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A Model for Correlated Paired Comparison Data

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Keywords: random effects; paired comparison data; pairwise likelihood; Thurstone-Mosteller model

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1 Paired comparison data

Paired comparison data arise from the comparison of objects in couples. This type of data occurs in many applications such as consumer behaviour, preference testing, sensory testing, biology, acoustics, sports and many other areas.

The traditional models developed for the analysis of paired comparison data are the Bradley-Terry model (Bradley and Terry, 1952) and the Thurstone-Mosteller model (Thurstone, 1927; Mosteller, 1951). In both models, the probability that an object is preferred over another is a function of the difference of the true "worth" of the objects. The main difference between the two models lies in the link function: logit for the Bradley-Terry model and probit for the Thurstone-Mosteller model. Extensions of these models have been developed to take into account specific features of paired comparison data as the existence of an order effect that advantages the object presented first, or situations in which there are three possible outcomes of the comparisons, i.e. preference for one of the two objects or impossibility to express a preference.

Commonly, models for paired comparison data are fitted by maximum likelihood under the assumption of independence among all paired comparisons. This assumption is rarely fulfilled in real applications. An example, later discussed, is sports tournaments where results of two matches involving a common player are naturally correlated. In the following section, we illustrate a model that allows for correlation between paired comparisons.

2 Mixed effects models for paired comparison data

Let Y_{ij}, $j > i = 1, \ldots, n$, be a binary random variable taking value 1 if object i is preferred to object j, and 0 otherwise. In traditional models for paired comparison data, the following generalised linear model is assumed. The density of Y_{ij} is distributed as a Bernoulli random variable whose mean is related to the worth of objects through

$$g\left\{\Pr\left(Y_{ij}=1\right)\right\} = \lambda_i - \lambda_j,$$

where g is a suitable link function and λ_i is the worth parameter for object $i = 1, \ldots, n$. The worth parameter may depend on explanatory variables through the relation

$$\lambda_i = x_i^{\mathrm{T}}\beta,$$

where x_i is a p-dimensional vector of explanatory variables related to object i and β is a vector of p regression parameters. Note that the linear predictor does not include an intercept because this is not identifiable in paired comparison models.

Correlation between observations with a common object can be introduced by including an object-specific zero mean random effect u_i

$$\lambda_i = x_i^{\mathrm{T}}\beta + u_i.$$

Accordingly, the conditional mean of an observation given the object-specific random effects is expressed as

$$g\left\{\Pr\left(Y_{ij}=1|u_i,u_j\right)\right\} = \left(x_i - x_j\right)^{\mathrm{T}}\beta + u_i - u_j.$$

The binary observation Y_{ij} is equivalently represented as a censored continuous random variable $Y_{ij} = I\left\{Z_{ij} > 0\right\}$, where $I\left\{A\right\}$ denotes the indicator function of the set A and

$$Z_{ij} = \left(x_i - x_j\right)^{\mathrm{T}}\beta + u_i - u_j + \epsilon_{ij},$$

where ϵ_{ij} are independent zero mean continuous random variables.

To proceed with likelihood inference, we assume that the random effects u_i are independent, identically distributed normal random variables with zero mean and variance σ^2, the latent errors ϵ_{ij} are independent, identically distributed standard normal variables and they are uncorrelated with the random effects. In other words, the proposed model is a mixed effects version of the Thurstone-Mosteller model. Accordingly, the correlation between a pair of censored random variables Z_{ij} and Z_{kl} is

$$\operatorname{corr}\left(Z_{ij}, Z_{kl}\right) = \begin{cases} \sigma^2/(1+2\sigma^2) & \text{if } i=k \text{ or } j=l, \\ 0 & \text{if } i\neq j\neq k\neq l, \\ -\sigma^2/(1+2\sigma^2) & \text{if } i=l \text{ or } j=k, \end{cases} \tag{1}$$

thus, the model allows for dependence between pairs of observations sharing an object.

