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A new competition model combining the Lotka-Volterra model and the Bass model in pharmacological market competition

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Keywords: Lotka-Volterra competition model, diachronic competition model, Bass model, diffusion processes.

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1 Introduction

The life cycle of a product in a target market is a strategic and fundamental marketing concept by firms that are planning new and innovative products take the place of older and outmoded ones. In concept, a product typically is launched, is adopted by consumers followed by a rapid growth period, reaches a deadlock, and then irremediably declines. These product phases may be quite different in both duration and framework.

The univariate analysis of the life cycle of a single product had a consolidated theoretical development starting from the works of Fourt and Woodlock (1960), Mansfield (1961), and Bass (1969). In particular, in the standard Bass model, the innovators, the first pioneering adopters, and the imitators (those who later adopted

through word of mouth) are the principal latent drivers leading the diffusion process of a product in the market.

The original work of Bass was successful in supporting an authentic research line that, in the sequel, was reorganized in books by Mahajan and Peterson (1985), Mahajan and Wind (1986), and Mahajan et al. (2000) as well as articles by Mahajan and Muller (1979), Mahajan et al.(1990), Mahajan et al.(1993), and Parker (1994). Recently, the Bass model was extended due to a number of issues that, in practice, limited its application. Bemmaor and Lee (2002) and Karmeshu and Goswami (2001) treated the problem of heterogeneity of agents; Guseo and Guidolin (2009) made the latent market potential time dependent and ascribed it a dynamic character; and Bass et al. (1994) introduced a general intervention function to include in the modeling marketing-mix variables under the control of managers, external accidents, incentives or policy measures, and so on.

In the beginning of competition modeling, competition was considered a synchronic process, which is a very simplistic assumption if we consider that this condition is anything but common in real life. In fact, products typically enter the market at different times, and the roles that they assume reciprocally are quite different depending on whether they stood alone for a little while, stood alone for a long time, or stood in competition all along or only later in their life cycle. It is natural that the dynamics change, and a competition model should have a suitable degree of flexibility to allow for changing parameters.

In the literature on competition, locating the first synchronic modeling approaches requires going back to the works of Peterson and Mahajan (1978), Parker and Gatignon (1948), Mahajan et al. (1993), and Kalish et al. (1995). The diachronic competition approach is divided into balanced and unbalanced models. In the balanced models, word of mouth affects the adoptions of both products without distinctions, while in the unbalanced models, the interpersonal communication is split into two components: within-brand effects and cross-brand effects. The distinction appears crucial as it provides further flexibility because the products are perceived as different by consumers.

There are not as many examples in literature about diachronic competition. We cite Krishnan et al. (2000) for a balanced model with a partial regime change in the model parameters; Guseo and Mortarino (2012) for a regime change after competition; Savin and Terwiesch (2005) for proposing an unbalanced model; and again Guseo and Mortarino (2014) for generalizing the model proposed earlier through a parameter that continuously allows one to shift from a balanced to an unbalanced model.

In addition to the specific literature about competition models cited above, there is also a consolidated research line about competitive models that numbers the Lotka-Volterra model as one of the most widely used synchronic competition models. It was extensively used to model the dynamics of growth, coexistence, and survival of two (or even more) populations that share the resources in the same environment. It is essentially grounded on an alteration of a logistic equation to incorporate the interspecific competition coefficients, which, when added to the intraspecific coefficients make the modeling more complete. In fact, the species, being different, also possess different characteristics about the reproductive behavior and the exploita-

tion of resources; therefore, it is natural that the influence that each species has on one another is different. In a notable number of applications, the study of the system of differential equations leads us to consider the possible longstanding scenarios for the analyzed species, such as extinction and coexistence.

In this paper, we propose two extensions of the Lotka-Volterra competition model. The first one is inspired by the innovation component that is a fundamental part of the standard Bass model and permits one of the principal drivers of adoptions in the markets to be taken into account. The second extension allows that the Lotka-Volterra model may become diachronic by simply adjoining a standard Bass model that is able to capture the diffusion of the first competitor in its stand-alone period.

