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A VARIATIONAL INTEGRATORS APPROACH TO SECOND ORDER MODELING AND IDENTIFICATION OF LINEAR MECHANICAL SYSTEMS

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"Ethical axioms are found and tested
not very differently from the axioms of science.
Truth is what stands the test of experience."
Albert Einstein

Abstract

The identification of linear second order models of mechanical systems has been object of intensive research and of several papers in the last decades. In this thesis the interest is focused on mechanical systems which can be described by a second order vector model of the following form:

$$M\ddot{q} + D\dot{q} + Kq = f \tag{1}$$

where M and K are symmetric positive definite while D is only symmetric positive semidefinite. All current identification techniques operate in discrete time. Noisy data obtained by sampling the system must be used to estimate the *continuous time* physical parameters M, K and D. Since identification operates in discrete time one needs to convert the discrete time identified system into a continuous time one. There are structural constraints that need to be imposed to obtain the second order structure (2).
In short, the procedure is composed of three main steps:

1. Discrete-time Identification, mostly using subspace methods, from sampled input-output data;

2. Implementation of a set of constraints which force the identified system to the form (2);

3. Conversion from the discrete to a continuous model and conversion of the relative system parameters.

The usual procedure assumes that the discrete time identified system is a Zero-Order-Hold (ZOH) discretization of the underlying continuous time system.

This assumption may lead to serious numerical problems, since the conversion discrete-to-continuous (`d2c`) requires the computation of the matrix logarithm of a $2n \times 2n$ matrix, which is well-known to be an ill conditioned problem resulting in a serious amplification of the noisy errors in the discrete estimates.

The proposed solution to this problem is to introduce a new discretization technique of the equations of motion of a mechanical systems introduced by Veselov, and further developed by J.Marsden and co-workers. This technique has been developed for general mechanical system and leads to discrete systems characterized by a sort of "discrete mechanical structure". Unlike the usual discretization procedures familiar in control, e.g. ZOH, it can lead to linear algebraic transformation formulas for the recovery of the continuous time parameters from the discretized model.

In this thesis variational integrators are applied to linear second order mechanical systems and it is shown that physically meaningful properties of the continuous-time model, like passivity, are preserved in the discretization.

Sommario

L'identificazione di modelli di sistemi meccanici del secondo ordine è stata oggetto di un'intensa attività di ricerca negli ultimi decenni. In questa tesi ci si focalizza nei sistemi meccanici che si posso descrivere con un modello del secondo ordine del seguente tipo:

$$M\ddot{q} + D\dot{q} + Kq = f \tag{2}$$

dove M e K sono matrici definite positive mentre D i solo semidefinita positiva. Tutte le attuali tecniche di identificazione operano a tempo discreto. I dati rumorosi ottenuti dal campionamento del sistema devono essere utilizzati per stimare i parametri fisici del sistema a tempo continuo M, K e D. Poichè il processo di identificazione opera a tempo discreto si rende necessaria una conversione del sistema discreto identificato in uno a tempo continuo. Ci sono vincoli strutturali che devono essere imposti per ottenere la struttura del secondo ordine (2).
In breve, la procedure si compone di tre parti principali:

1. Identificazione a tempo discreto, per lo pi metodi a sottospazi, dai dati ingresso-uscita campionati;

2. Implementazione di un set di vincoli che forzi il sistema identificato alla forma (2);

3. Conversione dal dominio di tempo discreto a quello continuo and conversione dei relativi parametri del sistema.

La procedura classica prevede che il sistema identificato a tempo discreto sia ottenuto per discretizzazione di tipo Zero-Order-Hold (ZOH) del sottostante modello continuo. Quest'assunzione porta a gravi problemi di tipo numerico, poichè la

conversione dal discreto al continuo (`d2c`) richiede il calcolo del logaritmo per una matrice $2n \times 2n$. E' noto che tale operazione comporta problemi di malcondizionamento numerico che producono un amplificazione degli errori di stima nel sistema discreto.

La soluzione proposta al problema è di introdurre una nuova tecnica di discretizzazione delle equazioni del moto per sistemi meccanici, introdotta da Veselov, e successivamente sviluppata da J.Marden e dai suoi collaboratori. Questa tecnica è stata sviluppata per sistemi meccanici generici e porta a sistemi discreti caratterizzati da una sorta di "struttura meccanica discreta". Diversamente dalle procedure di discretizzazione classiche, familiari nel mondo del controllo, e.g. ZOH, tale metodo porta a una formula di trasformazione algebrica lineare per il recupero dei parametri continui da quelli discreti.
Nella tesi gli integratori variazionali sono applicati ai sistemi meccanici lineari del secondo ordine e verrà provato che nella discretizzazione vengono preservate proprietà con intrinseco significato fisico del modello a tempo continuo, ad esempio la passività.

Contents

Chapter 1

Introduction and Problem Statement

Of particular interest are systems which can be described by a second order vector model of the following form:

$$M\ddot{q} + D\dot{q} + Kq = f \tag{1.1}$$

where M and K, both symmetric positive definite matrices in $\mathbb{R}^{n\times n}$, have the interpretation of generalized mass (or inertia) and generalized stiffness coefficient matrices respectively, while $D \in \mathbb{R}^{n\times n}, D = D^\top$ is a linear (viscous) damping coefficient which is at least positive semidefinite. The generalized forces f acting on the system can be expressed as a linear function of a vector of independently assignable generalized input forces u of dimension $k \leq n$; namely

$$f = Lu$$

where the matrix L, which will be assumed to be known, describes the physical locations at which the input forces u act on the system. Without loss of generality it may be assumed that L is of full column rank; i.e.

$$\operatorname{rank} L = k\,. \tag{1.2}$$

For simplicity and for mathematical convenience it will be assumed that a full set of linear sensors is available to the experimenter; i.e. that all n degrees of freedom are measured via linear sensors. In particular we shall assume that the measurement equation is of the form $y = Cq$ where C is a square invertible matrix, which is clearly equivalent to assuming that the full generalized displacements vector q is measured. The case in which the generalized velocities vector $\dot{q}$, or n independent linear combinations of the q, $\dot{q}$ variables are measured, can be given an essentially equivalent treatment. System (1.1) can also be represented in state space form; for example defining

$$x := [q,\, \dot{q}]^\top, \tag{1.3}$$

one gets

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1}L \end{bmatrix} u \\ y &= \begin{bmatrix} I & 0 \end{bmatrix} x. \end{aligned} \tag{1.4}$$

We shall comment later on the special (passive Hamiltonian) structure of this realization which leads to the inverse second-order polynomial transfer function of the model (1.1).

Throughout the thesis we shall assume that the system (1.1) with input u is controllable. See [12] for a direct test of controllability/observability of second order models of the type considered in this paper. Note that under our assumptions the system is automatically controllable and observable and hence minimal. This is a necessary condition for parameter identifiability

Now, system identification deals almost exclusively with discrete-time data and discrete time models. Nevertheless in several areas of engineering, and especially in mechanical or structural engineering, the estimation of physical parameters which pertain to the underlying (physical) *continuous time* model of the type (1.1) is very often required. A typical example is the estimation of the proper modes of vibration of a mechanical structure. The proper modes are the eigenvalues of a linear vector second order *continuous time* system, i.e. are solutions of an algebraic equation of the form:

$$\det\left(Ms^2 + Ds + K\right) = 0. \tag{1.5}$$

It is a well-known fact that accurate information on these proper values and on the associated proper vectors may be hard to get from an estimated discretized system, no matter how accurate the estimates may be. The reason of this difficulty may be attributed to the ill-conditioning of the discrete-to-continuous conversion (see the next section for some details).

Moreover in the presence of noise, even with the use of anti-aliasing filters (which must necessarily be approximated since the true bandwidth of the signal is not known), oversampling has also the well-known effect of bringing in noise alias in the estimates and further deteriorates the identification of the discrete-time model.

Another difficulty with the inverse ZOH discretization is that it is highly non-linear so that, even when the exponential is theoretically invertible, it does introduce *bias* in the estimates of the continuous-time parameters, even when the discrete-time parameters are unbiased and accurate. For this reason a linear (or "approximately linear") discrete-to-continuous conversion would be highly desirable[1].

One may add that ZOH does not in general preserve the basic physical properties of the underlying continuous system such as *passivity*, which may then be impossible to recapture when transforming back the discrete to a continuous model. We are in particular seeking transformations which preserve the second order input-output structure of the type (1.1), which is indeed a basic characteristic of linear models of fully observed mechanical systems (Newton law). In general a continuous state-space realization obtained by the `d2c` routine from an identified discrete model (1.7) of a mechanical system, say by standard subspace methods, will never possess the passive-Hamiltonian structure which is necessary for the input-output relation of the system to have the second-order form (1.1) and hence to allow for the recovery of the physical parameters M, D, K. That this is not of purely academic interest is witnessed by the interest in this problem in the recent mechanical engineering literature, see e.g. [1, 16, 15] and the references therein.

[1] One may argue that the Euler discretization is a well known instance of linear conversion map but unfortunately the Euler discretization is way too rough to be of use in most situations.

Of course one may argue that one should use continuous-time identification directly. Unfortunately according to the current literature on continuous time identification, see e.g. [24, 7] and the references therein, the existing continuous-time algorithms do not seem to be of much help for accurate physical parameters identification. In many cases continuous-time identification algorithms eventually end up to relay on logarithmic transforms, like inverting the relation $z = \exp\{sh\}$ which turns out to be equivalent to the MATLAB `d2c` transformation. For the reasons given above, these methods are not always reliable and should be avoided. There is also a quite popular approach based on filtering the continuous time data by a family of test functions [20], which may or may not be orthonormal. Besides facing the problem of numerical integration for computing the inner products over a long period of time, to reach reasonable accuracy these methods require the computation of inner products of the signals with a large number of test functions. This is so since each inner product plays eventually the role of a single discrete-time sample value of the signal. To our knowledge, reliable continuous-time identification methods which can be applied to concrete multivariable real-world problems seem still to be missing. Progress still to be made in this area and for the time being we may have to stick to discrete-time identification.

1.1 Continuous to Discrete conversion

The sampling of the continuous system gives a set of discrete data with sampling time h

$$(f(k), q(k))\,. \tag{1.6}$$

Assume data are fitted, by some identification algorithm, by a discrete-time state space model of the form

$$\begin{aligned} x(k+1) =& Fx(k) + Gf(k) \\ y(k) =& Hx(k) + Jf(k)\,. \end{aligned} \tag{1.7}$$

If h is short enough one can naturally imagine (1.7) to be related to an underlying (unknown) continuous-time state space model by some discretization rule. A simple example is the standard zero-order-hold (ZOH). The ZOH sampler transforms a continuous time system into a discrete time one by synchronously sampling the output of the continuous system once the input signal is approximated by a piecewise constant function on each sampling interval. The matrices that characterize the transformation in the discrete state space form are

$$\begin{aligned}
F_{ZOH} &= e^{AT} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}_{t=T}, \\
G_{ZOH} &= \left(\int_{\tau=0}^{T} e^{A\tau} d\tau\right) B = A^{-1}(F - I)B, \\
H_{ZOH} &= C, \\
J_{ZOH} &= D.
\end{aligned} \tag{1.8}$$

The state and output signals turn out to be *approximate* discretizations of the continuous counterparts and are exact discretizations only when the input is actually a piecewise constant function of time. If the input function can be well approximated by a function which is piecewise constant on each sampling interval the (ZOH) sampler describes the relation between (1.7) and the underlying continuous time model. The original parameters, say the matrices A and B of a $2n$ dimensional continuous time model, may then be recovered from estimates of the parameters (F, G) of the discrete time model (1.7), by inverting the relations $F = \exp Ah$, $G = \int_0^h \exp As\, ds\, B$. This is what is implemented in the `d2c` routine in MATLAB. In certain circumstances this may however turn into a very ill-conditioned problem. In particular the recovery of matrix A from the estimated F involves the computation of the logarithm of F which may be a complex matrix or, for a large sampling period, be undefined as requiring the inversion of the exponential map in a region of the complex plane where it is not invertible. A common belief is that the problem should be solvable by choosing a suitably high sampling frequency, but actually it is easy to see that, even in the trivial example of a scalar F subject to a perturbation δF, the relative error incurred when computing $A + \delta A := \dfrac{1}{h} \log(F + \delta F)$ is

$$\frac{\delta A}{A} = \frac{1}{\log F} \frac{\delta F}{F}$$

a similar formula holding in the matrix case, see [5, Formula 2.3]. Since for $h \to 0$ $F \to I$, the condition number of computing $A = \frac{1}{h} \log F$ tends to infinity when

$h \to 0$. This means that when the sampling frequency is very high, the effect of unavoidable random errors on the estimates of F (and G) could be dramatically amplified in computing A by the logarithmic transformation. See [5] and the references therein. A deeper analysis of this problem will be given in the following chapters.

The strategy to convert discrete time model into continuous time domain is the nodal point in the estimation of continuous time model parameters. Giving the above analysis some specific features of the desired conversion method can detected:

- Simple, well-conditioned, possible linear conversion functions;
- Hamiltonian like discrete mechanical structure;
- Preservation of characteristic properties of the mechanical system, e.g. passivity.

The content of the thesis has the following layout:

1. It will be introduced a discretization technique of mechanical systems based on the idea of variational integrators. This technique leads to linear conversion formulas from a discrete identified model to the corresponding continuous *input-output* model.
2. It will be shown that in an important special case the variational discretization leads to a well-know continuous-to-discrete transformation used in system and control, namely the Cayley-Tustin discretization. This discretization is different from the usual periodic sampling (ZOH). This alternative sampling technique will be discussed and some related computational problems will be addressed.
3. The above technique will be used to attack the mechanical system identification from noisy discrete input-output data. As a preliminary step a standard discrete-time subspace identification technique will be discussed and used in order to supply good starting values to a successive Prediction Error optimization-based algorithm.

4. A refinement of the subspace identification estimates by a Prediction Error algorithm which complies with the constraints of second order mechanical structure will be described. This is the final step of the procedure.

5. Finally, some simulation results are shown and compared with the results obtained by state of the art identification methods.

