The maximum line length problem

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Abstract

We consider a queueing system, which is constituted by a set of $M/M/1$ (sub–)systems, sharing the same scarce resources, but otherwise running independently. We analyse the nonlinear programming problem of minimizing the expectation of the maximum line length among the subsystems, with the service rates as the decision variables. Furthermore, we introduce three different nonlinear programming problems, which have natural interpretations with reference to the same queueing system and whose optimal solutions are useful to solve the original problem.

Key Words: Optimization, queues, nonlinear programming, resource allocation.

1 Introduction

We consider a system constituted by several facilities which purvey different kinds of service (or the same service at different locations) to different customers streams. An example is offered by a health service system which operates through different service units which are distributed in a definite region. The demand of service at the different service units is known and we consider the problem of choosing the different service capacities in order to purvey an overall good quality service while satisfying some resource constraints.

This is a typical problem of optimal design of a queueing system as classified in [1] and in particular a static problem, because we are interested in choosing

Supported by the Italian Ministry of University and Research.
Published in JORBEL (The Belgian Journal of Operations Research, Statistics and Computer Science), 37, 1997, 23-39.

constant service rates of the different facilities which are optimal in some sense in equilibrium.

A natural performance indicator of the multiservice system is the maximum line length, i.e. the maximum among the numbers of customers found in all the facilities (either in queue or being served). Therefore we are led to formulate the problem of minimizing the expectation of the maximum line length (the MLL problem), assuming that the different services/facilities require different amounts of the same economic resources, which are available to the complex system in finite quantities.

It is rather intuitive that the different problem of minimizing the maximum among the expectations of the line lengths is strictly related to the MLL problem. In fact the latter problem has been studied in [8], where it has been named the LLP problem, and has optimal solutions which are easy to compute by an exact algorithm. On the contrary, the search for an optimal solution to the MLL problem presents some computational difficulties.

A third problem, that of minimizing the sum of the expectations of the line lengths (the $SELL$ problem), or, equivalently, the problem of minimizing the average expected line length among the different facilities, is related to the MLL problem and has a unique optimal solution which can be computed easily. The relation between the MLL and $SELL$ problems is still intuitive and it is also suggested by an approximation of the objective function of the MLL problem.

Finally, a fourth problem, that of maximizing the idle system probability, (the ISP problem), is related to the MLL problem. This is less intuitive and is suggested, like in the case of the $SELL$ problem, by an approximation of the objective function of the MLL problem. Also the ISP problem has a unique optimal solution which is easy to compute numerically.

In the next Section we give a formal definition of the multiservice system and in Section 3 we introduce and discuss the Maximum Line Length problem. In Section 4 we consider the three problems LLP, $SELL$ and ISP, which are related with MLL in different ways. In Section 5 we suggest using Rosen's gradient projection method in order to solve numerically the linearly constrained MLL problem. Finally, In Section 6 we present and discuss some results concerning numerical experiments on a few significant instances of the problem and evaluate the optimal solutions of the problems LLP, $SELL$ and ISP as approximations of optimal solutions of the MLL problem.

2 The m–service system

Let us consider m independent facilities which operate in a unique system and offer m different kinds of service to m independent streams of customers. The ith facility and its customers stream constitute an $M/M/1$ queueing system with arrival rate $\lambda_i > 0$ and service rate $\mu_i \geq 0$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)'$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)'$; λ is considered an exogenous parameter, whereas μ is a design parameter. Let $N_i = N_i(\mu_i)$ be the state of the ith facility, defined as the line length, i.e. the number of customers at that facility, whether being served or waiting for service. We know that, if $\mu > \lambda$ (i.e. if $\mu_i > \lambda_i, \quad 1 \leq i \leq m$), then the state variables $N_1, N_2, \ldots, N_m$ have an equilibrium distribution.

