Towards a Formal Account of Identity Criteria

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Abstract

Identity criteria are used to confer ontological respectability: Only entities with clearly determined identity criteria are ontologically acceptable. From a logical point of view, identity criteria should mirror the identity relation in being reflexive, symmetrical, and transitive. However, this logical constraint is only rarely met. More precisely, in some cases, the relation representing the identity condition fails to be transitive. We consider the proposals given so far to give logical adequacy to inadequate identity conditions. We focus on the most refined proposal and expand its formal framework by taking into account two further aspects that we consider essential in the application of identity criteria to obtain logical adequacy: *contexts* and *granular levels* .

Introduction

Consider the following thesis characterizing a strong ontological realism (for an overview on this topic see Devitt [1]):

(SOR) There is a mind-independent world and it is structured: there are distinct objects, properties, etc.

If an ontological realist adopts (SOR), there is a problem of selecting, from among the many entities such as objects, properties, events, facts, etc., the real entities i.e. those entities existing independently of our mental states. Hence, with respect to objects of a specific kind, one can be a realist, if one takes them to be real entities, or an antirealist, if one takes them to be mere projections of one's thoughts.

Adopting a different jargon, we can say that the problem for ontological realists is selecting those objects that have ontological respectability. One standard (Quinian) solution in analytic philosophy is to argue that identity criteria are required for ontological respectability: only entities that have clearly determined identity criteria are ontologically respectable, i.e. acceptable. Think, for example, of the case of properties: following Quine [12] , properties would not be ontologically acceptable because they do not have any no suitable identity criterion.

Question: are there general constraints to identity criteria for the individuation of real substances?

We distinguish between two kinds of constraints: *formal constraints* and *metaphysical constraints*. *Metaphysical constraints* normally derive from the theses of the general framework adopted, for example absolute identity vs. relative identity or four-dimensionalism vs. three-dimensionalism. Conversely, *formal constraints* are specified on the basis of the logical form of the identity criteria and some properties induced by it.

In the present work we focus only on formal constraints, or requirements on identity criteria; more specifically, we focus on a specific formal constraint: equivalence. The main goal in our paper is to make some steps towards a formal characterization of identity criteria.

The paper is divided into four sections. In the first section we present the problem at issue, that is, the logical requirements that identity criteria are supposed to meet and some, commonly used identity criteria failing to meet one of those requirements. In the second section, we will present Williamson's and De Clerq and Horsten's treatment (in [8] and [10]) of logically inadequate identity criteria. In the third section, we will try to embody De Clercq and Horsten's proposal in an enlarged framework that takes into account contexts and levels of granularity too. In section 4 we then conclude with some general remarks.

1 Logical adequacy of identity criteria

The credit for introducing the notion of an identity criterion (from now on, IC) is usually attributed to Frege . In his *Foundations of Arithmetic* Frege introduces the idea of IC in a context where he wonders how we can grasp or formulate the concept of numbers (see [2], §62):

> If we are to use the symbol a to signify an object, we must have a criterion for deciding in all cases whether b is the

same as a, even if it is not always in our power to apply this criterion.

Even if it is not completely clear whether or not Frege thought of ICs as related only to abstract entities, his considerations about ICs seem to adapt both for concrete and abstract objects. He suggests that an IC has the function of providing a general way of answering the following question, with a and b objects in a given domain:

Fregean Question: How can we know whether a is identical to b?

Consider two famous examples of ICs provided by Frege in [2]:

- *IC for directions*: if a and b are lines, then the direction of line a is identical to the direction of line b if and only if a is parallel to b;
- *Hume's principle*: for any concepts F and G, the number of F-things is equal to the number of G-things if and only if there is a one-to-one correspondence between the F-things and the G-things.

In the philosophical literature, the Fregean question has been reformulated in the following ways:

***Ontological Question* (OQ):** If a and b are Ks, what is it for the object a to be identical to b

***Epistemic Question* (EQ):** If a and b are Ks, how can we know that a is the same as b?

***Semantic Question* (SQ):** If a and b are Ks, when do 'a' and 'b' refer to the same object?

The difference between an answer to (EQ) and an answer to (OQ) is not purely formal. When answering (EQ), we think of conditions associated with a procedure for deciding the identity questions concerning objects of some kind K. In answering (OQ), we think of conditions which are meant to provide an ontological analysis of the identity between objects of kind K. Finally, an answer to (SQ) concerns sameness and difference of reference of simple or complex names.