The inclusion of the random effects is useful not only to model dependence in paired comparison data, but it also allows to account for the imperfect representation of the worth λ_i by the linear predictor $x_i^{\mathrm{T}}\beta$.

Unfortunately, the mixed effects Thurstone-Mosteller model has an intractable likelihood function, which results from integrating out all the random effects

$$\mathcal{L}(\theta; y) = \int_{\mathbb{R}^n} \left\{ \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} P(Y_{ij} = y_{ij} | u_i, u_j; \theta) \right\} \left\{ \prod_{i=1}^{n} \frac{1}{\sigma} \phi\left(\frac{u_i}{\sigma}\right) \right\} \, \mathrm{d}u_1 \cdots \, \mathrm{d}u_n, \quad (2)$$

where $\theta = (\beta^{\mathrm{T}}, \sigma^2)^{\mathrm{T}}$ is the parameter vector and $\phi(\cdot)$ denotes the density function of a standard normal variable. Thus, the full likelihood consists in a complicated integral of dimension equal to the number of objects being compared. Except for small n, a direct approximation of the likelihood can yield numerical difficulties, or even be impractical. In the following, we propose to resort to pairwise likelihood inference to achieve reduction in computational complexity while retaining part of the likelihood properties.

3 Pairwise likelihood inference

A composite likelihood is a class of pseudo-likelihood constructed by compounding marginal or conditional probabilities for subsets of events (Lindsay, 1988; Varin *et al.*, 2011). In our specific case, it is convenient to consider a particular example of composite likelihood known as pairwise likelihood (Le Cessie and Van Houwelingen, 1994). This consists of the product of bivariate marginal probabilities associated with each pair of observations

$$\mathcal{L}_{\text{pair}}(\theta; y) = \prod_{\{i,j,k,l \in \mathcal{D}\}} \Pr(Y_{ij} = y_{ij}, Y_{kl} = y_{kl}; \theta),$$

where $\mathcal{D}$ denotes the set of indexes i, j, k, l identifying two different observations, that is with $i < j$, $k < l$, excluding the case in which both $i = k$ and $j = l$, and with $k \geqslant i$ in order to include all couples of observations only once. Under the model assumptions each of the above bivariate marginal probabilities is a two dimensional normal integral. Indeed, the joint distribution of the pair of censored random variables (Z_{ij}, Z_{kl}) is bivariate normal with zero mean, variance $1+2\sigma^2$ and correlation as in equation (1). Then, the probability that object i loses against both j and k is

$$\begin{aligned} \Pr(Y_{ij} = 0, Y_{ik} = 0; \theta) &= \Pr(Z_{ij} < 0, Z_{ik} < 0; \theta) \\ &= \Phi_2\left(-\frac{(x_i - x_j)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}}, -\frac{(x_i - x_k)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}}; \frac{\sigma^2}{1+2\sigma^2} \right), \end{aligned}$$

where $\Phi_2(\cdot, \cdot; \rho)$ denotes the cumulative distribution function of a bivariate normal random variable with standardised marginals and correlation ρ. The probabilities of the other possible outcomes (win-loss, loss-win and win-win) are

$$\begin{aligned}
\Pr(Y_{ij}=1, Y_{ik}=0;\theta) &= \Phi\left(-\frac{(x_i-x_k)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}}\right)+\\
&\quad -\Phi_2\left(-\frac{(x_i-x_j)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}},-\frac{(x_i-x_k)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}};\frac{\sigma^2}{1+2\sigma^2}\right),\\
\Pr(Y_{ij}=0, Y_{ik}=1;\theta) &= \Phi\left(-\frac{(x_i-x_j)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}}\right)+\\
&\quad -\Phi_2\left(-\frac{(x_i-x_j)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}},-\frac{(x_i-x_k)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}};\frac{\sigma^2}{1+2\sigma^2}\right),\\
\Pr(Y_{ij}=1, Y_{ik}=1;\theta) &= 1-\Phi\left(-\frac{(x_i-x_j)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}}\right)-\Phi\left(-\frac{(x_i-x_k)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}}\right)+\\
&\quad +\Phi_2\left(-\frac{(x_i-x_j)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}},-\frac{(x_i-x_k)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}};\frac{\sigma^2}{1+2\sigma^2}\right),
\end{aligned}$$