We assume that the services offered by the m different utilities require different amounts of several scarce resources. Let there exist $r \geq 1$ goods (resources) and let a quantity b_j of the jth resource be available per time unit. Running the service system at the vector rate μ requires a quantity $g_j(\mu)$ of the jth resource, $1 \leq j \leq r$. Thus we have to consider a budget constraint $g(\mu) \leq b$, where $g = (g_1, g_2, \ldots, g_r)'$, $b \in \mathbb{R}^r$ and $g(\mu)$ satisfies the following conditions:

- i) $g(\lambda) < b$;
- ii) g is continuous;
- iii) for each $j \in \{1, 2, \ldots, r\}$, g_j is a monotonically non–decreasing function and

  $(\mu^1 \leq \mu^2, \quad \mu^1 \neq \mu^2) \quad \Rightarrow \quad (g(\mu^1) \leq g(\mu^2), \quad g(\mu^1) \neq g(\mu^2))$;
- iv) g exceeds b with respect to every component of μ, in the sense that for all $i \in \{1, 2, \ldots, m\}$, there exists $j \in \{1, 2, \ldots, r\}$ such that

$$\lim_{\mu_i \to +\infty} g_j(\mu) > b_j, \quad \text{for all } \mu_n \geq \lambda_n, \ \ n \neq i.$$

3 The Maximum Line Length problem

We want to address the problem of minimizing the expectation of the maximum line length in the multiservice system. The motivation for it is that long queues may offer a poor image of the overall service system to possible customers. Moreover long queues may constitute an additional cost to the service system.

For all $\mu \in \mathbb{R}^m, \ \mu > \lambda$, let us define the random variable

$$M_\mu = \max_i N_i(\mu_i), \tag{3.1}$$

i.e. the maximum among the line lengths of the m subsystems, and let us consider the problem

$$\begin{aligned} MLL: \ \text{minimize} \quad & f(\mu) \ = \ \mathrm{E}(M_\mu), \\ \text{subject to} \quad & g(\mu) \leq b, \\ & \mu > \lambda, \end{aligned} \tag{3.2}$$

where λ and b are exogenous parameters, with $\lambda > 0$ and $g(\lambda) < b$. We notice that the feasible set of MLL is a nonempty and bounded set. For any fixed $\mu > \lambda$, the distribution function of M_μ is given by

$$\begin{aligned} F(x) &= \Pr\{M_\mu \le x\} = \textstyle\prod_{i=1}^m \Pr\{N_i \le x\} \\ &= \textstyle\prod_{i=1}^m \left(1 - \frac{\lambda_i^{x+1}}{\mu_i^{x+1}}\right), \quad x = 0, 1, \ldots \end{aligned} \tag{3.3}$$

Hence, from a known characteristic of the expectation of positive random variables (see [2], pp.265–266), we obtain that

$$f(\mu) = \sum_{n=0}^{\infty} \Big[1 - \prod_{i=1}^{m} \Big(1 - \frac{\lambda_i^{n+1}}{\mu_i^{n+1}}\Big)\Big]. \tag{3.4}$$

After observing that the function μ_i^{-n-1} is strictly decreasing and strictly convex, for all $n \ge 0$, we obtain from formula (3.4) that the expectation $\mathrm{E}(M_\mu) = f(\mu)$ is a strictly decreasing and strictly convex function of μ_i, for all $i \in \{1, \ldots, m\}$. Nevertheless, we are not able to prove that $f(\mu)$ is convex, nor that it is not convex. A different representation of $f(\mu)$, as a finite sum which we call the subset enumeration form, is the following:

$$f(\mu) = \sum_{\emptyset \neq J \subseteq \{1,2,\ldots,m\}} \frac{(-1)^{|J|+1} \prod_{j\in J} \lambda_j}{(\prod_{j\in J} \mu_j - \prod_{j \in J} \lambda_j)}. \tag{3.5}$$

We verify the formula (3.5) in the Appendix. From (3.5) it is clear that $f(\mu)$ is a continuously differentiable function of μ in the feasible set and its partial derivative with respect to $\mu_t,\ t = 1, 2, \ldots, m$, is

$$\frac{\partial f(\mu)}{\partial \mu_t} = \sum_{\{t\} \subseteq J \subseteq \{1,2,\ldots,m\}} \frac{(-1)^{|J|} \mu_t^{-1} \prod_{j\in J} \lambda_j \mu_j}{(\prod_{j\in J} \mu_j - \prod_{j\in J} \lambda_j)^2}. \tag{3.6}$$

As $f(\mu)$ is a continuous and monotonic function, we have that problem MLL admits an optimal solution $\tilde{\mu}$, at which some component of the generalized budget constraint is active: $g_j(\tilde{\mu}) = b_j$ for some j.