It is worthwhile considering what the logical form of ICs looks like even if different ways of conceiving the form have been proposed. The reason is that there are some requirements that ICs must satisfy to provide acceptable identity conditions, and part of those requirements are formal, i.e. given by their logical form. Among various formulations of IC, we consider the following ones:

$$\forall x \forall y((x \in K \wedge y \in K) \rightarrow (f(x) = f(y) \leftrightarrow R(x, y))), \tag{IC*}$$

and:

$$\forall x \forall y((x \in K \wedge y \in K) \rightarrow (x = y \leftrightarrow R(x, y))), \tag{IC}$$

where R constitutes the identity condition for $f(x)$s or for xs and is a relation holding between objects belonging to some kind K, and, in the (IC*) case, f is a function whose domain is K itself and the range is a set of elements which constitutes a different set, $f(K)$. The intuitive reading of (IC*) is the following: if x and y are K, then x is the same f as y if and only if R holds between x and y [1]. Sometime, (IC*) is formulated in the following way (without a reference to K):

$$\forall x \forall y(x' = y' \leftrightarrow R(x, y)), \tag{IC**}$$

where 'x'' and 'y'' are terms representing entities of the kind K suitably connected with x and y.

For Williamson (IC*), or (IC**), is the logical form of a *two-level identity criterion* (see [9], pp. 145-146). Frege's criterion of identity for directions is an example of a two-level identity criterion:

$$\forall x \forall y(o(x) = o(y) \leftrightarrow P(x, y)) \tag{O}$$

where x and y range over lines, o is a letter for "the direction of" and P for "is parallel to". In (O) the identity sign is flanked by terms constructed with a functional letter, and the right-hand side of the biconditional introduces a relation among entities different from the entities for which the criterion is formulated. On the contrary, the *Axiom of extensionality* for sets:

$$\forall x \forall y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)) \tag{A}$$

is an example of *one-level identity criterion.* In (A) the identity sign is flanked by terms for sets, and the right-hand side states a

[1] Brand has given a different characterization for the *logical form* of ICs in terms of second order modal logic:

$\exists F \forall x \forall y$(if x and y are ϕs then $\Box(x = y \leftrightarrow R(x, y))$)

relation equivalent to identity between sets. In the case of *two-level* ICs the conditions of identity concern objects which are not of the same kind of objects for which the IC is provided. On the contrary, in the case of *one-level* ICs the conditions of identity concern objects which are of the same kind of objects for which the identity criterion is provided.

Williamson ([9], p. 147) points out that

> The idea of a two-level criterion of identity has an obvious advantage. No formula could be more basic (in any relevant sense) than '$x = y$', but some might be more basic than '$ox = oy$', by removing the symbol 'o' and inserting something more basic than it

In such cases one can speak of a reductivist conception of identity criteria because identity among objects of a certain kind depends on relations among more basic objects[2].

In the next section of our paper we limit our analysis to the formal constraints on the relation R in (IC).

1.1 Requirements for R

In this section, some constraints for the relation R are listed and discussed. The relation R is what the identity condition consists of or, put otherwise, given an identity statement $a = b$, R is a relation that holds between a and b, is other than identity and analyzes what it is for the referents of a and b to be identical (See Linnebo in [3], p. 206). How should R look to be a good candidate for being the identity condition of objects of some kind K? To answer this question, we take into account three contributions: Carrara and Giaretta [4], Brand [5] and Lombard [6].

Non-vacuousness The identity condition cannot have parts that are vacuously satisfiable. Consider the following example (see [6], p. 32-33). Let PO be the set of physical objects, S the set of sets,

[2]It is debatable if there is a real distinction between *two-level* and *one-level* ICs. J. Lowe has suggested that a *two-level* IC can be recast as *one-level.* For example (O) can be so reformulated:

$\forall x \forall y((\text{Direction}(x) \wedge \text{Direction}(y)) \rightarrow (x = y \leftrightarrow \exists w \exists z(L(w) \wedge L(z) \wedge Of(x, w) \wedge Of(y, z) \wedge P(w, z))))$

where 'Direction' is 'to be a direction', 'L' 'to be a line', and 'Of' 'to be of' (Lowe discusses *one-level* and *two-level* identity criteria in [13], [14]).

$R(x, y)$ the identity condition for PO and $R'(x, y)$ the identity condition for S:

$$\forall x \forall y(((x \in PO \vee x \in S) \wedge (y \in PO \vee y \in S)) \rightarrow (x = y \leftrightarrow (R(x, y) \vee R'(x, y)))).$$

The condition given above for the identity of x and y is not associated with a kind of entities in a metaphysically interesting sense, since the members of the alleged kind do not share an essence. The identity condition must specify a relation that holds between elements of a certain kind such that all of them are alike with respect to the properties associated to that kind. In such a perspective, the identity condition supplies a property of properties. Lombard calls this property *determinable* since it determines a class of properties, called *determinates*, having that property.

Informativeness R should contribute to specify the nature of the kind K of objects for which R acts as an identity condition. If the role of an IC is to specifie some non-trivial essential properties for objects of kind K the form of the relation cannot be tautological, for instance, it cannot have the following form:

$$R(x, y) \vee \neg R(x, y)$$

Unfortunately, the identity condition does not completely characterize the nature of instances of K: to decide about identity questions concerning a K we need the concept of K, that is not provided by the ICs. The above observation is due to Frege. He argues that in:

> "the direction of line a is identical to the direction of line b" the direction of a plays the part of an object, and our definition affords us a means of recognising this object as the same again, in case it should happen to crop up in some other guise, say as the direction of b. But this means does not provide for all cases... That says nothing as to whether the proposition: the direction of line a is identical to q should be affirmed or denied, except for the one case where q is given in the form of the direction of b (see [2], §66).