In Section 2, an introductory history about contents and developments of Lotka-Volterra equations is treated. In Section 3, the extensions of LVC models are presented and discussed. In Section 4, an application to pharmaceutical drugs data assesses the goodness-of-fit of modeling. As a concluding section, the obtained results are discussed and commented in Section 5.

2 The Lotka-Volterra competition models: A brief history

The roots of the Lotka-Volterra model hail from the Malthus model first published in 1978 that is a basic step of the theory of population growth. The equation of the Malthus model is

$$X_{t+1} = X_t(1 + \alpha),$$

where X_t is the population in time t, and α is the population growth index.
The continuous representation of previous discrete equation is

$$\frac{\delta X}{\delta t} = \alpha X.$$

Its solution is an exponential function. The simplicity of this model is its strength and, at the same time, also its weakness because, in this case, the unique evolutionary prospects for a population may be either extinction or unlimited growth. Obviously, this is not true in practice. For example, small populations often grow heavily, and large populations diminish, but the trend is to reach a natural equilibrium despite no significant changes occurring in the meantime. The carrying capacity is the key factor that is not included in the Malthus model, and it is connected to the capacity of the environment to absorb population growth, given the limited available resources. Therefore, the carrying capacity is the amount of population in which the number of births and deaths are equal. In other words, it is the greatest number of individuals that may be sustained given the available resources. In these conditions, population is in a stable equilibrium. Verhulst (1838) adopted a logistic conception and proposed the following differential equation for population at time t,

$$\frac{\delta X}{\delta t} = \alpha X \left(1 - \frac{X}{K}\right), \tag{1}$$

where K is the carrying capacity; in this case, the population progressively increases to the limit K when $t \to \infty$. This equation was also rediscovered by Pearl and Reed (1920) to explain the growth of the population of the United States.

The logistic model (1) deals only with internal competition between individuals of a species. Later, Lotka (1925) and Volterra (1926) introduced, independently, a mathematical competition model (LVC) that describes the dynamics of two species coexisting in the same environment and that act as prey and predator, competing for resources. In this sense, the LVC model is an extension of the logistic model because it incorporates a part that describes the effects of external competition among individuals of different species. The concept of environment and the dynamics of growth and death are intentionally simplified and, together with the assumption of no external special interventions result in a framework that it is still a convincing model. The differential equations of the LVC model are:

$$\begin{aligned} \frac{dX_1}{dt} &= \frac{\alpha_1 X_1}{K_1}(K_1 - X_1 - \alpha_{12} X_2) \\ \frac{dX_2}{dt} &= \frac{\alpha_2 X_2}{K_2}(K_2 - X_2 - \alpha_{21} X_1), \end{aligned} \tag{2}$$

where X_1 and X_2 are populations of species 1 and 2, respectively; α_1 and α_2 are the growth rates of species 1 and 2, respectively; K_1 and K_2 are the carrying capacities of the populations; α_{12} is the competition coefficient that measures the competitive effect that population 2 has on population 1 and vice versa for α_{21}. In other words, α_{12} measures the external competition with respect to the internal one, that is, how many individuals of species 1 are equivalent to individuals of species 2. The interspecific coefficients identify the different influences that the two species reciprocally exercise and that condition intensively their whole competitive evolution.

The system of Equation (2) has no analytics solution, but approximate solutions are easily numerically computed.

The applications of this model and also its extensions are innumerable in biology, demography, and physics. There is also a long history of the use of these equations in economics theory. We could cite, among others, a work by Morris and Pratt (2003) that proposed an application of the Lotka-Volterra competition model in a market in which populations are competitors that contend for market shares to obtain a competitive advantage. Their analysis describes the evolution of the diffusion model of the second competitor that invades the market with respect to the first competitor as a function of parameters of the models, classifying the final situation in defined classes. In this case, they consider forcedly the two competitors as if they were synchronous, even if it is not clearly so. In Section 3, we try to overtake this disadvantageous issue by proposing an alternative model.

Moreover, in the literature, the majority of the applications of the LVC model are properly concentrated on the mathematical approach, which is focused on describing the development of the trajectory traced by the system of differential equations. In this case, it is necessary to find the equilibrium points, if they exist, to establish whether they are stable or not, to discuss the phase plane, and to determine the evolution of the competition between species. In this paper, we estimate the parameters of the model as a linear model, and then in Section 4, we discuss the estimates

from a statistical point of view.