In fact, from (3.4) we obtain the following inequalities:

$$f(\mu) \ge \sum_{n=0}^{\infty} \frac{\lambda_i^{n+1}}{\mu_i^{n+1}} = \frac{\lambda_i}{(\mu_i - \lambda_i)}, \quad \text{for all } i \in \{1, 2, \ldots, m\}. \tag{3.7}$$

Now, let us choose a feasible solution $\overline{\mu}$ of MLL: from the inequalities (3.7) it follows that there exists $\eta > 0$, such that

$$f(\mu) > f(\overline{\mu}), \quad \text{for all } \mu > \lambda \text{ such that } \min_j (\mu_j - \lambda_j) < \eta. \tag{3.8}$$

A suitable choice for η is, e.g.:

$$\eta \;=\; \min_j \lambda_j / f(\overline{\mu}). \tag{3.9}$$

Therefore, we may replace the strict inequality $\mu > \lambda$, in the statement of the constraints of problem MLL, with the inequality

$$\mu \;\geq\; \lambda + \eta e, \tag{3.10}$$

where $e = (1, 1, \ldots, 1)$. The resulting restricted feasible set is a compact set. Moreover, for each feasible solution μ which does not satisfy (3.10) there exists a feasible solution μ' which satisfies it and is better than μ: $\mu' \geq \lambda + \eta e$ and $f(\mu') < f(\mu)$. We observe that the continuity of the objective function and the compactness of the restricted feasible set guarantee the existence of an optimal solution to problem MLL. Finally, an optimal solution cannot be an internal point of the feasible set in view of the strict monotonicity of the objective function.

In the case that g is continuously differentiable and that an optimal solution $\tilde{\mu}$ satisfies the constraint qualification conditions (see [5], p.177), $\tilde{\mu}$ must satisfy the Kuhn–Tucker conditions:

$$\begin{gathered} \nabla f(\mu) + u \nabla g(\mu) = 0, \\ g(\mu) \leq b, \quad u(g(\mu) - b) = 0, \\ \mu > \lambda, \quad u \geq 0. \end{gathered} \tag{3.11}$$

In view of the form (3.5) of $f(\mu)$, we may also consider the MLL problem as a special (multi–ratio) fractional programming problem (see [7]). Nevertheless, it seems that no standard fractional programming method can be used to solve MLL.

4 Different problems related with MLL

The analysis of the features of the MLL problem in the previous Section suggests using the subset enumeration form (3.5) of the objective function and the corresponding form (3.6) of its derivatives in order to search for candidate solutions for optimality, using the Kuhn–Tucker conditions (3.11). Of course this is a reasonable suggestion only if m is small, because the numbers of terms in the formulas (3.5) and (3.6) are exponential functions of m. Therefore it is important in practice to find either significant approximations of the MLL problem, or approximate methods to solve MLL. Here we present some approximations of the MLL problem and, in the next Section, we will discuss their role in the initialization of numerical methods to solve the MLL problem. On the other hand, approximate methods to solve MLL are discussed in [6].

