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Periodic Long Memory GARCH models

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Keywords: GARCH models, long memory, seasonality, volatility.

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1 Introduction

A large body of research suggests that the volatility of many financial time series is strongly persistent. In particular, it has been observed that the autocorrelation functions of the squared, log-squared and absolute returns are best characterized by a slowly mean-reverting hyperbolic rate of decay and the periodogram of the transformed returns shows a peak at zero frequency. These empirical evidences are not consistent with standard ARCH/GARCH (see Engle 1982 and Bollerslev 1986) or with stochastic volatility (SV) models (see Taylor 1986). In light of these empirical findings, models with long memory in the volatility process have been proposed to match these characteristics of the returns. Among these models, it is worth men-

tioning the Fractionally Integrated GARCH model (FIGARCH) proposed in Baillie *et al.* (1996) and in Bollerslev and Mikkelsen (1996), the long memory ARCH(∞) model introduced in Robinson (1991). In the SV context, a long memory version has been proposed in Breidt *et al.* (1998) and in Harvey (1998).
The increased availability of ultra-high frequency data has provided new insights to empirical analysis. One relevant characteristic of such data is given by the presence of a strong evidence of cyclical pattern in the volatility of the series, mainly due to the so called time-of-the-day phenomena (as for example market opening and closing operations and lunch hours effects). In fact, the effect of a distinct inverse-J shaped pattern in the variance of stock returns over the trading day is well documented (see for example Wood, McInish and Ord (1985) and Andersen and Bollerslev 1997). Similarly, equally pronounced patterns in the volatility of intra-day foreign exchange rates were emphasized by Baillie and Bollerslev (1991) and by Dacorogna *et al.* (1993). Focusing again on squared, log-squared and absolute returns, such periodic pattern appears as a cyclical persistent behaviour on the autocorrelation functions. Furthermore, some pronounced peaks at one or more non-zero frequencies on the periodograms are observed.
The empirical evidence cumulated so far suggests the importance of taking into account the periodic intra-daily dynamics for modelling the volatility properly. Standard GARCH and SV models can account only for an exponential decay in the autocorrelation structure of the conditional variance. The long memory extensions of both the models, i.e. FIGARCH and long memory SV models can properly account for long-range persistence in the volatility. However all of these models cannot account for cyclical behaviour, especially when long-range dependencies are characteristics of the periodic pattern. Ignoring this kind of persistent cyclical behaviour may lead to erroneous conclusions on the persistence properties of the shocks to the conditional variance and the results of modelling and forecasting the volatility dynamics can be affected (see for instance the empirical evidence provided by Andersen and Bollerslev 1997a, 1997b, 1998).
Several models have been considered to describe such a cyclical behaviour. Dacorogna *et al.* (1993), analyzing high frequency foreign exchange rates, introduce a new time scale, called θ-time, characterized by non-periodic volatility. Guillaume *et al.* (1995) consider θ-time GARCH models, i.e. GARCH models based on data measured on θ-time. In Andersen and Bollerslev (1997a, 1998) the volatility process is viewed as the interaction of a long-memory intra-daily factor and an intra-daily periodic component. In their paper, the second component has been filtered out through a flexible Fourier form regression whereas the former component has been modelled by using a FIGARCH dynamic. It is worth noting that their filtering procedure was unable to extract all of the periodic signal from the data.
Instead of filtering out the intra-day periodic factor, one can consider the possibility of modelling this component by introducing periodic lags in the conditional variance. This procedure is in general used in the ARMA literature when dealing with seasonality in the mean. Such extension leads to seasonal GARCH models used for instance in Bollerslev and Hodrick (1992). In this framework it is worth mentioning the natural generalization provided by Bollerslev and Ghysels (1996). Their periodic GARCH or P-GARCH model, characterized by periodically varying parameters al-

lows for a greater degree of flexibility when modelling periodicity in the conditional variance.
The models described earlier stress how important is to take into account a periodic component in the volatility dynamics. However, such models do not consider the presence of long range dependencies at periodic frequencies. This characteristic is typical in the study of the intra-daily movements of the conditional variance. Recently some authors (see Guegan (2000) and Bisaglia *et al.* (2003) among others) proposed the use of $k-$factor GARMA models (Gegenbauer ARMA), introduced by Woodward *et al.* (1998) in order to model this feature. This class of parameterizations allows for long memory together with multiple periodicity. More precisely Bisaglia *et al.* (2003), in order to describe the volatility of an intra-daily financial time series, consider a proxy for the true conditional variance and use a $k-$factor GARMA model on this transformed series. This approach seems really promising, even though it can be ineffective when the main goal of the analysis is to forecast the original series.
In this paper we cope with the issue of long and periodic persistence in the volatility process through the GARCH family models, that is a more standard tool in financial econometrics. A similar problem has been recently tackled by Arteche (2004) in a stochastic volatility framework.

In particular we introduce a new class of models, called PLM-GARCH (Periodic Long Memory GARCH), which represents an extension to the well known FIGARCH model, proposed in Baillie *et al.* (1996).
The rest of the paper is organized as follows. In Section 2 several specifications of the PFIGARCH dynamics are introduced and discussed. In Section 3 the problem of building a PFIGARCH model is outlined. Furthermore some simulation results on the modelling procedure are presented. In Section 4 some empirical results on real intra-daily financial series are shown.

2 Periodic Long Memory GARCH models

In this section we introduce a class of periodic long memory models, that allow to describe periodic patterns with long memory behavior in the conditional variance.
All the models considered in this section are defined by the following equations

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t \\ \varepsilon_t &= \sigma_t \, z_t \end{aligned} \tag{1}$$

where μ_t is the (possibly time varying) conditional mean of y_t that can be described, for example, by an ARMA model, z_t is an i.i.d. random variable with zero mean while σ_t^2 is, as usual, the conditional variance whose specification is considered in the next paragraph. The error term ε_t is such that $E\left(\varepsilon_t | I^{t-1}\right) = 0$ and $E\left(\varepsilon_t^2 | I^{t-1}\right) = \sigma_t^2$, $E\left(\cdot | I^{t-1}\right)$ denoting the conditional expectation with respect to the history of the process until time $t-1$.

