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Control theory / *Théorie du contrôle*

Some Regularity Properties on Bolza problems in the Calculus of Variations

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Abstract. The paper summarizes the main core of the last results that we obtained in [4, 8, 17] on the regularity of the value function for a Bolza problem of a one-dimensional, vectorial problem of the calculus of variations. We are concerned with a nonautonomous Lagrangian that is possibly highly discontinuous in the state and velocity variables, nonconvex in the velocity variable and non coercive. The main results are achieved under the assumption that the Lagrangian is convex on the one-dimensional lines of the velocity variable and satisfies a local Lipschitz continuity condition w.r.t. the time variable, known in the literature as Property (S), and strictly related to the validity of the Erdmann–Du-Bois Reymond equation.

Under our assumptions, there exists a minimizing sequence of Lipschitz functions. A first consequence is that we can exclude the presence of the Lavrentiev phenomenon. Moreover, under a further mild growth assumption satisfied by the minimal length functional, fully described in the paper, the above sequence may be taken with the same Lipschitz rank, even when the initial datum and initial value vary on a compact set. The Lipschitz regularity of the value function follows.

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1. Introduction

The paper summarizes some recent results obtained in [4, 8, 17] on the Bolza problem $(\mathrm{P}_{t,x})$ of the calculus of variations which consists in minimizing an integral functional

$$J_t(y) = \int_t^T \Lambda(s, y(s), y'(s))\,\mathrm{d}s + g(y(T))$$

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as y varies among the absolutely continuous functions on $[t, T]$ satisfying a given initial condition $y(t) = x \in \mathbb{R}^n$. Here Λ is a positive, Lebesgue–Borel Lagrangian and g is a positive function, possibly extended valued. The value function $V(t, x)$ is defined for $t \in [0, T]$ and $x \in \mathbb{R}^n$, by

$$V(t,x) = \inf(\mathrm{P}_{t,x}).$$

Our results are aimed at the study of the regularity of V. Usually one assumes either the validity of Tonelli's existence assumptions or at least the a priori existence of a minimizer *for every* initial pair (t, x), associated with a superlinearity condition as in the autonomous case studied by G. Dal Maso and H. Frankowska in [15]; there the result is obtained via a Lipschitz regularity study of the (hypothetical) minimizers.

However, several problems in the applications concern Lagrangians that are possibly discontinuous, non convex and even non coercive in the velocity variable y'. In such a situation the hypotheses of the celebrated Tonelli's existence theorem are far from being satisfied. The systematic study of problems with slow growth had a strong impulse after the existence and regularity results of F. Clarke in [13], who introduced the growth condition (henceforth named of type (H)) including the length functional corresponding to $\Lambda(u) = \sqrt{1+|u|^2}$.

We propose here another approach to the study of the value function, that enlarges the class of admissible Lagrangians and does not require neither convexity in the velocity variable, nor coercivity, nor continuity in the state or velocity variables. The existence of a minimizer is not required for this approach, but merely a minimizing sequence of functions that are equi-Lipschitz. The result is stated in full generality in [17, Theorem 5.1], in a more general framework of controlled-linear problems, allowing the Lagrangian to be extended valued. It is inspired by the work by A. Cellina and A. Ferriero in [11]. One of the aims of the paper is to describe a shorter self-contained proof of Theorem 12 in the real valued case of the calculus of variations. There are several improvements and new technical details with respect to [11]:

- We require the convexity of $0 < r \mapsto \Lambda(s, y, ru)$ instead of global convexity of Λ w.r.t. u: the role of such a radial convexity appeared in [5–7];
- We do not impose continuity of the Lagrangian in the variables (y, u);
- We consider Clarke's growth condition of type (H), less restrictive than the one (below called of type (G)) considered in [11] (though both are satisfied in the superlinear case);
- We deal with a non autonomous Lagrangian, under a local Lipschitz condition on $s \mapsto \Lambda(s, y, u)$, which is not merely technical.

The growth condition of type (H) requires a lot of care. Quite surprisingly, this effort brings a new result, not considered in [11], namely the nonoccurrence of the Lavrentiev phenomenon without assuming any kind of growth other than the linear one from below: The fact that

$$\inf\{J_t(y) : y \in \mathrm{AC}([t,T];\mathbb{R}^n)\} = \inf\{J_t(y) : y \in \mathrm{Lip}([t,T];\mathbb{R}^n)\}.$$

is well known in the autonomous case (see [2]), but seems new to the authors in the general case without assuming any kind of regularity in the state or velocity. The Maximum Principle, often invoked in similar problems, is not used here due to the lack of Lipschitz continuity properties in the state variable.

2. Basic setting and notation

Let $0 \le t < T$ and $x \in \mathbb{R}^n$. We consider the *Bolza type problem*

$$\min J_t(y) := \int_t^T \Lambda(s, y(s), y'(s))\,\mathrm{d}s + g(y(T)) \qquad (\mathrm{P}_{t,x})$$

subject to

$$\begin{cases} y \in \mathrm{AC}([t,T];\mathbb{R}^n),\ y(t) = x \\ y'(s) \in \mathscr{U} \text{ a.e. } s \in [t,T],\ y(s) \in \mathscr{S} \text{ for all } s \in [t,T], \end{cases} \tag{D}$$

with the following basic assumptions.

Basic Assumptions and Notation. *The following conditions hold.*

- *The* Lagrangian $\Lambda : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty[,\ (s,y,u) \mapsto \Lambda(s,y,u)$ *is* Lebesgue–Borel *measurable (i.e., measurable with respect to the* $\mathscr{L}([0,T]) \times \mathscr{B}_{\mathbb{R}^n \times \mathbb{R}^n}$ *measurable sets);*
- *The Lagrangian* Λ *is bounded on bounded sets;*
- *The state constraint set* $\mathscr{S}$ *is a nonempty subset of* $\mathbb{R}^n$*;*
- *The* velocity set $\mathscr{U} \subset \mathbb{R}^n$ *is a nonempty cone, i.e. if* $u \in \mathscr{U}$ *then* $\lambda u \in \mathscr{U}$ *whenever* $\lambda \geq 0$*;*
- (Linear growth from below) *There are* $\alpha > 0$ *and* $d \geq 0$ *satisfying, for a.e.* $s \in [0,T]$ *and every* $y \in \mathbb{R}^n, u \in \mathscr{U}$,
$$\Lambda(s,y,u) \geq \alpha|u| - d; \tag{1}$$
- *The* cost function $g : \mathbb{R}^n \to [0,+\infty[\, \cup \{+\infty\}$ *is a given positive function, not identically equal to* $+\infty$.