Chapter 2

The Variational Integrators approach to discretization

A novel twist to the discretization of mechanical systems has been provided by the theory of *variational integrators*, see [29], and the recent work of J. Marsden and co-workers, see e.g. [17]. These techniques seem to be fairly well known to numerical analysts working with mechanical models but not so familiar to the system and control community. The key idea is that the discrete equations of motion should not be derived by attempting a direct discretization of the equations (1.1) or (1.4) but rather derived by paraphrasing what happens in continuous time; i.e. by making stationary a *discrete action integral* defined in terms of a suitable *discrete Lagrangian function.* The (discrete) equations of motion should then follow just like the Euler-Lagrange equations in continuous time. In short, the variational integrators paradigm is to build from scratching a theory of Lagrangian *Discrete Mechanics.*

In (continuous-time) Lagrangian mechanics we are given a Lagrangian function $L(q(t), \dot{q}(t))$ and external forces $f_L(q(t), \dot{q}(t), t)$ and the equations of motion follow from the *Lagrange-d'Alembert principle*, equivalent in the conservative case to the stationary action principle. The Lagrange-D'Alembert principle states that the trajectory of a mechanical system starting at time t_0 at position q_0 and arriving at time t_1 at position q_1 must satisfy the variational principle

$$\delta \int_{t_0}^{t_1} L(q,\dot{q})\,dt + \int_{t_0}^{t_1} f(q,q,t)\,\delta q(t)\,dt = 0 \tag{2.1}$$

for arbitrary variations $\delta q(t)$, while holding the endpoints q_0 and q_1 of the curve $t \mapsto q(t)$ fixed. This leads to the well-known forced *Euler Lagrange equations*(see e.g. [17, p. 421]):

$$\frac{\partial L}{\partial q}(q,\dot{q}) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) + f_L(t) = 0\,. \tag{2.2}$$

For a quadratic Lagrangian,

$$L(q(t),\dot{q}(t)) = \frac{1}{2}\dot{q}^\top M\dot{q} - \frac{1}{2}q^\top Kq \tag{2.3}$$

and an external force composed by a dissipation force $f_D = -D\dot{q}$ and the actual (generalized) external force $f(t)$:

$$f_L(t) := -D\dot{q}(t) + f(t)\,, \tag{2.4}$$

one obtains a linear second order vector differential equation of the form (1.1).

2.1 Brief review of variational integrators theory

In order to mimic this procedure in discrete time one may first consider a discretization $\{q_k = q(kh) \;\; k \in [0,N]\}$ and a curve segment $\{q_{k,k+1}(t)\,;\, t \in \, [kh,\,(k+1)h\,)\}$ between two configuration points, $q_k = q(kh)$ and $q_{k+1} = q((k+1)h)$, in the configuration space $Q \subset \mathbb{R}^n$, placed h units of time apart.

The *discrete (exact) Lagrangian* increment $L_d^E(q_k, q_{k+1}, h)$ must contribute to the action integral along the above curve segment. One defines the *exact (forced)* *discrete Lagrangian* and the *exact discrete forces* on that curve segment as:

$$L_d^E(q_k, q_{k+1}, h) := \int_{kh}^{(k+1)h} L(q_{k,k+1}(t), \dot{q}_{k,k+1}(t))dt\,, \tag{2.5}$$

$$f_d^{E-}(q_k, q_{k+1}, h) := \int_{kh}^{(k+1)h} f_L(q_{k,k+1}(t), \dot{q}_{k,k+1}(t))\frac{\partial q_{k,k+1}}{\partial q_k}(t)dt\,, \tag{2.6}$$

$$f_d^{E+}(q_k, q_{k+1}, h) := \int_{kh}^{(k+1)h} f_L(q_{k,k+1}(t), \dot{q}_{k,k+1}(t))\frac{\partial q_{k,k+1}}{\partial q_{k+1}}(t)dt\,. \tag{2.7}$$

where $q_{k,k+1}$ is the solution of the forced Euler-Lagrange equations (2.2) with endpoint conditions $q_{k,k+1}(kh) = q_k$ and $q_{k,k+1}((k+1)h) = q_{k+1}$. See [17, p. 427] for details.

Consider then the following *Discrete Lagrange-D'Alembert principle*

$$\delta \sum_{k=0}^{N-1} L_d^E(q_k, q_{k+1}, h) + \sum_{k=1}^{N-1}\Big(f^{E+}(q_{k-1}, q_k, h) + f^{E-}(q_k, q_{k+1}, h)\Big)\delta q_k = 0 \tag{2.8}$$

where the variation $\delta q(t)$ of a continuous curve is replaced by a discrete (finite) sequence of variations $\{\delta q_k\}_{k=0,\dots,N}$, for arbitrary δq_k's. The variation is computed with *fixed* end points.

The discrete variational principle leads to the *(Exact) Discrete Euler-Lagrange* (EDEL) equations

$$\mathrm{D}_2 L_d^E(q_{k-1}, q_k, h) + \mathrm{D}_1 L_d^E(q_k, q_{k+1}, h) + f^{E+}(q_{k-1}, q_k, h) + f^{E+}(q_k, q_{k+1}, h) = 0\,. \tag{2.9}$$

where D_i stands for the partial derivative operator applied to the i-th argument of the function on which it is acting. These equations should be interpreted as an algorithm mapping the pair $(q_k, q_{k+1}) \in Q \times Q$ to the next configuration pair $(q_{k+1}, q_{k+2}) \in Q \times Q$, i.e.,

$$\mathrm{DEL} : (q_k, q_{k+1}) \mapsto (q_{k+1}, q_{k+2})\,. \tag{2.10}$$

If it were possible to compute the integrals (2.5) explicitely, we would have a discrete model which describes exactly the continuous dynamic at the discrete time instants $t = kh$. In general this computation is not possible and we need to use an approximation both for the discrete Lagrangian and for the discretized external forces. These approximations are denoted $L_d(q_k, q_{k+1})$, $f_d^+(q_k, q_{k+1}, k)$, $f_d^-(q_k, q_{k+1}, k)$ without superscripts, i.e:

$$L_d(q_k, q_{k+1}, h) \approx \int_{kh}^{(k+1)h} L(q(t), \dot{q}(t))d\,, \tag{2.11}$$

$$f_d^-(q_k, q_{k+1}, h) \approx \int_{kh}^{(k+1)h} f(q(t), \dot{q}(t)) \frac{\partial q}{\partial q_k}(t)dt\,, \tag{2.12}$$

$$f_d^+(q_k, q_{k+1}, h) \approx \int_{kh}^{(k+1)h} f(q(t), \dot{q}(t)) \frac{\partial q}{\partial q_{k+1}}(t)dt\,. \tag{2.13}$$

It is remarkable that although many approximations are possible, the "stationary action" principle leads in any case to *Discrete Euler Lagrange Equations* of a standard form

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + f_d^+(q_{k-1}, q_k, k) + f_d^-(q_k, q_{k+1}, k+1) = 0 \tag{2.14}$$

The specific form of the approximations depends on the specific discretization rule used for approximating the integrals.

N.B.: The solution of the Discrete Euler Lagrange Equations derived from an approximate Lagrangian will not any longer be equal to the true configuration variable sampled at the discrete time instants $t = kh$. Now q_k will just be an approximation of the true $q(kh)$. It is known [17] that in any case, when no external forces are applied, the approximation of the flow preserves *symplecticity* but does not preserve (generally speaking) energy.

2.2 Midpoint rule discretization

Probably the simplest way to approximate the Lagrangian and the external forces, is by the so-called (forward) "midpoint rule" which defines the approximate discrete flow $\{q_k\}$ by the substitution

$$q \simeq \frac{q_k + q_{k+1}}{2}, \qquad \dot{q} \simeq \frac{q_{k+1} - q_k}{h} \tag{2.15}$$

in the (2.11), (2.12) and (2.13) it holds

$$L_d(q_k, q_{k+1}, hk, h(k+1)) := hL\left(\frac{q_{k+1} + q_k}{2}, \frac{q_{k+1} - q_k}{h}\right), \tag{2.16}$$

$$f^-(q_k, q_{k+1}, h\,k, h\,(k+1)) := \frac{h}{2} f\left(\frac{q_{k+1} + q_k}{2}, \frac{q_{k+1} - q_k}{h}, \frac{h\,k + h\,(k+1)}{2}\right), \tag{2.17}$$

$$f^+(q_k, q_{k+1}, h\,k, h\,(k+1)) := \frac{h}{2} f\left(\frac{q_{k+1} + q_k}{2}, \frac{q_{k+1} - q_k}{h}, \frac{h\,k + h\,(k+1)}{2}\right). \tag{2.18}$$

In the quadratic Lagrangian (2.5) this leads to:

$$L_d(q_k, q_{k+1}) = h[(\frac{q_{k+1} - q_k}{h})^\top \frac{M}{2} (\frac{q_{k+1} - q_k}{h}) - (\frac{q_{k+1} + q_k}{2})^\top \frac{K}{2} (\frac{q_{k+1} + q_k}{2})]. \tag{2.19}$$

As for the external forces (2.4), the midpoint rule discretization of the general exact expressions (2.6), (2.7), yields[1]

$$f_d^+(q_{k-1}, q_k, k) = -D\frac{q_k - q_{k-1}}{2} + \frac{h}{4}[f(h(k-1)) + f(hk)]$$
$$f_d^-(q_k, q_{k+1}, k+1) = -D\frac{q_{k+1} - q_k}{2} + \frac{h}{4}[f(hk) + f(h(k+1))].$$

By putting together the above equations with

$$D_1 L_d(q_k, q_{k+1}) = -M\frac{q_{k+1} - q_k}{h} - \frac{h}{2}K\frac{q_{k+1} + q_k}{2},$$
$$D_2 L_d(q_{k-1}, q_k) = M\frac{q_k - q_{k-1}}{h} - \frac{h}{2}K\frac{q_k + q_{k-1}}{2},$$

and rearranging the time index, we find the *forced discrete Euler Lagrange equations*

[1] Details of the approximation will be given in the Appendix.

which are the discrete-time counterpart to system (1.1):

$$\left(\frac{M}{h}+\frac{hK}{4}+\frac{D}{2}\right)q_k-\left(\frac{2M}{h}-\frac{hK}{2}\right)q_{k-1} \\ +\left(\frac{M}{h}+\frac{hK}{4}-\frac{D}{2}\right)q_{k-2}=f_d(k) \quad (2.20)$$

where

$$f_d(k):=\frac{h}{4}\left[\,f(hk)+2f(h(k-1))+f(h(k-2))\,\right], \quad (2.21)$$

is an equivalent discrete force. Introducing the *discrete mass, damping and stiffness matrices*,

$$M_d:=\frac{M}{h}+\frac{hK}{4}+\frac{D}{2}\,, \quad (2.22a)$$

$$D_d:=-[\frac{2M}{h}-\frac{hK}{2}]\,, \quad (2.22b)$$

$$K_d:=\frac{M}{h}+\frac{hK}{4}-\frac{D}{2}\,, \quad (2.22c)$$

equation (2.20) can be rewritten in a convenient second-order form as

$$M_dq_k+D_dq_{k-1}+K_dq_{k-2}=f_d(k)\,, \quad (2.23)$$

where $f_d(k)$ is defined by (2.21) or, equivalently, by

$$f_d(k)=L\frac{h}{4}\left[\,u(hk)+2u(h(k-1))+u(h(k-2))\,\right]\;:=Lu_d(k)\,, \quad (2.24)$$

the matrix L being the same as in the continuous-time model. Naturally $u(k)$ denotes the sampled value of the input force at $t=kh$. Note that the computation of the discrete forcing function $\{f_d(k)\}$ (or $u_d(k)$) requires adjacent samples at times k, $k-1$ and $k-2$ of the sampled external force f (or u) so the input-output model (2.23) has *zeros* (or numerator dynamics), contrary to the continuous time model (1.1). Note that the matrices M_d, K_d and D_d are symmetric, hence they need a reduced number of parameter to be completely described. Moreover if h is small enough we have

$$M_d > 0 \quad M_d = M_d^T\,, \tag{2.25a}$$
$$K_d > 0 \quad K_d = K_d^T\,, \tag{2.25b}$$
$$D_d < 0 \quad D_d = D_d^T\,. \tag{2.25c}$$

The relations (2.22) are linear and invertible. By inverting them, the original continuous time parameters (M, D, K) can be easily recovered from the parameters of the discretized model (2.20) by means of the *linear relations*

$$M := \frac{h}{4}[M_d + K_d - D_d]\,, \tag{2.26a}$$
$$D := M_d - K_d\,, \tag{2.26b}$$
$$K := \frac{1}{h}[M_d + K_d + D_d]\,. \tag{2.26c}$$

These are nice linear relations much in the spirit of what we wanted to achieve. Naturally, it must be kept in mind that the solution of (2.20) provides only an *approximation* of the exact flow $t \mapsto q(t)$ sampled at $t = kh$. The approximation error for the midpoint rule is of the order of $O(h^2)$ see [17, p. 402]. More about this will be said in the next section. Use of more complicated approximation schemes than (2.15) can provide approximations of the exact flow of arbitrarily high order, see [8]. We can conclude that the (2.22) and (2.26) define linear invertible continuous-to-discrete conversion for linear mechanical systems.

Chapter 3

The Midpoint discretization and the Cayley transform

A discrete system with a reduced number of parameters has been described, and a discrete "'mechanical structure"' has been pointed out. However some questions need to be answered before using it for identification purpose. It is not clear, in fact, what kind (if any) of sampling operation on the continuous time data (q, u) generated by the model (1.1) would lead to the discrete difference equation (2.23). Still, is the conversion operation better conditioned then the ZOH? In the following a look into these problems is proposed.

It is a remarkable fact that discretization by the midpoint rule (2.15) applied to a general linear time-invariant system is equivalent to the well-known Cayley transformation. Relations with the Cayley transform seem to have been noticed before; e.g. see [2], but in a rather different context.