2.1 The PLM-GARCH model

First, let us recall the FIGARCH model as specified by Baillie *et al.* (1990). According to these authors, let us start from the GARCH(p, q) specification

$$\sigma_t^2 = \omega + \alpha\,(L)\,\varepsilon_t^2 + \beta(L)\sigma_t^2 \tag{2}$$

where L is the lag operator, $\alpha\,(L) = \sum_{i=1}^{p} \alpha_i L^i$ and and $\beta\,(L) = \sum_{i=1}^{q} \beta_i L^i$ are polynomials of order q and q, respectively.
Therefore, the implied ARMA(m, q) process in ε_t^2 is given by

$$\phi_1(L)\varepsilon_t^2 = [1 - \beta(L)]\,\nu_t \tag{3}$$

where $\phi_1(L) = [1 - \alpha(L) - \beta(L)]$, $m = max\{p, q\}$ and $\nu_t = \varepsilon_t^2 - \sigma_t^2$ is a martingale difference.
Allowing $\phi_1(L)$ to be fractionally integrated of order d, we obtain the following ARFIMA representation for ε_t^2

$$(1 - L)^d \phi(L)\varepsilon_t^2 = [1 - \beta(L)]\,\nu_t \tag{4}$$

where $\phi_1(L) = (1 - L)^d \phi(L)$, $0 \le d \le 1$ to ensure positivity and strict stationarity (Bollerslev and Mikkelsen, 1996). The corresponding FIGARCH model for the conditional variance is finally given by

$$\sigma_t^2 = \omega + \beta(L)\sigma_t^2 + \Big[1 - \beta(L) - (1 - L)^d \phi(L)\Big]\,\varepsilon_t^2. \tag{5}$$

Following similar arguments, in order to allow for periodic long memory behaviour let us consider Seasonal ARFIMA, or SARFIMA$(p, d, q)_S$, specification (see also Porter-Hudak (1990)) for ε_t^2,

$$(1 - L^S)^d \phi(L)\varepsilon_t^2 = [1 - \beta(L)]\,\nu_t \tag{6}$$

where S is the length of the periodic component, d is the long memory parameter and $\phi(L)$ is a polynomial of order $max\{p, q\} - S$ such that $[1 - \alpha(L) - \beta(L)] = (1 - L^S)^d \phi(L)$. Then, it is not difficult to show that the conditional variance is given by

$$\begin{aligned} \sigma_t^2 &= \omega + \beta(L)\sigma_t^2 + \Big[1 - \beta(L) - (1 - L^S)^d \phi(L)\Big]\,\varepsilon_t^2 && (7) \\ &= \omega + \beta(L)\sigma_t^2 + \xi(L)\varepsilon_t^2 && (8) \end{aligned}$$

with the constraints $max\{p, q\} \ge S$.
The first two terms in the conditional variance reproduce the traditional GARCH(p,q) structure while the third one can describe both short and long memory components. The latter, in particular, operates at zero and periodic frequencies. The parameter S represents the length of the cycle and d modulates the (long) memory degree. If $\phi(L) = 1$ then only coefficients ξ relative to lags that are multiple of S are not zero. We denote the the model given by (1) and (12) as a Periodic Long Memory GARCH

or PLM-GARCH(p, d, q, S) model, where p is the order of $\phi(L)$, q is the order of $\beta(L)$, d is the long memory parameter and S is the period of the cyclical component. Clearly, a PLM-GARCH model represents a generalization of the standard FIGARCH model, because it allows for long memory dependent periodic patterns.
As in the standard FIGARCH model, it must be $0 < d < 1$ to ensure positivity of the conditional variance and strictly stationarity. The additonal GARCH parameters must satisfy the standard restrictions for positivity ($\omega > 0$, $\xi_j \geq 0$ for $j = 1, 2, ...,$ and $\beta_i \geq 0$ for $i = 1, 2, ...q$). The model is not covariance stationary but it is strictly stationary under specific assumptions that can be derived adapting some results reported in Zaffaroni (2004). These assumptions imply a number of non linear constraints on the model parameters due to the products between functions of d and $\phi(L)$.
The theoretical spectrum of the ε_t^2 (after mean removal) can be obtained from (6) as

$$f_\eta(\lambda) = \frac{\mid 1 - \beta\left(e^{-i\lambda}\right) \mid^2}{\mid \left(1 - e^{-i\lambda S}\right)^d \phi\left(e^{-i\lambda}\right) \mid^2} f_\nu(\lambda) \tag{9}$$

where $f_\nu(\lambda)$ is the spectrum of the innovation.
When $p = q = 0$ the spectrum in (9) assumes the more simple form

$$\begin{aligned} f_\eta(\lambda) &= \frac{1}{\mid \left(1 - e^{-i\lambda S}\right)^d \mid^2} f_\nu(\lambda) \\ &= (2 - 2\cos S\lambda)^{-d} f_\nu(\lambda) \end{aligned} \tag{10}$$

In this case the spectrum shows peaks at the frequencies $\omega \in \left\{0, \frac{2\pi}{S}, \frac{4\pi}{S}, \frac{6\pi}{S} ... \frac{2k\pi}{S}\right\}$ with $\frac{2k\pi}{S} < \pi$. For example, figure 1 reports the theoretical spectrum of a PLM-GARCH(0,0.2,0,7).

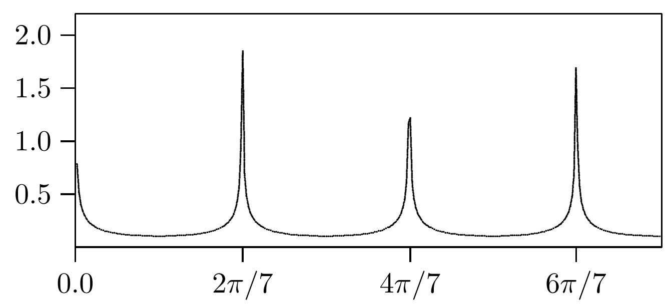

Figure 1: Theoretical spectrum of a PLM-GARCH(0,0.2,0,7).

