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Introduction

The syntomic cohomology for smooth varieties of finite type over a DVR of mixed characteristic with perfect residue field was defined by Besser in [?] by gluing the rigid cohomology of the special fiber with the de Rham cohomology of the generic one. We want to develop the syntomic cohomology as a cohomological theory equipped with a definition of cohomology with compact support allowing to construct the Poincaré duality and the Gysin morphism. A possible way to obtain a functorial association between those varieties and complexes calculating their syntomic cohomology is working in the category of diagrams. For this purpose we will define the category *pHD* of *p-adic Hodge diagrams* similar to that studied first in [Beĭ86] (to glue Betti and Hodge cohomologies for varieties over $\mathbb{C}$) and then in [Hub95]. The syntomic diagram and the syntomic diagram with compact support will be defined as *p*-adic Hodge diagrams in order to get a pairing between them. The construction of the category *pHD* is modeled on the work of Bannai [Ban02]: as in our case, he wants to relate the rigid cohomology and the de Rham cohomology of the two fibers, the main difference being that in our hypothesis the comparison maps are not necessarily quasi-isomorphisms (we do not require the variety to be proper). The possibility of regarding the category *pHC* as a gluing of categories (construction developed in [Hub95, §4]) will allow us to compare our work with that of Bannai [Ban02]: Bannai's category of *p*-adic Hodge complexes will be a subcategory of ours, as expected.

The cohomologies involved in the construction of the syntomic theory are equipped with additional structures: in the case of the special fiber, its rigid cohomology has a canonical Frobenius endomorphism while, in the characteristic zero framework, the de Rham cohomology of a smooth scheme is a filtered vector space. In the first part of chapter 1 we will deal with the algebraic stuctures in which the previous objects live in order to find the suitable definition for the categories of diagrams. The category of filtered vector spaces with strict morphisms is abelian, as this would be an advantage because we could construct the derive category of it. The problem with this approach is that the complex $R\Gamma_{\text{dR}}$ is in general not functorial on this category, i.e. the image by $R\Gamma_{\text{dR}}$ of a morphism of smooth varieties over K is not in general a strict morphism of filtered vector spaces, hence this is not the good category to deal with. Following the results of [Hub95], we will see that the non abelian category of filtered K-vector spaces with compatible morphisms can be endowed with a structure of exact category (the definition of exact sequences will be related to strict morphisms), and consequently with a notion of acyclicity and quasi isomorphism that will allow us to generalize the construction of derived category in this non abelian setting. The following step will be the definition of the category *pHD* of *p*-adic Hodge diagrams, containing as objects the syntomic diagrams, as the derived category of a very exact category $\mathcal{H}$, a category of *diagrams of objects*, inheriting its very exactness structure by that of its specializations. One thing we would like to stress concerning the category *pHD* is that the natural compatibility requirement for homotopies and quasi isomorphisms between complexes of objects in $\mathcal{H}$ will not allow us to get as specializations the

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derived categories obtained by the process of derivation of each specialization of *pHC* (the reason is that the multiplicative system by which we localize and the set of homotopies by which we quotient is smaller in the case of diagrams). Anyway, what is essential is that the property of the de Rham complexes to be strict [PS08] will allow the syntomic diagrams to have a good notion of cohomology, i.e. the cohomology of each specialization will coincide to what we expect if we don't consider the compatibility requirement due to the diagram construction. This is because for a complex of objects of a very exact category, the naive cohomology coincides with the cohomology defined by the *t*-structure, as in the abelian case. In the last part of chapter 1 we solve the problem that dealing with the various objects and morphisms for the construction of the syntomic cohomology as a functor at level of complexes, we first obtain a diagram made up by many complexes: thanks to proposition 1.2.26, we can define a functor from large diagrams to smaller ones (i.e. *p*-adic Hodge complexes) that preserves cohomology and the tensor product. Using this functor we will be able to define our syntomic diagram as a more manageable object still preserving the cohomological structures and informations needed to develop the syntomic theory (the approach will be the same also for the compact support case).

The first step in the construction of the syntomic diagrams will be the construction of a serie of complexes of sheaves (whose hypercohomology is the desired one), and maps between those complexes. In order to compute their hypercohomology by keeping the structure of diagrams we will need a functorial flasque resolution for complexes of sheaves over topos. Generalized Godement resolution [**?**] is flasque if made with respect to a conservative family of points of the topos (i.e. a set of points that distinguishes sheaves). The conservative family for the topos of rigid analytic spaces will be made of prime filters (van der put scnheider..)

Chapter 1

Categories of diagrams

1.1 Preliminars

1.1.1 Exact categories

For first we recall some definitions concerning a generic category (see fo example [Freyd03]).

Definition 1.1.1. Let C be a category. Let $f : A \to B$ be a morphism in C. We will say that

- f is a monomorphism if the only pairs $x : C \to A$, $y : C \to A$ of maps in C such that $f \circ x = f \circ y$ are the obvious ones: $x = y$. We will also say that f is left effaceable.

- f is an epimorphism if the only pairs $x : B \to C$, $y : B \to C$ of maps in C such that $x \circ f = y \circ f$ are the obvious ones: $x = y$. We will also say that f is right effaceable.

- f is an isomorphism if there are maps $x : B \to A$, $y : B \to A$ such that $f \circ x$ and $y \circ f$ are identity maps.

Definition 1.1.2. Let C be a category and let A_1, A_2 be objects of C. We say that

- an object P is the product of A_1 and A_2 if there exist maps $p_1 : P \to A_1$ and $p_2 : P \to A_2$ such that for every pair of maps $p'_1 : X \to A_1$ and $p'_2 : X \to A_2$ there is a unique $u : X \to P$ such that the diagram

$$
\begin{array}{ccc}
 & & A_1 \\
 & \nearrow^{p_1'} & \uparrow_{p_1} \\
X & \xrightarrow{\ u\ } & P \\
 & \searrow_{p_2'} & \downarrow_{p_2} \\
 & & A_2
\end{array}
$$

commutes.

- an object S is the sum of A_1 and A_2 if there exist maps $s_1 : A_1 \to P$ and $s_2 : A_2 \to S$ such that for every pair of maps $s'_1 : A_1 \to X$ and $s'_2 : A_2 \to X$ there is a unique $u : S \to X$

such that the diagram

$$
\begin{array}{ccc}
A_1 & & \\
\ \ \downarrow s_1 & \searrow s_1' & \\
X & \xrightarrow{u} & S \\
\ \ \uparrow s_2 & \nearrow s_2' & \\
A_2 & &
\end{array}
$$

commutes.

In the category of k-vector spaces, product and sum are given by the direct sum.

Definition 1.1.3. Let C be a category, let $f : A \to C$ and $g : B \to C$ be morphisms in C. The pullback of f and g consists of an object P and two morphisms $p_1 : P \to A$, $p_2 : P \to B$ such that the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{p_1} & A \\
\downarrow p_2 & & \downarrow f \\
B & \xrightarrow{g} & C
\end{array}
$$

commutes, and (P, p_1, p_2) is universal with respect to this diagram.

Definition 1.1.4. Let C be a category, let $f : C \to A$ and $g : C \to B$ be morphisms in C. The pushout of f and g consists of an object P and two morphisms $i_1 : A \to P$, $i_2 : B \to P$ such that the diagram

$$
\begin{array}{ccc}
P & \xleftarrow{i_1} & A \\
\uparrow i_2 & & \uparrow f \\
B & \xleftarrow{g} & C
\end{array}
$$

commutes, and (P, i_1, i_2) is universal with respect to this diagram.

A zero object is an object with precisely one map to and from each object. If the category C has a zero object, we define the zero map $A \to B$ to be the unique map $A \to 0 \to B$ for any A, B in C.

Let C be a category with a zero object, let $f : A \to B$ be a morphism in C.

- The kernel of f is a map $i : K \to A$ such that the diagram

$$
\begin{array}{ccc}
K & & \\
\downarrow i & \searrow 0 & \\
A & \xrightarrow{f} & B
\end{array}
$$

commutes, and for all $i' : K' \to A$ such that

$$
\begin{array}{ccc}
K' & & \\
\downarrow i' & \searrow 0 & \\
A & \xrightarrow{f} & B
\end{array}
$$

commutes, there exists a unique $u : K' \to K$ such that

$$
\begin{array}{ccc}
 & & K' \\
 & \nearrow^{u} & \downarrow^{i'} \\
K & \xrightarrow{\;i\;} & A
\end{array}
$$

commutes.

- The cokernel of f is a map $p : B \to C$ such that the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & B \\
 & \searrow^{0} & \downarrow^{p} \\
 & & C
\end{array}
$$

commutes, and for all $p' : B \to C'$ such that

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & B \\
 & \searrow^{0} & \downarrow^{p'} \\
 & & C'
\end{array}
$$

commutes, there exists a unique $u : C \to C'$ such that

$$
\begin{array}{ccc}
B & \xrightarrow{\;p'\;} & C' \\
\downarrow^{p} & \nearrow^{u} & \\
C & &
\end{array}
$$

commutes.

- If the map f has a kernel $i : K \to A$ admitting a cokernel, the cokernel of i is called the image of f.

- If the map f has a cokernel $p : B \to C$ admitting a kernel, the kernel of p is called the coimage of f.

Before giving the definition of abelian and exact category we recall when a category is said to be additive.

Definition 1.1.5. A category C is additive if
i)There exists a zero object.
ii)For every A, B in C the set $\mathrm{Hom}_C(A, B)$ has an addition endowing it with the structure of an abelian group and such that the composition is bilinear.
iii)There exist finite sums and products.

In an additive category we can define the cone of a morphism between complexes.

Definition 1.1.6. Let C be an additive category, let $A^\bullet \to B^\bullet$ be a map of complexes of objects in C. The mapping cone of f is the complex $MC(f) = A^\bullet[1] \oplus B^\bullet$ with differentials defined as

$$
d_{MC(f)}^i : A^{i+1} \oplus B^i \to A^{i+2} \oplus B^{i+1}, \quad (a, b) \mapsto (-d_{A^\bullet}^{i+1}(a), d_{B^\bullet}(b) - f^{i+1}(a)).
$$

The definition of mapping cone endows the homotopy category $K(C)$ with a structure of triangulated category.

Definition 1.1.7. A category $\mathcal{A}$ is abelian if

A0. $\mathcal{A}$ has a zero object

A1. For every pair of objects there exists a product and

A1*. a sum.

A2. Every map has a kernel and

A2*. a cokernel.

A3. Every monomorphism is a kernel of a map.

A3*. Every epimorphism is a cokernel of a map

If $\mathcal{A}$ is an abelian category, there is a notion of naive cohomology of a complex $A^\bullet$ defined as

$$H^i(A^\bullet) = \ker d_A^n / \operatorname{Im} d_A^{n-1}.$$

If we consider $A^\bullet$ as an object of the category $K(\mathcal{A})$ then we can define the cohomology of $A^\bullet$ as its image by the cohomological functor existing for the triangulated structure of the homotopy category $K(\mathcal{A})$. The two definitions are equivalent by definition of truncation functors[1]. Furthermore, the cone of a map $f : A^\bullet \to B^\bullet$ is acyclic if and only if f is a quasi isomorphism (consider the long exact sequence of cohomology associated to the triangle $A^\bullet \xrightarrow{f} B^\bullet \to MC(f) \to$ in $K(\mathcal{A})$).

Now we are going to give the definition of exact category. It can be considered as a generalization of the concept of abelian category.

$\boxed{\texttt{exactcat}}$ **Definition 1.1.8.** An exact category is an additive category $\mathcal{E}$ equipped with a family E of sequences of the form

$$0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0 \qquad\qquad (1.1) \;\boxed{\texttt{ex1}}$$

called the short exact sequences of C, such that

i)Any sequence in $\mathcal{E}$ isomorphic to a sequence in E is in E. For any C', C'' in $\mathcal{E}$ the sequence

$$0 \to C' \xrightarrow{(id,0)} C' \oplus C'' \xrightarrow{p_2} C'' \to 0$$

is in E. For any sequence 1.1 in E, i is a kernel for j and j is a cokernel for i in the additive category $\mathcal{E}$.

ii)The class of admissible epimorphisms (i.e. of maps that occurs as second arrow of an exact sequence 1.1) is closed under composition and under pullback by arbitrary maps in $\mathcal{E}$. Dually, the class of admissible monomorphisms (i.e. of maps that occurs as first arrow of an exact sequence 1.1) is closed under composition and under pushout by arbitrary maps in $\mathcal{E}$.

iii)Let $C \to C''$ be a map possessing a kernel in $\mathcal{E}$. If there exists a map $D \to C$ in $\mathcal{E}$ such that $D \to C \to C''$ is an admissible epimorphism, then $C \to C''$ is an admissible epimorphism. Dually for admissible monomorphisms.

Every abelian category $\mathcal{A}$ is obviously exact if we take as E the set of exact sequences of $\mathcal{A}$.

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[1]referenza?

Definition 1.1.9. An exact functor $F : \mathcal{E} \to \mathcal{E}'$ between exact categories is an additive functor carrying exact sequences in $\mathcal{E}$ into exact sequences in $\mathcal{E}'$.

We recall the construction of the derived category of an exact category (see [Hub95, 2.1] and [Lau]).

`funtore_h` *Theorem* 1.1.10. Let $\mathcal{E}$ be an exact category. There exists an abelian category $\mathcal{F}(\mathcal{E})$ and a fully faithful functor

$$h : \mathcal{E} \to \mathcal{F}(\mathcal{E})$$

making $\mathcal{E}$ a full subcategory of $\mathcal{F}(\mathcal{E})$, stable by extension, such that a sequence

$$0 \to E' \to E \to E'' \to 0$$

of $\mathcal{E}$ is exact if and only if its image by h is an exact sequence of $\mathcal{F}(\mathcal{E})$.

Proof. See [cf. [Lau, 1.0.3]]. $\qquad\qquad\square$

We stress the fact that the functor h canonically extends to the category of complexes $C(\mathcal{E})$.

`s_exact_cat` **Definition 1.1.11.** A complex $E^\bullet$ of objects of an exact category $\mathcal{E}$ is said to be acyclic in degree n if $h(E^\bullet)$ is acyclic in degree n. If $E^\bullet$ is acyclic in any degree, we will simply say that $E^\bullet$ is acyclic. We say that a morphism f in $C(\mathcal{E})$ is a quasi isomorphism if $MC(f)$ is acyclic.

`sucatesatta` *Remark* 1.1.12. Let $\mathcal{E}$ be an exact category. In addition, we assume that all morphisms in $\mathcal{E}$ have kernel and cokernel and that for $u : E \to F$ the sequence

$$0 \to \ker u \to E \to \operatorname{Coim} u \to 0$$

is exact.

`def_acyclic` *Remark* 1.1.13. In chapter 2 of [Hub95] Huber defines a morphism in an exact category to be strict if its image and coimage are isomorphic (by remark 1.1.12, we can always speak about the strictness of a morphism in an exact category, because we suppose that every map has image and coimage). We want to verify that Huber's definition of strict epimorphism (i.e. a map right effaceable in which image and coimage are isomorphic) is equivalent to 1.1.8. Let

$$0 \to E' \xrightarrow{u} E \xrightarrow{p} E'' \to 0$$

be an exact sequence. We want to show that p is epimorphic and that $\operatorname{Im}p \simeq \operatorname{Coim}p$. For i) of definition 1.1.8, we have $p = \operatorname{coker}u$, hence p is epimorphic by proposition 4.34 of [Kn07]. For the second claim, by remark 1.1.12, the sequence

$$0 \to \ker p \to E \to \operatorname{Coim}p = \ker \operatorname{Coker}p \to 0$$

is exact hence, by i) of definition 1.1.8, the second piece of the sequence is the cokernel of the first one. Viceversa, let $p : E \to F$ be an epimorphism such that $\operatorname{Im}p \simeq \operatorname{Coim}p$. By the assumption in remark 1.1.12, the sequence

$$0 \to \ker p \to E \to \operatorname{Im}p \to 0$$

is exact. Every exact category is balanced, hence by proposition 10.2 of [Mit65] F is the image of the epimorphism p.

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Huber also defines a complex $E^\bullet$ in $K(\mathcal{E})$ to be acyclic in degree n if $d^{n-1} : E^{n-1} \to \ker(d^n)$ is a strict epimorphism, i.e. (just seen) if there exists a short exact sequence in $\mathcal{E}$

$$0 \to E' \to E^{n-1} \xrightarrow{d^{n-1}} \ker d^n \to 0. \qquad (1.2) \boxed{\text{shs}}$$

The image of 1.2 by the exact functor h is a short exact sequence

$$0 \to h(E') \to h(E^{n-1}) \xrightarrow{h(d^{n-1})} h(\ker d^n) \to 0$$

in the abelian category $\mathcal{F}(\mathcal{E})$, hence

$$\mathrm{Im} h(d^{n-1}) \simeq h(\ker d^n).$$

By definition of $hE = \mathrm{Hom}_\mathcal{E}(-, E)$ and by the universal property of the kernel in $\mathcal{E}$ (the kernel always exists by the assumption in remark 1.1.12) we have

$$\ker h(d^n) = \{f \in \mathrm{Hom}_\mathcal{E}(-, E^n) \,|\, d^n \circ f = 0\} = \mathrm{Hom}_\mathcal{E}(-, \ker d^n) = h(\ker d^n)$$

and we can conclude that Huber's definition of acyclicity is equivalent to definition 1.1.11.

$\boxed{\text{localizzare}}$ *Proposition* 1.1.14 (Corollary 1.2.5 of [Lau]). The full subcategory $K^\phi(\mathcal{E})$ of $K(\mathcal{E})$ whose objects are the acyclic complexes is *epaisse* ([Verd], §2 definition 1.1).

By the previous proposition, we can consider the derived category

$$D(\mathcal{E}) = K(\mathcal{E})/K^\phi(\mathcal{E}).$$

We can similarly define $D^b(\mathcal{E})$.

Remark 1.1.15. In the notations of [Verd, §2], the multiplicative system $\varphi(K^\phi(\mathcal{E}))$ is equal to the set of quasi isomorphisms of $K(\mathcal{E})$. This shows that $D(\mathcal{E})$ can be also constructed by localizing $K(\mathcal{E})$ with respect to the multiplicative system of quasi isomorphisms, as in the case of abelian categories. This holds more generally for any triangulated category (loc.cit.).

$\boxed{\text{heart}}$ *Remark* 1.1.16. There are canonical truncation functors on $C(\mathcal{E})$: for a complex $E^\bullet$ in $C(\mathcal{E})$ let

$$\tau_{\leq n}(E^\bullet) = [\ldots E^{n-2} \to E^{n-1} \to \ker d^n \to 0 \ldots];$$

$$\tau_{\geq n}(E^\bullet) = [\ldots 0 \to \mathrm{Coim}(d^n) \to E^n \to E^{n+1} \ldots].$$

They induce functors on the homotopy category which commute with quasi isomorphisms. Hence they give functors on the derived category. By [Lau] 1.4.2 they define a non-degenerated t-structure on $D(\mathcal{E})$. Its heart is an abelian category $\tilde{\mathcal{E}}$ equivalent to the full subcategory of $D(\mathcal{E})$ of complexes of the form

$$\ldots \to 0 \to K^{-1} \xrightarrow{d} K^0 \to 0 \ldots$$

where d is a (not necessarily strict) monomorphism.

The problem with complexes of exact categories (that are not abelian) is that we cannot say that the naive cohomology is isomorphic to the cohomology of the t-structure of $D(\mathcal{E})$. To solve this problem we need to introduce the definition of very exact category.

Definition 1.1.17. An object of $C(\mathcal{E})$ is strict if all differentials are strict morphisms, i.e. each factors as a strict epimorphism followed by a strict monomorphism[2]. The full subcategory of strict complexes is denoted by $C_{(\mathrm{str})}(\mathcal{E})$.

Let $C_{\mathrm{str}}(\mathcal{E})$ be the full subcategory of $C_{(\mathrm{str})}(\mathcal{E})$ where only those morphisms are used which are strict on the naive cohomology objects (the naive cohomology of a complex $E^\bullet$ in $C(\mathcal{E})$ exists, even if it does not induce a functor on the derived category).

We will also use the notations D_{str} and $D_{(\mathrm{str})}$ for the image of the categories of complexes under the canonical functor.

<div style="float:left">eryexactcat</div>

Definition 1.1.18 (cf. [Hub95, 2.1.2])**.** The category of complexes is called derivabily strict if the following conditions hold:
1. Strictness of complexes is invariant under quasi isomorphisms.
2. Let $f : E \to F$ be a morphism between strict complexes which induces strict morphisms on all naive cohomology objects. Then the cone of f is also a strict complex.

An exact category is called very exact if all morphisms have a kernel and a cokernel and its complex category is derivably strict.

<div style="float:left">o_coh_agree</div>

Proposition 1.1.19. If $\mathcal{E}$ is a very exact category, on the full subcategory of strict complexes $C_{\mathrm{str}}(\mathcal{E})$ the naive cohomology objects agree with the cohomology objects of the canonical t-structure.

Proof. See proposition 2.1.3 of [Hub95]. □

1.1.2 Filtered objects of an abelian category

<div style="float:left">FA</div>

Let $\mathcal{A}$ be a fixed abelian category. We recall the definition of filtered objects of $\mathcal{A}$.

Definition 1.1.20 (cf. [Del71, 1.1])**.** A (decreasing) filtration F of an object A of $\mathcal{A}$ is a family of subobjects

$$\{F^n A\}_{n \in \mathbb{Z}}$$

such that $F^m A \subset F^n A$ for every $n \leq m$. A filtered object is an object provided with a filtration.

A (compatible) morphism from a filtered object (A, F) to a filtered object (B, F) is a map $f : A \to B$ such that $f(F^n A) \subset F^n B$ for every $n \in \mathbb{Z}$.

The category $F\mathcal{A}$ of filtered objects of $\mathcal{A}$ and compatible morphisms is not abelian. To show this it is sufficient to find a morphism that is both monomorphic and epimorphic without being an isomorphism (see [Freyd03], theorem 2.12). Consider for example A an object in $\mathcal{A}$ different from the zero object, two filtrations $F, \bar{F}$ on A defined by

$$\ldots = F^2 A = F^1 A = 0 \subset A = F^0 A = \ldots$$

$$\ldots = \bar{F}^2 A = 0 \subset A = \bar{F}^1 A = \bar{F}^0 A = \ldots$$

and let $f : (A, F) \to (A, \bar{F})$ be the compatible morphism of filtered spaces induced by the identity on A. The map f is clearly monomorphic and epimorphic, but it is not an isomorphism: in fact ever compatible morphism $x : (A, \bar{F}) \to (A, F)$ such that $f \circ x = id_{(A,\bar{F})}$ should be the identity at level of $\mathcal{A}$, then we would have $x(\bar{F}^1 A) \subset 0$, i.e. $x(\bar{F}^1 A) = 0$ whence $f(x(\bar{F}^1 A)) = 0 \neq A = \bar{F}^1 A$.

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[2]Huber's definition of strict monomorphism coincides with Lamon's one, the proof is similar to that in the first part of remark 1.1.13

Definition 1.1.21. A morphism $f : (A, F) \to (B, F)$ is strict if the canonical arrow

$$\mathrm{Coim} f \to \mathrm{Im} f$$

is an isomorphism of filtered objects.

Remark 1.1.22. In a module category[3], a morphism f is strict if and only if

$$f(F^n A) = f(A) \cap F^n B$$

for every $n \in \mathbb{Z}$ (see [Del71] after proposition 1.1.11[4]).

Proposition 1.1.23. Let K be a field. The category $V_K^{F_s}$ of filtered K-vector spaces with strict morphisms is abelian.

Proof. The zero object $(0, F)$ is obviously defined.

Let $(A_1, F), (A_2, F)$ in $V_K^{F_s}$, then $A_1 \oplus A_2$ is a product in the category Vect_K, i.e. for any vector space X, for any $p'_1 : X \to A_1, p'_2 : X \to A_2$ there exists a unique $u : X \to A_1 \oplus A_2$ such that the diagram

(1.3) `diagraminve`

commutes. We define a filtration F on the product by setting $F^n(A_1 \oplus A_2) = F^n A_1 \oplus F^n A_2$. The projections $p_i : (A_1 \oplus A_2, F) \to (A_i, F)$ are strict by definition, so they are morphisms in $V_K^{F_s}$. Let (X, F) be a filtered vector space. To show that the object $(A_1 \oplus A_2, F)$ is a product in $V_K^{F_s}$ we must verify that $u : (X, F) \to (A_1 \oplus A_2, F)$ is strict. Let $u(X) = A'_1 \oplus A'_2$, we must prove that

$$u(F^n X) = F^n(A_1 \oplus A_2) \cap (A'_1 \oplus A'_2) \qquad (1.4) \quad \boxed{\texttt{u\_strict}}$$

for every $n \in \mathbb{Z}$. The first member of the previous equality can be written as $u(F^n X) = p_1(u(F^n X)) \oplus p_2(u(F^n X)) = p'_1(F^n X) \oplus p'_2(F^n X)$ by definition of projection in Vect_K and by the commutativity of the diagram 1.3. On the other hand $F^n(A_1 \oplus A_2) \cap (A'_1 \oplus A'_2) = (F^n A_1 \cap A'_1) \oplus (F^n A_2 \cap A'_2)$. The strictness of p'_i implies that $p'_i(F^n X) = F^n A_i \cap p'_i X = F^n A_i \cap p_i(u(X)) = F^n A_i \cap p_i(A'_1 \oplus A'_2) = F^n A_i \cap A'_i$ whence the result.

The existence of the sum can be similarly shown.

Now we must prove that if $f : (A, F) \to (B, F)$ is a morphism in $V_K^{F_s}$ we can define the kernel of f by $i : (\ker f, F) \to (A, F)$ where $\ker f$ is the kernel in Vect_K endowed with the filtration induced by the filtration on A and i is the canonical injection. The map i is strict by definition of induced filtration, and the composition $f \circ i$ is clearly the null object. We must prove that for every $i' : (K, F) \to (A, F)$ in $V_K^{F_s}$ such that $f \circ i' = 0$ the unique map $u : K \to \ker f$ such that the diagram

(1.5) `diagramkenr`

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[3]definizione?
[4]dove per non dimostrato!

commutes is strict with respect to the filtrations just defined, i.e. that $u(F^n K) = F^n \ker f \cap u(K)$. By the strictness of i', the left side can be written as $i'(F^n K) = i'(K) \cap F^n A$, and the second term is $i(u(K)) \cap i(F^n A) =$ by the injectivity of i in Vect_K and the commutativity of the diagram 1.5.

The existence of cokernel in $V_K^{F_s}$ analogously follows by the existence of it in Vect_K and the definition of filtration on the quotient.

The fact that every monomorphism $i : (K, F) \to (A, F)$ is the kernel of a morphism $f_i : (A, F) \to (B, F)$ in $V_K^{F_s}$ follows by the same statement in Vect_K if we prove that the map f_i is strict. It is sufficient to consider $B = A/K$ endowed with the quotient filtration. A similar argument shows that every epimorphism is the cokernel of a morphism. $\qquad\square$

In order to define the category of p-adic Hodge diagrams we would like to deal with abelian categories because we need to derive each of them. The problem with the category $V_K^{F_s}$ is that we cannot be sure that the de Rham cohomology (as defined in 3.2) will be functorial if considered with values in the category $V_K^{F_s}$. The solution is endowing the non abelian category $F\text{Vect}_K$ with the structure of an exact category (definition 1.1.8), in order to be able to derive it.