In Frege's opinion, the nature of certain objects is entirely clarified only if one can find a way to refer to them such that it would determine the truth-value of any identity sentence concerning the given objects, without any restriction. What do we need to obtain the universal definiteness of identity questions concerning a K? Frege is absolutely clear about this: we need the concept of K ("What we lack is the concept of direction"(Frege [2], §66)).

Partial exclusivity An identity condition for a kind K of objects cannot be so general that it can be applied to other kinds of objects. The example provided by Lombard is the following:
'If x and y are both non-physical objects, x and y are identical iff they have the same individual essence' ([6], p. 36).
Now, the properties falling under the 'large' property 'having an individual essence' do not apply only to non-physical objects and can be part of the identity conditions for many kinds of objects.

Minimality The identity condition for K-objects is required to specify the smallest number of determinables such that the determinates falling under them turn out to be necessary and sufficient to ensure identity between two objects of kind K. The determinables specified in the identity condition cannot be superfluous.

Non-circularity The identity condition for K-objects cannot make use of the concept of K itself, otherwise it is circular. There has been a long debate about the circularity of the IC for events proposed by Davidson (see [7]):

If x and y are events, $x = y$ iff x and y have the same causes and effects.

Since some causes and effects are events, the identity condition for events involves identity between events: in fact, to determine whether two events are the same we are required to determine, first, the identity of events taken as their causes or effects.

Non-tautologicity R cannot be a property that every two objects of kind K share. Formally:

$$R \subset K \times K$$

The formula says that the relation R is a *proper* subset of the set $K \times K$, that is, there is some pair of objects that are K such that the objects of the pair are not in the extension of R.

K**-Maximality** R must be maximal with respect to K. In other words, R is required to be the widest dyadic property that makes an identity condition true. A dyadic property G is wider than a property G' iff for any x and y, if $G'(x, y)$, then $G(x, y)$, but not vice versa. That means that the ordered pairs of G' are a subset of the set of ordered pairs of G. Formally, for all the relations R' that are possible candidates for the identity condition Φ:

$$R' \subseteq R$$

Uniqueness R is unique with respect to K. That means, if there are $R_1, R_2, ...R_n$, such that (i) each R_i satisfies IC and (ii) each R_k is independent of each R_j (that is, every R_k is neither narrower nor wider than each R_j), then at most one of $R_1, R_2, ...R_n$ provides a correct identity criterion for K-objects.

Equivalence R must be an equivalence relation. In the left side of the biconditional in (IC), there is an identity relation, which is an equivalence relation. Consequently, the relation R on the right side of the biconditional must be an equivalence relation, too. In order to be logically adequate, then, an identity criterion is required to exhibit an equivalence relation as identity condition.

In this paper we want to focus on identity criteria which fail to meet the equivalence constraint and show how this problem can be overcome by logical means.

1.2 Failure of transitivity

As has been observed in the philosophical debate about identity criteria, some relations considered as intuitively good candidates for R often fail to be transitive. Consider some examples offered by Williamson [8]:

- Let $x, y, z, ...$ range over color samples and f be the function that maps color samples to perceived colors. A plausible candidate for R might be the relation of indistinguishability. It is easy to verify, though, that such an R is not necessarily transitive: it might happen that x is indistinguishable from y and y from z, but x and z can be perceived different in color.
- If $f(x)$ is a physical magnitude, to determine $f(x) = f(y)$ you measure x and y. If x and y differed by little, the measurement

operation could give the identity of the physical magnitudes as a result. If R were defined on the basis of the measurement operations, it would turn out to be not transitive, since the sum of many little differences is not itself little.

The examples above show how some relations that are intuitively plausible candidates to be identity conditions do not meet the logical constraint of Equivalence that IC demands. However, instead of refusing this kind of plausible but inadequate identity criteria, it has been suggested to approximate the relation R whenever it is not transitive. That means that, given a non-transitive R, we can obtain equivalence relations that approximate R by some operations. Some approaches have been suggested: two of them are due to (Williamson [8], [9]), while a third approach is due to (De Clercq and Horsten [10]).

2 Approximations of identity conditions

2.1 Williamson's approaches

Williamson's suggestion about the best approximation to a non transitive relation consists in giving up the requirement for the identity condition to be both necessary and sufficient. Consider R a non transitive relation that we take to be the best candidate for being R, for some kind of objects $f(x)$s. Consider such an R a constant. Consider then variables on relations $R', R'', ...$ as possible approximations to R. To determine the best approximation R' to R Williamson suggests two constraints that R' must meet:

***Weak constraint*:** no candidate relation R'' should approximate R better than R'.

***Strong constraint*:** R' should approximate R better than any other candidate R''.

Williamson proposes two ways to find an adequate equivalence relation to substitute a non transitive R: an *approach form above* and an *approach from below*.

