamical fields, we need to know only the first order expressions of N and N_i. The demonstration of this fact is quite simple. Imagine to have in the Lagrangian of the theory a term that depends on N

$$\mathcal{L} = \mathcal{L}(N)\,. \tag{2.11}$$

We can expand this Lagrangian around the first order value $N = N^{(1)}$ as

$$\mathcal{L}(N) = \mathcal{L}(N^{(1)}) + \frac{\partial \mathcal{L}}{\partial N}\Big|_{N^{(1)}} \cdot \left(\sum_{n=2}^{\infty} N^{(n)} \right) + \frac{\partial^2 \mathcal{L}}{\partial^2 N}\Big|_{N^{(1)}} \cdot \left(\sum_{n,n'=2}^{\infty} N^{(n)} N^{(n')} \right) + \dots\ . \tag{2.12}$$

The second term in this expansion vanishes because it multiplies $\partial\mathcal{L}/\partial N|_{N^{(1)}}$, that is zero if we evaluate the Euler-Lagrange equation for N at first order. Then, only the first and the third terms remain. But the third term contains a summation which starts with a quartic order term. All the other terms in the expansion are of higher order. Then, if we are interested in the contributions until the cubic order, we can take only $\mathcal{L}(N^{(1)})$. An analogous result can be found introducing the dependence over N_i.

We conclude this section remarking that in an inflationary context transverse vector perturbations are usually not considered and put to zero. In fact, they are found to decay during inflation and then to be substantially irrelevant in this context (see e.g. [32]). Thus, in the following only scalar and TT tensor perturbations will be considered.

2.2 Useful gauges

In this section, we introduce two convenient gauge fixings at first order in the perturbations admitting simple non-linear generalizations.

The first one is the so-called *spatially flat* gauge. In this gauge, the scalar perturbations $\Phi^{(1)}$ and $h^{||(1)}$ are removed in the 3-metric. Then, one is free to remove also the vector perturbation $h_i^{(1)}$. We leave only with $h_{ij}^{TT(1)}$ and the first order inflaton perturbation $\delta\phi^{(1)}$. So, the 3-metric takes the form

$$h_{ij}^{\text{flat}} = a^2(\delta_{ij} + \gamma_{ij}) \tag{2.13}$$

where for convenience of notation we have fixed $\gamma_{ij} = h_{ij}^{TT(1)}$.

On the other hand, another very convenient gauge is the so-called *comoving* gauge. Here, the inflaton perturbation is reabsorbed together with $h^{||(1)}$ and $h_i^{(1)}$, and the 3-metric reads

$$h_{ij}^{\text{com}} = a^2\left[(1 + 2\Phi^{(1)})\delta_{ij} + \gamma_{ij}\right] . \tag{2.14}$$

Since we want a final description in terms of quantities that do not depend by the gauge fixing, we need to define gauge invariant variables. It is possible to show that, at linear order, γ_{ij} variable defined here is gauge invariant [30, 36]. Moreover, we can define the following scalar gauge invariant variable [30, 36]

$$\zeta^{(1)} = \Phi^{(1)} - \frac{H}{\dot{\phi}}\delta\phi^{(1)} . \tag{2.15}$$

This quantity is the so-called *curvature perturbation on slices of uniform energy density*. By construction, the meaning of ζ is that it represents the gravitational potential on slices of uniform energy density. In the inflationary context, ζ and γ_{ij} are the two gauge invariant variables that are used to link theoretical predictions with the observations.

The gauge fixings (2.13) and (2.14) admit non-linear generalizations which allows for a computation of the higher order statistics of the perturbations. At non-linear level, we can generalize the 3-metric of the spatially flat gauge as [39]

$$h_{ij}^{\text{flat}} = a^2 \exp[\gamma]_{ij} \tag{2.16}$$

where

$$\exp[\gamma]_{ij} = \delta_{ij} + \gamma_{ij} + \gamma_{il}\gamma_{lj} + \dots \,. \tag{2.17}$$

Instead, we can generalize the comoving gauge as [39][1]

$$h_{ij}^{\rm com} = a^2 e^{2\zeta} \exp[\gamma]_{ij} \tag{2.18}$$

where

$$e^{2\zeta} = 1 + 2\zeta + 4\zeta^2 + \dots \,. \tag{2.19}$$

Notice that in these last cases the quantities ζ, γ_{ij} and $\delta\phi$ are no longer first order fields, but they include all their perturbative expansion. In addition, we see that the gauge invariant variable ζ explicitly appears in the 3-metric tensor of the comoving gauge. This makes the latter a good gauge to do the computations, specially in standard gravity and at linear level, since we can directly find the quadratic action for ζ. However, as we will see later on in this work, the spatially flat gauge may help in simplifying some computations when dealing with non-linearities.

We end this section providing the non-linear (second order) relation between the variable ζ and the inflaton perturbation $\delta\phi$ in spatially flat gauge [39].[2]

$$\zeta = -\frac{H}{\dot\phi}\delta\phi + \frac{1}{2}\frac{\ddot\phi}{\dot\phi H}\left(\frac{H}{\dot\phi}\delta\phi\right)^2 + \frac{1}{4}\frac{\dot\phi^2}{H^2}\left(\frac{H}{\dot\phi}\delta\phi\right)^2 . \tag{2.20}$$

Identifying $\zeta_1 = -H/\dot\phi\,\delta\phi$ as the first order value of ζ, we can rewrite Eq. (2.20) in terms of slow-roll parameters as

$$\zeta = \zeta_1 - \frac{\eta}{2}\,\zeta_1^2 \,. \tag{2.21}$$

As we can see, the non-linear part of this relation is slow-roll suppressed, letting us to consider $\zeta \simeq \zeta_1$ if we are interested in slow-roll dominant physics only. This is useful when one wants to work in spatially flat gauge and express the final results in terms of the gauge invariant variable ζ.

2.3 Scalar perturbations

We start by reviewing the dynamical evolution of the scalar gauge invariant quantity ζ in the context of slow-roll models of inflation. The starting point is to rewrite action (1.30) with the metric in the ADM form (2.9) in a way that only Latin contractions with the 3-metric h_{ij} remain. We find (see e.g. [39])

$$S = \frac{1}{2}\int d^4x\sqrt{h}\left[N(R^{(3)} - h^{ij}\partial_i\phi\partial_j\phi - 2V) + N^{-1}(\dot\phi - N^i\partial_i\phi)^2 + N(K_{ij}K^{ij} - K^2)\right] , \tag{2.22}$$

[1] The quantity γ_{ij} in this gauge in principle differs from the one in the spatially flat gauge. But, it can be shown that the two ones can be considered the same for perturbations that exceed the Hubble horizon during inflation. Thus, we can still consider γ_{ij} as a gauge invariant quantity. See the Appendix of Ref. [39] for more details.

[2] Also in this case one should consider more terms in the relation (2.20), but it is possible to show that they are irrelevant for perturbation modes that go outside the Hubble horizon during inflation. For the complete relation, go to the Appendix of Ref. [39].

where h_{ij} denotes the 3-metric, $\sqrt{h} = -det[h_{ij}]$, $R^{(3)}$ is the scalar curvature associated to the 3-metric, and K_{ij} is the extrinsic curvature tensor of constant time spatial hypersurfaces given by

$$K_{ij} = \frac{1}{2N}(\dot{h}_{ij} - D_i N_j - D_j N_i)\,, \qquad K = K_{ij}h^{ij}\,, \tag{2.23}$$

where the quantity D_i is the three-dimensional covariant derivative.

The algebraic equations of motion for the fields N and N_i are given by the functional derivatives $\delta S/\delta N_i$ and $\delta S/\delta N$. They read

$$D_j(K_i^j - K\delta_i^j) - N^{-1}\partial_i\phi\,(\dot{\phi} - N^j\partial_j\phi) = 0\,, \tag{2.24}$$
$$R^{(3)} - 2V - (K_{ij}K^{ij} - K^2) - N^{-2}(\dot{\phi} - N^i\partial_i\phi)^2 - h^{ij}\partial_i\phi\partial_j\phi = 0\,. \tag{2.25}$$

These equations can be solved perturbatively once fixing a gauge and setting

$$N = 1 + N^{(1)} + \ldots\,, \qquad N^i = \partial^i\psi^{(1)} + N_T^{i(1)} + \ldots\,, \quad (\partial_i N_T^{i(1)} = 0)\,. \tag{2.26}$$

Here, we have stopped our expansion at first order in the perturbations ζ and γ_{ij}. In fact, we have previously showed that for a computation until cubic order in the Lagrangian, first order solutions for N and N^i are enough. These first order solutions in comoving gauge (2.18) are given by [39]

$$N^{(1)}_{\rm com} = \frac{\dot{\zeta}}{H}\,, \tag{2.27}$$
$$\psi^{(1)}_{\rm com} = -\frac{\zeta}{H} + \frac{a^2}{H}\partial^{-2}\zeta\,, \tag{2.28}$$
$$N^{i(1)}_{T,\rm com} = 0\,. \tag{2.29}$$

Substituting them into action (2.22) (and fixing $\gamma_{ij} = 0$), after performing some integrations by parts, one can finally obtain the following second order action for ζ

$$S = M_{\rm Pl}^2 \int d^4x\,\epsilon\, a^3 \left(\dot{\zeta}^2 - \frac{(\partial_i\zeta)^2}{a^2}\right)\,. \tag{2.30}$$

Now, it is useful to go to conformal time τ,

$$d\tau = \frac{dt}{a}\,, \tag{2.31}$$

where the action take the form

$$S = M_{\rm Pl}^2 \int d^4x\,\epsilon a^2 \left(\zeta'^2 - (\partial_i\zeta)^2\right)\,, \tag{2.32}$$

the primes denoting derivatives with respect to conformal time. After performing the field redefinition

$$\zeta = \frac{\Phi}{a\sqrt{2\epsilon}M_{\rm Pl}}\,, \tag{2.33}$$

the action for the auxiliary field Φ becomes

$$S = \frac{1}{2} \int d^4x \left(\Phi'^2 - (\partial_i \Phi)^2 + \frac{(a\sqrt{\epsilon})''}{a\sqrt{\epsilon}} \Phi^2 \right) , \tag{2.34}$$

which is the action for a Klein-Gordon scalar field with an effective mass term

$$m_\Phi = \frac{(a\sqrt{\epsilon})''}{a\sqrt{\epsilon}} . \tag{2.35}$$

Thus, we can Fourier expand and canonically quantize the field Φ as[3]

$$\Phi = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_\mathbf{k} \tag{2.36}$$

$$\phi_\mathbf{k} = a_\mathbf{k} u_\mathbf{k} + a^\dagger_{-\mathbf{k}} u^*_{-\mathbf{k}} , \tag{2.37}$$

where $a_\mathbf{k}$ and $a^\dagger_\mathbf{k}$ are the so-called annihilation and creation operators satisfying the common relations

$$[a_\mathbf{k}, a_{\mathbf{k}'}] = 0 , \qquad [a_\mathbf{k}, a^\dagger_{\mathbf{k}'}] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') . \tag{2.38}$$

The equation of motion for the mode function $u_\mathbf{k}$ is simply the Klein-Gordon equation with time dependent effective mass

$$u''_\mathbf{k} + \left(k^2 - m_\Phi\right) u_\mathbf{k} = 0 . \tag{2.39}$$

During a quasi-de Sitter stage of inflationary expansion, the scale factor can be rewritten as

$$a(\tau) = -\frac{1}{H\tau(1-\epsilon)} \simeq -\frac{1}{H\tau} + \mathcal{O}(\epsilon) . \tag{2.40}$$

Thus, neglecting sub-leading terms proportional to slow-roll parameters ϵ and η, the previous equation of motion simply becomes

$$u''_\mathbf{k} + \left(k^2 - \frac{2}{\tau^2}\right) u_\mathbf{k} = 0 . \tag{2.41}$$

To get physical insight on the behaviour of $u_\mathbf{k}$, let us analyze the two different regimes of Eq. (2.41):

- On sub-horizon scales, i.e. $-k\tau \gg 1$, we recover a free massless scalar field in conformal time, whose solution is

$$u^{\text{sub}}_\mathbf{k}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} . \tag{2.42}$$

  Thus, fluctuations with wavelength well within the horizon oscillates as they behave like they are in a Minkowski spacetime.

[3]Here and in the following, the momentum k in the Fourier expansions will indicate a comoving momentum.

- On super-horizon scales, i.e. $-k\tau \ll 1$, the effective mass term dominates and the equation of motion becomes

$$u''_{\mathbf{k}} - \frac{2}{\tau^2} u_{\mathbf{k}} = 0\,. \tag{2.43}$$

  This equation is solved by

$$u_{\mathbf{k}} = aB(k)\,, \tag{2.44}$$

  where the constant of integration $B(k)$ can be obtained making a rough matching with the sub-horizon solution at the horizon crossing $aH = k$, so that

$$u_{\mathbf{k}}^{\text{super}} = \frac{H}{\sqrt{2k^3}}\,. \tag{2.45}$$

  Thus, fluctuations with wavelengths much larger than the horizon freeze-out and their amplitude becomes constant.

Now, we can go into the exact solution of Eq. (2.39). If we consider the full expression of the scale factor $a(\tau)$ including slow-roll corrections, the mass term (2.35) at leading order in slow-roll parameters reads

$$\frac{(a\sqrt{\epsilon})''}{a\sqrt{\epsilon}} = \frac{d}{d\tau}\left(\frac{a}{2}\frac{\partial_t(a^2\epsilon)}{a^2\epsilon}\right) + \left(\frac{a}{2}\frac{\partial_t(a^2\epsilon)}{a^2\epsilon}\right)^2 \tag{2.46}$$

$$= \frac{d}{d\tau}\left[aH(1+2\epsilon-\eta)\right] + \left[aH(1+2\epsilon-\eta)\right]^2 \tag{2.47}$$

$$\simeq \frac{d}{d\tau}\left[\frac{1+3\epsilon-\eta}{-\tau}\right] + \left[\frac{1+3\epsilon-\eta}{-\tau}\right]^2 \tag{2.48}$$

$$\simeq \frac{2+9\epsilon-3\eta}{\tau^2}\,. \tag{2.49}$$

Thus, the exact equation of motion at leading order in slow-roll parameters reads

$$u''_{\mathbf{k}} + \left[k^2 - \frac{1}{\tau^2}\left(\nu_s^2 - \frac{1}{4}\right)\right] u_{\mathbf{k}} = 0\,. \tag{2.50}$$

where

$$\nu_s^2 = \frac{9}{4} + 9\epsilon - 3\eta\,. \tag{2.51}$$

Since slow-roll parameters are approximately constant during inflation, Eq. (2.50) is a Bessel equation and its solution can be written in terms of Hankel functions of first and second kind

$$u_{\mathbf{k}}(\tau) = \sqrt{-\tau}\left[c_1(k) H_{\nu_s}^{(1)}(-k\tau) + c_2(k) H_{\nu_s}^{(2)}(-k\tau)\right]\,. \tag{2.52}$$

The final particular solution can be determined by specifying a certain initial condition for the mode function $u_{\mathbf{k}}(\tau)$ when the mode k is well within the horizon. This choice automatically fixes also the initial quantum vacuum state. Above we saw that, asymptotically in the sub-horizon limit, the mode function $u_{\mathbf{k}}(\tau)$ takes the form (2.42), which we will take as initial condition. The vacuum associated to this initial condition is the so-called *Bunch Davies* vacuum state.[4]

In the end, the exact solution reads (see e.g. [32])

$$u_{\mathbf{k}}(\tau) = \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{2}(\nu_s+1/2)} \sqrt{-\tau}\, H^{(1)}_{\nu_s}(-k\tau)\,. \tag{2.53}$$

We are interested to the super-horizon expression of the perturbations. So, using the asymptotic behaviour of the Hankel function, we find out the following super-horizon limit

$$u_{\mathbf{k}}(\tau) = 2^{\nu_s-3/2} e^{i\frac{\pi}{2}(\nu_s+1/2)} \frac{\Gamma(\nu_s)}{\Gamma(3/2)} \left(\frac{(-k\tau)^{1/2-\nu_s}}{\sqrt{2k}} \right) \qquad (-k\tau \ll 1)\,, \tag{2.54}$$

where Γ is the Euler Gamma function. Since slow-roll parameters are very small during inflation, we can expand $\nu_s = (\frac{9}{4}+9\epsilon-3\eta)^{1/2}$ and take only the leading order contribution

$$\nu_s = \frac{3}{2} + 3\epsilon - \eta\,. \tag{2.55}$$

In the following, using the same approach, we will analyze the dynamics of TT tensor perturbations.

2.4 Tensor perturbations

The other gauge invariant dynamical perturbation during inflation is the so-called TT tensor perturbation γ_{ij}, also called *primordial gravitational waves* (PGW). This variable encodes two degrees of freedom, that are the two helicitites of the gravitational waves. More precisely, we can expand γ_{ij} in Fourier space as

$$\gamma_{ij} = \int \frac{d^3k}{(2\pi)^3} \sum_s \gamma^s_{ij\,\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}} = \int \frac{d^3k}{(2\pi)^2} \sum_s e^s_{ij}(\vec{k}) \gamma^s_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{2.56}$$

where $e^s_{ij}(\vec{k})$ is the polarization tensor, s is the polarization index and $\gamma^s_{\mathbf{k}}$ is the so-called *graviton mode*. For our later purposes it is convenient to use the so-called circular left (L) and right (R) polarization states of PGW, which are defined as

$$e^R_{ij} = \frac{1}{\sqrt{2}}(e^+_{ij} + ie^\times_{ij})\ , \tag{2.57}$$

$$e^L_{ij} = \frac{1}{\sqrt{2}}(e^+_{ij} - ie^\times_{ij})\ , \tag{2.58}$$

[4]This is the common vacuum choice used to study inflationary models. However, in literature we can find many physical mechanisms that lead to deviations from Bunch Davies vacuum at the inset of inflation (see e.g. [40–44] for some examples). We will not go through this point in the rest of the Thesis.

where $e^{\times}_{ij}$ and ϵ^{+}_{ij} are the usual two linear independent polarizations. It is possible to show the validity of the following relations (see, e.g., Ref. [45])

$$
\begin{aligned}
k^i e^s_{ij} &= 0 = e^i_i\,, \\
e^L_{ij}(\vec{k})e^{ij}_L(\vec{k}) &= e^R_{ij}(\vec{k})e^{ij}_R(\vec{k}) = 0\ , \\
e^L_{ij}(\vec{k})e^{ij}_R(\vec{k}) &= 2\ , \\
e^{R*}_{ij}(\vec{k}) &= e^L_{ij}(\vec{k})\ , \\
e^s_{ij}(-\vec{k}) &= e^s_{ij}(\vec{k})\ , \\
k_l\epsilon^{mlj}\ {e_{(s)}}_j{}^i(\vec{k}) &= -i\ \lambda_s k\ e^{im}_{(s)}(\vec{k})\ , \\
\gamma^{R*}_{\mathbf{k}} &= \gamma^L_{-\mathbf{k}}\ ,
\end{aligned}
\tag{2.59}
$$

where k_i is the i-th component of the momentum $\mathbf{k}$, $\lambda_R = +1$ and $\lambda_L = -1$. Here ϵ_{ijk} is the Levi-Civita pseudo-tensor with three Latin indices. Also we recall that s is the polarization index and not a tensor index. In Eqs. (2.59) Latin contractions are made with the δ_{ij}.

In computing the quadratic level contribution to tensor perturbations, we can fix $N_i = 0$ and $N = 1$ in the action (2.22) since at first order they do not depend on tensor modes (together with $\zeta = 0$). In the end, the action for tensor perturbations reads (see e.g. [39])

$$
S = \frac{M^2_{\mathrm{Pl}}}{8}\int d^4x\, a^3\left(\dot{\gamma}_{ij}\dot{\gamma}^{ij} - \frac{\partial_k\gamma_{ij}\partial^k\gamma^{ij}}{a^2}\right)\,. \tag{2.60}
$$

Using the expansion (2.56) and Eqs. (2.59), we can derive the corresponding action in Fourier space for the graviton modes $\gamma^s_{\mathbf{k}}$. This reads

$$
S = \frac{M^2_{\mathrm{Pl}}}{4}\sum_s\int\frac{dt\,d^3k}{(2\pi)^3}\,a^3\left(|\dot{\gamma}^s_{\mathbf{k}}|^2 - \frac{k^2|\gamma^s_{\mathbf{k}}|^2}{a^2}\right)\,. \tag{2.61}
$$

This has the same form of the scalar perturbation case apart for some coefficients. In particular, we can proceed in the same way, i.e. by passing to conformal time τ and making the following field redefinition for the graviton mode

$$
\gamma^s_{\mathbf{k}} = \sqrt{2}\frac{\mu^s_{\mathbf{k}}}{aM_{\mathrm{Pl}}}\,, \tag{2.62}
$$

which gives the following Klein-Gordon action in Fourier space for both the two polarizations of the field μ

$$
S = \frac{1}{2}\int\frac{d\tau\,d^3k}{(2\pi)^3}\left(\mu'^2 - (\partial_i\mu)^2 + m_\mu\,\mu^2\right)\,, \tag{2.63}
$$

where now the time dependent effective mass reads

$$
m_\mu = \frac{a''}{a}\,. \tag{2.64}
$$

Also in this case the field $\mu_{\mathbf{k}}$ is canonically quantized as

$$\mu_{\mathbf{k}} = a_{\mathbf{k}} u_{\mathbf{k}} + a^{\dagger}_{-\mathbf{k}} u^{*}_{-\mathbf{k}} \,. \tag{2.65}$$

The mode function $u_{\mathbf{k}}$ still obeys an analogous equation as (2.50) but with a slightly different mass term given by

$$\frac{a''}{a} = \frac{d}{d\tau}\left(\frac{a}{2}\frac{\partial_t(a^2)}{a^2}\right) + \left(\frac{a}{2}\frac{\partial_t(a^2)}{a^2}\right)^2 \tag{2.66}$$

$$= \frac{d}{d\tau}\left[aH\right] + \left[aH\right]^2 \tag{2.67}$$

$$\simeq \frac{d}{d\tau}\left[\frac{1+\epsilon}{-\tau}\right] + \left[\frac{1+\epsilon}{-\tau}\right]^2 \tag{2.68}$$

$$\simeq \frac{2+3\epsilon}{\tau^2} \,. \tag{2.69}$$

Thus, the equation of motion turns out to be

$$u''_{\mathbf{k}} + \left[k^2 - \frac{1}{\tau^2}\left(\nu_T^2 - \frac{1}{4}\right)\right] u_{\mathbf{k}} = 0 \,, \tag{2.70}$$

where this time the coefficient ν_T reads

$$\nu_T^2 = \frac{9}{4} + 3\epsilon \,. \tag{2.71}$$

The final solution for $u_{\mathbf{k}}$ is the same as (2.53), after replacing ν_s with the following leading order expression of ν_T

$$\nu_T = \frac{3}{2} + \epsilon \,. \tag{2.72}$$

In the next section, we will use the results derived so far to find out the expected spectra statistics of primordial perturbations within slow-roll models of inflation.

2.5 Power spectra

Primordial perturbations, viewed as function of the position at fixed time, have a random distribution, whose statistical properties are exactly what we would like to constrain with the observations. In general, we can compute all the n-th order correlators associated to a certain random variable. In this section, we will show the results for the power spectra case, which is the Fourier transform of the two-point function. In fact, as we will see better later on, when we are within single-field slow-roll models of inflation we can stop our analysis to the power spectrum, since the primordial perturbations turn out to be almost Gaussian. As a consequence, all the odd-n correlators almost vanish and the even-n correlators are well described as sums of products of two-point functions. However, we will discuss later on about the importance to evaluate non-Gaussianities, in particular for those models which go beyond single field slow-roll inflation. We want to remark that we are interested only to the super-horizon statistics of primordial perturbations, which is what we can link to the current observations. In fact, those scales that do not exceed the Hubble horizon during inflation can not give observational imprints in the late time Universe, as we will better see in Chap. 4.

Using the definition of ζ in terms of Φ (2.33) and the super-horizon limit of the linear solution for the mode function $u_{\mathbf{k}}$ (2.54), the super-horizon two point function in the Fourier space for the scalar quantity ζ is given by

$$\langle \zeta_{\mathbf{k}_1} \zeta^*_{\mathbf{k}_2} \rangle_{(-k\tau \ll 1)} = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \frac{|u_{\mathbf{k}_1\,(-k\tau \ll 1)}|^2}{a^2\, 2\epsilon\, M_{\rm Pl}^2} \,, \tag{2.73}$$

where

$$P_\zeta(k_1) = \frac{|u_{\mathbf{k}_1\,(-k\tau \ll 1)}|^2}{a^2\, 2\epsilon\, M_{\rm Pl}^2} \tag{2.74}$$

is the super-horizon power spectrum of the curvature perturbation. Usually, we refer to its dimensionless version

$$\Delta_\zeta(k) = \frac{k^3}{2\pi^2} P_\zeta(k) \,. \tag{2.75}$$

By a straightforward computation, we find

$$\Delta_\zeta(k) = \frac{H_*^2}{8\pi^2 M_{\rm Pl}^2 \epsilon_*} \left(\frac{k}{k_*} \right)^{n_s - 1} \tag{2.76}$$

where

$$\frac{H_*^2}{8\pi^2 M_{\rm Pl}^2 \epsilon_*} = \mathcal{A}_s \tag{2.77}$$

is the amplitude of curvature perturbation at some pivot scale k_*. Expression (2.76) is found taking in consideration only leading order contribution in slow-roll parameters. The quantity n_s denotes the so-called scalar spectral index ans is given in terms of slow-roll parameters as

$$n_s - 1 = -6\epsilon + 2\eta \,. \tag{2.78}$$

What we learn from this last equation is that the power spectrum of curvature perturbations generated during inflation is almost scale-invariant, i.e. the amplitude of a fluctuation at a scale k is almost independent on the scale itself.

In exactly the same way we derive the power spectrum of PGW γ_{ij} as

$$\langle \gamma^{ij}_{\mathbf{k}_1} \gamma^*_{ij\,\mathbf{k}_2} \rangle_{(-k\tau \ll 1)} = \sum_s e^s_{ij}(\vec{k}) e^{ij*}_s(\vec{k}) \langle \gamma^s_{\mathbf{k}_1} \gamma^{s*}_{\mathbf{k}_2} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \frac{4\,|u_{\mathbf{k}_1\,(-k\tau \ll 1)}|^2}{a^2\, M_{\rm Pl}^2} \times 2 \,, \tag{2.79}$$

where an additional factor of 2 comes from the fact that we are summing over the 2 polarization states which turns out to have the same power spectrum statistics. In the end, the dimensionless tensor power spectrum reads

$$\Delta_T(k) = \frac{k^3}{2\pi^2} P_T(k) = \frac{2H_*^2}{\pi^2 M_{\rm Pl}^2} \left(\frac{k}{k_*} \right)^{n_T - 1} \tag{2.80}$$

where

$$\frac{2H_*^2}{\pi^2 M_{\rm Pl}^2} = \mathcal{A}_T \tag{2.81}$$

is the amplitude of tensor perturbations at some pivot scale k_*. n_T is the so-called tensor spectral index and reads

$$n_T = -2\epsilon\,. \tag{2.82}$$

Also the tensor power spectrum turns out to be almost scale invariant. Notice that the amplitude of tensor modes depend only on the value of the Hubble parameter during inflation, which, if measured would provide important information about the energy scale of inflation. In fact, from Eq. (1.39) we have

$$H \propto V^{1/2}\,. \tag{2.83}$$

Moreover, it is very interesting to compare the amplitudes of the tensor and scalar perturbations, which is done defining the so-called *tensor-to-scalar ratio* as

$$r = \frac{\mathcal{A}_T}{\mathcal{A}_s} = 16\epsilon\,. \tag{2.84}$$

As we can see from Eqs. (2.78) and (2.84) constraints on r and n_s are also constraints on slow-roll parameters ϵ and η. Moreover, slow-roll parameters are by definition directly related to the first two derivatives of the inflaton potential, so that a measurement of r and n_s can put bounds on the shape of the scalar potential given by a particular slow-roll inflationary model (see Fig. 4.2 in the next chapter). Notice also that

$$r = -8n_T\,, \tag{2.85}$$

which is known as the "consistency relation" of slow roll models of inflation [46–48]. In fact, a probe of the violation of this relation would immediately falsify any of the possible slow-roll models of inflation [49, 50]. If futuristic measurements will falsify this relation, this would mean that inflation is not driven by the simple one scalar field model we have analyzed so far, but we would have to look to alternative more complicated scenarios.

Chapter 3

Primordial non-Gaussianity

As we anticipated in the previous chapter, in the framework of the single field slow-roll models of inflation, non-linear analyses (see e.g. [39, 51–53]) have shown that primordial non-Gaussianities are negligible.

In particular, we can formulate a sort of "no-go theorem" [52], which states that every model of slow-roll single field inflation in Bunch-Davies vacuum and in presence of Einstein gravity deviates from Gaussianity in a negligible way, as the amount of produced non-Gaussianity is suppressed by slow-roll parameters. It would seem that this argument discourages the search for primordial non-Gaussianity. On the contrary, this is exactly the reason why it is very interesting to develop the topic. If, for instance, non-Gaussianity was revealed by observations, it would make us to discard slow-roll models. In addition, these data may help in disentangle the degeneracies between different theoretical models that have similar predictions in the power spectrum statistics. On the other hand, as there can be large differences in size and shape of non-Gaussianities between different models, the detection of such features may reveal the underlying exact physics and particle content of inflation (see e.g. [32, 54–61]). Finally, even if we eventually do not discover important signals of non-Gaussianities, we still can use constraints on non-Gaussianities to constrain the parameters of those models of inflation for which the power spectrum statistics alone does not provide any useful information (in this work, we will see a specific example of this situation).

3.1 Primordial bispectra

The lowest order correlator beyond the two-point function is the three point function, which in Fourier space we write as

$$\langle X_{\mathbf{k}_1} X_{\mathbf{k}_2} X_{\mathbf{k}_3} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B_{X_1 X_2 X_3}(k_1, k_2, k_3) \, , \tag{3.1}$$

where each X_i can be either the curvature perturbation ζ or a graviton mode γ_s. Under the assumption of statistical homogeneity and isotropy, the bispectrum $B(k_i)$ is a function of the magnitude of the momenta $\mathbf{k}_i$'s that form a closed triangle configuration (as it is shown in Fig. 3.1). Usually, in order to put constraints on a the strength of a given

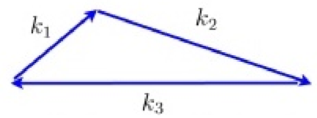


Figure 3.1. Triangle configuration of the bispectrum momenta $\mathbf{k}_i$.

bispectrum, it is defined a dimensionless non-Gaussianity coefficient, denoted by $f_{\rm NL}$, associated to each specific parametrization of the underlying bispectrum in terms of the momenta k_i's. This parameter roughly is given by [62]

$$f_{\rm NL} \sim \frac{B(k,k,k)}{P(k)^2}\,, \tag{3.2}$$

where $P(k)$ denotes the power spectrum of a given primordial perturbation and the bispectrum is evaluated in the so-called equilateral configuration, i.e. when all the momenta have equal magnitude. Thus, $f_{\rm NL}$ quantifies the strength of the bispectrum with respect to a given power spectrum statistics. Usually, the convention is to choose the power spectrum of the curvature perturbation $P_\zeta(k)$. Moreover, for a given bispectrum we can define the so-called shape function [62–64]

$$S(k_1,k_2,k_3) = \frac{1}{N}(k_1k_2k_3)^2 B(k_1,k_2,k_3)\,, \tag{3.3}$$

where N is a normalization factor which is defined such that $S(k,k,k) = 1$. The shape function contains the information regarding the explicit dependence over the momenta k_i for a given bispectrum. This is particulary important to distinguish among different models, as distinct scenarios for inflation can predict completely different shapes. Notice that, due to the fact that the momenta form a closed triangle, once we specify two of the three momenta, the last one is automatically fixed. As a consequence, the shape function depends only on the ratios between two of the three momenta and the third one, i.e. $x_2 = k_2/k_1$ and $x_3 = k_3/k_1$.

The standard technique employed to compute these higher-order correlators is the so-called *In-In formalism* which is based on the interaction picture (see App. B for a brief review of this formalism and relative references). The full exact computation of primordial bispectra within the single field slow-roll model scenario has been made for the first time in Ref. [39]. It gave the following results for the cross-bispectra of primordial perturbations

$$B_{\zeta\zeta\zeta}(k_1,k_2,k_3) = \frac{H^4}{M_{\rm Pl}^4}\frac{1}{2\epsilon^2}\frac{1}{\prod_i(2k_i^3)} \times\left[(\epsilon-\eta)\sum_i k_i^3+\epsilon\left(\frac{1}{2}\sum_i k_i^3+\frac{1}{2}\sum_{i\neq j}k_ik_j^2+4\frac{\sum_{i>j}k_i^2k_j^2}{k_T}\right)\right]\,, \tag{3.4}$$

$$B_{\gamma_s\zeta\zeta}(k_1,k_2,k_3) = \frac{H^4}{M_{\rm Pl}^4}\frac{2}{\epsilon}\frac{1}{\prod_i(2k_i^3)}e^s_{ij}(k_1)k_2^ik_3^j\left[-k_T+\frac{\sum_{i>j}k_ik_j}{k_T}+\frac{k_1k_2k_3}{k_T^2}\right]\,, \tag{3.5}$$

$$B_{\gamma_{s_1}\gamma_{s_2}\zeta}(k_1,k_2,k_3) = \frac{H^4}{M_{\rm Pl}^4}\frac{1}{\prod_i(2k_i^3)}e^{s_1}_{ij}(k_1)e^{ij}_{s_2}(k_2)\left[-\frac{1}{4}k_3^3+\frac{1}{2}k_3(k_1^2+k_2^2)+4\frac{k_1^2k_2^3}{k_T}\right]\,, \tag{3.6}$$

$$B_{\gamma_{s_1}\gamma_{s_2}\gamma_{s_3}}(k_1,k_2,k_3) = -4\frac{H^4}{M_{\rm Pl}^4}\frac{1}{\prod_i(2k_i^3)}e^{s_1}_{ij}(k_1)e^{s_2}_{mn}(k_2)e^{s_3}_{kl}(k_3)\,t^{imk}t^{jnl} \times\left[-k_T+\frac{\sum_{i>j}k_ik_j}{k_T}+\frac{k_1k_2k_3}{k_T^2}\right]\,, \tag{3.7}$$

where $k_T = k_1+k_2+k_3$, ϵ and η are slow-roll parameters and

$$t^{ijl} = k_2^i\delta^{jl}+k_3^j\delta^{il}+k_1^l\delta^{ij}\,. \tag{3.8}$$

From the above expressions, it is clear that the slow-roll dominant bispectra in the fundamental scenario are $B_{\zeta\zeta\zeta}$ and $B_{\gamma_s\zeta\zeta}$ which have the same order of magnitude and scale like $1/\epsilon$. Let us focus on $B_{\zeta\zeta\zeta}$. It turns out that it can be rewritten as the sum between two different bispectra shapes

$$B_{\zeta\zeta\zeta}(k_1,k_2,k_3) = B_{\rm loc}(k_1,k_2,k_3)+B_{\rm equil}(k_1,k_2,k_3)\,. \tag{3.9}$$

where $B_{\rm loc}(k_1,k_2,k_3)$ refers to the local shape (see Eq. (3.11)), while $B_{\rm equil}(k_1,k_2,k_3)$ has an equilateral shape (see Eq. (3.13)). The final prediction for non-Gaussianity coefficients associated to these two shapes turn out to be [64]

$$f_{\rm NL}^{\rm loc} = 6\epsilon-2\eta\,,\qquad f_{\rm NL}^{\rm equil} = 5\epsilon\,. \tag{3.10}$$

We immediately see that they are proportional to slow-roll parameters. Non-Gaussianity coefficients associated to the other primordial bispectra are of the same order of magnitude or smaller. Thus, if slow-roll models represent the correct description of inflation, then primordial bispectra are expected to be negligible.

3.2 Shapes of bispectra

In this section, we present the most studied forms of shape functions we can find in literature.

- **Local Shape**: the bispectrum of this shape reads [62]

$$B_{\text{loc}}(k_1,k_2,k_3) = 2f_{\text{NL}}^{\text{loc}}\mathcal{A}_S^2\left(\frac{1}{k_1^{4-n_s}k_2^{4-n_s}}+\frac{1}{k_1^{4-n_s}k_3^{4-n_s}}+\frac{1}{k_2^{4-n_s}k_3^{4-n_s}}\right)\,, \tag{3.11}$$

where $\mathcal{A}_S$ (2.77) is the scalar power spectrum amplitude and n_s the scalar tensor tilt (2.78). This bispectrum is larger in the so-called *squeezed* configuration, i.e. when one of the momenta is much smaller than the others. Usually, this kind of shape is generated in those models in which the quantity ζ takes the following non-linear correction in real space

$$\zeta(\mathbf{x}) = \zeta_g(\mathbf{x}) - \frac{3}{5}f_{\text{NL}}^{\text{loc}}(\zeta_g(\mathbf{x})^2 - \langle\zeta_g(\mathbf{x})\rangle^2)\,. \tag{3.12}$$

Examples of these models are e.g. [65–68]. In particular, it turns out that these non-linearities become relevant when they take place on super-horizon scales, like for those models of multi-field inflation where additional light scalar fields contribute to the curvature perturbation, namely the so-called *curvaton scenarios* (see e.g. [69] for a recent review).

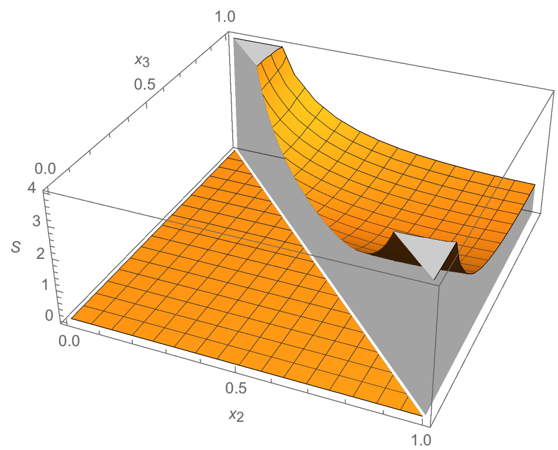


Figure 3.2. 3D-Plot of the local shape (3.11) in terms of $x_2 = k_2/k_1$ and $x_3 = k_3/k_1$.

- **Equilateral Shape**: the bispectrum of this shape reads [62, 70]

$$\begin{aligned}B_{\text{equil}}(k_1,k_2,k_3) = 6f_{\text{NL}}^{\text{equil}}\mathcal{A}_S^2\Bigg[&-\frac{1}{k_1^{4-n_s}k_2^{4-n_s}}-\frac{1}{k_1^{4-n_s}k_3^{4-n_s}}-\frac{1}{k_2^{4-n_s}k_3^{4-n_s}}\\ &-\frac{2}{(k_1k_2k_3)^{2(4-n_s)/3}}+\left(\frac{1}{k_1^{(4-n_s)/3}k_2^{2(4-n_s)/3}k_3^{4-n_s}}+(5\text{ perm.})\right)\Bigg]\,.\end{aligned} \tag{3.13}$$

This bispectrum is larger in the so-called *equilateral* configuration, i.e. when all the momenta have equal magnitude. Usually, this kind of shape is generated in those models with non-canonical kinetic terms described by an effective Lagrangian $\mathcal{L}_{\rm kin} = P(X, \phi)$ where $X = \partial_\mu \phi \partial^\mu \phi$ (see e.g. [71]). Other examples in literature are models with general higher-derivative interactions, as *ghost inflation* (see e.g. [72]), effective field theories of inflation (see e.g. [73]), and *Galileon-like* models (see e.g. [74]).

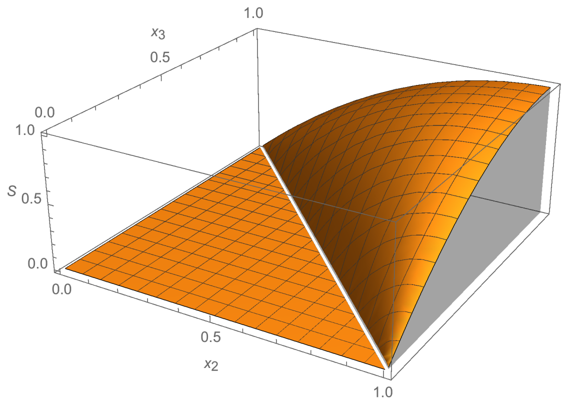


Figure 3.3. 3D-Plot of the equilateral shape (3.13) in terms of $x_2 = k_2/k_1$ and $x_3 = k_3/k_1$.

- **Orthogonal Shape**: the bispectrum of this shape reads [62, 75]

$$B_{\rm ortho}(k_1, k_2, k_3) = 6 f_{\rm NL}^{\rm ortho} \mathcal{A}_S^2 \left[-\frac{3}{k_1^{4-n_s} k_2^{4-n_s}} - \frac{3}{k_1^{4-n_s} k_3^{4-n_s}} - \frac{3}{k_2^{4-n_s} k_3^{4-n_s}} - \frac{8}{(k_1 k_2 k_3)^{2(4-n_s)/3}} + \left(\frac{3}{k_1^{(4-n_s)/3} k_2^{2(4-n_s)/3} k_3^{4-n_s}} + (5 \text{ perm.}) \right) \right] . \tag{3.14}$$

This kind of shape arises in the same models of the equilateral shape, in particular in effective field theories of inflation (se e.g. [75, 76]).

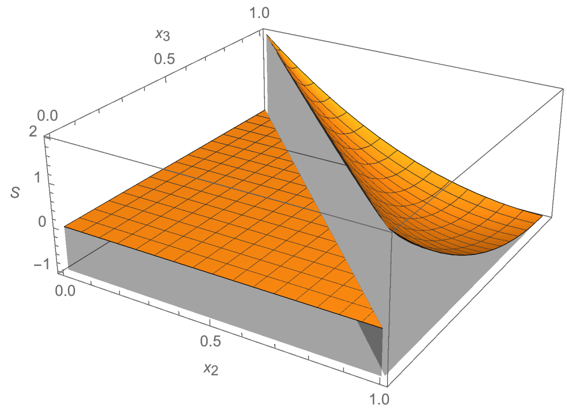


Figure 3.4. 3D-Plot of the orthogonal shape (3.14) in terms of $x_2 = k_2/k_1$ and $x_3 = k_3/k_1$.

3.3 Squeezed limit of bispectra

In this section, we very briefly review the so-called "squeezed issue" related to primordial bispectra, leaving all the technical details to the literature (see Refs. [77–86]). In fact, it is well known that primordial bispectra are not completely physical in the squeezed limit where we take one of the three momenta much smaller than the others. In the real space this corresponds to taking the cross-correlation between two fields evaluated at close points $\mathbf{x}_1$ and $\mathbf{x}_2$, and a third field evaluated at a point $\mathbf{x}_3$ that is far away to the infinite. It is possible to show that the physical signal of this cross-correlation is the one computed in the so-called Conformal Fermi Coordinate (CFC) frame centered in the point $\mathbf{x}_0$ which stays in the middle of $\mathbf{x}_1$ and $\mathbf{x}_2$ (see e.g. [81]). This local frame is constructed by imposing that the metric becomes unperturbed FRW along the time-like geodesic passing through $\mathbf{x}_0$ (the so-called *central geodesic*), with corrections that go as the spatial distance from the central geodesic squared and involve second order derivatives of metric perturbations, as we would expect by the virtue of the equivalence principle.

Here we consider a generic isotropic bispectrum $B_{XXY}(k_1, k_2, k_3)$ evaluated on super-horizon scales in global coordinates, where X, Y can be either ζ or γ_s variables, and we take the mathematical limit in which the momentum of the Y variable, k_3, is much smaller than the momenta k_1 and k_2 of X variables: in this limit, $k_3 = k_L \simeq 0$ and $k_1 \simeq k_2 \simeq k_S$. It is possible to show that, expanding the given bispectrum in powers of $(k_L/k_S) \ll 1$, the leading order effects of the long-wavelength perturbation k_L on the short modes can be removed by transforming to the CFC local frame, leaving only contributions starting from the order $(k_L/k_S)^2$. In particular, in co-moving gauge and in single field inflation the formula giving the physical squeezed bispectra up to order $(k_L/k_S)^2$ reads [81]

$$
\begin{aligned}
B_{XXY}(k_S, k_S, k_L)_{\text{squeezed,ph}} = & \Bigg[\frac{1}{2} P_{Yh}(k_L) P_{XX}(k_S) \frac{d\log(k_S^3 P_{XX}(k_S))}{d\log k_S} \\
& + \frac{1}{2} P_{Y h^T_{ij}}(k_L) \frac{k_S^i k_S^j}{k_S^2} \frac{d\log(P_{XX}(k_S))}{d\log k_S}\Bigg] \\
& + B_{XXY}(k_1, k_2, k_3)^{\vec{k}_1 = \vec{k}_S - \frac{1}{2}\vec{k}_L,\ \vec{k}_2 = -\vec{k}_S - \frac{1}{2}\vec{k}_L,\ \vec{k}_3 = \vec{k}_L}_{\text{squeezed}} \\
& + \mathcal{O}\left(\frac{k_L}{k_S}\right)^2 ,
\end{aligned}
\tag{3.15}
$$

where $B_{XXY}(k_1, k_2, k_L)_{\text{squeezed}}$ is the bispectrum in global coordinates, $P_{AB}(k)$ gives the cross-power spectrum of A and B variables, all the power spectra are evaluated at the time of horizon crossing of the short momenta k_S, $h = h_i{}^i/3$ is the trace of the spatial metric perturbation, while h^T_{ij} is its trace-less part.

The terms in the square bracket of (3.15) are such that they exactly cancel out the leading order value of bispectrum $B_{XXY}(k_1, k_2, k_3)_{\text{squeezed}}$, leaving terms of the order $(k_L/k_S)^2$. For this reason, Eq. (3.15) is also known in literature as *consistency relation*, as it gives an instrument to compute the expected leading order value of the squeezed bispectra, starting from the power spectra.

Chapter 4

Constraints from CMB anisotropies

The CMB is a back body radiation with an average temperature of $T = 2.7255 \pm 0.0006\ K$ [87, 88] which, according to the standard paradigm, was released at the time of recombination epoch, when the matter became transparent to photons. It was discovered casually about 50 years ago [89], and from that moment it has been one of the main probes of the standard cosmological model. Initially, it was found to be an extremely uniform radiation, whose nature is one of the main probes of the cosmological principle. Only later on, some tiny temperature fluctuations were discovered [90]. These are one of the main reasons why we need to set quantum perturbations during inflation. The CMB data remain currently the most precise and accurate way we have to test our knowledge of the primordial Universe [7, 62, 91].

In this chapter, we will briefly review the physics of the CMB according to the standard radiation transport models. In particular, we will focus into the connection between early time perturbations and the late time CMB temperature anisotropies we observe now. Finally, we will show the current constraints on the physics of inflation and on the six fundamental parameters describing the full ΛCDM cosmological model. For more details about the physics of the CMB we refer the reader to many reviews and books in the literature, like e.g. [92–101].

4.1 Decomposition in Stokes parameters

In general, a sea of photons like the CMB can have four different polarization states which are encoded into a 2×2 density matrix[1]

$$\rho_{ij} = \frac{1}{2} \begin{pmatrix} \Delta_T + \Delta_Q & \Delta_U - i\Delta_V \\ \Delta_U + i\Delta_V & \Delta_T - \Delta_Q \end{pmatrix} , \tag{4.1}$$

where Δ_T, Δ_Q, Δ_U, and Δ_V are the so-called Stokes parameters (see e.g. [93]).

Unpolarized CMB radiation is characterized by $\Delta_Q = \Delta_U = \Delta_V = 0$, and the parameter Δ_T describes the overall radiation intensity. In literature, the quantity Δ_T

[1] When we refer to the Stokes parameters, we take only the relative fluctuations over the respective mean value, e.g. $\Delta_T = (\Delta T - T_0)/T_0$ and so on.

is usually called "temperature fluctuation" or "temperature modes". The Δ_Q and Δ_U Stokes parameters describe the linear polarization of the CMB. In particular, taking two orthogonal (x, y) axes on the polarization plane, the Q-mode gives the difference in intensity between CMB photons with polarization vectors along the x and y axes respectively, while the U-mode gives the difference in intensity between CMB photons with a polarization vector along axes rotated by 45 degrees with respect to the x and y axes. Finally, the Δ_V Stokes parameter describes the CMB circular polarization or, better to say, it gives the difference in intensity between the two circular polarization modes of the CMB radiation.

CMB fluctuations (both temperature and polarization) are functions of the position and direction on the sky $\hat{\mathbf{n}}$, and they can be expanded on the sphere in terms of a spin-weighted basis as [94]

$$\Delta_T(\hat{\mathbf{n}}) = \sum_{\ell,m} a^I_{\ell m} Y_{\ell m}(\hat{\mathbf{n}}) \ , \tag{4.2}$$

$$\Delta_V(\hat{\mathbf{n}}) = \sum_{\ell,m} a^V_{\ell m} Y_{\ell m}(\hat{\mathbf{n}}) \ , \tag{4.3}$$

$$\Delta^{\pm}_P(\hat{\mathbf{n}}) = (\Delta_Q \pm i\Delta_U)(\hat{\mathbf{n}}) = \sum_{\ell,m} a^{\pm 2}_{\ell m} \ {}_{\pm 2}Y_{\ell m}(\hat{\mathbf{n}}) \, , \tag{4.4}$$

where ${}_sY_{\ell m}$ denotes the weighted spin-s harmonic sphere (see App. C). This decomposition is possible since the Δ_T and Δ_V polarization fields turn out to be spin-0 fields on the sphere, while the $(\Delta_Q \pm i\Delta_U)$ combination is a spin± 2 field [94]. In particular, this last feature implies that Δ_Q and Δ_U polarization modes are not invariant under a rotation on the polarization plane (while Δ_T and Δ_V modes are). In general, we would prefer a description of the CMB polarization in terms of spin-0 quantities that are invariant under rotations. In order to define these quantities, we need to act on $\Delta^{\pm}_P$ the spin raising and lowering operators $\eth$ and $\bar{\eth}$ (see again App. C) as

$$\Delta_E(\hat{\mathbf{n}}) = -\frac{1}{2}\left[\bar{\eth}^2 \Delta^+_P(\hat{\mathbf{n}}) + \eth^2 \Delta^-_P(\hat{\mathbf{n}})\right] \ , \tag{4.5}$$

$$\Delta_B(\hat{\mathbf{n}}) = \frac{i}{2}\left[\bar{\eth}^2 \Delta^+_P(\hat{\mathbf{n}}) - \eth^2 \Delta^-_P(\hat{\mathbf{n}})\right] \ . \tag{4.6}$$

Here, we have introduced the so-called E and B polarization modes. These modes offer an alternative description of CMB linear polarization which, differently from Q and U modes, is invariant under a rotation on the polarization plane. In the following, we will use a description of the radiation transfer both in terms of Q/U and E/B modes.

4.2 Boltzmann equations

The connection between primordial perturbations and CMB anisotropies is made through the so-called Boltzmann equations, which describe the time dependent evolution of CMB polarization modes at linear level and predict the expected amount of each polarization mode today. These equations take care of two main contributions: the Compton scattering between CMB photons and electrons and gravitational redshift which relates CMB anisotropies to primordial perturbations. The explicit full expressions of Boltzmann equations in Fourier space and in presence of only scalar perturbations of the metric tensor turn out to be [99]

$$\frac{d}{d\eta}\Delta_T^{(s)} + iK\mu\,\Delta_T^{(s)} = -\Phi' - iK\mu\Psi - \tau'\Big[\Delta_T^{0(s)} - \Delta_T^{(s)} + \mu v_B - \frac{1}{2}P_2(\mu)\,\Pi\Big] , \tag{4.7}$$

$$\frac{d}{d\eta}\Delta_P{}^{\pm(s)} + iK\mu\Delta_P{}^{\pm(s)} = -\tau'\Big[- \Delta_P{}^{\pm(s)} + \frac{1}{2}[1 - P_2(\mu)]\,\Pi\Big] , \tag{4.8}$$

$$\frac{d}{d\eta}\Delta_V{}^{(s)} + iK\mu\Delta_V{}^{(s)} = -\tau'\left[-\Delta_V{}^{(s)} + \frac{3}{2}\mu\Delta_V{}^{1(s)}\right] , \tag{4.9}$$

$$\Pi = \Delta_I{}^{2(s)} + \Delta_P{}^{2(s)} + \Delta_P{}^{0(s)} , \tag{4.10}$$

where $\mathbf{K}$ denotes the Fourier conjugate of $\mathbf{x}$ and we have fixed a coordinate system where $\mathbf{K} \parallel \hat{\mathbf{z}}$ axis. Here, η denotes the conformal time, the prime denotes differentiation with respect to conformal time, Ψ and Φ are scalar cosmological perturbations of the metric tensor (produced during inflation), v_B is the electrons average velocity, $\mu = \hat{n}\cdot\hat{K} = \cos\theta$ is the cosine of the angle between the line of sight and the Fourier mode and $P_\ell(\mu)$ is the Legendre polynomial of rank ℓ. The quantities $\Delta_T^{\ell(s)}$, $\Delta_P^{\ell(s)}$ and $\Delta_V^{\ell(s)}$ represent the ℓ-th order terms in the Legendre polynomial expansion of the corresponding modes.[2] Finally, the quantity $\tau(\eta)$ is the so-called *optical depth* defined as

$$\tau(\eta) = \int_\eta^{\eta_0} d\eta'\, a(\eta') n_B x_e \sigma_T \ , \qquad\qquad \tau'(\eta) = -a(\eta) n_B x_e \sigma_T \, , \tag{4.12}$$

where n_B is the electron density, x_e is the ionization fraction and σ_T is the so-called Thomson cross-section. $\tau'(\eta)$ has the physical meaning of *collision rate* in conformal time.