Definition 1.1.24. A sequence

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$$

in $F\mathcal{A}$ is called short exact sequence if a is a strict monomorphism and b a strict epimorphism with kernel a [Hub95, 3.1.1].

Proposition 1.1.25. The category $F\mathcal{A}$ (with the previous definition of exact sequence) is very exact.

Proof. By [Hub95, 3.1.2] we can say that $F\mathcal{A}$ (with the previous definition of exact sequence) is exact. By proposition 3.1.7 and 3.1.8 of [Hub95] we can conclude. $\qquad\square$

Motivations for these constructions: a jump to next chapters. We will not consider directly the abelian category of filtered complexes with (compatible differentials and) strict morphisms because we cannot be sure that the de Rham complex is functorial with this category as target. By [Hub95] (proposition 7.1.2) we will only know the de Rham complexes to be a functor with values on the category $C_{str}(FVect_{K_0})$ (strict differentials and maps that are strict on the naive cohomology).

1.1.3 Frobenius structure

Definition 1.1.26 (Frobenius morphism)**.** Let k be a field of characteristic p. The Frobenius endomorphism of k is the following map

$$\sigma : k \to k, \quad \alpha \mapsto \alpha^p.$$

If σ is an automorphism, the field k is said to be perfect. Let $\mathcal{V}$ be the ring of Witt vectors of a perfect field k, and let K be the fraction field of $\mathcal{V}$. There exists a unique homomorphism $\sigma' : \mathcal{V} \to \mathcal{V}$ such that the diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\sigma'} & \mathcal{V} \\ \downarrow & & \downarrow \\ k & \xrightarrow{\sigma} & k \end{array}$$

commutes (see [Se], or also theorem 1.2 of [Rab07] for a more update reference). By construction, the map σ' is bijective, hence it is an automorphism of $\mathcal{V}$. This implies that also the extension of σ' to the fraction field

$$\sigma'' : K \to K$$

is an automorphism.

Notation. Throughout this work $\mathcal{V}$ is a complete discrete valuation ring with maximal ideal $\mathbf{p}$, quotient field K and residue field k of characteristic p. We assume k is perfect and we denote $\mathcal{V}_0 \subset \mathcal{V}$ the Witt ring of k, K_0 its quotient field and $\sigma : K_0 \to K_0$ the isomorphism which is the lifting in characteristic zero of the Frobenius automorphism on k.

If C is an abelian category, we will denote by $C(C)$ the category of complexes of objects in C, $K(C)$ the category $C(C)$ where morphisms are considered modulo null homotopic maps, $D(C)$ the derived category of C.

We use $X, Y \ldots$ for k-schemes; $\mathcal{X}, \mathcal{Y}, \ldots$ for K-analytic spaces; $X, Y, \ldots$ for K-schemes; $\mathscr{X}, \mathscr{Y}, \ldots$ for $\mathcal{V}$-schemes.

1.2 Diagrams

1.2.1 The category pHC of p-adic Hodge complexes

Definition 1.2.1. Let C be a category and let $F : C \to C$ be a covariant functor. The pair category of C with respect to the functor F is the category C_F defined as follows:

- an object in C_F is a pair (M, φ) where M is an object of C and $\varphi : M \to F(M)$ is a morphism in C

- a morphism in C_F from (M, φ) to (N, ψ) is map $f : M \to N$ in C such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & F(M) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F(f)} \\ N & \xrightarrow{\psi} & F(N) \end{array}$$

commutes.

pairop | *Lemma* 1.2.2. Let C be a category and let $F, G : C \to C$ be covariant functors such that $GF = FG = id_C$. The category $(C_F)^{op}$ is equivalent to the pair category $(C^{op})_G$.

Proof. The equivalence of categories is given by the functor

$$\alpha : (C_F)^{op} \to (C^{op})_G$$

which sends an object (M, φ) to $(GM, G\varphi)$ and a morphism f to Gf. □

mono | *Lemma* 1.2.3. Let C be a category and let $f : (M, \varphi) \to (N, \psi)$ be a morphism in C_F.
i)If f is a monomorphism in C then f is a monomorphism in C_F.
ii)If C is abelian and f is a monomorphism in C_F, then f is a monomorphism in C.

Proof. The first part of the lemma is trivial by definition of monomorphism.

Suppose that for any $g_1, g_2 : (M', \varphi') \to (M, \varphi)$ in C_F such that $f \circ g_1 = f \circ g_2$ we have $g_1 = g_2$. The category C is abelian, hence it is equivalent to prove that the kernel of $f : M \to N$ is equal to zero (see 2.17* of [Freyd03]). Let $g_1 : (\ker f, \varphi_{|\ker f}) \to (M, \varphi)$ be the morphism in C_F induced by the inclusion of $\ker f$ into M, and let $g_2 : (\ker f, \varphi_{|\ker f}) \to (M, \varphi)$ be the zero morphism. The composition $f \circ g_1$ equals $f \circ g_2$ (because both equal zero). Hence we can conclude that the inclusion g_1 is the zero map, i.e. that $\ker f = 0$.

$\square$

CFabelian *Proposition* 1.2.4. If C is an abelian category and $F, G : C \to C$ are covariant functors such that $GF = FG = id_C$, then the pair category C_F is abelian.

Proof. We must verify that C_F has a null object, that it has pullbacks and pushouts and that every monomorphism (resp epimorphism) is a kernel (resp cokernel) of some morphism.

The object 0 of C provided with the null endomorphism is clearly an initial and terminal object of C^F.

Let $\{f_i : (M_i, \varphi_i) \to (N, \varphi)\}_{i \in I}$ be a family of morphisms in C_F. Consider a pullback (P, p_i) of the family $\{f_i\}$ in C (where $p_i : M_i \to N$ are the projections). The functor F is an equivalence of categories, hence $(F(P), F(p_i))$ is a pullback of $F(f_i)$ in C. The big diagram

commutes because its componing smaller diagram commutes; by the universal property of the pullback $F(P)$ there exists a unique

$$\psi : P \to F(P)$$

in C such that the diagram

commutes for every $i \in I$. This shows that (P, ψ) is an object of C_F and that the maps p_i are morphisms in C_F from (P, ψ) to (M_i, φ_i) for every $i \in I$. Furthermore, (P, ψ) makes the diagram

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commute because of the properties of P as a pullback in C. Let (P', ψ') and $p'_i : (P', \psi') \to (M_i, \varphi_i)$ in C_F such that

$$
\begin{array}{ccc}
(P', \psi') & \xrightarrow{\ p'_j\ } & (M_j, \varphi_j) \\
{\scriptstyle p'_i}\downarrow & & \downarrow{\scriptstyle f_j} \\
(M_i, \varphi_i) & \xrightarrow{\ f_i\ } & (N, \varphi)
\end{array}
$$

commutes. By the properties of P as a pullback in C there exists a unique $u : P' \to P$ such that the diagram

$$
\begin{array}{ccc}
P' & \xrightarrow{\ u\ } & P \\
& {\scriptstyle p'_i}\searrow & \downarrow{\scriptstyle p_i} \\
& & M_i
\end{array}
\qquad (1.6) \quad \boxed{\text{coM}}
$$

commutes for every $i \in I$. We must prove that u is a morphism in C_F, i.e. that the diagram

$$
\begin{array}{ccc}
P' & \xrightarrow{\ u\ } & P \\
{\scriptstyle \psi'}\downarrow & & \downarrow{\scriptstyle \psi} \\
F(P') & \xrightarrow{\ F(u)\ } & F(P)
\end{array}
\qquad (1.7)
$$

commutes. For every $i \in I$ we set $\alpha_i = F(p_i) \circ \psi \circ u$. The diagram

$$
\begin{array}{ccc}
P' & & \\
& \searrow{\scriptstyle \alpha_j} & \\
{\scriptstyle \psi \circ u}\searrow & F(P) \xrightarrow{\ F(p_j)\ } & F(M_j) \\
{\scriptstyle \alpha_i} & {\scriptstyle F(p_i)}\downarrow \qquad & \downarrow{\scriptstyle F(f_j)} \\
& F(M_i) \xrightarrow{\ F(f_i)\ } & F(N)
\end{array}
$$

commutes by definition. By the commutativity of 1.6 and the fact that p_i and p'_i are morphisms in C_F, also the diagram

$$
\begin{array}{ccc}
P' & & \\
& \searrow{\scriptstyle \alpha_j} & \\
{\scriptstyle F(u)\circ\psi'}\searrow & F(P) \xrightarrow{\ F(p_j)\ } & F(M_j) \\
{\scriptstyle \alpha_i} & {\scriptstyle F(p_i)}\downarrow \qquad & \downarrow{\scriptstyle F(f_j)} \\
& F(M_i) \xrightarrow{\ F(f_i)\ } & F(N)
\end{array}
$$

commutes hence, by the properties of $F(P)$ as a pullback, we can conclude that $\psi \circ u = F(u) \circ \psi'$.

Let $f : (M, \varphi) \to (N, \psi)$ be a monomorphism in C_F. By lemma 1.2.3, $f : M \to N$ is a monomorphism in the abelian category C, so there exists a map $g : N \to N'$ such that f is a kernel of g in C, i.e. $g : N \to N'$ is a cokernel of f. The fact that $F(g) \circ \psi \circ f = F(g) \circ F(f) \circ \varphi = F(fg) \circ \varphi = 0$ implies, by the universal property of N' as a cokernel of f in C, the existence of

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a unique $\psi' : N' \to FN'$ such that the diagram

$$
\begin{array}{c}
\end{array}
$$

commutes, whence $g : (N, \psi) \to (N', \psi')$ is a morphism in C_F. The composition $g \circ f$ is obviously zero also in C_F. It remains to verify that $((M, \varphi), f)$ satisfies the universal property of the kernel of g. Let $f' : (M', \varphi') \to (N, \psi)$ in C_F such that $g \circ f' = 0$. By the universal property of (M, f) as kernel of g in C there exists a unique $u : M' \to M$ such that $f \circ u = f'$. We must prove that u is a morphism in C_F, i.e. that the diagram

$$
\begin{array}{ccc}
M' & \xrightarrow{\ u\ } & M \\
\downarrow{\varphi'} & & \downarrow{\varphi} \\
FM' & \xrightarrow{F(u)} & FM
\end{array}
$$

commutes. In the diagram

we have $F(g) \circ [F(f') \circ \varphi'] = 0$, and the map $F(u) \circ \varphi'$ is the unique map α such that

$$
F(f) \circ \alpha = F(f') \circ \varphi' \tag{1.8}
$$

forthekerne

(by the universal property of $(FM, F(f))$ as a kernel of $F(g)$). But also $\varphi \circ u$ satisfies the property 1.8, whence the result.

A category has pushouts if and only if its opposite category has pullbacks, and we have just proved that $(C^{op})_G$ admits pullbacks, hence we can conclude by lemma 1.2.2. We can use the same strategy to say that C_F has cokernels.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Definition 1.2.5. Let S_{rig} be the category of pairs (M, φ), where
    i) M is the tensor product $M_0 \otimes_{K_0} K$ for some object M_0 in $Vect_{K_0}$.
    ii) (Frobenius structure) $\varphi : (M_0)^\sigma \to M_0$ is a K_0-linear morphism.
The morphisms $(M, \varphi) \to (N; \psi)$ in S_{rig} are morphisms in $Vect_{K_0}$ compatible with respect to

the Frobenius structures, i.e. the K_0-linear maps f such that the diagram

$$
\begin{array}{ccc}
M_0^\sigma & \xrightarrow{\ \varphi\ } & M_0 \\
{\scriptstyle f^\sigma}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
N_0^\sigma & \xrightarrow[\ \psi\ ]{} & N_0
\end{array}
$$

commutes.

Proposition 1.2.6. The category S_{rig} is abelian.

Proof. Let

$$\sigma : Vect_K \to Vect_K$$

be the equivalence of categories that to every vector space M associates M^σ and let σ^{-1} be its inverse. The category S_{rig} is clearly equivalent to the pair category $(Vect_{K_0})_{\sigma^{-1}}$. The result follows by proposition 1.2.4. □

Definition 1.2.7. To uniformize the notation, let S_K be the category $Vect_K$ of K-vector spaces, and let S_{dR} be the category $FVect_K$ of filtered K-vector spaces with compatible morphisms. We consider S_{dR} with its structure of very exact category (proposition 1.1.25).

The category pHC will be a category of complexes of a very exact category $\mathcal{H}$, so for first we must give the definition of $\mathcal{H}$.

$\boxed{\text{H}}$ **Definition 1.2.8.** Let $\mathcal{H}$ be the category defined as follows:

- An object is a system $M = (M_{\mathrm{rig}}, M_{\mathrm{dR}}, M_K, c, s)$ where
  i) (M_{rig}, ϕ) is an object of S_{rig}
  ii) (M_{dR}, F) is an object of S_{dR}
  iii) M_K is an object of S_K
  iv) The map $c : M_{\mathrm{rig}} \to M_K$ (resp. $s : M_{\mathrm{dR}} \to M_K$) is a morphism in S_K. We will call c, s the *comparison morphisms* of M.

  Notice that the morphisms c, s give a diagram in S_K

  $$M_{\mathrm{rig}} \xrightarrow{c} M_K \xleftarrow{s} M_{\mathrm{dR}}. \tag{1.9}$$

- A morphism $f : M \to M'$ in $\mathcal{H}$ is given by a system $f = (f_{\mathrm{rig}}, f_{\mathrm{dR}}, f_K)$ where $f_? : M_? \to M'_?$ is a morphism in $S_{\mathrm{rig}}, S_{\mathrm{dR}}, S_K$, for $? = \mathrm{rig}, \mathrm{dR}, K$ respectively, and such that they are compatible with respect to the comparison morphisms. i.e. such that the diagram

  $$
  \begin{array}{ccccc}
  M_{\mathrm{rig}} & \xrightarrow{\ c\ } & M_K & \xleftarrow{\ s\ } & M_{\mathrm{dR}} \\
  {\scriptstyle f_{\mathrm{rig}}}\big\downarrow & & {\scriptstyle f_K}\big\downarrow & & \big\downarrow{\scriptstyle f_{\mathrm{dR}}} \\
  M'_{\mathrm{rig}} & \xrightarrow[\ c'\ ]{} & M'_K & \xleftarrow[\ s'\ ]{} & M'_{\mathrm{dR}}
  \end{array}
  $$

  commutes.

The category $\mathcal{H}$ inherits the structure of additive category by the categories $S_?$ for $? = \mathrm{rig}, K, \mathrm{dR}$ in the following sense.

Definition 1.2.9. A sequence

$$M' \to M \to M''$$

in $\mathcal{H}$ is exact if $M'_? \to M_? \to M''_?$ is exact for every $? = \mathrm{rig}, K, \mathrm{dR}$.

Remember that in the abelian specializations exact sequences are the *usual* exact sequences, while for the rigid side we deal with a filtered category of an abelian category, hence we refer to definition 1.1.24. We stress the fact that there can be some exact sequence $M'_? \to M_? \to M''_?$ in some specializations $? = \mathrm{rig}, K, \mathrm{dR}$ that does not extend to an exact sequence in $\mathcal{H}$ because for morphisms in $\mathcal{H}$ we have a request of compatibility. Anyway the following result holds.

Proposition 1.2.10. The category $\mathcal{H}$ is very exact.

Proof. It is easy to verify the axioms of definition 1.1.8 for the family of sequences just defined. The very exactness of $\mathcal{H}$ (definition 1.1.18) is induced by the very exactness of the specializations. $\qquad\square$

Notation. We will denote $C_?^b := C^b(S_?)$ for any $? = \mathrm{rig}, K, \mathrm{dR}$.

```
def:pHC
```
 Definition 1.2.11. The category of *p-adic Hodge complexes pHC* is the complex category $C^b(\mathcal{H})$.

```
_def_of_pHC
```
 Remark 1.2.12. It is useful to write an explicit description of the category *pHC*:

- An object of *pHC* is a system $M = (M_\mathrm{rig}^\bullet, M_\mathrm{dR}^\bullet, M_K^\bullet, c, s)$, where

  i) $(M_\mathrm{rig}^\bullet, \phi)$ is an object of C_rig^b.

  ii) $(M_\mathrm{dR}^\bullet, F)$ is an object of C_dR^b.

  iii) $M_K^\bullet$ is an object of C_K^b and $c : M_\mathrm{rig}^\bullet \to M_K^\bullet$ (resp. $s : M_\mathrm{dR}^\bullet \to M_K^\bullet$) is a morphism in C_K^b. We will also call c, s the *comparison morphisms* of M.

  Notice that the morphisms c, s give a diagram in C_K^b

  $$M_\mathrm{rig}^\bullet \xrightarrow{c} M_K^\bullet \xleftarrow{s} M_\mathrm{dR}^\bullet. \tag{1.10} \boxed{\texttt{csdiagr}}$$

- A morphism of *p*-adic Hodge complexes is given by a system $f = (f_\mathrm{rig}, f_\mathrm{dR}, f_K)$ where $f_? : M_?^\bullet \to M_?'^\bullet$ is a morphism in $C_\mathrm{rig}^b(K), C_\mathrm{dR}^b(K), C^b(K)$, for $? = \mathrm{rig}, \mathrm{dR}, K$ respectively, and such that they are compatible with respect to the comparison morphisms. i.e. such that the diagram

$$
\begin{array}{ccccc}
M_\mathrm{rig}^\bullet & \xrightarrow{c} & M_K^\bullet & \xleftarrow{s} & M_\mathrm{dR}^\bullet \\
\downarrow{\scriptstyle f_\mathrm{rig}} & & \downarrow{\scriptstyle f_K} & & \downarrow{\scriptstyle f_\mathrm{dR}} \\
M'^\bullet_\mathrm{rig} & \xrightarrow{c'} & M'^\bullet_K & \xleftarrow{s'} & M'^\bullet_\mathrm{dR}
\end{array}
$$
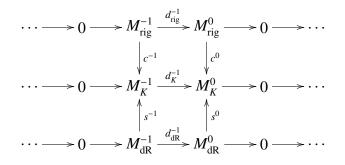
  commutes.

1.2.2 The category pHD of p-adic Hodge diagrams

The category pHC is a category of complexes of objects in the additive category $\mathcal{H}$, hence the definition of homotopy and of the homotopy category $pHK := K(pHC)$ follows. From the exactness of $\mathcal{H}$, we have the definition of acyclic object of pHK and that of quasi isomorphism between complexes of pHK (see definition 1.1.11), and we can localize pHK by quasi isomorphisms (see proposition 1.1.14) and finally give the following definition.

Definition 1.2.13. The category pHD of p-adic Hodge diagrams is the derived category of the very exact category $\mathcal{H}$.

It is easy for us to find the heart of the category pHD: thanks to the point of view of deriving the exact category $\mathcal{H}$, we can simply refer to remark 1.1.16 and say that the heart of pHD is an abelian category $\mathcal{A}$ equivalent to the full subcategory of pHD of diagrams of the form (explicit)

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & M_{\mathrm{rig}}^{-1} & \xrightarrow{d_{\mathrm{rig}}^{-1}} & M_{\mathrm{rig}}^{0} & \longrightarrow & 0 & \longrightarrow & \cdots \\
& & & & \downarrow{\scriptstyle c^{-1}} & & \downarrow{\scriptstyle c^{0}} & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & M_{K}^{-1} & \xrightarrow{d_{K}^{-1}} & M_{K}^{0} & \longrightarrow & 0 & \longrightarrow & \cdots \\
& & & & \uparrow{\scriptstyle s^{-1}} & & \uparrow{\scriptstyle s^{0}} & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & M_{\mathrm{dR}}^{-1} & \xrightarrow{d_{\mathrm{dR}}^{-1}} & M_{\mathrm{dR}}^{0} & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

where d is a monomorphism (and all squares must commute because d^{-1} is a morphism in $\mathcal{H}$).

We can also directly start from the definition in remark 1.2.12 an make an explicit (step by step) construction of pHD.

$\boxed{\texttt{mor(pHC)}}$ **Definition 1.2.14.** Let M, M' be objects of pHC. An *homotopy* of p-adic Hodge complexes from M to M' is a system of homotopies h_i compatible with the comparison maps. The category pHK is defined as the category pHC modulo morphisms homotopic to zero. We say that a morphism $f = (f_{\mathrm{rig}}, f_{\mathrm{dR}}, f_K)$ in pHC (or pHK) is a *quasi-isomorphism* if every $f_?$ is a quasi-isomorphism. Finally we say that an object M in pHC (or pHK) is *acyclic* if $H^j(M_?) = 0$ for any $? = \mathrm{rig}, \mathrm{dR}, K$.

We stress the fact that the definition of homotopy is the *usual* one for any $? = \mathrm{rig}, K, \mathrm{dR}$ (the requirement to be additive is enough to define homotopy), while when we speak about quasi isomorphisms in the framework of the very exact category $FVect_K$ (i.e. for $? = \mathrm{dR}$) we refer to definition 1.1.11.

Lemma 1.2.15. i) The category pHK is a triangulated category.

ii) The localization of pHK with respect to the class of quasi isomorphisms exists. This category, denoted by pHD, is a triangulated category.

Proof. i) For any morphism $f : M \to N$ in pHK we define the (mapping) cone $MC(f)$ to be the object made of complexes $MC(f_?)$, $? = \mathrm{rig}, \mathrm{dR}, K$, and the natural comparison maps. Then a triangle in pHK is a triple isomorphic to $(M, N, MC(f))$ as above. The axioms of a triangulated category ([KS90, Ch.I, §1.5]) are easily verified by the componentwise construction (by *component* we mean the complexes involved in the diagram).

ii) According with [KS90, Ch.I, Prop.1.6.7] it is enough to prove that the subcategory of acyclic objects is a null system. In particular it is sufficient to note that the cone $MC(f)$ of a morphism $f : M \to N$ between acyclic objects is acyclic too. By definition of cone in the category pHK, this can be proven on each of the complexes $M_?^\bullet, N_?^\bullet$ for ? = rig, K, dR. Consider the distinguished triangle

$$M_?^\bullet \to N_?^\bullet \to MC(f). \tag{1.11} \quad \boxed{\text{ex+}}$$

$\boxed{\text{ex+}}$ In the case ? = rig, K the associated long exact sequence

$$\cdots \to H^i(M_?^\bullet) \to H^i(N_?^\bullet) \to H^i(MC(f)) \to H^{i+1}(M_?^\bullet) \to \ldots,$$

and the fact that $H^i(M_?^\bullet) = H^i(N_?^\bullet) = 0$ imply $H^i(MC(f)) = 0$. In the case ? = dR we can conclude by iii) of lemma 1.2.3 of [Lau].

$\square$

1.2.3 Comparison with other constructions

Glueing of categories

We recall some definitions and results developed in [Hub95, §4].

Let $\mathcal{A}, \mathcal{B}, C$ be very exact categories, and let $F_\mathcal{A} : \mathcal{A} \to C$, $F_\mathcal{B} : \mathcal{B} \to C$ be good comparison functors, i.e. left exact functors such that $F_? \simeq H^0 \widetilde{F_?}$ for some exact functor $\widetilde{F_?} : ? \to C^{\geq 0}(C)$ (in particular, a comparison funtor is good if it is exact).

Definition 1.2.16. The glued exact category of $\mathcal{A}, \mathcal{B}$ via C will be the following: an object is a tuple

$$(M_\mathcal{A}, M_\mathcal{B}, M_C, m_\mathcal{A}, m_\mathcal{B})$$

with $M_? \in \mathrm{Ob}(?)$ and $m_? : F_?(M_?) \to M_C$ morphisms in C. A morphism is a triple $(f_\mathcal{A}, f_\mathcal{B}, f_C)$ of morphisms ($f_?$ in ?) which are compatible under the comparison morphisms.

The rigid glued exact category is the full subcategory of objects for which the comparison morphisms are isomorphisms.

$\boxed{\text{acca}}$ *Remark* 1.2.17. Let $F_{\mathrm{dR}} : S_{\mathrm{dR}} \to S_K$ and $F_{\mathrm{rig}} : S_{\mathrm{rig}} \to S_K$ be the forgetful functors. The glued exact category of $S_{\mathrm{rig}}, S_{\mathrm{dR}}$ via S_K coincides with the category $\mathcal{H}$.

Definition 1.2.18. a) Let $C^+(\mathcal{A}, \mathcal{B}, C)$ be the glued complex category of $\mathcal{A}, \mathcal{B}$ via C. An object is a tuple

$$(M_\mathcal{A}, M_\mathcal{B}, M_C, m_\mathcal{A}, m_\mathcal{B})$$

where $M_? \in \mathrm{Ob}C^+(?)$ and $m_? : \widetilde{F_?}(M_?) \to M_C$ morphisms in $C^+(C)$. A morphism is a triple $(f_\mathcal{A}, f_\mathcal{B}, f_C)$ of morphisms ($f_?$ in $C^+(?)$) which are compatible under the comparison morphisms.
b) The full subcategory $C^+_{iso}(\mathcal{A}, \mathcal{B}, C)$ of objects where the comparison morphisms are quasi isomorphisms will be called the rigid glued complex category.
c) The canonical forgetful functors from $C^+(\mathcal{A}, \mathcal{B}, C)$ to the categories $C^+(\mathcal{A}), C^+(\mathcal{B}), C^+(C)$ are called specializations functors (or specializations for short).

Notice that if we consider the very exact categories S_K, S_{rig} and S_{dR} with the good comparison functors $F_{\mathrm{dR}} : S_{\mathrm{dR}} \to S_K$ and $F_{\mathrm{rig}} : S^b_{\mathrm{rig}}(K) \to S_K$ and we construct the glued complex category, we obtain exactly the category of p-adic Hodge complexes, i.e.

$$C^+(S_{\mathrm{rig}}, S_{\mathrm{dR}}, S_K) = pHC.$$

The reason is that the comparison functor are exact so $\widetilde{F}_? = F_?$, hence $C^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ coincides with the complex category of the glued exact category, that is (by remark 1.2.17) with pHC.

The definitions of mapping cone, homotopy category, quasi isomorphism for the glued complex category ([Hub95, 4.1.5, 4.1.7]) coincide with those of 1.2.14 (for the definition of homotopic category this follows once more by the exactness of the comparison functors) . Huber defines the glued derived category $D^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ as we espect [Hub95, 4.1.7], so we finally get

$$D^+(S_{\text{rig}}, S_{\text{dR}}, S_K) = pHD.$$

We could also have directly justified the previous fact by remark [Hub95, p. 31]: the forgetful functors $F_{\text{rig}}, F_{\text{dR}}$ are exact, hence the glued derived category $D^+(S_{\text{dR}}, S_{\text{rig}}, S_K)$, defined as the localization of the glued homotopy category at acyclic objects, coincides with the derived category of the glued exact category.

A subcategory of pHD

In the paper [Ban02] there is a construction of the category of p-adic Hodge complexes fitted to the special case when X has a smooth proper compactification over $\mathcal{V}$ such that the complement is a relative simple normal crossing divisor over $\mathcal{V}$ (this hypothesis forces the comparison morphisms to be quasi isomorphisms). We first recall some definitions.

Definition 1.2.19. [Ban02, 2.1] Let C^b_{rig} be the category defined as follows: an object in C^b_{rig} is a pair $(M^\bullet, \varphi)$ where
i) $M^\bullet_{\text{rig}} = M^\bullet_0 \otimes K$ for some $M^\bullet_0$ in $C^b(Vect_{K_0})$
ii)$(M^\bullet_0)^\sigma \to M^\bullet_0$ is a quasi isomorphism in $C^b(Vect_{K_0})$.
The morphisms in this category are morphisms in $C^b(Vect_{K_0})$ compatible with φ.