where Φ denotes the cumulative distribution function of a standard normal random variable. Hence, pairwise likelihood considerably reduces the computational effort as it involves a set of bivariate normal integrals in place of the high-dimensional integral of the full likelihood. Bivariate normal integrals are computed with very high numerical accuracy using routines in the `R` (`R` Development core team, 2009) package `mvtnorm` (Genz *et al.*, 2010).

The logarithm of the pairwise likelihood is denoted by $\ell_{\mathrm{pair}}(\theta; y) = \log \mathcal{L}_{\mathrm{pair}}(\theta; y)$ and its maximum, $\hat{\theta}$, is the maximum pairwise likelihood estimator. Under mild regularity conditions, the maximum pairwise likelihood estimator is consistent and asymptotically normally distributed with mean θ and covariance matrix $\mathrm{G}(\theta) = \mathrm{H}(\theta)^{-1}\mathrm{J}(\theta)\mathrm{H}(\theta)^{-1}$, where $\mathrm{J}(\theta) = \mathrm{var}\ \{\nabla \ell_{\mathrm{pair}}(\theta; Y)\}$ and $\mathrm{H}(\theta) = \mathrm{E}\left\{-\nabla^2 \ell_{\mathrm{pair}}(\theta; Y)\right\}$, see Cox and Reid (2004).

Hypothesis testing and interval estimation can be based on the parwise likelihood analogue of the likelihood ratio statistic. Suppose that δ is a q-dimensional subvector of the whole parameter vector $\theta = (\delta^{\mathrm{T}}, \lambda^{\mathrm{T}})^{\mathrm{T}}$ and that it is of interest to test hypothesis $\mathrm{H}_0 : \delta = \delta_0$. This hypothesis can be assessed through the pairwise likelihood ratio statistic defined as

$$\mathcal{W}_{\mathrm{pair}}(Y) = 2\left[\ell_{\mathrm{pair}}(\hat{\theta}; Y) - \ell_{\mathrm{pair}}(\delta_0, \hat{\lambda}(\delta_0); Y)\right],$$

where $\hat{\lambda}(\delta_0)$ denotes the maximum pairwise likelihood estimator in the subspace where $\delta = \delta_0$. The pairwise log-likelihood ratio statistic has asymptotic distribution given by the weighted sum $\sum_{i=1}^{q} \xi_i \chi^2_{i(1)}$, where $\chi^2_{i(1)}$ are independent chi-square random variables with 1 degree of freedom and the ξ_i are the eigenvalues of $(\mathrm{H}^{\delta\delta})^{-1}\mathrm{G}_{\delta\delta}$, where $\mathrm{H}^{\delta\delta}$ denotes the block of the inverse of $\mathrm{H}(\theta)$ pertaining to δ and $\mathrm{G}_{\delta\delta}$ is the block of the matrix $\mathrm{G}(\theta)$ pertaining to δ.

3.1 Simulations

The performance of the pairwise likelihood estimator is evaluated through a simulation study. Data is simulated from a single round robin tournament in which each

Table 1: Empirical means and standard deviations of 500 estimates of the parameters of the mixed effects Thurstore-Mosteller model for increasing values of $\sigma \in \{0.2, 0.4, 0.6, 0.8, 1\}$. True values for β_1 and β_2 are -2 and 1, respectively.