On the other hand, transverse and traceless tensor perturbations γ_{ij} alter only the unpolarized mode T through the following linear equation [99]

$$\frac{d}{d\eta}\Delta_{T,i}^{(t)} + iK\mu\,\Delta_{T,i}^{(t)} = -\frac{1}{2}\gamma_i' + \tau'\Big[- \frac{1}{10}\Delta_{T,i}^{0(t)} + \Delta_{T,i}^{(t)} - \frac{1}{7}\Delta_{T,i}^{2(t)} - \frac{3}{70}\Delta_{T,i}^{4(t)}\Big] , \tag{4.13}$$

where $i = +, \times$ denotes the two independent linear polarizations of tensor perturbations and $\Delta_{T,+}^{(t)}$ and $\Delta_{T,\times}^{(t)}$ are the components of $\Delta_T^{(t)}$ in the following azimuthal decomposition

$$\Delta_T^{(t)} = \Delta_{T,+}^{(t)}(1-\mu^2)\cos(2\phi) + \Delta_{T,\times}^{(t)}(1-\mu^2)\sin(2\phi) \, , \tag{4.14}$$

[2] Here, we adopt the convention

$$\Delta_T^\ell = \frac{1}{(-i)^\ell}\int_{-1}^{1}\frac{d\mu'}{2}\Delta_T(k,\mu')P_\ell(\mu') \tag{4.11}$$

and an analogous expression for Δ_V^ℓ and Δ_P^ℓ.

where ϕ is the azimuthal angle of $\hat{n}$ in polar coordinates and μ is as stated above. For more details about the derivation of Eqs. (4.7)-(4.9) and (4.13), see e.g. Refs. [93, 94, 99].[3]

In particular, once we define the harmonic coefficients of each CMB mode as

$$a_{\ell m}^{X} = \int d^2\hat{\mathbf{n}}\, Y_{\ell m}(\hat{\mathbf{n}})\, \Delta_X(\hat{\mathbf{n}}) \ , \tag{4.15}$$

where $X = T, E, B, V$, we can form a huge set of coupled differential equations for each $a_{\ell m}^{X}$, where each ℓ multipole is coupled to the $\ell - 1$ and $\ell + 1$ by the so-called *free streaming term*, which is the second term on the left-hand side of Boltzmann equations (i.e. $iK\mu\,\Delta_X$) and describes the projection of the CMB inhomogeneities produced during and after the recombination epoch onto anisotropies in the sky. Solving this coupled system of differential equations, we can find the today expected strength of the CMB inhomogeneities for each multipole scale ℓ which has been left imprinted during and after the recombination epoch. Here, we are not going to enter into the details of the exact numerical solutions, but in the following we will limit to give the qualitative description of the expected amplitude of $a_{\ell m}^{X}$'s coefficients referring the reader to the literature for more computational details.

First of all, a particular angular scale $\Delta\theta$ on the CMB sphere is linked to the multipole scale ℓ through [99]

$$\ell = \frac{2\pi}{\Delta\theta} \, . \tag{4.16}$$

Thus, small multipole moments correspond to large angular scales. In particular, the largest scales are the ones that were outside the Hubble horizon at the time of recombination, when the CMB fluctuations formed. As no physics could have affected them until they eventually re-enter the horizon, these modes carry almost unaltered information about the amplitude of primordial perturbations at the end of inflation. Instead, smaller scales evolve in a more complicated way.

Before neutral hydrogen formed (much before the recombination epoch), the Compton scattering between electrons and photons was very efficient, and photons and baryons were *tightly coupled* in a cosmological plasma forming a photon-baryon fluid. During this phase, cosmological perturbations of the metric tensor led to the formation of "potential wells" (i.e. regions of the space with slightly less gravitational potential than others) where the cosmological fluid got attracted creating local compression zones. But, since this fluid had its own radiation pressure, the latter acted as a restoring force, leading also to decompression in these zones. Physically, we had the formation of acoustic waves propagating at the speed of sound in the cosmic medium, where the energy density locally oscillated in time passing from overdensity to underdensity moments. This process predicts the presence of oscillations in CMB temperature (and polarization) fluctuations due to the compression and rarefaction of a standing acoustic wave. Therefore, the peaks we observe in the CMB temperature power spectrum (see Fig. 4.1) correspond to those

[3] Boltzmann equations, usually, do not exactly coincide between the different references. This is due to the fact that each reference uses his own formal conventions. Here, we have followed the conventions of Ref. [99].

modes that have undergone these acoustic oscillations and are caught at both their minima or maxima (the power spectrum statistics is sensitive to the modulus square of a given amplitude) depending by their phase at the time of recombination. Moreover, notice that we do not see this peak structure before a certain multipole scale, i.e. above a certain angular scale. This scale corresponds to the so-called comoving sound horizon r_* quantifying the maximum distance the acoustic waves could cover before the recombination. On the other side, on very small scales, photons have less and less distance where they can travel before making a second scattering, and we start to see the effects of shear viscosity and heat conduction in the fluid. This causes the damping at high multipoles of the $a^X_{\ell m}$ amplitudes, causing a damping in the CMB power spectra that we can see in Fig. 4.1 (this is the so-called *diffusion damping*).

In addition, in the standard scenario only unpolarized and some level of linear polarized anisotropies are produced. In fact, circular polarization does not have any source term on the right-hand side of Eq. (4.9) and, with an initial condition of zero V modes, no V modes are produced as well. The final harmonic coefficients of the unpolarized ($X = T$) and E/B-mode polarization ($X = E/B$) anisotropies given by the scalar (ζ) and the tensor perturbations ($\gamma^{R/L}$), are expressed, respectively, as [102, 103]

$$\begin{aligned} a^{(s)X}_{\ell m} &= 4\pi(-i)^\ell \int \frac{d^3\vec{k}}{(2\pi)^3} \mathcal{T}^X_{\ell(s)}(k)\, Y^*_{\ell m}(\hat{k})\, \zeta_{\vec{k}} \ , \\ a^{(t)X}_{\ell m} &= 4\pi(-i)^\ell \int \frac{d^3\vec{k}}{(2\pi)^3} \mathcal{T}^X_{\ell(t)}(k) \sum_{\lambda=R/L} \left[{}_{-2}Y^*_{\ell m}(\hat{k})\, \gamma^R_{\vec{k}} + (-1)^x \, {}_{+2}Y^*_{\ell m}(\hat{k})\, \gamma^L_{\vec{k}} \right] \ , \end{aligned} \tag{4.17}$$

where $\mathcal{T}^X_{\ell(s)}(k)$ and $\mathcal{T}^X_{\ell(t)}(k)$ are the scalar and tensor transfer functions, respectively, and x takes 0 (1) for $X = T, E$ ($X = B$). It is clear from these equations that CMB fluctuations are tightly bound to initial primordial perturbations, which are set by the inflationary epoch, and thus they are a direct probe of the physics of the early Universe. We will not give here the detailed expressions of the transfer functions which can be found in the literature above, but just qualitatively describe them.

In particular, we have that scalar perturbations left their main imprints on CMB modes only during the recombination epoch and the respective transfer function is mainly sensitive, together with projection effects, to the time evolution of the curvature perturbation ζ after inflation. This instantaneous imprint is commonly called *Sachs-Wolfe* effect. On the contrary, tensor perturbations gave to the mode T also an integrated imprint $\propto \gamma'$ from recombination epoch until nowadays. This is the so-called *Sachs-Wolfe integrated* effect. This is possible because tensor perturbations at a given scale remain constant in time until they re-enter the Hubble horizon. At this point, they start to oscillate around the 0 value with a maximum amplitude decreasing rapidly in time, until they finally decay. In general, also scalar perturbations have a similar integrated effect $\propto \zeta'$, but after recombination epoch they remain generally constant if they re-enter the Hubble horizon during the matter dominated epoch. A very small integrated contribution may arise only from the recombination to the equivalence epoch or during the late time dark energy dominated Universe, when scalar perturbations re-entering the Hubble horizon take some time evolution (see [19] for more details).

Moreover, it is interesting to mention that, for very large scales that were outside the Hubble horizon at the recombination epoch, the transfer function is given only by the projection of inhomogeneities onto the spherical sky, since no other physics could affect it. In particular, for the case of T modes, the projection is given by

$$\mathcal{T}^{T}_{\ell(s)}(k) \propto j_\ell[k(\eta_0 - \eta_*)] \,, \tag{4.18}$$

where j_ℓ denotes the spherical Bessel function and η_* is the conformal time at the recombination epoch, so that $(\eta_0 - \eta_*)$ gives the conformal distance to recombination. Once performing the integral (4.25) in this limit, we can find that the power spectrum almost goes like

$$C^{TT}_\ell \simeq \frac{c}{\ell(\ell+1)} \,, \tag{4.19}$$

where c is a certain constant. This explains why CMB power spectra are plotted in literature together with the additional factor $\ell(\ell+1)/2\pi$. In fact, according to our theoretical argument, the quantity $\ell(\ell+1)C^{TT}_\ell/2\pi$ is expected to be almost flat on the largest scales, exactly as we can see in Fig. 4.1.

To conclude this section, we explicitly show that scalar perturbations at linear order are not able to source B modes, contrary to tensor perturbations. In fact, it is possible to show that Eq. (4.8) admits the general integral solution [19]

$$\Delta_P{}^{\pm(s)}(\eta_0, K, \mu) = \frac{3}{4}(1-\mu^2)\int_0^{\eta_0} d\eta\, e^{iK(\eta-\eta_0)\mu-\tau}\,\tau'\,\Pi(\eta, K) \,. \tag{4.20}$$

Since scalar perturbations are invariant under rotations and so axially-symmetric around $\hat{\mathbf{z}}$ (so they excite only the CMB multipoles with $m=0$), we get $\Delta_P^+ = \Delta_P^-$. Thus, from Eq. (4.4) ,we have $\Delta_U^{(s)} = 0$, and scalar perturbations source only Q modes. Moreover, in this case (i.e. for scalar perturbations) the spin raising and lowering operators act like (see App. C, Eq. (C.2) for m=0)

$$\bar{\eth}^2\,\Delta_P{}^{\pm(s)} = \eth^2\Delta_P^{\pm(s)} = \partial^2_\mu[(1-\mu^2)\ \Delta_P{}^{\pm(s)}(\eta_0, K, \mu)] \ . \tag{4.21}$$

Therefore, using the definitions (4.5) and (4.6), we easily get

$$\Delta_E{}^{(s)}(\eta_0, K, \mu) = -\frac{3}{4}\int_0^{\eta_0} d\eta\, e^{-\tau}\tau'\Pi(\eta, K)\partial^2_\mu\left[(1-\mu^2)^2 e^{iK(\eta-\eta_0)\mu}\right] \ , \tag{4.22}$$

and

$$\Delta_B{}^{(s)}(\eta_0, K, \mu) = 0 \ , \tag{4.23}$$

concluding that linear scalar perturbations cannot source B-mode polarization, but only the E-mode.[4] In fact, it is well known that the B-mode polarization in the CMB is

[4]Notice also that the quantity $g(\eta) = e^{-\tau(\eta)}\tau'(\eta)$, called *visibility function*, has a narrow peak at the period of the recombination and is zero elsewhere (see e.g. [19]). Thus, the main contributions to E modes from scalar perturbations in Eq. (4.22) come from the recombination, as we stated above.

generated mainly by weak gravitational lensing and by tensor perturbations [99, 104]. In particular, the weak lensing contribution becomes dominant on high multipoles ($\ell \gtrsim 100$), while the tensor perturbation signal is dominant on low multipoles (larger CMB angular scales). Alternatively, a very small amount of B modes can be generated also by second-order vector and tensor modes sourced by scalar perturbations (see e.g. Refs. [105, 106]), but it is expected to be a subdominant higher-order effect. Moreover, since the T- and E-mode signal from scalar perturbations is theoretically expected to give the most important contribution, covering the tensor modes signal, the CMB B modes represent the only clean channel to search for primordial gravitational waves. For this reason, they are considered a sort of "smoking-gun" in the detection of primordial tensor modes. In particular, a significant signal following the predicted large-scale shape in the CMB B-mode spectrum would reveal us the complete physics of inflation.

4.3 Constraints on primordial power spectra

The cosmological observable most directly related to the primordial power spectra is the angular power spectra of the CMB fluctuations. In Eq. (4.15), we have shown that CMB anisotropies (both temperature and polarization) are described with the spherical harmonics coefficients (4.17). The CMB 2-point functions are written in terms of these coefficients as

$$\left\langle a^{(p)X_1}_{\ell_1 m_1} a^{(p)X_2 *}_{\ell_2 m_2} \right\rangle = \delta_{\ell_1,\ell_2}\delta_{m_1,m_2} C^{X_1X_2}_{\ell_1,p}\,, \tag{4.24}$$

where the angular power spectra $C^{X_1X_2}_{\ell,p}$ read

$$C^{X_1X_2}_{\ell,p} = \frac{2}{\pi}\int dk\, k^2 P_p(K) \mathcal{T}^{X_1}_{\ell(p)}(k) \mathcal{T}^{X_2 *}_{\ell(p)}(k) \times \begin{cases} 1 & : X_1X_2 = TT, EE, BB, TE \\ \chi & : X_1X_2 = TB, EB\,; \qquad p = t\,, \end{cases} \tag{4.25}$$

where $p = s, t$ depending if we are evaluating the scalar or tensor contribution and χ is the so-called chirality of primordial gravitational waves, defined as

$$\chi = \frac{P_R - P_L}{P_R + P_L}\,, \tag{4.26}$$

where R and L refer to the respective chiral gravitational waves. As we have seen above, in the standard single field slow-roll scenario, the two chiralities have equal amplitude, thus $\chi = 0$ and we would expect TB and EB cross-correlators to be vanishing.

In Fig. 4.1, the last TT, EE and TE CMB power spectra measured by the *Planck* mission are shown. Instead, the power spectra concerning the B modes are still missing as it has not been found yet a significant signal consistent with a primordial origin from inflation; nedless to say, this is the main aim of the proposed polarization satellite projects to follow up on the *Planck* mission, as the LiteBIRD experiment (see e.g. [107, 108]).

The most recent experimental results concerning the power spectrum statistics of primordial perturbations during inflation are given by the *Planck* 2018 publications[5] [7,

[5]Constraint on r has been obtained combining the *Planck* data with the BICEP2/Keck Array BK15 data.

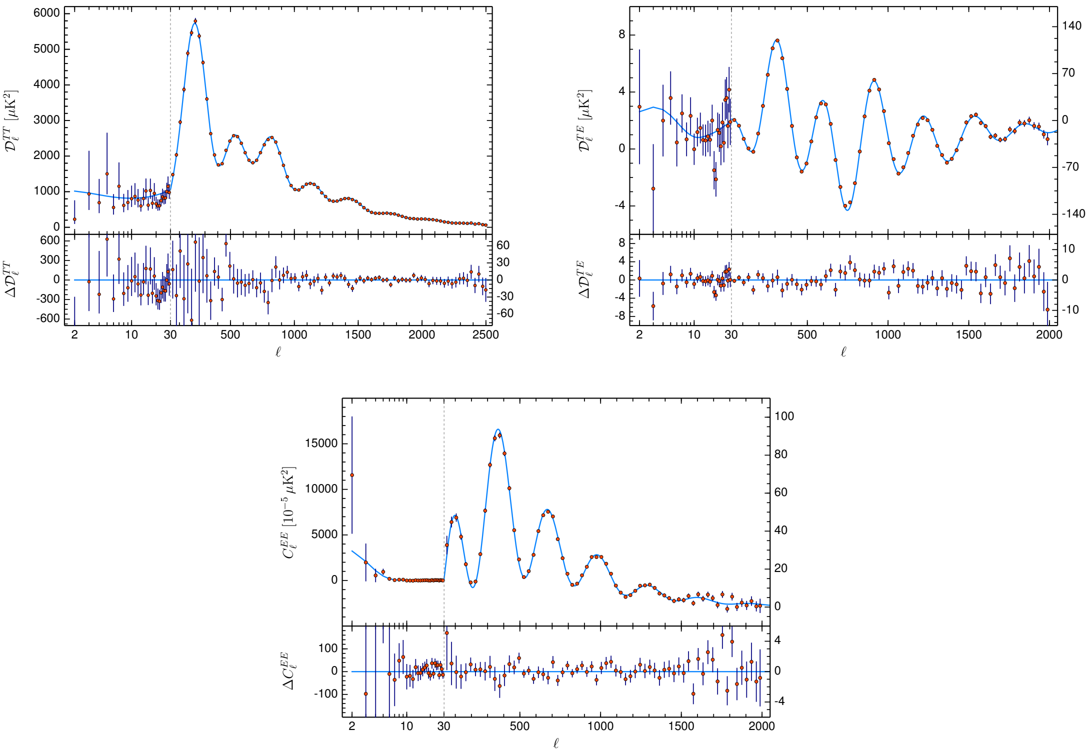


Figure 4.1. CMB TT, ET and EE power spectra (from [7]). On the vertical axis a $\ell(\ell+1)/2\pi$ factor is understood. We observe that low multipoles have larger errors than high multipoles. In fact, the predicted power spectrum is the average power in the multipole ℓ an observer would see in an ensemble of Universes. So, in principle we could even not extract a statistics, because a real observer can see only one Universe with its own set of $a_{\ell m}$'s. However, assuming statistical isotropy, we can collect $2\ell+1$ m-samples of power for each multipole, leading to the unavoidable error $\Delta C_\ell = \sqrt{2/(2\ell+1)}\, C_\ell$, which is the so-called *Cosmic Variance Limit* (CVL) error. This decreases with the square root of ℓ and puts a lower bound on the uncertainty that an hypothetical CMB experiment would commit in measuring CMB power spectra.

91, 109, 110]. In Tab. 4.1, we summarize the constraints for the primordial cosmological parameters $\mathcal{A}_s$, r and n_s that we have introduced in the previous chapter. One of the most

Parameter	*Planck* 2018 results
$\ln(10^{10}\mathcal{A}_s)$	3.04 ± 0.01 (68% CL)
r	< 0.056 (95% CL, pivot scale $k_* = 0.002$ Mpc^{-1})
n_s	0.965 ± 0.004 (68% CL)

Table 4.1. Constraints on primordial cosmological parameters obtained combining *Planck TT*, *TE*, *EE* +low*E*+lensing data [91].

important results is a small departure from exact scale invariance of scalar perturbations (i.e. $n_s = 1$) at more than 5σ. This represents a strong confirmation of the slow-roll models expectation of small deviations from scale invariance, proportional to the slow roll parameters ϵ and η. Another very important result is the constraint on the tensor-to-scalar ratio r. As we have showed in the previous chapter, the amplitude of tensor perturbations can be uniquely related to the Hubble parameter which labels the energy scale of inflation. Thus, the current upper bound on r gives us the following upper bound of the Hubble parameter

$$\frac{H}{M_{\rm Pl}} < 2.5 \times 10^{-5} \quad (95\%\ \text{CL})\,, \tag{4.27}$$

which translates in the following bound on the energy density during inflation

$$V < (1.6 \times 10^{16}\ \text{GeV})^4 \quad (95\%\ \text{CL})\,. \tag{4.28}$$

The couple of parameters (n_s, r) can be used to put constraints on inflationary models, comparing observational results with theoretical predictions. In Fig. 4.2, it is shown the allowed region in the (n_s, r) space together with the predictions of certain single field inflationary models.[6] In the following, we very briefly summarize the basics of the models considered, referring the reader to the original *Planck* papers for additional details.

- **Natural inflation**: in this model, the expansion is driven by a pseudo-scalar field with a typical axion potential

$$V(\phi) = \Lambda^4 \left[1 + \cos\left(\frac{\phi}{f}\right)\right]\,, \tag{4.29}$$

  f being the decay constant of ϕ. The flatness of the potential is protected by the (nearly exact) axionic shift symmetry. This model agrees with *Planck* observations

[6] The predictions of models are different depending by the number of e-folds to the end of inflation. This is related to the uncertainty about the reheating process which connects inflation to radiation dominated epoch (see e.g. [111, 112]). The details of this mechanism go beyond the scope of this Thesis.

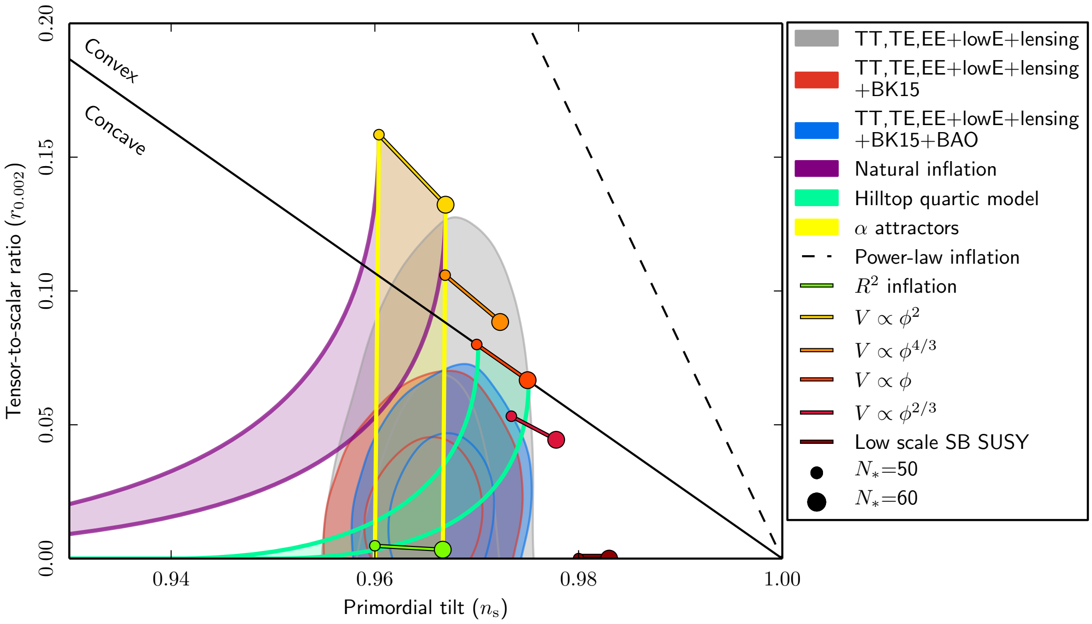


Figure 4.2. Marginalized joint 68% CL and 95% CL regions for n_s and r from *Planck* in combination with other data sets. It is included a comparison to the theoretical predictions of selected inflationary models [91]. These are different depending by the number of e-folds to the end of inflation. This is related to the uncertainty about details of the reheating process which connects inflation to radiation dominated epoch (see e.g. [111, 112]).

for $f/M_{\rm Pl} \gtrsim \mathcal{O}(1)$, which for various reasons is an undesirable constraint on the theory, if we want to identify this axion with the QCD axion (see e.g. [113, 114] for more details).

- **Hilltop inflation**: in this model, the slow-roll potential takes the form [115]

$$V(\phi) = \Lambda^4 \left(1 - \frac{\phi^p}{\mu^p} + ...\right) , \tag{4.30}$$

where the dots stand for higher order terms that are negligible during inflation, but are important at the end of inflationary epoch to ensure positiveness of the potential. In Fig. 4.2, it is shown the $p = 2$ case, which is currently compatible with data.

- **α-attractors models**: This is a class of models with a potential of the form [116–119]

$$V(\phi) = \Lambda^4 \tanh^{2m}\left(\frac{\phi}{\sqrt{6\alpha} M_{\rm Pl}}\right) . \tag{4.31}$$

- **Power law inflation**: In these models, the scale factor grows in time as a power-law, $a \sim t^2$. In order to realize this exact analytic power-law solution, the slow-roll potential has to take the form

$$V(\phi) = \Lambda^4 e^{-\lambda\phi/M_{\rm Pl}} . \tag{4.32}$$

As we can see from Fig. 4.2, this model now is highly discouraged, as it lies outside the joint 99% CL contour.

- **Spontaneously broken SUSY**: this is an example of the so-called Hybrid models (see e.g. [120, 121]). Usually, these kind of models are highly disfavoured by the observations predicting $n_s > 1$. But, spontaneously broken SUSY model given by the potential [122]

$$V(\phi) = \Lambda^4 \left[1 + \alpha_h \log\left(\frac{\phi}{M_{\rm Pl}}\right)\right] \tag{4.33}$$

predicts $n_s < 1$. From Fig. 4.2, we see that this model is currently almost disfavoured by the recent tighter constraints on n_s. Notice that for $\alpha_h \ll 1$ this coincides with power-law potential with $p \ll 1$.

- **R^2 inflation**: this is the first inflationary model proposed [123, 124] . The action of this model reads

$$S = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g} \left[R + \frac{R^2}{6M}\right] . \tag{4.34}$$

However, it is possible to show that this modified gravity scenario corresponds to a single field slow-roll model with potential given by

$$V(\phi) = \Lambda^4 \left[1 - \exp\left(\sqrt{\frac{2}{3}} \frac{\phi}{M_{\rm Pl}}\right)\right]^2 . \tag{4.35}$$

Notice, from Fig. 4.2, that this model is at the center of the area favored by *Planck* data.

- **Chaotic Inflation**: This is a class of models with monomial potential

$$V(\phi) = \Lambda^4 \left(\frac{\phi}{M_{\rm Pl}}\right)^n , \tag{4.36}$$

where inflation happens when $\phi > M_{\rm Pl}$. They are also called "large field models" because the field excursion $\Delta\phi$ is typically large. It can be shown that cubic and quartic potentials, which are not shown in Fig. 4.2, are well outside the 95% CL region and are strongly disfavoured. Even the quadratic potentials lies just outside the 95% CL region. On the contrary, linear and fractional values like $n = 4/3$ or $2/3$ (see e.g. [125, 126]) are still compatible.

4.4 Constraints on primordial non-Gaussianities

The cosmological observable most directly related to the primordial bispectra is the angular bispectra of the CMB fluctuations, i.e. the three-point cross correlator of all the possible $a^X_{\ell m}$'s in Eq. (4.17). This reads in formula

$$\begin{aligned}\left\langle a^{(p_1)X_1}_{\ell_1 m_1} a^{(p_2)X_2}_{\ell_2 m_2} a^{(p_3)X_3}_{\ell_3 m_3} \right\rangle &= \left(\frac{2}{\pi}\right)^3 \int dx\, x^2 \int dk_1 dk_2 dk_3\, (k_1k_2k_3)^2 B_{p_1p_2p_3}(k_1,k_2,k_3) \\ &\quad \times T^{p_1}_{\ell_1}(k_1) T^{p_2}_{\ell_2}(k_2) T^{p_3}_{\ell_3}(k_3)\, j_{\ell_1}(k_1x) j_{\ell_2}(k_2x) j_{\ell_3}(k_3x)\, \mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3} \\ &= b_{\ell_1\ell_2\ell_3}\, \mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3} , \end{aligned} \tag{4.37}$$

where the quantity $\mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3}$ is known as Gaunt integral and it represents the integral over the angular part of $\mathbf{x}$. This can be written in terms of Wigner 3-j symbols as (see e.g. [127, 128])

$$\mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3} = \int d^2\Omega_x Y_{\ell_1 m_1}(\mathbf{x}) Y_{\ell_2 m_2}(\mathbf{x}) Y_{\ell_3 m_3}(\mathbf{x}) \tag{4.38}$$

$$= \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} . \tag{4.39}$$

The function $b_{\ell_1\ell_2\ell_3}$, which multiplies the Gaunt integral in Eq. (4.37), is the so-called *reduced bispectrum*. The estimation of the $f_{\rm NL}$ non-Gaussian parameter associated to a given primordial shape is achieved through a fit of a theoretical ansatz for the reduced bispectrum to the observed CMB angular bispectrum. The final optimal statistical estimator of $f_{\rm NL}$ is found with an efficient numerical implementation.[7] In the latest *Planck* analysis (see [62]), three different techniques have been employed to estimate $f_{\rm NL}$ because comparing different methods improves the robustness of the results. The final

[7]The details of it go beyond the aim of the Thesis, see e.g. [54] for more details.

constraints on local, equilateral and orthogonal non-Gaussianity coefficients associated to the bispectrum of the curvature perturbation ζ read [62]

$$f_{\rm NL}^{\rm loc} = -0.9 \pm 5.1 \quad (68\% \text{ CL})\,, \tag{4.40}$$
$$f_{\rm NL}^{\rm equil} = -26 \pm 47 \quad (68\% \text{ CL})\,, \tag{4.41}$$
$$f_{\rm NL}^{\rm ortho} = -38 \pm 24 \quad (68\% \text{ CL})\,. \tag{4.42}$$

These results are compatible with a weak non-Gaussianity, telling us that at least the primordial perturbation ζ has a Gaussian statistics to a very high level of accuracy and suggest that initial fluctuations were linear and weakly interacting, in accordance with the prediction of slow-roll models of inflation. Of course, there is still space for alternative physically-motivated models that go beyond a simple Gaussian statistics. On the other hand, constraints on non-Gaussianities can put stringent bounds on the allowed parameter space for theories predicting important levels of non-Gaussianities in the curvature perturbation. Moreover, we currently lack of significant constraints on primordial bispectra concerning the tensor perturbations, which makes the analysis of CMB non-Gaussianities involving tensor modes (in particular involving the B modes, see e.g. the recent papers [129, 130]) particularly interesting in view of the next decade experiments involving the searching for the B-mode polarization of the CMB (like e.g. LiteBIRD [107, 108]).

4.5 Constraints on ΛCDM parameters

For completeness regarding the physics of the CMB, we end this chapter by presenting, in Tab. 4.2, the *Planck* constraints on the six fundamental parameters (which we already encountered thorough the chapter) that are used to fit the CMB data with the current ΛCDM model of cosmology. These are the baryon density parameter Ω_b, the (cold) dark matter density parameter Ω_c, the amplitude of primordial scalar perturbations $\mathcal{A}_s$, the scalar spectral index n_s, the optical depth τ (Eq. (4.12)) and the acoustic angular scale on the sky, given by $\theta_* = r_*/D_M$, where r_* is the comoving sound horizon, and D_M is the comoving angular diameter distance that maps this distance into an angle on the sky. We will not go into the details on how the constraints on these parameters are derived since it is not the purpose of this work. See the *Planck* paper [7] for more details. Needless to say, this is a clear example on the power of the CMB in inferring a large quantity of cosmological parameters, labelling it as one of the most important probe of the modern cosmology.

Cosmological parameter	68 % interval
$\Omega_b h^2$	0.0224 ± 0.0001
$\Omega_c h^2$	0.1193 ± 0.0009
$\ln(10^{10}\mathcal{A}_s)$	3.05 ± 0.01
n_s	0.967 ± 0.004
τ	0.056 ± 0.007
$100\theta_*$	1.0410 ± 0.0003

Table 4.2. *Planck* 2018 constraints for the base-ΛCDM model from CMB TT, TE, EE +low$E$ power spectra, in combination with CMB lensing reconstruction and baryonic acoustic oscillations (BAO) [7]. The quantity $h = H_o/(100\,\mathrm{km\,Mpc^{-1}s^{-1}})$ is the so-called reduced Hubble constant.

Chapter 5

Effective field theory of inflation

Until now we have worked within the framework of the standard slow-roll inflationary scenario. However, as discussed above, we must be open-minded about alternative more complicated scenarios. In general, there are many ways in which we can modify the standard lore. For example, we can either modify Einstein gravity, add additional field content, or introduce non-canonical kinetic terms for fields. All these possibilities must rely on a well-formulated high-energy quantum field theory which does not face with any instability or inconsistency (like e.g. Ostrogradsky ghosts [131]). In particular, we must ensure that the background evolution of the metric is consistent with an inflationary epoch (which is crucial to solve the fine tuning on the initial conditions). Once we find a well-defined model, we have to perturb around the background and study the dynamics of fluctuations in the same way as we do within the slow-roll scenario. The endpoint of this approach gives a late time statistics of primordial perturbations that can be linked to the CMB fluctuations. Then, observations test the physics of these perturbations and hopefully are able to put constraints on the model. Nevertheless, we need also a physical motivation to study a certain model, which explains why that framework is more interested than others. Following this path, we would like to find a general approach which explains the standard slow-roll scenario operators and allows for a huge number of new terms in the action of the theory at the same time. Inside an unified model, we could put constraints on each of the new terms, resulting as less model-dependent as possible. This approach, which directly studies primordial perturbations with as less assumptions as possible about the model-dependent microphysics of the background, is the so-called Effective Field Theory of Inflation (EFTI) (see e.g. [76, 132]). In this chapter, we will briefly review the basic concepts of this method following the original paper [76], and show how we can find the single field slow-roll models action within this frame.

5.1 EFT action for inflation

In general, an effective field theory approach consists in the description of a system in terms of light degrees of freedom with the systematic construction of all the lowest dimension operators compatible with the underlying symmetries. This method can be very powerful when applied to the perturbation theory during inflation, specially because we lack of a fundamental theory of high-energy physics and gravity, and we do not need any specific assumption on the fields driving inflation.

In constructing our effective field theory, the fundamental step is to identify the relevant degrees of freedom and symmetries of interest. When studying the early Universe perturbations, independently of what is driving the inflationary expansion, we introduced a scalar perturbation $\delta\phi$ which can be seen as a common local time shift of the "inflaton field" ϕ

$$\delta\phi(\mathbf{x}, t) = \phi(t + \pi(\mathbf{x})) - \phi_0(t)\,, \tag{5.1}$$

where the time-dependent background value ϕ_0 correspond to a FRW quasi-de Sitter solution for the background metric. This background solution spontaneously breaks time-translation invariance, and the scalar field $\pi(\mathbf{x})$ giving the local time shift is the Goldstone boson associated with the spontaneous breakdown of this symmetry. Moreover, we know that general relativity has invariance under a generic diffeomorphism

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(t, \mathbf{x})\,. \tag{5.2}$$

This invariance leads to the gauge invariance of the perturbations of the metric. In particular, it is easy to notice that under a generic time diffeomorphism of the form

$$t' = t + \xi^0(t, \mathbf{x}) \tag{5.3}$$

the scalar perturbation $\delta\phi$ transforms as

$$\delta\phi' = \delta\phi + \dot{\phi}_0(t)\xi^0\,. \tag{5.4}$$

We notice that, if we choose $\xi^0 = -\delta\phi/\dot{\phi}_0$, we set $\delta\phi' = 0$, fixing the so-called *unitary gauge*.[1] In this gauge, the scalar perturbation formally disappears from the action and is "eaten" by the metric, in exact analogy with the Higgs mechanism of the Standard Model of particle physics. Since with this choice we have broken time diffeomorphisms, our theory will be invariant only under the remaining spatial diffeomorphisms of the form

$$(x^i)' = x^i + \xi^i(t, \mathbf{x})\,. \tag{5.5}$$

We can now define a perturbation theory which breaks time-diffeomorphism invariance (reflecting the time-translation breaking of the background). The action of such a theory allows many more terms than the full-diffeomorphism invariant theory, where the only 2 derivatives operator built out of the metric that we could have written in the action would have been the Ricci scalar. In the following, we discuss in details all the terms that we can now admit in the action.

- First of all, we admit all those terms which are invariant under the full diffeomorphisms: this includes scalars given by the contraction of the Riemann tensor $R_{\mu\nu\rho\sigma}$ and its covariant derivative.

[1] This gauge is characterized only by the condition $\delta\phi = 0$, and has a residual gauge invariance, not coinciding with the comoving gauge (2.18), where the gauge is completely fixed.

- Under a generic spatial diffeomorphism from a set of coordinates $(t, \mathbf{x})$ to $(t, \mathbf{x}')$, the quantity $\partial_\mu t'$ is equal to δ^0_μ. As a consequence, every free upper 0 index does not transform. Therefore, we can use the 00 component of the full metric, i.e. g^{00}, and functions of it.

- We can define the so-called unit vector to surfaces of constant time

$$n_\mu = \frac{\partial_\mu t'}{\sqrt{-g^{\mu\nu}\partial_\mu t' \partial_\nu t'}} \,. \tag{5.6}$$

  With this quantity, we can construct the induced 3D spatial metric on surfaces of constant time, $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. This metric is used to derive the tensor components projected on the 3D surfaces (like e.g. the 3D Riemann tensor $R^{(3)}_{\mu\nu\rho\sigma}$). Thus, covariant derivatives of n_μ are invariant under space-diffeomorphisms and they can be used in the action. In particular, we can decompose these into a part projected on the surface of constant time and a part perpendicular to it. The first one is the so-called *extrinsic curvature* of spatial surfaces

$$K_{\mu\nu} = h_\mu{}^\sigma \nabla_\sigma n_\nu \,, \tag{5.7}$$

  and the other one does not give rise to new terms because it can be rewritten as

$$n^\sigma \nabla_\sigma n_\nu = -\frac{1}{2}(-g^{00})^{-1} h^\mu{}_\nu \partial_\mu (-g^{00}) \,. \tag{5.8}$$

- Using at the same time the Riemann tensor of the induced 3D spatial metric and the extrinsic curvature is redundant because one can be rewritten with the other and the 3D metric as

$$R^{(3)}_{\alpha\beta\gamma\delta} = h^\mu_\alpha h^\nu_\beta h^\rho_\gamma h^\sigma_\delta R_{\mu\nu\rho\sigma} - K_{\alpha\gamma}K_{\beta\delta} + K_{\beta\gamma}K_{\alpha\delta} \,. \tag{5.9}$$

  We can also avoid to use the 3-metric $h_{\mu\nu}$ explicitly, writing it in terms of $g_{\mu\nu}$ and n_μ.

- In front of any operator we can allow a generic function of time $f(t)$.

Now, we can conclude that the most generic action in unitary gauge and invariant under spatial diffeomorphisms is given by [76]

$$S = \int d^4x \sqrt{-g} F(g^{00}, R_{\alpha\beta\gamma\delta}, \nabla_\mu, K_{\mu\nu}, t) \,, \tag{5.10}$$

where all the free indexes inside the generic function F are upper 0's. We can can now start to write explicitly all the quantities that we can form with the specified terms, obtaining at the higher order in the perturbative expansion

$$S = \int d^4x \sqrt{-g} \left[\frac{M^2_{\rm Pl}}{2} R - c(t) g^{00} - \Lambda(t) + ... \right] \,, \tag{5.11}$$

where the dots stand for terms from quadratic order in the perturbations and depend by the perturbed quantities

$$\delta g^{00} = g^{00} + 1 \,, \qquad \delta R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - R^{(0)}_{\alpha\beta\gamma\delta} \,, \qquad \delta K_{\mu\nu} = K_{\mu\nu} - K^{(0)}_{\mu\nu} \,, \tag{5.12}$$

which start at first order in the perturbations. The functions $c(t)$ and $\Lambda(t)$ in the action can be uniquely determined by imposing that they correspond to a FRW solution of the background Einstein equations

$$G^{(0)}_{\mu\nu} = 8\pi G T^{(0)}_{\mu\nu} \,, \tag{5.13}$$

where the zero-th order energy momentum tensor is given by

$$T^{(0)}_{\mu\nu} = -\left(\frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}}\right)^{(0)} = 2c(t)u^\mu u^\nu + (c(t) - \Lambda(t))g^{\mu\nu} \,. \tag{5.14}$$

This is the energy momentum tensor of a fluid with energy density $\rho = c(t) + \Lambda(t)$ and pressure density $p = c(t) - \Lambda(t)$. Thus, we finally get the following Einstein equations

$$H^2 = \frac{1}{3M_{\rm Pl}^2}(c(t) + \Lambda(t)) \tag{5.15}$$

$$\dot{H} + H^2 = -\frac{1}{3M_{\rm Pl}^2}(c(t) - \Lambda(t)) \,. \tag{5.16}$$

We can solve algebraically these equations in terms of $c(t)$ and $\Lambda(t)$ and substitute the solutions back into Eq. (5.11), obtaining the following most generic action in unitary gauge with broken time diffeomorphisms describing perturbations around a flat FRW background with a Hubble parameter $H(t)$ [76]

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Bigg[&\frac{M_{\rm Pl}^2}{2}R + M_{\rm Pl}^2\dot{H}g^{00} - M_{\rm Pl}^2(3H^2(t) + \dot{H}(t)) \\ &+\frac{M_2(t)^4}{2!}(g^{00}+1)^2 + \frac{M_3(t)^4}{3!}(g^{00}+1)^3 \\ &-\frac{\bar{M}_2(t)^3}{2}(g^{00}+1)\delta K^\mu{}_\mu - \frac{\bar{M}_2(t)^2}{2}\delta K^{\mu\,2}_{\;\mu} - \frac{\bar{M}_3(t)^2}{2}\delta K^\mu{}_\nu \delta K^\nu{}_\mu + \ldots\Bigg] \,, \end{aligned} \tag{5.17}$$

where the dots stand for terms which are of higher order in the perturbed tensors (5.12), i.e. at higher order in the perturbations and with an increasing number of derivatives. Here, all the time dependent coefficients $M_a(t)$ and $\bar{M}_b(t)$ are free parameters characterizing all the possible different effects on perturbations of any single-field model of inflation.

5.2 Stuekelberg trick

In this section, we show how we can formally restore the full diffeomorphism invariance in the action (5.17), employing the so-called *Stuekelberg trick*. We start by considering the following terms in the action

$$S = \int d^4x\sqrt{-g}\left[A(t) + B(t)g^{00}\right] \,, \tag{5.18}$$

which formally corresponds to the first line of Eq. (5.17). Under a broken time diffeomorphism

$$t' = t + \xi^0(x)\,, \tag{5.19}$$

the components of the metric transform as

$$g'^{\rho\sigma}(x'(x)) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} g^{\mu\nu}(x)\,. \tag{5.20}$$

The "old" components of the metric can be rewritten in terms of the new ones as

$$g^{\rho\sigma}(x) = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g'^{\mu\nu}(x(x'))\,. \tag{5.21}$$

If we insert this last equation inside action (5.18), the latter becomes

$$S = \int d^4x \sqrt{-g'(x'(x))}\left|\frac{\partial x'}{\partial x}\right|\left[A(t) + B(t)\frac{\partial x^0}{\partial x'^\mu}\frac{\partial x^0}{\partial x'^\nu} g'^{\mu\nu}(x(x'))\right]\,. \tag{5.22}$$

Then, if we make a change of variable passing to the transformed coordinates x', we get

$$\begin{aligned} S = \int d^4x' \sqrt{-g'(x')}\, \Big[A(t' - \xi^0(x(x'))) + B(t' - \xi^0(x(x'))) \\ \times \frac{\partial(t' - \xi^0(x(x')))}{\partial x'^\mu}\frac{\partial(t' - \xi^0(x(x')))}{\partial x'^\nu} g'^{\mu\nu}(x')\Big]\,. \end{aligned} \tag{5.23}$$

Now, we promote the parameter $\xi^0(x)$ to a field through the definition

$$\pi(x) = -\xi^0(x)\,. \tag{5.24}$$

Moreover, we change the name of variable from x' to x. Thus, action (5.23) finally becomes

$$S = \int d^4x \sqrt{-g(x)}\left[A(t + \pi(x)) + B(t + \pi(x)))\frac{\partial(t + \pi(x))}{\partial x^\mu}\frac{\partial(t + \pi(x))}{\partial x^\nu} g^{\mu\nu}(x)\right]\,. \tag{5.25}$$

This action differs from (5.18). This is not surprising, since the starting action was not invariant under time diffeomorphism. However, if we assign to the field $\pi(x)$ the additional transformation rule

$$\pi'(x'(x)) = \pi(x) - \xi^0(x) \tag{5.26}$$

under a generic time diffeomorphism (5.19), then action (5.25) is now invariant under a generic time diffeomorphism, thus acquiring invariance under a full spacetime diffeomorphism. In the Standard Model of particle physics this is called Stuekelberg trick, and in this context the field $\pi(x)$ is the Goldstone boson associated with the breakdown of time diffeomorphisms. This mode becomes explicit in the action after one formally "restore" full diffeomorphism invariance through the additional transformation rule (5.26) for $\pi(x)$.

This is the same procedure that is done in the context of non-Abelian SU(2) gauge theories, where adding mass terms for the vector gauge fields break the gauge invariance of the theory. However, this invariance can be restored by including an extra field into the action (see e.g. [133]).

Returning to our case, we can apply this prescription to the full unitary gauge action (5.17). In this case, under time reparametrization (5.19) with $\xi^0 = -\pi(x)$, the metric components transform as

$$g'^{ij} = g^{ij} \tag{5.27}$$

$$g'^{0i} = (1+\dot{\pi})g^{0i} + \partial_j\pi\, g^{ij} \tag{5.28}$$

$$g'^{00} = (1+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})\partial_i\pi\, g^{0i} + \partial_i\pi\partial_j\pi\, g^{ij}\,. \tag{5.29}$$

Applying these transformation rules into action (5.17), we get the following action

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Bigg[&\frac{M_{\rm Pl}^2}{2}R + M_{\rm Pl}^2\dot{H}g^{00} - M_{\rm Pl}^2\left(3H^2(t+\pi)+\dot{H}(t+\pi)\right) \\ &+M_{\rm Pl}^2\dot{H}(t+\pi)\left((1+\dot{\pi})^2g^{00}+2(1+\dot{\pi})\partial_i\pi\, g^{0i}+\partial_i\pi\partial_j\pi\, g^{ij}\right) \\ &+\frac{M_2(t+\pi)^4}{2!}\left((1+\dot{\pi})^2g^{00}+2(1+\dot{\pi})\partial_i\pi\, g^{0i}+\partial_i\pi\partial_j\pi\, g^{ij}+1\right)^2 \\ &+\frac{M_3(t+\pi)^4}{3!}\left((1+\dot{\pi})^2g^{00}+2(1+\dot{\pi})\partial_i\pi\, g^{0i}+\partial_i\pi\partial_j\pi\, g^{ij}+1\right)^3+...\Bigg]\,. \end{aligned} \tag{5.30}$$

This action now describes the same physics of (5.17) but, assuming π to obey the transformation rule as in Eq. (5.26), it is invariant under a full diffeomorphism.

5.3 Slow-roll models

In this section we show how standard slow-roll models of inflation arise from the EFT action (5.30). We start by defining a decoupling limit, where $\pi(x)$ is decoupled by the metric perturbations $\delta g^{\mu\nu}$, and all the dynamics is described by the Goldstone boson only. This decoupling limit is given by the formal limits

$$M_{\rm Pl}\to\infty\,,\qquad\qquad \dot{H}\to 0\,. \tag{5.31}$$

keeping the combination $M_{\rm Pl}^2\dot{H}$ = const. Let us give an explicit example to convince ourself of this fact. Assuming all the $M_a = \bar{M}_b = 0$, the kinetic term for $\pi(x)$ is given only by

$$\mathcal{L}^{\pi}_{\rm kin} = M_{\rm Pl}^2\dot{H}\dot{\pi}^2\,. \tag{5.32}$$

Thus, the canonically normalized Goldstone boson reads

$$\pi_c = M_{\rm Pl}\dot{H}^{\frac{1}{2}}\pi\,. \tag{5.33}$$

The 00 metric perturbation is canonically normalized by

$$\delta g_c^{00} = M_{\rm Pl}\,\delta g^{00}\,. \tag{5.34}$$

For instance, at quadratic order in the perturbations, in action (5.30), we get the following coupling between the metric and the Goldstone boson:

$$2M_{\rm Pl}^2\dot{H}\dot{\pi}^2\delta g^{00} = 2\dot{H}^{\frac{1}{2}}\dot{\pi}_c\delta g_c^{00}\,. \tag{5.35}$$

Once we rewrite the term in terms of canonically normalized fields, we immediately see that in the limit $\dot{H} = 0$ this term disappears. Similarly, in action (5.30) we can take the following cubic term

$$M_{\rm Pl}^2\dot{H}\dot{\pi}^2\delta g^{00} = M_{\rm Pl}^{-1}\,\dot{\pi}_c\,\delta g_c^{00}\,. \tag{5.36}$$

In this case, the term disappears in the formal limit $M_{\rm Pl} = \infty$. Iterating this reasoning for all the coupling terms in (5.30) at any perturbative order, we get the only 2 conditions (5.31).

Physically, in the context of inflation, the limits in (5.31) must be interpreted in the following way: as we have seen in Chap. 2, during inflation we can always define 2 variables (ζ and γ_{ij}) which are constant, at any order in perturbation theory, when the corresponding physical mode λ_{phs} is outside the horizon. For this reason, we are interested to compute correlation functions just after horizon crossing, thus in making predictions for physical wave-numbers of order H during inflation. Therefore, the formal limit $M_{\rm Pl} = \infty$ corresponds to the assumption that the energy scale of inflation, i.e. H, is much smaller than $M_{\rm Pl}$. On the other hand, the condition $\dot{H} = 0$ is equivalent to impose a quasi-De Sitter background dynamics during inflation. Later on, we will better convince of these facts.

Now, we make a further assumption on the time dependence of the coefficients of any operator in action (5.30). In fact, although they can depend generically on time, we are interested in a background solution where they do not vary significantly in one Hubble time. If it was the case, the rapid time dependence of this coefficients could win against the friction created by the exponential expansion, so that inflation may cease to be a dynamical attractor. Thus, we assume that

$$f(t+\pi) \simeq f(t)\,. \tag{5.37}$$

This assumption allows us to neglect in (5.30) all the terms in π without a derivative, i.e. we are assuming an approximate continuous shift symmetry for π. Now, putting together conditions (5.31) and (5.37), action (5.30) simplifies into [76]

$$\begin{aligned} S = \int d^4x\sqrt{-g}\,\Bigg[&\frac{M_{\rm Pl}^2}{2}R - M_{\rm Pl}^2\dot{H}\left(\dot{\pi}^2 - \frac{(\partial_i\pi)^2}{a^2}\right) \\ &+2M_2^4\left(\dot{\pi}^2 + \dot{\pi}^3 - \dot{\pi}\frac{(\partial_i\pi)^2}{a^2}\right) - \frac{4}{3}M_3^4\dot{\pi}^3 + ...\Bigg]\,. \end{aligned} \tag{5.38}$$

Now, we take the limit in which $M_a = \bar{M}_b = 0$. In this limit, our action simply becomes

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - M_{\rm Pl}^2\dot{H}\left(\dot{\pi}^2 - \frac{(\partial_i\pi)^2}{a^2}\right)\right] . \tag{5.39}$$

Thus, in the limits we considered (decoupling limit+shift symmetry for π) the Goldstone boson turns out to be a Gaussian random field described by the simple quadratic Lagrangian

$$S_\pi = \int d^4x\sqrt{-g}\left[-M_{\rm Pl}^2\dot{H}\left(\dot{\pi}^2 - \frac{(\partial_i\pi)^2}{a^2}\right)\right] . \tag{5.40}$$

Of course, this would be the correct action for π only in the exact decoupling limit (5.31). In particular, reminding the definition of the slow-roll parameter ϵ

$$\epsilon = -\frac{\dot{H}}{H^2} \tag{5.41}$$

and expressing the ratio between H and $M_{\rm Pl}$ in the context of slow-roll models of inflation as[2]

$$\frac{H}{M_{\rm Pl}} \simeq 4\times 10^{-4}\sqrt{\epsilon}\,, \tag{5.43}$$

we realize that the decoupling limit is equivalent to the limits

$$\epsilon \to 0 \qquad\qquad \eta \to 0 \tag{5.44}$$

in terms of slow-roll parameters. Therefore, the decoupling limit is equivalent to the slow-roll condition for the inflaton field. But, physically speaking, slow-roll parameters can not be exactly equal to 0 (otherwise inflation would never end), thus in our effective field theory the reduced Planck mass can never be infinite and $\dot{H} \neq 0$. For this reason, action (5.40) is not complete, but it must contain other terms which are slow-roll suppressed (but effectively not equal to 0). These will be a small mass term for the Goldstone boson (proportional to slow-roll parameters), and slow-roll suppressed interaction terms leading to negligible non-Gaussianity [76]. After we have convinced of these facts, it is straightforward to get that π took the role of the inflaton perturbation (apart for the field redefinition $\pi = \delta\phi/\dot{\phi}$) and, matching the kinetic terms of actions (2.32) and (5.40), we get the following slow-roll leading relation between π and the curvature perturbation ζ[3]

$$\zeta = -H\pi + \mathcal{O}(\epsilon,\eta)\,. \tag{5.45}$$

[2]This can be derived putting together Eqs. (2.81) and (2.84), finding

$$\frac{H}{M_{\rm Pl}} = \sqrt{8\pi^2\epsilon\,\mathcal{A}_s} \simeq 4\times 10^{-4}\sqrt{\epsilon}\,, \tag{5.42}$$

where we have used the latest value of $\mathcal{A}_s$ as measured by the *Planck* satellite (see Tab. 4.1).

[3]Notice that this relation is equivalent to the first order limit of Eq. (2.21) once we identify $\pi = \delta\phi/\dot{\phi}$. This is a confirmation that π corresponds to the inflaton perturbation. Higher order terms in (2.21) are slow-roll suppressed with respect to first order one, as predicted also by Eq. (5.45).

On other hand, the action for the TT tensor perturbations γ_{ij} is simply obtained by the Ricci scalar term in (5.39), apart for slow-roll suppressed terms. In the end, we were able to find out the standard single field slow-roll model action for primordial perturbations ζ and γ_{ij} starting from an EFT approach.