From (12) it is easy to see that the PLM-GARCH model contains, as particular cases, several other well known GARCH models: the standard GARCH (for $d = 0$ and $S = 1$) and thus the IGARCH model, the FIGARCH model (for $0 < d < 1$ and $S = 1$) and the Seasonal GARCH, or SGARCH, model(for $d = 0$, $p = kS$, $q = rS$). The Periodic GARCH, or PGARCH, model is not nested in the PLM-GARCH model. However the latter generalizes the PGARCH because it includes long memory at the non-periodic and periodic frequencies.
It is worth noting that long memory modelling (both periodic and non periodic) is carried out with just a single parameter.

2.2 The PLM-EGARCH and PLM-LGARCH models

The estimation of the PLM-GARCH model can result cumbersome due to the positivity constraints on the parameters. An advantageous alternative is to model $log(\sigma_t^2)$ instead of σ_t^2. This permits to consider (at least) two kinds of models. The first one can be obtained generalizing the EGARCH model: in this case the dynamics of the conditional variance is given by

$$\left(1-L^S\right)^d \phi(L)\left[\ln\left(\sigma_t^2\right)-\bar{\omega}\right]=\alpha(L)z_t+\gamma(L)\left(|z_t|-E|z_t|\right) \tag{11}$$

where $\bar{\omega}=E\left[\ln(\sigma_t^2)\right]$, $\phi(L)$ is a polynomial of order $p-S$ and such that $[1-\beta(L)]=(1-L^S)^d\phi(L)$. The polynomial $\gamma(L)$ is of order m and models possible asimmetrical effects, being zero when conditional variance reacts simmetrically to shocks. The model is stationary if $0\leq d\leq 0.5$, while it is not stationary when $d>0.5$ (Bollerslev and Mikkelsen, 1996). We refer to this model as PLM-EGARCH(p,m,d,q,S), where p,d,q and S have the same meaning than in the PLM-GARCH and m is the order of $\gamma(L)$.
A second possibility is considering a log-GARCH-type model. In this case we obtain the PLM-LGARCH model given by

$$\ln\left(\sigma_t^2\right) = \omega+\beta(L)\ln\left(\sigma_t^2\right)+\left[1-\beta(L)-\left(1-L^S\right)^d\phi(L)\right]\left[\ln\left(\varepsilon_t^2\right)-\bar{\omega}\right] \tag{12}$$

$$= \omega+\beta(L)ln(\sigma_t^2)+\xi(L)\left[\ln\left(\varepsilon_t^2\right)-\bar{\omega}\right] \tag{13}$$

where $\phi(L)=1+\sum_{i=1}^{\max(p,q)-1}\phi_i L^i$ is such that $[1-\alpha(L)-\beta(L)]=(1-L)^d\phi(L)$, and $\bar{\omega}=E\left[ln(z_t^2)\right]$. If z_t is assumed to be gaussian then $\bar{\omega}=-1.27$.
The choice of considering $ln(_t^2)$ instead of the usual $|_t|$ in (13) is due to practical and computational convenience. In fact in this way it is easier to obtain the martingale difference defining the ARMA representation of the model; furthermore the term in $_t$ has the same magnitude order of $ln(\sigma_t^2)$. Finally, model (13) can be easily extended to include leverage effects. The ARMA representation corresponding is:

$$\left(1-L^S\right)^d\phi\left(L\right)\left[\ln\left(\varepsilon_t^2\right)-\bar{\omega}\right]=\omega+\left[1-\beta\left(L\right)\right]\nu_t \tag{14}$$

From the ARMA representations is possible to derive the spectra both for the PLM-EGARCH and for the PLM-LGARCH. The main advantage of the representations (11) and (13) is of computational nature because they do not require any constraints on parameters to ensure variance positivity. Note that the long memory structure applies to $ln(\sigma_t^2)$ in the PLM-EGARCH model and to $ln(_t^2)$ in the PLM-LGARCH.

2.3 K-factors PLM-GARCH models

All the models described in the previous section assume that there is a unique periodic component and that the long memory components shares the same memory degree both at the seasonal and non-seasonal frequencies. This assumption may be too restrictive in those situations where it is reasonable that different features of the markets generate different periodic, and possibly long memory, behaviours.

For example, it is not obvious that the degree of memory at the standard long memory frequency $f = 0$ must be equal to that of peridic frequencies. Whenever we find more periodic peaks with very different heigths this may be a signal of different degrees of memory. Our propose is thus generalizing the PLM-GARCH model in order to manage different periodicities with different degrees of persistence.
To this end we start with the following generalization of the ARFIMA representation given in (6) for ^2_t

$$\left[\prod_{i=1}^{k}\left(1-L^{S_i}\right)^{d_i}\right]\phi\left(L\right)\varepsilon_t^2=\omega+\left[1-\beta\left(L\right)\right]\nu_t \tag{15}$$

where k is a non negative integer indicating the number of cycles; S_j $(j = 1, \ldots, k)$ are positive integers defining the length of cycle; $0 \leq d_j \leq 1$ $(j = 1, \ldots, k)$ specify the degree of memory of the cyclical components S_j; finally,$\phi(L)$ is such that $[1-\alpha\left(L\right)-\beta\left(L\right)] = \left[\prod_{i=1}^{k}\left(1-L^{S_i}\right)^{d_i}\right]\phi\left(L\right)$. The first factor in (15), $\prod_{i=1}^{k}(1-L^{S_i})^{d_i}$, operates only at periodic frequencies and is related to long memory periodic components; the second fator, $\beta(L)$, modulates the short memory cyclical and non cyclical autoregressive dynamics. Finally, $\alpha(L)$ and $\gamma(L)$ describes the short period, and possibly asimmetrical, dynamics related to the innovations.
Then, we obtain the corresponding GARCH model

$$\sigma_t^2=\omega+\beta\left(L\right)\sigma_t^2+\left[1-\beta\left(L\right)-\left[\prod_{i=1}^{k}\left(1-L^{S_i}\right)^{d_i}\right]\phi\left(L\right)\right]\varepsilon_t^2 \tag{16}$$