3.1 Cayley transformation

Starting from the simple integrator's differential equation

$$\dot{x}(t) = u(t) \tag{3.1}$$

and applying the trapezoidal rule with a time step length h, the trivial following steps

$$\int_0^h \dot{x} dt = \int_0^h u dt \tag{3.2}$$

$$x(h) - x(0) = \int_0^h u dt \tag{3.3}$$

$$x(h) - x(0) = \frac{h}{2}(u(h) + u(0)) \tag{3.4}$$

lead to an approximation of integral action in the time step. Passing through the zeta transformation one can write

$$\frac{1}{s} = \frac{h}{2}\frac{z+1}{z-1}\,. \tag{3.5}$$

Inverting, the well known expression that defines the bilinear transformation map (also named Tustin transformation [11, 9]) is obtained

$$s = \frac{2}{h}\frac{z-1}{z+1}. \tag{3.6}$$

It is known that the state space transformation corresponding to the Tustin transform (3.6) on transfer function is the so-called *Cayley transform*. Hence applying the trapezoidal rule to the standard state space model

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{3.7}$$

the Cayley transform can be derived. In the interval $[0, h]$ one can write

$$x(h) - x(0) \simeq \frac{h}{2}\left[Ax(h) + Bu(h) + Ax(0) + Bu(0)\right] \tag{3.8}$$

$$= hA\frac{x(h) + x(0)}{2} + hB\frac{u(h) + u(0)}{2}\,, \tag{3.9}$$

which leads to a discrete linear equation:

$$\left(I - \frac{h}{2}A\right)\bar{x}((k+1)h) = \left(I + \frac{h}{2}A\right)\bar{x}(kh) + \frac{hB}{2}\left(u((k+1)h) + u(kh)\right) \tag{3.10}$$

where $\bar{x}(kh)$ is an approximation of the sampled original continuous state $x(kh)$. Note that $\bar{x}(kh) \neq x(kh)$ even if the input function is piecewise-linear on each sampling interval (in which case the integration of u by the trapezoidal rule would be exact). Now, $I - \frac{h}{2}A$ is certainly invertible if h is small enough and we can solve the equation for $\bar{x}((k+1)h)$. Defining a corresponding approximate output by $\bar{y}(kh) := C\bar{x}(kh) + Du(kh)$ and the "midpoint rule" sequences

$$u_{\frac{1}{2}}(kh) := \frac{u((k+1)h) + u(kh)}{2}, \quad \bar{y}_{\frac{1}{2}}(kh) := \frac{\bar{y}((k+1)h) + \bar{y}(kh)}{2} \tag{3.11}$$

the discretized state space model is derived, i.e. the Cayley transform of (3.7)

$$\begin{cases} \bar{x}((k+1)h) & = A_{\frac{1}{2}}\bar{x}(kh) + B_{\frac{1}{2}}u_{\frac{1}{2}}(kh) \\ \bar{y}_{\frac{1}{2}}(kh) & = C_{\frac{1}{2}}\bar{x}(kh) + D_{\frac{1}{2}}u_{\frac{1}{2}}(kh) \end{cases} \tag{3.12}$$

with the matrices

$$\begin{aligned} A_{\frac{1}{2}} &= \left(I - \frac{hA}{2}\right)^{-1}\left(I + \frac{hA}{2}\right) & B_{\frac{1}{2}} &= h\left(I - \frac{hA}{2}\right)^{-1} B\,, \\ C_{\frac{1}{2}} &= C\left(I - \frac{hA}{2}\right)^{-1}, & D_{\frac{1}{2}} &= \frac{h}{2}C\left(I - \frac{hA}{2}\right)^{-1} B + D\,. \end{aligned} \tag{3.13}$$

It's interesting to note that the system (3.12) describes also the relation from $u(kh)$ and $\bar{y}(kh)$. In this case the state is defined by

$$x^*(hk) = \left(I - \frac{hA}{2}\right)\bar{x}(hk) - \frac{hB}{2}u(hk) \tag{3.14}$$

and the system

$$\begin{cases} x^*((k+1)h) & = A_c x^*(kh) + B_c u(kh) \\ \bar{y}(kh) & = C_c{}^* x(kh) + D_c u(kh) \end{cases} \tag{3.15}$$

has the same matrices of (3.12)

$$A_c = A_{\frac{1}{2}}\,,\; B_c = B_{\frac{1}{2}}\,,\; C_c = C_{\frac{1}{2}}\,,\; D_c = D_{\frac{1}{2}}\,. \tag{3.16}$$

The just given relations lead to the following proposition.

Proposition 3.1.1. *Variational integration by the midpoint rule* (2.15) *applied to the linear mechanical system* (1.1) *coincides with the Cayley-Tustin discretization. In other words, the difference equation* (2.23) *with numerator polynomial defined by* (2.24) *acting on the input $f(k)$, is the input-output counterpart of the Cayley transform* (3.13) *applied to any (minimal) state space realization of* (1.1).

Proof. Let us show that the Tustin transform (3.6) applied to the transfer function of (1.1) produces the discrete transfer function of the difference equation (2.23). Denote

$$G(s) := [Ms^2 + Ds + K]^{-1}; \tag{3.17}$$

then it is immediate to check that

$$\begin{aligned} G(s)^{-1}|_{s=\frac{2}{h}\frac{z-1}{z+1}} &= M\frac{4}{h^2}\frac{(z-1)^2}{(z+1)^2} + D\frac{2}{h}\frac{z-1}{z+1} + K \\ &= [M_d z^2 + D_d z + K_d]\,[I_n\frac{h}{4}(z^2+2z+1)]^{-1}\,, \end{aligned} \tag{3.18}$$

which is precisely the inverse transfer function of the model (2.20), i.e.

$$\left(\frac{M}{h} + \frac{hK}{4} + \frac{D}{2}\right) q_k - \left(\frac{2M}{h} - \frac{hK}{2}\right) q_{k-1} + \left(\frac{M}{h} + \frac{hK}{4} - \frac{D}{2}\right) q_{k-2} = f_d(k)$$

where

$$f_d(k) := \frac{h}{4}\,[\,f(hk) + 2f(h(k-1)) + f(h(k-2))\,]\,,$$

. □

It is remarkable that the spurious zeros in the Tustin-discretized transfer function[1] of the system (1.1) are produced by the midpoint discretization (2.21) of

[1] These are the analog of the Euler-Frobenius polynomials for ZOH discretization, [26].

the external force. This observation will be useful later on.

Remark 1. Going back to the question posed at the beginning of this section, we shall now address the problem of how one may compute the signal q_k.
Denote by $T(f)$ the Tustin transform of a continuous time signal f and let $G_c(z)$ be the Tustin transform of the transfer function $G(s)$; i.e. the discrete transfer function of (3.15). Since it must apparently be true that

$$T(q) = G_c(z)T(u)$$

and since the discrete input function in (3.15) is the ordinary sampled input $u(k) \equiv u(kh)$ (and not the Tustin transform thereof), the signal q_k is not equal to the Tustin transform of the continuous-time flow $q(t)$. On the other hand forcing the input function $u(k) = \mathcal{Z}^{-1}\{T(u)\}$ it must be true that $q_k = \mathcal{Z}^{-1}\{T(q)\}$.

Practical schemes for computing the Tustin transform are discussed in the literature. See e.g. [21, 28]. These schemes are however computationally demanding.

It is well-known that for a wide class of continuous-time functions, for $h \to 0$ the Tustin discretization becomes arbitrarily close to the ordinary sampling discretization [9] (and in fact both signals tend to the exact discretization q^E). Equivalently, when h is chosen small enough the approximation error of the Cayley-Tustin discretization will be of the same order of magnitude of that of the ZOH transform. We shall compare approximation errors later on.

Right now, we are mostly interested in comparing the relative error amplification (conditioning) of the inverse transforms $(M_d, D_d, K_d) \mapsto (M, D, K)$ with that of the ZOH discretization.

3.2 Ill-conditioning of discrete to continuous transformation

In the section 1.1 a rough analysis of the ill-conditioning has been shown; this gave an intuition about the ill-conditioning issue using the ZOH conversion method. What happens with the just found variational methods? Here a simulative analysis is proposed with the aim to describe the situation in which the ill-conditioning is visible, comparing the two different methods.

Consider a scalar second order mechanical system

$$\ddot{q} + \frac{d}{m}\dot{q} + \frac{k}{m}q = \frac{1}{m}f \tag{3.19}$$

and its state space form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ q &= x \end{aligned} \qquad A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}. \tag{3.20}$$

The interest is focused on the discrete to continuous conversion. The analysis is based on measuring the amplification of a perturbation on the discrete model, in the conversion to the continuous domain. Note that on the contrary of the ZOH the midpoint conversion is applied directly on the matrices (M_d, K_d, D_d) and not on the state space form. This requires a different treatment for the 2 methods and the comparison will result to give a partial information. However the linearity of the variational map suggests the possibility of a complete and quite simple analysis of the ill-conditioning. ZOH sampling is essentially based on logarithmic function and an analytic analysis is really difficult (see [5]), therefore only a simulation result will be shown. The procedure to extract a measure of the ill-conditioning can be summurized by the following points:

1. Pick the discretization of (3.20);
2. Add a perturbation on the parameters. Note that for ZOH means to perturb all the state space matrices.
3. Convert to continuous time and measure the "distance" from the original parameter.

In order to give a fair comparison the sampling time must be chosen as multiple of the in the bandwidth of the system. Hence, given

$$\omega_n^2 := \frac{k}{m}, \qquad 2\zeta\omega_n := \frac{d}{m}\,, \tag{3.21}$$

pick $\zeta < 1$, as for most mechanical systems, and compute the 3 dB bandwidth

$$\omega_B := \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}, \tag{3.22}$$

Now set sampling frequencies and sampling time

$$\omega_i := i\omega_B, \quad h_i := \frac{2\pi}{\omega_i} \quad i = 0, 1, 2, 3.$$

3.2.1 ZOH

The following procedure describes how to obtain a simulated characterization of the ill-conditioning of the ZOH based on model (3.20).

1. Compute $\{F_i, g_i\} := \mathtt{c2d}\,\{A, b\}\,, \quad for \quad i = 0, 1, 2, 3.$

2. On each entry of F_i, g_i add Gaussian errors with standard deviation equal to 5 % of the Frobenius norm of F (pretending subspace method is used to identify them). Get 50 samples of

$$\tilde{F}_i = F_i + \delta F, \qquad \tilde{g}_i := gi + \delta g \tag{3.23}$$

3. Compute

$$\mathtt{d2c}\left\{\tilde{F}_i, \tilde{g}_i\right\} := A + \delta A_i, b + \delta b_i \tag{3.24}$$

   and take the sample averages $\overline{\delta A_i}, \overline{\delta b_i}$, where

$$\overline{\delta A_i} = \sqrt{\frac{1}{N}\sum_{k=1}^{N}((A_i)_k - A).^2} \qquad N = 50,$$

$$\overline{\delta b_i} = \sqrt{\frac{1}{N}\sum_{k=1}^{N}((b_i)_k - b).^2}\,.$$

   The operator $.^2$ is the square computed element-wise.

4. Compute

$$\frac{\|\overline{\delta A_i}\|}{\|A\|}, \quad \frac{\|\overline{\delta b_i}\|}{\|b\|}, \quad i = 0, 1, 2, 3. \tag{3.25}$$

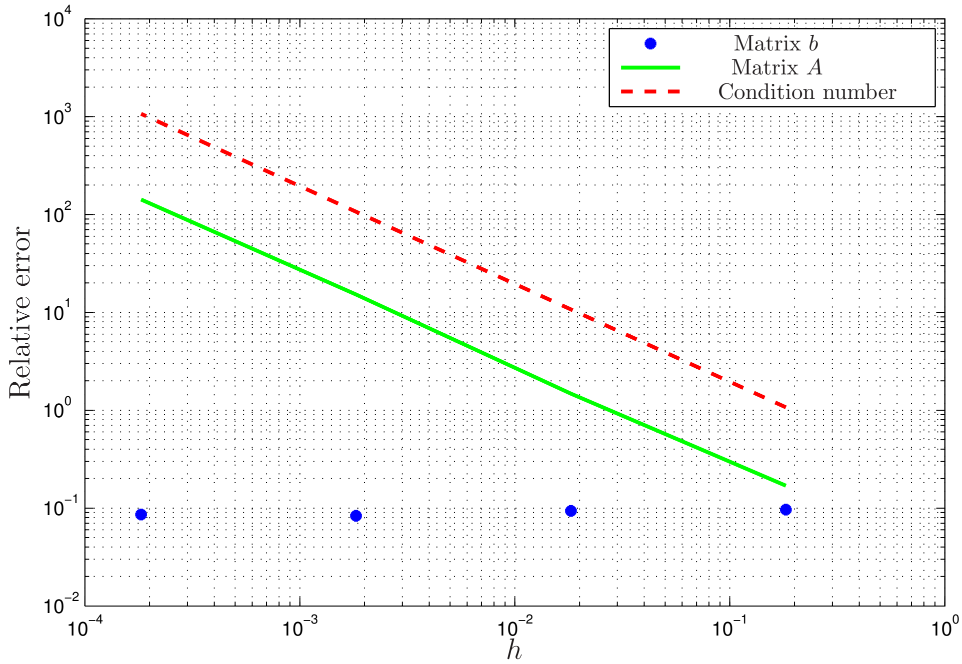

Figure 3.1: Relative error on the matrices A and B compared with $\frac{1}{\|log(F_i)\|}$

using Frobenius norms.

Figure 3.1 shows the result using the ZOH discretization compared with the condition number of the logarithm that is in this simple scalar case proportional to

$$\frac{1}{\|log(F_i)\|}.$$

3.2.2 Conditioning of the Midpoint Rule transformation

Define Γ and Γ_d as

$$\Gamma = \begin{bmatrix} M \\ K \\ D \end{bmatrix}, \Gamma_d = \begin{bmatrix} M_d \\ K_d \\ D_d \end{bmatrix} \tag{3.26}$$

and the linear operator T

$$T = \begin{bmatrix} \frac{1}{h}I & \frac{h}{4}I & \frac{1}{2}I \\ \frac{1}{h}I & \frac{h}{4}I & -\frac{1}{2}I \\ -2\frac{1}{h}I & \frac{h}{2}I & 0 \end{bmatrix}, \tag{3.27}$$

where I is the identity matrix of proper dimensions. The inverse of T is, in accordance with 2.26,

$$R = T^{-1} = \begin{bmatrix} \frac{h}{4}I & \frac{h}{4}I & -\frac{h}{4}I \\ \frac{1}{h}I & \frac{1}{h}I & \frac{1}{h}I \\ I & -I & 0 \end{bmatrix}. \tag{3.28}$$

With this definition the map $(M, K, D) \mapsto (M_d, K_d, D_d)$ becomes

$$\Gamma_d = T\,\Gamma\,, \tag{3.29}$$

and the inverse

$$\Gamma = R\,\Gamma_d\,. \tag{3.30}$$

With this reformulation the ill-conditioning issue can be as a well known linear algebra problem, i.e the computation of the condition number of a matrix. In our specific the problem it is the maximum amplification for a perturbation on Γ_d, namely $\delta\Gamma_d$, through the linear operator R. Formally

$$\kappa(R) = \max_{\delta\Gamma_d, \Gamma_d} \frac{\|R^{-1}\Gamma_d\|/\|R^{-1}\delta\Gamma_d\|}{\|\Gamma_d\|/\|\delta\Gamma_d\|}. \tag{3.31}$$

Using the SVD decomposition the solution to the maximization is immediate. R can be written using the classical notation as

$$R = U\Sigma V^T\,. \tag{3.32}$$

The maximum of (3.31) is reached when Γ_d is the right singular vector relative to the smallest singular value (last row of V) and $\delta\Gamma_d$ is the right singular vector relative to the biggest singular value (first row of V). Hence the condition number becomes function of the singular values

$$\kappa(R) = \kappa(h) = \frac{\sigma_{\max}(R)}{\sigma_{\min}(R)}, \tag{3.33}$$

where it's highlighted that the interest is on the dependence by the sampling time h.