Notice that in our derivation we have assumed all the "extra" parameters $M_a = \bar{M}_b = 0$, corresponding to the so-called *vanilla scenario*. In fact, these coefficients parametrize all the possible EFT single field models of inflation that depart from standard slow-roll models.

5.4 Validity of EFT

In this section, we briefly comment on the regimes of validity of EFT approach when we search for alternative models of inflation. In fact, the main reason why one introduces this formalism is to modify single field slow-roll models of inflation by admitting some of the M_a and $\bar{M}_b$ parameters to be different by 0 in action (5.38). However, as we are going to see, these additional terms may introduce in the theory pathologies and inconsistencies which usually theoretically constrain the new space of parameters even before the matching with the experiment.

We start by expanding action (5.38) in terms of π until cubic order. The most generic action for the Goldstone boson in the decoupling limit (thus neglecting higher order terms in slow-roll) up to cubic order in π turns out to be [76]

$$S = \int d^4x\sqrt{-g}\left[-\frac{M_{\rm Pl}^2\dot{H}}{c_s^2}\left(\dot{\pi}^2 - c_s^2\frac{(\partial_i\pi)^2}{a^2}\right)\right.$$
$$\left. -\frac{M_{\rm Pl}^2\dot{H}}{c_s^2}\dot{\pi}\frac{(\partial_i\pi)^2}{a^2} - \frac{2}{3}M_{\rm Pl}^2\dot{H}\frac{\tilde{c}_3}{c_s^4}\dot{\pi}^3\right], \tag{5.46}$$

where we have defined the new speed of sound of π as

$$c_s^2 = \left(1 - \frac{2M_2^4}{M_{\rm Pl}^2\dot{H}}\right)^{-1} \tag{5.47}$$

and

$$\tilde{c}_3 = c_s^2\frac{M_3^4}{M_2^4}\,. \tag{5.48}$$

We immediately see from action (5.46) that, in order to prevent the Goldstone boson to become an unstable field, the coefficient of the $\propto \dot{\pi}^2$ kinetic term must be positive, i.e. we need $c_s^2 > 0$. From Eq. (5.47) this translates into the following condition on M_2^4

$$M_2^4 > \frac{M_{\rm Pl}^2\dot{H}}{2}\,. \tag{5.49}$$

Moreover, we can still have superluminal propagation when $c_s^2 > 1$. If we want to avoid this scenario we must assume the stronger condition

$$-\frac{2M_2^4}{M_{\rm Pl}^2\dot{H}} > 0\,, \tag{5.50}$$

which, since $\dot{H} < 0$, gives the condition $M_2^4 > 0$. However, notice that superluminal propagation in effective field theories may not be a problem per se (see e.g. [134]), but simply implies that the theory can not have a Lorentz invariant UV completion [135].

Furthermore, the two cubic operators in Eq. (5.46) give rise to two different 3 scalars bispectra that are given by a linear combination of equilateral and orthogonal shapes of the same type we have seen in Chap. 3. The corresponding non-Gaussian coefficients associated to these amplitudes are given by (see e.g. [71, 136] for the full computation)

$$f_{\rm NL}^{\dot{\pi}^3} = \frac{10}{243}\left(1-\frac{1}{c_s^2}\right)\left(\tilde{c}_3 + \frac{3}{2}c_s^2\right)\,, \qquad f_{\rm NL}^{\dot{\pi}(\partial_i\pi)^2} = \frac{85}{324}\left(1-\frac{1}{c_s^2}\right)\,. \tag{5.51}$$

As we can see, we generate non-Gaussianity of the order

$$f_{\rm NL} \sim \frac{1}{c_s^2}\,. \tag{5.52}$$

This means that in the limit $c_s^2 \to 0$ we can have in principle huge non-Gaussianities. However, even before looking to the constraints on $f_{\rm NL}$ given by the experiments, it is possible to show that theoretically speaking c_s^2 can not be indefinite small. In fact, in the limit $c_s^2 \to 0$, interaction terms in action (5.46) become large, leading to a strong coupling regime that would spoil the perturbativity of the theory. However, since these interactions are non-renormalizable (the corresponding coupling constants have the physical dimensions of M^{-2}), there must exist an UV cut-off energy scale Λ setting the energy limit at which the π's self interactions enter in the strong coupling regime. To identify this cut-off, we have to rewrite the action in terms of canonically normalized fields

$$\pi_c = (-2M_{\rm Pl}^2\dot{H}c_s)^{1/2}\pi\,, \tag{5.53}$$

and rescale spatial coordinates as

$$x_i \to \frac{x_i}{c_s}\,. \tag{5.54}$$

If we do so, action (5.46) becomes

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Bigg[&\frac{1}{2}\left(\dot{\pi}_c^2 - \frac{(\partial_i\pi_c)^2}{a^2}\right) \\ &-\frac{1}{\sqrt{8M_{\rm Pl}^2|\dot{H}|c_s^5}}\left(\dot{\pi}_c\frac{(\partial_i\pi_c)^2}{a^2} + \frac{2}{3}\tilde{c}_3\,\dot{\pi}_c^3\right)\Bigg]\,. \end{aligned} \tag{5.55}$$

Now, the strong coupling scale can be read in the denominator of the cubic interactions as

$$\Lambda^2 = \sqrt{8M_{\rm Pl}^2|\dot{H}|c_s^5}\,. \tag{5.56}$$

As we can see, when $c_s = 1$, the strong coupling scale becomes higher as in this limit the theory loses their possibly dangerous interaction. On the contrary, when c_s becomes smaller, approaching 0, the cut-off scale decreases and the theory can become strongly coupled even at the characteristic energy scale of inflation. Since we are interested in making predictions for cosmological perturbations at energies of the order of the Hubble scale H, we must impose that Λ is bigger than H, which implies

$$H^2 < \sqrt{8M_{\rm Pl}^2|\dot{H}|c_s^5}\,. \tag{5.57}$$

If we solve this constraint in terms of c_s, we get the theoretical constraint[4]

$$c_s > \left(\frac{H^2}{8\pi^2 c_s M_{\rm Pl}^2\epsilon}\,\pi^2\right)^{1/4} = \left(\mathcal{A}_s\pi^2\right)^{1/4} \simeq 0.012\,, \tag{5.58}$$

where in the current effective field theory formalism

$$\mathcal{A}_s \equiv \frac{H^2}{8\pi^2 c_s M_{\rm Pl}^2\epsilon}\,. \tag{5.59}$$

Of course, a better estimate of the scale Λ can be found by computing the exact energy scale where the scattering of π loses perturbative unitarity, i.e. when higher order terms in the loop perturbative expansion become more important than the lower ones, signalling the breakdown of the loop expansion. This leads to the following estimate for the cut-off scale (see e.g. [76, 137])

$$\Lambda^2 = \sqrt{16\pi^2 M_{\rm Pl}^2|\dot{H}|\frac{c_s^5}{1-c_s^2}}\,, \tag{5.60}$$

which, for the pathological limit $c_s \to 0$, gives the following constraint on c_s:

$$c_s > \left(\frac{1}{2}\frac{H^2}{8\pi^2 c_s M_{\rm Pl}^2\epsilon}\right)^{1/4} = (\mathcal{A}_s/2)^{1/4} \simeq 0.006\,, \tag{5.61}$$

which differs from (5.58) only for a factor 2.

5.5 Constraints from observations

We can use the latest experimental constraints on $f_{\rm NL}$ from the *Planck* mission (see Eq. (4.40)) to make constraints in the parameter space of this effective field theory, as it can be shown in Fig. 5.1. In particular, after marginalizing over the parameter $\tilde{c}_3$, we can

[4] Here, we use $\mathcal{A}_s$ as tabulated in Tab. 4.1.

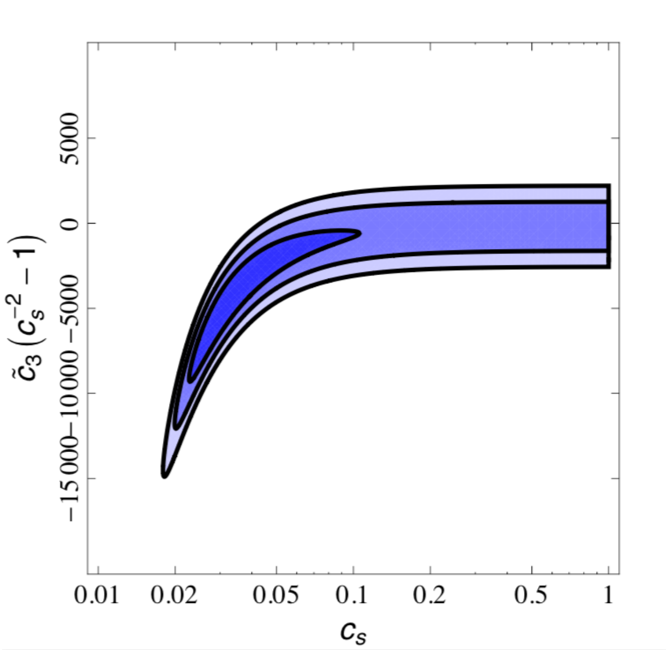


Figure 5.1. 68%, 95%, and 99.7% confidence regions in EFT single field inflation parameter space $(c_s, \tilde{c}_3)$ [62].

find the following bound on the speed of sound of scalar fluctuations

$$c_s \geqslant 0.021\,, \qquad 95\%\text{CL } (T{+}E)\,. \tag{5.62}$$

Since the EFTI action (5.38) describes the leading interaction terms of all single-field models of inflation, with the constraints in Fig. 5.1 we are able to put bounds on different specific models of inflation at the same time. For example, in the so-called *DBI scenario* (see e.g. [138, 139]) corresponding to $\tilde{c}_3 = 3(1 - c_s^2)/2$, we get the following constraints for c_s [62]

$$c_s^{\text{DBI}} \geqslant 0.086\,, \qquad 95\%\text{CL } (T{+}E)\,. \tag{5.63}$$

The same happens for many other models (that we will not review here). The most recent observational constraints, where the EFTI methods have been used, can be found in the recent *Planck* paper on non-Gaussianites [62].

5.6 Considerations and extensions

What we have seen in this chapter is a powerful method to generalize the single field slow-roll model of inflation in the most possible model independent way. In fact, in the action (5.38), we can "switch on or off" some particular operators in order to recover various single field models of inflation. This is probably the most important point, because it allows us to generically study very different models with a unifying formalism. Experiments will put bounds on the size of the various operators (for example with measurements of non-Gaussianities of primordial perturbations), that generically describe deviations from the standard scenario. In some sense, this is similar to what one does in particle physics, where we put constraints on the size of the operators describing deviations from the Standard Model, constraining in this way the effects of new physics. Moreover, it is possible to clearly determine the regime of validity of any new term and where an UV completion of the theory is required. As we have seen in Sec. 5.4, this typically leads to theoretical constraints over the new parameters of a specified theory. Another interesting aspect is that in evaluating the new EFT operators in terms of the perturbations, we can both work in gauge (2.16) and (2.18). In fact, even if the EFT expansion is defined in unitary gauge (where the inflaton field is "eaten" by the metric), in Sec. 5.2 we have seen that we can perform the "Stuekelberg trick", going back to the gauge where a scalar field π (which turns out to be the inflaton field, apart for field redefinition) explicitly reappears in the action (and disappears from the metric).

Here, in introducing this method, we have focused only on the scalar perturbation case, but analogous arguments arise when one is interested in computing the EFTI modifications on the statistics involving tensor perturbations. In particular, in this last case, we can allow for parity breaking terms in the action (while, considering scalar perturbations only, we could not build parity breaking operators). Thus, we can add the following rule, generalizing the list of allowed terms in Sec. 5.1, in order to allow for parity breaking terms:

- We can include odd powers of the Levi-Civita covariant pseudo-tensor $\epsilon^{\mu\nu\rho\sigma}/\sqrt{-g}$ and the projected Levi Civita pseudo-tensor $n_\mu \epsilon^{\mu\nu\rho\sigma}/\sqrt{-g} = \epsilon^{0\nu\rho\sigma}/\sqrt{-h}$ which is invariant under space diffeomorphisms.

If we implement this additional condition, then the most general parity breaking action we can build (stopping at quadratic level in the perturbations) reads

$$\begin{aligned} S_{\not{P}} = \int d^4x \, [&\alpha(t)\, \epsilon^{\mu\nu\rho\sigma}\, n_\mu \left(\nabla_\nu \delta K_{\rho\alpha}\right) \delta K^\alpha_\sigma \\ &+\beta(t)\, \epsilon^{\mu\nu\rho\sigma} \delta R_{\mu\nu}{}^{\lambda\alpha}\, \delta R_{\rho\sigma\lambda\alpha} + \gamma(t)\, \epsilon^{\mu\nu\rho\sigma}\, n_\mu \left(\nabla^\alpha \delta K^\lambda_\nu\right) \delta R_{\rho\sigma\lambda\alpha} + \delta(t)\, \epsilon^{\mu\nu\rho\sigma}\, n_\mu\, \delta K^\lambda_\nu \left(\nabla^\alpha \delta R_{\rho\sigma\lambda\alpha}\right) \\ &+...]\,, \end{aligned} \tag{5.64}$$

where the dots stand for higher order terms in derivatives and perturbations.

In the first line, there is the only cubic derivative term one can build, while starting from the second line there are four derivatives terms. Contrary to the parity preserving case, we can not build any term with less than 3 derivatives. However, notice that the term in the second line given by the contractions of the 2 Riemann tensors and the Levi Civita pseudo-tensor is a topological term that can be rewritten as

$$\epsilon^{\mu\nu\rho\sigma} \delta R_{\mu\nu}{}^{\lambda\alpha}\, \delta R_{\rho\sigma\lambda\alpha} = \partial_\mu A^\mu \,, \tag{5.65}$$

where we introduced the current

$$A^\mu = 2\epsilon^{\mu\alpha\beta\gamma} \left(\frac{1}{2}\, \delta\Gamma^\xi_{\alpha\nu}\, \partial_\beta \delta\Gamma^\nu_{\gamma\xi} + \frac{1}{3}\, \delta\Gamma^\xi_{\alpha\nu}\, \delta\Gamma^\nu_{\beta\eta}\, \delta\Gamma^\eta_{\gamma\xi} \right) . \tag{5.66}$$

Therefore, the derivative ∂_μ can be integrated by parts and moved to the coupling function $\beta(t)$, yielding to an effective cubic derivative term on cosmological perturbations. Moreover, notice that the quantity A^0 alone is invariant under a spatial diffeomorphism, thus we can include it separately in the action as an additional three derivatives term. So, it is more appropriate to rewrite action (5.64) as

$$S_{\not{\mathcal{P}}} = \int d^4x \; \big[\alpha(t)\, \epsilon^{\mu\nu\rho\sigma} n_\mu \left(\nabla_\nu \, \delta K_{\rho\alpha}\right) \delta K^\alpha_\sigma - \left(\partial_\mu \beta_1(t)\right) A^\mu + \beta_2(t)\, A^0 \\ + \gamma(t)\, \epsilon^{\mu\nu\rho\sigma}\, n_\mu \left(\nabla^\alpha \delta K^\lambda_\nu\right) \delta R_{\rho\sigma\lambda\alpha} + \delta(t)\, \epsilon^{\mu\nu\rho\sigma}\, n_\mu\, \delta K^\lambda_\nu \left(\nabla^\alpha \delta R_{\rho\sigma\lambda\alpha}\right) \\ + ...\big] \, . \tag{5.67}$$

This is the most general EFT action we can build to describe parity breaking physics in the primordial perturbations during inflation.[5] Starting from the following chapter, we will focus mainly in the Chern-Simons operator (5.65), which is the only term being explicitly invariant under a full diffeomorphism of the metric, and arises from a slightly different effective field theories approach describing modifications in the gravity sector during inflation (like e.g. [6, 143]).

[5]In literature, in the context of quantum Horava-Lifschitz gravity, it is usually considered as a 3 derivatives parity breaking term also the following 3D Chern-Simons operator [140, 141]

$$-4 \int d^4x \, \epsilon^{ijk} \left(\frac{1}{2}\, {}^{(3)}\Gamma^p_{iq} \partial^{(3)}_j \Gamma^q_{kp} + \frac{1}{3}\, {}^{(3)}\Gamma^p_{iq}\, {}^{(3)}\Gamma^q_{jr}\, {}^{(3)}\Gamma^r_{kp} \right) , \tag{5.68}$$

where ${}^{(3)}\Gamma^k_{ij}$ is the Christoffel symbol computed with the 3D metric. However, it is possible to show that this term does not add new physics in our EFT context, because it can be rewritten in terms of the 3 derivatives operators in (5.67) (see Ref. [142]).

Part II
Parity breaking in the gravity sector from the primordial Universe

In the first part of this work, we have seen that slow-roll models of inflation, the actual inflationary paradigm, describe successfully many observed features of the Universe, its homogeneity, flatness, and in particular the origin of the first curvature (density) perturbations. In most of such models, Einstein gravity is usually assumed to describe the theory of gravity. However, because of the possibly very high energies underlying the inflationary dynamics, it might be that remnant signatures of modification to Einstein gravity are left imprinted in the inflationary quantum fluctuations. In literature, the examples are different, from the first model of inflation [123], based on R^2-higher order gravitational terms, to more recent scenarios, such as, e.g., [6, 45, 59, 74, 143–167].[6] In such modified gravity models of inflation, we can have modifications in both the background dynamics and the evolution of primordial perturbations. In particular, production of primordial non-Gaussianities is an interesting feature of these models. In fact, we have seen that in the simplest models of inflation with standard gravity, the non-linearity parameter $f_{\rm NL}$, which measures the level of non-Gaussianity in the primordial curvature perturbations, is proportional to slow-roll parameters (ϵ, η). Thus, in the standard scenario, primordial non-Gaussianity is highly suppressed. Instead, in a modified gravity model we can have enhancement of non-Gaussianities (see e.g. [150] and Refs. therein). This is due to the fact that a modification of Einstein gravity can bring new degrees of freedom as well as new interactions among fundamental fields of the theory.

In this second part, we will focus on modifications in slow-roll models of inflation provided by Chern-Simons gravity, which is the highest order parity breaking operator in the gravity sector that arises from EFT modifications of Einstein gravity. In the literature, effects of this kind of gravity on gravitational waves in an inflationary context have been studied for the first time in Ref. [177], and in Ref. [178] more details have been elaborated on more general aspects of the theory, while more recent works include Refs. [45, 158–161, 179–182]. In particular, Chern-Simons gravity is obtained adding the so-called Chern-Simons gravitational term coupled to the inflaton field in the action of the slow-roll inflationary models. This term can be written in terms of the Riemann

[6]For more general inflationary frameworks, see also [142, 168–175]. For a review on inflationary gravitational waves containing other examples, see Ref. [176].

tensor as

$$S_{\not{P}} = \int d^4x \left[f(\phi)\, \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{\kappa\lambda} R_{\rho\sigma\kappa\lambda} \right] \,, \tag{5.69}$$

where $f(\phi)$ is a generic function of the scalar inflaton field ϕ only, $\epsilon^{\mu\nu\rho\sigma}$ is the total antisymmetric Levi-Civita pseudo-tensor (with $\epsilon^{1230} = 1$), $R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor. As we have seen in Sec. 5.6, this operator naturally arises from an effective field theory generalization of inflation. In Eq. (5.69) one can replace the Riemann tensor with the Weyl tensor, Eq. (6.3), without introducing any modification to the action (see, e.g. [6, 183]). For this reason, the Chern-Simons gravitational term belongs to the so-called Weyl-square terms, typically abbreviated with the symbol $\tilde{C}C$. The form of the coupling function $f(\phi)$ could be determined a priori by a quantum theory of gravity, which is still incomplete for the moment. For this reason, the coupling can be taken totally general. A fundamental result is the polarization of primordial gravitational waves (PGW) into chiral-eigenstates, the so-called left (L) and right (R) polarization states and the possibility to detect such parity breaking signatures looking at CMB polarization. The reason is that a parity transformation changes the chirality of PGW, and so, in a theory in which parity is broken, we expect a difference in the dynamical evolution of these chiral-eigenstates.

There are several observational studies on parity breaking signatures of PGW (and on cosmological birefringence) using CMB data [184–202]. At present, CMB power spectra are not able to efficiently constrain the level of parity breaking. Moreover, it has been shown recently, on very general ground, that even from future CMB data it will be almost hopeless to constrain the chirality of PGW exploiting only the two-point correlation statistics, specifically the cross-correlation of B polarization modes with temperature T anisotropies and E polarization modes [203]. Only in the most optimistic cases some weak constraints might be placed. Therefore, it has become even more interesting and crucial to ask whether relevant parity breaking signatures can arise in higher-order correlators.

Thus, motivated by this possibility, we will investigate parity breaking signatures in primordial bispectra that come from the term (5.69). We will analyze the tensor fluctuations (gravitational waves) bispectra $\langle\gamma\gamma\gamma\rangle$ and their mixed correlators with the scalar curvature perturbation ζ, $\langle\gamma\zeta\zeta\rangle$ and $\langle\gamma\gamma\zeta\rangle$. Analyzing the scalar bispectrum $\langle\zeta\zeta\zeta\rangle$ is not interesting for our goals, because it is insensitive to parity breaking signatures, being sourced only by scalar fields. In particular, we will focus on the bispectrum $\langle\gamma\gamma\zeta\rangle$. In fact, we will show that contributions to other bispectra are highly suppressed. The first part of this investigation is the topic of Ref. [204] (Chaps. 6-7). In addition, we will present a forecast on the possibility to observe such a signal with CMB bispectra statistics. This forecast is the subject of Ref. [205] (Chap. 8).[7]

[7]The scientific results presented in this part contain some differences and original considerations with respect to the content of the corresponding published papers [204, 205]. These are the results of scientific discussions having during a research period of 4 months at the University of Amsterdam.

Chapter 6

Chiral gravitational waves production

In this chapter, we see the origin of Chern-Simons modified gravity from EFT of gravity, we specify the computational conventions adopted to derive our results, and describe in which sense, at linear level, gravitational waves acquire chirality.

6.1 Chern-Simons gravity from EFT of gravity

The Hilbert-Einstein (H-E) action describing standard gravity is built by admitting covariant terms with a maximum number of two derivatives of the metric tensor. The only covariant term which obeys to this constraint is the scalar curvature R and the result is the well-known density Lagrangian

$$\mathcal{L}_{\mathrm{HE}} = \sqrt{g}\left(\frac{M_{\mathrm{Pl}}^2}{2}R\right) . \tag{6.1}$$

A standard way to modify Einstein gravity is to relax this condition and consider a more general theory of gravity in which the action is built with an expansion in series of covariant terms that contains an increasing number of derivatives of the metric tensor. The first correction to the Lagrangian (6.1) is obtained by considering the most general covariant terms with four derivatives of the metric tensor. We build these terms doing tensor contractions between two tensors with two derivatives of the metric tensor. Such tensors are the fundamental curvature tensors of general relativity: the Riemann tensor $R_{\mu\nu\rho\sigma}$, the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R. Then, the most general expression for the additional Lagrangian we want to focus on is

$$\Delta\mathcal{L} = \sqrt{g}\left(f_1 R^2 + f_2 R_{\mu\nu}R^{\mu\nu} + f_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) + f_4 \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^{\kappa\lambda} R_{\rho\sigma\kappa\lambda} + \dots . \tag{6.2}$$

where the dots stand for terms with more than 4 derivatives and f_n are some arbitrary dimensionless coefficients. From Lagrangian (6.2) we see that the Chern-Simons term (the one which multiplies f_4) is appearing naturally by this (effective field theory) expansion.

It is convenient to rewrite the Lagrangian (6.2) in terms of the Weyl tensor $C_{\mu\nu\rho\sigma}$ which is the traceless part of the Riemann tensor. This because we can understand better some properties of the Chern-Simons term that will be useful to perform the computations later on. The Weyl tensor is defined by

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}\right) + \frac{R}{6}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}\right) . \tag{6.3}$$

Thus, by doing a simple redefinition of the coefficients f_n, Lagrangian (6.2) becomes

$$\Delta\mathcal{L} = \sqrt{g}\left(f_1 R^2 + f_2 R_{\mu\nu}R^{\mu\nu} + f_3 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right) + f_4 \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu}{}^{\kappa\lambda} C_{\rho\sigma\kappa\lambda} . \tag{6.4}$$

This if we consider only the metric tensor $g_{\mu\nu}$. If we also introduce a scalar field ϕ which plays the role of the inflaton field we should include also covariant terms up to four derivatives of the inflaton field itself. The result is the full Lagrangian (see e.g. Ref. [6])

$$\begin{aligned}
\mathcal{L} = \sqrt{g}\Bigg[&\frac{1}{2}M_{\mathrm{Pl}}^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial^\mu\phi - V(\phi) + \\
&+ f_3(\phi)\Big(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big)^2 + f_4(\phi)g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\Box\phi + f_5(\phi)\Big(\Box\phi\Big)^2 + f_7(\phi)R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \\
&+ f_8(\phi)R\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + f_9(\phi)R\,\Box\phi + f_{10}(\phi)R^2 + f_{11}(\phi)R^{\mu\nu}R_{\mu\nu} + f_{12}(\phi)C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\Bigg] + \\
&+ f_{13}(\phi)\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu}{}^{\kappa\lambda}C_{\rho\sigma\kappa\lambda} ,
\end{aligned} \tag{6.5}$$

where we allow the various dimensionless coefficients f_n to depend on the inflaton field. In the first line, we recognize the standard action of slow-roll models of inflation. The Chern-Simons term coupled to the inflaton field stays in the last line. Since we are interested only in signatures of parity breaking modifications of Einstein gravity during the inflationary epoch, we restrict to the following Lagrangian

$$\mathcal{L} = \sqrt{g}\left[\frac{1}{2}M_{\mathrm{Pl}}^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] + f(\phi)\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu}{}^{\kappa\lambda}C_{\rho\sigma\kappa\lambda} . \tag{6.6}$$

The Chern-Simons term, in addition to introducing parity breaking signatures in the gravity sector, has other interesting peculiarities. First of all, this term is zero if computed on the background metric (the background metric is the Friedmann-Robertson-Walker (FRW) metric, which is conformally invariant and thus the Weyl tensor is zero in FRW). As a consequence, the Chern-Simons term does not modify the background dynamics of inflation, but modifies only the statistics of primordial perturbations. In addition, it is invariant under a Weyl transformation of the metric. This fact will also be useful in performing the computations. As a final consideration, we want to emphasize that the Chern-Simons term is a total derivative term (as we have shown in Eq. (5.65)); for this reason the coupling with the inflaton $f(\phi)$ is necessary to achieve a non trivial signature in the theory. Therefore, Chern-Simons gravity could have left signatures only on primordial perturbations, since at the end of the inflationary epoch the inflaton decays and the Chern-Simons term becomes a surface term, restoring standard gravity. We end this chapter by remarking that this term is expected to be the highest order parity breaking correction to Einstein gravity, or, better to say, it is the covariant parity breaking term with the lowest number of derivatives. In fact, as explained in [6], within an effective field theory approach more derivatives has a certain operator, less is expected to be important in the EFT expansion. In literature, also parity breaking operators with more than 4 derivatives have been studied. For instance, in Refs. [164–166] possible parity violating signatures in graviton non-Gaussianities due to a Weyl-cubic parity breaking term $\tilde{C}C^2$ (which contains 6 derivatives) have been considered, however reveling in this particular case a negligible contribution.

6.2 Gauge fixing

The gauge in which we will work is the spatially flat gauge (2.16) and with the metric in the ADM form (2.9). We remind that in this gauge, besides the fields N and N_i, we remain with only the scalar perturbation of the inflaton field $\delta\phi$, and the transverse and traceless modes γ_{ij} inside the spatial part of the metric tensor

$$h_{ij} = a^2[\delta_{ij} + \gamma_{ij} + \gamma_{il}\gamma_{lj} + ...] \, , \; \gamma_i{}^i = 0 \, , \; \partial^i\gamma_{ij} = 0 \; . \tag{6.7}$$

As said in Chap. 2, N and N_i are auxiliary fields in standard gravity, which means that they can be removed by solving the corresponding equations of motion and substituting the solutions back into the action. However, when we introduce a modified gravity term in the action, we have to take care of possible modifications of the equations of motion of such fields. In our case, we are interested to perform an analysis of the bispectrum of primordial perturbations. In order to do this, it is necessary to expand the action of the theory until third order in cosmological perturbations. As discussed in Chap. 2, in this case we need to know only the first order values of N and N_i. In gauge (2.16), the standard gravity solutions for auxiliary fields until first order in the perturbations read [39]

$$N^{(1)}_{\text{flat}} = \frac{\dot{\phi}_0}{2H}\delta\phi \; , \quad \psi^{(1)}_{\text{flat}} = -a^2\frac{\dot{\phi}_0{}^2}{2H^2}\partial^{-2}\left[\frac{d}{dt}\left(\frac{H\delta\phi}{\dot{\phi}_0}\right)\right] \; , \quad N^{i(1)}_{T,\text{flat}} = 0 \; . \tag{6.8}$$

Now, let us discuss briefly if, in the case gravity is described by action (6.6), modifications w.r.t. standard gravity are introduced by the Chern-Simons term to the first order values of the fields N and N_i. We notice immediately that N is a scalar field and at first order it does not take any contribution from parity breaking terms. In addition, N_i can be splitted in a scalar perturbation mode ψ and a transverse vector perturbation $N^{i(1)}_T$. The Chern-Simons term can not modify at linear level the scalar perturbation mode ψ for the same reason as for the field N; instead, a priori, it can give a contribution to the transverse vector perturbation. In particular, there are two possibilities: the first possibility is that $N^{i(1)}_T$ remains an auxiliary field. However, at first order in the perturbations $\delta\phi$ and γ_{ij}, there is no way to build any non-trivial transverse vector. So, in this case, at first order we can take as usual $N_T^{i(1)} = 0$ during inflation (as it happens in standard gravity, see Eq. (6.8)). The second possibility is that the Chern-Simons term turns the field $N_T^{i(1)}$ into a dynamical field: however, as we said at the end of Sec. 2.1, vector perturbations in an inflationary context can be safety put to zero. Thus, for what we have stated, we can take for the fields N and ψ the same values as in Eq. (6.8) and put $N_T^{i(1)} = 0$.

Now, we have all the tools we need to study the dynamical evolution of primordial perturbations.

6.3 Signatures in primordial power-spectra

In this section, we review the results about parity breaking signatures at linear level in the primordial power-spectra, providing also some original considerations. This is also useful to clarify the conventions and the assumptions we adopted.

In the quadratic part of the Lagrangian, the Chern-Simons term does not provide any contribution to the inflaton perturbation. In fact, as we have stated in the previous section, there is no way to build parity breaking operators with only scalar perturbations. Instead, it is interesting to analyze the parity breaking effects on the power spectra of the transverse and traceless tensor perturbations γ_{ij} (i.e. PGW). Let us remember the following Fourier decomposition (2.56)

$$\gamma_{ij} = \int \frac{d^3k}{(2\pi)^2} \sum_{s=R/L} e^s_{ij}(\vec{k}) \gamma^s_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \,, \tag{6.9}$$

where we adopt circular left (L) and right (R) polarization states of PGW as made in Sec. 2.4. In addition, it is convenient to pass to the conformal time $d\tau = a^{-1}dt$ instead of adopting cosmological time t. In this way, the metric reduces in the form $g_{\mu\nu} = a^2\gamma_{\mu\nu}$, where $\gamma_{\mu\nu}$ is the perturbed Minkowski metric tensor. Since the Chern-Simons term is invariant in form under a Weyl transformation of the metric $g' = e^{-2w(x,t)}g$, it is sufficient to choose $w = \ln a$, to understand that we can compute the Chern-Simons term using the metric $\gamma_{\mu\nu}$. We achieve this metric simply setting $a = 1$ and substituting cosmological time t with conformal time τ.

Now, using the gauge (6.7), we compute the density Lagrangian (6.6) at second order in tensor perturbations. Because of the fact that the Weyl tensor is zero on the background, the correction to the quadratic action from the Chern-Simons term comes from the computation of the tensor contraction $f(\phi_0)\epsilon^{\mu\nu\rho\sigma} C_{\mu\nu}{}^{\kappa\lambda}|_T^{(1)} C_{\rho\sigma\kappa\lambda}|_T^{(1)}$, where ϕ_0 denotes the background value of the inflaton field. The indices (1) and T indicate that we need to compute the corresponding tensors at first order in tensor perturbations. For simplicity of notation, in the following we will use ϕ to refer to the background value of the inflaton instead of ϕ_0. In performing this computation, we can set $N = 1$ and $N_i = 0$ because we are not interested in scalar perturbations. The first order values of the components of the Weyl tensor read (computed taking the scale factor $a = 1$)

$$C_{0i0j}|_T^{(1)} = -\frac{1}{4}[\gamma_{ij}'' + \partial^2\gamma_{ij}]\,, \tag{6.10}$$

$$C_{0ijk}|_T^{(1)} = -\frac{1}{2}[\partial_j\gamma_{ik}' + \partial_k\gamma_{ij}']\,, \tag{6.11}$$

$$C_{ijkl}|_T^{(1)} = \frac{1}{2}[-\partial_i\partial_k\gamma_{il} + \partial_i\partial_l\gamma_{jk} + \partial_j\partial_k\gamma_{il} - \partial_j\partial_l\gamma_{ik}] - \frac{1}{4}[-\delta_{ik}\Box\gamma_{il} + \delta_{il}\Box\gamma_{jk} + \delta_{jk}\Box\gamma_{il} - \delta_{jl}\Box\gamma_{ik}]\,, \tag{6.12}$$

where $\Box = \partial_\tau^2 + \partial^2$ and the prime denotes derivative with respect to the conformal time. Thus, the quadratic modification to the tensor action reads

$$\Delta S|_{\gamma\gamma} = \int d^4x\, \epsilon^{ijk}\, f(\phi)\frac{\partial}{\partial\tau}\left[(\gamma_{il}')(\partial_j\gamma_k'^{\,l}) - (\partial_r\gamma_{il})(\partial_j\partial^r\gamma_k{}^l)\right]\,, \tag{6.13}$$

where the Latin contractions are made with the Dirac delta. We can integrate the conformal time derivative by parts, obtaining

$$\Delta S|_{\gamma\gamma} = -\int d^4x\, \epsilon^{ijk}\, f'(\phi)\left[(\gamma_{il}')(\partial_j\gamma_k'^{\,l}) - (\partial_r\gamma_{il})(\partial_j\partial^r\gamma_k{}^l)\right]\,. \tag{6.14}$$

Thus, using the relations (2.59), we get the following new total action for the graviton modes

$$S|_{\gamma\gamma} = \sum_{s=L,R}\int d\tau\, \frac{d^3k}{(2\pi)^3}\, A_{T,s}^2\left[\, |\gamma_s'(\tau,k)|^2 - k^2|\gamma_s(\tau,k)|^2\,\right]\,, \tag{6.15}$$

where

$$A_{T,s}^2 = \frac{M_{\rm Pl}^2}{2}a^2\left(1 - 8\lambda_s\frac{k}{a}\frac{\dot f(\phi)}{M_{\rm Pl}^2}\right) = a^2\left(1 - \lambda_s\frac{k_{\rm phys}}{M_{\rm CS}}\right) \tag{6.16}$$

and

$$M_{\rm CS} = \frac{M_{\rm Pl}^2}{8\dot f(\phi)} \tag{6.17}$$

is the so called Chern-Simons mass (the dot represents derivative with respect to cosmological time). Let us recall that the coefficient λ_s is $+1$ for R polarization modes and -1 for L modes. Thus, there are some values of the physical wave number $k_{\rm phys} = k/a$ for which the factor $A_{T,R}^2$ becomes negative. In particular, from Eq. (6.16), this happens for $k_{\rm phys} > M_{\rm CS}$. The R modes with physical wave-numbers larger than the $M_{\rm CS}$ acquire a negative kinetic energy becoming unstable. Similar instabilities at quantum level and at high energies can be very problematic (see e.g. [206]). In order to avoid this problem, we set an ultra-violet cut-off $\Lambda < M_{\rm CS}$ to the theory. This cut-off is well motivated within an effective field theory approach (Λ can be taught as a scale of new physics). Since we want to study the statistics of perturbations that go outside the Hubble horizon during inflation, we have to assume $\Lambda > H$ (this forces $M_{\rm CS} > H$).

Now, instead of deriving the equations of motion for the fields γ_s, it is more convenient to make before the field redefinition

$$\mu_s = A_{T,s}\gamma_s \ . \tag{6.18}$$

Following the same steps of Sec. 2.4, we compute the new effective potential term for the fields μ_s which, at leading order in slow-roll, turns out to be

$$\frac{A''_{T,s}}{A_{T,s}} = \frac{2}{\tau^2}\left(1 - \frac{\lambda_s}{2}k\tau\frac{H}{M_{\rm CS}}\mathcal{A}\right) = \frac{2}{\tau^2}\left(1 + \frac{\lambda_s}{2}\frac{k_{\rm phys}}{H}\frac{H}{M_{\rm CS}}\mathcal{A}\right) . \tag{6.19}$$

In eq. (6.19) we have defined the following quantity

$$\mathcal{A} = \frac{1}{(1-\lambda_s k_{\rm phys}/M_{\rm CS})^2}\left\{\left[1-\xi+\frac{\omega}{2}-\frac{\xi}{2H\tau}\right]\left(1-\lambda_s\frac{k_{\rm pyhs}}{M_{\rm CS}}\right) - \lambda_s\frac{k_{\rm phys}}{2M_{CS}}\left[\frac{1}{2}+\xi+\frac{\xi^2}{2}\right]\right\} , \tag{6.20}$$

where [204]

$$k_{\rm phys} = \frac{k}{a} , \tag{6.21}$$

$$\xi = \frac{\dot{M}_{\rm CS}}{M_{\rm CS}H} , \tag{6.22}$$

and

$$\omega = \frac{\ddot{M}_{\rm CS}}{M_{\rm CS}H^2} , \tag{6.23}$$

where the dots stand for derivatives with respect to cosmological time t. The new equations of motion read

$$\mu''_s(\tau,k) + \left(k^2 - \frac{A''_{T,s}}{A_{T,s}}\right)\mu_s(\tau,k) = 0 . \tag{6.24}$$

Now, we assume for simplicity that the Chern-Simons mass is approximately constant in time during inflation, i.e. $\xi = \omega \simeq 0$. In this case, the effective potential simplifies into

$$\mu''_s + \left(k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \lambda_s\frac{k}{\tau}\frac{H}{M_{\rm CS}}\right)\mu_s = 0 \ , \tag{6.25}$$

where we recall that $\nu_T = \frac{3}{2} + \epsilon$, and ϵ is the slow-roll parameter (1.48).

Eq. (6.25) differs by the one of standard slow-roll models of inflation (2.70) by an additional term in the effective mass (proportional to τ^{-1}) which gives a correction to the sound of speed of the μ_s fields. In particular, we get the following new sound of speed:

$$c^2_{R/L} = 1 \pm \frac{1}{k\tau}\frac{H}{M_{\rm CS}} . \tag{6.26}$$

We can notice that the L-handed graviton field acquires a superluminar velocity ($\tau < 0$), signaling the breaking of Lorentz invariance in this theory. Now, as usual, we canonically quantize the fields $\mu_{R,L}$ in creation and annihilation operators as

$$\hat{\mu}_s(k,\tau) = u_s(k,\tau)\hat{a}_s(\vec{k}) + u_s^*(k,\tau)\hat{a}_s^\dagger(-\vec{k}) \ . \tag{6.27}$$

Thus, the equations of motion for the mode functions $u_s(k,\tau)$ are

$$u_s'' + \left(k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \lambda_s \frac{k}{\tau}\frac{H}{M_{\rm CS}}\right) u_s = 0 \ . \tag{6.28}$$

Eq. (6.28) is the so-called Whittaker equation whose exact solution with the Bunch Davies initial condition reads (see e.g. [160])

$$u_s(k,\tau) = 2\sqrt{\frac{(-k\tau)^3}{k}} e^{-i(\frac{\pi}{4} - \pi\nu_T/2)}\ e^{-ik\tau}\ U\left(\frac{1}{2} + \nu_T - \lambda_s \frac{H}{M_{\rm CS}}, 1 + 2\nu_T, 2ik\tau\right)\ e^{+\frac{\pi}{4}\lambda_s H/M_{\rm CS}} \ , \tag{6.29}$$

where U is the confluent hypergeometric function [207]. On super-horizon scales (i.e. $z = -k\tau \ll 1$) the solution (6.29) simplifies, becoming

$$u_s(k,\tau)_{z\ll 1} = \sqrt{\frac{1}{2k^2\tau^2 k}} e^{i(-\frac{\pi}{4} + \frac{\pi}{2}\nu_T)} \frac{\Gamma(\nu_T)}{\Gamma(3/2)}\left(\frac{-k\tau}{2}\right)^{3-2\nu_T}\quad e^{+\frac{\pi}{4}\lambda_s H/M_{\rm CS}} \ . \tag{6.30}$$

Now, we can compute the expected super-horizon power spectra of each polarization mode. We have

$$P_T^L = \langle 0|\hat{\gamma}_{ij}^L(\vec{k})\hat{\gamma}_L^{ij}(\vec{k}')|0\rangle = 2\frac{|u_L(z)_{z\ll 1}|^2}{A_{T,L}^2} \ , \tag{6.31}$$

$$P_T^R = \langle 0|\hat{\gamma}_{ij}^R(\vec{k})\hat{\gamma}_R^{ij}(\vec{k}')|0\rangle = 2\frac{|u_R(z)_{z\ll 1}|^2}{A_{T,R}^2} \ . \tag{6.32}$$

At leading order in the slow-roll parameters we find

$$P_T^L = \tfrac{P_T}{2} e^{-\frac{\pi}{4}H/M_{\rm CS}} \ , \tag{6.33}$$
$$P_T^R = \tfrac{P_T}{2} e^{+\frac{\pi}{4}H/M_{\rm CS}} \ , \tag{6.34}$$

where

$$P_T = \frac{4}{k^3}\frac{H^2}{M_{\rm Pl}^2}\left(\frac{z}{2}\right)^{3-2\nu_T} \tag{6.35}$$

is the standard gravity total tensor power spectrum in the standard slow-roll models without the Chern-Simons correction. As we explained above, the dimensionless coefficient $H/M_{\rm CS}$ has to be smaller than 1 due to the energy cut-off. However, a scenario in which $H \lesssim \Lambda \lesssim M_{\rm CS}$ is highly unnatural. It is more natural to assume an energy cut-off $\Lambda \gg H$, thus forcing $H/M_{\rm CS} \ll 1$. For this reason, we can expand in series the exponentials in (6.33) and take only the first order value. At the end, the relative difference between the power spectrum of right (R) and left (L) helicity states reads

$$\chi = \frac{P_T^R - P_T^L}{P_T^R + P_T^L} = \frac{\pi}{2}\left(\frac{H}{M_{\rm CS}}\right) \, . \tag{6.36}$$

This observable (that usually is called *chirality* of gravitational waves) quantifies the expected differences between the power spectrum of the R and L polarizations of the primordial gravitational waves at the end of inflation. We expect that its value is small (i.e. $\ll 1$) for the considerations made in developing the theory. Moreover, we expect that it is almost scale independent in a scenario where $M_{\rm CS}$ is almost constant in time during inflation. This scale independence drops when we consider a scenario where $M_{\rm CS}$ is time varying during inflation. In this case, instead of solving the exact equation of motion (6.24), it is more convenient to adopt an alternative method to compute the chirality of primordial gravitational waves.

This alternative approach relies in the In-In formalism method (see App. B), which allows to perturbatively estimate the corrections to primordial correlators induced by new small interaction terms in the Lagrangian. In particular, using the first order limit of Eq. (B.18) in Fourier space, we find the following correction to the tensor 2-point function:

$$\delta\langle\gamma_{R/L}(0,\vec{k})\gamma_{R/L}(0,\vec{k}')\rangle = -i\int_{-\infty}^{0} d\tau' \, \langle\gamma_{R/L}(0,\vec{k})\gamma_{R/L}(0,\vec{k}')H_{R/L}(p,\tau')\rangle \tag{6.37}$$

where

$$H_{R/L}(p,\tau) = \mp\int \frac{d^3p}{(2\pi)^3}\, a\frac{p}{M_{\rm CS}}\left[\, |\gamma'_{R/L}(\tau,p)|^2 - p^2|\gamma_{R/L}(\tau,p)|^2 \,\right] \, . \tag{6.38}$$

The choice of evaluation time $\tau = 0$ in (6.37) corresponds to the late-time limit on super-horizon scales. In the evaluation of integral (6.37), we can use in first approximation the de Sitter solutions for the mode-functions $\gamma_{R/L}(0,k)$ (Eq. (7.7)), $a = -1/(H\tau)$, and evaluate the parameters inside the integral at the time of horizon-crossing of the mode k.[1] The final result reads (see e.g. [142])

$$\delta\langle\gamma_{R/L}(0,\vec{k})\gamma_{R/L}(0,\vec{k}')\rangle \simeq \pm(2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')P_T\,\frac{\pi}{4}\left(\frac{H}{M_{\rm CS}}\right)^{*} , \tag{6.39}$$

where the star $*$ denotes the value of the parameters at the horizon crossing of the mode k. The chirality in this case approximately reads

$$\chi(k_*) \simeq \frac{\pi}{2}\left(\frac{H}{M_{\rm CS}}\right)^{*} \tag{6.40}$$

and acquires scale dependence, still remaining $\ll 1$ as it is proportional to the ratio $H/M_{\rm CS}$ (Notice that this result is a confirmation of (6.36) for the scale invariant case).

[1] see Ref. [39] for more details. These approximations are discussed in Chap. 7.

As we already mentioned, even in the most optimistic cases, only weak constraints on the parameter χ could be achieved from CMB power spectra (see e.g. Refs. [201, 203]). In particular, in Ref. [203], it is shown that ideally the 1σ detection of $\chi \neq 0$ is possible only if $\chi \gtrsim 0.4$ (which would require the unnatural condition $H \lesssim M_{\rm CS}$). The possibility to put constraints on the value of χ with experiments involving the direct detection of polarized primordial gravitational waves through interferometers regards only futuristic experiments (see e.g. Refs. [208, 209]). In fact, current interferometers are very far from being sensitive to the primordial gravitational waves amplitude as predicted by slow-roll models of inflation.[2] Moreover, in Refs. [211, 212], it was shown that three-dimensional informations, like the tidal imprints of primordial gravitational waves fossilized into the large-scale structure of the Universe, could be critical in the future to discerning the effects of parity violation in the primordial gravitational-wave background. In particular, in Ref. [212], it is claimed that, with futuristic galaxy surveys, constraints on primordial chirality at the percentage level could be derived. Unfortunately, the use of both interferometers or galaxy surveys to constrain the chirality of primordial gravitational waves is expected to be very futuristic.

Briefly speaking, in the model we have considered, the formation of a very small parity violation in the PGW power spectrum is expected. This might be very difficult to observe even with the next experiments involving the polarization of the CMB. Any other alternative experiment sensitive to the chirality of primordial gravitational waves is still futuristic. For this reason it is interesting to investigate if a significant parity breaking signature can arise from a higher order statistics of primordial perturbations, i.e. looking to the cubic interaction terms among fields. Thus, in the following chapter, we will study how the Chern-Simons term coupled to the inflaton affects the bispectra of primordial perturbations.

[2]This can be possible only in particular models of inflation where a significant blue tensor tilt is achieved (see the review [210]). On the contrary, Chern-Simons gravity only induces opposite corrections to the power spectrum of gravitons with opposite helicities. No significant modifications w.r.t. standard slow-roll scenario are provided to both the tensor tilt n_T and tensor-to-scalar-ratio r (from the prediction of Eq. (2.82), in the current model the amplitude of primordial gravitational waves is almost scale invariant). In Ref. [204], it was claimed that modifications of order χ^2 could have provided to both r and n_T. However, due to the smallness of χ, these modifications can be safely neglected.

Chapter 7

Signatures in primordial bispectra

In this chapter, we are interested to understand whether the effects of the new interaction terms coming from the Chern-Simons term can bring to a relevant parity breaking effect in the bispectra of the primordial perturbations. In the subsequent three sections, we will estimate the primordial three-point functions $\langle\gamma\gamma\gamma\rangle$ and $\langle\gamma\zeta\zeta\rangle$ and compute explicitly $\langle\gamma\gamma\zeta\rangle$. In fact, as we have anticipated in the introduction of this part, we will show that the parity breaking contribution to the correlator $\langle\gamma\gamma\zeta\rangle$ is the only one that is not suppressed as the power spectrum case. The readers who are not interested in the technical details of the computation of such three-point functions can go directly to Sec. 7.4 where we will provide our main results, including the explicit expression of $\langle\gamma\gamma\zeta\rangle$, and a discussion on its parity breaking signatures.

For parity invariance reasons, the only primordial bispectra that can arise from the Chern-Simons term are the ones associated to the three-point functions $\langle\gamma\gamma\gamma\rangle$, $\langle\gamma\gamma\delta\phi\rangle$ and $\langle\gamma\delta\phi\delta\phi\rangle$.

The master formula used to compute this kind of correlators at first perturbative order, in the quantum field theory ensamble, is provided by the In-In formalism formula switched in Fourier space (see App. B, Eq. (B.19))

$$\langle\delta_a(\vec{k_1})\delta_b(\vec{k_2})\delta_c(\vec{k_3})\rangle = -i\int_{t_0}^{t} dt'\langle 0|\Big[\delta_a(\vec{k_1},t)\delta_b(\vec{k_2},t)\delta_c(\vec{k_3},t)\ ,\ H_{\rm int}(t')\Big]|0\rangle\ , \tag{7.1}$$

where $H_{\rm int} = -L_{\rm int}$ is the cubic interaction Hamiltonian between fields δ_a, δ_b and δ_c, t_0 is the time at which this interaction is switched on and t is the time at which we evaluate the correlator. In the formula just introduced t is a totally arbitrary time coordinate; it will be very convenient for the computations to adopt the conformal time τ instead of the cosmological time. The interactions are switched on on sub-horizon scales, that correspond to the limit $\tau_0 = -\infty$; on the other hand we evaluate the correlator on super-horizon scales, thus in the limit $\tau = 0$. Now, we will start to evaluate the interaction Lagrangians for the bispectra in which we are interested in.

7.1 Three gravitons non-Gaussianities

This interaction comes from the following contribution to the Lagrangian

$$\mathcal{L}_{\rm int}^{\gamma\gamma\gamma} = f(\phi)\epsilon^{\mu\nu\rho\sigma}\left[C_{\mu\nu}{}^{\kappa\lambda}|_T^{(2)} C_{\rho\sigma\kappa\lambda}|_T^{(1)} + C_{\mu\nu}{}^{\kappa\lambda}|_T^{(1)} C_{\rho\sigma\kappa\lambda}|_T^{(2)}\right], \tag{7.2}$$

where the suffix T indicates that we have to evaluate the contribution of the tensor perturbations only and the index (n) means that we have to consider the n-th order of the perturbations contributing to the corresponding Weyl tensor. We will use an identical notation also for the other interaction vertices. As we can see, the explicit computation of this term requires to compute the Weyl tensor up to second order in the tensor perturbations. Instead of performing a direct computation, we can have an idea of the strength of this interaction vertex using only the tensorial properties of the Chern-Simons term. In fact, as we said above, the Chern-Simons term is a total derivative term for a constant f, so that we can write

$$\mathcal{L}_{\rm int}^{\gamma\gamma\gamma} = f(\phi)\partial_\mu A^\mu\,, \tag{7.3}$$

where A^μ is the four-vector (5.66).[1] As we said in Sec. 6.3, we can compute the Weyl tensor adopting the conformal time instead of cosmological time and setting the scale factor $a = 1$. The result is that A^μ has to depend only by γ_{ij} and derivatives of γ_{ij} and by no additional factors a or H. Thus, A^μ will be a cubic combination of tensor perturbations and their derivatives only. In particular, integrating by parts the Lagrangian (7.3), it follows

$$\mathcal{L}_{\rm int}^{\gamma\gamma\gamma} = -f'(\phi)A^0\,, \tag{7.4}$$

where, for our purposes, we can just evaluate the coupling $f(\phi)$ on the background. Here the prime $'$ refers to the derivative w.r.t. conformal time. If we put this interaction into formula (7.1), we find

$$\langle\gamma_{s_1}(\vec{k}_1)\gamma_{s_2}(\vec{k}_2)\gamma_{s_3}(\vec{k}_3)\rangle = i\int_{-\infty}^{0} d\tau' a\dot{f}(\phi)\langle 0|\left[\gamma_{s_1}(\vec{k}_1,0)\gamma_{s_2}(\vec{k}_2,0)\gamma_{s_3}(\vec{k}_3,0)\ ,\ A^0\right]|0\rangle\ , \tag{7.5}$$

where we have used the fact that $f'(\phi) = a\dot{f}(\phi)$.

