On optimal advertising policies and equilibria

Bruno Viscolani

Department of Pure and Applied Mathematics, University of Padua
Via Trieste, 63
I-35121 Padova, Italy
e-mail: viscolani@math.unipd.it

Abstract The objective of this paper is to present a special application field of optimal control and differential game theories which is of interest to management scientists. After introducing the Vidale-Wolfe's and Nerlove-Arrow's dynamic models for advertising processes, I will focus on a few exemplary problems. At a first investigation level, we find some optimal control problems, then the more interesting and challenging problems in the framework of game theory. The view of all problems is unified by the common reference to the same two basic dynamic models.

Key words Advertising; optimal control; differential games.

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1 Introduction

Advertising is a special tool of Marketing, and is an object of study in Management Science. Most of the literature dealing with advertising has an empirical, descriptive and qualitative nature, nevertheless there exists a stream of research in quantitative marketing which has produced interesting models for both descriptive and optimization purposes.

Here I want to provide some ideas on

- dynamic models of relevant advertising phenomena,
- decision problems of a monopolistic firm which admit optimal control formulations,
- decision problems of an oligopoly which admit game theory formulations.

I will draw only a sketch of a wide landscape, with the aim of stimulating curiosity, if not attention, about an application field to which mathematics can give important contributions. On the other hand I address the reader to review papers [38,14,29], and systematic books [10,26] to find a thorough presentation of models and methods available.

The outline of the paper is as follows. In Section 2 I introduce the Vidale-Wolfe's and Nerlove-Arrow's dynamic models for advertising processes. In Section 3 some typical optimal control problems are formulated using the dynamic models of Section 2: they concern a single *monopolist* firm. Finally, in Section 4, differential game theory is used to formulate some interesting and challenging problems which concern the interaction of several firms in an *oligopoly*.

2 Dynamic advertising models

Two main models are at the basis of the literature on optimal control applications to advertising and they have been proposed in a five years interval around 1960: the first one, dated 1957, is due to Vidale and Wolfe [42], whereas the second one, dated 1962, is due to Nerlove and Arrow [33]. Both of them focus on the relationship between advertising and demand for a product, the former in a direct way, the latter in a mediated way. The two review articles [38] and [14] present the general context of the optimal control models in advertising until 1994: there we see how those two models have become the pivots of the mathematical research on advertising.

2.1 Vidale and Wolfe's model

Vidale and Wolfe in [42] aim at modelling the sales response to advertising and try to represent some characteristic behaviors as observed in real data. They observe two main facts concerning the relation between sales and advertising. Sales intensity decreases with time if no advertising is done and, if an adequate advertising effort is done over a time period, then sales intensity increases, but a saturation effect may emerge. Hence they suggest that the sale intensity $s(t)$ satisfies the differential equation

$$\dot{s}(t) \ = \ \theta u(t)[m - s(t)] - \delta s(t) \,, \tag{1}$$

where $u(t)$ is the advertising intensity, and $\theta, m, \delta > 0$ are parameters. This is a linear differential equation in the state variable $s(t)$, where the term $-\delta s(t)$ represents the spontaneous decay of the state. The parameter m is the saturation threshold, an upper bound to the sales intensity: when $m - s(t)$ is small and positive, the advertising intensity $u(t)$ must be large to sustain the sale level.

2.2 Nerlove and Arrow's model

Nerlove and Arrow assume in [33] that the demand of a product (hence its sale intensity) depends on a state variable, called *goodwill*, that represents the effects of a firm investment in advertising. The goodwill affects the demand of the product together with price and other exogenous factors. Goodwill is therefore seen as a stock of productive capital: it is subject to depreciation, i.e. spontaneous decay proportional to its value, and, on the other hand, it is sustained by the investment flow controlled by the firm. Nerlove and Arrow focus on the most elementary differential equation which describes an investment phenomenon in a capital stock subject to depreciation, i.e. the linear equation

$$\begin{cases} \dot{G}(t) &=& u(t) - \delta G(t)\,, \\ G(0) &=& \alpha\,, \end{cases} \tag{2}$$

where $G(t)$ and $u(t)$ are the capital stock (goodwill) and the advertising investment intensity at time t, $\delta > 0$ is a decay parameter, which represents the capital depreciation over time, and α is the known value of the capital at the initial time 0. In fact equation (2) represents the capital dynamics of the neoclassical aggregate growth model [40, p. 432].