3 Extensions of the LVC model: The LVBC diachronic model

Our main idea is to obtain a new competition model that overtakes some limitations of LVC models (2) by introducing a framework that, on one hand, emphasizes the role of innovators in the start-up of competition processes and that, on the other hand, considers that products competing in the market often are not synchronous. These extensions can be introduced separately and give rise to the models that we present in the following.

Let us consider two different brands competing in a market and sharing the same group of adopters. We propose an extension of model (2), denoted as LVBC model,

$$\begin{aligned} \frac{dX_1}{dt} &= \left(p_1 + \frac{\alpha_1 X_1}{K_1}\right)(K_1 - X_1 - \alpha_{12} X_2) \\ \frac{dX_2}{dt} &= \left(p_2 + \frac{\alpha_2 X_2}{K_2}\right)(K_2 - X_2 - \alpha_{21} X_1), \end{aligned} \tag{3}$$

where p_1 and p_2 are parameters that identify the innovation effect of the two different brands on the market. As in LVC model (2), X_1 and X_2 are cumulative adoptions of products 1 and 2, α_1 and α_2 are respectively the growth rates of product 1 and product 2, K_1 and K_2 are the carrying capacities of products 1 and 2, and α_{12} is the competition coefficient that measures the competitive effect that product 2 has on product 1, and vice versa for α_{21}. In other words, in the case of products instead of populations, α_{12} is interpreted as the reduction of growth in the diffusion of product 1 due to the presence of product 2 and vice versa for α_{21}. The competition between the two products can lead also to the exclusion of one of the products or the coexistence of both products.

The contribution given in model (3) by innovation parameters p_1 and p_2, which represent a fundamental part of standard Bass models, is relevant because they start the mechanism of the diffusion process by highlighting the role of pioneers functioning by a subgroup of individuals.

With respect to other competition models, model (3) extends the local perception of the residual market of each brand $(K_i - X_i), i = 1, 2$ through a quantity $\alpha_{ij} X_j$, with $i, j = 1, 2$, and $i \neq j$ that measures the strength of inhibition of the antagonist in the competition.

However, because the class of Lotka-Volterra models is conceived for simultaneous processes, the application of model (3) cannot be considered completely satisfactory when the two products enter the market at different times. Unfortunately, the condition of not equal launch times for products coexisting in the market is a recurring case in practice and not only in particular cases or for special products. In fact, it is quite common that a brand launches a new product into the marketplace, and thereafter a competitor responds with a new product that is the antagonist of the preceding one. Marketing strategies of the firms address eroding market shares of other brands producing similar but not equivalent products. Waiting times may be short or long according to the reaction times and the peculiarities of the firms.

In the presence of a non trivial delay, it is not advantageous to avoid the market evolution of the first product because it implies a distorted estimation of competition between competitors. In this way, the two local perceived residual markets are dynamically modified to also consider external competitive effects with respect to the internal ones.

A possible solution is to modify the LVBC model by introducing a standard Bass model to capture the growth evolution in the stand-alone period for the first competitor. Because it is the initial period of a product's evolution, the choice of a simple Bass model does not appear simple, but is appropriate. Therefore, in diachronic competition, model (3) is modified as

$$\begin{aligned} \frac{dX_1}{dt} &= \left[\left(p_{1A} + \frac{\alpha_{1A}X_1}{K_{1A}}\right)(K_{1A} - X_1)\right] I_{[t<t_0]} + \left[\left(p_1 + \frac{\alpha_1 X_1}{K_1}\right)(K_1 - X_1 - \alpha_{12}X_2)\right] I_{[t\geq t_0]} \\ \frac{dX_2}{dt} &= \left(p_2 + \frac{\alpha_2 X_2}{K_2}\right)(K_2 - X_2 - \alpha_{21}X_1), \end{aligned} \tag{4}$$

where p_{1A}, α_{1A}, and K_{1A} are respectively the coefficients of innovation, imitation, and potential market for the first product in the stand-alone period until time t_0 when a second competitor gets in competition with the first already existing product, and I_η indicates a dummy function that is equal to 1 when condition η is satisfied.