4.1 Maximizing the idle system probability

Using the first term in the series of positive terms (3.4), to approximate $f(\mu)$, transforms problem MLL into the problem of minimizing a special lower bound of $f(\mu)$. Moreover, after noticing that

$$1-\prod_{i=1}^{m}\left(1-\frac{\lambda_i}{\mu_i}\right) \;=\; 1-\prod_{i=1}^{m}\Pr\{N_i=0\} \;=\; 1-\Pr\{\text{idle system}\}, \tag{4.1}$$

we have that the latter (approximate) problem is equivalent to the problem of maximizing the "idle system" probability:

$$\begin{array}{rll} ISP: & \text{maximize} & \prod_{i=1}^{m}\left(1-\frac{\lambda_i}{\mu_i}\right), \\ & \text{subject to} & g(\mu)\le b, \\ & & \mu>\lambda. \end{array} \tag{4.2}$$

Now, the ISP objective function $\Pr\{\text{idle system}\}$ is a strictly increasing and strictly concave function of μ_i, for all $i \in \{1,\ldots,m\}$. A reasoning similar and symmetric to that used for the MLL problem allows us to state that ISP has an optimal solution which is on the boundary of the feasible set. Moreover, taking the logarithm of $\Pr\{\text{idle system}\}$ we obtain the following problem:

$$\begin{array}{rll} LISP: & \text{maximize} & \phi(\mu) \;=\; \sum_{i=1}^{m}\ln\left(1-\frac{\lambda_i}{\mu_i}\right), \\ & \text{subject to} & g(\mu)\le b, \\ & & \mu>\lambda. \end{array} \tag{4.3}$$

Problem $LISP$ is equivalent to ISP and, furthermore, it has a concave objective function. Therefore, if the components of $g(\mu)$ are convex functions, problem ISP, like $LISP$, has a unique optimal solution.

4.2 Minimizing the sum of expected line lengths

In this Section we consider using the terms associated with the singleton sets of components of μ in the finite representation (3.5) of $f(\mu)$, in order to approximate the objective function of MLL. We obtain the following approximation of $f(\mu)$:

$$\begin{aligned} \psi(\mu) \;&=\; \sum_{i=1}^{m}\frac{\lambda_i}{(\mu_i-\lambda_i)} \;=\; \sum_{i=1}^{m}L_i(\mu_i) \\ &=\; \sum_{i=1}^{m}\mathrm{E}(N_i(\mu_i)). \end{aligned} \tag{4.4}$$

The function $\psi(\mu)$ is the sum of the expected line lengths of the m subsystems, which constitute the m–service system. Using the approximation $\psi(\mu)$ of the objective function $f(\mu)$ transforms the MLL problem into the problem of minimizing the sum of expected line lengths:

$$\begin{array}{rll} SELL: \text{ minimize} & \psi(\mu) & = \sum_{i=1}^{m} L_i(\mu_i), \\ \text{subject to} & & g(\mu) \le b, \\ & & \mu > \lambda, \end{array} \tag{4.5}$$

We notice that $\psi(\mu)$ is a strictly decreasing and strictly convex function. Moreover, a reasoning similar to that used for the MLL problem allows us to state that $SELL$ has an optimal solution, which is on the boundary of the feasible set. Furthermore, if the components of $g(\mu)$ are convex functions, then the optimal solution is unique.

It is interesting to notice that $SELL$ is equivalent to the problem of minimizing the average expected line length of the multiservice system,

$$\frac{1}{m}\sum_{i=1}^{m} \mathrm{E}(N_i(\mu_i)) \;=\; \frac{1}{m}\psi(\mu), \tag{4.6}$$

and that the function $\psi(\mu)/m$ is a lower bound of $f(\mu)$, because of the inequalities (3.7). Therefore, also the $SELL$ problem, like ISP, is equivalent to a problem of minimizing a special lower bound of $f(\mu)$.

4.3 The line length problem

A last problem related with MLL is that of minimizing the maximum among the expected line lengths of the m utilities, subject to the same constraints of MLL. This problem, named LLP, is a special case of a family of problems which have been studied and solved in [8] and is formulated as follows:

$$\begin{array}{rl} LLP: \text{ minimize} & \max_i L_i(\mu_i), \\ \text{subject to} & g(\mu) \le b, \\ & \mu > \lambda. \end{array} \tag{4.7}$$

The optimal value of the objective function of LLP is the maximum, z^*, among the solutions of the equations

$$g_j((1 + 1/z)\lambda) = b_j, \quad 1 \le j \le r, \tag{4.8}$$

and an optimal solution is the vector $\mu^* = (1 + 1/z^*)\lambda$.