The goodwill variable may assume different meanings, depending on the particular context: a first example is sales intensity (see [8]), a second one, provided by [28], is *reservation price*, a third one, used by [47] in the discrete time version of the model, is *awareness*.

3 Optimal control problems

At a first level of analysis, one may assume that a firm acts alone, as a *monopolist*, in its environment, the market. The firm decision problems concerning its advertising investments find a convenient framework in optimal control theory. Let us consider some examples of typical problems, considering advertising intensity as the unique control variable, for simplicity, although one can find in the literature several problems in which e.g. price is also a control variable. To this purpose, let $[0,T]$ be the programming interval (with $T \in (0,+\infty]$), recalling that in some cases a finite horizon is well justified by a concrete marketing situation, but quite often an infinite horizon is considered.

3.1 Optimal advertising and Vidale-Wolfe's model

The basic optimal control problem represents a firm which wants to maximize its profit (discounted at rate $\rho \geq 0$)

$$J\left[u(t)\right] \;=\; \int_0^T \left[\pi s(t) - C(u(t))\right] e^{-\rho t} dt\,, \tag{3}$$

under an initial state condition $s(0) = s_0$, a constraint on the control $u(t) \in [0,\bar{u}]$ and, in finite horizon, a terminal state constraint $s(t) \in [s_T^1, s_T^2]$. Here, $\pi > 0$ is the constant marginal revenue and $C(u) \geq 0$ is the cost intensity associated with the advertising intensity u, an increasing and convex function, possibly linear or quadratic. Such problems are studied by Sethi [35] and [36], who reformulates the model in terms of *market share* $x = s/m \in [0, 1]$, obtaining the motion equation

$$\dot{x}(t) \;=\; \theta u(t)[1 - x(t)] - \delta x(t) \,. \tag{4}$$

This point of view is particularly interesting in practice, because one may measure the market share empirically in real markets.

An interesting variant of the Vidale-Wolfe's model has been proposed still by Sethi [39]: he replaces the complement market share $1 - x(t)$ factor in equation (4) by means of its square root and obtains the motion equation

$$\dot{x}(t) \;=\; \theta u(t)\sqrt{1 - x(t)} - \delta x(t) \,; \tag{5}$$

hence he considers also a stochastic extension of it,

$$dx(t) \;=\; \Big(\theta u(t)\sqrt{1 - x(t)} - \delta x(t)\Big)\, dt + \sigma(x(t)) dw(t) \,, \tag{6}$$

where $\sigma(x(t))dw(t)$ is a white noise component. After assuming an infinite horizon and quadratic advertising costs $C(u) = u^2$, he solves the deterministic problem, as well as the stochastic problem of maximizing the expectation of the discounted profit.

3.2 Optimal advertising and Nerlove-Arrow's model

Using the Nerlove-Arrow's model requires to specify how the demand for the product depends on the goodwill, or how the revenue depends on it. Let the function R, increasing and concave, represents the firm profit intensity, gross of the advertising costs. We assume that a firm aims at controlling the goodwill evolution in order to maximize its profit (discounted at rate $\rho \geq 0$)

$$J\,[u(t)] \;=\; \int_0^T \left[R\,(G(t)) - C\,(u(t))\right] e^{-\rho t} dt + S\,(G(T))\, e^{-\rho T} \,. \tag{7}$$

The expression $R\,(G(t)) - C\,(u(t))$ represents the net profit rate and the function S summarizes the effects of the final goodwill $G(T)$ on the profit to be obtained at time T or later on. It is consistent to assume that the salvage function vanishes, $S\,(G(T))\, e^{-\rho T} = 0$, in problems with $T = +\infty$.

