Nonparametric Estimation for accelerated life testing under imperfect repair

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ABSTRACT This paper considers nonparametric estimation of lifetime distribution of a system subject to imperfect repair, based on data from constant stress accelerated life tests. By assuming as time transformation function relating stress to lifetime, a version of the inverse power law, a method of estimating the lifetime distribution at use condition stress has been recently proposed by Diana and Giordan. This method, based on the Brown-Proschan imperfect repair model, is nonparametric in that it does not make any assumptions about the underlying distribution of life length. Some simulations to understand if accelerated life tests can be used instead of normal tests evaluate the behaviour of the test procedure.

1 Introduction

Suppose that one desires to estimate the distribution function of the lifetime of a device under normal use conditions. If the lifetimes increase, the time consumed in testing a sample of devices may be excessive. The usual solution for this problem involves the use of accelerated life tests (ALTs). Samples of devices are subjected to conditions of greater stress than that of normal use, and from the results in these high-stress environments, an estimate of performance of the device in the use condition is formed. The problem has been widely considered in literature. Under the papers on this subject we quote those of Ball, Shaked and Zimmer (1979) and of Diana and Giordan (2003). In the first a nonparametric model for ALTs has been introduced, while in the second the model has been extended to imperfect repair (IR). Unfortunately, from a practical point of view, the ALTs provide results that don't agree with tests at normal use conditions. The aim of this paper is to try to understand if the "rescaled" failure times arising from ALTs can be used for estimation of the parameters of the underline lifetime distribution at use conditions. To gain this end in section 3 the IR model is briefly summarized and in section 4 a simulation study using Weibull as lifetime distribution is performed. The fifth section is devoted to the conclusions.

2 Notation

k	Number of stress levels
V_i	Accelerated stress, $i = 1, \ldots, k$
V_0	Use condition stress
$S_i(\cdot)$	Reliability function at stress $V_i, i = 0, \ldots, k$
n_i	Number of items on test at stress level i
N	$N = \sum_{i=1}^{k} n_i$
$f_i(\cdot)$	probability density function at stress level i
$g(\cdot)$	Time Transformation Function (TTF)
α, β	Parameters of the TTF
$\theta_{ii'}$	Scale factor between S_i and $S_{i'}$
p	perfect repair probability
T_{ij}	j-th failure time at stress level i
$T_{ij} - T_{i,j-1}$	j-th inter-failure time at stress i
Z_{ij}	Type of j-th repair at stress level i
b_{ij}	Age of system prior to the j-th failure at stress level i, $j = 1, \ldots, n_i, i = 1, \ldots, k$

3 Model and Estimation

For the purposes of the present paper the IR model can be summarized in this way. At stress V_i, $i = 1, \ldots, k$ a system is put on test at time 0 and runs until n_i failures are observed. After each failure a repair is made. The type of repair is identified by a Bernoulli variable:

$$Z_{ij} = \begin{cases} 1 \text{ if the } j\text{-th repair is perfect} \\ 0 \text{ if the } j\text{-th repair is minimal} \end{cases} \quad i = 1, \ldots, k$$

$j = 1, \ldots, n_i$. With this notations the age of a system just prior to the j-th failure, at stress level i, can be defined as:

$$b_{ij} = \begin{cases} t_{i1} & \text{for } j = 1 \\ (1 - z_{i,j-1})\, b_{i,j-1} + (t_{ij} - t_{i,j-1}) & \text{for } j = 2, \ldots, n_i \end{cases}$$

$i = 1, \ldots, k$.

Let us denote with

$$S_i(t) = S(g(V_i)t), \quad t \geq 0 \tag{1}$$

the reliability function of a system at constant stress V_i, where $S(\cdot)$ is an unknown reliability function independent of stress and $g(\cdot)$ is a TTF:

$$g(V_i) = \alpha V_i^{\beta} \tag{2}$$

where $\alpha > 0$ and $\beta > 0$ are unknown parameters.