The bra-ket contractions into Eq. (7.5) produces products of three Green functions of the type

$$\langle\gamma_{s_1}(\vec{p}_1,0)\gamma_{s_2}(\vec{p}_2,\tau')\rangle = (2\pi)^3\delta^{(3)}(\vec{p}_1+\vec{p}_2)\,u_{s_1}(p_1,0)u_{s_2}(p_2,\tau')\,, \tag{7.6}$$

with $u_s(k,\tau)$ given as in Eq. (6.29), apart for the contribution of field redefinition (6.18). As a first approximation, we can use the expression of u_s setting the slow-roll parameters and $H/M_{\rm CS}$ to be vanishing. This is justified by the fact that, under our assumptions,

[1] With the notation used in (7.3), we indicate that we will take the part of the Chern-Simons interaction that gives rise to a graviton cubic interaction only.

such parameters are much smaller than 1 during inflation, and thus we can expand in series the solution (6.29), taking the zero-th order value only. The solution becomes

$$u_s(k,\tau) = \frac{iH}{M_{\rm Pl}\sqrt{k^3}}(1+ik\tau)e^{-ik\tau} \;, \tag{7.7}$$

which is the solution for the mode function of a scalar field in a de Sitter space.

Then, substituting (7.7) into Eq. (7.5), we will arrive to an integral of the type

$$\langle\gamma_{s_1}(\vec{k_1})\gamma_{s_2}(\vec{k}_2)\gamma_{s_3}(\vec{k}_3)\rangle = i\int_{-\infty}^{0} d\tau' \left(-\frac{1}{H\tau}\right)\dot{f}(\phi)\left(\prod_i \frac{H^2}{M_{\rm Pl}^2 k_i^3}\right) f(k_i,\tau')e^{-ik_T\tau'} \;, \tag{7.8}$$

where $k_T = k_1+k_2+k_3$, and $f(k_i,\tau')$ is a function polynomial in the arguments. We have used $a = -1/(H\tau)$, a relation which holds apart for small slow-roll corrections. In particular, we can minimize the errors committed in evaluating the integral taking the value of parameters H and $\dot{f}(\phi)$ during the horizon crossing of the momentum $K = k_T$. It is possible to show that, after and much before horizon crossing, the integral in Eq. (7.8) vanishes (see Ref. [39] and also, e.g., [53] for more details). In fact, we know that much before horizon crossing the integrand function has a high oscillatory behaviour because of the imaginary exponential $e^{-ik_T\tau'}$, which mediates the integral to zero; moreover, after horizon crossing the tensor perturbations γ_{ij} become constant and then the commutator operator with the interaction Lagrangian in Eq. (7.5) becomes zero. Thus, we have

$$\langle\gamma_{s_1}(\vec{k_1})\gamma_{s_2}(\vec{k}_2)\gamma_{s_3}(\vec{k}_3)\rangle \simeq i\dot{f}_*(\phi)\left(\prod_i \frac{H_*^2}{M_{\rm Pl}^2 k_i^3}\right)\frac{1}{H_*}\int_{-\infty}^{0} d\tau' \left(-\frac{1}{\tau}\right) f(k_i,\tau')e^{-ik_T\tau'} \;, \tag{7.9}$$

where the $*$ refers to the time of horizon crossing of the momentum k_T. In the following, we will omit the $*$ for simplicity of notation. Integrals of the type in Eq. (7.9) can be computed by parts after passing to complex plane and performing a Wick rotation (see e.g. Refs. [39, 53, 71]). The result is an imaginary function which depends on the momenta k_i and has the physical dimensions of a M^3. In addition, from Eq. (6.17) , it follows

$$\dot{f}(\phi) = \frac{M_{\rm Pl}^2}{8M_{\rm CS}} \;. \tag{7.10}$$

Thus, if we multiply and divide Eq. (7.9) by $\sum_i k_i^3$, we find out the following estimation

$$\langle\gamma_{s_1}(\vec{k_1})\gamma_{s_2}(\vec{k}_2)\gamma_{s_3}(\vec{k}_3)\rangle \sim \frac{H}{M_{\rm CS}}\left(\sum_{i\neq j} P_T(k_i)P_T(k_j)\right) M(k_i)\,, \tag{7.11}$$

where $M(k_i)$ is a dimensionless function of the momenta k_i and $P_T(k)$ the power spectrum of tensor perturbations, Eq. (6.35). Due to momentum conservation, the function $M(k_i)$ has to be of order 1 and gives only the momentum shape of the bispectrum. As we can see from this final result, parity breaking in graviton non-Gaussianities are suppressed by the ratio $H/M_{\rm CS}$ which is $\ll 1$, exactly as the power spectrum case.

7.2 Two scalars and a graviton non-Gaussianities

Here, we are interested in the following interaction Lagrangian

$$\begin{aligned}\mathcal{L}_{\rm int}^{\delta\phi\delta\phi\gamma} = \epsilon^{\mu\nu\rho\sigma}\Big[&\frac{\partial f(\phi)}{\partial\phi}\delta\phi\ C_{\mu\nu}{}^{\kappa\lambda}|_S^{(1)}C_{\rho\sigma\kappa\lambda}|_T^{(1)} + \frac{\partial f(\phi)}{\partial\phi}\delta\phi\ C_{\mu\nu}{}^{\kappa\lambda}|_T^{(1)}C_{\rho\sigma\kappa\lambda}|_S^{(1)}\\ &+ f(\phi)C_{\mu\nu}{}^{\kappa\lambda}|_S^{(2)}C_{\rho\sigma\kappa\lambda}|_T^{(1)} + f(\phi)C_{\mu\nu}{}^{\kappa\lambda}|_T^{(1)}C_{\rho\sigma\kappa\lambda}|_S^{(2)}\\ &+ f(\phi)C_{\mu\nu}{}^{\kappa\lambda}|_{ST}^{(2)}C_{\rho\sigma\kappa\lambda}|_S^{(1)} + f(\phi)C_{\mu\nu}{}^{\kappa\lambda}|_S^{(1)}C_{\rho\sigma\kappa\lambda}|_{ST}^{(2)}\Big]\ ,\end{aligned} \tag{7.12}$$

where the suffix S means that we have to evaluate the contribution only of the scalar perturbations to the corresponding Weyl tensor, and the double suffix ST means that we have to evaluate the quadratic contribution in which both scalar and tensor perturbations appear.

In Eq. (7.12) the first two terms come from the expansion in series of the function $f(\phi)$ around the background value of the inflaton multiplied by the contraction of two Weyl tensors at first order in tensor perturbations. The other terms instead come from the contraction between the Weyl tensor at second order in scalar perturbations and the Weyl tensor at first order in tensor perturbations. Finally, there are also contributions coming from the contraction between the Weyl tensor at second order sourced by mixed scalar and tensor perturbations and the Weyl tensor at first order in scalar perturbations. In particular, in the spatially flat gauge, scalar perturbations appear only through first order expressions of the fields N and N_i. If we take these explicit expressions (Eq. (6.8)) and we use the definition of slow-roll parameter ϵ (1.48), it follows

$$N^{(1)} \sim \sqrt{\epsilon}\,\delta\phi, \qquad N_i^{(1)} \sim \sqrt{\epsilon}\,\delta\phi\ . \tag{7.13}$$

Thus, N and N_i are sub-dominant in the slow-roll hypothesis in comparison with the inflaton perturbation $\delta\phi$. For this reason in the slow-roll limit the terms dominant in Eq. (7.12) are the first two. These terms depend only by the Weyl tensor at first order and can be easily computed using (6.10) and the following first order scalar contribution to the Weyl tensor components (computed taking the scale factor $a = 1$):

$$\begin{aligned}C_{0i0j}|_S^{(1)} &= \frac{1}{2}[\partial_i\partial_j N^{(1)} - \frac{1}{3}\delta_{ij}\,\partial^2 N^{(1)}]\,,\\ C_{0ijk}|_S^{(1)} &= 0\,,\\ C_{ijkl}|_S^{(1)} &= \frac{1}{2}[\delta_{jl}\partial_i\partial_k - \delta_{jk}\partial_i\partial_l - \delta_{il}\partial_j\partial_k + \delta_{ik}\partial_j\partial_l]N^{(1)} - \frac{1}{3}[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}](\partial^2 N^{(1)})\,,\end{aligned} \tag{7.14}$$

where $N^{(1)}$ is as in (6.8). In the end, we obtain

$$\mathcal{L}_{\rm int}^{\delta\phi\delta\phi\gamma} = -8\frac{\partial f(\phi)}{\partial\phi}\sqrt{\epsilon}\ (\partial^l\delta\phi)\epsilon^{ijk}\Big[(\partial_k\delta\phi)\partial_i\gamma'_{lj}\Big]\ . \tag{7.15}$$

Using again (6.17), we can express

$$\frac{\partial f(\phi)}{\partial \phi} = \frac{M_{\rm Pl}^2}{8M_{\rm CS}\dot{\phi}} \ , \tag{7.16}$$

and, inserting this last equation into Eq. (7.15), we find

$$\mathcal{L}_{\rm int}^{\delta\phi\delta\phi\gamma} = -\frac{M_{\rm Pl}^2}{M_{\rm CS}\dot{\phi}}\sqrt{\epsilon}\ (\partial^l \delta\phi)\epsilon^{ijk}\Big[(\partial_k\delta\phi)\partial_i\gamma'_{lj}\Big] \ . \tag{7.17}$$

If we use (7.1), we can perform an estimation similar for what we have done for the three-graviton non-Gaussianities. The result is

$$\langle \delta\phi(\vec{k}_1)\delta\phi(\vec{k}_2)\gamma_s(\vec{k}_3)\rangle \sim \frac{H}{M_{\rm CS}}\left(\sum_{i\neq j}\Delta_T(k_i)\Delta_T(k_j)\right)F(k_i) \ , \tag{7.18}$$

where $F(k_i)$ is another dimensionless function of order 1. The result is that parity breaking signature in such a bipectrum is still suppressed by the ratio $H/M_{\rm CS}$.

7.3 One scalar and two graviton non-Gaussianities

The interaction Lagrangian which gives contributions to this bispectrum is

$$\begin{aligned}\mathcal{L}_{\rm int}^{\delta\phi\gamma\gamma} = \epsilon^{\mu\nu\rho\sigma}\Big[\left(\frac{\partial}{\partial\phi}f(\phi)\right)\delta\phi\ C^{(1)\ \kappa\lambda}_{\mu\nu}|_T C^{(1)}_{\rho\sigma\kappa\lambda}|_T + f(\phi)C^{(1)\ \kappa\lambda}_{\mu\nu}|_S C^{(2)}_{\rho\sigma\kappa\lambda}|_T + f(\phi)C^{(2)\ \kappa\lambda}_{\mu\nu}|_T C^{(1)}_{\rho\sigma\kappa\lambda}|_S \\ + f(\phi)C^{(1)\ \kappa\lambda}_{\mu\nu}|_T C^{(2)}_{\rho\sigma\kappa\lambda}|_{ST} + f(\phi)C^{(2)\ \kappa\lambda}_{\mu\nu}|_{ST} C^{(1)}_{\rho\sigma\kappa\lambda}|_T\Big] \ ,\end{aligned} \tag{7.19}$$

where the notations are the same of Eqs. (7.2), (7.12).

Here, the first term comes from the expansion in series of the function $f(\phi)$ around the background value of the inflaton multiplied by the contraction of two Weyl tensors at first order in tensor perturbations. Moreover, there are terms coming from the contraction between the Weyl tensor at second order in tensor perturbations and the Weyl tensor at first order in scalar perturbations. Finally there are some "mixed'" terms. In this case, contributions coming from the contraction between the Weyl tensor at second order sourced by mixed scalar and tensor perturbations and the Weyl tensor at first order in tensor perturbations appear. In analogy with the previous case, in the slow-roll hypothesis, the term dominant in Eq. (7.19) is the first one. Thus, making a straightforward computation, we obtain the following cubic Lagrangian in Fourier space:

$$\begin{aligned}L_{\rm int}^{\delta\phi\gamma\gamma}(\tau) = -\lambda_s \times \int d^3K\frac{\delta^{(3)}(\vec{k}+\vec{p}+\vec{q})}{(2\pi)^6}\Bigg\{\left(\frac{\partial f(\phi)}{\partial\phi}\right)p\ \delta\phi'(\vec{k})\Big[\gamma'^s_{ij}(\vec{p})\gamma'^{ij}_s(\vec{q}) + (\vec{p}\cdot\vec{q})\,\gamma^s_{ij}(\vec{p})\gamma^{ij}_s(\vec{q})\Big] \\ +a\left(\dot{\phi}\frac{\partial^2 f(\phi)}{\partial^2\phi}\right)p\ \delta\phi(\vec{k})\Big[\gamma'^s_{ij}(\vec{p})\gamma'^{ij}_s(\vec{q}) + (\vec{p}\cdot\vec{q})\,\gamma^s_{ij}(\vec{p})\gamma^{ij}_s(\vec{q})\Big]\Bigg\} \ ,\end{aligned} \tag{7.20}$$

where we have used the notation $\int d^3k\ d^3p\ d^3q = \int d^3K$ and a sum over the polarization index s is understood for simplicity of notation.

In this Lagrangian, we see that there are some interaction vertices that depend on the second derivative of the coupling $f(\phi)$ w.r.t. the inflaton field. These interactions might give a contribution which is not suppressed directly by the ratio $H/M_{\rm CS}$, as we have seen in the previous cases. In fact, in those cases, we have dealt with the first derivative of the coupling function $f(\phi)$ w.r.t. the inflaton. For this reason, we perform now a detailed computation, using the In-In formalism, in order to show that these new interaction vertices can bring a potentially relevant parity breaking signature to the bispectra statistics we are analysing. At the end, we will quantify if this signature is still highly suppressed or not. In the next steps we will omit the time argument $\tau = 0$ again for simplicity of notation.

First of all, we notice that, due to the form of the interaction Lagrangian (7.20), the only non-vanishing 2 gravitons-1 scalar correlators are

$$\langle \gamma_R(\vec{k_1})\gamma_R(\vec{k_2})\delta\phi(\vec{k_3})\rangle\,, \qquad \langle \gamma_L(\vec{k_1})\gamma_L(\vec{k_2})\delta\phi(\vec{k_3})\rangle\ . \tag{7.21}$$

In fact, it is straightforward to verify that

$$\langle \gamma_R(\vec{k_1})\gamma_L(\vec{k_2})\delta\phi(\vec{k_3})\rangle = 0\ . \tag{7.22}$$

We start from the computation of the correlator $\langle \gamma_R(\vec{k_1})\gamma_R(\vec{k_2})\delta\phi(\vec{k_3})\rangle$. The computation of the other correlator will be analogous.

Using Eq. (7.1), we have

$$\begin{aligned}\langle \gamma_R(\vec{k}_1)\gamma_R(\vec{k}_2)\delta\phi(\vec{k}_3)\rangle = -\frac{i}{(2\pi)^6}\int d^3K\ \delta^{(3)}(\vec{k}+\vec{p}+\vec{q})\ \int_{-\infty}^{0} d\tau' \Bigg[\frac{\partial f(\phi)}{\partial\phi}\Big(\mathcal{B}_1(\tau')+\mathcal{B}_2(\tau')\Big) \\ +a\ \dot{\phi}\frac{\partial^2 f(\phi)}{\partial^2\phi}\Big(\mathcal{B}_3(\tau')+\mathcal{B}_4(\tau')\Big)\Bigg]\ ,\end{aligned} \tag{7.23}$$

where

$$\mathcal{B}_1 = p\,\langle 0|\ \Big[\delta\phi(\vec{k_1},0)\gamma_R(\vec{k_2},0)\gamma_R(\vec{k_3},0)\ ,\ \delta\phi'(\vec{k},\tau')\gamma'^{R}_{ij}(\vec{p},\tau')\gamma'^{ij}_{R}(\vec{q},\tau')\Big]\ |0\rangle\,, \tag{7.24}$$

$$\mathcal{B}_2 = p\,(\vec{p}\cdot\vec{q})\,\langle 0|\ \Big[\delta\phi(\vec{k_1},0)\gamma_R(\vec{k_2},0)\gamma_R(\vec{k_3},0)\ ,\ \delta\phi'(\vec{k},\tau')\gamma^{R}_{ij}(\vec{p},\tau')\gamma^{ij}_{R}(\vec{q},\tau')\Big]\ |0\rangle\,, \tag{7.25}$$

$$\mathcal{B}_3 = p\langle 0|\ \Big[\delta\phi(\vec{k_1},0)\gamma_R(\vec{k_2},0)\gamma_R(\vec{k_3},0)\ ,\ \delta\phi(\vec{k},\tau')\gamma'^{R}_{ij}(\vec{p},\tau')\gamma'^{ij}_{R}(\vec{q},\tau')\Big]\ |0\rangle\,, \tag{7.26}$$

$$\mathcal{B}_4 = p\,(\vec{p}\cdot\vec{q})\,\langle 0|\ \Big[\delta\phi(\vec{k_1},0)\gamma_R(\vec{k_2},0)\gamma_R(\vec{k_3},0)\ ,\ \delta\phi(\vec{k},\tau')\gamma^{R}_{ij}(\vec{p},\tau')\gamma^{ij}_{R}(\vec{q},\tau')\Big]\ |0\rangle\,. \tag{7.27}$$

Here the symbol $\Big[\ \cdot\ ,\ \cdot\ \Big]$ denotes the commutator operator. We can compute these expressions using the Wick theorem. We need to compute preliminary the following contractions between fields:

$$\langle 0|\delta\phi(\vec{k},\tau)\delta\phi(\vec{k}',\tau')|0\rangle = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')\, u(k,\tau)u^*(k,\tau')\,, \tag{7.28}$$

$$\langle 0|\delta\phi(\vec{k},\tau)\delta\phi'(\vec{k}',\tau')|0\rangle = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')\, u(k,\tau)\frac{d}{d\tau}u^*(k,\tau')\,, \tag{7.29}$$

$$\langle 0|\gamma^R_{ij}(\vec{k},\tau)\gamma_R(\vec{k}',\tau')|0\rangle = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')\, u_R(k,\tau)u^*_R(k,\tau')\epsilon^R_{ij}(\vec{k})\,, \tag{7.30}$$

$$\langle 0|\gamma^R(\vec{k},\tau)\gamma^R_{ij}(\vec{k}',\tau')|0\rangle = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')\, u_R(k,\tau)u^*_R(k,\tau')\epsilon^{R*}_{ij}(\vec{k})\,, \tag{7.31}$$

$$\langle 0|\gamma^R_{ij}(\vec{k},\tau)\gamma'^R(\vec{k}',\tau')|0\rangle = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')\, u_R(k,\tau)\left(\frac{du^*_R(k,\tau')}{d\tau}\right)\epsilon^R_{ij}(\vec{k})\,, \tag{7.32}$$

$$\langle 0|\gamma'^R(\vec{k},\tau)\gamma^R_{ij}(\vec{k}',\tau')|0\rangle = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')\left(\frac{du_R(k,\tau)}{d\tau}\right)u^*_R(k,\tau')\epsilon^{R*}_{ij}(\vec{k})\,, \tag{7.33}$$

where $u(k,\tau)$ is the mode function of the inflaton perturbation and $u_s(k,\tau)$ is the one of the tensor perturbations. Thus, performing all the contractions, Eq. (7.23) becomes

$$\begin{aligned}\langle\gamma_R(\vec{k}_1)\gamma_R(\vec{k}_2)\delta\phi(\vec{k}_3)\rangle = &-i(2\pi)^3\delta^3(k_1+k_2+k_3)\ \mathrm{Im}\{[k_1(I_1+I_2)+k_1(\vec{k}_1\cdot\vec{k}_2)(I_3+I_4)]\\ &\times e^{R*}_{ij}(\vec{k}_1)e^{ij*}_R(\vec{k}_2)-c.c.\}+(\vec{k}_1\longleftrightarrow\vec{k}_2)\ ,\end{aligned} \tag{7.34}$$

where the I_n are the integrals

$$I_1 = u(k_1,0)u_R(k_1,0)u_R(k_2,0)\int_{-\infty}^0 d\tau'\left(\frac{\partial}{\partial\phi}f(\phi)\right)\left[\frac{d}{d\tau}u^*(k_1,\tau')\frac{d}{d\tau}u^*_R(k_1,\tau')\frac{d}{d\tau}u^*_R(k_2,\tau')\right]\,, \tag{7.35}$$

$$I_2 = u(k_1,0)u_R(k_2,0)u_R(k_3,0)\int_{-\infty}^0 d\tau' a\left(\frac{\partial}{\partial\phi}\dot{f}(\phi)\right)\left[u^*(k_1,\tau')\frac{d}{d\tau}u^*_R(k_2,\tau')\frac{d}{d\tau}u^*_R(k_3,\tau')\right]\,, \tag{7.36}$$

$$I_3 = u(k_1,0)u_R(k_2,0)u_R(k_3,0)\int_{-\infty}^0 d\tau'\left(\frac{\partial}{\partial\phi}f(\phi)\right)\left[\frac{d}{d\tau}u^*(k_1,\tau')u^*_R(k_2,\tau')u^*_R(k_3,\tau')\right]\,, \tag{7.37}$$

$$I_4 = u(k_1,0)u_R(k_2,0)u_R(k_3,0)\int_{-\infty}^0 d\tau' a\left(\frac{\partial}{\partial\phi}\dot{f}(\phi)\right)\left[u^*(k_1,\tau')u^*_R(k_2,\tau')u^*_R(k_3,\tau')\right]\,. \tag{7.38}$$

We can compute analytically these integrals with the same approximations we have already discussed above. In particular, the Hubble parameter H and the function $f(\phi)$ and its derivatives can be evaluated at the time of horizon crossing of momentum $k_T = \sum_i k_i$ and put outside the integral. The second approximation is about the cosmological evolution of the scale factor a. At leading order in slow-roll, we have $a \simeq -1/(H\tau)$. Finally, we take the value of the functions u and u_s setting the slow-roll parameters and the ratio $H/M_{\rm CS}$ equal to zero. This approximation is justified by the fact that these parameters are very small during inflation. Thus, as a result of the last approximation, we can use the mode function of a scalar field in a de Sitter space (7.7)

$$u_s(k,\tau) = \frac{iH}{M_{\rm Pl}\sqrt{k^3}}(1+ik\tau)e^{-ik\tau} \ , \tag{7.39}$$

$$u(k,\tau) = \frac{iH}{\sqrt{2k^3}}(1+ik\tau)e^{-ik\tau} \ , \tag{7.40}$$

where the different normalizations of the variables γ and $\delta\phi$ come out from the study of the action of standard slow-roll inflationary models (see e.g. [39]).

With the prescriptions just explained, we start now the explicit computation of the first integral I_1. It reads

$$I_1 = -4M_{\rm Pl}^2\left(\prod_{i=1,2,3}\frac{H^2}{M_{\rm Pl}^2 2k^3}\right)\left(\frac{\partial}{\partial\phi}f(\phi)\right)k_1^2k_2^2k_3^2\int_{-\infty}^0 d\tau'\tau'^3 e^{-iK_T\tau'} \ , \tag{7.41}$$

where $K_T = k_1+k_2+k_3$.

This integral can be performed by parts, after passing to the complex plane and performing a Wick rotation of the integration contour. We obtain

$$I_1 = -4M_{\rm Pl}^2\left(\prod_{i=1,2,3}\frac{H^2}{M_{\rm Pl}^2 2k^3}\right)\left(\frac{\partial}{\partial\phi}f(\phi)\right)k_1^2k_2^2k_3^2\left(-\frac{3!}{K_T^4}\right) \ . \tag{7.42}$$

We see that the final result is real. Thus, it does not give any contributions to the correlator (7.34). For the same reason, also the integrals I_2 and I_3 do not give any contribution. The only integral which is not trivial is I_4. Let us see its computation in details

$$I_4 = 4\frac{M_{\rm Pl}^2}{H}\left(\prod_{i=1,2,3}\frac{H^2}{M_{\rm Pl}^2 2k^3}\right)\left(\dot\phi\frac{\partial^2}{\partial^2\phi}f(\phi)\right)\int_{-\infty}^0\frac{d\tau'}{\tau'}(1+ik_1\tau')(1+ik_2\tau')(1+ik_3\tau')e^{-iK_T\tau'} \ . \tag{7.43}$$

We write down the integral which appears in (7.43) as

$$\int_{-\infty}^0\frac{d\tau'}{\tau'}(1+ik_1\tau')(1+ik_2\tau')(1+ik_3\tau')e^{-iK_T\tau'} \ , \tag{7.44}$$

which can be splitted into the sum of four integrals

$$\begin{aligned}&\left(\int_{-\infty}^0\frac{d\tau'}{\tau'}e^{-iK_T\tau'}\right)+\left(iK_T\int_{-\infty}^0 d\tau' e^{-iK_T\tau'}\right)-\left(\prod_{i\neq j}k_ik_j\int_{-\infty}^0 d\tau'\tau' e^{-iK_T\tau'}\right)\\ &-\left(ik_1k_2k_3\int_{-\infty}^0 d\tau'\tau'^2 e^{-iK_T\tau'}\right) \ .\end{aligned} \tag{7.45}$$

All the integrals, apart the first one, can be computed by parts and give a real contribution. For this reason, they do not give any contribution to the correlator (7.34). Instead, the first integral can be written in terms of the exponential integral $Ei(z)$ by promoting the real variable τ' to a complex variable and performing a Wick rotation of the integration contour (i.e. a change of variable $\tau' = -i\tau''$). It becomes:

$$\lim_{\tau \longrightarrow 0-} \int_{-i\infty}^{i\tau} \frac{d\tau''}{\tau''} e^{-K_T \tau''} \; . \tag{7.46}$$

The *complex exponential integral* is defined as [207]

$$Ei(z) = \int_{\infty}^{z} \frac{dz'}{z'} e^{-z'} \quad |Arg(z)| < \pi \; . \tag{7.47}$$

It is well defined for all complex numbers z that are off the real negative axis. A good characteristic of this integral is that it is independent by the integration contour and depends only by z. In particular, it can be expressed in terms of the following series representation [207]

$$Ei(z) = -\gamma - \ln z - \sum_{k=1}^{\infty} \frac{(-z)^k}{k\; k!} \; , \tag{7.48}$$

where γ is the Euler-Mascheroni constant and $\ln z$ is the principal complex logarithm of the complex number z. This series converges for all z that are not in the real axis. Applying formula (7.48), the integral (7.46) becomes

$$\lim_{K_T\tau \longrightarrow 0-} \left[-\gamma - \ln(iK_T\tau) - \sum_{k=1}^{\infty} \frac{(-iK_T\tau)^k}{kk!} \right] = -\gamma + \left(\lim_{K_T\tau \longrightarrow 0} \; \ln|K_T\tau| \right) + i\frac{\pi}{2} \; , \tag{7.49}$$

where the $\ln|K_T\tau|$ in this case represents a real logarithm. Thus, at the end, we get

$$\mathrm{Im}(I_4) = 4\frac{M_{\rm Pl}^2}{H} \left(\prod_{i=1,2,3} \frac{H^2}{M_{\rm Pl}^2 2k_i^3} \right) \left(\dot{\phi} \frac{\partial^2}{\partial^2 \phi} f(\phi) \right) \times \left(i\frac{\pi}{2} \right) \; . \tag{7.50}$$

If we substitute this result into Eq. (7.34) and we consider also the contributions of the permutations, we find the final result

$$\begin{aligned} \langle \gamma_R(\vec{k}_1)\gamma_R(\vec{k}_2)\delta\phi(\vec{k}_3)\rangle =& (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3) 4\pi \frac{M_{\rm Pl}^2}{H} \left(\prod_{i=1,2,3} \frac{H^2}{M_{\rm Pl}^2 2k_i^3} \right) \left(\dot{\phi} \frac{\partial^2}{\partial^2 \phi} f(\phi) \right) \\ & \times (k_1+k_2)(\vec{k}_1 \cdot \vec{k}_2)\, e_{ij}^R(\vec{k}_1) e_R^{ij}(\vec{k}_2) \; . \end{aligned} \tag{7.51}$$

Following the same steps, we are able to compute the correlator $\langle \gamma_L(\vec{k}_1)\gamma_L(\vec{k}_2)\delta\phi(\vec{k}_3)\rangle$ as well. It is sufficient to substitute in the previous steps R with L and take a relative factor -1 due to the $\lambda_L = -\lambda_R$ relation in the interaction Lagrangian (7.20). Thus, we have:

$$\langle \gamma_L(\vec{k}_1)\gamma_L(\vec{k}_2)\delta\phi(\vec{k}_3)\rangle = -\,(2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)4\pi\frac{M_{\rm Pl}^2}{H}\left(\prod_{i=1,2,3}\frac{H^2}{M_{\rm Pl}^2 2k_i^3}\right)\left(\dot{\phi}\frac{\partial^2}{\partial^2\phi}f(\phi)\right)$$
$$\times\,(k_1+k_2)(\vec{k}_1\cdot\vec{k}_2)\,e_{ij}^L(\vec{k}_1)e_L^{ij}(\vec{k}_2)\ . \tag{7.52}$$

We can try to express the final result in a way in which we write explicitly the dependence over the wave numbers k_i. As we have seen in Sec. 3.1, because of momentum conservation, $\delta^{(3)}\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right)=0$, the three momenta inside the bispectrum form a triangle in the momentum space. For invariance under rotations, we can put this triangle in the (x,y)-plane without losing any generality. It follows that a triangle can be constructed by

$$\vec{k}_1 = k_1(1,0,0)\ ,\quad \vec{k}_2 = k_2(\cos\theta,\sin\theta,0),\qquad \vec{k}_3 = k_3(\cos\Phi,\sin\Phi,0)\ , \tag{7.53}$$

where θ and Φ are the angles that the momenta $\vec{k}_2$ and $\vec{k}_3$ form respectively with the momentum $\vec{k}_1$. With these choices of the momenta, we can write down the explicit expressions of L and R polarization tensors. They read (see e.g. [165])

$$e_{ij}^s(\vec{k}_1) = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & i\lambda_s\\ 0 & i\lambda_s & -1\end{pmatrix}\ , \tag{7.54}$$

$$e_{ij}^s(\vec{k}_2) = \frac{1}{\sqrt{2}}\begin{pmatrix}\sin^2\theta & -\sin\theta\cos\theta & -i\lambda_s\sin\theta\\ -\sin\theta\cos\theta & \cos^2\theta & i\lambda_s\cos\theta\\ -i\lambda_s\sin\theta & i\lambda_s\cos\theta & -1\end{pmatrix}\ . \tag{7.55}$$

Thus, through an explicit calculation, we find

$$\vec{k}_1\cdot\vec{k}_2 = k_1k_2\cos\theta,\qquad e_{ij}^s(\vec{k}_1)e_s^{ij}(\vec{k}_2) = \frac{1}{2}(1-\cos\theta)^2\ , \tag{7.56}$$

where θ is the angle between the two momenta $\vec{k}_1$ and $\vec{k}_2$. By the cosine theorem, we can express this angle as a function of the three wave numbers k_i

$$\cos\theta = \frac{k_3^2-k_2^2-k_1^2}{2k_1k_2}\ . \tag{7.57}$$

In the end, the correlator (7.51) becomes

$$\langle \gamma_R(\vec{k}_1)\gamma_R(\vec{k}_2)\delta\phi(\vec{k}_3)\rangle = (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)4\pi\frac{\dot{\phi}}{H}\left(\prod_{i=1,2,3}\frac{H^2}{M_{\rm Pl}^2 2k_i^3}\right)\left(M_{\rm Pl}^2\frac{\partial^2}{\partial^2\phi}f(\phi)\right)$$
$$\times\,(k_1+k_2)k_1k_2\frac{\cos\theta(1-\cos\theta)^2}{2}\ . \tag{7.58}$$

We can rewrite this correlator in a more convenient way in which the product of two power spectrum of PGW (6.35) appears. It is sufficient to multiply and divide for $\sum_i k_i^3$ to obtain the final result

$$\langle \gamma_R(\vec{k}_1)\gamma_R(\vec{k}_2)\delta\phi(\vec{k}_3)\rangle =(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{\pi}{64}\frac{\dot{\phi}}{H}\left(\sum_{i\neq j}P_T(k_i)P_T(k_j)\right)\left(H^2\frac{\partial^2}{\partial^2\phi}f(\phi)\right) \times\frac{(k_1+k_2)k_1k_2}{\sum_i k_i^3}\cos\theta(1-\cos\theta)^2\ . \tag{7.59}$$

7.4 Main results

As we have understood in the previous sections, the two gravitons-one scalar bispectrum (7.59) is the most relevant for parity breaking signatures. Before starting to analyze such signatures, we switch to the gauge invariant variable ζ, using the local relation (2.21). Thus, in the coordinate space we have

$$\langle \gamma_R(\vec{x}_1)\gamma_R(\vec{x}_2)\zeta(\vec{x}_3)\rangle \simeq -\frac{H}{\dot{\phi}}\langle \gamma_R(\vec{x}_1)\gamma_R(\vec{x}_2)\delta\phi(\vec{x}_3)\rangle\ . \tag{7.60}$$

The symbol $\simeq$ means that we are evaluating the correlator at first order in the slow-roll parameters.

Thus, passing in Fourier space and substituting Eq. (7.59) into Eq. (7.60), we have

$$\begin{aligned}\langle \gamma_R(\vec{k}_1)\gamma_R(\vec{k}_2)\zeta(\vec{k}_3)\rangle &\simeq -\frac{H}{\dot{\phi}}\langle \gamma_R(\vec{k}_1)\gamma_R(\vec{k}_2)\delta\phi(\vec{k}_3)\rangle\\ &= -(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{\pi}{64}\left(\sum_{i\neq j}P_T(k_i)P_T(k_j)\right)\left(H^2\frac{\partial^2 f(\phi)}{\partial^2\phi}\right)\\ &\times\frac{(k_1+k_2)k_1k_2}{\sum_i k_i^3}\cos\theta(1-\cos\theta)^2\,,\end{aligned} \tag{7.61}$$

where we recall that θ is the angle between the two momenta $\vec{k}_1$ and $\vec{k}_2$. Proceeding with the same reasoning for computing the vertex $\langle \gamma_L(\vec{k}_1)\gamma_L(\vec{k}_2)\zeta(\vec{k}_3)\rangle$, we find

$$\langle \gamma_L(\vec{k}_1)\gamma_L(\vec{k}_2)\zeta(\vec{k}_3)\rangle = -\langle \gamma_R(\vec{k}_1)\gamma_R(\vec{k}_2)\zeta(\vec{k}_3)\rangle\,. \tag{7.62}$$

Notice that we have obtained a result which differs for a sign in the passage from left to right gravitons. This result is not inconsistent and can be explained with a symmetry argument: if parity was a symmetry of the theory, there would be not difference between the statistics of L and R gravitons. And thus the correlators $\langle\gamma_R\gamma_R\zeta\rangle$ and $\langle\gamma_L\gamma_L\zeta\rangle$ would be equal. But the Chern-Simons term breaks parity symmetry and, as a consequence, we have a difference between the two bispectra. This is achievable e.g. by a sign difference,

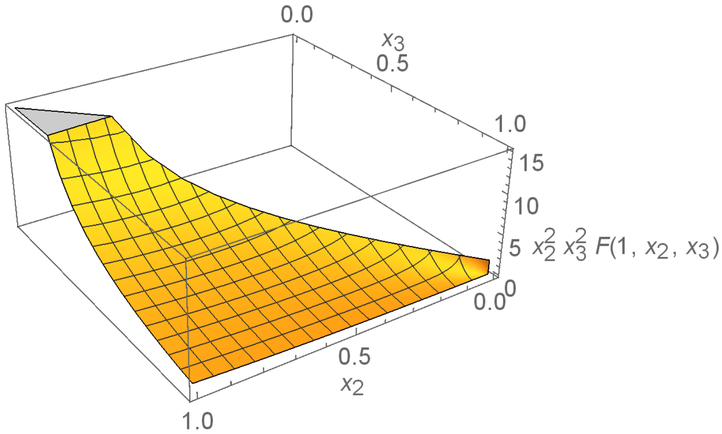


Figure 7.1. 3D-plot of the shape function of the correlator $\langle\gamma_R(\vec{k}_1)\gamma_R(\vec{k}_2)\zeta(\vec{k}_3)\rangle$. The quantity $F(1,x_2,x_3)x_2^2x_3^2$ in terms of $x_2 = k_2/k_1$ and $x_3 = k_3/k_1$ is shown. The figure is normalized to have value 1 for equilateral configurations $x_2 = x_3 = 1$.

as it happens specifically in our case.[2] If we take the dependence of the bispectrum (7.61) over the momenta k_i's, we obtain the shape function

$$F(k_1,k_2,k_3) = \left(\sum_{i\neq j}\frac{1}{k_i^3k_j^3}\right)\frac{(k_1+k_2)k_1k_2}{\sum_i k_i^3}\cos\theta(1-\cos\theta)^2, \tag{7.63}$$

where $\cos\theta$ is defined in Eq. (7.57). As it is customary for scale-independent bispectra, the shape function is plotted as the quantity $F(1,x_1,x_2)x_2^2x_3^2$ in terms of the variables $x_2 = k_2/k_1$ and $x_3 = k_3/k_1$. From Fig. 7.1, we see that the shape function peaks when $x_2 = 1$ and $x_3 = 0$. This corresponds to the so-called squeezed limit, in which the momenta of the gravitons, k_1 and k_2, are much larger than the momentum k_3 of the scalar perturbation.

At this point, we can start to analyze the parity breaking signatures induced by our result (7.61) in the bispectra we are analysing. For this purpose, we can define a parity

[2]Notice that this is what happens similarly in the analysis of [166].

breaking coefficient[3]

$$\Pi = \frac{\langle\gamma_R(\vec{k})\gamma_R(\vec{k})\zeta(\vec{k})\rangle_{\rm TOT} - \langle\gamma_L(-\vec{k})\gamma_L(-\vec{k})\zeta(-\vec{k})\rangle_{\rm TOT}}{\langle\gamma_R(\vec{k})\gamma_R(\vec{k})\zeta(\vec{k})\rangle_{\rm TOT} + \langle\gamma_L(-\vec{k})\gamma_L(-\vec{k})\zeta(-\vec{k})\rangle_{\rm TOT}}, \tag{7.67}$$

where the suffix "TOT" stands for the total bispectra, when we include also the contribution of standard gravity (see Eq. (3.6)). Notice that Π has been defined so that it is equal to zero when the R- and L-handed bispectra coincide, thus when parity is a symmetry of the theory. In Eq. (7.67), the numerator represents the difference between R and L contributions to the bispectrum $\langle\gamma\gamma\zeta\rangle$ which ultimately will determine the amplitude of observable quantities, such as, e.g., CMB bispectra involving (mixed) temperature fluctuations (T) and polarization (E and B) correlators, like, e.g., 3-point correlations of the type $\langle TBB\rangle$ or $\langle EBB\rangle$ (see Chap. 8). Moreover, in defining the amplitude Π, we have evaluated the bispectra in the equilateral configuration of the three momenta. This is a standard convention which is adopted in literature (see e.g. [62]). To evaluate the quantity Π notice that in the model considered[4]

$$\langle\gamma_R(\vec{k}_1)\gamma_R(\vec{k}_2)\zeta(\vec{k}_3)\rangle = -\langle\gamma_L(-\vec{k}_1)\gamma_L(-\vec{k}_2)\zeta(-\vec{k}_3)\rangle\,. \tag{7.68}$$

Thus, using Eq. (3.6) and our results, Eq. (7.61) and (7.62), we find:

$$\Pi = \frac{96\pi}{25}H^2\frac{\partial^2 f(\phi)}{\partial^2\phi}\ . \tag{7.69}$$

Now, let us comment about the result we have obtained in Eq. (7.69). The parity-breaking amplitude Π depends on the strength of the second derivative of the coupling

[3]This coefficient is not properly an amplitude of primordial non-Gaussianity $f_{\rm NL}$, but an estimate of how large parity breaking is in our bispectra. Nevertheless, one can define separately for L and R polarizations a coefficient of non-Gaussianity $f_{\rm NL}^{R,L}$ as

$$f_{\rm NL}^{R,L} = \frac{\langle\gamma_{R,L}(k)\gamma_{R,L}(k)\zeta(k)\rangle}{P_T^2(k)}\,, \tag{7.64}$$

finding

$$f_{\rm NL}^{R} = \frac{\pi}{256}H^2\frac{\partial^2 f(\phi)}{\partial^2\phi}, \tag{7.65}$$

and the same result for $f_{\rm NL}^{L}$ apart a sign difference. From this definition, it is clear that we have normalized the L and R bispectra using the tensor power-spectrum (6.35). An alternative definition can be adopted by normalizing with the scalar power-spectrum, which would simply give the same result as above, rescaled by the inverse of the tensor-to-scalar perturbation ratio. In this context, we do prefer the definition (7.64) because the parity breaking signatures arise in the tensor sector. Using the result (7.64) and Eq. (7.69), we can link such coefficient of non-Gaussianity to the coefficient Π finding

$$f_{\rm NL}^{R} = \frac{25}{24576}\Pi \simeq 10^{-3}\Pi\ . \tag{7.66}$$

[4]In Eq. (7.62), we have showed that $\langle\gamma_R(\vec{k}_1)\gamma_R(\vec{k}_2)\zeta(\vec{k}_3)\rangle = -\langle\gamma_L(\vec{k}_1)\gamma_L(\vec{k}_2)\zeta(\vec{k}_3)\rangle$. Eq. (7.68) follows after noticing that the bispectrum (7.61) depends only on the modulus of the momenta $\vec{k}_i$'s and not by their direction (In fact, $\cos\theta$ can be expressed in terms of the k_i's as showed in Eq. (7.57)).

$f(\phi)$ w.r.t. the inflaton which is something new with respect to the power-spectrum case.[5]

In fact, in the latter case only the first derivative of the coupling $f(\phi)$ appeared (see Eqs. (6.36) and (7.16)). Thus, apparently we can say that our result is independent of the parity breaking in the power-spectrum case which is labelled by the parameter χ in Eq. (6.36). However, a theoretical constraint occurs when we look to the radiative stability of the theory. Following the same reasoning of Sec. 5.4, we switch to the following canonically normalized gravitons in de Sitter space

$$\gamma_c^s(\vec{k}) = \left(\frac{M_{\rm Pl}^2}{2}\right)^{1/2} \gamma^s(\vec{k}) \,, \tag{7.70}$$

and we rewrite the last term of interaction Lagrangian (7.20) (which gives the strength of our bispectrum). We obtain

$$\begin{aligned}\mathcal{L} =& - \lambda_s \int d^3K \, \frac{\delta^3(\vec{k}+\vec{p}+\vec{q})}{(2\pi)^6} \, a \frac{2}{M_{\rm Pl}^2} \left(\dot{\phi}\frac{\partial^2 f(\phi)}{\partial^2\phi}\right) p\,(\vec{p}\cdot\vec{q}) \;\gamma_c^s(\vec{p})\gamma_c^s(\vec{q})\delta\phi_c(\vec{k})\, e_{ij}^s(\vec{p})e_s^{ij}(\vec{q}) \\ =& - \lambda_s \int d^3K \, \frac{\delta^3(\vec{k}+\vec{p}+\vec{q})}{(2\pi)^6} \, a \frac{1}{\Lambda_S^2} \, p\,(\vec{p}\cdot\vec{q}) \;\gamma_c^s(\vec{p})\gamma_c^s(\vec{q})\delta\phi_c(\vec{k})\, e_{ij}^s(\vec{p})e_s^{ij}(\vec{q}) \,, \end{aligned} \tag{7.71}$$

where

$$\Lambda_S^2 = \frac{M_{\rm Pl}^2}{2}\left(\dot{\phi}\frac{\partial^2 f(\phi)}{\partial^2\phi}\right)^{-1} \,. \tag{7.72}$$

To avoid a strong coupling regime (which would spoil the perturbativity of the theory), we must impose

$$H^2 < \Lambda_S^2 \,, \tag{7.73}$$

which gives the following constraint on the strength of the second order derivative:

$$H^2 \frac{\partial^2 f(\phi)}{\partial^2\phi} < \frac{M_{\rm Pl}}{H}\frac{1}{2\sqrt{2\epsilon}} \simeq \frac{2}{\sqrt{2}}\left(\frac{0.01}{r}\right) \times 10^6 \,. \tag{7.74}$$

Thus, recalling the definition of Π, Eq. (7.69), we get the theoretical constraint

$$\Pi \lesssim \left(\frac{0.01}{r}\right) \times 10^7 \,. \tag{7.75}$$

From Eq. (7.75), it would seem that, by decreasing r, one would get less stringent bounds on Π. In fact, one should keep in mind that the bispectrum of Eq. (7.61) is proportional to r^2 coming from the 2 tensor power spectra, attenuating the power of the bispectrum

[5] Arrived at this point, one might ask whether the appearance of the second derivative is something of inevitable. In fact, it is interesting to notice that if the first derivative of the coupling $f(\phi)$ w.r.t. the inflaton would be exactly a constant in the first term of Eq. (7.19), then one would obtain a vanishing result for the correlators of interest. This confirms the result that we find in Eq. (7.61), namely that these correlators are actually sensitive to the second derivative $f(\phi)$ w.r.t. the inflaton field.

in the $r \to 0$ limit independently by the value of Π (which is determined by the strength of the second order derivative of the coupling function $f(\phi)$).

In the next section, we will make a brief comment about the physical meaning of the squeezed limit in the current bispectrum statistics and, after that, in Chap. 8, we will perform a numerical evaluation of the minimum 1σ error committed in measuring the parameter Π with an ideal CMB experiment. The aim of this investigation is to assess the usage of CMB bispectra for constraining parity breaking signatures of Chern-Simons gravity during inflation.

7.5 Comments about the squeezed limit

A useful cross-check of our results, (7.61) and (7.62), comes from considering its squeezed limit, in which we take the momentum of the scalar fluctuation $\vec{k}_3 = \vec{k}_L \simeq 0$ and the other two momenta of the tensor fields such that $\vec{k}_1 = \vec{k}_S \simeq -\vec{k}_2$.

If we use the squeezed consistency relation (3.15), we get that the expected leading order value of the R-handed squeezed bispectrum reads (in our conventions $h = 2\zeta$ and $h^T_{ij} = \gamma_{ij}$)

$$\langle \gamma_R(\vec{k}_S)\gamma_R(-\vec{k}_S)\zeta(\vec{k}_L)\rangle|_{\text{squeezed}} = -\frac{d\log(k_S^3 P_T^R(k_S))}{d\log k_S} P_\zeta(k_L) P_T^R(k_S)\,, \tag{7.76}$$

where

$$P_\zeta(k) = \frac{H^2}{4\epsilon M_{\text{Pl}}^2 k^3} \tag{7.77}$$

is the scalar power spectrum of slow-roll models of inflation and $P_T^R(k)$ is the right-handed tensor power-spectrum as in Eq. (6.33). On super-horizon scales, we can rewrite Eq. (7.76) in terms of the derivative with respect to the cosmological time as

$$\langle \gamma_R(\vec{k}_S)\gamma_R(-\vec{k}_S)\zeta(\vec{k}_L)\rangle|_{\text{squeezed}} = -P_\zeta(k_L)\left(3P_T^R(k(t)) + \frac{1}{H}\frac{d}{dt}P_T^R(k(t))\right)\Big|_{t=t_S}\,, \tag{7.78}$$

where we used the fact that

$$d\log(k_S^3 P_T^R(k_S)) = 3(d\log k_S) + \frac{dP_T^R(k(t_S))}{P_T^R(k(t_S)}\,, \tag{7.79}$$

and that each short mode k_S is related to the time t_S of horizon crossing by the relation

$$k_S = k(t_S) = a(t_S)H(t_S)\,. \tag{7.80}$$

In fact, since $a \sim e^{Ht}$, then we have (apart for slow-roll corrections)

$$d\log k_S = H dt_S\,. \tag{7.81}$$

Computing the time derivative term in Eq. (7.78), we find

$$\frac{1}{H}\frac{d}{dt_S}P_T^R(k_S) \simeq \left[-3 - 2\epsilon + \frac{\pi}{4}\left(\frac{\dot{M}_{\text{CS}}}{M_{\text{CS}}^2}\right)\right] \times \frac{P_T(k_S)}{2}\,. \tag{7.82}$$

Here, $\simeq$ refers to the dominant contributions in slow-roll parameters and in $(H/M_{\rm CS})$, and $P_T(k)$ is the tensor power spectrum as in Eq. (6.35). Since we are interested only in contributions of the parity breaking part of the bispectrum, we consider in Eq. (7.82) only the term that depends on the Chern-Simons mass. Using the definition (6.17), we find

$$\frac{\dot{M}_{\rm CS}}{M_{\rm CS}^2} \simeq -16H^2 f''(\phi)\epsilon\,, \tag{7.83}$$

where the prime $'$ denotes derivative w.r.t. the inflaton field. Here, similarly as above, we have neglected another term that is proportional to the ratio $H/M_{\rm CS}$ which is much smaller than 1 in our theory. This term, in fact, has not been considered in our computation of the bispectrum performed in the previous section.

Thus, gathering all together in Eq. (7.78), bispectrum (7.61) in the squeezed limit becomes

$$\langle\gamma_R(\vec{k}_S)\gamma_R(-\vec{k}_S)\zeta(\vec{k}_L)\rangle|_{\rm squeezed} = \frac{\pi}{8}H^2 f''(\phi)P_T(k_L)P_T(k_S)\ . \tag{7.84}$$

Now, it is easy to verify that, if we take our complete result (7.61) and expand it around the squeezed configuration, then at leading order in (k_L/k_S) we obtain exactly Eq. (7.84), as we would expect.

However, as we have seen in Sec. 3.3, the strict squeezed limit result corresponds to a gauge artifact. In fact, after passing to CFC frame, we expect our physical squeezed bispectrum to be (see e.g. Refs. [84, 85, 213])

$$\langle\gamma_R(\vec{k}_S)\gamma_R(-\vec{k}_S)\zeta(\vec{k}_L)\rangle|_{\rm squeezed,ph} = \mathcal{O}\left(\frac{k_L^2}{k_S^2}\right)\times H^2 f''(\phi)P_T(k_L)P_T(k_S)\ . \tag{7.85}$$

The quantity $\mathcal{O}\left(k_L^2/k_S^2\right)$[6] is the first physical correction term to our squeezed bispectrum. An analogous argument is valid (apart for a sign difference) also for the L-handed squeezed bispectrum $\langle\gamma_L(\vec{k}_S)\gamma_L(-\vec{k}_S)\zeta(\vec{k}_L)\rangle$.

A problematic consequence of this fact appears when one tries to compute the signal-to-noise ratio for the bispectrum $B(k_1,k_2,k_3)$. This in formula reads

$$\frac{S}{N} = \left(\sum_{l_1\leq l_2\leq l_3}\frac{B^2_{l_1,l_2,l_3}}{C_{l_1}C_{l_2}C_{l_3}}\right)^{\frac{1}{2}} \approx \left(\int dx_2 dx_3 B^2(1,x_2,x_3)x_2^4x_3^4\right)^{\frac{1}{2}}\ . \tag{7.86}$$

Here, the integration over x_3 goes from $x_{min} = l_{min}/l_{max}$ to 1, while the integration over x_2 goes from $1-x_3$ to 1. Since for *Planck* we can take $l_{min} = 2$ and $l_{max} \approx 2000$ (see Ref. [62]), $x_{min} = 10^{-3}$. In the squeezed limit, our physical bispectrum goes as in Eq. (7.85). But this is valid only when x_3 is sufficiently small. For example, when x_3 is greater than 10^{-1}, we are already too far from the squeezed configuration to use the

[6] We have not computed explicitly the coefficient which multiplies the factor k_L^2/k_S^2, because it is not the purpose of this research (for such a computation in the case of the scalar bispectrum, see e.g. [85]). Our aim, in fact, is to show that one has to pay attention when constraining with observations a bispectrum from single-clock inflation that peaks in the squeezed configuration.

expression (7.85). On the contrary, in this case we have to use the result (7.61). For this reason, it is interesting to compare the signal-to-noise ratio calculated in two different cases: in a first case, we substitute the bispectrum (7.61) into Eq. (7.86) without taking into consideration spurious signatures that come from the squeezed configurations; in a second case, instead, we split the integration over x_3 into two parts: when $x_3 < 10^{-1}$ we use the squeezed expression (7.85) for the bispectrum,[7] instead when $x_3 \geq 10^{-1}$ we use expression (7.61). If we compare the two S/N ratios we obtain that in the second case, we lose approximately 50% of the signal that we would have if we take into considerations also spurious signatures. This implies that to produce the same S/N ratio, the primordial parity breaking amplitude Π should increase by a factor two w.r.t to the case where we do consider spurious effects in the squeezed limit, something which however is not difficult to obtain within the parameter space of the model. Of course, this is just an estimate, because one should define better the value of x_3 at which the very squeezed limit ends.

[7] We have verified that our conclusions are not sensitive to the precise value of the term $\mathcal{O}(k_L^2/k_S^2)$. In fact, in the squeezed limit ($x_3 \approx 0$) the integrand of Eq. (7.86) is suppressed as x_3^2 and thus it gives a negligible contribution over squeezed configurations.