We recall the definition of C^b_{MF} considering [Ban02, remark 2.3].

Definition 1.2.20. [Ban02, 2.2] Let C^b_{MF} be the category defined as follows

- An object consists of objects $M^\bullet_{\text{dR}}$ in $C^b(S_{\text{dR}})$, $M^\bullet_{\text{rig}}$ in $C^b(S_{\text{rig}})$, $M^\bullet_K$ in $C^b(S_K)$ and a diagram

$$M^\bullet_{\text{rig}} \to M^\bullet_K \leftarrow M^\bullet_{\text{dR}}$$

  of quasi isomorphisms, called comparison maps, in $C^b(S_K)$.

- A morphism f is a set $(f_?)$ for $? \in \{\text{rig}, K, \text{dR}\}$ of morphisms in the respective categories which are compatible with the comparison maps.

In definition [Ban02, 2.2] Bannai gives the definitions homotopies, quasi isomorphisms and acyclic maps between objects in C^b_{MF} with respect to specializations, and he proves that the homotopic category K^b_{MF} of C^b_{MF} is triangulated. The problem arises in the proof of the existence of the localization D^b_{MF} of K^b_{MF} by the subcategory of acyclic objects [Ban02, 2.6]: to prove that the subcategory of acyclic objects is epaisse he cites the result 1.2.5 of [Lau], but the category C^b_{rig} is not the category of complexes of any exact category (this implies that we can neither see C^b_{MF} as a glued category). Our idea is to subsitute the category C^b_{MF} with the complex category of a very exact one. This will permit also to get the expected comparison between Bannai's and Huber's constructions.

Bannai should have defined C_{MF}^b as a subcategory of the rigid complex category $C_{iso}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$: this requirement satisfies the condition of quasi isomorphisms as comparison maps. For the requirement of the Frobenius to be a quasi isomorphism in the rigid specialization, we can substitute C_{MF}^b with the full subcategory

$$C_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$$

of $C_{iso}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ made by objects whose Frobenius in the rigid specialization is a quasi isomorphism. Now we want to see how to construct D_{MF}^b.

Proposition 1.2.21. The homotopy category $K_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ of $C_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ is a triangulated category.

Proof. The rigid glued homotopy category $K_{iso}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ is triangulated (proposition 4.1.6 of [Hub95]) and the subcategory $K_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ is full, hence it is enough to show that if $A \to B \to C$ is a triangle in $K_{iso}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ with A, B objects of $K_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$, then also C is in $K_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$. This is true by definition of Frobenius on the cone in the rigid specialization as direct sum of the Frobenius of A_{rig} and B_{rig}, that are quasi isomorphisms. □

Lemma 1.2.22. The set of quasi isomorphisms in $K_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ is a multiplicative system.

Proof. By the proof of lemma 4.1.8 of [Hub95] the set of quasi isomorphisms is a multiplicative system of $K_{iso}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$. The category $K_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ is a full subcategory of $K_{iso}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ hence, by definition of multiplicative system, it is enough to notice that for any M in $K_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$, if $f : M \to N$ is a quasi isomorphism in $K_{iso}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$, then N is in $K_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$. If the Frobenius φ of $M_{\text{rig}}^\bullet$ is a quasi isomorphism and M is quasi isomorphic to N, then also the Frobenius ψ of $N_{\text{rig}}^\bullet$ is an isomorphism in cohomology, because for any i the diagram

$$
\begin{array}{ccc}
H^i(M_{\text{rig}}^\bullet) & \xrightarrow{H^i(f)} & H^i(N_{\text{rig}}^\bullet) \\
{\scriptstyle H^i(\varphi)} \big\uparrow & & {\scriptstyle H^i(\psi)} \big\uparrow \\
H^i(M_{\text{rig}}^{\bullet\,\sigma}) & \xrightarrow{H^i(f^\sigma)} & H^i(N_{\text{rig}}^{\bullet\,\sigma})
\end{array}
$$

commutes, and we can conclude. □

Proposition 1.2.23. The category $D_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$, obtained localizing $K_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ by the system of its quasi isomorphisms, is a full subcategory of the rigid glued derived category $D_{iso}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$.

Proof. By the previous lemma we can apply proposition 3.3 of [HartshRD]. □

By the fact that the triangulated structure of $K_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ is induced by that of $K_{iso}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ (see the proof of proposition 1.2.21), the cohomology induced by the t-structure coincides with the naive one (see 1.1.19) also on the strict objects of $D_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$.

Thanks to the previous proposition, we can take the category $D_{iso,B}^+(S_{\text{rig}}, S_{\text{dR}}, S_K)$ as an alternative definition for Bannai's D_{MF}^b.

1.2.4 Tensor product in pHC

First of all we state a general result that will be useful to define the tensor product in the category of p-adic Hodge complexes.

If V is a vector space over a field L and $\tau : L \to L$ is an automorphism, we will denote by V^τ the L-vector space whose additive group is the additive group underlying V and whose multiplication is defined by

$$\alpha \cdot_\tau v := \tau(\alpha)v$$

where the second term is a multiplication in V. If V is a complex of L-vector spaces, we define a complex V^τ by setting

$$d^i_{V^\tau} := d^i_V.$$

Proposition 1.2.24. Let V and W be vector spaces over a field L, let $\tau : L \to L$ be an automorphism. Then there is an isomorphism of L-vector spaces

$$V^\tau \otimes W^\tau \xrightarrow{\sim} (V \otimes W)^\tau.$$

This result still holds in the case we consider $V^\bullet, W^\bullet \in C^b(L)$.

Proof. We begin fixing some notations: the tensor products we consider are

$$V \times W \xrightarrow{\cdot \otimes \cdot} V \otimes W$$

and

$$V^\tau \times W^\tau \xrightarrow{\cdot \otimes_\tau \cdot} V^\tau \otimes W^\tau. \tag{1.12}$$

It is easy to verify that the map

$$\beta : V^\tau \times W^\tau \to (V \otimes W)^\tau, \quad (v, w) \mapsto v \otimes w$$

is L-bilinear hence, by the universal property of the tensor product 1.12, there exists a unique L-linear map

$$f : V^\tau \otimes W^\tau \to (V \otimes W)^\tau$$

such that the diagram

$$
\begin{array}{ccc}
V^\tau \times W^\tau & \xrightarrow{\cdot \otimes_\tau \cdot} & V^\tau \otimes W^\tau \\
& \searrow{\scriptstyle \beta} & \downarrow{\scriptstyle f} \\
& & (V \otimes W)^\tau
\end{array}
$$

commutes. If $v_1, \dots, v_m$ is a base of V (hence[5] of V^τ) and $w_1, \dots, w_n$ is a base of W (hence of W^τ), then $\{v_i \otimes w_j\}_{i,j}$ is a base of $V^\tau \otimes W^\tau$, and it is sent by f in a base of $(V \otimes W)^\tau$ because of the commutativity of the diagram and the definition of β. Hence f is an isomorphism.

Let now $V^\bullet$ and $W^\bullet$ be complexes of L-vector spaces. By definition, the differentials of $V^{\bullet\tau}$ are equal to the differentials of $V^\bullet$, hence the diagram

$$
\begin{array}{ccc}
(V^i)^\tau \otimes (W^i)^\tau & \xrightarrow{f^i} & ((V \otimes W)^i)^\tau \\
\downarrow{\scriptstyle d^i_{V^\tau \otimes W^\tau}} & & \downarrow{\scriptstyle d^i_{(V \otimes W)^\tau}} \\
(V^{i+1})^\tau \otimes (W^{i+1})^\tau & \xrightarrow{f^{i+1}} & ((V \otimes W)^{i+1})^\tau
\end{array}
$$

---

[5]because τ is an isomorphism

commutes by definition of differentials on the tensor product. Hence f is an isomorphism also in the category of complexes. $\square$

Definition 1.2.25. Given two p-adic Hodge complexes M and M' we define their tensor product $M \otimes M'$ component-wise, i.e. $(M_{rig}^\bullet \otimes M_{rig}'^\bullet, M_{dR}^\bullet \otimes M_{dR}'^\bullet, M_K^\bullet \otimes M_K'^\bullet, c \otimes c', s \otimes s')$. The tensor product is an object of pHC endowed with the following structure

- By using the isomorphism of proposition 1.2.24, we can consider the map

$$(M_0^\bullet \otimes M_0'^\bullet)^\sigma \simeq M_0^{\bullet\sigma} \otimes M_0'^{\bullet\sigma} \xrightarrow{\phi \otimes \phi'} M_0^\bullet \otimes M_0'^\bullet$$

  as Frobenius structure on the tensor product.

- The filtration $F_\otimes$ on $M_{dR}^\bullet \otimes M_{dR}'^\bullet$ is defined by

$$F_\otimes^n(M_{dR}^\bullet \otimes M_{dR}'^\bullet) = \sum_{i+j=n} F^i(M_{dR}^\bullet) \otimes F'^j(M_{dR}'^\bullet).$$

The natural definition of tensor product 1.2.25 on pHC extends to the category pHD, and we denote by $\mathbb{K}$ the unit object:

$$\mathbb{K}_{rig}^\bullet \xrightarrow{id} \mathbb{K}_K^\bullet \xleftarrow{id} \mathbb{K}_{dR}^\bullet$$

where

- $\mathbb{K}_?^\bullet$ is equal to K concentrated in degree zero for every $? = rig, K, dR$

- $\mathbb{K}_{rig}^\bullet$ is the object $(\mathbb{K}_{rig}^\bullet, \sigma)$ of $C_{rig}^b(K)$

- the filtration F on $\mathbb{K}_{dR}^\bullet$ is defined by $F^i(\mathbb{K}_{dR}^\bullet) = \mathbb{K}_{dR}^\bullet$ for every $i \leq 0$, $F^i(\mathbb{K}_{dR}^\bullet) = 0$ otherwise.

1.2.5 Shortening a diagram

The following procedure is a generalization of [Lev98, Ch. V, 2.3.3].

Proposition 1.2.26. Let $A^\bullet \xleftarrow{\alpha} C^\bullet \xrightarrow{\beta} B^\bullet$ be a diagram of complexes in $C^b(K)$. Assume that one between α, β is a quasi-isomorphism, then the *quasi push-out* complex $P^\bullet = MC((\alpha, \beta) : C^\bullet \to A^\bullet \oplus B^\bullet))$ is quasi-isomorphic to $B^\bullet$:

(in the picture we have assumed the quasi-isomorphism to be α).

An analogous result holds for the *quasi pull-back* complex $MC(\alpha - \beta : A^\bullet \oplus B^\bullet \to C^\bullet)[-1]$ of a diagram $A^\bullet \xrightarrow{\alpha} C^\bullet \xleftarrow{\beta} B^\bullet$.

Proof.   • *quasi push-out*

Let $p : A^\bullet \times B^\bullet \to A^\bullet$ be the canonical projection. By the octahedral axiom (see for example remark 1.4.5 of [KS90]) there exist two dotted arrows

and the upper part of this diagram is a distinguished triangle. It is sufficient now to notice that $MC(p)[-1] \simeq B^\bullet$ by definition and that $MC(\alpha)[-1] \simeq 0$ by hypothesis.

• *quasi pull-back*

Let $i : A^\bullet \to A^\bullet \oplus B^\bullet$ be the canonical injection. The diagram is the following

In this case the result follows by the fact that $MC(i)[-1] \simeq B^\bullet$ by definition.

$\square$

| pHC' | **Definition 1.2.27.** Let pHC' be a category of systems $(M_{\mathrm{rig}}^\bullet, M_{\mathrm{dR}}^\bullet, M_1^\bullet, M_2^\bullet, M_3^\bullet, c, s, f, g)$ where

  i) $(M_{\mathrm{rig}}^\bullet, \phi)$ is an object of $C_{\mathrm{rig}}^b(K)$ and $H^*(M_{\mathrm{rig}}^\bullet)$ is finitely generated over K.

  ii) $(M_{\mathrm{dR}}^\bullet, F)$ is an object of $C_{\mathrm{dR}}^b(K)$ and $H^*(M_{\mathrm{dR}}^\bullet)$ is finitely generated over K.

  iii) $M_i^\bullet$ are objects of $C^b(K)$ for any $i = 1, 2, 3$, $c : M_{\mathrm{rig}}^\bullet \to M_K^\bullet$ $s : M_{\mathrm{dR}}^\bullet \to M_K^\bullet$, $f :$, $g :$ are morphisms in $C^b(K)$.

Notice that an object in pHC' is related to a diagram in $C^b(K)$ of the type

$$M_{\mathrm{rig}}^\bullet \xrightarrow{c} M_1^\bullet \xleftarrow{f} M_2^\bullet \xrightarrow{g} M_3^\bullet \xleftarrow{s} M_{\mathrm{dR}}^\bullet. \qquad (1.13) \quad \boxed{\texttt{diagramforG}}$$

Assume that c, s and one between f and g are quasi-isomorphisms. Via quasi push-out, we can define a functor

$$F : pHC' \to pHC$$

$$(M_{\mathrm{rig}}^\bullet \xrightarrow{c} M_1^\bullet \xleftarrow{f} M_2^\bullet \xrightarrow{g} M_3^\bullet \xleftarrow{s} M_{\mathrm{dR}}^\bullet) \mapsto (M_{\mathrm{rig}}^\bullet \to P^\bullet \leftarrow M_{\mathrm{dR}}^\bullet)$$

compatible with tensor product.

Remark 1. By proposition 1.2.26, if we consider a diagram of the form

$$(M_{\mathrm{rig}}^\bullet \xrightarrow{\sim} M_1^\bullet \leftarrow M_2^\bullet \xrightarrow{\sim} M_3^\bullet \xleftarrow{\sim} M_{\mathrm{dR}}^\bullet)$$

in pHC', then we have

$$F(M') = (M_{\mathrm{rig}}^\bullet \xrightarrow{\sim} P^\bullet \leftarrow M_{\mathrm{dR}}^\bullet).$$

For the same reason we have

$$F((M_{\mathrm{rig}}^\bullet \xrightarrow{\sim} M_1^\bullet \xleftarrow{\sim} M_2^\bullet \to M_3^\bullet \xleftarrow{\sim} M_{\mathrm{dR}}^\bullet)) = (M_{\mathrm{rig}}^\bullet \to P^\bullet \xleftarrow{\sim} M_{\mathrm{dR}}^\bullet).$$
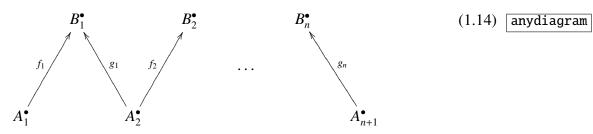
We can clearly begin with a diagram of the form

of objects and maps are in the category of complexes of an additive category, and then apply the procedure of the quasi push-out to make the diagram smaller. We have just noticed how quasi-isomorphisms "jump" by this procedure.

1.2.6 Diagrams and cohomology

We recall a construction due to Bannai ([Ban02], proposition 2.16). For any pair M, M' of objects of pHC, let $\mathcal{D}_{M,M'}$ be the following diagram (which is not in our categories because $\mathrm{Hom}_{K_0}^\bullet(M_0^\sigma, M_0')$ has not a Frobenius structure)

$$
\begin{array}{ccccc}
\mathrm{Hom}_{K_0}^\bullet(M_0^\sigma, M_0') & & \mathrm{Hom}_K^\bullet(M_{\mathrm{rig}}, M_K') & & \mathrm{Hom}_K^\bullet(M_{\mathrm{dR}}, M_K') \\
& \nwarrow h_0 \quad h_1 \nearrow & & \nwarrow h_2 \quad h_3 \nearrow & & \nwarrow h_4 \\
& \mathrm{Hom}_{K_0}^\bullet(M_0, M_0') & & \mathrm{Hom}_K^\bullet(M_K, M_K') & & \mathrm{Hom}_K^{\bullet, F}(M_{\mathrm{dR}}, M_{\mathrm{dR}}')
\end{array}
\tag{1.15}
$$

where $h_0(x_0) = x_0 \circ \phi - \phi' \circ x_0^\sigma$, $h_1(x_0) = c' \circ (x_0 \otimes \mathrm{id}_K)$, $h_2(x_K) = x_K \circ c$, $h_3(x_K) = x_K \circ s$, $h_4(x_{\mathrm{dR}}) = s' \circ x_{\mathrm{dR}}$. Then define the complexes of abelian groups $\Gamma_0(M, M') :=$direct sum of the bottom row, $\Gamma_1(M, M') :=$direct sum of the top row. Consider the morphism

$$\psi_{M,M'} : \Gamma_0(M, M') \to \Gamma_1(M, M') \quad (x_0, x_K, x_{\mathrm{dR}}) \mapsto (-h_0(x_0), h_1(x_0) - h_2(x_K), h_3(x_K) - h_4(x_{\mathrm{dR}}))$$

Definition 1.2.28. For any M, M' objects of pHC, we define

$$\Gamma(M, M') := MC(\psi_{M,M'})[-1]. \tag{1.16}$$

We will sometimes also use the following more general notations (see 1.2 of [Beĭ86]).

Definition 1.2.29. Let $\mathcal{D}$ be a diagram of the form 1.14. Put $\widetilde{\Gamma}^0(\mathcal{D}) = \bigoplus_i A_i^\bullet$, $\widetilde{\Gamma}^1(\mathcal{D}) = \bigoplus_i B_i^\bullet$. One has two morphisms $\varphi_{1,2} : \widetilde{\Gamma}^0(\mathcal{D}) \to \widetilde{\Gamma}^1(\mathcal{D})$, $\varphi_1 = \Sigma_i f_i$, $\varphi_2 = \Sigma_i g_i$. Put

$$\widetilde{\Gamma}(\mathcal{D}) = MC(\varphi_1 - \varphi_2 : \widetilde{\Gamma}^0(\mathcal{D}) \to \widetilde{\Gamma}^1(\mathcal{D}))[-1].$$

Hence we will also use the notation $\widetilde{\Gamma}(\mathcal{D}_{M,M'})$ to indicate the complex 1.16.

Definition 1.2.30. Let $\mathbb{K}(-n)$ be the Tate twist p-adic Hodge complex: $\mathbb{K}(-n)^\bullet_{\mathrm{rig}}$ (resp. $\mathbb{K}(-n)^\bullet_{\mathrm{dR}}$, $\mathbb{K}(-n)^\bullet$) is equal to K concentrated in degree zero; the Frobenius is $\phi(\lambda) := p^n \sigma(\lambda)$; the filtration is $F^i = K$ for $i \leq n$ and zero otherwise. For any complex p-adic Hodge complex M and for any integer n we denote by $M(n)$ the p-adic Hodge complex $M \otimes \mathbb{K}(n)$.

Remark 1.2.31. The cone $\Gamma(\mathbb{K}(-n), M)$ is quasi isomorphic to

$$MC(M_0 \oplus F^n M_{\mathrm{dR}} \xrightarrow{\eta} M_0 \oplus M_K)[-1] \quad \eta(x_0, x_{\mathrm{dR}}) = (\phi(x_0) - p^n x_0, (x_0 \otimes \mathrm{id}_K) - s(x_{\mathrm{dR}}))$$

where $x_0 \in M_0$, $x_{\mathrm{dR}} \in F^n M_{\mathrm{dR}}$.

Proposition 1.2.32 (Ext-formula). With the above notation there is a canonical morphism of abelian groups

$$\mathrm{Hom}_{pHD}(M, M'[n]) \cong H^n(\Gamma(M, M')) .$$

Proof. By the octahedron axiom we have the following triangle in $D^b(Ab)$

$$\mathrm{Ker}\,\psi_{M,M'} \to \Gamma(M, M') \to \mathrm{Coker}\,\psi_{M,M'}[-1] \xrightarrow{+} .$$

Its cohomological long exact sequence is

$$\xrightarrow{\partial} H^n(\mathrm{Ker}\,\psi_{M,M'}) \to H^n(\Gamma(M, M')) \to H^n(\mathrm{Coker}\,\psi_{M,M'}[-1]) \xrightarrow{\partial}$$

Note that by construction $H^n(\mathrm{Ker}\,\psi_{M,M'}) = \mathrm{Hom}_{pHK}(M, M'[n])$. Let I be the family of quasi-isomorphisms $g : M' \to M''$, then $\mathrm{Hom}_{pHD}(M, M'[n]) = \varinjlim_I \mathrm{Hom}_{pHK}(M, M''[n])$. Thus the result is proven if we show that:

i) $H^n(\Gamma(M, M')) \cong H^n(\Gamma(M, M''))$ for any $g : M' \to M''$ quasi-isomorphism.

ii) $\varinjlim_I H^n(\mathrm{Coker}\,\psi_{M,M''}[-1]) = 0$.

The first claim follows from the exactness of $\Gamma(M, -)$ and the second is proven in [Beĭ86, 1.7,1.8] (and with more details in [Hub95, Lemma 4.2.8] or [Ban02, Lemma 2.15]) with the assumption that all the gluing maps are quasi-isomorphisms, but this hypothesis is not necessary. $\qquad\square$

Lemma 1.2.33 (Tensor Product). Let M, N, I be p-adic Hodge complexes. For any $\alpha \in K$ there is a morphism of complexes

$$\cup_\alpha : \Gamma(I, M) \otimes \Gamma(I, N) \to \Gamma(I, M \otimes N).$$

All such $\cup_\alpha$ are equivalent up to homotopy.

Proof. It is a particular case of the more general result [Beĭ86, 1.11] for diagrams. $\qquad\square$

Chapter 2

Godement resolutions

god

Here we recall some facts about *generalized* Godement resolutions (see the construction of the *simplicial canonical resolution* in 6.4 of [God58]). We need to verify that, under certain conditions, the generalized Godement resolution of a complex of sheaves $\mathcal{F}^\bullet$ over some topological spaces, or more generally of objects in some topoi, indeed provides a flasque resolution of $\mathcal{F}^\bullet$, that is also in some sense *functorial*, in order to link the cohomologies of different complexes of sheaves arising in the syntomic diagrams. Furthermore, we will also prove the compatibility of Godement resolution with tensor product, needed to define the syntomic pairing.

We can also consider the generalized Godement resolution in the more general framework of *monads*; we will introduce this concept in the last section, the reference for this part is [Ivo05].

2.1 Topoi

In order to construct a Godement resolution for sheaves defined over a rigid analytic space, we need to deal with the concept of topos. We recall here some definitions about the category of topoi (see [SGAIV] and [Lev98]).

Definition 2.1.1. Let C be a category. A *Grothendieck pre-topology* on C consists of giving, for each object X of C, a collection of families of morphisms

$$\text{Cov}(X) := \{\{f_\alpha : U_\alpha \to X \,|\, \alpha \in A\}\}$$

satisfying the following axioms:

- If $\{f_\alpha : U_\alpha \to X \,|\, \alpha \in A\}$ is in $\text{Cov}(X)$, and if $Y \to X$ is in C, then the fiber product $U_\alpha \times_X Y$ exists for each $\alpha \in A$, and the family $\{p_2 : U_\alpha \times_X Y \to Y \,|\, \alpha \in A\}$ is in $\text{Cov}(Y)$

- If $\{f_\alpha : U_\alpha \to X \,|\, \alpha \in A\}$ is in $\text{Cov}(X)$, and if $\{g_\beta : V_{\alpha\beta} \to U_\alpha \,|\, \beta \in B_\alpha\}$ is in $\text{Cov}(U_\alpha)$ for each $\alpha \in A$ then $\{f_\alpha \circ g_\beta : V_{\alpha\beta} \to X \,|\, \alpha \in A, \beta \in B_\alpha\}$ is in $\text{Cov}(X)$

- The identity map $id_X : X \to X$ is in $\text{Cov}(X)$.

By abuse of notation, for us a *Grothendieck site* is a category with a Grothendieck pre-topology.

In the framework of sites we can define what is a presheaf, and the condition for which it is a sheaf.

Definition 2.1.2. Let C be a Grothendieck site. A presheaf on C, with values in a category $\mathcal{A}$, is a contravariant functor $\mathcal{F} : C \to \mathcal{A}$; with morphisms of presheaves being natural transformations, this form the category of presheaves on C with values in $\mathcal{A}$.

A presheaf $\mathcal{F}$ with values in **Sets** is a sheaf if for each object U of C and each covering family $\{f_\alpha : U_\alpha \to U | \alpha \in A\}$ in $\mathrm{Cov}(U)$, the sequence of sets

$$\emptyset \to \mathcal{F}(U) \xrightarrow{\Pi_\alpha \mathcal{F}(f_\alpha)} \prod_\alpha \mathcal{F}(U_\alpha) \underset{\mathcal{F}(p_2)}{\overset{\mathcal{F}(p_1)}{\rightrightarrows}} \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \times_U U_\beta)$$

is exact (where $p_1 : U_\alpha \times_U U_\beta \to U_\alpha$ and $p_2 : U_\alpha \times_U U_\beta \to U_\beta$ are the canonical projections).

Definition 2.1.3 (The category of topoi). Let C be a site. We will call $\widetilde{C}$ the category of sheaves of sets over C. A topos T is a category which is equivalent to the category $\widetilde{C}$ for some Grothendieck site C. A morphism of topoi $u : T_1 \to T_2$ is a triple consisting of functors $u_* : T_1 \to T_2$, $u^* : T_2 \to T_1$, and a natural isomorphism (adjunction map)

$$\mathrm{Hom}_{T_1}(u^*(-), -) \to \mathrm{Hom}_{T_2}(-, u_*(-)),$$

with the additional requirement that the functor u^* is left exact, i.e. preserves finite projective limits.

With its unique topology, the one-point category forms a site, and the category of sheaves on this site is the same as the category of presheaves, which in turns is equivalent to the category of sets; thus the category **Sets** is a topos.

$\boxed{\text{point}}$ **Definition 2.1.4.** Let T be a topos. A point on T is a morphism of topoi

$$p : \mathbf{Sets} \to T.$$

If $\mathcal{F}$ is an object of T, we will write $\mathcal{F}_p$ for $p^*(\mathcal{F})$.

2.2 Cosimplicial objects

$\boxed{\text{osimplicial}}$ Let Δ be the category whose objects are the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for integers $n \geq 0$, and whose morphisms are the nondecreasing monotone functions. If C is any category, a cosimplicial object of C is a covariant functor

$$B^* : \Delta \to C.$$

We write B^n for $B^*([n])$, and we denote the codegeneracy maps by $\sigma_i^n : B^{n+1} \to B^n$ and the cofaces by $\delta_i^{n-1} : B^n \to B^{n+1}$ (see corollary 8.1.4 of [Wei94]) . As in 8.2.1 of [Wei94], if the category C is abelian we can associate to $B^* : \Delta \to C$ a complex $C = s(B^*)$ where $C^n = B^n$ and its boundary morphism $d^n : C^n \to C^{n+1}$ is the alternating sum of the coface operators:

$$d^n = \delta_0^{n-1} - \delta_1^{n-1} + \cdots + (-1)^n \delta_n^{n-1}.$$

2.3 Generalized Godement resolutions

Consider two Grothendieck sites $\mathcal{P}, \mathcal{X}$ and a morphism $u : \mathcal{P} \to \mathcal{X}$. Let $\widetilde{\mathcal{P}}$ (resp. $\widetilde{\mathcal{X}}$) be the category of abelian sheaves on $\mathcal{P}$ (resp. on $\mathcal{X}$). Then we have a pair of adjoint functors (u_*, u^*) where $u_* : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{P}}$ and $u^* : \widetilde{\mathcal{P}} \to \widetilde{\mathcal{X}}$. For any object $\mathcal{F}$ of $\widetilde{\mathcal{X}}$ we can define a cosimplicial object $B^*(\mathcal{F}) : \Delta \to \widetilde{\mathcal{X}}$ in the following way. First let $\eta : \mathrm{id}_{\mathcal{X}} \to u_* u^*$ and $\epsilon : u^* u_* \to \mathrm{id}_{\mathcal{P}}$ respectively the unit and the counit of the adjunction. Define $B^n(\mathcal{F}) := (u_* u^*)^n(\mathcal{F})$ with co-degeneracy maps

$$\sigma_i^n := (u_* u^*)^i u_* \epsilon u^* (u_* u^*)^{n-1-i} : B^{n+1}(\mathcal{F}) \to B^n(\mathcal{F}) \quad i = 0, ..., n-1$$

and co-faces

$$\delta_i^{n-1} := (u_* u^*)^i \eta (u_* u^*)^{n-i} : B^n(\mathcal{F}) \to B^{n+1}(\mathcal{F}) \quad i = 0, ..., n \ .$$

Lemma 2.3.1. With the above notation let $sB^*(F)$ be the associated complex of objects of $\mathcal{F}$. Then there is a canonical map $b_{\mathcal{F}} : \mathcal{F} \to sB^*(\mathcal{F})$ such that $u^*(b_F)$ is a quasi-isomorphism. Moreover, if u^* is exact and conservative, then $\eta_{\mathcal{F}}$ is a quasi-isomorphism.