To estimate the lifetime distribution at use condition, the ALT data should be transformed to the corresponding data which have the same statistical properties as test data at use condition. Let $T_{i'}$ and T_i be the lifetimes of system at stress $V_{i'}$ and V_i, and $S_{i'}(t)$ and $S_i(t)$ be the corresponding reliability functions, $i = 0, 1, \ldots, k$. From assumptions (??) and (??) the scale factor between S_i and $S_{i'}$ is defined by

$$\theta_{ii'} = \left(\frac{V_{i'}}{V_i}\right)^{\beta} \quad \text{for } i, i' = 0, 1, \ldots, k \tag{3}$$

Therefore

$$T_i = \left(\frac{V_{i'}}{V_i}\right)^{\beta} T_{i'} \quad \text{for } i, i' = 0, 1, \ldots, k \tag{4}$$

and

$$\beta = \frac{\ln \theta_{ii'}}{\ln (V_{i'}/V_i)}$$

A well known nonparametric estimator of S_i, $i = 1, \ldots, k$ is given by:

$$\widehat{S}_i(t) = \begin{cases} 1 & \text{for} \quad t < b_{i(1)} \\ \prod_{s=1}^{j} \frac{k_{is}}{k_{is}+1} & \text{for} \quad b_{i(j)} \le t < b_{i,(j+1)} \\ & j = 1, \ldots, n_i - 1 \\ 0 & \text{for} \quad t \ge b_{i(n_i)} \end{cases} \tag{5}$$

with $k_{is} = \sum_{l=s}^{n_i-1} z_{i(l)}$.

A possible estimator of the scale factor $\theta_{ii'}$ is

$$\widehat{\theta}_{ii'} = \frac{\int_{u_{ii'}}^{1} \widehat{S}_i^{-1}(u)\, du}{\int_{u_{ii'}}^{1} \widehat{S}_{i'}^{-1}(u)\, du} \tag{6}$$

where

$$u_{ii'} = \max\left\{\widehat{S}_i\left(b_{i,(n_i)}\right), \widehat{S}_{i'}\left(b_{i',(n_{i'})}\right)\right\} \tag{7}$$

$i, i' = 0, \ldots, k$ and an estimator of β is

$$\widehat{\beta} = \frac{\sum_{i=1}^{k-1} \sum_{i'=i+1}^{k} \left[\ln (V_{i'}/V_i)\right] \left[\ln \widehat{\theta}_{ii'}\right]}{\sum_{i=1}^{k-1} \sum_{i'=i+1}^{k} \left[\ln (V_{i'}/V_i)\right]^2} \tag{8}$$

Once we have an estimate $\widehat{\beta}$ of β, we transform the observed lifetimes T_{ij}, $j = 1, \ldots, n_i$, $i = 1, \ldots, k$ in the rescaled ones at use condition:

$$\widetilde{T}_{ij} = \left(\frac{V_i}{V_0}\right)^{\widehat{\beta}} T_{ij} \tag{9}$$

and so

$$\widetilde{b}_{ij} = \begin{cases} \widetilde{t}_{i1} & \text{for } j = 1 \\ (1 - z_{i,j-1})\,\widetilde{b}_{i,j-1} + (\widetilde{t}_{ij} - \widetilde{t}_{i,j-1}) & \text{for } j = 2, \ldots, n_i \end{cases} \tag{10}$$

$i = 1, \ldots, k$.

Now we write $u_1 = \widetilde{b}_{11}$, $u_2 = \widetilde{b}_{12}, \ldots, u_{n_1} = \widetilde{b}_{1,n_1}$, $u_{n_1+1} = \widetilde{b}_{21}, \ldots, u_N = \widetilde{b}_{k,n_k}$ and $\gamma_1 = z_{11}, \ldots, \gamma_{n_1} = z_{1,n_1}, \ldots, \gamma_N = z_{k,n_k}$.

Let $u_{(1)} \leq \ldots \leq u_{(N)}$ be the ordered values of u_i's and let $\gamma_{(1)}, \ldots, \gamma_{(N)}$ be the induced order statistics generated by ordering the u_i's; so an estimator of $S_0(\cdot)$ is given by the following statistic:

$$\widehat{S}_0(t) = \begin{cases} 1 & \text{for} \quad t < u_{(1)} \\ \prod_{s=1}^{j} \frac{k_s}{k_s+1} & \text{for} \quad u_{(j)} \leq t < u_{(j+1)} \\ & \quad j = 1, \ldots, N-1 \\ 0 & \text{for} \quad t \geq u_{(N)} \end{cases} \tag{11}$$

where $k_s = \sum_{l=s}^{N-1} \gamma_{(l)}$.