If $t \in [0,T[$ and $x \in \mathbb{R}^n$ an *admissible trajectory* for $(\mathrm{P}_{t,x})$ is a function $y \in \mathrm{AC}([t,T];\mathbb{R}^n)$ such that $y' \in \mathscr{U}$ a.e., $y(s) \in \mathscr{S}$ for all $s \in [t,T]$ and $J_t(y) < +\infty$. We assume the existence of an admissible trajectory for every $t \in [0,T[$ and $x \in \mathbb{R}^n$; this is certainly true, for instance, if $\mathscr{S}$ is convex and $\mathscr{U} = \mathbb{R}^n$, or if g is real valued.

A *minimizing sequence* $(y_j)_j$ for $(\mathrm{P}_{t,x})$ is a sequence of admissible trajectories such that

$$\lim_{j \to +\infty} J_t(y_j) = \inf(\mathrm{P}_{t,x}).$$

If $z \in \mathbb{R}^n$ we shall denote by $B_\delta(z)$ the open ball of center z and radius δ. The norm in L^1 is denoted by $\|\cdot\|_1$, and the norm in L^∞ by $\|\cdot\|_\infty$.

We will denote by $|\cdot|$ both the norm in euclidean spaces and the Lebesgue measure in $\mathbb{R}$; the precise meaning will be clear from the context.

3. Structure assumptions (A) and (S)

In what follows, we assume the structure Assumption (A) on $\Lambda(s,y,u)$ with respect to u and the local Lipschitz condition (S) on $\Lambda(s,y,u)$ with respect to s.

Structure Assumption (A). *For a.e.* $s \in [0,T]$ *and every* $y \in \mathbb{R}^n, u \in \mathscr{U}$*, the map* $0 < r \mapsto \Lambda(s,y,ru)$ *is convex.*

The main results involve the following local Lipschitz condition on the Lagrangian Λ with respect to the time variable.

Condition (S). *There are* $\kappa, A \geq 0, \gamma \in L^1([0,T]), \varepsilon_* > 0$ *satisfying, for a.e.* $s \in [0,T]$

$$|\Lambda(s_2,y,u) - \Lambda(s_1,y,u)| \leq \big(\kappa\Lambda(s,y,u) + A|u| + \gamma(s)\big)|s_2 - s_1| \tag{2}$$

whenever $s_1, s_2 \in [s-\varepsilon_*, s+\varepsilon_*] \cap [0,T]$, $y \in \mathbb{R}^n$, $u \in \mathscr{U}$.

Remark 1. Condition (S) is satisfied if $\Lambda(s,y,u) = \Lambda(y,u)$ is *autonomous*. Indeed in that case (2) holds with $\kappa = A = 0, \gamma \equiv 0$ and $\varepsilon_* = T$. Condition (S) is satisfied if, for all $[0,T]$, $y \in \mathbb{R}^n$, $u \in \mathscr{U}$, the map $s \mapsto \Lambda(s,y,u)$ is differentiable and

$$|D_s\Lambda(s,y,u)| \leq \beta\big(\Lambda(s,y,u) + |u| + 1\big). \tag{3}$$

We refer to [17, Proposition 3.3] for a proof.

4. Growth conditions

We discuss here some weak growth conditions that were considered, starting from [13], for the basic problem of the calculus of variations. In the smooth setting, the conditions concern the behavior of the *Hamiltonian* defined by

$$\forall\ (s,y,u,\xi)\in[0,T]\times\mathbb{R}^{3n}\qquad H(s,y,u,\xi):=\xi\cdot u-\Lambda(s,y,u)$$

at $\xi=\nabla_u\Lambda(s,y,u)$, as $|u|\to+\infty$. In order to fully understand the motivations of the conditions below (H^δ_B) and (G), it is important to observe that $-H(s,y,u,\nabla_u\Lambda(s,y,u))$ represents the ordinate of the intersection of the tangent line to the graph of

$$0<r\mapsto z=\Lambda(s,y,ru)$$

at $r=1$ with the z axis.

In the nonsmooth setting, under the validity of Condition (A), the quantity $\Lambda(s,y,u)-u\cdot\nabla_u\Lambda(s,y,u)$ is replaced by

$$\Lambda(s,y,u)-Q(s,y,u),$$

where $Q(s,y,u)$ is a convex subgradient of $0<r\mapsto\Lambda(s,y,ru)$ at $r=1$, i.e., a real valued function such that

$$\forall\ r>0\qquad\Lambda(s,y,ru)-\Lambda(s,y,u)\geq Q(s,y,u)(r-1).$$

4.1. *Some bounds*

We begin with a recent result formulated in [17], showing that, under suitable assumptions, for any given $c>0$ and $\overline{\nu}>0$, the term $L(s,y,u)-Q(s,y,u)$ is bounded above as $|u|\geq\overline{\nu}$ and bounded below as $|u|<c$, even in the nonsmooth setting. We observe that this property is valid without assuming any growth conditions, neither convexity nor regularity. Quite surprisingly it is the key point in the proof of the non occurrence of the Lavrentiev Phenomenon (Claim (1) of Theorem 12). We refer to [17, Proposition 4.24] for the proof of Proposition 2.

Proposition 2 (Some bounds for $\Lambda-\partial_r\Lambda(s,y,ru)_{r=1}$). *Assume that Λ satisfies Condition* (A). *Let $Q(s,y,u)\in\partial_r\Lambda(s,y,ru)_{r=1}$. For all $K\geq0,c>0,\overline{\nu}>0$ we have*

$$\inf_{\substack{s\in[0,T],|y|\leq K\\|u|<c,u\in\mathcal{U}}}\left\{\Lambda(s,y,u)-Q(s,y,u)\right\}\in\mathbb{R},\tag{4}$$

$$\sup_{\substack{s\in[0,T],|y|\leq K\\|u|\geq\overline{\nu},u\in\mathcal{U}}}\{\Lambda(s,y,u)-Q(s,y,u)\}<+\infty.\tag{5}$$

4.2. *Growth Condition* (H^δ_B)

A weak growth condition called (H) was first introduced in [13] under the assumption that Λ is convex in the velocity variable and, more recently, considered in [6, 7] in a more general non convex setting. Condition (H^δ_B) below, introduced in [17], is the natural generalization of the above (H) for problems of the calculus of variations: B is an upper bound of $\inf(\mathrm{P}_{t,x})$, where the initial time t may vary on an interval $[0,\delta]$ and the initial point x varies in a compact set.