In the scalar case the matrices become

$$R = \begin{bmatrix} \frac{h}{4} & \frac{h}{4} & -\frac{h}{4} \\ \frac{1}{h} & \frac{1}{h} & \frac{1}{h} \\ 1 & -1 & 0 \end{bmatrix}, \Gamma = \begin{bmatrix} m \\ k \\ d \end{bmatrix}, \Gamma_d = \begin{bmatrix} m_d \\ k_d \\ d_d \end{bmatrix}. \tag{3.34}$$

and the singular value of R can be computed explicitly giving the set

$$\sigma(R) = \left\{ 1/8\,\sqrt{2}\sqrt{\frac{48 + 3\,h^4 + \sqrt{2304 - 224\,h^4 + 9\,h^8}}{h^2}}, \sqrt{2}, \right.$$
$$\left. 1/8\,\sqrt{2}\sqrt{\frac{48 + 3\,h^4 - \sqrt{2304 - 224\,h^4 + 9\,h^8}}{h^2}} \right\}. \tag{3.35}$$

Therefore the condition number is

$$\kappa(h) = \frac{\sigma_{\max}(R)}{\sigma_{\min}(R)} = \frac{1/8\,\sqrt{2}\sqrt{\frac{48+3\,h^4+\sqrt{2304-224\,h^4+9\,h^8}}{h^2}}}{1/8\,\sqrt{2}\sqrt{\frac{48+3\,h^4-\sqrt{2304-224\,h^4+9\,h^8}}{h^2}}} \propto \frac{1}{h^2} \tag{3.36}$$

This result points out that the ill-conditioning of the midpoint rule may be even worse than the ZOH. But some more information can be extracted from the decomposition computing U and V. They can be written regardless the normalization as

$$V = \begin{bmatrix} 1/2\,\frac{-8-1/2\,h^4-1/2\,\sqrt{2304-224\,h^4+9\,h^8}}{h^4-16} & -1 & 1/2\,\frac{-8-1/2\,h^4+1/2\,\sqrt{2304-224\,h^4+9\,h^8}}{h^4-16} \\ 1/2\,\frac{-8-1/2\,h^4-1/2\,\sqrt{2304-224\,h^4+9\,h^8}}{h^4-16} & 1 & 1/2\,\frac{-8-1/2\,h^4+1/2\,\sqrt{2304-224\,h^4+9\,h^8}}{h^4-16} \\ 1 & 0 & 1 \end{bmatrix} \tag{3.37}$$

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 1/8\,\frac{48+3\,h^4+\sqrt{2304-224\,h^4+9\,h^8}}{h^2} - 3/4\,h^2 & 0 & 1/8\,\frac{48+3\,h^4-\sqrt{2304-224\,h^4+9\,h^8}}{h^2} - 3/4\,h^2 \\ 0 & 1 & 0 \end{bmatrix}. \tag{3.38}$$

Now it is interesting to note that for $h \to 0$ the singular vectors become really simple. When normalized these become

$$V_0 = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \end{bmatrix} \qquad U_0 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{3.39}$$

Remark 2. *This means that if* $\delta\Gamma_d = [\delta m_d, \delta k_d, \delta d_d]$ *is proportional to* $[\,\frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\,]^T$*, say* $\alpha[\,\frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\,]^T$*, and* $\Gamma_d = [m_d, k_d, d_d] = [\,-\frac{1}{\sqrt{6}}\ \ -\frac{1}{\sqrt{6}}\ \ \sqrt{\frac{2}{3}}\,]^T$*,* $\delta\Gamma = [\delta m, \delta k, \delta d]$ *is* $\alpha\kappa(h)[\,0\ 1\ 0\,]^T$*, that is actually*

$$\delta k \propto \frac{\alpha}{h^2}.$$

Reasoning in the same way it holds:

$$\delta d \propto \frac{\alpha}{h} \qquad \delta m \propto \alpha$$

Concluding this theoretical analysis, the parameter k is shown to be the most affected by the disturbance. On the other hand ill-conditioning on m is actually close to zero.

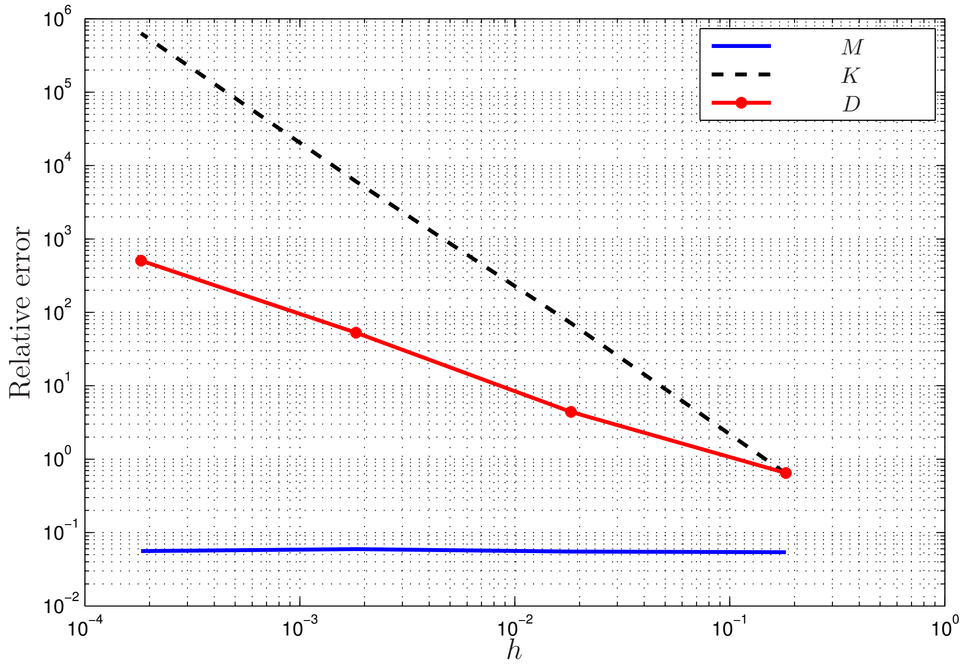

Figure 3.2: Ill-conditioning with midpoint method on the 3 parameters m,d,k

3.2.2.1 Relative errors in simulation

As in the subsection 3.2.1 a simulation procedure is developed for the sake of validation of the theoretical analysis just given. Remembering the steps proposed, in this case the `c2d` conversion is made by the linear operator T and the disturbances are applied directly on the discrete parameters m_d, k_d and d_d. Then, going back to the continuous domain through R, one can compute

$$\frac{\|\overline{\delta m_i}\|}{\|m\|}, \quad \frac{\|\overline{\delta d_i}\|}{\|d\|}, \frac{\|\overline{\delta k_i}\|}{\|k\|}, \quad i = 0, 1, 2, 3. \tag{3.40}$$

In Figure 3.2 is shown the ill-conditioning for the 3 parameters (m, d, k). The parameter k is the most affected as seen in the theoretical analysis and the slope of the corresponding line is in accordance with the result in (3.36). Note that m is not affected by the ill-conditioning as expected.

In general the linear map $(M_d, K_d, D_d) \mapsto (M, K, D)$ cannot be considered globally better conditioned than the log function. However, the different structure of the midpoint model leads to different conditioning behavior for the matrices M, K, D introducing a new element to be considered. Finally it's important to note that it is not known what happens using the midpoint transformation in identification since it is not known how the estimation errors are distributed in relation to the singular vectors of R. In short an algorithm has to be pointed out and then tested to verify the effectiveness in using the midpoint conversion method.

3.3 Preservation of passivity under midpoint sampling

There is a sizable literature on passivity of sampled (i.e. discretized) continuous linear systems, see [4, 25, 19]. Even if there is a clear axiomatic definition of passive discrete linear (and nonlinear) systems, it is not immediately clear how to do sampling in such a way as to preserve passivity. For example, it is well known that with the standard definition of sampled input-output functions, neither Euler method nor the Zero-Order-Hold sampling in general preserve passivity, see [10].

Now assume that the linear system (3.7) is passive; i.e. $\dim y(t) = \dim u(t)$ and there exist a quadratic energy function $V(x) = \frac{1}{2}x^\top Px$ such that

$$V(x(t)) - V(x(0)) \leq \int_0^t y^\top(s)u(s)\,ds\,. \tag{3.41}$$

It is a basic fact of linear systems theory [30] that dissipativity is *equivalent* to the existence of symmetric positive semidefinite matrices P solutions of the linear matrix inequality (LMI)

$$\begin{bmatrix} A^\top P + PA & PB - C^\top \\ C - B^\top P & D + D^\top \end{bmatrix} \leq 0 \tag{3.42}$$

which is in turn equivalent to the fact that the transfer function of a passive system: $G(s) := C(sI - A)^{-1}B + D$ should be *positive-real*, see [30] .

Lossless systems are an important special case. For these systems the inequality in (3.41) is replaced by an equality sign. It can be shown that, under natural minimality assumptions for the realization (A, B, C), the LMI (3.42) has a *unique* solution $P = P^\top$ which is strictly positive definite. This function is a bona-fide *total energy* of the system. Linear port-controlled Hamiltonian systems (see [27]) are a special case: they are lossless systems with an Hamiltonian structure. It is shown in [27] that the energy function of these systems is in fact the Hamiltonian function.

Passivity for discrete linear systems is defined as for the continuous-time case. A discrete linear system,

$$\begin{aligned} x(k+1) &= A_d\, x(k) + B_d\, u(k) \\ y(k) &= C_d\, x(k) + D_d\, u(k) \end{aligned} \tag{3.43}$$

is *passive* if there exist a quadratic energy function $V(x) = \frac{1}{2}x^\top P x$ such that

$$V\left(x(k+1)\right) - V\left(x(k)\right) \leq y(k)^\top u(k)\,. \tag{3.44}$$

It is shown that a linear discrete system in the form (3.43) is passive if and only if the discrete linear matrix inequality (DLMI):

$$\begin{bmatrix} A_d^\top P A_d - P & A_d^\top P B_d - C_d^\top \\ B_d^\top P A_d - C_d & B_d^\top P B_d - (D_d + D_d^\top) \end{bmatrix} \leq 0 \tag{3.45}$$

admits symmetric positive semidefinite solution matrices P. The discrete LMI condition can be generalized to nonlinear systems as reported for example in [13].

The following fact was apparently first discovered by P. Faurre in 1973 and can be found in an unpublished INRIA report [6].

Theorem 3.3.1 (P. Faurre)**.** *Consider a (minimal) linear system* (3.7) *and its discrete-time counterpart obtained by the Cayley transform formulas* (3.13)*. Then one system is passive if and only if the other is, and the energy functions are the same. Moreover the set of solution of the DLMI is the same of the LMI for the continuous time model.*

Note that this statement *per se* does not tell how the inputs and outputs of the continuous system should be "sampled" in order to preserve passivity nor what relation the discrete state of the sampled system has with the continuous state. The midpoint rule interpretation of the bilinear transformation given above answers these questions.

In particular, the midpoint rule variational integrator is a passive discrete mechanical system which is conservative (lossless) if and only if the original continuous-time system was.

In the discrete domain the lossless systems, in analogy to the continuous case, are characterized by the equality in (3.44). The definition of energy in this context is not straightforward but through the discrete Lagrangian a possible definition is

$$E = \frac{\partial L_d}{\partial h} \tag{3.46}$$

In variational integrator theory this quantity has an important role and is related to the simplectic structure that is guaranteed for each variational integrator. In particular it is proved that every variational integrator of a lossless system has a bounded energy that oscillates around the value of the original system. This is a crucial property in the astronomical field for which the classical integrators does not preserve a coherent energy behavior.

It is interesting to note that the midpoint rule applied to a lossless linear mechanical system behaves peculiarly, because it does not present the classical oscillation but the energy is perfectly conserved.

Chapter 4

Identification

Assume one collects sampled data measurements from our continuous system over a suitably long (discrete) time interval T

$$\{q(k)\,;\, k = 1, 2, \ldots, T\}\,, \quad \{u(k)\,;\, k = 1, 2, \ldots, T\} \tag{4.1}$$

with sampling period h. From these data one wants to estimate the mechanical parameters (M, D, K) of the underlying model (1.1). Following the idea exposed in chapter 1 and in chapter 3, a natural identification procedure should be in three steps:

1. Estimates the samples of the Tustin transform of $u(t)$ and $q(t)$ (Remark 1);

2. Estimate the parameters (M_d, D_d, K_d), of the discrete-time variational model (2.23);

3. Recover the corresponding estimates of the continuous time parameters by using the inverse "input-output midpoint transform" (2.26);

Before discussing this however, a reasonable model needs to be set up for describing the actual sampled data (4.1). For this reason initially the ZOH model derived by sampling (1.1) is discussed and later on the identification of the discrete-time variational model (2.23) will be taken up.

4.1 Identification of the ZOH discretization

Since the measurements of $q(k)$ will invariably be affected by noise, a stochastic model needs to be set up. Assuming additive white measurement errors $y(k) = q(k) + w(k)$ ($w(k)$ white stationary) the input-output ZOH discretization of (1.1) can be rewritten as a second order stochastic vector difference equation model of the form

$$y(k) = A_1 y(k-1) + A_2 y(k-2) + B_1 f(k-1) + B_2 f(k-2) + e(k) \tag{4.2}$$

where the process $\{e(k)\}$ is given by $e(k) = w(k) - A_1 w(k-1) - A_2 w(k-2)$, which is colored. Since $f(k)$ acts on the system through the one step delay in the state equation, in this model there is no direct coupling (no $B_0 f(k)$ term) between the external force input and the output variable. In essence the model (4.2) is a so-called *output error* (OE) model whose predictor depends non-linearly on the parameters and gives rise to a nonlinear estimation problem.

For OE models a PEM identification method which can incorporate various constraints on the system parameters such as symmetry of the various matrices etc. is the "grey box" `IDGREY` algorithm described in the MATLAB System Identification Toolbox guide [14]. This algorithm however is very sensitive to noise and to the choice of initial values for the parameters and tends to get easily stuck into local minima. To obtain reasonable results accurate initial parameter estimates are absolutely necessary.

In order to compute good initial estimates a natural choice is to run a preliminary subspace algorithm, say the `n4sid` algorithm, on the data (4.1), using ordinarily sampled input-output data $(y(k),\, f(k))$ or $(y(k),\, u(k))$. This will yield a discrete innovation model of the type

$$\begin{aligned} x(k+1) &= Fx(k) + Gf(k) + Ke(k) \\ y(k) &= Hx(k) + e(k)\,. \end{aligned} \tag{4.3}$$

Since for OE models the state-output dynamics of the stochastic and deterministic subsystems is the same and we are interested only in the estimation of the "deterministic" subsystem, the estimated Kalman gain K and the innovation covariance matrices in the `n4sid` function, can be forced to zero [14]. Also, since the number of degrees of freedom is known a priori, the order estimation is not necessary and the algorithm can be pre-set to return a $2n$-dimensional discrete realization (H, F, G) of the deterministic subsystem. Let us note that in the algorithm the direct coupling term $Jf(k)$ is forced to be zero.

Now consider the model (2.23) describing the variational midpoint approximation of $q(k)$. By the argument exposed in Remark 1, the ideal input and output data set should be the Tustin transformation of the continuous time signals $u(t)$ and $y(t)$. Let $\bar{y}(k) = \mathcal{Z}^{-1}\{T(q)\}$ and $\bar{f}(k) = \mathcal{Z}^{-1}\{T(u)\}$.

