UNIVERSITÀ DEGLI STUDI DI PADOVA

Università degli Studi di Padova
Dipartimento di Matematica
Scuola di Dottorato di Ricerca in Scienze Matematiche
Indirizzo: Matematica Computazionale
XXVI Ciclo

Investigation of new conditions for steepness from a former result by Nekhoroshev

Direttore della Scuola:
Ch.mo Prof. Paolo Dai Pra

Coordinatore dell'indirizzo:
Ch.ma Prof.ssa Michela Redivo Zaglia

Supervisore:
Ch.mo Prof. Massimiliano Guzzo

Dottoranda:
Gabriella Schirinzi

ABSTRACT. In questa tesi viene presentata la costruzione di nuove condizioni sufficienti per la verifica di una proprietà delle funzioni denominata steepness. Tale proprietà è un'ipotesi fondamentale per l'applicazione del teorema di Nekhoroshev ad un sistema Hamiltoniano quasi integrabile, e la sua formulazione viene fornita da Nekhoroshev in maniera implicita. Per questo motivo è necessario avere a disposizione delle condizioni sufficienti per la verifica della stepness.

Nekhoroshev formulò negli anni settanta il suo celebre teorema, il quale garantisce sotto opportune ipotesi una forte stabilità per quei sistemi dinamici che non sono integrabili, ma possono scriversi come una piccola perturbazione di un sistema integrabile. Il teorema di Nekhoroshev costituisce un risultato fondamentale nell'ambito della Teoria delle Perturbazioni, in particolar modo per le sue importanti applicazioni nella meccanica celeste.

Per la costruzione delle nuove condizioni sufficienti per la steepness viene utilizzato un risultato dimostrato da Nekhoroshev. Le nuove condizioni sono più deboli di quelle conosciute fino ad ora, e di conseguenza permettono di individuare una classe più ampia di funzioni steep. In particolare, le nuove condizioni riguardano funzioni di due, tre e quattro variabili rispettivamente.

Nell'ultimo capitolo di questa tesi viene costruito un algoritmo generale per la verifica della steepness di funzioni di tre o quattro variabili. Inoltre, allo scopo di fornire qualche esempio concreto di applicazione delle nuove condizioni, tale algoritmo viene applicato a due sistemi fisici: l'Hamiltoniana del problema dei tre corpi ristretto circolare, e l'Hamiltoniana di una catena di quattro oscillatori armonici, con l'energia potenziale del problema di Fermi-Pasta-Ulam. In entrambi i casi le nuove condizioni sufficienti permettono di dimostrare numericamente la steepness.

ABSTRACT. This Thesis presents the construction of new sufficient conditions for the verification of a property of functions called steepness. It is a peculiar property required for the application of the Nekhoroshev Theorem to a quasi-integrable Hamiltonian system, and its formulation is given by Nekhoroshev in an implicit way. Therefore sufficient conditions are necessary for the verification of the steepness.

Nekhoroshev formulated his celebrated Theorem in the seventies, providing under suitable hypothesis a strong stability result for those dynamical systems which are not integrable, but can be considered as a small perturbation of an integrable system. The Nekhoroshev Theorem is a fundamental result in the framework of the Perturbation Theory, especially for its important applications in Celestial Mechanics.

For the construction of new sufficient conditions for steepness, a result proved by Nekhoroshev is used. The new conditions are weaker than the ones known up to now, hence they allow to detect a larger class of steep functions. In particular, the new conditions concern functions of two, three and four variables respectively.

In the last Chapter of this Thesis a general algorithm for the verification of the steepness of functions of three or four variables is constructed. Moreover, in order to provide some concrete examples of applicability of the new conditions, such algorithm is applied to two physical systems: the Hamiltonian of the circular restricted three-body problem, and the Hamiltonian of a chain of four harmonic oscillators, with the potential energy of the Fermi-Pasta-Ulam problem. In both cases the new sufficient conditions allow to prove numerical evidence of the steepness.

to my mother

Contents

Introduction

The development of the Hamiltonian Perturbation Theory received a strong impulse from the study of long-term stability problems of Celestial Mechanics. Since the fundamental pioneering works of Laplace, Lagrange and Poincaré, the old problem of the stability of the Solar System motivated most of the research in this field.

Many investigations of mechanical stability concern the Hamiltonian systems which are not integrable, but are in some sense "close" to an integrable one, the so-called quasi-integrable systems. There are important examples from Physics of systems which can be described by quasi-integrable Hamiltonians, including the Planetary problem. The stability of quasi-integrable systems is object of study of the Hamiltonian Perturbation Theory, whose most important results are the ce-lebrated KAM[**38, 2, 47**] and Nekhoroshev[**49, 50**] Theorems.

Let us consider an analytic quasi-integrable system

$$H(I,\varphi) = h(I) + \varepsilon f(I,\varphi), \tag{0.1}$$

where $(I,\varphi) \in D \times \mathbb{T}^n$, $D \subseteq \mathbb{R}^n$ open, are action-angle variables, and ε is sufficiently small. The KAM Theorem refers to results proved by Kolmogorov, Arnol'd and Moser between the fifties and the sixties of the last century, and ensures, under suitable hypotheses, perpetual stability of the motions of H, for almost all initial conditions in the phase-space. Only a subset of the phase-space of small Lebesgue measure, commonly called Arnol'd web, is not included in those stable solutions. The KAM Theorem provides a strong stability result and had many applications in Celestial Mechanics (see for example [**3, 14, 15, 16, 13, 17, 20, 26, 42, 43, 61**]). The stability of the orbits in the so-called Arnol'd web remains an open problem, which is known under the name of Arnol'd diffusion [**4**].

The Nekhoroshev Theorem [**49, 50**], formulated by Nekhoroshev in the seventies of the last century, provides an upper bound to the stability times in the Arnol'd web. Precisely, if the Hamiltonian H is analytic and its integrable approximation h satisfies a non-degeneracy assumption called steepness, any possible instability of the action variables may occur only after times which increase exponentially with an inverse power of ε. Since the Nekhoroshev Theorem uniformly applies to all initial conditions of the phase-space, by providing finite but very long stability times, it has important consequences for the stability of the motions of systems of interest from Physics (see for example [**6, 9, 10, 11, 24, 25, 18, 12, 22, 23, 28, 31, 32, 36, 35, 34, 33, 40, 41, 46, 53, 63**]).

A peculiar hypothesis of the Nekhoroshev Theorem concerns the geometric properties of the integrable approximation, called steepness.

Precisely, Nekhoroshev defined a function h to be steep at a point $\bar{I} = (\bar{I}_1, \dots, \bar{I}_n) \in D$ if $\nabla h(\bar{I}) \neq \underline{0}$, and if for each $m = 1, \dots, n-1$, there exist constants $C_m > 0$, $\delta_m > 0$ and $\alpha_m \geq 1$ such that, given any m-dimensional linear space λ orthogonal to $\nabla h(\bar{I})$, we have:

$$\max_{0 \leq \eta \leq \xi} \left(\min_{I \in \bar{I} + \lambda : \|I - \bar{I}\| = \eta} \|\nabla (h|_{\bar{I}+\lambda})(I)\| \right) > C_m \, \xi^{\alpha_m}, \ \ \forall \xi \in (0, \delta_m],$$

where $\nabla h|_{\bar{I}+\lambda}$ denotes the gradient of the restriction of h to the affine space through $\bar{I}$ spanned by λ. In order to apply the Nekhoroshev Theorem to a specific Hamiltonian, we therefore need to verify the steepness of its integrable approximation, and it may be very difficult because the definition of steepness is formulated in an implicit way.

Nekhoroshev indicated as the simplest examples of steep functions the quasi-convex ones, and the functions satisfying a non-degeneracy property on the 3-jet, namely the so-called 3-jet non-degenerate functions. Moreover, he formulated a general result about the steepness of functions whose generic r-jet satisfies certain conditions defined by systems of equalities and inequalities.[1]

We remark that quasi-convexity represents a special case of steepness. In fact, for quasi-convex functions, the proof of the Nekhoroshev Theorem greatly simplifies, due to the simpler geometry of resonances, and also to the possibility of using energy conservation to provide the confinement of the motions [**9, 8, 56, 45**]. Also, the stability times provided by the Nekhoroshev Theorem, are longer for the steep functions which are quasi-convex.

Unfortunately, quasi-convexity is a strong property rarely verified in real physical systems, while steepness is, in some sense, a generic property for an integrable Hamiltonian. Therefore the study of the steepness is very important in view of the applications to systems of interest from Physics. It happens, in fact, that Hamiltonians describing real systems have an integrable approximation which is neither quasi-convex nor 3-jet non-degenerate, as in the case of the circular restricted three-body problem in a neighborhood of the Lagrangian points L_4 and L_5 for a specific value of the mass ratio [**6**], and in the case of the Riemann ellipsoids [**24**]. We need, therefore, conditions which let us identify the steepness also of a 3-jet degenerate function.

In [**49, 50**] Nekhoroshev constructed sufficient conditions for steepness of a function, based on the solvability of collections of systems $\mathscr{C}^r(n)$ of equalities and inequalities, depending on the number n of degrees of freedom, the derivatives of the function up to a certain order r, and some auxiliary parameters. For each $n \geq 2$, $r \geq 2$ and a fixed value $\bar{I} \in D$, we denote by $\sigma^r(n)$ the set of r-jets at $\bar{I}$ of functions with n degrees of freedom, such

[1] We follow the definition of r-jet used by Nekhoroshev in [**49**]. The r-jet of a function h at a point $\bar{I} = (\bar{I}_1, \ldots, \bar{I}_n)$ is the vector $P_r(h)$ consisting of the coefficients of the Taylor polynomial of order r of the function h at $\bar{I}$, with the exception of the constant term, that is

$$P_r(h) = \{h_\mu, 1 \leq |\mu|_1 \leq r\}, \qquad h_\mu := \frac{1}{\mu!} \frac{\partial^{|\mu|_1} h}{\partial I^\mu}(\bar{I}),$$

where $\mu = (\mu_1, \ldots, \mu_n)$ is a multi-index, $\mu_i \geq 0$ are integers and $|\mu|_1 = \sum_{i=1}^n \mu_i$.

that at least one of the systems in $\mathscr{C}^r(n)$ is solvable. Nekhoroshev proved that if $\nabla h(\bar{I}) \neq \underline{0}$ and if the r-jet of h at $\bar{I}$ lies outside the closure of $\sigma^r(n)$, then h is steep in a neighborhood of $\bar{I}$.

Such conditions are really explicit for $r = 2$, corresponding to quasi-convexity for all $n \geq 2$. For $r = 3$ the conditions are only a slight modification of the 3-jet non-degeneracy, and do not let us identify a class of steep functions larger than the 3-jet non-degenerate ones. Therefore, weaker sufficient conditions must involve also the 4-jet of the function.

For $r \geq 4$, an explicit expression for such conditions has not yet been investigated and formulated.

In order to produce explicitly the conditions provided by Nekhoroshev, from a collection $\mathscr{C}^r(n)$, one needs to construct a new collection of systems describing the closure of $\sigma^r(n)$ or, when this is not possible, a closed set containing $\sigma^r(n)$. Actually the new collection will represent the explicit sufficient condition for steepness: if a r-jet at $\bar{I}$ does not solve any of the systems in this collection, it means it lies outside the closure of $\sigma^r(n)$, and consequently the function is steep in a neighborhood of $\bar{I}$.

We remark that, performing the closure of a certain set whose elements satisfy equalities and inequalities, involves operations like limits of sequences, and the limit values not necessarily satisfy the same inequalities.

The main result of this Thesis is the investigation of the case $r = 4$. Precisely, we construct new sufficient conditions for the steepness of functions of $n = 2, 3, 4$ degrees of freedom. We also prove that when $n \geq 5$, the Nekhoroshev result does not provide any extension of the 3-jet condition, therefore the only interesting cases are $n = 2, 3, 4$.

In the construction of the new conditions, we first show that in some cases one can formulate the systems of the collections $\mathscr{C}^r(n)$ in a simplified form, by reducing the number of the auxiliary parameters. Then, for the case $n = 2$, we find the closure of the set $\sigma^4(2)$, while for the cases $n = 3$ and $n = 4$, we construct two closed sets containing respectively $\sigma^4(3)$ and $\sigma^4(4)$ (see Propositions 2.3, 2.4 and 2.5, Chapter 2, and [**59**]).

The new conditions may be effectively investigated numerically [**60**]. In order to test the new sufficient conditions for steepness, in Chapter 3 we construct an algorithm for the verification of the steepness of a given function with $n = 3,4$, which extends the algorithm of [**6**] for functions of three degrees of freedom. The extension consists in the fact that our algorithm is able to verify the steepness also of 3-jet degenerate functions, and can be used also for functions with four degrees of freedom. Moreover, in Chapter 3 we illustrate the numerical investigation of the steepness of two selected examples, one with $n = 3$ and one with $n = 4$.

The first example is the Hamiltonian of the circular restricted three-body problem, whose stability properties in a neighborhood of the elliptic equilibria L_4 and L_5 are still not completely known. In [**6**] Benettin, Fassò and Guzzo provided numerical evidence of exponential stability for all the values of the reduced mass μ below the Routh critical mass, except a finite number of values. For a special value μ_3, the authors could not prove stability, because the integrable approximation of the Hamiltonian is 3-jet degenerate. Therefore, we decided to test the new sufficient condition for steepness in the case $\mu = \mu_3$, and we obtained that, also for this value of the reduced mass, the Hamiltonian is steep in a neighborhood of the equilibria.

The second example we considered is the Hamiltonian of a chain of four oscillators, with the potential of the famous Fermi-Pasta-Ulam problem [**27**]. The Hamiltonian depends on two parameters α, β, that we assumed in $(0,1]$. We investigated the steepness of the Hamiltonian in a neighborhood of the origin for different values of α and β. In particular, we chose a certain number of couples α, β, and found that, for all of them, the Hamiltonian is never quasi-convex neither 3-jet non-degenerate. Therefore, we selected a couple of values, precisely $\alpha = 0.1$ and $\beta = 0.9$, and we tested the new sufficient condition for steepness. We obtained that the Hamiltonian is steep in a neighborhood of the origin.

The Thesis is organized as follows. In Chapter 1 we provide a general description of the main results of the Hamiltonian Perturbation Theory, that is the KAM and Nekhoroshev Theorems, and we discuss the notion of steepness. In Chapter 2 we report the sufficient conditions for steepness provided by Nekhoroshev in [**49**], with the detailed description of the systems in the collections $\mathscr{C}^r(n)$. Then, we formulate and prove the Propositions 2.3, 2.4 and 2.5, which are the new sufficient conditions for steepness for functions of respectively two, three and four degrees of freedom. Finally, in Chapter 3, we construct the algorithm for the verification of the steepness of functions with three and four degrees of freedom, and we use it for the verification of the steepness of two specific Hamiltonians.

CHAPTER 1

Long-term stability in Hamiltonian systems

In this Chapter we review the main ideas at the basis of the Hamiltonian Perturbation Theory, and we discuss the celebrated KAM [**38, 2, 47**] and Nekhoroshev [**49, 50**] Theorems. Then, we focus our attention on the steepness, which is the main argument of this Thesis.

1.1. Integrable and quasi-integrable Hamiltonian systems

A Hamiltonian system with n degrees of freedom is integrable when it admits n first integrals satisfying suitable conditions, which in particular allow us to describe the dynamics by linear motions along n-dimensional invariant tori. Such a characterization is provided by the Liouville-Arnol'd Theorem, which can be stated as follows.

THEOREM 1.1 (**Liouville-Arnol'd** [**1**]). *Let $H : B \to \mathbb{R}$ be a Hamiltonian, with B an open subset of $\mathbb{R}^{2n}$ provided with canonical variables (p,q), and assume there exist n first integrals $F_1,\dots,F_n : B \to \mathbb{R}$, such that $\{F_i,F_j\} = 0$ in B for all i, j. Assume $F_1,\dots,F_n$ are linearly independent, that is:*

$$\operatorname{rank}\Big(\frac{\partial(F_1,\dots,F_n)}{\partial(p_1,\dots,p_n,q_1,\dots,q_n)}\Big) = n,$$

on a level set

$$\Sigma_c = \{(p,q) \in B : F_i(p,q) = c_i\}, \qquad c = (c_1,\dots,c_n) \in \mathbb{R}^n,$$

so that Σ_c is a n-dimensional sub-manifold of B. Further assume that Σ_c is compact (or contains a compact component, and restrict the attention to it). Then

i. *Σ_c is diffeomorphic to the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$*

ii. *there exist a neighborhood $\mathscr{C}$ of c, an open set $D \subseteq \mathbb{R}^n$ and a canonical diffeomorphism*

$$\begin{aligned} w: \bigcup_{c' \in \mathscr{C}} \Sigma_{c'} &\longrightarrow D \times \mathbb{T}^n \\ (p,q) &\longmapsto (I,\varphi) \end{aligned}$$

such that $H \circ w^{-1}(I,\varphi) = h(I)$ and $F_i \circ w^{-1}(I,\varphi) = \mathscr{F}_i(I)$.

Hence, when a Hamiltonian H satisfies the hypotheses of the Liouville-Arnol'd Theorem, we can introduce new variables (I,φ), which are called action-angle variables, such that the actions I are first integrals, and the Hamilton's equations assume the very simple form:

$$\begin{cases} \dot{I} = 0 \\ \dot{\varphi} = \frac{\partial h}{\partial I} =: \omega(I)\,. \end{cases}$$

Such equations can be immediately integrated, providing for each initial condition $(I(0),\varphi(0))$ a linear motion with constant velocity $\omega(I(0))$ on the invariant torus $I = I(0)$:

$$\begin{cases} I(t) = I(0) \\ \varphi(t) = \omega(I(0))\,t + \varphi(0)\,. \end{cases}$$

The presence of a set of linearly independent first integrals in mutual involution is a very restrictive hypothesis, and corresponds to suitable symmetry properties of the system.

Most of the real physical systems do not satisfy this hypothesis, hence they are not integrable. Nevertheless they can be often described by a quasi-integrable Hamiltonian, that is a Hamiltonian that in action-angle variables $(I,\varphi) \in D \times \mathbb{T}^n$, with $D \subseteq \mathbb{R}^n$ open, is of the form:

$$H(I,\varphi) = h(I) + \varepsilon f(I,\varphi) \tag{1.1}$$

where $h(I)$ is the integrable approximation, and ε is a small parameter ($|\varepsilon| \ll 1$) that measures the intensity of the perturbation $f(I,\varphi)$. The correspondent Hamilton's equations are:

$$\begin{cases} \dot{I} = -\varepsilon \frac{\partial f}{\partial \varphi} \\ \dot{\varphi} = \frac{\partial h}{\partial I} + \varepsilon \frac{\partial f}{\partial I} \,. \end{cases} \tag{1.2}$$

In this case the actions are not expected to be first integrals anymore, and the Hamilton's equations cannot be integrated by quadratures.

In a quasi-integrable system the action variables, in principle, may evolve in time with a speed of order ε. Despite this, it typically turns out that after a long time, even grater than $1/\varepsilon$, the actions still differ only slightly from their initial values. Motivated by this fact, we say that a solution $(I(t),\varphi(t))$ of (1.2) is stable in a given finite or infinite interval of time $[0,T_\varepsilon]$ if we have

$$||I(t) - I(0)|| < c(\varepsilon) \tag{1.3}$$

for all $t \in [0,T_\varepsilon]$, where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. The value T_ε is called the stability time of the solution $(I(t),\varphi(t))$. This notion of stability becomes non-trivial if T_ε grows at least as $1/\varepsilon$, and when the interval of time is infinite, we say that the solution is perpetually stable.

A fundamental question of the Hamiltonian Perturbation Theory is if perpetually stable solutions of (1.2) exist and, in case, how many of them there are; for solutions which are not perpetually stable, one would like to estimate their time of stability T_ε.

1.2. The fundamental equation of the Perturbation Theory

Let us consider an analytic Hamiltonian (1.1), and assume $f(I,\varphi)$ is bounded in $\mathscr{C}^1$-norm by A, then from the Hamilton's equations (1.2) the so-called (trivial) a priori estimate for the variation of the actions follows:

$$||I(t) - I(0)|| \leq \varepsilon A |t| \,.$$

It means that up to times of order $1/\sqrt{\varepsilon}$ the variation of the actions remains $\sqrt{\varepsilon}$-limited.

The Hamiltonian Perturbation Theory consists in some techniques which try to improve the a priori estimate, that is to extend the stability time to values grater than $1/\sqrt{\varepsilon}$. Precisely, the classical approach is to search for a near to identity canonical change of variables such that, in the new variables, the perturbation appears reduced, for example to the order ε^2. This way the time of stability would be extended to the order $1/\varepsilon$. When such a transformation exists, it means we can perform a so-called perturbation step. The best we can expect is to be able to iterate the procedure, extending as much as possible the stability time of the system.

In Hamiltonian mechanics a way to obtain near to the identity canonical transformations is the Lie series method: the transformation is given by the flow at a fixed time of an autonomous auxiliary Hamiltonian system. Precisely, given a Hamiltonian $\chi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$, its time-$\varepsilon$ Hamiltonian flow $\Phi^{\varepsilon}_{\chi}$ defines a near to identity canonical change of variables:

$$(I,\varphi) = \Phi^{\varepsilon}_{\chi}(I',\varphi').$$

The idea is to search χ such that the flow $\Phi^{\varepsilon}_{\chi}$ conjugates H to a new Hamiltonian $H' = H \circ \Phi^{\varepsilon}_{\chi}$ of the form

$$H'(I',\varphi') = h(I') + \varepsilon g(I') + \varepsilon^2 f'(I',\varphi')\,.$$

Standard computations show that χ must satisfy the equation

$$\omega(I) \cdot \frac{\partial \chi}{\partial \varphi}(I,\varphi) = f(I,\varphi) - \langle f \rangle_{\varphi}\,, \tag{1.4}$$

where $\langle f \rangle_{\varphi}$ denotes the average over φ of f. Equation (1.4) is called the fundamental equation of the Perturbation Theory, because the study of the stability of a quasi-integrable system is based on the existence of a solution of such equation.

Let us expand f and χ in Fourier series:

$$f = \sum_{k\in\mathbb{Z}^n} f_k(I) e^{ik\cdot\varphi} \qquad \chi = \sum_{k\in\mathbb{Z}^n} \chi_k(I) e^{ik\cdot\varphi}$$

and denote by $\hat{f} = \{k \in \mathbb{Z}^n : f_k \neq 0\}$ the spectrum of f. Then equation (1.4) implies:

$$\chi = -i \sum_{k \in \hat{f}} \frac{f_k(I)}{\omega(I) \cdot k} e^{ik \cdot \varphi}, \tag{1.5}$$

which is well defined only for those actions I such that:

$$\omega(I) \cdot k \neq 0 \qquad \forall k \in \hat{f},$$

that is actions corresponding to non-resonant values (we say that I corresponds to a resonant value if $\omega(I) \cdot k = 0$ for some $k \in \hat{f}$).

Actually, for the convergence of the series (1.5), we should keep sufficiently far from resonances, hence the actions should satisfy a stronger condition. For some positive constants γ, $\tau \in \mathbb{R}$, for example, this is granted by the so-called Diophantine condition:

$$|\omega(I) \cdot k| \geq \frac{\gamma}{\|k\|^{\tau}} \qquad \forall k \in \hat{f}. \tag{1.6}$$

1.2.1. The Poincaré difficulty. The presence of small divisors in equation (1.5) represents an essential problem in the Hamiltonian Perturbation Theory. In particular Poincaré proved that in a general situation, small divisors prevent the solvability of equation (1.4) in any open subset of the action domain [**54**]. In fact, by assuming a local non-degeneracy of the frequency map:

$$\det\left(\frac{\partial \omega}{\partial I}\right) = \det\left(\frac{\partial^2 h}{\partial I^2}\right) \neq 0$$

at all $I \in D$, and a genericity condition on the spectrum of f, the relevant resonances form a dense set in the action-space. Therefore we cannot even define the series (1.5) in any open subset of the phase-space, and consequently we cannot perform the perturbation step.

Nevertheless the Hamiltonian Perturbation Theory developed some techniques to escape the Poincaré difficulty, and to extend the stability time for both resonant and non-resonant motions.

The main idea is that the perturbation of a Hamiltonian (1.1) can be written as the sum of two parts: one part has a finite spectrum, the other one is sufficiently small, and does not influence significantly the motion of the actions for a very long time. This is the idea at the basis of the proofs of the two main Theorems of the Perturbation Theory, the KAM and the Nekhoroshev Theorems, leading on the one hand to perpetual stability in closed sets, on the other hand to exponential stability in all the phase-space.

1.3. Perpetual stability of non-resonant motions

In [**2, 5, 38, 48**] Arnol'd, Moser and Kolmogorov proved that if h is analytic and non-degenerate, in the phase-space of a Hamiltonian (1.1) there exists a set of invariant tori which are only a slight deformation of the invariant tori of the unperturbed system.

Such set, which is called Kolmogorov set, almost fills the whole phase-space. In fact, the measure of the complementary of the Kolmogorov set tends to zero as $\varepsilon \to 0$. In particular, all the solutions on the Kolmogorov set are perpetually stable.

This is the celebrated KAM Theorem, that we recall here in the version by Arnol'd, with a statement improved by Lazutkin, Chierchia and Gallavotti, and Pöschel [**39, 19, 55**].

THEOREM 1.2 (**KAM**). *Let the Hamiltonian (1.1) be analytic and bounded in a complex neighborhood of the domain $D \times \mathbb{T}^n$, with $D \subseteq \mathbb{R}^n$ open, and let h satisfy the non-degeneracy condition*

$$\det\left(\frac{\partial^2 h}{\partial I^2}\right) \neq 0 \tag{1.7}$$

at all $I \in D$. Then we can find positive constants ε_0, a_1, a_2 such that for all $0 \leq \varepsilon < \varepsilon_0$ there exist

- *a near to identity smooth canonical transformation*

$$w_\varepsilon : D' \times \mathbb{T}^n \longrightarrow D \times \mathbb{T}^n$$
$$(I', \varphi') \longmapsto (I, \varphi)$$

with $D' \subseteq \mathbb{R}^n$ open, such that $\|I - I'\| \leq a_1\sqrt{\varepsilon}, \|\varphi - \varphi'\| \leq a_2\sqrt{\varepsilon}$
- *a subset $D_\varepsilon \subseteq D \cap D'$ with large Lebesgue measure, that is*

$$\mathrm{meas}(D - D_\varepsilon) \sim \sqrt{\varepsilon}$$

- *an integrable Hamiltonian $h_\varepsilon(I')$ defined on $D' \times \mathbb{T}^n$*

such that the new Hamiltonian $H \circ w_\varepsilon$ coincides, together with all its derivatives, with h_ε for all $I' \in D_\varepsilon$.

On the different invariant tori corresponding to the different $I' \in D_\varepsilon$, the motions are quasi-periodic with frequencies $\omega' = \nabla h_\varepsilon(I')$, and it turns out that such frequencies are Diophantine with suitable $\gamma \sim \sqrt{\varepsilon}$ and $\tau = n$.

We remark that several formulations of the KAM Theorem exist in literature, with different hypotheses on h and f. In particular, Arnol'd proved that the non-degeneracy assumption (1.7) may be replaced by the so-called iso-energetic non-degeneracy:

$$\det \begin{pmatrix} \frac{\partial^2 h}{\partial I^2} & \frac{\partial h}{\partial I} \\ \frac{\partial h}{\partial I} & 0 \end{pmatrix} \neq 0\,,$$

which ensures abundance of invariant tori in each energy level surface.

This is particularly important for $n = 2$. In fact, in this case the energy level is a 3-dimensional surface which contains a large measure set of invariant 2-dimensional tori. Such tori separate the energy level, so that a generic trajectory either lies on a torus or is trapped between two of them. In both cases the trajectory can not escape, and we have perpetual stability.

We remark that the conditions for non-degeneracy and iso-energetic non-degeneracy are independent from one another, that means that a non-degenerate system may be iso-energetically degenerate, and a iso-energetically non-degenerate system may be degenerate.

Finally, we remark that the KAM Theorem concerns stability for infinite times, but limited to open subsets of the phase-space. An important question is related to what times of stability characterize fixed open domains.

The Nekhoroshev Theorem [**49, 50**], which is another fundamental Theorem of the Perturbation theory, gives an answer to this question.

1.4. Exponential stability

When the integrable approximation h of an analytic Hamiltonian (1.1) satisfies suitable geometric conditions, a remarkable result of the Perturbation Theory ensures that the stability time T_ε, for an estimate as in (1.3), is much larger than $1/\varepsilon$ for all initial conditions in the phase-space.

The mentioned result is the celebrated Nekhoroshev Theorem, which applies under the hypothesis of steepness for h (see Section 1.5 for the definition), and ensures stability times increasing exponentially with an inverse power of ε.

The Nekhoroshev Theorem [**49, 50**] is not a perpetual stability result, but since it uniformly applies to all initial conditions of the phase-space, by providing finite but very long stability times, it has important consequences for the stability of systems of interest from Physics (see for example [**6, 9, 10, 11, 24, 25, 18, 12, 28, 31, 32, 36, 46, 53**]).

A possible statement of the Nekhoroshev Theorem is the following.

THEOREM 1.3 (**Nekhoroshev Theorem** [**49, 50**]). *Let the Hamiltonian (1.1) be analytic in the domain $D \times \mathbb{T}^n$, with $D \subseteq \mathbb{R}^n$ open, and let h satisfy in D a non-degeneracy condition called steepness. Then there exist positive constants $a, b, \varepsilon_0, I_0, t_0$ such that if $0 \le \varepsilon < \varepsilon_0$, for every initial datum $(I(0), \varphi(0)) \in D \times \mathbb{T}^n$ the solutions of system (1.2) satisfy*

$$\|I(t) - I(0)\| \le I_0 \varepsilon^a \tag{1.8}$$

for all times t such that

$$|t| \le t_0 \exp\left(\frac{\varepsilon_0}{\varepsilon}\right)^b . \tag{1.9}$$

The parameters a and b depend on the steepness properties of h (see Section 1.5).

The simplest classes of steep functions are the convex and the quasi-convex ones. We recall that h is convex at $I \in D$ if $\sum_{i=1}^n \frac{\partial^2 h}{\partial I_i \partial I_j}(I) v_i v_j > 0$ or < 0 for all $v = (v_1, \dots, v_n) \in \mathbb{R}^n \setminus \{\underline{0}\}$. A convex function is steep at all points $I \in D$ such that $\omega(I) \neq \underline{0}$.

Instead, following Nekhoroshev, h is quasi-convex at $I \in D$, if $\omega(I) \neq \underline{0}$ and the only solution $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ of the system

$$\begin{cases} \sum_{i=1}^n \frac{\partial h}{\partial I_i}(I) v_i = 0 \\ \sum_{i,j=1}^n \frac{\partial^2 h}{\partial I_i \partial I_j}(I) v_i v_j = 0 \end{cases} \tag{1.10}$$

is $v = (0, \dots, 0)$. Quasi-convexity is clearly a generalization of convexity, hence a convex function is in particular quasi-convex at all points I where $\omega(I) \neq \underline{0}$.

Nekhoroshev in [**49**] indicates also another class of steep functions: the 3-jet non-degenerate functions, which are defined as follows.

A function h is 3-jet non-degenerate at $I \in D$, if $\omega(I) \neq \underline{0}$ and the only solution $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ of the system

$$\begin{cases} \sum_{i=1}^n \frac{\partial h}{\partial I_i}(I)\, v_i = 0 \\ \sum_{i,j=1}^n \frac{\partial^2 h}{\partial I_i \partial I_j}(I)\, v_i v_j = 0 \\ \sum_{i,j,k=1}^n \frac{\partial^3 h}{\partial I_i \partial I_j \partial I_k}(I)\, v_i v_j v_k = 0 \end{cases} \tag{1.11}$$

is $v = (0, \dots, 0)$.