Definition 3 ($c_t(B)$ and $\Phi(B)$). *Let $t\in[0,T[$, $B\geq0$ and assume the linear growth from below* (1), *i.e., for a.e. $s\in[0,T]$, for all $y\in\mathbb{R}^n$, $u\in\mathcal{U}$,*

$$\Lambda(s,y,u)\geq\alpha|u|-d\quad(\alpha>0,d\geq0).$$

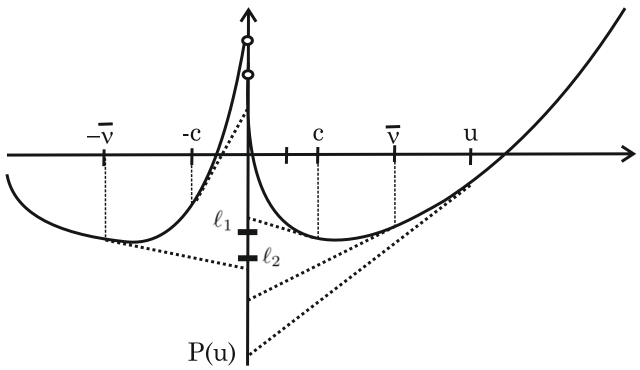

Figure 1. $\inf_{|u|<c}\left\{\Lambda(u)-Q(u)\right\}\geq \ell_1;\sup_{|u|\geq\overline{\nu}}\{\Lambda(u)-Q(u)\}\leq \ell_2$

Let

$$c_t(B):=\frac{B+d(T-t)}{\alpha\,(T-t)}.$$

Moreover, if Condition (S) *holds, we define*

$$\Phi(B):=\kappa B+\frac{A}{\alpha}(B+d\,T)+\|\gamma\|_1, \tag{6}$$

where we set κ, A, γ *equal to 0 if* Λ *is* autonomous.

Remark 4. Notice that, in Definition 3, $t\in[0,T[\mapsto c_t(B)$ and $0\leq B\mapsto c_t(B)$ are increasing. In the autonomous case, since κ, A and γ may be chosen to be equal to 0, we consider $\Phi(B):=0$.

We refer to [17, Proposition 4.10] for the proof of Proposition 5, a key tool in the proof of the main result.

Proposition 5 (The roles of $\Phi(B)$ and $c_t(B)$). *Assume the linear growth from below* (1) *and the validity of Condition* (S). *Let* $t\in[0,T[$, $x\in\mathbb{R}^n$, y *be admissible for* $(\mathrm{P}_{t,x})$ *with* $J_t(y)\leq B$ *for some* $B\geq 0$. *Then*

(1) $\displaystyle\int_t^T |y'(s)|\,\mathrm{d}s\leq\frac{B+\mathrm{d}(T-t)}{\alpha}=(T-t)c_t(B).$

(2) *For every* $\sigma>c_\delta(B)$ *the set* $\{s\in[t,T]:|u(s)|<\sigma\}$ *is non negligible.*

(3) $\displaystyle\int_t^T\left\{\kappa\Lambda(s,y(s),y'(s))+A|y'(s)|+\gamma(s)\right\}\mathrm{d}s\leq\Phi(B).$

Assume that Condition (S) holds and let B be an upper bound of the values of a prescribed family of admissible trajectories for $(\mathrm{P}_{t,x})$ as t varies in an interval $[0,\delta]$. In light of Proposition 2, the next Condition (H^δ_B) imposes for any $K\geq 0$ and suitable $\overline{\nu}>0$ and $c>c_\delta(B)$, a specific gap between the terms in (4).

Growth Condition ($\mathbf{H^\delta_B}$). *Assume that* Λ *satisfies Conditions* (A) *and* (S). *Let* $B\geq 0$ *and* $0\leq\delta<T$. *We say that* Λ *satisfies* (H^δ_B) *if there are a selection* $Q(s,y,u)$ *of* $\partial_r\Lambda(s,y,ru)_{r=1}$, $\overline{\nu}>0$ *and* $c>c_\delta(B)$ *satisfying, for all* $K\geq 0$,

$$\sup_{\substack{s\in[0,T],|y|\leq K\\ |u|\geq\overline{\nu},u\in\mathscr{U}}}\{\Lambda(s,y,u)-Q(s,y,u)\}+2\Phi(B)<\inf_{\substack{s\in[0,T],|y|\leq K\\ |u|<c,u\in\mathscr{U}}}\{\Lambda(s,y,u)-Q(s,y,u)\}. \tag{7}$$

Remark 6. If $u\mapsto\Lambda(s,y,u)$ is differentiable, (7) is equivalent to

$$\sup_{\substack{s\in[0,T],|y|\leq K\\ |u|\geq\overline{\nu},u\in\mathscr{U}}}\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\Lambda\left(s,y,\frac{u}{\mu}\right)\mu\right]_{\mu=1}+2\Phi(B)<\inf_{\substack{s\in[0,T],|y|\leq K\\ |u|<c,u\in\mathscr{U}}}\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\Lambda\left(s,y,\frac{u}{\mu}\right)\mu\right]_{\mu=1}; \tag{8}$$

which may be rewritten in terms of the *Hamiltonian* as

$$\inf_{\substack{s\in[0,T],|y|\le K\\ |u|<c,u\in\mathscr{U}}} H(s,y,u,\nabla_u\Lambda(s,y,u)) > \sup_{\substack{s\in[0,T],|y|\le K\\ |u|\ge\overline{\nu},u\in\mathscr{U}}} H(s,y,u,\nabla_u\Lambda(s,y,u)) + 2\Phi(B). \tag{9}$$

Remark 7 (Interpretation of Condition ($\mathrm{H}_{\mathbf{B}}^{\delta}$)). Consider for simplicity a Lagrangian $\Lambda(u)$ of the variable u. Let $\Lambda(u) < +\infty$ and let $Q(u) \in \partial_r\Lambda(u)$. Notice that

$$\forall\ r>0 \quad \Lambda(ru) \ge \phi_u(r) := \Lambda(u) + Q(u)(r-1).$$

The value $\phi_u(0) = P(u) := \Lambda(u) - Q(u)$ represents the intersection of the "tangent" line $z = \phi_u(r)$ to $0 < r \mapsto \Lambda(ru)$ at $r = 1$ with the z axis.