The consideration of nonlinear and convex advertising costs is sometimes translated into the assumption that the goodwill productivity term is nonlinear and concave in the advertising intensity [17], instead of being linear as in the original Nerlove-Arrow's equation (2). For instance, we may find a goodwill motion equation as (see e.g. [5], [28])

$$\dot{G}(t) \;=\; b\sqrt{u(t)} - \delta G(t) \,, \tag{8}$$

where b is a positive parameter, or as (see e.g. [37], [20])

$$\dot{G}(t) \;=\; b \ln u(t) - \delta G(t) \,, \quad (u(t) \geq 1) \,. \tag{9}$$

3.2.1 Infinite horizon Let $T = +\infty$, $R(G) = (q - \varepsilon_1)G^\gamma/\gamma - \varepsilon_2 G^{2\gamma}/2\gamma^2$, and $C(u) = \kappa u^2/2$, where $\gamma \in [1/2, 1)$, $q, \varepsilon_1, \varepsilon_2, \kappa > 0$ and $\varepsilon_1 < q$. We think of q as the sale price and G^γ/γ as the demand rate, so that qG^γ/γ is the revenue intensity. Furthermore, $\varepsilon_1 y + \varepsilon_2 y^2/2$ is the production cost intensity associated with the production intensity y. The Hamiltonian function is

$$H(G, u, p, t) \;=\; (q - \varepsilon_1)G^\gamma/\gamma - \varepsilon_2 G^{2\gamma}/2\gamma^2 - \kappa u^2/2 + p\,(u - \delta G)\;, \quad (10)$$

where p is the adjoint variable, so that $H_{uu} < 0$ and an optimal control is unique and must satisfy the condition

$$\kappa u(t) \;=\; p(t)\,, \quad (11)$$

provided that it exists. Hence we obtain the differential system

$$\begin{cases} \dot{G}(t) \;=\; p(t)/\kappa - \delta G(t)\,, \\ \dot{p}(t) \;=\; -(q - \varepsilon_1)G^{\gamma-1}(t) + \varepsilon_2 G^{2\gamma-1}(t)/\gamma + (\delta + \rho)\,p(t)\,, \end{cases} \quad (12)$$

which admits a unique equilibrium point with coordinates (G^*, p^*) such that

$$p^* \;=\; \kappa\delta G^* \quad (13)$$

and G^* is the solution of the equation

$$\gamma(q - \varepsilon_1)G^{\gamma-1} \;=\; \varepsilon_2 G^{2\gamma-1} + \gamma\,(\delta + \rho)\,\kappa\delta G\,. \quad (14)$$

At the equilibrium, the optimal advertising policy $u^* = p^*/\kappa = \delta G^*$ is chosen precisely to compensate the goodwill decay.

3.2.2 Finite horizon - Advertising an event Let $T < +\infty$, $R(G) = 0$, $C(u) = \kappa u^2/2$ and $S(G) = -\nu\left(G - \bar{G}\right)^2/2$ (where κ, $\nu > 0$, $\bar{G} > \alpha$), and $\rho = 0$. This instance does not account for sales before the final time T. Here we aim at programming the advertising campaign for an event (or the introduction of a new product in the market). The function $S(G)$ describes the payoff obtained by the organizers of an event like a concert or a theatre performance. For such events the number of available seats is a crucial parameter. We may think of a goodwill threshold $\bar{G}$ such that

- if the final goodwill exceeds it, $G(T) > \bar{G}$, then the demand is greater than the available seats, and there are some unsatisfied customers;
- if the final goodwill is less than it, $G(T) < \bar{G}$, then the demand is less than the available seats, and some tickets remain unsold.

The quadratic penalty function $S(G)$ is a symmetric representation of two kinds of loss: in the first case a loss of reputation, in the latter a loss of revenue. A similar problem has been analyzed in [21], although with reference to a different model with nonlinear dynamics.