The approach from above seeks the smallest equivalence relation R^+ such that $R \subseteq R^+$. That means, some $f(x)$ and $f(y)$ that are not identical under R turn out to be identical under R^+ or, equivalently, R^+ is a super-relation of R. The equivalence classes given by R^+ are numerically more than the equivalence classes given by R. R^+ always exists and is unique. The identity criterion of this form

$$\forall x \forall y (f(x) = f(y) \leftrightarrow R^{+}(x, y)) \tag{IC$^+$}$$

provides a sufficient, but not necessary, condition for the identity of $f(x)$s.

The approach from below seeks the largest equivalence relation R^{-} such that $R^{-} \subseteq R$. That means, R^{-} is a sub-relation of R since not all the ordered pairs in R are ordered pairs in R^{-}. R^{-} always exists on the assumption of the Axiom of Choice but it is not unique. To decide which relation can be preferable over others some constraints can be put. One of it is what Williamson calls *Minimality Constraint*. According to it the relation R^{-} to be preferred is the one with the minimum number of equivalence classes. The identity criterion of this form

$$\forall x \forall y (f(x) = f(y) \leftrightarrow R^{-}(x, y)) \tag{IC$^-$}$$

provides a necessary, but not sufficient, condition for the identity of $f(x)$s.

There are cases where a proposed identity condition is necessary for some kind of entities. For instance, the condition of being perceptually indistinguishable is a plausible identity condition for colors. On the contrary, there are other kinds of entities for which a good identity criterion is sufficient: certain forms of mental continuity can be considered as a sufficient condition for personal identity. But this is not so obviously sufficient. There are not always good reasons to consider a condition as obviously necessary or sufficient for the identity of some kinds of entities. There is a third option that is worthy to be considered: to regard the condition as neither necessary nor sufficient for the identity of the $f(x)$s.

2.2 De Clercq and Horsten's approach

De Clercq and Horsten [10] suggest an approach to find approximating relations that is alternative to the one proposed by Williamson and is called *overlapping approach*: the equivalence relation that is sought partially overlaps R, instead of being a sub- or a super-relation with respect to R.

The advantages of such an approach are (i) it can be used for cases where the most plausible identity condition is neither sufficient not necessary and (ii) it can generate closer approximations than Williamson's approach.

The proposal is based on the assumption that R is not indeterminate: any two objects either stand in the relation R or they do not. This assumption serves the scope to avoid difficulties that are not necessary to face, but it can be given up in case of a refinement of the approach.

The authors propose to define an equivalence relation $R^{\pm}$ that closely approximates R and achieves that task better that R^{+} or R^{-}. For the sake of clarity, consider an example.

Given a function f, let the domain of objects for f be the following:

$$\mathcal{D} = \{a, b, c, d, e\}$$

Assume there is a candidate relation R, reflexive and symmetric, for the identity condition for $f(x)s$. When R holds between two objects x and y we denote this as $\overline{xy}$ (as De Clercq and Horsten do). Put otherwise, $\overline{xy}$ means $R(x, y)$ and $R(y, x)$. Let R on $\mathcal{D}$ be the following:

$$R = \{\overline{ac}, \overline{ad}, \overline{bc}, \overline{bd}, \overline{cd}, \overline{de}\}$$

R is not an equivalence relation. In fact, it fails to be transitive. For instance, R holds between a and d and between d and e, but it does not hold between a and e.
Consider now how R^{+} looks like in this case. It is unique and it is the smallest equivalence relation that is a superset of R, that is:

$$R = \{\overline{ab}, \overline{ac}, \overline{ad}, \overline{ae}, \overline{bc}, \overline{bd}, \overline{be}, \overline{cd}, \overline{ce}, \overline{de}\}$$

On the contrary, R^{-} is not unique. For instance, one of the largest equivalence relations included in R is the following:

$$R^{-} = \{\overline{bc}, \overline{bd}, \overline{cd}\}$$

To determine whether R^{+} or R^{-} is the best approximation to R, first you measure the degree of unfaithfulness of R^{+} and R^{-} with respect to R. Such a degree is the number of revisions you make to get R^{+}, R^{-} from R. A revision is any adding or removing of an ordered pair to or from R. In the example considered above, R^{+} is obtained by adding four ordered pairs to R and R^{-} by removing three ordered pairs. The degree of unfaithfulness of R^{+} is 4 and the degree of R^{-} is 3. Thus, R^{-} is closer to R than R^{+}. That means, with R^{-} you stay closer to your intuitive identity condition R because R^{-} modifies R less than R^{+} does.