Condition (H_B^{δ}) means that there is a gap of at least $2\Phi(B)$ between the above value of $P(u)$ as $|u| \ge \overline{\nu}$ and the value of $P(u)$ as $|u| < c$, more precisely we have (see Figure 2):

$$\sup_{|u|\ge\overline{\nu}} P(u) + 2\Phi(B) < \inf_{|u|<c} P(u).$$

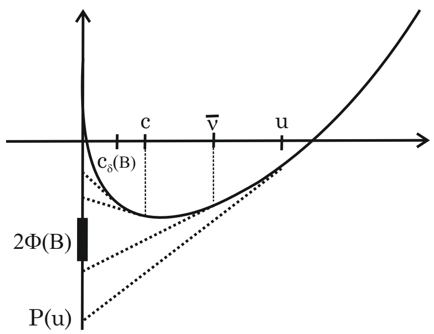

Figure 2. Condition (H_B^{δ}) and the gap of at least $2\Phi(B)$

4.3. *The Growth Condition* (G)

In the growth condition (G) it is required that, for each $K \ge 0$, the term

$$\sup_{\substack{s\in[0,T],|y|\le K\\ |u|\ge\nu,u\in\mathscr{U}}} \{\Lambda(s,y,u) - Q(s,y,u)\}$$

in the left-hand side part of (7) tends to $-\infty$ as $\nu \to +\infty$. The condition (for Lagrangians that are convex in the velocity variable) was introduced by Cellina, Treu and Zagatti in [12], while the version presented here, requiring just radial convexity in the velocity variable, appeared in [18] in the autonomous case.

Growth Condition (G). *We say that* Λ *satisfies* (G) *if* Λ *satisfies Condition (A) and there is a selection* $Q(s,y,u)$ *of the convex subgradient* $\partial_r\Lambda(s,y,ru)_{r=1}$ *such that, for each* $K \ge 0$ *fixed,*

$$\lim_{\substack{|u|\to+\infty\\ u\in\mathscr{U}}} \{\Lambda(s,y,u) - Q(s,y,u)\} = -\infty \ \textit{unif.}\ |y| \le K \tag{10}$$

i.e., if for all $M \in \mathbb{R}$ *there exists* $R > 0$ *such that,*

$$(s,y,u) \in [0,T]\times\mathbb{R}^n\times\mathscr{U},\ |y|\le K,\ |u|>R \Rightarrow \Lambda(s,y,u) - Q(s,y,u) < M. \tag{11}$$

Remark 8.

(1) Superlinearity implies the validity of Condition (G) (see [7, Proposition 2]);
(2) If $u \mapsto \Lambda(s,y,u)$ is differentiable, (10) becomes

$$\lim_{|u|\to+\infty} \Lambda(s,y,u) - u\cdot\nabla_u\Lambda(s,y,u) = -\infty \text{ unif. } |y|\le K. \tag{12}$$

Example 9. Let

$$\forall\,(s,y,u)\in[0,1]\times\mathbb{R}^2 \qquad \Lambda(s,y,u) := h(s,y)\big(|u|-\sqrt{|u|}\big),$$

where h is Borel and bounded on bounded sets. Then Λ satisfies Condition (G). Indeed, $0<r\mapsto h(s,y)(r|u|-\sqrt{r|u|})$ is convex for all $u\in\mathbb{R}$ and, for all $u\ne 0$,

$$\Lambda(s,y,u) - u\frac{d}{du}\Lambda(s,y,u) = -h(s,y)\frac{\sqrt{|u|}}{2}\to-\infty$$

as $|u|\to+\infty$ uniformly for $s\in[0,1]$ and y in bounded sets.

The next Example, that illustrates the importance of Condition (H^δ_B), is taken from [13, Example 4.3].

Example 10. [13, Example 4.3] The function

$$\forall\,(s,y,u)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n \qquad \Lambda(s,y,u)=\Lambda(u):=\sqrt{1+|u|^2}$$

satisfies Condition (H^δ_B) for any choice of $\delta\ge0$ and B. Notice that Λ does not satisfy Condition (G).

Condition (G) implies the validity of Condition (H^δ_B), no matter what $B\ge0$ and $\delta\ge0$ are.

Proposition 11. *Assume that Λ satisfies Condition* (A). *Let*

$$Q(s,y,u)\in\partial_r\Lambda(s,y,ru)_{r=1}.$$

If Λ satisfies Condition (G) *and Condition* (S) *then Λ satisfies Hypothesis* (H^δ_B)*, independently of the choice of $\delta\in[0,T[$ and $B\ge0$.*

Proof. Let $\delta\in[0,T[$ and $B\ge0$. Fix any $K\ge0$. Let $Q(s,y,u)\in\partial_r\Lambda(s,y,ru)_{r=1}$ be such that

$$\lim_{\substack{|u|\to+\infty\\ |y|\le K,\,u\in\mathcal{U}}} \big\{\Lambda(s,y,u)-Q(s,y,u)\big\} = -\infty \ \text{ unif. } s\in[0,T].$$

Then

$$\lim_{\nu\to+\infty}\ \sup_{\substack{s\in[0,T]\\ |u|\ge\nu,\,u\in\mathcal{U}}} \big\{\Lambda(s,y,u)-Q(s,y,u)\big\} = -\infty. \tag{13}$$

Fix $c>c_\delta(B)$. Now, (7) of Proposition 2 implies that

$$\inf_{\substack{s\in[0,T]\\ |u|<c,\,u\in\mathcal{U}}} \big\{\Lambda(s,y,u)-Q(s,y,u)\big\} > -\infty.$$

It follows from (13) that, for $\overline{\nu}$ big enough,

$$\sup_{\substack{s\in[0,T]\\ |u|\ge\overline{\nu},\,u\in\mathcal{U}}} \big\{\Lambda(s,y,u)-Q(s,y,u)\big\} + 2\Phi(B) < \inf_{\substack{s\in[0,T]\\ |u|<c,\,u\in\mathcal{U}}} \big\{\Lambda(s,y,u)-Q(s,y,u)\big\},$$

proving the validity of Condition (H^δ_B). $\square$

5. Existence of minimizing sequences of Lipschitz functions

When the basic problem $(\mathrm{P}_{t,x})$ admits a minimizer, one important issue is establishing its regularity. Lipschitz continuity is a first step in this direction: several results have been obtained in the last decades, long time after the pioneer works of Tonelli (see [2, 6, 7, 10, 13–15, 18]). When Lipschitz minimizers are not expected to exist, an important property to investigate is the occurrence of the Lavrentiev phenomenon, i.e., the fact that despite Lipschitz functions are dense in the absolutely continuous functions, it might happen that

$$\inf_{\substack{z \text{ admissible}\\ z\in \mathrm{AC}([t,T];\mathbb{R}^n)}} J_t(z) < \inf_{\substack{z \text{ admissible}\\ z\in \mathrm{Lip}([t,T];\mathbb{R}^n)}} J_t(z).$$