Given that all periodic components act at the zero frequency with different long memory degrees, in order to avoid explosive patterns we must constraint the long memory coefficients imposing that $\sum_{i=1}^{k} d_i < 1$. This specification can account for k periodic component of periods S_i, each one with a different degree of memory d_i. Again, if $\phi(L) = 1$ then $\beta(L)$ has non null coefficients only at lags corresponding to multiple of S_i and to their cross products.
Since the model given by (1) and (16) can account for k periodic components with different degrees of persistence we denote it as k-PLM-GARCH(p, $\{d_1, ..., d_k\}, q, \{S_1, ..., S_k\}$).
The representation (15) allows us to derive the theoretical spectrum for ε_t^2.
Depending on the number of factors and on the degrees of memory it is possible to describe a large variety of periodic behaviours. For example, Figure 2 shows the theoretical spectrum of a k-PLM-GARCH((0,{0.05,0.1},0,{6,9})). The periodic component of period 9 acts at the frequencies $2\pi/9$, $4\pi/9$, $6\pi/9$ and $8\pi/9$, while the second one of period 6 produces peaks at $2\pi/6$ and $4\pi/6$. At the frequency $2\pi/3$ the two cycles work togheter giving a higher peak.
As in the case of the single factor models it is not easy to satisfy all the constraints, thus more manageable representations are given by the multifactor versions of the PLM-EGARCH and PLM-LGARCH models which are, respectively

$$\left[\prod_{i=1}^{k}\left(1-L^{S_i}\right)^{d_i}\right]\phi\left(L\right)\left[\ln\left(\sigma_t^2\right)-\bar{\omega}\right]=\alpha(L)z_t+\gamma(L)\left(|z_t|-E|z_t|\right) \tag{17}$$

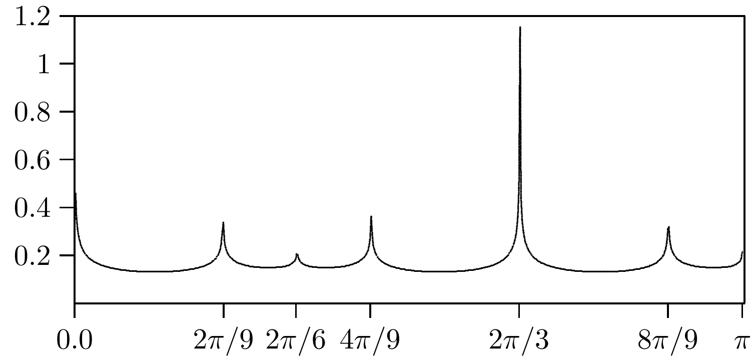

Figure 2: Theoretical spectrum of a k-PLM-GARCH(0,{0.05,0.1},0,{6,9})

and

$$\left[\prod_{i=1}^{k}\left(1-L^{S_i}\right)^{d_i}\right]\phi(L)\left[\ln\left(\varepsilon_t^2\right)-k(\theta)\right]=\omega+\left[1-\beta(L)\right]\nu_t \tag{18}$$

The formula (18) can also be written as

$$\ln\left(\sigma_t^2\right)=\omega+\beta(L)\ln\left(\sigma_t^2\right)+\left[1-\beta(L)\left[\prod_{i=1}^{k}\left(1-L^{S_i}\right)^{d_i}\right]\phi(L)\right]\left[\ln\left(\varepsilon_t^2\right)-k(\theta)\right].$$

3 Building a PLM-GARCH model

A simple way to build a PLM-GARCH model for on observed time series can be implemented by a Box-Jenkins-like method. This is obtained by a four steps iterative procedure based on:

1. filter out the conditional mean dependence;
2. identification;
3. estimation;
4. diagnostic checking of the PLM-GARCH model.

As usual, the identification and estimation steps can overlap. The first phase consists on removing the conditional mean dependence, for example by fitting an ARMA model on the series y_t and use the residual series $e_t = y_t - \hat{\mu}_t$ for building the real PLM-GARCH model. Of course if y_t is uncorrelated this step is not necessary.
In the second step the whole PLM-GARCH model or, more frequently, some of its components are identified. The identification can be achieved by exploiting the ARMA representation of the PLM-GARCH models. In practice it requires to jointly consider the ACF and the periodogram of e_t^2, for PLM-GARCH and PLM-EGARCH, and of $\ln e_t^2$, for PLM-LGARCH. With respect to the ACF a persistent behavior indicates the presence of a long memory components, while peaks at specific frequencies

dominating the other ones are a signal of a periodic component. When the peak is relative to the zero frequency it is typical of non periodic long memory bahaviour.
Once a model has been identified the model parameters have to be estimated. The estimation may be based on the standard Quasi Maximum Likelihood approach widely used in the GARCH literature. This imlpies to maximize with respect to a vector of unknown parameters $\theta \equiv (\omega, \alpha_1, ..., \alpha_p, \gamma_1, ..., \gamma_p,\ \phi_1, ...\phi_r,\ \beta_1, ...\beta_q, d_1, ..., d_k)$ the normal log-likelihood function

$$l\left(\theta \mid y_1 \dots y_n\right) \approx \sum_{j=1}^{n} l\left(y_t \mid I^{t-1}, \theta\right) = \sum_{j=h}^{t} \left(-\frac{1}{2} \ln\left(\sigma_t^2\right) - \frac{1}{2} \frac{\left(y_t - \mu_t\right)^2}{\sigma_t^2} \right) \tag{19}$$

