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On the constitutive relations for second sound in thermo-electroelasticity

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In papers [1] and [2] of 1982, Coleman, Fabrizio and Owen gave a derivation of implications of the second law of thermodynamics to describe second sound in rigid heat conductors, by using a natural extension to anisotropic media of the well-known Cattaneo's relation. Later, in 1992, Öncü and Moodie [3] gave a derivation of the constitutive relations of an elastic heat conductor for which the heat flux and the temperature obey a frame-invariant form of a generalized Cattaneo's equation. Recently, in 2004, Rybalko [4] has shown that a second-sound wave is accompanied by the appearance of electric induction. Here, we extend the theory [3]: following the standard Coleman–Noll procedure [5], we derive the thermodynamic restrictions on the constitutive relations for an electrically polarizable and finitely deformable, heat conducting elastic continuum which interacts with the electric field. The constitutive equations include an evolution equation for the heat flux; the latter and the temperature obey a frame-invariant form of Cattaneo's equation.

1. Introduction

1.1. Second sound and generalized thermoelasticity

It is observed experimentally that at low temperature, heat propagates as a thermal wave. This phenomenon is named 'second sound' from the wave motion of heat being similar to the propagation of sound in air.

The wave nature of heat propagation has been observed e.g. in superfluids [6] and in very pure crystals, where second sound occurs at low temperature [7–10]. Peshkov [6] also suggested that second sound might take place in materials that have a phonon gas.

Fourier's law of heat conduction fails to model second sound since it yields heat conduction equations of the diffusion types, which lead to infinite speeds of propagation for heat waves, contrary to physical observations.

Cattaneo [11] eliminated the 'paradox of instantaneous propagation of thermal disturbances' by substituting the constitutive Fourier's law for the heat flux with the well-known evolution equation

$$\tau(\theta)\dot{\mathbf{q}} + \mathbf{q} = -\kappa(\theta)\,\mathrm{grad}\,\theta, \tag{1.1}$$

where $\tau(\theta)$ and $\kappa(\theta)$ are positive. It has been observed that Eq. (1.1) is not frame-invariant. Hence Fox [12] for isotropic materials proposed the frame-invariant equation

$$\tau(\dot{\mathbf{q}} - \mathbf{W}\mathbf{q}) + \mathbf{q} = -\kappa\,\mathrm{grad}\,\theta, \tag{1.2}$$

where

(i) $\mathbf{q}$ and θ satisfy the heat conduction inequality

$$\mathbf{q}\cdot\mathrm{grad}\,\theta \le 0, \tag{1.3}$$

(ii) $\mathbf{W}$ is the skew-symmetric part of the velocity gradient,

(iii) τ and κ are positive functions of θ and the joint invariants of $\mathbf{q}$ and $\mathrm{grad}\,\theta$.

As noted in [3, p. 89], starting from (1.2) implies that inequality (1.3) is not valid in all admissible processes. Hence the classical Coleman–Noll procedure of continuum thermodynamics cannot be set up.

Several papers were written to present continuum theories capable of predicting thermal waves propagating at finite speeds in various media; such theories are often referred to as *generalized thermoelasticity*.

The interesting paper [13] deserves a mention: it develops a general theory of heat conduction, for nonlinear rigid materials with memory, that has associated with it finite propagation speeds. In Gurtin–Pipkin's theory, the constitutive functionals are assumed to depend on the summed histories of θ and $\mathrm{grad}\,\theta$. Then the restrictions that thermodynamics places on constitutive relations are determined. Hence Chen and Gurtin [14] extends [13] to deformable media. They show that there exist two speeds of propagation for acceleration waves: the 'first sound speed' is mechanical in nature and lies near the isothermal and isentropic sound speeds of the material, while the 'second sound speed' is associated with a predominantly thermal wave.

The interested reader can find in [3], on pp. 88–90, a history of second sound literature from the beginning until 1990. A more detailed history is written in [15, Sec. VIII], [16]. Among such papers, [12–14, 17–19] and [20] are very important. The above theories, except the last two, are related with Cattaneo's equation.

1.1.1. Thermodynamic theories for second sound with internal variables. Paper Caviglia *et al.* [21], which extends Morro and Ruggeri [22] from the rigid case to the deformable one, develops a thermoelasticity theory in which the

Coleman–Noll procedure is followed, an internal vector variable together with an evolution equation for it are used, and the classical Fourier law is postulated under stationary conditions. CIMMELLI [23] develops a thermoelasticity theory in the framework of a gradient extension of thermodynamics with internal state variables, and in [24] presents 'a generalization of the classical Coleman–Noll procedure in the presence of first-order non-local constitutive functions', to model nonlinear heat conduction in solids in the presence of a dynamical empirical temperature.

We note that in the theory presented here, the heat flux can also be interpreted as an internal variable.

1.1.2. Second sound within extended irreversible thermodynamics. The analysis of second sound phenomena has been successfully developed also in the frame of extended irreversible thermodynamics [25, 26]). A variational principle, based on extended irreversible thermodynamics, has been proposed by LEBON and DAUBY [27] to describe heat-wave propagation in dielectric crystals at low temperature.

1.1.3. Second sound within the Green–Naghdi theory. Another approach is that proposed in [28, 29]. It is based on an integral thermodynamic equality rather than on entropy inequality and uses the notion of thermal displacement associated with empirical temperature.

1.2. Second sound in thermo-electroelastic media

Recently it has been shown that second sound, that is a temperature wave, is accompanied by the appearance of electric induction; indeed, RYBALKO [4] in 2004 studies the electric response induced by second sound in superfluid helium: experimentally he shows that 'the relative motion of the superfluid and normal components of He II in a second-sound wave is accompanied by the appearance of electric induction'. Hence RYBALKO *et al.* [30] also perform experiments that show the connections between the mechanical motion and electric induction.

Later PASHITSKII *et al.* [31, 32] try to explain the experimantal data of electric polarization of superfluid helium during the second-sound excitation as the inertial polarization of a dielectric medium. They use Landau's two-fluid model [33], in which the liquid is regarded as a mixture of a 'normal' fluid carrying entropy and a 'superfluid' carrying none.

Note that ATKIN *et al.* [34, 35] criticize Landau's two-fluid model and present a continuum approach for a heat conducting elastic body that is not electrically polarizable 'in which an additional vector field is introduced to represent the flow of the microscopic excitations, from which the effect is thought to originate'.

Piezoelectric ceramics and composites are extensively used in many engineering applications such as sensors, actuators, intelligent structures, etc. Firstly, MINDLIN [36] proposed a thermo-piezoelectricity theory and also derived the governing equations of a thermo-piezoelectric plate [37]. NOWACKI [38, 39] has studied the physical laws for the thermo-piezoelectric materials. A generalized linear thermoelasticity theory for piezoelectric media has been developed by CHANDRASEKHARAIAH [40], where 'a theory of thermoelasticity for piezoelectric materials, which includes heat flux among the independent constitutive variables, is formulated. It is found that the linearized version of the theory admits a finite speed of the thermal signals.'

Even if [40] is concerned with the linear theory, it starts by considering the non-linear theory (in Section 2, pp. 41–43), and uses the Coleman–Noll procedure to deduce from the fundamental field equations the constitutive restrictions, which the entropy production inequality imposes. Differently from the present paper,

(i) it uses a different free-energy function (see Eq. (2.8) there),

(ii) it adopts an assumption ([40, Eq. (2.10)]), that is not justified there and that is not needed here, since it is not necessary for the deduction of the constitutive restrictions,

(iii) it does not use any restriction to the heat-flux evolution law similar to the invertibility condition w.r.t. $\mathbf{q}$, introduced by [3] and extended here,

(iv) it does not contain any consideration of frame-indifference.

1.3. On generalized thermo-magnetoelectroelasticity

There are several papers that study, or use for applications, generalized theories for thermo-magneto-electro-elasticity. For instance, [41, 42, 43].

These papers explain a theoretic interest for such topic, motivated by technological applications; but they always use linear constitutive equations and thus, put in evidence the need for more general non-linear theories.

1.4. On Coleman–Fabrizio–Owen and Öncü–Moodie papers

COLEMAN, FABRIZIO and OWEN in [1, 2] gave a derivation of implications of the second law of thermodynamics for a rigid heat conductor, for which the heat flux vector $\mathbf{q}$ and the temperature θ obey the relation

$$\hat{\mathbf{T}}(\theta)\dot{\mathbf{q}} + \mathbf{q} = -\hat{\mathbf{K}}(\theta)\,\mathrm{grad}\,\theta, \tag{1.4}$$

with $\hat{\mathbf{T}}(\theta)$ and $\hat{\mathbf{K}}(\theta)$ being non-singular, which extends to anisotropic media the well-known Cattaneo's relationship (1.1).

Later ÖNCÜ and MOODIE [3] extended COLEMAN *et al.* [1, 2] to the case of a deformable thermoelastic body; in fact, the authors observed that [3, pp. 89–90])

'*... it is apparent that the experimentally observed thermal effects, though at varying degrees, are coupled with deformation. It is for this reason that in this paper we extend the validity of the formulation of Coleman, Fabrizio and Owen to the case when the role played by deformations is appreciable. ...*'

Then, [3] shows that when the referential heat flux $\mathbf{Q}$, the deformaton gradient $\mathbf{F}$ and the temperature θ, obey the relation

$$\mathbf{T}(\mathbf{F},\theta)\dot{\mathbf{Q}} + \mathbf{Q} = -\mathbf{K}(\mathbf{F},\theta)\,\mathrm{Grad}\,\theta, \tag{1.5}$$

with $\mathrm{Grad}\,\theta = \mathbf{F}^T\,\mathrm{grad}\,\theta$, and $\mathbf{T}(\mathbf{F},\theta)$ and $\mathbf{K}(\mathbf{F},\theta)$ being non-singular, then the second law of thermodynamics requires that the specific internal energy ε and the first Piola–Kirchhoff stress $\mathbf{S}$ should satisfy the relations

$$\rho_R\varepsilon = \rho_R\hat{\varepsilon}_o(\mathbf{F},\theta) + \mathbf{Q}\cdot\mathbf{A}(\mathbf{F},\theta)\mathbf{Q}, \tag{1.6}$$

$$\mathbf{S} = \mathbf{S}_o(\mathbf{F},\theta) + \mathbf{Q}\cdot\mathbf{P}_Z(\mathbf{F},\theta)\mathbf{Q}, \tag{1.7}$$

where (i) ρ_R is the referential mass density, (ii) $\varepsilon_o(\mathbf{F},\theta)$ and $\mathbf{S}_o(\mathbf{F},\theta)$ are, respectively, the classical specific internal energy per unit mass and Piola–Kirchhoff stress tensor, (iii) $\mathbf{A}(\mathbf{F},\theta)$, $\mathbf{P}_Z(\mathbf{F},\theta)$ are defined by

$$\mathbf{A}(\mathbf{F},\theta) = -\frac{\theta^2}{2}\frac{\partial}{\partial\theta}\left[\frac{\mathbf{Z}(\mathbf{F},\theta)}{\theta^2}\right],$$

$$\mathbf{P}_Z(\mathbf{F},\theta) = \frac{1}{2\theta}\frac{\partial}{\partial\mathbf{F}}\mathbf{Z}(\mathbf{F},\theta), \qquad \mathbf{Z}(\mathbf{F},\theta) = \mathbf{K}(\mathbf{F},\theta)^{-1}\mathbf{T}(\mathbf{F},\theta); \tag{1.8}$$

lastly, (iv) $\mathbf{Z}$ is symmetric and $\mathbf{K}$ is positive-definite.