Conditions (1.11) are weaker than (1.10), thus quasi-convex functions are also 3-jet non-degenerate.

Moreover, Nekhoroshev formulated a general result about the steepness of functions whose generic r-jet satisfies certain conditions, defined by systems of equalities and inequalities. Such result provides some sufficient conditions for steepness, and we will see later that it turns out to be very useful because the steepness is a property implicitly defined.

<u>Remark</u>. We follow the definition of r-jet used by Nekhoroshev in [**49**]. The r-jet of a function h at a point $\bar{I} = (\bar{I}_1, \dots, \bar{I}_n)$ is the vector $P_r(h)$ consisting of the coefficients of the Taylor polynomial of order r of the function h at $\bar{I}$, with the exception of the constant term, that is

$$P_r(h) = \{h_\mu, 1 \leq |\mu|_1 \leq r\}, \qquad h_\mu := \frac{1}{\mu!} \frac{\partial^{|\mu|_1} h}{\partial I^\mu}(\bar{I}),$$

where $\mu = (\mu_1, \ldots, \mu_n)$ is a multi-index, $\mu_i \geq 0$ are integers and $|\mu|_1 = \sum_{i=1}^n \mu_i$.

The quasi-convexity represents a special case of steepness. In fact, for quasi-convex functions, the proof of the Nekhoroshev Theorem greatly simplifies, due to the simpler geometry of resonances and also to the possibility of using energy conservation to provide the confinement of the motions [**9, 8, 56, 45**]. In fact, in the early eighties, Benettin and Gallavotti [**9**] showed that when h is quasi-convex, the conservation of the energy plays a fundamental role in the confinement of the actions near resonances. We briefly report here the argument they used.

Let us consider a Hamiltonian (1.1) and a resonance of order d. Precisely, let $\mathscr{L}$ be a d-dimensional lattice which admits a basis of integer n-dimensional vectors $k^{(1)}, k^{(2)}, \ldots k^{(d)}$, such that $\|k^{(i)}\| \leq (\varepsilon_0/\varepsilon)^b$ for all $i \in \{1, 2, \ldots, d\}$. The correspondent resonant manifold $M_{\mathscr{L}}$ is clearly:

$$M_{\mathscr{L}} = \{I \in D : k \cdot \omega(I) = 0 \, \forall k \in \mathscr{L}\}\,.$$

In a small neighborhood of $M_{\mathscr{L}}$, and sufficiently far from other resonances, Nekhoroshev proves that it is possible to define a near to identity canonical transformation which conjugates the Hamiltonian H to a resonant normal form adapted to the resonance $\mathscr{L}$ and with exponentially small remainder, that is:

$$H' = h(I) + \varepsilon g(I, \varphi) + e^{-\left(\frac{\varepsilon_0}{\varepsilon}\right)^b} f'(I, \varphi)\,. \tag{1.12}$$

In H' there is still a non-integrable term of order ε, but it contains only the Fourier harmonics in $\mathscr{L}$:

$$g(I, \varphi) = \sum_{k \in \mathscr{L}} g_k(I) e^{ik \cdot \varphi}\,.$$

If we neglect the term $e^{-\left(\frac{\varepsilon_0}{\varepsilon}\right)^b} f'(I, \varphi)$, from the Hamilton's equations we obtain that the new actions move on a plane $\Pi_{\mathscr{L}}(I(0))$ parallel to $\mathscr{L}$ and containing the initial point $I(0)$. In fact $\dot{I}$ is a linear combination of vectors of $\mathscr{L}$:

$$\dot{I} = -\varepsilon \sum_{k \in \mathscr{L}} i g_k(I) k \, e^{ik\cdot\varphi} .$$

The plane $\Pi_{\mathscr{L}}(I(0))$ is called fast drift plane (see Figure 1).

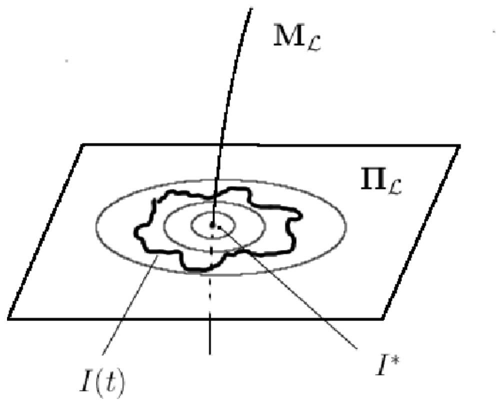

FIGURE 1. The confinement of the actions in the fast drift plane

Since h is quasi-convex, then the resonant manifold $M_{\mathscr{L}}$ is transversal to the plane of fast drift $\Pi_{\mathscr{L}}$. We call I^* their intersection point: such point is an extremal for h restricted to $\Pi_{\mathscr{L}}$. This is easy to see if we expand h around I^*:

$$h(I) = h(I^*) + \omega(I^*)\cdot(I-I^*) + \frac{1}{2}h''(I^*)(I-I^*)\cdot(I-I^*) + \dots$$

and we observe that the linear term vanishes when $I \in \Pi_{\mathscr{L}}$ ($h''(I^*)$ denotes the Hessian matrix of h computed at I^*). As a consequence, in a neighborhood of I^*, and on the plane $\Pi_{\mathscr{L}}$, the level surfaces of h are concentric ellipsoids around I^*.

Now the energy conservation provides the confinement. In fact, we observe that the term εg has oscillations bounded by ε. Therefore the motion of the actions is practically confined between two nearby level surfaces of h. Such behavior persists as long as the neglected perturbation term,

$e^{-\left(\frac{\varepsilon_0}{\varepsilon}\right)^b} f'$, does not sufficiently move the actions, that is up to exponentially long times of the order $e^{\left(\frac{\varepsilon_0}{\varepsilon}\right)^b}$.

After Benettin and Gallavotti, in the nineties Pöschel and Lochak [**56, 44**], by combining the argument on the conservation of the energy with new ideas about the treatment of resonances, proved that, for quasi-convex functions, the parameters a and b appearing in the estimates (1.8) and (1.9), assume the best possible values: $a = b = \frac{1}{2n}$. Therefore we say that convex functions are the most stable among the steep functions.

In literature the Nekhoroshev Theorem is often used with reference to the convex case, therefore we recall here the statement of the Theorem provided by Pöschel in [**56**].

THEOREM 1.4 (**Nekhoroshev Theorem in the convex case**). *Let the Hamiltonian (1.1) be analytic in a complex neighborhood of $D \times \mathbb{T}^n$ and let h be m-convex, that is*

$$h''(I)u \cdot u \geq m u \cdot u$$

for any $u \in \mathbb{R}^n$, at any $I \in D$, where h'' denotes the Hessian matrix of h. There exist $\varepsilon_0, a_0, t_0, \varepsilon_* > 0$ such that if $\varepsilon < \varepsilon_0$, then any motion $(I(t), \varphi(t))$ satisfies

$$\|I(t) - I(0)\| \leq a_0 \varepsilon^{\frac{1}{2n}}$$

for any time t such that

$$|t| \leq t_0 \exp\left(\frac{\varepsilon_*}{\varepsilon}\right)^{\frac{1}{2n}}.$$

If the complex neighborhood of $D \times \mathbb{T}^n$ where (1.1) is analytic has the form $D_{\rho_I} \times \mathbb{T}^n_{\rho_\varphi}$, where

$$D_{\rho_I} = \{I \in \mathbb{C}^n : \text{ there exists } I_0 \in D \text{ with } \|I - I_0\| \leq \rho_I\}$$
$$\mathbb{T}^n_{\rho_\varphi} = \{\varphi \in (\mathbb{C}/2\pi\mathbb{Z})^n : |\Im\, \varphi_i| \leq \rho_\varphi \text{ for any } i = 1, \dots, n\},$$

then possible values for ε_0, t_0, a_0 and ε_* are:

$$\varepsilon_0 = \frac{m\rho_I^2}{2^{10}(11\frac{M}{m})^{2n}}, \quad a_0 = \rho_I \left(11\frac{M}{m}\right)^{-1}$$

$$t_0 = \left(11\frac{M}{m}\right)^2 \frac{\rho_\varphi}{\Omega}, \quad \varepsilon_* = \left(\frac{\rho_\varphi}{6}\right)^{2n} \varepsilon_0\,,$$

where M is such that $\|h''(I)u\| \le M\|u\|$ for any $u \in \mathbb{R}^n$ and any $I \in D_{\rho_I}$, and $\Omega = \sup_{I \in D_{\rho_I}} \|\nabla h(I)\|$.

It's important to remark that convexity is a strong property rarely verified in real physical systems. On the other hand, steepness seems to be in some sense generic for analytic functions [**50, 52**]. Therefore the study of the steepness is very important in view of the applications to systems of interest from Physics.

1.5. Steepness

Given a Hamiltonian (1.1), the steepness of the integrable approximation h at a point $\bar{I} \in D$ concerns lower estimates of the gradient of h restricted to any linear space λ orthogonal to $\nabla h(\bar{I})$. The definition is motivated by the following simpler estimate [**51**]. Let us consider the real polynomial:

$$P(x) = a_k x^k + a_{k-1} x^{k-1} + \ldots + a_0 \ \ , \ \ a_k \neq 0,$$

then there exists a constant $c(a_k)$ such that

$$\max_{0 \le \eta \le \xi} \min_{|x|=\eta} |P(x)| \ge c(a_k)\, \xi^k$$

for all $a_0, \ldots, a_{k-1}$.

Steepness is a multivariable generalization of this estimate, and is defined as follows.

DEFINITION 1.5 (Steepness [**49, 50, 51**]). *Let $h : D \to \mathbb{R}$ be a smooth function, with $D \subseteq \mathbb{R}^n$ open. We say that h is steep at a point $\bar{I} = (\bar{I}_1, \ldots, \bar{I}_n) \in D$ if the following two conditions hold:*

- $\nabla h(\bar{I}) \neq \underline{0}$ *;*
- *for each $m = 1, \ldots, n-1$, there exist constants $C_m > 0$, $\delta_m > 0$ and $\alpha_m \ge 1$ such that, given any m-dimensional linear space λ orthogonal to $\nabla h(\bar{I})$, we have:*

$$\max_{0 \le \eta \le \xi} \left(\min_{I \in \bar{I} + \lambda : \|\tilde{I} - \bar{I}\| = \eta} \|\nabla (h|_{\bar{I}+\lambda})(I)\| \right) > C_m\, \xi^{\alpha_m}, \ \ \forall \ \ \xi \in (0, \delta_m],$$

where $\nabla h|_{\bar{I}+\lambda}$ denotes the gradient of the restriction of h to the affine space through $\bar{I}$ spanned by λ.

For each $\bar{I}$, the numbers $0 < g \leq \|\nabla h(\bar{I})\|, C_1, \ldots, C_{n-1}$ and $\delta_1, \ldots, \delta_{n-1}$ are called the coefficients, and $\alpha_1, \ldots, \alpha_{n-1}$ the indices of steepness of the function h at $\bar{I}$.

We also say that h is steep in D with coefficients $g, C_1, \ldots, C_{n-1}, \delta_1, \ldots, \delta_{n-1}$ and indices $\alpha_1, \ldots, \alpha_{n-1}$, if h is steep at each point of D with these coefficients and indices.

The definition of steepness is given implicitly, thus it cannot be used to verify the steepness of a function. Therefore, we need some sufficient conditions for steepness, and we will see in Chapter 2 a possible way to construct them, following a suggestion by Nekhoroshev.

But first, let's try to understand the geometrical meaning of steepness. Let us suppose that a function h is steep at I and let us consider a m-dimensional linear space λ orthogonal to $\nabla h(I)$. Let γ be any curve on $I+\lambda$ that joins I to another point at a distance $d < \delta_m$ from I.

Then, on this curve we can find a point $\widetilde{I}$ such that the norm of $\nabla(h|_{I+\lambda})$ at $\widetilde{I}$ is bounded from below by a power of d: $\|\nabla(h|_{I+\lambda})(\widetilde{I})\| > C_m d^{\alpha_m}$.

This fact implies, in particular, that if a function h is steep at I, then on every plane λ orthogonal to $\nabla h(I)$ there does not exist any curve γ that joins I to some other point, such that $\nabla(h|_{I+\lambda})$ identically vanishes along γ.

We also observe that if the plane λ in the definition is not perpendicular to $\nabla h(I)$, then $\nabla(h|_{I+\lambda})(I) \neq \underline{0}$ and the coefficients and the indices of steepness can be always found.

The indices of steepness are essential for the stability estimates (1.8) and (1.9), because the parameters a and b depend only on them according to the following expressions:

$$a = \frac{2}{12z+3n+14}, \quad b = \frac{3a}{2\alpha_{n-1}}, \tag{1.13}$$

where

$$z = [\alpha_1(\alpha_2 \cdot \cdot \cdot (\alpha_{n-3}(\alpha_{n-2} \cdot n + n - 2) + n - 3) + \ldots + 2) + 1] - 1$$

for $n > 2$, whereas $z = 1$ for $n = 2$.

It is easy to see that the best estimates (1.8) and (1.9), and consequently the best stability, correspond to the lowest possible values of the indices: $\alpha_1 = \ldots = \alpha_{n-1} = 1$. These are the values that the indices of steepness assume for the quasi-convex functions, which, as we already said, represent the simplest class of steep functions.

Nekhoroshev asserts it is possible to prove that all the steep functions which are not quasi-convex have at least one of their indices of steepness greater than 1, for this reason we can say that the quasi-convex functions are the "steepest".

One may wonder if the exponential stability ensured by the Nekhoroshev Theorem is valid for an arbitrary Hamiltonian (1.1), with non-steep integrable approximation. Nekhoroshev proved that it is not, in particular he proved the existence of a rather large set $\mathscr{M}$ of non-steep functions with the following property. Let $h \in \mathscr{M}$, then a system with Hamiltonian (1.1) and with an appropriate perturbation εf, $f = f(h)$, has for any $\varepsilon > 0$ solutions $(I_\varepsilon(t), \varphi_\varepsilon(t))$ such that $I_\varepsilon(t)$ leaves its initial position $I_\varepsilon(0)$ with a speed of order ε during the time interval $1/\varepsilon$. Therefore the time of stability of these solutions is much less than $1/\varepsilon$, and coincides with the a priori estimate.

For example, the Hamiltonian system

$$H = \frac{1}{2}(I_1^2 - I_2^2) + \varepsilon \sin(\varphi_1 - \varphi_2)$$

admits the special solution $I_1 = -\varepsilon t$, $I_2 = \varepsilon t$, $\varphi_1 = -\frac{1}{2}\varepsilon t^2$, $\varphi_2 = -\frac{1}{2}\varepsilon t^2$, which satisfies:

$$\|I(t) - I(0)\| = \sqrt{2}\varepsilon t\,.$$

Therefore we have instability over a time of the order $1/\varepsilon$. In particular this solution moves along the line of equation $I_1 + I_2 = 0$, on which the integrable approximation of H is not steep.

The set of the steep functions is considerably larger than the quasi-convex and 3-jet non-degenerate functions. Moreover, many Hamiltonians describing real physical systems, are neither quasi-convex nor 3-jet non-degenerate. This is the case, for example, of the circular restricted three-body problem in a neighborhood of the Lagrangian points L_4 and L_5 for a specific value of the mass ratio [**6**], and also of the Riemann ellipsoids [**24**].

Hence, in order to apply the Nekhoroshev Theorem to systems of this kind, it would be useful to have sufficient conditions for steepness weaker than the 3-jet non-degeneracy.

Nekhoroshev provided a way to construct some sufficient conditions involving the r-jets of the function, from the following general result that he proved in [**51, 50**].

In what follows we denote by $J^r(n)$ the space of the r-jets of all smooth functions of n variables at a fixed point $\bar{I}$.

THEOREM 1.6. *For any $r \geq 2$ and $n \geq 2$, in $J^r(n)$ there exists a semi-algebraic set $\Sigma^r(n)$ with the following properties:*

a) *let h be an arbitrary function of class C^{2r-1} in a neighborhood of $\bar{I}$ with $\nabla h(\bar{I}) \neq \underline{0}$, and let $P_r(h)$ lie outside $\Sigma^r(n)$. Then h is steep in some neighborhood of $\bar{I}$;*
b) *for each $m = 1, \ldots, n-1$, the steepness index α_m of h in this neighborhood is not larger than $\overline{\alpha}_m$, where*

$$\overline{\alpha}_m = \begin{cases} \max\left[1, 2r-3-\frac{n(n-2)}{2}+2m(n-m-1)\right] & \textit{when } n \textit{ is even} \\ \\ \max\left[1, 2r-3-\frac{(n-1)^2}{2}+2m(n-m-1)\right] & \textit{when } n \textit{ is odd}; \end{cases} \tag{1.14}$$

c) *the co-dimension of* $\Sigma^r(n)$ *in* $J^r(n)$ *satisfies the following estimate*

$$\text{co-dim}\,\Sigma^r(n) \geq \begin{cases} \max\left[0, r-1-\frac{n(n-2)}{4}\right] & \textit{when } n \textit{ is even} \\ \\ \max\left[0, r-1-\frac{(n-1)^2}{4}\right] & \textit{when } n \textit{ is odd}. \end{cases} \tag{1.15}$$

In particular this co-dimension tends to infinity as $r \to \infty$.

If we are able to explicitly represent the set $\Sigma^r(n)$ by some algebraic conditions, then we have the possibility of verifying the steepness of a function through its r-jet.

Unfortunately, Nekhoroshev did not provide an explicit expression of $\Sigma^r(n)$, but provided a way to construct a certain subset $\sigma^r(n) \subseteq J^r(n)$ whose closure coincides with $\Sigma^r(n)$.

For each $r, n \geq 2$, the set $\sigma^r(n)$ is defined through a collection of systems $\mathscr{C}^r(n)$, and such collection is the starting point for constructing some explicit sufficient conditions for steepness.

Finally, we remark that there exist also alternative characterizations of the steepness of a function. In [**52**] Niederman proved a geometric criterion for steepness based on the existence of critical points. Precisely, he proved that an analytic function h in an open set $D \subseteq \mathbb{R}^n$ is steep on any compact set $\Sigma \subset D$, if and only if its restriction to any affine subspace of $\mathbb{R}^n$ admits only isolated critical points.

Before Niederman, a similar sufficient condition for steepness had been proved by Ilyashenko in [**37**]. There the author proves that a complex-valued holomorphic function on a domain of $\mathbb{C}^n$, whose restriction to any affine subspace admits only $\mathbb{C}$-isolated critical points, is steep on $\mathbb{C}^n$.

CHAPTER 2

New explicit sufficient conditions for steepness

This Chapter is devoted to the steepness of a function.

As anticipated in the previous Chapter, up to now the only known classes of steep functions are the quasi-convex and the 3-jet non-degenerate ones. Hence there is the problem of determining if a 3-jet degenerate function is steep or not.

Here we first review the sufficient conditions for steepness introduced by Nekhoroshev in [**51, 49, 50**]. Precisely, we give the complete characterization of the sets $\sigma^r(n)$ introduced at the end of Chapter 1, and we restrict to the cases $r = 4$, $n = 2,3,4$.

For the case $n = 2$ we give the explicit expression of the systems forming the collection $\mathscr{C}^4(2)$ and find the closure of the set $\sigma^4(2)$, instead for the cases $n = 3$ and $n = 4$, after giving the explicit expression of the systems in $\mathscr{C}^4(3)$ and $\mathscr{C}^4(4)$ respectively, for both sets we construct closed sets in which they are contained. This way, we formulate new explicit sufficient conditions for the steepness of functions with two, three and four degrees of freedom [**59**].

2.1. The sufficient conditions for steepness formulated by Nekhoroshev

In this Section we provide the characterization of the sets $\sigma^r(n) \subseteq J^r(n)$ introduced in Chapter 1, Section 1.5, for $r \geq 2$ and $n \geq 2$.

These sets are defined through collections $\mathscr{C}^r(n)$ of systems of equalities and inequalities, depending on the number n of degrees of freedom, the r-jet of a function and some auxiliary parameters. A certain r-jet belongs to $\sigma^r(n)$ if and only if it satisfies at least one of the systems of $\mathscr{C}^r(n)$. From such collections it is possible to construct sufficient conditions for steepness, one for each choice of r and n. In fact the following Proposition holds.

PROPOSITION 2.1. [**49, 50**] *For each $n \geq 2$ and $r \geq 2$ we have*

$$\overline{\sigma^r(n)} \equiv \Sigma^r(n)$$

where $\overline{\sigma^r(n)}$ denotes the closure of $\sigma^r(n)$.

As a consequence of Proposition 2.1, if h is a smooth function with non-zero gradient at $\bar{I}$, and $P_r(h)$ lies outside $\overline{\sigma^r(n)}$, then h is steep in some neighborhood of the point $\bar{I}$.

Hence, in order to construct the sufficient conditions for steepness, we need to compute the closure of $\sigma^r(n)$, when possible, or eventually a closed set containing $\sigma^r(n)$.

Before describing in details the set $\sigma^r(n)$, we introduce some notations. Given a smooth function $h : D \to \mathbb{R}$, with $D \subseteq \mathbb{R}^n$, and a point $\bar{I} \in D$, for any $v^1, \dots, v^k \in \mathbb{R}^n$, $k \geq 1$, we define

$$h^k[v^1, \dots, v^k] := \sum_{i_1, \dots, i_k = 1}^{n} \frac{\partial^k h}{\partial I_{i_1} \dots \partial I_{i_k}}(\bar{I}) v^1_{i_1} \dots v^k_{i_k}, \tag{2.1}$$

and we also denote by $h^{k+1}[v^1, \dots, v^k, \cdot]$ the vector such that, for any $v \in \mathbb{R}^n$,

$$h^{k+1}[v^1, \dots, v^k, v] = h^{k+1}[v^1, \dots, v^k, \cdot] \cdot v. \tag{2.2}$$

Then if $\nabla h(\bar{I}) \neq \underline{0}$, we denote by Λ the $(n-1)$-dimensional linear space orthogonal to $\nabla h(\bar{I})$ and by Π_Λ the orthogonal projection on Λ.

Now let us precisely see the way the set $\sigma^r(n)$ is defined. It is the set which contains the r-jets at $\bar{I}$ of functions h of n variables such that:

- $\nabla h(\bar{I}) \neq \underline{0}$;
- in the action-space there exists a curve $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$ of the type:

$$\begin{aligned} \gamma_l(t) &= \bar{I}_l + t \\ \gamma_i(t) &= \bar{I}_i + \textstyle\sum_{j=1}^{\beta_m-1} b_{ij} t^j \qquad i \neq l \end{aligned} \tag{2.3}$$

with the following properties: first, it is contained in a m-dimensional space $\lambda \subseteq \Lambda$, moreover the gradient of the restriction of the function h to λ has a zero of order not smaller than $\beta_m - 1$ with respect to the parameter t on the curve γ at $t = 0$, that is:

$$\frac{d^p}{dt^p}\bigg|_{t=0} \|\nabla(h|_\lambda)|_{I=\gamma(t)}\| = 0,\ p = 0, 1, \ldots, \beta_m - 1\,,$$

where $\beta_m \in \{2, \ldots, r\}$ is a parameter which depends on r, n and m, and is defined as follows:

$$\beta_m = \frac{1}{2}(\overline{\alpha}_m + 3) \tag{2.4}$$

where $\overline{\alpha}_m$ is defined in (1.14).

2.1.1. Algebraic characterization of $\sigma^r(n)$. Nekhoroshev gave an algebraic characterization to $\sigma^r(n)$ by introducing $(n-1)$ systems $\mathbf{S_m}(h)$, $m = 1, \ldots, n-1$, one for each possible dimension of the space $\lambda \subseteq \Lambda$.

Each of these systems consists of four subsystems of polynomial equations and inequalities:

$$\mathbf{S_m}(h) := \begin{cases} \mathscr{S}_m 1(h) \\ \mathscr{S}_m 2(h, A^i) \\ \mathscr{S}_m 3(h, A^i) \\ \mathscr{S}_m 4(h, A^i, b_{ij}) \end{cases} \qquad m = 1, \ldots, n-1 \tag{2.5}$$

defined below. The systems $\mathscr{S}_m 1, \ldots, \mathscr{S}_m 4$ depend on:

- linearly independent vectors $A^1, \ldots, A^m \in \mathbb{R}^n$, which represent a basis for λ;
- real coefficients b_{ij} ($i = 2, \ldots, m$; $j = 1, \ldots, \beta_m - 1$) which determine a curve γ as in (2.3);

and are defined as follows:

- $\mathscr{S}_m 1$ imposes the gradient of h to be non-zero at the point $\bar{I}$:

$$\nabla h(\bar{I}) \neq \underline{0};$$

- $\mathscr{S}_m 2$ imposes the vectors $A^1,\ldots,A^m$ to be linearly independent:

$$\text{rank}[A^1,\ldots,A^m]=m\,;$$

- $\mathscr{S}_m 3$ imposes the vectors $A^1,\ldots,A^m$ to belong to Λ:

$$\begin{cases} h^1[A^1]=0 \\ \vdots \\ h^1[A^m]=0\,; \end{cases}$$

- $\mathscr{S}_m 4$ contains a system of $m(\beta_m-1)$ equations that we obtain in the following way. First we write the restriction of the Taylor polynomial p of order r of the function h to the space $\bar{I}+\lambda$, where λ is spanned by $A^1,\ldots,A^m$, and we truncate it at order β_m. Let us first represent this polynomial by $p(y)=\sum_{1\le|\nu|_1\le r} h_\nu\, y^\nu$, where $y=I-\bar{I}\in\mathbb{R}^n$ and $y^\nu=y_1^{\nu_1}y_2^{\nu_2}\cdots y_n^{\nu_n}$.

  In order to compute the restriction of $p(y)$ to λ, we introduce the coordinates $x\in\mathbb{R}^m$ on λ by $y=y(x)=\mathbf{A}x$, where the columns of the matrix $\mathbf{A}$ are the vectors A^i : $\mathbf{A}:=((A^1)^T,\ldots,(A^m)^T)$. We obtain the polynomial $\widetilde{p}(x)=p(y(x))$ in $x_1,\ldots,x_m$.

  Then, by truncating $\widetilde{p}(x)$ at the order β_m, we obtain $f(x)=\sum_{1\le|\mu|_1\le\beta_m} f_\mu\, x^\mu$. The coefficients f_μ form the β_m-jet of the restriction of h to λ. We find useful to represent the coefficients f_μ by using also the notation introduced in (2.1).

  Precisely, from

$$\begin{aligned}\tilde{p}(x) \;=\; & \sum_{i=1}^{m} h^1[A^i]\,x_i+\frac{1}{2}\sum_{i,j=1}^{m} h^2[A^i,A^j]\,x_ix_j+\frac{1}{6}\sum_{i,j,k=1}^{m} h^3[A^i,A^j,A^k]\,x_ix_jx_k \\ & +\frac{1}{24}\sum_{i,j,k,l=1}^{m} h^4[A^i,A^j,A^k,A^l]\,x_ix_jx_kx_l+\ldots \end{aligned} \tag{2.6}$$

  we immediately obtain that:
  - the coefficients f_μ with $|\mu|_1=1$ are $h^1[A^i]$, with $i=1,\ldots m$;
  - the coefficients f_μ with $|\mu|_1=2$ are $h^2[A^i,A^j]/2$, with $i,j=1,\ldots m$;

– if $\beta_m \geq 3$, the coefficients f_μ with $|\mu|_1 = 3$ are $h^3[A^i, A^j, A^k]/6$, with $i, j, k = 1, \dots m$;
– if $\beta_m \geq 4$, the coefficients f_μ with $|\mu|_1 = 4$ are $h^4[A^i, A^j, A^k, A^l]/24$, with $i, j, k, l = 1, \dots m$.

Then we consider the curve $x(t)$, $t \in \mathbb{R}$, defined by:

$$\begin{cases} x_1(t) = t \\ x_i(t) = \sum_{j=1}^{\beta_m - 1} b_{ij} t^j \qquad \text{for } i = 2, \dots, m \end{cases}$$

and we compute the partial derivatives $\frac{\partial f}{\partial x_i}$ at $x(t)$, $i = 1, \dots, m$.

In such a way, we obtain m polynomials in t: by setting all the coefficients of $t, t^2, \dots, t^{\beta_m - 1}$ of these polynomials to zero we obtain the system $\mathscr{S}_m 4$ of $m(\beta_m - 1)$ equations.

We say that $\mathbf{S_m}(h)$ is solvable for h if there exist $A^1, \dots, A^m$ and b_{ij} such that all the subsystems $\mathscr{S}_m 1, \dots, \mathscr{S}_m 4$ are verified.

The systems $\mathbf{S_m}(h)$, $m = 1, \dots, n-1$, form the collection $\mathscr{C}^r(n)$, hence we say that the r-jet $P_r(h)$ belongs to $\sigma^r(n)$ if there exists $m \in \{1, \dots, n-1\}$ such that $\mathbf{S_m}(h)$ is solvable for h.

We will focus our attention on the case $r = 4$ for the following reason: if $r = 2$ or $r = 3$, Theorem 1.6 is not useful to produce new sufficient conditions for steepness. In fact, when $r = 2$ the conditions provided by Theorem 1.6 correspond to quasi-convexity for all $n \geq 2$, while for $r = 3$ they are only a slight modification of the 3-jet non-degeneracy. In particular, for $r = 3$ the conditions provided by Theorem 1.6 do not let us identify a class of steep functions larger than the 3-jet non-degenerate ones.

2.1.2. Explicit expression of the systems defining $\sigma^r(n)$. Now let us see the explicit expression of the systems defining $\sigma^r(n)$. For any $r, n \geq 2$, the set $\sigma^r(n)$ contains the r-jets of all functions $h(I)$, smooth in a neighborhood of $\bar{I}$, for which at least one of the systems $\mathbf{S_1}(h), \dots, \mathbf{S_{n-1}}(h)$ described previously, is solvable.

As we saw before, each system $\mathbf{S_m}(h)$, $m = 1, \ldots, n-1$, is formed by four subsystems $\mathscr{S}_m 1, \ldots, \mathscr{S}_m 4$. The subsystems $\mathscr{S}_m 1, \mathscr{S}_m 2, \mathscr{S}_m 3$ are easily expressed for all n, m (see Subsection 2.1.1), while the expression of the subsystem $\mathscr{S}_m 4$ depends on the value of β_m: the lower is β_m, the simpler is $\mathscr{S}_m 4$.

Let us consider generic $n \geq 2$ and $r \geq 2$.

The special case $m = 1$

We first observe that $2 \leq \beta_1 \leq r$. In fact,

- if $n = 2k$ with $k \geq 1$ integer, the function

$$f_p(k) := 2r - 3 - \frac{n(n-2)}{2} + 2(n-2) = 2r - 7 - 2k^2 + 6k \qquad (2.7)$$

  is strictly monotone decreasing for $k \geq 2$, and has its maximum $2r-3$ both for $k = 1$ and $k = 2$. Correspondingly, from (2.4) we have the maximum value of $\beta_1 = r$;
- if $n = 2k+1$ with $k \geq 1$, the function

$$f_d(k) := 2r - 3 - \frac{(n-1)^2}{2} + 2(n-2) = 2r - 3 - 2(k-1)^2 \qquad (2.8)$$

  is strictly monotone decreasing for $k \geq 1$, and has its maximum $2r-3$ for $k = 1$. Correspondingly, from (2.4) we have the maximum value of $\beta_1 = r$.