Actually using the filtered input $\bar{f}_d$ defined by (2.24) with $\bar{f}$, avoids including the spurious zero polynomial $\frac{h}{4}(1 + 2z^{-1} + z^{-2})$ in the input-output model. In this case, the state space model for $\bar{y}(k)$ has the form

$$x(k+1) = Fx(k) + G\bar{f}_d(k) \tag{4.4}$$

$$\bar{y}(k) = Hx(k) + J\bar{f}_d(k) + e(k)\,. \tag{4.5}$$

and correspond s to a difference equation

$$\bar{y}(k) = \hat{A}_1\bar{y}(k-1) + \hat{A}_2\bar{y}(k-2) + J\bar{f}_d(k) + e(k) \tag{4.6}$$

where,

$$\hat{A}_1 := -M_d^{-1}D_d\,, \quad \hat{A}_2 := -M_d^{-1}K_d\,, \quad J := M_d^{-1}\,. \tag{4.7}$$

Using the filtered input $\bar{f}_d$ defined by (2.24), the model (4.6) is of the purely autoregressive (AR) type with $J \neq 0$ since in the midpoint approximation model (2.23) there is a direct coupling between $f_d(k)$ and q_k. In fact J must actually be equal to M_d^{-1}.

Naturally one should account for the fact that the signals $\bar{y}(k)$ and $\bar{f}(k)$ are not available. In fact reliable estimate of Tustin transformation using sample data are difficult to get. A possible procedure based on a cascade of filtering operations is discussed in [21, p. 688-690] but this approach is not reliable and is very time-consuming. However the difference between (f, y) and $(\bar{f}, \bar{y})$ is visible only with increasing h thus the issue of reconstructing the Tustin transformation can be ignored if h is very small.

Now we present a procedure to compute a preliminary estimate of the parameters M_d, D_d, K_d from the parameters of the ZOH model (4.2) once the latter is identified by the subspace algorithm. To this end let us recall the following result; see e.g. [18].

Lemma 4.1.1. *A necessary and sufficient condition for a* $2n$ *dimensional discrete state space model* (1.7) *with* $\dim y(k) = n$*, to have an input-output relation described by a second order vector difference equation is that*

$$\operatorname{rank}\begin{bmatrix} H \\ HF \end{bmatrix} = 2n \tag{4.8}$$

in other words, all the observability indices of the system must be equal to 2.

In practice, for our ZOH system identified by a subspace method, the matrix

$$\Omega := \begin{bmatrix} H \\ HF \end{bmatrix}$$

will almost always be of rank $2n$ (invertible). Using Ω^{-1} as a similarity transformation one gets the block-companion form

$$\Omega F \Omega^{-1} = \begin{bmatrix} 0 & I \\ F_{21} & F_{22} \end{bmatrix}, \qquad H\Omega^{-1} = \begin{bmatrix} I & 0 \end{bmatrix}$$

where the blocks F_{21}, F_{22} can be computed by solving the equation [1]

$$HF^2 = \begin{bmatrix} F_{21} & F_{22} \end{bmatrix} \Omega \,.$$

Hence the deterministic identified ZOH model can be transformed into the form

$$\begin{aligned} x_d(k+1) &= \begin{bmatrix} 0 & I \\ F_{21} & F_{22} \end{bmatrix} x_d(k) + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} f(k) && (4.9) \\ y(k) &= \begin{bmatrix} I & 0 \end{bmatrix} x_d(k) \,. && (4.10) \end{aligned}$$

[1]This is often called the "shift-invariance" condition in the identification literature.

Note that the transformed G matrix, equal to

$$\hat{G} := \Omega G = \begin{bmatrix} HG \\ HFG \end{bmatrix} \tag{4.11}$$

will in general not have the structure necessary to yield a second order input-output representation of the form (4.2). In general the input-output difference equation of the system, instead of being of the AR-type as in (2.23), will be of a special ARMAX form,

$$y(k) = A_1 y(k-1) + A_2 y(k-2) + B_1 f(k-1) + B_2 f(k-2) \tag{4.12}$$

with $B_1 = \hat{G}_1$ and $B_2 = \hat{G}_2 - F_{22}\hat{G}_1$ as it follows from the general expressions

$$B_1 = \hat{G}_1 - F_{22}J, \qquad B_2 = \hat{G}_2 - F_{21}J - F_{22}\hat{G}_1 \tag{4.13}$$

by setting $J = 0$. Now for $h \to 0$ the ZOH model (4.12) and the deterministic part of (4.6) should approximately coincide. This means that

$$\begin{aligned}&[I + A_1 z^{-1} + A_2 z^{-2}]^{-1}[B_1 z^{-1} + B_2 z^{-2}] \simeq \\ &\qquad [I + M_d^{-1} D_d z^{-1} + M_d^{-1} K_d z^{-2}]^{-1}\,[M_d^{-1}\frac{h}{4}(1 + 2z^{-1} + z^{-2})]\end{aligned} \tag{4.14}$$

In particular the gain matrices of the corresponding transfer functions[2] should be the same and equal to the continuous time gain, K^{-1}, of (1.1). The discrete mechanical parameters (M_d, D_d, K_d) can then be obtained by solving

$$J = M_d^{-1} = B_1 + B_2, \quad M_d^{-1} K_d = -F_{21}, \quad M_d^{-1} D_d = -F_{22}\,. \tag{4.15}$$

These relations yield approximate estimates of (M_d, D_d, K_d), which can be used as initial values for the parameter updating recursion of the `IDGREY` algorithm. Of course the smaller h the better the approximation.

The deterministic model structure for the `IDGREY` algorithm is taken to be a block-companion state space structure of the type (4.9) parametrized directly in

[2]Obtained by evaluating the two members of (4.14) at $z = 1$.

terms of the unknown parameters (M_d, D_d, K_d),

$$\bar{x}(k+1) = \begin{bmatrix} 0 & I \\ F_{21} & F_{22} \end{bmatrix} \bar{x}(k) + \begin{bmatrix} 0 \\ G \end{bmatrix} f_d(k+2) \tag{4.16}$$

$$\bar{y}(k) = \begin{bmatrix} I & 0 \end{bmatrix} \bar{x}(k)\,, \tag{4.17}$$

where the filtered input f_d is shifted two steps ahead in order to get a zero direct coupling term in the output equation. The model parameters are

$$F_{21} := -M_d^{-1} K_d\,, \quad F_{22} := -M_d^{-1} D_d\,, \quad G := M_d^{-1}\,. \tag{4.18}$$

clearly in a one-to-one relation with (M_d, D_d, K_d). It is easy to check that this model corresponds to an input-output difference equation of the type (4.6) with parameters given by (4.7).

One should keep in mind that the parameter estimates are subjected to two kinds of errors. The first is the unavoidable stochastic estimation error while the second is the error due to the (deterministic) model approximation by the midpoint rule inherent in the variational integration approach. In brief we shall refer to this last source of errors as being “due to the Cayley transform”. Obviously this (relative) error increases with h as coarse sampling generally corresponds to bad approximation by the trapezoidal rule. The first kind of error *on the discrete parameters*, depends on the measurement noise variance and on a (information) matrix which describes the sensitivity of the model class to parameter variations. In the transition from discrete to continuous systems this error can be amplified by a bad conditioning of the discrete-to-continuous transform as discussed in the previous chapters. The aim in the following is to prove, that despite the ill-conditioning , for a wide range of sampling intervals the linear inverse midpoint discrete-to-continuous transform (2.26), induces in general smaller relative errors on the continuous parameters than the logarithmic transform. In the next section, the procedure will be compared with the continuous time state of the art procedure proposed in [16] in a couple of simulation examples.

4.2 The Identification algorithm in 5 steps

Given the previous analysis the whole algorithm can be summarized into the following sequence (scheme on Figure 4.1).

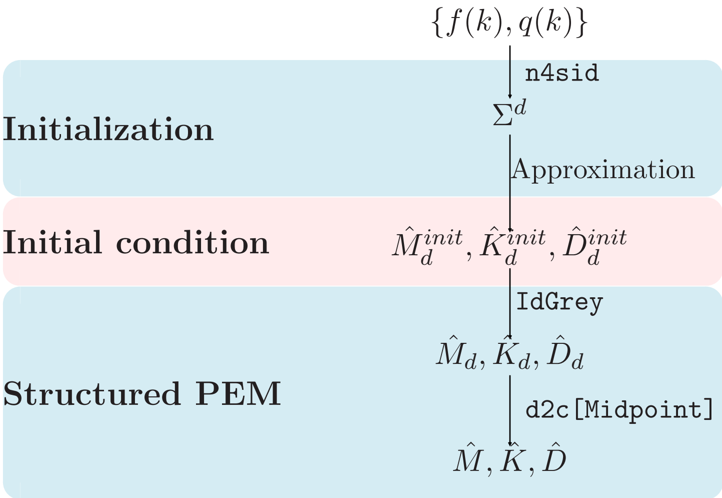

Figure 4.1: Scheme of Identification procedure

1. Are given the sampled data $\{f(k), q(k)\}$ from the continuous time mechanical linear system (1.4) with sampling time h. The data are assumed to be good approximation of the ideal Tustin transformation of $\{f(t), q(t)\}$;

2. Perform `n4sid` over the data $\{f(k), q(k)\}$ and get the system (4.3); (Σ^d in the Figure). This system has no specific property or structure;

3. Exploit the condition (4.14) and compute the matrices $\hat{M}_d^{init}, \hat{K}_d^{init}, \hat{D}_d^{init}$, that are initialization point for the constrained optimization procedure based on PEM. Note that they are rough estimation of the M_d and D_d, K_d;

4. Initialize the `Idgrey` function with the just found parameters and perform identification using the data set $\{f_d(k), q(k)\}$ where $f_d(k)$ is (2.24). The so obtained $\hat{M}_d, \hat{D}_d, \hat{K}_d$ are the best estimates for the "'discrete mechanical"' parameter for the model ((2.23)).

5. Now let's use the midpoint transformation and convert in the continuous time domain. The estimation of the continuous time parameters $\hat{M}, \hat{D}, \hat{K}$ are by construction symmetric.

Chapter 5

Results

The results are shown comparing the variational integrator procedure with a continuous time identification based on the `d2c` conversion. The latter should be applied to a discrete-time state-space model which we have chosen to identify by the subspace algorithm implemented in the identification toolbox (`n4sid`). For comparison, we need to extract a second order model of the type (1.1) from the continuous state space model obtained in this way. Unfortunately, for this to be possible certain Hamiltonian-like structural conditions should be imposed on the identification algorithm. This, although in principle possible in the continuous version of `IDGREY`, slows down the algorithm to the extent to make it practically unusable. The structural conditions need to be forced upon the identified system by a suitable "projection" procedure such as that devised by [15, 1] found in the literature. Such a procedure is summarized by the following points (Figure 5.1):

1. Are given sampled data $\{f(k), q(k)\}$ from a continuous time linear system;
2. Perform a subspace identification method, say a `n4sid`, over the data $\{f(k), q(k)\}$ and get the system Σ^d;
3. Convert the system Σ^d into $\hat{\Sigma}^c$ in the continuous domain using the Matlab function `d2c`;
4. Manipulate $\hat{\Sigma}^c$ in order to obtain a second order continuous model of the system, following the algebraic procedure described in [15]. This produces an estimation of the continuous time parameters $\hat{M}, \hat{D}, \hat{K}$.

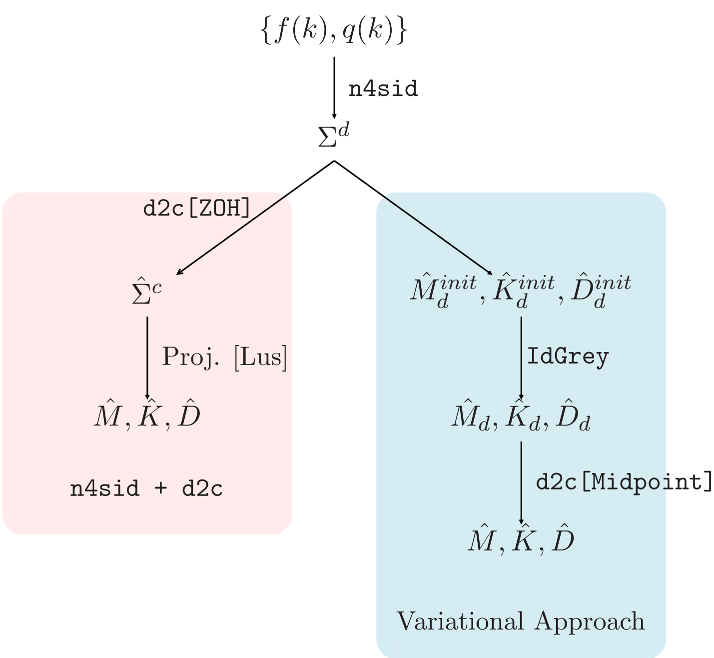

Figure 5.1: Scheme of the two compared identification procedures: `n4sid + d2c` is taken from [15, 16]
.

5.1 Simulation setting

In the previous chapter a specific procedure has been pointed out but the specific implementation requires some more descriptions.
The first step of the algorithm is the sampling of a continuous time signal that is output of a continuous time input-output model. In computed aided simulations this situation cannot be handled directly , except when the analytic description of signals is known. In order to approximate the ideal situation a very little simulation time T_s is defined and a ZOH discretization of the continuous time mechanical system (1.4) is used. In such a way the approximated solutions of the linear differential equation are good enough to consider this discretization a hidden layer in the following description.

A well known difficulty in the identification simulation is the choice of the input sequence and the present case includes even more difficulties. First of all the input signals need to be chosen in the continuous domain though the identification is discrete time based. This fact, and the multiple identification steps required for the proposed procedure, do not allow to use any canonical identification inputs (PBRS or whatever), even if they are proved to be optimal in a sense. Assume the input signal $u(k)$ is a PBRS that necessarily has to be interpreted in continuous time using the ZOH principle with time step h. In this framework the relation from $u(kh)$ and $y(kh)$ is *perfectly* modeled by the ZOH discretization of (1.4) with time step h. Hence, in noise free conditions with enough samples, the identification of such a data-set gives the same result regardless the values of h, i.e. there is no need to reduce the sampling time to increase performances. In fact no ill-condition affects the parameters thanks to the noise free condition, and there is no error in the input approximation. This is far from a realistic situation and may compromise the comparison of the two procedures. Since the real setting is a motivation for this study a different choice has to be taken. In this direction the input signal is built using an actuation time step T_a which is multiple of the generic sampling time h. Then the actuation signal is chosen as a white noise Gaussian sequence interpolated cubicly. The greater is h and worse the signal is described by a ZOH approximation The cubic interpolation of the actuation signal has the aim to reproduce a realistic situation in which a real actuator works without discontinuities. It's important to note that, using such a formulation, there are not discretization rules that produce models that describe the data, even if without added noise.

5.2 Simulation results

In the following a comparison of the described identification methods is presented. To show the quality of the identification, relative errors from the original continuous time parameters are computed for different values of h. Moreover a Monte Carlo method is applied for each h in order to avoid sample variations. The order of the model is expected to be a crucial element hence two different models taken from literature are proposed .