This phenomenon was highlighted by Lavrentiev in early 1900's (cf. [9, 16]) with an example. In [3] Ball and Mizel exhibited the phenomenon in the case of a polynomial non-autonomous Lagrangian, both superlinear and convex in the velocity variable (thus satisfying Tonelli's existence result). In the autonomous case, Alberti and Serra Cassano proved in [1] that the Lavrentiev phenomenon does not occur. Adding the requirement that Λ satisfies the growth condition (G), Cellina and Ferriero proved in [11] that the infimum of $(\mathrm{P}_{t,x})$ may be reached through a sequence of equi-Lipschitz functions.

Theorem 12 extends the above results; it summarizes the contents of [17, Theorem 5.1] and [17, Corollaries 5.7 and 5.9] in the special case of the calculus of variations for real valued, non autonomous Lagrangians.

Theorem 12 (Existence of nice admissible trajectories [17]). *Assume that Λ satisfies Conditions* (A) *and* (S). *Then*

(1) (No Lavrentiev phenomenon) *For every $t \in [0,T[$ and $x \in \mathbb{R}^n$ there is a minimizing sequence of Lipschitz functions for* $(\mathrm{P}_{t,x})$.

(2) (Minimizing sequences of equi-Lipschitz functions) *Let $0 \le \delta < T$, $\delta_* \ge 0$, $x_* \in \mathbb{R}^n$ and assume that Λ satisfies Condition* (H_B^δ) *for every $B \ge 0$, $\delta \in [0,T[$. Suppose, moreover, one of the two additional hypotheses:*
- *$\mathscr{S}$ is convex and $\mathscr{U} = \mathbb{R}^n$, or*
- *g is real valued and locally bounded, and* $(0 \in \mathscr{U})$ *or* $(\mathscr{S} = \mathbb{R}^n)$.

Then, for every $t \in [0,\delta]$ and $x \in B_{\delta_*}(x_*)$, there is a minimizing sequence for $(\mathrm{P}_{t,x})$ *of equi-Lipschitz functions with rank depending only on B, δ, x_*, δ_*.*

The proof of Theorem 12 is postponed to Section 7.

Remark 13. Condition (S) plays a central role here. The celebrated example of Ball and Mizel [3] shows that if Condition (S) does not hold, then the Lavrentiev phenomenon might occur.

6. Lipschitz continuity of the value function

Recall that the *value function V* associated with problems $(\mathrm{P}_{t,x})$ is the function defined by

$$\forall\ t \in [0,T], \forall\ x \in \mathbb{R}^n \qquad V(t,x) = \inf(\mathrm{P}_{t,x}).$$

Since g is not identically $+\infty$ it follows that $V(t,x) < +\infty$ for every (t,x). Typically, the regularity of the value fuction is obtained by assuming the a priori existence of minimizers for the problems $(\mathrm{P}_{t,x})$ and superlinearity of the Lagrangian (see [15]). None of these conditions is needed in the following regularity result for the value function (see [4, 8]).

Theorem 14 (Lipschitz continuity of the value function [4, 8]). *Assume that Λ satisfies Assumptions* (A) *and* (S) *and satisfies the growth condition* (H_B^δ) *for every $B \ge 0$, $\delta \in [0,T[$ and that the conditions of Claim* (2) *of Theorem 12 are in force. Then the value function $V(t,x)$ is locally Lipschitz.*

Sketch of the proof. Let $t_0 \in [0, T[$ and $0 < \varepsilon < T - t_0$. The application of Theorem 12 with $\delta = t_0 + \varepsilon, \delta_* = \varepsilon$ shows that there is $K \geq 0$ and minimizing sequences for $(\mathrm{P}_{t,x})$ that are Lipschitz with rank less than K as (t, x) vary in $[t_0 - \varepsilon, t_0 + \varepsilon] \times B_\varepsilon(x_*)$.

The arguments of [4, Lemma 6.1] (cf. also [8, Section 5]) or, in the autonomous case, of [15], allow to conclude. □

7. Proof of Theorem 12

The proof of Theorem 12 adapts and simplifies the proof of [17, Theorem 5.1] to the less technical case of a real valued Lagrangian of the calculus of variations. Fix an admissible trajectory y for $(\mathrm{P}_{t,x})$ such that $J_t(y) \leq \inf(\mathrm{P}_{t,x}) + 1$. For Claim (1) we set $B := \inf(\mathrm{P}_{t,x}) + 1$. Under the assumptions of Claim (2), where t and x vary, it is easy to show (see [17, Lemma 5.3]) that there is $B \geq 0$ such that

$$\forall\, t \in [0, \delta], \forall\, x \in B_{\delta_*}(x_*) \qquad \inf(\mathrm{P}_{t,x}) + 1 \leq B. \tag{14}$$

Let

$$\begin{cases} \eta = 0 & \text{if } (\mathrm{H}_B^\delta) \text{ holds,} \\ \eta > 0 & \text{otherwise.} \end{cases}$$

A crucial point of the proof consists in constructing a Lipschitz function $\overline{y}$ such that

$$J_t(\overline{y}) \leq J_t(y) + \eta.$$

This is built up in several steps. We make use of similar arguments for the proofs of Claims (1) and (2) of Theorem 12. We will highlight the differences in the final step.

(i). *Let α, d be as in* (1). *Then*

$$\int_t^T |y'(s)| \,\mathrm{d}s \leq (T - t) c_t(B) \leq R = R(B) := \frac{B + \mathrm{d}T}{\alpha}.$$

The claim follows immediately from (1) of Proposition 5.