According to the specific value of β_1, the system $\mathbf{S_1}(h)$ is defined by:

$$\mathbf{S_1}(h) := \begin{cases} \nabla h(\bar{I}) \neq \underline{0} \\ h^1[A] = 0 \\ h^2[A, A] = 0 \\ \ldots \\ h^{\beta_1}[A, \ldots, A] = 0 \end{cases} \qquad (2.9)$$

where $A \in \mathbb{R}^n$. In fact,

- $\mathscr{S}_1 1$ provides: $\nabla h(\bar{I}) \neq \underline{0}$;
- $\mathscr{S}_1 2$ provides the condition: $A := A^1 \neq \underline{0}$;

- $\mathscr{S}_1 3$ provides the condition: $h^1[A] = 0$;
- $\mathscr{S}_1 4$: the polynomial $f(x)$ is defined by:

  $$f(x) = h^1[A]\,x + \frac{1}{2}h^2[A,A]\,x^2 + \ldots + \frac{1}{\beta_1!}h^{\beta_1}[A,\ldots,A]x^{\beta_1},$$

  hence by setting to zero the coefficients in $t, t^2, \ldots, t^{\beta_1 - 1}$ of $\partial f/\partial x$ computed at $x(t) = t$, we obtain the conditions $h^2[A,A] = 0, \ldots, h^{\beta_1}[A,\ldots,A] = 0$.

According to the definitions given previously, the system $\mathbf{S_1}(h)$ is solvable for h if it has a solution vector $A \in \mathbb{R}^n \setminus \{\underline{0}\}$.

The special cases $\beta_1 = 2$ and $\beta_1 = 3$

It is possible to prove that when $\beta_1 = 2$ or $\beta_1 = 3$, the conditions $P_r(h) \notin \Sigma^r(n)$ and $\nabla h(\bar{I}) \neq \underline{0}$ correspond respectively to the quasi-convexity and the 3-jet non-degeneracy of h at $\bar{I}$. Therefore in such cases Theorem 1.6 is not useful to produce new sufficient conditions for steepness.

In particular, when $r = 4$ we have $\beta_1 = 4$ for $n = 2, 3, 4$, and $\beta_1 \leq 3$ in all the other cases. In fact

- if $n = 2k$, with $k \geq 1$, the function $f_p(k)$ defined in (2.7) for $r = 4$ is

  $$f_p(k) = 1 - 2k^2 + 6k,$$

  which is equal to 5 if $k = 1$ and $k = 2$, that is if $n = 2$ and $n = 4$. Correspondingly, in both cases from (2.4) we have $\beta_1 = 4$. Instead for $k \geq 3$ we have $f_p(k) \leq 1$, and then from (2.4) $\beta_1 = 2$;
- if $n = 2k + 1$, with $k \geq 1$, the function $f_d(k)$ defined in (2.8) for $r = 4$ is

  $$f_d(k) = 5 - 2(k-1)^2$$

  which is equal to 5 if $k = 1$, that is if $n = 3$. Correspondingly from (2.4) we have $\beta_1 = 4$. Instead when $k \geq 2$, it is $f_d(k) \leq 3$, and consequently $\beta_1 \leq 3$.

For these reasons, we will investigate in details only the cases $r = 4$ and $n = 2, 3, 4$, that is we will construct new sufficient conditions for steepness involving the 4-jet of functions of two, three and four variables.

Example: the set $\sigma^4(3)$

As an example, we report here the explicit expression of the systems defining $\sigma^r(n)$ when $r = 4$ and $n = 3$. Since $n = 3$, the collection $\mathscr{C}^4(3)$ contains the two systems $\mathbf{S_1}(h)$ and $\mathbf{S_2}(h)$, defined by:

$$\mathbf{S_1}(h) := \begin{cases} \nabla h(\bar{I}) \neq \underline{0} \\ h^1[A] = 0 \\ h^2[A,A] = 0 \\ h^3[A,A,A] = 0 \\ h^4[A,A,A,A] = 0 \end{cases} \qquad \mathbf{S_2}(h) := \begin{cases} \nabla h(\bar{I}) \neq \underline{0} \\ h^1[A^1] = 0 \\ h^1[A^2] = 0 \\ h^2[v,A^1] = 0 \\ h^2[v,A^2] = 0 \\ 2\, b_{22}\, h^2[A^1,A^2] + h^3[v,v,A^1] = 0 \\ 2\, b_{22}\, h^2[A^2,A^2] + h^3[v,v,A^2] = 0 \end{cases}$$

where $A, A^1, A^2 \in \mathbb{R}^3$, $b_{21}, b_{22} \in \mathbb{R}$ and $v = A^1 + b_{21}A^2$.

The system $\mathbf{S_1}(h)$ is solvable for h if it is verified by some vector $A \neq \underline{0}$, while system $\mathbf{S_2}(h)$ is solvable for h if there exist two linearly independent vectors A^1, A^2 and real coefficients b_{21}, b_{22} such that $\mathbf{S_2}(h)$ is verified.

2.2. Statement of new sufficient conditions for steepness

Consider a smooth function $h : D \longrightarrow \mathbb{R}$, with $D \subseteq \mathbb{R}^n$ open. Before stating the results, we give the following

DEFINITION 2.2. *For any $\bar{I} \in D$, the function h is called 4-jet non-degenerate at $\bar{I}$ if the system*

$$\begin{cases} h^1[v] = 0 \\ h^2[v,v] = 0 \\ h^3[v,v,v] = 0 \\ h^4[v,v,v,v] = 0 \end{cases} \tag{2.10}$$

has only the trivial solution $v = (v_1, \dots, v_n) = \underline{0}$. *Otherwise, it is called 4-jet degenerate.*

It is important to remark that, except for the special case $n = 2$, the 4-jet non-degeneracy is not a sufficient condition for steepness.

As an example, we consider the following function of three degrees of freedom $h(I_1, I_2, I_3) = \frac{1}{2}(I_1 - \frac{I_2^2}{2})^2 + I_3$: it is 4-jet non-degenerate and non-steep at the point $\bar{I} = (0,0,0)$. In fact, the gradient of the restriction of h to the 2-dimensional space Λ orthogonal to $\nabla h(\bar{I}) = (0,0,1)$ vanishes on the curve

$$\gamma := \begin{cases} I_1 = \frac{I_2^2}{2} \\ I_3 = 0. \end{cases}$$

Similarly, the function $h(I) = \frac{1}{2}(I_1 - \frac{I_2^2}{2})^2 + I_3^2 + I_4$, with four degrees of freedom, is 4-jet non-degenerate but non-steep at $\bar{I} = (0,0,0,0)$.

The sufficient conditions for steepness we propose are stated in the following three Propositions, which refer to functions of two, three and four variables respectively.

PROPOSITION 2.3. [**59**] *Let* $h : D \longrightarrow \mathbb{R}$, *with* $D \subseteq \mathbb{R}^2$ *open, be a smooth function, and let* $\bar{I} \in D$ *satisfy* $\nabla h(\bar{I}) \neq \underline{0}$. *If h is 4-jet non-degenerate at* $\bar{I}$, *then h is steep in a neighborhood of* $\bar{I}$.

PROPOSITION 2.4. [**59**] *Let* $h : D \longrightarrow \mathbb{R}$, *with* $D \subseteq \mathbb{R}^3$ *open, be a smooth function, and let* $\bar{I} \in D$ *satisfy* $\nabla h(\bar{I}) \neq \underline{0}$. *If*

1. *h is 4-jet non-degenerate at* $\bar{I}$*;*
2. *the system*

$$\begin{cases} h^1[v] = 0 \\ \Pi_\Lambda h^2[v, \cdot] = \underline{0} \\ h^3[v,v,v] = 0 \end{cases} \tag{2.11}$$

has the only solution $v = (v_1, v_2, v_3) = \underline{0}$*;*

then h is steep in a neighborhood of $\bar{I}$.

We remark that an equivalent formulation of condition 2 of Proposition 2.4, is the following one: *for all* $v \in \mathbb{R}^3 \setminus \{\underline{0}\}$ *such that*

$$\begin{cases} h^1[v] = 0 \\ h^2[v,v] = 0 \\ h^3[v,v,v] = 0 \end{cases} \tag{2.12}$$

the system

$$\begin{cases} v \cdot w = 0 \\ h^1[w] = 0 \\ h^2[v,w] = 0 \end{cases} \tag{2.13}$$

has the only solution $w = (w_1, w_2, w_3) = \underline{0}$. [1]

PROPOSITION 2.5. [**59**] *Let* $h : D \longrightarrow \mathbb{R}$*, with* $D \subseteq \mathbb{R}^4$ *open, be a smooth function, and let* $\bar{I} \in D$ *satisfy* $\nabla h(\bar{I}) \neq \underline{0}$*. If*

1. *h is 4-jet non-degenerate at* $\bar{I}$*;*
2. *the restriction of the Hessian operator* $h''(\bar{I})$ *to the linear space* Λ *is non-degenerate, that is to say it is a non-singular matrix;*
3. *for all* $v \in \mathbb{R}^4 \setminus \{\underline{0}\}$ *such that*

$$\begin{cases} h^1[v] = 0 \\ h^2[v,v] = 0 \\ h^3[v,v,v] = 0 \end{cases} \tag{2.14}$$

   the system

$$\begin{cases} v \cdot w = 0 \\ h^1[w] = 0 \\ h^2[v,w] = 0 \\ h^2[w,w]\, h^4[v,v,v,v] = 3(h^3[v,v,w])^2 \end{cases} \tag{2.15}$$

   has the only solution $w = (w_1, w_2, w_3, w_4) = \underline{0}$*;*

then h is steep in some neighborhood of $\bar{I}$*.*

[1] Notice that from Proposition 2.4 it follows that if *h* is 3-jet non-degenerate, than it is steep. Analogous evident implications of the same result hold for any number of degrees of freedom *n*.

2.3. Examples of 3-jet degenerate steep functions

Example 1. The function of three variables $h(I) = I_1 + \frac{I_2^2}{2} - \frac{I_3^2}{2} - \frac{I_3^4}{4}$ is 3-jet non-degenerate at all points $I = (I_1, I_2, I_3)$ with $I_3 \neq 0$. Then, we consider I with $I_3 = 0$ and prove that at these points h satisfies the hypotheses of Proposition 2.4. Clearly, h is 3-jet degenerate and 4-jet non-degenerate at I.

Since $\nabla h(I_1, I_2, 0) = (1, I_2, 0)$, the space Λ is spanned by the orthonormal vectors $e' = (0,0,1)$, $e'' = (-I_2, 1, 0)/\sqrt{1+I_2^2}$.

The restriction of the Hessian matrix $h''(I)$ to Λ in the basis e', e'' is represented by the matrix

$$h''_\Lambda(I) = \begin{pmatrix} h''e' \cdot e' & h''e' \cdot e'' \\ h''e'' \cdot e' & h''e'' \cdot e'' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & (1+I_2^2)^{-1} \end{pmatrix}$$

which is non-degenerate. Therefore, the only solution of system (2.11) is $v = (0,0,0)$.

Example 2. The function of four variables $h(I) = I_1 + \frac{I_2^2}{2} + \frac{I_3^2}{2} - \frac{I_4^2}{2} + \frac{I_4^4}{4}$ is 3-jet non-degenerate at all points $I = (I_1, I_2, I_3, I_4)$ with $I_4 \neq 0$. Then, we consider I with $I_4 = 0$ and prove that at these points h satisfies the hypotheses of Proposition 2.5.

Clearly, h is 3-jet degenerate and 4-jet non-degenerate at I. For $I_4 = 0$, the space Λ is generated by the orthogonal vectors $e' = (0,0,0,1)$, $e'' = \rho_1(-I_2 - I_3, 1, 1, 0)$ and $e''' = \rho_2(-I_2 + I_3, 1 + I_2 I_3 + I_3^2, -1 - I_2^2 - I_2 I_3, 0)$, with

$$\rho_1 = 1/\sqrt{1+(I_2+I_3)^2}$$
$$\rho_2 = 1/\sqrt{(I_3-I_2)^2 + (1+I_2I_3+I_3^2)^2 + (1+I_2^2+I_2I_3)^2}$$

By direct computation we obtain the restriction of the Hessian matrix $h''(I)$ to Λ in the basis e', e'', e'''

$$h''_\Lambda(I) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2\rho_1^2 & (I_3^2 - I_2^2)\rho_1\rho_2 \\ 0 & (I_3^2 - I_2^2)\rho_1\rho_2 & ((1+I_2I_3+I_2^2)^2 + (1+I_2I_3+I_3^2)^2)\rho_2^2 \end{pmatrix}$$

whose determinant is $-\rho_1^2\rho_2^2(2+(I_2+I_3)^2)^2 \neq 0$, so that $h''_\Lambda(I)$ is non-degenerate, with eigenvalues $\lambda_1 = -1$, $\lambda_3 \geq \lambda_2 > 0$ (since $\lambda_2\lambda_3$, $\lambda_2+\lambda_3 > 0$). Being $h''(I)$ a symmetric matrix, we introduce an orthonormal basis e_1, e_2, e_3, e_4 in $\mathbb{R}^4$ with e_4 parallel to $\nabla h(I)$, and e_1, e_2, e_3 eigenvectors of $h''_\Lambda(I)$, related to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ respectively. Then, we consider $v \neq \underline{0}$ such that $h^1[v] = h^2[v,v] = h^3[v,v,v] = 0$, and a vector $w \in \Lambda\backslash\{\underline{0}\}$ such that $v \cdot w = 0$, $h^2[v,w] = 0$, $h^2[w,w] = 0$ (for $I_4 = 0$, the last equation of (2.15) is $h^2[w,w] = 0$).

By denoting $v = \sum_{i=1}^3 v'_i e_i$, $w = \sum_{i=1}^3 w'_i e_i$, from (2.14) and (2.15) we obtain:

$$\begin{cases} v'_1 w'_1 + v'_2 w'_2 + v'_3 w'_3 = 0 \\ v'^2_1 = \lambda_2 v'^2_2 + \lambda_3 v'^2_3 \\ w'^2_1 = \lambda_2 w'^2_2 + \lambda_3 w'^2_3 \\ v'_1 w'_1 = \lambda_2 v'_2 w'_2 + \lambda_3 v'_3 w'_3 . \end{cases} \tag{2.16}$$

We can consider $v'_1 \neq 0$ and $w'_1 \neq 0$, otherwise from the second and third equations of (2.16) we would obtain also $v'_2 = v'_3 = 0$ and $w'_2 = w'_3 = 0$. Then, from the same equations, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\frac{v'_2}{v'_1} = \frac{1}{\sqrt{\lambda_2}}\cos\alpha \ , \ \frac{v'_3}{v'_1} = \frac{1}{\sqrt{\lambda_3}}\sin\alpha \ , \ \frac{w'_2}{w'_1} = \frac{1}{\sqrt{\lambda_2}}\cos\beta \ , \ \frac{w'_3}{w'_1} = \frac{1}{\sqrt{\lambda_3}}\sin\beta .$$

From the first and last equations of (2.16) we obtain

$$(\lambda_3 - \lambda_2)\sin\alpha\sin\beta = (1+\lambda_2)\lambda_3$$

which cannot be satisfied by any value of α, β (this is evident if $\lambda_2 = \lambda_3$, while if $\lambda_3 > \lambda_2 > 0$ we have $(1+\lambda_2)\lambda_3/(\lambda_3-\lambda_2) > 1$).

2.4. Proofs of Propositions 2.3, 2.4 and 2.5

To prove Propositions 2.3, 2.4 and 2.5 we need to apply the Theorem 1.6, that is to compute the closure of $\sigma^r(n)$, or eventually a closed set containing $\sigma^r(n)$, for specific values of r and n.

We call $\psi_1(n)$ the set which contains the r-jets at $\bar{I}$ of all functions h such that the set $\mathbf{S_1}(h)$, given by (2.9), is solvable with respect to h. Let us introduce also the larger set $\psi_1^*(n)$ containing the r-jets at $\bar{I}$ of all functions h such that the system

$$\begin{cases} h^1[A] = 0 \\ h^2[A,A] = 0 \\ \dots \\ h^{\beta_1}[A,\dots,A] = 0 \end{cases} \tag{2.17}$$

has a non-trivial solution $A \in \mathbb{R}^n \setminus \{\underline{0}\}$. We prove the following

LEMMA 2.6. *We have:*

$$\overline{\psi_1(n)} = \psi_1^*(n). \tag{2.18}$$

PROOF. We first prove that $\psi_1^*(n)$ is closed. In fact, let us consider a convergent sequence of elements of $\psi_1^*(n)$, that is r-jets $P_r(h_k)$ such that for each $k \geq 0$ there exists a vector $A_k \in \mathbb{R}^n \setminus \{\underline{0}\}$ that verifies:

$$\begin{cases} h_k^1[A_k] = 0 \\ h_k^2[A_k,A_k] = 0 \\ \dots \\ h_k^{\beta_1}[A_k,\dots,A_k] = 0 \end{cases} \tag{2.19}$$

and such that $\lim_{k\to\infty} P_r(h_k) = P_r(h)$, where $P_r(h)$ is the r-jet of some function h. We prove that $P_r(h)$ belongs to $\psi_1^*(n)$.

Since $A_k \neq \underline{0}\ \forall k$, we consider the sequence $\overline{A}_k = \frac{A_k}{\|A_k\|} \in \mathbb{S}^{n-1}$ which still verifies (2.19). We can extract from $\overline{A}_k$ a convergent subsequence $\overline{A}_{k_j}$: $\lim_{j\to\infty} \overline{A}_{k_j} = A \in \mathbb{S}^{n-1}$, hence from system (2.19) we have

$$\begin{cases} \lim_{j\to\infty} h_{k_j}^1[\overline{A}_{k_j}] = h^1[A] = 0 \\ \lim_{j\to\infty} h_{k_j}^2[\overline{A}_{k_j},\overline{A}_{k_j}] = h^2[A,A] = 0 \\ \dots \\ \lim_{j\to\infty} h_{k_j}^{\beta_1}[\overline{A}_{k_j},\dots,\overline{A}_{k_j}] = h^{\beta_1}[A,\dots,A] = 0. \end{cases}$$

Since $A \neq \underline{0}$, we have proved that $P_r(h) \in \psi_1^*(n)$.

Finally, we prove $\psi_1^*(n) = \overline{\psi_1(n)}$. It is evident that $\psi_1(n) \subseteq \psi_1^*(n)$, therefore since $\psi_1^*(n)$ is closed, we immediately obtain $\overline{\psi_1(n)} \subseteq \psi_1^*(n)$.

It remains to prove that $\psi_1^*(n) \subseteq \overline{\psi_1(n)}$, that is for each element $P_r(h) \in \psi_1^*(n)$ there exists a sequence of elements of $\psi_1(n)$ convergent to $P_r(h)$.

If $\nabla h(\bar{I}) \neq \underline{0}$, there is nothing more to prove. Therefore, we consider the case $\nabla h(\bar{I}) = \underline{0}$, and we denote by $A \in \mathbb{R}^n \setminus \{\underline{0}\}$ the solution of (2.17).
We consider $A^\perp \in \mathbb{R}^n \setminus \{\underline{0}\}$ such that $A^\perp \cdot A = 0$, and we define $f(I) := A^\perp \cdot I$. Since $\nabla f(\bar{I}) = A^\perp \neq \underline{0}$ and $f^1[A] = f^2[A,A] = \ldots = f^{\beta_1}[A,\ldots,A] = 0$, we take the following sequence of functions

$$h_k := h + \frac{1}{k} f$$

and we observe that each function h_k satisfies

$$\begin{cases} \nabla h_k(\bar{I}) = \frac{A^\perp}{k} \neq \underline{0} \\ h_k^1[A] = h_k^2[A,A] = \ldots = h_k^{\beta_1}[A,\ldots,A] = 0\,. \end{cases}$$

Moreover we have $\lim_{k\to\infty} P_r(h_k) = P_r(h)$, and this completes the proof. □

2.4.1. Proof of Proposition 2.3. We fix $r = 4$ and $n = 2$, and consider the set $\sigma^4(2)$. The collection of systems which defines $\sigma^4(2)$ contains only the system $\mathbf{S_1}(h)$. Since $\beta_1 = 4$, from (2.9) we have:

$$\mathbf{S_1}(h) := \begin{cases} \nabla h(\bar{I}) \neq \underline{0} \\ h^1[A] = 0 \\ h^2[A,A] = 0 \\ h^3[A,A,A] = 0 \\ h^4[A,A,A,A] = 0 \end{cases} \tag{2.20}$$

where $A \in \mathbb{R}^2$.

Hence, $\sigma^4(2)$ is the set of the 4-jets $P_4(h)$ such that there exists $A \in \mathbb{R}^2 \setminus \{\underline{0}\}$ satisfying $\mathbf{S_1}(h)$, that is $\sigma^4(2) = \psi_1(2)$, where $\psi_1(2)$ has been defined at the beginning of this Section.

From Lemma 2.6, we have

$$\overline{\sigma^4(2)} = \overline{\psi_1(2)} = \psi_1^*(2)\,.$$

The proof of Proposition 2.3 follows immediately from Theorem 1.6 and Proposition 2.1.

2.4.2. Proof of Proposition 2.4. We fix $r = 4$ and $n = 3$, and consider the set $\sigma^4(3)$. We first formulate the explicit expression of the collection of systems that defines $\sigma^4(3)$ and provide a more compact formulation. Then, we find a closed set containing $\sigma^4(3)$ and conclude the proof of Proposition 2.4.

2.4.2.1. *Explicit formulation of* $\sigma^4(3)$. The collection of systems which defines $\sigma^4(3)$ contains the two systems $\mathbf{S_1}(h)$ and $\mathbf{S_2}(h)$.

Since $\beta_1 = 4$ and $\beta_2 = 3$, we have:

$$\mathbf{S_1}(h) := \begin{cases} \nabla h(\bar{I}) \neq \underline{0} \\ h^1[A] = 0 \\ h^2[A,A] = 0 \\ h^3[A,A,A] = 0 \\ h^4[A,A,A,A] = 0 \end{cases} \qquad \mathbf{S_2}(h) := \begin{cases} \nabla h(\bar{I}) \neq \underline{0} \\ h^1[A^1] = 0 \\ h^1[A^2] = 0 \\ h^2[v,A^1] = 0 \\ h^2[v,A^2] = 0 \\ 2\, b_{22}\, h^2[A^1,A^2] + h^3[v,v,A^1] = 0 \\ 2\, b_{22}\, h^2[A^2,A^2] + h^3[v,v,A^2] = 0 \end{cases} \tag{2.21}$$

where $A, A^1, A^2 \in \mathbb{R}^3$, $v = A^1 + b_{21}A^2$ and $b_{21}, b_{22} \in \mathbb{R}$.

For any h, we say that $\mathbf{S_1}(h)$ is solvable for h if the system $\mathbf{S_1}(h)$ is satisfied by some $A \neq \underline{0}$; we say that $\mathbf{S_2}(h)$ is solvable for h if $\mathbf{S_2}(h)$ is satisfied by two linearly independent vectors A^1, A^2 and real coefficients b_{21}, b_{22}.

The expression of $\mathbf{S_1}(h)$ has been already obtained in Section 2.1. Therefore, it remains to show how we obtained the expression of $\mathbf{S_2}(h)$.

In such a case:

- $\mathscr{S}_2 1$ provides: $\nabla h(\bar{I}) \neq \underline{0}$;
- $\mathscr{S}_2 2$ provides the condition $\mathrm{rank}[A^1, A^2] = 2$, which means that the vectors A^1, A^2 must be linearly independent;
- $\mathscr{S}_2 3$ provides the conditions: $h^1[A^1] = 0$, $h^1[A^2] = 0$;
- $\mathscr{S}_2 4$: since for $n = 3$ we have $\beta_2 = 3$, the polynomial $f(x)$ is defined by

$$\begin{aligned} f(x) &= \sum_{i=1}^{2} h^1[A^i]\,x_i + \frac{1}{2}\Big(h^2[A^1,A^1]x_1^2 + 2h^2[A^1,A^2]x_1x_2 + h^2[A^2,A^2]x_2^2\Big) \\ &+ \frac{1}{6}\Big(h^3[A^1,A^1,A^1]x_1^3 + 3h^3[A^1,A^1,A^2]x_1^2x_2 + 3h^3[A^1,A^2,A^2]x_1x_2^2 \\ &+ h^3[A^2,A^2,A^2]x_2^3\Big). \end{aligned}$$

We set to zero the coefficients of t,t^2 of $\partial f/\partial x_i$ computed at $x_1(t) = t$, $x_2(t) = b_{21}t + b_{22}t^2$.

The coefficients of t provide the additional conditions

$$\begin{array}{l} h^2[A^1,A^1] + b_{21}h^2[A^1,A^2] = h^2[A^1,A^1 + b_{21}A^2] = 0 \\ h^2[A^2,A^1] + b_{21}h^2[A^2,A^2] = h^2[A^2,A^1 + b_{21}A^2] = 0 \end{array} \quad ,$$

that is, by introducing the compact notation $v = A^1 + b_{21}A^2$,

$$h^2[v,A^1] = 0 \ , \ h^2[v,A^2] = 0.$$

The coefficients of t^2 provide the additional conditions

$$\begin{array}{l} b_{22}h^2[A^1,A^2] + \frac{1}{2}h^3[A^1,A^1,A^1] + b_{21}h^3[A^1,A^1,A^2] + \frac{1}{2}b_{21}^2h^3[A^1,A^2,A^2] = 0 \\ b_{22}h^2[A^2,A^2] + \frac{1}{2}h^3[A^2,A^1,A^1] + b_{21}h^3[A^2,A^2,A^1] + \frac{1}{2}b_{21}^2h^3[A^2,A^2,A^2] = 0 \end{array} \quad ,$$

that is

$$\begin{array}{l} b_{22}h^2[A^1,A^2] + \frac{1}{2}h^3[A^1,v,v] = 0 \\ b_{22}h^2[A^2,A^2] + \frac{1}{2}h^3[A^2,v,v] = 0 \end{array} \quad .$$

Therefore, the expression of $\mathbf{S_2}(h)$ in (2.21) follows.

2.4.2.2. *A reduced formulation of* $\mathbf{S_2}(h)$. The system defining $\mathbf{S_2}(h)$ in (2.21) depends on the variables $A^1, A^2 \in \mathbb{R}^3$ and $b_{21}, b_{22} \in \mathbb{R}$. We here rewrite $\mathbf{S_2}(h)$ in a more compact form, by reducing also the number of variables. Precisely, the system $\mathbf{S_2}(h)$ can be also formulated as:

$$\mathbf{S_2}(h) = \begin{cases} \nabla h(\bar{I}) \neq \underline{0} \\ h^1[v] = 0 \\ h^1[u] = 0 \\ \Pi_\Lambda h^2[v,\cdot] = \underline{0} \\ \Pi_\Lambda(2\alpha h^2[u,\cdot] + h^3[v,v,\cdot]) = \underline{0}, \end{cases} \tag{2.22}$$

and it is solvable for h if there exist two linearly independent vectors $v,u \in \mathbb{R}^3$, and a real number α, which verify (2.22).

In fact, let us consider two linearly independent vectors A^1, A^2, and $b_{21}, b_{22} \in \mathbb{R}$ satisfying the system $\mathbf{S_2}(h)$ in (2.21) for some function h.

In particular, A^1, A^2 is a basis for Λ, and using the notation introduced in (2.2), we have

$$\begin{matrix} h^2[v,A^1] = h^2[v,\cdot]\cdot A^1 = 0 \\ h^2[v,A^2] = h^2[v,\cdot]\cdot A^2 = 0 \end{matrix} \quad \Longleftrightarrow \quad \Pi_\Lambda h^2[v,\cdot] = \underline{0}.$$

Similarly we obtain $\Pi_\Lambda(2b_{22}h^2[A^2,\cdot] + h^3[v,v,\cdot]) = \underline{0}$. Therefore, the vectors $u = A^2$, $v = A^1 + b_{21}A^2$ are linearly independent and, with the parameter $\alpha = b_{22}$, they solve the system (2.22) for h.

Vice versa, if $\alpha \in \mathbb{R}$ and u,v linearly independent vectors are such that they solve the system (2.22), then $A^2 = u$, $A^1 = v - u$, $b_{21} = 1$, $b_{22} = \alpha$ solve the system in (2.21) and $A^1, A^2 \in \Lambda$ are linearly independent.

2.4.2.3. *A closed set containing* $\sigma^4(3)$. The set $\sigma^4(3)$ contains the 4-jets $P_4(h)$ such that at least one of the systems $\mathbf{S_1}(h)$ and $\mathbf{S_2}(h)$ is solvable for h.

We will prove that the closure of $\sigma^4(3)$ is contained in the union of two closed sets: the set $\psi_1^*(3)$ defined at the beginning of this Section, and a set $\psi_2^*(3)$ that we define below.

LEMMA 2.7. *We denote by $\psi_2^*(3)$ the set of 4-jets $P_4(h)$ such that there exist a 2-dimensional space λ and a vector $v \in \lambda \setminus \{\underline{0}\}$ verifying:*

$$\begin{cases} \Pi_\lambda \nabla h(\bar{I}) = \underline{0} \\ \Pi_\lambda\, h^2[v,\cdot] = \underline{0} \\ h^3[v,v,v] = 0\,. \end{cases} \tag{2.23}$$

The set $\psi_2^*(3)$ is closed.

PROOF. Let us consider a sequence of 4-jets $P_4(h_k)$ in $\psi_2^*(3)$, convergent to some $P_4(h)$. Hence, for any k there exist a 2-dimensional space λ_k and a vector $v_k \in \lambda_k \setminus \{\underline{0}\}$, which verify:

$$\begin{cases} \Pi_{\lambda_k} \nabla h_k(\bar{I}) = \underline{0} \\ \Pi_{\lambda_k} h_k^2[v_k,\cdot] = \underline{0} \\ h_k^3[v_k,v_k,v_k] = 0\,, \end{cases} \tag{2.24}$$

and $\lim_{k\to\infty} P_4(h_k) = P_4(h)$. We prove that $P_4(h) \in \psi_2^*(3)$.

For each k, let us choose $u_k \in \lambda_k \setminus \{\underline{0}\}$ orthogonal to v_k. Then the system (2.24) implies:

$$\begin{cases} h_k^1[v_k] = h_k^1[u_k] = 0 \\ h_k^2[v_k,v_k] = h_k^2[v_k,u_k] = 0 \\ h_k^3[v_k,v_k,v_k] = 0 \\ v_k \cdot u_k = 0\,. \end{cases} \tag{2.25}$$

Since v_k, $u_k \neq \underline{0}$, we consider the two sequences $\overline{v}_k = \frac{v_k}{\|v_k\|}$ and $\overline{u}_k = \frac{u_k}{\|u_k\|}$ in the unit sphere $\mathbb{S}^2$, still verifying (2.25).