5.2.1 3×3 system

Consider the example 1 of [1]. A three point masses system (three degrees of freedom) is simulated moving along a fixed direction in space, with parameters

$$M = \begin{bmatrix} 0.8 & 0.0 & 0.0 \\ 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 1.2 \end{bmatrix}, \qquad K = \begin{bmatrix} 4.0 & -1.0 & -1.0 \\ -1.0 & 4.0 & -1.0 \\ -1.0 & -1.0 & 4.0 \end{bmatrix}$$
$$D = \begin{bmatrix} 0.4 & -0.1 & -0.1 \\ -0.1 & 0.4 & -0.1 \\ -0.1 & -0.1 & 0.4 \end{bmatrix} \qquad .$$

Three sensors measure and record the sampled displacements of the three point masses with independent (white) measurement noise with SNR 15. Note that such a noise mimics a difficult realistic condition. An array of $3 \times N = 3 \times 6000$ data points is collected and used for system identification. The sampling time interval is chosen to be the most significant range. The results are shown in Figures 5.2, 5.3, 5.4 and 5.5 and the relative errors are presented for the matrices $\hat{M}$, $\hat{K}$, $\hat{D}$ and $\hat{A}$. Formally, usign the matrix M as example, the relative error is computed as

$$\frac{\|\delta M\|}{\|M\|}, \tag{5.1}$$

where

$$\hat{M} = M + \delta M. \tag{5.2}$$

The matrix $\hat{A}$ is estimated by reconstruction using the state space model 1.4.

It is apparent that the relative error with the variational integrator method is much smaller than that obtained by the `n4sid` + `d2c` procedure. Note how, with this last procedure, the error clearly blows up when $h \to 0$ in accordance with the ill-conditioning of the `c2d` conversion. Note that the estimation of the matrix D takes a great advantage from the proposed method, contrary to the estimation of the composed matrix A. This last discrepancy suggests that the global system A final observation concerns the lower limit of the sampling time. As described in the previous chapter the initialization for the constrained PEM is crucial, and if the preliminary `n4sid` does not give a reliable result, the PEM cannot be performed. This defines a limit in the application of the discrete mechanical structure and suggests to search for some constraining procedures based on subspace methods.

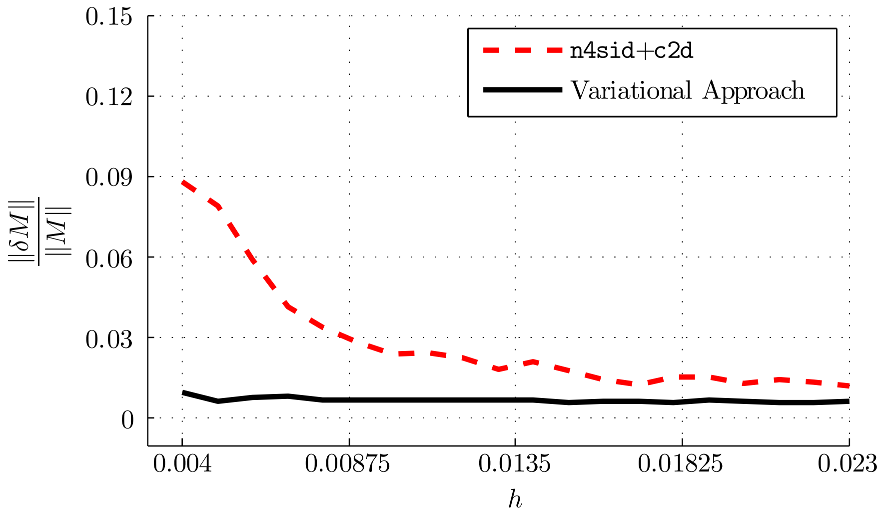

Figure 5.2: Comparing relative error of $\hat{M}$ based on data of the 3×3 model.

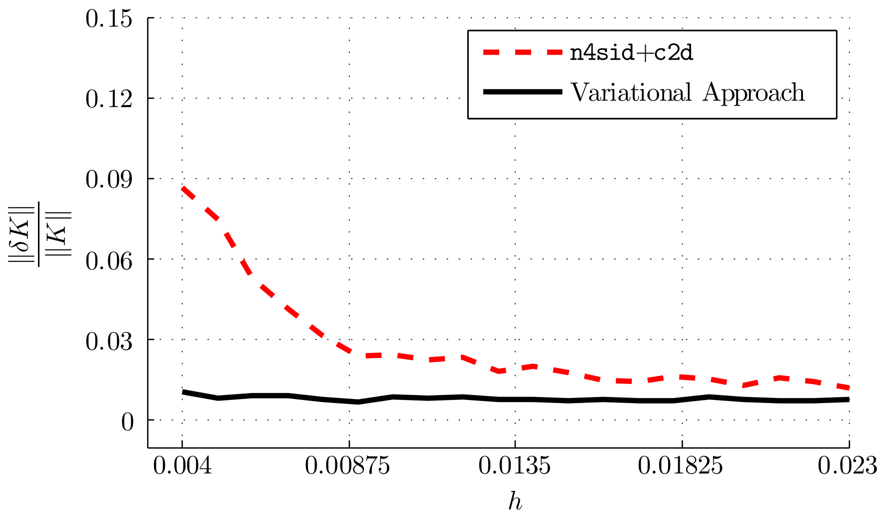

Figure 5.3: Comparing relative errors of $\hat{K}$ based on data of the 3×3 model.

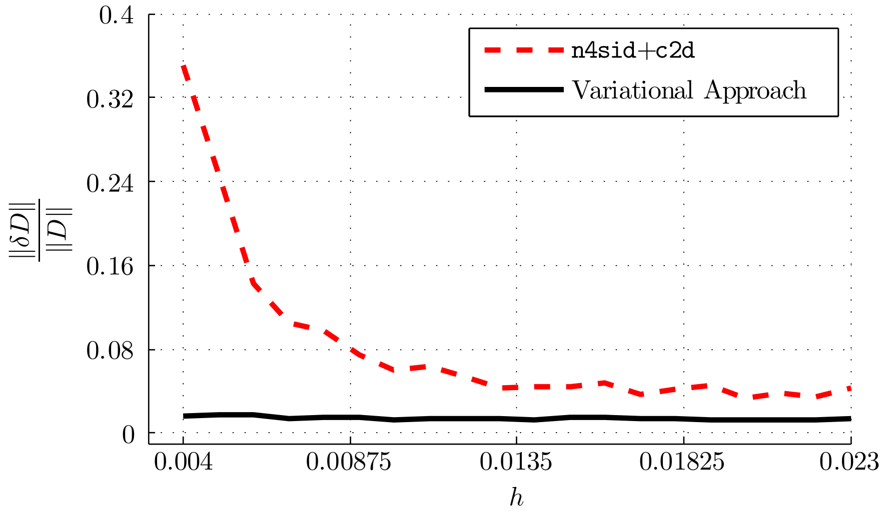

Figure 5.4: Comparing relative errors of $\hat{D}$ based on data of the 3×3 model.

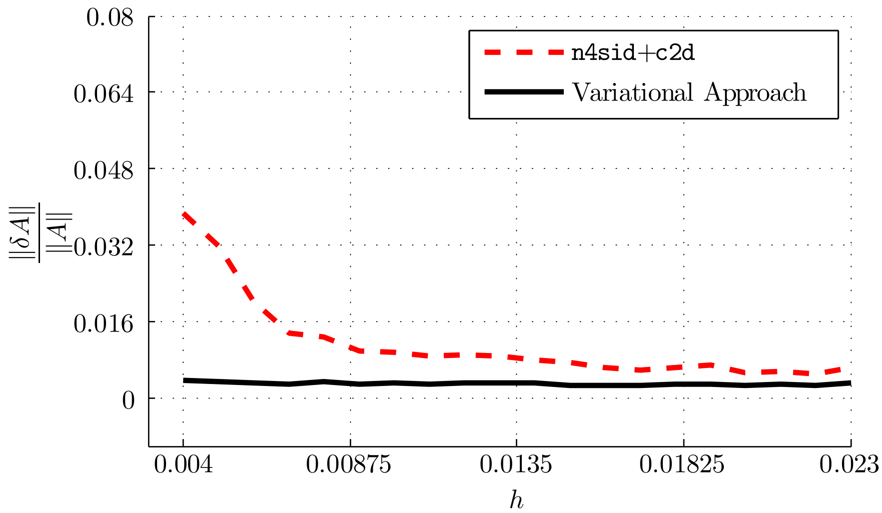

Figure 5.5: Comparing relative errors of $\hat{A}$ based on data of the 3×3 model.

5.2.2 8×8 system

The second model is taken from [1] and has order 8 (In appendix A.5). Simulations are performed exactly in the same way as before, changing only the range of sampling times in which the relative errors are observed. The use of a model with a high order is an interesting test for the algorithm proposed because the advantage in the reduced number of parameters is expected to increase. In this more critical condition the peculiarities of midpoint approach will be better shown. First of all in Figures 5.6, 5.7, 5.8 and 5.9 the results of an identificaiton over 6000 samples and a $20dB$ added noise are reported. The graphs show a similar behavior than in the 3×3 model, and, observing the values of the relative errors, it is even more clear the advantage in using the midpoint approach. In particular in the estimation of matrix D a 5% error is obtained against the 20% of the unstructured approach.

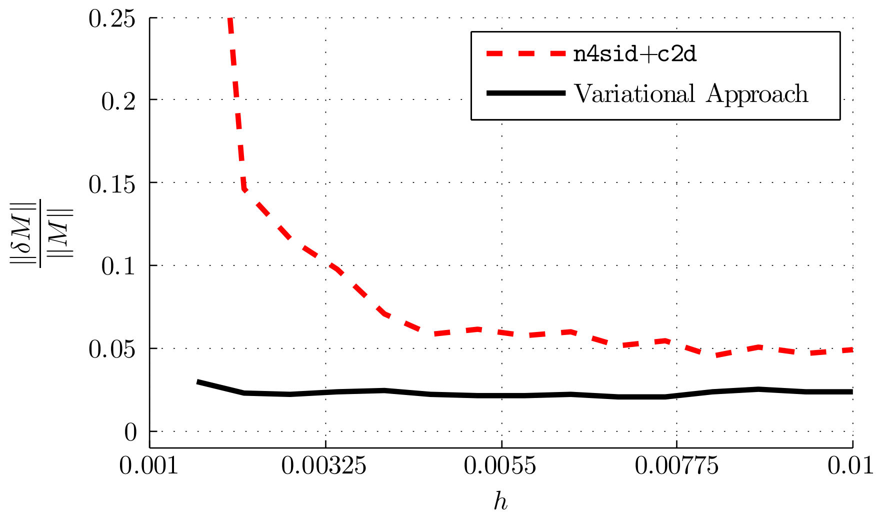

Figure 5.6: Comparing relative errors of $\hat{M}$ based on data of the 8×8 model.

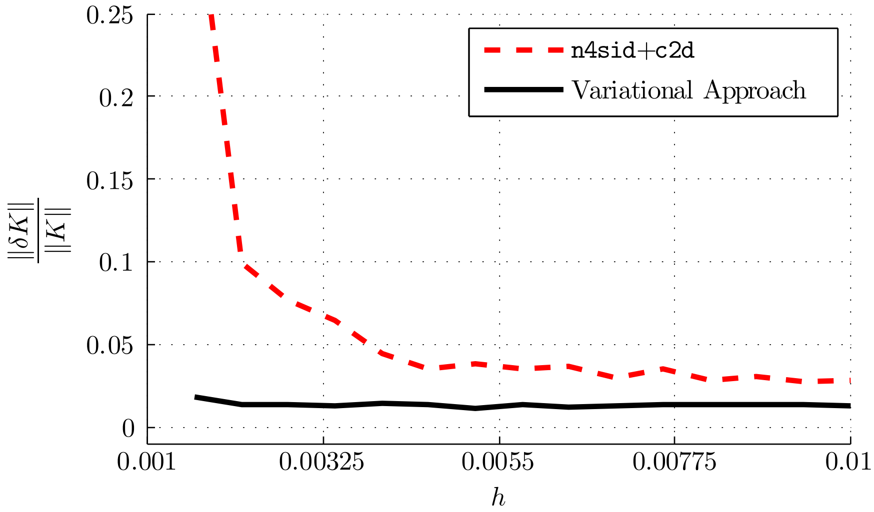

Figure 5.7: Comparing relative errors of $\hat{K}$ based on data of the 8×8 model.

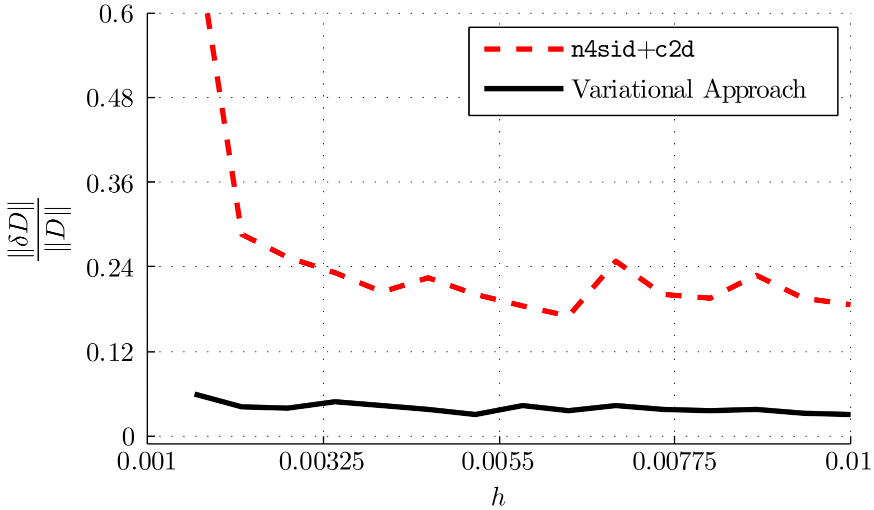

Figure 5.8: Comparing relative errors of $\hat{D}$ based on data of the 8×8 model.

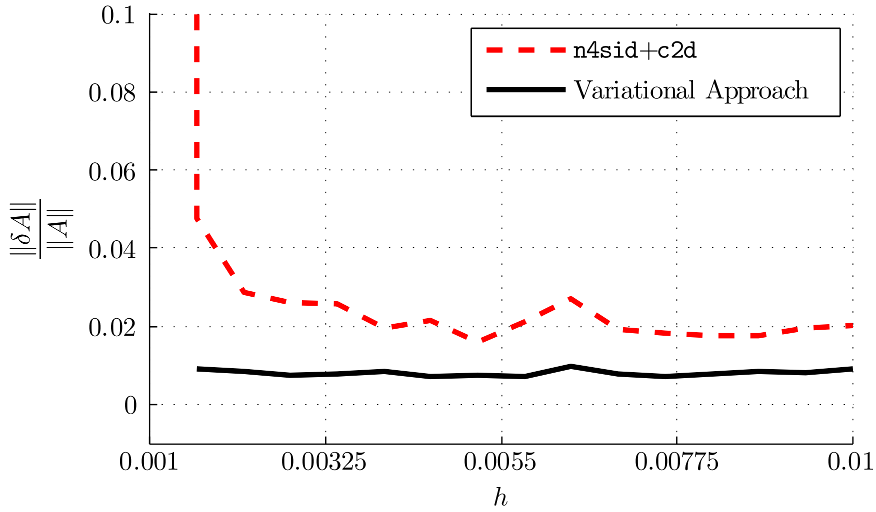

Figure 5.9: Comparing relative errors of $\hat{A}$ based on data of the 8×8 model.