Chapter 8

Measuring parity breaking signatures with CMB bispectra

8.1 Notations

To begin with, we recall the bottom-line expression of the tensor-tensor-scalar bispectrum $\langle\gamma\gamma\zeta\rangle$ we found in the previous chapter

$$
\begin{aligned}
\left\langle \gamma_{R/L}(\vec{k}_1)\gamma_{R/L}(\vec{k}_2)\zeta(\vec{k}_3)\right\rangle &= (2\pi)^3\delta^{(3)}\left(\sum_{n=1}^{3}\vec{k}_n\right) \\
&\quad \times (+/-)F_{k_1k_2k_3}\left(\hat{k}_1\cdot\hat{k}_2\right)\left[e_{ij}^{R/L}(\vec{k}_1)e_{ij}^{R/L}(\vec{k}_2)\right]^* , \\
\left\langle \gamma_R(\vec{k}_1)\gamma_L(\vec{k}_2)\zeta(\vec{k}_3)\right\rangle &= 0\,,
\end{aligned}
\tag{8.1}
$$

where $e_{ij}^{R/L}(\vec{k})$ is the polarization tensor of R(L)-handed gravitational waves,

$$
F_{k_1k_2k_3} = -\frac{\pi H^6}{2M_{\rm Pl}^4}\frac{\partial^2 f(\phi)}{\partial\phi^2}\frac{(k_1+k_2)}{k_1^2k_2^2k_3^3} = -\frac{25\pi^4}{768}\mathcal{P}_\zeta^2\left(r^2\Pi\right)\frac{(k_1+k_2)}{k_1^2k_2^2k_3^3}\,,
\tag{8.2}
$$

where we have used the definition of Π in our model, Eq. (7.69), and

$$
\left(\frac{H}{M_{\rm Pl}}\right)^2 = \frac{\pi^2}{2}r\mathcal{P}_\zeta\,,
\tag{8.3}
$$

r being the tensor-to-scalar ratio. Equation (8.3) holds since the standard relation $r = 16\epsilon$ is still valid in the current model apart for a very small correction in the total tensor power spectrum which is negligible in the $\chi \ll 1$ regime in which we are working. According to our results, the bispectrum (8.1) peaks in the squeezed configurations $k_1 \sim k_2 \gg k_3$. As we will show in the next section, this fact will lead to a signal-to-noise ratio enhancement when using BBT or BBE CMB bispectra to make a measurement of the parameter Π.

For CMB bispectrum computations, we will follow the procedure already developed in Refs. [102, 103]. In order to do so, we have to express the tensor-tensor-scalar bispectrum with the polarization tensors used in Ref. [103], satisfying $e_{ij}^{(+2/-2)}(\hat{k}) = -e_{ij}^{R/L}(\vec{k})$. Since we now consider the tensor-mode decomposition using [103]

$$\gamma_{ij}(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \sum_{\lambda=\pm 2} \gamma_{\vec{k}}^{(\lambda)} e_{ij}^{(\lambda)}(\hat{k}) \ , \tag{8.4}$$

we have $\gamma_{\vec{k}}^{(+2/-2)} = -\gamma_{R/L}(\vec{k})$. Also, changing the notation of $\zeta(\vec{k}_3)$ to $\zeta_{\vec{k}_3}$ for simplicity of notation, Eq. (8.1) can be rewritten as

$$\left\langle \gamma_{\vec{k}_1}^{(\lambda_1)} \gamma_{\vec{k}_2}^{(\lambda_2)} \zeta_{\vec{k}_3} \right\rangle = (2\pi)^3 \delta^{(3)}\left(\sum_{n=1}^{3} \vec{k}_n \right) F_{k_1 k_2 k_3} \left(\frac{\lambda_1}{2} \right) \delta_{\lambda_1,\lambda_2} (\hat{k}_1 \cdot \hat{k}_2) e_{ij}^{(-\lambda_1)}(\hat{k}_1) e_{ij}^{(-\lambda_2)}(\hat{k}_2) \, . \tag{8.5}$$

The spherical harmonic coefficients of the temperature ($X = T$) and E/B-mode polarization ($X = E/B$) anisotropies from the scalar (ζ) and the tensor ($\gamma^{(\pm 2)}$) perturbations are expressed respectively as [102, 103]

$$a_{\ell m}^{(s)X} = 4\pi(-i)^\ell \int \frac{d^3\vec{k}}{(2\pi)^3} \mathcal{T}_{\ell(s)}^X(k) \zeta_{\vec{k}} Y_{\ell m}^*(\hat{k}) \ , \tag{8.6}$$

$$a_{\ell m}^{(t)X} = 4\pi(-i)^\ell \int \frac{d^3\vec{k}}{(2\pi)^3} \mathcal{T}_{\ell(t)}^X(k) \sum_{\lambda=\pm 2} \left(\frac{\lambda}{2} \right)^x \gamma_{\vec{k}}^{(\lambda)} {}_{-\lambda}Y_{\ell m}^*(\hat{k}) \ , \tag{8.7}$$

where ${}_\lambda Y_{\ell m}(\hat{k})$ is the spin-weighted spherical harmonic function, $\mathcal{T}_{\ell(s)}^X(k)$ and $\mathcal{T}_{\ell(t)}^X(k)$ are respectively the scalar and tensor transfer functions, and x takes 0 (1) for $X = T, E$ ($X = B$). Using these conventions, the CMB bispectra sourced by the primordial tensor-tensor-scalar correlators can be written as

$$\begin{aligned} \left\langle a_{\ell_1 m_1}^{(t)X_1} a_{\ell_2 m_2}^{(t)X_2} a_{\ell_3 m_3}^{(s)X_3} \right\rangle &= \left[\prod_{n=1}^{2} 4\pi(-i)^{\ell_n} \int \frac{d^3\vec{k}_n}{(2\pi)^3} \mathcal{T}_{\ell_n(t)}^{X_n}(k_n) \sum_{\lambda_n=\pm 2} \left(\frac{\lambda_n}{2} \right)^{x_n} {}_{-\lambda_n}Y_{\ell_n m_n}^*(\hat{k}_n) \right] \\ &\quad \times 4\pi(-i)^{\ell_3} \int \frac{d^3\vec{k}_3}{(2\pi)^3} \mathcal{T}_{\ell_3(s)}^{X_3}(k_3) Y_{\ell_3 m_3}^*(\hat{k}_3) \left\langle \gamma_{\vec{k}_1}^{(\lambda_1)} \gamma_{\vec{k}_2}^{(\lambda_2)} \zeta_{\vec{k}_3} \right\rangle . \end{aligned} \tag{8.8}$$

8.2 Allowed harmonic-space configurations

First of all, we check the ℓ-space configurations where a nonvanishing signal of the amplitude Π (Eq. (7.67)) lies in by following the procedure discussed in Ref. [214].

Using the fact that ${}_{-\lambda}Y_{\ell m}(-\hat{k}) = (-1)^\ell {}_\lambda Y_{\ell m}(\hat{k})$, we can rewrite Eq. (8.8) as

$$\begin{aligned} \left\langle a_{\ell_1 m_1}^{(t)X_1} a_{\ell_2 m_2}^{(t)X_2} a_{\ell_3 m_3}^{(s)X_3} \right\rangle &= \left[\prod_{n=1}^{2} 4\pi(-i)^{\ell_n} \int \frac{d^3\vec{k}_n}{(2\pi)^3} \mathcal{T}_{\ell_n(t)}^{X_n}(k_n) \sum_{\lambda_n=\pm 2} \left(\frac{\lambda_n}{2} \right)^{x_n} {}_{-\lambda_n}Y_{\ell_n m_n}^*(\hat{k}_n) \right] \\ &\quad \times 4\pi(-i)^{\ell_3} \int \frac{d^3\vec{k}_3}{(2\pi)^3} \mathcal{T}_{\ell_3(s)}^{X_3}(k_3) Y_{\ell_3 m_3}^*(\hat{k}_3) \\ &\quad \times (-1)^{x_1+x_2+\ell_1+\ell_2+\ell_3} \left\langle \gamma_{-\vec{k}_1}^{(-\lambda_1)} \gamma_{-\vec{k}_2}^{(-\lambda_2)} \zeta_{-\vec{k}_3} \right\rangle . \end{aligned} \tag{8.9}$$

In our model the parity breaking tensor-tensor-scalar bispectrum under consideration (8.5) has odd parity. Namely, it obeys

$$\left\langle \gamma_{\vec{k}_1}^{(\lambda_1)} \gamma_{\vec{k}_2}^{(\lambda_2)} \zeta_{\vec{k}_3} \right\rangle = - \left\langle \gamma_{-\vec{k}_1}^{(-\lambda_1)} \gamma_{-\vec{k}_2}^{(-\lambda_2)} \zeta_{-\vec{k}_3} \right\rangle . \tag{8.10}$$

Due to this peculiarity, Π turns out to be proportional to the amplitude of this bispectrum (see e.g. Eq. (8.2)).

Now, comparing Eq. (8.8) with Eq. (8.9) under the parity-odd condition (8.10), we find that

$$\left\langle a_{\ell_1 m_1}^{(t)X_1} a_{\ell_2 m_2}^{(t)X_2} a_{\ell_3 m_3}^{(s)X_3} \right\rangle \left[1 + (-1)^{x_1 + x_2 + \ell_1 + \ell_2 + \ell_3}\right] = 0 \tag{8.11}$$

always holds. Thus, nonvanishing signal is confined to

$$x_1 + x_2 + \ell_1 + \ell_2 + \ell_3 = \mathrm{odd} . \tag{8.12}$$

This means that in our case a nonvanishing signal for Π in $X_1X_2X_3$ and BBX_3 (BX_2X_3 and X_1BX_3), where $X_1, X_2, X_3 = T, E$, arises from odd (even) $\ell_1 + \ell_2 + \ell_3$ components. It is worth stressing that these combinations are not realized under the usual parity-conserving theories like Einstein gravity and, therefore, they can become distinctive indicators of Chern-Simons gravity if they are detected. The restriction given by Eq. (8.12) is, of course, confirmed also in the following CMB bispectrum formulation.

8.3 CMB bispectrum formulation

Plugging Eq. (8.5) into Eq. (8.8) yields

$$\begin{aligned} \left\langle a_{\ell_1 m_1}^{(t)X_1} a_{\ell_2 m_2}^{(t)X_2} a_{\ell_3 m_3}^{(s)X_3} \right\rangle &= \left[\prod_{n=1}^{3} \frac{(-i)^{\ell_n}}{\pi} \int_0^\infty k_n^2 dk_n \int d^2\hat{k}_n \right] \mathcal{T}_{\ell_1(t)}^{X_1}(k_1) \mathcal{T}_{\ell_2(t)}^{X_2}(k_2) \mathcal{T}_{\ell_3(s)}^{X_3}(k_3) \\ &\times \sum_{\lambda_1 = \pm 2} \left(\frac{\lambda_1}{2}\right)^{x_1 + x_2 + 1} {}_{-\lambda_1}Y_{\ell_1 m_1}^*(\hat{k}_1) {}_{-\lambda_1}Y_{\ell_2 m_2}^*(\hat{k}_2) Y_{\ell_3 m_3}^*(\hat{k}_3) \\ &\times \delta^{(3)}\left(\sum_{n=1}^{3} \vec{k}_n\right) F_{k_1 k_2 k_3} \left(\hat{k}_1 \cdot \hat{k}_2\right) e_{ij}^{(-\lambda_1)}(\hat{k}_1) e_{ij}^{(-\lambda_1)}(\hat{k}_2) . \end{aligned} \tag{8.13}$$

Here, the angular-dependent parts are decomposed using the spin-weighted spherical harmonics as

$$(\hat{k}_1 \cdot \hat{k}_2) e_{ij}^{(-\lambda_1)}(\hat{k}_1) e_{ij}^{(-\lambda_1)}(\hat{k}_2) = \frac{2^5 \pi^2}{15} \sum_{j\mu} {}_{\lambda_1}Y_{j\mu}^*(\hat{k}_1) {}_{\lambda_1}Y_{j-\mu}^*(\hat{k}_2) (-1)^{\mu+1+j} \frac{\left(h_{1\,2\,j}^{0\lambda_1 -\lambda_1}\right)^2}{2j+1}, \tag{8.14}$$

$$\begin{aligned} \delta^{(3)}\left(\sum_{n=1}^{3} \vec{k}_n\right) &= 8 \int_0^\infty y^2 dy \left[\prod_{n=1}^{3} \sum_{L_n M_n} (-1)^{\frac{L_n}{2}} j_{L_n}(k_n y) Y_{L_n M_n}^*(\hat{k}_n) \right] \\ &\times \begin{pmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \end{pmatrix} h_{L_1 L_2 L_3}^{0\,0\,0} , \end{aligned} \tag{8.15}$$

where

$$h^{s_1 s_2 s_3}_{l_1 l_2 l_3} \equiv \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ s_1 & s_2 & s_3 \end{pmatrix} \tag{8.16}$$

with $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ the Wigner $3j$ symbol. The products of the resulting (spin-weighted) spherical harmonics are integrated with respect to $\hat{k}_n$ according to the identities:

$$\int d^2\hat{k} \prod_{n=1}^{2} Y^*_{l_n m_n}(\hat{k}) = (-1)^{m_1} \delta_{l_1, l_2} \delta_{m_1, -m_2} , \tag{8.17}$$

$$\int d^2\hat{k} \prod_{n=1}^{3} {}_{s_n}Y^*_{l_n m_n}(\hat{k}) = h^{-s_1 - s_2 - s_3}_{l_1 \ \ l_2 \ \ l_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} . \tag{8.18}$$

Adding the angular momenta in the induced Wigner symbols by use of, e.g.,

$$\begin{aligned} \sum_{m_4 m_5 m_6} (-1)^{\sum_{n=4}^{6}(l_n - m_n)} & \begin{pmatrix} l_5 & l_1 & l_6 \\ m_5 & -m_1 & -m_6 \end{pmatrix} \begin{pmatrix} l_6 & l_2 & l_4 \\ m_6 & -m_2 & -m_4 \end{pmatrix} \begin{pmatrix} l_4 & l_3 & l_5 \\ m_4 & -m_3 & -m_5 \end{pmatrix} \\ &= \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{Bmatrix} , \end{aligned} \tag{8.19}$$

where $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$ is the Wigner $6j$ symbol, and performing the summations with respect to λ_1, we obtain the bottom-line expression $\left\langle a^{(t)X_1}_{\ell_1 m_1} a^{(t)X_2}_{\ell_2 m_2} a^{(s)X_3}_{\ell_3 m_3} \right\rangle = B^{X_1 X_2 X_3}_{(tts)\ell_1 \ell_2 \ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ where the angle-averaged bispectrum turns out to be

$$\begin{aligned} B^{X_1 X_2 X_3}_{(tts)\ell_1 \ell_2 \ell_3} &= \delta^{\rm odd}_{x_1 + x_2 + \ell_1 + \ell_2 + \ell_3} (-i)^{\ell_1 + \ell_2 + \ell_3} \sum_{L_1 L_2} (-1)^{\frac{L_1 + L_2 + \ell_3}{2}} h_{L_1 L_2 \ell_3} \\ &\times \frac{2^6 \pi^2}{15} \sum_j \frac{(-1)^{1 + j + L_2 + \ell_1}}{2j+1} h^{0 2 -2}_{L_1 \ell_1 j} h^{0 2 -2}_{L_2 \ell_2 j} \left(h^{0 2 -2}_{1\, 2\, j} \right)^2 \begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ L_2 & L_1 & j \end{Bmatrix} \\ &\times \int_0^\infty y^2 dy \left[\prod_{n=1}^{2} \frac{2}{\pi} \int_0^\infty k_n^2 dk_n \mathcal{T}^{X_n}_{\ell_n (t)}(k_n) j_{L_n}(k_n y) \right] \\ &\times \frac{2}{\pi} \int_0^\infty k_3^2 dk_3 \mathcal{T}^{X_3}_{\ell_3 (s)}(k_3) j_{\ell_3}(k_3 y) F_{k_1 k_2 k_3} , \end{aligned} \tag{8.20}$$

with

$$\delta^{\rm odd}_{\ell} = \begin{cases} 1 & (\ell = {\rm odd}) \\ 0 & (\ell = {\rm even}) \end{cases} . \tag{8.21}$$

In fact, the ranges of the summations with respect to L_1, L_2 and j are limited to a few modes by the selection rules of the Wigner symbols.

8.4 Fisher matrix forecasts

From Eq. (8.2), one can see that $F_{k_1k_2k_3}$ depends linearly on the chirality parameter Π, given by Eq. (7.69), and quadratically on the tensor-to-scalar ratio r. Thus, $B^{X_1X_2X_3}_{(tts)\ell_1\ell_2\ell_3} \propto r^2\Pi$.

We now evaluate the detectability of Π by computing the Fisher matrix from XXX ($\mathcal{F}^{XXX}$) or BBX ($\mathcal{F}^{BBX}$), with $X = T, E$, according to

$$\mathcal{F}^{XXX} = \sum_{\ell_1,\ell_2,\ell_3=2}^{\ell_{\rm max}} \frac{\left(\hat{B}^{XXX}_{\ell_1\ell_2\ell_3}\right)^2}{6C^{XX}_{\ell_1}C^{XX}_{\ell_2}C^{XX}_{\ell_3}}(-1)^{\ell_1+\ell_2+\ell_3}\,, \tag{8.22}$$

$$\mathcal{F}^{BBX} = \sum_{\ell_1,\ell_2,\ell_3=2}^{\ell_{\rm max}} \frac{\left(\hat{B}^{BBX}_{\ell_1\ell_2\ell_3}\right)^2}{2C^{BB}_{\ell_1}C^{BB}_{\ell_2}C^{XX}_{\ell_3}}(-1)^{\ell_1+\ell_2+\ell_3}\,, \tag{8.23}$$

where

$$\begin{aligned}
&\hat{B}_{\ell_1\ell_2\ell_3} \equiv \partial B_{\ell_1\ell_2\ell_3}/\partial\Pi = B_{\ell_1\ell_2\ell_3}/\Pi\,,\\
&B^{XXX}_{\ell_1\ell_2\ell_3} \equiv B^{XXX}_{(tts)\ell_1\ell_2\ell_3} + B^{XXX}_{(tst)\ell_1\ell_2\ell_3} + B^{XXX}_{(stt)\ell_1\ell_2\ell_3} = B^{XXX}_{(tts)\ell_1\ell_2\ell_3} + B^{XXX}_{(tts)\ell_3\ell_1\ell_2} + B^{XXX}_{(tts)\ell_2\ell_3\ell_1}\,,\\
&B^{BBX}_{\ell_1\ell_2\ell_3} \equiv B^{BBX}_{(tts)\ell_1\ell_2\ell_3}\,.
\end{aligned} \tag{8.24}$$

We focus on these cases, since they represent the best representative combinations to be compared. Here we have assumed a weak non-Gaussian signal (we will comment later on this assumption), so that the variance can be expressed with the products of the angular power spectra C^{XX}_ℓ and C^{BB}_ℓ. Furthermore, in order to derive $\mathcal{F}^{BBX}$, we have used the fact that the expected signal of the cross-correlation XB is undetectably small, i.e., $C^{XX}_\ell C^{BB}_\ell \gg (C^{XB}_\ell)^2$, which is a consequence of the $\chi \ll 1$ condition. In the following, we consider a full-sky noiseless cosmic-variance-limited-level (CVL-level) experiment. In this case, C^{XX}_ℓ and C^{BB}_ℓ are determined by the signal computed from theory (and when specified it includes the contribution from lensing, see later). The expected 1σ errors on Π for $r = 10^{-2}$ and 10^{-3}, computed according to $\Delta\Pi^{X_1X_2X_3} = 1/\sqrt{\mathcal{F}^{X_1X_2X_3}}$, are shown in Fig. 8.1.

From the TTT results, we find that the usual scaling relation for the squeezed-type non-Gaussianity case, $\Delta\Pi^{TTT} \propto \ell_{\rm max}^{-1}$ [68, 103, 215], stops at $\ell_{\rm max} \sim 100$ corresponding to the end of the large-scale amplification due to the integrated Sachs-Wolfe effect induced by gravitational waves [216]. The same suppression was confirmed in the tensor-tensor-tensor bispectrum case [166, 217–220]. At very low ℓ's, $\mathcal{T}^T_{\ell(t)}$ is comparable in size to $\mathcal{T}^T_{\ell(s)}$, so, e.g.,

$$B^{TTT}_{(tts)\ell_1\ell_2\ell_3}/B^{TTT}_{(sss)\ell_1\ell_2\ell_3} \sim \langle\gamma\gamma\zeta\rangle\,/\,\langle\zeta\zeta\zeta\rangle \tag{8.25}$$

becomes a good approximation. Considering the comparison with the usual scalar-mode local-type non-Gaussianity case [68], we therefore have

$$B^{TTT}_{(tts)\ell_1\ell_2\ell_3}/B^{TTT}_{(sss)\ell_1\ell_2\ell_3} \sim 10^{-3} \times (r^2\Pi)/f_{\rm NL} \tag{8.26}$$

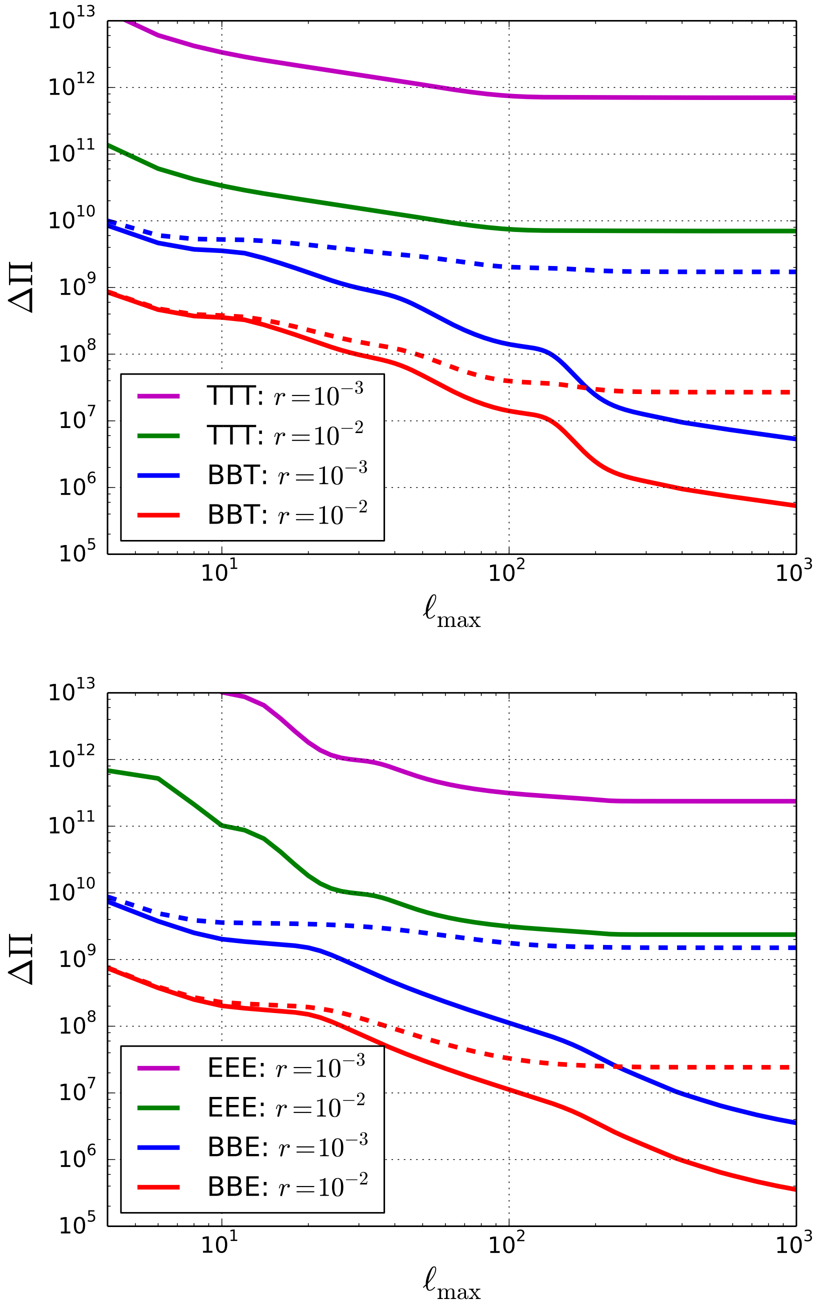


Figure 8.1. Expected 1σ errors on the nonlinear chirality parameter Π from TTT, EEE, BBT and BBE for $r = 10^{-2}$ and 10^{-3} as a function of $\ell_{\rm max}$. Here, we assume a full-sky measurement without any instrumental uncertainties. Solid (dashed) lines for BBT and BBE are computed assuming a perfectly delensed (undelensed) B-mode polarization data.

(see, e.g., Eq. (5.62) of Ref. [204]). From this, we find the following transforming formula

$$\Delta\Pi^{TTT} \sim 10^3 r^{-2} \Delta f_{\rm NL}^{TTT} \,. \tag{8.27}$$

Substituting the expected 1σ error on $f_{\rm NL}$ obtained in the literature, $\Delta f_{\rm NL}^{TTT}(\ell_{\rm max} = 10) \sim 10^3$ [68] into this last equation, we can derive

$$\Delta\Pi^{TTT}(\ell_{\rm max} = 10) \sim 10^6 r^{-2} \,. \tag{8.28}$$

As expected, this fully agrees with the results in Fig. 8.1. Notice that $\Delta\Pi^{TTT} \propto r^{-2}$ exactly holds because of $\hat{B}^{TTT}_{\ell_1\ell_2\ell_3} \propto r^2$. The similar features are also seen in the EEE case.

The Fisher matrix from BBT or BBE is computed assuming two kinds of cases where the B-mode polarization data is fully-delensed or undelensed. In the former case, the errors are expected to scale like

$$\Delta\Pi^{BBX} \propto r^{-1} \,, \tag{8.29}$$

because of $C_\ell^{BB} \propto r$ and $\hat{B}^{BBX}_{\ell_1\ell_2\ell_3} \propto r^2$.

Moreover, at very low ℓ's,

$$\hat{B}^{BBX}_{\ell_1\ell_2\ell_3} / \hat{B}^{XXX}_{\ell_1\ell_2\ell_3} \sim 1 \tag{8.30}$$

and

$$C_\ell^{BB} / C_\ell^{XX} \sim r \tag{8.31}$$

hold because of the small shape difference between the transfer functions $\mathcal{T}^T_{\ell(t)}$, $\mathcal{T}^E_{\ell(t)}$, $\mathcal{T}^B_{\ell(t)}$, $\mathcal{T}^T_{\ell(s)}$ and $\mathcal{T}^E_{\ell(s)}$. Thus,

$$\Delta\Pi^{BBX} / \Delta\Pi^{XXX} \sim r \tag{8.32}$$

is expected to hold. All these scaling are actually confirmed from the solid lines in Fig. 8.1, justifying our numerical results for BBT and BBE.

The former analysis tells us that if delensing perfectly works, the squeezed-type scaling

$$\Delta\Pi^{BBX} \propto \ell_{\rm max}^{-1} \tag{8.33}$$

is still maintained beyond $\ell_{\rm max} \sim 100$.[1] From the above estimates and our numerical results, we can therefore expect that

$$\Delta\Pi^{BBT} \sim \Delta\Pi^{BBE} \sim 10^6 \left(\frac{0.01}{r}\right) \left(\frac{500}{\ell_{\rm max}}\right) \,. \tag{8.34}$$

Even in the perfectly-delensed situation, such a rapid sensitivity improvement by increasing $\ell_{\rm max}$ is not expected when estimating the usual power spectrum chirality parameter,

$$\chi \equiv \frac{\left\langle \gamma^{(+2)}\gamma^{(+2)} \right\rangle - \left\langle \gamma^{(-2)}\gamma^{(-2)} \right\rangle}{\left\langle \gamma^{(+2)}\gamma^{(+2)} \right\rangle + \left\langle \gamma^{(-2)}\gamma^{(-2)} \right\rangle}, \tag{8.35}$$

[1] Very roughly, the effects of the transfer functions in $\hat{B}^{BBX}_{\ell_1\ell_2\ell_3}$ and $\sqrt{C^{BB}_{\ell_1} C^{BB}_{\ell_2} C^{XX}_{\ell_3}}$ of Eq. (8.23) cancel each other out. Thus, one can obtain this result following closely the estimate for local bispectra given in Refs. [68, 221].

from C_ℓ^{TB} and C_ℓ^{EB} [203, 219, 222]. The Fisher matrix in this case is expressed as

$$\mathcal{F}^{XB} = \sum_{\ell=2}^{\ell_{\max}} (2\ell+1) \frac{\left(\hat{C}_\ell^{XB}\right)^2}{C_\ell^{XX} C_\ell^{BB}}, \tag{8.36}$$

where

$$\hat{C}_\ell^{XB} \equiv \partial C_\ell^{XB} / \partial \chi = C_\ell^{XB} / \chi \,. \tag{8.37}$$

Especially for high ℓ, $\hat{C}_\ell^{XB}$ and C_ℓ^{BB} (that are sourced by the tensor modes alone) are subdominant compared with C_ℓ^{XX} (that is also generated from the scalar mode). Hence, $(\hat{C}_\ell^{XB})^2/(C_\ell^{XX} C_\ell^{BB})$ is highly suppressed causing a saturation at high ℓ (see Fig. 8.2). This actually prevents $\mathcal{F}^{XB}$ from growing. In contrast, in Eq. (8.23), both $\hat{B}_{\ell_1\ell_2\ell_3}^{BBX}$ and $\sqrt{C_{\ell_1}^{BB} C_{\ell_2}^{BB} C_{\ell_3}^{XX}}$ come from two tensors and one scalar, and therefore the ratio $(\hat{B}_{\ell_1\ell_2\ell_3}^{BBX})^2/(C_{\ell_1}^{BB} C_{\ell_2}^{BB} C_{\ell_3}^{XX})$ does not decay for high ℓ, enhancing $\mathcal{F}^{BBX}$. This indicates that CMB bispectra represent a promising observable to test and measure the chirality of gravitational waves arising from parity-violation effects. In contrast, the presence of the lensing B-mode can degrade the sensitivity, as expected and as shown in Fig. 8.1. Thus, a delensing of the CMB map is required.

There is a further aspect to point out. In real data analysis, one may be concerned about the contaminations due to late-time secondary contributions such as the so-called ISW-lensing and polarization-lensing bispectra [223, 224]. These effects completely vanish in the multipole domain under consideration (8.12) (because these late-time effects are not parity-violating sources). Hence, the extra process of subtracting such secondary contributions in the CMB bispectra is indeed not required. However, additional contributions to the covariance of CMB bispectra may arise when we Wick expand it and include the CMB higher order correlators (i.e. the connected 4- and 6-points functions) coming from the late-time physics. Also in this case, a delensing of the CMB map is the only alternative to remove this extra contribution from the covariance.

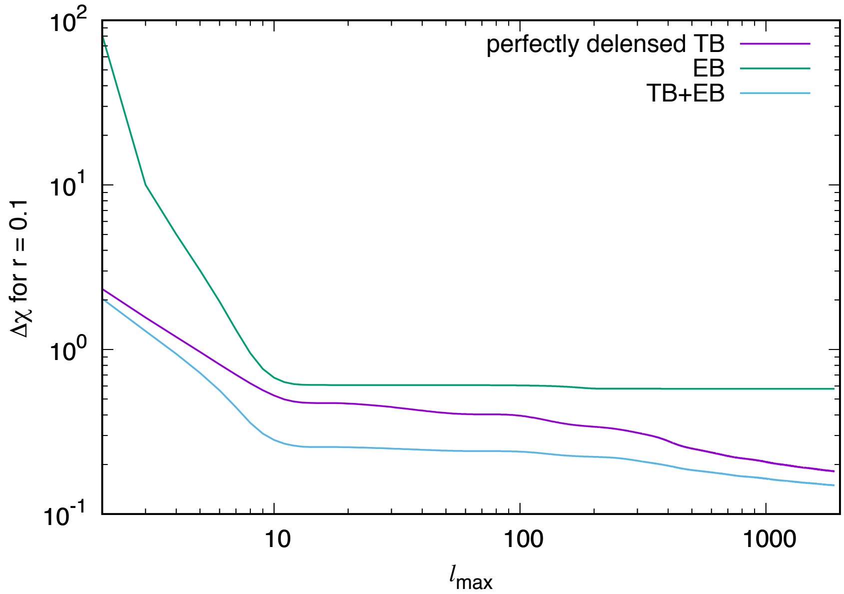


Figure 8.2. Expected 1σ ideal errors on the chirality parameter χ from TB and EB power spectra for $r = 10^{-1}$ as a function of $\ell_{\max}$. A perfectly delensed CMB map is assumed. We can see that, after a certain maximum multipole scale ($\ell_{max} \approx 10$), we get a saturation of $\Delta\chi$ at the value ≈ 0.2 (combining the TB and EB channels). Thus, there is no hope of improvement in the measure of χ even in futuristic CMB experiments with very high angular resolution and complete lensing subtraction.

8.5 Comments on the weak non-Gaussian signal assumption

In this section, we comment on the assumption of weak non-Gaussian signals. In performing our Fisher matrix forecasts, we implicitly assumed that the CMB non-Gaussianities due to primordial perturbations are weak and the Gaussian statistics is predominant. In fact, this is what we would expect by the constraints on CMB non-Gaussianities already made in literature (see Sec. 4.4), even tough no constraints on primordial tensor non-Gaussianities have currently been made.

The request that non-linear effects from the 2 gravitons-1 scalar correlator do not overcome in strength the linear ones translate into the following condition

$$\langle \gamma\gamma\zeta \rangle < A_T^2 A_\zeta \,, \tag{8.38}$$

where A here refers to the linear amplitude of the respective tensor/scalar perturbations (which is the square root of the power spectrum). Since from Eq. (7.64) we have

$$\langle \gamma\gamma\zeta \rangle = f_{\mathrm{NL}}^{R,L} A_T^4 \,, \tag{8.39}$$

this means that the upper value of f_{NL} denoting the passage to a high non-linear regime reads

$$f_{\mathrm{NL}}^{R,L} < \frac{A_\zeta}{A_T^2} = \frac{1}{r A_\zeta} \,. \tag{8.40}$$

Thus, using the experimental value $A_\zeta \simeq 10^{-5}$ (see [91]), we get

$$f_{\mathrm{NL}}^{R,L} \lesssim \left(\frac{0.01}{r} \right) \times 10^7 \,, \tag{8.41}$$

or in terms of $\Pi \simeq 10^3 f_{\mathrm{NL}}^{R,L}$ (see (7.66))

$$\Pi \lesssim \left(\frac{0.01}{r} \right) \times 10^{10} \,. \tag{8.42}$$

The upper bound on Π just derived is 3 orders of magnitude weaker than the one that we imposed in order to preserve the perturbativity of the theory (see Eq. (7.75)). For this reason, in the allowed region of the parameter space for Π, we are in the weak non-Gaussianities regime where the forecasts made are legit.

8.6 Conclusions

If we match the allowed values in the parameter space, Eq. (7.75), with our final Fisher matrix forecasts, Eq. (8.34) and Fig. 8.1, we get that there is a window of 1 order of magnitude in the parameter Π where we can hope to detect parity breaking effects with the measurement of BBT or BBE CMB angular bispectra. On the contrary, CMB angular bispectra not involving B modes show much worse sensitivity to Π, resulting quite useless in making a constraint on Π in the weak non-Gaussianity regime.[2] Moreover, we showed that an improvement in the angular resolution of the experiment could in principle enhance the minimum testable value of Π. In fact, the 1σ ideal error on Π scales as $\ell_{\rm max}^{-1}$ (contrary to what happens when estimating chirality from power spectra C_ℓ^{TB} and C_ℓ^{EB}). However, realistically speaking, the B modes coming from the gravitational lensing degrade the small scale contribution to the S/N ratio in a way that a saturation is achieved for $\ell > 10^2$ (see Fig. 8.1). For this reason, the CMB delensing is a necessary condition to improve the sensitivity of the experiment. Nevertheless, we showed that BBT and BBE CMB angular bispectra could become in the future essential observables for testing Chern-Simons gravity during inflation, as well as other parity breaking theories of inflation with bispectra satisfying the parity-odd condition (8.10).

[2] What we found is in accordance with e.g. Refs. [129, 225], where it is claimed the importance of cross-correlation with CMB B modes to significantly improve constraints on non-Gaussianities involving tensor modes.

Part III
CMB V modes from physics beyond standard scenario

CMB radiation represents a crucial observational tool of modern cosmology. As we have seen in Sec. 4.2, the standard Boltzmann equations describing the radiation transfer of the CMB from the recombination epoch until today predict the presence of some level of linear polarization, the so-called E and B modes, which has been widely studied and reviewed in the literature (see e.g. Refs. [93, 94, 99, 104, 226–229]). This is the result of the Compton scattering between CMB photons and electrons and gravitational redshift, induced by cosmological perturbations of the metric. Instead, the generation of CMB circular polarization is usually not considered, because the electron-photon Compton scattering cannot generate it at the classical level.

However, some models have been proposed that can lead to the generation of CMB circular polarization. One possible way is via Faraday conversion of the linear polarization generated at the surface of last scattering by various sources of cosmic birefringence (see e.g. Refs. [230, 231] for a recent review). For instance, in Refs. [232–241], V-mode formation due to magnetic fields is discussed. In Refs. [242–245], V-mode formation due to photon-photon interactions via Heisenberg-Euler interaction is considered. V-mode generation due to interactions coming from extensions of QED is studied, in particular, in Refs. [238, 246, 247], where Lorentz-violating operators are considered. In Ref. [248], it is shown that a cosmological pseudoscalar field may generate circular polarization in the CMB, while, in Ref. [249], it is shown that V-mode generation can be obtained in axion inflation. Moreover, in Refs. [250, 251], it is shown that forward scattering between CMB photons and neutrinos can source V modes through Standard Model interactions. In Ref. [252], circular polarization of CMB photons via their Compton scattering with polarized cosmic electrons is considered. In Ref. [253], it is shown that V modes in the CMB may arise from primordial vector and tensor perturbations. In particular, in Refs. [254, 255], the case of chiral gravitational waves is considered.

Despite the fact that CMB circular polarization has not been observationally explored so much up to now, these many examples show how its detection might reveal interesting phenomena occurring in the evolution of the Universe. In particular, from the observational point of view, CMB circular polarization is not excluded. As an example, the SPIDER collaboration has recently provided new constraints on the Stokes parameter V at 95 and 150 GHz, by observing angular scales corresponding to $33 < \ell < 307$ [202]. The constraints on the circular polarization power-spectrum $\ell(\ell+1)C^{\ell}_{VV}/(2\pi)$ are reported in a range from 141 μK^2 to 255 μK^2 at 150 GHz for a thermal CMB spectrum. Also, in Ref. [256], some interesting detection prospects are discussed.

Motivated by all these theoretical and observational reasons, in this third part of the Thesis we will investigate about innovative physics which may lead to the production of CMB V modes, working with the formalism of the so-called *quantum Boltzmann equation* (see App. D for a brief review). In this formalism, the master equation governing the time evolution of the CMB photon polarization matrix (4.1) reads (D.15)

$$(2\pi)^3\delta^{(3)}(\vec{0})2k^0\frac{d}{dt}\rho_{ij}(\mathbf{k}) = i\left\langle [H_I(t), \mathcal{D}^I_{ij}(\mathbf{k})]\right\rangle - \int_{t_0}^{t} dt' \, \left\langle [H_I(t), [H_I(t'), \mathcal{D}^I_{ij}(\mathbf{k})]]\right\rangle \,, \quad (8.43)$$

where $H_I(t)$ denotes a certain interaction Hamiltonian (in perturbative regime) involving photons and other fields, and $D^I_{ij}(\mathbf{k}) = a_i(\mathbf{k})a^\dagger_j(\mathbf{k})$ is the photon number operator. In particular, we will focus on the effects of the *forward scattering term* (the first term in the right-hand side of (8.43)) which is able to generate couplings between different CMB polarization states.[3] In fact, Eq. (8.43) is derived adopting a perturbative approach so that increasing powers of the interaction Hamiltonian $H_I(t)$ reduce the strength of the corresponding term. For this reason, in any fundamental interaction in perturbative regime in which the forward scattering term is non-zero, a-priori it is expected to give the relevant physical effects on the CMB polarizations. Of course, this is not the case of the standard QED interaction between photons and electrons, where such a forward scattering term vanishes (see e.g. [93]), and all the relevant effects arise from the *damping term* only (the second term in the right-hand side of (8.43)).

First of all, in Chap. 9, we will analyze the effects on CMB polarization from the polarization mixing induced by the forward scattering with primordial gravitons [259]. Then, in Chap. 10, we will consider the forward scattering mixing in the case of photon-fermion interactions that go beyond the Standard Model of particle physics [260].

[3] This is the same physical mechanism that generates neutrino flavor mixings, see e.g. Ref. [257]. Such mixing in the CMB polarization states has been previously considered e.g. in Refs. [238, 243, 245, 247, 250, 258].

Chapter 9

Effects of photon-graviton forward scattering on CMB polarization

9.1 Introduction

From an effective field theory (EFT) point of view, general relativity can be considered as the low energy limit of a well-defined renormalizable theory of gravity. In this low energy theory, gravitational waves are (transverse-traceless) fluctuations around a suitable background space-time; expanding the Einstein-Hilbert Lagrangian in a power series of these fluctuations, we are able to extract a positive-definite kinetic term describing the free propagation of gravitational waves. Thus, we are able to canonically quantize gravitational waves, introducing creation and annihilation operators, as it is done with the particle content and gauge bosons of the standard model of fundamental interactions. In such a formalism, field operators describing gravitational waves are called gravitons. In particular, expanding the full covariant Lagrangian of a generic theory in power series of all the field content, we can derive interaction terms between gravitons and other fields.

In this chapter, using this quantum field theory (QFT) approach (as well as the quantum Boltzmann equation (8.43)), we will study the forward scattering mixing induced to a CMB photon due to the so-called "gravitational Compton scattering" with gravitons. In this case, the full Lagrangian we have to consider is the sum of the Einstein-Hilbert Lagrangian and the covariant electromagnetic Lagrangian. Feynman amplitudes describing gravitational Compton scattering have already been computed in the literature (see e.g. Refs. [261–263]). Thus, after a brief review, we use these results to analyze the CMB polarization mixing induced.

9.2 Computation of the forward scattering term

The effects of photon-graviton forward scattering are described by the following Boltzmann equation:

$$(2\pi)^3\delta^{(3)}(0)(2k^0)\frac{d\rho_{ij}^{(\gamma)}(k)}{dt} = i\left\langle\left[H_{\gamma g}(t), \mathcal{D}_{ij}^{(\gamma)}(k)\right]\right\rangle , \tag{9.1}$$

where $\rho_{ij}^{(\gamma)}$ is the polarization matrix of the photon [see Eq. (9.31)], $\mathcal{D}_{ij}^{(\gamma)} = a_i^\dagger a_j$ is the photon number operator and $H_{\gamma g}(t)$ is the *quantum* interaction Hamiltonian between photons and gravitons, defined in terms of the second order S-matrix as

$$S^{(2)} = -i \int_{-\infty}^{\infty} dt\, H_{\gamma g}(t)\,. \tag{9.2}$$

In order to derive the photon-graviton interaction Hamiltonian, we follow the same convention of Refs. [261, 264]. The gravitational dynamics is given by the Einstein-Hilbert Lagrangian

$$\mathcal{L}_g = \frac{\sqrt{-g}}{\kappa^2} g^{\alpha\lambda} \left(\Gamma^\beta_{\alpha\lambda}\Gamma^\mu_{\beta\mu} - \Gamma^\mu_{\alpha\beta}\Gamma^\beta_{\lambda\mu}\right)\,, \tag{9.3}$$

where $\kappa^2 = 16\pi G$ and we have dropped irrelevant surface terms. We consider the weak-field limit and expand the metric around Minkowski space-time in powers of κ, as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}(x)\,, \tag{9.4}$$

where $h_{\mu\nu}(x)$ is the graviton field. Now, we have to expand the Lagrangian (9.3) around the background in powers of κ and $h_{\mu\nu}(x)$. The terms linear in $h_{\mu\nu}(x)$ vanish by virtue of Einstein's equations. We keep only the nonlinear terms in $h_{\mu\nu}(x)$, up to first order in κ. In fact, it is possible to show that terms with higher powers of κ give a gradually suppressed contribution, when inserted into Eq. (9.1). Thus, we have

$$\mathcal{L}_g = \mathcal{L}_g^{(0)} + \mathcal{L}_g^{(1)} + \mathcal{O}(\kappa^2)\,, \tag{9.5}$$

where $\mathcal{L}_g^{(0)}$ is the Lagrangian describing the free propagation of gravitons and reads

$$\mathcal{L}_g^{(0)} = \frac{1}{4}\left\{h_\alpha^{\alpha,\beta}\left(2h^{;\lambda}_{\beta\lambda} - h^\alpha_{\alpha,\beta}\right) + h^{\sigma\lambda,\alpha}\left(-2h_{\alpha\lambda,\sigma} + h_{\sigma\lambda,\alpha}\right)\right\}\,, \tag{9.6}$$

$\mathcal{L}_g^{(1)}$ is the three-gravitons interaction Lagrangian and is given by

$$\begin{aligned}\mathcal{L}_g^{(1)} = \kappa\Big\{&\frac{1}{2}h^\alpha_\alpha\mathcal{L}_g^{(0)} - \frac{1}{4}h^{\lambda\rho}\Big[2h^\alpha_{\alpha,\sigma}\left(h^\sigma_{\lambda,\rho} - h^{;\sigma}_{\lambda\rho}\right) + \left(-2h_{\alpha\lambda,\sigma}h_\rho^{\sigma,\alpha} + 2h_{\sigma\lambda,\alpha}h_\rho^{\sigma,\alpha} + h_{\alpha\sigma,\lambda}h^{\sigma\alpha}_{,\rho}\right)\\ &+h^\alpha_{\alpha,\lambda}\left(2h^\nu_{\rho,\nu} - h^\alpha_{\alpha,\rho}\right) + 2h^{\sigma\nu}_{,\nu}h_{\lambda\rho,\sigma} - 4h_\rho^{\sigma,\alpha}h_{\alpha\sigma,\lambda}\Big]\Big\}\,. \end{aligned}\tag{9.7}$$

Here and in the following, Greek indices are raised and lowered using the Minkowski metric $\eta_{\mu\nu}$. Now, we make the following spatial Fourier expansion for the graviton field:

$$h_{\mu\nu}(x) = \int \frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\sum_{r=+,\times}\left[b_r(\mathbf{q})\, h^{(r)}_{\mu\nu}\, e^{iqx} + b^\dagger_{r'}(\mathbf{q})\, h^{(r)\,*}_{\mu\nu}\, e^{-iqx}\right]\,, \tag{9.8}$$

where $b_{\mathbf{q}}^{(r)}$ and $b_{\mathbf{q}}^{(r)\,\dagger}$ are graviton annihilation and creation operators, respectively, obeying the canonical commutation relations

$$\left[b_r(\mathbf{q}), b^\dagger_{r'}(\mathbf{q}')\right] = (2\pi)^3 2q^0\delta^{(3)}(\mathbf{q}-\mathbf{q}')\delta_{rr'}\,, \tag{9.9}$$

and $h^{(r)}_{\mu\nu}$ are the polarization tensors with the well-known properties

$$h^{(r)}_{\mu\nu}(q) q^{\mu} = 0 \,, \qquad h^{\mu}_{\mu}(q) = 0 \,, \qquad h^{(r)}_{\mu\nu}(q) \left(h^{(r')\,\mu\nu}(q) \right)^{*} = \delta^{rr'} \,. \tag{9.10}$$

It is also convenient to represent the polarization tensor $h^{(r)\,\mu\nu}$ in terms of a direct product of unit spin polarization vectors

$$h^{(r)}_{\mu\nu} = e^{(r)}_{\mu} e^{(r)}_{\nu} \;, \qquad e^{(r)}_{\mu} q^{\mu} = 0 \;, \qquad \left[e^{(r)}_{\mu}(q) \left(e^{(r')\,\mu}(q) \right)^{*} \right]^{2} = \delta^{rr'} \,. \tag{9.11}$$

The explicit expression for the graviton propagator in the harmonic (de Donder) gauge is given by [261, 264]

$$D^{(g)}_{\mu\nu\alpha\beta}(q) = \frac{i}{q^2} \left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta} \right) . \tag{9.12}$$

To compute the S-matrix (9.2), we need to consider also the covariant electromagnetic Lagrangian density [261, 264]

$$\mathcal{L}_{\gamma} = \mathcal{L}^{(0)}_{\gamma} + \mathcal{L}^{(1)}_{\gamma} + \mathcal{L}^{(2)}_{\gamma} + \mathcal{O}(\kappa^3) \,, \tag{9.13}$$

where $\mathcal{L}^{(0)}_{\gamma}$ is the part of the Lagrangian that describes the free photon propagation and is given by

$$\mathcal{L}^{(0)}_{\gamma} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \,, \tag{9.14}$$

where, as usual, $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ is the gauge-field strength and A^{μ} is the photon field. The remaining terms on the right-hand side of Eq. (9.13) give the photon-graviton couplings

$$\mathcal{L}^{(1)}_{\gamma} = -\frac{\kappa}{16\pi} \left(\frac{1}{2} h^{\alpha}_{\alpha} F_{\mu\nu} F^{\mu\nu} - 2 h^{\mu\nu} F^{\alpha}_{\mu} F_{\alpha\nu} \right) , \tag{9.15}$$

and

$$\mathcal{L}^{(2)}_{\gamma} = -\frac{\kappa^2}{16\pi} \left\{ -h^{\alpha}_{\alpha} h^{\mu\nu} F^{\alpha}_{\mu} F_{\alpha\nu} + \left[\frac{1}{8} \left(h^{\alpha}_{\alpha} \right)^2 - \frac{1}{4} h_{\mu\nu} h^{\mu\nu} \right] F_{\alpha\beta} F^{\alpha\beta} + h^{\mu\nu} h^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} \right.$$
$$\left. + 2 h^{\mu\alpha} h^{\nu}_{\alpha} F_{\mu\beta} F^{\beta}_{\nu} \right\} . \tag{9.16}$$

In particular, $\mathcal{L}^{(1)}_{\gamma}$ gives the two photons-one graviton interaction vertex, while $\mathcal{L}^{(2)}_{\gamma}$ gives the two photons-two gravitons vertex. Notice that in Eq. (9.13) we considered also a coupling term which is quadratic in κ. This should not create any confusion, since the important fact is that, when we will apply the formula (9.1), the term (9.16) will give a contribution of the same order in powers of κ.

As we made for the graviton field, we expand also the photon field in Fourier space as

$$A_{\mu}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2p^0} \sum_{s=1,2} \left[a_s(\mathbf{p})\, \epsilon^{(s)}_{\mu}\, e^{ipx} + a^{\dagger}_{s'}(\mathbf{p})\, \epsilon^{(s)\,*}_{\mu}\, e^{-ipx} \right] , \tag{9.17}$$

where $\epsilon_\mu^{(s)}$ are the real photon polarization four-vectors, s labels the two physical transverse photon polarizations, a_s and $a^\dagger_{s'}$ are photon annihilation and creation operators, respectively, satisfying the canonical commutation relation

$$\left[a_s(\mathbf{p}), a^\dagger_{s'}(\mathbf{p}')\right] = (2\pi)^3 2p^0 \delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{ss'} \,. \tag{9.18}$$

We fix the Feynman gauge for the photon field, thus the photon propagator is given by [261, 264]

$$D^{(\gamma)}_{\mu\nu}(p) = -4\pi \frac{i}{p^2}\, \eta_{\mu\nu} \,. \tag{9.19}$$

Now, we have all the elements to evaluate the right-hand side of Eq. (9.1). In fact, the expression of the second order S-matrix contribution describing the photon-graviton scattering is

$$\begin{aligned} S^{(2)} = & -\frac{1}{2}\int d^4x_1 d^4x_2\, T\left\{\mathcal{L}^{(1)}_\gamma(x_1)\mathcal{L}^{(1)}_\gamma(x_2)\right\} - \frac{1}{2}\int d^4x_1 d^4x_2\, T\left\{\mathcal{L}^{(1)}_\gamma(x_1)\mathcal{L}^{(1)}_g(x_2)\right\} \\ & + i\int d^4x\, \mathcal{L}^{(2)}_\gamma(x)\,, \end{aligned} \tag{9.20}$$

where T denotes the time-ordering operator.