4 AN ILLUSTRATIVE EXAMPLE

Table 1 illustrates an application from industrial life-testing reported in Cox, Oakes (1984). Springs are tested and failure time is the number of cycles to failure (in units of 10^3 cycles). 60 springs were allocated, 10 to each of six different stress levels. At the lower stress levels (700 and 750) many springs are censored (*) and the corresponding numbers of cycles are not considered.

Table1. Cycles to failure (in units of 10^3 cycles) of springs

stress										
950	225	171	198	189	189	135	162	135	117	162
900	216	162	153	216	225	216	306	225	243	189
850	324	321	432	252	279	414	396	379	351	333
800	627	1051	1434	2020	525	402	463	431	365	715
750	3402	9417	1802	4326	11520*	7152	2969	3012	1550	11211
700	12510*	12505*	3027	12505*	6253	8011	7795	11604*	11604*	12470*

For the purpose of this illustration, we take those of four test stress at $V_1 = 800$, $V_2 = 850$, $V_3 = 900$, $V_4 = 950$ and the use condition stress is assumed $V_0 = 700$, where at each level of stress the probability of perfect repair is one. Suppose that the TTF is a version of the inverse power law.

The estimation procedure is as follows.

1. The estimates $\widehat{S}_i(\cdot)$, $i = 1,2,3,4$ are obtained
2. The estimate $\widehat{\beta}$ is 8.338
3. All failure times at accelerated stress are transformed in failure times at use conditions stress using $\widehat{\beta} = 8.338$ and $V_0 = 700$
4. From this rescaled failure times we obtain $\widehat{S}_0(\cdot)$. See Fig. 1 for graphical representation of $\widehat{S}_0(\cdot)$.

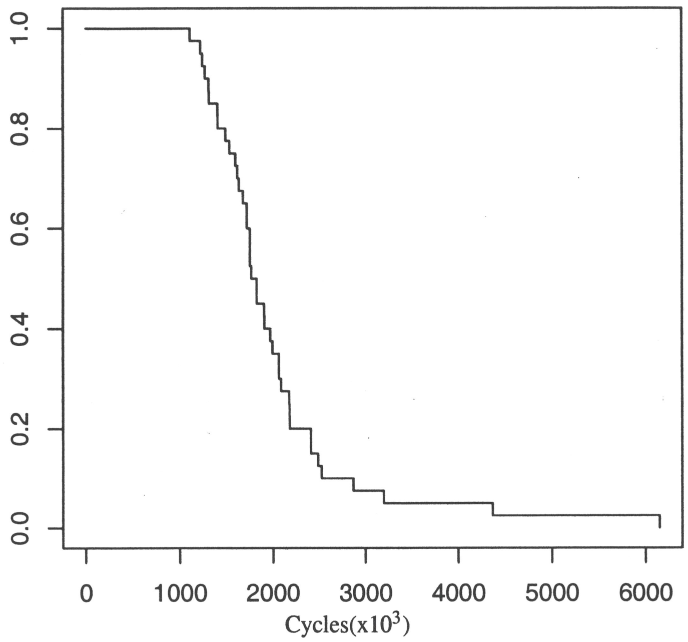

Figure1. $\widehat{S}_0(\cdot)$

5 SIMULATION

In this section the least-squares estimators (LSEs) based on failure-times arising from normal tests are compared with those obtained from rescaled failure-times arising from ALTs, assuming a Weibull lifetime distribution. The comparison is performed with respect to biases and means squared errors (MSEs) when the true lifetime distribution is exponential or Weibull.

To find the LSEs we use the linear model:

$$\log\left(-\log\left(\hat{S}_0(t)\right)\right) = -\eta\log\gamma + \eta\log t\ ,$$

where η and γ are shape and scale parameters of the Weibull distribution. We considered four values of perfect repair probability (0.5,0.7,0.9,1) and three accelerated stress levels $V_1 = 2$, $V_2 = 4$, $V_3 = 6$, while use condition stress is $V_0 = 1$. For each p and for each accelerated stress V_i, $20,000$ samples of size 25, 50 and 100 from Weibull distribution with scale parameter V_i^{-1} and shape parameter $\eta = 1$ and 2, respectively, are generated. Consequently the total sample sizes for ALTs and normal test are 75, 150 and 300, respectively. So we are able to estimate LSEs for the shape and scale parameters for the considered lifetime distribution and the relative biases and MSEs. Finally we repeat the same process for use conditions test. Table 2 shows biases and MSEs of LSEs of shape parameter of Weibull distribution under normal (NT) or accelerated life test, while Table 3 shows the same quantities for scale parameter.