We know, from the Jensen's inequality (see [3], p.153), that

$$f(\mu) \;=\; \mathrm{E}(M_\mu) \;\ge\; \max_i \mathrm{E}(N_i(\mu_i)) \;=\; \max_i L_i(\mu_i), \tag{4.9}$$

because $\max\{x_1, x_2, \ldots, x_m\}$ is a convex function of $(x_1, x_2, \ldots, x_m)$. Therefore, also LLP, like ISP and $SELL$, is equivalent to a problem of minimizing a special lower bound of $f(\mu)$.

5 Numerical solution for the linearly constrained MLL problem

Here we consider the special and important case in which the constraint function is linear, i.e. $g(\mu) = A\mu$, where A is an $r \times m$ matrix. This is the natural case to consider as a first approach to the general problem. On the other hand, linear constraints are typical of many resourse allocation models, which in the case of linear objectives give rise to linear programming problems which are so important in practice. The general assumptions of Section 2 on the constraint function g imply that

- i$'$) $A\lambda < b$;
- iii$'$) $A \geq 0$ and for each $i \in \{1, 2, \ldots, m\}$ there exists $j \in \{1, 2, \ldots, r\}$, such that $a_{ji} > 0$.

In this case the constraint qualification conditions hold ([5], p.177) and the Kuhn–Tucker conditions (3.11) are necessary for any optimal solution $\tilde{\mu}$. In particular, the first equation in (3.11) reads

$$-\nabla f(\mu) \;=\; uA. \tag{5.1}$$

It is rather natural to consider the Rosen's gradient projection method (see [4], p.330, or [5], p.197) in order to determine the optimal solutions of the problem MLL numerically. Assuming to know a feasible point, the method requires to project the negative gradient of the objective function on the "working surface" in order to define the direction of movement. Then a best movement is made in that direction and a new iteration takes place until a solution to the Kuhn–Tucker conditions is found. In the case of linear constraints the computation of the projection of the negative gradient on the boundary of the feasible set is simple. The method requires, at each iteration, the computation of the gradient $\nabla f(\mu)$ of the objective function and the minimization of $f(\mu)$ on a special line segment.

5.1 Initialization of numerical solution algorithms

A first question to answer in order to implement Rosen's algorithm is how to choose the initial feasible solution μ^0. It may help the convergence of the algorithm that the initial feasible point μ^0 has some characteristics of the optimal solutions and a condition that can be easily imposed on μ^0 is that it belongs to the boundary of the feasible set. In fact, any such point can be obtained by choosing a positive vector (direction) $d \in \mathbb{R}^m$, $d > 0$, and then setting

$$\mu^0 \;=\; \lambda + t_0 d, \tag{5.2}$$

where

$$t_0 \; = \; \max\{t > 0 \mid A(\lambda + td) \le b\}. \tag{5.3}$$

We still remain with a wide range of choices, depending on the direction $d > 0$, which may affect the convergence of the algorithm.

It would be convenient to find a μ^0 which is near to the unknown optimal solution and a reasonable answer to this aspiration is offered by the optimal solutions of the problems ISP ($LISP$), $SELL$ and LLP, that we have considered in Section 4. In fact, for the last of these problems, LLP, we know an explicit optimal solution, whereas for the first two, $LISP$ and $SELL$, we know that they have unique optimal solutions and that such solutions are easier to find (using e.g. Rosen's algorithm) than that of problem MLL. This is clear because of the strict convexity (concavity) of the objective functions and because of the computational simplicity of such functions and their gradients.

6 Numerical results

We report here on the results of some computational experiments in order to illustrate the above considerations on the relations existing among the optimal solutions to the problems MLL, ISP ($LISP$), $SELL$ and LLP.