Now, calling p and p' the incoming and outgoing momenta of the photon, q and q' the incoming and outgoing momenta of the graviton, we can evaluate the second-order S-matrix (9.20) using Feynman's rules (see Refs. [261, 262]). In particular, Feynman diagrams for the photon-graviton scattering are shown in Fig. 9.1. The result is such that the photon-graviton interaction Hamiltonian defined in (9.2) turns out to be

$$\begin{aligned} H_{\gamma g}(t) = & \int d\mathbf{q}d\mathbf{q}'d\mathbf{p}d\mathbf{p}'(2\pi)^3\delta^{(3)}(\mathbf{q}'+\mathbf{p}'-\mathbf{q}-\mathbf{p})\exp\left[it(q'^0+p'^0-q^0-p^0)\right] \\ & \times\left[b^\dagger_{r'}(\mathbf{q}')a^\dagger_{s'}(\mathbf{p}')(M_1+M_2+M_3)a_s(\mathbf{p})b_r(\mathbf{q})\right]\,, \end{aligned} \tag{9.21}$$

where for simplicity of notation

$$d\mathbf{q} = \frac{d^3\mathbf{q}}{(2\pi)^3 2q^0}\,, \qquad\qquad d\mathbf{p} = \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0}\,, \tag{9.22}$$

and the three different Feynman amplitudes appearing in Eq. (9.21) are given by [261]

$$\begin{aligned} M_1 = & \frac{\kappa^2}{p\cdot q}\left[p^\mu(e\cdot\epsilon) - \epsilon^\mu(p\cdot e)\right]\left[p'^\nu(\epsilon'^*\cdot e'^*) - \epsilon'^{*\nu}(p'\cdot e'^*)\right] \\ & \times\left[g^{(0)}_{\mu\nu}(e\cdot p)(e'^*\cdot p') + q_\mu q'_\nu(e\cdot e'^*) - e'^*_\mu q'_\nu(e\cdot p) - q_\mu e_\nu(e'^*\cdot p')\right] + (\epsilon, p \leftrightarrow \epsilon'^*, -p')\,, \end{aligned} \tag{9.23}$$

$$\begin{aligned} M_2 = & \frac{\kappa^2}{2q\cdot q'}\Big\{\left[(p\cdot p')(\epsilon\cdot\epsilon'^*) - (\epsilon'^*\cdot p)(\epsilon\cdot p')\right]\left[2(e\cdot q')(e'^* q) - (e\cdot e'^*)(q\cdot q')\right](e\cdot e'^*) \\ & - \left[(p\cdot p')\epsilon^\mu\epsilon'^{*\nu} + (\epsilon\cdot\epsilon'^*)p^\mu p'^\nu - (p\cdot\epsilon'^*)\epsilon^\mu p'^\nu - (\epsilon\cdot p')p^\mu\epsilon'^{*\nu} + (\epsilon, p\leftrightarrow\epsilon'^*, -p')\right] \\ & \times\left[(e\cdot e'^*)^2 q_\mu q'_\nu - 2(e\cdot e'^*)(q\cdot q')e_\mu e'^*_\nu + 2(e\cdot q')(e'^*\cdot q)e_\mu e'^*_\nu + (e\cdot q')^2 e'^*_\mu e'^*_\nu \right. \\ & \left. + (e'^*\cdot q)^2 e_\mu e_\nu - 2(e\cdot e'^*)(e\cdot q')e'^*_\nu q'_\mu - 2(e\cdot e'^*)(e'^*\cdot q)e_\nu q_\mu\right]\Big\}\,, \end{aligned} \tag{9.24}$$

$$
\begin{aligned}
M_3 =& \kappa^2 \big\{(e\cdot e'^*)^2[(p\cdot p')(\epsilon\cdot\epsilon'^*) - (p\cdot\epsilon'^*)(p'\cdot\epsilon)] - 2[(p\cdot e)(\epsilon\cdot e'^*) - (p\cdot e'^*)(\epsilon\cdot e)] \\
&\times [(p'\cdot e)(\epsilon'^*\cdot e'^*) - (p'\cdot e'^*)(\epsilon'^*\cdot e)] - 2(e\cdot e'^*)[(p\cdot e)(p'\cdot e'^*)(\epsilon\cdot\epsilon'^*) \\
&+ (\epsilon\cdot e)(\epsilon'^*\cdot e'^*)(p\cdot p') - (p\cdot e)(\epsilon'^*\cdot e'^*)(\epsilon\cdot p') - (e\cdot\epsilon)(p'\cdot e'^*)(p\cdot\epsilon'^*) \\
&+ (\epsilon, p \leftrightarrow \epsilon'^*, -p')]\big\} \ , \qquad (9.25)
\end{aligned}
$$

where $e \equiv e^{(r)}(q)$, $e' \equiv e^{(r')}(q')$, $\epsilon \equiv \epsilon^{(s)}(p)$ and $\epsilon' \equiv \epsilon^{(s')}(p')$. At the end, we will see that the contribution of $M_1 + M_3$ amplitudes in the forward scattering vanishes. Hence, we focus only on M_2. After expanding M_2 and doing some algebra we obtain

$$
\begin{aligned}
M_2 = -& \frac{\kappa^2}{2(q\cdot q')} \big\{(e\cdot e'^*)^2 \big[(q\cdot\epsilon'^*)\big((p\cdot p')(\epsilon\cdot q') - (p\cdot q')(\epsilon\cdot p')\big) + (\epsilon\cdot\epsilon'^*)\big((p\cdot q')(q\cdot p') \\
&+(p\cdot p')(q\cdot q') + (p\cdot q)(p'\cdot q')\big) - (p\cdot\epsilon'^*)\big((q\cdot q')(\epsilon\cdot p') + (q\cdot p')(\epsilon\cdot q') + (q\cdot\epsilon)(p'\cdot q')\big) \\
&+(q'\cdot\epsilon'^*)\big((p\cdot p')(q\cdot\epsilon) - (p\cdot q)(\epsilon\cdot p')\big)\big] - 2(e\cdot e'^*)\big\{-(e\cdot p')(p\cdot\epsilon'^*)(q\cdot\epsilon)(q\cdot e'^*) \\
&-(e\cdot\epsilon)(p\cdot\epsilon'^*)(q\cdot p')(q\cdot e'^*) + (e\cdot\epsilon)(p\cdot p')(q\cdot\epsilon'^*)(q\cdot e'^*) - (e\cdot q')(p\cdot\epsilon'^*)(\epsilon\cdot p')(q\cdot e'^*) \\
&-(e\cdot p)(q\cdot\epsilon'^*)(\epsilon\cdot p')(q\cdot e'^*) - (e\cdot p')(p\cdot\epsilon'^*)(q\cdot q')(\epsilon\cdot e'^*) + (e\cdot\epsilon'^*)\big[(p\cdot p')\big((q\cdot\epsilon)(q\cdot e'^*) \\
&+(q\cdot q')(\epsilon\cdot e'^*)\big) - (\epsilon\cdot p')\big((p\cdot q)(q\cdot e'^*) + (p\cdot e'^*)(q\cdot q')\big)\big] - (e\cdot\epsilon)(p\cdot\epsilon'^*)(q\cdot q')(p'\cdot e'^*) \\
&-(e\cdot q')(p\cdot\epsilon'^*)(\epsilon\cdot q')(p'\cdot e'^*) + (e\cdot\epsilon)(p\cdot p')(q\cdot q')(e'^*\cdot\epsilon'^*) - (e\cdot q')(p\cdot q')(\epsilon\cdot p')(e'^*\cdot\epsilon'^*) \\
&-(e\cdot p)(q\cdot q')(\epsilon\cdot p')(e'^*\cdot\epsilon'^*) + (e\cdot q')(p\cdot p')(\epsilon\cdot q')(e'^*\cdot\epsilon'^*) - (e\cdot q')(p\cdot\epsilon'^*)(\epsilon\cdot e'^*)(p'\cdot q') \\
&+(\epsilon\cdot\epsilon'^*)\big[(e\cdot p')\big((p\cdot q)(q\cdot e'^*) + (p\cdot e'^*)(q\cdot q')\big) + (e\cdot p)\big((q\cdot e'^*)(q\cdot p') + (q\cdot q')(e'^*\cdot p')\big) \\
&+(e\cdot q')\big((p\cdot p')(q\cdot e'^*) + (p\cdot q')(e'^*\cdot p') + (p\cdot e'^*)(p'\cdot q')\big)\big] + (e\cdot q')(p\cdot p')(\epsilon\cdot e'^*)(q'\cdot\epsilon'^*) \\
&-(e\cdot q')(p\cdot e'^*)(\epsilon\cdot p')(q'\cdot\epsilon'^*)\big\} + 2\big\{(e\cdot q')^2\big[(p'\cdot e'^*)\big((\epsilon\cdot\epsilon'^*)(p\cdot e'^*) - (p\cdot\epsilon'^*)(\epsilon\cdot e'^*)\big) \\
&+(e'^*\cdot\epsilon'^*)\big((p\cdot p')(\epsilon\cdot e'^*) - (p\cdot e'^*)(\epsilon\cdot p')\big)\big] + (q\cdot e'^*)(e\cdot q')\big[(e\cdot p')\big((\epsilon\cdot\epsilon'^*)(p\cdot e'^*) \\
&-(p\cdot\epsilon'^*)(\epsilon\cdot e'^*)\big) + (e\cdot\epsilon'^*)\big((p\cdot p')(\epsilon\cdot e'^*) - (p\cdot e'^*)(\epsilon\cdot p')\big) + (e\cdot p)\big((\epsilon\cdot\epsilon'^*)(e'^*\cdot p') \\
&-(\epsilon\cdot p')(e'^*\cdot\epsilon'^*)\big)\big] + (e\cdot p)(q\cdot e'^*)^2\big((\epsilon\cdot\epsilon'^*)(e\cdot p') - (e\cdot\epsilon'^*)(\epsilon\cdot p')\big) \\
&+(e\cdot\epsilon)(q\cdot e'^*)\big[(e\cdot\epsilon'^*)(p\cdot p')(q\cdot e'^*) - (p\cdot\epsilon'^*)\big((e\cdot p')(q\cdot e'^*) + (e\cdot q')(e'^*\cdot p')\big) \\
&+(e\cdot q')(p\cdot p')(e'^*\cdot\epsilon'^*)\big]\big\}\big\} \ . \qquad (9.26)
\end{aligned}
$$

Now, using the four-momentum conservation $q \cdot q' = p \cdot p'$, we can rewrite M_2 as

$$
\begin{aligned}
M_2 = &-\frac{\kappa^2}{2}\left\{(e\cdot e'^*)^2\left((q\cdot\epsilon'^*)(\epsilon\cdot q')+(\epsilon\cdot\epsilon'^*)(p\cdot p')-(p\cdot\epsilon'^*)(\epsilon\cdot p')+(q'\cdot\epsilon'^*)(q\cdot\epsilon)\right)\right.\\
&-2(e\cdot e'^*)\left[(e\cdot\epsilon)(q\cdot\epsilon'^*)(q\cdot e'^*)-(e\cdot p')(p\cdot\epsilon'^*)(\epsilon\cdot e'^*)+(e\cdot\epsilon'^*)\left((q\cdot\epsilon)(q\cdot e'^*)\right.\right.\\
&\left.+(p\cdot p')(\epsilon\cdot e'^*)-(\epsilon\cdot p')(p\cdot e'^*)\right)-(e\cdot\epsilon)(p\cdot\epsilon'^*)(p'\cdot e'^*)+(e\cdot\epsilon)(p\cdot p')(e'^*\cdot\epsilon'^*)\\
&-(e\cdot p)(\epsilon\cdot p')(e'^*\cdot\epsilon'^*)+(e\cdot q')(\epsilon\cdot q')(e'^*\cdot\epsilon'^*)+(\epsilon\cdot\epsilon'^*)\left((e\cdot p')(p\cdot e'^*)\right.\\
&\left.\left.+(e\cdot p)(e'^*\cdot p')+(e\cdot q')(q\cdot e'^*)\right)+(e\cdot q')(\epsilon\cdot e'^*)(q'\cdot\epsilon'^*)\right]+2\left[(e\cdot q')^2(e'^*\cdot\epsilon'^*)(\epsilon\cdot e'^*)\right.\\
&\left.\left.+(q\cdot e'^*)(e\cdot q')(e\cdot\epsilon'^*)(\epsilon\cdot e'^*)+(e\cdot\epsilon)(q\cdot e'^*)\left((e\cdot\epsilon'^*)(q\cdot e'^*)+(e\cdot q')(e'^*\cdot\epsilon'^*)\right)\right]\right\}\\
&-\frac{\kappa^2}{2(q\cdot q')}\left\{(e\cdot e'^*)^2\left[-(q\cdot\epsilon'^*)(p\cdot q')(\epsilon\cdot p')+(\epsilon\cdot\epsilon'^*)\left((p\cdot q')(q\cdot p')+(p\cdot q)(p'\cdot q')\right)\right.\right.\\
&\left.-(p\cdot\epsilon'^*)\left((q\cdot p')(\epsilon\cdot q')+(q\cdot\epsilon)(p'\cdot q')\right)-(q'\cdot\epsilon'^*)(p\cdot q)(\epsilon\cdot p')\right]-2(e\cdot e'^*)\\
&\times\left\{-(e\cdot p')(p\cdot\epsilon'^*)(q\cdot\epsilon)(q\cdot e'^*)-(e\cdot\epsilon)(p\cdot\epsilon'^*)(q\cdot p')(q\cdot e'^*)-(e\cdot q')(p\cdot\epsilon'^*)(\epsilon\cdot p')(q\cdot e'^*)\right.\\
&-(e\cdot p)(q\cdot\epsilon'^*)(\epsilon\cdot p')(q\cdot e'^*)-(e\cdot\epsilon'^*)(\epsilon\cdot p')(p\cdot q)(q\cdot e'^*)-(e\cdot q')(p\cdot\epsilon'^*)(\epsilon\cdot q')(p'\cdot e'^*)\\
&-(e\cdot q')(p\cdot q')(\epsilon\cdot p')(e'^*\cdot\epsilon'^*)-(e\cdot q')(p\cdot\epsilon'^*)(\epsilon\cdot e'^*)(p'\cdot q')+(\epsilon\cdot\epsilon'^*)\left[(e\cdot p')(p\cdot q)(q\cdot e'^*)\right.\\
&\left.+(e\cdot p)(q\cdot e'^*)(q\cdot p')+(e\cdot q')\left((p\cdot q')(e'^*\cdot p')+(p\cdot e'^*)(p'\cdot q')\right)\right]\\
&\left.-(e\cdot q')(p\cdot e'^*)(\epsilon\cdot p')(q'\cdot\epsilon'^*)\right\}+2\left\{(e\cdot q')^2\left[(p'\cdot e'^*)\left((\epsilon\cdot\epsilon'^*)(p\cdot e'^*)-(p\cdot\epsilon'^*)(\epsilon\cdot e'^*)\right)\right.\right.\\
&\left.-(e'^*\cdot\epsilon'^*)(p\cdot e'^*)(\epsilon\cdot p')\right]+(q\cdot e'^*)(e\cdot q')\left[(e\cdot p')\left((\epsilon\cdot\epsilon'^*)(p\cdot e'^*)-(p\cdot\epsilon'^*)(\epsilon\cdot e'^*)\right)\right.\\
&\left.-(e\cdot\epsilon'^*)(p\cdot e'^*)(\epsilon\cdot p')+(e\cdot p)\left((\epsilon\cdot\epsilon'^*)(e'^*\cdot p')-(\epsilon\cdot p')(e'^*\cdot\epsilon'^*)\right)\right]\\
&+(e\cdot p)(q\cdot e'^*)^2\left((\epsilon\cdot\epsilon'^*)(e\cdot p')-(e\cdot\epsilon'^*)(\epsilon\cdot p')\right)-(p\cdot\epsilon'^*)(e\cdot\epsilon)(q\cdot e'^*)\left((e\cdot p')(q\cdot e'^*)\right.\\
&\left.\left.\left.+(e\cdot q')(e'^*\cdot p')\right)\right\}\right\}\,. \qquad (9.27)
\end{aligned}
$$

Now, in order to compute the forward scattering term in (9.1), we need to take the expectation value of the following expression:

$$
\begin{aligned}
i\left[H_{\gamma g}(t),\mathcal{D}_{ij}^{(\gamma)}(k)\right] = i\int d\mathbf{q}d\mathbf{q}'d\mathbf{p}d\mathbf{p}'(2\pi)^3\delta^{(3)}(\mathbf{q}'+\mathbf{p}'-\mathbf{q}-\mathbf{p})M_2\left[b_{r'}^\dagger(q')b_r(q)a_{s'}^\dagger(p')a_j(k)\right.\\
\left.\times 2p^0(2\pi)^3\delta_{is}\delta^{(3)}(\mathbf{p}-\mathbf{k})-b_{r'}^\dagger(q')b_r(q)a_i^\dagger(k)a_s(p)2p'^0(2\pi)^3\delta_{js'}\delta^{(3)}(\mathbf{p}'-\mathbf{k})\right]\,. \qquad (9.28)
\end{aligned}
$$

The expectation value of a generic operator A is defined as

$$\langle A\rangle = tr(\hat{\rho}^{(i)}A)\,, \qquad (9.29)$$

$\hat{\rho}^{(i)}$ being the density operator of a system of particles. Applying this definition to the case of photons, one can find the following expression for the expectation value of the product between photon creation and annihilation operators [93]

$$\left\langle a_m^\dagger(\mathbf{p}')a_n(\mathbf{p})\right\rangle = 2p^0(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\rho_{mn}^{(\gamma)}(\mathbf{p})\,, \qquad (9.30)$$

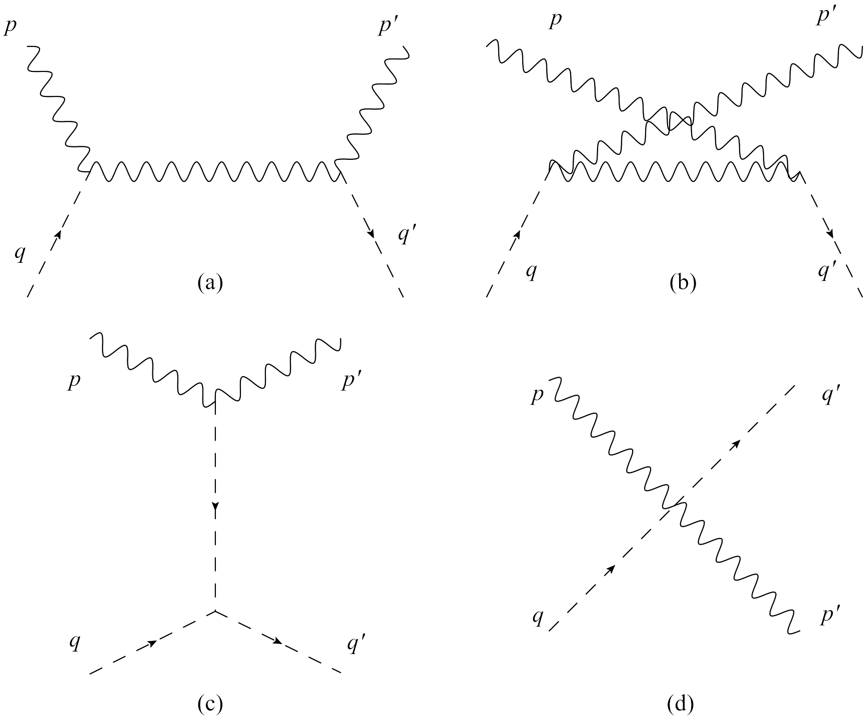


Figure 9.1. Feynman diagrams for the photon-graviton scattering, dashed lines represent gravitons, wavy lines represent photons. Diagrams (a) and (b) give the amplitude M_1, Eq. (9.23); diagram (c) gives the amplitude M_2, Eq. (9.24); diagram (d) gives the amplitude M_3, Eq. (9.25).

where $\rho_{ij}^{(\gamma)}$ is the polarization density matrix of the electromagnetic radiation (4.1)[1]

$$\rho_{ij}^{(\gamma)} = \frac{1}{2}\begin{pmatrix} \Delta_I^{(\gamma)} + \Delta_Q^{(\gamma)} & \Delta_U^{(\gamma)} - i\Delta_V^{(\gamma)} \\ \Delta_U^{(\gamma)} + i\Delta_V^{(\gamma)} & \Delta_I^{(\gamma)} - \Delta_Q^{(\gamma)} \end{pmatrix}, \tag{9.31}$$

$\Delta_I^{(\gamma)}$, $\Delta_Q^{(\gamma)}$, $\Delta_U^{(\gamma)}$, and $\Delta_V^{(\gamma)}$ being the photon Stokes parameters ($I \equiv T$ of Eq. (4.1)).

By the same reasoning, one can find an analogous relation for gravitons

$$\left\langle b_m^\dagger(\mathbf{q}')b_n(\mathbf{q})\right\rangle = 2q^0(2\pi)^3\delta^{(3)}(\mathbf{q}-\mathbf{q}')\rho_{mn}^{(g)}(\mathbf{q}), \tag{9.32}$$

where $\rho_{mn}^{(g)}$ is the polarization density matrix of gravitons [265, 266]

$$\rho_{mn}^{(g)} = \frac{1}{2}\begin{pmatrix} \Delta_I^{(g)} + \Delta_Q^{(g)} & \Delta_U^{(g)} - i\Delta_V^{(g)} \\ \Delta_U^{(g)} + i\Delta_V^{(g)} & \Delta_I^{(g)} - \Delta_Q^{(g)} \end{pmatrix}, \tag{9.33}$$

in which $\Delta_I^{(g)}, \Delta_Q^{(g)}, \Delta_U^{(g)}$ and $\Delta_V^{(g)}$ are the Stokes parameters associated to gravitons; in general, these are given in terms of the power-spectrum statistics of the gravitons. As an example, let us consider $\Delta_I^{(g)}$. The total tensor power-spectrum P_h of gravitons is defined as

$$\langle h_{\mu\nu}^r(\mathbf{x})h_r^{\mu\nu}(\mathbf{x}+\mathbf{r})\rangle = \int \frac{d^3q}{(2\pi)^3}P_h(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{r}}. \tag{9.34}$$

If we insert Eq. (9.8) into the left-hand side of the previous equation and we use Eqs. (9.9) and (9.10) we find

$$\Delta_I^{(g)}(\mathbf{q}) = 2q^0P_h(\mathbf{q}). \tag{9.35}$$

Now, using the expectation values (9.30) and (9.32) and performing the integration over $\mathbf{p}$, $\mathbf{p}'$ and $\mathbf{q}'$, we find that the forward scattering term is given by

$$\begin{aligned} i\left\langle\left[H_f(t), \mathcal{D}_{ij}^{(\gamma)}(k)\right]\right\rangle = i\,(2\pi)^3\delta^{(3)}(0)\int d\mathbf{q}\left(\delta_{is}\rho_{s'j}^{(\gamma)}(\mathbf{k}) - \delta_{js'}\rho_{is}^{(\gamma)}(\mathbf{k})\right)\rho_{rr'}^{(g)}(\mathbf{q}) \\ \times M_2^{r,r',s,s'}(\mathbf{q}'=\mathbf{q}, \mathbf{p}=\mathbf{p}'=\mathbf{k}), \end{aligned} \tag{9.36}$$

where the contraction between Latin indices is made with Kronecker delta. Thus, recalling the expression of M_2, Eq. (9.27), and inserting Eq. (9.36) back into Eq. (9.1), after some straightforward algebra we finally get the forward scattering mixings between the

[1] Here, we use the index γ to distinguish it from the polarization matrix of gravitons (9.33) (with index g).

photon Stokes parameters in the following form

$$\dot{\Delta}_I^{(\gamma)} = 0\,, \tag{9.37}$$

$$\dot{\Delta}_Q^{(\gamma)} = \frac{\kappa^2}{k^0}\int d\mathbf{q}\,\Delta_I^{(g)}(\mathbf{q})\,(q\cdot\epsilon^1(\mathbf{k}))\,(q\cdot\epsilon^2(\mathbf{k}))\Delta_V^{(\gamma)}\ , \tag{9.38}$$

$$\dot{\Delta}_U^{(\gamma)} = -\frac{\kappa^2}{2k^0}\int d\mathbf{q}\,\Delta_I^{(g)}(\mathbf{q})\left[(q\cdot\epsilon^1(\mathbf{k}))^2-(q\cdot\epsilon^2(\mathbf{k}))^2\right]\Delta_V^{(\gamma)}\ , \tag{9.39}$$

$$\dot{\Delta}_V^{(\gamma)} = \frac{\kappa^2}{2k^0}\int d\mathbf{q}\,\Delta_I^{(g)}(\mathbf{q})\left[\left((q\cdot\epsilon^1(\mathbf{k}))^2-(q\cdot\epsilon^2(\mathbf{k}))^2\right)\Delta_U^{(\gamma)}-2\,(q\cdot\epsilon^1(\mathbf{k}))\,(q\cdot\epsilon^2(\mathbf{k}))\Delta_Q^{(\gamma)}\right]\ , \tag{9.40}$$

where dots stand for time derivatives.

Then, in order to take the integral over $d\mathbf{q}$, we fix a coordinate system where the z-axis is aligned with the three-momentum of the scattered photon $\mathbf{k}$ and photon polarization vectors $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ stay along x and y axes. In such a case, we can write the photon and graviton kinematic variables in the following form:

$$\mathbf{k} = k^0\,(0,0,1)\ , \tag{9.41}$$

$$\mathbf{q} = q^0(\sin\theta'\cos\phi',\sin\theta'\sin\phi',\cos\theta')\ , \tag{9.42}$$

$$\boldsymbol{\epsilon}_1 = (1,0,0)\ , \tag{9.43}$$

$$\boldsymbol{\epsilon}_2 = (0,1,0)\ , \tag{9.44}$$

where θ' and ϕ' are the polar angles defining the direction of the three-momentum of the graviton in space. Thus, we can rewrite the previous set of equations as

$$\dot{\Delta}_I^{(\gamma)} = 0\,, \tag{9.45}$$

$$\dot{\Delta}_Q^{(\gamma)} = \frac{\Delta_V^{(\gamma)}}{4k^0}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,q^0\sin^2\theta'\sin 2\phi'\kappa^2\Delta_I^{(g)}(\mathbf{q})\,, \tag{9.46}$$

$$\dot{\Delta}_U^{(\gamma)} = -\frac{V^{(\gamma)}}{4k^0}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,q^0\sin^2\theta'\cos 2\phi'\kappa^2\Delta_I^{(g)}(\mathbf{q})\,, \tag{9.47}$$

$$\dot{\Delta}_V^{(\gamma)} = \frac{1}{4k^0}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,q^0\sin^2\theta'[\cos 2\phi'{\Delta_U}^{(\gamma)}-\sin 2\phi'{\Delta_Q}^{(\gamma)}]\kappa^2\Delta_I^{(g)}(\mathbf{q})\,. \tag{9.48}$$

Finally, using the results of Appendix E, we can write

$$\dot{\Delta}_I^{(\gamma)} = 0\,, \tag{9.49}$$

$$\dot{\Delta}_Q^{(\gamma)} = \frac{\Delta_V^{(\gamma)}}{2\pi k^0}\kappa^2\bar{\rho}_{gw}\sum_{l,m}\int d^2\hat{\mathbf{q}}\, c_{lm}^I \sin^2\theta' \sin 2\phi'\, Y_l^m(\theta',\phi')\,, \tag{9.50}$$

$$\dot{\Delta_U}^{(\gamma)} = -\frac{\Delta_V^{(\gamma)}}{2\pi k^0}\kappa^2\bar{\rho}_{gw}\sum_{l,m}\int d^2\hat{\mathbf{q}}\, c_{lm}^I \sin^2\theta' \cos 2\phi'\, Y_l^m(\theta',\phi')\,, \tag{9.51}$$

$$\dot{\Delta}_V^{(\gamma)} = \frac{1}{2\pi k^0}\kappa^2\bar{\rho}_{gw}\sum_{l,m}\int d^2\hat{\mathbf{q}}\, c_{lm}^I \sin^2\theta'[\cos 2\phi' {\Delta_U}^{(\gamma)} - \sin 2\phi' {\Delta_Q}^{(\gamma)}]\, Y_l^m(\theta',\phi')\,, \tag{9.52}$$

where $\bar{\rho}_{gw}$ is the energy density of gravitons averaged over all directions, Eq. (E.9) and c_{lm}^I are the harmonic coefficients in the decomposition of $\Delta_I^{(g)}(\mathbf{q})$ in terms of spherical harmonics, Eq. (E.6).

Let us briefly comment on this final set of equations: it is straightforward to verify that the source terms appearing in the right-hand sides all identically vanish when photons interact with gravitons that are characterised by a statistically isotropic power-spectrum. Thus, to achieve a nontrivial result, we need the photon to interact with an anisotropic background of gravitons. In the latter case, Q and U photon polarization states couple with the V polarization state, while the I unpolarized state remains unchanged. Notice that this result can be applied in full generality to the interactions involving gravitons and photons of whatever origin. In the next section, we will give some examples applying our results to study the effect on the photon polarization due to the forward scattering with primordial gravitons generated during inflation.

9.3 Forward scattering with inflationary gravitons

As we have seen in Chap. 2, standard slow-roll models of inflation predict an isotropic power-spectrum of primordial gravitons. Therefore, in this case inflationary gravitons have no effect in Eqs. (9.50)-(9.52). In this section, we will briefly review some alternative models of inflation where a certain level of anisotropy in the tensor power-spectrum is generated and, using Eq. (9.35), we will link the power-spectrum predicted by these models to the results found at the end of Sec. 9.2. In the next section, we will provide also a general estimate of the effects on CMB polarization.

Anisotropic solid inflation

Anisotropic solid inflation is a novel inflationary model studied in Refs. [267–269], based on the original model of solid inflation [270]. According to this model, the inflationary period is driven by a configuration which behaves like a solid: the space is fragmented into cells whose location is defined by a triplet of scalar fields $\phi^I(t,\mathbf{x})$, where $I = 1, 2$ or 3. The three scalars can be viewed as the three coordinates that give the position, at time t, of the cell element that, at the time $t = 0$, was in the position $\mathbf{x}$. At the background level one has

$$\langle \phi^I \rangle = x^I \quad , \quad I = 1, 2, 3. \tag{9.53}$$

From the previous equation, we understand that the scalar fields ϕ^I are time-independent at the background level and give a sort of "average position" of each cell. In order to require isotropy and homogeneity of the background, in the Lagrangian of the theory the following internal symmetries are imposed

$$\phi^I \to \phi^I + C^I \tag{9.54}$$

and

$$\phi^I \to O^I_J \phi^J \quad , \quad O^I_J \in SO(3)\,. \tag{9.55}$$

The most general action consistent with the previous symmetries and minimally coupled to gravity is given by [270]

$$S = \int d^4x\sqrt{-g}\left\{\frac{M_P^2}{2}R + F[X, Y, Z]\,,\right\}, \tag{9.56}$$

where

$$X \equiv \mathrm{Tr}\, B = B^{ii} \quad , \quad Y \equiv \frac{\mathrm{Tr}\,\left(B^2\right)}{\left(\mathrm{Tr}\, B\right)^2} \quad , \quad Z \equiv \frac{\mathrm{Tr}\,\left(B^3\right)}{\left(\mathrm{Tr}\, B\right)^3}\,, \quad B^{IJ} \equiv g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J\,. \tag{9.57}$$

Writing down the background cosmological equations, the slow-roll parameters turn out to be [268, 270]

$$\epsilon = \frac{XF_X}{F} \quad , \quad \eta = 2\left(\epsilon - 1 - \frac{X^2F_{XX}}{XF_X}\right), \tag{9.58}$$

where $F_X = \partial F/\partial X$ and the same for Y and Z.

The scalar field perturbations are given by

$$\phi^I = x^I + \pi^I(t, \mathbf{x})\,. \tag{9.59}$$

In particular, it is possible to decompose the perturbations $\pi^I(t, \mathbf{x})$ into a transverse and a longitudinal part, as

$$\pi^I(t, \mathbf{x}) = \partial_I \pi_L(t, \mathbf{x}) + \pi^I_T(t, \mathbf{x})\,, \qquad \partial_I \pi^I_T = 0\,. \tag{9.60}$$

The field $\pi_L(t, \mathbf{x})$ labels the “phonons” for the longitudinal fluctuations of the solid. The sound speeds of longitudinal and transverse excitations are given by [268, 270]

$$c_L^2 \equiv 1 + \frac{2F_{XX}X^2}{3F_XX} + \frac{8(F_Y + F_Z)}{9F_XX}\,, \tag{9.61}$$

and

$$c_T^2 = 1 + \frac{2(F_Y + F_Z)}{3XF_X} = \frac{3}{4}\left(1 + c_L^2 - \frac{2\epsilon}{3} + \frac{\eta}{3}\right). \tag{9.62}$$

The anisotropic version of this model is achieved introducing a preferred direction in the background metric; for example, we can consider a Bianchi type-I background geometry with residual 2d isotropy

$$\begin{aligned} & ds^2 = -dt^2 + a^2\,(t)\,dx^2 + b^2\,(t)\left[dy^2 + dz^2\right] \ , \\ & a \equiv \mathrm{e}^{\alpha - 2\sigma} \quad , \quad b \equiv \mathrm{e}^{\alpha + \sigma} \ , \end{aligned} \tag{9.63}$$

where the field σ labels the anisotropy. Here, the x-axis is labeled as the preferred direction. Einstein's equations for this kind of background applied to solid inflation are given by [268]

$$H^2 - \dot{\sigma}^2 = -\frac{F}{3M_P^2} \ , \tag{9.64}$$

$$\dot{H} + 3\dot{\sigma}^2 = \frac{e^{4\sigma} + 2e^{-2\sigma}}{3M_P^2} e^{-2\alpha} F_X \ , \tag{9.65}$$

$$\ddot{\sigma} + 3H\dot{\sigma} = \frac{2(e^{4\sigma} - e^{-2\sigma})}{3M_P^2} e^{-2\alpha} F_X - \frac{4e^{6\sigma}(1 - e^{6\sigma})F_Y}{(2 + e^{6\sigma})^3 M_P^2} - \frac{6e^{6\sigma}(1 - e^{12\sigma})F_Z}{(2 + e^{6\sigma})^4 M_P^2} \ . \tag{9.66}$$

If we rewrite Eq. (9.66) in the small anisotropy limit (i.e. $\sigma \ll 1$), it becomes

$$\ddot{\sigma} + 3H\dot{\sigma} + 4\epsilon H^2 c_T^2 \sigma = 0 \ . \tag{9.67}$$

Assuming nearly constant values of ϵ and c_T, one can solve Eq. (9.67) finding [268]

$$\sigma(t) \simeq \sigma_1 e^{-\int dt\,\left[(3-(2+c_L^2)\epsilon)\right]H} + \sigma_2 e^{-\int dt\,\frac{4}{3}c_T^2 \epsilon H} \ , \tag{9.68}$$

where σ_1 and σ_2 are two constants.

From Eq. (9.68), we understand that the general solution is a superposition of two kinds of solutions: the solution proportional to σ_1 is fast-decaying, while the solution proportional to σ_2 is slowly decaying. The result is that, immediately after the beginning of inflation, only the second solution survives. Moreover, if inflation does not last longer than a time $1/\sqrt{\epsilon}c_T$, a residual anisotropic deformation of the background is preserved [268]; thus, solid inflation is not efficient in diluting the initial anisotropy, contrary to what happens in the standard slow-roll inflationary scenario.[2] This is due to the fact that to produce inflation, the solid must be insensitive to the spatial expansion, but, at the same time, this makes it rather inefficient in erasing anisotropic deformations of the

[2] In fact, in slow-roll models of inflation with a Bianchi type-I background Eq. (9.67) simplifies, becoming

$$\ddot{\sigma} + 3H\dot{\sigma} = 0 \ .$$

The solution of this equation reads

$$\sigma(t) \simeq \sigma_0 e^{-\int dt\, 3H} \ .$$

Thus, a rapid dilution of all the initial small anisotropy follows. This result is also known as *cosmic no hair theorem*, and states that slow-roll inflation rapidly erases all kinds of anisotropies. For this reason, a Bianchi type-I slow-roll model of inflation is not sufficient to lead to anisotropies in the primordial perturbations.

geometry. In fact, Ref. [268] shows that it is rather general to expect anisotropic evolution in these scenarios. For this reason, during a solid inflationary period, signatures of primordial anisotropies can be left imprinted into primordial perturbations [268, 270]. In particular, in Ref. [269] the anisotropy in the power-spectrum statistics of primordial gravitons has been computed. The final result is

$$P_h^{\text{solid}}(\mathbf{q}) = P_h^{(0)}(q)\left[1+\frac{16N\sigma F_Y}{9\epsilon c_L^5 F}\left[\epsilon c_L^5(1-3\cos^2\alpha)+2\sigma N\frac{F_Y}{F}\sin^4\alpha\right]\right], \tag{9.69}$$

where $P_h^{(0)}(q) = 4H^2/q^3M_{\text{Pl}}^2$ is the total power-spectrum of gravitons in the standard slow-roll inflationary models as computed in Sec. 2.5, $N = -\ln(-q\eta_e)$ is the number of e-folds when the mode $\mathbf{q}$ leaves the horizon until the end of inflation, and α is the angle between the preferred direction and the momentum $\mathbf{q}$ of the graviton.

Now, let us assume that the preferred direction is completely general in the (x, y) plane,[3] so that $\hat{n} = (\cos\phi, \sin\phi, 0)$. The angle α between the preferred direction $\hat{n}$ and the generic graviton momentum $\hat{q} = (\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$ is given by

$$\cos\alpha = \hat{n}\cdot\hat{q} = \cos\phi'\sin\theta'\cos\phi + \sin\phi'\sin\theta'\sin\phi\,, \tag{9.70}$$

$$\sin^2\alpha = 1-\cos^2\alpha = 1-\cos^2\phi'\sin^2\theta'\cos^2\phi-\sin^2\phi'\sin^2\theta'\sin^2\phi-2\cos\phi'\sin^2\theta'\cos\phi\sin\phi'\sin\phi\,. \tag{9.71}$$

Thus, apart from some coefficients, the integrals we have to compute turn out to be

$$I_1 = \frac{1}{4}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,q^0\sin^2\theta'\sin 2\phi'\kappa^2\Delta_{I_{\text{solid}}}^{(g)}(\mathbf{q}) \;= -B\,\sin(2\phi)\,, \tag{9.72}$$

and

$$I_2 = \frac{1}{4}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,q^0\sin^2\theta'\cos 2\phi'\kappa^2\Delta_{I_{\text{solid}}}^{(g)}(\mathbf{q}) \;= -B\,\cos(2\phi)\,, \tag{9.73}$$

where

$$B = \kappa^2\bar{\rho}_{gw}\frac{4N\sigma F_Y}{9\epsilon c_L^5 F}\left[\frac{8}{5}\epsilon c_L^5+\frac{128}{105}\sigma N\frac{F_Y}{F}\right], \tag{9.74}$$

$\bar{\rho}_{gw}$ being the energy density of primordial gravitons averaged over all directions, as defined in Eq. (E.9).

At the end, inserting Eqs. (9.72) and (9.73) into Eqs. (9.46)-(9.48), we obtain

$$\dot{\Delta}_Q^{(\gamma)} = \frac{I_1}{k^0}\Delta_V^{(\gamma)}\,, \tag{9.75}$$

$$\dot{\Delta_U}^{(\gamma)} = -\frac{I_2}{k^0}\Delta_V^{(\gamma)}\,, \tag{9.76}$$

$$\dot{\Delta}_V^{(\gamma)} = \frac{1}{k^0}\left[I_2\Delta_U{}^{(\gamma)} - I_1\Delta_Q{}^{(\gamma)}\right]. \tag{9.77}$$

[3] Notice that, in principle, we could choose completely general $\hat{n}$. However, from angular integrals in Eqs. (9.46)-(9.48) it follows that it is sufficient to introduce in $\Delta_I^{(g)}$ a dependence on the polar angle ϕ' to achieve a non-trivial result.

Squeezed non-Gaussianity anisotropic imprint

Squeezed tensor bispectra can lead to anisotropic modulations of the tensor power-spectrum in some inflationary scenarios, similarly to the effect induced in the curvature power-spectrum by the so-called "tensor fossils'" (see [78, 84, 211, 271–274]). This can happen, e.g., in models of inflation where space-time diffeomorphisms are spontaneously broken (see, e.g., [84, 274]). Here, spontaneous breaking means that one or more scalar fields driving inflation admit a background value which is not invariant under a generic space-time reparametrization.

In general, the squeezed limit of the three-gravitons bispectrum is given by (see, e.g., [274])

$$\langle h_{s_1}(\mathbf{q_1})h_{s_2}(\mathbf{q_2})h_s(\mathbf{Q})\rangle \xrightarrow[q_1\simeq q_2]{Q\to 0} (2\pi)^3\delta^{(3)}(\mathbf{q_1}+\mathbf{q_2}+\mathbf{Q})P_h(Q)P_h(q)\left(\frac{3}{2}+f_{\rm NL}\right)\epsilon^s_{ij}(\mathbf{Q})\,\mathbf{\hat{q}_1^i}\mathbf{\hat{q}_2^j}\,\delta_{s_1 s_2}\,, \tag{9.78}$$

where $\mathbf{Q}$ and $\mathbf{q}$ are the momenta of the long and short modes, respectively. The parameter $f_{\rm NL}$ characterizes how much we are violating the so-called Maldacena's consistency relation (see Ref. [39]), due to the spontaneously breaking of space-time reparametrizations.

It is possible to show that a single soft graviton $\tilde{h}_s(\mathbf{Q})$ is able to modulate the tensor power-spectrum as follows [274]

$$P_h(\mathbf{q})|_h = P_h(q)^{(0)} + \tilde{h}_s(\mathbf{Q})\frac{\langle h_{s_1}(\mathbf{q_1})h_{s_2}(\mathbf{q_2})h_s(\mathbf{Q})\rangle'}{P_h(Q)}\,, \tag{9.79}$$

where $P_h^{(0)}$ is the unmodulated total tensor power-spectrum and the prime $'$ means that we have to drop the Dirac-delta in the corresponding expression. Moreover, if we want to look for this modulation at a given position $\mathbf{x}$ in a cube of volume V of physical space, we should consider the cumulative effect of all soft graviton modes with minimum wavelength $\lambda_L = V^{1/3}$. This leads to a local quadrupolar anisotropy in the total tensor power-spectrum which can be parametrized as

$$P_h^{\rm squeezed}(\mathbf{q},\mathbf{x}) = P_h^{(0)}(q)\left(1+\gamma_{ij}(\mathbf{x})\hat{q}^i\hat{q}^j\right)\,, \tag{9.80}$$

where

$$\gamma_{ij}(\mathbf{x}) = \frac{f_{\rm NL}}{V_L}\int_{|Q|<Q_L} d^3Q\; e^{i\mathbf{Q}\cdot\mathbf{x}}\; h_s(\mathbf{Q})\epsilon^s_{ij}(\mathbf{Q})\,. \tag{9.81}$$

From its definition $\gamma_{ij}(\mathbf{x})$ depends on the position in space and is a stochastic Gaussian tensor field, with variance given by

$$\langle\gamma_{ij}(\mathbf{x})\gamma_{ij}(\mathbf{x})\rangle = \frac{f_{\rm NL}^2}{V_L^2}\int_{|Q|<Q_L} d^3Q\, P_h(\mathbf{Q})\,. \tag{9.82}$$

From Eq. (9.80) we understand that $\gamma_{ij}(\mathbf{x})$ labels the local preferred directions of the tensor power-spectrum, due to the modulation provided by long modes. For simplicity, we can assume that $\gamma_{ij}(\mathbf{x})$ is a constant, thus $\gamma_{ij}(\mathbf{x}) = \gamma_{ij}$. In this case, γ_{ij} is a three-dimensional symmetric constant matrix whose entries are proportional to $f_{\rm NL}$ and fix the form of the quadrupolar angular dependence. Notice that, a priori, γ_{ij} takes a random value extracted from a Gaussian distribution with variance given by Eq. (9.82). So, the precise angular dependence of the quadrupolar anisotropy is not completely fixed by the theory, but depend on which particular realization γ_{ij} is fixed during inflation.

Now, we take the modulated power-spectrum

$$P_h^{\rm squeezed}(\mathbf{q}) = P_h^{(0)}(q)\left(1 + \bar{\gamma}_{ij}\hat{q}^i\hat{q}^j\right) , \tag{9.83}$$

with a fixed value $\bar{\gamma}_{ij}$, and we substitute Eq. (9.83) into Eqs. (9.46)-(9.48). Recalling Eq. (9.35) and performing some simple integrals, we finally get

$$\dot{\Delta}_Q^{(\gamma)} = \bar{\gamma}_{12}\frac{4}{15}\frac{\kappa^2\bar{\rho}_{gw}}{k^0}\Delta_V^{(\gamma)} , \tag{9.84}$$

$$\dot{\Delta}_U^{(\gamma)} = -\left(\bar{\gamma}_{11} - \bar{\gamma}_{22}\right)\frac{2}{15}\frac{\kappa^2\bar{\rho}_{gw}}{k^0}\Delta_V^{(\gamma)} , \tag{9.85}$$

$$\dot{\Delta}_V^{(\gamma)} = \frac{2}{15}\frac{\kappa^2\bar{\rho}_{gw}}{k^0}\left[\left(\bar{\gamma}_{11} - \bar{\gamma}_{22}\right)\Delta_U^{(\gamma)} - 2\bar{\gamma}_{12}\Delta_Q^{(\gamma)}\right] . \tag{9.86}$$

9.4 Effects on CMB polarization

In this section, we evaluate the effect of the (quantum) Boltzmann equations we derived on the CMB photon polarization.

As we have seen in Sec. 4.2, before the recombination epoch CMB photons were basically unpolarized, thus $\Delta_Q = \Delta_U = \Delta_V = 0$. In this case, it is straightforward to show that no new physics is provided by the forward scattering mixing we derived. At the time of recombination, a small amount of Δ_Q polarization modes due to the Compton scattering of photons with baryons is formed [19]. Thus, after the recombination epoch, we start with initial conditions where only the Δ_Q polarization mode is nonzero, i.e. $\Delta_Q(0) = \Delta_{Q_{\rm init}}$ and $\Delta_U(0) = \Delta_V(0) = 0$. In such a case, according to our mixing, CMB V modes are initially coupled only with Q modes, through a set of differential equations like

$$\dot{\Delta}_V = \left(\frac{\kappa^2\bar{\rho}_{gw}}{k^0}A\right)\Delta_Q , \tag{9.87}$$

$$\dot{\Delta}_Q = -\left(\frac{\kappa^2\bar{\rho}_{gw}}{k^0}A\right)\Delta_V , \tag{9.88}$$

where

$$A = \frac{1}{2\pi}\sum_{l,m}\int d^2\hat{\mathbf{q}}\, c_{lm}^I \sin^2\theta' \sin 2\phi' Y_l^m(\theta', \phi') \tag{9.89}$$

is a dimensionless parameter depending on the underlying theory. In Eqs. (9.87) and (9.88) we have dropped the U modes. In fact, in our Boltzmann equation U modes vanish at the beginning of the time evolution and they are coupled only with V modes, hence they cannot be produced until a reasonable amount of V modes is produced in turn. If we neglect the time dependence of k^0 and $\bar{\rho}_{gw}$ due to the CMB gravitational redshift, this kind of coupled set of differential equations can be easily solved leading to the oscillatory behavior

$$\Delta_V = \Delta_{Q_{\rm init}} \sin(\omega t)\,, \tag{9.90}$$
$$\Delta_Q = \Delta_{Q_{\rm init}} \cos(\omega t)\,, \tag{9.91}$$

where

$$\omega = \frac{\kappa^2 \bar{\rho}_{gw}}{k^0} A \tag{9.92}$$

is the frequency of the oscillation.

The result is that the value of Δ_Q starts to decrease like a cosine, while the value of Δ_V grows like a sine. At a certain time, when the value of Δ_V becomes important, the coupling between V and U modes should be taken into consideration and can potentially lead also to the generation of U modes, modifying our general solution. However, let us neglect for simplicity the U modes in all the discussion. This is equivalent to make a fine-tuning in the parameters of the models we considered, in order to decouple U and V modes in the Boltzmann equations.[4]

From Eqs. (9.87) and (9.88), we understand that the effect on CMB polarization is greater for photons with smaller wave number k^0, thus we consider CMB photons with comoving frequency of 1 GHz ($k^0 = 2\pi f$), which is the order of magnitude of the smallest CMB frequency that has been measured.[5] From the constraint on the total energy density of gravitational waves today, provided by nucleosynthesis ($\bar{\rho}_{gw} \lesssim 10^{-5}\rho^o_{crit}$, where ρ^o_{crit} is the critical energy density of the Universe today, see Ref. [276]) and from the definition of $\kappa^2 = 16\pi G$, it follows that, at the time of recombination epoch (at redshift $z \approx 1100$), we have

$$\omega_{rec}(f = 1\,\text{GHz}) \lesssim 5 \times 10^{-41} A\ s^{-1}\,. \tag{9.93}$$

From this constraint it follows that, if we take $A \sim 1$, neglect the gravitational red-shift, and evaluate Eq. (9.90) after 10^9 years (the time interval from the recombination epoch until today), we get

$$\Delta_{V_{\rm today}} \lesssim 10^{-25} \Delta_{Q_{\rm init}}\,. \tag{9.94}$$

In this last case, neglecting in 10^9 years the effect of gravitational redshift on the frequency k^0 and on the energy density $\bar{\rho}_{gw}$ is indeed not a good approximation. However, if we account for it, we would expect that the upper bound on Δ_V is even smaller than the

[4] In the case of *anisotropic solid inflation* it is enough to choose $\phi = \pi/4$ in Eqs. (9.72) and (9.73); in the case of *squeezed non-Gaussianity theories* we need $(\bar{\gamma}_{11} - \bar{\gamma}_{22}) = 0$ in Eqs. (9.85) and (9.86).

[5] More precisely, the lowest measured CMB frequency corresponds to 0.6 GHz as measured by the TRIS instrument (see e.g. Ref. [275]).

one shown in Eq. (9.94).[6] Therefore, these estimations show that the forward scattering between CMB photons and (primordial) anisotropic gravitons leads to the production of circular polarization in the CMB, but this in general is very inefficient. This brief analysis suggests that unfortunately the CMB does not seem to be the best source of photons to be used in searching for a signature of anisotropic (primordial) gravitons, and one should look for other kinds of sources.

9.5 Conclusions

In this chapter, we studied the forward scattering of photons with gravitons, focusing on the effect that this scattering has on photon polarization. We derived fully general equations which display a coupling among the Δ_Q, Δ_U linear polarization states and the Δ_V circular polarization state of photons. These couplings are not vanishing only if photons interact with gravitons that have anisotropies in their power-spectrum statistics. As an application of our general results, we have considered some models of inflation where primordial anisotropic gravitons are generated and we linked our Boltzmann equations to these models. Finally, we evaluated the effect of a primordial anisotropic background of gravitons on the CMB polarization. We saw that the effect on the CMB is expected to be very small (leading to a very inefficient conversion of linear polarization in circular polarization) in a way that it is probably impossible to measure it via CMB polarization measurements. However, we have to be open-minded and think about alternative scenarios where "artificial" polarized photons can be employed: in this case, playing with the initial values of the Stokes parameters and with the frequency of the photons we could enhance the effects on the photons polarization. Thus, we could use controlled photons to probe statistical anisotropies in (primordial) backgrounds of gravitational waves, e.g. constraining free parameters of alternative models of inflation. More in general, we could use our result to search for any source of anisotropic gravitons in the Universe (in particular, in Ref. [277], our general results have been used to show that significant circular polarization on photons from a binary merger can be generated due to the forward scattering with the gravitational waves counterpart). In the future, we could use this new tool to get insight into the physics of gravitational waves and provide an innovative way to look for gravitational-wave events.

[6] In fact, the energy density of gravitons and the physical wave number of CMB photons in terms of gravitational redshift z are given respectively by

$$\rho_{crit}(z) = (1+z)^4 \rho^o_{crit}$$

and

$$k^0(z) = (1+z)\, k^0_{\text{com}} \, .$$

Thus, the parameter $\omega(z)$ in Eq. (9.92) scales as

$$\omega(z) = (1+z)^3 \omega_o \, .$$

Inserting the latter equation into Eqs. (9.87) and (9.88), we understand that the strength of the coupling between Q and V modes in the past was larger than the one today. Thus, if we neglect the effect of gravitational redshift, we overestimate the final amount of V modes today.

Chapter 10

CMB circular and B-mode polarization from photon-fermion forward scattering

10.1 Introduction

In this chapter, we will study V-mode polarization generation in the CMB radiation from its direct coupling with linear polarization states induced by the forward scattering of photons with generic fermions at or after the recombination epoch. In particular, we will assume a completely general photon-fermion interaction which may also go beyond QED, but still preserving the combination of charge conjugation, parity and time reversal (CPT), which up to now is observed to be an exact symmetry of nature at a fundamental level. In order to do so, we will use a generic parametrization of the photon-fermion scattering amplitude which follows only by the imposition of gauge-invariance (see e.g. Refs. [278–280]).

We will show that V modes can be produced by forward scattering for a generic interaction preserving all the C (charge conjugation), P (parity), and T (time-reversal) discrete symmetries, if the stress tensor of the fermion contains anisotropies. In addition, we will show that V modes can be sourced also from an interaction violating C and P symmetries, but preserving the CP combination. In this case, together with the anisotropies in the fermionic stress tensor, we need the fermion to interact with the photon only in the L- or R-handed helicity state, like the L-handed neutrino in the Standard Model interactions. In particular, this last case confirms and generalizes the results found in Ref. [250]. We will also analyze the cases in which C, T and P, T symmetries are violated individually, while preserving, respectively, the combinations CT and PT. We will show that, in these cases, it is impossible to generate V modes by forward scattering, but we can have formation of CMB B modes. In particular, in the case of a generic interaction which violates P, T symmetries, it is possible to generate B modes with no conditions on the fermions the photons interact with, while in the case in which C, T are violated we need the fermion to be in the L- or R-handed helicity state. All these conclusions are summarized in Tab. 10.1.

10.2 General parametrization of photon-fermion scattering amplitude

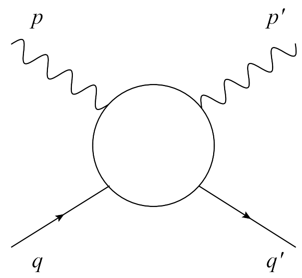


Figure 10.1. Figurative representation of photon-fermion interaction.

We are interested in the Compton scattering of a photon by a fermion (Fig. 10.1)

$$\gamma(p) + f(q) \to \gamma(p') + f(q') \,, \tag{10.1}$$

where $p\,(p')$ is the initial (final) momentum of the photon and $q\,(q')$ is the initial (final) momentum of the fermion. It is possible to construct the invariant amplitude of this process using a general method. The amplitude of such a process can be written in the form [278–280])

$$M_{fi} = F^{\lambda\mu} \epsilon_{\lambda}^{s'*} \epsilon_{\mu}^{s} \,. \tag{10.2}$$

where ϵ_{μ}^{s} and $\epsilon_{\nu}^{s'}$ are the polarization vectors of incoming and outgoing photons and $s, s' = 1, 2$ label the physical transverse polarization of the photons. Gauge-invariance requires $\epsilon^{s} \cdot p = \epsilon^{s'} \cdot p' = 0$. Moreover, the rank-2 tensor $F^{\mu\nu}$, which is called "Compton tensor", must satisfy the conserved current condition $p_{\mu} F^{\mu\nu} = p'_{\nu} F^{\mu\nu} = 0$, as a consequence of gauge-invariance. It is possible to provide a general parametrization of $F^{\mu\nu}$ satisfying the previous condition from the linear combination of basis vectors defined below.