5.2.2.1 Ill-conditioning consequences in the variational approach

In section 3.2 it has been shown and proved that the midpoint discretization is ill-conditioned but there is no evidence of this in the results shown up to now. In order to check these two apparently contrasting facts, a simulation with an *exact* initialization of the PEM procedure has been performed. In such a way the lower bound on the sampling time is avoided and the sampling time range is extended. Figures 5.10, 5.11, 5.12 and 5.13 show the results obtained in the critical condition of SNR equal to $8dB$ and a very small sample time. It is very interesting to note the accordance with the analysis of the section 3.2. The matrix M is completely insensitive to the ill-conditioning problem and the estimation increases its accuracy with decreasing h. The other parameters, on the contrary, present a degradation of the relative errors similar to the ones of the unstructured approach described above.

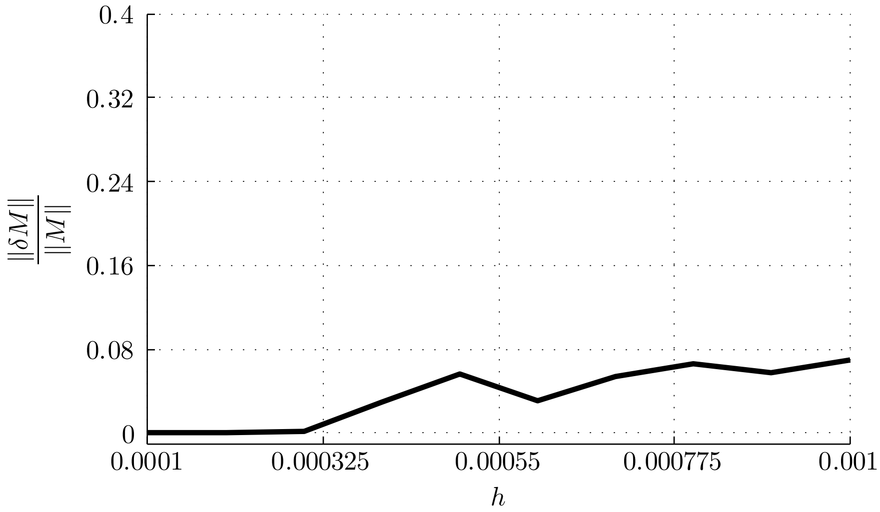

Figure 5.10: Relative error of $\hat{M}$ based on data of the 8×8 model with variational approach. The PEM is initialized exactly.

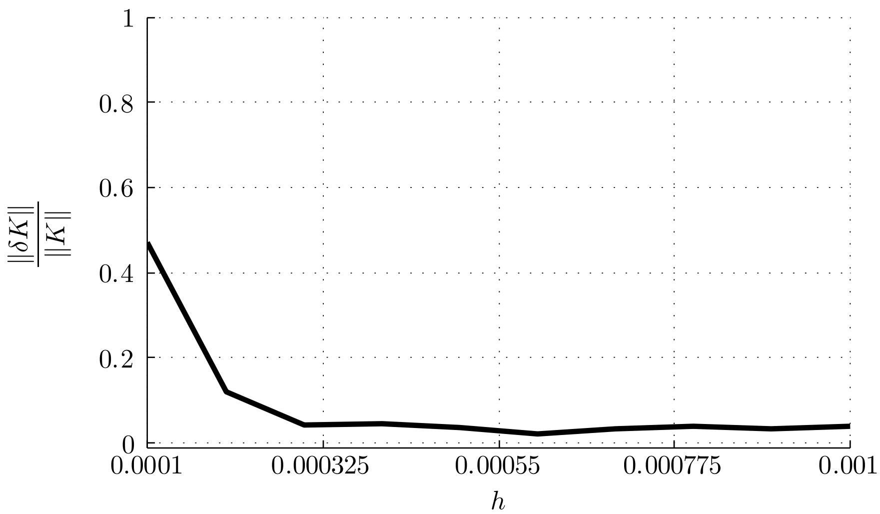

Figure 5.11: Relative error of $\hat{K}$ based on data of the 8×8 model with variational approach. The PEM is initialized exactly.

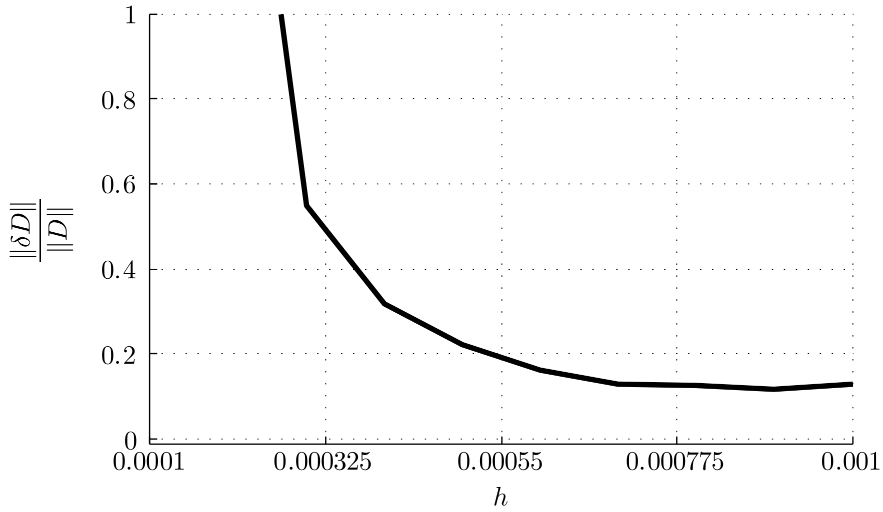

Figure 5.12: Relative error of $\hat{D}$ based on data of the 8×8 model with variational approach. The PEM is initialized exactly.

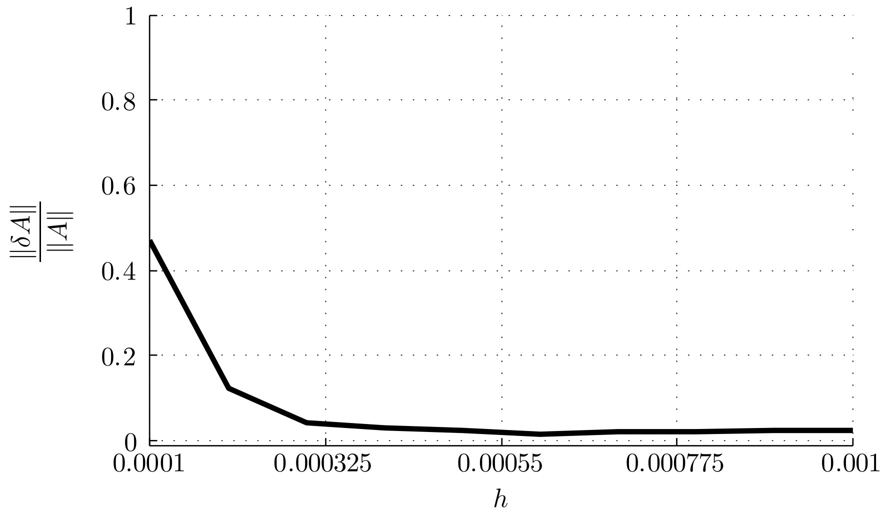

Figure 5.13: Relative error of $\hat{A}$ based on data of the 8×8 model with variational approach. The PEM is initialized exactly.

5.2.2.2 Model error with increasing h

The last open problem, more times addressed in the thesis(see Remark (1)), concerns the consequences of approximating the Tustin transformation with the sampled signals. The midpoint structure model does not fit the data exactly and with increasing values of h the parameters estimation deteriorates. In Figures 5.14 and 5.15 the relative errors are shown in a time sampling range that includes greater value of h for the most significants matrices identified in M and D. The SNR is here very low, equal to $8dB$. A gradual increase of the relative error is evident for the matrix M at the point that the old procedure gives a better result. For the matrix D the good quality of the estimation is confirmed.

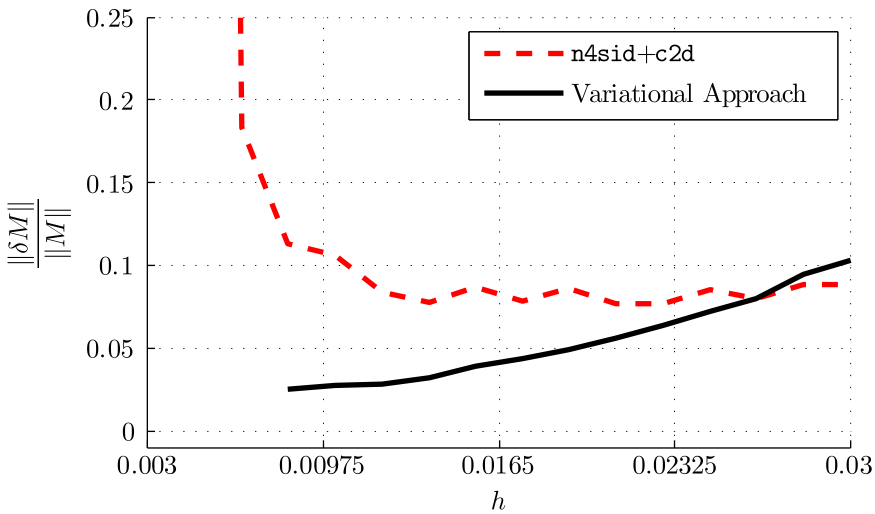

Figure 5.14: Comparing relative errors of $\hat{M}$ based on data of the 8×8 model. The model error is visible.

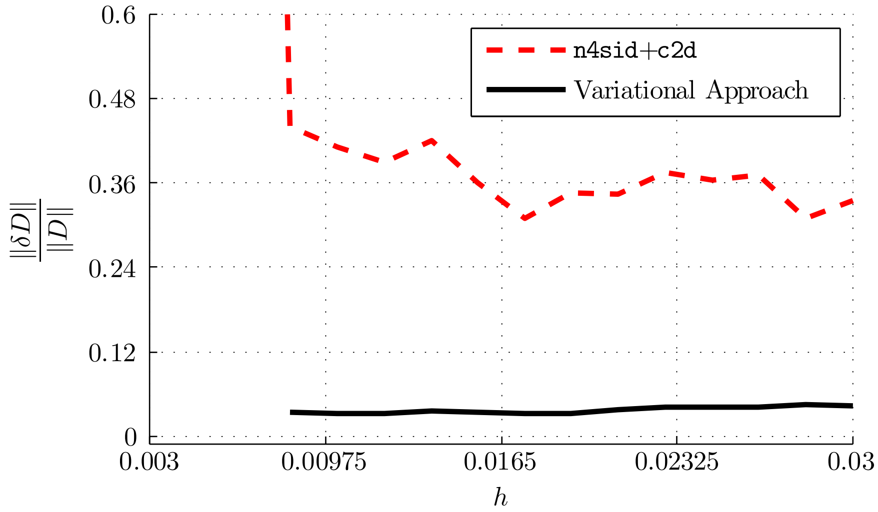

Figure 5.15: Comparing relative errors of $\hat{M}$ based on data of the 8×8 model. The model error is not visible.

Chapter 6

Conclusion

In this thesis the variational integrator theory has been recovered and applied for the first time in the mechanical identification field. This powerful tool defines a natural way to discretize continuous time mechanical systems, conserving many of their properties in the discrete domain. Exploiting this prerogative, a new discretization procedure for the second order Lagrangian equations of a linear mechanical systems has been described and a discrete mechanical structure has been defined.
The proposed procedure is based on a rather elementary discretization rule (the midpoint rule) but more elaborate variational discretizations are possible which lead to approximation errors of higher order than $O(h^2)$. On the other hand this simple approximation gives a transformation function that is linear. The use of these higher order schemes has not been explored and is left to future investigations. Note that in general also non linear identification can be performed using this kind of approach.

The ill-conditioning of the linear operator that defines the time domain conversion, has been analyzed and a peculiar behavior has been detected. A different error amplification has been observed for the three matrices M, K and D; this evidence has underlined the tricky nature of the continuous to discrete conversion. However the simplicity of the conversion function and the reduced parameters discrete model are proven to be effective properties.

The simulation results have shown a great advantage in using the proposed algorithm, mostly in the extreme condition of the classical approach.

Two main limits are found and narrow the range of applications of the algorithm:

- The necessity of an initialization step before the constrained optimization;
- The necessity to estimate the Tustin transform of continuous time signal by means of sampled data for accurate result with greater h.

These give the stimuli for future extensions and enhancements. A new procedure completely based on subspace methods should solve the initialization problem skipping the curse of the non linear optimization. Also exploiting the principle described in [21] to obtain a discrete version of the filtering procedure, could lead to estimate accurately the Tustin transform.

Also the proposed identification procedure should possibly be completed with order estimation.