Consider now the following equivalence relation:

$$R^{\pm} = \{\overline{ab}, \overline{ac}, \overline{ad}, \overline{bc}, \overline{bd}, \overline{cd}\}$$

With respect to R, $R^{\pm}$ adds one ordered pair and takes off another one. So the degree of unfaithfulness of $R^{\pm}$ is 2, that is, less than both R^{+} and R^{-}. Formally, the *degree of unfaithfulness* (DOU) is given by the symmetric difference $\triangle$:

$$(R^{\pm}, R) = |R \triangle R^{\pm}| \tag{DOU}$$

How does $R^{\pm}$ looks like? It is an overlapping relation with respect to R and it is a kind of hybrid relation between R^{+} and R^{-}, since it both adds and removes one ordered pair. An overlapping relation can be closer to R than the relations obtained with the approach from below and from above.

3 Contexts and Levels of Granularity

Let us consider and revise the example about phenomenal colors given by Williamson. The case of colors is a well-known example of failure of transitivity and it has been discussed also in other places in the philosophical literature. Some observations by Hardin [11] on this issue are remarkable.

Hardin observes that many philosophers endorse a view according to which the following principle (that I will call **NT** from now on) holds:

> There exist triples of phenomenal colors x, y and z, such that x is indiscriminable from y and y is indiscriminable from z, but x is discriminable from z.

By 'indiscriminability between colors' Hardin means 'perceptual indistinguishability'. By **NT** the relation of perceptual indistinguishability fails to be transitive; therefore, the IC for colors based on the relation of indistinguishability is incoherent: on the left side of the biconditional there is a necessarily equivalent relation (identity), on the right side a not necessarily equivalent relation. This problem seems to affect any semantic account of color terms relying on everyday uses of color predicates. Hardin argues that phenomenal colors are themselves indeterminate, that is, there is no sharp color-discrimination threshold;

since the truth of **NT** is based on the assumption that there is a discrimination threshold, refusing this assumption implies refusing **NT** too.

In non problematic cases, that is, when we have to make judgments on very different colors, we report our observations using coarse-grained predicates because we do not need to express shade differences. When we have to deal with borderline cases of colors, we tend to be more precise in using color predicates. More fine-grained color predicates are used in color science and technology, but in everyday life people do not use them; that is not just because there are limits of hue discriminability, but because of "something like the limit of useful naming of phenomenal hues for the purposes of communicating between people" ([11], p. 221). Put otherwise, the number of possible color discriminations is much higher than the number of color terms normally used. Why? First of all, there is a variability in discrimination between observers; second, people observe colors under normal conditions such as changing light, contrast, shadows, and not under standard conditions, and normal conditions make color comparisons problematic; third, it is more difficult to compare a color with a mental standard (like the standard of 'red' that one could have seen in the Munsell Chart) than with another color perceived at the same time.

So, color perception is influenced by many factors and the use of color predicates is somewhat sloppy. Hardin suggests that to answer a question like "What are the boundaries of red?" we must first

> specify, explicitly or tacitly, a context and a level of precision and [...] realize the margin of error or indeterminacy which that context and level carry with them. ([11], p. 230.)

In the following analysis, we wish to show that De Clercq and Horsten's framework can be improved if you consider the use of ICs in a *context* and in a *level of precision*. Moreover, we agree with Hardin's belief that the nature of our purposes imposes limits on the precision of our utterances: a too large set of color predicates would make our judgments more precise, but would also hinder a profitable communication among agents.

The IC for phenomenal colors is an example of an IC that has mostly an epistemic function: we do not know precisely whether two colors are identical. We only rely on our perception which is fallible. So, we express an IC for colors in a logically inadequate way.

Williamson and De Clercq and Horsten believe that there are logically adequate ICs and try to capture them by approximating our intuitively good, but logically inadequate ICs.

Consider now the following variations of the example of IC for perceived colors:

1. You see two monochromatic spots, A and B, and you do not detect any difference with respect to their color. Following Williamson, you claim that they have the same color, because they are perceptually indistinguishable. Now, suppose you add two further monochromatic spots, C and D, such that they are perceptually distinguishable. However, A is indistinguishable from C and B from D. In such a scenario, you can accept to revise your previous judgement and say that A and B are distinct.
2. You see two color samples A and B from a distant point of view such that you are not able to distinguish A-color from B-color. You say that A and B have the same color. Now, you get closer to them and detect a difference between them. So, you revise your previous judgement and say that A and B are distinct.
3. You see two monochromatic spots again, A and B. You perceive them as equally, say, orange. Nevertheless a friend of yours, who is a painter, tells you that she perceives them actually different: B is more yellowish than A. According to her color perception, which is more refined than yours, there are more differences among color samples than you detect.