From the sequence $(\overline{v}_k, \overline{u}_k)$, defined on the compact set $\mathbb{S}^2 \times \mathbb{S}^2$, we can extract a convergent subsequence $(\overline{v}_{k_j}, \overline{u}_{k_j})$: $\lim_{j\to\infty}(\overline{v}_{k_j}, \overline{u}_{k_j}) = (v,u) \in \mathbb{S}^2 \times \mathbb{S}^2$, and from system (2.25) it follows:

$$\begin{cases} \lim_{j\to\infty} h_{k_j}^1[\overline{v}_{k_j}] = h^1[v] = 0 \\ \lim_{j\to\infty} h_{k_j}^1[\overline{u}_{k_j}] = h^1[u] = 0 \\ \lim_{j\to\infty} h_{k_j}^2[\overline{v}_{k_j},\overline{v}_{k_j}] = h^2[v,v] = 0 \\ \lim_{j\to\infty} h_{k_j}^2[\overline{v}_{k_j},\overline{u}_{k_j}] = h^2[v,u] = 0 \\ \lim_{j\to\infty} h_{k_j}^3[\overline{v}_{k_j},\overline{v}_{k_j},\overline{v}_{k_j}] = h^3[v,v,v] = 0 \\ \lim_{j\to\infty} \overline{v}_{k_j} \cdot \overline{u}_{k_j} = v \cdot u = 0 \end{cases} \iff \begin{cases} \Pi_\lambda \nabla h(\bar{I}) = \underline{0} \\ \Pi_\lambda h^2[v,\cdot] = \underline{0} \\ h^3[v,v,v] = 0 \end{cases}$$

where λ is the 2-dimensional space generated by v and u, which are linearly independent. □

Finally, we prove the following

LEMMA 2.8. *We have:*

$$\overline{\sigma^4(3)} \subseteq \psi_1^*(3) \cup \psi_2^*(3)\,. \tag{2.26}$$

PROOF. $\sigma^4(3)$ is the union of two sets: the set of 4-jets $P_4(h)$ such that $\mathbf{S_1}(h)$ is solvable for h, and the set of 4-jets $P_4(h)$ such that $\mathbf{S_2}(h)$ is solvable for h. The first one is the set $\psi_1(3)$ defined at the beginning of this Section, and we call $\psi_2(3)$ the second one.

From Lemma 2.6 we immediately obtain $\overline{\psi_1(3)} = \psi_1^*(3)$. Therefore, we prove $\overline{\psi_2(3)} \subseteq \psi_2^*(3)$. It is sufficient to prove that $\psi_2(3) \subseteq \psi_2^*(3)$. Then, since $\psi_2^*(3)$ is closed, $\overline{\psi_2(3)} \subseteq \psi_2^*(3)$ follows immediately.

Let $P_4(h) \in \psi_2(3)$, and let us consider linearly independent vectors $u, v \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$ satisfying $\mathbf{S_2}(h)$. From the last two equations of $\mathbf{S_2}(h)$ it follows that $v \in \Lambda$ satisfies also $h^3[v,v,v] = 0$, hence it is also a solution of (2.23) with $\lambda = \Lambda$. □

We now consider h with $\nabla h(\bar{I}) \neq \underline{0}$, so that $P_4(h) \in \psi_2^*(3)$ if and only if condition (2.23) is satisfied by $\lambda = \Lambda$ and some $v \in \Lambda \setminus \{\underline{0}\}$. Then, Proposition 2.4 follows from Theorem 1.6 and Proposition 2.1.

2.4.3. Proof of Proposition 2.5. We fix $r = 4$ and $n = 4$, and consider the set $\sigma^4(4)$. We first formulate the explicit expression of the collection of systems defining $\sigma^4(4)$, and provide a more compact formulation. Then, we find a closed set containing $\sigma^4(4)$ and conclude the proof of Proposition 2.5.

2.4.3.1. *Explicit formulation of* $\sigma^4(4)$. The collection $\mathscr{C}^4(4)$ contains the three systems $\mathbf{S_1}(h)$, $\mathbf{S_2}(h)$ and $\mathbf{S_3}(h)$.

Since $\beta_1 = 4$, $\beta_2 = 4$ and $\beta_3 = 2$, we have:

$$\mathbf{S_1}(h) := \begin{cases} \nabla h(\bar{I}) \neq \underline{0} \\ h^1[A] = 0 \\ h^2[A,A] = 0 \\ h^3[A,A,A] = 0 \\ h^4[A,A,A,A] = 0 \end{cases} \tag{2.27}$$

with $A \in \mathbb{R}^4$;

$$\mathbf{S_2}(h) := \begin{cases} \nabla h(\bar{I}) \neq \underline{0} \\ h^1[A^1] = 0 \\ h^1[A^2] = 0 \\ h^2[v,A^1] = 0 \\ h^2[v,A^2] = 0 \\ 2\,b_{22}\,h^2[A^1,A^2] + h^3[v,v,A^1] = 0 \\ 2\,b_{22}\,h^2[A^2,A^2] + h^3[v,v,A^2] = 0 \\ 6\,b_{23}\,h^2[A^1,A^2] + 6\,b_{22}\,h^3[v,A^2,A^1] + h^4[v,v,v,A^1] = 0 \\ 6\,b_{23}\,h^2[A^2,A^2] + 6\,b_{22}\,h^3[v,A^2,A^2] + h^4[v,v,v,A^2] = 0 \end{cases} \tag{2.28}$$

with $A^1, A^2 \in \mathbb{R}^4$, $v = A^1 + b_{21}A^2$, and $b_{21}, b_{22}, b_{23} \in \mathbb{R}$;

$$\mathbf{S_3}(h) := \begin{cases} \nabla h(\bar{I}) \neq \underline{0} \\ h^1[A^1] = 0 \\ h^1[A^2] = 0 \\ h^1[A^3] = 0 \\ h^2[v,A^1] = 0 \\ h^2[v,A^2] = 0 \\ h^2[v,A^3] = 0 \end{cases} \tag{2.29}$$

with $A^1, A^2, A^3 \in \mathbb{R}^4$, $v = A^1 + b_{21}A^2 + b_{31}A^3$, and $b_{21}, b_{31} \in \mathbb{R}$.

According to the definitions given in Section 2.1, we say that $\mathbf{S_1}(h)$ is solvable for h if it has a non-trivial solution $A \neq \underline{0}$; $\mathbf{S_2}(h)$ is solvable for h if there exist two linearly independent vectors A^1, A^2 and real coefficients b_{21}, b_{22}, b_{23} such that it is verified; $\mathbf{S_3}(h)$ is solvable for h if there exist three linearly independent vectors A^1, A^2, A^3 and real coefficients b_{21}, b_{31} such that it is verified.

In fact, the expression of $\mathbf{S_1}(h)$ follows from $\beta_1 = 4$ (see Section 2.1), while the system $\mathbf{S_2}(h)$ is obtained the following way:

- $\mathscr{S}_2 1$ provides: $\nabla h(\bar{I}) \neq \underline{0}$;
- $\mathscr{S}_2 2$ provides the condition $\mathrm{rank}[A^1, A^2] = 2$, that means that the vectors A^1, A^2 must be linearly independent;
- $\mathscr{S}_2 3$ provides the conditions: $h^1[A^1] = 0$, $h^1[A^2] = 0$;
- $\mathscr{S}_2 4$: since for $n = 4$ we have $\beta_2 = 4$, the polynomial $f(x)$ is defined by

$$\begin{aligned}
f(x) &= \sum_{i=1}^{2} h^1[A^i]\, x_i + \frac{1}{2}\Big(h^2[A^1,A^1]x_1^2 + 2h^2[A^1,A^2]x_1x_2 + h^2[A^2,A^2]x_2^2\Big) \\
&+ \frac{1}{6}\Big(h^3[A^1,A^1,A^1]x_1^3 + 3h^3[A^1,A^1,A^2]x_1^2x_2 + 3h^3[A^1,A^2,A^2]x_1x_2^2 \\
&+ h^3[A^2,A^2,A^2]x_2^3\Big) + \frac{1}{24}\Big(h^4[A^1,A^1,A^1,A^1]x_1^4 + 4h^4[A^1,A^1,A^1,A^2]x_1^3x_2 \\
&+ 6h^4[A^1,A^1,A^2,A^2]x_1^2x_2^2 + 4h^4[A^1,A^2,A^2,A^2]x_1x_2^3 + h^4[A^2,A^2,A^2,A^2]x_2^4\Big).
\end{aligned}$$

We set to zero the coefficients of t, t^2, t^3 of $\partial f / \partial x_i$ computed at $x_1(t) = t$, $x_2(t) = b_{21}t + b_{22}t^2 + b_{23}t^3$.

The coefficients of t, t^2 can be computed exactly as for the case $n = 3, m = 2$. In particular, by introducing the compact formulation $v = A^1 + b_{21}A^2$, the coefficients of t provide the conditions

$$h^2[v, A^1] = 0 \ , \ h^2[v, A^2] = 0$$

while from the coefficients of t^2 we obtain

$$\begin{aligned}
b_{22}h^2[A^1,A^2] + \tfrac{1}{2}h^3[v,v,A^1] &= 0 \\
b_{22}h^2[A^2,A^2] + \tfrac{1}{2}h^3[v,v,A^2] &= 0 \quad .
\end{aligned}$$

Finally, the coefficients of t^3 provide the additional conditions

$$
\begin{aligned}
&b_{23}h^2[A^1,A^2]+b_{22}h^3[A^1,A^1,A^2]+b_{21}b_{22}h^3[A^1,A^2,A^2]+\frac{1}{6}h^4[A^1,A^1,A^1,A^1]\\
&\quad+\frac{1}{2}b_{21}h^4[A^1,A^1,A^1,A^2]+\frac{1}{2}b_{21}^2h^4[A^1,A^1,A^2,A^2]+\frac{1}{6}b_{21}^3h^4[A^1,A^2,A^2,A^2] \quad =0
\end{aligned}
$$

$$
\begin{aligned}
&b_{23}h^2[A^2,A^2]+b_{22}h^3[A^1,A^2,A^2]+b_{21}b_{22}h^3[A^2,A^2,A^2]+\frac{1}{6}h^4[A^1,A^1,A^1,A^2]\\
&\quad+\frac{1}{2}b_{21}h^4[A^1,A^1,A^2,A^2]+\frac{1}{2}b_{21}^2h^4[A^1,A^2,A^2,A^2]+\frac{1}{6}b_{21}^3h^4[A^2,A^2,A^2,A^2] \quad =0
\end{aligned}
$$

which can be written in the simplified form

$$
b_{23}h^2[A^1,A^2]+b_{22}h^3[v,A^1,A^2]+\frac{1}{6}h^4[A^1,v,v,v]=0
$$

$$
b_{23}h^2[A^2,A^2]+b_{22}h^3[v,A^2,A^2]+\frac{1}{6}h^4[A^2,v,v,v]=0\,.
$$

It remains to show how we obtained the expression of $\mathbf{S_3}(h)$:

- $\mathscr{S}_3 1$ provides: $\nabla h(\bar{I})\neq \underline{0}$;
- $\mathscr{S}_3 2$ provides the condition $\mathrm{rank}[A^1,A^2,A^3]=3$, that means that the vectors A^1,A^2,A^3 must be linearly independent;
- $\mathscr{S}_3 3$ provides the conditions: $h^1[A^1]=0$, $h^1[A^2]=0$, $h^1[A^3]=0$;
- $\mathscr{S}_3 4$: since for $n=4$ we have $\beta_3=2$, the polynomial $f(x)$ is defined by

$$
\begin{aligned}
f(x) \;&=\; \sum_{i=1}^{3} h^1[A^i]\,x_i+\frac{1}{2}\Big(h^2[A^1,A^1]x_1^2+h^2[A^2,A^2]x_2^2\\
&+\; h^2[A^3,A^3]x_3^2+2h^2[A^1,A^2]x_1x_2+2h^2[A^1,A^3]x_1x_3+2h^2[A^2,A^3]x_2x_3\Big).
\end{aligned}
$$

We set to zero the coefficient of t of $\partial f/\partial x_i$ computed at $x_1(t)=t$, $x_2(t)=b_{21}t$, $x_3(t)=b_{31}t$, obtaining the additional conditions

$$
h^2[A^1,A^1]+b_{21}h^2[A^1,A^2]+b_{31}h^2[A^1,A^3]=0
$$

$$
b_{21}h^2[A^2,A^2]+h^2[A^1,A^2]+b_{31}h^2[A^2,A^3]=0
$$

$$
b_{31}h^2[A^3,A^3]+h^2[A^1,A^3]+b_{21}h^2[A^2,A^3]=0,
$$

which can be written in the simplified form

$$
h^2[v,A^1]=0\ ,\ h^2[v,A^2]=0\ ,\ h^2[v,A^3]=0,
$$

where $v = A^1 + b_{21}A^2 + b_{31}A^3$.

2.4.3.2. *A reduced formulation of* $\mathbf{S_2}(h)$ *and* $\mathbf{S_3}(h)$. We formulate $\mathbf{S_2}(h)$ and $\mathbf{S_3}(h)$ in the more compact forms:

$$\mathbf{S_2}(h) = \begin{cases} \nabla h(\bar{I}) \neq \underline{0} \\ h^1[v] = 0 \\ h^1[u] = 0 \\ \Pi_\lambda\, h^2[v,\cdot] = \underline{0} \\ \Pi_\lambda\,(2\alpha\, h^2[u,\cdot] + h^3[v,v,\cdot]) = \underline{0} \\ \Pi_\lambda\,(6\beta\, h^2[u,\cdot] + 6\alpha\, h^3[v,u,\cdot] + h^4[v,v,v,\cdot]) = \underline{0} \end{cases} \tag{2.30}$$

where λ denotes the linear space spanned by u and v, and $\alpha, \beta \in \mathbb{R}$;

$$\mathbf{S_3}(h) = \begin{cases} \nabla h(\bar{I}) \neq \underline{0} \\ h^1[v] = 0 \\ \Pi_\Lambda\, h^2[v,\cdot] = \underline{0} \end{cases} \tag{2.31}$$

where Λ is the linear space orthogonal to $\nabla h(\bar{I})$.

With such formulations, we say that $\mathbf{S_2}(h)$ is solvable for h if it is satisfied by two linearly independent vectors $v, u \in \mathbb{R}^4$, and coefficients $\alpha, \beta \in \mathbb{R}$, while $\mathbf{S_3}(h)$ is solvable for h if it is satisfied by a vector $v \in \mathbb{R}^4 \setminus \{\underline{0}\}$.

In the formulation (2.30), the system $\mathbf{S_2}(h)$ depends on the vectors $v, u \in \mathbb{R}^4$ and on $\alpha, \beta \in \mathbb{R}$, while in (2.28) it depends on the vectors $A^1, A^2 \in \mathbb{R}^4$ and on $b_{21}, b_{22}, b_{23} \in \mathbb{R}$. In the formulation (2.31), the system $\mathbf{S_3}(h)$ depends on the vector $v \in \mathbb{R}^4$, while in (2.29) it depends on the vectors $A^1, A^2, A^3 \in \mathbb{R}^4$ and on $b_{21}, b_{31} \in \mathbb{R}$. Thus, in both systems we have reduced the number of parameters.

2.4.3.3. *A closed set containing* $\sigma^4(4)$. The set $\sigma^4(4)$ contains the 4-jets $P_4(h)$ such that at least one of the systems $\mathbf{S_1}(h), \mathbf{S_2}(h)$ and $\mathbf{S_3}(h)$ is solvable for h.

We will make use of the following technical remarks.

Remark 1. If $\mathbf{S_2}(h)$ is solvable for h with $\alpha = 0$, then the vector $v \neq \underline{0}$ is also a solution of $\mathbf{S_1}(h)$.

Remark 2. If $\mathbf{S_2}(h)$ is solvable for h and the restriction $\Pi_\lambda h'' \Pi_\lambda$ of the Hessian matrix of h to λ is completely degenerate, then the vector $v \neq \underline{0}$ is also a solution of $\mathbf{S_1}(h)$. In fact, from $\mathbf{S_2}(h)$, we have

$$h^2[v,v] = h''v \cdot v = (\Pi_\lambda h'' \Pi_\lambda v) \cdot v = 0$$

$$h^3[v,v,v] = h^3[v,v,\cdot] \cdot v = (\Pi_\lambda h^3[v,v,\cdot]) \cdot v = -2\alpha h'' u \cdot v = 0,$$

and similarly $h^3[v,v,u] = 0$. Finally, we have

$$h^4[v,v,v,v] = h^4[v,v,v,\cdot] \cdot v = -6\beta h^2[u,v] - 6\alpha h^3[u,v,v] = 12\,\alpha^2\, h''[u,u] = 0.$$

Consequently the set $\sigma^4(4)$ can be represented as the union of the three sets:

$$\sigma^4(4) = \psi_1(4) \cup \psi_2(4) \cup \psi_3(4)$$

where $\psi_1(4)$ is the set defined at the beginning of this Section with $\beta_1 = 4$; $\psi_2(4)$ is the set of 4-jets $P_4(h)$ such that $\mathbf{S_2}(h)$ is solvable for h with $\alpha \neq 0$ and $\Pi_\lambda h'' \Pi_\lambda$ not completely degenerate; $\psi_3(4)$ is the set of 4-jets $P_4(h)$ such that $\mathbf{S_3}(h)$ is solvable for h.

We will prove that the closure of $\sigma^4(4)$ is contained in the union of three closed sets: the set $\psi_1^*(4)$ defined at the beginning of this Section, and the two sets $\psi_2^*(4)$ and $\psi_3^*(4)$ that we define below.

LEMMA 2.9. *We denote by $\psi_2^*(4)$ the set of 4-jets $P_4(h)$ such that there exist two linearly independent vectors $v, u \in \mathbb{R}^4$ that verify:*

$$\begin{cases} v \cdot u = 0 \\ h^1[v] = 0 \\ h^1[u] = 0 \\ h^2[v,v] = 0 \\ h^2[v,u] = 0 \\ h^3[v,v,v] = 0 \\ h^2[u,u]\, h^4[v,v,v,v] = 3(h^3[v,v,u])^2\,. \end{cases} \tag{2.32}$$

The set $\psi_2^*(4)$ is closed.

PROOF. Let us consider a convergent sequence of elements in $\psi_2^*(4)$, that is 4-jets $P_4(h_k)$ such that for each $k \geq 0$ there exist two linearly independent vectors $v_k, u_k \in \mathbb{R}^4$ which verify:

$$\begin{cases} v_k \cdot u_k = 0 \\ h_k^1[v_k] = 0 \\ h_k^1[u_k] = 0 \\ h_k^2[v_k, v_k] = 0 \\ h_k^2[v_k, u_k] = 0 \\ h_k^3[v_k, v_k, v_k] = 0 \\ h_k^2[u_k, u_k]\, h_k^4[v_k, v_k, v_k, v_k] = 3(h_k^3[v_k, v_k, u_k])^2 \end{cases} \tag{2.33}$$

and such that $\lim_{k\to\infty} P_4(h_k) = P_4(h)$, where $P_4(h)$ is the 4-jet of some function h. We prove that $P_4(h) \in \psi_2^*(4)$.

Since $v_k, u_k \neq \underline{0}$ $\forall k$ and all equations of (2.33) are homogeneous in $\|u_k\|, \|v_k\|$, we define the sequences of unit vectors $\bar{v}_k = \frac{v_k}{\|v_k\|}$, $\bar{u}_k = \frac{u_k}{\|u_k\|}$ which still satisfy (2.32) $\forall k$.

The sequence $(\bar{v}_k, \bar{u}_k)$ is defined on the compact set $\mathbb{S}^3 \times \mathbb{S}^3$, therefore we can extract a subsequence $(\bar{v}_{k_j}, \bar{u}_{k_j})$ convergent to some $(u, v) \in \mathbb{S}^3 \times \mathbb{S}^3$.

From system (2.33) it follows

$$\begin{cases} \lim_{j\to\infty} \bar{v}_{k_j} \cdot \bar{u}_{k_j} = v \cdot u = 0 \\ \lim_{j\to\infty} h_{k_j}^1[\bar{v}_{k_j}] = h^1[v] = 0 \\ \lim_{j\to\infty} h_{k_j}^1[\bar{u}_{k_j}] = h^1[u] = 0 \\ \lim_{j\to\infty} h_{k_j}^2[\bar{v}_{k_j}, \bar{v}_{k_j}] = h^2[v, v] = 0 \\ \lim_{j\to\infty} h_{k_j}^2[\bar{v}_{k_j}, \bar{u}_{k_j}] = h^2[v, u] = 0 \\ \lim_{j\to\infty} h_{k_j}^3[\bar{v}_{k_j}, \bar{v}_{k_j}, \bar{v}_{k_j}] = h^3[v, v, v] = 0 \\ \lim_{j\to\infty} h_{k_j}^2[\bar{u}_{k_j}, \bar{u}_{k_j}]\, h_{k_j}^4[\bar{v}_{k_j}, \bar{v}_{k_j}, \bar{v}_{k_j}, \bar{v}_{k_j}] - 3(h_{k_j}^3[\bar{v}_{k_j}, \bar{v}_{k_j}, \bar{u}_{k_j}])^2 \quad = \\ h^2[u, u]\, h^4[v, v, v, v] - 3(h^3[v, v, u])^2 \qquad = 0 \end{cases}$$

with v, u both non-zero. Hence $P_4(h) \in \psi_2^*(4)$. □

LEMMA 2.10. *We denote by $\psi_3^*(4)$ the set of 4-jets $P_4(h)$ such that there exist a 3-dimensional space λ and a vector $v \in \lambda \setminus \{\underline{0}\}$ which verify:*

$$\begin{cases} \Pi_\lambda \nabla h(\bar{I}) = \underline{0} \\ \Pi_\lambda\, h^2[v,\cdot] = \underline{0}\,. \end{cases} \tag{2.34}$$

The set $\psi_3^*(4)$ is closed.

PROOF. Let us consider a convergent sequence of elements in $\psi_3^*(4)$, that is 4-jets $P_4(h_k)$ such that for each $k \geq 0$ there exist a 3-dimensional space λ_k and a vector $v_k \in \lambda_k \setminus \{\underline{0}\}$ which satisfy

$$\begin{cases} \Pi_{\lambda_k} \nabla h_k(\bar{I}) = \underline{0} \\ \Pi_{\lambda_k}\, h_k^2[v_k,\cdot] = \underline{0}\,, \end{cases} \tag{2.35}$$

and $\lim_{k\to\infty} P_4(h_k) = P_4(h)$ for some h. We prove that $P_4(h) \in \psi_3^*(4)$.

Let us first choose two arbitrary vectors $u_k, w_k \in \lambda_k \setminus \{\underline{0}\}$ such that v_k, u_k, w_k are mutually orthogonal (they always exist). Hence the system (2.35) implies:

$$\begin{cases} h_k^1[v_k] = h_k^1[u_k] = h_k^1[w_k] = 0 \\ h_k^2[v_k,v_k] = h_k^2[v_k,u_k] = h_k^2[v_k,w_k] = 0 \\ v_k \cdot u_k = v_k \cdot w_k = u_k \cdot w_k = 0\,. \end{cases} \tag{2.36}$$

Since $v_k, u_k, w_k \neq \underline{0}\ \forall k$, we can consider the sequences of unit vectors $\bar{v}_k = \frac{v_k}{\|v_k\|}$, $\bar{u}_k = \frac{u_k}{\|u_k\|}$ and $\bar{w}_k = \frac{w_k}{\|w_k\|}$ which still verify (2.36) $\forall k$.

The sequence $(\bar{v}_k, \bar{u}_k, \bar{w}_k)$ is defined on the compact set $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$, therefore we can extract a subsequence $(\bar{v}_{k_j}, \bar{u}_{k_j}, \bar{w}_{k_j})$ convergent to some $(v,u,w) \in \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$.

From system (2.36) we obtain

$$\left\{\begin{array}{l}\lim_{j\to\infty} h^1_{k_j}[\bar{v}_{k_j}] = h^1[v] = 0\\ \lim_{j\to\infty} h^1_{k_j}[\bar{u}_{k_j}] = h^1[u] = 0\\ \lim_{j\to\infty} h^1_{k_j}[\bar{w}_{k_j}] = h^1[w] = 0\\ \lim_{j\to\infty} h^2_{k_j}[\bar{v}_{k_j},\bar{v}_{k_j}] = h^2[v,v] = 0\\ \lim_{j\to\infty} h^2_{k_j}[\bar{v}_{k_j},\bar{u}_{k_j}] = h^2[v,u] = 0\\ \lim_{j\to\infty} h^2_{k_j}[\bar{v}_{k_j},\bar{w}_{k_j}] = h^2[v,w] = 0\\ \lim_{j\to\infty} \bar{v}_{k_j}\cdot\bar{u}_{k_j} = v\cdot u = 0\\ \lim_{j\to\infty} \bar{v}_{k_j}\cdot\bar{w}_{k_j} = v\cdot w = 0\\ \lim_{j\to\infty} \bar{u}_{k_j}\cdot\bar{w}_{k_j} = u\cdot w = 0\end{array}\right. \Longleftrightarrow \left\{\begin{array}{l}\Pi_\lambda \nabla h(\bar{I}) = \underline{0}\\ \Pi_\lambda h^2[v,\cdot] = \underline{0}\end{array}\right.$$

where λ is the 3-dimensional space spanned by the linearly independent vectors v, u and w. □

LEMMA 2.11. *We have:*

$$\overline{\sigma^4(4)} \subseteq \psi_1^*(4) \cup \psi_2^*(4) \cup \psi_3^*(4)\,.$$

PROOF. From Lemma 2.6 we have $\overline{\psi_1(4)} = \psi_1^*(4)$.

We proceed by proving $\overline{\psi_2(4)} \subseteq \psi_2^*(4)$. First, we prove that $P_4(h) \in \psi_2(4)$ if and only if there exist two linearly independent vectors $v, w \in \mathbb{R}^4$ which verify:

$$\left\{\begin{array}{l}\nabla h(\bar{I}) \neq \underline{0}\\ v\cdot w = 0\\ h^1[v] = h^1[w] = 0\\ h^2[v,v] = h^2[v,w] = 0\\ h^2[w,w] \neq 0\\ h^3[v,v,v] = 0\\ h^3[v,v,w] \neq 0\\ 3(h^3[v,v,w])^2 = h^2[w,w]h^4[v,v,v,v]\,.\end{array}\right. \tag{2.37}$$

As a consequence $\psi_2(4) \subseteq \psi_2^*(4)$: in fact, if $P_4(h) \in \psi_2(4)$, then it satisfies system (2.37), and therefore $P_4(h) \in \psi_2^*(4)$. Finally, since $\psi_2^*(4)$ is closed, $\overline{\psi_2(4)} \subseteq \psi_2^*(4)$ follows immediately.

Let us assume $P_4(h) \in \psi_2(4)$, and consider $v, u \in \mathbb{R}^4$ linearly independent, $\beta \in \mathbb{R}$, $\alpha \in \mathbb{R} \setminus \{0\}$ which solve the system $\mathbf{S_2}(h)$ in (2.30).

From $\Pi_\lambda h^2[v,\cdot] = \underline{0}$ we obtain $\Pi_\lambda h'' \Pi_\lambda v = \underline{0}$, that is v is eigenvector with eigenvalue $\lambda_1 = 0$. Since $\Pi_\lambda h'' \Pi_\lambda$ is symmetric non-completely degenerate, it has a second real eigenvalue $\lambda_2 \neq 0$, with eigenvector $w \in \lambda \setminus \{\underline{0}\}$ which is orthogonal to v.

Therefore, (v,w) is an orthogonal basis on λ, and we have

$$h^2[v,v] = h^2[v,w] = 0 \;\; , \;\; h^2[w,w] = \lambda_2\,|w|^2 \neq 0 \;\; , \;\; v \cdot w = 0 \;\; , \;\; u = u_1 v + u_2 w \tag{2.38}$$

with some $u_1 \in \mathbb{R}$, $u_2 \in \mathbb{R} \setminus \{0\}$.

From $\mathbf{S_2}(h)$ we have

$$\begin{cases} \nabla h(\bar{I}) \neq \underline{0} \\ h^1[v] = 0 \\ h^1[w] = 0 \\ h^2[v,v] = 0 \\ h^2[v,w] = 0 \\ h^3[v,v,v] = 0 \\ 2\alpha u_2\, h^2[w,w] + h^3[v,v,w] = 0 \\ 6\alpha u_2\, h^3[v,v,w] + h^4[v,v,v,v] = 0 \\ 6\beta u_2\, h^2[w,w] + 6\alpha u_1\, h^3[v,v,w] + 6\alpha u_2\, h^3[v,w,w] + h^4[v,v,v,w] = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} \nabla h(\bar{I}) \neq \underline{0} \\ h^1[v] = 0 \\ h^1[w] = 0 \\ h^2[v,v] = 0 \\ h^2[v,w] = 0 \\ h^3[v,v,v] = 0 \\ u_2 = -\frac{h^3[v,v,w]}{2\alpha\, h^2[w,w]} \neq 0 \\ 3(h^3[v,v,w])^2 = h^2[w,w] h^4[v,v,v,v] \\ 6\beta u_2\, h^2[w,w] + 6\alpha u_1\, h^3[v,v,w] + 6\alpha u_2\, h^3[v,w,w] + h^4[v,v,v,w] = 0\,. \end{cases} \tag{2.39}$$

From (2.38) and (2.39) it follows that $P_4(h)$ satisfies all conditions in (2.37) with respect to the vectors v and w.

Let us now assume that $P_4(h)$ satisfies all the conditions (2.37) with respect to two linearly independent vectors v and w. Let us denote by λ the 2-dimensional space spanned by v, w. Since $h^2[v,v] = h^2[v,w] = 0$, we have $\Pi_\lambda h^2[v,\cdot] = \underline{0}$, so that $\Pi_\lambda h''\Pi_\lambda$ is degenerate; since $h^2[w,w] \neq 0$, the linear operator $\Pi_\lambda h''\Pi_\lambda$ is not completely degenerate.

We claim that $P_4(h)$ satisfies also all the conditions defining $\mathbf{S_2}(h)$ in (2.30) with v and $u = w$, with suitable definition of $\alpha \neq 0$ and β. We first project the last two equations of the system in (2.30) over the vectors v, u:

$$
\begin{gathered}
h^3[v,v,v] = 0 \\
2\alpha h^2[u,u] + h^3[v,v,u] = 0 \\
6\alpha h^3[v,v,u] + h^4[v,v,v,v] = 0 \\
6\beta h^2[u,u] + 6\alpha h^3[v,u,u] + h^4[v,v,v,u] = 0 \quad .
\end{gathered}
\tag{2.40}
$$

From (2.37), we immediately obtain that the first of these equations is satisfied by v; since $h^2[u,u] \neq 0$, the second and third equations are satisfied by $\alpha = -h^3[v,v,u]/(2h^2[u,u]) \neq 0$; finally, the fourth equation is satisfied by $\beta = -(6\alpha h^3[v,u,u] + h^4[v,v,v,u])/(6h^2[u,u])$.

We conclude by proving $\overline{\psi_3(4)} = \psi_3^*(4)$. First, we prove that the limit of each convergent sequence of elements of $\psi_3(4)$ belongs to $\psi_3^*(4)$. It is sufficient to observe that $\psi_3(4)) \subseteq \psi_3^*(4)$ and, since $\psi_3^*(4)$ is a closed set for Lemma 2.10, the thesis follows.

Then, we prove that for a given element $P_4(h) \in \psi_3^*(4)$, there always exists a sequence of elements of $\psi_3(4)$ convergent to it.

If $\nabla h(\bar{I}) \neq \underline{0}$, then $P_4(h) \in \psi_3(4)$. Therefore we consider the case $\nabla h(\bar{I}) = \underline{0}$. We take $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4 \setminus \{\underline{0}\}$ solving (2.34), and define $f(I) := \alpha_1 I_1 + \alpha_2 I_2 + \alpha_3 I_3 + \alpha_4 I_4$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is the non-zero vector orthogonal to the space λ.

The vector v is a solution of (2.34) also for $h = f$, moreover $\nabla f(\bar{I}) \neq \underline{0}$. Thus the sequence of functions:

$$h_k := h + \frac{1}{k} f$$

is such that $P_4(h_k) \in \psi_3(4)$ for each k, and $\lim_{k\to\infty} P_4(h_k) = P_4(h)$. $\square$

The proof of Proposition 2.5 follows from Theorem 1.6 and Proposition 2.1.