1.5. On the present paper

Recent papers by Rybalko [4] and [30] have pointed out that a 'previously unknown effect' has been 'observed experimentally: the appearance of an electric field in superfluid helium during the propagation of second-sound waves or in the presence of induced oscillations of the velocity of the normal component' [31, p. 8]. In particular, Rybalko [4] experimentally confirms that a second-sound wave can be accompanied by the appearance of electric induction in a reversible fashion; the author declares: 'The idea of a possible relationship between the internal electric field and undamped superfluid flows of liquid helium below (the lambda point) T_λ is developed experimentally for the first time'.

Later, Pashitskii *et al.* [31, 32] have given a theoretical justification to such effect, using the two-fluid Landau's model [33] for helium II. Yet (i) Landau's model has been criticized, e.g., by Atkin *et al.* [35, p. 115]:

'The superfluid and normal fluid cannot exist independently, and therefore helium II may not be regarded as a mixture in the usual sense, although the equations governing the motion are derived as if it were.'

Moreover (ii) Pashitskii *et al.* [31, p. 9] recognize a limit in their theoretical analysis:

'However, quantitative estimates of the effect give a value substantially lower than that observed experimentally'.

Hence a natural question arises: can the generalized thermo-elasticity theory by Öncü and Moodie [3], be extended to thermo-electroelasticity? The present paper gives a positive reply to this question: a generalized thermodynamic theory of an electrically polarizable and finitely deformable heat conducting elastic continuum, interacting with an electric field, is developed here. Unlike the classical theories, here the heat flux is an independent variable and it is ruled by a rate-type constitutive equation, namely by a first-order (in time) differential equation which, as a particular case, neglecting all electrical quantities, can reduce to the thermo-elasticity theory of Öncü and Moodie [3] and to the classical Maxwell–Cattaneo relationship (1.1). In this way, finite speed of propagation of disturbances is guaranteed. The whole theory is developed within the frame of Rational Thermodynamics with rate-type constitutive equations. Thermodynamic restrictions are derived both in the spatial and referential configurations, by applying the Coleman–Noll procedure [5], Truesdell [44]. A generalized free energy is defined and it is proved that this function determines the entropy, the first Piola–Kirchhoff stress tensor and the polarization vector. Compatibility with the Principle of Material Indifference is considered as well. The case in which the rate equation for the heat flux can be put in the Maxwell–Cattaneo's form is investigated in Section 6, where the explicit form of the thermodynamic potentials and of the first Piola–Kirchhoff stress tensor are derived. From this general model the classical parabolic theory, where the heat flux is given by a constitutive equation, is derived as a particular case.

The classical results of Tiersten's paper [45] are discussed.

Lastly, it is pointed out that the rate equation for the heat flux can involve also heat flux gradients (see Remark 2).

2. Preliminary definitions

Here we extend Öncü–Moodie [3] from thermo-elasticity to thermo-electroelasticity; we treat topics in parallel with the corresponding ones in [3] and mainly using similar notations. By dropping any reference to electricity, the present theory exactly reduces to the one in [3].

Following [3], here the heat flux is treated as an independent variable (as well as e.g. the deformation gradient and the absolute temperature), which is determined by a rate-type evolution equation.

We consider a body B whose particles are identified with the positions $\mathbf{X} \in \mathcal{E}$ they occupy in a fixed reference configuration $\mathcal{B}$ of a three-dimensional Euclidean point space $\mathcal{E}$. A referential mass density $\rho_R(.) : \mathcal{B} \to (0,\infty)$ is given, so that $m(\mathcal{P}) := \int_{\mathcal{P}} \rho_R dV$ is the mass of the part $\mathcal{P}$ of $\mathcal{B}$. We assume that the material filling B is characterized by a given process class $\mathbb{P}(B)$ of B as a set of ordered 10-tuples of functions on $\mathcal{B} \times \mathbb{R}$:

$$p = \big(\mathbf{x}(.), \theta(.), \varphi(.), \varepsilon(.), \eta(.), \boldsymbol{\tau}(.), \mathbf{P}(.), \mathbf{q}(.), \mathbf{b}(.), r(.)\big) \in \mathbb{P}(B) \tag{2.1}$$

defined with respect to $\mathcal{B}$, satisfying the balance laws of linear momentum, moment of momentum, energy, the entropy inequality, and the field equations of electrostatics, where

$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ is the *motion*,
$\theta = \theta(\mathbf{X}, t) \in (0,\infty)$ is the *absolute temperature*,
$\varphi = \varphi(\mathbf{X}, t)$ is the *electric potential*,
$\varepsilon = \varepsilon(\mathbf{X}, t)$ is the specific *internal energy* per unit mass,
$\eta = \eta(\mathbf{X}, t)$ is the specific *entropy* per unit mass,
$\boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{X}, t)$ ($\mathbf{S} = \mathbf{S}(\mathbf{X}, t)$) is the *Cauchy* (*first Piola–Kirchhoff*) *stress tensor*,
$\mathbf{P} = \mathbf{P}(\mathbf{X}, t)$ ($\mathbb{P} = \mathbb{P}(\mathbf{X}, t)$) is the spatial (referential) *polarization vector*,
$\mathbf{q} = \mathbf{q}(\mathbf{X}, t)$ ($\mathbf{Q} = \mathbf{Q}(\mathbf{X}, t)$) is the spatial (referential) *heat flux vector*,
$\mathbf{b} = \mathbf{b}(\mathbf{X}, t)$ is the external specific *body force* per unit mass,
$r = r(\mathbf{X}, t)$ is the *radiation heating* per unit mass.

The referential and spatial heat flux and polarization vectors are related by

$$\mathbb{P} = J\mathbf{F}^{-1}\mathbf{P}, \qquad \mathbf{Q} = J\mathbf{F}^{-1}\mathbf{q}. \tag{2.2}$$

Any motion $\mathbf{x}(.,.)$, temperature field $\theta(.,.)$ and electric potential field $\varphi(.,.)$ of B are *regular* enough functions, in the sense that they have all the derivatives needed for writing the local balance laws for B.

We use Grad and Div (grad and div) to denote material (spatial) gradient and divergence, respectively; a superposed dot denotes the material time derivative.

The *deformation gradient* $\mathbf{F}$ at $\mathbf{X}$ at time t is given by

$$\mathbf{F} = \mathbf{F}(\mathbf{X}, t) = \operatorname{Grad} \mathbf{x}(\mathbf{X}, t), \tag{2.3}$$

and the invertibility of the deformation is assured by the condition

$$J = \det \mathbf{F} > 0.$$

The *velocity* $\mathbf{v}$ of $\mathbf{X}$ at time t is given by

$$\mathbf{v} = \mathbf{v}(\mathbf{X}, t) = \dot{\mathbf{x}}(\mathbf{X}, t). \tag{2.4}$$

The local law of conservation of mass is expressed by

$$\rho_R = \rho J, \qquad \dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \tag{2.5}$$

where $\rho = \rho(\mathbf{X}, t)$ is the mass density of $\mathbf{X}$ at time t.

The electric potential φ, together with the polarization vector $\mathbf{P}$, in Gaussian units, determines the *Eulerian electric displacement* $\mathbf{D}$ by the equality

$$\mathbf{D} = \mathbf{E}^M + 4\pi\mathbf{P}, \tag{2.6}$$

where $\mathbf{E}^M = -\nabla_{\mathbf{x}}\varphi$ is the *(Maxwellian) spatial electric vector*. Hence the *referential electric displacement* is

$$\boldsymbol{\Delta} = J\mathbf{F}^{-1}\mathbf{D} = J\mathbf{F}^{-1}\mathbf{E}^M + 4\pi\mathrm{I\!P}. \tag{2.7}$$

Any two corresponding referential and spatial 'energy-flux' vectors, that are related by

$$\mathrm{I\!H} = J\mathbf{F}^{-1}\mathbf{h}, \tag{2.8}$$

have spatial and referential divergences which are related by

$$\operatorname{Div}\mathrm{I\!H} = J\operatorname{div}\mathbf{h}. \tag{2.9}$$

The spatial and referential polarization vectors per unit volume are respectively defined by

$$\boldsymbol{\pi} = \mathbf{P}/\rho, \qquad \boldsymbol{\Pi} = \mathrm{I\!P}/\rho_R. \tag{2.10}$$

3. Spatial description

3.1. Local balance laws in spatial form

The *total stress tensor* $\boldsymbol{\sigma}$ is defined by

$$\boldsymbol{\sigma} = \boldsymbol{\tau} + \mathbf{T}^E, \tag{3.1}$$

where

$$\mathbf{T}^E = \frac{1}{4\pi}\left[\mathbf{E}^M \otimes \mathbf{D} - \frac{1}{2}(\mathbf{E}^M \cdot \mathbf{E}^M)\mathbf{I}\right], \tag{3.2}$$

is the *Maxwell stress tensor* (cf. [45, Eq. (3.19)], [46]). The use of $\boldsymbol{\sigma}$ allows to write the field equations of momentum and angular momentum in a form as if the electric fields were missing (see the equalities between brackets on the right of Eqs. (3.3) and (3.4) below).

Under suitable assumptions of regularity and using (2.5), the usual integral forms of the balance laws of linear momentum, moment of momentum, energy, the field equations of electrostatics and the entropy inequality, are equivalent to the spatial field equations

$$\rho\dot{\mathbf{v}} = \operatorname{div}\boldsymbol{\tau} + \mathbf{P}\cdot\nabla_x\mathbf{E}^M + \rho\mathbf{b}, \qquad (\rho\dot{\mathbf{v}} = \operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{b}), \tag{3.3}$$

$$\operatorname{skw}\boldsymbol{\tau} + \operatorname{skw}\mathbf{T}^E = \mathbf{O}, \qquad (\boldsymbol{\sigma} = \boldsymbol{\sigma}^T), \tag{3.4}$$

$$\rho\dot{\varepsilon} = \boldsymbol{\tau}\cdot\nabla\mathbf{v} - \operatorname{div}\mathbf{q} + \mathbf{E}^M\cdot\rho\dot{\boldsymbol{\pi}} + \rho r, \tag{3.5}$$

$$\mathbf{E}^M = -\nabla_{\mathbf{x}}\varphi, \qquad \operatorname{div}\mathbf{D} = 0, \tag{3.6}$$

$$\rho\dot{\eta} \geq \rho(r/\theta) - \operatorname{div}(\mathbf{q}/\theta). \tag{3.7}$$

We note that Eqs. (3.3), (3.5) and (3.7) respectively coincide with Eqs. (3.23), (3.40) and (3.43) of [45]. In Eq. (3.3) the term $\mathbf{P}\cdot\nabla_x\mathbf{E}^M$ can be interpreted as the portion of the body force due to the electric field quantities (e.g., see [47, p. 10], [48, p. 33] and [45]).

Incidentally, note that, following [45], the additional power term $\mathbf{E}^M\cdot\rho\dot{\boldsymbol{\pi}}$ in Eq. (3.5) can be obtained as follows: one postulates the integral equation of conservation of energy in the form ([45, Eq.(3.30)]), that contains the contribution of a ‘rate of supply of energy to the material from the quasi-static electric field’, σ; hence, by using the divergence theorem, the equation of balance of mass and using the expression in [45, Eq. (3.39)] for σ, that is,

$$\sigma = \mathbf{P}\cdot\nabla_x\mathbf{E}^M\cdot\mathbf{v} + \mathbf{E}^M\cdot\rho\dot{\boldsymbol{\pi}}, \tag{3.8}$$

one arrives at the local form (3.5) of the energy law.