This model denoted by LVBC-diac recovers those data that are normally censored using a synchronic modeling and that, on the contrary, are essential to describe the whole evolution of the product first coming to the market. Furthermore, this clearly affects the final differential equations system by returning a more balanced and reliable competition model. This is an essential feature, especially when the stand-alone period is particularly wide so that the preceding history for the first competitor, which in practice is a monopoly period, is too long and complex to be recovered. This special modeling with two distinct regimes allows for this diachronic competition by giving the correct weight to competitors with different sales histories.

Because LVBC-diac (4) and also LVC (2) and LVBC (3) models do not have closed-form solutions, they have to be applied directly beginning with the differential form, which will be used to estimate the parameters involved.

4 An application to pharmaceutical drugs

The need for testing the performance of our model with respect to the preceding models introduced in the literature drove us to make comparisons using the same data as Guseo and Mortarino (2012) and Guseo and Mortarino (2014). Data were provided by IMS-Health, Italy, and consist of a cumulative quarterly number of packages of Cimetidine and Ranitidine sold in Italy. Cimetidine was introduced in the second quarter of 1979; Ranitidine was commercialized by the fourth quarter of 1981. Data are available until the third quarter of 1991, and on the whole, there are 50 observations for Cimetidine and 40 observations for Ranitidine. These drugs are histamine H_2-receptor antagonists and inhibit the production of acid in the stomach. Cimetidine was marketed by Smith, Kline & French in the United Kingdom in 1976 and in the United States three years later. It was an authentic success with

more the $1 billion a year in sales. In 1981, Glaxo, in response to Cimetidine, developed Ranitidine, which was found to have fewer adverse drug reactions, longer-lasting action, and 10 times the activity of Cimetidine. By 1988, it was the world's best-selling prescription drug.

Table 1: Multivariate estimation results for model LVBC-diac.

Parameters	Estimate	Standard Error
p_{1A}	0.006396	0.01099
α_{1A}	0.3923	0.09459
K_{1A}	11280	2067
p_1	0.03246	6.894
α_1	-0.03374	7.483
K_1	38280	8157000
α_{12}	0.05714	1.027
α_2	0.09758	0.03641
p_2	0.0003021	0.001916
K_2	4803000	30220000
α_{21}	90.47	616.7

These data represent a case of diachronic competition: Ranitidine was introduced in the Italian pharmaceutical market when Cimetidine was the leader of the market for 2 years and also in a descending phase of its life cycle.

Our first aim is to assess the performance of the LVBC-diac model (4). To estimate the parameters of the LVBC-diac model, it is necessary algebraically to simplify the model to reduce it in minimal terms. We consider that dX_1/dt is in practice a function of X_1, X_1^2 and X_1X_2 and that dX_2/dt is equally a function of X_2, X_2^2 and X_1X_2. Therefore, we can use a multiple linear regression analysis method using R software.

Figure 1 shows the correspondence between the observed trends of the number of packages sold for each pharmaceutical drug and the values fitted by the LVBC model whose parameter estimates are summarized in Table 1. The difference between the two trends is clearly evident: Ranitidine at the beginning came into the market at the highest degree with respect to Cimetidine, and afterwards its growth was impressive, carrying the overall potential market at a very high level. On the other hand, Cimetidine, which was in its descending phase at the moment of the entry of Ranitidine, surely took advantage of the competitor's entrance, extending the duration of its life cycle. These considerations are justified also by the estimated parameters of Table 1. The greatly different orders of magnitude between K_1 and K_2 highlight the enormous potential market achieved by the second competitor Ranitidine (4803000) with respect to the first competitor Cimetidine (49560). The estimated coefficient of innovation p_1 is evidently and correctly greater than p_2 given that the innovation process was started just by Cimetidine when Ranitidine was missing. Moreover, the relevant weight in the competition is confirmed also by the different values of the competition coefficients α_{12} and α_{21}, proving the inhibition strength of Ranitidine. The fit of model (4) seems quite good, but it has to be

assessed by also evaluating other existing models in literature.