Proof. It is a particular case of lemma 2.4.2 of section 2.4. $\square$

Definition 2.3.2. In the previous notations, let $\mathcal{F}$ be a sheaf [1] in $\widetilde{\mathcal{X}}$. The Godement complex of $\mathcal{F}$ with respect to $\widetilde{\mathcal{P}}$ (the morphism u being implied) is the complex of sheaves

$$G_{\widetilde{\mathcal{P}}} \mathcal{F} := sB^*(\mathcal{F}).$$

Remark 2.3.3. By lemma 2.3.1, in the case u^* is exact and conservative, the complex $G_{\widetilde{\mathcal{P}}} \mathcal{F}$ is a resolution of $\mathcal{F}$.

The following proposition will be crucial to define the maps involved the syntomic diagrams we will construct in the next two chapters: in fact it justifies the needed *functoriality* of the generalized Godement resolution.

[1] **Proposition 1.** *If*

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \xrightarrow{f} & \widetilde{\mathcal{Y}} \\ {\scriptstyle u}\big\uparrow & & \big\uparrow{\scriptstyle v} \\ \widetilde{\mathcal{P}} & \xrightarrow{g} & \widetilde{\mathcal{Q}} \end{array}$$

is a commutative diagram of topoi, $\mathcal{F}$ a sheaf in $\widetilde{\mathcal{X}}$, $\mathcal{G}$ a sheaf in $\widetilde{\mathcal{Y}}$ with a map $a : \mathcal{G} \to f_* \mathcal{F}$, then there exists a morphism

$$G_{\widetilde{\mathcal{Q}}} \mathcal{G} \to f_* G_{\widetilde{\mathcal{P}}} \mathcal{F}$$

such that the diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{a} & f_* \mathcal{F} \\ {\scriptstyle \eta_{\mathcal{G}}}\big\downarrow & & \big\downarrow{\scriptstyle f_*(\eta_{\mathcal{F}})} \\ G_{\widetilde{\mathcal{Q}}} \mathcal{G} & \longrightarrow & f_* G_{\widetilde{\mathcal{P}}} \mathcal{F} \end{array}$$

commutes.

---

[1] or a complex of sheaves

Proof. We will just show that there is a canonical map $\beta : v_* v^* \mathcal{G} \to f_* u_* u^* \mathcal{F}$ lifting a. First consider the composition $\mathcal{G} \to f_* \mathcal{F} \to f_* u_* u^* \mathcal{F}$. Then we get a map $\alpha : \mathcal{G} \to v_* g_* u^* \mathcal{F}$ because $v_* g_* = f_* u_*$. By adjunction this gives $v^* \mathcal{G} \to g_* u^* \mathcal{F}$. Then we apply v_* and use the equality $v_* g_* = f_* u_*$ to obtain the desired map. Now that we have constructed $\beta := v_*(\mathrm{adj}(\alpha))$, we must show that the diagram

$$
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\;a\;} & f_* \mathcal{F} \\
\downarrow{\scriptstyle \eta_{\mathcal{G}}} & & \downarrow{\scriptstyle f_*(\eta_{\mathcal{F}})} \\
v_* v^* \mathcal{G} & \xrightarrow{\;\beta\;} & f_* u_* u^* \mathcal{F}
\end{array}
$$

commutes, i.e. that $\alpha = (v_*(\mathrm{adj}(\alpha))) \circ \eta_{\mathcal{G}}$. By the construction of the adjunction isomorphism for the couple of functors v^*, v_* by mean of the unit and counit maps, the second term of the last equality is $\mathrm{adj}(\mathrm{adj}(\alpha)) = \alpha$.

$\square$

Now that we have solved the problem of G to be a functorial resolution, we see that, in the case the topos $\widetilde{\mathcal{P}}$ is related to some particular set of points of X, then the Godement complex is even made of flasque sheaves.

Definition 2.3.4 (see 1.3.5 of [Lev98])**.** Let $\widetilde{X}$ be a topos. We say it has *enough points* if there is a set P of points of X such that a map $f : \mathcal{F} \to \mathcal{G}$ is an isomorphism (resp. monomorphism, resp. epimorphism) if and only if the maps $f_p : \mathcal{F}_p \to \mathcal{G}_p$ are isomorphisms (resp. monomorphisms, resp. epimorphisms) for all $p \in P$ (see definition 2.1.4). A set P of points of X which satisfies the above condition is called a conservative family of points of $\widetilde{X}$. If C is a Grothendieck site, we call a conservative family of points of $\widetilde{C}$ a conservative family of points of C.

Example 2.3.5. Let X be a topological space. Then there is a canonical way to consider X as a Grothendieck site X_{top}, and the set of points of X (i.e. the elements of X) forms a conservative family of points of $\widetilde{X}_{\mathrm{top}}$ (see example 1.3.4 and remark 1.3.6 in chapter IV of [Lev98]).

By the previous example, if X be a complex algebraic scheme, the classical site of the analytic space $X(\mathbb{C})$ has enough points. For the same reason, if X is a scheme over a field L, the Zariski site of X, denoted by X^{zar}, has enough points. We are going to prove now that in the case of L-varieties, i.e. for reduced separated L-schemes of finite type, it is enough to restrict to closed points to separate sheaves. More precisely, we will see in proposition 2.3.11 that the reduction assumption is not necessary. We will call $P(X^{\mathrm{zar}})$ this conservative family of points of the Zariski site X^{zar} of the scheme X.

In order to prove a more general result about the set of closed points as a conservative family for the Zariski site of a scheme, we first need to study the affine case.

Definition 2.3.6. A ring R is a Jacobson ring if every prime ideal of R is the intersection of maximal ideals.

It is obvious that any field is a Jacobson ring. The following general form of the Nullstellensatz shows in particular that any algebra over a field is a Jacobson ring.

Theorem 2.3.7. (cf. [Eis95], 4.19] Let R be a Jacobson ring. If S is a finitely generated R-algebra, then S is a Jacobson ring. Further, if $\mathfrak{n} \subset S$ is a maximal ideal, then $\mathfrak{m} := \varphi^{-1}(\mathfrak{n})$ is a maximal ideal of R, where $\varphi : R \to S$ is the canonical morphism.

Proof. See theorem 4.19 of [Eis95].

□

Now we can state the conditions for an affine scheme to have the set of closed points as a conservative family.

`affinecase` *Lemma* 2.3.8 (The affine case). Let A be a Jacobson ring and let $X = \operatorname{Spec} A$. Let $\mathcal{F}$ be an abelian sheaf on X such that $\mathcal{F}_{\mathfrak{m}} = 0$ for every $\mathfrak{m} \in \operatorname{Max} A$. Then $\mathcal{F} = 0$.

In other words, for any Jacobson ring A, the set $\operatorname{Max} A$ is a conservative family of points of $X = \operatorname{Spec} A$.

Proof. Recall that for any abelian sheaf $\mathcal{F}$ on a topological space X and for any point $x \in X$ the stalk $\mathcal{F}_x$ is zero if for any open $U \ni x$ and for any section s of $\mathcal{F}$ on U there exists an open neighborhood $V \subset U$ of x such that $s_{|V} = 0$.

Now let $\mathfrak{p} \in \operatorname{Spec} A$ and let U be an open neighborhood of $\mathfrak{p}$ in $\operatorname{Spec}(A)$; so U has the form

$$U = D(\mathfrak{a}) := \{\mathfrak{q} \in \operatorname{Spec} A \mid \mathfrak{q} \not\supset \mathfrak{a}\}$$

for some ideal $\mathfrak{a} \leq A$ such that $\mathfrak{p} \not\supset \mathfrak{a}$. Moreover

$$\mathfrak{p} = \bigcap_{\operatorname{Max} A \ni \mathfrak{m} \supset \mathfrak{p}} \mathfrak{m}$$

because A is a Jacobson ring. Hence there exists a maximal ideal $\overline{\mathfrak{m}} \supset \mathfrak{p}$ such that $\overline{\mathfrak{m}} \not\supset \mathfrak{a}$, i.e. such that $U \ni \overline{\mathfrak{m}}$. This implies by hypothesis the existence, for any section $s \in \mathcal{F}(U)$, of an open neighborhood $V \subset U$ of $\overline{\mathfrak{m}}$ such that $s_{|V} = 0$. The open V is also a neighborhood of $\mathfrak{p}$: we can suppose $V = D(\mathfrak{b})$ for an ideal $\mathfrak{b} \leq A$ such that $\overline{\mathfrak{m}} \not\supset \mathfrak{b}$, but $\overline{\mathfrak{m}} \supset \mathfrak{p}$ hence $\mathfrak{p} \not\supset \mathfrak{b}$ as claimed.

□

Before proving an analogous of lemma 2.3.8 for a scheme that is not necessarily affine, we need to state a general lemma about topological spaces.

`toglue` *Lemma* 2.3.9. Let X be a topological space and let $\{U_i\}$ be an open cover of X. If F_i is a family of conservative points of U_i for any i, then $\bigcup_{i \in I} F_i$ is a conservative family for X.

Proof. Let $\mathcal{F}$ be a sheaf on X such that $\mathcal{F}_x$ is zero for every $x \in \bigcup_{i \in I} F_i$. Then $\mathcal{F}_{|U_i} = 0$ for every $i \in I$, whence the result.

□

`atedalgebra` *Lemma* 2.3.10. Let R be a ring, let X be a separated R-scheme, let $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ be open affine subsets of X. Then $U \cap V = \operatorname{Spec} C$ where $C \simeq (A \otimes B)/I$ for some ideal $I \leq A \otimes_R B$.

Proof. By definition of separated scheme, the diagonal morphism $\Delta : X \to X \times_R X$ is a closed immersion. The open subset $U \cap V$ is affine (because X is separated, see [Shaf94], chapter V, proposition 3) and it is clearly isomorphic to the image of the immersion $\Delta_{|U \cap V} : U \cap V \to U \times_R V$, i.e. to a closed subset of $U \times V = \operatorname{Spec} A \otimes_R B$, whence the result.

□

Proposition 2.3.11. Let L be a field. For every separated L-scheme X locally of finite type, the set
$$X^{cl} := \{\text{closed points of } X\}$$
is a conservative family for the Zariski site of X.

Proof. Consider an affine cover $\{U_i\} = \mathrm{Spec} A_i$ of X, where A_i is a finitely generated L-algebra for every i. By theorem 2.3.7 and lemma 2.3.8, the set $U_i^{cl} = \mathrm{Max} A_i$ is a conservative family of U_i for every i. Hence, by lemma 2.3.9, it is sufficient to prove that $\bigcup_{i \in I} \mathrm{Max} A_i = X^{cl}$. By definition of induced topology on a subset, we only need to show $\bigcup_{i \in I} \mathrm{Max} A_i \subset X^{cl}$.

Let x be a maximal ideal of A_j for some $j \in I$. For $i = j$ and for any i such that $U_i \not\ni x$, the set $U_i \setminus \{x\}$ is clearly open in X. Consider $i \in I$ such that $x \in U_i$. The set $\{x\}$ is closed in U_j, so $\{x\}$ is closed also in $U_i \cap U_j$. Let $\alpha : U_i \cap U_j \hookrightarrow U_i$ be the canonical inclusion. The scheme X is separated, so the intersection of affine subsets is affine, and we call A_{ij} the ring such that $U_i \cap U_j = \mathrm{Spec} A_{ij}$. So we can consider the morphism

$$\varphi : A_i \to A_{ij}$$

associated to α. The ring A_{ij} is a finitely generated L-algebra by lemma 2.3.10, so A_{ij} is finitely generated also as A_i-algebra. Hence, by theorem 2.3.7, the ideal $\varphi^{-1}(x)$ is also maximal in A_i, i.e. $\{x\}$ is closed in U_i too.

Hence $X \setminus \{x\} = \bigcup_{i \in I}(U_i \setminus \{x\})$ is open in X as claimed.

$\square$

Now that we have found the conditions for closed points to be a conservative family in the particular case of the Zariski site, we want to show that the existence of such a family for a generic site allows the Godement complex to be made of flasque sheaves.

Remark 2.3.12 (see remark 1.3.6 of [Lev98]). Let P be a set of points of a topos $\widetilde{X}$. The collection of morphisms $p : \mathbf{Sets} \to \widetilde{X}$ for $p \in P$ defines the morphism of topoi

$$i : \coprod_{p \in P} \mathbf{Sets} \to \widetilde{X}. \tag{2.1}$$

In particular, for every set of points P of $\widetilde{X}$ we will call $\widetilde{\mathcal{P}}$ the topos $\coprod_{p \in P} \mathbf{Sets}$. Also, for any $\mathcal{F} \in \widetilde{X}$ we will often simply write $G_P \mathcal{F}$ for the Godement complex of $\mathcal{F}$ with respect to the morphism 2.1.

Proposition 2.3.13. Let $\widetilde{X}$ be a topos with enough points and let P be a family of conservative points of $\widetilde{X}$. For any $\mathcal{F} \in \widetilde{X}$, the Godement complex $G_P \mathcal{F}$ defined by the morphism of topoi i of 2.1 is a flasque resolution of $\mathcal{F}$.

Proof. The functor i^* is exact and conservative by definition. Hence, by remark 2.3.3, we must only show that $G_{\widetilde{\mathcal{P}}} \mathcal{F}$ is flasque. The first term of the Godement resolution is the product of the direct image of sheaves over a point:

$$\prod_{p \in \mathcal{P}} p_* \mathcal{F}_p. \tag{2.2}$$

A sheaf over a point is clearly flasque, the direct image of a flasque sheaf is flasque (theorem 3.1.1 of [God58]) and the product of flasque sheaves is flasque, hence 2.2 is a flasque sheaf. The other terms of the resolution are constructed with analogous procedure. $\square$

2.3.1 Compatibility with tensor product

In order to define a pairing in the framework of syntomic cohomology, we will need some compatibilities between generalized Godement resolutions and tensor product.

Notation. If $A^\bullet$ is a complex of abelian sheaves on a topological space X, then we denote by $GA^\bullet$ the generalized Godement resolution with respect to the points of X (see example 2.3.5).

Lemma 1. *Let*

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \xrightarrow{\bar{g}} & \widetilde{\mathcal{Y}} \\ \downarrow f & & \downarrow \bar{f} \\ \widetilde{\mathcal{T}} & \xrightarrow{g} & \widetilde{\mathcal{Z}} \end{array}$$

(2.3)

be a commutative diagram of topoi. For every $\mathcal{F} \in \widetilde{\mathcal{Z}}$ there exists a map

$$\alpha : g^*(\bar{f}_*\bar{f}^*(\mathcal{F})) \to f_*f^*(g^*\mathcal{F}).$$

Proof. By the commutativity of the diagram 2.3 we have $f_*f^*(g^*\mathcal{F}) = f_*\bar{g}^*(\bar{f}^*\mathcal{F})$, so it is enough to prove that there exists a map

$$g^*\bar{f}_*\mathcal{G} \to f_*\bar{g}^*\mathcal{G}$$

for every $\mathcal{G} \in \widetilde{\mathcal{Y}}$. Let U be an open of $\mathcal{Y}$, we must construct a morphism

$$\varinjlim_{S \supset g(U)} \mathcal{G}(\bar{f}^{-1}(S)) \to \varinjlim_{M \supset \bar{g}(f^{-1}(U))} \mathcal{G}(M).$$

By the universal property of the direct limit, it is sufficient to prove that $S \supset g(U)$ implies $\bar{f}^{-1}(S) \supset \bar{g}(f^{-1}(U))$ and this easily follows by the fact that $\bar{f}^{-1}(S) \supset \bar{f}^{-1}(g(U)) \supset \bar{g}(f^{-1}(U))$. $\square$

Proposition 2.3.14. Let $A^\bullet$ and $B^\bullet$ be two complexes of sheaves on a topological space X. There exists a natural morphism of resolutions

$$f_{A,B} : GA^\bullet \otimes GB^\bullet \to G(A^\bullet \otimes B^\bullet)$$

Proof. By [God58], chapter II, 6.4, we have

$$GA^\bullet \widetilde{\otimes} GB^\bullet \to G(A^\bullet \widetilde{\otimes} B^\bullet)$$

where the symbol $\widetilde{\otimes}$ denotes the total tensor product of the complexes $A^\bullet$ and $B^\bullet$ (denoted $\hat{\otimes}$ in [God58], chapter II, 2.10), and $A^\bullet \widetilde{\otimes} B^\bullet$ is a complex of sheaves on $X \times X$. Applying the functor Δ^*, where $\Delta : X \to X \times X$ is the diagonal map, we obtain the desired map as

$$GA^\bullet \otimes GB^\bullet \simeq \Delta^*(GA^\bullet \widetilde{\otimes} GB^\bullet) \to \Delta^*(G(A^\bullet \widetilde{\otimes} B^\bullet)) \xrightarrow{\alpha} G(\Delta^*(A^\bullet \widetilde{\otimes} B^\bullet)) \simeq G(A^\bullet \otimes B^\bullet)$$

The map α follows by the previous lemma[2] and the other isomorphisms are justified by the example in 2.11. of [God58]. $\square$

---

[2]consider the diagram

$$\begin{array}{ccc} P & \xrightarrow{\Delta} & P \times P \\ \downarrow u & & \downarrow (u,u) \\ X & \xrightarrow{\Delta} & X \times X. \end{array}$$

The last result still holds in the case of generalized Godement resolutions:

Corollary 2.3.15. Let X be a site and let P be a conservative family on X, then for any pair of complexes $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ of sheaves in $\widetilde{X}$, there is a canonical morphism

$$G_P\mathcal{F}^\bullet \otimes G_P\mathcal{G}^\bullet \to G_P(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet).$$

2.3.2   Points on rigid analytic spaces

In order to define our syntomic theory, we must deal also with rigid analytic spaces. In particular we will need to apply the tool of Godement resolutions to sheaves on K-rigid analytic spaces. One of the problem that arises in the developement of our construction is to find a conservative family of points on a rigid analytic space: as a matter of fact, these spaces are defined as Grothendieck sites, not simply as topological spaces (see [Bosch]), hence we can not procede as in example 2.3.5 for the case of the Zariski site: in the rigid setting, the set of *classic* points does not provide a conservative family, as shown in example 2.3.16 below. We are going to use Van der Put and Schneider's definition of prime filter [vdPS95], that will provide a conservative family of points in the rigid analytic framework. The definition of filter is inspired by the properties of the system of neighborhoods containing a point of a topological space. This is the reason why the notion of filter is a generalization of the concept of point for a Grothendieck site.

Example 2.3.16 (cf. [FresVdP04, 7.0.11]). **A sheaf of O_X-modules with trivial stalks**

Let K be a complete non archimedean algebraically closed field, let $X = \mathrm{Sp}(K\langle T \rangle)$. One considers the presheaf $\mathcal{F}$ on X defined for connected affinoid subsets U of X by:

(a) $\mathcal{F}(U) = 0$ if U is contained in a closed disk of radius < 1.

(b) If U is not contained in a closed disk of radius < 1, then $\mathcal{F}(U) = L$ where L is the completion, with respect to the spectral norm, of the field of fractions of $K\langle T \rangle$.

If U is the disjoint union $U_1 \cup \cdots \cup U_s$ of connected affinoids, then one defines $\mathcal{F}(U) = \mathcal{F}(U_1) \oplus \cdots \oplus \mathcal{F}(U_s)$.

First of all we must verify that the presheaf $\mathcal{F}$ is a sheaf. Let U be an affinoid subset of X, let $\{V, W\}$ be a covering of U made by affinoid subsets. We must prove that the sequence

$$0 \to \mathcal{F}(U) \longrightarrow \mathcal{F}(V) \oplus \mathcal{F}(W) \to \mathcal{F}(V \cap W) \tag{2.4}$$

is exact. Every affinoid subset is, in a unique way, the finite union of disjoint connected affinoid subsets ([FresVdP04, 2.1.1]), so we can suppose U connected affinoid. If U is contained in some closed disk of radius < 1 the same holds for V and W, so the sequence 2.4 is clearly exact because all the terms involved are zero. Now suppose that U is not contained in any closed disk of radius < 1. If exactly one between V and W is, then the sequence 2.4 is

$$0 \to L \longrightarrow 0 \oplus L \to 0$$

so it is obviously exact. So suppose that V (resp. W) is not contained in any closed disk of radius < 1. By 9.7.2 of [BGR] there exist $b_0, a_0, \ldots, a_{m+n}, \in X$ and $s_0, r_0, \ldots, r_{m+n} \in |K^*|$, less or equal than 1, such that $V = D^+(a_0, r_0) \setminus \bigcup_{i=1}^n D^-(a_i, r_i)$ and $W = D^+(b_0, s_0) \setminus \bigcup_{i=n+1}^{n+m} D^-(a_i, r_i)$. By our assumption, $a_0 = b_0 = 1$, hence $V \cap W = D^+(x, 1) \setminus \bigcup_{i=1}^{m+n} D^-(a_i, r_i)$ with $x \in V \cap W$. This means that $V \cap W$ is not contained in any closed disk of radius < 1, so the sequence 2.4 is

$$0 \to L \longrightarrow L \oplus L \to L$$

and it is clearly exact.

The case U not contained in any closed disk of radius < 1 and both V, W contained is not possible, because $V \subset D^+(x, \alpha)$ and $W \subset D^+(x, \beta)$ (we can suppose both disks to have any $x \in V \cap W$ as center) imply $U \subset D^+(x, \max\{\alpha, \beta\})$.

The fact that $\mathcal{F}$ is a sheaf of O_X-modules easily follows by the definition of L.

Now we must show that for any $x \in X$ the stalk $\mathcal{F}_x$ is zero.

Let $x \in X$. By definition of direct limit and by definition of the sheaf $\mathcal{F}$, it is enough to verify that for every affinoid subset $U \ni x$ of X there exists an affinoid subset $U \supset V \ni x$ such that $V \subset D^+(a, \alpha)$ for some $a \in X$ and some $\alpha < 1$. Let $x \in X$ such that $|x| \neq 1$, and let $U \ni x$ be an affinoid subset. There exists $0 < \alpha < 1$ such that $x \in D^+(x, \alpha) \subset X$ (we can think of α as the distance $|x - n|$ where n is the nearest point of the boundary of X). Then we can take $V = U \cap D^+(x, \alpha)$. We stress that the disk $D^+(x, \alpha)$ is rational[3], hence affinoid (see [Bosch], proposition 11 of 1.6). We can conclude V is indeed an affinoid subset because the intesection of two affinoid subsets is an affinoid subset (see [Bosch], proposition 14 of 1.6).

If $|x| = 1$ then, for any $0 < \alpha < 1$ such that $\alpha = |y'|$ for some $y' \in X$, the disk $D^+(x, \alpha) \subset D^-(x, 1) \subset \partial X \subset X$ is an affinoid subset containing x[4], and we can procede as before.

Definition 2.3.17. Let X be a rigid analytic space over K. We recall that a *filter f* on X is a collection $(U_\alpha)_\alpha$ of admissible open of X satisfying:

i) $X \in f, \varnothing \notin f$;

ii) if $U_\alpha, U_\beta \in f$ then $U_\alpha \cap U_\beta \in f$;

iii) if $U_\alpha \in f$ and the admissible open V contains U_α, then $V \in f$.

A *prime filter* on X is a filter p satisfying moreover

iv) if $U \in p$ and $U = \cup_{i \in I} V_i$ is an admissible covering of U, then $V_{i_0} \in p$ for some $i_0 \in I$.

Let $P(X)$ be the set of all prime filters of X. We can give to $P(X)$ a Grothendieck topology and define a morphism of sites $\sigma : P(X) \to X$ [vdPS95, §4]. Also we denote by $Pt(X)$ with the set of prime filters with the discrete topology. Let $i : Pt(X) \to P(X)$ be the canonical inclusion and $\xi = \sigma \circ i$.

It is possible to associate to every filter f on the rigid analytic space X a point of the topos $\widetilde{X}$ (in the sense of definition 2.1.4): consider $\widetilde{P(X)}$ as the topos associated to the topological space $P(X)$, then a point $f \in P(X)$, i.e. a filter, corresponds to a morphism of topoi

$$\mathbf{Sets} \to \widetilde{P(X)}$$

so we can consider its composition with σ to obtain a point of the topos X.

`lmm:vdpsch` *Lemma 2.3.18.* With the above notation the functor $\xi^* : Sh(X) \to Sh(Pt(X))$ is exact and conservative. In other words the set $Pt(X)$ is a conservative family of points for the site X.

Proof. See [vdPS95, §4] after the proof of the theorem 1. □

The previous result allows to get a functorial flasque resolution also in the rigid analytic setting.

Notation. If X is a rigid analytic space (resp. X a K-scheme), and $\mathcal{F}$ is an abelian sheaf over X (resp. over X), when the underlying space will be clear by the context, we will denote also by $G_{an}\mathcal{F}$ (resp. $G_{zar}\mathcal{F}$) the Godement resolution $G_{Pt(X)}\mathcal{F}$ (resp. $G_{P(X)}\mathcal{F}$).

---

[3] define $y' = x - n$, we can write $D^+(x, \alpha)$ as $\{y \in X; |f_1(y)| \leq |f_0(y)|\}$ where $f_0 = y', f_1 = x - T \in K < T >$.

[4] the second inclusion follows by [Bosch], proposition 2 of 1.1

2.4 Monads

The natural framework of Generalized Godement resolutions is the more general theory of monads. We recall here some definitions and results of [Ivo05].

Definition 2.4.1. Let C be a category. A monad over C is the data of

- an endofunctor $M : C \to C$

- a natural transformation $\mu : MM \to M$

- a natural transformation $\eta : \mathrm{id} \to M$

such that the diagrams

and

commute.

Once we have fixed a monad (M, μ, η) in C, we can find a way to associate to every object C of C a cosimplicial object $B^*(M, C)$ of C with a coaugmentation $C \to B^*(M, C)$: we define the objects in degree $n + 1$ as $M^n C$, the codegeneration maps as

$$\sigma_i^n := M^i \eta M^{n-1-i} : M^{n+1} C \to M^n C \qquad i = 0, \ldots, n - 1$$

and the cofaces as

$$\delta_i^{n-1} := M^i \mu M^{n-i} : M^n C \to M^{n+1} C \qquad i = 0, \ldots, n.$$

The coaugmentation is given by $0 \to C \xrightarrow{\eta_C} B^0(M, C) = MC$.