(ii). *There is $K := K(B, x_*, \delta_*)$ such that $|y(s)| \leq K$ for every $s \in [t, T]$.* Indeed, it follows from Step (i) that for all s,

$$|y(s)| \leq |x| + \int_t^s |y'(\tau)| \,\mathrm{d}\tau \leq |x_*| + \delta_* + R.$$

(iii). *Choice of μ.* Let $c > c_\delta(B)$ and $Q(s, y, u) \in \partial_r \Lambda(s, y, r\,u)_{r=1}$; if Condition (H_B^δ) holds we assume moreover the validity of (7). We fix $\mu = \mu(\delta, B) \in]0, 1[$ in such a way that

$$\frac{c_\delta(B)}{\mu} < c. \tag{15}$$

(iv). *Definition of* Ω. Define

$$\Omega := \left\{ s \in [t, T] : \frac{|y'(s)|}{\mu} < c \right\}.$$

Then

$$|\Omega| \geq \left(1 - \frac{c_\delta(B)}{\mu c} \right) (T - t). \tag{16}$$

Indeed, from Step (i),

$$(T - t) c_\delta(B) \geq (T - t) c_t(B) \geq \int_{[t,T] \setminus \Omega} |y'(s)| \,\mathrm{d}s \geq c\mu |[t, T] \setminus \Omega|,$$

the claim follows.

(v). *Definition of* $\Xi(\nu)$, Υ. We set $P(s,z,v) := \Lambda(s,z,v) - Q(s,z,v)$. For $\nu > 0$ we define

$$\Xi(\nu) := \sup_{\substack{s\in[0,T],|z|\le K\\ |v|\ge\nu, v\in\mathscr{U}}} P(s,z,v), \quad \Upsilon := \inf_{\substack{s\in[0,T],|z|\le K\\ |v|<c, v\in\mathscr{U}}} P(s,z,v).$$

From Proposition 2 we know that $\Xi(\nu) < +\infty$ and $\Upsilon \in \mathbb{R}$. We may assume that $\Xi(\nu) > -\infty$ for every ν, otherwise $\mathscr{U}$ is bounded and the result follows trivially.

(vi). *Choice of* $\overline{\nu}$. If Condition (H_B^δ) holds let $\overline{\nu} = \overline{\nu}(B,K) = \overline{\nu}(B,\delta,x_*,\delta_*) > 0$ be such that

$$\forall\, \nu \ge \overline{\nu} \qquad \Xi(\nu) + 2\Phi(B) < \Upsilon. \tag{17}$$

If, as in Claim (1), we do not assume Condition (H_B^δ), we choose $\overline{\nu} = \overline{\nu}(B,K) = \overline{\nu}(B,x_*,\delta_*,\eta)$ large enough in such a way that

$$\frac{R}{\overline{\nu}}(2\Phi(B) + \Xi(\overline{\nu}) - \Upsilon) \le \eta,$$

so that, Ξ being decreasing,

$$\forall\, \nu \ge \overline{\nu} \quad \frac{R}{\nu}(2\Phi(B) + \Xi(\nu) - \Upsilon) \le \eta. \tag{18}$$

(vii). *For a.e.* $s \in \Omega$ *and a.e.* $\widetilde{s} \in [t,T]$,

$$\Lambda\left(\widetilde{s}, y(s), \frac{y'(s)}{\mu}\right)\mu - \Lambda(\widetilde{s}, y(s), y'(s)) \le -(1-\mu)\Upsilon. \tag{19}$$

Indeed for $\widetilde{s} \in [t,T]$ and a.e. $s \in \Omega$,

$$\Lambda\left(\widetilde{s}, y(s), \frac{y'(s)}{\mu}\right)\mu - \Lambda(\widetilde{s}, y(s), y'(s)) = -\mu\left[\Lambda\left(\widetilde{s}, y(s), \frac{y'(s)}{\mu}\mu\right)\frac{1}{\mu} - \Lambda\left(\widetilde{s}, y(s), \frac{y'(s)}{\mu}\right)\right]. \tag{20}$$

We have

$$\Lambda\left(\widetilde{s}, y(s), \frac{y'(s)}{\mu}\mu\right)\frac{1}{\mu} - \Lambda\left(\widetilde{s}, y(s), \frac{y'(s)}{\mu}\right) \ge P\left(\widetilde{s}, y(s), \frac{y'(s)}{\mu}\right)\frac{1-\mu}{\mu},$$

and since on Ω, from Step (iv), $\frac{|y'(s)|}{\mu} < c$, we obtain, $\mathscr{U}$ being a cone, that

$$\Lambda\left(\widetilde{s}, y(s), \frac{y'(s)}{\mu}\mu\right)\frac{1}{\mu} - \Lambda\left(\widetilde{s}, y(s), \frac{y'(s)}{\mu}\right) \ge \frac{1-\mu}{\mu} \inf_{\substack{\widetilde{s}\in[0,T],|z|\le K\\ |v|<c, v\in\mathscr{U}}} P(\widetilde{s}, z, v) = \frac{1-\mu}{\mu}\Upsilon, \tag{21}$$

and thus the conclusion follows from (20).

(viii). *For every* $\nu > 0$ *define*

$$S_\nu := \{s \in [t,T] : |y'(s)| > \nu\}, \quad \varepsilon_\nu := \int_{S_\nu}\left(\frac{|y'(s)|}{\nu} - 1\right) \mathrm{d}s.$$

Then

$$|S_\nu| \to 0, \quad \varepsilon_\nu \le \frac{R}{\nu} \to 0 \text{ as } \nu \to +\infty$$

uniformly with respect to $t \in [0,\delta]$ *and* $x \in B_{\delta_*}(x_*)$. Indeed, it follows from Step (i) that

$$\nu|S_\nu| \le \int_{S_\nu} |y'(s)|\,\mathrm{d}s \le R.$$

(ix). *Choice of* ν *and of* $\Sigma_\nu \subset \Omega$. Taking into account Step (viii), we choose $\nu = \nu(B,\delta,x_*,\delta_*) \ge \max\{\overline{\nu}, c\}$ in such a way that

$$(\varepsilon_\nu \le)\ \frac{R}{\nu} \le \min\left\{(1-\mu)\left(1 - \frac{c_\delta(B)}{\mu c}\right)(T-\delta), \frac{\varepsilon_*}{2}\right\}. \tag{22}$$

Choose a measurable subset $\Sigma_\nu \subseteq \Omega$ in such a way that $|\Sigma_\nu| = \frac{\varepsilon_\nu}{1-\mu}$: this is possible since, from (22) and Step (iv),

$$\frac{\varepsilon_\nu}{1-\mu} \le \left(1 - \frac{c_\delta(B)}{\mu c}\right)(T-\delta) \le |\Omega|.$$

From now on we set $\Xi := \Xi(\nu)$.