Example 1 shows how our perception of colors can be different, depending on the range of colors we see at the same moment. Better said, comparing a color sample with one or more color samples makes our judgements about colors differ. Thus, a relation R expressed by a criterion of identity can vary across contexts of judgment. For instance, consider a domain $\mathcal{D} = \{a, b, c, d, e\}$ and a context o, that is a subset of $\mathcal{D}$: $o = \{a, b\}$. Suppose $R = \{\overline{ab}\}$ in the context o. Consider now an enlarged context, o' containing a and b plus other elements, c and d: $o' = \{a, b, c, d\}$. In o' you may have the following R-pairs: $\overline{ac}, \overline{ad}$, but not $\overline{ab}$.

Example 2 and 3 present a different issue than 1. Given the same context, R varies along different granular levels of observation. When you are distant from the objects for which you have to make an identity statement, you are looking at them from a coarse point of view. Anyway, you make an identity statement. Getting closer to the elements of the context, you reach a more fine-grained observational level and so you can make a different identity statement. The point of view of the painter can be seen as well as a fine-grained observational level. In short, you can look at the elements of a context under different standards of precision, each of them corresponding to a granular level of observation. The finer the level is, the more differences between the individuals are detected.

In the following paragraph we try to formalise the notions of contexts and granular levels and integrate them with De Clercq and Horsten's formal treatment of approximate relations.

3.1 Granular Models

Let $\mathcal{L}$ be a formal language through which we can represent English expressions. $\mathcal{L}$ consists of:

- individual constant symbols: $\overline{a}, \overline{b}, ...$ (there is a constant symbol for each element of the domain);
- individual variables: $x_0, x_1, x_2, ...$ (countably many);
- 2-arity predicate symbols $P_1, P_2, ...$;
- usual logical connectives with identity, quantifiers.

The set of terms consists of individual constant and individual variable symbols.
Formulas are defined as follows:

1. If t_1, t_2 are terms, then $P_1(t_1, t_2), P_2(t_1, t_2), ...$ are formulas;
2. If t_1, t_2 are terms, then $t_1 = t_2$ is a formula;
3. If ϕ, ψ are formulas, then $\phi \Box \psi$ is a formula, where $\Box$ is one of the usual logical connectives;
4. If ϕ is a formula, then $\neg\phi$ is a formula;
5. If ϕ is a formula, then $\forall x_i \phi, \exists x_i \phi$ are formulas.

Let us give now an interpretation to $\mathcal{L}$. Let $\mathcal{D}$ be a fixed non empty domain of objects. We define a context o as a subset of domain $\mathcal{D}$. So, the set of all contexts O in $\mathcal{D}$ is the powerset of $\mathcal{D}$:

Definition 1 $O = \wp(\mathcal{D})$.

We interpret, then, R as a binary relation, which is reflexive and symmetric, but not necessarily transitive. Moreover, R is a primitive relation and pairs the elements that are indistinguishable according to the identity condition it represents. For instance, in the case of color samples R gives rise to a set of ordered pairs, each of them consisting of elements that are indistinguishable with regard to their (perceived) color. R is then the relation that makes identity statements about the elements of the domain possible, according to IC.

Let $\mathcal{M} = \langle \mathcal{D}, R \rangle$ be a granular structure. Put otherwise, $\mathcal{M}$ is a structure consisting of the domain $\mathcal{D}$, together with all the contexts in $\mathcal{D}$, and a binary relation R (a two-arity predicate).

To account for example 1 formulated above, we postulate that R varies across contexts. Before providing a formal definition, let us consider a further example. Given a certain domain $\mathcal{D}$, let us isolate three subsets of it, i.e. three contexts:

- $o_1 = \{a, b\}$
- $o_2 = \{a, b, c\}$
- $o_3 = \{a, b, c, d\}$

Observe that some elements, namely a and b, are in all the contexts, while c is in two of them. Consider now a granular structure $\mathcal{M}_1 = \langle \mathcal{D}, R \rangle$ and assume it not be a very fine structure; suppose that R come out with the following sets - each of them corresponding to one context:

- $R_{o_1}^{\mathcal{M}_1} = \{\overline{ab}\}$
- $R_{o_2}^{\mathcal{M}_1} = \{\overline{ab}\}$
- $R_{o_3}^{\mathcal{M}_1} = \{\overline{ac}, \overline{bd}\}$

The relation R in the granular structure $\mathcal{M}_1$ holds between a and b in contexts o_1 and o_2, but not in context o_3. This means that, given a certain granular structure M_i, R can vary across contexts. Formally:

Definition 2 *Given a granular structure $\mathcal{M}_i$ and given two contexts o_l and o_k, R varies across o_l and o_k iff there is a non empty intersection $o^* = o_l \cap o_k \neq \emptyset$ such that $\exists x \in o^* \exists y \in o^* : (\overline{xy} \in R_{o_l}^{\mathcal{M}_i} \wedge \overline{xy} \notin R_{o_k}^{\mathcal{M}_i}) \vee (\overline{xy} \in R_{o_k}^{\mathcal{M}_i} \wedge \overline{xy} \notin R_{o_l}^{\mathcal{M}_i})$.*

If in a granular structure $\mathcal{M}_i$ the relation R fails to be transitive with respect to some (if not all) contexts $o \subseteq O$, then the formal framework given by De Clercq and Horsten is applied. That means, for each R in each context o an equivalence overlapping relation $R^{\pm}$ can be defined[3]. If a relation R is transitive in a context o, then in that case $R^{\pm}$ coincides with the given R. In contexts where R is not transitive, $R^{\pm}$ denote a relation that differs from R in the fact that it adds and/or remove some ordered pairs from R, as described by De Clercq and Horsten.