Suppose now that $\mathcal{A}$ and $\mathcal{B}$ are abelian categories, $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ adjoint functors. As before, we will call η and ϵ respectively the unit and the counit of the adjunction. In this setting is always possible to define a monad (M, μ, η) of $\mathcal{A}$ where $M := GF$ and $\mu := G\epsilon F$. By the previous construction we can associate to every object A in $\mathcal{A}$ the cosimplicial object $B^*(M, A)$. This object gives us a complex $sB^*(M, A)$ of objects in $\mathcal{A}$, as explained in 2.2, with a coaugmentation

$$A \xrightarrow{\eta_A} sB^*(M, A). \tag{2.5}$$

Lemma 2.4.2. With the above notation, the map $F(\eta_A)$ is a quasi-isomorphism. In particular, if F is exact and conservative, the coaugmentation η_A is a quasi-isomorphism.

Proof. See [Ivo05, Ch. III, Lemma 3.4.1]. □

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Chapter 3

The syntomic diagram

For any smooth $\mathcal{V}$-scheme $\mathscr{X}$ we want to define a *p*-adic Hodge complex $R\Gamma(\mathscr{X})$ computing its syntomic cohomology (defined in [Bes00]). We will call $R\Gamma(\mathscr{X})$ the syntomic diagram of $\mathscr{X}$. The tool of generalized Godement resolutions, developed in chapter 2, will be crucial to define the morphisms arising in the diagram. On the other hand, those morphisms will be all quasi-isomorphisms but one, that is the specialization morphism defined by ? in [?]: this permits the compatibility with the syntomic theory developed by [Bes00]. Another crucial result we want to stress is the existence of a $\mathcal{V}$-compactification generically smooth and normal crossing: in the construction of the diagram, it will allow to relate the rigid and the de Rham setting by mean of functoriality.

3.1 Besser's complexes

Let X be a *k*-scheme. We want to associate to X a complex of K-vector spaces, whose cohomology is the rigid cohomology defined by Berthelot, in a functorial way (in a suitable category). It turns out that the definition of a functor $\mathbb{R}\Gamma_{\mathrm{rig}}(-) : \mathsf{Sch}_k \to \mathsf{Compl}_K$ is not enough for our purpose of defining a syntomic diagram, because we need something more to compare the rigid setting with the characteristic zero one. Similarly to Besser's situation [Bes00], we will need to construct three rigid functors, and obtain

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X) \leftarrow \widetilde{\mathbb{R}\Gamma}_{\mathrm{rig}}(X)_{\overline{X},\mathscr{P}} \to \mathbb{R}\Gamma_{\mathrm{rig}}(X)_{\overline{X},\mathscr{P}} \qquad (3.1)$$

where $\overline{X}$ is a compactification of X and $\mathscr{P}$ a *p*-adic formal $\mathcal{V}$-scheme which is smooth in a neighborhood of X. In order to associate a canonical resolution to a complex of sheaves we use generalized Godement resolutions, as developed in chapter 2, instead of the techniques of [SD72]. This represents the main difference of our approach with respect to Besser's definitions [Bes00].

3.1.1 The functor $j^\dagger$

Definition 3.1.1. Let V be a strict neighborhood of $]X[_{\mathscr{P}}$ in $]\overline{X}[_{\mathscr{P}}$. We define the functor $j_V^\dagger$ from the category of sheaves over V to itself by

$$j_V^\dagger(\mathcal{F}) := \varinjlim_U j_{U*} j_U^{-1} \mathcal{F},$$

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where the direct limit is over all U which are strict neighborhoods of $]X[_{\mathscr{P}}$ in $]\overline{X}[_{\mathscr{P}}$ contained in V and $j_U : U \hookrightarrow V$ is the canonical embedding.

We will simply write $j^\dagger$ instead of $j^\dagger_{]\overline{X}[_{\mathscr{P}}}$. Explicitely, $j^\dagger$ is defined for every sheaf $\mathcal{F}$ over $]\overline{X}[_{\mathscr{P}}$ as

$$j^\dagger(\mathcal{F}) = \varinjlim_U j_{U*} j_U^{-1} \mathcal{F},$$

where the direct limit is over all U which are strict neighborhoods of $]X[_{\mathscr{P}}$ in $]\overline{X}[_{\mathscr{P}}$ and $j_U : U \hookrightarrow]\overline{X}[_{\mathscr{P}}$ is the canonical embedding.

3.1.2 The complex $\mathbb{R}\Gamma_{rig}(X)_{\overline{X},\mathscr{P}}$

We introduce the definition of some categories that will be necessary to construct the main of Besser's rigid complexes.

$\boxed{\text{RT}}$ **Definition 3.1.2.** A rigid triple is a system $(X, \overline{X}, \mathscr{P})$ where:

- X is a k-scheme;

- $j : X \to \overline{X}$ is an open embedding into a proper k-scheme;

- $\overline{X} \to \mathscr{P}$ is a closed embedding into a p-adic formal $\mathcal{V}$-scheme $\mathscr{P}$ which is smooth in a neighborhood of X.

$\boxed{\text{blemorphism}}$ **Definition 3.1.3.** [Bes00] Let $(X, \overline{X}, \mathscr{P}), (Y, \overline{Y}, \mathscr{Q})$ be two rigid triples and let $f : X \to Y$ be a morphism of k-schemes. Let $U \subset]\overline{X}[_{\mathscr{P}}$ be a strict neighborhood of $]X[_{\mathscr{P}}$ and $F : U \to \mathscr{Q}$ be a morphism of K-rigid spaces. We say that F is compatible with f if it induces the following commutative diagram

$$
\begin{array}{ccc}
]X[_{\mathscr{P}} & \xrightarrow{\ F\ } &]Y[_{\mathscr{Q}} \\
\downarrow{\scriptstyle sp} & & \downarrow{\scriptstyle sp} \\
X & \xrightarrow{\ f\ } & Y.
\end{array}
$$

We write $\mathrm{Hom}_f(U, \mathscr{Q})$ for the collection of morphisms $U \to \mathscr{Q}_K$ compatible with f.

If we set

$$\mathrm{Hom}((X, \overline{X}, \mathscr{P}), (Y, \overline{Y}, \mathscr{Q})) = \{(f, F) | f \in \mathrm{Hom}_{\mathrm{Sch}_k}(X, Y), F \in \varinjlim_U \mathrm{Hom}_f(U, \mathscr{Q})\},$$

the collection of rigid triples forms a category, denoted it by RT.

$\boxed{\text{rct}}$ **Definition 3.1.4.** Let $(X, \overline{X}, \mathscr{P})$ be a rigid triple, we define the rigid complex associated to the triple $(X, \overline{X}, \mathscr{P})$ as

$$\mathbb{R}\Gamma_{rig}(X)_{\overline{X},\mathscr{P}} = \varinjlim_U \Gamma(U, G_{Pt(U)} j_U^\dagger G_{Pt(U)} \Omega_U^\bullet)$$

where the direct limit is over the strict neighborhoods of the tube $]X[_{\mathscr{P}}$ (with respect to inclusion).

Remark 2. Let $U' \overset{i}{\hookrightarrow} U$ be an inclusion between strict neighborhoods of $]X[_P$. Consider the canonical map $\Omega_U^\bullet \to i_*\Omega_{U'}^\bullet$, and apply proposition 1 to get

$$G_{Pt(U)}\Omega_U^\bullet \to i_*G_{Pt(U')}\Omega_{U'}^\bullet.$$

Then apply the functor $j_U^\dagger$ and proposition 5.1.13 of [LS07] to obtain

$$j_U^\dagger G_{Pt(U)}\Omega_U^\bullet \to j_U^\dagger i_* G_{Pt(U')}\Omega_{U'}^\bullet = i_* j_{U'}^\dagger G_{Pt(U')}\Omega_{U'}^\bullet.$$

By proposition 1 there exists a a map

$$G_{Pt(U)}j_U^\dagger G_{Pt(U)}\Omega_U^\bullet \to i_*G_{Pt(U')}j_{U'}^\dagger G_{Pt(U')}\Omega_{U'}^\bullet$$

hence we have the corresponding map at level of global sections

$$\Gamma(U, G_{Pt(U)}j_U^\dagger G_{Pt(U)}\Omega_U^\bullet) \to \Gamma(U, i_*G_{Pt(U')}j_{U'}^\dagger G_{Pt(U')}\Omega_{U'}^\bullet) = \Gamma(U', G_{Pt(U')}j_{U'}^\dagger G_{Pt(U')}\Omega_{U'}^\bullet).$$

This shows that the direct limit of definition 3.1.4 is well defined.

Lemma 2. *Let $\mathcal{A}$ be a filtered category, A a ring and consider a functor*

$$C_-^\bullet : \mathcal{A} \to C(\mathrm{Mod}_A), \quad \alpha \mapsto C_\alpha^\bullet.$$

If for any $\alpha \overset{f}{\to} \beta \in \mathrm{Mor}(\mathcal{A})$ the map $C_f^\bullet$ is a quasi-isomorphism, then the complex

$$\varinjlim_{\alpha \in \mathcal{A}} C_\alpha^\bullet$$

is quasi-isomorphic to each term of the limit.

Proof. The result follows by the exactness of the functor $\varinjlim : Fct(\mathcal{A}, \mathrm{Mod}_A) \to \mathrm{Mod}_A$ (see proposition 3.3.3 of [SH07]). $\qquad\square$

Proposition 2. *The assignment*

$$(X, \overline{X}, \mathscr{P}) \mapsto \mathbb{R}\Gamma_{rig}(X)_{\overline{X}, \mathscr{P}}$$

defines a functor from RT to $Compl_K$. Furhermore the cohomology of the complex $\mathbb{R}\Gamma_{rig}(X)_{\overline{X}, \mathscr{P}}$ is the rigid cohomology of X.

Proof. Let $(f, F) : (X, \overline{X}, \mathscr{P}) \to (Y, \overline{Y}, \mathscr{Q})$ be a morphism in RT, and also call $F : U \to \mathscr{Q}$ a representative of the germ F. Let $V \supset]Y[_{\mathscr{Q}}$ a strict neighborhood, we must find a strict neighborhood $U' \subset]X[_{\mathscr{P}}$ and a map $\Gamma(V, G_{Pt(V)}j_V^\dagger G_{Pt(V)}\Omega_V^\bullet) \to \Gamma(U', G_{Pt(U')}j_{U'}^\dagger G_{Pt(U')}\Omega_{U'}^\bullet)$. By Lemma 4.3 of [Bes00], $F^{-1}(V) = U'$ is a strict neighbordood of $]X[_{\mathscr{P}}$, hence we can consider $F_{|U'} := F : U' \to V$ and the map $F^{-1}\Omega_V^\bullet \to \Omega_{U'}^\bullet$. Now we use proposition 1 and apply the functor $j_{U'}^\dagger$ to obtain

$$j_{U'}^\dagger F^{-1}G_{Pt(V)}\Omega_V^\bullet \to j_{U'}^\dagger G_{Pt(U')}\Omega_{U'}^\bullet.$$

Composing with the canonical morphism constructed in proposition 5.1.14 of [LS07] we finally obtain

$$F^{-1}j_V^\dagger G_{Pt(V)}\Omega_V^\bullet \longrightarrow j_{U'}^\dagger G_{Pt(U')}\Omega_{U'}^\bullet$$
$$\downarrow$$
$$j_{U'}^\dagger F^{-1}G_{Pt(V)}\Omega_V^\bullet$$

By adjunction there exists a map

$$j_V^\dagger G_{Pt(V)}\Omega_V^\bullet \to F_* j_{U'}^\dagger G_{Pt(U')}\Omega_{U'}^\bullet$$

whence, by applying proposition 1, the desired map

$$G_{Pt(V)} j_V^\dagger G_{Pt(V)}\Omega_V^\bullet \to F_* G_{Pt(U')} j_{U'}^\dagger G_{Pt(U')}\Omega_{U'}^\bullet.$$

By [Berth86], the cohomology of every complex in the direct limit is the rigid cohomology of X, hence by lemma 2 also the cohomology of $\mathbb{R}\Gamma_{rig}(X)_{\overline{X},\mathscr{P}}$ is the rigid cohomology of X. □

3.1.3 The complex $\mathbb{R}\Gamma_{rig}(X)$

Now that we have the building block of the construction we can apply the same procedue as in [Bes00] to get the other two functors that we need.

$\boxed{\text{SET}}$ **Definition 3.1.5.** Let X be a k-scheme.

- The category of rigid data for X, denoted $RD(X)$, is the collection of all triples $(X', \overline{X'}, \mathscr{P})$ with $X' = X$, with morphisms whose first component is the identity map of X.

- The set PT_X is the set of all pairs $(f, (\overline{Y}, \mathscr{P}))$ where $f : X \to Y$ is a morphism of k-schemes and $(\overline{Y}, \mathscr{P}) \in RD(Y)$. The subset PT_X^0 contains all pairs where the first component is the identity map of X.

- The category SET_X is the category whose objects are all finite subsets of PT_X and whose morphisms are inclusions. We denote by SET_X^0 the full subcategory whose objects are all subsets with a non-empty intersection with PT_X^0.

Remark 3. The category $RD(X)$ is not filtered. For this reason, in order to define a complex that is independent from auxiliary data and computes the rigid cohomology we can not simply take the direct limit

$$\varinjlim_{(X,\overline{X},\mathscr{P})\in RD(X)} \mathbb{R}\Gamma_{rig}(X)_{\overline{X},\mathscr{P}},$$

because this complex could not be quasi isomorphic to the complexes $\mathbb{R}\Gamma_{rig}(X)_{\overline{X},\mathscr{P}}$.

We recall, as in lemma 4.12 of [Bes00], that to every $A \in \mathcal{SET}_X^0$ we can associate a rigid triple $(X, \overline{X}_A, \mathscr{P}_A)$:

$$X \hookrightarrow \prod_{a\in A} X_a \hookrightarrow \prod_{a\in A} \overline{X}_a \hookrightarrow \mathscr{P}_A := \prod_{a\in A} \mathscr{P}_a,$$

where $\overline{X}_A$ is the closure of X in $\prod_{a\in A} \overline{X}_a$. We stress that the first inclusion is closed because there is some $a \in A$ such that $X_a = X$. Furthermore X is locally closed in $\prod_{a\in A} \overline{X}_a$, and that means that X is an open of $\overline{X}_A$. For that reason, $(X, \overline{X}_A, \mathscr{P}_A)$ is a rigid triple.

From now on we will denote elements of PT_X by letters like a and the associated auxiliary datum by $(f_a, (\overline{Y}_a, \mathscr{P}_a))$.

To simplify notation, we will now define

$$\mathcal{F}_X(A) := \mathbb{R}\Gamma_{rig}(X)_{\overline{X}_A,\mathscr{P}_A}.$$

This is then a contravariant functor $SET_X^0 \to \text{Compl}_K$, and all morphisms of complexes one obtains are quasi-isomorphisms.

Definition 3.1.6. We define the rigid complex

$$\mathbb{R}\Gamma_{rig}(X) := \varinjlim_{A \in SET_X^0} \mathcal{F}_X(A).$$

Proposition 3. *The assignment* $X \mapsto \mathbb{R}\Gamma_{rig}(X)$ *defines a contravariant functor* $\mathsf{Sch}_k \to \mathsf{Compl}_K$.

Proof. [Bes00], proposition 4.14. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

frobenius | *Proposition* 3.1.7. There exists a canonical σ-linear endomorphism of $\mathbb{R}\Gamma_{rig}(X/K_0)$ inducing the Frobenius on cohomology.

Proof. [Bes00], corollary 4.22. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.1.4 The complex $\widetilde{\mathbb{R}\Gamma}_{rig}(X)_{\overline{X},\mathscr{P}}$

rctilde | We recall the result explained in section 4 of [Bes00]. Suppose there are sets PT_X' and $PT_X^{0'}$ with a projection map $\Pi : PT_X' \to PT_X$ sending $PT_X^{0'}$ to PT_X^0. Then for $a \in PT_X'$ we can define $\overline{X}_a := \overline{X}_{\Pi(a)}, \overline{\mathscr{P}}_a := \overline{\mathscr{P}}_{\Pi(a)}$, and define $\overline{X}_{A'}$ and $\overline{\mathscr{P}}_{A'}$ for $A' \subset PT_X'$ in the same way as before. We obtain a functor

$$\widetilde{\mathcal{F}}_X : SET_X^{0'} \to \mathsf{Compl}_K, \quad A' \mapsto \mathbb{R}\Gamma_{rig}(X)_{\overline{X}_{A'},\overline{\mathscr{P}}_{A'}}$$

as before, where $SET_X^{0'}$ denotes the obvious construction. For $A' \in SET_X^{0'}$, the canonical projection $\Pi : A' \to \Pi(A')$ induces canonical maps $\Delta_{A'} : (\overline{X}_{\Pi(A')}, \overline{\mathscr{P}}_{\Pi(A')}) \to (\overline{X}_{A'}, \mathscr{P}_{A'})$ which induce a quasi-isomorphism

$$\mathbb{R}\Gamma_{rig}(\Delta_{A'}) : \widetilde{\mathcal{F}}_X(A') \to \mathcal{F}_X(\Pi(A'))$$

by proposition 4.7 of [Bes00]. Going to the limit we obtain a map

$$\varinjlim_{A' \in SET_X^{0'}} \widetilde{\mathcal{F}}_X(A') \to \varinjlim_{A' \in SET_X^{0'}} \mathcal{F}_X(\Pi(A'))$$

which is again a quasi-isomorphism as in both limits all maps are. Composing with the canonical map

$$\varinjlim_{A' \in SET_X^{0'}} \widetilde{\mathcal{F}}_X(\Pi(A')) \to \varinjlim_{A \in SET_X^0} \mathcal{F}_X(A)$$

we obtain

$$\Delta : \varinjlim_{A' \in SET_X^{0'}} \widetilde{\mathcal{F}}_X(A') \to \varinjlim_{A' \in SET_X^{0'}} \mathcal{F}_X(A).$$

In our specific case, for every fixed rigid triple $(X, \overline{X}, \mathscr{P})$ consider the set $PT_{X,\overline{X},\mathscr{P}}$ of all morphism of rigid triples from $(X, \overline{X}, \mathscr{P})$ to another rigid triple and the subset $PT_{X,\overline{X},\mathscr{P}}^0$ consisting of the identity morphism. There is a canonical forgetful projection

$$\Pi : PT_{X,\overline{X},\mathscr{P}} \to PT_X$$

and $\Pi(PT_{X,\overline{X},\mathscr{P}}^0) \subset PT_X^0$.

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Definition 3.1.8. For every rigid triple $(X, \overline{X}, \mathscr{P})$ we define

$$\widetilde{\mathbb{R}\Gamma}_{rig}(X)_{\overline{X},\mathscr{P}} := \varinjlim_{A' \in SET_X^{0'}} \widetilde{\mathcal{F}}_X(A').$$

Proposition 4. *The assignment* $X \mapsto \widetilde{\mathbb{R}\Gamma}_{rig}(X)_{\overline{X},\mathscr{P}}$ *defines a contravariant functor* $RT_X \to$ *Compl$_K$.*

Proof. See lemma 4.15 of [Bes00]. □

3.1.5 Natural transformations between rigid complexes

nattransf

Now we are ready to describe the rigid part of the syntomic diagram, introduced in 3.1. The natural transformation to the complex $\mathbb{R}\Gamma_{rig}(X)$ is an immediate consequence of lemma 4.15 of [Bes00].

Now we must construct the map q. For the universal property of the direct limit, it is enough to have, for every $A' \in \mathcal{SET}^0_{X,\overline{X},\mathscr{P}}$, a map

$$\widetilde{\mathcal{F}}_X(A') = \mathbb{R}\Gamma_{\mathsf{rig}}(X)_{\overline{X}_{A'},\mathscr{P}_{A'}} \to \mathbb{R}\Gamma_{\mathsf{rig}}(X)_{\overline{X},\mathscr{P}}.$$

This means that we must find a map $(X, \overline{X}, \mathscr{P}) \to (X, \overline{X}_{A'}, \mathscr{P}_{A'})$ in the category where the functor $\mathbb{R}\Gamma_{rig}(-)_{-,-}$ is defined, i.e. in $\mathcal{RT}$ (see proposition 2). This map is the projection. Naturality is clearly verified.

3.2 The de Rham cohomology and the complex $\mathbb{R}\Gamma_{dR}(X)_Y$

rhamcomplex

We must then compare the previous rigid setting with the characteristic zero one, i.e. with the de Rham cohomology of the special fiber. We recall here a variation of Huber's definition of de Rham complex ([Hub95], chapter 7). We need to know not only a complex computing de Rham cohomology, but also complexes computing all the filtered parts. Here K can be any field of characteristic zero. Let X be a smooth K-scheme. A de Rham datum for X is an injection $i : X \hookrightarrow Y$ where Y is a smooth and proper K-scheme and $D := Y \setminus X$ is a divisor with normal crossing. We denote by $\Omega_Y\langle D\rangle$ the de Rham complex of Y with logarithmic poles along D (in the Zariski topology) (See [Jan90, 3.3]).

deRhCompl **Definition 1.** *To a de Rham datum* (Y) *and to every* $k \in \mathbb{Z}_{\geq 0}$ *we associate a complex, called k-th filtered part of the de Rham complex of* X *with respect to the datum* (Y)*, defined by*

$$F^k\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)_Y := \Gamma(Y, G_{zar}^2\Omega_{Y/K}^{\geq k}\langle D\rangle).$$

We will write $\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)_Y$ *for* $F^0\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)_Y$ *and we will call it the de Rham complex of* X *associated to the de Rham datum* Y.

Remark 3.2.1. Even if it is not necessary for our purpose, we want to recall here that is also possible to give a definition of such a complex that does not depend on the de Rham datum (see definition 5.1 of [Bes00]): the *k*-th filtered part of the de Rham complex of X is defined by

$$F^k\mathbb{R}\Gamma_{\mathrm{dR}}(X/K) := \varinjlim_Y F^k\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)_Y$$

where the limit is over all de Rham data. According to this definition the F^k, in spite of their name, are not subcomplexes of $\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)$ but there are natural maps

$$F^k\mathbb{R}\Gamma_{\mathrm{dR}}(X/K) \to \mathbb{R}\Gamma_{\mathrm{dR}}(X/K),$$

obtained by appliyng the (right exact) functor $\varinjlim_Y$ to the injective map $F^k\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)_Y \to \mathbb{R}\Gamma_{\mathrm{dR}}(X/K)_Y$. We stress the fact that by using definition 1 we obtain instead a filtered complex.

`def:derham` **Definition 3.2.2.** The De Rham cohomology groups of X are

$$H^i_{\mathrm{dR}}(X) := H^i(Y, \Omega^\bullet_Y\langle D\rangle).$$

These are filtered K-vector spaces and the *Hodge filtration* is

$$F^j H^i_{\mathrm{dR}}(X) := H^i(Y, \Omega^{\geq j}_Y\langle D\rangle)$$

`nctoriality` *Remark* 3.2.3. Using the same argument of [Del71, 3.2.11] we get the functoriality of $H^i_{\mathrm{dR}}(X)$ (as filtered K-vector space) in X and the independence of the choice of Y. Moreover if $\tau : K \to \mathbb{C}$ is an embedding we get an isomorphism of filtered vector spaces $H^i(Y, \Omega_Y\langle D\rangle)\otimes_K \mathbb{C} \cong H^i(Y^{\mathrm{an}}, \Omega_{Y^{\mathrm{an}}}\langle D\rangle)$ by GAGA. Hence the above definition is compatible with Hodge theory as developed in [ElZein].

We stress the fact that, from the cohomological point of view, our definition of de Rham complex is equivalent to definition 5.1 of [Bes00] and 7.1 of [Hub95] : they all compute the de Rham cohomology because they are all defined by mean of acyclic or injective resolutions of the complex of sheaves $\Omega^\bullet_Y\langle D\rangle$.

`dRstrict` *Proposition* 3.2.4. The filtered complex $R\Gamma_{\mathrm{dR}}(X)$ is strict.

Proof. By 3) of lemma 4.9 of [PS08] and theorem 3.18 of [PS08][1], the spectral sequence related to $\Omega^\bullet_Y\langle D\rangle$ with respect to the trivial filtration degenerates at E_1. By definition of Godement resolution as total complex associated to a double complex formed by injective resolutions, also the spectral sequence related to $R\Gamma_{\mathrm{dR}}(X)$ with respect to the trivial filtration degenerates at E_1, whence the result. $\qquad\square$

3.3 Existence of a $\mathcal{V}$-compactification with generic normal crossings

We recall that we consider $\mathcal{V}$ a discrete valuation ring of mixed characteristic $(0, p)$, we denote by K its fraction field and by k its residue field. Let $\mathcal{X}$ be a smooth $\mathcal{V}$-scheme.

Now that we have defined the rigid complexes for a k-scheme and the De Rham complexes for a K-scheme, we would like to apply these constructions respectively to the special fiber $\mathcal{X}_k$ and to the generic fiber $\mathcal{X}_K$ of $\mathcal{X}$, in order to be able to define a p-adic Hodge diagram that relates the two settings. For this purpose we need to fix a compactification

$$\mathcal{X} \to \mathcal{Y}$$

---

[1]parla di $\mathbb{C}$..??????

of the variety $\mathscr{X}$ such that the generic fiber $\mathscr{Y}_K$ is smooth and the complement $\mathscr{Y}_K \setminus \mathscr{X}_K$ is a normal crossing divisor[2]. The following results guarantee the existence of such an embedding.

Lemma 3.3.1. Let $\mathscr{X}$ be a smooth $\mathscr{V}$-scheme and let $Z \subset \mathscr{X}_K$ be a closed subscheme of the generic fiber. There exists a unique $\mathscr{Z} \subset \mathscr{X}$ closed subscheme which is flat over $\mathscr{V}$ and such that $\mathscr{Z}_K = Z$. The scheme $\mathscr{Z}$ is called the Zariski closure of Z.

Proof. See 2.8.5 of [EGAIV]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We recall a very important result by Hironaka (see [Wlod05]).

<div style="border:1px solid; display:inline-block; padding:2px">Hironaka</div> *Theorem* 3.3.2 (Canonical Resolution of Singularities). Let Y be an algebraic variety over a field of characteristic zero.

There exists a canonical desingularization of Y that is a smooth variety $\widetilde{Y}$ toghether with a proper birational morphism $\mathrm{res}_Y : \widetilde{Y} \to Y$ which is functorial with respect to proper morphisms. For any smooth morphism $\phi : Y' \to Y$ there is a natural lifting $\phi : \widetilde{Y}' \to \widetilde{Y}$ which is a smooth morphism.

In particular $\mathrm{res}_Y : \widetilde{Y} \to Y$ is an isomorphism over the nonsingular part of Y.

Moreover res_Y commutes with (separable) ground field extensions.

Proof. See [Wlod05]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

<div style="border:1px solid; display:inline-block; padding:2px">ctification</div> *Proposition* 3.3.3. Let $\mathscr{X}$ be a smooth $\mathscr{V}$-scheme. There exists a compactification

$$\mathscr{X} \to \mathscr{Y}$$

such that the generic fiber $\mathscr{Y}_K$ is smooth and the complement $\mathscr{Y}_K \setminus \mathscr{X}_K$ is a normal crossing divisor.

Proof. By Nagata (see [Conr91]) there exists an open embedding

$$\mathscr{X}_K \to Y$$

where Y is a proper K-scheme. By the Hironaka resolution theorem 3.3.2, we can assume that Y is smooth and that $Y \setminus \mathscr{X}_K$ is a normal crossing divisor.