Under the previous assumptions, we have a linear quadratic deterministic optimal control problem and we can study it using the standard method of the completions of the squares [46]. After defining $z(t) = G(t) - \bar{G}$ the problem becomes

$$\begin{aligned} \min \int_0^T \kappa u(t)^2/2\,dt + \nu z(T)^2/2\,, \\ \dot{z}(t) \;=\; u(t) - \delta z(t) - \delta \bar{G}\,. \end{aligned} \tag{15}$$

We consider the associated Riccati equations and eventually obtain that a solution to them requires that there exists a unique feedback optimal control that can be written as

$$u^*(t) \;=\; -\frac{q(t)\left(G^*(t) - \bar{G}\right) + s(t)}{\kappa}\,. \tag{16}$$

If $\delta = 0$, the solution is $q(t) = \nu\kappa/\left(\kappa + \nu\left(T - t\right)\right)$, and the optimal control is $u^*(t) = -\nu\left(G^*(t) - \bar{G}\right)/\left(\kappa + \nu\left(T - t\right)\right)$. The function $q(t)$ is strictly increasing: the weight of the reaction to the deviation of the goodwill from the target, $x^*(t) - \bar{G}$, is higher and higher as time approaches T.

A variant of this problem has been studied in [3], where the goodwill is assumed to satisfy a linear stochastic differential equation. There the control (advertising intensity) enters directly the diffusion term of the motion SDE. Therefore, some uncertainty is introduced in the advertising effectiveness and this fact modifies the structure of the optimal solution.

3.2.3 Finite horizon - Sale of a seasonal product A special finite horizon problem concerns the sale of a seasonal product. Let $y(t)$ represent the quantity of product sold by time t and let the demand be a linear function of goodwill

$$\dot{y}(t) \;=\; G(t)\,, \quad y(0) \;=\; 0\,. \tag{17}$$

Let $C(u) = \kappa u^2/2$, $\kappa > 0$, as for the above problem, and let us consider explicitly the constraint $u \geq 0$ on the advertising intensity. Let $R(G) = qG$ and $S(G, y) = -w(y)$, where $q > 0$ is the price of the product and $w(y)$ is a positive, increasing and convex function, representing the production cost of the quantity y of product. Let us further assume that the discount rate is $\rho = 0$, as the sale interval is short for a seasonal product.

The problem Hamiltonian function is

$$H(G, y, u, p_1, p_2, t) \;=\; qG - \kappa u^2/2 + p_1(u - \delta G) + p_2 G\,, \tag{18}$$

so that $H_{uu} < 0$ and the unique optimal control must be

$$u(t) \;=\; \max\{0,\, p_1(t)/\kappa\}\,, \tag{19}$$

provided that it exists. As in the infinite horizon case we find that, at the optimum, marginal advertising cost equals marginal goodwill value,

$\kappa u(t) = p_1(t)$, provided that advertising is active, $u(t) > 0$. From the adjoint equations and the trasversality conditions, we obtain that

$$p_1(t) = \frac{q + p_2}{\delta} \left(1 - e^{\delta(t-T)}\right) , \tag{20}$$

$$p_2(t) = p_2 = -w'(y(T)) < 0 . \tag{21}$$

Therefore, after defining

$$y_0 = \frac{\alpha}{\delta} \left(1 - e^{-\delta T}\right) \tag{22}$$

as the *free* sale level, which is the sale quantity attainable without any advertising effort, we have that

- either $q \leq w'(y_0)$, and then $u^*(t) \equiv 0$, $y^*(T) \equiv y_0$;
- or there exists $y^* > y_0$ such that $-p_2 = w'(y^*) < q$, and then $u^*(t) > 0$ for all $t < T$, $y^*(T) = y^*$ ($u^*(t)$ is strictly decreasing and $u^*(T) = 0$).