		σ				
		0.2	0.4	0.6	0.8	1
β_1	mean	-1.975	-2.098	-2.021	-2.070	-2.018
	s.e.	0.781	1.028	1.659	2.190	2.597
β_2	mean	1.008	1.006	1.021	1.013	1.023
	s.e.	0.130	0.201	0.237	0.338	0.415
σ	mean	0.185	0.376	0.592	0.787	0.985
	s.e.	0.061	0.087	0.111	0.132	0.186

of n objects is compared once with all the other objects. The worth parameter of the objects is assumed to be

$$\lambda_i = \beta_1 x_{1i} + \beta_2 x_{2i} + u_i,$$

where covariates x_{1i} are independently simulated from a normal distribution with mean 0 and standard deviation 0.1 and covariates x_{2i} are independently simulated from a Bernoulli distribution with probability of success 0.6. Table 1 reports empirical means and standard deviations of 500 simulated parameter estimates in data sets involving $n = 30$ for various values of the random effects standard deviation $\sigma \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$.

The results of the simulations seem satisfactory. Biases of the regression parameters are relatively small. The estimate of σ is slightly downward biased, as expected, in variance components models. In fact, also the full likelihood is known to produce downward biased estimates of this parameter. Finally, as expected, inflating σ implies higher variability in the estimates of the regression coefficients.

4 Applications

In this section, we illustrate pairwise likelihood inference in the mixed effects Thurstone-Mosteller model with application to two sports tournaments.

4.1 Volleyball

Sports data are a natural field of application of models for paired comparison data. The first application considered regards the results of the 2006/2007 Italian men's volleyball A1 league. The league is composed of 14 teams that compete in a double round-robin tournament, that is, each team competes twice against all the other teams in the league, for a total of 182 competitions. The matches cannot end in a tie, so there are only two possible outcomes for each contest. At the end of the

Table 2: Estimates (`est.`) and standard errors (`s.e.`) of independence (first two columns) and mixed effects (last two columns) models for the volleyball data.

	independence		mixed effects	
	est.	s.e.	est.	s.e.
playoffs	0.101	0.032	0.115	0.056
home effect	0.446	0.098	0.516	0.107
mean age	0.136	0.059	0.149	0.105
σ	-	-	0.379	0.116

regular season, the best eight teams access to the playoffs to compete for the title of Italian Champion. The information available about the volleyball teams are the number of accesses to the playoffs in the previous eight years and the mean age of the players. The home effect is a further covariate which accounts for the advantage deriving from playing in a home field. In fact, it is commonly recognised that a team playing at home enjoys the benefits of the acquaintance with the playing field and a larger number of supporters. None of the matches played during the season took place in a neutral field, so it seems important to account for this effect. The interest lies in determining whether these covariates – previous access to playoffs, mean age and home effect – affect the result of the matches.

The first two columns of Table 2 display the estimates of the traditional Thurstone-Mosteller model termed independence model and corresponding to the restriction $\sigma^2 = 0$, and the last two columns present the estimates of the proposed mixed effects model. The estimated home effect and its standard error confirm that, in this volleyball tournament, teams playing at home have an important advantage over the guest teams. The significance of the parameter relating to the number of accesses to the playoffs in the previous 8 years reveals that teams which were strong in the past tend to remain strong also in the present season. Finally, the independence model states that the mean age of the team has a positive influence on its ability. This covariate has a narrow range, indeed it lies between 25.25 and 29.31 years old, but it seems that teams with older players, who are probably more experienced, have higher probability of winning. However, the fitted mixed effects model leads to a different conclusion. In fact, the inclusion of the random effects increases the standard error of the estimates of the regression coefficients and the mean age effect is not significant anymore, while the other covariates remain significant. Finally, the estimate of the random effect standard deviation σ is 0.379 with standard error 0.116.