We have implemented Rosen's algorithm on a personal computer to solve problem MLL with linear constraints. We intend to evaluate the optimal solutions of ISP ($LISP$), $SELL$ and LLP as approximations of the optimal solution of MLL and also compare them and randomly chosen points on the boundary of the feasible set as initial solutions while using Rosen's algorithm.

In order to evaluate the objective function $f(\mu)$ and its derivatives we use the finite sum forms (3.5) and (3.6) and we observe that they require the computation of $2^m - 1$ and 2^{m-1} terms respectively. Therefore the computational time increases rapidly with m and we have seen that $m = 10$ is a limit size in order to run the algorithm in a reasonable time. This observation poses the question of the existence of a good numerical treatment of the function $f(\mu)$ and of its derivatives so that instances of the MLL problem with a size $m > 10$ can be solved. The question is the object of discussion in [6].

In each of the instances of the MLL problem which follow we have run Rosen's algorithm using different choices of the initial solution and in the tables we show the value $f(\mu^0)$ of the objective function at the initial solution, the approximate optimal value f^* which is obtained by the algorithm (which stops after finding two consecutive values of the objective function $f(\mu)$ that differ by less than 10^{-4}), the percentage relative error, $100[f(\mu^0) - f^*]/f^*$, which is incurred when using the initial solution μ^0 as an approximation of an optimal solution, and, finally, the number of iterations of the algorithm which were necessary to obtain the approximate optimal solution.

As initial solutions we have used:

- 10 points on the boundary, using formulas (5.2) and (5.3) with randomly chosen directions d: in this case the numbers in the table are average values;
- a point on the boundary, using formulas (5.2) and (5.3) with the special direction $d = e = (1, 1, \dots, 1)$;
- the optimal solutions of the problems ISP ($LISP$), $SELL$ and LLP, which we denote by μ^{LLP}, μ^{SELL} and μ^{ISP}.

Example 1.1

The number of service units is $m = 10$, the vector of demand intensities, i.e. the arrival rates vector, is $\lambda = (10, 2, 5, 4, 2, 8, 10, 10, 5, 2)$, the constraint coefficients matrix and the resource bounds are

$$A = \begin{pmatrix} 1 & 1 & 3 & 1 & 2 & 1 & 1 & 3 & 1 & 1 \\ 2 & 2 & 3 & 2 & 5 & 1 & 2 & 1 & 2 & 3 \end{pmatrix}, \qquad b = \begin{pmatrix} 250 \\ 110 \end{pmatrix}.$$

Results are summarised in the following table.

Table 1: results of Example 1.1

μ^0	$\lambda + t_0 d$ averages	$\lambda + t_0 e$	μ^{LLP}	μ^{SELL}	μ^{ISP}
$f(\mu^0)$	9.7911	2.5416	2.3672	2.1999	2.2682
f^* appr.	2.1670	2.1669	2.1670	2.1669	2.1669
% err.	352.	17.3	9.2	1.5	4.7
n. iter.	14.6	6	6	5	6

Example 1.2

$m = 10$, $\lambda = (20, 2, 10, 5, 4, 30, 10, 20, 10, 40)$, matrix A and vector b as in Example 1.1.

Table 2: results of Example 1.2

μ^0	$\lambda + t_0 d$ averages	$\lambda + t_0 e$	μ^{LLP}	μ^{SELL}	μ^{ISP}
$f(\mu^0)$	100.7682	26.6913	18.9199	17.9418	22.3592
f^* appr.	17.2077	17.2077	17.2077	17.2077	17.2077
% err.	486.	55.1	10.	4.3	30.
n. iter.	27.0	18	21	20	21