Several problems of this kind have been studied (see [12] and [13]), but with linear advertising costs and, on the other hand, with further constraints on the advertising intensity (control) $u(t)$: in that case the optimal policies are bang-bang controls because of the special problem structure.

3.2.4 Heterogeneous markets and segmentation If a consumer population is heterogeneous, the firm has to determine the part of the consumers which may have an interest in buying the product, i.e. the *target market* [27, p. 379]. This is obtained by first dividing the market into distinct *segments*, consumer groups which exhibit special needs and behaviors [27, p. 379] and which require specific products and marketing mix. Then the firm has to decide to which consumer groups the product should be proposed and how to reach each segment using the available communication tools, while considering that different advertising media entail specific costs and different segments provide different marginal revenues.

The treatment of a segmented market requires a generalization of Nerlove-Arrow's model which considers one goodwill component for each segment of the population. Let A be the (finite) segment set let $G_a(t)$ represent the stock of goodwill of the product at time t, for the (consumers in the) a segment, $a \in A$. A possible model, considering the use of one advertising medium for all segments, is the following

$$\dot{G}_a(t) = \gamma_a \varphi(u(t)) + \sum_{b \in A \setminus \{a\}} \delta_{ab} G_b(t) - \delta_a G_a(t) , \qquad a \in A , \tag{23}$$

where $\delta_a > 0$ represents the goodwill depreciation rate for the members of the consumer group a, δ_{ab} represents the influence of segment b on segment a, $\gamma_a \varphi(u(t))$ is the effective advertising intensity at time t directed to that same group, $\gamma_a > 0$, and finally $\varphi(u)$ is an increasing and concave function of the advertising medium level u.

In the special case that the influence parameters δ_{ab} are all 0, equation (23) reduces to a set of independent ordinary differential equations. Under such assumptions and the initial state condition $G_a(0) = \alpha_a \geq 0$, a problem of optimal advertising for a new product introduction will have the objective functional

$$J = -\int_0^T C(u(t))\, dt + \sum_{a \in A} \pi_a G_a(T) \ , \tag{24}$$

whose first term represents advertising costs, whereas the second one represents revenue (see [4]).

A further step in the analysis of different behaviors of a heterogeneous population is addressed by the *age-segmentation*, which has been profitably considered in contexts different from that of advertising, e.g. social analysis and drug addiction (see [1], [19]). In this case, we are led to consider an age-distributed goodwill variable which evolves according to a linear partial differential equation as the following (see [18])

$$\begin{aligned} \partial_t G(t,a) + \partial_a G(t,a) &= \gamma(a) u(t) - \delta(a) G(t,a) \ , \\ G(0,a) &= \alpha(a) \ , \\ G(t,0) &= \beta(t) \ , \end{aligned} \tag{25}$$

where $\delta(a) > 0$ represents the depreciation rate for the members of age a, $\gamma(a) \geq 0$ is the age-spectrum of the advertising medium, $\int_0^\infty \gamma(a) da = 1$, $\alpha(a) \geq 0$ is the known goodwill level at the initial time for the age a class of the population and $\beta(t)$ is the goodwill level at all times for the age 0 class. Special optimal control tools for the treatment of such models are presented in [15].

4 Game theory problems

Often an oligopolistic framework which considers several manufacturers in the market is a better representation of real life situations. In that case each manufacturer's profit depends on the decisions of all manufacturers. The natural models to deal with such situations are provided by the theory of games and, in the last two decades, a substantial literature on differential games in marketing has been produced. A comprehensive synthesis is the book by Jørgensen and Zaccour [26]. Advertising models have an important part in that matter and address two key issues: "the strategic, intertemporal interdependencies among the firm and its competitors, and between the firm and its customers"[26, p. 29].

Among the variety of models which have been proposed we consider here the *market share models*, which are derived from the Lanchester model of combat, the *sale response models*, which are based on the Vidale-Wolfe's dynamics, and the *advertising goodwill models*, which are based on the Nerlove-Arrow's dynamics.