Let $\psi = \psi()$ be the specific *free energy* per unit mass defined by

$$\psi = \varepsilon - \theta\eta - \mathbf{E}^M\cdot\boldsymbol{\pi}. \tag{3.9}$$

Note that the free energy (3.9) coincides with the ‘thermodynamic function’ χ defined in [45, Eq. (4.2)]; in an equivalent fashion, in a thermo-electroelasticity theory it is customary to introduce the enthalpy $G = \psi - \mathbf{E}^M\cdot\mathbf{P}$, with $\psi = e - \theta\eta$ (e.g., see [49, p. 90]).

Then, by using ψ defined in (3.9), Eqs. (3.5) and (3.7) yield the *reduced dissipation inequality*

$$\rho(\dot{\psi} + \eta\dot{\theta}) - \boldsymbol{\tau}\cdot\nabla\mathbf{v} + \frac{1}{\theta}\mathbf{q}\cdot\mathbf{g} + \rho\boldsymbol{\pi}\cdot\dot{\mathbf{E}}^M \leq 0, \tag{3.10}$$

where $\mathbf{g} = \operatorname{grad}\theta(\mathbf{X},t)$ is the spatial temperature gradient.

3.2. Spatial constitutive assumptions

Let $\mathcal{D}$ be an open and simply connected domain consisting of 5-tuples $(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{q},\mathbf{g})$ and assume that

$$\textit{if} \quad (\mathbf{F},\theta,\mathbf{E}^M,\mathbf{q},\mathbf{g})\in\mathcal{D}, \quad \textit{then} \quad (\mathbf{F},\theta,\mathbf{E}^M,\mathbf{0},\mathbf{0})\in\mathcal{D}.$$

Below we use the time derivative for the heat flux vector

$$\overset{\circ}{\mathbf{q}} = \dot{\mathbf{q}} - \mathbf{L}\mathbf{q} + (tr\mathbf{L})\mathbf{q}, \qquad \mathbf{L} = \operatorname{grad}\mathbf{v}, \tag{3.11}$$

which is used in [3] on page 97. The equality $\dot{\mathbf{Q}} = J\mathbf{F}^{-1} \overset{\circ}{\mathbf{q}}$ (see [3, Eq. (5.8)]), where $\mathbf{Q}$ is the material heat flux vector, shows that the spatial counterpart of the material derivative $\dot{\mathbf{Q}}$ is represented by $\overset{\circ}{\mathbf{q}}$ rather than by $\dot{\mathbf{q}}$.

Assumption 1. *For every $p \in \mathbb{P}(B)$ the specific free energy $\psi(\mathbf{X}, t)$, the specific entropy $\eta(\mathbf{X}, t)$, the Cauchy stress tensor $\boldsymbol{\tau}(\mathbf{X}, t)$, the polarization vector $\mathbf{P}(\mathbf{X}, t)$, and the time rate of the heat flux $\overset{\circ}{\mathbf{q}}$ $(\mathbf{X}, t)$ are given by continuously differentiable functions on $\mathcal{D}$ such that*

$$\psi = \overline{\psi}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{q}, \mathbf{g}), \tag{3.12}$$

$$\eta = \overline{\eta}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{q}, \mathbf{g}), \tag{3.13}$$

$$\boldsymbol{\tau} = \overline{\boldsymbol{\tau}}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{q}, \mathbf{g}), \tag{3.14}$$

$$\mathbf{P} = \overline{\mathbf{P}}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{q}, \mathbf{g}), \tag{3.15}$$

$$\overset{\circ}{\mathbf{q}} = \mathbf{h}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{q}, \mathbf{g}). \tag{3.16}$$

Further, the tensors $\partial_{\mathbf{q}}\mathbf{h}(.)$ and $\partial_{\mathbf{g}}\mathbf{h}(.)$ are non-singular.

Of course, once $\rho(.)$, $\overline{\psi}(.)$, $\overline{\eta}(.)$ and $\overline{\mathbf{P}}(.)$ are known, then equality (3.9) gives the continuously differentiable function $\overline{\varepsilon}(.)$ determining $\varepsilon = \varepsilon(\mathbf{X}, t)$ such that

$$\varepsilon = \overline{\varepsilon}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{q}, \mathbf{g}). \tag{3.17}$$

The assumed properties of the *heat flux evolution function* $\mathbf{h}(.)$ indicate that it is invertible for $\mathbf{q}$ and also for $\mathbf{g}$. The inverse of $\mathbf{h}(.)$ with respect to $\mathbf{q}$ is denoted by

$$\mathbf{q} = \mathbf{h}^*(\mathbf{F}, \theta, \mathbf{E}^M, \overset{\circ}{\mathbf{q}}, \mathbf{g}). \tag{3.18}$$

Note that

$$\partial_{\mathbf{g}}\mathbf{h}^*(.) = -[\partial_{\mathbf{q}}\mathbf{h}]^{-1}(.)\partial_{\mathbf{g}}\mathbf{h}(.), \tag{3.19}$$

so that the tensor $\partial_{\mathbf{g}}\mathbf{h}^*(.)$ is also continuous and non-singular.

Also note that the dependence upon $\mathbf{X}$ is not written for simplicity; it is implicit and understood if the body is not materially homogeneous.

3.3. Coleman–Noll method and thermodynamic restrictions in the spatial description

Given any motion $\mathbf{x}(\mathbf{X}, t)$, temperature field $\theta(\mathbf{X}, t)$ and electric potential field $\varphi(\mathbf{X}, t)$, the constitutive equations (3.12)–(3.16) determine $e(\mathbf{X}, t)$, $\eta(\mathbf{X}, t)$, $\boldsymbol{\tau}(\mathbf{X}, t)$, $\mathbf{P}(\mathbf{X}, t)$, $\overset{\circ}{\mathbf{q}}$ $(\mathbf{X}, t)$, and the local laws (3.3) and (3.5) determine $\mathbf{b}(\mathbf{X}, t)$

and $r(\mathbf{X},t)$. Hence, for any given motion, temperature field and electric potential field, a corresponding process p is constructed.

The method of Coleman–Noll [5] is based on the postulate that every process p so constructed belongs to the process class $\mathbb{P}(B)$ of B; that is, on the assumption that the constitutive assumptions (3.12)–(3.16) are compatible with thermodynamics, in the sense of the following

DISSIPATION PRINCIPLE. *For any given motion, temperature field and electric potential field, the process p constructed from the constitutive equations* (3.12)–(3.16) *belongs to the process class* $\mathbb{P}(B)$ *of* B. *Therefore the constitutive functions* (3.12)–(3.16) *are compatible with the second law of thermodynamics in the sense that they satisfy the dissipation inequality* (3.7).

It is a matter of routine to extend to thermo-electroelasticity Coleman's remark for thermoelasticity written in [5], on page 1119, lines 8–30 from the top; such extension, which includes the electric field, is written here just by paraphrasing Coleman.

REMARK 1. Let $\mathbf{A}(t)$ be any time-dependent invertible tensor, $\alpha(t)$ any time-dependent positive scalar, $\mathbf{a}(t)$ any time-dependent vector, $\beta(t)$ be any time-dependent scalar, $\mathbf{b}(t)$ any time-dependent vector, and Y any material point of B, whose spatial position in the reference configuration $\mathcal{B}$ is $\mathbf{Y}$. We can always construct at least one admissible electro-thermodynamic process in $\mathcal{B}$ such that

$$\mathbf{F}(\mathbf{X},t), \theta(\mathbf{X},t), \mathbf{g}(\mathbf{X},t), \mathbf{E}^M(\mathbf{X},t)$$

have, respectively, the values $\mathbf{A}(t), \alpha(t), \mathbf{a}, \mathbf{b}$ at $\mathbf{X} = \mathbf{Y}$.

An example of such a process is the one determined by the following deformation function, temperature distribution and electric potential:

$$\mathbf{x} = \mathbf{x}(\mathbf{X},t) = \mathbf{Y} + \mathbf{A}(t)[\mathbf{X} - \mathbf{Y}], \tag{3.20}$$

$$\theta = \theta(\mathbf{X},t) = \alpha(t) + [\mathbf{A}^T(t)\mathbf{a}(t)] \cdot [\mathbf{X} - \mathbf{Y}], \tag{3.21}$$

$$\varphi = \varphi(\mathbf{X},t) = \beta(t) + [\mathbf{A}^T(t)\mathbf{b}(t)] \cdot [\mathbf{X} - \mathbf{Y}]. \tag{3.22}$$

Thus, at a given time t, we can arbitrarily specify not only $\mathbf{F}$, θ, $\mathbf{g}$ and $\mathbf{E}^M$ but also their time derivatives $\dot{\mathbf{F}}$, $\dot{\theta}$, $\dot{\mathbf{g}}$ and $\dot{\mathbf{E}}^M$ at a point $\mathbf{Y}$ and to be sure that there exists at least one electro-thermodynamic process corresponding to this choice.

The next theorem is proved following the Coleman–Noll procedure [5], that is, by using Remark 1.

THEOREM 1. *The Dissipation Principle is satisfied if and only if the following conditions hold:*

(i) *the free energy response function* $\overline{\psi}(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{q},\mathbf{g})$ *is independent of the temperature gradient* $\mathbf{g}$ *and determines the entropy, the Cauchy stress tensor,* *and the polarization vector through the relations*

$$\overline{\eta}(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{q}) = -\frac{\partial\overline{\psi}}{\partial\theta}(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{q}), \tag{3.23}$$

$$\overline{\boldsymbol{\tau}}(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{q}) = \rho\mathbf{F}\frac{\partial\overline{\psi}}{\partial\mathbf{F}}(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{q}), \tag{3.24}$$

$$\overline{\boldsymbol{\pi}}(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{q}) = -\frac{\partial\overline{\psi}}{\partial\mathbf{E}^M}(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{q}); \tag{3.25}$$

(ii) *the reduced dissipation inequality*

$$\rho\theta\frac{\partial\overline{\psi}}{\partial\mathbf{q}}(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{q})\cdot\dot{\mathbf{q}} + \mathbf{q}\cdot\mathbf{g} \le 0, \tag{3.26}$$

where

$$\dot{\mathbf{q}} = \mathbf{h}(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{q},\mathbf{g}) + [\mathbf{L} - (tr\mathbf{L})\mathbf{I}]\,\mathbf{q}, \tag{3.27}$$

is satisfied.

P r o o f. By the chain rule we have

$$\dot{\psi} = \partial_{\mathbf{F}}\overline{\psi}\cdot\dot{\mathbf{F}} + \partial_\theta\overline{\psi}\dot{\theta} + \partial_{\mathbf{E}^M}\overline{\psi}\cdot\dot{\mathbf{E}}^M + \partial_{\mathbf{q}}\overline{\psi}\cdot\dot{\mathbf{q}} + \partial_{\mathbf{g}}\overline{\psi}\cdot\dot{\mathbf{g}}. \tag{3.28}$$

Now, we have

$$\frac{\partial v^i}{\partial x^j} = \frac{\partial v^i}{\partial X^K}\frac{\partial X^K}{\partial x^j} = \frac{\partial X^K}{\partial x^j}\frac{d}{dt}\frac{\partial x^i}{\partial X^K}, \tag{3.29}$$

that is,

$$\nabla\mathbf{v} = \mathbf{F}^{-\mathbf{T}}\dot{\mathbf{F}}, \tag{3.30}$$

and thus

$$\boldsymbol{\tau}\cdot\nabla\mathbf{v} = \mathbf{F}^{-\mathbf{1}}\boldsymbol{\tau}\cdot\dot{\mathbf{F}}. \tag{3.31}$$

Thus by substituting Eqs. (3.28), (3.31) and the constitutive equations (3.12)–(3.16) into the dissipation inequality (3.10), we obtain

$$\begin{aligned}(\rho\partial_{\mathbf{F}}\overline{\psi} - \mathbf{F}^{-1}\overline{\boldsymbol{\tau}})\cdot\dot{\mathbf{F}} + \rho(\partial_\theta\overline{\psi} + \overline{\eta})\dot{\theta} + \rho(\partial_{\mathbf{E}^M}\overline{\psi} + \overline{\boldsymbol{\pi}})\cdot\dot{\mathbf{E}}^M \\ + \rho\partial_{\mathbf{q}}\overline{\psi}\cdot\dot{\mathbf{q}} + \rho\partial_{\mathbf{g}}\overline{\psi}\cdot\dot{\mathbf{g}} + \frac{1}{\theta}\mathbf{q}\cdot\mathbf{g} \le 0.\end{aligned} \tag{3.32}$$

Now, by Remark 1, to the time derivatives $\dot{\mathbf{F}}, \dot{\theta}, \dot{\mathbf{E}}^M$ and $\dot{\mathbf{g}}$ can be assigned arbitrary values independently of the other variables; this implies theses (i) and (ii). □

The next theorem extends Theorem 2 of [3].