Example 2

$m = 10$, $\lambda = (1, 2, 1, 1, 1, 8, 8, 8, 6, 6)$,

$$A = \begin{pmatrix} 3 & 4 & 1 & 5 & 0 & 10 & 10 & 10 & 9 & 9 \\ 0 & 1 & 0 & 0 & 1 & 5 & 6 & 10 & 10 & 10 \\ 1 & 2 & 1 & 1 & 1 & 5 & 5 & 5 & 8 & 8 \\ 1 & 0 & 1 & 0 & 0 & 2 & 2 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 & 0 & 5 & 4 & 6 & 10 & 8 \\ 1 & 1 & 1 & 1 & 0 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}, \qquad b = \begin{pmatrix} 600 \\ 400 \\ 350 \\ 150 \\ 350 \\ 400 \end{pmatrix}.$$

Table 3: results of Example 2

μ^0	$\lambda + t_0 d$ averages	$\lambda + t_0 e$	μ^{LLP}	μ^{SELL}	μ^{ISP}
$f(\mu^0)$	10.2450	7.1768	8.7065	6.9335	7.0928
f^* appr.	6.9244	6.8907	6.8755	6.9335	6.9884
% err.	48.	4.2	26.6	0.0	1.5
n. iter.	13.5	6	7	1	7

Example 3.1

$m = 10$, $\lambda = (5, 1, 2, 4, 1, 5, 5, 4, 3, 2)$, the constraint coefficients matrix $A = (a_{ij})$ has 11 rows with

- $a_{ij} = 0,\ i \neq j,\ 1 \leq i \leq 10$;
- $(a_{11}, a_{22}, \ldots, a_{10\,10}) = (1, 1, 3, 1, 2, 1, 1, 3, 1, 1)$;
- $(a_{11\,1}, a_{11\,2}, \ldots, a_{11\,10}) = (1, 1, 3, 1, 2, 1, 1, 3, 1, 1)$;

the resource bounds vector is $b = (25, 3, 20, 10, 5, 40, 15, 45, 15, 50, 155)$.

Table 4: results of Example 3.1

μ^0	$\lambda + t_0 d$ averages	$\lambda + t_0 e$	μ^{LLP}	μ^{SELL}	μ^{ISP}
$f(\mu^0)$	47.9251	8.0187	2.6965	2.0394	2.0454
f^* appr.	2.0391	2.0352	2.0352	2.0350	2.0349
% err.	2250.	294.	32.5	0.2	0.5
n. iter.	11.7	7	6	2	4

Example 3.2

$m = 10$, $\lambda = (20, 1, 1, 4, 1, 30, 10, 5, 10, 40)$, matrix A and vector b as in Example 3.1.

Table 5: results of Example 3.2

μ^0	$\lambda + t_0 d$ averages	$\lambda + t_0 e$	μ^{LLP}	μ^{SELL}	μ^{ISP}
$f(\mu^0)$	68.4536	43.8396	20.7013	19.2873	25.1359
f^* appr.	18.5643	18.3292	18.3292	18.3292	18.3295
% err.	269.	139.	12.9	5.2	37.1
n. iter.	152.1	136	135	107	63

Example 4.1

$m = 10$, $\lambda = (10, 2, 5, 4, 2, 8, 10, 10, 5, 2)$, matrix A and vector b are constituted of the first 10 rows (elements) of those of Example 3.1. In this case we know the optimal solution, $\mu^* = (25, 3, 20/3, 10, 5/2, 40, 15, 15, 15, 50)$, and then the exact optimal value of the objective function is $f^* = 6.834754$.

Table 6: results of Example 4.1

μ^0	$\lambda + t_0 d$ averages	$\lambda + t_0 e$	μ^{LLP}	μ^{SELL}	μ^{ISP}
$f(\mu^0)$	306.8759	42.1986	12.6259	6.8349	6.8349
f^* appr.	6.8353	6.8353	6.8355	6.8349	6.8349
% err.	4390.	517.	84.7	0.0	0.0
n. iter.	9.7	9	8	1	1

Example 4.2

$m = 10$, $\lambda = (5, 1, 2, 4, 1, 5, 5, 4, 3, 2)$, matrix A and vector b as in Example 4.1. The optimal solution is the same as that of Example 4.1 and then the exact optimal value of the objective function is $f^* = 1.908701$.