R does not vary only across contexts. As the examples 2 and 3 above show, R also varies across granular levels*Granularity.* While the notion of context refers to the number of elements considered and can be extensionally characterized, as proposed, we characterize the epistemic notion of granular level in an indirect way. Each granular structure belongs to a certain granular level, which corresponds to the level of precision of R in ordinating elements in contexts. Put otherwise, given the same context $o \subseteq O$, different granular structures can give different sets of ordered pairs generated by R with resect to o. If the relation R of a certain granular structure holds among all the elements of the context considered, no difference is detected among them (with respect to some property), so all of them are considered indistinguishable. The granular structure is then considered coarse-grained. On the contrary, a more fine-grained granular structure shall have a relation R holding between a less number of elements of o.

Consider a further example. Fix the context $o_2 = \{a, b, c\}$ as above. The relation R in the granular structure $\mathcal{M}_1$ only holds between a and b. Consider now another granular structure, $\mathcal{M}_2 = \{\mathcal{D}, R\}$[4]. The relation R in $\mathcal{M}_2$ does not hold between any elements in the context o_2, and so neither between a and b. This means that the granular structure $\mathcal{M}_2$ is more fine-grained, since it is able to detect more differences among elements in contexts.

To determine whether two or more granular structures belong to different granular levels you apply the following definition:

Definition 3 *Given a context o_i and two granular structures $\mathcal{M}_l$ and $\mathcal{M}_k$, $\mathcal{M}_l$ and $\mathcal{M}_k$ belong to different granular levels iff $\exists x \in o_i \exists y \in o_i : (\overline{xy} \in R_{o_i}^{\mathcal{M}_l} \wedge \overline{xy} \notin R_{o_i}^{\mathcal{M}_k}) \vee (\overline{xy} \in R_{o_i}^{\mathcal{M}_k} \wedge \overline{xy} \notin R_{o_i}^{\mathcal{M}_l})$.*

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[3]If you prefer to maintain Williamson's approaches, instead of $R^{\pm}$ you can get R^{+} or R^{-}.

[4]Note that the domain $\mathcal{D}$ remains fixed in all the granular structures, and so the set of contexts O. The relation R also is the same - for example, perceptual indiscriminability - but its interpretation can differ along the grain size of the structure, as we see in the example.

Finally, let us define the relation *to be at least as fine as* between granular structures. First, we define the relation, formally: $\leq^c$, between cardinality of sets:

Definition 4 *Given an $o \in O$, for all the pairs $\overline{xy}$ in M and $\overline{xy}$ in M', $|\{\overline{xy}^{M'}\}| \leq^c |\{\overline{xy}^{M}\}|$ iff the number of $\overline{xy}^{M'}$ is less than or equal to the number of $\overline{xy}^{M}$ in o.*

Then, we define the relation between granular structures with respect to some context $o \in O$:

Definition 5 *Given a context $o \in O$, M'* is at least as fine as *than M iff the number of $\overline{xy}^{M'}$ is less than the number of $\overline{xy}^{M}$, that is:*

$$M' \leq^* M \textit{ iff } |\{\overline{xy}^{M'}\}| \leq^c |\{\overline{xy}^{M}\}|.$$

Example: let $o = \{a, b, c, d, e\}$ a given context. Consider two granular structures, $\mathcal{M}_1 = \langle \mathcal{D}, R \rangle$, $\mathcal{M}_2 = \langle \mathcal{D}, R \rangle$. According to $\mathcal{M}_1$, we have: $R_o = \{\overline{ab}, \overline{bc}, \overline{de}\}$. It is not transitive ($a$ is indistiguishable from b and b from c, but a is not indistinguishable from c). The best overlapping approximations is the following: $R^{\pm} = \{\overline{ab}, \overline{bc}, \overline{ac}\}$. The pair $\overline{ac}$ has been added. The degree of unfaithfulness of $R^{\pm}$ is 1. According to $\mathcal{M}_2$, we have: $R_o = \{\overline{ab}, \overline{bc}, \overline{cd}, \overline{de}, \overline{ce}\}$. In this case R it is not transitive either. The best overlapping approximation removes the pairs $\overline{ab}, \overline{bc}$ and it is the following: $R^{\pm} = \{\overline{cd}, \overline{de}, \overline{ce}\}$. According to definitions 4 and 5 and given the context o, $\mathcal{M}_1$ is finer than $\mathcal{M}_2$ because its relation R gives a less number of pairs than the relation R in $\mathcal{M}_2$.

