Construction of MV - and SMV - optimum designs for binary response models

J. Lopez-Fidalgo, C. Tommasi

2002.6

Dipartimento di Scienze Statistiche
Università degli Studi
Via C. Battisti 241-243
35121 Padova

Aprile 2002

CONSTRUCTION OF MV– AND SMV–OPTIMUM DESIGNS FOR BINARY RESPONSE MODELS

Jesús López–Fidalgo
Department of Statistics. University of Salamanca
Plaza de los Caídos s/n, 37002–Salamanca (Spain)
fidalgo@usal.es, Tel: 34 923 294458, Fax: 34 923 294514
Chiara Tommasi
Department of Statistics. University of Padova
Via Cesare Battisti, 241-243, 35121–Padova (Italy)
microbo@stat.unipd.it, Tel: 39 827 4176

April 15, 2002

Abstract

Recently, Dette and Sahm (1998) have put forward a procedure to construct MV– and SMV–optimum designs for binary response models. In order to implement computationally this procedure some assumptions have to be made and some theoretical results must be proved. This paper provides the background to produce a computer code for computing local designs for different regions of the parameters of the model. Designs for some models used in practice are also provided as well as the efficiencies for estimating the parameters.

Key words and phrases: Binary response models, Fisher information matrix, Locally optimum design, Optimum design of experiments, Standardized criteria.

1 Introduction

In nonlinear models the information matrix depends on some unknown parameters, as it is the case of the binary response model. Sometimes explicit *local* optimum designs depending on the parameters can be derived. Then, for a given initial value of the parameter the optimum design is completely determined (see e.g. Atkinson and Donev, 1992). Explicit formulae for designs and efficiencies and for all possible values of the parameters are of interest for robustness studies on the nominal values.

Optimum experimental designs for models with binary response have been studied widely. Ford, Torsney and Wu (1992), Wu (1985, 1988) and Sitter and Wu (1993) use local optimality criteria. On the other hand, Abdelbasit and Plackett (1983) and Chaloner and Larntz (1989) used Bayesian approaches to solve the problem of the optimum design dependence on the unknown parameters. Torsney and López Fidalgo (2001) give MV-optimal designs for general compact interval design spaces. López-Fidalgo and Wong (2000) provide a study of the sensitivity to the nominal values comparing efficiencies of the MV– and SMV–optimal designs.

Dette and Sahm (1998) have recently proposed a procedure for binary response models with symmetric pdf. They have proved that there is always a symmetric MV–optimum design with 2, 3 or 4 points in its support. In this paper a k-point symmetric design will be a design with k distinct points in its support and such that if it has weight at point x then it has the same weight at point $-x$. In the first step the equivalence theorem is checked for the 2-point design. If it is not satisfied the equivalence theorem is checked for 3-point design. If it is not satisfied the 4-point design must be optimum. In most of the models used in practice this last possibility never shows up. In this paper an example of a model needing a 4–point design will be provided.

The theory and results given in this paper are developed for computational purposes. A code for computing local MV– and SMV–optimal designs for binary models is available from the second author. Optimal designs for some classic models and efficiencies for estimating each parameter will be provided in general as well as for some specific models.

2 Symmetric MV– and SMV–optimum designs for symmetric density functions

Let y be a binary response variable with probability of success

$$p(y=1;\theta,x) = F[\beta(x-\mu)], \qquad \theta = (\mu,\beta)^T, \quad x \in \mathbb{R} \tag{1}$$

where $F(z)$ is a known cdf with a symmetric pdf $f(z)$ differentiable at $z \neq 0$. The possible lack of differentiability at $z = 0$, allows some useful models in practice, as it is the case of linear increasing for $z < 0$ and linear decreasing for $z > 0$. The variable x denotes the experimental condition, that in this paper will not be submitted to any restriction.

For this model the Fisher information matrix at θ of a design ξ is well known (see e.g. Fedorov and Hackl, 1997),

$$I(\theta,\xi) = \int h(\beta(x-\mu)) \begin{pmatrix} \beta^2 & -\beta(x-\mu) \\ -\beta(x-\mu) & (x-\mu)^2 \end{pmatrix} d\xi(x),$$

where

$$h(z) = \frac{f^2(z)}{F(z)[1-F(z)]}.$$

The MV–criterion, introduced by Elfving (1959), is defined by

$$\Phi_{MV}[I(\theta,\xi)] = \max_i \, e_i' \, I^{-1}(\theta,\xi) \, e_i,$$

where e_i denotes the unit vector. Very often, the diagonal elements of the Fisher information matrix are of a very different scale. Then MV–optimality gives too much importance for instance to one of the parameters yielding 1–point optimal designs. In order to give a proportional importance to all the parameters in the model Dette (1997) proposes to standardize the optimality criteria. The corresponding standardized MV–criterion is the SMV–criterion,

$$\Phi_{SMV}[I(\theta,\xi)] = \max_i \frac{e_i' \, I^{-1}(\theta,\xi) \, e_i}{\min_\eta \, e_i' \, I^{-}(\theta,\eta) \, e_i}.$$