THEOREM 2. *The frame-invariant time derivative of the heat flux,* $\overset{\circ}{\mathbf{q}}$, *vanishes for all thermal equilibrium states* $(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{0}, \mathbf{0}) \in \mathcal{D}$ *and the tensor*

$$\mathbf{K}(\mathbf{F}, \theta, \mathbf{E}^M) = \partial_{\mathbf{q}}\mathbf{h}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{0}, \mathbf{0})^{-1}\partial_{\mathbf{g}}\mathbf{h}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{0}, \mathbf{0}) \tag{3.33}$$

is positive-definite.

P r o o f. Making, in the proof of Theorem 2 of [3, p.94], the three changes below, transforms it in a proof for the present theorem:

1. insert the additional variable $\mathbf{E}^M$ in each occurrence of $\mathbf{H}^*$, $\mathbf{H}$, f, $\mathbf{K}, \mathbf{K}^*$;
2. replace $(\mathbf{F}, \theta, \mathbf{Q}, \mathbf{G})$ with $(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{q}, \mathbf{g})$;
3. replace $\mathbf{H}^*$, $\mathbf{H}$, and $\dot{\mathbf{Q}}$ with $\mathbf{h}^*, \mathbf{h}$ and $\overset{\circ}{\mathbf{q}}$, respectively. □

REMARK 2. In certain theories it is assumed a dependence of the evolution equation for the heat flux $\mathbf{q}$ on the gradients $\nabla\mathbf{q}$ and $\nabla^2\mathbf{q}$ too. For instance, for a rigid heat conductor [25] we obtain the balance equation

$$\frac{\partial q_i}{\partial t} + N_{ij,j} = -\frac{1}{\tau_R}q_i, \tag{3.34}$$

which specializes to the Guyer–Krumhansl equation [50, 51, 52]

$$\tau_R\frac{\partial q_i}{\partial t} + q_i = -k\theta_i + \ell_1^2(q_{i,jj} + 2q_{j,ji}), \tag{3.35}$$

and which, in the linear case, has the following expression for N_{ij} [53]:

$$N_{ij} = \frac{k}{\tau_R}\theta\delta_{ij} - L_1 q_{i,j} - L_2 q_{j,i} - L_3 q_{k,k}\delta_{ij}, \tag{3.36}$$

where L_i are constants.

The exploitation of the entropy inequality through the Coleman–Noll procedure can be set up when in Assumption 1 the constitutive equations (3.12)–(3.16) also have

$$\nabla\mathbf{q}, \dots, \ \nabla^n\mathbf{q} \qquad (n > 0)$$

as arguments.

Indeed, the proof of Theorem 1 uses Remark 1, that does not depend on the function $\mathbf{q} = \mathbf{q}(\mathbf{X}, t)$. Thus such a proof can be repeated simply by replacing in (3.28) the term $\partial_{\mathbf{q}}\overline{\psi} \cdot \dot{\mathbf{q}}$ with

$$\sum_{j=0}^{n} \partial_{(\nabla^j\mathbf{q})}\overline{\psi} \cdot (\nabla^j\dot{\mathbf{q}}),$$

where $\nabla^0 \mathbf{q} = \mathbf{q}$ and $\dot{\mathbf{q}}$ is given by (3.27) with $\mathbf{h}$ containing also $\nabla^i \mathbf{q}$, $i = 1, \ldots, n$, as variables.

Hence, Theorem 1 remains valid also when the constitutive equations (3.12)–(3.16) are of the type

$$\chi = \overline{\chi}(\mathbf{F}, \theta, \mathbf{E}^M, \nabla^i \mathbf{q}, \mathbf{g}), \qquad (i = 0, 1, \ldots, n) \tag{3.37}$$

and the reduced dissipation inequality (3.26) in its thesis (ii) is replaced by

$$\rho\theta \sum_{j=0}^{n} \frac{\partial \overline{\psi}}{\partial(\nabla^j \mathbf{q})}(\mathbf{F}, \theta, \mathbf{E}^M, \nabla^i \mathbf{q}) \cdot (\nabla^j \dot{\mathbf{q}}) + \mathbf{q} \cdot \mathbf{g} \leq 0. \tag{3.38}$$

3.4. Use of invariant response functions

In order to satisfy the principle of material objectivity, the constitutive functions must be scalar invariant under rigid rotations of the deformed and polarized body. The invariance of ψ in a rigid rotation is assured when ψ is an arbitrary function of the referential quantities $E_{LM}, \theta, W_L, Q_L, G_L$, where

$$E_{LM} = \frac{1}{2}(x_{k,L} x_{k,M} - \delta_{LM}) \qquad \left(\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I})\right), \tag{3.39}$$

$$W_L = -\frac{\partial \varphi}{\partial X_L} = -\frac{\partial \varphi}{\partial x^p} \frac{\partial x^p}{\partial X_L} \qquad (\mathbf{W} = \mathbf{F}^T \mathbf{E}^M), \tag{3.40}$$

$$Q_L = J \frac{\partial X^L}{\partial x^\ell} q_\ell, \qquad G_L = \frac{\partial x^i}{\partial X^L} g_i \qquad (\mathbf{Q} = J \mathbf{F}^{-1} \mathbf{q}, \quad \mathbf{G} = \mathbf{F}^T \mathbf{g}), \tag{3.41}$$

where a comma denotes a partial derivative. Hence we assume that

$$\psi = \tilde{\psi}(\mathbf{E}, \theta, \mathbf{W}, \mathbf{Q}, \mathbf{G}). \tag{3.42}$$

Next we calculate the time derivatives in Eq. (3.28) by using $\tilde{\psi}$, that is, by using the identity

$$\psi = \overline{\psi}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{q}, \mathbf{g}) = \tilde{\psi}(\mathbf{E}, \theta, \mathbf{W}, \mathbf{Q}, \mathbf{G}), \tag{3.43}$$

where $(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{q}, \mathbf{g})$ and $(\mathbf{E}, \theta, \mathbf{W}, \mathbf{Q}, \mathbf{G})$ are related by (3.39)-(3.41); we find

$$\begin{aligned} \frac{\partial \overline{\psi}}{\partial \mathbf{F}} \cdot \dot{\mathbf{F}} = \Bigg[& \frac{\partial \tilde{\psi}}{\partial E_{RS}} \frac{\partial E_{RS}}{\partial(\partial x^i / \partial X_K)} + \frac{\partial \tilde{\psi}}{\partial W_R} \frac{\partial W_R}{\partial(\partial x^i / \partial X_K)} \\ & + \frac{\partial \tilde{\psi}}{\partial Q_R} \frac{\partial Q_R}{\partial(\partial x^i / \partial X_K)} + \frac{\partial \tilde{\psi}}{\partial G_R} \frac{\partial G_R}{\partial(\partial x^i / \partial X_K)} \Bigg] \frac{d}{dt} \frac{\partial x^i}{\partial X_K}. \end{aligned} \tag{3.44}$$

Now, by (3.39)–(3.41),

$$\frac{\partial\tilde{\psi}}{\partial E_{RS}}\frac{\partial E_{RS}}{\partial(\partial x^i/\partial X_K)}\frac{d}{dt}\frac{\partial x^i}{\partial X_K}=\frac{\partial\tilde{\psi}}{\partial E_{RS}}\frac{1}{2}\left(\delta_{RK}\frac{\partial x^i}{\partial X_S}+\frac{\partial x^i}{\partial X_R}\delta_{SK}\right)\frac{\partial\dot{x}^i}{\partial X_K}\tag{3.45}$$
$$=\frac{1}{2}\left(\frac{\partial\tilde{\psi}}{\partial E_{KS}}\frac{\partial x^i}{\partial X_S}+\frac{\partial\tilde{\psi}}{\partial E_{RK}}\frac{\partial x^i}{\partial X_R}\right)\frac{\partial\dot{x}^i}{\partial X_K}=\frac{\partial\tilde{\psi}}{\partial E_{RK}}\frac{\partial x^i}{\partial X_R}\frac{\partial\dot{x}^i}{\partial X_K}=\left(\frac{\partial\tilde{\psi}}{\partial\mathbf{E}}\mathbf{F}^T\right)\cdot\dot{\mathbf{F}},$$

$$\frac{\partial\tilde{\psi}}{\partial W_R}\frac{\partial W_R}{\partial(\partial x^i/\partial X_K)}\frac{d}{dt}\frac{\partial x^i}{\partial X_K}=\frac{\partial\tilde{\psi}}{\partial W_R}\delta_{KR}E_i^M\frac{d}{dt}\frac{\partial x^i}{\partial X_K}\tag{3.46}$$
$$=\left(\frac{\partial\tilde{\psi}}{\partial\mathbf{W}}\otimes\mathbf{E}^M\right)\cdot\dot{\mathbf{F}},$$

and, similarly,

$$\frac{\partial\tilde{\psi}}{\partial G_R}\frac{\partial G_R}{\partial(\partial x^i/\partial X_K)}\frac{d}{dt}\frac{\partial x^i}{\partial X_K}=\frac{\partial\tilde{\psi}}{\partial G_R}\delta_{KR}g_i\frac{d}{dt}\frac{\partial x^i}{\partial X_K}=\left(\frac{\partial\tilde{\psi}}{\partial\mathbf{G}}\otimes\mathbf{g}\right)\cdot\dot{\mathbf{F}}.\tag{3.47}$$

Now, recall the well-known identities

$$\begin{aligned}&\frac{\partial J}{\partial(\partial x^p/\partial X_M)}=J\frac{\partial X^M}{\partial x^p},\qquad Q_R=J\frac{\partial X^R}{\partial x^\ell}q^\ell,\\&\frac{\partial(\partial X^R/\partial x^\ell)}{\partial(\partial x^i/\partial X^K)}=-\left(\frac{\partial X^R}{\partial x^i}\right)\left(\frac{\partial X^K}{\partial x^\ell}\right).\end{aligned}\tag{3.48}$$

Hence we have

$$\frac{\partial\tilde{\psi}}{\partial Q_R}\frac{\partial Q_R}{\partial\left(\frac{\partial x^i}{\partial X^K}\right)}=\frac{\partial\tilde{\psi}}{\partial Q_R}\frac{\partial\left(J\frac{\partial X^R}{\partial x^\ell}q^\ell\right)}{\partial\left(\frac{\partial x^i}{\partial X^K}\right)}=\frac{\partial\tilde{\psi}}{\partial Q_R}\left[\frac{\partial J}{\partial\left(\frac{\partial x^i}{\partial X^K}\right)}\frac{\partial X^R}{\partial x^\ell}+J\frac{\partial\left(\frac{\partial X^R}{\partial x^\ell}\right)}{\partial\left(\frac{\partial x^i}{\partial X^K}\right)}\right]q^\ell$$
$$=J\frac{\partial\tilde{\psi}}{\partial Q_R}\left(\frac{\partial X^K}{\partial x^i}\frac{\partial X^R}{\partial x^\ell}-\frac{\partial X^R}{\partial x^i}\frac{\partial X^K}{\partial x^\ell}\right)q^\ell.$$