with σ_t^2 defined by some PLM-GARCH model. Some initial conditions are required to start up the recursion for the variance function and, possibly, the conditional mean. Folowing Baillie *et al.* (1996), we maximize the likelihood conditionally on a set of start-up values. These are fixed to zero if referred to the mean and to the uncondtional variance when referred to th GARCH dynamics. The diagnostic check of the fitted model is similar to that of a standard GARCH model and is based on significativity of parameters, likelihood, uncorrelation of simple and squared standardized residuals, gaussianity and so on. Note that when PLM-LGARCH models are involved the analyses have to be made on $\ln(z_t^2)$ instead of z_t^2. For PLM-GARCH models particular care has to be payed to checking for residual periodic and/or long memory components. To this end the main tools are the ACF and the periodogram od the standardized residuals.
If the fitted model is not adequate other components have to be added and the procedure re-start from step 2.
When multifactor models are considered, computational drawbacks can derive from the interactios between different long memory components. First of all, depending on the length of the cycles S_i, the components can overlap partially or totally with possible identification and specification problems. Furthermore, we are dealing with components having a dependence on the infinite past with possible convergence problems specially for large values of S, which may means few available past values for short time series. Finally, fitting models on very long time series is time demanding and could require some hours of CPU time.
We now turn to show the above procedure by using a simulated series. To this end we generated a time series of length $n = 2000$ from a k-PLM-LGARCH with model parameters $\mu_t = 0$, $z_t \sim N(0, 1)$ and

$$\ln\left(\sigma_t^2\right) = -0.09 + \left[1 - (1 - L)^{0.2}\left(1 - L^7\right)^{0.3}\right]\left[\ln\left(y_t^2\right) - k(\theta)\right].$$

The model thus has no mean component, no short memory components and two long memory components with different degrees of persistence ($d_1 = 0.2$ and $d_2 = 0.3$). The first is the traditional kind of long memory behavior ($S_1 = 1$), while the second one describes a cyclical persistence of period seven ($S_2 = 7$).
Let now observe the time series y_t generated by the above model. The series of the level results to be uncorrelated while the autocorrelation function of $\ln(y_t^2)$ is clearly correlated and shows cyclical and persistent behavior (Figure 3). Observing the

periodogram of $\ln(y_t^2)$ (Figure3) a peak corresponding to the frequency $f_1 = 0$ and others three peaks relative, respectively, to the frequencies $f_2 = 0.1428$,$f_3 = 0.2857$ and $f_4 = 0.4347$ are easily distinguishable. Note that we are considering $\ln(y_t^2)$ because we want to apply a PLM-LGARCH model, but analogous characteristics can be seen in the ACF and in the periodogram of y_t^2. Given these findings, we start

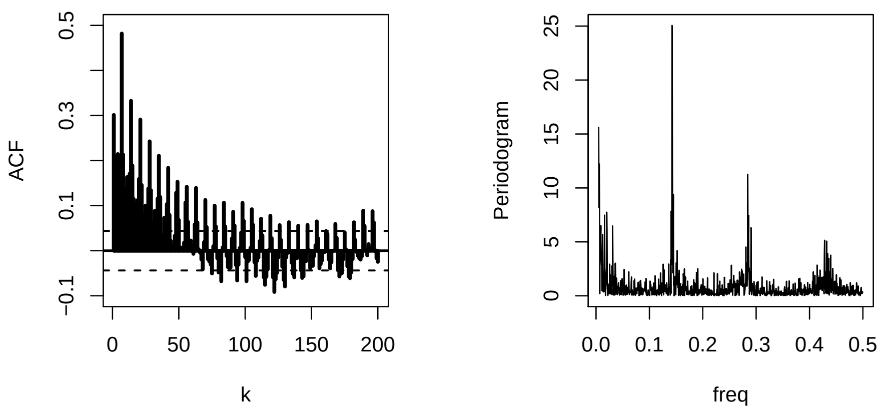

Figure 3: Autocorrelation function and periodograms of $\ln(y_t^2)$

by estimating a model which contains just the traditional long memory component

$$\ln\left(\sigma_t^2\right) = \omega + \left[1 - (1-L)^{d_1}\right]\left[\ln\left(y_t^2\right) - k(\theta)\right]$$

where z_t is gaussian and thus $k(\theta) = -1.27$.
This first step produces the estimates $\hat{\omega} = -0.499$ and $\hat{d}_1 = 0.1916$, both significant. The standardized residuals e_t are uncorrelated but $\ln e_t^2$ are correlated (Figure 4) and the autocorrelation tends to be persistent and periodic. Furthermore, a peak at frequency $f = 0.1428$, corresponding to a cyclical component of period 7, is clearly present in the periodogram of $\ln e_t^2$ (Figure 4).

The previous considerations suggest to introduce a second long memory periodic component of period 7, obtaining the following equation for conditional variance

$$\ln\left(\sigma_t^2\right) = \omega + \left[1 - (1-L)^{d_1}(1-L^7)^{d_2}\right]\left[\ln\left(y_t^2\right) - k(\theta)\right]. \tag{20}$$

The estimated parameter of model (20) are all significant and are reported togheter with their standar errors in table 1. Now, both the simple standardized residuals e_t and $\ln(e_t^2)$ are uncorrelated (Figure 5) and the periodogram of the latter does not show any dominant peak.