CHAPTER 3

Numerical verification of the steepness of a function

In this Chapter we provide an algorithm for the verification of the steepness of a smooth integrable Hamiltonian $h(I) : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, $n = 3,4$

$$h(I) = \sum_{i=1}^{n} \omega_i I_i + \frac{1}{2}\sum_{i,j=1}^{n} A_{ij} I_i I_j + \frac{1}{6}\sum_{i,j,k=1}^{n} B_{ijk} I_i I_j I_k + \frac{1}{24}\sum_{i,j,k,l=1}^{n} C_{ijkl} I_i I_j I_k I_l\,, \tag{3.1}$$

with $\omega_i, A_{ij}, B_{ijk}, C_{ijkl}$ known coefficients, in a neighborhood of $\bar{I} = \underline{0}$.

We also assume $\omega = (\omega_1, \ldots, \omega_n) \neq \underline{0}$.

Our algorithm represents an extension of an algorithm already provided by Benettin, Fassò and Guzzo in [**6**]. There the authors, in order to study the stability of the Hamiltonian of the circular restricted three-body problem in a neighborhood of the elliptic equilibria L_4 and L_5, constructed an algorithm for the verification of the steepness of a function $h(I)$ with 3 degrees of freedom of the form:

$$h(I) = \sum_{i=1}^{n} \omega_i I_i + \frac{1}{2}\sum_{i,j=1}^{n} A_{ij} I_i I_j + \frac{1}{6}\sum_{i,j,k=1}^{n} B_{ijk} I_i I_j I_k\,,$$

precisely identifying if h is quasi-convex, 3-jet non-degenerate, or satisfies a property called directional quasi-convexity.

We recall that a function $h(I)$ is directionally quasi-convex at $\bar{I} = \underline{0}$ if the restriction of the quadratic form $h^2[v,v]$ to the space orthogonal to ω does not vanish in the first octant, that is if the system

$$\begin{cases} h^1[v] = 0 \\ h^2[v,v] = 0 \\ v_1, \ldots, v_n \geq 0 \end{cases}$$

admits the only solution $v = (v_1, \ldots, v_n) = \underline{0}$.

The directional quasi-convexity is not a sufficient condition for steepness, but it is an important property for the study of the stability of a quasi-integrable Hamiltonian in a neighborhood of an elliptic equilibrium. Here, in fact, due to the singularity of the actions, we can not introduce action-angle variables, therefore the Nekhoroshev Theorem [**49, 50**] can not be applied.

In [**25, 32, 6**] a Nekhoroshev like stability for analytic Hamiltonians in a neighborhood of an elliptic equilibrium has been proved, under the assumption that the Birkhoff normal form of order four of the Hamiltonian exists and is convex, quasi-convex or directionally quasi-convex. In particular, in [**6**], Benettin, Fassò and Guzzo proved similar results also for Hamiltonians with three degrees of freedom whose Birkhoff normal form of order 8 exists and is 3-jet non-degenerate.

Our algorithm extends the algorithm provided in [**6**] in the following way. For $n = 3$, with a minor modification of the algorithm of [**6**], we introduce the verification of the steepness of functions (3.1) which are 3-jet degenerate but satisfy the hypotheses of Proposition 2.4.

Instead for $n = 4$, we need to specifically adapt the algorithm to the higher dimensionality and to the trickiness of the hypotheses of Proposition 2.5.

We also apply these algorithms for the verification of the steepness of two specific Hamiltonians. The first one is the Hamiltonian of the circular restricted three-body problem in a neighborhood of the elliptic equilibria L_4 and L_5, for the only value of the reduced mass $\mu = \mu_3$ whose 6th order Birkhoff normal form was found 3-jet degenerate [**6**].

The second one is the Hamiltonian of a chain of $n = 4$ harmonic oscillators with potential energy derived from the well known Fermi-Pasta-Ulam problem (see, for example, [**7, 57, 58**]).

In both cases, since we are dealing with elliptic equilibria, we verify the steepness of a suitable Birkhoff normal form. Precisely for each problem, we construct the Birkhoff normal form of order eight:

$$h^{(8)} = k_2(I) + k_4(I) + k_6(I) + k_8(I) + O(9)$$

where k_i are homogeneous polynomials of degree $i/2$ in the actions I (see Appendix A for the construction of the Birkhoff normal forms in a neighborhood of an elliptic equilibrium).

The verification of the steepness, as well as the computation of the normal forms, are performed numerically by the software Mathematica.

3.1. Description of the algorithm

We verify the steepness in a neighborhood of the origin of a Hamiltonian (3.1) such that $\omega \neq \underline{0}$. We denote by $A = (A_{ij})_{i,j=1,\dots,n}$ the Hessian matrix of h computed at the origin, and by Λ the linear space orthogonal to ω.

3.1.1. The algorithm in the case $n = 3$.

The first three steps constitute the algorithm constructed in [**6**]. The last one represents the extension of the algorithm to the case of a function h which is 3-jet degenerate and satisfies the hypotheses of Proposition 2.4 at the origin.

(**1**) We perform a rotation of the coordinates I in order to carry the vector ω into the first coordinate axis, and we denote by R the rotation matrix. Then we take the appropriate 2×2 sub-matrix A_Λ of the rotation of A, which represents the restriction of the Hessian matrix to the space Λ. We compute the two eigenvalues of A_Λ: if they are both positive or negative, then we conclude that $h(I)$ is quasi-convex at the origin.

(**2**) We suppose $h(I)$ is not quasi-convex at the origin, so that we compute the vectors $v \in \Lambda \backslash \{\underline{0}\}$ such that $h^2[v,v] = 0$.

Let $\lambda_1 \leq \lambda_2$ be the eigenvalues of A_Λ, and let $x = (x_1, x_2)$, $y = (y_1, y_2)$ be eigenvectors of λ_1 and λ_2 respectively. According to the definition (see

Chapter 1), the function $h(I)$ is 3-jet non-degenerate at the origin if and only if all vectors $v \in \Lambda \setminus \{\underline{0}\}$ such that

$$h^2[v,v] = 0, \tag{3.2}$$

satisfy also $h^3[v,v,v] \neq 0$. Given any $v \in \Lambda \setminus \{\underline{0}\}$ such that (3.2) holds, we can write

$$v = R^T(0,d)$$

where $d \in \mathbb{R}^2$ solves

$$A_\Lambda d \cdot d = 0\,. \tag{3.3}$$

We search the solutions of (3.3). Since the matrix A_Λ is symmetric, we can diagonalize it by an orthogonal matrix S:

$$S^T A_\Lambda S = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} =: D.$$

Then $A_\Lambda = SDS^T$, and equation (3.3) becomes:

$$\lambda_1 w_1^2 + \lambda_2 w_2^2 = 0 \tag{3.4}$$

where $w = (w_1, w_2) \in \mathbb{R}^2$ is such that $w = S^T d$.

We distinguish between different cases, depending on the values of λ_1, λ_2.

A) $\underline{\lambda_1 < 0 < \lambda_2}$

Equation (3.4) determines two lines through the origin, which contain the vectors $w = \left(\sqrt{-\frac{\lambda_2}{\lambda_1}}\,w_2, w_2\right)$, and $w = \left(-\sqrt{-\frac{\lambda_2}{\lambda_1}}\,w_2, w_2\right)$, with $w_2 \in \mathbb{R}$. We can fix $\|w\| = 1$, therefore equation (3.4) has the two solutions:

$$w^A = \frac{1}{\sqrt{\lambda_1 - \lambda_2}}(\sqrt{-\lambda_2}, \sqrt{\lambda_1})$$
$$w^B = \frac{1}{\sqrt{\lambda_1 - \lambda_2}}(-\sqrt{-\lambda_2}, \sqrt{\lambda_1})\,.$$

Consequently, the two vectors $v^A, v^B \in \Lambda\backslash\{\underline{0}\}$ defined by

$$\begin{aligned} v^A &= R^T(0,d^A) \quad \text{with} \quad d^A = \sqrt{\tfrac{\lambda_1}{\lambda_1-\lambda_2}}\,y + \sqrt{\tfrac{-\lambda_2}{\lambda_1-\lambda_2}}\,x \\ v^B &= R^T(0,d^B) \quad \text{with} \quad d^B = \sqrt{\tfrac{\lambda_1}{\lambda_1-\lambda_2}}\,y - \sqrt{\tfrac{-\lambda_2}{\lambda_1-\lambda_2}}\,x\,, \end{aligned}$$

satisfy (3.2). If it happens that

$$\begin{aligned} h^3[v^A,v^A,v^A] &\neq 0 \\ h^3[v^B,v^B,v^B] &\neq 0\,, \end{aligned}$$

then $h(I)$ is 3-jet non-degenerate at the origin. If it happens that $h^3[v^A,v^A,v^A]=0$ or $h^3[v^B,v^B,v^B]=0$, then the function is 3-jet degenerate, and its steepness will be tested using Proposition 2.4.

B) One of the two eigenvalues λ_1,λ_2 vanishes

We first suppose $\lambda_1 = 0$. Then $\lambda_2 > 0$, and equation (3.4) is solved by the vectors $w=(w_1,0)$ with $w_1 \in \mathbb{R}$, and in particular by $w=(1,0)$. Consequently the vector $v \in \Lambda\setminus\{\underline{0}\}$ defined by

$$v = R^T(0,x)\,,$$

satisfies (3.2), and therefore $h(I)$ is 3-jet non-degenerate at the origin if and only if $h^3[v,v,v]\neq 0$.

We suppose now $\lambda_2 = 0$. Then $\lambda_1 < 0$, and equation (3.4) is solved by the vectors $w=(0,w_2)$ with $w_2 \in \mathbb{R}$, and in particular by $w=(0,1)$. Consequently the vector $v \in \Lambda\setminus\{\underline{0}\}$ defined by

$$v = R^T(0,y)\,,$$

satisfies (3.2), and therefore $h(I)$ is 3-jet non-degenerate at the origin if and only if $h^3[v,v,v]\neq 0$.

C) $\lambda_1 = \lambda_2 = 0$

In this case equation (3.4) is solved by all the vectors $w \in \mathbb{R}^2$. We can fix $\|w\|=1$, therefore equation (3.4) has the solutions:

$$w^\gamma = (\cos\gamma,\sin\gamma) \quad \gamma\in[0,2\pi)\,.$$

Then the vectors $v \in \Lambda \setminus \{\underline{0}\}$ such that (3.2) holds are:

$$v^\gamma = R^T(0, d^\gamma) \quad \text{with} \quad d^\gamma = x\cos\gamma + y\sin\gamma$$

for all $\gamma \in [0, 2\pi)$. As a consequence, $h(I)$ is 3-jet non-degenerate at the origin if and only if for each $\gamma \in [0, 2\pi)$, the vector v^γ satisfies $h^3[v^\gamma, v^\gamma, v^\gamma] \neq 0$.

(**3**) From step (**2**), we obtained all the vectors $v \in \Lambda \setminus \{\underline{0}\}$ satisfying (3.2). We can therefore check if $h(I)$ is directionally quasi-convex at the origin, which is verified if and only if all such vectors v have two components with opposite signs.

(**4**) We finally suppose that $h(I)$ is 3-jet degenerate at the origin, so that there exists at least one vector $v \in \Lambda \setminus \{\underline{0}\}$ such that

$$\begin{cases} h^2[v,v] = 0 \\ h^3[v,v,v] = 0\,. \end{cases} \tag{3.5}$$

The function $h(I)$ satisfies the hypotheses of Proposition 2.4 at the origin if, for all vectors $v \in \Lambda \setminus \{\underline{0}\}$ verifying (3.5), the following two conditions hold:

(1) $h^4[v,v,v,v] \neq 0$

(2) the system

$$\begin{cases} v \cdot w = 0 \\ h^1[w] = 0 \\ h^2[v,w] = 0 \end{cases} \tag{3.6}$$

admits the only solution $w = (w_1, w_2, w_3) = \underline{0}$.

Since Λ is a 2-dimensional space, for any $v \in \Lambda \setminus \{\underline{0}\}$ there is only one vector $v^\perp \in \Lambda \setminus \{\underline{0}\}$ such that $v \cdot v^\perp = 0$. Hence, if for all vectors $v \in \Lambda \setminus \{\underline{0}\}$ which verify (3.5), it holds:

$$\begin{aligned} h^4[v,v,v,v] &\neq 0 \\ h^2[v,v^\perp] &\neq 0\,, \end{aligned}$$

then $h(I)$ satisfies the hypotheses of Proposition 2.4 at the origin.

3.1.2. The algorithm in the case $n = 4$.

The first three steps constitute the generalization to functions with four degrees of freedom of the algorithm constructed in [**6**]. The last step represents the extension of the algorithm to the case of a function h which is 3-jet degenerate and satisfies the hypotheses of Proposition 2.5 at the origin.

(**1**) We proceed as in the case $n = 3$. We perform a rotation of the coordinates I in order to carry the vector ω into the first coordinate axis, and we denote by R the rotation matrix. Then we take the appropriate 3×3 sub-matrix A_Λ of the rotation of A, which represents the restriction of the Hessian matrix to the space Λ.

We compute the three eigenvalues of A_Λ: if they are all positive or all negative, then $h(I)$ is quasi-convex at the origin.

(**2**) We suppose $h(I)$ is not quasi-convex at the origin, so that we compute the vectors $v \in \Lambda \backslash \{\underline{0}\}$ such that $h^2[v,v] = 0$.

Let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be the eigenvalues of A_Λ, and let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, $z = (z_1, z_2, z_3)$ be eigenvectors of λ_1, λ_2 and λ_3 respectively. According to the definition (see Chapter 1), $h(I)$ is 3-jet non-degenerate at the origin if and only if all vectors $v \in \Lambda \setminus \{\underline{0}\}$ such that:

$$h^2[v,v] = 0 \tag{3.7}$$

satisfy also $h^3[v,v,v] \neq 0$. Given any $v \in \Lambda \setminus \{\underline{0}\}$ such that (3.7) holds, we can write

$$v = R^T(0,d)$$

where $d \in \mathbb{R}^3$ solves

$$A_\Lambda d \cdot d = 0\,. \tag{3.8}$$

We search the solutions of (3.8). Since the matrix A_Λ is symmetric, we can diagonalize it by an orthogonal matrix S:

$$S^T A_\Lambda S = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} =: D\,.$$

Then $A_\Lambda = SDS^T$, and equation (3.8) becomes:

$$\lambda_1 w_1^2 + \lambda_2 w_2^2 + \lambda_3 w_3^2 = 0 \tag{3.9}$$

where $w = (w_1, w_2, w_3) \in \mathbb{R}^3$ is such that $w = S^T d$. We distinguish between different cases, depending on the values of $\lambda_1, \lambda_2, \lambda_3$.

A) $\underline{\lambda_1, \lambda_2, \lambda_3 \neq 0}$

In particular, since $h(I)$ is not quasi-convex, we have $\lambda_1 < 0$ and $\lambda_3 > 0$.

- $\lambda_2 > 0$

  In this case equation (3.9) describes two elliptical cones with vertex in the origin and height along w_1: one with positive values of w_1 and one with negative values of w_1. Because of the symmetry, we can restrict to the first cone, hence we take $w_1 \geq 0$. We can fix $\|w\| = 1$, and we also introduce polar coordinates, so that equation (3.9) is solved by:

  $$w^\gamma = \left(\sqrt{1-r^2}, r\cos\gamma, r\sin\gamma\right)$$

  with $r = \sqrt{\frac{-\lambda_1}{\lambda_2\cos^2\gamma + \lambda_3\sin^2\gamma - \lambda_1}}$ and $\gamma \in [0, 2\pi)$. Consequently, the vectors in the space Λ such that (3.7) holds are

$$v^\gamma = R^T(0, d^\gamma) \quad \text{with} \quad d^\gamma = x\sqrt{1-r^2} + y\,r\cos\gamma + z\,r\sin\gamma$$

  for all $\gamma \in [0, 2\pi)$. Then, $h(I)$ is 3-jet non-degenerate at the origin if and only if, for each $\gamma \in [0, 2\pi)$, the vector v^γ satisfies $h^3[v^\gamma, v^\gamma, v^\gamma] \neq 0$.

- $\lambda_2 < 0$
  In this case equation (3.9) describes two elliptical cones with vertex in the origin and height along w_3: one with positive values of w_3 and one with negative values of w_3. Because of the symmetry, we can restrict to the first cone, hence we take $w_3 \geq 0$. We can fix $\|w\| = 1$, and we also introduce polar coordinates, therefore equation (3.9) has the solutions:

$$w^\gamma = \left(r\cos\gamma, r\sin\gamma, \sqrt{1-r^2}\right)$$

  with $r = \sqrt{\frac{\lambda_3}{\lambda_3-\lambda_1\cos^2\gamma-\lambda_2\sin^2\gamma}}$ and $\gamma \in [0, 2\pi)$. Consequently, the non-vanishing vectors in the space Λ such that (3.7) holds are

$$v^\gamma = R^T(0, d^\gamma) \quad \text{with} \quad d^\gamma = x\,r\cos\gamma + y\,r\sin\gamma + z\sqrt{1-r^2}$$

  for all $\gamma \in [0, 2\pi)$. Then, $h(I)$ is 3-jet non-degenerate at the origin if and only if for each $\gamma \in [0, 2\pi)$, the vector v^γ satisfies $h^3[v^\gamma, v^\gamma, v^\gamma] \neq 0$.

B) <u>One of the three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ vanishes</u>

- $\lambda_1 = 0$ or $\lambda_3 = 0$
  We suppose $\lambda_1 = 0$. Then $0 < \lambda_2 \leq \lambda_3$, and equation (3.9) is solved by the vectors $w = (w_1, 0, 0)$ with $w_1 \in \mathbb{R}$. We fix $\|w\| = 1$, therefore equation (3.9) has the solution:

$$w = (1, 0, 0)\,.$$

  Consequently the vector $v \in \Lambda \setminus \{\underline{0}\}$ such that (3.7) holds is:

$$v = R^T(0, x)\,,$$

  and if $h^3[v, v, v] \neq 0$, then $h(I)$ is 3-jet non-degenerate at the origin.
  We suppose now $\lambda_3 = 0$. Then $\lambda_1 \leq \lambda_2 < 0$, and equation

(3.9) is solved by the vectors $w=(0,0,w_3)$ with $w_3\in\mathbb{R}$. We fix $\|w\|=1$, therefore equation (3.9) has the solution:

$$w=(0,0,1)\,.$$

Consequently the vector $v\in\Lambda\setminus\{\underline{0}\}$ such that (3.7) holds is:

$$v=R^T(0,z)\,,$$

and if $h^3[v,v,v]\neq 0$, then $h(I)$ is 3-jet non-degenerate at the origin.

- $\lambda_2=0$

  Then $\lambda_1<0$ and $\lambda_3>0$. In this case equation (3.9) describes two planes through the origin in $\mathbb{R}^3$: one containing the vectors $w=\left(\sqrt{\frac{\lambda_3}{-\lambda_1}}w_3,w_2,w_3\right)$, and the other one containing the vectors $w=\left(-\sqrt{\frac{\lambda_3}{-\lambda_1}}w_3,w_2,w_3\right)$, with $w_2,w_3\in\mathbb{R}$. We can fix $\|w\|=1$ and consider in both cases $w_2\geq 0$, then the solutions of (3.9) are $w=(\pm\sqrt{\frac{\lambda_3}{-\lambda_1}}w_3,\sqrt{\frac{\lambda_1+(\lambda_3-\lambda_1)w_3^2}{\lambda_1}},w_3)$, with $w_3\in\mathbb{R}$.

  Introducing polar coordinates, the solutions of (3.9) can be written as:

$$w^\gamma=\begin{cases}\left(\sqrt{\frac{\lambda_3}{\lambda_3-\lambda_1}}\cos\gamma,\sin\gamma,\sqrt{\frac{-\lambda_1}{\lambda_3-\lambda_1}}\cos\gamma\right) & \gamma\in[0,\pi)\\ \left(-\sqrt{\frac{\lambda_3}{\lambda_3-\lambda_1}}\cos\gamma,-\sin\gamma,\sqrt{\frac{-\lambda_1}{\lambda_3-\lambda_1}}\cos\gamma\right) & \gamma\in[\pi,2\pi)\,.\end{cases}$$

  Then the non-vanishing vectors in the space Λ such that (3.7) holds are $v^\gamma=R^T(0,d^\gamma)$, with

$$d^\gamma=\begin{cases}x\sqrt{\frac{\lambda_3}{\lambda_3-\lambda_1}}\cos\gamma+y\sin\gamma+z\sqrt{\frac{-\lambda_1}{\lambda_3-\lambda_1}}\cos\gamma & \gamma\in[0,\pi)\\ -x\sqrt{\frac{\lambda_3}{\lambda_3-\lambda_1}}\cos\gamma-y\sin\gamma+z\sqrt{\frac{-\lambda_1}{\lambda_3-\lambda_1}}\cos\gamma & \gamma\in[\pi,2\pi)\,.\end{cases}$$

  If for each $\gamma\in[0,2\pi)$ the vector v^γ satisfies $h^3[v^\gamma,v^\gamma,v^\gamma]\neq 0$, then $h(I)$ is 3-jet non-degenerate at the origin.

C) $\underline{\lambda_1=\lambda_2=0 \text{ or } \lambda_2=\lambda_3=0}$

We suppose $\lambda_1 = \lambda_2 = 0$. Then equation (3.9) is solved by the vectors $w = (w_1, w_2, 0)$ with $w_1, w_2 \in \mathbb{R}$. We fix $\|w\| = 1$ and we introduce polar coordinates, therefore equation (3.9) has the solutions:

$$w^\gamma = (\cos\gamma, \sin\gamma, 0) \quad \gamma \in [0, 2\pi).$$

Then the non-vanishing vectors in the space Λ such that (3.7) holds are:

$$v^\gamma = R^T(0, d^\gamma) \quad \text{with} \quad d^\gamma = x\cos\gamma + y\sin\gamma$$

for all $\gamma \in [0, 2\pi)$. If for each $\gamma \in [0, 2\pi)$ the vector v^γ satisfies $h^3[v^\gamma, v^\gamma, v^\gamma] \neq 0$, then $h(I)$ is 3-jet non-degenerate at the origin.

We suppose now $\lambda_2 = \lambda_3 = 0$. Then equation (3.9) is solved by the vectors $w = (0, w_2, w_3)$ with $w_2, w_3 \in \mathbb{R}$. We fix $\|w\| = 1$ and we introduce polar coordinates, therefore equation (3.9) has the solutions:

$$w^\gamma = (0, \cos\gamma, \sin\gamma) \quad \gamma \in [0, 2\pi).$$

Then the non-vanishing vectors in the space Λ such that (3.7) holds are:

$$v^\gamma = R^T(0, d^\gamma) \quad \text{with} \quad d^\gamma = y\cos\gamma + z\sin\gamma$$

for all $\gamma \in [0, 2\pi)$. If for each $\gamma \in [0, 2\pi)$ the vector v^γ satisfies $h^3[v^\gamma, v^\gamma, v^\gamma] \neq 0$, then $h(I)$ is 3-jet non-degenerate at the origin.

D) $\underline{\lambda_1 = \lambda_2 = \lambda_3 = 0}$

In this case equation (3.9) is solved by all the vectors $w \in \mathbb{R}^3$. We can fix $\|w\| = 1$, therefore equation (3.9) has the solutions:

$$w^{\theta,\gamma} = (\sin\theta\cos\gamma, \sin\theta\sin\gamma, \cos\theta) \quad \theta \in [0, \pi), \gamma \in [0, 2\pi).$$

Then the non-vanishing vectors in the space Λ such that (3.7) holds are:

$$v^{\theta,\gamma} = R^T(0, d^{\theta,\gamma}) \quad \text{with} \quad d^{\theta,\gamma} = x\sin\theta\cos\gamma + y\sin\theta\sin\gamma + z\cos\theta$$

for all $\theta \in [0,\pi), \gamma \in [0,2\pi)$. If for each couple (θ,γ) the vector $v^{\theta,\gamma}$ is such that $h^3[v^{\theta,\gamma}, v^{\theta,\gamma}, v^{\theta,\gamma}] \neq 0$, then $h(I)$ is 3-jet non-degenerate at the origin.

(**3**) From step (**2**), we obtained all the vectors $v \in \Lambda \setminus \{\underline{0}\}$ satisfying (3.7). We can therefore check if $h(I)$ is directionally quasi-convex at the origin, that is verified if and only if all such vectors v have two components with opposite signs.

(**4**) We suppose $h(I)$ is 3-jet degenerate at the origin. Therefore there exists at least one vector $v \in \Lambda \setminus \{\underline{0}\}$ such that

$$\begin{cases} h^2[v,v] = 0 \\ h^3[v,v,v] = 0\,. \end{cases} \tag{3.10}$$

The function $h(I)$ satisfies the hypotheses of Proposition 2.5 at the origin if and only if:

(1) the matrix A_Λ is non-degenerate, that is all eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A_Λ are non-vanishing;

(2) for all vectors $v \in \Lambda \setminus \{\underline{0}\}$ satisfying (3.10), the following two conditions hold:

(2a) $h^4[v,v,v,v] \neq 0$

(2b) the system

$$\begin{cases} v \cdot w = 0 \\ h^1[w] = 0 \\ h^2[v,w] = 0 \\ h^2[w,w]h^4[v,v,v,v] = 3(h^3[v,v,w])^2 \end{cases} \tag{3.11}$$

admits only the solution $w = (w_1, w_2, w_3, w_4) = \underline{0}\,.$

Let $v \in \Lambda \setminus \{\underline{0}\}$ be such that (3.10) is verified. We consider the system formed by the first three equations of (3.11), which is linear in the variables

w_1, w_2, w_3 and w_4. Precisely:

$$B := \begin{cases} v \cdot w = 0 \\ h^1[w] = 0 \\ h^2[v,w] = 0\,. \end{cases}$$

The system B has maximal rank. The first two equations are linearly independent because the vectors v and ω are orthogonal, while the third equation is linearly independent from each of the first two, because the vector $h^2[v,\cdot]$ is orthogonal both to v and ω. In fact, if it was $h^2[v,\cdot] \parallel \omega$, then $\Pi_\Lambda h^2[v,\cdot] = \underline{0} \iff A_\Lambda v = \underline{0}$, and this is impossible because A_Λ is non-degenerate.We can fix $\|w\| = 1$, therefore system B admits a unique solution $\bar{w} \in \mathbb{R}^4 \setminus \{\underline{0}\}$.

Finally, we can state what follows. If A_Λ is non-degenerate and if, for each $v \in \Lambda \setminus \{\underline{0}\}$ verifying (3.10), it holds:

$$\begin{aligned} &h^4[v,v,v,v] \neq 0 \\ &h^2[\bar{w},\bar{w}]h^4[v,v,v,v] \neq 3(h^3[v,v,\bar{w}])^2\,, \end{aligned}$$

where $\bar{w}$ is the unique non-zero solution of B, then $h(I)$ satisfies the hypotheses of Proposition 2.5 at the origin.

3.2. Verification of the steepness of a function with three degrees of freedom: the Hamiltonian of the circular restricted three-body problem

In this Section we implement the algorithm described in Section 3.1 for the verification of the steepness on the Hamiltonian of the circular restricted three-body problem.

The system consists in two primary bodies M_1 and M_2 with masses m_1 and m_2, and a third body M. The two primaries perform circular orbits around their common center of mass, while the third body M moves under the effect of the force field generated by the primaries.

We consider a coordinate system x_1, x_2, x_3 with the origin in the center of mass, so that at time $t = 0$ the two primaries are both on the x_1 axis. The motion of M_1 and M_2 takes place in the plane x_1, x_2. Then we choose the units of length, mass and time such that:

- the reciprocal distance between M_1 and M_2 is 1;
- $m_1 + m_2 = 1$; then denoting by

$$\mu = \frac{m_2}{m_1 + m_2}$$

  the reduced mass, we have $m_1 = 1 - \mu$ and $m_2 = \mu$;
- the period of rotation of M_1 and M_2 is 2π.

By denoting with $\tilde{Q}$ the barycentric rotating coordinates and with $\tilde{P}$ the conjugate momenta, the Hamiltonian of the system is:

$$\tilde{H}(\tilde{Q}, \tilde{P}; \mu) = \frac{1}{2}(\tilde{P}_1^2 + \tilde{P}_2^2 + \tilde{P}_3^2) - \tilde{Q}_1 \tilde{P}_2 + \tilde{Q}_2 \tilde{P}_1 - \frac{1-\mu}{d_1} - \frac{\mu}{d_2}. \qquad (3.12)$$

For all the values of the reduced mass $\mu > 0$, the system admits 5 equilibrium points L_1, L_2, L_3, L_4, L_5, which are called Lagrangian equilibria and are represented in Figure 1.

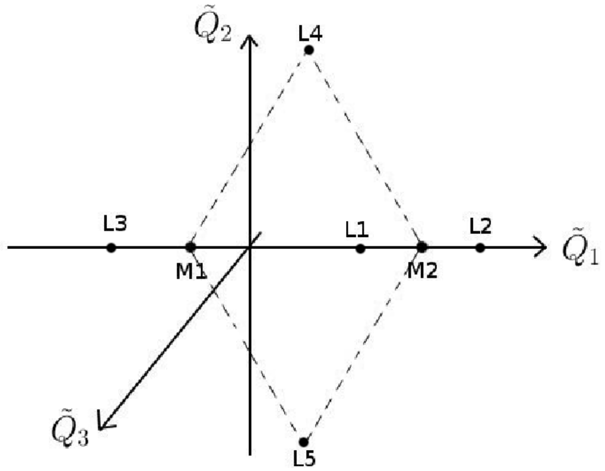

FIGURE 1. Configurations of the Lagrangian equilibrium points in the baricentric rotating system.

3.2 Verification of the steepness of a function with three degrees of freedom

The triangular points L_4 and L_5 are elliptic for all values of the reduced mass μ below the Routh's critical mass

$$\mu_R = \frac{1}{2}(1 - \frac{\sqrt{69}}{9}) \approx 0.0385209\,.$$

We will investigate the steepness of the Birkhoff normal form of order eight of the Hamiltonian (3.12) in a neighborhood of the elliptic equilibrium L_4, for a certain value of the reduced mass lower than μ_R.

In [**25, 32, 6**] Nekhoroshev like stability results for analytic Hamiltonians in a neighborhood of an elliptic equilibrium have been proved. Precisely, the stability has been proved for Hamiltonians whose Birkhoff normal form of order four exists and is convex, quasi-convex or directionally quasi-convex, and for Hamiltonians of three degrees of freedom whose Birkhoff normal form of order eight exists and is 3-jet non-degenerate.

In [**6**] Guzzo, Fassò and Benettin use these results to provide numerical evidence of the stability of the Hamiltonian (3.12) in a neighborhood of L_4, for all $\mu < \mu_R$ except a finite number of values of the reduced mass. Precisely:

- for the following values of μ the Birkhoff normal form of order four is neither quasi-convex nor directionally quasi-convex, and the Birkhoff normal form of order eight does not exist:

  $\mu_{(1,3,0)} \approx 0.0135160$ having the resonance $(1,3,0)$
  $\mu_{(1,2,0)} \approx 0.0242939$ having the resonance $(1,2,0)$
  $\mu_{(0,3,1)} \approx 0.0148525$ having the resonance $(0,3,1)$
  $\mu_{(3,3,-2)} \approx 0.0115649$ having the resonance $(3,3,-2)$

- for $\mu_3 \approx 0.0147808$ the Birkhoff normal form of order four is neither quasi-convex nor directionally quasi-convex, and the Birkhoff normal form of order eight exists, but is 3-jet degenerate.