Dette and Sahm (1998) give a general procedure for finding MV– and SMV–optimum designs for the binary response model (1), under the hypotheses

$$\max_{z\in \mathbb{R}} h(z) = h(0) \qquad \text{and} \qquad k^2 = \max_{z\in\mathbb{R}} z^2 h(z) = c^2 h(c). \tag{2}$$

The graphical Elfving's method (1952), provides the optimum design for estimating β,

$$\xi_\beta = \left\{ \begin{array}{cc} \mu - c/\beta & \mu + c/\beta \\ 1/2 & 1/2 \end{array} \right\}.$$

It is MV–optimum if and only if $\beta^2 \geq c$. Otherwise ($\beta^2 < c$), equal variance designs are needed. This leads to information matrices with equal diagonal elements. To verify whether an equal variance design ξ is MV–optimum the following equivalence theorem inequality must be checked,

$$\sum_{i=1}^{2} \alpha_i \frac{e_i' \, I^{-1}(\theta,\xi) \, I(\theta,\xi_x) \, I^{-1}(\theta,\xi) \, e_i}{e_i' \, I^{-1}(\theta,\xi) \, e_i} \leq 1, \qquad x \in \mathbb{R} \tag{3}$$

where equality holds on the support of ξ, and ξ_x is the 1–point design at x. The details are provided by Dette and Sahm (1998) who have also proved that a symmetric MV–optimum design with at most 4 support points always exists. Therefore, when $\beta^2 < c$, the first step of Dette and Sahm's procedure is to check whether the 2–point symmetric design with equal diagonal elements in the information matrix is MV–optimum using the equivalence theorem inequality. This design is

$$\xi_{MV}^{(2)} = \left\{ \begin{array}{cc} \mu - \beta & \mu + \beta \\ \frac{1}{2} & \frac{1}{2} \end{array} \right\}$$

and after easy but tedious algebra, the equivalence theorem inequality (3) becomes

$$\beta^4 h'(\beta^2) h(x) + 2\beta^2 h(\beta^2) h(x) - h'(\beta^2) x^2 h(x) - 2\beta^2 h^2(\beta^2) \leq 0 \quad x \in \mathbb{R}$$

If $\xi_{MV}^{(2)}$ is not MV–optimum a 3–point symmetric design with equal diagonal elements in the information matrix is the next candidate. This design has to be,

$$\xi_{MV}^{(3)} = \begin{Bmatrix} \mu - \frac{z_0}{\beta} & \mu & \mu + \frac{z_0}{\beta} \\ \frac{1-w}{2} & w & \frac{1-w}{2} \end{Bmatrix}, \quad w = \frac{h(z_0)(z_0^2 - \beta^4)}{\beta^4[h(0) - h(z_0)] + z_0^2 h(z_0)}, \tag{4}$$

where $H(x) = [h(0) - h(x)]/x^2 h(x)$ and

$$\min_{x \geq \beta^2} H(x) = H(z_0). \tag{5}$$

The function H will be of paramount importance in the determination of MV–optimal designs for different values of β^2. In order to check the MV–optimality of $\xi_{MV}^{(3)}$ the equivalence theorem inequality (3) has now a simple form,

$$H(x) \geq H(z_0), \qquad x \in \mathbb{R}. \tag{6}$$

Actually, Dette and Sahm (1998) propose to take v_0 where the global minimum point of $H(x)$ is reached instead of z_0 restricted to $x \geq \beta^2$. Both procedures are equivalent but z_0 is more convenient for computational purposes.

If inequality (6) is not satisfied either, then the last step is to compute the MV–optimum design in the class of all the 4–point symmetric designs with equal diagonal elements in the information matrix. This design is

$$\xi_{MV}^{(4)} = \begin{Bmatrix} \mu - \frac{z_1}{\beta} & \mu - \frac{z_2}{\beta} & \mu + \frac{z_2}{\beta} & \mu + \frac{z_1}{\beta} \\ \frac{w}{2} & \frac{1-w}{2} & \frac{1-w}{2} & \frac{w}{2} \end{Bmatrix}, \tag{7}$$

where

$$w = \frac{h(z_2)(\beta^4 - z_2^2)}{h(z_2)(\beta^4 - z_2^2) + h(z_1)(z_1^2 - \beta^4)}$$

and (z_1, z_2) is the minimum point for a fixed β of the function

$$G(x, y, \beta) = \frac{h(y)(\beta^4 - y^2) + h(x)(x^2 - \beta^4)}{(x^2 - y^2)h(x)\, h(y)},$$

subject to the constraint $0 < y < \beta^2 < x$.