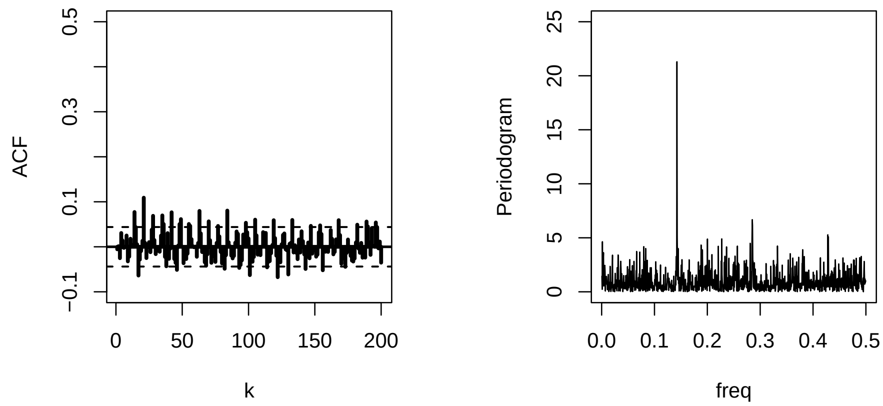

Figure 4: Autocorrelation function and periodograms of the log squared residuals $\ln(e_t^2)$ of the model with only one component

PLM-LGARCH	ω	d_1	d_2
par. est	-0.1366	0.1829	0.2493
std.err	0.04535	0.01273	0.0266
t-stat	-3.0120	14.368	9.366

Table 1: Estimated parameters of model (20).

4 Empirical examples

To illustrate the empirical relevance of Periodic Long Memory GARCH models, in this section we present the estimation results for two real financial time series. The first application concerns the half-hourly time series of the S&P500 US stock index. In this case only single factor models are enough and we compare the performance of various PLM-GARCH with a periodic (but short-memory) GARCH and with a Periodic GARCH (PGARCH). The second example is relative to the hourly time series of the TIM stock returns, a share of the italian stock market rated to the Milan exchange and belonging to the MIB30 index. For this series a two factor PLM-LGARCH is the most appropriate.

Comparison among the various models will be based on three criteria: the value of the loglikelihood, the autocorrelation function of the square (log square for PLM-LGARCH) standardized residuals and the periodogram of the same quantities. In both the examples, the models identification was carried out by choosing the model that, on the whole, gave the best performance in terms of significance of the parameters and in terms of periodogram and autocorrelation function of the squared standardized residuals, z_t^2.

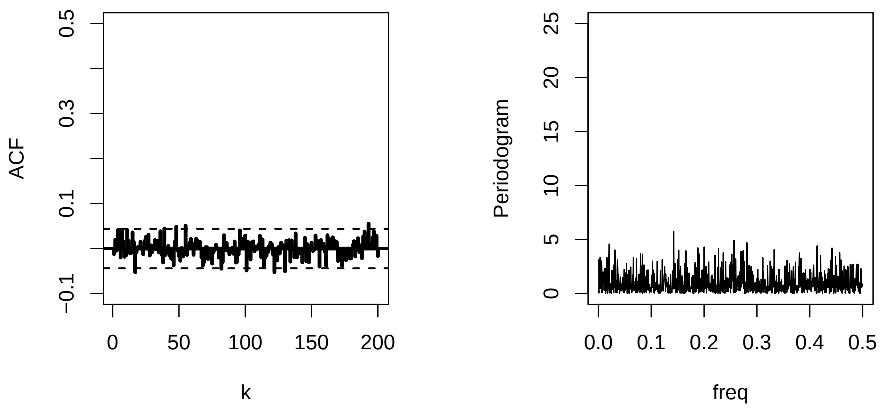

Figure 5: Autocorrelation function and periodograms of the log squared residuals $\ln(e_t^2)$ of the model with two components

4.1 PLM-EGARCH for the intraday Standard & Poor 500 stock index market

This section analyses the behavior of the intraday volatility of the Standard & Poor 500 US stock index in the period 6/3/2000 - 25/2/2005. The frequency of the original series was five minutes, but the five minutes returns have been aggregated to obtain the half-hour return time series. This is the series that will be studied.
Before modeling the series, some adjustments and corrections were necessary. First, the data have been adjusted for the daylight-save time. Then, observations in the interval 8 - 30 september 2001 have been omitted because from 11 to 30 september there were no data and to avoid distorsion in the periodic component. For this reason, when a national holyday occured within the week the observations of that day were replaced by the mean of the corresponding day interval.

Figure 6 shows the mean behavior of the volatility, as measured by $| r_t |$, each 30 minutes inside the week. Every working week – from monday to friday – has 70 half-hours. In the considered period there are 257 weeks: every point of Figure 6 is the mean of the 257 first half-hours, of the second half-hours, . . . , of the 70-th half-hours. There is a strong intradayly peridodic component, kwnown in the literature as "inverse J effect", and a weak weekly periodic component.

To accounts for serial correlation on log-returns level we identified and fitted the constrained AR model $r_t = \phi_2 r_{t-2} + \phi_7 r_{t-7} + \phi_{14} r_{t-14} + \phi_{70} r_{t-70}$, whose estimated parameters are $\hat{\phi_2} = 0.03$ (s.e. 0.007), $\hat{\phi_7} = 0.02$ (s.e. 0.007), $\hat{\phi_{14}} = -0.019$ (s.e. 0.007), $\hat{\phi_{70}} = 0.04$ (s.e. 0.007). The residuals e_t of this model do not show any serial correlation, while, as usual, are correlated in the squared (see Figure 7) with a clear periodic patterns of period 14 (one day). In the spectrum (Figure 7) is also present a peak at zero frequency typical of a non periodic long memory behavior. To describe these empirical evidences founded in the data we fitted to the series e_t a

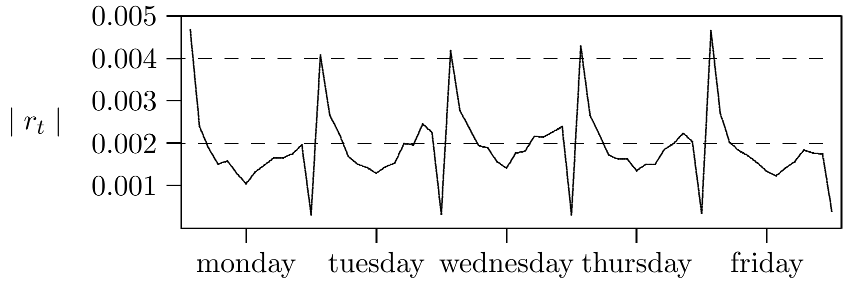

Figure 6: Intraday and the day-of-the-week periodicities.