Now, eliminating q^ℓ by

$$q^\ell=J^{-1}\frac{\partial x^\ell}{\partial X^M}Q^M,\tag{3.49}$$

we find

$$\frac{\partial\tilde{\psi}}{\partial Q_R}\frac{\partial Q_R}{\partial\left(\frac{\partial x^i}{\partial X^K}\right)}=J\frac{\partial\tilde{\psi}}{\partial Q_R}\left(\frac{\partial X^K}{\partial x^i}\frac{\partial X^R}{\partial x^\ell}-\frac{\partial X^R}{\partial x^i}\frac{\partial X^K}{\partial x^\ell}\right)J^{-1}\frac{\partial x^\ell}{\partial X^M}Q^M$$
$$=\frac{\partial\tilde{\psi}}{\partial Q_R}\left(\frac{\partial X^K}{\partial x^i}\delta^R_M-\frac{\partial X^R}{\partial x^i}\delta^K_M\right)Q^M=\frac{\partial\tilde{\psi}}{\partial Q_R}\left(\frac{\partial X^K}{\partial x^i}Q^R-\frac{\partial X^R}{\partial x^i}Q^K\right),$$

thus in absolute notation, by $(2.2)_2$, we have

$$\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\cdot\frac{\partial\mathbf{Q}}{\partial\mathbf{F}}=\left(\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\cdot\mathbf{Q}\right)\mathbf{F}^{-1}-\mathbf{F}^{-T}\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\otimes\mathbf{Q}$$
$$=J\left[\left(\mathbf{F}^{-T}\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\cdot\mathbf{q}\right)\mathbf{F}^{-1}-\mathbf{F}^{-T}\left(\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\otimes\mathbf{q}\right)\mathbf{F}^{-T}\right]. \tag{3.50}$$

Hence, we can rewrite the terms in the right-hand side of (3.28) by the right-hand sides of the equalities shown below:

$$\frac{\partial\overline{\psi}}{\partial\mathbf{F}}\cdot\dot{\mathbf{F}}=\Bigg[\frac{\partial\tilde{\psi}}{\partial\mathbf{E}}\mathbf{F}^{T}+\frac{\partial\tilde{\psi}}{\partial\mathbf{W}}\otimes\mathbf{E}^{M}$$
$$+J\left(\left(\mathbf{F}^{-T}\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\cdot\mathbf{q}\right)\mathbf{F}^{-1}-\mathbf{F}^{-T}\left(\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\otimes\mathbf{q}\right)\mathbf{F}^{-T}\right)+\frac{\partial\tilde{\psi}}{\partial\mathbf{G}}\otimes\mathbf{g}\Bigg]\cdot\dot{\mathbf{F}}, \tag{3.51}$$

$$\frac{\partial\overline{\psi}}{\partial\mathbf{E}^{M}}\cdot\dot{\mathbf{E}}^{M}=\left(\mathbf{F}\frac{\partial\tilde{\psi}}{\partial\mathbf{W}}\right)\cdot\dot{\mathbf{E}}^{M}, \tag{3.52}$$

$$\frac{\partial\overline{\psi}}{\partial\mathbf{q}}\cdot\dot{\mathbf{q}}=\left(\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\frac{\partial\mathbf{Q}}{\partial\mathbf{q}}\right)\cdot\dot{\mathbf{q}}=\left(J\mathbf{F}^{-T}\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\right)\cdot\dot{\mathbf{q}}, \tag{3.53}$$

$$\frac{\partial\overline{\psi}}{\partial\mathbf{g}}\cdot\dot{\mathbf{g}}=\frac{\partial\tilde{\psi}}{\partial G_R}\frac{\partial G_R}{\partial g^i}\frac{dg^i}{dt}=\frac{\partial\tilde{\psi}}{\partial G_R}F^i_R\frac{dg^i}{dt}=\left(\mathbf{F}\frac{\partial\tilde{\psi}}{\partial\mathbf{G}}\right)\cdot\dot{\mathbf{g}}. \tag{3.54}$$

Consequently, the *dissipation inequality* (3.32) becomes

$$\Bigg[\frac{\partial\tilde{\psi}}{\partial\mathbf{E}}\mathbf{F}^{T}+\frac{\partial\tilde{\psi}}{\partial\mathbf{W}}\otimes\mathbf{E}^{M}+J\left(\left(\mathbf{F}^{-T}\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\cdot\mathbf{q}\right)\mathbf{F}^{-1}-\mathbf{F}^{-T}\left(\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\otimes\mathbf{q}\right)\mathbf{F}^{-T}\right)$$
$$+\frac{\partial\tilde{\psi}}{\partial\mathbf{G}}\otimes\mathbf{g}-\rho^{-1}\mathbf{F}^{-1}\boldsymbol{\tau}\Bigg]\cdot\dot{\mathbf{F}}+\left(\frac{\partial\tilde{\psi}}{\partial\theta}+\eta\right)\dot{\theta}$$
$$+\left(\mathbf{F}\frac{\partial\tilde{\psi}}{\partial\mathbf{W}}+\boldsymbol{\pi}\right)\cdot\dot{\mathbf{E}}^{M}+\left(J\mathbf{F}^{-T}\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\right)\cdot\dot{\mathbf{q}}+\left(\mathbf{F}\frac{\partial\tilde{\psi}}{\partial\mathbf{G}}\right)\cdot\dot{\mathbf{g}}+\frac{\rho^{-1}}{\theta}\mathbf{q}\cdot\mathbf{g}\leq 0. \tag{3.55}$$

Now we apply Coleman–Noll's procedure [5], that is, Remark 1: by the arbitrariness of $\dot{\mathbf{g}}$ we have $\partial\tilde{\psi}/\partial\mathbf{G}=\mathbf{O}$ and by the arbitrariness of the time derivatives $\dot{\mathbf{F}}$, $\dot{\theta}$, $\dot{\mathbf{E}}^M$, from (3.55) we obtain

$$\rho^{-1}\mathbf{F}^{-1}\boldsymbol{\tau}=\frac{\partial\tilde{\psi}}{\partial\mathbf{E}}\mathbf{F}^{T}+\frac{\partial\tilde{\psi}}{\partial\mathbf{W}}\otimes\mathbf{E}^{M}$$
$$+J\left[\left(\mathbf{F}^{-T}\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\cdot\mathbf{q}\right)\mathbf{F}^{-1}-\mathbf{F}^{-T}\left(\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\otimes\mathbf{q}\right)\mathbf{F}^{-T}\right], \tag{3.56}$$

$$\tilde{\eta}=-\frac{\partial\tilde{\psi}}{\partial\theta}, \tag{3.57}$$

$$\tilde{\boldsymbol{\pi}} = -\mathbf{F}\frac{\partial\tilde{\psi}}{\partial\mathbf{W}}, \tag{3.58}$$

$$\rho J\mathbf{F}^{-T}\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\cdot\dot{\mathbf{q}} + \frac{1}{\theta}\mathbf{q}\cdot\mathbf{g} \le 0, \tag{3.59}$$

with $\dot{\mathbf{q}}$ given by Eq. (3.27) and where $\tilde{\psi}$ and its derivatives depend on $(\mathbf{E}, \theta, \mathbf{W}, \mathbf{Q})$. We have proved the version of Theorem 1 that employs the objective free-energy response function $\tilde{\psi}$:

THEOREM 3. *The Dissipation Principle is satisfied if and only if the following conditions hold:*

(i) *the objective free energy response function* $\tilde{\psi}(\mathbf{E}, \theta, \mathbf{W}, \mathbf{Q}, \mathbf{G})$ *is independent of the temperature gradient* $\mathbf{G}$ *and determines the Cauchy stress tensor, the entropy and the polarization vector per unit mass through the relations* (3.56)–(3.58);

(ii) *the reduced dissipation inequality* (3.59) *is satisfied with* $\dot{\mathbf{q}}$ *given by* (3.27).

We point out that Eqs. (3.56), (2.10) and (3.58) yield the expression for the Cauchy stress

$$\boldsymbol{\tau} = \rho\mathbf{F}\frac{\partial\tilde{\psi}}{\partial\mathbf{E}}\mathbf{F}^T - \mathbf{P}\otimes\mathbf{E}^M + \rho_R\left(\mathbf{F}^{-T}\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\cdot\mathbf{q}\right)\mathbf{I} - \rho_R\mathbf{F}\mathbf{F}^{-T}\left(\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\otimes\mathbf{q}\right)\mathbf{F}^{-T}. \tag{3.60}$$

Hence, for the antisymmetric portion $\boldsymbol{\tau}^A$ of $\boldsymbol{\tau}$ we obtain the expression

$$\boldsymbol{\tau}^A = \frac{1}{2}\left(\mathbf{E}^M\otimes\mathbf{P} - \mathbf{P}\otimes\mathbf{E}^M\right) - \rho_R\,\mathrm{skw}\left[\mathbf{F}\mathbf{F}^{-T}\left(\frac{\partial\tilde{\psi}}{\partial\mathbf{Q}}\otimes\mathbf{q}\right)\mathbf{F}^{-T}\right], \tag{3.61}$$

where $\mathrm{skw}\,\mathbf{T} = (\mathbf{T}-\mathbf{T}^T)/2$ denotes the skew part of a tensor $\mathbf{T}$.

Note that when $\partial\tilde{\psi}/\partial\mathbf{Q} = \mathbf{0}$, Eqs. (3.60) and (3.61) respectively coincide with Eqs. (4.19) and (4.22) of [45].