Benettin, Fassò and Guzzo obtain the stability results by a numerical verification of the steepness of the Hamiltonian (3.12). Since for the case $\mu = \mu_3$, because of the 3-jet degeneracy of the 6th order Birkhoff normal form, they could not provide numerical evidence of stability, we decided to verify the hypotheses of Proposition 2.5 of the 8th order Birkhoff normal form. Actually, we find numerical evidence of such steepness.

3.2.1. The Hamiltonian in a neighborhood of L_4. Following [**6**], we perform some changes of variables which are needed in order to perform effectively the Birkhoff steps around L_4. We consider the Hamiltonian (3.12) for $\mu = \mu_3$. The equilibrium L_4 has the following coordinates

$$(\tilde{Q}^{L_4}, \tilde{P}^{L_4}) = \Big(\frac{1-2\mu_3}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, \frac{1-2\mu_3}{2}, 0\Big).$$

We first introduce the coordinates $(Q,P) = (\tilde{Q} - \tilde{Q}^{L_4}, \tilde{P} - \tilde{P}^{L_4})$, which conjugate the Hamiltonian (3.12) to:

$$H(Q,P) = \frac{1}{2}(P_1^2 + P_2^2 + P_3^2) - Q_1P_2 + Q_2P_1 - \frac{1-2\mu_3}{2}Q_1 - \frac{\sqrt{3}}{2}Q_2 - \frac{1-\mu_3}{\rho_+(Q)} - \frac{\mu_3}{\rho_-(Q)} \tag{3.13}$$

where $\rho_\pm(Q) = \sqrt{Q_1^2 + Q_2^2 + Q_3^2 \pm Q_1 + \sqrt{3}Q_2 + 1}\,$.

The linearization of the Hamiltonian vector field of (3.13) in a neighborhood of the equilibrium $Q = P = \underline{0}$, gives a system of the form:

$$\dot{X} = LX\,,$$

where $X = (Q,P)$, and L is a 6×6 matrix. The eigenvalues of L are all imaginary, according to the ellipticity of the equilibrium, and are $\pm i f_+$, $\pm i f_-$ and $\pm i$, with:

$$f_\pm = \sqrt{\frac{1 \pm f}{2}}, \qquad f = \sqrt{1 - 27\mu_3 + 27\mu_3^2}\,.$$

In a neighborhood of the equilibrium there exists a linear change of variables $(q,p) = (q(Q,P), p(Q,P))$ which conjugate the quadratic part $k_2(q,p)$ of the Hamiltonian (3.13) to:

$$k_2(I) = \omega \cdot I, \tag{3.14}$$

where $\omega = (f_+, -f_-, 1)$, and $I = (I_1, I_2, I_3)$, $I_i = \frac{p_i^2+q_i^2}{2}$. In [**6**] the matrix of the transformation $(q,p) = T\,(Q,P)$ has been explicitly computed:

$$T = \begin{pmatrix} -3kf_+g_+ & (7+2f)g_- & 0 & (7-2f)g_+ & -3kf_-g_- & 0 \\ (2f-3)f_+g_+ & -3kg_- & 0 & -3kg_+ & -(2f+3)f_-g_- & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -4f_+g_+ & 3kf_+^2g_- & 0 & 3kf_-^2g_+ & -4f_-g_- & 0 \\ 0 & (4ff_-g_-)^{-1} & 0 & (4ff_+g_+)^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $k = \sqrt{3}(1-2\mu_3)$ and $g_\pm = 1/\sqrt{ff_\pm(9k^2 \mp 10f - 1)}$.

It is convenient to introduce the complex variables (see [**6, 25, 32**]):

$$z_j := \frac{p_j - iq_j}{i\sqrt{2}}, \quad w_j := \frac{p_j + iq_j}{\sqrt{2}}, \qquad j = 1, \dots, 3$$

where $z = (z_1, z_2, z_3)$ are the coordinates and $w = (w_1, w_2, w_3)$ the momenta. Hence the Hamiltonian of which we will compute the Birkhoff normal form of order eight is:

$$h(z,w) = H(T^{-1}C\,(z,w)) \tag{3.15}$$

where

$$C = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

is the matrix such that $(q,p) = C\,(z,w)$.

3.2.2. The verification of the steepness. We construct the Birkhoff normal form of order eight of the Hamiltonian (3.15), according to the procedure described in Appendix A. Precisely, we take the Taylor expansion of order eight around the origin of (3.15)

$$h^{(2)} = k_2(I) + h_3^{(2)} + h_4^{(2)} + h_5^{(2)} + h_6^{(2)} + h_7^{(2)} + h_8^{(2)} + O(9)$$

and we construct

$$h^{(8)} = h^{(2)} \circ \phi_{\chi_3} \circ \phi_{\chi_4} \circ \phi_{\chi_5} \circ \phi_{\chi_6} \circ \phi_{\chi_7} \circ \phi_{\chi_8}$$

where the functions χ_j, $j = 3,\ldots,8$, are defined in Appendix A, and ϕ_{χ_j} denotes the time-1 Hamiltonian flow of χ_j. By suitable choice of the χ_j, we obtain the polynomial

$$h^{(8)} = k_2(I) + k_4(I) + k_6(I) + k_8(I) + O(9)$$

where

$$\begin{aligned} k_4(I) &= \langle h_4^{(3)} \rangle_0 \\ k_6(I) &= \langle h_6^{(5)} \rangle_0 \\ k_8(I) &= \langle h_8^{(7)} \rangle_0 \end{aligned}$$

with

$$\begin{aligned}
h_4^{(3)} &= h_4^{(2)} + \frac{1}{2}\mathscr{L}_{\chi_3} h_3^{(2)} \\
h_6^{(5)} &= h_6^{(3)} + \frac{1}{2}\mathscr{L}_{\chi_4}(h_4^{(3)} + k_4) \\
h_6^{(3)} &= h_6^{(2)} + \mathscr{L}_{\chi_3} h_5^{(2)} + \frac{1}{2}\mathscr{L}_{\chi_3}^2 h_4^{(2)} + \frac{1}{8}\mathscr{L}_{\chi_3}^3 h_3^2 \\
h_8^{(7)} &= h_8^{(4)} + \frac{1}{2}\mathscr{L}_{\chi_5} h_5^{(3)} + \mathscr{L}_{\chi_6} k_4 \\
h_5^{(3)} &= h_5^{(2)} + \mathscr{L}_{\chi_3} h_4^{(2)} + \frac{1}{3}\mathscr{L}_{\chi_3}^2 h_3^{(2)} \\
h_8^{(4)} &= h_8^{(3)} + \mathscr{L}_{\chi_4} h_6^{(3)} + \frac{1}{3}\mathscr{L}_{\chi_4}^2 h_4^{(3)} + \frac{1}{6}\mathscr{L}_{\chi_4}^2 k_4 \\
h_8^{(3)} &= h_8^{(2)} + \mathscr{L}_{\chi_3} h_7^{(2)} + \frac{1}{2}\mathscr{L}_{\chi_3}^2 h_6^{(2)} + \frac{1}{6}\mathscr{L}_{\chi_3}^3 h_5^{(2)} + \frac{1}{24}\mathscr{L}_{\chi_3}^4 h_4^{(2)} + \frac{1}{144}\mathscr{L}_{\chi_3}^5 h_3^{(2)} .
\end{aligned}$$

We report here the explicit expression of the integrable approximation

$$\begin{aligned}
h(I) &= k_2(I) + k_4(I) + k_6(I) + k_8(I) \\
&= a_1 I_1 + a_2 I_2 + I_3 + b_1 I_1^2 + b_2 I_1 I_2 + b_3 I_2^2 + b_4 I_1 I_3 + b_5 I_2 I_3 + b_6 I_3^2 + c_1 I_1^3 \\
&\quad + c_2 I_1^2 I_2 + c_3 I_1 I_2^2 + c_4 I_2^3 + c_5 I_1^2 I_3 + c_6 I_1 I_2 I_3 + c_7 I_2^2 I_3 + c_8 I_1 I_3^2 + c_9 I_2 I_3^2 \\
&\quad + c_{10} I_3^3 + d_1 I_1^4 + d_2 I_1^3 I_2 + d_3 I_1^2 I_2^2 + d_4 I_1 I_2^3 + d_5 I_2^4 + d_6 I_1^3 I_3 + d_7 I_1^2 I_2 I_3
\end{aligned}$$

$$+d_8 I_1 I_2^2 I_3 + d_9 I_2^3 I_3 + d_{10} I_1^2 I_3^2 + d_{11} I_1 I_2 I_3^2 + d_{12} I_2^2 I_3^2 + d_{13} I_1 I_3^3 + d_{14} I_2 I_3^3$$
$$+d_{15} I_3^4$$

where

$$a_1 = 0.943129 \quad a_2 = -0.332428 \quad b_1 = 0.161402 \quad b_2 = -2.75156 \quad b_3 = 0.127405$$
$$b_4 = 0.114703 \quad b_5 = 0.260248 \quad b_6 = -0.00270827 \quad c_1 = -0.540594 \quad c_2 = 2.89302$$
$$c_3 = 1122.67 \quad c_4 = 137.613 \quad c_5 = 0.670007 \quad c_6 = 24.4582$$
$$c_7 = 4.32674 \quad c_8 = -0.0812255 \quad c_9 = -0.217382 \quad c_{10} = -0.0000752361$$
$$d_1 = 3.16392 \quad d_2 = -187.931$$
$$d_3 = -118322. \quad d_4 = -105166.$$
$$d_5 = -6680.33 \quad d_6 = 4.49506$$
$$d_7 = 255.299 \quad d_8 = 5179.18$$
$$d_9 = 465.806 \quad d_{10} = 3.362 \quad d_{11} = -106.782$$
$$d_{12} = -23.1704 \quad d_{13} = 0.0651032 \quad d_{14} = 0.105165 \quad d_{15} = 0.000631065\,.$$

The explicit expression of the functions χ_j will be reported in Appendix B.

In confirmation of the results obtained in [**6**], we first show that $h(I)$ is not quasi-convex, not directionally quasi-convex and 3-jet degenerate at $I = \underline{0}$. After that, we show that $h(I)$ satisfies the hypotheses of Proposition 2.5 at the origin.

(1) Verification of the quasi-convexity

We denote by Λ the 2-dimensional linear space orthogonal to ω and by A the Hessian matrix of $h(I)$ computed at the origin.

We construct an orthonormal vector basis $\{e_1, e_2, e_3\}$ such that $e_1 \parallel \omega$, and we perform a rotation of the coordinates I. We denote by R the rotation matrix. Then we take the appropriate 2×2 sub-matrix A_Λ of the rotation of A, which represents the restriction of the Hessian matrix to the space Λ. We compute the eigenvalues λ_1, λ_2 of A_Λ:

$$\lambda_1 = -2.33242, \qquad \lambda_2 = 1.86534\,.$$

Since they have opposite signs, $h(I)$ is not quasi-convex at the origin.

(2) Verification of the 3-jet non-degeneracy

The two eigenvalues of A_Λ are both non-vanishing, therefore there are two directions in the space Λ on which the quadratic form $h^2[v,v]$ vanishes, and they are:

$$\begin{aligned} v^A &= R^T\left(0, \sqrt{\tfrac{\lambda_2}{\lambda_2-\lambda_1}}\,x_1 + \sqrt{\tfrac{-\lambda_1}{\lambda_2-\lambda_1}}\,y_1, \sqrt{\tfrac{\lambda_2}{\lambda_2-\lambda_1}}\,x_2 + \sqrt{\tfrac{-\lambda_1}{\lambda_2-\lambda_1}}\,y_2\right) \\ v^B &= R^T\left(0, \sqrt{\tfrac{\lambda_2}{\lambda_2-\lambda_1}}\,x_1 - \sqrt{\tfrac{-\lambda_1}{\lambda_2-\lambda_1}}\,y_1, \sqrt{\tfrac{\lambda_2}{\lambda_2-\lambda_1}}\,x_2 - \sqrt{\tfrac{-\lambda_1}{\lambda_2-\lambda_1}}\,y_2\right), \end{aligned}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are eigenvectors of the eigenvalues λ_1 and λ_2 respectively. We computed the vectors v^A, v^B, and we obtained:

$$\begin{aligned} v^A &= (0.729513, 0.0125508, -0.683852) \\ v^B &= (-0.0697574, -0.964467, -0.254826)\,. \end{aligned}$$

Since it holds:

$$\begin{aligned} h^3[v^A, v^A, v^A] &= 0 \\ h^3[v^B, v^B, v^B] &= -197.75 \neq 0\,, \end{aligned}$$

we can conclude that h is 3-jet degenerate at the origin, according to the results obtained in [**6**].

(3) Verification of the directional quasi-convexity

Since the vector v^A has two components with opposite signs, then $h(I)$ is not directionally quasi-convex at the origin.

(4) Verification of the hypotheses of Proposition 2.4

The vector v^A is the only non-vanishing vector in Λ verifying:

$$\begin{aligned} h^2[v^A, v^A] &= 0 \\ h^3[v^A, v^A, v^A] &= 0\,. \end{aligned}$$

We computed the unique vector $v^{A^\perp} \in \Lambda \setminus \{\underline{0}\}$ such that $v^A \cdot v^{A^\perp} = 0$:

$$v^{A^\perp} = (0.745154, 4.76855, 0.882426)\,.$$

Since it holds:

$$h^4[v^A, v^A, v^A, v^A] = -12.4955 \neq 0$$
$$h^2[v^A, v^{A^\perp}] = -10.234 \neq 0\,,$$

then $h(I)$ satisfies the hypotheses of Proposition 2.5, and therefore it is steep at the origin.

3.3. Verification of the steepness of a function with four degrees of freedom: the Hamiltonian of a chain of four harmonic oscillators

In this Section we implement the algorithm for the verification of the steepness on the Hamiltonian of a system of four particles connected each other by non-linear springs. We also introduce two additional particles whose position is fixed. Precisely, if x_j denotes the displacement of the j-th particle from its equilibrium position, for $j = 0, \ldots, 5$, then the following condition holds:

$$x_0 = x_5 = 0\,.$$

We denote by $y = (y_1, y_2, y_3, y_4)$ the momenta conjugated to the coordinates $x = (x_1, x_2, x_3, x_4)$, hence the Hamiltonian of the system can be written as:

$$H(x,y;\alpha,\beta) = \frac{1}{2}\sum_{j=1}^{4} y_j^2 + \sum_{j=0}^{4}\frac{1}{2}(x_{j+1}-x_j)^2 + \sum_{j=0}^{4}\left(\frac{\alpha}{2}(x_{j+1}-x_j)^3 + \frac{\beta}{3}(x_{j+1}-x_j)^4\right) \tag{3.16}$$

where α and β are positive parameters which measure the non-linearity in the forces between the particles in the chain. For the sake of simplicity, we set equal to 1 the mass of the particles and the harmonic constant of the springs.

The system has $n = 4$ degrees of freedom, and in the limit on big values of n, it represents the famous Fermi-Pasta-Ulam problem [**27**].

In the fifties Fermi decided to perform numerical experiments on such a model, with the collaboration of Pasta and Ulam. His intention was to prove the ergodic property of non-linear systems, which states that a non-linear dynamical system behaves as ergodic. He wanted to prove such property, in particular, for a system subject to an arbitrarily small non-linear perturbation. The results of the numerical experiments, however, turned out to be in contradiction with the ergodic hypothesis: the system, in fact, showed an integrable-like behavior. This fact gave rise to the so-called Fermi-Pasta-Ulam paradox, which is still largely under investigation.

We are interested in small perturbations of the system, therefore we will take $\alpha,\beta \leq 1$.

The origin of the system is an equilibrium point, in particular it can be proved that such equilibrium is elliptic. Hence in a neighborhood of the origin there exists a linear change of variables $(q,p) = (q(x,y),p(x,y))$, such that the quadratic part of the Hamiltonian (3.16) in the new variables is:

$$k_2(I) = \omega \cdot I$$

where $\omega = (\omega_1,\omega_2,\omega_3,\omega_4)$, with $\omega_j = 2\sin\frac{j\pi}{10}$, and $I = (I_1,I_2,I_3,I_4)$, with $I_j = \frac{p_j^2+q_j^2}{2}$.

The transformation is given by:

$$x_j = \sqrt{\tfrac{2}{5}}\textstyle\sum_{k=1}^4 q_k \sin\frac{kj\pi}{5}$$

$$y_j = \sqrt{\tfrac{2}{5}}\textstyle\sum_{k=1}^4 p_k \sin\frac{kj\pi}{5}, \qquad j = 1,\ldots,4$$

and we denote by T the matrix such that $(q,p) = T\,(x,y)$.

We will investigate the steepness of the Birkhoff normal form of order eight of (3.16) in a neighborhood of the origin. For this reason we introduce the complex variables $(z,w) = (z_1,z_2,z_3,z_4,w_1,w_2,w_3,w_4)$, and the Hamiltonian of which we will construct the normal form is:

$$h(z,w;\alpha,\beta) = H(T^{-1}C\,(z,w);\alpha,\beta) \tag{3.17}$$

where C is the matrix such that $(q,p) = C\,(z,w)$.

3.3 Verification of the steepness of a function with four degrees of freedom

3.3.1. The verification of the steepness. We construct the Birkhoff normal form of order eight of the Hamiltonian (3.17), according to the procedure described in Appendix A. Precisely we take the Taylor expansion of (3.17) around the origin, which is of degree four:

$$h^{(2)}(Z;\alpha,\beta) = k_2(I) + h_3^{(2)}(Z;\alpha,\beta) + h_4^{(2)}(Z;\alpha,\beta)$$

where $Z = (z,w)$, and we construct:

$$h^{(8)}(I;\alpha,\beta) = h^{(2)} \circ \phi_{\chi_3} \circ \phi_{\chi_4} \circ \phi_{\chi_5} \circ \phi_{\chi_6} \circ \phi_{\chi_7} \circ \phi_{\chi_8}$$

where the functions χ_j, $j = 3,\ldots,8$, are defined in Appendix A. We obtain the polynomial:

$$h^{(8)}(I;\alpha,\beta) = k_2(I) + k_4(I;\alpha,\beta) + k_6(I;\alpha,\beta) + k_8(I;\alpha,\beta) + O(9)$$

where:

$$k_4 = \langle h_4^{(3)} \rangle_0 \qquad k_6 = \langle h_6^{(5)} \rangle_0 \qquad k_8 = \langle h_8^{(7)} \rangle_0$$

with

$$\begin{aligned}
h_4^{(3)} &= h_4^{(2)} + \frac{1}{2}\mathscr{L}_{\chi_3} h_3^{(2)}, & h_6^{(5)} &= h_6^{(3)} + \frac{1}{2}\mathscr{L}_{\chi_4}(h_4^{(3)} + k_4)\\
h_6^{(3)} &= \frac{1}{2}\mathscr{L}_{\chi_3}^2 h_4^{(2)} + \frac{1}{8}\mathscr{L}_{\chi_3}^3 h_3^2, & h_8^{(7)} &= h_8^{(4)} + \frac{1}{2}\mathscr{L}_{\chi_5} h_5^{(3)} + \mathscr{L}_{\chi_6} k_4\\
h_5^{(3)} &= \mathscr{L}_{\chi_3} h_4^{(2)} + \frac{1}{3}\mathscr{L}_{\chi_3}^2 h_3^{(2)}, & &\\
h_8^{(4)} &= h_8^{(3)} + \mathscr{L}_{\chi_4} h_6^{(3)} + \frac{1}{3}\mathscr{L}_{\chi_4}^2 h_4^{(3)} + \frac{1}{6}\mathscr{L}_{\chi_4}^2 k_4 & &\\
h_8^{(3)} &= \frac{1}{24}\mathscr{L}_{\chi_3}^4 h_4^{(2)} + \frac{1}{144}\mathscr{L}_{\chi_3}^5 h_3^{(2)} . & &
\end{aligned}$$

We denote by $h(I;\alpha,\beta)$ the integrable approximation of $h^{(8)}(I;\alpha,\beta)$.

We consider many couples $\alpha,\beta \in (0,1]$, and we find that $h(I;\alpha,\beta)$ is never quasi-convex but always directionally quasi-convex at the origin. To test the 3-jet degeneracy, we restrict our attention to a sample of 11 of such couples, and for all of them we find that $h(I;\alpha,\beta)$ is 3-jet degenerate at the origin. We continue by choosing one of the couples, precisely $\alpha = 0.1$, $\beta = 0.9$, and we show that for these values $h(I;\alpha,\beta)$ satisfies the hypotheses of Proposition 2.5 at the origin.

(1) Verification of the quasi-convexity

We denote by Λ the 3-dimensional linear space orthogonal to ω and by $A_{\alpha,\beta}$ the Hessian matrix of $h(I;\alpha,\beta)$ computed at the origin.

We construct an orthonormal vectors basis $\{e_1, e_2, e_3, e_4\}$ such that $e_1 \parallel \omega$, and we perform a rotation of the coordinates I. We denote by R the rotation matrix.

Then we take the appropriate 3×3 sub-matrix $A_{\Lambda,\alpha,\beta}$ of the rotation of $A_{\alpha,\beta}$, which represents the restriction of the Hessian matrix to the space Λ. For fixed α, β, we denote by $\lambda_1 \leq \lambda_2 \leq \lambda_3$ the eigenvalues of $A_{\Lambda,\alpha,\beta}$, and by $x = (x_1,x_2,x_2)$, $y = (y_1,y_2,y_3)$, $z = (z_1,z_2,z_3)$ eigenvectors of λ_1, λ_2 and λ_3 respectively.

We analyzed the sign of the function $P(\alpha,\beta) := \lambda_1\,\lambda_3$ for α, β varying in $(0,1]$ and we found that $P(\alpha,\beta)$ is always negative for all the values considered of α and β, so that the eigenvalues of $A_{\Lambda,\alpha,\beta}$ never have the same sign.

In figure 2(a) we represent an example of our results for $\alpha \in (0,1]$ and fixed $\beta = 0.1$. To better appreciate the sign of $P(\alpha,\beta)$ near $\alpha \sim 0$, in figure 2(b) we also represent a zoom of figure 2(a).

In figure 3(a) we represent the function $P(\alpha,\beta)$ for $\alpha = 0.1$ and $\beta \in (0,1]$. Also in this case, to better appreciate the sign of $P(\alpha,\beta)$ near $\beta \sim 0$, in figure 3(b) we represent a zoom of figure 3(a).

Therefore, for all the couples α, β considered, $h(I;\alpha,\beta)$ is never quasi-convex at the origin.

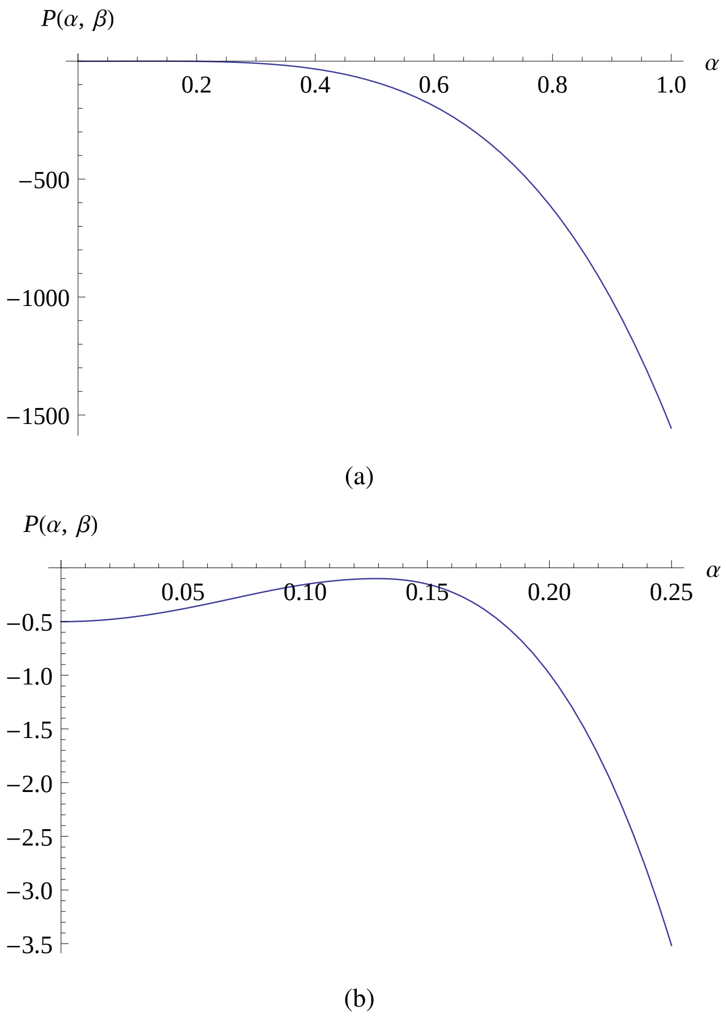

FIGURE 2. $P(\alpha,\beta)$ for $\beta = 0.1$, and for α in $(0,1]$ (a), α in $(0,0.25]$ (b)

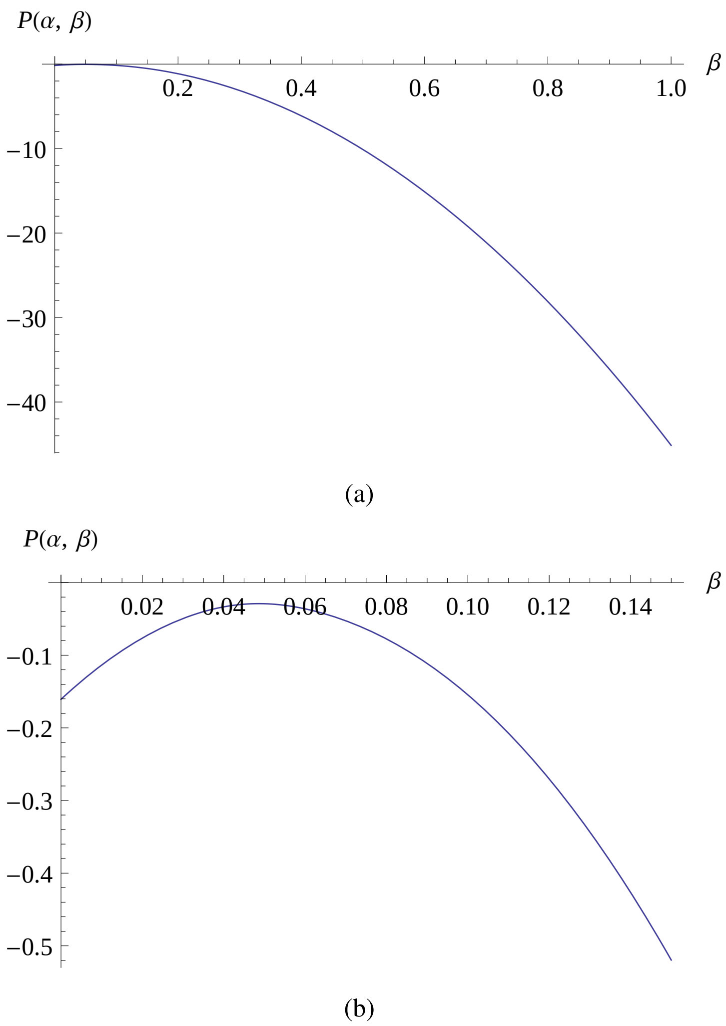

FIGURE 3. $P(\alpha,\beta)$ for $\alpha = 0.1$, and for β in $(0,1]$ (a), β in $(0,0.15]$ (b)

(2) Verification of the 3-jet non-degeneracy

From point A), we obtained $\lambda_1 < 0$ and $\lambda_3 > 0$ for all the couples α,β we considered.

We also found that the intermediate eigenvalue λ_2 changes its sign depending on the values of α and β, and accordingly one has to refer to the corresponding method of computation of the vectors $v \in \Lambda\backslash\{\underline{0}\}$ such that $h^2[v,v] = 0$.

Precisely, when $\lambda_2 \neq 0$, the vectors of the space Λ on which the quadratic form $h^2[v,v]$ vanishes are:

- $\lambda_2 > 0$

$$v^\gamma = R^T(0, d^\gamma) \qquad d^\gamma = x\sqrt{1-r^2} + y\,r\cos\gamma + z\,r\sin\gamma$$

with $r = \sqrt{\frac{\lambda_1}{\lambda_2\cos^2\gamma + \lambda_3\sin^2\gamma - \lambda_1}}$ and $\gamma \in [0, 2\pi)$.

- $\lambda_2 < 0$

$$v^\gamma = R^T(0, d^\gamma) \qquad d^\gamma = x\,r\cos\gamma + y\,r\sin\gamma + z\sqrt{1-r^2}$$

with $r = \sqrt{\frac{\lambda_3}{\lambda_3 - \lambda_1\cos^2\gamma - \lambda_2\sin^2\gamma}}$ and $\gamma \in [0, 2\pi)$.

For the verification of the 3-jet non-degeneracy, we fixed several values of the parameters $\alpha, \beta \in (0,1]$, all such that $\lambda_2 \neq 0$, and we considered the function $F(\gamma) := h^3[v^\gamma, v^\gamma, v^\gamma] : [0, 2\pi) \to \mathbb{R}$. For each choice of α and β, there is always at least one value $\bar{\gamma} \in [0, 2\pi)$ (usually there are two of them) such that $F(\bar{\gamma}) = 0$. In figures 4 and 5 we represent $F(\gamma)$ for some values of $\alpha, \beta \in (0,1]$.

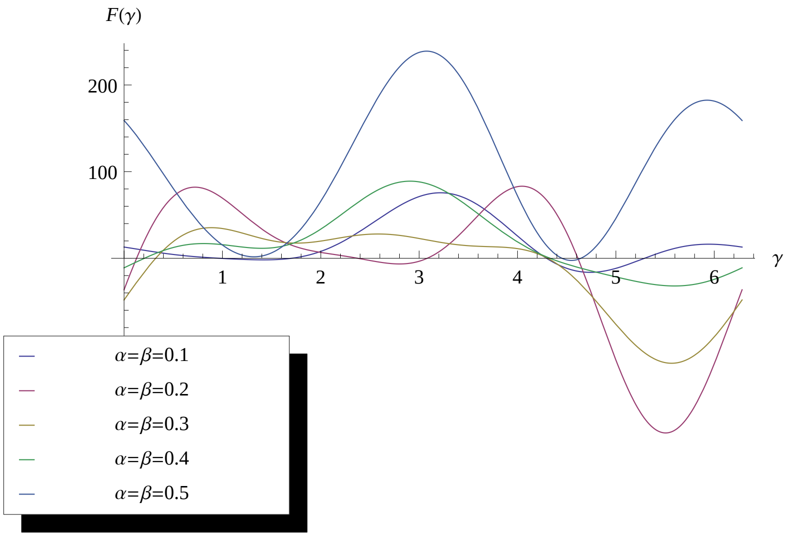

FIGURE 4. $F(\gamma)$ for $\alpha = \beta = 0.1, 0.2, 0.3, 0.4, 0.5$

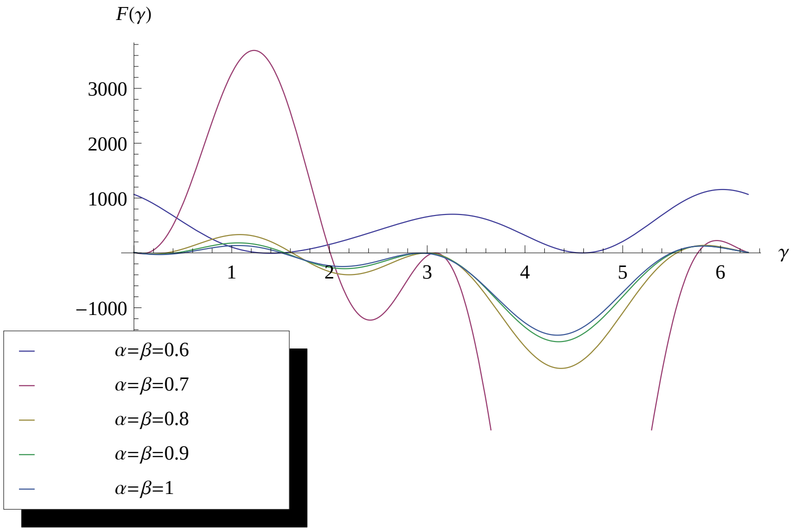

FIGURE 5. $F(\gamma)$ for $\alpha = \beta = 0.6, 0.7, 0.8, 0.9, 1$

Therefore, for all the couples α, β considered, and presumably for all $\alpha, \beta \in (0,1]$, $h(I;\alpha,\beta)$ is always 3-jet degenerate at the origin.