PLM-EGARCH model, with a periodic long memory component of period 14 (one day) and a short memory component accounting also for leverage effects.
For this series we identified and estimated a Periodic GARCH, a Seasonal (short memory) GARCH, a PLM-GARCH, a PLM-LGARCH and a PLM-EGARCH model. For all seasonal models the periodic long memory component has period $S = 14$. We tried to model also a possible weekly cycle by introducing a short memory periodic component of period $S = 35$. The identification step led us to consider models for which the dynamics of the conditional variance is defined by the parameters reported, togheter with their estimates, in table 2. In the table we did not report the estimates for the PGARCH model because it did not give good results but, insted, it gave a number of convergence problems.

From Table 2 we can see that the weekly component is generally not significant, apart from for the PLM-LGARCH specification, where it is weak but significant. The analysis of the periodogram and of the autocorrelation function of the squared or log squared standardized residuals z_t^2 indicates that the model with the best performances is the PLM-EGARCH. For example, the SEGARCH model is able to describe a part of the periodic structure but fails to give a complete description.
The fitted PLM models provide very close results. However, the PLM-GARCH has more restrictive short-term dynamics compared to the others and, as expected, provides a lower loglikelihood. Differently, PLM-EGARCH and PLM-LGARCH have short term components but the former has a neatly better loglikelihood. When comparing the long memory coefficients we observe that PLM-GARCH and PLM-LGARCH estimates are lower and within the stationary bounds ($0 < d < 1$ for these two models). On the contrary, the PLM-EGARCH provides a higher value of $\hat{d}$ and located outside the stationarity region ($0 < d < 0.5$). This result in consistent with the work of Bollerslev and Mikkelsen (1996) that found evidence on non-stationary long memory on the daily returns of the S&P500 index.
On the contrary, in the periodogram of the standardized residuals of the PLM-EGARCH model, there are not dominating frequencies, in particular at the periodic ones (see Figure 8). Also considering the autocorrelation of z_t^2, residuals the best result is that of the PLM-EGARCH model, where the squared residuals resulted to be substantially uncorrelated. Finally with regard the value of the likelihood the best model is again the PLM-EGARCH, followed by PLM-LGARCH, PLM-GARCH, SEGARCH and PGARCH.

Model	estimate	std. error	$t-$stat	LogLik
S-EGARCH				14891.4
ω	0.0624	0.0253	2.462	
β_1	0.0358	0.0121	2.962	
β_7	0.8879	0.0166	53.345	
α_1	0.0433	0.0494	0.875	
γ_1	0.0268	0.0588	0.4550	
α_7	-0.1092	0.0215	-5.080	
γ_7	0.4752	0.0485	9.789	
PLM-GARCH				17318.5
ω	0.00017	0.0002	0.8334	
d_1	0.3688	0.0204	18.101	
β_1	0.3225	0.0345	9.3483	
PLM-LGARCH				17355.9
ω	0.0262	0.0089	3.3275	
d_1	0.5352	0.0679	7.8747	
β_1	-0.0214	0.00832	-2.5683	
β_7	0.7927	0.0363	21.808	
β_35	0.0649	0.0089	7.2821	
ϕ_1	0.03773	0.01844	2.0464	
ϕ_7	0.3091	0.0362	8.5241	
ϕ_35	0.0663	0.0112	5.9033	
PLM-EGARCH				17637.75
$\bar{\omega}$	0.1125	0.3303	2.3407	
d_1	0.8600	0.1640	5.2245	
ϕ_1	0.7352	0.0388	18.926	
ϕ_7	-0.2511	0.0489	-5.1326	
α_1	-0.0140	0.0161	-2.1722	
γ_1	0.0611	0.0265	2.3054	
α_7	-0.0255	0.0153	-2.6612	
γ_7	0.1614	0.0513	3.1470	

Table 2: S&P500: estimated parameters of the conditional variance of the models.

4.2 k-PLM-LGARCH for intraday TIM stock returns

As an example of a multifactor model we consider the hourly time series of TIM (Telecom Italia Mobile), a share of the italian stock market rated to the Milan exchange. This time series 1 october 1997 - 30 November 1998, a period in which the Milan exchange was opened from 10 a.m. to 5 p.m. and thus there were 7 hourly observation for every day. This seris was already used in Bordignon, Bisaglia e Lisi (2003): in that work the authors considered a k-factor GARMA model and applied it to a tranfosrmation of the squared returns. Now models of the class k-PLM-GARCH are applied directly to the simple return. Since these latter appear to be uncorrelated no mean component was considered and thus the mean residuals e_t coincide with the simple returns r_t.
In our exercise, as before, we considered k-PLM-GARCH, k-PLM-EGARCH and k-PLM-LGARCH models. Since this latter resulted the best one in the following, for sake of brevity, we will describe only this model.
Let thus we consider the periodogram and the ACF of $log(e_t^2)$.

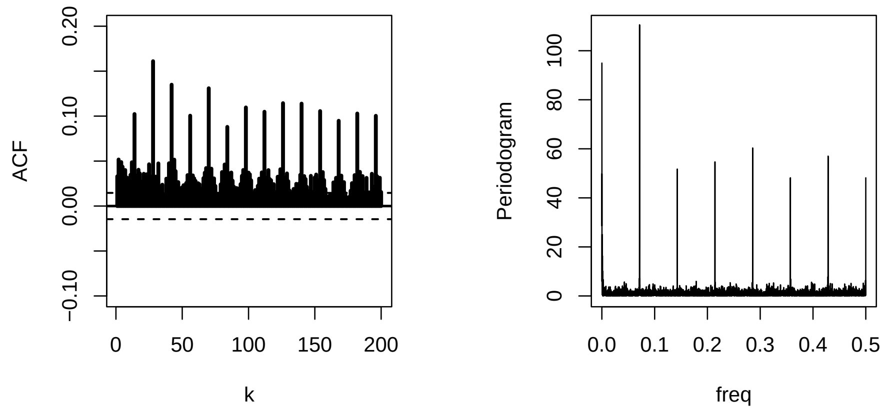

Figure 7: Autocorrelation function and periodogram of the squared residuals of the linear model for S&P500 data.