4.1 Games of the Lanchester's type

The first model of competitive advertising decisions is a variation of the Lanchester model of combat, already used to describe populations evolution [9, p. 19], and leads to the so-called *market share models*. As market shares of firms are observable, such models are particularly interesting for empirical studies (see e.g. [44] and [45]). In the case of a duopoly, the market shares $x_1(t)$, $x_2(t)$ of the two firms are subject to the dynamics

$$\dot{x}_i(t) \;=\; \gamma_i u_i(t)\,(1 - x_i(t)) - \gamma_j u_j(t) x_i(t)\,, \quad \{i,j\} = \{1,2\}\,, \tag{26}$$

where $u_1(t)$, $u_2(t)$ are the firms advertising intensities and $\gamma_1, \gamma_2 > 0$. One observes that the sales rates may be expressed as

$$s_i(t) \;=\; m x_i(t)\,, \qquad i \in \{1,2\}\,, \tag{27}$$

where $m > 0$ is the *market potential*, possibly a constant. Hence, as for the objective functionals of the players (firms), the profits are

$$J_i\,[u(t)] \;=\; \int_0^T \left[\pi s_i(t) - C_i(u_i(t))\right] e^{-\rho t} dt\,, \quad i \in \{1, \ldots, N\}\,, \tag{28}$$

where $u(t) = (u_1(t), u_2(t))$ and $C_i(u_i)$ is the cost intensity of the manufacturer i associated with the advertising intensity u_i.

The two important solution concepts are the *open loop Nash equilibrium*, where the controls $u_i(t)$ depend on time t only, and the *Markovian (or feedback) Nash equilibrium*, where the controls $u_i(t)$ depend on the information on the state $x(t) = (x_1(t), x_2(t))$ at time t, or equivalently on $s(t) = (s_1(t), s_2(t))$.

In this stream of research we find also some stochastic variant, as in [34]. Moreover, we may see the market share models as a formal generalization of the sale response models presented in the following section, as suggested by Little [30].

4.2 Games with Vidale-Wolfe's dynamics

Let N be the number of firms (players) and let $u_i(t)$ and $s_i(t)$, $i \in \{1, \ldots, N\}$, be the advertising intensity (i.e. the i-th firm strategy) and the sale intensity of the i-th firm. Consistently with the monopolistic Vidale-Wolfe's dynamics (1), it is natural to assume that the system dynamics is

$$\dot{s}_i(t) \;=\; \theta_i u_i(t) \left[m - \sum_{j=1}^{N} s_j(t) \right] - \delta_i s_i(t)\,, \qquad i \in \{1, \ldots, N\}\,, \tag{29}$$

where $\theta_i, m, \delta_i > 0$ and m is the saturation threshold, i.e. the *market potential*. The objective functionals representing manufacturers' profits are again (28) where $u(t) = (u_1(t), \ldots u_N(t))$.

Variants of the dynamics (29) have been introduced to admit a varying market potential and a concave advertising term, i.e. decreasing marginal effects of advertising. An example is the Erickson [11] model

$$\dot{s}_i(t) \;=\; \theta_i \sqrt{u_i(t)} \left[m(t) - \sum_{j=1}^{N} s_j(t) \right] - \delta_i s_i(t) \,, \qquad i \in \{1, \ldots, N\} \,, \quad (30)$$

with a market potential $m(t)$ given exogenously.

4.3 Games with Nerlove-Arrow's dynamics

Let N and $u_i(t)$, $i \in \{1, \ldots, N\}$, be defined as in Section 4.2. Typically, two kinds of game formulations, based on the Nerlove-Arrow's dynamics, have been considered. A third one has attracted recently some interest, because of special interpretations, and poses some new analytical challenges.