6 Conclusions

Table 2 shows that biases and MSEs of the estimators of shape parameter of Weibull lifetime distribution, coming from NTs and ALTs, are both of a size. If we consider the behaviour of shape parameter through the different values of perfect repair probability, we see that it is almost constant. It is interesting to observe that bias/η and MSE/η^2 seem independent from η.
From Table 3 it is clear a greater variability in the estimates of scale parameter. This little worse behaviour do not penalize ALTs in respect of NTs.
Further simulation results for couples of stress levels, not included here, also show patterns very like to those previously exhibited.
As final consideration, it seems to us the rescaled failure times arising from ALTs can be used for estimation of the parameters of the underline lifetime distribution at use condition.

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Table2. Biases and MSEs of LSEs of shape parameter of Weibull distribution

n		$\eta=1$				$\eta=2$			
		p=0.5	p=0.7	p=0.9	p=1	p=0.5	p=0.7	p=0.9	p=1
25	biasNT	-0.043	-0.044	-0.046	-0.045	-0.084	-0.089	-0.090	-0.088
	MSENT	0.017	0.017	0.017	0.016	0.067	0.067	0.066	0.065
	biasALT	-0.036	-0.036	-0.038	-0.040	-0.070	-0.075	-0.077	-0.079
	MSEALT	0.016	0.016	0.016	0.016	0.064	0.064	0.063	0.064
50	biasNT	-0.029	-0.030	-0.029	-0.030	-0.057	-0.060	-0.061	-0.061
	MSENT	0.009	0.008	0.008	0.008	0.033	0.033	0.033	0.033
	biasALT	-0.025	-0.027	-0.028	-0.029	-0.051	-0.055	-0.054	-0.055
	MSEALT	0.008	0.008	0.008	0.008	0.033	0.033	0.032	0.032
100	biasNT	-0.0	-0.020	-0.020	-0.021	-0.038	-0.038	-0.040	-0.041
	MSENT	0.004	0.004	0.004	0.004	0.017	0.016	0.017	0.017
	biasALT	-0.017	-0.018	-0.020	-0.020	-0.036	-0.036	-0.037	-0.038
	MSEALT	0.004	0.004	0.004	0.004	0.016	0.016	0.016	0.016

Table3. Biases and MSEs of LSEs of scale parameter of Weibull distribution

n		$\eta=1$				$\eta=2$			
		p=0.5	p=0.7	p=0.9	p=1	p=0.5	p=0.7	p=0.9	p=1
25	biasNT	-0.067	-0.037	-0.016	-0.006	-0.038	-0.023	-0.010	-0.006
	MSENT	0.027	0.019	0.016	0.016	0.007	0.005	0.004	0.004
	biasALT	-0.013	0.021	0.041	0.048	-0.028	-0.007	0.003	0.008
	MSEALT	0.023	0.018	0.018	0.018	0.007	0.005	0.004	0.004
50	biasNT	-0.038	-0.021	-0.007	-0.001	-0.021	-0.011	-0.004	-0.001
	MSENT	0.013	0.009	0.008	0.008	0.003	0.002	0.002	0.002
	biasALT	-0.015	0.009	0.021	0.027	-0.016	-0.005	0.002	0.006
	MSEALT	0.012	0.009	0.009	0.009	0.003	0.002	0.002	0.002
100	biasNT	-0.022	-0.011	-0.003	0.002	-0.011	-0.006	-0.002	0.
	MSENT	0.006	0.005	0.009	0.004	0.002	0.001	0.001	0.001
	biasALT	-0.007	0.001	0.004	0.015	-0.008	-0.003	0.	0.003
	MSEALT	0.006	0.005	0.004	0.004	0.002	0.001	0.001	0.001