(3) Verification of the directional quasi-convexity

For fixed α and β, we considered the following function: $B(\gamma) := M_\gamma m_\gamma$, where M_γ and m_γ are respectively the greatest and the smallest component of the vector v^γ.

We analyzed the sign of $B(\gamma)$ for fixed values of α, β in $(0,1]$, and we found that $B(\gamma)$ is always strictly negative for all the couples α, β considered, so that $h(I;\alpha,\beta)$ is always directional quasi-convex at the origin.

In figure 6 we represent an example of our results.

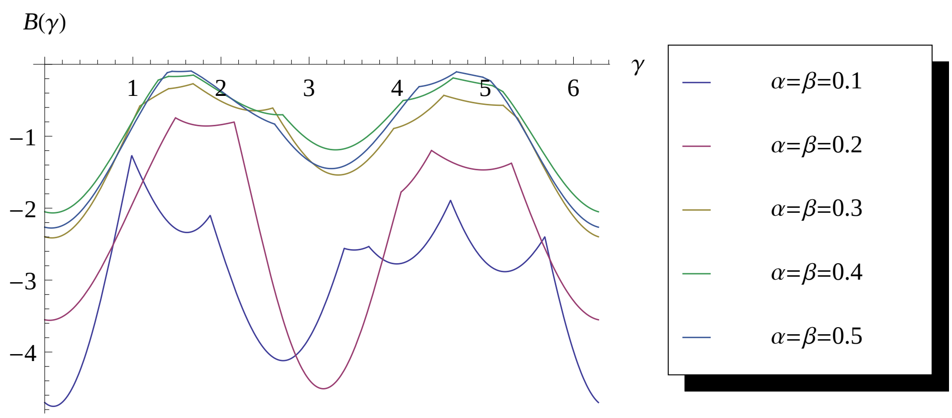

FIGURE 6. $B(\gamma)$ for $\alpha = \beta = 0.1, 0.2, 0.3, 0.4, 0.5$

(4) Verification of the hypotheses of Proposition 2.5

We select $\alpha = 0.1$ and $\beta = 0.9$, and in figure 7 we represent the function $F(\gamma) = h^3[v^\gamma, v^\gamma, v^\gamma]$: as we can see, $h(I;\alpha,\beta)$ is 3-jet degenerate at the origin.

Precisely, there are two values $\gamma_1, \gamma_2 \in [0, 2\pi)$ such that $F(\gamma_1) = F(\gamma_2) = 0$. We computed approximatively these values:

$$\gamma_1 = 2.29830744307\,, \qquad \gamma_2 = 5.71819341588\,.$$

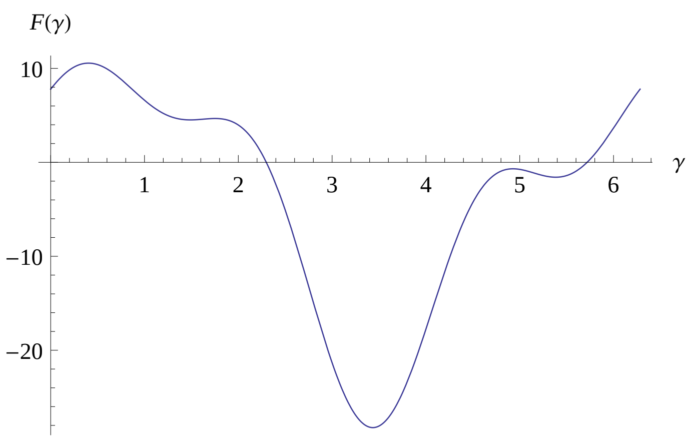

FIGURE 7. $F(\gamma)$ for $\alpha = 0.1$ and $\beta = 0.9$

The eigenvalues of $A_{\Lambda,\alpha,\beta}$ are:

$$\lambda_1 = -9.60123, \quad \lambda_2 = 1.48302, \quad \lambda_3 = 3.76194,$$

hence they are all non-vanishing, and in particular $\lambda_2 > 0$. Therefore the two vectors $v^{\gamma_1}, v^{\gamma_2} \in \mathbb{R}^4 \setminus \{\underline{0}\}$ which solve:

$$\begin{cases} h^1[v] = 0 \\ h^2[v,v] = 0 \\ h^3[v,v,v] = 0 \end{cases}$$

are:

$$\begin{aligned} v^{\gamma_1} &= (-0.553501, -0.164378, -0.176342, 0.43144) \\ v^{\gamma_2} &= (0.775441, -0.212722, -0.120282, -0.0181689). \end{aligned}$$

The unique solution $w^{\gamma_1} \in \mathbb{R}^4$, with $\|w^{\gamma_1}\| = 1$, of the system:

$$\begin{cases} v^{\gamma_1} \cdot w = 0 \\ h^1[w] = 0 \\ h^2[v^{\gamma_1}, w] = 0 \end{cases}$$

is $w^{\gamma_1} = (0.208442, -0.873288, 0.426641, 0.109072)$, and is such that

$$h^2[w^{\gamma_1}, w^{\gamma_1}] h^4[v^{\gamma_1}, v^{\gamma_1}, v^{\gamma_1}, v^{\gamma_1}] \neq 3(h^3[v^{\gamma_1}, v^{\gamma_1}, w^{\gamma_1}])^2 .$$

In fact we have:

$$h^2[w^{\gamma_1}, w^{\gamma_1}] h^4[v^{\gamma_1}, v^{\gamma_1}, v^{\gamma_1}, v^{\gamma_1}] = -658.539 \quad \text{and} \quad 3(h^3[v^{\gamma_1}, v^{\gamma_1}, w^{\gamma_1}])^2 = 231.187 .$$

The unique solution $w^{\gamma_2} \in \mathbb{R}^4$, with $\|w^{\gamma_2}\| = 1$, of the system:

$$\begin{cases} v^{\gamma_2} \cdot w = 0 \\ h^1[w] = 0 \\ h^2[v^{\gamma_2}, w] = 0 \end{cases}$$

is $w^{\gamma_2} = (0.110936, 0.761750, -0.637325, 0.035308)$, and is such that

$$h^2[w^{\gamma_2}, w^{\gamma_2}] h^4[v^{\gamma_2}, v^{\gamma_2}, v^{\gamma_2}, v^{\gamma_2}] \neq 3(h^3[v^{\gamma_2}, v^{\gamma_2}, w^{\gamma_2}])^2 .$$

In fact we have:

$$h^2[w^{\gamma_2}, w^{\gamma_2}] h^4[v^{\gamma_2}, v^{\gamma_2}, v^{\gamma_2}, v^{\gamma_2}] = 41.2909 \quad \text{and} \quad 3(h^3[v^{\gamma_2}, v^{\gamma_2}, w^{\gamma_2}])^2 = 90.6655 .$$

Finally, since

$$h^4[v^{\gamma_1}, v^{\gamma_1}, v^{\gamma_1}, v^{\gamma_1}] = -551.733 \neq 0$$
$$h^4[v^{\gamma_2}, v^{\gamma_2}, v^{\gamma_2}, v^{\gamma_2}] = 27.3614 \neq 0\,,$$

we can conclude that for $\alpha = 0.1$ and $\beta = 0.9$, the function $h(I;\alpha,\beta)$ satisfies the hypotheses of Proposition 2.5, and therefore it is steep in a neighborhood of the origin.

Conclusions

Among the most useful and interesting applications of the Nekhoroshev Theorem there are problems from Celestial Mechanics. When the perturbing parameter ε is sufficiently small, in fact, the stability time proved by Nekhoroshev may be comparable to the age of the Universe, and therefore can exceed the lifetime of a real system [**18, 23, 27, 29, 30**].

Starting from the Planetary problem, many physical systems of interest can be studied by the Theory of Perturbations, and for some of them an investigation of the stability has been already performed, using the notions and the instruments known up to now. Among the most recent works I mention as an example [**14, 16, 18, 20, 21, 28, 42, 61**].

I think the results proved in my Thesis may be useful to perform a step forward in the study of the stability of such systems.

In Chapter 3 I already provided two examples of applicability of my results, which are the Hamiltonian of the circular restricted three-body problem and the Hamiltonian of a chain of four harmonic oscillators, with the potential energy derived from the famous Fermi-Pasta-Ulam problem. For such systems I showed that the new sufficient conditions for steepness obtained in this Thesis are useful to prove steepness, which is a fundamental property for the applicability of the Nekhoroshev Theorem.

A widely investigated physical system, for which the steepness represents a relevant problem, is the motion of an asteroid in the main belt, which is the region of the space between Mars and Jupiter. Such system, like the most of the astronomical systems of interest, is degenerate, that is to say the

Hamiltonian has a number of first integrals of the motion which is strictly greater than the number of degrees of freedom.

In [**36, 46**] a Nekhoroshev like stability result has been proved for the asteroids problem, providing long time stability in particular for the eccentricity and the inclination of an asteroid. In [**53**] the authors investigated the fulfillment of the conditions for the validity of such result, precisely they considered the Koronis and Veritas families of asteroids, and analized their steepness properties.

They obtained that almost all asteroids in those families are convex, quasi-convex or 3-jet non-degenerate: 71 elements in the Koronis family and 13 in the Veritas family turned out to be 3-jet degenerate, therefore their steepness properties are still unknown. Such particular asteroids may represent a concrete example on which testing the new sufficient condition for the steepness of functions with three degrees of freedom, proved in this Thesis.

The Riemann ellipsoids are another interesting example of physical system whose stability properties have been widely investigated for a long time, and still have some open questions.

The Riemann ellipsoids are steady motions of an ideal, incompressible and self-gravitating fluid with ellipsoidal shape. The interest in such motions originated from the attempt to explain the shape of the planets. Already Newton and McLaurin discovered the first examples of such ellipsoides. From a mathematical point of view, the Riemann ellipsoids represent elliptic equilibria of a quasi-integrable Hamiltonian system.

In [**24**] the authors investigated the stability of those ellipsoids which are spectrally stable, but of unknown Lyapunov stability. They obtained that all the ellipsoids which are non-resonant up to the fourth order, are directionally quasi-convex, and therefore stable. But still the stability of some different kinds of ellipsoids, such as the axisymmetric Riemann ellipsoids or the McLaurin spheroids, has to be investigated. Therefore such ellipsoids may represent a possible example of system on which testing the new sufficient conditions for the steepness of functions with four degrees of freedom, proved in this Thesis.

Finally it is interesting to mention a system with four degrees of freedom which has not been yet investigated from the point of view of the stability properties: the elliptic restricted three-body problem. A possible investigation of the applicability of the Nekhoroshev Theorem to such system, may need the new sufficient conditions for steepness obtained in this Thesis.

The Nekhoroshev's result that I used for the construction of new sufficient conditions for steepness, leaves open the possibility of many other developments. It is possible to construct new sufficient conditions for steepness for all values of n, being n the number of degrees of freedom of the system, and for all values of r (I recall that r denotes the maximum derivative order of the function involved). But the construction of such conditions becomes rapidly very elaborated with n and r increasing. Therefore it would be interesting to investigate the existence of some recursion in the sufficient conditions, for example depending on n and for fixed values of r, or correspondingly for r increasing and fixed values of n.

APPENDIX A

The Birkhoff normal forms in a neighborhood of an elliptic equilibrium

We consider a quasi-integrable analytic Hamiltonian system H with n degrees of freedom, having an elliptic equilibrium at the origin. That means that there exists a set of canonical variables $(p,q) \in \mathbb{R}^{2n}$ (defined in a neighborhood of the origin) such that in these variables H takes the form

$$H(p,q) = k_2(p,q) + f^{(3)}(p,q) \tag{A.1}$$

where

$$k_2(p,q) = \sum_{j=1}^{n} \omega_j \frac{p_j^2 + q_j^2}{2},$$

$\omega = (\omega_1, \ldots, \omega_n)$ is the frequency vector of H at the origin, and $f^{(3)}$ is a smooth function having a zero of order three at the origin.

Introducing the action functions $I = (I_1, \ldots, I_n)$ such that $I_j(p,q) = \frac{p_j^2+q_j^2}{2}$, $j = 1, \ldots, n$, then the Hamiltonian (A.1) may be represented as

$$H = k_2(I) + f^{(3)}. \tag{A.2}$$

The Birkhoff Theorem ensures that, under suitable non-resonance conditions on the frequency vector, in a neighborhood of the elliptic equilibrium it is possible to construct normal forms of (A.2). Precisely

THEOREM A.1 (Birkhoff). *Let us fix an integer $N \geq 2$ and suppose the frequency vector ω of (A.2) does not satisfy any resonance condition up to the order N, that is*

$$\omega \cdot \nu \neq 0 \qquad \textit{for all } \nu \in \mathbb{Z}^n \setminus \{\underline{0}\},\ |\nu| \leq N, \textit{ where } |\nu| = \sum_{j=1}^{n} |\nu_j|,$$

then there exists a neighborhood $\mathscr{U}_N$ of the origin and a canonical transformation $w_N : \mathscr{U}_N \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ which puts the Hamiltonian (A.2) in Birkhoff normal form up to order N, namely such that

$$H \circ w_N = k_2(I) + \sum_{j=2}^{[N/2]} k_{2j}(I) + f^{(N+1)} \tag{A.3}$$

where each $k_{2j}(I)$ is a homogeneous polynomial of degree j in $I_1, \dots, I_n$. The remainder $f^{(N+1)}$ is a Taylor series in p, q which starts at order $N+1$ and is convergent in $\mathscr{U}_N$.

The Birkhoff normal form of order N is in a quasi-integrable form, in fact the term $f^{(N+1)}$ represents a small perturbation of the integrable Hamiltonian $k_2(I) + \sum_{j=2}^{[N/2]} k_{2j}(I)$.

Constructing a Birkhoff normal form consists in a sequence of perturbation steps which are not performed in the standard way, since we are not working in action-angle variables. It is convenient to introduce, in place of the coordinates (p, q), the conjugate complex variables:

$$z_j := \frac{p_j - iq_j}{i\sqrt{2}}\,, \quad w_j := \frac{p_j + iq_j}{\sqrt{2}}\,, \qquad j = 1, \dots, n$$

($z = (z_1, \dots, z_n)$ are the coordinates and $w = (w_1, \dots, w_n)$ the momenta), so that $I_j = i w_j z_j\,,\ j = 1, \dots, n\,.$

In view of finding a solution of the fundamental equation of the Perturbation Theory (see Chapter 1), we need to define a Fourier expansion for analytic functions $g(z, w)$.

We decide to use an idea of Siegel (see [**62, 6, 25**]), who defined a suitable Fourier series in the variables (z, w) which, for a quasi-integrable analytic Hamiltonian coincides, out of the manifolds $I_j = 0$, to the Fourier series in action-angle variables.

Precisely, if we consider the Taylor expansion of an analytic function $g(z, w)$:

$$g(z, w) = \sum_{p, m \in \mathbb{N}^n} g_{pm} z^p w^m\,,$$

then for any integer vector $k \in \mathbb{Z}^n$, the k-th Fourier harmonic of g is defined by:

$$\langle g \rangle_k(z,w) = \sum_{\substack{p,m \in \mathbb{N}^n \\ p-m=k}} g_{pm} z^p w^m \,. \tag{A.4}$$

Each step of the construction of the Birkhoff normal form of order N, consists in a canonical change of variables performed by the time-1 flow of a suitable auxiliary Hamiltonian. Precisely, we consider the Taylor expansion of H around the origin up to the order N, and we denote it by $h^{(2)}$:

$$h^{(2)} = k_2(I) + h_3^{(2)} + \ldots + h_N^{(2)} + O(N+1) \tag{A.5}$$

where $h_j^{(2)}$ are homogeneous polynomial of degree j in (z,w).

For all $j = 3, \ldots, N$, the Birkhoff normal form of order j will be

$$h^{(j)} = k_2(I) + \ldots + k_{2[\frac{j}{2}]}(I) + h_{j+1}^{(j)} + \ldots + h_N^{(j)} + O(N+1)$$

with $h_l^{(j)}$ homogeneous polynomial of degree l in (z,w), and k_l homogeneous polynomial of degree $l/2$ in I. The normal forms $h^{(j)}$ are obtained through an iterative procedure:

$$h^{(j)} = h^{(j-1)} \circ \phi_{\chi_j} \quad j = 3, \ldots, N$$

where ϕ_{χ_j} is the time-1 flow of the Hamiltonian χ_j, and χ_j is the solution of the fundamental equation of the perturbation theory:

$$\{k_2, \chi_j\} = h_j^{(j-1)} - \langle h_j^{(j-1)} \rangle_0$$

that is

$$\chi_j = -i \sum_{k \in \mathbb{Z}^n \setminus \{\underline{0}\}} \frac{\langle h_j^{(j-1)} \rangle_k}{i\omega \cdot k} \,.$$

With evidence, the definition of all the auxiliary Hamiltonians is possible when the frequency vector ω does not satisfy any resonance condition up to order N.

APPENDIX B

The 8th order Birkhoff normal form of the Hamiltonian of the circular restricted three-body problem

In this Appendix we report the explicit expressions of the functions χ_3, χ_4, χ_5 and χ_6, involved in the construction of the 8th order Birkhoff normal form of the Hamiltonian of the circular restricted three-body problem (see Chapter 3).

The normal form is computed in a neighborhood of the elliptic equilibrium L_4 and for a fixed value of the reduced mass $\mu = \mu_3$.

We remark that, even though χ_7 and χ_8 are required to construct the normal form of order eight, they do not contribute to $k_8(I)$, and therefore do not need to be explicitly constructed. We only check that no resonances of order seven or eight prevent their construction.

$$
\begin{aligned}
&\chi_3(z,w) = \\
&(0.314326+1.19549\,i)w_1^3+(2.74776-2.79607\,i)w_1^2w_2 \\
&-(5.32268+0.319699\,i)w_1w_2^2+(0.767883+4.62586\,i)w_2^3 \\
&-(0.179638+0.0889783\,i)w_1w_3^2-(0.0236156+0.234035\,i)w_2w_3^2 \\
&-(0.567936-1.54841\,i)w_1^2z_1+(6.27098+0.897071\,i)w_1w_2z_1 \\
&-(4.40193+7.13861\,i)w_2^2z_1+(0.247783+0.500247\,i)w_3^2z_1 \\
&-(1.54841-0.567936\,i)w_1z_1^2+(4.36103+3.22364\,i)w_2z_1^2 \\
&-(1.19549+0.314326\,i)z_1^3-(3.22364+4.36103\,i)w_1^2z_2 \\
&-(4.8931-12.1586\,i)w_1w_2z_2+(12.4272+0.916175\,i)w_2^2z_2 \\
&+(0.167323+0.016884\,i)w_3^2z_2-(0.897071+6.27098\,i)w_1z_1z_2
\end{aligned}
$$

$$-(12.1586-4.8931\,i)w_2z_1z_2+(2.79607-2.74776\,i)z_1^2z_2$$
$$+(7.13861+4.40193\,i)w_1z_2^2-(0.916175+12.4272\,i)w_2z_2^2$$
$$+(0.319699+5.32268\,i)z_1z_2^2-(4.62586+0.767883\,i)z_2^3$$
$$-(0.555332-1.12116\,i)w_1w_3z_3+(2.34799-0.236928\,i)w_2w_3z_3$$
$$-(1.12116-0.555332\,i)w_3z_1z_3+(0.236928-2.34799\,i)w_3z_2z_3$$
$$-(0.500247+0.247783\,i)w_1z_3^2-(0.016884+0.167323\,i)w_2z_3^2$$
$$+(0.0889783+0.179638\,i)z_1z_3^2+(0.234035+0.0236156\,i)z_2z_3^2$$

$$\chi_4(z,w)=$$
$$(1.69379+0.233854\,i)w_1^4+(4.28931-11.5592\,i)w_1^3w_2$$
$$-(26.4783+1.53214\,i)w_1^2w_2^2-(44.3035-34.995\,i)w_1w_2^3$$
$$+(14.3266-2.60582\,i)w_2^4-(0.197511-0.14474\,i)w_1^2w_3^2$$
$$+(0.189755+0.489293\,i)w_1w_2w_3^2-(0.456577-0.435692\,i)w_2^2w_3^2$$
$$-0.000338534\,i\,w_3^4+(5.48859-2.22304\,i)w_1^3z_1-(4.3988+$$
$$+9.80995\,i)w_1^2w_2z_1+(0.864678-4.2146\,i)w_1w_2^2z_1-(19.2549$$
$$+0.284727\,i)w_2^3z_1+(0.0323889-0.0127796\,i)w_1w_3^2z_1-(0.483593$$
$$-0.0898874\,i)w_2w_3^2z_1-(11.4384+10.4269\,i)w_1w_2z_1^2+(16.1037$$
$$+7.55428\,i)w_2^2z_1^2+(0.492813-0.151455\,i)w_3^2z_1^2+(5.48859$$
$$+2.22304\,i)w_1z_1^3-(9.13472+3.27905\,i)w_2z_1^3+(1.69379-0.233854\,i)z_1^4$$
$$-(9.13472-3.27905\,i)w_1^3z_2+(17.4053+30.0956\,i)w_1^2w_2z_2$$
$$+(13.3553-30.8111\,i)w_1w_2^2z_2+(5.51231+17.4136\,i)w_2^3z_2$$
$$+(0.292424-0.543855\,i)w_1w_3^2z_2+(0.0715228-0.0362568\,i)w_2w_3^2z_2$$
$$-(11.4384-10.4269\,i)w_1^2z_1z_2-(38.3978+0.903033\,i)w_2^2z_1z_2$$
$$-(0.184434+0.656401\,i)w_3^2z_1z_2-(4.3988-9.80995\,i)w_1z_1^2z_2$$
$$+(17.4053-30.0956\,i)w_2z_1^2z_2+(4.28931+11.5592\,i)z_1^3z_2$$

$$+(16.1037-7.55428\,i)w_1^2z_2^2-(38.3978-0.903033\,i)w_1w_2z_2^2$$
$$-(0.352382-0.137904\,i)w_3^2z_2^2+(0.864678+4.2146\,i)w_1z_1z_2^2$$
$$+(13.3553+30.8111\,i)w_2z_1z_2^2-(26.4783-1.53214\,i)z_1^2z_2^2$$
$$-(19.2549-0.284727\,i)w_1z_2^3+(5.51231-17.4136\,i)w_2z_2^3$$
$$-(44.3035+34.995\,i)z_1z_2^3+(14.3266+2.60582\,i)z_2^4$$
$$+(0.0934021+0.551296\,i)w_1^2w_3z_3+(4.26194-0.157583\,i)w_1w_2w_3z_3$$
$$-(0.416793+3.10622\,i)w_2^2w_3z_3-0.00135414w_3^3z_3-(1.0184$$
$$-1.25\,i)w_2w_3z_1z_3+(0.0934021-0.551296\,i)w_3z_1^2z_3-(1.0184$$
$$+1.25\,i)w_1w_3z_2z_3+(4.26194+0.157583\,i)w_3z_1z_2z_3-(0.416793$$
$$-3.10622\,i)w_3z_2^2z_3+(0.492813+0.151455\,i)w_1^2z_3^2-(0.184434$$
$$-0.656401\,i)w_1w_2z_3^2-(0.352382+0.137904\,i)w_2^2z_3^2+(0.0323889$$
$$+0.0127796\,i)w_1z_1z_3^2+(0.292424+0.543855\,i)w_2z_1z_3^2-(0.197511$$
$$+0.14474\,i)z_1^2z_3^2-(0.483593+0.0898874\,i)w_1z_2z_3^2+(0.0715228$$
$$+0.0362568\,i)w_2z_2z_3^2+(0.189755-0.489293\,i)z_1z_2z_3^2-(0.456577$$
$$+0.435692\,i)z_2^2z_3^2-0.00135414w_3z_3^3+0.000338534\,i\,z_3^4$$

$$\chi_5(z,w)=$$
$$(23.6776+5.60465\,i)w_1^5-(47.409+175.03\,i)w_1^4w_2-(447.191$$
$$-329.705\,i)w_1^3w_2^2+(774.449+444.385\,i)w_1^2w_2^3+(92.4234$$
$$-715.858\,i)w_1w_2^4-(340.8-70.0965\,i)w_2^5-(1.30314-0.850712\,i)w_1^3w_3^2$$
$$+(0.186131+0.439028\,i)w_1^2w_2w_3^2+(3.15832+6.6943\,i)w_1w_2^2w_3^2$$
$$+(8.70484-5.34965\,i)w_2^3w_3^2+(0.0237389-0.006909\,i)w_1w_3^4$$
$$+(0.146066-0.107917\,i)w_2w_3^4+(26.6093+11.8102\,i)w_1^4z_1$$
$$+(35.3907-252.565\,i)w_1^3w_2z_1-(762.357-88.6466\,i)w_1^2w_2^2z_1$$
$$+(449.728+893.539\,i)w_1w_2^3z_1+(424.948-445.808\,i)w_2^4z_1$$

$$
\begin{aligned}
&-(3.98847+1.37492\,i)w_1^2w_3^2z_1-(1.32442-8.96575\,i)w_1w_2w_3^2z_1\\
&+(4.03999+6.11297\,i)w_2^2w_3^2z_1-(0.0480885-0.000195137\,i)w_3^4z_1\\
&-(1.94202-11.0964\,i)w_1^3z_1^2+(139.325-100.215\,i)w_1^2w_2z_1^2\\
&-(397.311+225.77\,i)w_1w_2^2z_1^2+(10.8179+396.674\,i)w_2^3z_1^2\\
&-(0.92526+4.31137\,i)w_1w_3^2z_1^2-(7.56516-7.3996\,i)w_2w_3^2z_1^2\\
&+(11.0964-1.94202\,i)w_1^2z_1^3+(63.3881-54.6346\,i)w_1w_2z_1^3\\
&-(166.916+2.95525\,i)w_2^2z_1^3+(1.66498-3.26142\,i)w_3^2z_1^3+(11.8102\\
&+26.6093\,i)w_1z_1^4+(41.3492-97.5736\,i)w_2z_1^4+(5.60465+23.6776\,i)z_1^5\\
&-(97.5736-41.3492\,i)w_1^4z_2+(479.807+438.42\,i)w_1^3w_2z_2+(392.695\\
&-1442.26\,i)w_1^2w_2^2z_2-(1724.98-315.6\,i)w_1w_2^3z_2+(502.146\\
&+905.474\,i)w_2^4z_2+(3.07518-3.50268\,i)w_1^2w_3^2z_2+(8.17429\\
&-1.37227\,i)w_1w_2w_3^2z_2-(12.2044+21.037\,i)w_2^2w_3^2z_2-(0.0957693\\
&-0.097198\,i)w_3^4z_2-(54.6346-63.3881\,i)w_1^3z_1z_2+(103.041\\
&+330.416\,i)w_1^2w_2z_1z_2+(842.528-390.174\,i)w_1w_2^2z_1z_2-(965.583\\
&+628.344\,i)w_2^3z_1z_2+(10.5457+1.00084\,i)w_1w_3^2z_1z_2+(5.65319\\
&-3.36995\,i)w_2w_3^2z_1z_2-(100.215-139.325\,i)w_1^2z_1^2z_2+(330.416\\
&+103.041\,i)w_1w_2z_1^2z_2+(197.685-529.568\,i)w_2^2z_1^2z_2+(0.920263\\
&+7.44498\,i)w_3^2z_1^2z_2-(252.565-35.3907\,i)w_1z_1^3z_2+(438.42\\
&+479.807\,i)w_2z_1^3z_2-(175.03+47.409\,i)z_1^4z_2-(2.95525\\
&+166.916\,i)w_1^3z_2^2-(529.568-197.685\,i)w_1^2w_2z_2^2+(552.051\\
&+608.995\,i)w_1w_2^2z_2^2+(630.523-168.261\,i)w_2^3z_2^2-(3.01354\\
&-3.09051\,i)w_1w_3^2z_2^2-(13.8104-4.65778\,i)w_2w_3^2z_2^2-(225.77\\
&+397.311\,i)w_1^2z_1z_2^2-(390.174-842.528\,i)w_1w_2z_1z_2^2+(608.995\\
&+552.051\,i)w_2^2z_1z_2^2-(4.87077-1.7315\,i)w_3^2z_1z_2^2+(88.6466\\
&-762.357\,i)w_1z_1^2z_2^2-(1442.26-392.695\,i)w_2z_1^2z_2^2+(329.705
\end{aligned}
$$

$$-447.191\,i)z_1^3z_2^2+(396.674+10.8179\,i)w_1^2z_2^3-(628.344$$
$$+965.583\,i)w_1w_2z_2^3-(168.261-630.523\,i)w_2^2z_2^3+(2.34491$$
$$+0.400813\,i)w_3^2z_2^3+(893.539+449.728\,i)w_1z_1z_2^3+(315.6$$
$$-1724.98\,i)w_2z_1z_2^3+(444.385+774.449\,i)z_1^2z_2^3-(445.808$$
$$-424.948\,i)w_1z_2^4+(905.474+502.146\,i)w_2z_2^4-(715.858$$
$$-92.4234\,i)z_1z_2^4+(70.0965-340.8\,i)z_2^5+(9.2822+18.2817\,i)w_1^3w_3z_3$$
$$+(54.1487-74.5425\,i)w_1^2w_2w_3z_3-(145.293+18.6019\,i)w_1w_2^2w_3z_3$$
$$+(27.9522+121.901\,i)w_2^3w_3z_3-(0.623552+0.244055\,i)w_1w_3^3z_3$$
$$-(0.703858+1.6666\,i)w_2w_3^3z_3+(7.7796+9.65092\,i)w_1^2w_3z_1z_3$$
$$+(65.1709-17.4315\,i)w_1w_2w_3z_1z_3-(72.6585+67.0003\,i)w_2^2w_3z_1z_3$$
$$-(0.516955+1.58614\,i)w_3^3z_1z_3+(9.65092+7.7796\,i)w_1w_3z_1^2z_3$$
$$+(0.320338-24.4621\,i)w_2w_3z_1^2z_3+(18.2817+9.2822\,i)w_3z_1^3z_3$$
$$-(24.4621-0.320338\,i)w_1^2w_3z_2z_3+(9.0056+112.04\,i)w_1w_2w_3z_2z_3$$
$$+(109.458-60.1262\,i)w_2^2w_3z_2z_3-(0.432843-0.207139\,i)w_3^3z_2z_3$$
$$-(17.4315-65.1709\,i)w_1w_3z_1z_2z_3+(112.04+9.0056\,i)w_2w_3z_1z_2z_3$$
$$-(74.5425-54.1487\,i)w_3z_1^2z_2z_3-(67.0003+72.6585\,i)w_1w_3z_2^2z_3$$
$$-(60.1262-109.458\,i)w_2w_3z_2^2z_3-(18.6019+145.293\,i)w_3z_1z_2^2z_3$$
$$+(121.901+27.9522\,i)w_3z_2^3z_3-(3.26142-1.66498\,i)w_1^3z_3^2$$
$$+(7.44498+0.920263\,i)w_1^2w_2z_3^2+(1.7315-4.87077\,i)w_1w_2^2z_3^2$$
$$+(0.400813+2.34491\,i)w_2^3z_3^2-(1.27385-3.81436\,i)w_1w_3^2z_3^2+$$
$$(11.0663-2.01585\,i)w_2w_3^2z_3^2-(4.31137+0.92526\,i)w_1^2z_1z_3^2+$$
$$(1.00084+10.5457\,i)w_1w_2z_1z_3^2+(3.09051-3.01354\,i)w_2^2z_1z_3^2$$
$$+(3.81436-1.27385\,i)w_3^2z_1z_3^2-(1.37492+3.98847\,i)w_1z_1^2z_3^2$$
$$-(3.50268-3.07518\,i)w_2z_1^2z_3^2+(0.850712-1.30314\,i)z_1^3z_3^2$$
$$+(7.3996-7.56516\,i)w_1^2z_2z_3^2-(3.36995-5.65319\,i)w_1w_2z_2z_3^2$$