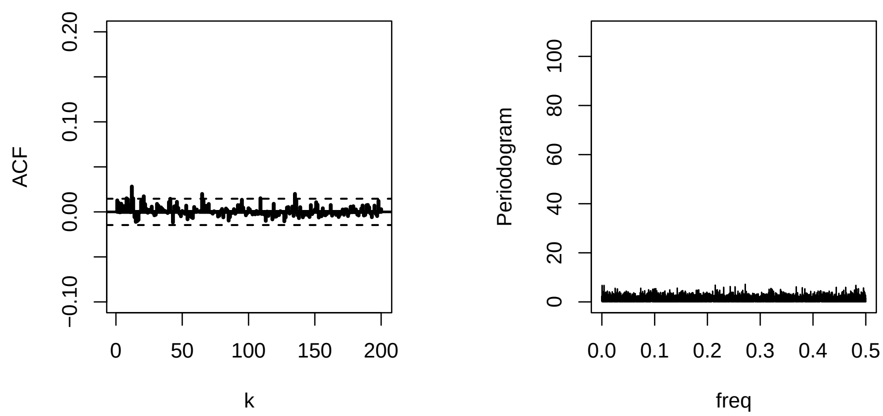

Figure 8: Autocorrelation function and periodogram of the standardized squared residuals of the PLM-EGARCH model.

The sample autocorrelation function dispayed in Figure 9 show clearly long-range dependence and a persistente repetitive pattern of period 7, the intradaily period connected to the intraday volatility pattern. The same kind of behavior can be seen in the periodogram (Figure 9) that exhibits a main peak at the frequency $f_2 = 0.1428 (= 1/7)$, a second one at the frequency $f_1 = 0$, the conventional long memory frequency, and a minor one at $f_3 = 0.2857 (= 1/3.5)$.

As a first step we assume the possibility to have two long memory components, the traditional one and one of period 7, with different degrees of persistence. Thus we apply a two factor PLM-LGARCH that, beyond a short memory component, has a traditional long memory component at the zero frequency ($S_1 = 1$) and a periodic long memory component at f_2 ($S_2 = 7$). The results of the estimation step are

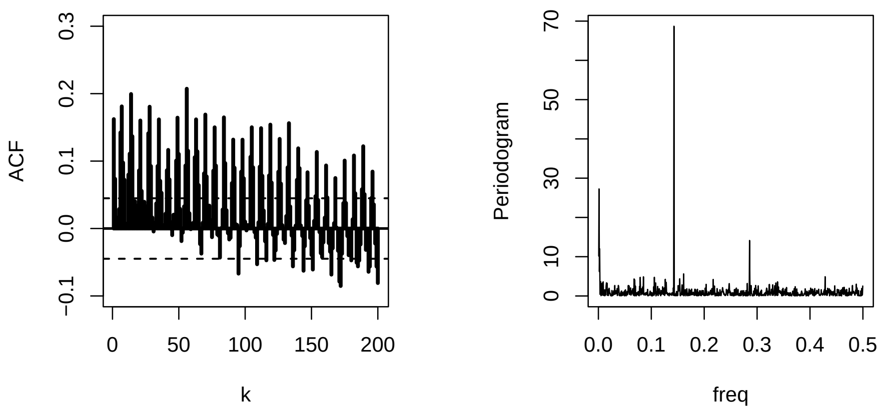

Figure 9: Autocorrelation function and periodograms of the squared returns $\ln(r_t^2)$ for TIM data.

reported in table 3.

Table 3 shows that both the long memory components are significant and that

PLM-LGARCH	ω	d_1	d_2	α_7	β_7	LogLik
par. est	-0.0979	0.1149	0.3414	0.4131	0.7224	6944.0
std.err	0.0806	0.0181	0.0571	0.0518	0.0633	
t-stat	-1.2133	6.3503	5.9831	7.9794	11.4123	

Table 3: TIM: estimated parameters of the conditional variance with the PLM-LGARCH model.

they have quite different degrees of peristence. When we considered a third long memory component at frequency f_3 it did not result to be significant. Both $\hat{d}_1$ and $\hat{d}_2$ indicate that the estimated model is stationary. The autocorrelation function and the periodogram of the log-square standardized residuals of the estimated model are shown in figure 10. Both these indicators suggest that the model is adequate and that there are not relevant residual correlations or periodic components.

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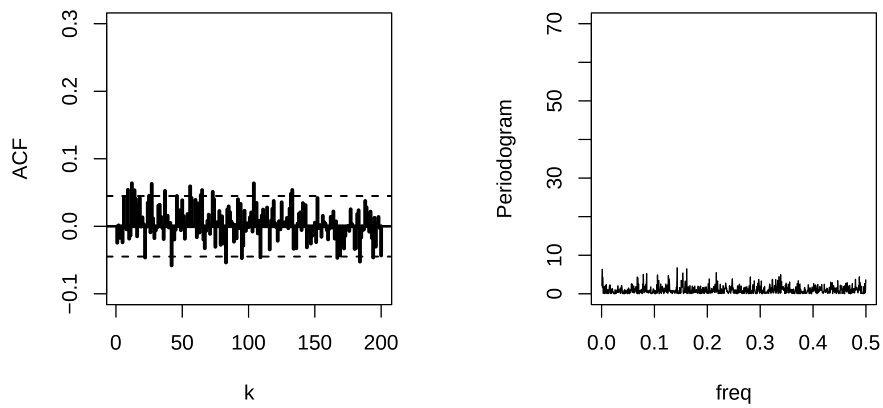

Figure 10: Autocorrelation function and periodograms of the squared residuals of the k-PLM-LGARCH model for TIM returns.

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