It is here of main interest to test whether the variance component is null, or in other terms if correlation between matches with a common player is relevant or not. The test of the hypothesis $\mathrm{H}_0 : \sigma^2 = 0$ against $\mathrm{H}_1 : \sigma^2 > 0$ is complicated because the parameter value under the null hypothesis lies on the boundary of the parameter space and thus standard asymptotic results do not apply. In this case it is convenient to resort to parametric bootstrap as in Bellio and Varin (2005). First, compute the pairwise likelihood ratio statistic

$$\mathcal{W}_{\text{pair}}(y) = 2\left[\ell_{\text{pair}}\{\hat{\theta}(y); y\} - \ell_{\text{pair}}\{\hat{\theta}_0(y); y\}\right],$$

where $\hat{\theta}_0(y)$ is the maximum pairwise likelihood estimator under the null hypothesis. Then, setting $\theta = \hat{\theta}_0(y)$, generate M data sets $y^{(1)}, \ldots, y^{(M)}$. For each simulated data set, compute the maximum pairwise likelihood estimator $\hat{\theta}(y^{(m)})$, the maximum pairwise likelihood estimator under the null hypothesis $\hat{\theta}_0(y^{(m)})$ and the relative pairwise log-likelihood ratio statistic $\mathcal{W}_{\text{pair}}(y^{(m)})$. The p-value of the test is then estimated by quantity

$$p = \frac{\sum_{m=1}^{M} I\left\{\mathcal{W}_{\text{pair}}(y^{(m)}) \geqslant \mathcal{W}_{\text{pair}}(y)\right\} + 1}{M+1}.$$

In the volleyball data, this parametric bootstrap test based on $1,000$ simulations yields a p-value smaller than 0.01, thus not supporting the null hypothesis $\mathrm{H}_0 : \sigma^2 = 0$.

4.2 Water Polo

The second data set considered here consists of the results of the water polo matches played by teams in the male A1 league during the 2008/2009 regular season. The water polo tournament has a double round robin structure, so in each half of the season every team competes once against all the other teams in the league. The A1 league includes 12 teams playing altogether 132 matches. At the end of the regular season, the best eight teams access to the playoffs to compete for the title of Italian Champion. The available covariate is the number of accesses to the playoffs in the previous six years. The analysis is focused on determining whether there is a sort of "tradition effect" in water polo, that is whether teams strong in past seasons tend to be strong also in the present one.

Water polo matches can also end in ties, hence the model presented so far needs a further extension in order to account for the three possible outcomes of the matches. This extension can be accomplished through the introduction of a threshold parameter τ. Thus, the probability that i loses against j is equal to the probability that the corresponding latent random variable Z_{ij} is smaller than $-\tau$. The probability of a tie between i and j is equal to the probability that the corresponding latent variable Z_{ij} is between $-\tau$ and τ. Finally, the probability that i wins against j is equal to the probability that Z_{ij} is larger than τ. Then, the probability that i loses both the matches against j and k is

$$P(Z_{ij} < -\tau, Z_{ik} < -\tau) = \Phi_2\left(-\frac{\tau + (x_i - x_j)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}}, -\frac{\tau + (x_i - x_k)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}}; \frac{\sigma^2}{1+2\sigma^2}\right),$$

while the probability that i and j draws and i loses against k is equal to

$$\begin{aligned} P(-\tau < Z_{ij} < \tau, Z_{ik} < -\tau) &= P(Z_{ij} < \tau, Z_{ik} < -\tau) - P(Z_{ij} < -\tau, Z_{ik} < -\tau) \\ &= \Phi_2\left(\frac{\tau - (x_i - x_j)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}}, -\frac{\tau + (x_i - x_k)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}}; \frac{\sigma^2}{1+2\sigma^2}\right) + \\ &\quad -\Phi_2\left(-\frac{\tau + (x_i - x_j)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}}, -\frac{\tau + (x_i - x_k)^{\mathrm{T}}\beta}{\sqrt{1+2\sigma^2}}; \frac{\sigma^2}{1+2\sigma^2}\right). \end{aligned}$$

Table 3: Estimates (`est.`) and standard errors (`s.e.`) of independence (first two columns) and mixed effects (last two columns) models for the water polo data.

	independence		mixed effects	
	est.	s.e.	est.	s.e.
playoffs	0.238	0.034	0.315	0.090
home effect	0.223	0.116	0.294	0.129
threshold	0.218	0.051	0.290	0.066
σ	-	-	0.616	0.134

The probabilities of the other possible outcomes are similarly computed.