I) There is one goodwill variable for each player/firm and it is affected by the advertising strategy of the same firm only.
Let $G_i(t)$, $i \in \{1, \ldots, N\}$, be the goodwill stock of the i-th firm, with dynamics described by the differential equation

$$\begin{cases} \dot{G}_i\,(t) \;=\; u_i\,(t) - \delta G_i\,(t) \;, \\ G_i\,(0) \;=\; \alpha_i \;. \end{cases} \quad (31)$$

The payoff of the $i-th$ player has the form

$$J_i\,[u(t)] \;=\; \int_0^T \left[R_i\,(G(t)) - C_i\,(u_i(t)) \right] e^{-\rho t} dt + S_i\,(G(T))\, e^{-\rho T} \;, \quad (32)$$

where $G(t)$ and $u(t)$ are the n-dimensional goodwill and advertising strategy variables.
The first examples are given by [41] and [16], within different oligopoly contexts. In the former the demand of each one of two firms depends on the goodwill of both firms. In the latter the market share of each one of N firms is assumed to be a function of the goodwill of all firms,

$$x_i(t) \;=\; \frac{G_i(t)^\epsilon}{\sum_{j=1}^{N} G_j(t)^\epsilon} \,, \quad (33)$$

where $\epsilon > 0$, and the ith firm revenue depends on $x_i(t)$.
Other examples of a type I game are given by [7] and [6]: in the latter a model of advertising competition is discussed in a dynamic duopoly with diminishing returns to advertising effort. More recently we find [31] and [32].
The optimal control problems arising from the game formulations can be tackled using either the Pontryagin Maximum Principle, to obtain open-loop Nash equilibria, or the Hamilton Jacobi Bellman Equation approach, to obtain Markovian Nash equilibria.

II) There is a unique goodwill variable, which is affected by the advertising efforts (controls) of the different firms

$$\begin{cases} \dot{G}(t) = & \sum_{i=1}^{N} \gamma_i u_i(t) - \delta G(t) \ , \\ G(0) = & \alpha \ , \end{cases} \tag{34}$$

with $\gamma_i > 0$, $i = 1, \dots N$. The payoff of the $i - th$ player has still the form (32), where $G(t)$ is now a 1-dimensional goodwill variable.
Examples are given by [23], [22], [24] and [25], which consider a simple distribution channel, where a retailer promotes the manufacturer product and the latter may possibly spend in advertising to sustain the retailer campaign. These are instances of the so-called *leader-follower* model: a two-player game where one of the players (the manufacturer - leader) makes his decision before the other player (the retailer - follower) makes hers. The solution of such games is sought in the form of a Stackelberg equilibrium [10, p. 111].

III) There is one goodwill variable for each player/firm and it is affected by the advertising strategy of all the firms, positively or negatively.
Using the type I games notation, let the dynamics be described by

$$\begin{cases} \dot{G}_i(t) = & u_i(t) - \sum_{j \neq i} \eta_{ij} u_j(t) - \delta G_i(t) \ , \\ G_i(0) = & \alpha_i \ , \end{cases} \tag{35}$$

where $\eta_{ij} \geq 0$ is the interference parameter of player j's advertising on player i's goodwill. In this case the goodwill of some firm may assume negative values and then the associated product demand may suitably be represented by a nonsmooth function.

An example is given by [2], which considers a distribution channel, where a retailer advertises a private label, at the level u_h, while selling also a national brand which is advertised by its manufacturer, at the level u_j. Then the national brand goodwill evolution is determined by the equation

$$\dot{G}_j(t) = u_j(t) - \eta_{ij} u_h(t) - \delta G_j(t) \ , \tag{36}$$

where $\eta_{ij} > 0$ is small. In a different situation, studied in [43], two competing manufacturers have their goodwill variables determined by equations similar to (36) and no constraint is present on the sign of the goodwill production term $u_j(t) - \eta u_h(t)$. Then one firm may have a negative goodwill at some times and with the assumption that the demand function is

$$D_j(G_j) = \beta_j \max\{0, G_j\} \ , \tag{37}$$

the manufacturers problems are nonsmooth optimal control problems.

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