(x). *The set $S_\nu \cap \Omega$ is negligible.* Indeed, if $s \in S_\nu$ then $|y'(s)| > \nu$, whereas if $s \in \Omega$ then $|y'(s)| < \mu c < c$ so that Step (ix) implies $|y'(s)| < \nu$, whence the claim.

(xi). *The change of variable φ.* We introduce the following absolutely continuous change of variable $\varphi : [t,T] \to \mathbb{R}$ defined by

$$\varphi(t) := t, \quad \text{for a.e. } \tau \in [t,T] \quad \varphi'(\tau) = \begin{cases} \dfrac{|y'(\tau)|}{\nu} & \text{if } \tau \in S_\nu \\ \mu & \text{if } \tau \in \Sigma_\nu \\ 1 & \text{otherwise.} \end{cases}$$

Notice that φ is well defined since $S_\nu \cap \Sigma_\nu$, a subset of $S_\nu \cap \Omega$, is negligible. Clearly φ is strictly increasing and, from Steps (viii) and (ix), the image of φ is $[t,T]$ and thus $\varphi : [t,T] \to [t,T]$ is bijective; let us denote by ψ its inverse, which is absolutely continuous and even Lipschitz, since $\|\psi'\|_\infty \le \frac{1}{\mu}$.

(xii). *Set $\overline{y} := y \circ \psi$. Then $\overline{y}$ is admissible, $\overline{y}(t) = y(t)$ and $\overline{y}(T) = y(T)$.* It follows from [20, Corollary 5] and [19, Chapter IX, Theorem 5] that $\overline{y}$ is absolutely continuous and, for a.e. $s \in [t,T]$,

$$\overline{y}'(s) = \frac{y'(\psi(s))}{\varphi'(\psi(s))}.$$

Since $\overline{y}$ is defined via a reparametrization of y, we still have that $\overline{y}(s) \in \mathscr{S}$ for all s. The fact that $\psi(t) = t, \psi(T) = T$ yields $\overline{y}(t) = y(t)$ and $\overline{y}(T) = y(T)$. Notice that $\overline{y}'(s) = \frac{1}{\varphi'(\psi(s))} y'(\psi(s)) \in \mathscr{U}$ for a.e. $s \in [t,T]$, the set $\mathscr{U}$ being a cone.

(xiii). *$\overline{y}$ is Lipschitz; if Condition (H_B^δ) holds then the Lipschitz rank of $\overline{y}$ depends just on B, δ, x_*, δ_*; otherwise it might depend also on η.*

It is convenient to write explicitly the function $\overline{y}'(s)$, which is given by

$$\overline{y}'(s) = \begin{cases} \nu \dfrac{y'(\psi(s))}{|y'(\psi(s))|} & \text{if } \psi(s) \in S_\nu, \\ \dfrac{y'(\psi(s))}{\mu} & \text{if } \psi(s) \in \Sigma_\nu, \\ y'(\psi(s)) & \text{otherwise.} \end{cases}$$

Since $|y'(s)| \le \nu$ a.e. out of S_ν it turns out from the fact that $\Sigma_\nu \subseteq \Omega$ and Step *ix)* that

$$|\overline{y}'(s)| \le \max\{\nu, c\} \le \nu.$$

The dependence of ν on B, δ, x_*, δ_* (and, possibly, on η if we do not assume (H_B^δ)) follows from Step (vi).

(xiv). $\|\varphi(\tau) - \tau\|_\infty \le 2\varepsilon_\nu \le \varepsilon_*$. Indeed, for all $\tau \in [t,T]$ we have

$$\begin{aligned} |\varphi(\tau) - \tau| &\le \int_t^\tau \left|\varphi'(s) - 1\right| \mathrm{d}s \\ &\le \int_{S_\nu} \left(\frac{|y'(s)|}{\nu} - 1\right) \mathrm{d}s + \int_{\Sigma_\nu} (1-\mu)\,\mathrm{d}s \\ &\le \varepsilon_\nu + (1-\mu)|\Sigma_\nu| = 2\varepsilon_\nu \le \varepsilon_*, \end{aligned}$$

where the last inequality follows from (22) in Step (ix).

(xv). *Estimate of $J_t(\overline{y})$ in terms of $\int_t^T \Lambda\big(\varphi(\tau), y(\tau), y'(\tau)\big)\,\mathrm{d}\tau$.* Since y and $\overline{y}$ share the same boundary values, we have

$$J_t(\overline{y}) = \int_t^T \Lambda(s, \overline{y}(s), \overline{y}'(s))\,\mathrm{d}s + g(y(T)). \tag{23}$$

The change of variables $s=\varphi(\tau)$ yields

$$\begin{aligned}\int_t^T \Lambda(s,\overline{y}(s),\overline{y}'(s))\,\mathrm{d}s &= \int_t^T \Lambda\left(\varphi(\tau),y(\tau),\frac{y'(\tau))}{\varphi'(\tau)}\right)\varphi'(\tau)\,\mathrm{d}\tau \\ &= I_{S_\nu}+I_{\Sigma_\nu}+I_1,\end{aligned}\tag{24}$$

where we set

$$\begin{aligned}I_{S_\nu} &:= \int_{S_\nu}\Lambda\left(\varphi(\tau),y(\tau),\nu\frac{y'(\tau)}{|y'(\tau)|}\right)\frac{|y'(\tau)|}{\nu}\,\mathrm{d}\tau,\\ I_{\Sigma_\nu} &:= \int_{\Sigma_\nu}\Lambda\left(\varphi(\tau),y(\tau),\frac{y'(\tau)}{\mu}\right)\mu\,\mathrm{d}\tau,\\ I_1 &:= \int_{[t,T]\setminus(\Sigma_\nu\cup S_\nu)}\Lambda\big(\varphi(\tau),y(\tau),y'(\tau)\big)\,\mathrm{d}\tau.\end{aligned}$$

- *Estimate of I_{S_ν}.*

$$I_{S_\nu}\le\int_{S_\nu}\Lambda\big(\varphi(\tau),y(\tau),y'(\tau)\big)\,\mathrm{d}\tau+\Xi\varepsilon_\nu.\tag{25}$$