Table 7: results of Example 4.2

μ^0	$\lambda + t_0 d$ averages	$\lambda + t_0 e$	μ^{LLP}	μ^{SELL}	μ^{ISP}
$f(\mu^0)$	73.3925	8.0187	2.6965	1.9087	1.9087
f^* appr.	1.9122	1.9130	1.9130	1.9087	1.9087
% err.	3738.	319.	40.1	0.0	0.0
n. iter.	10.6	9	8	1	1

From the above results we first observe that in all the instances considered the algorithm converges essentially to the same value of the objective function, no matter what initial solution has been used. This suggests that the MLL problem has a unique optimal solution.

It is clear that a randomly chosen point on the boundary of the feasible set is a poor initial solution in general and that the point $\lambda + t_0 e$ is a better choice. The latter has the good feature of maximizing the distance between μ and λ, under the condition that $\mu_i - \lambda_i = \mu_j - \lambda_j$ for all $i, j \in \{1, 2, \ldots, m\}$.

The optimal solution μ^{LLP} of the line length problem, which is as easy to compute as $\lambda + t_0 e$, results better than $\lambda + t_0 e$ in all respects.

Finally, the optimal solutions μ^{SELL} and μ^{ISP} to the problems of minimizing the sum of expected line lengths and of maximizing the idle system probability, respectively, appear to be the best choices as initial solutions of the Rosen's algorithm. The computational effort required to determine either μ^{SELL} or μ^{ISP} is a little greater than that required by μ^{LLP}, but, on the other hand, μ^{SELL} and μ^{ISP} appear to be very good approximations of the optimal solution to the MLL problem.

7 Conclusion

The analysis of the maximum line length (MLL) problem and of its relations with the ISP, $SELL$ and LLP problems has shown the essential similarities of such different optimization problems for a multiservice system. Those similarities have some interesting consequences also from a computational viewpoint. This aspect is now being explored in [6], in order to propose an efficient algorithm for the solution of the MLL problem.

The results which have been obtained here concern the simplest case of a multiservice system, which is constituted by independent $M/M/1$ service units. A similar analysis for multiservice systems constituted by more general service units would be more interesting for the possible practical applications. On the other hand, it is clear that the consequent higher analytical complexity would render the MLL problem impossible to study directly in general. Then an interesting question is that of finding classes of service units to which the present analysis can be applied. More generally, one could try to understand to what extent the results for the multi–$M/M/1$ service system can give information on the behaviour of more general multiservice systems.

A different limitation of the present study and of that of [6] is that the numerical analysis has only been applied to problems with linear constraints. On one hand, this is the obvious and necessary starting point of such analysis, because of its simplicity and of its reasonable interpretation. On the other hand, more general constraints should be considered in the case that specific nonlinear cost (resource requirement) functions were suggested by special applications.

Appendix The subset enumeration form of $f(\mu)$

Here we verify the formula (3.5) which provides a finite sum representation of $f(\mu)$.

After setting $\rho_i = \lambda_i/\mu_i$, $i = 1, 2, \ldots, m$, we have from (3.4) that

$$\begin{aligned} f(\mu) &= \sum_{n=0}^{\infty} \Big[1 - \prod_{i=1}^{m} \Big(1 - \rho_i^{n+1}\Big)\Big] \\ &= \sum_{n=0}^{\infty} \Big[\sum_{\emptyset \neq J \subseteq \{1,2,\ldots,m\}} (-1)^{|J|+1} \prod_{j \in J} \rho_j^{n+1} \Big] \\ &= \sum_{\emptyset \neq J \subseteq \{1,2,\ldots,m\}} (-1)^{|J|+1} \sum_{n=0}^{\infty} \Big(\prod_{j \in J} \rho_j \Big)^{n+1} \\ &= \sum_{\emptyset \neq J \subseteq \{1,2,\ldots,m\}} \frac{(-1)^{|J|+1} \prod_{j \in J} \rho_j}{1 - \prod_{j \in J} \rho_j}, \end{aligned}$$

which is equivalent to formula (3.5).

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