Now we have to find a model $\mathscr{Y}$ of Y over $\mathscr{V}$. Let $\mathscr{P}$ be the glueing of the schemes Y (considered as a $\mathscr{V}$-scheme) and $\mathscr{X}$ along $\mathscr{X}_K$. We have $\mathscr{P}_K = Y$. The problem is that $\mathscr{P}$ could not be proper. The $\mathscr{V}$-scheme $\mathscr{P}$ is separated and of finite type, hence by Nagata (see [Conr91]) there exist a **proper** $\mathscr{V}$-scheme $\mathscr{Y}$ and an open embedding

$$\mathscr{P} \to \mathscr{Y}.$$

There is a composition of **open immersions**

$$\mathscr{X} \to \mathscr{Y}$$

hence $\mathscr{X}$ is dense in $\mathscr{Y}$ so $\mathscr{X}_K$ is dense in $\mathscr{Y}_K$. But the fact that X_K is dense also in Y_K implies $\mathscr{Y}_\mathbf{K} = Y$. So The $\mathscr{V}$-scheme $\mathscr{Y}$ is the model we were looking for. $\qquad\quad$ □

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[2]we need a compactification over $\mathscr{V}$ because two of the rigid functors are defined over *RT* so we need a compactification over *k* and then a formal scheme, that will be the formal completion of $\mathscr{Y}$ (compactification over $\mathscr{V}$) along its special fiber.

3.4 Specialization

Let $\mathscr{X}$ be a smooth variety over $\mathcal{V}$. We fix once for all

$$g : \mathscr{X} \to \mathscr{Y}$$

a compactification with generic normal crossing divisor D (see proposition 3.3.3). Let

$$u : \mathscr{Y}_K^{an} \to \mathscr{Y}_K^{zar}$$

be the canonical map, considered as a morphism of topoi, i.e.

$$u^* : Sh_{/\mathscr{Y}_K} \to Sh_{/\mathscr{Y}_K^{an}}$$

$$u_* : Sh_{/\mathscr{Y}_K^{an}} \to Sh_{/\mathscr{Y}_K^{zar}}.$$

The specialization map links the de Rham complex of $\mathscr{X}_K$ and the rigid complex of $\mathscr{X}_k$. The 'heart' of this morphism is the canonical map

$$j_M^{\dagger} : M \to j^{\dagger} M$$

where M is a $O_{\mathscr{Y}_K^{an}}$-module.

First we recall the construction of this canonical map.

3.4.1 The functor $j^{\dagger}$ and the canonical map

Let N be an O_U-module. Consider the direct image j_{U_*} and the inverse image j_U^{-1} as adjoint functors:

$$\mathrm{Hom}(M, j_{U_*}N) = \mathrm{Hom}(j_U^{-1}M, N).$$

If we choose $N = j_U^{-1}M$, we can consider α as corresponding to the identity morphism by the identification $\mathrm{Hom}(M, j_{U_*}j_U^{-1}M) = \mathrm{Hom}(j_U^{-1}M, j_U^{-1}M)$. Then the canonical map we are looking for is the composition

$$M \xrightarrow{\alpha} j_{U_*}j_U^{-1}M \to \varinjlim_{U'} j_{U'_*}j_{U'}^{-1}M = j^{\dagger}M.$$

If W is an open in $\mathscr{Y}_K^{an}$, then $\alpha(W) : M(W) \to (j_{U_*}j_U^{-1}M)(W) = j_U^{-1}M(W \cap U) = \varinjlim_{T \supset j_U(W \cap U)} M(T) = M(W \cap U)$. That means that we simply restrict the sections over W to the strict neighborhood U (then we take the direct limit over all U).

3.4.2 The construction of sp

Notation. Let $u : \widetilde{\mathcal{P}} \to \widetilde{\mathcal{X}}$ be a morphism of topoi and consider a complex of sheaves $\mathcal{F}^{\bullet}$ in $\widetilde{\mathcal{X}}$ (same situation of definition 2.3.2). We will denote by $G_{\mathcal{P}}^2\mathcal{F}^{\bullet}$ the complex of sheaves $G_{\mathcal{P}}(G_{\widetilde{\mathcal{P}}}(\mathcal{F}^{\bullet}))$.

At level of sheaves, we can construct the specialization map by applying the functorial Godement resolution to the following canonical morphism

$$G_{Pt(\mathscr{Y}_K^{an})}\Omega_{\mathscr{Y}_K}^{\bullet}\langle D\rangle^{an} \to j^{\dagger}G_{Pt(\mathscr{Y}_K^{an})}\Omega_{\mathscr{Y}_K}^{\bullet}\langle D\rangle^{an},$$

obtaining the map

$$G^2_{Pt(\mathscr{Y}^{an}_K)}\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{an} \xrightarrow{sp} G_{Pt(\mathscr{Y}^{an}_K)}j^\dagger G_{Pt(\mathscr{Y}^{an}_K)}\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{an}.$$

Classically[3] the specialization morphism of a smooth variety links de Rham cohomology of its generic fiber with the rigid cohomology of its special fiber. We stress the fact that this is the case, even more, that simply the cohomology of the global sections of the complexes of sheaves involved computes what we expect:

- The cohomology of the complex of vector spaces $\Gamma(\mathscr{Y}^{an}_K, G^2_{Pt(\mathscr{Y}^{an}_K)}\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{an})$ is the de Rham cohomology because of the coherence of the sheaves and the compactness of $\mathscr{Y}_K$ (GAGA).

- The fact that the cohomology of $\Gamma(\mathscr{Y}^{an}_K, G_{Pt(\mathscr{Y}^{an}_K)}j^\dagger G_{Pt(\mathscr{Y}^{an}_K)}\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{an})$ is the rigid cohomology defined in [Berth97] follows by 2.5 of [Berth86] ($\mathscr{X}^{an}_K$ is a strict neighborhood, hence $\Omega^\bullet_{\mathscr{X}^{an}_K}$ computes the rigid cohomology and it is isomorphic to $\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{an}$ at level of overconvergent sections).

3.5 Other maps

othermaps1

In order to define the sintomic diagram of the variety $\mathscr{X}$ we need some more complexes and morphisms.

3.5.1 Link between Besser's complexes and the de Rham analytic

The construction of a map that links the de Rham side to the rigid one reduces to the following proposition.

[a] **Proposition 5.** *With the above notations we have the following diagram of complexes of vector spaces*

$$\Gamma(\mathscr{Y}^{an}_K, G_{an}j^\dagger G_{an}\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{an}) \to \Gamma(\mathscr{X}^{an}_K, G_{Pt(\mathscr{X}^{an}_K)}j^\dagger G_{Pt(\mathscr{X}^{an}_K)}\Omega^\bullet_{\mathscr{X}^{an}_K}) \to \mathbb{R}\Gamma_{rig}(\mathscr{X}_k)_{\mathscr{Y}_k,\hat{\mathscr{Y}}}$$

where $\hat{\mathscr{Y}}$ is the p-adic completion of $\mathscr{Y}$. We denote by a the composition of these maps.

Proof. By definition 3.1.4 we have

$$\mathbb{R}\Gamma_{rig}(\mathscr{X}_k)_{\mathscr{Y}_k,\hat{\mathscr{Y}}} := \varinjlim_U \Gamma(U, G_{Pt(U)}j^\dagger G_{Pt(U)}\Omega^\bullet_U)$$

where the limit is over the strict neighborhoods of the tube of $\mathscr{X}_k$. We know that $\mathscr{X}^{an}_K$ is one of such strict neighborhoods, hence the map on the right comes from the universal property of the direct limit.

For the map on the left consider the canonical inclusion

$$\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{an} \to g^{an}_*\Omega^\bullet_{\mathscr{X}^{an}_K}$$

---

[3]referenza? Baldassarri Cailotto Fiorot dicono Hartshorne "On the de rham...". io non trovo.

and apply proposition 1 to the diagram

$$\begin{array}{ccc} \mathscr{X}_K^{an} & \xrightarrow{g_*^{an}} & \mathscr{Y}_K^{an} \\ \uparrow & & \uparrow \\ Pt(\mathscr{X}_K^{an}) & \longrightarrow & Pt(\mathscr{Y}_K^{an}) \end{array}$$

to obtain

$$G_{Pt(\mathscr{Y}_K^{an})} \to g_*^{an} G_{Pt(\mathscr{X}_K^{an})} \Omega^\bullet_{\mathscr{X}_K^{an}}.$$

Now we can apply $j^\dagger$ and proposition 5.1.13 of [LS07]:

$$j^\dagger G_{Pt(\mathscr{Y}_K^{an})} \to j^\dagger g_*^{an} G_{Pt(\mathscr{X}_K^{an})} \Omega^\bullet_{\mathscr{X}_K^{an}} \simeq g_*^{an} j^\dagger G_{Pt(\mathscr{X}_K^{an})} \Omega^\bullet_{\mathscr{X}_K^{an}}.$$

We use once more proposition 1 to obtain the result. □

3.5.2 Link with the algebraic setting

alg

To link the analytic side of the diagram with the algebraic setting we must prove the following result.

comparison

Proposition 6. *Let* Y *be a proper K-scheme. Let* $u : Y^{an} \to Y^{zar}$ *be the canonical map from the rigid analytic site to the Zariski site of* Y. *Then for any Zariski sheaf* $\mathcal{F}$ *on* Y^{zar} *there is a diagram*

$$G_{P(Y^{zar})}\mathcal{F} \leftarrow G_{Pt(Y^{an}) \sqcup P(Y^{zar})}\mathcal{F} \to u_* G_{Pt(Y^{an})}(u^*\mathcal{F}).$$

If X $\to$ Y *is a smooth compactification with normal crossing divisor (in the category of K-schemes) and we consider* $\mathcal{F} = \Omega^\bullet_Y\langle D\rangle$, *then the diagram is functorial with respect to the pairs* (X, Y). *The same holds true with* $G_?^2$ *instead of* $G_?$.

Proof. The first claim follows from proposition 1 applied to the following commutative diagram of sites

$$\begin{array}{ccccc} Pt(Y^{an}) & \longrightarrow & Pt(Y^{an}) \sqcup P(Y^{zar}) & \longleftarrow & P(Y^{zar}) \\ \downarrow & & \downarrow & & \downarrow \\ Y^{an} & \xrightarrow{w} & Y^{zar} & \underset{id}{\longleftarrow} & Y^{zar} \end{array}$$

with respect to the canonical map $\mathcal{F} \to u_* u^* \mathcal{F}$. We recall that the Zariski points of Y^{zar} provide a conservative family of the Zariski site (see section 2.3 after example 2.3.5), hence *a fortiori* $Pt(Y^{an}) \sqcup P(Y^{zar})$ is a conservative family too.

The second claim follows from the functoriality of the complex $\Omega^\bullet_Y\langle D\rangle$. □

Notation. In the situation of proposition 6 we will denote by $G_{an+zar}\mathcal{F}$ the Godement resolution $G_{Pt(Y^{an}) \sqcup P(Y^{zar})}(\mathcal{F})$.

3.6 Syntomic diagram and syntomic cohomology

Now we put together all we have done getting a diagram $R\Gamma'(\mathscr{X})$ in the category pHC' (see definition 1.2.27)

$$R\Gamma_{\mathrm{rig}}(\mathscr{X}_k/K) \qquad R\Gamma_{\mathrm{rig}}(\mathscr{X}_k/K)_{\mathscr{Y}_k,\hat{\mathscr{Y}}} \qquad \Gamma(\mathscr{Y}_K^{an}, G_{an}^2 u^* \Omega_{\mathscr{Y}_K}^\bullet \langle D \rangle) \qquad \Gamma(\mathscr{Y}_K, G_{zar}^2 \Omega_{\mathscr{Y}_K}^\bullet \langle D \rangle)$$

$$\overset{\alpha_1}{\nearrow} \qquad \overset{\alpha_2}{\nwarrow} \qquad \overset{\alpha_3}{\nearrow} \qquad \overset{\alpha_4}{\nwarrow} \qquad \overset{\alpha_5}{\nearrow} \qquad \overset{\alpha_6}{\nwarrow} \qquad \overset{\alpha_7}{\nearrow} \qquad \overset{\alpha_8}{\nwarrow}$$

$$R\Gamma_{\mathrm{rig}}(\mathscr{X}_k/K_0) \quad \widetilde{R\Gamma}_{\mathrm{rig}}(\mathscr{X}_k/K)_{\mathscr{Y}_k,\hat{\mathscr{Y}}} \quad \Gamma(\mathscr{Y}_K^{an}, G_{an}^2 u^* \Omega_{\mathscr{Y}_K}^\bullet \langle D \rangle) \quad \Gamma(\mathscr{Y}_K, G_{an+zar}^2 \Omega_{\mathscr{Y}_K}^\bullet \langle D \rangle) \, \Gamma(\mathscr{Y}_K, G_{zar}^2 \Omega_{\mathscr{Y}_K}^\bullet \langle D \rangle)$$

where $\alpha_1, \alpha_5, \alpha_8$ are the identity maps; α_2, α_3 are the maps of 3.1; α_4 is the composition $a \circ sp$ (see proposition 5 and 3.4.2); α_6, α_7 are defined in 6. Notice that all α_i but α_4 are quasi-isomorphisms. By applying repeatedly the quasi push-out construction we obtain a diagram of the following shape

$$R\Gamma_{\mathrm{rig}}(\mathscr{X}/K_0) \overset{\sim}{\to} R\Gamma_K(\mathscr{X}) \overset{s}{\leftarrow} R\Gamma_{\mathrm{dR}}(\mathscr{X}) \tag{3.2} \quad \boxed{\text{RGAMMA}}$$

as explained in remark 1.

Definition 3.6.1. We define the syntomic diagram of $\mathscr{X}$ as the object of pHD corresponding to the diagram 3.2. We will denote it by $R\Gamma(\mathscr{X})$.

Notice that the previous diagram is indeed a p-adic Hodge complex whose cohomology in the specialization is the expected one by proposition 3.2.4.

Proposition 7. *Let $Sm/\mathcal{V}$ be the category of smooth $\mathcal{V}$-schemes. The previous construction induces a contravariant functor*

$$R\Gamma(-) : Sm/\mathcal{V} \to pHD.$$

Proof. Let $f : \mathscr{X} \to \mathscr{X}'$ be a morphism of smooth $\mathcal{V}$-schemes. By proposition 6 and the definition of the rigid part of the syntomic diagram (section 3.1), to get the functoriality of $R\Gamma(-)$ it is enough to show that we can find two generic normal crossing compactifications $g : \mathscr{X} \to \mathscr{Y}$ and $g' : \mathscr{X}' \to \mathscr{Y}'$ and a map $h : \mathscr{Y} \to \mathscr{Y}'$ extending f, i.e. such that $hg = g'f$. We argue as in 3.2.11 of [Del71]. Fix two generic normal crossing compactifications $l : \mathscr{X} \to \mathscr{Z}$ and $g' : \mathscr{X}' \to \mathscr{Y}'$. Then consider the canonical embedding $\mathscr{X} \to \mathscr{Z} \times \mathscr{Y}'$ induced by l and $g'f$. Let $\overline{\mathscr{X}}$ be the closure of $\mathscr{X}$ in $\mathscr{Z} \times \mathscr{Y}'$. We can take $\mathscr{Y}$ as a generically resolution of singularities of $\overline{\mathscr{X}}$. $\square$

Remark 3.6.2. We need the first piece of the rigid side to be a complex functorial over the category of smooth schemes, i.e. independent from the compactification, in order to define a Frobenius morphism on it (proposition 3.1.7). It will be the same also for the compact support case, developed in chapter 4.

Definition 3.6.3. Let $\mathscr{X}$ be a smooth algebraic scheme over $\mathcal{V}$. For any n, i integers we define the (rigid) syntomic cohomology groups of $\mathscr{X}$

$$H_{\mathrm{syn}}^n(\mathscr{X}, i) := Hom_{pHD}(\mathbb{K}, R\Gamma(\mathscr{X})(i)[n]) = H^n(\Gamma(\mathbb{K}, R\Gamma(\mathscr{X})(i)))$$

(last equality holds by proposition 1.2.32).

Proposition 8. *The definition [3.6.3] of syntomic cohomology coincides with the definition 6.1 of [Bes00].*

Proof. We have $\Gamma(\mathbb{K}, R\Gamma(\mathscr{X})(i)) \simeq \Gamma(\mathbb{K}(-i), R\Gamma(\mathscr{X}))$, therefore we can conclude by remark 1.2.31. $\qquad\square$

Corollary 3.6.4. There is a long exact sequence
$$\cdots \to H^{n-1}_{rig}(\mathscr{X}_k/K_0) \oplus F^i H^{n-1}_{dR}(\mathscr{X}_K/K) \to H^{n-1}_{rig}(\mathscr{X}_k/K_0) \oplus H^{n-1}_{rig}(\mathscr{X}_k/K) \to$$
$$\to H^n_{syn}(\mathscr{X}, i) \to$$
$$\to H^n_{rig}(\mathscr{X}_k/K_0) \oplus F^i H^n_{dR}(\mathscr{X}_K/K) \to H^n_{rig}(\mathscr{X}_k/K_0) \oplus H^n_{rig}(\mathscr{X}_k/K) \to \cdots$$

Proof. See the proof of proposition 6.3 of [Bes00]. $\qquad\square$

Chapter 4

The syntomic diagram with compact support

In this chapter we will adapt the construction of chapter 3 for syntomic cohomology with compact support. The definition of the syntomic diagram with compact support $R\Gamma_c(\mathscr{X})$ of a smooth $\mathcal{V}$-scheme $\mathscr{X}$ will be completely analogous to that of syntomic diagram. The notions of syntomic cohomology with compact support and of syntomic homology will consequently follow.

4.1 Besser's complexes with compact support

We want to repeat the construction of section 3.1 for rigid cohomology with compact support.

4.1.1 Rigid cohomology with compact support

We recall here the construction of the rigid cohomology with compact support for a scheme X defined over the field k [Berth86].

Consider a rigid triple $(X, \overline{X}, \mathscr{P})$ and let $i :]\overline{X} \setminus X[_{\mathscr{P}} \longrightarrow]\overline{X}[_{\mathscr{P}}$ be the open inclusion. For every abelian sheaf E over $]\overline{X}[_{\mathscr{P}}$ we define the left exact functor

$$\underline{\Gamma}_{]X[}(E) := \ker(E \longrightarrow i_*i^*E).$$

Proposition 9. *If E is coherent, the derived functor $\mathbb{R}\underline{\Gamma}_{]X[}E$ is isomorphic, in derived category, to the complex $[E \longrightarrow i_*i^*E]$.*

Proof. Consider the short exact sequence

$$0 \to \Gamma_{]X[}E \to E \to i_*i^*E \to 0$$

in the category of complexes. In derived category this corresponds to the triangle

$$\mathbb{R}\Gamma_{]X[}E \to E \to i_*i^*E \overset{+1}{\to}$$

because of the exactness of the functor i_*i^* on coherent sheaves. □

The rigid cohomology with compact support of X is defined as in 3.3 of [Berth86]

$$H_c^*(X) := H^*(]\overline{X}[_{\mathscr{P}}, \mathbb{R}\underline{\Gamma}_{]X[}(\Omega_{]\overline{X}[}^\bullet)).$$

We can prove that these cohomology groups are independent from the choices, contravariant with respect to proper morphisms and covariant with respect to open immersions [Berth86].

4.1.2 The complex $\mathbb{R}\Gamma_{rig,c}(X)_{\overline{X},\mathscr{P}}$

We introduce a variation of the category RT (definition 3.1.2) in order to define the suitable category on which $\mathbb{R}\Gamma_{rig,c}(X)_{\overline{X},\mathscr{P}}$ will be a functor.

rictdiagram **Definition 4.1.1.** In the same situation of definition 3.1.3, we say that a morphism F compatible with f is strict if there is a commutative diagram

$$
\begin{array}{ccccc}
]X[_{\mathscr{P}} & \longrightarrow & U & \longleftarrow & U\backslash]X[_{\mathscr{P}} \\
\downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\
]Y[_{\mathscr{Q}} & \longrightarrow & V & \longleftarrow & V\backslash]Y[_{\mathscr{Q}}
\end{array}
$$

where V is a strict neighborhood of $]Y[_{\mathscr{Q}}$ in $]\overline{Y}[_{\mathscr{Q}}$

We defined $\mathcal{RT}^s$ as the category in which objects are rigid triples and morphisms are couples $(f, F) \in \mathrm{Mor}(RT)$ with F germ of strict morphism compatible with f.

Remark 4. We must check that the composition in $\mathcal{RT}^s$ is well defined. Let $(f, F) : (X, \overline{X}, \mathscr{P}) \to (Y, \overline{Y}, \mathscr{Q})$ and $(g, G) : (Y, \overline{Y}, \mathscr{Q}) \to (Z, \overline{Z}, \mathscr{R})$ be morphisms in $\mathcal{RT}^s$. We will call $F : U \to V$ and $G : V' \to W$ the representatives of the respective germs too ($U \supset]X[_{\mathscr{P}}, W \supset]Z[_{\mathscr{R}}$ and $V, V' \supset]Y[_{\mathscr{Q}}$ are strict neighborhoods). By definition, the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle sp}\uparrow & & \uparrow{\scriptstyle sp} \\
]X[_{\mathscr{P}} & \longrightarrow &]Y[_{\mathscr{Q}} \\
\downarrow & & \downarrow \\
U & \xrightarrow{F} & V \\
\uparrow & & \uparrow \\
U\backslash]X[_{\mathscr{P}} & \longrightarrow & V\backslash]Y[_{\mathscr{Q}}
\end{array}
$$

and

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
{\scriptstyle sp}\uparrow & & \uparrow{\scriptstyle sp} \\
]Y[_{\mathscr{Q}} & \longrightarrow &]Z[_{\mathscr{R}} \\
\downarrow & & \downarrow \\
V' & \xrightarrow{G} & W \\
\uparrow & & \uparrow \\
V'\backslash]Y[_{\mathscr{Q}} & \longrightarrow & W\backslash]Z[_{\mathscr{R}}
\end{array}
$$

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commute. If we set $V'' = V \cap V'$, $\widetilde{F} = F_{|F^{-1}(V'')}$ and $\widetilde{G} = G_{|V''}$, the diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & W \\
{\scriptstyle sp}\uparrow & & {\scriptstyle sp}\uparrow & & {\scriptstyle sp}\uparrow \\
]X[_P & \longrightarrow &]Y[_{\mathscr{Q}} & \longrightarrow &]Z[_{\mathscr{R}} \\
\downarrow & & \downarrow & & \downarrow \\
F^{-1}(V'') & \xrightarrow{\ \widetilde{F}\ } & V'' & \xrightarrow{\ \widetilde{G}\ } & W \\
\uparrow & & \uparrow & & \uparrow \\
F^{-1}(V'')\backslash]X[_{\mathscr{P}} & \longrightarrow & V''\backslash]Y[_{\mathscr{Q}} & \longrightarrow & W\backslash]Z[_{\mathscr{R}}
\end{array}
$$

commutes (we only restrict to smaller neighborhoods, so the complement of the tube always maps to the complement of the tube). Therefore, if we define H as the germ of $\widetilde{G} \circ \widetilde{F}$ and $h = g \circ f$, the morphism $(f, F) \circ (g, G) := (h, H)$ is in RT^s.

$\boxed{\text{rcc}}$ **Definition 4.1.2.** Let $(X, \overline{X}, \mathscr{P}) \in RT^s$, we define the rigid complex with compact support associated to this triple as

$$
\mathbb{R}\Gamma_{rig,c}(X)_{\overline{X},\mathscr{P}} = \varinjlim_U \Gamma(U, G_{Pt(U)}\Gamma_{]X[}G_{Pt(U)}\Omega_U^\bullet)
$$

where the direct limit is over the strict neighborhoods of the tube $]X[_{\mathscr{P}}$ (with respect to inclusion).

Remark 5. To show that the direct limit of definition 4.1.2 is well defined, we can procede as in remark 2, applyng the functor $\Gamma_{]X[}$ instead of $j^\dagger$ and using proposition 5.2.15 of [LS07].

$\boxed{\text{RCC}}$ **Proposition 10.** *The assignment*

$$
(X, \overline{X}, \mathscr{P}) \mapsto \mathbb{R}\Gamma_{rig,c}(X)_{\overline{X},\mathscr{P}}
$$

defines a functor from RT^s to Compl_K. Furhermore the cohomology of the complex $\mathbb{R}\Gamma_{rig,c}(X)_{\overline{X},\mathscr{P}}$ is the rigid cohomology with compact support of X.

Proof. Let $(f, F) : (X, \overline{X}, P) \to (Y, \overline{Y}, Q)$ be a morphism in RT^s, and also call $F : U \to Q$ a representative of the germ F such that for some $V \supset]Y[_P$ the map $F : U \to V$ makes the diagram of definition 4.1.1 commute.

Let $V' \supset]Y[_{\mathscr{Q}}$ be a strict neighborhood, we must find a strict neighborhood $U' \supset]X[_{\mathscr{P}}$ and a map

$$
\Gamma(V', G_{Pt(V')}\Gamma_{]Y[}G_{Pt(V')}\Omega_{V'}^\bullet) \to \Gamma(U', G_{Pt(U')}\Gamma_{]X[}G_{Pt(U')}\Omega_{U'}^\bullet).
$$

For this purpose, by proposition 1 it is sufficient to find $U' \supset]X[_P$, $\widetilde{F} : U' \to V'$ and an arrow

$$
\Gamma_{]Y[}G_{Pt(V')}\Omega_{V'}^\bullet \to \widetilde{F}_*\Gamma_{]X[}G_{Pt(U')}\Omega_{U'}^\bullet. \tag{4.1} \boxed{\text{goal}}
$$

By definition of direct limit, nothing is lost assuming that $V' \subset V$, so we can define U' as $F^{-1}(V')$ and

$$
\widetilde{F} := F_{|U'} : U' \to V'.
$$

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The map $\widetilde{F}$ is strict (because is the restriction of a strict map), so by proposition 5.2.17 of [LS07] there exists a morphism

$$\widetilde{F}^{-1}\Gamma_{]Y[}G_{V'}\Omega^{\bullet}_{V'} \to \Gamma_{]X[}\widetilde{F}^{-1}G_{V'}\Omega^{\bullet}_{V'}. \tag{4.2}$$

By adjunction, the morphism 4.1 is equivalent to the composition of 4.2 with a map

$$\Gamma_{]X[}\widetilde{F}^{-1}G_{V'}\Omega^{\bullet}_{V'} \to \Gamma_{]X[}G_{Pt(U')}\Omega^{\bullet}_{U'}.$$

This follows by applying the functor $\Gamma_{]X[}$ to a morphism

$$\widetilde{F}^{-1}G_{V'}\Omega^{\bullet}_{V'} \to G_{Pt(U')}\Omega^{\bullet}_{U'}$$
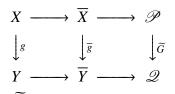
that can be easily obtained using the result of proposition 1.

All maps in the direct limit are quasi-isomorphisms, for proposition 6.4.1 of [LS07], hence we can apply lemma 2 to conclude that $\mathbb{R}\Gamma_{\mathrm{rig,c}}(X)_{\overline{X},\mathscr{P}}$ indeed computes the rigid cohomology with compact support.