Summarizing the process, when $\beta^2 \geq c$, the 2–point design ξ_β is MV–optimum. If $\beta^2 < c$, it may have 2, 3 or 4 support points. The aim of this work is to identify the parametric subsets where the symmetric MV–optimum design coincides with $\xi_{MV}^{(2)}$, $\xi_{MV}^{(3)}$ and $\xi_{MV}^{(4)}$, respectively. In the next section some theoretical results are provided in order to give sufficient conditions to implement this method computationally. The identification of these parametric regions will be used to compute SMV–optimum designs, ξ_{SMV}. In fact if $\xi_{0,s}$ is the symmetric MV–optimum design for $\mu = 0$ and $\beta = s = k^{1/2}h(0)^{-1/4}$ then the weight of the SMV-optimum at point x is $\xi_{SMV}(x) = \xi_{0,s}[\beta/s(x-\mu)]$ (Dette and Sahm, 1998). Thus, depending on the parametric subset where lies s^2, ξ_{SMV} will be completely determined.

3 Subsets of the parametric space producing different MV-optimum designs

Since $\beta \geq c$ leads always to ξ_β as MV–optimum, it shall be considered only the case $\beta^2 < c$ henceforth. The next theorem identifies parametric regions where a symmetric MV–optimum design cannot have either 2 or 3 support points. Since $h(x)$ and $H(x)$ are symmetric respect to zero, we will only consider the case $x > 0$.

Theorem 3.1 *If $v_0 \in [0, c)$ is the unique global minimum point of $H(x)$, then*

a) *if $\beta^2 \in (0, v_0)$, a 2–point symmetric MV–optimum design is not possible and the 3–point design*

$$\xi_{MV}^{(3)} = \begin{Bmatrix} \mu - \frac{v_0}{\beta} & \mu & \mu + \frac{v_0}{\beta} \\ \frac{1-w}{2} & w & \frac{1-w}{2} \end{Bmatrix}, \quad w = \frac{h(v_0)(v_0^2 - \beta^4)}{\beta^4[h(0) - h(v_0)] + v_0^2 h(v_0)} \tag{8}$$

is MV–optimum;

b) *if $\beta^2 \in [v_0, c)$, a 3–point symmetric MV–optimum is not possible.*

Proof

a) If $\beta^2 \in (0, v_0)$ then $z_0 = v_0$ and the 3–point designs (4) and (8) are the same. Comparing the diagonal elements of the inverse information matrices of $\xi_{MV}^{(3)}$ and $\xi_{MV}^{(2)}$ we have

$$\frac{v_0^2\, h(v_0) + \beta^4\, [h(0) - h(v_0)]}{\beta^2\, h(0)\, v_0^2\, h(v_0)} < \frac{1}{\beta^2\, h(\beta^2)},$$

since $H(v_0) < H(\beta^2)$. Thus there always exists a 3–point design that is better than $\xi_{MV}^{(2)}$. Moreover, $\xi_{MV}^{(3)}$ is MV–optimum since (6) is satisfied ($z_0 = v_0$).

b) If $\beta^2 \in [v_0, c)$, there are essentially three possible cases.

b.1) If $H(x)$ is non decreasing on $(v_0, +\infty)$, then the solution of (5) is $z_0 = \beta^2$ and the 3–point design (4) becomes $\xi_{MV}^{(2)}$.

b.2) If $H(x)$ tends to some limit greater than $H(v_0)$, say $H(v_1)$ as in Figure 1(a), then for any $\beta^2 \in [v_0, v_1]$ the 3–point design (4) and $\xi_{MV}^{(2)}$ are the same, while for any $\beta^2 > v_1$ there is not solution for (5).

b.3) If v_2 is a local minimum point of $H(x)$ such that $v_0 < v_2 < c$, as in Figure 1 (b), then for any $\beta^2 \in [v_0, v_1] \cup [v_2, c)$ the 3–point design (4) and $\xi_{MV}^{(2)}$ are the same. For any $\beta^2 \in (v_1, v_2)$ the solution of (5) is $z_0 = v_2$. The 3–point design (4) cannot be MV–optimum. This can be proved taking into account that v_2 is a local minimum point of $H(x)$ and replacing v_2 for z_0 in the equivalence theorem inequality (6).