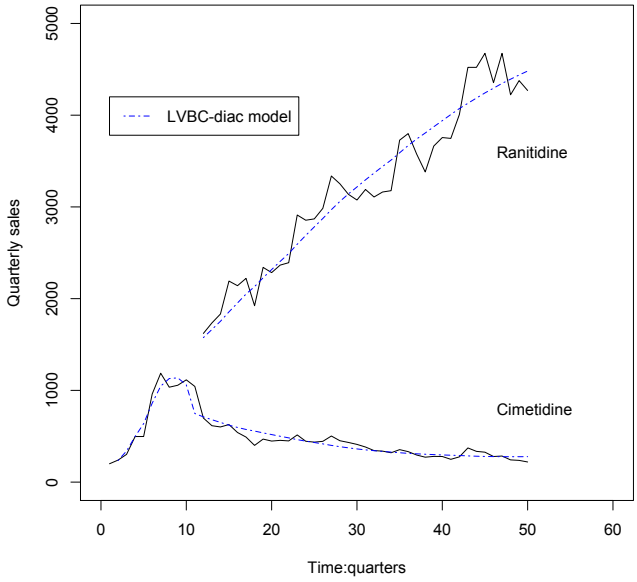

Figure 1: The LVBC-diac model: observed versus fitted values for Cimetidine and Ranitidine

So, as our second aim, we compare the performance of model (4) with the CRCD model introduced in Guseo and Mortarino (2012) and also with the UCRCD model introduced in Guseo and Mortarino (2014), both of them tested on the same data. In Guseo and Mortarino (2012), the authors compared the CRCD model with the Bass model and the joint model KBKD introduced by Krishnan et al. (2000). This model can be obtained by the CRCD model as a particular case, and it was the only alternative balanced diachronic model available in the literature. The authors showed that the CRCD model had a better goodness-of-fit in the specific case studied. Figure 2 and Figure 3 exhibit separately the fit of the LVBC model versus the CRCD and UCRCD models, which for this data, until now, are considered the better models with respect to KBKD.

Figure 2 shows that, for Cimetidine, the LVBC model is perfectly intermediate between the other two models and seems to follow the observed trend better, especially in the final period where CRCD and UCRCD take different directions. In the initial part, the three models are quite perfectly overlapped given that they all lie on the same Bass model, but afterwards, the difference is quite evident. In Figure 3, the differences are notably reduced, and the curves are overlapped, so the trend is notably good for each of the considered models. It is not redundant to observe that because CRCD and UCRCD models were obtained by a closed-form solution, it is perfectly possible for them to give forecasts for any number of quarters, as visible in Figure 2 and Figure 3 in which the CRCD and UCRCD curves are longer than

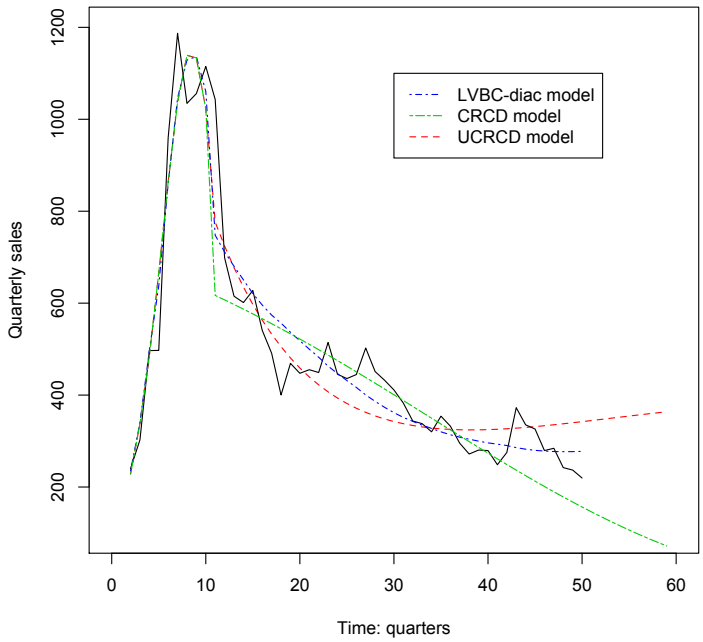

Figure 2: Comparison between observed and fitted values for the LVBC-diac model, CRCD model, and UCRCD model with respect to Cimetidine

the LVBC curve. This is not a prosaic difference, and it shows that the forecasts are a natural completion for these models rather than for the LVBC and LVBC-diac models, which are estimated directly from the differential form through a numerical minimization procedure. In particular, future research should polarise on this peculiar aspect in order to enable this model to do forecasts.