$+(4.65778-13.8104\,i)w_2^2z_2z_3^2-(2.01585-11.0663\,i)w_3^2z_2z_3^2$
$+(8.96575-1.32442\,i)w_1z_1z_2z_3^2-(1.37227-8.17429\,i)w_2z_1z_2z_3^2$
$+(0.439028+0.186131\,i)z_1^2z_2z_3^2+(6.11297+4.03999\,i)w_1z_2^2z_3^2$
$-(21.037+12.2044\,i)w_2z_2^2z_3^2+(6.6943+3.15832\,i)z_1z_2^2z_3^2-(5.34965$
$-8.70484\,i)z_2^3z_3^2-(1.58614+0.516955\,i)w_1w_3z_3^3+(0.207139$
$-0.432843\,i)w_2w_3z_3^3-(0.244055+0.623552\,i)w_3z_1z_3^3-(1.6666$
$+0.703858\,i)w_3z_2z_3^3+(0.000195137-0.0480885\,i)w_1z_3^4+(0.097198$
$-0.0957693\,i)w_2z_3^4-(0.006909-0.0237389\,i)z_1z_3^4-(0.107917$
$-0.146066i)z_2z_3^4$

$\chi_6(z,w)=$
$(135.483-115.076\,i)w_1^6-(1373.47+770.464\,i)w_1^5w_2-(227.438$
$-5712.53\,i)w_1^4w_2^2+(10772.5-4955.\,i)w_1^3w_2^3-(9615.74$
$+10712.4\,i)w_1^2w_2^4-(7204.29-8053.42\,i)w_1w_2^5+(3689.23$
$+1328.72\,i)w_2^6+(0.130953-11.711\,i)w_1^4w_3^2-(43.0304$
$-43.1233\,i)w_1^3w_2w_3^2+(116.8+113.165\,i)w_1^2w_2^2w_3^2+(22.8293$
$-157.607\,i)w_1w_2^3w_3^2-(28.0997+3.91441\,i)w_2^4w_3^2+(0.74729$
$+0.473462\,i)w_1^2w_3^4-(0.107668+2.17556\,i)w_1w_2w_3^4-(2.98394$
$-0.735748\,i)w_2^2w_3^4-(0.000546859-0.0075\,i)w_3^6+(531.179$
$-187.445\,i)w_1^5z_1-(2716.23+3472.61\,i)w_1^4w_2z_1-(6911.44$
$-11088.\,i)w_1^3w_2^2z_1+(20301.9+12533.6\,i)w_1^2w_2^3z_1+(1640.55$
$-22090.4\,i)w_1w_2^4z_1-(9136.41-2402.9\,i)w_2^5z_1-(1.3949$
$+1.57867\,i)w_1^3w_3^2z_1-(41.8893-67.0125\,i)w_1^2w_2w_3^2z_1+(134.265$
$+94.4878\,i)w_1w_2^2w_3^2z_1+(38.3338+6.92625\,i)w_2^3w_3^2z_1+(0.225633$
$+0.739753\,i)w_1w_3^4z_1+(2.51306-2.17243\,i)w_2w_3^4z_1+(792.47$

$$+423.232\,i)w_1^4z_1^2+(160.899-6375.63\,i)w_1^3w_2z_1^2-(15597.$$
$$-5221.23\,i)w_1^2w_2^2z_1^2+(12263.5+18462.4\,i)w_1w_2^3z_1^2+(7639.14$$
$$-8860.24\,i)w_2^4z_1^2+(2.07331+18.9239\,i)w_1^2w_3^2z_1^2-(14.6496$$
$$-5.81483\,i)w_1w_2w_3^2z_1^2+(13.9082-10.9149\,i)w_2^2w_3^2z_1^2-(0.0397215$$
$$-0.256686\,i)w_3^4z_1^2+(4770.82-3503.83\,i)w_1^2w_2z_1^3-(10652.6$$
$$+6284.49\,i)w_1w_2^2z_1^3-(2257.19-9193.53\,i)w_2^3z_1^3-(17.3743$$
$$-13.5795\,i)w_1w_3^2z_1^3+(24.7532-10.4885\,i)w_2w_3^2z_1^3-(792.47$$
$$-423.232\,i)w_1^2z_1^4+(3363.65+1359.7\,i)w_1w_2z_1^4-(505.897$$
$$+4858.13\,i)w_2^2z_1^4-(3.7256-9.43303\,i)w_3^2z_1^4-(531.179$$
$$+187.445\,i)w_1z_1^5+(614.372+1305.02\,i)w_2z_1^5-(135.483+115.076\,i)z_1^6$$
$$-(614.372-1305.02\,i)w_1^5z_2+(10481.7-645.264\,i)w_1^4w_2z_2+$$
$$-(17862.1+26745.9\,i)w_1^3w_2^2z_2-(25268.9-40845.8\,i)w_1^2w_2^3z_2$$
$$+(25616.4+16503.2\,i)w_1w_2^4z_2-(699.657+15384.\,i)w_2^5z_2+(23.1547$$
$$+57.821\,i)w_1^3w_3^2z_2+(151.931-249.266\,i)w_1^2w_2w_3^2z_2-(402.712$$
$$+22.7688\,i)w_1w_2^2w_3^2z_2+(114.904+219.009\,i)w_2^3w_3^2z_2-(2.40376$$
$$+0.294013\,i)w_1w_3^4z_2+(1.15723+3.71107\,i)w_2w_3^4z_2-(3363.65$$
$$-1359.7\,i)w_1^4z_1z_2+(19394.1+15156.1\,i)w_1^3w_2z_1z_2+(11842.3$$
$$-63082.3\,i)w_1^2w_2^2z_1z_2-(76867.3-13954.8\,i)w_1w_2^3z_1z_2+(17169.9$$
$$+30704.1\,i)w_2^4z_1z_2-(6.10228-47.5006\,i)w_1^2w_3^2z_1z_2+(178.847$$
$$-93.6383\,i)w_1w_2w_3^2z_1z_2-(79.7468+112.521\,i)w_2^2w_3^2z_1z_2-(0.675672$$
$$+2.02693\,i)w_3^4z_1z_2-(4770.82+3503.83\,i)w_1^3z_1^2z_2+(56557.1$$
$$-30916.3\,i)w_1w_2^2z_1^2z_2-(30042.4+27871.5\,i)w_2^3z_1^2z_2-(46.9745$$
$$-35.4328\,i)w_1w_3^2z_1^2z_2+(38.4941+176.606\,i)w_2w_3^2z_1^2z_2-(160.899$$
$$+6375.63\,i)w_1^2z_1^3z_2-(19394.1-15156.1\,i)w_1w_2z_1^3z_2+(25234.9$$
$$+12707.7\,i)w_2^2z_1^3z_2-(54.5268+4.62775\,i)w_3^2z_1^3z_2+(2716.23$$

$$
\begin{aligned}
&-3472.61\,i)w_1z_1^4z_2-(10481.7+645.264\,i)w_2z_1^4z_2+(1373.47\\
&-770.464\,i)z_1^5z_2+(505.897-4858.13\,i)w_1^4z_2^2-(25234.9\\
&-12707.7\,i)w_1^3w_2z_2^2+(60240.1+37702.5\,i)w_1^2w_2^2z_2^2+(9954.13\\
&-73218.3\,i)w_1w_2^3z_2^2-(32513.5-14246.9\,i)w_2^4z_2^2-(105.404\\
&+88.4464\,i)w_1^2w_3^2z_2^2-(121.883-491.01\,i)w_1w_2w_3^2z_2^2+\\
&+(314.509-228.742\,i)w_2^2w_3^2z_2^2+(1.97852-0.666286\,i)w_3^4z_2^2\\
&+(10652.6-6284.49\,i)w_1^3z_1z_2^2-(56557.1+30916.3\,i)w_1^2w_2z_1z_2^2\\
&+(69697.5-29225.5\,i)w_2^3z_1z_2^2-(18.6338+100.418\,i)w_1w_3^2z_1z_2^2+\\
&-(334.512-117.713\,i)w_2w_3^2z_1z_2^2+(15597.+5221.23\,i)w_1^2z_1^2z_2^2\\
&-(11842.3+63082.3\,i)w_1w_2z_1^2z_2^2-(60240.1-37702.5\,i)w_2^2z_1^2z_2^2\\
&+(49.607-84.0962\,i)w_3^2z_1^2z_2^2+(6911.44+11088.\,i)w_1z_1^3z_2^2\\
&+(17862.1-26745.9\,i)w_2z_1^3z_2^2+(227.438+5712.53\,i)z_1^4z_2^2\\
&+(2257.19+9193.53\,i)w_1^3z_2^3+(30042.4-27871.5\,i)w_1^2w_2z_2^3\\
&-(69697.5+29225.5\,i)w_1w_2^2z_2^3+(135.93+40.2501\,i)w_1w_3^2z_2^3\\
&+(2.22645-255.609\,i)w_2w_3^2z_2^3-(12263.5-18462.4\,i)w_1^2z_1z_2^3\\
&+(76867.3+13954.8\,i)w_1w_2z_1z_2^3-(9954.13+73218.3\,i)w_2^2z_1z_2^3\\
&+(48.1672+101.933\,i)w_3^2z_1z_2^3-(20301.9-12533.6\,i)w_1z_1^2z_2^3\\
&+(25268.9+40845.8\,i)w_2z_1^2z_2^3-(10772.5+4955.\,i)z_1^3z_2^3-(7639.14\\
&+8860.24\,i)w_1^2z_2^4-(17169.9-30704.1\,i)w_1w_2z_2^4+(32513.5\\
&+14246.9\,i)w_2^2z_2^4-(58.3927+7.80757\,i)w_3^2z_2^4-(1640.55\\
&+22090.4\,i)w_1z_1z_2^4-(25616.4-16503.2\,i)w_2z_1z_2^4+(9615.74\\
&-10712.4\,i)z_1^2z_2^4+(9136.41+2402.9\,i)w_1z_2^5+(699.657-15384.\,i)w_2z_2^5\\
&+(7204.29+8053.42\,i)z_1z_2^5-(3689.23-1328.72\,i)z_2^6+(217.048\\
&+31.7914\,i)w_1^4w_3z_3-(324.174+1194.6\,i)w_1^3w_2w_3z_3-(2276.03\\
&-1886.27\,i)w_1^2w_2^2w_3z_3+(1765.01+1397.63\,i)w_1w_2^3w_3z_3+(465.867
\end{aligned}
$$

$$
\begin{aligned}
&-1632.35\,i)w_2^4w_3z_3+(6.1319-7.14195\,i)w_1^2w_3^3z_3-(22.2392\\
&+2.3829\,i)w_1w_2w_3^3z_3+(0.990667+16.3723\,i)w_2^2w_3^3z_3+(0.0496894\\
&+0.408979\,i)w_3^5z_3+(380.848+290.639\,i)w_1^3w_3z_1z_3+(501.129\\
&-2410.45\,i)w_1^2w_2w_3z_1z_3-(4520.19-826.708\,i)w_1w_2^2w_3z_1z_3\\
&+(1231.41+2839.84\,i)w_2^3w_3z_1z_3+(0.940242-5.02873\,i)w_1w_3^3z_1z_3\\
&-(4.82972+3.79455\,i)w_2w_3^3z_1z_3+(1930.83-1248.97\,i)w_1w_2w_3z_1^2z_3\\
&-(2156.66+1684.88\,i)w_2^2w_3z_1^2z_3+(1.74825+4.26134\,i)w_3^3z_1^2z_3\\
&-(380.848-290.639\,i)w_1w_3z_1^3z_3+(1189.36+357.75\,i)w_2w_3z_1^3z_3\\
&-(217.048-31.7914\,i)w_3z_1^4z_3-(1189.36-357.75\,i)w_1^3w_3z_2z_3\\
&+(4079.15+3451.41\,i)w_1^2w_2w_3z_2z_3+(2153.4-9085.62\,i)w_1w_2^2w_3z_2z_3\\
&-(5359.92-1090.32\,i)w_2^3w_3z_2z_3-(9.051-24.3137\,i)w_1w_3^3z_2z_3+\\
&(42.5684-9.89061\,i)w_2w_3^3z_2z_3-(1930.83+1248.97\,i)w_1^2w_3z_1z_2z_3\\
&+(7647.19-3210.88\,i)w_2^2w_3z_1z_2z_3-(16.4571-10.0844\,i)w_3^3z_1z_2z_3\\
&-(501.129+2410.45\,i)w_1w_3z_1^2z_2z_3-(4079.15-3451.41\,i)w_2w_3z_1^2z_2z_3\\
&+(324.174-1194.6\,i)w_3z_1^3z_2z_3+(2156.66-1684.88\,i)w_1^2w_3z_2^2z_3\\
&-(7647.19+3210.88\,i)w_1w_2w_3z_2^2z_3+(1.57014-21.2178\,i)w_3^3z_2^2z_3\\
&+(4520.19+826.708\,i)w_1w_3z_1z_2^2z_3-(2153.4+9085.62\,i)w_2w_3z_1z_2^2z_3\\
&+(2276.03+1886.27\,i)w_3z_1^2z_2^2z_3-(1231.41-2839.84\,i)w_1w_3z_2^3z_3\\
&+(5359.92+1090.32\,i)w_2w_3z_2^3z_3-(1765.01-1397.63\,i)w_3z_1z_2^3z_3\\
&-(465.867+1632.35\,i)w_3z_2^4z_3+(3.7256+9.43303\,i)w_1^4z_3^2+(54.5268\\
&-4.62775\,i)w_1^3w_2z_3^2-(49.607+84.0962\,i)w_1^2w_2^2z_3^2-(48.1672\\
&-101.933\,i)w_1w_2^3z_3^2+(58.3927-7.80757\,i)w_2^4z_3^2+(60.1746\\
&+76.3395\,i)w_1^2w_3^2z_3^2+(98.4543-231.676\,i)w_1w_2w_3^2z_3^2-(271.425\\
&+21.1344\,i)w_2^2w_3^2z_3^2+(1.63096-0.0769577\,i)w_3^4z_3^2+(17.3743\\
&+13.5795\,i)w_1^3z_1z_3^2+(46.9745+35.4328\,i)w_1^2w_2z_1z_3^2+(18.6338
\end{aligned}
$$

$$-100.418\,i)w_1w_2^2z_1z_3^2-(135.93-40.2501\,i)w_2^3z_1z_3^2+(276.371$$
$$-129.747\,i)w_2w_3^2z_1z_3^2-(2.07331-18.9239\,i)w_1^2z_1^2z_3^2+(6.10228$$
$$+47.5006\,i)w_1w_2z_1^2z_3^2+(105.404-88.4464\,i)w_2^2z_1^2z_3^2-(60.1746$$
$$-76.3395\,i)w_3^2z_1^2z_3^2+(1.3949-1.57867\,i)w_1z_1^3z_3^2-(23.1547$$
$$-57.821\,i)w_2z_1^3z_3^2-(0.130953+11.711\,i)z_1^4z_3^2-(24.7532$$
$$+10.4885\,i)w_1^3z_2z_3^2-(38.4941-176.606\,i)w_1^2w_2z_2z_3^2+(334.512$$
$$+117.713\,i)w_1w_2^2z_2z_3^2-(2.22645+255.609\,i)w_2^3z_2z_3^2-(276.371$$
$$+129.747\,i)w_1w_3^2z_2z_3^2+(14.6496+5.81483\,i)w_1^2z_1z_2z_3^2-(178.847$$
$$+93.6383\,i)w_1w_2z_1z_2z_3^2+(121.883+491.01\,i)w_2^2z_1z_2z_3^2-(98.4543$$
$$+231.676\,i)w_3^2z_1z_2z_3^2+(41.8893+67.0125\,i)w_1z_1^2z_2z_3^2-(151.931$$
$$+249.266\,i)w_2z_1^2z_2z_3^2+(43.0304+43.1233\,i)z_1^3z_2z_3^2-(13.9082$$
$$+10.9149\,i)w_1^2z_2^2z_3^2+(79.7468-112.521\,i)w_1w_2z_2^2z_3^2-(314.509$$
$$+228.742\,i)w_2^2z_2^2z_3^2+(271.425-21.1344\,i)w_3^2z_2^2z_3^2-(134.265$$
$$-94.4878\,i)w_1z_1z_2^2z_3^2+(402.712-22.7688\,i)w_2z_1z_2^2z_3^2-(116.8$$
$$-113.165\,i)z_1^2z_2^2z_3^2-(38.3338-6.92625\,i)w_1z_2^3z_3^2-(114.904$$
$$-219.009\,i)w_2z_2^3z_3^2-(22.8293+157.607\,i)z_1z_2^3z_3^2+(28.0997$$
$$-3.91441\,i)z_2^4z_3^2-(1.74825-4.26134\,i)w_1^2w_3z_3^3+(16.4571$$
$$+10.0844\,i)w_1w_2w_3z_3^3-(1.57014+21.2178\,i)w_2^2w_3z_3^3-(0.940242$$
$$+5.02873\,i)w_1w_3z_1z_3^3+(9.051+24.3137\,i)w_2w_3z_1z_3^3-(6.1319$$
$$+7.14195\,i)w_3z_1^2z_3^3+(4.82972-3.79455\,i)w_1w_3z_2z_3^3-(42.5684$$
$$+9.89061\,i)w_2w_3z_2z_3^3+(22.2392-2.3829\,i)w_3z_1z_2z_3^3-(0.990667$$
$$-16.3723\,i)w_3z_2^2z_3^3+(0.0397215+0.256686\,i)w_1^2z_3^4+(0.675672$$
$$-2.02693\,i)w_1w_2z_3^4-(1.97852+0.666286\,i)w_2^2z_3^4-(1.63096$$
$$+0.0769577\,i)w_3^2z_3^4-(0.225633-0.739753\,i)w_1z_1z_3^4+(2.40376$$
$$-0.294013\,i)w_2z_1z_3^4-(0.74729-0.473462\,i)z_1^2z_3^4-(2.51306$$

$$+2.17243\,i)w_1z_2z_3^4-(1.15723-3.71107\,i)w_2z_2z_3^4+(0.107668$$
$$-2.17556\,i)z_1z_2z_3^4+(2.98394+0.735748\,i)z_2^2z_3^4-(0.0496894$$
$$-0.408979\,i)w_3z_3^5+(0.000546859+0.0075\,i)z_3^6$$

Bibliography

[1] Arnol'd V.I.: "Méthodes Mathématiques de la Méchanique Classique". MIR, Moscow (1976).

[2] Arnol'd V.I.: "Proof of a theorem of A.N. Kolmogorov on the preservation of conditionally periodic motions under a small perturbation of the Hamiltonian". Uspekhi Mat. Nauk. **18**, no. 5, pp. 13-40 (1963).

[3] Arnol'd V.I.: "Small denominators and problems of stability of motion in classical and celestial mechanics". Russ. Math. Surveys **18**, pp. 85-191 (1963).

[4] Arnol'd V.I.: "Instability of dynamical systems with several degrees of freedom". Sov. Math. Dokl. **6**, pp. 581-585 (1964).

[5] Arnol'd V.I. and Avez A.: "Ergodic problems of classical mechanics". Benjamin, New York-Amsterdam (1968).

[6] Benettin G., Fassò F. and Guzzo M.: "Nekhoroshev stability of L_4 and L_5 in the spatial restricted three body problem". Regul. Chaotic Dyn. **3**, pp. 56-72 (1998).

[7] Benettin G., Carati A., Galgani L. and Giorgilli A.: "The Fermi-Pasta-Ulam Problem and the Metastability Perspective". Lect. Notes Phys. **728**, pp. 151-189 (2008).

[8] Benettin G., Galgani L. and Giorgilli A.: "A proof of Nekhoroshev's theorem for the stability times in nearly integrable Hamiltonian systems". Celestial Mech. **37**, no. 1, pp. 1-25 (1985).

[9] Benettin G. and Gallavotti G.: "Stability of Motions near Resonances in Quasi-Integrable Hamiltonian Systems". Journ. Stat. Phys. **44**, pp. 293-338 (1986).

[10] Benettin G. and Fassò F.: "Fast rotations of the rigid body: A study by Hamiltonian perturbation theory. Part I". Nonlinearity **9**, pp. 137-186 (1996).

[11] Benettin G., Fassò F. and Guzzo M.: "Fast rotations of the rigid body: A study by Hamiltonian perturbation theory. Part II: Gyroscopic rotations". Nonlinearity **10**, pp. 1695-1717 (1997).

[12] Biasco L. and Chierchia L.: "Exponential stability for the resonant d'Alembert model of celestial mechanics". Discrete Contin. Dyn. Syst. **12**, no. 4, pp. 569-594 (2005).

[13] Celletti A.: "Construction of librational invariant tori in the spin-orbit problem". J. Appl. Math. Phys. (ZAMP) **45**, pp. 61-80 (1993).

[14] Celletti A. and Chierchia L.: "Rigorous estimates for a computer-assisted KAM theory". J. Math. Phys. **28**, pp. 2078-2086 (1987).

[15] Celletti A. and Chierchia L.: "Construction of analytic KAM surfaces and effective stability bounds". Commun. Math. Phys. **118**, pp. 119-161 (1988).

[16] Celletti A. and Chierchia L.: "On the stability of realistic three-body problems". Commun. Math. Phys. **186**, pp. 413-449 (1997)

[17] Celletti A. and Chierchia L.: "KAM stability and celestial mechanics". Mem. Amer. Math. Soc. **187**, no. 878 (2007).

[18] Celletti A. and Ferrara L.: "An application of the Nekhoroshev theorem to the restricted three-body problem". Celestial Mech. Dynam. Astronom. **64**, no. 3, pp. 261-272 (1996).

[19] Chierchia L. and Gallavotti G.: "Smooth prime integrals for quasi-integrable Hamiltonian systems". (Italian summary) Nuovo Cimento B **67**, no. 2, pp. 277-295 (1982).

[20] Chierchia L. and Pinzari G.: "The planetary N-body problem: symplectic foliation, reductions and invariant tori". Invent. Math. **186**, no. 1 (2011).

[21] Chierchia L. and Pinzari G.: "Planetary Birkhoff normal forms". J. Mod. Dyn. **5**, no. 4, pp. 623-664 (2011).

[22] Efthymiopoulos C.: "On the connection between the Nekhoroshev theorem and Arnold diffusion". Celestial Mech. Dynam. Astronom. **102**, pp. 49-68 (2008).

[23] Efthymiopoulos C. and Harsoula M.: "The speed of Arnold diffusion". Physica D **251**, pp. 19-38 (2013).

[24] Fassò F. and Lewis D.: "Stability properties of the Riemann ellipsoids". Archive for Rational Mechanics and Analysis **158**, pp. 259-292 (2001).

[25] Fassò F., Guzzo M. and Benettin G.: "Nekhoroshev-Stability of Elliptic Equilibria in Hamiltonian Systems". Communications in Mathematical Physics **197**, pp. 347-360 (1998).

[26] Féjoz J.: "Démonstration du théorème d'Arnold sur la stabilité du système planétaire (d'après Michael Herman)". Ergod. Th. Dyn. Sys. **24**, pp. 1-62 (2004).

[27] Fermi E., Pasta J. and Ulam S.: "Los Alamos Report". LA-1940, in E. Fermi, Collected Papers, **2**, pp. 977-988 (1955).

[28] Giorgilli A., Delshams A., Fontich E., Galgani L. and Simó C.: "Effective Stability for a Hamiltonian System near an Elliptic Equilibrium Point, with an Application to the Restricted three Body Problem". J. Diff. Eq. **77**, pp. 167-198 (1989).

[29] Giorgilli A., Locatelli U. and Sansottera M.: "Kolmogorov and Nekhoroshev theory for the problem of three bodies". Celestial Mechanics and Dynamical Astronomy **104**, Issue 1-2, pp. 159-173 (2009).

[30] Giorgilli A., Skokos C.: “On the stability of the Trojan asteroids”. Astronomy and Astrophysics **317**, pp.254-261 (1997).

[31] Guzzo M.: “Nekhoroshev stability of quasi-integrable degenerate Hamiltonian systems”. Regular and chaotic dynamics **4**, no. 2 (1999).

[32] Guzzo M., Fassò F. and Benettin G.: “On The Stability of Elliptic Equilibria”. Mathematical Physics Electronic Journal **4** (1998).

[33] Guzzo M., Knezevic Z. and Milani A.: “Probing the Nekhoroshev Stability of Asteroids”. Celest. Mech. Dyn. Astron. **83**, pp. 121-140 (2002).

[34] Guzzo M., Lega E. and Froeschlé C.: “First numerical investigation of a conjecture by N.N. Nekhoroshev about stability in quasi-integrable systems”. Chaos **21**, no. 3 (2011).

[35] Guzzo M., Lega E. and Froeschlé C.: “Diffusion and stability in perturbed nonconvex integrable systems”. Nonlinearity **19**, pp. 1049-1067 (2006).

[36] Guzzo M. and Morbidelli A.: “Construction of a Nekhoroshev like result for the asteroid belt dynamical system”. Celestial Mechanics **66**, pp. 255-292 (1997).

[37] Ilyashenko I.S.: “A steepness test for analytic functions”. Russian Math. Surveys **41**, pp. 229-230 (1986).

[38] Kolmogorov A.N.: “On conservation of conditionally periodic motions for a small change in Hamilton’s function”. Dokl. Akad. Nauk SSSR (N.S.) **98**, pp. 527-530 (1954).

[39] Lazutkin V.F.: “Existence of caustics for the billiard problem in a convex domain”. (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **37**, pp. 186-216 (1973).

[40] Lega E., Guzzo M. and Froeschlé C.: “Detection of Arnold diffusion in Hamiltonian systems”. Physica D **182**, 179-187 (2003).

[41] Lega E., Froeschlé C. and Guzzo M.: “Diffusion in Hamiltonian quasi-integrable systems”, in Topics in Gravitational Dynamics, edited by D. Benest, C. Froeschlé and E. Lega, LNP 729 (Springer, Heidelberg, 2007).

[42] Locatelli U. and Giorgilli A.: “Invariant tori in the secular motions of the three-body planetary systems”. Celest. Mech. Dyn. Astr. **78**, pp. 47-74 (2000).

[43] Locatelli U. and Giorgilli A.: “Construction of the Kolmogorov’s normal form for a planetary system”. Regular Chaotic Dyn. **10**, no.2, pp. 153-171 (2005).

[44] Lochak P. and Neishtadt A.I.: “Estimates of stability time for nearly integrable systems with a quasi-convex Hamiltonian”. Chaos **2**, pp. 495-499 (1992).

[45] Lochak P.: “Canonical perturbation theory via simultaneous approximation”. Russ. Math. Surv. **47**, pp. 57-133 (1992)

[46] Morbidelli A. and Guzzo M.: “The Nekhoroshev theorem and the asteroid belt dynamical system”. Celest. Mech. Dyn. Astron. **65**, pp. 107-136 (1997).

[47] Moser J.: “New aspects in the theory of stability of Hamiltonian systems”. Comm. Pure Appl. Math **11**, pp. 81-114 (1958).

[48] Moser J.: “On invariant curves of area-preserving maps of an anulus”. Nach. Akad. Wiss. Göttingen, Math. Phys. Kl. **1**, pp. 1-20 (1962).

[49] Nekhoroshev N.N.: “An exponential estimate of the time of stability of nearly integrable Hamiltonian systems. I”. Usp. Mat. Nauk **32**, no. 6, pp. 5-66 (1977); Russ. Math. Surv. **32**, pp. 1-65 (1977).

[50] Nekhoroshev N.N.: “An exponential estimate of the time of stability of nearly integrable Hamiltonian systems. II”. Tr. Semin. Petrovsk. **5**, pp. 5-50 (1979); In: Oleinik, O.A. (ed.) Topics in Modern Mathematics, Petrovskii Semin., no. 5. New York: Consultant Bureau (1985).

[51] Nekhoroshev N.N.: “Stable lower estimates for smooth mappings and for gradients of smooth functions”. Mat. Sb. **90**, pp. 432-478 (1973).

[52] Niederman L.: ”Hamiltonian stability and subanalytic geometry“. Annales de l’Institut Fourier, **56**, no. 3, pp. 795-813 (2006).

[53] Pavlovic R. and Guzzo M.: “Fulfillment of the conditions for the application of the Nekhoroshev theorem to the Koronis and Veritas asteroid families”. Monthly Notices of the Royal Astronomical Society. **384**, pp. 1575-1582 (2008).

[54] Poincaré H.: “Les Méthodes Nouvelles de la Méchanique Céleste”. Vol **1**, Gautier-Villars, Paris (1892).

[55] Pöschel J.: “Integrability of Hamiltonian systems on Cantor sets”. Comm. Pure Appl. Math. **35**, no. 5, pp. 653-696 (1982).

[56] Pöschel J.: “Nekhoroshev estimates for quasi-convex Hamiltonian systems”. Math. Z. **213**, pp. 187-217 (1993).

[57] Rink B. and Verhulst F.: “Near-integrability of periodic FPU-chains”. Physica A. **285**, pp. 467-482 (2000).

[58] Rink B.: “Symmetry and resonance in periodic FPU chains”. Comm. Math. Phys. **218**, no. 3, pp. 665-685 (2001).

[59] Schirinzi G. and Guzzo M.: “On the formulation of new explicit conditions for steepness from a former result of N.N.Nekhoroshev”. J. Math. Phys. **54**, 22 pp. (2013).

[60] Schirinzi G. and Guzzo M., in preparation (2014).

[61] Sansottera M., Locatelli U. and Giorgilli A.: “On the stability of the secular evolution of the planar Sun-Jupiter-Saturn-Uranus system”. Mathematics and Computers in Simulation **88**, pp. 1-14 (2013).

[62] Siegel C. and Moser J.: “Lectures on Celestial Mechanics”. Springer, Berlin (1971).

[63] Todorović N., Guzzo M., Lega E. and Froeschlè C.: "A numerical study of the stabilization effect of steepness". Celest. Mech. and Dyn. Astron. **110**, pp. 389-398 (2011).

Acknowledgements

I thank my advisor Prof. Massimiliano Guzzo, for having guided me during these years, for the effort he has put in my formation as a researcher, and for many useful discussions. He has been also always helpful when I needed, and for this I really thank him.

I also wish to thank Prof. Giancarlo Benettin, for all the useful advices he gave me, and particularly for having been always so kind with me.

I would like to thank Prof. Francesco Fassò, for letting me use the Mathematica program he wrote for the verification of the steepness of a function, and also for the useful suggestions he gave me after my seminar.

And I also sincerely thank Prof. Franco Cardin, who was the first one who believed in me.