$\square$

4.1.3 The complex $\mathbb{R}\Gamma_{\mathbf{rig,c}}(X)$

Lemma 3. *Let* $(X, \overline{X}, \mathscr{P})$, $(Y, \overline{Y}, \mathscr{Q})$ *be rigid triple, and let U be a strict neighborhood of* $]X[_{\mathscr{P}}$ *in* $]\overline{X}[_{\mathscr{P}}$ *and V a strict neighborhood of* $]Y[_{\mathscr{Q}}$ *in* $]\overline{Y}[_{\mathscr{Q}}$. *Suppose there is a commutative diagram*

$$
\begin{array}{ccccc}
X & \longrightarrow & \overline{X} & \longrightarrow & \mathscr{P} \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \overline{g}} & & \downarrow{\scriptstyle \widetilde{G}} \\
Y & \longrightarrow & \overline{Y} & \longrightarrow & \mathscr{Q}
\end{array}
$$

with g proper, and suppose $G := (\widetilde{G}_K)_{|U} : U \to V$. *Then G is compatible with g, and G is strict. Therefore* (g, G) *is a morphism in* $\mathcal{RT}^s$ *(here by G we mean the corresponding germ of morphism).*

Proof. The fact that G is compatible with g immediately follows from the commutativity of the previous diagram.

If $x \in V \backslash]X[_{\mathscr{P}}$, that is $\mathrm{sp}(x) \in \overline{X} \setminus X$, then $\overline{g}(\mathrm{sp}(x)) = \mathrm{sp}(G(x))$ belongs to $\overline{Y} \setminus Y$, because g proper implies $\overline{g}$ strict (lemma 15.2.3 of [Hub95]). Equivalently $G(x) \in V \backslash]Y[_{\mathscr{Q}}$. $\square$

To simplify the notation, we define

$$\mathcal{F}^c_X(A) := \mathbb{R}\Gamma_{\mathrm{rig,c}}(X)_{\overline{X}_A, \mathscr{P}_A}.$$

If $A \subset B$, the natural projections $\overline{X}_B \to \overline{X}_A$ and $\mathscr{P}_B \to \mathscr{P}_A$ induce a map of rigid data $(\overline{X}_B, \mathscr{P}_B) \to (\overline{X}_A, \mathscr{P}_A)$, that is strict by lemma 3 (in this case the proper map g is the identity over X). The induced morphism

$$\mathcal{F}^c_X(A) \to \mathcal{F}^c_X(B)$$

is a quasi-isomorphism because of Berthelot's results concerning the independence of rigid cohomology with compact support from the auxiliary choices. Hence $\mathcal{F}^c_X(A)$ is a covariant functor $\mathcal{SET}^0_X \to \mathsf{Compl}_K$, and all morphisms of complexes one obtains are quasi-isomorphisms.

Now we can define

$$\mathbb{R}\Gamma_{\mathrm{rig,c}}(X) := \varinjlim_{A \in \mathcal{SET}^0_X} \mathcal{F}^c_X(A).$$

Proposition 11. *The assignment $X \mapsto \mathbb{R}\Gamma_{rig,c}(X)$ defines a contravariant functor $\mathsf{Sch}_k^p \to$ Compl_K, where Sch_k^p is the category of schemes over k with proper morphisms. In particular, by functoriality, there exists a canonical σ-linear endomorphism of $\mathbb{R}\Gamma_{rig,c}(X/K_0)$ induced by the Frobenius morphism on X.*

Proof. Let $g : Z \to X$ a proper morphism of k-schemes. The idea is to follow Besser's proof of proposition 4.14 [Bes00]. The existence and uniqueness of a map $g^* : \mathbb{R}\Gamma_{\mathrm{rig},c}(X) \to \mathbb{R}\Gamma_{\mathrm{rig},c}(Z)$ follows immediately if we show that there is a map $g_\circ^*$ making the following diagram commute

$$
\begin{array}{ccc}
\varinjlim_{(A,B)} \mathcal{F}_X^c(B) & \xrightarrow{\ g_\circ^*\ } & \varinjlim_{(A,B)} \mathcal{F}_Z^c(A \cup g^\circ(B)) \\
{\scriptstyle (A,B) \mapsto B} \downarrow & & {\scriptstyle (A,B) \mapsto A \cup g^\circ(B)} \downarrow \\
\varinjlim_B \mathcal{F}_X^c(B) & \xrightarrow{\ g^*\ } & \varinjlim_A \mathcal{F}_Z^c(A)
\end{array}
$$

(where one takes the limits over $A \in \mathcal{SET}_Z^0$ and $B \in \mathcal{SET}_X^0$) and that the left vertical map is an isomorphism (not just a quasi-isomorphism). We need this construction because, given $B \in \mathcal{SET}_X^0$, then $g^\circ(B) \in \mathcal{SET}_Z \setminus \mathcal{SET}_Z^0$ so we can't directly define g^*.

To define $g_\circ^*$ we must exhibit, for every A, B, a map $(Z, \overline{Z}_{A \cup g^\circ(B)}, \mathscr{P}_{A \cup g^\circ(B)}) \to (X, \overline{X}_B, \mathscr{P}_B)$ in the category for which $\mathbb{R}\Gamma_{\mathrm{rig},c}(-)_{-,-}$ is a functor, i.e. a morphism in $\mathcal{RT}^s$, in other words, a couple (g, G) where g is a morphism of k-schemes and G is a germ of strict compatible morphism $V \to (\mathscr{P}_A)_K$. Consider the commutative diagram

$$
\begin{array}{ccc}
Z & \longrightarrow & \overline{Z}_{A \cup g^\circ(B)} & \longrightarrow & \mathscr{P}_{A \cup g^\circ(B)} \\
g \downarrow & & \overline{g} \downarrow & & \widetilde{G} \downarrow \\
X & \longrightarrow & \overline{X}_B & \longrightarrow & \mathscr{P}_B
\end{array}
$$

where the last two vertical maps are the projections. Because of the properness of the morphism g, we can conclude by using lemma 3. To show that for any two composable morphisms $W \xrightarrow{h} X \xrightarrow{g} Y$ we have $(gh)^* = h^* \circ g^*$ see Besser's proof of 4.14 of [Bes00]. $\qquad \square$

Remark about properness. What is the importance of properness for the functoriality of $\mathbb{R}\Gamma_{\mathrm{rig},c}(X)$? The functor $\mathbb{R}\Gamma_{\mathrm{rig},c}(X)$ is defined on the category of schemes over k, but by mean of $\mathbb{R}\Gamma_{\mathrm{rig},c}(X)_{\overline{X}, \mathscr{P}}$, that is a functor defined for strict morphisms. We can't directly ask anything over F (characteristic zero), because it doesn't make sense to speak about F (characteristic zero) when we deal with morphisms in Sch_k. Anyway, we need some requirements on f that force a "potential" F to be strict. This requirement is properness.

4.1.4 The complex $\widetilde{\mathbb{R}\Gamma}_{\mathsf{rig},c}(X)_{\overline{X}, \mathscr{P}}$

In the notations of subsection 3.1.4, once we have fixed a rigid triple $(X, \overline{X}, \mathscr{P})$, we can define a functor

$$
\widetilde{\mathcal{F}}_X^c : \mathcal{SET}_{(X,\overline{X},\mathscr{P})}^0 \to \mathsf{Compl}_K, \quad A' \mapsto \mathbb{R}\Gamma_{rig,c}(X)_{\overline{X}_{A'}, \mathscr{P}_{A'}}.
$$

We can now set

$$
\widetilde{\mathbb{R}\Gamma}_{\mathsf{rig},c}(X)_{\overline{X}, \mathscr{P}} := \varinjlim_{A' \in \mathcal{SET}_{X,\overline{X},\mathscr{P}}^0} \widetilde{\mathcal{F}}_X^c(A').
$$

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This is a contravariant functor for proper morphisms of rigid triples, i.e. for (f, F) such that f is proper (because we forget, the construction is as $\mathbb{R}\Gamma_{\mathrm{rig,c}}(X)$ (see proposition 11)):

Proposition 12. *The assignment $(X, \overline{X}, \mathscr{P}) \mapsto \widetilde{\mathbb{R}\Gamma}_{\mathrm{rig,c}}(X)_{\overline{X}, \mathscr{P}}$ defines a contravariant functor $\mathcal{RT}^p \to \mathrm{Compl}_K$, where $\mathcal{RT}^p$ is the category of rigid triples with morphisms of rigid triples (f, F) with f proper.*

Proof. Let (g, G) be a morphism in $\mathcal{RT}^p$. Here we consider $(g, G)^\circ : \mathcal{PT}_{X, \overline{X}, \mathscr{P}} \to \mathcal{PT}_{Z, \overline{Z}, \mathscr{Q}}$ as the composition with (g, G). The idea of the proof is the same of proposition 11, the only difference here is the definition of the functor $(g, G)^\circ$, but essentially we don't care about the germ G, we forget it.

The existence and uniqueness of a map $(g, G)^* : \widetilde{\mathbb{R}\Gamma}_{\mathrm{rig,c}}(X)_{\overline{X}, \mathscr{P}} \to \widetilde{\mathbb{R}\Gamma}_{\mathrm{rig,c}}(Z)_{\overline{Z}, \mathscr{Q}}$ follows immediately if we show that there is a map $(g, G)^*_\circ$ making the following diagram commute

$$
\begin{array}{ccc}
\varinjlim_{(A', B')} \widetilde{\mathcal{F}}^c_X(B') & \xrightarrow{(g,G)^*_\circ} & \varinjlim_{(A', B')} \widetilde{\mathcal{F}}^c_Z(A' \cup (g, G)^\circ(B')) \\
{\scriptstyle (A', B') \mapsto B'} \downarrow & & {\scriptstyle (A', B') \mapsto A' \cup \overline{g}^\circ(B')} \downarrow \\
\varinjlim'_B \widetilde{\mathcal{F}}^c_X(B') & \xrightarrow{(g,G)^*} & \varinjlim'_A \widetilde{\mathcal{F}}^c_Z(A')
\end{array}
$$

(where one takes the limits over $A' \in \mathcal{SET}^0_{Z, \overline{Z}, \mathscr{P}}$ and $B' \in \mathcal{SET}^0_{X, \overline{X}, \mathscr{P}}$), and that the left vertical map is an isomorphism (not just a quasi-isomorphism). We need this construction because, given $B' \in \mathcal{SET}^0_{X, \overline{X}, \mathscr{P}}$, then $(g, G)^\circ(B') \in \mathcal{SET}_{Z, \overline{Z}, \mathscr{Q}} \setminus \mathcal{SET}^0_{Z, \overline{Z}, \mathscr{Q}}$ so we can't directly define $(g, G)^*$.

To define $(g, G)^*_\circ$ we must exibit, for every A', B', a map

$$(Z, \overline{Z}_{A' \cup (g,G)^\circ(B')}, \mathscr{P}_{A' \cup (g,G)^\circ(B')}) \to (X, \overline{X}_{B'}, \mathscr{P}_{B'})$$

in the category for which $\mathbb{R}\Gamma_{rig,c}(-)_{-,-}$ is a functor, i.e. a morphism in $\mathcal{RT}^s$, in other words, a couple (f, F) where f is a morphism of k-schemes and F is a germ of strict compatible morphism $V \to (\mathscr{P}_A)_K$. Consider the commutative diagram

$$
\begin{array}{ccc}
Z & \longrightarrow \overline{Z}_{A' \cup (g,G)^\circ(B')} & \longrightarrow \mathscr{P}_{A' \cup (g,G)^\circ(B')} \\
{\scriptstyle f=g} \downarrow & {\scriptstyle \overline{f}} \downarrow & {\scriptstyle \widetilde{F}} \downarrow \\
X & \longrightarrow \overline{X}_{B'} & \longrightarrow \mathscr{P}_{B'}
\end{array}
$$

where the last two vertical maps are the projections. Because of the properness of the morphism g, we can conclude by using lemma 3. About the compatibility of this construction with the composition see again Besser's proof of 4.14 of [Bes00]. $\qquad \square$

We have defined the functor $\widetilde{\mathbb{R}\Gamma}_{\mathrm{rig,c}}(X)_{\overline{X}, \mathscr{P}}$ over the category $\mathcal{RT}^p$, but we must construct a natural transformation to the functor $\mathbb{R}\Gamma_{\mathrm{rig,c}}(X)$. For this purpose we should think about the functor $\widetilde{\mathbb{R}\Gamma}_{\mathrm{rig,c}}(X)_{\overline{X}, \mathscr{P}}$ as restricted, at the level of morphisms, to the morphisms (f, F) where F is strict.

Anyway, we can also define the functor $\widetilde{\mathbb{R}\Gamma}_{\mathrm{rig,c}}(X)_{\overline{X}, \mathscr{P}}$ in a more intrinsic way with respect to strictness, as follows.

Fix a rigid triple $(X, \overline{X}, \mathscr{P})$. Consider then the set $\mathcal{PT}^s_{X,\overline{X},\mathscr{P}}$ of all strict morphisms of rigid triples from $(X, \overline{X}, \mathscr{P})$ to another rigid triple, and the subset $\mathcal{PT}^{0,s}_{X,\overline{X},\mathscr{P}}$ consisting of the identity morphism. As before, we can define the category $\mathcal{SET}^{0,s}_{X,\overline{X},\mathscr{P}}$. Now define

$$\widetilde{\mathcal{F}}^{c,s}_X : \mathcal{SET}^{0,s}_{X,\overline{X},\mathscr{P}} \to \mathrm{Compl}_K, \quad A' \mapsto \mathbb{R}\Gamma_{\mathsf{rig},c}(X)_{\overline{X}_{A'},\mathscr{P}_{A'}} \tag{4.3}$$

and

$$\widetilde{\mathbb{R}\Gamma}_{rig,c}(X)_{\overline{X},\mathscr{P}} := \varinjlim_{A' \in \mathcal{SET}^{0,s}_{X,\overline{X},\mathscr{P}}} \widetilde{\mathcal{F}}^{c,s}_X(A'). \tag{4.4}$$

To prove functoriality we can do as in the previous proposition, the only thing we remark is that we can apply the functor $\widetilde{\mathcal{F}}^{c,s}_Z$ to $(A' \cup (g,G)^\circ(B'))$ because the composition of strict morphisms is strict.

The advantage of definition 4.4 is that there will be no need to restrict the set of morphisms when we deal with natural transformation in section 4.1.5.

4.1.5 Natural transformations between rigid complexes with compact support

The first "piece" of the syntomic diagram with compact support is

$$\mathbb{R}\Gamma_{\mathsf{rig},c}(X) \xleftarrow{p} \widetilde{\mathbb{R}\Gamma}_{\mathsf{rig},c}(X)_{\overline{X},\mathscr{P}} \xrightarrow{q} \mathbb{R}\Gamma_{\mathsf{rig},c}(X)_{\overline{X},\mathscr{P}}.$$

We will construct the natural transformations, and show that they induce quasi-isomorphisms.

For $A' \in \mathcal{SET}^0_{X,\overline{X},\mathscr{P}}$ the canonical projection $\Pi : A' \to \Pi(A')$ induces a canonical map, diagonals on identical elements, $\Delta_{A'} : (\overline{X}_{\Pi(A')}, \mathscr{P}_{\Pi(A')}) \to (\overline{X}_{A'}, \mathscr{P}_{A'})$. This map is strict for lemma 3, hence it induces a quasi-isomorphism

$$\widetilde{\mathcal{F}}^{c,s}_X(A') = \mathbb{R}\Gamma_{\mathsf{rig},c}(X)_{\overline{X}_{A'},\mathscr{P}_{A'}} \xrightarrow{\mathbb{R}\Gamma_{\mathsf{rig},c}(\Delta_{A'})} \mathbb{R}\Gamma_{\mathsf{rig},c}(X)_{\overline{X}_{\Pi(A')},\mathscr{P}_{\Pi(A')}} = \mathcal{F}^{c,s}_X(\Pi(A')).$$

Going to the limit we obtain a map, which is again a quasi-isomorphism as in both limits all maps are

$$\Delta : \varinjlim_{A'} \widetilde{\mathcal{F}}^{c,s}_X(A') \to \varinjlim_{A'} \mathcal{F}^{c,s}_X(\Pi(A')).$$

By the composition with the map

$$\varinjlim_{A'} \mathcal{F}^{c,s}_X(\Pi(A')) \to \varinjlim_{A} \mathcal{F}^{c,s}_X(A)$$

we obtain a quasi-isomorphism $p : \widetilde{\mathbb{R}\Gamma}_{\mathsf{rig},c}(X)_{\overline{X},\mathscr{P}} \to \mathbb{R}\Gamma_{\mathsf{rig},c}(X)$. It is easy to verify that p is a natural transformation between functors, i.e. that for every morphism $(X, \overline{X}, \mathscr{P}) \to (Y, \overline{Y}, \mathscr{Q})$ in $\mathcal{RT}^{p,s}$ the diagram

$$
\begin{array}{ccc}
\widetilde{\mathbb{R}\Gamma}_{\mathsf{rig},c}(Y) & \xrightarrow{\quad p(Y) \quad} & \mathbb{R}\Gamma_{\mathsf{rig},c}(Y)_{\overline{Y},\mathscr{Q}} \\
\downarrow & & \downarrow \\
\widetilde{\mathbb{R}\Gamma}_{\mathsf{rig},c}(X) & \xrightarrow{\quad p(X) \quad} & \mathbb{R}\Gamma_{\mathsf{rig},c}(X)_{\overline{X},\mathscr{P}}
\end{array}
$$

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commutes. We remark that also in this case we must change the class of morphisms considered for the functoriality of $\mathbb{R}\Gamma_{\mathrm{rig,c}}(X)$: this functor must be considered over the category C whose object are schemes over k and $\mathrm{Hom}_C(X,Y) = \mathrm{Hom}_{\mathcal{RT}^{p,s}}((X,\overline{X},\mathscr{P}),(Y,\overline{Y},\mathscr{Q}))$. Obviously the action of the functor $\mathbb{R}\Gamma_{\mathrm{rig,c}}(X)$ on a morphism $(f,F):(X,\overline{X},\mathscr{P}) \to (Y,\overline{Y},\mathscr{Q})$ in C only depends on the proper map f.

Now we must construct the map q. For the universal property of the direct limit, it is enough to have, for every $A' \in \mathcal{SET}_{X,\overline{X},\mathscr{P}}^{0,s}$ a map

$$\widetilde{\mathcal{F}}_X^{c,s}(A') = \mathbb{R}\Gamma_{\mathrm{rig,c}}(X)_{\overline{X}_{A'},\mathscr{P}_{A'}} \to \mathbb{R}\Gamma_{\mathrm{rig,c}}(X)_{\overline{X},\mathscr{P}}.$$

This means that we must find a map $(X,\overline{X},\mathscr{P}) \to (X,\overline{X}_{A'},\mathscr{P}_{A'})$ in the category where the functor $\mathbb{R}\Gamma_{rig,c}(-)_{-,-}$ is defined, i.e. in $\mathcal{RT}^s$ (see proposition 10). This map is the projection (identity is proper, Lemma 3). Naturality is clearly verified.

4.2 The de Rham cohomology with compact support and the complex $\mathbb{R}\Gamma_{\mathrm{dR},c}(X/K)_Y$

In the notations of 3.2, consider a smooth K-scheme X and a de Rham datum (Y) for X, and let $I \subset O_Y$ be the defining sheaf of ideals of the related normal crossing divisor D.

Definition 4.2.1. Consider a smooth K-scheme X. To a de Rham datum (Y) and to every $k \in \mathbb{Z}_{\geq 0}$ we associate a complex, called k-th filtered part of the de Rham complex with compact support of X with respect to the datum (Y), defined by

$$\mathrm{F}^k\mathbb{R}\Gamma_{\mathrm{dR},c}(X/K)_Y = \Gamma(Y, G_{zar}^2 I\Omega_{Y/K}^{\geq k}\langle D\rangle).$$

We will write $\mathbb{R}\Gamma_{\mathrm{dR},c}(X/K)_Y$ for $\mathrm{F}^0\mathbb{R}\Gamma_{\mathrm{dR},c}(X/K)_Y$ and we will call it the de Rham complex with compact support of X with respect to the datum (Y).

Definition 4.2.2. Consider a smooth K-scheme X. In the notations of section 3.2, the i-th de Rham cohomology with compact support of X is

$$H_{\mathrm{dR},c}^i(X) := H^i(Y, I\Omega_Y^\bullet\langle D\rangle).$$

These are filtered K-vector spaces and the *Hodge filtration* is defined as

$$F^j H_{\mathrm{dR},c}^i(X) := H^i(Y, I\Omega_Y^{\geq j}\langle D\rangle).$$

Remark 4.2.3. The functoriality of $H_{\mathrm{dR},c}^i(X)$ is justified as in remark 3.2.3 if we consider only proper maps using [Hub95, Lemma 15.2.3]. For the compatibility with the definition of de Rham cohomology with compact support defined in [PS08] see [PS08, Part II, Example 7.25].

`dRcstrict` *Proposition* 4.2.4. The filtered complex $\mathbb{R}\Gamma_{\mathrm{dR},c}(X/K)_Y$ is strict.

Proof. We can procede as in the proof of proposition 3.2.4 simply by using the fact that the spectral sequence associated to the complex of sheaves $I\Omega_Y^\bullet\langle D\rangle$ with respect to the trivial filtration degenerates at E_1 by [PS08, Part II, Example 7.25]. $\qquad\square$

4.3 Cospecialization

We refer to the setting of section 3.4.

The cospecialization map is the following morphism

$$G_{Pt(\mathscr{Y}_K^{an})}\Gamma_{]\mathscr{X}_k[}G_{Pt(\mathscr{Y}_K^{an})}I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{an} \xrightarrow{cosp} G^2_{Pt(\mathscr{Y}_K^{an})}I\Omega^\bullet_{\mathscr{Y}}\langle D\rangle^{an}$$

that is obtained by applying the Godement resolution to the canonical map

$$\Gamma_{]\mathscr{X}_k[}G_{Pt(\mathscr{Y}_K^{an})}I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{an} \to G_{Pt(\mathscr{Y}_K^{an})}I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{an}.$$

As in section 3.4, the global sections of the complexes of sheaves involved in the cospecialization morphisms do compute the "expected" cohomologies:

- The cohomology of the complex of vector spaces

$$G_{Pt(\mathscr{Y}_K^{an})}\Gamma_{]\mathscr{X}_k[}G_{Pt(\mathscr{Y}_K^{an})}I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{an}$$

  is the rigid cohomology with compact support defined in [Berth97] because of an excision argument and proposition 13 below.

Lemma 4. *If $f : A^\bullet \to B^\bullet$ is a surjective morphism of complexes, then there exists a quasi isomorphism* $\ker f \widetilde{\to} Cone(f)$.

Proof. Let $K^\bullet = \ker f$ and $C^\bullet = Cone(f)$. The morphism

$$\alpha^i : K[1]^i = K^{i+1} \to C^i = A^{i+1} \oplus B^i, \quad a \mapsto (a, 0)$$

induces an injective morphism of complexes

$$\alpha : K^\bullet[1] \to C^\bullet.$$

Let $Z^\bullet = \operatorname{coker}\alpha$. The map f is surjective, hence

$$Z^i = (A^{i+1}/K^{i+1}) \oplus B^i \simeq B^{i+1} \oplus B^i$$

and the differential is defined as $d(b^{i+1}, b^i) = (-d(b^{i+1}), b^{i+1} + d(b^i))$. In the homotopy category $id_Z = 0$, by mean of the chain homotopy

$$l^i : B^{i+2} \oplus B^{i+1} \to B^{i+1} \oplus B^i \quad (b^{i+2}, b^{i+1}) \mapsto (b^{i+1}, 0),$$

because $dl - ld = id_Z$. Therefore $Z^\bullet = 0$ in the derived category.

$\square$

Proposition 13. *If $\mathcal{F}^\bullet$ is a complex of flasque sheaves over $\mathscr{Y}_K^{an}$, then $\mathbb{R}\Gamma_{]\mathscr{X}_k[}\mathcal{F}^\bullet \overset{q.i.}{\simeq} \Gamma_{]\mathscr{X}_k[}\mathcal{F}^\bullet$.*

Proof. Let $i : \mathscr{Y}_K^{an}\setminus]\mathscr{X}_k[\longrightarrow \mathscr{Y}_K^{an}$ be the open inclusion. We must construct a quasi-isomorphism

$$\ker(\mathcal{F}^\bullet \xrightarrow{f} i_*i^*\mathcal{F}^\bullet) \overset{q.i.}{\simeq} Cone(\mathcal{F}^\bullet \xrightarrow{f} i_*i^*\mathcal{F}^\bullet).$$

It is easy to prove that f is surjective (because by hypothesis it is surjective on each open set), hence the result follows from lemma 4.

$\square$

- The global sections of $G^2_{Pt(\mathscr{Y}_K^{an})}I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{an}$ compute the de Rham cohomology with compact support because of GAGA results.

4.4 Other maps

Also in the case of compact support we need some more complexes and morphisms to define the whole diagram, as we did in section 3.5.

4.4.1 Link with Besser complexes with compact support via cospecialization

We must link on the left of to the rigid part of the diagram described in section 4.1. In 10 we have defined

$$\mathbb{R}\Gamma_{rig,c}(\mathscr{X}_k)_{\mathscr{Y}_k,\hat{\mathscr{Y}}} := \varinjlim_U \Gamma(U, G_{Pt(U)}\Gamma_{]\mathscr{X}_k[}G_{Pt(U)}\Omega_U^\bullet)$$

where the limit is over the strict neighborhoods of the tube. We know that X_K^{an} is one of such strict neighborhoods, hence the construction of the map reduces to the following proposition.

Proposition 14. *With the above notations, there exists a map*

$$\Gamma(\mathscr{Y}_K^{an}, G_{an}\Gamma_{]\mathscr{X}_k[}G_{an}w^*I\Omega_{\mathscr{Y}_K}^\bullet\langle D\rangle) \to \Gamma(\mathscr{X}_K^{an}, G_{an}\Gamma_{]\mathscr{X}_k[}G_{an}\Omega_{\mathscr{X}_K^{an}}^\bullet) \to R\Gamma_{rig,c}(\mathscr{X}_k/K)_{\mathscr{Y}_k,\hat{\mathscr{Y}}} .$$

We will call it b.

Proof. As in the proof of proposition 5, considering the functor $\Gamma_{]\mathscr{X}_k[}$ instead of $j^\dagger$ and applying proposition 5.2.15 of [LS07].

□

4.4.2 Link with the algebraic setting

The construction that links the analytic part of the diagram with the algebraic de Rham setting is completely analogous to the construction of 3.5.2: in the proof of proposition 6 we consider the functorial complex $\mathcal{F} = I\Omega_Y^\bullet\langle D\rangle$.