3.5. Internal dissipation and entropy equality

The local *internal dissipation* δ_o in a thermoelastic body is defined by TRUESDELL ([44, p. 112])

$$\delta_o = \theta\dot{\eta} - (r - \frac{1}{\rho}\,\mathrm{div}\,\mathbf{q}); \tag{3.62}$$

then one proves that $\delta_o \equiv 0$ along every local thermoelastic process. Within thermo-electroelasticity here we define the *internal dissipation* just by (3.62), and hence we extend the afore-mentioned theorem by the

THEOREM 4. *Along any local process of B we have*

$$\delta_o = -\frac{\partial\overline{\psi}}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} \geq \frac{1}{\rho\theta}\mathbf{q}\cdot\mathbf{g}. \tag{3.63}$$

P r o o f. By inserting the energy equation (3.5) in (3.62), we obtain the equality

$$\delta_o = \theta\dot{\eta} - \frac{1}{\rho}\big(\rho\dot{\varepsilon} - \boldsymbol{\tau}\cdot\nabla\mathbf{v} - \mathbf{E}^M\cdot\rho\dot{\boldsymbol{\pi}}\big); \tag{3.64}$$

now, taking the material derivative of (3.9) yields

$$\theta\dot{\eta} = -\dot{\psi} + \dot{\varepsilon} - \dot{\theta}\eta - \dot{\mathbf{E}}^M\cdot\boldsymbol{\pi} - \mathbf{E}^M\cdot\dot{\boldsymbol{\pi}} \tag{3.65}$$

and by replacing the latter into (3.64), we obtain

$$\delta_o = -\dot{\psi} - \dot{\theta}\eta - \dot{\mathbf{E}}^M\cdot\boldsymbol{\pi} + \frac{1}{\rho}\boldsymbol{\tau}\cdot\nabla\mathbf{v}; \tag{3.66}$$

substituting (3.31) and (3.28) in the latter, we find

$$\delta_o = -\Big(\partial_{\mathbf{F}}\overline{\psi}\cdot\dot{\mathbf{F}} + \partial_\theta\overline{\psi}\cdot\dot{\theta} + \partial_{\mathbf{E}^M}\overline{\psi}\cdot\dot{\mathbf{E}}^M + \partial_{\mathbf{q}}\overline{\psi}\cdot\dot{\mathbf{q}} + \partial_{\mathbf{g}}\overline{\psi}\cdot\dot{\mathbf{g}}\Big) - \dot{\theta}\eta - \dot{\mathbf{E}}^M\cdot\boldsymbol{\pi} + \partial_{\mathbf{F}}\overline{\psi}\cdot\dot{\mathbf{F}}; \tag{3.67}$$

thus, the constitutive restrictions in Theorem 1 yield $(3.63)_1$ and the the reduced dissipation inequality (3.26) yields $(3.63)_2$. □

In thermoelasticity one shows that any thermoelastic process is locally reversible, in the sense that the entropy equality

$$\dot{\eta} = \frac{r}{\theta} - \frac{1}{\rho\theta}\operatorname{div}\mathbf{q} \tag{3.68}$$

holds. Here this result is extended to thermo-electroelasticity by the following

THEOREM 5. *Along any local process of B, the following entropy equality*

$$\dot{\eta} = \frac{r}{\theta} - \frac{1}{\rho\theta}\operatorname{div}\mathbf{q} - \frac{\partial\psi}{\partial\mathbf{q}}\cdot\dot{\mathbf{q}} \tag{3.69}$$

holds.

P r o o f. Equalities (3.62) and $(3.63)_1$ imply (3.69). □

4. Referential description

4.1. Preliminaries

The referential description allows a more close comparison between the results of the present paper and their analogs in [3], where it is used. Hence next we adopt it and we use the first Piola–Kirchhoff stress tensor

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\tau}, \tag{4.1}$$

the Piola-transform

$$\boldsymbol{\Sigma} = J\mathbf{F}^{-1}\boldsymbol{\sigma} \tag{4.2}$$

of the total stress tensor $\boldsymbol{\sigma}$ defined in (3.1), the well-known equalities (2.2), (2.7)–(2.9), $(2.10)_2$ and

$$\operatorname{Div}\mathbf{S} = J\operatorname{div}\boldsymbol{\tau}. \tag{4.3}$$

Now the process class $\mathbb{P}(B)$ of B in Section 2, containing the processes (2.1), must be substituted for $\mathbb{P}_R(B)$, that is the set of ordered 10-tuples of functions on $\mathcal{B}\times\mathbb{R}$

$$p_R = \big(\mathbf{x}(.), \theta(.), \varphi(.), \varepsilon(.), \eta(.), \mathbf{S}(.), \mathbf{I\!P}(.), \mathbf{Q}(.), \mathbf{b}(.), r(.)\big) \in \mathbb{P}_R(B) \tag{4.4}$$

defined with respect to $\mathcal{B}$, satisfying the material versions of the balance laws of linear momentum, moment of momentum, energy, the entropy inequality and the field equations of electrostatics.

4.2. Local balance laws in material form

The local field laws (3.3)–(3.7) in the referential description are written in the form:

$$\rho_R\dot{\mathbf{v}} = \operatorname{Div}\mathbf{S} + \rho_R\mathbf{b}, \tag{4.5}$$

$$\mathbf{F}\boldsymbol{\Sigma}^T = \boldsymbol{\Sigma}\mathbf{F}^T, \tag{4.6}$$

$$\rho_R\dot{\varepsilon} = \mathbf{S}\cdot\dot{\mathbf{F}} - \operatorname{Div}\mathbf{Q} + \mathbf{W}\cdot\dot{\mathbf{I\!P}} + \rho_R r, \tag{4.7}$$

$$\mathbf{W} = -\nabla_{\mathbf{X}}\varphi \ (= -\mathbf{F}^T\nabla_{\mathbf{x}}\varphi), \qquad \operatorname{Div}\boldsymbol{\Delta} = 0, \tag{4.8}$$

$$\rho_R\dot{\eta} \geq \rho_R(r/\theta) - \operatorname{Div}(\mathbf{Q}/\theta). \tag{4.9}$$

The specific *free energy* per unit mass is defined by

$$\psi = \varepsilon - \theta\eta - \mathbf{W}\cdot\boldsymbol{\Pi}. \tag{4.10}$$

For justification of the terms $\mathbf{W}\cdot\dot{\mathbf{I\!P}}$ in (4.7) and $\mathbf{W}\cdot\boldsymbol{\Pi}$ in (4.10), one may read the comments between (3.7) and (3.10) in Section 3.1.

Then (4.7) and (4.9) yield the *reduced dissipation inequality*

$$\rho_R(\dot{\psi} + \eta\dot{\theta}) - \mathbf{S}\cdot\dot{\mathbf{F}} + \frac{1}{\theta}\mathbf{Q}\cdot\mathbf{G} + \mathbf{I\!P}\cdot\dot{\mathbf{W}} \leq 0, \tag{4.11}$$

where $\mathbf{G} = \operatorname{Grad}\theta(\mathbf{X},t)$ is the referential temperature gradient.

REMARK 3. Note that by the equalities

$$\mathbf{W}\cdot\boldsymbol{\Pi} = \mathbf{W}\cdot\mathbf{I\!P}/\rho_R = (\mathbf{F}^T\mathbf{E}^M)\cdot(J\mathbf{F}^{-1}\mathbf{P})/\rho_R = \mathbf{E}^M\cdot\mathbf{P}(J/\rho_R) = \mathbf{E}^M\cdot\boldsymbol{\pi},$$

the spatial and material versions (4.10) and (3.9) of ψ agree.

4.3. Referential constitutive assumptions

Let $\mathcal{D}_R$ be the open, simply-connected domain consisting of 5−tuples $(\mathbf{F}, \theta, \mathbf{W}, \mathbf{Q}, \mathbf{G})$ such that $(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{q}, \mathbf{g}) \in \mathcal{D}$; hence

$$\textit{if} \quad (\mathbf{F}, \theta, \mathbf{W}, \mathbf{Q}, \mathbf{G}) \in \mathcal{D}_R, \quad \textit{then} \quad (\mathbf{F}, \theta, \mathbf{W}, \mathbf{0}, \mathbf{0}) \in \mathcal{D}_R.$$

Assumption 2. For every $p \in \mathbb{P}_R(B)$, the specific free energy $\psi(\mathbf{X}, t)$, the specific entropy $\eta(\mathbf{X}, t)$, the first Piola–Kirchhoff stress tensor $\mathbf{S}(\mathbf{X}, t)$, the specific polarization vector $\mathbf{I\!P}(\mathbf{X}, t)$, and the time rate of the heat flux $\dot{\mathbf{Q}}(\mathbf{X}, t)$, are given by continuously differentiable functions on $\mathcal{D}_R$ such that

$$\psi = \hat{\psi}(\mathbf{F}, \theta, \mathbf{W}, \mathbf{Q}, \mathbf{G}), \tag{4.12}$$

$$\eta = \hat{\eta}(\mathbf{F}, \theta, \mathbf{W}, \mathbf{Q}, \mathbf{G}), \tag{4.13}$$

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{W}, \mathbf{Q}, \mathbf{G}), \tag{4.14}$$

$$\mathbf{I\!P} = \hat{\mathbf{I\!P}}(\mathbf{F}, \theta, \mathbf{W}, \mathbf{Q}, \mathbf{G}), \tag{4.15}$$

$$\dot{\mathbf{Q}} = \mathbf{H}(\mathbf{F}, \theta, \mathbf{W}, \mathbf{Q}, \mathbf{G}). \tag{4.16}$$

Further, the tensors $\partial_{\mathbf{Q}}\mathbf{H}(.)$ and $\partial_{\mathbf{G}}\mathbf{H}(.)$ are non-singular.

Of course, once $\rho_R(.)$, $\hat{\psi}(.)$, $\hat{\eta}(.)$ and $\hat{\mathbf{I\!P}}(.)$ are known, then equality (4.10) gives the continuously differentiable function $\hat{\varepsilon}(.)$ determining $\varepsilon(\mathbf{X}, t)$ such that

$$\varepsilon = \hat{\varepsilon}(\mathbf{F}, \theta, \mathbf{W}, \mathbf{Q}, \mathbf{G}). \tag{4.17}$$

The assumed properties of the *heat flux evolution function* $\mathbf{H}(.)$ indicate that it is invertible for $\mathbf{Q}$ and also for $\mathbf{G}$. We denote the inverse of $\mathbf{H}(.)$ with respect to $\mathbf{Q}$ by

$$\mathbf{Q} = \mathbf{H}^*(\mathbf{F}, \theta, \mathbf{W}, \mathbf{G}, \dot{\mathbf{Q}}). \tag{4.18}$$

Note that

$$\partial_{\mathbf{G}}\mathbf{H}^*(.) = -[\partial_{\mathbf{Q}}\mathbf{H}]^{-1}(.)\partial_{\mathbf{G}}\mathbf{H}(.), \tag{4.19}$$

so that the tensor $\partial_{\mathbf{G}}\mathbf{H}^*(.)$ is also continuous and non-singular. The dependence upon $\mathbf{X}$ is implicit and understood if the body is not materially homogeneous.

4.4. Coleman–Noll method and thermodynamic restrictions in the material description

Given any motion $\mathbf{x}(\mathbf{X}, t)$, temperature field $\theta(\mathbf{X}, t)$ and electric potential field $\varphi(\mathbf{X}, t)$, the constitutive equations (4.12)–(4.16) determine $e(\mathbf{X}, t)$, $\eta(\mathbf{X}, t)$,

$\mathbf{S}(\mathbf{X},t)$, $\mathbb{P}(\mathbf{X},t)$, $\dot{\mathbf{Q}}(\mathbf{X},t)$, and the local laws (4.5) and (4.7) determine $\mathbf{b}(\mathbf{X},t)$ and $r(\mathbf{X},t)$. Hence for any given motion, temperature field and electric potential field, a corresponding process p is constructed.

The method of Coleman–Noll [5] is based on the postulate that every process p so constructed belongs to the process class $\mathbb{P}_R(B)$ of B; that is, on the assumption that the constitutive assumptions (4.12)–(4.16) are compatible with thermodynamics, in the sense of the following

DISSIPATION PRINCIPLE. *For any given motion, temperature field and electric potential field, the process p constructed from the constitutive equations* (4.12)–(4.16) *belongs to the process class $\mathbb{P}_R(B)$ of B. Therefore the constitutive functions* (4.12)–(4.16) *are compatible with the second law of thermodynamics in the sense that they satisfy the dissipation inequality* (4.9).

THEOREM 6. *The Dissipation Principle is satisfied if and only if the following conditions hold:*

(i) *the free energy response function $\hat{\psi}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q},\mathbf{G})$ is independent of the temperature gradient $\mathbf{G}$ and determines the entropy, the first Piola–Kirchhoff stress and the polarization vector through the relations*

$$\hat{\eta}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q}) = -\frac{\partial\hat{\psi}}{\partial\theta}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q}), \tag{4.20}$$

$$\hat{\mathbf{S}}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q}) = \rho_R\frac{\partial\hat{\psi}}{\partial\mathbf{F}}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q}), \tag{4.21}$$

$$\hat{\mathbf{\Pi}}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q}) = -\frac{\partial\hat{\psi}}{\partial\mathbf{W}}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q}); \tag{4.22}$$

(ii) *the reduced dissipation inequality*

$$\rho_R\theta\frac{\partial\hat{\psi}}{\partial\mathbf{Q}}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q})\cdot\hat{\mathbf{H}}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q},\mathbf{G}) + \mathbf{Q}\cdot\mathbf{G} \le 0 \tag{4.23}$$

is satisfied.