These three cases show that a symmetric MV–optimum design has 2 or 4 support points. In fact (5) produces the design (4) that becomes $\xi_{MV}^{(2)}$, otherwise it is not MV–optimum. The case of H with more local optima after v_0 does not

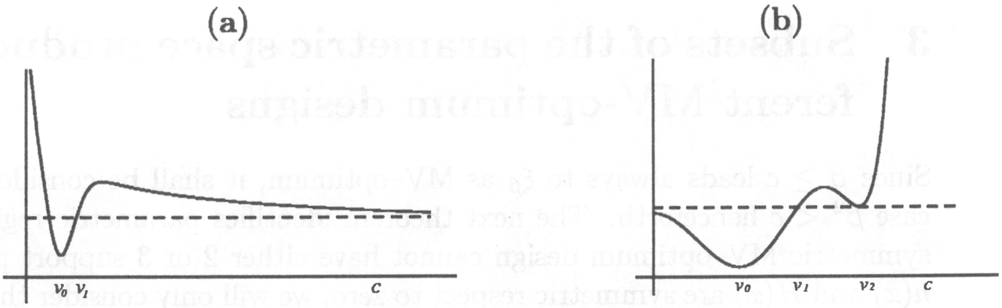

Figure 1: Two possible shapes of $H(x)$, for $x > 0$.

add more to the argument given above. □

Remark 1. If the global minimum point of $H(x)$ is not unique, then setting v_0 equal to the largest of such global minimum points, all the previous considerations remain valid.

The next theorem gives sufficient conditions for which a 4–point symmetric MV–optimum design is not possible. In this case if $\beta^2 \in (0, v_0)$ the MV–optimum design is $\xi_{MV}^{(3)}$ and for any $\beta^2 \in [v_0, c)$ the MV–optimum is $\xi_{MV}^{(2)}$. The interest of this theorem is to identify the cases where the search for a 4-point design is not needed that is obviously very convenient from the computational point of view.

Let us assume now that the 4–point symmetric MV–optimum design exists. The point (z_1, z_2) given in (7) minimizes the function $G(x, y, \beta)$ for a fixed value of β. Thus, setting the gradient of $G(x, y, \beta)$ equal to zero and after some algebra, (z_1, z_2) is a solution of one of the following systems of equations,

$$h(z_1) = h(z_2), \quad h'(z_1) = h'(z_2) = 0$$

or

$$t(z_1) = t(z_2), \quad t(z_1) = \frac{h(z_1) - h(z_2)}{z_1^2\, h(z_1) - z_2^2\, h(z_2)}, \tag{9}$$

where $t(x) = h'(x)/[\partial x^2 h(x)/\partial x]$ and $0 < z_2 < \beta^2 < z_1$. The symmetric design $\xi_{MV}^{(4)}$ will be MV–optimum if and only if the equivalence theorem inequality (3) is satisfied. After some algebra this inequality can be written as

$$[z_1^2 h(z_1) - z_2^2 h(z_2)]\, h(x) + [h(z_2) - h(z_1)]\, x^2 h(x) \leq (z_1^2 - z_2^2)\, h(z_1)\, h(z_2), \quad x \in \mathbb{R} \tag{10}$$

If (z_1, z_2) is a solution of the first system, then $h(z_1) = h(z_2)$ and then inequality (10) becomes $h(x) \leq h(z_1)$, for any $x \in \mathbb{R}$. But this is only possible if $h(z_1) = h(z_2) = h(0)$, since $h(x)$ has a unique maximum point at $x = 0$. Therefore, if a symmetric MV–optimum design with 4 support points exists, (z_1, z_2) will be necessarily a solution of the system (9).

Theorem 3.2 *Assuming that the symmetric function $h(x)$ is decreasing for $x > 0$ ($h'(x) < 0$) and $x^2 h(x)$ has a unique local maximum point on $(0, +\infty)$ at $x =$*

c, being increasing before ($\partial x^2h(x)/\partial x > 0$) and decreasing all the way after ($\partial x^2h(x)/\partial x < 0$) then a 4-point symmetric MV-optimum design is not possible in any of the two cases,

a) *the function $t(x)$ is decreasing on $(0,c)$ or*

b) *the function $H(x)$ has a unique local optimum point that is a local minimum at v_0 and $t(x)$ has a unique local optimum point that is a maximum on $(0,c)$.*