Table 2: Squared Pearson correlation coefficient between observed and fitted values for the LVBC-diac model, CRCD model, and UCRCD model.

ρ^2	LVBC-diac model	CRCD model	UCRCD model
Cimetidine (n=50)	0.9143862	0.874174	0.899239
Ranitidine (n=40)	0.9383331	0.937024	0.939933

Moreover, in order to assess the goodness-of-fit of the different models considered, an easy but effective measurement is given by the squared Pearson correlation coefficient ρ^2 between observed and fitted values. The results presented in Table 2 are split with respect to the two competing pharmaceutical drugs Cimetidine and Ranitidine. The differences between the performance of the alternative models with respect to these drugs are self-evident. In particular, it is not surprising that the ρ^2

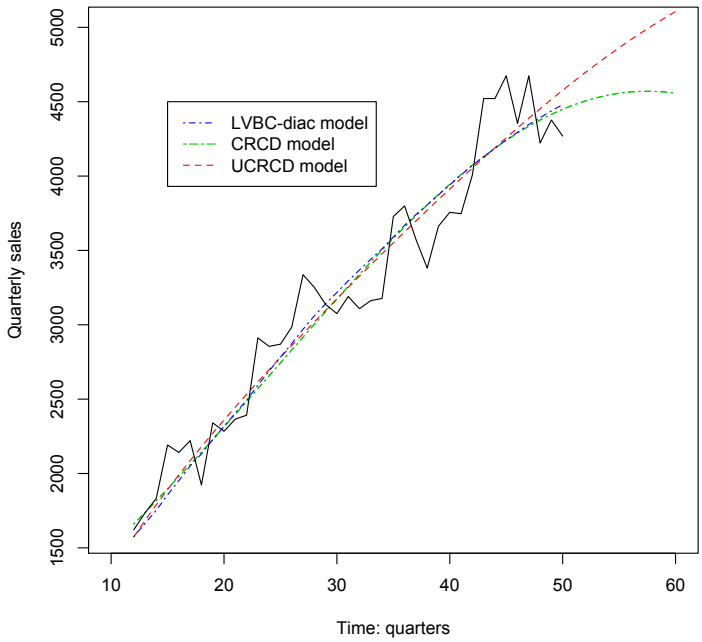

Figure 3: Comparison between observed and fitted values for the LVBC-diac model, CRCD model, and UCRCD model with respect to Ranitidine

is quite similar for each model when Ranitidine is considered, with a slight better fit for UCRCD model, while for Cimetidine the dissimilarities are notable even if not dramatic. In this specific case, the LVBC-diac model performs better than the CRCD and UCRCD models, determining a clear ranking among models, in which the worst fit is associated with the CRCD model. Therefore, it seems to be confirmed that the introduction of substantial changes in LVC competition model (2) obtaining the LVBC-diac model was essential, especially to capture the particular behavior of the first competitor in a more accurate way than the CRCD and UCRCD models which were already shown to be superior. The differences are very slight for the second competitor and become absolutely inappreciable. It seems inappropriate to give a ρ^2 to assess the global fit of both competitors because the risk is the lack of a reliable indicator that attempts to put together two different trends and thereby distorts the reality.

Moreover, we consider it worthwhile to point out that the values of ρ^2 in Table 2, which are smaller for CRCD and UCRCD models than those found in Guseo and Mortarino (2012) and Guseo and Mortarino (2014), were obtained by an analysis based upon instantaneous data, so these differences are perfectly normal. In fact, it is well known that the nonlinear squares algorithm for cumulative data determines high values of standard determination indexes that, in case of a good performance, are characterized by values higher than 0.99. The analysis of residuals highlights

the good performance of the LVBC-diac model (4) as shown in Figures 4 and 5.