3.2 Objections and replies

Some objections can be raised against the proposed formal characterization of ICs , as well as some problems in the account are to be underlined. We try to outline here some objections and problems, and sketch a reply to them.

- ICs are usually associated with sortal concepts, that is, with concepts that answer the question "What is x?". The examples of non-transitive ICs considered are associated to kinds of objects like colors and physical magnitudes. It is not clear, though, whether *colors* or *physical magnitudes* are to be considered sortal concepts. For instance, the adjective 'red' does not correspond to a sortal concept: we do not individuate an object x saying "x is a red".

This first objection seems to attack the notion of IC itself or, better said, the thesis that ICs are necessarily associated with sortal concepts. We accepted the standard thesis according to which only concepts associated with ICs are sortals. Being associated with an IC is a necessary condition for concepts to be sortals, but not a sufficient one. The possibility for some concepts to be associated with ICs without being sortals is not excluded.

Moreover, what happens if we consider 'red' as a substantive standing for the color red, e.g. "Red suits you"? In this case, 'red' can be considered as a sortal noun and, therefore, it would be easy to accommodate the problem via a revision of the formulation of the IC. We can formulate a one-level IC for colors as follows: given two perceived colors x and y, x is identical to y iff x is indistinguishable from y.

- A second objection runs as follows: what changes from context to context or from granular level to granular level is the extension of the relation. But we are also dealing with epistemic issues: In a certain context and granular level we make an identity judgment according to a certain relation R. When the context or the granular level changes we make a sort of revision of our previous identity judgment. So, if we want to be faithful to our intuitions and account for epistemic issues, an intensional treatment is more appropriate.

  We decided to provide an extensional model following Williamson's and De Clercq and Horsten's approaches. However, this second objection is very important. An intensional treatment of ICs would be interesting to be provided especially if you consider not only the ontological function of ICs, but also the epistemic one. If the goal is to model how we know and use ICs, we should think of an intensional formal framework. That is a possible further development of the account.

- The proposed model for accounting for ICs is not suitable for an infinite domain. The domain of objects must be finite. The applicability of the model is then reduced to some specific cases, while it should be generalized.

  As has already mentioned, the model for approximating ICs has been developed to face logical problems arising from the intuitive use of ICs for everyday problems (color comparisons and the like). De Clercq and Horsten too are aware of the problem that their approach is applicable only to finite domains. However, they attempt to accommodate the problem and suggest reducing infinite graphs to finite graphs. In a nutshell, it is worthwhile considering infinite graphs because we deal with relations that are potentially infinite, for instance the relations underlying the Sorites paradox. However, the transitivity failure of some relations is here at concern. Such a problem is shown by finite graphs, so there is nothing bad to represent the problem and the solution only using finite graphs. Moreover, it is rare that people in ordinary life make inferences with a great (even infinite) number of steps. Since we are dealing with ICs as they are commonly used by people and not by logicians, the infinity issue does not play a relevant role in the treatment of ICs.

- Consider the following problem: If ICs have the function of answering questions (EQ), (OQ), and (SQ), which of those questions is answered by an intuitive IC that contains a non-transitive relation R? Moreover, does an IC with an approximated relation like $R^{\pm}$ answer the same question or a different one?

  It seems plausible to claim that an IC containing a non-transitive relation R answers (EQ). Consider the IC for phenomenal colors: as we have seen, we do not know precisely whether two perceived colors are identical. We only rely on our perception, which is fallible. Therefore, the IC for colors we express is not logically adequate, but is sufficient for our pragmatic or epistemic purposes of color comparison.

  Which question does an IC containing an approximated relation such as $R^{\pm}$ answer? The relation $R^{\pm}$ is logically adequate; therefore, thank to it we can determine whether or not two items are actually identical in reality. So, it is plausible to think that an IC containing an approximating relation answers (OQ).

4 Conclusion

ICs are very often matter of philosophical discussion. However, the formal requirements that they must meet to be acceptable are rarely taken into account. In this paper we listed some formal requirements and focused on some ICs that fail to meet one of them: transitivity.

Instead of giving up ICs failing to meet the transitivity requirement, we considered the approaches proposed by Williamson and De Clercq and Horsten, by means of which transitive approximations to non-transitive ICs are defined.

Our purpose has been to improve De Clercq and Horsten's formal framework. Given a non transitive relation R, standing for an identity condition for some objects, we suggest fixing a context and a granular level of observation (a granular structure) *Granularity.* We allow R varying across *contexts* and *granular levels.* If in a context and according to some level R fails to be transitive, you can apply De Clercq and Horsten's approach and build the closest approximation to R for that context and that level.

By the framework developed in this paper, we wish to have been able to make a short step towards a formal account of identity criteria.

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