We first construct a general form for the Compton tensor $F^{\mu\nu}$ and then study its parity conserving and parity violating aspects. Using the procedure of Refs. [278–280]), we can write

$$\begin{aligned} F^{\mu\nu} =& G_0 \left(\hat{e}^{(1)\mu}\hat{e}^{(1)\nu} + \hat{e}^{(2)\mu}\hat{e}^{(2)\nu}\right) + G_1 \left(\hat{e}^{(1)\mu}\hat{e}^{(2)\nu} + \hat{e}^{(2)\mu}\hat{e}^{(1)\nu}\right) + G_2 \left(\hat{e}^{(1)\mu}\hat{e}^{(2)\nu} - \hat{e}^{(2)\mu}\hat{e}^{(1)\nu}\right) \\ &+ G_3 \left(\hat{e}^{(1)\mu}\hat{e}^{(1)\nu} - \hat{e}^{(2)\mu}\hat{e}^{(2)\nu}\right) \,, \end{aligned} \tag{10.3}$$

where G_i are invariant functions and $e^{(1)}$ and $e^{(2)}$ are two four-vectors satisfying the orthogonality condition $\hat{e}^{(1)} \cdot \hat{e}^{(2)} = 0$. In order to construct these two vectors, we have to use only the kinematic variables p, p', q, and q' and define a system of orthogonal vector basis of the form

$$Q^{\lambda} = (q^{\lambda} + q'^{\lambda}) - \frac{P^{\lambda}}{P^2}\,(q + q') \cdot P \,, \tag{10.4}$$

$$P^\lambda = p^\lambda + p'^\lambda\,, \tag{10.5}$$
$$N^\lambda = \epsilon^{\lambda\mu\nu\rho} Q_\mu t_\nu P_\rho\,, \tag{10.6}$$

where t^λ, for the tree-level contribution to the scattering amplitude, is given by

$$t^\lambda = q^\lambda - q'^\lambda = p'^\lambda - p^\lambda\,. \tag{10.7}$$

A possible choice of the normalized $\hat{e}^{(1)}$ and $\hat{e}^{(2)}$ four-vectors is given by (see e.g. [278])

$$\hat{e}^{(1)\lambda} = \frac{N^\lambda}{\sqrt{-N^2}}\,, \tag{10.8}$$

and

$$\hat{e}^{(2)\lambda} = \frac{Q^\lambda}{\sqrt{-Q^2}}\,. \tag{10.9}$$

From these definitions it is easy to verify the conserved current condition as

$$(P_\nu + t_\nu)F^{\mu\nu} = (P_\mu - t_\mu)F^{\mu\nu} = 0\,. \tag{10.10}$$

Here, we are interested in the forward scattering limit in which $t^\lambda = 0$ and $P^2 = 4p^2 = 0$. Under this condition, N^λ vanishes and the second term in Q^λ becomes singular. Therefore, $\hat{e}^{(1)\lambda}$ and $\hat{e}^{(2)\lambda}$ are not well-defined. In order to overcome these problems, we firstly change the normalization in $\hat{e}^{(2)\lambda}$ as

$$\hat{e}^{(2)\lambda} = \frac{Q^\lambda}{\sqrt{-4q^2}} = \frac{Q^\lambda}{\sqrt{-4m_f^2}}\,, \tag{10.11}$$

by noting that the second term in Q^λ does not contribute to the amplitude. Second, we introduce a new general quantity Δ^λ replacing t^λ in Eq. (1.51). This quantity Δ^λ has to be expressed in terms of kinematic variables and invariants of the interaction. However, in the forward scattering limit, any linear combination of the photon and electron four-momenta p^μ and q^μ leads to a vanishing value of $\hat{e}^{(1)\lambda}$ when doing the contractions with the Levi-Civita pseudotensor in Eq. (1.51). Hence, Δ^λ has to be given only in terms of scalar invariant quantities. Thus, the only possibility to define Δ^λ reads

$$\Delta^\lambda = (\Delta^0, 0)\,, \tag{10.12}$$

where Δ^0 is a generic function of scalar invariants in the interaction. Therefore, in the forward scattering limit, the four-vector N^λ becomes

$$\begin{aligned} N^\lambda &= \epsilon^{\lambda\mu 0\rho} Q_\mu \Delta^0 P_\rho \\ &= 4\epsilon^{\lambda\mu 0\rho} q_\mu \Delta^0 p_\rho\,, \end{aligned} \tag{10.13}$$

and $\hat{e}^{(1)}$ is now defined as

$$\hat{e}^{(1)i} = \frac{N^i}{\sqrt{-\mathbf{N}^2}} \qquad \hat{e}^{(1)0} = 0\,, \tag{10.14}$$

where $\mathbf{N}^2$ will stand for the modulus square of the three-vector

$$N^i = 4\epsilon^{ij0k} q_j \Delta^0 p_k \,, \tag{10.15}$$

which gives

$$\mathbf{N}^2 = 16(\Delta^0)^2 |\mathbf{p}\times\mathbf{q}|^2 \,. \tag{10.16}$$

It is easy to verify that $F_{\mu\nu}$, with the new definitions of $\hat{e}^{(1)\lambda}$ and $\hat{e}^{(2)\lambda}$ in Eqs. (10.14) and (10.11), satisfies the conserved current condition.

Before proceeding, it is worth to rewrite the factor of G_2 in Eq. (10.3) in a new form for the case of forward scattering. Using the following identity regarding the Levi-Civita pseudotensor [281]

$$g_{\lambda\mu}\epsilon_{\nu\alpha\beta\gamma} - g_{\lambda\nu}\epsilon_{\mu\alpha\beta\gamma} + g_{\lambda\alpha}\epsilon_{\mu\nu\beta\gamma} - g_{\lambda\beta}\epsilon_{\mu\nu\alpha\gamma} + g_{\lambda\gamma}\epsilon_{\mu\nu\alpha\beta} = 0 \,, \tag{10.17}$$

we obtain

$$\begin{aligned}
\hat{e}^{(1)\mu}\hat{e}^{(2)\nu} - \hat{e}^{(2)\mu}\hat{e}^{(1)\nu} =& \frac{4}{\sqrt{m_f^2\mathbf{N}^2}} q_\lambda \Delta_\beta q_\alpha p_\gamma \left(g^{\nu\lambda}\epsilon^{\mu\alpha\beta\gamma} - g^{\mu\lambda}\epsilon^{\nu\alpha\beta\gamma}\right) \\
=& \frac{4}{\sqrt{m_f^2\mathbf{N}^2}} q_\lambda \Delta_\beta q_\alpha p_\gamma \left(g^{\lambda\alpha}\epsilon^{\mu\nu\beta\gamma} - g^{\lambda\beta}\epsilon^{\mu\nu\alpha\gamma} + g^{\lambda\gamma}\epsilon^{\mu\nu\alpha\beta}\right) \\
=& \frac{4}{\sqrt{m_f^2\mathbf{N}^2}} \left(q^2\Delta_\beta p_\gamma \epsilon^{\mu\nu\beta\gamma} - q\cdot\Delta\, q_\alpha p_\gamma \epsilon^{\mu\nu\alpha\gamma} + q\cdot p\, q_\alpha \Delta_\beta \epsilon^{\mu\nu\alpha\beta}\right) .
\end{aligned} \tag{10.18}$$

Thus, in the end we have

$$\hat{e}^{(1)\mu}\hat{e}^{(2)\nu} - \hat{e}^{(2)\mu}\hat{e}^{(1)\nu} = \frac{4}{\sqrt{m_f^2\mathbf{N}^2}} \left(q^2\Delta_\alpha p_\beta - q\cdot\Delta\, q_\alpha p_\beta + q\cdot p\, q_\alpha \Delta_\beta\right) \epsilon^{\mu\nu\alpha\beta} \,. \tag{10.19}$$

Notice that the second term on the right-hand side of Eq. (10.19) is equal in form to the one that appears at quantum level in the interaction of a photon with the magnetic moment of a neutrino (see e.g. Refs. [282, 283]). Using the definition of Δ^λ in Eq. (10.12), we can further simplify Eq. (10.19) into[1]

$$\begin{aligned}
\hat{e}^{(1)\mu}\hat{e}^{(2)\nu} - \hat{e}^{(2)\mu}\hat{e}^{(1)\nu} =& \frac{4}{\sqrt{m_f^2\mathbf{N}^2}} \left[(q_0^2 - q^2)\Delta^0\, p_k + (q\cdot p - q_0\, p_0)\Delta^0\, q_k\right] \epsilon^{\mu\nu k0} \\
=& \frac{4\Delta^0}{\sqrt{m_f^2\mathbf{N}^2}} \left[|\mathbf{q}|^2\, p_k - (\mathbf{q}\cdot\mathbf{p})\, q_k\right] \epsilon^{\mu\nu k0} \,.
\end{aligned} \tag{10.20}$$

[1]Here we are implicitly assuming that Greek indices take only Latin values. In fact, as we will see later on, only the Latin components of the Compton tensor will be important.

Moreover, it is worth noticing that $\hat{e}^{(1)}$ is an axial vector and $\hat{e}^{(2)}$ is a vector. Using this property, it is straightforward to verify that the second and third brackets in Eq. (10.3) change sign under parity transformation, while the first and fourth brackets remain unchanged. Both these two combinations of the Compton tensor satisfy the crossing symmetry and gauge-invariance. However, $F^{\mu\nu}$ can be even or odd under parity.

In order to discuss these cases, we first provide the general expression for the coefficients G_i, and then we start from the parity-invariant case by deriving all nonvanishing terms of each G_i under the parity-invariance condition of the scattering amplitude. The coefficients can be represented in terms of the following bilinear covariant terms [278–280])

$$G_0 = \bar{u}_{r'}\left[f_1 + f_2\not{P} + f_3\gamma^5 + f_4\gamma^5\not{P}\right] u_r\,, \tag{10.21}$$

$$G_1 = \bar{u}_{r'}\left[f_5 + f_6\not{P} + f_7\gamma^5 + f_8\gamma^5\not{P}\right] u_r\,, \tag{10.22}$$

$$G_2 = \bar{u}_{r'}\left[f_9 + f_{10}\not{P} + f_{11}\gamma^5 + f_{12}\gamma^5\not{P}\right] u_r\,, \tag{10.23}$$

$$G_3 = \bar{u}_{r'}\left[f_{13} + f_{14}\not{P} + f_{15}\gamma^5 + f_{16}\gamma^5\not{P}\right] u_r\,, \tag{10.24}$$

where u_r and $\bar{u}_{r'}$ are Dirac spinors associated to the fermion; r, r' label fermion spin, $\not{P} = P_\mu\gamma^\mu$, γ^μ, and γ^5 are Dirac matrices and f_i are constant coefficients.

The invariant functions G_i involve four possibilities. One can show that the $\not{Q}$ and $\not{t}$ terms are nothing else than numbers due to the Dirac equation and hence they do not appear in the G_i invariants. Similarly, all higher powers of the γ^μ matrices are reduced to the above four possibilities. With the above representation, the time-reversal and parity transformations of each bilinear term are evident.

Even-parity amplitude

In this subsection, we determine the form of the fermion-photon scattering amplitude with the condition that the amplitude is even under parity transformation. As we have seen, the photon scattering amplitude is represented by (10.2). Since under parity transformation, the polarization vectors change as

$$(\epsilon_0, \boldsymbol{\epsilon}) \leftrightarrow (\epsilon_0, -\boldsymbol{\epsilon})\,, \tag{10.25}$$

the condition of parity invariance of scattering amplitude M_{fi} implies

$$(F^{00}, F^{i0}, F^{ik}) \to (F^{00}, -F^{i0}, F^{ik})\,. \tag{10.26}$$

Using the fact that $\hat{e}^{(1)}$ and $\hat{e}^{(2)}$ are a pseudovector and a vector, respectively, G_0 and G_3 must be scalars and G_1 and G_2 must be pseudoscalars. Consequently, we can obtain the following constraints, as a result of the even-parity condition

$$f_3 = f_4 = f_5 = f_6 = f_9 = f_{10} = f_{15} = f_{16} = 0\,. \tag{10.27}$$

Then, we impose the condition of time-reversal invariance. Under time-reversal we have

$$(q_0, \mathbf{q}) \leftrightarrow (q_0', -\mathbf{q}') , \qquad (p_0, \mathbf{p}) \leftrightarrow (p_0', -\mathbf{p}') , \tag{10.28}$$

and

$$(\epsilon_0, \boldsymbol{\epsilon}) \leftrightarrow (\epsilon_0'^{*}, -\boldsymbol{\epsilon}'^{*}) . \tag{10.29}$$

Hence, invariance of the scattering amplitude M_{fi} under time-reversal yields

$$(F^{00}, F^{i0}, F^{ik}) \rightarrow (F^{00}, -F^{0i}, F^{ki}) . \tag{10.30}$$

Similarly, the relations in Eq. (10.28) imply

$$\begin{aligned} &(Q_0, \mathbf{Q}) \rightarrow (Q_0, -\mathbf{Q}) , && (t_0, \mathbf{t}) \rightarrow (-t_0, \mathbf{t}) , \\ &(P_0, \mathbf{P}) \rightarrow (P_0, -\mathbf{P}) , && (N_0, \mathbf{N}) \rightarrow (N_0, -\mathbf{N}) , \end{aligned} \tag{10.31}$$

so that

$$\left(\hat{e}_0^{(1,2)}, \hat{\boldsymbol{e}}^{(1,2)}\right) \rightarrow \left(\hat{e}_0^{(1,2)}, -\hat{\boldsymbol{e}}^{(1,2)}\right) . \tag{10.32}$$

Thus, invariance under time-reversal implies

$$G_{0,1,3} \rightarrow G_{0,1,3} , \qquad G_2 \rightarrow -G_2 , \tag{10.33}$$

and based on the following properties of spinor bilinear terms under a time-reversal transformation

$$\bar{u}'\gamma^5 u \rightarrow -\bar{u}'\gamma^5 u , \qquad \bar{u}'\gamma^5 \not{P} u \rightarrow \bar{u}'\gamma^5 \not{P} u , \tag{10.34}$$

one can verify the following additional conditions

$$f_7 = f_{12} = 0 . \tag{10.35}$$

Consequently, under parity and time reversal invariance the number of free coefficients is reduced to

$$\begin{aligned} &G_0 = \bar{u}_{r'} \left[f_1 + f_2 \not{P}\right] u_r , && G_1 = \bar{u}_{r'} f_8 \gamma^5 \not{P} u_r , \\ &G_2 = \bar{u}_{r'} f_{11} \gamma^5 u_r , && G_3 = \bar{u}_{r'} \left[f_{13} + f_{14} \not{P}\right] u_r . \end{aligned} \tag{10.36}$$

For further investigation, we analyze the transformation under charge conjugation and crossing. The charge conjugation leads to

$$(\epsilon_0, \boldsymbol{\epsilon}) \leftrightarrow -(\epsilon_0^{*}, \boldsymbol{\epsilon}^{*}) . \tag{10.37}$$

As a result, invariance of the scattering amplitude M_{fi} under C transformation leads to

$$(F^{00}, F^{i0}, F^{ik}) \rightarrow (F^{00}, F^{0i}, F^{ki}) . \tag{10.38}$$

On the other hand, the crossing leads to

$$p \leftrightarrow -p' \qquad \text{and} \qquad \mu \leftrightarrow \nu , \tag{10.39}$$

and

$$\hat{e}^{(1)\lambda} \leftrightarrow \hat{e}^{(1)\lambda} \qquad \hat{e}^{(2)\lambda} \leftrightarrow -\hat{e}^{(2)\lambda}\,, \tag{10.40}$$

and we find that under charge conjugation and crossing

$$G_{0,2,3} \to G_{0,2,3} \qquad G_1 \to -G_1\,, \tag{10.41}$$

that is satisfied by the results presented in (10.36). Therefore, we can claim that the amplitude is invariant under CPT and crossing symmetry.

In particular, let us discuss the standard Compton scattering amplitude, which is based on QED. Using the standard Feynman rules, the amplitude of Compton scattering is given by

$$M_{fi} = -e^2 \epsilon^s_\mu(p)\ \epsilon^{s'*}_\nu(p')[\bar{u}(q')Q^{\mu\nu}u(q)]\,, \tag{10.42}$$

where

$$Q^{\mu\nu} = \frac{1}{s-m^2}\gamma^\nu(\not{p}+\not{q}+m)\gamma^\mu + \frac{1}{u-m^2}\gamma^\mu(\not{q}-\not{p}'+m)\gamma^\nu\,, \tag{10.43}$$

and the kinematic invariants are

$$\begin{aligned} s &= (p+q)^2 = (p'+q')^2 = m^2 + 2p\cdot q = m^2 + 2p'\cdot q'\,, \\ u &= (p-q')^2 = (p'-q)^2 = m^2 - 2p\cdot q' = m^2 - 2p'\cdot q\,. \end{aligned} \tag{10.44}$$

After some straightforward algebra we can find the following values of the coefficients f_i's [280]

$$f_1 = -ma_+\,, \qquad f_2 = 0\,, \qquad f_8 = \frac{1}{2}ia_+\,, \qquad f_{11} = -ma_+\,, \qquad f_{13} = ma_+\,, \qquad f_{14} = \frac{1}{2}a_-\,, \tag{10.45}$$

where

$$a_\pm = \frac{1}{s-m^2} \pm \frac{1}{u-m^2}\,. \tag{10.46}$$

Odd-parity amplitude

In this subsection, we impose the odd-parity condition. In this case $F^{\mu\nu}$ is a pseudotensor that under parity operation must transform as

$$(F^{00}, F^{i0}, F^{ij}) \to -(F^{00}, -F^{i0}, F^{ij})\,. \tag{10.47}$$

Imposing the odd-parity condition and using the properties of bilinear terms under parity transformation we get

$$f_1 = f_2 = f_7 = f_8 = f_{11} = f_{12} = f_{13} = f_{14} = 0\,. \tag{10.48}$$

Therefore, those terms that remain after imposing the above condition are

$$\begin{aligned} G_0 &= \bar{u}_{r'}\left[f_3\gamma^5 + f_4\gamma^5\not{P}\right]u_r\,, \qquad & G_1 &= \bar{u}_{r'}\left[f_5 + f_6\not{P}\right]u_r\,, \\ G_2 &= \bar{u}_{r'}\left[f_9 + f_{10}\not{P}\right]u_r\,, \qquad & G_3 &= \bar{u}_{r'}\left[f_{15}\gamma^5 + f_{16}\gamma^5\not{P}\right]u_r\,. \end{aligned} \tag{10.49}$$

Afterward, imposing the even-time-reversal condition, we find

$$f_3 = f_9 = f_{10} = f_{15} = 0\,. \tag{10.50}$$

Thus, we remain with

$$G_0 = \bar{u}_{r'} f_4 \gamma^5 \not{P} u_r\,, \qquad G_1 = \bar{u}_{r'} \left[f_5 + f_6 \not{P} \right] u_r\,, \qquad G_2 = 0\,, \qquad G_3 = \bar{u}_{r'} f_{16} \gamma^5 \not{P} u_r\,. \tag{10.51}$$

One can show that the resulting amplitude is odd under charge conjugation. Therefore, the final form of the amplitude is even under CPT combination. In this case, the $F^{\mu\nu}$ tensor is determined in terms of four free parameters.

10.3 Computation of the forward scattering term

In this section, we will provide a general expression to compute the forward scattering term on the right-hand side of the Boltzmann equation (8.43) for the photon-fermion scattering case. We will specialize in particular interactions in the next sections.

Similarly to Eq. (9.21) for the photon-graviton scattering case, it is possible to show that the general form of the photon-fermion interaction Hamiltonian we have to insert in Eq. (8.43) reads [93]

$$\begin{aligned} H_I(t) = \int & d\mathbf{q} d\mathbf{q}' d\mathbf{p} d\mathbf{p}' (2\pi)^3 \delta^{(3)}(\mathbf{q}' + \mathbf{p}' - \mathbf{q} - \mathbf{p}) \exp\left[it \left(q'^0 + p'^0 - q^0 - p^0 \right) \right] \\ & \times \left[b^\dagger_{r'}(q') a^\dagger_{s'}(p') \bar{u}_{r'}(q') F^{\mu\nu}(qr, q'r', ps, p's') u_r(q) \epsilon^s_\mu(\mathbf{p}) \epsilon^{s'}_\nu(\mathbf{p}') a_s(p) b_r(q) \right]\,, \end{aligned} \tag{10.52}$$

where

$$d\mathbf{q} = \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{m_f}{q^0}\,, \qquad\qquad d\mathbf{p} = \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0}\,. \tag{10.53}$$

a_s and $a^\dagger_{s'}$ are photon annihilation and creation operators respectively, which satisfy the canonical commutation relations

$$\left[a_s(\mathbf{p}), a^\dagger_{s'}(\mathbf{p}') \right] = (2\pi)^3 2p^0 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta_{ss'}\,, \tag{10.54}$$

and $b^{(r)}$ and $b^{(r)\,\dagger}$ are fermion annihilation and creation operators, respectively, obeying the canonical anti-commutation relations

$$\left\{ b_r(\mathbf{q}), b^\dagger_{r'}(\mathbf{q}') \right\} = (2\pi)^3 \frac{q^0}{m_f} \delta^{(3)}(\mathbf{q} - \mathbf{q}') \delta_{rr'}\,, \tag{10.55}$$

where m_f is the fermion mass.

Using Eq. (10.52), the commutation relation in the forward scattering term of Eq. (8.43) becomes

$$[H_I(0), \mathcal{D}_{ij}(k)] = \int d\mathbf{q}d\mathbf{q}'d\mathbf{p}d\mathbf{p}'(2\pi)^3\delta^{(3)}(\mathbf{q}'+\mathbf{p}'-\mathbf{q}-\mathbf{p})\bar{u}_{r'}(q')F^{\mu\nu}(qr, q'r', ps, p's')$$
$$\times\, u_r(q)\epsilon^s_\mu(\mathbf{p})\epsilon^{s'}_\nu(\mathbf{p}')\Big[b^\dagger_{r'}(q')b_r(q)a^\dagger_{s'}(p')a_j(k)2p^0(2\pi)^3\delta_{is}\delta^{(3)}(\mathbf{p}-\mathbf{k})$$
$$-b^\dagger_{r'}(q')b_r(q)a^\dagger_i(k)a_s(p)2p'^0(2\pi)^3\delta_{js'}\delta^{(3)}(\mathbf{p}'-\mathbf{k})\Big]\,. \tag{10.56}$$

After this step, in order to evaluate the forward scattering term, we will need to take the expectation value of Eq. (10.56). For this purpose, we provide the following expectation values [93]:

$$\left\langle a^\dagger_m(p')a_n(p)\right\rangle = 2p^0(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\rho_{mn}(\mathbf{p})\,, \tag{10.57}$$

and

$$\left\langle b^\dagger_m(q')b_n(q)\right\rangle = \frac{q^0}{m_f}(2\pi)^3\delta^{(3)}(\mathbf{q}-\mathbf{q}')\delta_{mn}\frac{1}{2}n_f(\mathbf{q})\,, \tag{10.58}$$

where ρ_{mn} is the photon beam polarization matrix (4.1) and n_f is the number density of fermions of momentum $\mathbf{q}$ per unit volume. After using the Dirac delta functions, one can easily perform the integrations over $\mathbf{p}$, $\mathbf{p}'$, and $\mathbf{q}'$ and obtain the limit $p = p'$ and $q = q'$ of the integrand, in agreement with the forward scattering condition.

At this point, we can fix the Coulomb gauge for the photon polarization vectors, where we have $\epsilon^\mu = (0, \boldsymbol{\epsilon})$. As a consequence of this gauge-fixing, we are interested in only "Latin" components of the Compton tensor $F^{\mu\nu}$ (thus, Latin components of the vector bases $\hat{e}^{(1)}$ and $\hat{e}^{(2)}$) to do the contractions in Eq. (10.56). In particular, using the definitions (10.14) and (10.11) and the result (10.19), the F^{ij} components in the forward scattering limit can be represented as

$$\begin{aligned}\bar{u}_{r'}(q')F^{ij}u_r(q) =& (G_0+G_3)\,\hat{e}^{(1)i}\hat{e}^{(1)j} + (G_0-G_3)\,\hat{e}^{(2)i}\hat{e}^{(2)j} + G_1\left(\hat{e}^{(1)i}\hat{e}^{(2)j}+\hat{e}^{(2)i}\hat{e}^{(1)j}\right)\\ &+ G_2\left(\hat{e}^{(1)i}\hat{e}^{(2)j}-\hat{e}^{(2)i}\hat{e}^{(1)j}\right)\\ =& (G_0+G_3)\,\frac{(4\Delta^0)^2}{\mathbf{N}^2}(\mathbf{q}\times\mathbf{p})^i(\mathbf{q}\times\mathbf{p})^j + (G_0-G_3)\,\frac{q^i}{m_f}\frac{q^j}{m_f}\\ &+ G_1\frac{4\Delta^0}{\sqrt{m_f^2\mathbf{N}^2}}\left[(\mathbf{q}\times\mathbf{p})^i q^j + q^i(\mathbf{q}\times\mathbf{p})^j\right] + G_2\frac{4\Delta^0}{\sqrt{m_f^2\mathbf{N}^2}}\left[|\mathbf{q}|^2 p_k - (\mathbf{q}\cdot\mathbf{p})\,q_k\right]\epsilon^{ijk0}\,.\end{aligned} \tag{10.59}$$

In the next sections, using Eq. (10.59), we will study the phenomenological consequences of the forward scattering term for CMB polarization in specific cases.

10.4 General conditions for generating circular polarization

In this section, we will give the most general conditions for generating circular polarization from photon-fermion forward scattering. Thus, we will consider specific expressions of the Compton tensor (10.59), evaluate Eq. (10.56), and study the effects of new interactions on the Stokes parameters.

Even-parity amplitude

We start by considering the even-parity terms. The general forms of the G_i coefficients invariant under time-reversal have been derived in the previous section. We have also determined the coefficients for the QED case.

The coefficients G_i read

$$G_0 + G_3 = \bar{u}_{r'}(\tilde{f}_1 + \tilde{f}_2 \not{P})u_r\,, \qquad G_1 = \bar{u}_{r'}(\tilde{f}_3\gamma^5 \not{P})u_r\,,$$
$$G_2 = \bar{u}_{r'}(\tilde{f}_4\gamma^5)u_r\,, \qquad G_0 - G_3 = \bar{u}_{r'}(\tilde{f}_5 + \tilde{f}_6 \not{P})u_r\,. \tag{10.60}$$

where $\tilde{f}_1 = f_1 + f_{13}$, $\tilde{f}_2 = f_2 + f_{14}$, $\tilde{f}_3 = f_8$, $\tilde{f}_4 = f_{11}$, $\tilde{f}_5 = f_1 - f_{13}$ and $\tilde{f}_6 = f_2 - f_{14}$.

Using the well-known spinorial relations

$$\bar{u}_{r'}(q)\gamma^5 u_r(q) = 0\,, \tag{10.61}$$

and

$$\bar{u}_{r'}(q)\gamma^\mu u_r(q) = \delta_{rr'}\frac{q^\mu}{m_f}\,, \tag{10.62}$$

we find

$$G_0 + G_3 = \left(\tilde{f}_1 + \tilde{f}_2\frac{P\cdot q}{m_f}\right)\delta_{rr'}\,, \qquad G_1 = \tilde{f}_3\bar{u}_{r'}\gamma^5\not{P}u_r\,, \tag{10.63}$$

$$G_2 = 0\,, \qquad G_0 - G_3 = \left(\tilde{f}_5 + \tilde{f}_6\frac{P\cdot q}{m_f}\right)\delta_{rr'}\,. \tag{10.64}$$

Using these results, the scattering amplitude (10.2) becomes

$$\begin{aligned} M_{fi} = &\left(\tilde{f}_1 + \tilde{f}_2\frac{P\cdot q}{m_f}\right)\frac{(4\Delta^0)^2}{\mathbf{N}^2}(\mathbf{q}\times\mathbf{p})\cdot\boldsymbol{\epsilon}^s(\mathbf{q}\times\mathbf{p})\cdot\boldsymbol{\epsilon}^{s'}\delta_{rr'} + \left(\tilde{f}_5 + \tilde{f}_6\frac{P\cdot q}{m_f}\right)\frac{(\mathbf{q}\cdot\boldsymbol{\epsilon}^s)}{m_f}\frac{(\mathbf{q}\cdot\boldsymbol{\epsilon}^{s'})}{m_f}\delta_{rr'} \\ &+\tilde{f}_3\bar{u}_{r'}\gamma^5\not{P}u_r\frac{4\Delta^0}{\sqrt{m_f^2\mathbf{N}^2}}\left[(\mathbf{q}\times\mathbf{p})\cdot\boldsymbol{\epsilon}^s\,(\mathbf{q}\cdot\boldsymbol{\epsilon}^{s'}) + (\mathbf{q}\cdot\boldsymbol{\epsilon}^s)(\mathbf{q}\times\mathbf{p})\cdot\boldsymbol{\epsilon}^{s'}\right]\,. \end{aligned} \tag{10.65}$$

In this equation, the main effects are expected to come from the term multiplying the $\tilde{f}_5$ and $\tilde{f}_6$ coefficients. In fact, other terms, containing at least one factor of Δ^0, will appear only when considering loop quantum effects. For this reason, in the next steps we will ignore them, since in a perturbation quantum field theory framework they are supposed to be an higher-order effect. Thus, the time evolution of polarization matrix elements is given by (from now on we will explicitly account for spatial dependence in the Boltzmann equations)

$$\begin{aligned} \frac{d}{dt}\rho_{ij}(\mathbf{x},\mathbf{k}) = &\frac{i}{2k^0 m_f^2}\int d\mathbf{q}\, n_f(\mathbf{x},\mathbf{q})\left(\tilde{f}_5 + 2\tilde{f}_6\frac{q\cdot k}{m_f}\right)\big(\delta_{is}\rho_{s'j}(\mathbf{x},\mathbf{k}) - \delta_{js'}\rho_{is}(\mathbf{x},\mathbf{k})\big)\,(\mathbf{q}\cdot\boldsymbol{\epsilon}^s)\,(\mathbf{q}\cdot\boldsymbol{\epsilon}^{s'}) \\ &+\text{standard Compton scattering terms (s.C.s.t.)}\,. \end{aligned} \tag{10.66}$$

Now, expressing Eq. (10.66) in terms of the different components, we have

$$\frac{d}{dt}\,\rho_{11}^{(1)}(\mathbf{x},\mathbf{k}) = \frac{i}{2k^0 m_f^2}\int d\mathbf{q}\,n_f(\mathbf{x},\mathbf{q})\left(\tilde{f}_5 + 2\tilde{f}_6\frac{q\cdot k}{m_f}\right)(\mathbf{q}\cdot\boldsymbol{\epsilon}_2)\,(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)\left[\rho_{21}^{(1)}(\mathbf{x},\mathbf{k}) - \rho_{12}^{(1)}(\mathbf{x},\mathbf{k})\right] + \text{s.C.s.t.}\,, \tag{10.67}$$

$$\frac{d}{dt}\,\rho_{22}^{(1)}(\mathbf{x},\mathbf{k}) = -\frac{d}{dt}\,\rho_{11}^{(1)}(\mathbf{x},\mathbf{k})\,, \tag{10.68}$$

$$\frac{d}{dt}\,\rho_{12}^{(1)}(\mathbf{x},\mathbf{k}) = \frac{i}{2k^0 m_f^2}\int d\mathbf{q}\,n_f(\mathbf{x},\mathbf{q})\left(\tilde{f}_5 + 2\tilde{f}_6\frac{q\cdot k}{m_f}\right)\Big[(\mathbf{q}\cdot\boldsymbol{\epsilon}_2)\,(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)(\rho_{22}^{(1)}(\mathbf{x},\mathbf{k}) - \rho_{11}^{(1)}(\mathbf{x},\mathbf{k})) + [(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)^2 - (\mathbf{q}\cdot\boldsymbol{\epsilon}_2)^2]\rho_{12}^{(1)}(\mathbf{x},\mathbf{k})\Big] + \text{s.C.s.t.}\,, \tag{10.69}$$

$$\frac{d}{dt}\,\rho_{21}^{(1)}(\mathbf{x},\mathbf{k}) = -\frac{i}{2k^0 m_f^2}\int d\mathbf{q}\,n_f(\mathbf{x},\mathbf{q})\left(\tilde{f}_5 + 2\tilde{f}_6\frac{q\cdot k}{m_f}\right)\Big[(\mathbf{q}\cdot\boldsymbol{\epsilon}_2)\,(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)(\rho_{22}^{(1)}(\mathbf{x},\mathbf{k}) - \rho_{11}^{(1)}(\mathbf{x},\mathbf{k})) + [(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)^2 - (\mathbf{q}\cdot\boldsymbol{\epsilon}_2)^2]\rho_{21}^{(1)}(\mathbf{x},\mathbf{k})\Big] + \text{s.C.s.t.}\,. \tag{10.70}$$

We can also convert the density matrix elements to the normalized Stokes brightness perturbations after changing momentum to the comoving one, $k_c = ak$, and going to the Fourier space. We find

$$\frac{d}{d\eta}\,\Delta_I^{(\gamma)}(\mathbf{K},\mathbf{k_c}) = \text{s.C.s.t.}\,, \tag{10.71}$$

$$\frac{d}{d\eta}\,\Delta_Q^{(\gamma)}(\mathbf{K},\mathbf{k_c}) = -\frac{a^2(\eta)}{k_c^0 m_f^2}\int d\mathbf{q}\,n_f(\mathbf{K},\mathbf{q})\left(\tilde{f}_5 + 2\tilde{f}_6\frac{q\cdot k_c}{a(\eta)m_f}\right)(\mathbf{q}\cdot\boldsymbol{\epsilon_2})\,(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)\Delta_V^{(\gamma)}(\mathbf{K},\mathbf{k_c}) + \text{s.C.s.t.}\,, \tag{10.72}$$

$$\frac{d}{d\eta}\,\Delta_U^{(\gamma)}(\mathbf{K},\mathbf{k_c}) = \frac{a^2(\eta)}{2k_c^0 m_f^2}\int d\mathbf{q}\,n_f(\mathbf{K},\mathbf{q})\left(\tilde{f}_5 + 2\tilde{f}_6\frac{q\cdot k_c}{a(\eta)m_f}\right)[(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)^2 - (\mathbf{q}\cdot\boldsymbol{\epsilon}_2)^2]\Delta_V^{(\gamma)}(\mathbf{K},\mathbf{k_c}) + \text{s.C.s.t.}\,, \tag{10.73}$$

$$\frac{d}{d\eta}\Delta_V^{(\gamma)}(\mathbf{K},\mathbf{k_c}) = -\frac{a^2(\eta)}{2k_c^0 m_f^2}\int d\mathbf{q}\,n_f(\mathbf{K},\mathbf{q})\left(\tilde{f}_5 + 2\tilde{f}_6\frac{q\cdot k_c}{a(\eta)m_f}\right)\Big[-2(\mathbf{q}\cdot\boldsymbol{\epsilon_2})(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)\Delta_Q^{(\gamma)}(\mathbf{K},\mathbf{k_c}) + [(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)^2 - (\mathbf{q}\cdot\boldsymbol{\epsilon}_2)^2]\Delta_U^{(\gamma)}(\mathbf{K},\mathbf{k_c})\Big] + \text{s.C.s.t.}\,, \tag{10.74}$$

where η denotes conformal time. From the last set of equations we see that the V modes in the CMB can be generated even with a parity preserving interaction. In particular, it is straightforward to verify that the fermionic number density $n_f(\mathbf{K},\mathbf{q})$ has to contain anisotropies in order to achieve a nontrivial coupling. In fact, under the assumption that $n_f(\mathbf{K},\mathbf{q})$ does not contain anisotropies, using the generic parametrizations (10.89), the angular integrals over the fermionic momentum $\mathbf{q}$ are vanishing as we show in the following:

$$\int_0^\pi d\theta' \sin\theta' \int_0^{2\pi} d\varphi' \, \left(\tilde{f}_5 + 2\tilde{f}_6 \frac{q\cdot k_c}{a(\eta)m_f}\right) (\mathbf{q}\cdot\boldsymbol{\epsilon}_2)(\mathbf{q}\cdot\boldsymbol{\epsilon}_1) \propto \int_0^\pi d\theta' \sin\theta' \int_0^{2\pi} d\varphi'$$
$$\times \left(\tilde{f}_5 + 2\tilde{f}_6 \frac{\cos\theta\cos\theta' + \cos(\varphi-\varphi')\sin\theta\sin\theta'}{a(\eta)m_f}\right) \sin(\varphi-\varphi')\sin\theta'$$
$$\times \left[\cos\theta'\sin\theta - \cos(\varphi-\varphi')\cos\theta\sin\theta'\right] = 0\,, \tag{10.75}$$

and

$$\int_0^\pi d\theta' \sin\theta' \int_0^{2\pi} d\varphi' \, \left(\tilde{f}_5 + 2\tilde{f}_6 \frac{q\cdot k_c}{a(\eta)m_f}\right) [(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)^2 - (\mathbf{q}\cdot\boldsymbol{\epsilon}_2)^2] \propto \int_0^\pi d\theta' \sin\theta' \int_0^{2\pi} d\varphi'$$
$$\times \left(\tilde{f}_5 + 2\tilde{f}_6 \frac{\cos\theta\cos\theta' + \cos(\varphi-\varphi')\sin\theta\sin\theta'}{a(\eta)m_f}\right)$$
$$\times \left[\,(\cos\theta'\sin\theta - \cos(\varphi-\varphi')\cos\theta\sin\theta')^2 - \sin^2(\varphi-\varphi')\sin^2\theta'\,\right] = 0\,. \tag{10.76}$$

Moreover, from the current model of particle physics, we know that a fermion can have a parity preserving interaction with a photon only through QED vertices. If we take the values of $\tilde{f}_5$ and $\tilde{f}_6$ for the case of QED, Eq. (10.45), and we evaluate them in the forward scattering limit, we find that $\tilde{f}_5 = \tilde{f}_6 = 0$. Thus, QED does not provide mixing terms among different polarizations, and only a parity preserving theory which goes beyond the standard paradigm could provide some kind of conversion from linear to circular polarization in the CMB.

Odd-parity amplitude

The general form of scattering amplitude for odd-parity was derived in Sec. 10.2. In that section, we found the general form of coefficients G_i for the odd-parity case

$$G_0 = \bar{u}_{r'}(f_4\gamma^5 \not{P})u_r\,,\quad G_1 = \bar{u}_{r'}(f_5 + f_6\not{P})u_r\,,\quad G_2 = 0\,,\quad G_3 = \bar{u}_{r'}(f_{16}\gamma^5\not{P})u_r\,. \tag{10.77}$$

The amplitude can be constructed using the tensor (10.59) and replacing the values of the coefficients (10.77). As in the previous subsection, we focus only on the terms which are expected to give the dominant contributions. Thus, our amplitude reads

$$M_{fi} = (f_4 - f_{16})\bar{u}_{r'}(q)\gamma^5\not{P}\,u_r(q)\,\frac{(\mathbf{q}\cdot\boldsymbol{\epsilon}^s)}{m_f}\,\frac{(\mathbf{q}\cdot\boldsymbol{\epsilon}^{s'})}{m_f}\,. \tag{10.78}$$

Using this result, we can find the time evolution of polarization matrix elements as

$$\frac{d}{dt}\rho_{ij}(\mathbf{x},\mathbf{k}) = i\frac{f_\mathrm{p}}{4k^0m_f^2}\int d\mathbf{q}\, n_f(\mathbf{x},\mathbf{q})\left(\delta_{is}\rho_{s'j}(\mathbf{x},\mathbf{k}) - \delta_{js'}\rho_{is}(\mathbf{x},\mathbf{k})\right)\bar{u}_r(q)\gamma^5\not{k}\,u_r(q)\,(\mathbf{q}\cdot\boldsymbol{\epsilon}^s)\,(\mathbf{q}\cdot\boldsymbol{\epsilon}^{s'})$$
$$+\mathrm{s.C.s.t.}\,, \tag{10.79}$$

where $f_{\rm p} \equiv 2(f_4 - f_{16})$. Therefore, we have

$$\frac{d}{dt}\,\rho_{11}^{(1)}(\mathbf{x},\mathbf{k}) = -\frac{if_{\rm p}}{4k^0 m_f^2}\int d\mathbf{q}\, n_f(\mathbf{x},\mathbf{q})\bar{u}_r \not{k}\gamma^5 u_r\,(\mathbf{q}\cdot\boldsymbol{\epsilon}_2)\,(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)\left[\rho_{21}^{(1)}(\mathbf{x},\mathbf{k}) - \rho_{12}^{(1)}(\mathbf{x},\mathbf{k})\right] + \text{s.C.s.t.}\,, \tag{10.80}$$

$$\frac{d}{dt}\,\rho_{22}^{(1)}(\mathbf{x},\mathbf{k}) = -\frac{d}{dt}\,\rho_{11}^{(1)}(\mathbf{x},\mathbf{k})\,, \tag{10.81}$$

$$\frac{d}{dt}\,\rho_{12}^{(1)}(\mathbf{x},\mathbf{k}) = -\frac{if_{\rm p}}{4k^0 m_f^2}\int d\mathbf{q}\, n_f(\mathbf{x},\mathbf{q})\bar{u}_r \not{k}\gamma^5 u_r\,\Big[(\mathbf{q}\cdot\boldsymbol{\epsilon}_2)\,(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)(\rho_{22}^{(1)}(\mathbf{x},\mathbf{k}) - \rho_{11}^{(1)}(\mathbf{x},\mathbf{k})) + [(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)^2 - (\mathbf{q}\cdot\boldsymbol{\epsilon}_2)^2]\rho_{12}^{(1)}(\mathbf{x},\mathbf{k})\Big] + \text{ s.C.s.t.}\,, \tag{10.82}$$

$$\frac{d}{dt}\,\rho_{21}^{(1)}(\mathbf{x},\mathbf{k}) = \frac{if_{\rm p}}{4k^0 m_f^2}\int d\mathbf{q}\, n_f(\mathbf{x},\mathbf{q})\bar{u}_r \not{k}\gamma^5 u_r\,\Big[(\mathbf{q}\cdot\boldsymbol{\epsilon}_2)\,(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)(\rho_{22}^{(1)}(\mathbf{x},\mathbf{k}) - \rho_{11}^{(1)}(\mathbf{x},\mathbf{k})) + [(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)^2 - (\mathbf{q}\cdot\boldsymbol{\epsilon}_2)^2]\rho_{21}^{(1)}(\mathbf{x},\mathbf{k})\Big] + \text{ s.C.s.t.}\,. \tag{10.83}$$

Here, we convert the density matrix elements to the normalized Stokes brightness perturbations and go to the Fourier space to obtain

$$\frac{d}{d\eta}\,\Delta_I{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) = \text{s.C.s.t.}\,, \tag{10.84}$$

$$\frac{d}{d\eta}\,\Delta_Q{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) = \frac{a(\eta)\,f_{\rm p}}{2k_c^0 m_f^2}\int d\mathbf{q}\, n_f(\mathbf{K},\mathbf{q})\,\bar{u}_r \not{k}_c\gamma^5 u_r\,(\mathbf{q}\cdot\boldsymbol{\epsilon_2})\,(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)\Delta_V{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) + \text{s.C.s.t.}\,, \tag{10.85}$$

$$\frac{d}{d\eta}\,\Delta_U{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) = -\frac{a(\eta)\,f_{\rm p}}{4k_c^0 m_f^2}\int d\mathbf{q}\, n_f(\mathbf{K},\mathbf{q})\,\bar{u}_r \not{k}_c\gamma^5 u_r\,[(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)^2 - (\mathbf{q}\cdot\boldsymbol{\epsilon}_2)^2]\Delta_V{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) + \text{s.C.s.t.}\,, \tag{10.86}$$

$$\frac{d}{d\eta}\,\Delta_V{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) = \frac{a(\eta)\,f_{\rm p}}{4k_c^0 m_f^2}\int d\mathbf{q}\, n_f(\mathbf{K},\mathbf{q})\,\bar{u}_r \not{k}_c\gamma^5 u_r\,\Big[-2(\mathbf{q}\cdot\boldsymbol{\epsilon_2})\,(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)\Delta_Q{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) + [(\mathbf{q}\cdot\boldsymbol{\epsilon}_1)^2 - (\mathbf{q}\cdot\boldsymbol{\epsilon}_2)^2]\Delta_U{}^{(\gamma)}(\mathbf{K},\mathbf{k_c})\Big] + \text{s.C.s.t.}\,. \tag{10.87}$$

The quantity $\sum_r \bar{u}_r\gamma^\mu\gamma^5 u_r$ vanishes when we sum over spins if the interacting fermion exists in both left- or right-handed helicity states. Thus, looking to this final set of equations, circular polarization in the CMB photons can be generated from a parity violating interaction only if the following condition is satisfied:

$$\sum_r \bar{u}_r \gamma^\mu \gamma^5 u_r \neq 0\,, \tag{10.88}$$

implying that the fermion particle must interact only in left- or right-handed helicity state.

Now, in order to perform the integral over $\mathbf{q}$, we choose the momentum and photon polarization vectors in the following form (see Fig. 10.2):

$$\begin{aligned}
\hat{\mathbf{K}} &= (0,0,1)\,,\\
\hat{\mathbf{k}} &= (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)\,,\\
\hat{\mathbf{q}} &= (\sin\theta'\cos\varphi',\ \sin\theta'\sin\varphi',\ \cos\theta')\,,\\
\boldsymbol{\epsilon}_1(k) &= (\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ -\sin\theta)\,,\\
\boldsymbol{\epsilon}_2(k) &= (-\sin\varphi,\ \cos\varphi,\ 0)\ .
\end{aligned} \tag{10.89}$$

Moreover, we can expand $n_f(\mathbf{K},\mathbf{q})$ in harmonic spheres as [6, 284]

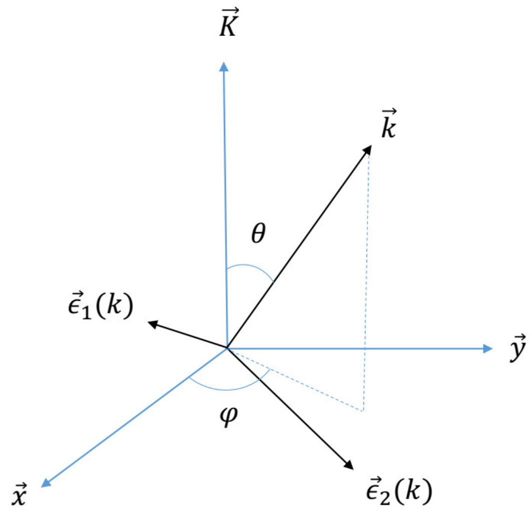


Figure 10.2. Pictorial representation of the polarizations and momentum direction of the photon.

$$n_f(\mathbf{K},\mathbf{q}) = n_f(\mathbf{K},|\mathbf{q}|)\sum_{\ell,m} c_{\ell m} Y_\ell^m(\hat{\mathbf{q}})\,. \tag{10.90}$$

In the Dirac representation, the helicity spinors are given by (see e.g. [285])

$$u_R(\mathbf{q}) = \sqrt{\frac{q^0+m_f}{2m_f}} \begin{pmatrix} \cos(\frac{\theta'}{2}) \\ \sin(\frac{\theta'}{2})e^{i\varphi'} \\ \frac{|\mathbf{q}|}{q^0+m_f}\cos(\frac{\theta'}{2}) \\ \frac{|\mathbf{q}|}{q^0+m_f}\sin(\frac{\theta'}{2})e^{i\varphi'} \end{pmatrix}, u_L(\mathbf{q}) = \sqrt{\frac{q^0+m_f}{2m_f}} \begin{pmatrix} -\sin(\frac{\theta'}{2}) \\ \cos(\frac{\theta'}{2})e^{i\varphi'} \\ \frac{|\mathbf{q}|}{q^0+m_f}\sin(\frac{\theta'}{2}) \\ -\frac{|\mathbf{q}|}{q^0+m_f}\cos(\frac{\theta'}{2})e^{i\varphi'} \end{pmatrix}. \tag{10.91}$$

Hence, assuming i.e. a left-handed fermion, the bilinear term $\bar{u}\gamma^\mu\gamma^5 u$ reads

$$\bar{u}\gamma^\mu\gamma^5 u = -\frac{1}{m}\left(|\mathbf{q}|, q^0\sin\theta'\,\cos\varphi', q^0\sin\theta'\,\sin\varphi', q^0\cos\theta'\right). \tag{10.92}$$

Now, using Eqs. (10.90), (10.92), and integrating over the fermion spherical angles φ' and θ', Eqs. (10.85)-(10.87) become

$$\begin{aligned}\frac{d}{d\eta}\,\Delta_Q{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) = {}& i\frac{\sqrt{\pi}}{420}\frac{a(\eta)\,f_{\rm p}}{16\pi^3 m_f^2}\int d|\mathbf{q}|\frac{|\mathbf{q}|^4}{q^0}n_f(\mathbf{K},|\mathbf{q}|)\left\{-28\sqrt{30}|\mathbf{q}|\left[\left(c_{22}e^{2i\varphi}-c_{2-2}e^{-2i\varphi}\right)\cos\theta\right.\right.\\ &\left.+\left(c_{21}e^{i\varphi}+c_{2-1}e^{-i\varphi}\right)\sin\theta\right]+10\sqrt{21}q^0\sin(2\theta)\left(c_{31}e^{i\varphi}+c_{3-1}e^{-i\varphi}\right)\\ &+4\sqrt{210}q^0\cos(2\theta)\left(c_{32}e^{2i\varphi}-c_{3-2}e^{-2i\varphi}\right)\\ &\left.-6\sqrt{35}q^0\sin(2\theta)\left(c_{33}e^{3i\varphi}+c_{3-3}e^{-3i\varphi}\right)\right\}\,\Delta_V{}^{(\gamma)}(\mathbf{K},\mathbf{k_c})+\text{ s.C.s.t.}\,,\end{aligned} \tag{10.93}$$

$$\begin{aligned}\frac{d}{d\eta}\,\Delta_U{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) = {}& -\frac{\sqrt{\pi}}{420}\frac{a(\eta)\,f_{\rm p}}{32\pi^3 m_f^2}\int d|\mathbf{q}|\frac{|\mathbf{q}|^4}{q^0}n_f(\mathbf{K},|\mathbf{q}|)\left\{-168\sqrt{5}|\mathbf{q}|\sin^2\theta\,c_{20}-14\sqrt{30}|\mathbf{q}|\right.\\ &\left[2\left(c_{21}e^{i\varphi}-c_{2-1}e^{-i\varphi}\right)\sin 2\theta+\left(c_{22}e^{2i\varphi}+c_{2-2}e^{-2i\varphi}\right)(\cos 2\theta+3)\right]\\ &+120\sqrt{7}q^0\cos\theta\sin^2\theta\,c_{30}+10\sqrt{21}q^0\left(c_{31}e^{i\varphi}-c_{3-1}e^{-i\varphi}\right)\sin\theta(3\cos 2\theta+1)\\ &+\sqrt{210}q^0\left(c_{32}e^{2i\varphi}+c_{3-2}e^{-2i\varphi}\right)(5\cos\theta+3\cos 3\theta)\\ &\left.-6\sqrt{35}q^0\left(c_{33}e^{3i\varphi}-c_{3-3}e^{-3i\varphi}\right)\sin\theta(\cos 2\theta+3)\right\}\,\Delta_V{}^{(\gamma)}(\mathbf{K},\mathbf{k_c})+\text{s.C.s.t.}\,,\end{aligned} \tag{10.94}$$

$$
\begin{aligned}
\frac{d}{d\eta}\,\Delta_V{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) = {} & \frac{\sqrt{\pi}}{420}\frac{a(\eta)\,f_{\mathrm{p}}}{32\pi^3 m_f^2}\int d|\mathbf{q}|\frac{|\mathbf{q}|^4}{q^0}n_f(\mathbf{K},|\mathbf{q}|)\Big[-2i\Big\{-28\sqrt{30}|\mathbf{q}|\left[\left(c_{22}e^{2i\varphi}\right.\right.\\
& \left.\left.-c_{2-2}e^{-2i\varphi}\right)\cos\theta + \left(c_{21}e^{i\varphi}+c_{2-1}e^{-i\varphi}\right)\sin\theta\right] + 10\sqrt{21}q^0\sin(2\theta)\\
& \times\left(c_{31}e^{i\varphi}+c_{3-1}e^{-i\varphi}\right) + 4\sqrt{210}q^0\cos(2\theta)\left(c_{32}e^{2i\varphi}-c_{3-2}e^{-2i\varphi}\right)\\
& -6\sqrt{35}q^0\sin(2\theta)\left(c_{33}e^{3i\varphi}+c_{3-3}e^{-3i\varphi}\right)\Big\}\,\Delta_Q{}^{(\gamma)}(\mathbf{K},\mathbf{k_c})\\
& -\Big\{168\sqrt{5}|\mathbf{q}|\sin^2\theta\,c_{20} + 14\sqrt{30}|\mathbf{q}|\left[2\left(c_{21}e^{i\varphi}-c_{2-1}e^{-i\varphi}\right)\sin 2\theta\right.\\
& \left.+\left(c_{22}e^{2i\varphi}+c_{2-2}e^{-2i\varphi}\right)(\cos 2\theta+3)\right] - 120\sqrt{7}q^0\cos\theta\sin^2\theta\,c_{30}\\
& -10\sqrt{21}q^0\left(c_{31}e^{i\varphi}-c_{3-1}e^{-i\varphi}\right)\sin\theta(3\cos 2\theta+1) - \sqrt{210}q^0\\
& \times\left(c_{32}e^{2i\varphi}+c_{3-2}e^{-2i\varphi}\right)(5\cos\theta+3\cos 3\theta)\\
& +6\sqrt{35}q^0\left(c_{33}e^{3i\varphi}-c_{3-3}e^{-3i\varphi}\right)\sin\theta(\cos 2\theta+3)\Big\}\,\Delta_U{}^{(\gamma)}(\mathbf{K},\mathbf{k_c})\Big]\\
& +\ \text{s.C.s.t.}\,. \qquad (10.95)
\end{aligned}
$$

From this set of equations, we find that quadrupolar or octupolar anisotropies in the quantity $n_f(\mathbf{K},\mathbf{q})$ have to appear (i.e. at least one of the $c_{2m}, c_{3m} \neq 0$) to have a coupling between Q and V modes. The same results hold considering a right-handed fermion apart for a negative overall sign in the Boltzmann equations.