Appendix A

A.1 Equivalence of the step-invariant response discretization and ZOH

Want to prove that the discretization method named step-invariant response [22] for a piece wise constant input $u(t)$ is equivalent to the classical c2d conversion (ZHO). The step-invariant response discretization is characterized by (A.1).

$$H(z) = \frac{Z\left\{y(kh)\right\}}{Z\left\{u(kh)\right\}} = (1 - z^{-1})Z\left\{L^{-1}\left\{\frac{H(s)}{s}\right\}\right\} \tag{A.1}$$

On the other hand the c2d conversion is defined w.r.t the standard state space form (A, B, C, D) in (A.2).

$$\begin{aligned} \mathbf{x}(k+1) &= e^{Ah}x(k) + (e^{Ah} - I)A^{-1}Bu(k); \\ y(k) &= Cx(k), \end{aligned} \tag{A.2}$$

Proof. To prove the equivalence we can apply the step-invariant transformation to the continuous time system with transfer function :

$$G(s) = C\left(sI - A\right)^{-1} B. \tag{A.3}$$

Hence

$$
\begin{aligned}
&\left(1-z^{-1}\right) Z\left\{L^{-1}\left\{\frac{G(s)}{s}\right\}\right\} \\
&= Z\left\{C\left(e^{At}-I\right) A^{-1} B\right\} \\
&= \left(1-z^{-1}\right) C A^{-1}\left(I-z^{-1} e^{Ah}\right)^{-1} B - C\left(I-z^{-1} I\right)^{-1} A^{-1} B \\
&= C\left(I-z^{-1} I\right)\left(\left(I-z^{-1} e^{Ah}\right)^{-1}-\left(I-z^{-1} I\right)^{-1}\right) A^{-1} B \\
&= C\left(\left(I-z^{-1} e^{Ah}\right)^{-1}-z^{-1}\left(I-z^{-1} e^{Ah}\right)^{-1}-\left(I-z^{-1} e^{Ah}\right)\left(I-z^{-1} e^{Ah}\right)^{-1}\right) A^{-1} B \\
&= z^{-1} C\left(e^{Ah}-I\right)\left(I-z^{-1} e^{Ah}\right)^{-1} A^{-1} B \\
&= z^{-1} C\left(I-z^{-1} e^{Ah}\right)^{-1}\left(e^{Ah}-I\right) A^{-1} B,
\end{aligned}
\tag{A.4}
$$

That is exactly the transfer function of (A.2). In the proof we exploit that

$$
\left(I-z^{-1} e^{Ah}\right)^{-1}\left(e^{Ah}-I\right)=\left(e^{Ah}-I\right)\left(I-z^{-1} e^{Ah}\right)^{-1}, \tag{A.5}
$$

and that

$$
A e^{Ah}=e^{Ah} A. \tag{A.6}
$$

□

A.2 More on Discrete Euler-Lagrange (DEL) equations with forcing

This is taken from [23]. Consider the extended action functional $\mathfrak{S}(q(\cdot), t(\cdot))$ defined (for strictly increasing function $t:[s_0, s_1] \rightarrow [t_0, t_1]$) as

$$
\mathfrak{S}(q(\cdot), t(\cdot))=\int_{s_0}^{s_1} L(q(s), q'(s)/t'(s))\, t'(s) ds\,. \tag{A.7}
$$

Its first variation can be computed as follow

$$
\begin{aligned}
\delta\mathfrak{S} &= \delta\int_{s_0}^{s_1} L\left(q(s), q'(s)/t'(s)\right) t'(s)ds \\
&= \int_{s_0}^{s_1}\left\{\left[\frac{\partial L}{\partial q}\left(q, q'/t'\right)\delta q + \frac{\partial L}{\partial \dot q}\left(q, q'/t'\right)\delta q'/t' - \frac{\partial L}{\partial \dot q}\left(q, q'/t'\right)q'\delta t'/(t')^2\right]t'\right. \\
&\qquad\qquad \left. + L\left(q, q'/t'\right)\delta t'\right\}ds \\
&= \int_{s_0}^{s_1}\left(\frac{\partial L}{\partial q}\left(q, q'/t'\right) - \frac{d}{ds}\frac{\partial L}{\partial \dot q}\left(q, q'/t'\right)\frac{1}{t'}\right)\delta q\, t' ds + \left.\frac{\partial L}{\partial \dot q}\delta q\right|_{s_0}^{s_1} \\
&+ \int_{s_0}^{s_1}\frac{d}{ds}\left(\frac{\partial L}{\partial \dot q}\left(q, q'/t'\right)\frac{q'}{t'} - L\left(q, q'/t'\right)\right)\frac{1}{t'}\,\delta t\, t' ds \\
&+ \left.\left(L\left(q, q'/t'\right) - \frac{\partial L}{\partial \dot q}\left(q, q'/t'\right)\frac{q'}{t'}\right)\delta t\right|_{s_0}^{s_1}.
\end{aligned}
\tag{A.8}
$$

Now, let $q^E(t, q_0, q_1, t_0, t_1, u(\cdot))$ be the solution of the forced Euler-Lagrangian equations that starts from q_0 at time t_0 and arrives in q_1 at time t_1 under the influence of the (generalized) force $u(\cdot)$. We have

$$
\delta\int_{t_0}^{t_1} L(q^E(\tau), \dot q^E(\tau))d\tau + \int_{t_0}^{t_1} u(\tau)\,\delta q(\tau)\,d\tau = 0\,. \tag{A.9}
$$

Define the exact discrete Lagrangian L_d^E as

$$
L_d^E(q_0, q_1, t_0, t_1, u(\cdot)) := \mathfrak{S}(q^E(t, q_0, q_1, t_0, t_1, u(\cdot))) \tag{A.10}
$$

and compute its partial derivative with respect to q_0, q_1, t_0, and t_1. From the expression of the first variation of $\mathfrak{S}$ we get

$$
\frac{\partial L_d^E}{\partial q_0}(q_0, q_1, t_0, t_1, u(\cdot)) = -\int_{t_0}^{t_1} u(\tau)\frac{\partial q^E(\tau)}{\partial q_0}d\tau - \frac{\partial L}{\partial \dot q}(q_0, \dot q_0) \tag{A.11a}
$$

and

$$
\frac{\partial L_d^E}{\partial q_1}(q_0, q_1, t_0, t_1, u(\cdot)) = -\int_{t_0}^{t_1} u(\tau)\frac{\partial q^E(\tau)}{\partial q_1}d\tau + \frac{\partial L}{\partial \dot q}(q_1, \dot q_1)\,. \tag{A.11b}
$$

where $\dot{q}_0 := \frac{d}{dt}\, q^E(t, q_0, q_1, t_0, t_1, u(\cdot))|_{t=t_0}$ and $\dot{q}_1 := \frac{d}{dt}\, q^E(t, q_0, q_1, t_0, t_1, u(\cdot))|_{t=t_1}$.

Remark 3. *From equations* (A.11a) *and* (A.11b)*, it is straightforward to derive the forced DEL equations presented Marsden and West 2001. Indeed, we have*

$$-\frac{\partial L}{\partial \dot{q}}(q^E(t_1), \dot{q}^E(t_1)) = \frac{\partial L_d^E}{\partial q_0}(q_1, q_2, t_1, t_2, u(\cdot)) + \int_{t_1}^{t_2} u(\tau)\frac{\partial q^E(\tau)}{\partial q_0} d\tau\,. \tag{A.12a}$$

$$\frac{\partial L}{\partial \dot{q}}(q^E(t_1), \dot{q}^E(t_1)) = \frac{\partial L_d^E}{\partial q_1}(q_0, q_1, t_0, t_1, u(\cdot)) + \int_{t_0}^{t_1} u(\tau)\frac{\partial q^E(\tau)}{\partial q_1} d\tau\,. \tag{A.12b}$$

Summing up the two equations above, one gets the desired result, i.e.,

$$\begin{split} D_2 L_d^E(q_0, q_1, t_0, t_1, u(\cdot)) + D_1 L_d^E(q_1, q_2, t_1, t_2, u(\cdot)) + \\ f_d^{E+}(q_0, q_1, t_0, t_1, u(\cdot)) + f_d^{E-}(q_0, q_1, t_0, t_1, u(\cdot)) = 0 \end{split} \tag{A.13}$$

where

$$f_d^{E+}(q_0, q_1, t_0, t_1, u(\cdot)) := \int_{t_0}^{t_1} u(\tau)\frac{\partial q^E(\tau)}{\partial q_1} d\tau\,, \tag{A.14a}$$

$$f_d^{E-}(q_1, q_2, t_1, t_2, u(\cdot)) := \int_{t_1}^{t_2} u(\tau)\frac{\partial q^E(\tau)}{\partial q_0} d\tau\,. \tag{A.14b}$$

A.3 Discretization of the external forces

It is here shown how, starting from

$$f_d^{E+}(q_0, q_1, t_0, t_1, f(\cdot)) := \int_{t_0}^{t_1} f(\tau)\frac{\partial q^E(\tau)}{\partial q_1} d\tau\,, \tag{A.15a}$$

$$f_d^{E-}(q_1, q_2, t_1, t_2, f(\cdot)) := \int_{t_1}^{t_2} f(\tau)\frac{\partial q^E(\tau)}{\partial q_0} d\tau\,. \tag{A.15b}$$

one arrives at the approximations $\frac{h}{4}[f(h(k{-}1)){+}f(hk))]$ and $\frac{h}{4}[f(h(k)){+}f(h(k{+}1))]$.

Consider the exact trajectory $q^E(t; q_0, q_1, f(\cdot))$ in the interval $t \in [t_0, t_1]$. This trajectory satisfies the boundary conditions

$$q^E(t_0; q_0, q_1, f(\cdot)) = q_0, \qquad q^E(t_1; q_0, q_1, f(\cdot)) = q_1$$

and hence

$$\begin{aligned} \frac{\partial}{\partial q_0} q^E(t_0; q_0, q_1, f(\cdot)) &= Id, & \frac{\partial}{\partial q_1} q^E(t_0; q_0, q_1, f(\cdot)) &= 0 \\ \frac{\partial}{\partial q_0} q^E(t_1; q_0, q_1, f(\cdot)) &= 0, , & \frac{\partial}{\partial q_1} q^E(t_1; q_0, q_1, f(\cdot)) &= Id \end{aligned}$$

Now to approximate

$$\int_{t_0}^{t_1} f(\tau) \frac{\partial q^E(\tau)}{\partial q_0} d\tau \qquad (*)$$

one can choose

$$(t_1 - t_0)/2 \times (f(t_0) \frac{\partial}{\partial q_0} q^E(t_0; q_0, q_1, f(\cdot)) + f(t_1) \frac{\partial}{\partial q_0} q^E(t_1; q_0, q_1, f(\cdot)))$$

which, based on the previous expressions for the derivatives, is equal to

$$(t_1 - t_0)/2 \times f(t_0).$$

A "midpoint rule" approximation of (*) is

$$(t_1 - t_0) \times ((f(t_0) + f(t_1))/2 \times (\frac{\partial}{\partial q_0} q^E(t_0; q_0, q_1, f(\cdot)) + \frac{\partial}{\partial q_0} q^E(t_1; q_0, q_1, f(\cdot)))/2)$$

which leads to

$$(t_1 - t_0) \times (f(t_0) + f(t_1))/4\,.$$

This is the formula we wanted to justify.

A.4 Relative errors measured on estimation of mechanical parameters

The analysis computed in the section 3.2 does face the ill-conditioning of the logarithmic function in the state space form. No words are spent for the the propagation of the errors after the projection used in the (`n4sid + d2c`) (see ??). In this direction the step 3 and 4 of subsection 3.2.1 are replaced with:

4. Compute
$$\texttt{d2c}\left\{\tilde{F}_i, \tilde{g}_i\right\} = \tilde{A}_i, \tilde{b}_i := A + \delta A_i, b + \delta b_i. \tag{A.16}$$
Then, with the same algebraic technique used in the identification procedure [15], manipulate $\tilde{A}_i, \tilde{b}_i$ and recover estimation of original mechanical parameters. Thus we have $\tilde{m}_i, \tilde{d}_i, \tilde{k}_i$ and we can compute
$$\overline{\delta m_i} = \sqrt{\frac{1}{N}\sum_{k=1}^{N}((m_i)_k - m)^2} \qquad N = 50$$
and the same for δd_i and δk_i.

5. Compute
$$\frac{\|\overline{\delta m_i}\|}{\|m\|}, \quad \frac{\|\overline{\delta d_i}\|}{\|d\|}, \frac{\|\overline{\delta k_i}\|}{\|k\|}, \quad i = 0, 1, 2, 3. \tag{A.17}$$

The result in Figure A.1 show that, the relative error is propagated on the three parameters in the same way. The relative errors amplification different for the parameters, shown in the theoretical analysis of the midpoint conversion is a peculiarity thereof.

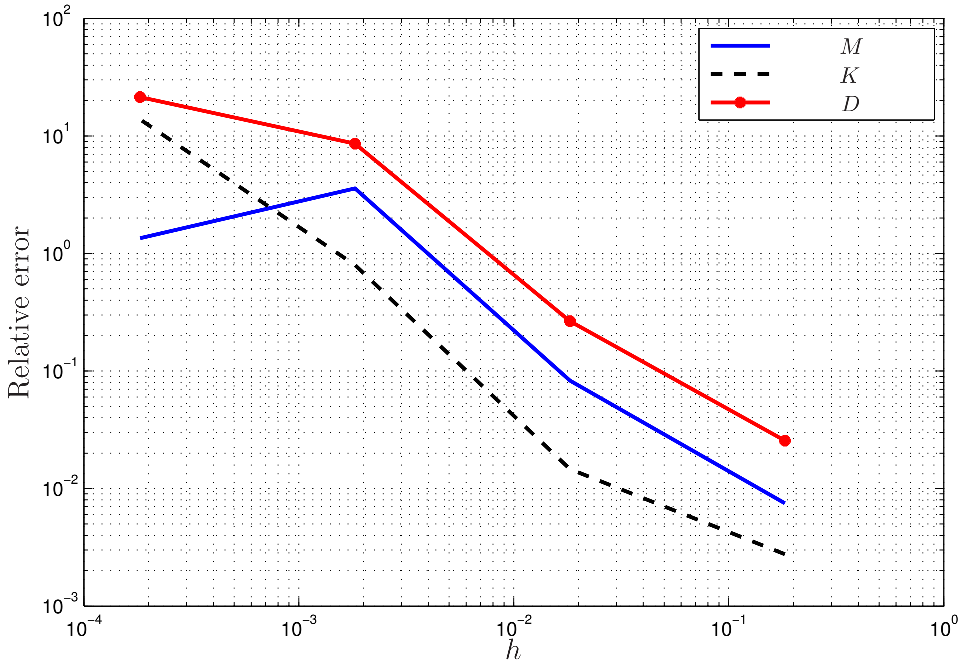

Figure A.1: Relative error with ZOH discretization on the original parameters

A.5 Matrices of the 8×8 system

$$M = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 \end{bmatrix} \tag{A.18}$$

$$K = \begin{bmatrix} 27071.1 & 0 & 0 & 0 & -10000 & 0 & -3535.5 & -3535.5 \\ 0 & 17071.1 & 0 & -10000 & 0 & 0 & -3535.5 & -3535.5 \\ 0 & 0 & 27071.1 & 0 & -3535.5 & 3535.5 & -10000 & 0 \\ 0 & -10000 & 0 & 17071.1 & 3535.5 & -3535.5 & 0 & 0 \\ -10000 & 0 & -3535.5 & 3535.5 & 27071.1 & 0 & 0 & 0 \\ 0 & 0 & 3535.5 & -3535.5 & 0 & 17071.1 & 0 & -10000 \\ -3535.5 & -3535.5 & -10000 & 0 & 0 & 0 & 27071.1 & 0 \\ -3535.5 & -3535.5 & 0 & 0 & 0 & -10000 & 0 & 17071.1 \end{bmatrix} \tag{A.19}$$

$$D = \begin{bmatrix} 136.4 & 0 & 0 & 0 & -50 & 0 & -17.7 & -17.7 \\ 0 & 86.4 & 0 & -50 & 0 & 0 & -17.7 & -17.7 \\ 0 & 0 & 136.4 & 0 & -17.7 & 17.7 & -50 & 0 \\ 0 & -50 & 0 & 86.4 & 17.7 & -17.7 & 0 & 0 \\ -50 & 0 & -17.7 & 17.7 & 136.4 & 0 & 0 & 0 \\ 0 & 0 & 17.7 & -17.7 & 0 & 86.4 & 0 & -50 \\ -17.7 & -17.7 & -50 & 0 & 0 & 0 & 136.4 & 0 \\ -17.7 & -17.7 & 0 & 0 & 0 & -50 & 0 & 86.4 \end{bmatrix} \tag{A.20}$$

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