4.5 The syntomic diagram with compact support

Similarly to 3.6, we can consider the p-adic Hodge complex $R\Gamma_c(\mathscr{X})$ associated to the diagram $R\Gamma_c'(\mathscr{X})$ defined as follows

$$
\begin{array}{ccccccc}
R\Gamma_{\text{rig},c}(\mathscr{X}_k/K) & & R\Gamma_{\text{rig},c}(\mathscr{X}_k/K)_{\mathscr{Y}_k,\hat{\mathscr{Y}}} & & \Gamma(\mathscr{Y}_K^{an}, G_{an}^2 u^* I\Omega_{\mathscr{Y}_K}^\bullet\langle D\rangle) & & \Gamma(\mathscr{Y}_K, G_{zar}^2 I\Omega_{\mathscr{Y}_K}^\bullet\langle D\rangle) \\
{\scriptstyle\beta_1}\nearrow & \nwarrow{\scriptstyle\beta_2} & {\scriptstyle\beta_3}\nearrow & \nwarrow{\scriptstyle\beta_4} & {\scriptstyle\beta_5}\nearrow & \nwarrow{\scriptstyle\beta_6} & {\scriptstyle\beta_7}\nearrow \nwarrow{\scriptstyle\beta_8} \\
R\Gamma_{\text{rig},c}(\mathscr{X}_k/K_0) & \widetilde{R\Gamma}_{\text{rig},c}(\mathscr{X}_k/K)_{\mathscr{Y}_k,\hat{\mathscr{Y}}} & \Gamma(\mathscr{Y}_K^{an}, G_{an}\Gamma_{]\mathscr{X}_k[}G_{an}u^* I\Omega_{\mathscr{Y}_K}^\bullet\langle D\rangle) & \Gamma(\mathscr{Y}_K, G_{an+zar}^2 I\Omega_{\mathscr{Y}_K}^\bullet\langle D\rangle) & \Gamma(\mathscr{Y}_K, G_{zar}^2 I\Omega_{\mathscr{Y}_K}^\bullet\langle D\rangle)
\end{array}
$$

where β_1, β_8 are the identity maps; β_2, β_3 are the maps of 4.1.5; β_4 is the map b of proposition 14; $\beta_5 = cosp$ of 4.3; β_6, β_7 are defined in 4.4.2. Notice that all β_i but β_5 are quasi-isomorphisms. By applying repeatedly the quasi pull-back construction we obtain a diagram of the following shape

$$R\Gamma_{\text{rig},c}(\mathscr{X}/K_0) \xrightarrow{c} R\Gamma_{K,c}(\mathscr{X}) \xleftarrow{\sim} R\Gamma_{\text{dR},c}(\mathscr{X}) \qquad (4.5) \quad \boxed{\text{RGAMMAC}}$$

as explained in remark 1.

Definition 4.5.1. We define the syntomic diagram with compact support of $\mathscr{X}$ as the object of *pHD* corresponding to the diagram 4.5. We will denote it by $R\Gamma_c(\mathscr{X})$.

Notice that the previous diagram is indeed a *p*-adic Hodge complex whose cohomology in the specialization is the expected one by proposition 4.2.4.

Proposition 15. *Let $Sm_c/\mathcal{V}$ be the category of smooth $\mathcal{V}$-schemes with proper morphisms. The previous construction induces a contravariant functor*

$$R\Gamma_c(-) : Sm_c/\mathcal{V} \to pHD.$$

Proof. We can argue as in the proof of proposition 7 to obtain two generic normal crossing compactifications $g : \mathscr{X} \to \mathscr{Y}$, $g' : \mathscr{X}' \to \mathscr{Y}'$ and an extension of the map f. If we further assume f to be proper, then the square

$$
\begin{array}{ccc}
\mathscr{X} & \xrightarrow{g} & \mathscr{Y} \\
\downarrow{\scriptstyle f} & & \downarrow \\
\mathscr{X}' & \xrightarrow{g'} & \mathscr{Y}'
\end{array}
$$

is also cartesian by lemma 15.2.3 of [Hub95]. From this fact, the result of subsection 4.4.2 and the definition of the rigid part of the syntomic diagram with compact support (section 4.1), we can conclude. $\qquad\square$

Definition 4.5.2. Let $\mathscr{X}$ be a smooth algebraic scheme over $\mathcal{V}$. For any n, i integers we define the (rigid) syntomic cohomology with compact support groups of $\mathscr{X}$

$$H^n_{\mathrm{syn,c}}(\mathscr{X}, i) := Hom_{pHD}(\mathbb{K}, R\Gamma_c(\mathscr{X})(i)[n]) = H^n(\Gamma(\mathbb{K}, R\Gamma_c(\mathscr{X})(i)))$$

(last equality holds by proposition 1.2.32).

Definition 4.5.3. Let $\mathscr{X}$ be a smooth scheme over $\mathcal{V}$. For any n, i integers we define the (rigid) syntomic homology groups of X

$$H_n^{\mathrm{syn}}(\mathscr{X}, i) := Hom_{pHD}(R\Gamma_c(\mathscr{X}), \mathbb{K}(-i)[-n]) = H^n(\Gamma(R\Gamma_c(\mathscr{X}), \mathbb{K}(-i)))$$

(last equality holds by proposition 1.2.32).

Chapter 5

The syntomic pairing and the Gysin map

We are going to work in the category of diagrams as we developed in chapter 1. We have constructed a diagram $\mathbb{R}\Gamma(\mathscr{X})$ computing the syntomic cohomology of $\mathscr{X}$ (definition 3.6.3) and a diagram $\mathbb{R}\Gamma_c(\mathscr{X})$ defining in the same way the compact support cohomology of $\mathscr{X}$ (definition 4.5.2).

In section 5.1 we will show that these diagrams have been defined in order to have a pairing

$$\mathbb{R}\Gamma(\mathscr{X}) \otimes \mathbb{R}\Gamma_c(\mathscr{X}) \to \mathbb{R}\Gamma_c(\mathscr{X}). \tag{5.1}$$

pairing

The advantage of working in the category of diagrams is that $\Gamma(\mathbb{K}, -)$ is a functor in the derived category pHD (see proposition 2.16 of [Ban02]), hence from the pairing 5.1 we can deduce a map[1]

$$\Gamma(\mathbb{K}, (\mathbb{R}\Gamma(\mathscr{X}) \otimes \mathbb{R}\Gamma_c(\mathscr{X}))(i)) \to \Gamma(\mathbb{K}, \mathbb{R}\Gamma_c(\mathscr{X})(i)).$$

We can finally compose the last map with the morphism

$$\Gamma(\mathbb{K}, \mathbb{R}\Gamma(\mathscr{X})(i)) \otimes \Gamma(\mathbb{K}, \mathbb{R}\Gamma_c(\mathscr{X})(i)) \to \Gamma(\mathbb{K}, \mathbb{R}\Gamma(\mathscr{X})(i) \otimes \mathbb{R}\Gamma_c(\mathscr{X})(i))$$

due to Beilinson [Beĭ86], to obtain the desired syntomic pairing.

In section 5.2 we will show that the syntomic theory is a Poincaré duality theory with support, and we will consequently be able to prove that for any proper morphism of smooth $\mathcal{V}$-schemes there exists a map in the same direction between the respective syntomic cohomologies: the so called Gysin morphism.

5.1   The syntomic pairing

We want to prove that there is a morphism of p-adic Hodge complexes

$$R\Gamma(\mathscr{X}) \otimes R\Gamma_c(\mathscr{X}) \to R\Gamma_c(\mathscr{X}).$$

This is what we need in order to have a pairing at the level of the complexes computing syntomic cohomology. The key point is the compatibility of the De Rham and rigid pairings with respect to the specialization and cospecialization maps.

---

[1]in the derived category

Lemma 5. *There exists a pairing*

$$p_{\mathrm{rig}} : j^\dagger \Omega^\bullet_{\mathscr{Y}_K}\langle D_K\rangle^{\mathrm{an}} \otimes \underline{\Gamma}_{]\mathscr{X}_k[} I\Omega^\bullet_{\mathscr{Y}_K}\langle D_K\rangle^{\mathrm{an}} \to \underline{\Gamma}_{]\mathscr{X}_k[} I\Omega^\bullet_{\mathscr{Y}_K}\langle D_K\rangle^{\mathrm{an}}$$

Proof. The wedge product of algebraic differentials induces the following pairing

$$p_{\mathrm{dR}} : \Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle \otimes I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle \longrightarrow I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle. \qquad (5.2) \quad \boxed{\text{pdR}}$$

The analytification of p_{dR} gives a pairing

$$\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{\mathrm{an}} \otimes I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{\mathrm{an}} \to I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{\mathrm{an}}.$$

Hence by [Berth97, Lemma 2.1] we get the result. $\qquad\square$

The following proposition, crucial for the purpose of constructing the syntomic pairing, solves the problem of the compatibility between the de Rham and the rigid pairings, at level of sheaves, with respect to specialization and cospecialization.

Proposition 16. *The following diagram commutes*

$$
\begin{array}{ccc}
G_{\mathrm{an}}{}^2(\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{\mathrm{an}}) \otimes G^2_{\mathrm{an}}(I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{\mathrm{an}}) & \xrightarrow{\;p^{an}_{dR}\;} & G^2_{\mathrm{an}}(I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{\mathrm{an}}) \\[2mm]
\Big\uparrow{\scriptstyle 1\otimes cosp} & & {\scriptstyle cosp}\Big\uparrow \\[2mm]
G^2_{\mathrm{an}}(\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{\mathrm{an}}) \otimes G_{\mathrm{an}}\underline{\Gamma}_{]\mathscr{X}_k[}G_{\mathrm{an}}(I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{\mathrm{an}}) & \xrightarrow{\;m\;} & G_{\mathrm{an}}\underline{\Gamma}_{]\mathscr{X}_k[}G_{\mathrm{an}}(I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{\mathrm{an}}) \\[2mm]
\Big\downarrow{\scriptstyle j^\dagger\otimes 1} & & \Big\downarrow{\scriptstyle 1} \\[2mm]
G_{\mathrm{an}}j^\dagger G_{\mathrm{an}}(\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{\mathrm{an}}) \otimes G_{\mathrm{an}}\underline{\Gamma}_{]\mathscr{X}_k[}G_{\mathrm{an}}(I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{\mathrm{an}}) & \xrightarrow{\;p_{\mathrm{rig}}\;} & G_{\mathrm{an}}\underline{\Gamma}_{]\mathscr{X}_k[}G_{\mathrm{an}}(I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{\mathrm{an}})
\end{array}
$$

where $m := p_{\mathrm{rig}} \circ (j^\dagger \otimes 1)$ by definition, $cosp$ is defined in section 4.3 and $j^\dagger$ is the canonical map described in subsection 3.4.1 (where the complex of sheaves involved is clear by the context).

Proof. The bottom square commutes by construction. Also we get $p_{\mathrm{rig}} \circ (j^\dagger \otimes 1) = p^{\mathrm{an}}_{\mathrm{dR}}$ restricted to $\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{\mathrm{an}} \otimes \underline{\Gamma}_{]\mathscr{X}_k[}(I\Omega^\bullet_{\mathscr{Y}_K}\langle D\rangle^{\mathrm{an}})$ by construction of the canonical map $j^\dagger$. $\qquad\square$

Before proving the whole pairing at level of diagrams we need the following result.

Remark 5.1.1. Let I be a filtered category and let $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ be direct systems. If for each i there is a pairing $A_i \otimes B_i \to B_i$ such that for every $i, j \in I$ the diagram

$$
\begin{array}{ccccc}
A_i & \otimes & B_i & \to & B_i \\
\uparrow & & \uparrow & & \uparrow \\
A_j & \otimes & B_j & \to & B_j
\end{array}
$$

commutes, then there exists a canonical pairing

$$\varinjlim_i A_i \otimes \varinjlim_i B_i \to \varinjlim_i B_i.$$

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Proposition 17. *Let $\mathcal{X}$ be an algebraic $\mathcal{V}$-scheme. Then there exists a morphism of p-adic Hodge complexes*

$$\pi : R\Gamma(\mathcal{X}) \otimes R\Gamma_c(\mathcal{X}) \to R\Gamma_c(\mathcal{X})$$

which is functorial with respect to $\mathcal{X}$ (as a morphism in pHD). Moreover taking the cohomology of π (at level of each complex) we get the following compatibility

$$
\begin{array}{ccc}
H^n_{dR}(\mathcal{X}_K) \otimes H^m_{dR,c}(\mathcal{X}_K) & \longrightarrow & H^{n+m}_{dR,c}(\mathcal{X}_K) \\
\uparrow{\scriptstyle id \otimes c} & & \uparrow{\scriptstyle c} \\
H^n_{dR}(\mathcal{X}_K) \otimes H^m_{rig,c}(\mathcal{X}_k) & \longrightarrow & H^{n+m}_{rig,c}(\mathcal{X}_k) \\
\downarrow{\scriptstyle s \otimes id} & & \downarrow{\scriptstyle id} \\
H^n_{rig}(\mathcal{X}_k) \otimes H^m_{rig,c}(\mathcal{X}_k) & \longrightarrow & H^{n+m}_{rig,c}(\mathcal{X}_k)
\end{array}
$$

Remark 5.1.2. Here by *s* and *c* we mean the cohomology of the maps *s* and *c* involved in the definitions of syntomic diagrams (see the diagrams 3.2 and 4.5).

Proof. It is sufficient to provide a pairing of enlarged diagrams (see 1.2.5). Thus we have to define a morphism of diagrams

$$\pi' : R\Gamma'(\mathcal{X}) \otimes R\Gamma'_c(\mathcal{X}) \to R\Gamma'_c(\mathcal{X})$$

(where $R\Gamma'(\mathcal{X})$ and $R\Gamma'_c(\mathcal{X})$ have been defined respectively in section 3.6 and 4.5).

- *Rigid side* (maps α_i, β_i for $i = 1, 2, 3$). For first we must construct a pairing

$$\mathbb{R}\Gamma_{rig}(\mathcal{X}_k)_{\mathcal{Y},\hat{\mathcal{Y}}} \otimes \mathbb{R}\Gamma_{rig,c}(\mathcal{X}_k)_{\mathcal{Y},\hat{\mathcal{Y}}} \to \mathbb{R}\Gamma_{rig,c}(\mathcal{X}_k)_{\mathcal{Y},\hat{\mathcal{Y}}}. \qquad (5.3) \quad \boxed{\text{pairingprim}}$$

Consider the pairing

$$G_{an} j^\dagger G_{an} \Omega^\bullet_U \otimes G_{an} \Gamma_{]\mathcal{X}_k[} G_{an} \Omega^\bullet_U \to G_{an} \Gamma_{]\mathcal{X}_k[} G_{an} \Omega^\bullet_U \qquad (5.4) \quad \boxed{\text{prima}}$$

following from the canonical pairing $\Omega^\bullet_U \otimes \Omega^\bullet_U \to \Omega^\bullet_U$, lemma 2.1 of [Berth97] and the compatibility of the Godement resolution with the tensor product (see subsection 2.3.1). To simplify the notations, let $\mathcal{F}^\bullet_U = G_{an} j^\dagger G_{an} \Omega^\bullet_U$ and $\mathcal{G}^\bullet_U = G_{an} \Gamma_{]\mathcal{X}_k[} G_{an} \Omega^\bullet_U$. We can consider the pairing 5.4 at level of global sections [2], to get

$$\Gamma(U, \mathcal{F}^\bullet_U) \otimes \Gamma(U, \mathcal{G}^\bullet_U) \to \Gamma(U, \mathcal{G}^\bullet_U). \qquad (5.5) \quad \boxed{\text{dopo}}$$

---

[2]formally: applying the global section functor we have

$$\Gamma(U, \mathcal{F}_U \otimes \mathcal{G}_U) \to \Gamma(U, \mathcal{G}_U)$$

and composing to the left with the canonical map of a presheaf into its sheafification for the tensor product, calculated at the level of global sections, i.e.

$$\Gamma(U, \mathcal{F}^\bullet_U) \otimes \Gamma(U, \mathcal{G}^\bullet_U) \to \Gamma(U, \mathcal{F}^\bullet_U \otimes \mathcal{G}^\bullet_U),$$

we finally obtain 5.5.

By remark 5.1.1 we get

$$\varinjlim_U \Gamma(U, \mathcal{F}_U^\bullet) \otimes \varinjlim_U \Gamma(U, \mathcal{G}_U^\bullet) \to \varinjlim_U \Gamma(U, \mathcal{G}_U^\bullet).$$

By definition of $\mathbb{R}\Gamma_{rig}(X)_{\overline{X},\mathcal{P}}$ and of $\mathbb{R}\Gamma_{rig,c}(X)_{\overline{X},\mathcal{P}}$ (see definitions 3.1.4 and 3.1.4), this is the construction of the pairing 5.3.

In order to define the pairing

$$\widetilde{\mathbb{R}\Gamma}_{rig}(\mathcal{X}_k)_{\mathcal{Y},\hat{\mathcal{Y}}} \otimes \widetilde{\mathbb{R}\Gamma}_{rig,c}(\mathcal{X}_k)_{\mathcal{Y},\hat{\mathcal{Y}}} \to \widetilde{\mathbb{R}\Gamma}_{rig,c}(\mathcal{X}_k)_{\mathcal{Y},\hat{\mathcal{Y}}}. \qquad (5.6) \quad \boxed{\text{pairingtild}}$$

we can apply the previous procedure to the pairing 5.3 just constructed, by applying the direct limit over the filtered category $SET^0_{X,\overline{X},\mathcal{P}}$ (see definition 3.1.5).

The compatibility between 5.3 e 5.6 is straightforward by the naturality of the construction (the natural transformations are described in subsections 3.1.5 and 4.1.5).

The same constructions lead to the definition of the pairing

$$\mathbb{R}\Gamma_{rig}(\mathcal{X}) \otimes \mathbb{R}\Gamma_{rig,c}(\mathcal{X}) \to \mathbb{R}\Gamma_{rig,c}(\mathcal{X})$$

and its compatibility with 5.6.

- *Mixed part* (maps α_i, β_i for $i = 4, 5$). It follows by proposition 16 (remember that there is a canonical map

$$\Gamma(U, \mathcal{F}_U^\bullet) \otimes \Gamma(U, \mathcal{G}_U^\bullet) \to \Gamma(U, \mathcal{F}_U^\bullet \otimes \mathcal{G}_U^\bullet)$$

for every complex of sheaves $\mathcal{F}_U^\bullet, \mathcal{G}_U^\bullet$ involved, because of the definition of tensor product of sheaves as a sheafification and the consequent canonical morphism).

- *De Rham side* (maps α_i, β_i for $i = 6, 7, 8$): it follows by the de Rham pairing p_{dR} (see 5.2) and by the functoriality of the map a (see proposition 5) for the $i = 4$ side.

$\square$

Corollary 5.1.3. There is a functorial pairing (induced by π of the previous proposition)

$$H^n_{syn}(\mathcal{X}, i) \otimes H^m_{syn,c}(\mathcal{X}, j) \to H^{n+m}_{syn,c}(\mathcal{X}, i + j). .$$

Proof. Consider the pairing of the Proposition 17. It induces a morphism $R\Gamma(\mathcal{X})(i) \otimes R\Gamma_c(\mathcal{X})(j) \to R\Gamma_c(\mathcal{X})(i + j)$ (more generally $R\Gamma(\mathcal{X})(i)[n] \otimes R\Gamma_c(\mathcal{X})(j)[m] \to R\Gamma_c(\mathcal{X})(i + j)[n + m]$). By 1.2.33 we get

$$\Gamma(\mathbb{K}, R\Gamma(\mathcal{X})(i)) \otimes \Gamma(\mathbb{K}, R\Gamma_c(\mathcal{X})(j)) \to \Gamma(\mathbb{K}, R\Gamma_c(\mathcal{X})(i + j)).$$

Then we can apply the functor H^0. By the definition of syntomic cohomology (3.6.3) and that of syntomic cohomology with compact support (4.5.2) we get the corollary. $\square$

5.2 Poincaré duality and Gysin map

We can now state a result of Poincaré duality for the syntomic theory with support. As an immediate consequence of this result we will be able to prove the existence of the Gysin map.

Proposition 5.2.1. Poincaré duality

Let $\mathscr{X}$ be a smooth algebraic $\mathcal{V}$-scheme of dimension d. For any n, i integers there is a canonical isomorphism

$$H^n_{syn}(\mathscr{X}, i) \simeq H^{syn}_{2d-n}(\mathscr{X}, d - i).$$

Proof. We must verify that

$$\mathrm{Hom}_{pHD}(\mathbb{K}, R\Gamma(\mathscr{X})(i)[n]) \simeq \mathrm{Hom}_{pHD}(R\Gamma_c(\mathscr{X}), \mathbb{K}(i - d)[n - 2d])$$

(see definitions 3.6.3 and 4.5.3), i.e., by proposition 1.2.32, that

$$H^n(\Gamma(\mathbb{K}, R\Gamma(\mathscr{X})(i))) \simeq H^n(\Gamma(R\Gamma_c(\mathscr{X}), \mathbb{K}(i - d)[-2d])).$$

In the alternative notation of definition 1.2.28, we must show that

$$H^n(\widetilde{\Gamma}(\mathcal{D}_{\mathbb{K}, R\Gamma(\mathscr{X})})) \simeq H^n(\widetilde{\Gamma}(\mathcal{D}_{R\Gamma_c(\mathscr{X}), \mathbb{K}(d)[-2d]})).$$

First recall that $\widetilde{\Gamma}(\mathcal{D}_{\mathbb{K}, R\Gamma(\mathscr{X})})$ is defined as

$$MC(R\Gamma_{\mathrm{rig}}(\mathscr{X}/K_0) \oplus R\Gamma_K(\mathscr{X}) \oplus R\Gamma_{\mathrm{dR}}(\mathscr{X}) \xrightarrow{\psi} R\Gamma_{\mathrm{rig}}(\mathscr{X}/K_0) \oplus R\Gamma_K(\mathscr{X}) \oplus R\Gamma_K(\mathscr{X})[-1]$$

where $\psi(x_0, x_K, x_{\mathrm{dR}}) = (\varphi(x_0) - x_0, c(x_0 \otimes \mathrm{id}_K) - x_K, x_k - s(x_{\mathrm{dR}}))$. In order to define the desired map we need to modify this complex replacing $R\Gamma_K(\mathscr{X})$ by $R\Gamma_{\mathrm{dR}}(\mathscr{X})$ as follows

$$MC(R\Gamma_{\mathrm{rig}}(\mathscr{X}/K_0) \oplus R\Gamma_{\mathrm{dR}}(\mathscr{X}) \oplus R\Gamma_{\mathrm{dR}}(\mathscr{X}) \xrightarrow{\psi'} R\Gamma_{\mathrm{rig}}(\mathscr{X}/K_0) \oplus R\Gamma_K(\mathscr{X}) \oplus R\Gamma_K(\mathscr{X})[-1]$$

where $\psi'(x_0, x'_{\mathrm{dR}}, x_{\mathrm{dR}}) = (\varphi(x_0) - x_0, c(x_0 \otimes \mathrm{id}_K) - s(x'_{\mathrm{dR}}), x'_{\mathrm{dR}} - x_{\mathrm{dR}})$. It is easy to see that this new complex, call it $M^\bullet$, is quasi isomorphic to $\widetilde{\Gamma}(\mathcal{D}_{\mathbb{K}, R\Gamma(\mathscr{X})})$.

The cup product induces a morphism of complexes $M^\bullet \to \widetilde{\Gamma}(\mathcal{D}_{R\Gamma(\mathscr{X}), \tau_{\geq 2d} R\Gamma(\mathscr{X})})$ which is a quasi isomorphism by the Poincaré duality theorems for rigid and de Rham cohomology[3] (see [Berth97] and [Hub95]). Explicitly, this map is induced by the following commutative diagram

$$R\Gamma_{\mathrm{rig}}(\mathscr{X}/K_0) \oplus R\Gamma_{\mathrm{dR}}(\mathscr{X}) \oplus R\Gamma_{\mathrm{dR}}(\mathscr{X}) \xrightarrow{\alpha} \mathrm{Hom}^\bullet_{K_0}(N_0, \tau_{\geq 2d} N_0) \oplus \mathrm{Hom}^\bullet_K(N_K \tau_{\geq 2d} N_K) \oplus \mathrm{Hom}^{\bullet, F}_K(N_{\mathrm{dR}}, \tau_{\geq 2d} N_{\mathrm{dR}})$$

$$\downarrow{\psi'} \qquad\qquad\qquad \downarrow{\theta}$$

$$R\Gamma_{\mathrm{rig}}(\mathscr{X}/K_0) \oplus R\Gamma_K(\mathscr{X}) \oplus R\Gamma_{\mathrm{dR}}(\mathscr{X}) \xrightarrow{\beta} \mathrm{Hom}^\bullet_{K_0}(N_0^\sigma, \tau_{\geq 2d} N_0) \oplus \mathrm{Hom}^\bullet_K(N_{\mathrm{rig}} \tau_{\geq 2d} N_K) \oplus \mathrm{Hom}^\bullet_K(N_{\mathrm{dR}}, \tau_{\geq 2d} N_K)$$

where $N := R\Gamma_c(\mathscr{X})$;

$$\theta(f_0, f_K, f_{\mathrm{dR}}) = (\varphi_c \circ f_0^\sigma - f_0 \circ \varphi_c, c \circ (f_0 \otimes \mathrm{id}_K) - f_K \circ c, f_K \circ s - s \circ f_{\mathrm{dR}});$$

$$\alpha(x_0, x'_{\mathrm{dR}}, x_{\mathrm{dR}}) : (y_0, y_K, y_{\mathrm{dR}}) \mapsto (x_0 \cup y_0, s(x'_{\mathrm{dR}}) \cup y_K, x_{\mathrm{dR}} \cup y_{\mathrm{dR}});$$

$$\beta(x_0, x_K, x_{\mathrm{dR}}) : (y_0, y_{\mathrm{rig}}, y_{\mathrm{dR}}) \mapsto (x_0 \cup \varphi_c(y_0), x_K \cup s(y_{\mathrm{rig}}), x_{\mathrm{dR}} \cup y_{\mathrm{dR}}).$$

---

[3]homology as the dual of cohomology with compact support

It is important to remark that the filtered complex $R\Gamma_{dR,c}(\mathcal{X})$ is strict (proposition 4.2.4), so that the truncation $\tau_{\geq 2d}R\Gamma_{dR,c}(\mathcal{X})$ is the usual truncation of complexes of K-vector spaces (see [Hub95], proposition 2.1.4]).

To conclude the proof it is sufficient to apply the exact functor $\mathcal{D}_{R\Gamma_c(\mathcal{X}),-}$ to the following quasi-isomorphisms

$$\tau_{\geq 2d}R\Gamma_c(\mathcal{X}) \leftarrow H^{2d}(R\Gamma_c(\mathcal{X})) \rightarrow \mathbb{K}(-d)[-2d]$$

(the term in the middle is made of complexes concentrated in degree zero as the top cohomology, so the first map is clearly q.i.; the second is because d is the dimension of X and the top cohomologies of that degree (not syntomic: rigid and de Rham) coincide with K).

$\square$

Corollary 5.2.2. Gysin map

Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of smooth algebraic $\mathcal{V}$-schemes of relative dimension $d_{\mathcal{X}}$ and $d_{\mathcal{Y}}$ respectively. Then there exists a canonical map

$$f_* : H^n_{syn}(\mathcal{X}, i) \to H^{n+2c}_{syn}(\mathcal{Y}, i+c)$$

where $c = d_{\mathcal{Y}} - d_{\mathcal{X}}$.

Proof. Proposition 5.2.1 and the functoriality of $R\Gamma_c(-)$ with respect to proper morphisms justify the following: $H^n_{syn}(\mathcal{X}, i) \simeq H^{syn}_{2d_{\mathcal{X}}-n}(\mathcal{X}, d_{\mathcal{X}} - i) := \mathrm{Hom}_{pHD}(R\Gamma_c(\mathcal{X}), \mathbb{K}[n - 2d_{\mathcal{X}}](i - d_{\mathcal{X}})) \to \mathrm{Hom}_{pHD}(R\Gamma_c(\mathcal{Y}), \mathbb{K}[n - 2d_{\mathcal{X}}](i - d_{\mathcal{X}})) =: H^{syn}_{2d_{\mathcal{X}}-n}(\mathcal{Y}, d_{\mathcal{X}} - i) \simeq H^{n+2c}_{syn}(\mathcal{Y}, i+c)$. $\square$

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