P r o o f. By the chain rule we have

$$\dot{\psi} = \partial_{\mathbf{F}}\hat{\psi}\cdot\dot{\mathbf{F}} + \partial_\theta\hat{\psi}\cdot\dot{\theta} + \partial_{\mathbf{W}}\hat{\psi}\cdot\dot{\mathbf{W}} + \partial_{\mathbf{Q}}\hat{\psi}\cdot\dot{\mathbf{Q}} + \partial_{\mathbf{G}}\hat{\psi}\cdot\dot{\mathbf{G}}. \tag{4.24}$$

Thus by substituting this equation together with the constitutive equations (4.12)–(4.16) into the dissipation inequality (4.11), we obtain

$$\begin{aligned}(\rho_R\partial_{\mathbf{F}}\hat{\psi} - \hat{\mathbf{S}})\cdot\dot{\mathbf{F}} + (\rho_R\partial_\theta\hat{\psi} + \hat{\eta})\dot{\theta} + (\rho_R\partial_{\mathbf{W}}\hat{\psi} + \hat{\mathbb{P}})\cdot\dot{\mathbf{W}}&\\ + \rho_R\partial_{\mathbf{Q}}\hat{\psi}\cdot\mathbf{H} + \rho_R\partial_{\mathbf{G}}\hat{\psi}\cdot\dot{\mathbf{G}} + \frac{1}{\theta}\mathbf{Q}\cdot\mathbf{G} \le 0.&\end{aligned} \tag{4.25}$$

Now we follow the Coleman–Noll [5] method: by Remark 1, written in referential form, we can state that $\dot{\mathbf{F}}, \dot{\theta}, \dot{\mathbf{W}}$ and $\dot{\mathbf{G}}$ can be assigned arbitrary values independently of the other variables; thus the theorem is proved. $\square$

Simply by inserting the additional variable $\mathbf{W}$ in each occurrence of $\mathbf{H}^*$, $\mathbf{H}$, f, $\mathbf{K}$, $\mathbf{K}^*$ in Theorem 2 of [3] and in its proof, we obtain the proof of the theorem below.

THEOREM 7. *The time derivative of the heat flux* $\dot{\mathbf{Q}}$ *vanishes for all thermal equilibrium states* $(\mathbf{F}, \theta, \mathbf{W}, \mathbf{0}, \mathbf{0}) \in \mathcal{D}$ *and the tensor*

$$\mathbf{K}(\mathbf{F}, \theta, \mathbf{W}) = \partial_{\mathbf{Q}}\mathbf{H}(\mathbf{F}, \theta, \mathbf{W}, \mathbf{0}, \mathbf{0})^{-1}\partial_{\mathbf{G}}\mathbf{H}(\mathbf{F}, \theta, \mathbf{W}, \mathbf{0}, \mathbf{0}) \tag{4.26}$$

is positive-definite.

Of course, now Remark 2 should be formulated in the material description too.

5. On Cattaneo's equation

The results and considerations in Sections 4, 5 of [3] remain true also in the context of the present theory, even if here we also have the referential electric field $\mathbf{W}$ as variable in each constitutive quantity. Hence the following theorem, which assumes the linearity of $\hat{\mathbf{H}}(\mathbf{F}, \theta, \mathbf{W}, \mathbf{Q}, \mathbf{G})$ in $\mathbf{Q}$ and $\mathbf{G}$, is true.

THEOREM 8. *Let the evolution equation of the heat flux be given by the following form of Cattaneo's equation:*

$$\hat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{W})\dot{\mathbf{Q}} + \mathbf{Q} = -\hat{\mathbf{K}}(\mathbf{F}, \theta, \mathbf{W})\mathbf{G}. \tag{5.1}$$

Then the Dissipation Principle is equivalent to the conditions:

(i) *the tensor* $\hat{\mathbf{K}}(\mathbf{F}, \theta, \mathbf{W})$ *is positive definite;*

(ii) *the tensor* $\hat{\mathbf{Z}}(\mathbf{F}, \theta, \mathbf{W})$ *is symmetric;*

(iii) *the response functions of the specific free energy, specific internal energy, specific entropy and first Piola-Kirchhoff stress are given by*

$$\rho_R\hat{\psi}(\mathbf{F}, \theta, \mathbf{W}, \mathbf{Q}) = \rho_R\hat{\psi}_o(\mathbf{F}, \theta, \mathbf{W}) + \frac{1}{2\theta}\mathbf{Q}\cdot\hat{\mathbf{Z}}(\mathbf{F}, \theta, \mathbf{W})\mathbf{Q}, \tag{5.2}$$

$$\rho_R\hat{\varepsilon}(\mathbf{F}, \theta, \mathbf{W}, \mathbf{Q}) = \rho_R\hat{\varepsilon}_o(\mathbf{F}, \theta, \mathbf{W}) + \mathbf{Q}\cdot\hat{\mathbf{A}}(\mathbf{F}, \theta, \mathbf{W})\mathbf{Q}, \tag{5.3}$$

$$\rho_R\hat{\eta}(\mathbf{F}, \theta, \mathbf{W}, \mathbf{Q}) = \rho_R\hat{\eta}_o(\mathbf{F}, \theta, \mathbf{W}) + \mathbf{Q}\cdot\hat{\mathbf{B}}(\mathbf{F}, \theta, \mathbf{W})\mathbf{Q}, \tag{5.4}$$

$$\hat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{W}, \mathbf{Q}) = \hat{\mathbf{S}}_o(\mathbf{F}, \theta, \mathbf{W}) + \mathbf{Q}\cdot\hat{\mathbf{P}}_Z(\mathbf{F}, \theta, \mathbf{W})\mathbf{Q}, \tag{5.5}$$

where

$$\hat{\psi}_o(\mathbf{F}, \theta, \mathbf{W}) = \hat{\psi}(\mathbf{F}, \theta, \mathbf{W}, \mathbf{0}), \tag{5.6}$$

$$\hat{\varepsilon}_o(\mathbf{F}, \theta, \mathbf{W}) = \hat{\psi}_o(\mathbf{F}, \theta, \mathbf{W}) - \theta\partial_\theta\hat{\psi}_o(\mathbf{F}, \theta, \mathbf{W}), \tag{5.7}$$

$$\hat{\eta}_o(\mathbf{F},\theta,\mathbf{W}) = -\partial_\theta \hat{\psi}_o(\mathbf{F},\theta,\mathbf{W}), \tag{5.8}$$

$$\hat{\mathbf{S}}_o(\mathbf{F},\theta,\mathbf{W}) = \rho_R \partial_{\mathbf{F}} \hat{\psi}_o(\mathbf{F},\theta,\mathbf{W}), \tag{5.9}$$

$$\hat{\mathbf{Z}}(\mathbf{F},\theta,\mathbf{W}) = \hat{\mathbf{K}}(\mathbf{F},\theta,\mathbf{W})^{-1}\hat{\mathbf{T}}(\mathbf{F},\theta,\mathbf{W}), \tag{5.10}$$

$$\hat{\mathbf{A}}(\mathbf{F},\theta,\mathbf{W}) = -\frac{\theta^2}{2}\frac{\partial}{\partial\theta}\left[\frac{\hat{\mathbf{Z}}(\mathbf{F},\theta,\mathbf{W})}{\theta^2}\right], \tag{5.11}$$

$$\hat{\mathbf{B}}(\mathbf{F},\theta,\mathbf{W}) = -\frac{1}{2}\frac{\partial}{\partial\theta}\left[\frac{\hat{\mathbf{Z}}(\mathbf{F},\theta,\mathbf{W})}{\theta}\right], \tag{5.12}$$

$$\hat{\mathbf{P}}_Z(\mathbf{F},\theta,\mathbf{W}) = \frac{1}{2\theta}\frac{\partial}{\partial\mathbf{F}}\hat{\mathbf{Z}}(\mathbf{F},\theta,\mathbf{W}). \tag{5.13}$$

We write the proof following the trace of the one of Theorem 3 in [3].

P r o o f. By Assumption 2, the tensors $\partial_{\mathbf{Q}}\mathbf{H}(.)$ and $\partial_{\mathbf{G}}\mathbf{H}(.)$ are non-singular. Let us define

$$\begin{aligned} \mathbf{T}(\mathbf{F},\theta,\mathbf{W})^{-1} &= -\partial_{\mathbf{Q}}\mathbf{H}(\mathbf{F},\theta,\mathbf{W},\mathbf{0},\mathbf{0}),\\ \mathbf{Z}(\mathbf{F},\theta,\mathbf{W})^{-1} &= -\partial_{\mathbf{G}}\mathbf{H}(\mathbf{F},\theta,\mathbf{W},\mathbf{0},\mathbf{0}). \end{aligned} \tag{5.14}$$

Hence, from (4.26) and (5.14) it follows that

$$\mathbf{K}(\mathbf{F},\theta,\mathbf{W}) = \mathbf{T}(\mathbf{F},\theta,\mathbf{W})\mathbf{Z}(\mathbf{F},\theta,\mathbf{W})^{-1}. \tag{5.15}$$

Now, linearity of $\hat{\mathbf{H}}(.)$ in $\mathbf{Q}$ and $\mathbf{G}$, i.e. the equality

$$\dot{\mathbf{Q}} = \partial_{\mathbf{Q}}\mathbf{H}(\mathbf{F},\theta,\mathbf{W},\mathbf{0},\mathbf{0})\mathbf{Q} + \partial_{\mathbf{G}}\mathbf{H}(\mathbf{F},\theta,\mathbf{W},\mathbf{0},\mathbf{0})\mathbf{G}, \tag{5.16}$$

is equivalent to the equality

$$\dot{\mathbf{Q}} = -\mathbf{T}(\mathbf{F},\theta,\mathbf{W})^{-1}\mathbf{Q} - \mathbf{Z}(\mathbf{F},\theta,\mathbf{W})^{-1}\mathbf{G}. \tag{5.17}$$

From the inequality (4.23) we find

$$\begin{aligned} &-\rho_R\theta\frac{\partial\hat{\psi}}{\partial\mathbf{Q}}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q})\cdot\mathbf{T}(\mathbf{F},\theta,\mathbf{W})^{-1}\mathbf{Q}\\ &\qquad -\rho_R\theta\frac{\partial\hat{\psi}}{\partial\mathbf{Q}}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q})\cdot\mathbf{Z}(\mathbf{F},\theta,\mathbf{W})^{-1}\mathbf{G} + \mathbf{Q}\cdot\mathbf{G} \le 0, \end{aligned} \tag{5.18}$$

and by the arbitrariness of $\mathbf{Q}$ and $\mathbf{G}$ this inequality holds if and only if

$$\begin{aligned} &\frac{\partial\hat{\psi}}{\partial\mathbf{Q}}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q}) = \frac{1}{\rho_R\theta}\mathbf{Z}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q})^T\mathbf{Q},\\ &\rho_R\frac{\partial\hat{\psi}}{\partial\mathbf{Q}}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q})\cdot\mathbf{T}(\mathbf{F},\theta,\mathbf{W})^{-1}\mathbf{Q} \ge 0. \end{aligned} \tag{5.19}$$

The symmetry of $\partial^2_{\mathbf{Q}}\hat{\psi}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q})$ implies that, for the Dissipation Principle to hold, $\mathbf{Z}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q})$ must be symmetric:

$$\mathbf{Z}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q}) = \mathbf{Z}(\mathbf{F},\theta,\mathbf{W},\mathbf{Q})^T. \tag{5.20}$$

Hence $(5.19)_2$ is equivalent to $\mathbf{Q}\cdot\mathbf{K}(\mathbf{F},\theta,\mathbf{W})^{-1}\mathbf{Q} \geq 0$, since $\mathbf{K}(\mathbf{F},\theta,\mathbf{W})$ is positive-definite.