Again, besides the accesses to the playoffs in the previous six years, the effect of playing at home is taken into account. Table 3 shows the results of the estimates of an independence model, corresponding to the restriction $\sigma^2 = 0$, (first two columns) and the mixed effects model (last two columns).

Both models confirm that the team playing at home has actually an advantage over the away team. The estimate of the accesses to the playoffs in the previous six years is also significant in both models, denoting that teams which were strong in the recent past tend to be strong also in the present season. Finally, the estimated random effect standard deviation is 0.616 with standard error 0.134. The bootstrap test for validating the hypothesis $\mathrm{H}_0 : \sigma^2 = 0$ yields a p-value smaller than 0.001 based on $1,000$ simulations. Therefore, also in water polo the hypothesis of null variance of the random effect may not be accepted.

5 Discussion

In this paper traditional models for paired comparison data are extended in order to introduce correlation among observations with common objects. In many instances, as for example in sports data, it is evident that a model which allows for correlation is more realistic. In the volleyball and water polo data analysed here, the presence of correlation between matches with common teams is borne out by the significance of the variance component. Modelling the dependence in the paired comparison model is important because of its impact on the standard errors of the regression parameters; in some cases accounting for dependence may change the significance of a parameter as was found in the volleyball application.

The mixed effects Thurstone-Mosteller model can be usefully applied also in other areas. For example, in biological studies scientists are interested in determining whether some specific covariates affect the outcomes of contests between animals (Stuart-Fox *et al.*, 2006). In this instance it seems important to account for dependence between fights involving the same animal in order to ascertain at which extent covariates are associated with the outcomes of fights.

The model for sports data can be further extended allowing for a temporal evolution of abilities of teams. For example, it is possible to include time-varying covariates which yield different abilities of teams in different matches. An alternative currently under study is the specification of a temporal evolution of the random effects which induce a temporal variation of abilities.

References

Bellio, R. and Varin, C. (2005). A pairwise likelihood approach to generalized linear models with crossed random effects. *Statistical Modelling* **5**, 217-227.

Bradley, R.A. and Terry, M.E. (1952). The rank analysis of incomplete block designs. I. The method of paired comparisons. *Biometrika* **39**, 324-345.

Cox, D.R. and Reid, N. (2004). A note on pseudolikelihood constructed from marginal densities. *Biometrika* **91**, 729-737.

Genz, A., Bretz, F., Miwa, T., Mi, X., Leisch, F., Scheipl, F. and Hothorn, T. (2010). `mvtnorm`: Multivariate normal and t distributions (http://CRAN.R-project.org/package=mvtnorm).

Le Cessie, S. and Van Houwelingen, J.C. (1994). Logistic regression for correlated binary data. *Applied Statistics* **43**, 95-108.

Lindsay, B.G. (1988). Composite likelihood methods. *Contemporary Mathematics* **80**, 221-239.

Mosteller, F. (1951). Remarks on the method of paired comparisons. I. The least square solution assuming equal standard deviation and equal correlations. *Psychometrika* **16**, 203-206.

`R` Development core team (2009). `R`: A language and environment for statistical computing. `R` foundation for statistical computing, Vienna, Austria. ISBN 3-900051-07-0 (http://www.R-project.org).

Stuart-Fox, D.M., Firth, D., Moussalli, A. and Whiting, M.J. (2006). Multiple signals in chameleon contests: designing and analysing animal contests as a tournament. *Animal Behavior* **71**, 1263-1271.

Thurstone, L.L. (1927). A law of comparative judgement. *Psychological Review* **34**, 273-286.

Varin, C., Reid, N. and Firth. D. (2011). An overview of composite likelihood methods. *Statistica Sinica* **21**, forthcoming.

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