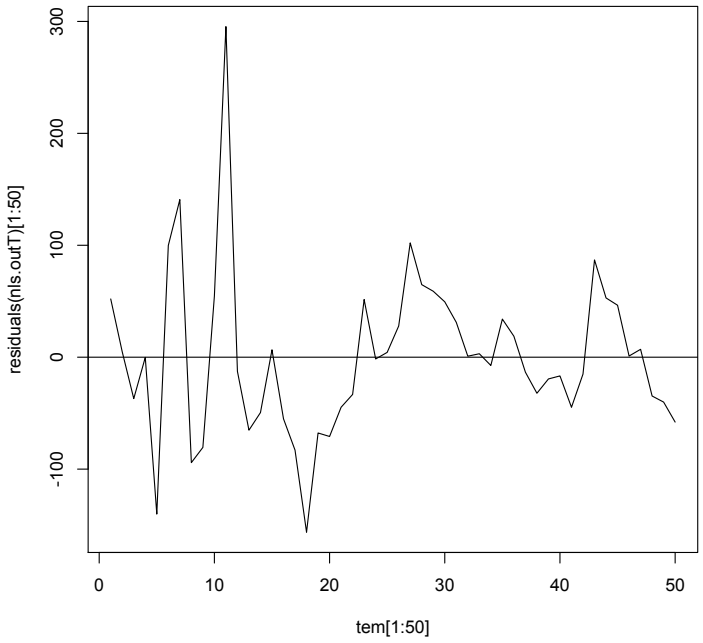

Figure 4: Residual analysis for LVBC-diac model for Cimetidine

5 Conclusion

The possibility of measuring the effects of competition processes for particular category products is an issue of strategic importance for firms and brands. In this paper, we introduce two models that are essentially extensions of the class of LVC competition models that have a consolidated history in literature. The need to recover an excellent competition framework that was applied in innumerable disciplines, from biology to physics and so on, has essentially motivated this extension. The principal and appealing feature in these models is that they offer a definition of residual markets that is more general with respect to the existing competition models available in literature, allowing that competing products contribute with different weights to perceived residual markets. On the other hand, both the lack of specific coefficients to capture the innovation components of competitors, which are essential to model the diffusion and competition model, and the heavy restriction of syncronicity in the entrance in the market by competitors lead us to propose the LVBC and LVBC-diac models. They are first discussed and then tested on pharmacological data already used to test among other existing competition models in the literature. The results are reassuring and appreciable, and they effectively confirm that the LVBC-diac model fits better than the considered CRCD, UCRCD, and KBKD models overall

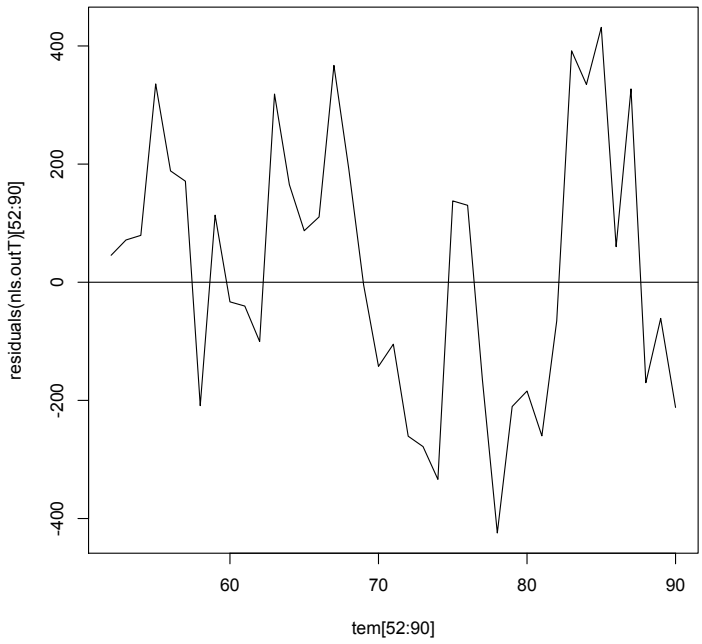

Figure 5: Residual analysis for the the LVBC-diac model for Ranitidine

when the first competitor enters the market considerably earlier than the second one. For the second competitor in this specific analyzed case, the improvements are not appreciable, but they are perfectly overlapped.

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