  Indeed, for a.e. $\tau\in S_\nu$ we have

$$\Lambda\big(\varphi(\tau),y(\tau),y'(\tau)\big)\frac{\nu}{|y'(\tau)|}-\Lambda\left(\varphi(\tau),y(\tau),\nu\frac{y'(\tau)}{|y'(\tau)|}\right)\ge P\left(\varphi(\tau),y(\tau),\nu\frac{y'(\tau)}{|y'(\tau)|}\right)\left(\frac{\nu}{|y'(\tau)|}-1\right)\tag{26}$$

  which implies

$$\Lambda\left(\varphi(\tau),y(\tau),\nu\frac{y'(\tau)}{|y'(\tau)|}\right)\frac{|y'(\tau)|}{\nu}-\Lambda\big(\varphi(\tau),y(\tau),y'(\tau)\big)\le P\left(\varphi(\tau),y(\tau),\nu\frac{y'(\tau)}{|y'(\tau)|}\right)\left(\frac{|y'(\tau)|}{\nu}-1\right).\tag{27}$$

  Since, for a.e. $\tau\in S_\nu$, $|y'(\tau)|>\nu$ and $\left|\nu\frac{y'(\tau)}{|y'(\tau)|}\right|=\nu$, we deduce from the fact that $\mathscr{U}$ is a cone that $\nu\frac{y'(\tau)}{|y'(\tau)|}\in\mathscr{U}$ and thus

$$\text{For a.e. } s\in S_\nu\quad P\left(\varphi(\tau),y(\tau),\nu\frac{y'(\tau)}{|y'(\tau)|}\right)\left(\frac{|y'(\tau)|}{\nu}-1\right)\le\left(\frac{|y'(\tau)|}{\nu}-1\right)\Xi.$$

  Therefore, for a.e. $\tau\in S_\nu$ inequality (27) yields

$$\Lambda\left(\varphi(\tau),y(\tau),\nu\frac{y'(\tau)}{|y'(\tau)|}\right)\frac{|y'(\tau)|}{\nu}\le\Lambda\big(\varphi(\tau),y(\tau),y'(\tau)\big)+\left(\frac{|y'(\tau)|}{\nu}-1\right)\Xi,$$

  whence (25).
- *Estimate of I_{Σ_ν}.* The function ψ being Lipschitz, the set of $\tau\in[t,T]$ such that $\widetilde{s}=\varphi(\tau)$ satisfies (19) is of full measure in $[t,T]$. Since $\Sigma_\nu\subset\Omega$ and $|\Sigma_\nu|=\frac{\varepsilon_\nu}{1-\mu}$, it is immediate from (19) that

$$I_{\Sigma_\nu}\le\int_{\Sigma_\nu}\Lambda\big(\varphi(\tau),y(\tau),y'(\tau)\big)\,\mathrm{d}\tau-\Upsilon\varepsilon_\nu.\tag{28}$$

Finally, from (23), (24), (25) and (28) we obtain

$$J_t(\overline{y})\le\int_t^T\Lambda\big(\varphi(\tau),y(\tau),y'(\tau)\big)\,\mathrm{d}\tau+\varepsilon_\nu\big(\Xi-\Upsilon\big)+g(y(T)).\tag{29}$$

(xvi). *Estimate of $\int_t^T\Lambda\big(\varphi(\tau),y(\tau),y'(\tau)\big)\,\mathrm{d}\tau$:*

$$\int_t^T\Lambda\big(\varphi(\tau),y(\tau),y'(\tau)\big)\,\mathrm{d}\tau\le\int_t^T\Lambda\big(\tau,y(\tau),y'(\tau)\big)\,\mathrm{d}\tau+2\Phi(B)\varepsilon_\nu.\tag{30}$$

Indeed, Condition (S) together with Step *xiv)* imply that, for a.e. $\tau\in[t,T]$,

$$\begin{aligned}\Lambda\big(\varphi(\tau),y(\tau),y'(\tau)\big)&\le\Lambda\big(\tau,y(\tau),y'(\tau)\big)+k(\tau)|\varphi(\tau)-\tau|\\ &\le\Lambda\big(\tau,y(\tau),y'(\tau)\big)+2k(\tau)\varepsilon_\nu,\end{aligned}$$

where we set

$$k(\tau):=\kappa\Lambda(\tau,y(\tau),y'(\tau))+A|y'(\tau)|+\gamma(\tau).$$

Notice that, from Proposition 5,

$$\int_t^T k(\tau)\,\mathrm{d}\tau \le \Phi(B).$$

Now (30) follows immediately.

(xvii). *Final estimate of* $J_t(\bar{y})$. From (29) and (30) of Steps (xv) and (xvi), we obtain

$$J_t(\bar{y}) \le J_t(y) + \varepsilon_\nu\,(2\Phi(B) + \Xi - \Upsilon)\,. \tag{31}$$

Two cases may occur.

- If Condition (H_B^δ) holds true, then the choice of ν in (17) of Step (v) implies that

  $$2\Phi(B) + \Xi - \Upsilon < 0.$$

  Thus, from (31) we obtain $J_t(\bar{y}) \le J_t(y)$. Notice here that the inequality is strict if $\varepsilon_\nu > 0$, and this occurs whenever $|y'| > \nu$ on a set of positive measure.
- Otherwise, the choice of ν in (18) of Step (v) implies that

  $$\varepsilon_\nu(2\Phi(B) + \Xi(\nu) - \Upsilon) \le \eta.$$

  Thus, from (31), we obtain

  $$J_t(\bar{y}) \le J_t(y) + \eta.$$

  The conclusion follows. □

Remark 15. The last part of the proof of Theorem 12 shows that if Condition (H_B^δ) holds then any absolutely continuous minimizer of J_t is necessarily Lipschitz. The regularity result is a particular case of the main theorem in [8].

Remark 16 (Explicit Lipschitz ranks). The knowledge of $\bar{\nu}$ and c in Condition (H_B^δ) correspondingly to the value K provided in Step (ii) of the proof of Theorem 12 allows to give an explicit bound of the Lipschitz constant in Claim (1) of Theorem 12. Indeed, referring to the proof of the theorem:

- Step (xiii) shows that ν is a suitable Lipschitz constant for $\bar{y}$;
- From Step *ix)*, one can choose

  $$\nu = \max\left\{\frac{R}{(1-\mu)\left(1-\frac{c_\delta(B)}{\mu c}\right)(T-\delta)}, c, \bar{\nu}, \frac{2R}{\varepsilon_*}\right\}, \tag{32}$$

  where $R = \frac{B+dT}{\alpha}$;
- Accordingly to (15), μ in (32) is any real number such that $\frac{c_\delta(B)}{c} < \mu < 1$.

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