Now, from the equalities (4.10), (4.20), (4.21), (5.19), (5.20), by using Taylor's expansions, it follows that all equalities (5.2) through (5.13) hold. □

Following [1] and [3] we may call $\hat{\mathbf{K}}(\mathbf{F},\theta,\mathbf{W})$ the *steady-state conductivity*, $\hat{\mathbf{T}}(\mathbf{F},\theta,\mathbf{W})$ the *tensor of relaxation times* and $\hat{\mathbf{Z}}^{-1}$ the *instantaneous conductivity.*

6. Hint at the classical theory, where the heat flux has a response function

For the electrically polarizable and deformable heat conducting elastic body B, we consider the (classical) theory, where the heat flux is treated as a dependent variable by a constitutive equation and there is no constitutive equation for its rate. In parallel with [3], [54] we follow the method of COLEMAN and NOLL [5] and find the thermodynamic restrictions on the constitutive relations of B.

We refer to the spatial description and proceed by listing the results in Sections 2 to 5, by showing their reductions in the classical theory and how they change. Sections 2 and 3.1 remain unchanged. Section 3.2 must be replaced by the section below.

6.1. Spatial constitutive assumptions in the classical theory

Let $\mathcal{D}$ be an open, simply-connected domain consisting of 4-tuples $(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{g})$, and assume that if $(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{g})\in\mathcal{D}$, then $(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{0})\in\mathcal{D}$.

ASSUMPTION 3. For every $p\in I\!P(B)$ the specific free energy $\psi(\mathbf{X},t)$, the specific entropy $\eta(\mathbf{X},t)$, the Cauchy stress tensor $\boldsymbol{\tau}(\mathbf{X},t)$, the specific polarization vector $\mathbf{P}(\mathbf{X},t)$, and the heat flux $\mathbf{q}(\mathbf{X},t)$, are given by continuously differentiable functions on $\mathcal{D}$ such that

$$\psi = \overline{\psi}(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{g}), \tag{6.1}$$

$$\eta = \overline{\eta}(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{g}), \tag{6.2}$$

$$\boldsymbol{\tau} = \overline{\boldsymbol{\tau}}(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{g}), \tag{6.3}$$

(6.4) $$\mathbf{P} = \overline{\mathbf{P}}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{g}),$$

(6.5) $$\mathbf{q} = \overline{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{g}).$$

Of course, once $\rho(.)$, $\overline{\psi}(.)$, $\overline{\eta}(.)$ and $\overline{\mathbf{P}}(.)$ are known, then Eq. (3.9) gives the continuously differentiable function $\overline{\varepsilon}(.)$ determining $\varepsilon(\mathbf{X}, t)$ such that

(6.6) $$\varepsilon = \overline{\varepsilon}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{g}).$$

The dependence upon $\mathbf{X}$ is not written only for brevity; when the body is not materially homogeneous, it becomes active.

6.2. Coleman–Noll method and thermodynamic restrictions in the classical theory

In Sec. 3.2 the constitutive law (3.16) must be replaced by (6.5). Thus the section rewrites as

ASSUMPTION 4. For every $p \in \mathbb{P}(B)$ the specific free energy $\psi(\mathbf{X}, t)$, the specific entropy $\eta(\mathbf{X}, t)$, the Cauchy stress tensor $\boldsymbol{\tau}(\mathbf{X}, t)$, the specific polarization vector $\mathbf{P}(\mathbf{X}, t)$, and the heat flux $\mathbf{q}(\mathbf{X}, t)$ are given by continuously differentiable functions on $\mathcal{D}$ such that

(6.7) $$\psi = \overline{\psi}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{g}),$$

(6.8) $$\eta = \overline{\eta}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{g}),$$

(6.9) $$\boldsymbol{\tau} = \overline{\boldsymbol{\tau}}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{g}),$$

(6.10) $$\mathbf{P} = \overline{\mathbf{P}}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{g}),$$

(6.11) $$\mathbf{q} = \overline{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{g}).$$

Of course, once $\rho(.)$, , $\overline{\psi}(.)$, $\overline{\eta}(.)$ and $\overline{\mathbf{P}}(.)$ are known, equality (3.9) gives the continuously differentiable function $\overline{\varepsilon}(.)$ determining $\varepsilon(\mathbf{X}, t)$ such that

(6.12) $$\varepsilon = \overline{\varepsilon}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{g}).$$

The dependence upon $\mathbf{X}$ is not written only for brevity.

The *dissipation principle* and Remark 1 must be understood here just as in Section 3.3; then Theorem 1 becomes the theorem below, whose proof has the same steps in the proof of the former by dropping there $\mathbf{h}$.

THEOREM 9. *The Dissipation Principle is satisfied if and only if the following conditions hold:*

(i) *the free energy response function* $\overline{\psi}(\mathbf{F}, \theta, \mathbf{E}^M, \mathbf{g})$ *is independent of the temperature gradient* $\mathbf{g}$ *and determines the entropy, the Cauchy stress and the polarization vector through the relations*

(6.13) $$\overline{\eta}(\mathbf{F}, \theta, \mathbf{E}^M) = -\frac{\partial \overline{\psi}}{\partial \theta}(\mathbf{F}, \theta, \mathbf{E}^M),$$

$$\overline{\boldsymbol{\tau}}(\mathbf{F},\theta,\mathbf{E}^M) = \rho\mathbf{F}\frac{\partial\overline{\psi}}{\partial\mathbf{F}}(\mathbf{F},\theta,\mathbf{E}^M), \tag{6.14}$$

$$\overline{\boldsymbol{\pi}}(\mathbf{F},\theta,\mathbf{E}^M) = -\frac{\partial\overline{\psi}}{\partial\mathbf{E}^M}(\mathbf{F},\theta,\mathbf{E}^M). \tag{6.15}$$

(ii) *the Fourier inequality*

$$\mathbf{q}\cdot\mathbf{g} \leq 0 \tag{6.16}$$

is satisfied.

A consequence of the Fourier inequality (6.16) is that, just as in thermoelasticity, the *static heat flux* vanishes:

Theorem 10. *The heat flux* $\mathbf{q}$ *vanishes for all thermal equilibrium states* $(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{0}) \in \mathcal{D}$*, that is,*

$$\overline{\mathbf{q}}(\mathbf{F},\theta,\mathbf{E}^M,\mathbf{0}) = \mathbf{0}. \tag{6.17}$$

6.3. Use of invariant response functions in the classical theory

The present section is rewritten following Sec. 3.4, by dropping in it each occurrence of $\dot{\mathbf{q}}$ and also of $\mathbf{q}$, when the latter appears as variable within a response function, and by putting $\partial\tilde{\psi}/\partial\mathbf{q} = \mathbf{0} = \partial\tilde{\psi}/\partial\mathbf{Q}$. Hence Theorem 3 reduces to the following

Theorem 11. *The Dissipation Principle is satisfied if and only if the following conditions hold:*

(i) *the objective free energy response function* $\tilde{\psi}(\mathbf{E},\theta,\mathbf{W},\mathbf{G})$ *is independent of the temperature gradient* $\mathbf{G}$ *and determines the entropy, the Cauchy stress tensor and the polarization vector per unit mass through the relations*

$$\rho^{-1}\mathbf{F}^{-1}\boldsymbol{\tau} = \frac{\partial\tilde{\psi}}{\partial\mathbf{E}}\mathbf{F}^T + \frac{\partial\tilde{\psi}}{\partial\mathbf{W}}\otimes\mathbf{E}^M, \tag{6.18}$$

$$\tilde{\eta} = -\frac{\partial\tilde{\psi}}{\partial\theta}, \tag{6.19}$$

$$\tilde{\boldsymbol{\pi}} = -\mathbf{F}\frac{\partial\tilde{\psi}}{\partial\mathbf{W}}, \tag{6.20}$$

(ii) *the Fourier inequality* (6.16) *is satisfied.*

Note that Eqs. (6.18)–(6.20) respectively coincide with Eqs. (4.19)–(4.21) of ([45]); moreover, (6.18), (2.10) and (6.20) yield the Cauchy stress expression (3.60) and for its antisymmetric portion they yield

$$\boldsymbol{\tau}^A = \frac{1}{2}\big(\mathbf{E}^M\otimes\mathbf{P} - \mathbf{P}\otimes\mathbf{E}^M\big), \tag{6.21}$$

that coincides with [45, Eq. (3.24)].

6.4. Internal dissipation and entropy equality in the classical theory

Within thermo-electroelasticity, the *internal dissipation* is defined by (3.62). Hence just the same proofs of Theorem 4 and Theorem 5 with no change yield the proofs of the theorems below.

THEOREM 12. *Along any local process of B we have*

$$\delta_o = 0. \tag{6.22}$$

THEOREM 13. *Along any local process of B the following entropy equality*

$$\dot{\eta} = \frac{r}{\theta} - \frac{1}{\rho\theta}\operatorname{div}\mathbf{q} \tag{6.23}$$

holds.

7. Conclusions and discussion

(a) The analysis of the Colemann–Noll procedure made here for of an electrically polarizable and finitely deformable heat-conducting elastic continuum, has shown that free energy and constitutive equations cannot depend on the gradients of the unknown fields, with the only exception of the heat flux evolution law, which can depend on heat flux gradients.

This implies that weak non-locality, i.e., the presence of the gradients of all the unknown fields as arguments of the constitutive functions, is not allowed in the present theory [24, p. 912]. Hence, a generalization of the classical Coleman–Noll procedure in the presence of first-order non-local constitutive functions seems to be useful in order to set up a weakly non-local theory for non-local materials, such as the Korteweg fluids. This generalization might extend the theory of CIMMELLI *et al.* [24] for rigid heat-conducting bodies to an electrically polarizable and finitely deformable heat-conducting elastic continuum.

(b) YANG [54] set up in a very general fashion the governing equations for small amplitude incremental fields superposed on finite biasing or initial fields in a thermoelectroelastic body in the classical theory with Fourier's law of heat propagation; he applied the linearization procedure referring to the (classical) thermoelectroelasticity theory TIERSTEN [45]. So Yang could write many successive papers devoted to thermo-electromechanic devices where the results in [54] are used.

Now, in order to solve the analogous problem in a theory where heat propagates at a finite speed, the preliminary task is to have at hand a generalized thermoelectroelasticity theory like e.g. the one presented here. Hence this might be a future work following the present paper.

(c) In agreement with ATKIN *et al.* [34, 35] criticism to the two-fluid Landau's model, the present theory may constitute a preliminary task towards a generalized thermo-electroelasticity theory which conceives helium as a unique continuum and does not use the micropolar technique used in those papers.

(d) Bargmann *et al.* [55] paper uses the Green–Naghdi generalized thermoelasticity theory to model 'cryovolcanism', *i.e.* the icy counterpart of Hearth volcanism. This phenomenon was discovered by the Cassini spacecraft in 2005 during its close fly-bys on Saturn's moon Enceladus.

That is to say that new experiments could arise in the future within new and non-intuitive contexts. And thus several theories, the more simple ones, are needed for the experimentalists.

Why limit the theories to those that have already been fully justified by the experiments already performed?

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