10.5 General conditions for generating B-mode polarization

As we have seen in the previous section, new interactions which are even or odd under party and even under time-reversal can generate V modes, but are unable to generate B-mode polarization through the forward scattering term. This is due to the fact that, in Eq. (10.59), the term multiplying the G_2 coefficient vanishes being the amplitude even under time-reversal. Let us briefly explain this fact. After doing the expectation value of Eq. (10.56), the forward scattering contribution to the Boltzmann equations schematically reads as

$$
\frac{d\rho_{ij}^{(\gamma)}(\mathbf{K},\mathbf{k_c})}{dt} \propto i\int d\mathbf{q}\left(\delta_{is}\rho_{s'j}^{(\gamma)}(\mathbf{k_c}) - \delta_{js'}\rho_{is}^{(\gamma)}(\mathbf{k_c})\right)\delta_{rr'}\,n_f(\mathbf{q})M^{r,r',s,s'}(\mathbf{q'}=\mathbf{q},\mathbf{p}=\mathbf{p'}=\mathbf{k})\,, \qquad (10.96)
$$

where M is the scattering amplitude of the process taken in the forward scattering limit. Now, we can express the Q-mode taking the difference between the $ij = 11$ and $ij = 22$ components of the polarization matrix. So, we have

$$
\begin{aligned}
\frac{d}{dt}\,\Delta_Q{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) \propto i\int d\mathbf{q}\,\Big[&\left(\rho_{s'1}^{(\gamma)}(\mathbf{k_c})M^{r,r,1,s'} - \rho_{1s}^{(\gamma)}(\mathbf{k_c})M^{r,r,s,1}\right)\\
&-\left(\rho_{s'2}^{(\gamma)}(\mathbf{k_c})M^{r,r,2,s'} - \rho_{2s}^{(\gamma)}(\mathbf{k_c})M^{r,r,s,2}\right)\Big]\,n_f(\mathbf{q})\,. \qquad (10.97)
\end{aligned}
$$

Now, summing over the remaining s and s' indices, the coupling with the U modes is given by the following term:

$$\frac{d}{dt}\,\Delta_Q{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) \propto i \int d\mathbf{q}\, U^{(\gamma)}(\mathbf{K},\mathbf{k_c}) \left(M^{r,r,1,2} - M^{r,r,2,1}\right) n_f(\mathbf{q})\,. \tag{10.98}$$

From this last equation, we get that only scattering amplitudes that are antisymmetric in the final s, s' photon polarization indices can give a direct coupling between Q and U modes. This coupling converts E modes directly into B modes and vice versa. The only term in the amplitude of the process to have this property is the one proportional to the G_2 coefficient due to the Levi-Civita tensor contracting the photon polarization vectors in Eq. (10.59). All the other terms turn out to be symmetric in the s and s' indices, thus not providing any direct coupling between the Q and U modes.

Now, in this section we will investigate the case in which the fermion-photon scattering amplitude is odd under time-reversal, leading for a non-negligible value of the G_2 term, thus providing a direct source term for B-mode polarization.

Even-parity and odd-time-reversal amplitude

As we discussed in Sec. 10.2, after imposing the even-parity condition, the G_2 coefficient is restricted to be [see Eqs. (10.23) and (10.27)]

$$G_2 = \bar{u}_{r'}(f_{11}\gamma^5 + f_{12}\gamma^5 \not{P})u_r\,. \tag{10.99}$$

After imposing the odd-time-reversal condition, the only nonzero coefficients are

$$G_1 = \bar{u}_{r'}(f_7\gamma^5)u_r \qquad \text{and} \qquad G_2 = \bar{u}_{r'}(f_{12}\gamma^5\not{P})u_r\,. \tag{10.100}$$

Moreover, we impose the odd charge conjugation condition, so that the amplitude is even under CPT. As a result we get $f_7 = 0$. Finally, the scattering amplitude is reduced to

$$M_{fi} = -4f_{12}\,\bar{u}_{r'}\not{P}\gamma^5 u_r\,\frac{\Delta^0}{m_f\sqrt{\mathbf{N}^2}}\left[|\mathbf{q}|^2\,\mathbf{p}\cdot(\boldsymbol{\epsilon}^s\times\boldsymbol{\epsilon}^{s'}) - (\mathbf{q}\cdot\mathbf{p})\left[\mathbf{q}\cdot(\boldsymbol{\epsilon}^s\times\boldsymbol{\epsilon}^{s'})\right]\right]\,. \tag{10.101}$$

The only term which survives multiplies a factor of Δ^0. Hence, the corresponding effect will be a loop quantum effect. The time evolution of the brightness Stokes parameters is given by

$$\begin{aligned}\frac{d}{d\eta}\,\Delta_Q{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) = &-\frac{a(\eta)f_{12}}{k_c^0 m_f}\int d\mathbf{q}\,n_f(\mathbf{K},\mathbf{q})\,\bar{u}_r\not{k}_c\gamma^5 u_r\,\frac{1}{\sin\psi}\left[\left(|\mathbf{q}|\,\hat{\mathbf{k}}_\mathbf{c} - (\hat{\mathbf{q}}\cdot\hat{\mathbf{k}}_\mathbf{c})\,\mathbf{q}\right)\cdot(\boldsymbol{\epsilon}_\mathbf{1}\times\boldsymbol{\epsilon}_\mathbf{2})\right]\\ &\times\Delta_U^{(\gamma)}(\mathbf{K},\mathbf{k_c}) + \text{s.C.s.t.}\,,\end{aligned} \tag{10.102}$$

and

$$\begin{aligned}\frac{d}{d\eta}\,\Delta_U{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) = &\frac{a(\eta)f_{12}}{k_c^0 m_f}\int d\mathbf{q}\,n_f(\mathbf{K},\mathbf{q})\,\bar{u}_r\not{k}_c\gamma^5 u_r\,\frac{1}{\sin\psi}\left[\left(|\mathbf{q}|\,\hat{\mathbf{k}}_\mathbf{c} - (\hat{\mathbf{q}}\cdot\hat{\mathbf{k}}_\mathbf{c})\,\mathbf{q}\right)\cdot(\boldsymbol{\epsilon}_\mathbf{1}\times\boldsymbol{\epsilon}_\mathbf{2})\right]\\ &\times\Delta_Q{}^{(\gamma)}(\mathbf{K},\mathbf{k_c}) + \text{s.C.s.t.}\,,\end{aligned} \tag{10.103}$$

and hence in terms of the $\Delta_P^{\pm(\gamma)}$ fields (see Eq. (4.4)) we have

$$\frac{d}{d\eta}\,\Delta_P^{\pm(\gamma)} + iK\mu\Delta_P^{\pm(\gamma)} = \mp i\alpha'\Delta_P^{\pm(\gamma)} + \text{s.C.s.t.}\,, \tag{10.104}$$

where $\mu = \hat{k}_c \cdot \hat{K} = \cos\theta$, and α' is defined as

$$\alpha'(\eta) = -\frac{a(\eta) f_{12}}{k_c^0 m_f} \int d\mathbf{q}\, n_f(\mathbf{K},\mathbf{q})\, \bar{u}_r \not{k}_c \gamma^5 u_r \frac{1}{\sin\psi} \left[\Big(|\mathbf{q}|\,\hat{\mathbf{k}}_\mathbf{c} - (\hat{\mathbf{q}}\cdot\hat{\mathbf{k}}_\mathbf{c})\,\mathbf{q}\Big)\cdot(\boldsymbol{\epsilon}_\mathbf{1}\times\boldsymbol{\epsilon}_\mathbf{2})\right]\,, \tag{10.105}$$

with

$$\alpha(\eta) = -\int_\eta^{\eta_0} \alpha'(\eta')\,d\eta' \tag{10.106}$$

and

$$\begin{aligned}\sin\psi =& [(\sin\theta\sin\theta'\sin(\varphi-\varphi'))^2 + (\cos\varphi\cos\theta'\sin\theta - \cos\theta\cos\varphi'\sin\theta')^2 \\ &+ (\cos\theta'\sin\theta\sin\varphi - \cos\theta\sin\theta'\sin\varphi')^2]^{1/2}\,.\end{aligned} \tag{10.107}$$

As a result, Eq. (10.104) can be rewritten as

$$\frac{d}{d\eta}\left[\Delta_P{}^{\pm(\gamma)}\, e^{iK\mu\eta \pm i\alpha(\eta) - \tau(\eta)}\right] = e^{iK\mu\eta \pm i\alpha(\eta) - \tau(\eta)}\left(\frac{1}{2}\tau'\,[1 - P_2(\mu)]\;\Pi\right)\,, \tag{10.108}$$

where again $\Pi = I^{2(S)} + P^{2(S)} - P^{0(S)}$. Integrating the last equation gives the general solution

$$\Delta_P{}^{\pm(\gamma)}(\eta_0,\mathbf{K},\mu) = \frac{3}{4}(1-\mu^2)\int_0^{\eta_0} d\eta\, e^{iK(\eta-\eta_0)\mu \pm i\alpha(\eta) - \tau(\eta)}\,\tau'(\eta)\,\Pi(\eta,\mathbf{K})\,. \tag{10.109}$$

Then, using Eqs. (4.5), (4.6), and (4.21), we get the following expressions for the E and B modes

$$\Delta_E{}^{(\gamma)}(\eta_0,\mathbf{K},\mu) = -\frac{3}{4}\int_0^{\eta_0} d\eta\, g(\eta)\,\Pi(\eta,\mathbf{K})\partial_\mu^2\left[(1-\mu^2)^2 e^{iK(\eta-\eta_0)\mu}\cos\alpha(\eta)\right]\,, \tag{10.110}$$

$$\Delta_B{}^{(\gamma)}(\eta_0,\mathbf{K},\mu) = -\frac{3}{4}\int_0^{\eta_0} d\eta\, g(\eta)\,\Pi(\eta,\mathbf{K})\partial_\mu^2\left[(1-\mu^2)^2 e^{iK(\eta-\eta_0)\mu}\sin\alpha(\eta)\right]\,. \tag{10.111}$$

Also in this case, we need the fermion to be left- or right-handed, otherwise $\alpha = 0$ since $\sum_r \bar{u}_r\gamma^\mu\gamma^5 u_r = 0$. However, in this case, the angular integral inside the definition of α, Eq. (10.105), is not equal to 0 if $n_f(\mathbf{K},\mathbf{q})$ is isotropic. Thus, we do not have to impose any particular condition to the fermionic stress tensor.

Odd-parity and odd-time-reversal amplitude

The expressions of the coefficients G_i's under odd-parity condition have been presented in Eq. (10.49). Hence, G_2 is restricted to

$$G_2 = \bar{u}_{r'}(f_9 + f_{10}\not{P})u_r \,. \tag{10.112}$$

Then applying the odd-time-reversal condition on G_i, we get

$$f_4 = f_5 = f_6 = f_{16} = 0 \,. \tag{10.113}$$

Therefore,

$$G_0 = \bar{u}_{r'}(f_3\gamma^5)u_r \,, \quad G_1 = 0 \,, \quad G_2 = \bar{u}_{r'}(f_9 + f_{10}\not{P})u_r \,, \quad G_3 = \bar{u}_{r'}(f_{15}\gamma^5)u_r \,, \tag{10.114}$$

which are all even under charge conjugation. Hence, the final form of amplitude will be even under CPT. Using these results, the final form of the amplitude is simplified to

$$M_{fi} = 4\frac{\Delta^0}{m_f\sqrt{N^2}}\left(f_9 + f_{10}\frac{P\cdot q}{m_f}\right)\left[|\mathbf{q}|^2\,\mathbf{p}\cdot(\boldsymbol{\epsilon}^s\times\boldsymbol{\epsilon}^{s'}) - (\mathbf{q}\cdot\mathbf{p})\left[\mathbf{q}\cdot(\boldsymbol{\epsilon}^s\times\boldsymbol{\epsilon}^{s'})\right]\right]\delta_{rr'} \,. \tag{10.115}$$

The corresponding E-mode and B-mode polarizations are derived using the same method that we used to derive Eqs. (10.110) and (10.111). The only difference is that the parameter $\alpha'(\eta)$ changes into the following form:

$$\alpha'(\eta) = \frac{a^2(\eta)}{k_c^0 m_f}\int d\mathbf{q}\, n_f(\mathbf{K},\mathbf{q})\left(f_9 + 2f_{10}\frac{q\cdot k_c}{a(\eta)m_f}\right)\frac{1}{\sin\psi}\left[\left(|\mathbf{q}|\,\hat{\mathbf{k}}_\mathbf{c} - (\hat{\mathbf{q}}\cdot\hat{\mathbf{k}}_\mathbf{c})\,\mathbf{q}\right)\cdot(\boldsymbol{\epsilon}_\mathbf{1}\times\boldsymbol{\epsilon}_\mathbf{2})\right] \,. \tag{10.116}$$

As a result, in this case there is no restriction on the handedness of the fermion. In fact, α' can be different from zero if the fermion interacts both in the left- and right-handed states. Moreover, also in this case, we do not have to impose any particular condition in the fermion stress tensor, since we do not need anisotropies for providing a value different from zero to the angular integral contained in the α' expression.

10.6 Majorana fermions

In the previous sections, we assumed the fermion to be a Dirac spinor. In this section, we will analyze what changes when the interacting fermion is a Majorana spinor, instead of a Dirac spinor. Analogous considerations have already been made in Ref. [251] for the case in which the fermion is a neutrino.

A Majorana fermion is a particle which coincides with its own antiparticle and hence it has no electric charge [286–288]. We remind that a Majorana spinor is defined as

$$\psi_M = \gamma^0 C\psi_M^* \,, \tag{10.117}$$

Symmetries broken	V-mode formation	B-mode formation
All preserved	Anisotropies in $n_f(\mathbf{K}, q)$	/
C and P	Anisotropies in $n_f(\mathbf{K}, q)$ Only R- or L-handed fermion	/
C and T	/	Only R- or L-handed fermion
P and T	/	No conditions

Table 10.1. The conditions one needs to impose on the fermion to directly convert CMB E modes into V and B modes through fermion-photon forward scattering in the different cases analyzed.

where C is the charge conjugation operator. The properties of Majorana bilinear terms under parity, charge conjugation, and time-reversal transformations have been summarized in Refs. [286–288]. The Majorana condition implies $\psi_M = \psi_M^c$. As a result, a Majorana spinor transforms under charge conjugation as

$$C^{-1}\psi_M\, C = \psi_M\,. \tag{10.118}$$

Thus, in general we can write

$$C^{-1}\left(\bar{\psi}_M A\, \psi_M\right) C = \bar{\psi}_M A\, \psi_M\,, \tag{10.119}$$

that for $A = \gamma^\mu$ becomes

$$\bar{\psi}_M \gamma^\mu\, \psi_M = 0\,. \tag{10.120}$$

However, one can show that the transformations of the Majorana bilinear terms under P, T, and C are the same as Dirac bilinear terms. It was discussed in Ref. [289] that the Compton scattering amplitude for Majorana fermions is given by

$$M_{fi} = \bar{u}_{r'}(q')\epsilon^s_\mu \left[F^{\mu\nu}(q, q', p, p') + C\left(F^{\nu\mu}(-q', -q, p, p')\right)^T C^{-1}\right] \epsilon^{s'}_\nu u_r(q)\,, \tag{10.121}$$

$F^{\mu\nu}$ being as in Eq. (10.3). Now, if in general

$$C\left(F^{\nu\mu}(-q', -q, p, p')\right)^T C^{-1} = -F^{\mu\nu}(q, q', p, p')\,, \tag{10.122}$$

we find that $M^M_{fi} = 0$ identically. However, if

$$C\left(F^{\nu\mu}(-q', -q, p, p')\right)^T C^{-1} = F^{\mu\nu}(q, q', p, p')\,, \tag{10.123}$$

then the scattering amplitude becomes

$$M^M_{fi} = 2M^D_{fi}\,. \tag{10.124}$$

Thus, when the Compton tensor $F^{\mu\nu}$ transforms like a pseudotensor under C, we get no fermion-photon forward scattering mixing. On the contrary, when $F^{\mu\nu}$ is invariant under C, we get the same coupling as discussed in the previous sections, but with an additional factor of 2 with respect to the Dirac fermion case.

10.7 Conclusions

In the standard lore, circular and B-mode polarization of CMB photons cannot be generated via Compton scattering with electrons from linear scalar perturbations. In this chapter, we studied the conversion of CMB E modes into V and B modes due to the forward scattering with a generic fermion. We assumed interactions which may also go beyond the Standard Model of particle physics, keeping only gauge-invariance and the preservation of CPT symmetry. We derived various sets of Boltzmann equations describing the radiation transfer of CMB polarization. Our final results are qualitatively summarized in Tab. 10.1. We can have conversion in V modes both preserving all the discrete symmetries and breaking the C and P symmetries. Instead, conversion into B modes may arise only from the breaking of the T symmetry. Since our results are expressed in terms of free parameters, they offer a viable tool to put constraints on fundamental physics properties beyond the standard paradigms.

Part IV
Overview and conclusion

In this work, first of all we have reviewed fundamental aspects and open questions of the standard model of cosmology, focusing on the description of slow-roll models of inflation and model independent generalizations of them.

Then, we investigated the signatures in the primordial bispectra induced by a Chern-Simons gravitational term coupled to the inflaton field through a generic coupling function $f(\phi)$. This is a modified gravity scenario of inflation that naturally arises from an effective field theory approach (Eqs. (5.67) and (6.2)). In particular, we showed that this term introduces parity breaking in the theory, polarizing PGW into left and right chiral eigenstates and leaving scalar modes unchanged at linear level. Our analysis of the bispectra statistics was motivated by the fact that, in such a scenario, the chirality of primordial gravitons in the $\langle\gamma\gamma\rangle$ power spectrum is suppressed by the ratio $H/M_{\rm CS}$ (where $M_{\rm CS}$ is the so called Chern-Simons mass, Eq. (6.17)), which is expected to be very small, thus making impossible to probe the theory with the CMB angular TB and EB power spectra only. We have shown that an asymmetry among the bispectra $\langle\gamma_R\gamma_R\zeta\rangle$ and $\langle\gamma_L\gamma_L\zeta\rangle$ (caused by the breaking of parity symmetry) arises as the only parity breaking signature that is not a-priori suppressed. Hence, we defined a dimensionless parameter, Π, labelling the relative difference between these two chiral bispectra (Eq. (7.67)). This parameter turned out to be proportional to the strength of the second order derivative of the coupling function $f(\phi)$ (Eq. (7.69)). For this reason, only a non-minimal coupling function (i.e. $f(\phi)$ which is not just proportional to ϕ) is able to produce such a signature. Therefore, if experimental observations will indicate such a signature, we would have a clear evidence during inflation of Chern-Simons gravity with a non-minimal coupling with the inflaton field. Within an effective field theory treatment, the value of Π is theoretically constrained (Eq. (7.75)). This constraint is alleviated when the tensor-to-scalar-ratio r is small. In this case, we can realize a relatively large parity breaking in the $\langle\gamma\gamma\zeta\rangle$ bispectra even if the parity breaking in the tensor power spectrum is very low. However, in the same limit we also reduce the amplitude of the bispectra (Eq. (7.61)), suggesting that at the same time we lose sensitivity on a possible detection of this parity breaking signal.

In the subsequent forecast about the detectability of this signature with an ideal CMB experiment, we showed that BBX (where $X = E, T$) CMB angular bispectra could become essential observables in probing this parity breaking signature. In particular, important future improvements regarding the sensitivity of the experiments on the measure of Π are expected with both lensing subtraction and increasing of the angular resolution (Eq. (8.34) and Fig. 8.1). On the contrary, we showed that the CMB power spectra statistics does not get any futuristic improvement in the sensitivity to parity breaking signatures (Fig. 8.2).

In the future, it would be interesting to study modifications induced in this theory when we couple the Chern-Simons term with an external scalar field χ not participating to the slow-roll dynamics. In fact, the insertion of this new degree of freedom could in principle lead to a modification in the strength of the parity breaking signatures during inflation. Another interesting alternative is to study the effects, during inflation, of the other parity breaking terms included in the effective field theory action (5.67). In this regards, it is worth to mention Ref. [290], where different chiral scalar-tensor theories (thought to be generalizations of the Chern-Simons modified gravity theory) are considered. In fact, while the effects of these operators on the power spectrum statistics are expected to be very similar to our scenario (see e.g. [291]), they could affect the bispectra statistics of primordial perturbations in a completely different way (in particular, providing different shape functions and strength of primordial non-Gaussianities). Finally, we notice that in our study we have completely neglected the effects of Chern-Simons gravity during the re-heating epoch, assuming an immediate transition between inflation and standard FRW evolution. However, during the re-heating epoch the inflaton field coupled to the Chern-Simons term has no more a slow-roll dynamics, suggesting important modifications of the effects of this modified gravity on primordial perturbations. Nevertheless, a detailed analysis should be done, as the dynamics of primordial perturbations could become very complicated in this regime.

Moreover, we studied how we can possibly exploit the CMB to probe new physics beyond the Standard Model of particle physics.

In this regards, first of all we considered the generation of CMB V modes induced by the forward scattering between CMB photons and gravitons. We showed that only gravitons with anisotropies in their statistics can couple the linear and circular CMB polarizations (Eqs. (9.50)-(9.52)). However, even assuming primordial gravitons with large anisotropies (generated in specific models of inflation), we have estimated that the final effect on CMB circular polarization is negligible. However, the formalism we developed is very general and can therefore be applied in other contexts, e.g. to measure gravitational waves of astrophysical origin.

Then, we studied the forward scattering mixing on CMB polarizations induced by photon-fermion interactions which may also go beyond the Standard Model of particle physics, keeping only gauge-invariance and the preservation of CPT symmetry. Our final results are qualitatively summarized in Tab. 10.1. We can have conversion of linear polarization in V modes both preserving all the discrete symmetries and breaking the C and P symmetries. Instead, the conversion of CMB E modes into B modes may arise only from the breaking of the T symmetry.

Since our results are expressed in terms of free parameters, it would be interesting in the future to use CMB data to directly put constraints on the alternative physics studied.

A natural extension of this study would be deriving the effects on CMB polarizations of the damping term in Eq. (8.43) for the different interactions we have considered.

All the intriguing and interesting extensions of our work mentioned here are left for future research.

Part V
Appendix

Appendix A

ADM expressions of general relativity curvature tensors

In this Appendix, we give the expressions in the ADM formalism of the fundamental tensors of general relativity. For more details regarding the derivations of these formulas, see e.g. Ref. [292].

Riemann tensor components

$$R_{ijkl} = R^{(3)}_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk} \ , \tag{A.1}$$

$$R_{0ijk} = N[D_j K_{ik} - D_k K_{ij}] + N^l[R^{(3)}_{lijk} + K_{lj}K_{ik} - K_{lk}K_{ij}] \ , \tag{A.2}$$

$$\begin{aligned} R_{0i0j} = {} & N[\dot{K}_{ij} + D_i D_j N + N K^k_i K_k j - (D_j K_{ik})N^k - (D_i N^k)K_{kj} - (D_i K_{kj})N^k \\ & + (D_k K_{ij})N^k] + N^l N^k[-R^{(3)}_{ljik} - K_{il}K_{jk} + K_{lk}K_{ij}] \ . \end{aligned} \tag{A.3}$$

Ricci tensor components

$$R_{ij} = R^{(3)}_{ij} + K_{lk}K_{ij}h^{lk} - 2K_{ik}K_{jl}h^{lk} - \frac{1}{N}[\dot{K}_{ij} + D_i D_j N - N^l \partial_l K_{ij} - K_{il}\partial_j N^l - K_{jl}\partial_i N^l] \ , \tag{A.4}$$

$$\begin{aligned} R_{0i} = {} & -\frac{N^j}{N}\dot{K}_{ij} - \frac{N^j}{N}(D_i D_j N) - N^j K^k_i K_{kj} + \frac{N^j}{N}[(D_j K_{ik})N^k + (D_i N^k)K_{kj} \\ & + (D_i K_{kj})N^k - (D_k K_{ij})N^k] + N^l h^{jk} R^{(3)}_{ljik} + N^l h^{jk} K_{il}K_{jk} - N^l h^{jk} K_{lk}K_{ij} \ , \end{aligned} \tag{A.5}$$

$$R_{00} = R_{0i0j}\left(h^{ij} - \frac{N^i N^j}{N^2}\right) \ . \tag{A.6}$$

Scalar curvature

$$R = R^{(3)} + K_{ij}K^{ij} + (K^i_i)^2 - \frac{2}{N}(\dot{K}^i_i) + \frac{2N^j}{N}(D_j K^i_i) - \frac{2}{N}\triangle N \ , \tag{A.7}$$

where $\triangle = h^{ij}\partial_i\partial_j$ denotes the covariant laplacian built with the 3-metric h_{ij}.

In particular, from this last equation it follows

$$R = R^{(3)} + K_{ij}K^{ij} + (K^i_i)^2 + (\text{surface terms}) \ . \tag{A.8}$$

This last equation explains the appearance of terms proportional to K^2 in the ADM expression of slow-roll models action (2.22).

Appendix B

In-In formalism

In this Appendix, we will briefly review the so-called In-In formalism [52, 293–295]. This technique allows to compute perturbatively the cosmological correlation functions. The method is quite different by what is used in standard quantum field theory. In fact, here we are not interested in the calculation of S-matrix elements, but rather in evaluating expectation values in a certain vacuum state of products of fields at a fixed time. In particular, we do not need to impose conditions on the fields at both very early and very late times (as in the calculation of S-matrix elements), but only on very early times: for instance, during the inflationary epoch, the early-time limit corresponds to the limit when the wavelength of the fields in Fourier space is deep inside the Hubble horizon. In this case, according to the equivalence principle, the interaction picture fields should have the same form as in Minkowski space-time (apart for diffeomorphisms), leading to the imposition of the *Bunch-Davies* vacuum state as initial vacuum state.[1]

Now, we start our derivation by considering a general Hamiltonian system, with canonical variable $\phi(\mathbf{x},t)$[2] and conjugate momentum $\pi(\mathbf{x},t)$ satisfying the commutations relations

$$[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\delta^{(3)}(x-y)\,, \qquad [\phi(\mathbf{x},t),\phi(\mathbf{y},t)] = [\pi(\mathbf{x},t),\pi(\mathbf{y},t)] = 0\,, \tag{B.1}$$

and the Heisenberg equations

$$\dot{\phi}(\mathbf{x},t) = i\,[H[\phi(t),\pi(t)],\phi(\mathbf{x},t)]\,, \qquad \dot{\pi}(\mathbf{x},t) = i\,[H[\phi(t),\pi(t)],\pi(\mathbf{x},t)]\,. \tag{B.2}$$

The Hamiltonian H does not depend explicitly on time, but it depends on time through the fields $\phi(t)$ and $\pi(t)$. Now, we expand the fields as:

$$\phi(\mathbf{x},t) = \phi_0(t) + \delta\phi(\mathbf{x},t)\,, \qquad \pi(\mathbf{x},t) = \pi_0(t) + \delta\pi(\mathbf{x},t) \tag{B.3}$$

where ϕ_0 and π_0 denotes background classical values satisfying Hamilton equations

$$\dot{\phi}_0(t) = \frac{\delta H[\phi_0(t),\pi_0(t)]}{\delta\pi_0(t)}\,, \quad \dot{\pi}_0(t) = -\frac{\delta H[\phi_0(t),\pi_0(t)]}{\delta\phi_0(t)} \tag{B.4}$$

[1] Notice that this is perfectly in agreement with what explicitly found in Sec. 2.3.

[2] In our inflationary context, $\phi(\mathbf{x},t)$ can correspond to both the inflaton and the metric fields.

and $\delta\phi$ and $\delta\pi$ denotes quantum perturbations w.r.t. the background. Substituting Eq. (B.3) into Eq. (B.1), it is straightforward to show that perturbations satisfy the same commutation relations as the total fields

$$[\delta\phi(\mathbf{x},t),\delta\pi(\mathbf{y},t)] = i\delta^{(3)}(x-y)\,, \quad [\delta\phi(\mathbf{x},t),\delta\phi(\mathbf{y},t)] = [\delta\pi(\mathbf{x},t),\delta\pi(\mathbf{y},t)] = 0\,. \tag{B.5}$$

Expanding the total Hamiltonian H in powers of the fluctuations

$$\begin{aligned} H[\phi(t),\pi(t)] = {} & H[\phi_0(t),\pi_0(t)] + \\ & + \left[\frac{\delta H[\phi_0(t),\pi_0(t)]}{\delta\phi_0(\mathbf{x},t)}\delta\phi(\mathbf{x},t) + \frac{\delta H[\phi_0(t),\pi_0(t)]}{\delta\pi_0(\mathbf{x},t)}\delta\pi(\mathbf{x},t)\right] + \\ & + \tilde{H}[\delta\phi(t),\delta\pi(t),t] \end{aligned} \tag{B.6}$$

we find terms of zero-th and first order in the perturbations, plus an additional term $\tilde{H}$ which stands for all higher-order contributions. Using Eqs. (B.4)-(B.6), we can show that the time evolution of the perturbations is generated by the time dependent Hamiltonian $\tilde{H}$:

$$\dot{\delta\phi}(\mathbf{x},t) = i\,[\tilde{H}[\delta\phi(t),\delta\pi(t),t],\delta\phi(\mathbf{x},t)]\,, \quad \dot{\delta\pi}(\mathbf{x},t) = i\,[\tilde{H}[\delta\phi(t),\delta\pi(t),t],\delta\pi(\mathbf{x},t)]\,. \tag{B.7}$$

The quantum evolution of the perturbations can be expressed as

$$\delta\phi(t) = U^{-1}(t,t_0)\delta\phi(t_0)U(t,t_0)\,, \qquad \delta\pi(t) = U^{-1}(t,t_0)\delta\pi(t_0)U(t,t_0)\,, \tag{B.8}$$

where t_0 is some early time and U is a unitary operator. Substituting Eq. (B.8) into Eq. (B.7), we find out that U must obey the equation of motion

$$\frac{d}{dt}U(t,t_0) = -i\,\tilde{H}[\delta\phi(t),\delta\pi(t),t]U(t,t_0) \tag{B.9}$$

with initial condition $U(t_0,t_0) = 1$. Notice that in cosmology the classical background solution would describe the FRW background and we can take $t_0 = -\infty$, by which we mean any time early enough so that the wavelengths of interest are deep inside the horizon. Now, we decompose $\tilde{H}$ into a kinematic term H_0 that is quadratic in the fluctuations, and an interaction part H_I as

$$\tilde{H}[\delta\phi(t),\delta\pi(t),t] = H_0[\delta\phi(t),\delta\pi(t),t] + H_I[\delta\phi(t),\delta\pi(t),t]\,. \tag{B.10}$$

As in standard quantum field theory, the interaction picture is introduced defining fluctuations operators whose time dependence is generated by the quadratic part of the Hamiltonian as

$$\dot{\delta\phi}_I(\mathbf{x},t) = i\,[H_0[\delta\phi_I(t),\delta\pi_I(t),t],\delta\phi_I(\mathbf{x},t)]\,, \quad \dot{\delta\pi}_I(\mathbf{x},t) = i\,[H_0[\delta\phi_I(t),\delta\pi_I(t),t],\delta\pi_I(\mathbf{x},t)]\,, \tag{B.11}$$

with the initial conditions

$$\delta\phi_I(t_0) = \delta\phi(t_0) \qquad \delta\pi_I(t_0) = \delta\pi(t_0)\,. \tag{B.12}$$

The time evolution of fluctuations in interaction picture can again be expressed as unitary transformations

$$\delta\phi(t) = U_0^{-1}(t,t_0)\delta\phi(t_0)U_0(t,t_0)\,, \qquad \delta\pi(t) = U_0^{-1}(t,t_0)\delta\pi(t_0)U_0(t,t_0)\,, \tag{B.13}$$

where this time $U_0(t,t_0)$ follows the differential equation

$$\frac{d}{dt}U_0(t,t_0) = -i\,H_0[\delta\phi(t_0),\delta\pi(t_0),t]U_0(t,t_0)\,, \tag{B.14}$$

with initial condition $U_0(t_0,t_0)=1$. Combining together Eqs. (B.9) and (B.14), it can be shown that if we define $F = U_0^{-1}(t,t_0)U(t,t_0)$, the operator $F(t,t_0)$ satisfies the equation

$$\frac{d}{dt}F(t,t_0) = -i\,H_I(t)U_0(t,t_0)\,, \tag{B.15}$$

where $H_I(t)$ is the interaction Hamiltonian in the interaction picture

$$H_I(t) = U_0^{-1}(t,t_0)H_I[\delta\phi_I(t),\delta\pi_I(t),t]U_0(t,t_0)\,. \tag{B.16}$$

The solution of Eq. (B.15) reads

$$F(t,t_0) = T\exp\left(-i\int_{t_0}^{t} H_I(t')dt'\right)\,, \tag{B.17}$$

where, as usual, "T exp(·)" means the time-ordered product of the operators once we do the expansion in series of the exponential. Now, we can express the expectation value of a products of any fluctuation field $Q(t)$ (in Heisenberg picture) as

$$\begin{aligned}\langle Q(t)\rangle &= \langle 0|\,U^{-1}(t,t_0)\,Q(t_0)\,U(t,t_0)\,|0\rangle\\ &= \langle 0|\,U^{-1}(t,t_0)U_0(t,t_0)\,Q_I(t)\,U_0^{-1}(t,t_0)U(t,t_0)\,|0\rangle\\ &= \langle 0|\,F^{-1}(t,t_0)\,Q_I(t)\,F(t,t_0)\,|0\rangle\\ &= \langle 0|\,\left[\bar{T}\exp\left(i\int_{t_0}^{t}H_I(t')dt'\right)\right]\,Q_I(t)\,\left[T\exp\left(-i\int_{t_0}^{t}H_I(t')dt'\right)\right]\,|0\rangle\,,\end{aligned} \tag{B.18}$$

where $\bar{T}$ denotes anti-time ordering and $|0\rangle$ denotes the vacuum state at the initial time t_0. Applying formula (B.18), one can stop the expansion at the desired order in the interaction Hamiltonian H_I. Moreover, notice that, in order to compute (B.18), we firstly need to apply the *Wick's theorem* to evaluate the expectation value of products of fields in interaction picture, and then perform the remnant integrals.

Just as an example, the tree-level expectation value for the product of three fields $\delta\phi(t)$ is given by

$$\langle\delta\phi_{\mathbf{x}_1}\delta\phi_{\mathbf{x}_2}\delta\phi_{\mathbf{x}_3}\rangle(t) = -i\int_{t_0}^{t}dt'\langle 0|\Big[\delta\phi^I_{\mathbf{x}_1}(t)\delta\phi^I_{\mathbf{x}_2}(t)\delta\phi^I_{\mathbf{x}_3}(t),H_I(t')\Big]|0\rangle\,. \tag{B.19}$$

Typically, assuming to work with interaction Hamiltonians in perturbative regime, tree level computations are sufficient to give a good estimation of the statistics of cosmological perturbations.

Appendix C

Spin-raising and lowering operators

Here, we briefly review the definitions of the spin-raising and lowering operators, giving an example on how we can use them to define the weighted spherical harmonics. We refer to e.g. [227] for more details. The spin raising ($\eth$) and lowering ($\bar{\eth}$) operators acting on a generic spin s function ${}_sf(\theta,\phi)$ defined on the 2D sphere are given by

$$\begin{aligned} \eth\ {}_sf(\theta,\phi) &= -\sin^s\theta\,[\partial_\theta + i\csc\theta\partial_\phi]\sin^{-s}\theta_s f(\theta,\phi)\ , \\ \bar{\eth}\ {}_sf(\theta,\phi) &= -\sin^{-s}\theta\,[\partial_\theta - i\csc\theta\partial_\phi]\sin^{s}\theta_s f(\theta,\phi)\ . \end{aligned} \tag{C.1}$$

In particular, the new functions $\eth\ {}_sf(\theta,\phi)$ and $\bar{\eth}\ {}_sf(\theta,\phi)$ have spin $s+1$ and $s-1$, respectively. For example, the spin raising and lowering operators acting twice on a generic spin-±2 function ${}_{\pm2}f(\mu,\phi)$ which is factorized as ${}_{\pm2}f(\theta,\phi) = {}_{\pm2}\tilde{f}(\mu)e^{im\phi}$ (i.e. the CMB polarization fields) can be expressed as

$$\begin{aligned} \bar{\eth}\,^2{}_2f(\theta,\phi) &= \left(-\partial_\mu + \frac{m}{1-\mu^2}\right)^2\left[(1-\mu^2)_2f(\mu,\phi)\right]\ , \\ \eth\,^2{}_{-2}f(\theta,\phi) &= \left(-\partial_\mu - \frac{m}{1-\mu^2}\right)^2\left[(1-\mu^2)_{-2}f(\mu,\phi)\right]\ , \end{aligned} \tag{C.2}$$

where $\mu \equiv \cos\theta$. In this way, just acting with a differential operator, we can easily define spin-0 quantities starting from spin-2 ones. This procedure is used in the case of CMB to pass from the $P^{\pm}$ spin-±2 linear polarization fields to the E and B modes, which are spin-0 fields.

Using Eqs. (C.1), we can even express the spin-weighted spherical harmonic functions on 2D sphere, ${}_sY_{lm}(\theta,\phi)$, in terms of the common harmonic spheres ${}_0Y_{lm}(\theta,\phi) = Y_{lm}(\theta,\phi)$ just acting with the spin raising/lowering operator as

$$\begin{aligned} {}_sY_{lm}(\theta,\phi) &= \left[\frac{(l-s)!}{(l+s)!}\right]^{\frac{1}{2}}\eth\,^sY_{lm}(\theta,\phi)\quad (0\le s\le l)\ , \\ {}_sY_{lm}(\theta,\phi) &= \left[\frac{(l+s)!}{(l-s)!}\right]^{\frac{1}{2}}(-1)^s\bar{\eth}\,^{-s}Y_{lm}(\theta,\phi)\quad (-l\le s\le 0)\ . \end{aligned} \tag{C.3}$$

Then, it is possible to show the validity of the following relations

$$
\begin{aligned}
\eth \; {}_sY_{lm}(\theta,\phi) &= [(l-s)(l+s+1)]^{\frac{1}{2}} \; {}_{s+1}Y_{lm}(\theta,\phi) \ , \\
\bar{\eth} \; {}_sY_{lm}(\theta,\phi) &= -\left[(l+s)(l-s+1)\right]^{\frac{1}{2}} \; {}_{s-1}Y_{lm}(\theta,\phi) \ , \\
\bar{\eth} \, \eth \; {}_sY_{lm}(\theta,\phi) &= -(l-s)(l+s+1) {}_sY_{lm}(\theta,\phi) \; m \, ,
\end{aligned}
\tag{C.4}
$$

which can be used to derive the following explicit expression of the weighted spherical harmonics

$$
\begin{aligned}
{}_sY_{lm}(\theta,\phi) \,&=\, e^{im\phi}\left[\frac{(l+m)!(l-m)!}{(l+s)!(l-s)!}\frac{(2l+1)}{4\pi}\right]^{1/2}\sin^{2l}(\theta/2) \\
&\times \sum_r \binom{l-s}{r}\binom{l+s}{r+s-m}(-1)^{l-r-s+m}\cot^{2r+s-m}(\theta/2) \ .
\end{aligned}
\tag{C.5}
$$

It is straightforward to verify the orthogonality and completeness conditions for the ${}_sY_{lm}(\theta,\phi)$ as

$$
\begin{aligned}
\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta {}_sY^*_{l'm'}(\theta,\phi) {}_sY_{lm}(\theta,\phi) &= \delta_{l',l}\delta_{m',m} \ , \\
\sum_{lm} {}_sY^*_{lm}(\theta,\phi) {}_sY_{lm}(\theta',\phi') &= \delta(\phi-\phi')\delta(\cos\theta-\cos\theta') \, ,
\end{aligned}
\tag{C.6}
$$

as well as the following properties regarding the transformation under conjugate and parity

$$
\begin{aligned}
{}_sY^*_{lm}(\theta,\phi) &= (-1)^{s+m} {}_{-s}Y_{l-m}(\theta,\phi) \ , \\
{}_sY_{lm}(\pi-\theta,\phi+\pi) &= (-1)^l_{-s}Y_{lm}(\theta,\phi) \ .
\end{aligned}
\tag{C.7}
$$

Appendix D

Quantum Boltzmann equation

In this Appendix, we will briefly review the derivation of the so-called *quantum Boltzmann equation* partially following Ref. [93]. This equation provides a quantum framework to study the effects of the collisions between CMB photons and other particles, and can be used to derive the standard Boltzmann equations (4.7)-(4.9).

The starting point is to adopt the second-quantized formalism, introducing creation and annihilation operators for CMB photons obeying the usual canonical commutation relations

$$[a_s(p), a^\dagger_{s'}(p')] = (2\pi)^3 2p^0 \delta^{(3)}(\mathbf{p} - \mathbf{p}')\delta_{ss'} \,, \tag{D.1}$$

where s, s' label the photon polarizations. Now, we introduce the so-called photon number operator

$$\mathcal{D}_{ij}(\mathbf{k}) = a^\dagger_i(\mathbf{k}) a_j(\mathbf{k}) \,. \tag{D.2}$$

The time evolution of this operator, in Heisenberg picture, is given by

$$\frac{d}{dt}\mathcal{D}_{ij} = i[H, \mathcal{D}_{ij}] \,, \tag{D.3}$$

where H is the full Hamiltonian of the theory which we can decompose in the free and interaction terms as

$$H = H_0 + H_I \,. \tag{D.4}$$

Here, the interaction term is a functional of the photon field and all the fields interacting with the photon. Now, we want to express the right-hand side of Eq. (D.3) in terms of fields in interaction picture. For this purpose, we remind that a generic operator $\xi_H(t)$ in Heisenberg picture can be expressed in terms of the same operator $\xi_I(t)$ in interaction picture through

$$\xi_H(t) = F^{-1}(t, t_0)\, \xi_I(t)\, F(t, t_0) \,, \tag{D.5}$$

where

$$F(t, t_0) = T \exp\left(-i \int_{t_0}^{t} H_I(t')dt' \right) \,. \tag{D.6}$$

Expanding Eq. (D.5) in series of the interaction Hamiltonian, we find the first order relation

$$\xi_H(t) \simeq \xi_I(t) + i\int_{t_0}^{t} dt'[H_I(t'), \xi_I(t')] \tag{D.7}$$

which is valid under the condition that the interaction Hamiltonian is a small correction in the total Hamiltonian (i.e. in perturbative regime). Writing this relation for the operator $\mathcal{D}_{ij}$, we get

$$\mathcal{D}^H_{ij}(t) \simeq \mathcal{D}^I_{ij}(t) + i\int_{t_0}^{t} dt'[H_I(t'), \mathcal{D}^I_{ij}(t')]\,. \tag{D.8}$$

Inserting this last equation in the right-hand side of (D.3), we find

$$\frac{d}{dt}\mathcal{D}_{ij}(t) = i[H_I(t), \mathcal{D}^I_{ij}(t)] - \int_{t_0}^{t} dt'[H_I(t), [H_I(t'), \mathcal{D}^I_{ij}(t')]]\,. \tag{D.9}$$

Now, we have to take the expectation value of both the members of the previous equation. In particular, the expectation value of the number operator on the left-side can be computed after introducing the density operator describing a system of photons as

$$\hat{\rho} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\rho_{ij}(\mathbf{p})a^\dagger_i(\mathbf{p})a_j(\mathbf{p})\,, \tag{D.10}$$

where ρ_{ij} is the density matrix (4.1). With this operator the expectation value of a generic operator $\mathcal{O}$ associated to CMB photons can be expressed as

$$\langle\mathcal{O}\rangle = \mathrm{tr}[\hat{\rho}\,\mathcal{O}]\,. \tag{D.11}$$

Thus, the expectation value of the number operator is given by

$$\langle\mathcal{D}_{ij}(\mathbf{k})\rangle = \mathrm{tr}[\hat{\rho}\,D_{ij}(\mathbf{k})] = \int \frac{d^3p}{(2\pi)^3}\langle\mathbf{p}|\,\hat{\rho}\,D_{ij}(\mathbf{k})\,|\mathbf{p}\rangle\,, \tag{D.12}$$

where

$$|\mathbf{p}\rangle = a^\dagger(\mathbf{p})|0\rangle \tag{D.13}$$

denotes the quantum state of a single photon of momentum $\mathbf{p}$. Using repetitively Eq. (D.1) into (D.12), then the expectation value (D.12) reads

$$\langle\mathcal{D}_{ij}(\mathbf{k})\rangle = (2\pi)^3\delta^{(3)}(\vec{0})2k^0\rho_{ij}(\mathbf{k})\,. \tag{D.14}$$

Thus, taking the expectation value of Eq. (D.9), we finally get

$$(2\pi)^3\delta^{(3)}(\vec{0})2k^0\frac{d}{dt}\rho_{ij}(\mathbf{k}) = i\left\langle[H_I(t), \mathcal{D}^I_{ij}(\mathbf{k})]\right\rangle - \int_{t_0}^{t} dt'\ \left\langle[H_I(t), [H_I(t'), \mathcal{D}^I_{ij}(\mathbf{k})]]\right\rangle\,. \tag{D.15}$$

This equation for the photon density matrix can be expressed in terms of Stokes parameters using the definition (4.1). The first term on the right hand side of (D.15) is the so-called *forward scattering term* and describes possible mixing between the different CMB polarizations due to the forward scattering of CMB photons with other particles. It is possible to show that in the standard scenario (i.e. Standard Model interactions between photons and electrons) this term is equal to 0 [93], thus no mixing is induced. The second term on the right hand side of (D.15) is the usual *scattering term* which describes how Stokes parameters vary in time when CMB photons get deflected after a collision with another sea of particles. If we compute the latter taking only the usual QED interactions between photons and electrons, then we can derive the full set of Boltzmann equations (4.7)-(4.9) (see [93] for the full computation).

Appendix E

The energy density of gravitational waves

In this Appendix, we derive the energy density of gravitons (i.e. gravitational waves) in terms of gravitational Stokes parameter $\Delta_I^{(g)}$. Starting from the Lagrangian (9.6), we can find the energy density of gravitational waves as[1]

$$\rho_{gw} = \frac{1}{2} \langle \dot{h}_{\mu\nu} \, \dot{h}^{\mu\nu} \rangle \ . \tag{E.1}$$

By inserting

$$\dot{h}_{\mu\nu}(x) = \frac{i}{2} \int \frac{d^3q}{(2\pi)^3} \sum_{r=+,\times} \left[b_{\mathbf{q}}^{(r)} \, h_{\mu\nu}^{(r)} \, e^{iqx} - b_{\mathbf{q}}^{(r)\,\dagger} \, h_{\mu\nu}^{(r)\,*} \, e^{-iqx} \right] \tag{E.2}$$

into (E.1), we get

$$\begin{aligned} \rho_{gw} = -\frac{1}{8} \Bigg\langle \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} \sum_{r,r'} & \left[b_{\mathbf{q}}^{(r)} \, h_{\mu\nu}^{(r)} \, e^{iqx} - b_{\mathbf{q}}^{(r)\,\dagger} \, h_{\mu\nu}^{(r)\,*} \, e^{-iqx} \right] \\ & \times \left[b_{\mathbf{q}'}^{(r')} \, h^{\mu\nu(r')} \, e^{iq'x} - b_{\mathbf{q}'}^{(r')\,\dagger} \, h^{\mu\nu(r')*} \, e^{-iq'x} \right] \Bigg\rangle \ , \end{aligned} \tag{E.3}$$

[1] Normally, it is impossible to define a full non-linear local energy-momentum tensor for the gravitational field $g_{\mu\nu}$. The definition (E.1) can be used in our context because we work in the weak field approximation, as stated in Eq. (9.4) (see Ref. [296] for more details).

which can be further simplified as

$$
\begin{aligned}
\rho_{gw} = &-\frac{1}{8}\int \frac{d^3q}{(2\pi)^3}\int \frac{d^3q'}{(2\pi)^3}\sum_{r=+,\times}\sum_{r'=+,\times}\Big\{h^{(r)}_{\mu\nu}\,h^{\mu\nu(r')}\,e^{i(q+q')(x)}\;\langle b^{(r)}_{\mathbf{q}}\,b^{(r')}_{\mathbf{q}'}\rangle \\
&-h^{(r)}_{\mu\nu}\,h^{\mu\nu(r')*}\,e^{i(q-q')(x)}\;\langle b^{(r)}_{\mathbf{q}}\,b^{(r')\,\dagger}_{\mathbf{q}'}\rangle - h^{(r)\,*}_{\mu\nu}\,h^{\mu\nu(r')}\,e^{i(q'-q)(x)}\;\langle b^{(r)\dagger}_{\mathbf{q}}\,b^{(r')}_{\mathbf{q}'}\rangle \\
&+h^{(r)\,*}_{\mu\nu}\,h^{\mu\nu(r')*}\,e^{-i(q'+q)(x)}\;\langle b^{(r)\dagger}_{\mathbf{q}}\,b^{(r')\,\dagger}_{\mathbf{q}'}\rangle\Big\} \\
&= \frac{1}{8}\int \frac{d^3q}{(2\pi)^3}\int \frac{d^3q'}{(2\pi)^3}\sum_{r=+,\times}\sum_{r'=+,\times} h^{(r)\,*}_{\mu\nu}\,h^{\mu\nu(r')}\,e^{i(q'-q)(x)}\;\langle b^{(r)\dagger}_{\mathbf{q}}\,b^{(r')}_{\mathbf{q}'}\rangle \\
&= \frac{1}{8}\int \frac{d^3q}{(2\pi)^3}\int \frac{d^3q'}{(2\pi)^3}\sum_{r=+,\times}\sum_{r'=+,\times} h^{(r)\,*}_{\mu\nu}\,h^{\mu\nu(r')}\,e^{i(q'-q)(x)}\;\Big(2q'^0(2\pi)^3\delta^{(3)}(\mathbf{q}'-\mathbf{q})\rho_{rr'}(\mathbf{q}')\Big)\,.
\end{aligned}
\tag{E.4}
$$

Now, using Eqs. (9.32) and (9.10), we get

$$
\begin{aligned}
\rho_{gw} &= \frac{1}{4}\int \frac{d^3q}{(2\pi)^3}\sum_{r=+,\times}\sum_{r'=+,\times} q^0\rho_{rr'}(\mathbf{q})\delta_{r,r'} = \frac{1}{4}\int \frac{d^3q}{(2\pi)^3}\,q^0\left[\rho_{++}(\mathbf{q})+\rho_{\times\times}(\mathbf{q})\right] \\
&= \frac{1}{4}\int \frac{d^3q}{(2\pi)^3}\,q^0\Delta^{(g)}_I(\mathbf{q})\,.
\end{aligned}
\tag{E.5}
$$

We can expand $\Delta^{(g)}_I(\mathbf{q})$ in terms of spherical harmonics as [297]

$$
\Delta^{(g)}_I(\mathbf{q}) = \Delta^{(g)}_I(q^0)\sum_{l,m} c^I_{lm}Y^m_l(\theta',\phi')\,. \tag{E.6}
$$

Therefore, we can write

$$
\rho_{gw} = \frac{\pi}{2}\int df\,f^3\Delta^{(g)}_I(f)\sum_{l,m}\int d^2\hat{\mathbf{q}}\,c^I_{lm}Y^m_l(\theta',\phi')\,, \tag{E.7}
$$

where $q^0 = 2\pi f$. The isotropic part of gravitational waves energy density, $\bar{\rho}_{gw}$, is given by

$$
\bar{\rho}_{gw} = \frac{\pi}{2}\frac{c^I_{00}}{\sqrt{4\pi}}\int df\,f^3\Delta^{(g)}_I(f)\int d^2\hat{\mathbf{q}}\,, \tag{E.8}
$$

and, after normalizing the monopole moment as $c^I_{00} = \sqrt{4\pi}$ and performing the angular integral, we finally get

$$
\bar{\rho}_{gw} = 2\pi^2\int df\,f^3\Delta^{(g)}_I(f)\,. \tag{E.9}
$$

171

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