Proof

From the assumptions of the theorem $t(x) < 0$ if $x < c$ and $t(x) > 0$ if $x > c$. Thus, if $t(z_1) = t(z_2)$ then $0 < z_2 < z_1 < c$ or $c < z_2 < z_1$. But $\beta^2 < c$ and for a 4-point design follows that $z_2 < \beta^2 < z_1$. Hence the only possible situation is $0 < z_2 < z_1 < c$.

a) If $t(x)$ is decreasing on $(0,c)$ then $t(z_1) = t(z_2)$ cannot be satisfied and then a 4-point symmetric MV–optimum design is not possible.

b) If $\xi_{MV}^{(4)}$ is MV–optimum for some $\beta^2 < c$, then the equivalence theorem inequality (10) must be satisfied for any x, in particular for $x = 0$,

$$H(z_2) \leq H(z_1). \tag{11}$$

If the function $H(x)$ has a unique local minimum point at $x = v_0$, then $H'(x) > 0$ for $x \in [v_0, c)$. This means $[h(x) - h(0)]\partial x^2h(x)/\partial x - h'(x)x^2h(x) \geq 0$, and then, $-H(x) \geq t(x)$ for $x \geq v_0$.

Thus, if $t(x)$ has a unique local maximum point on $(0,c)$, it has to be less than or equal to v_0. Since $0 < z_2 < z_1 < c$ is a solution of $t(z_1) = t(z_2)$, then either $z_2 < z_1 < v_0$ or $z_2 < v_0 < z_1 < c$. In the first case $-H(z_2) < -H(z_1)$. If $z_2 < v_0 < z_1 < c$ then $-H(z_2) < t(z_2) = t(z_1) < -H(z_1)$. Thus both cases are in contradiction with (11) and $\xi_{MV}^{(4)}$ cannot be MV–optimum. □

Remark 2. The hypotheses made in these theorem are stronger than assumptions (2), which Dette and Sahm (1998) made. But under these hypotheses, as shown in the previous theorem, any singular point of $G(x, y, \beta)$ where the derivative vanishes for a fixed value of β is in the set $\{(x, y) : 0 < y < x < c\}$. Taking into account that $\partial x^2h(x)/\partial x = xh(0)[2 - x^3H'(x)]/[x^2H(x) + 1]^2$ and the hypothesis of the last theorem any singular point of $H(x)$, where the derivative vanishes, has to be in $(0,c)$. These properties are also useful from the computational point of view.

4 Efficiencies of MV– and SMV–optimum designs

In this section we give the efficiencies of MV– and SMV–optimum designs with respect to the best designs for estimating μ and β, respectively. From,

$$\min_{\xi} e_1' I^-(\theta, \xi) e_1 = 1/h(0)\beta^2 \quad \text{and} \quad \min_{\xi} e_2' I^{-1}(\theta, \xi) e_2 = \beta^2/k^2,$$

then

$$\text{eff}_\mu(\xi_{MV}) = \frac{1/[\beta^2 h(0)]}{e_1' I^{-1}(\theta, \xi_{MV}) e_1} \quad \text{and} \quad \text{eff}_\beta(\xi_{MV}) = \frac{\beta^2/k^2}{e_2' I^{-1}(\theta, \xi_{MV}) e_2}.$$

Table 1 shows efficiencies for the possible MV–optimum designs in order to estimate each parameter. As β^2 tends to zero, the efficiency of the MV–optimum design

β^2	$\text{eff}_\mu(\xi_{MV})$	$\text{eff}_\beta(\xi_{MV})$
$(0, v_0)$	$\omega + (1-\omega)h(v_0)/h(0)$	$(1-\omega)\, v_0^2\, h(v_0)/k^2$
$[v_0, z_2]$	$h(\beta^2)/h(0)$	$\beta^4\, h(\beta^2)/c^2\, h(c)$
(z_2, z_1)	$\frac{h(z_1)h(z_2)(z_1^2 - z_2^2)}{h(0)[h(z_2)(\beta^4 - z_2^2) + h(z_1)(z_1^2 - \beta^4)]}$	$\frac{\beta^4 h(z_1)h(z_2)(z_1^2 - z_2^2)}{c^2 h(c)[h(z_2)(\beta^4 - z_2^2) + h(z_1)(z_1^2 - \beta^4)]}$
$[z_1, c)$	$h(\beta^2)/h(0)$	$\beta^4\, h(\beta^2)/c^2\, h(c)$
$[c, \infty)$	$h(c)/h(0)$	1

Table 1: Efficiencies of MV–optimum designs for a general function $h(z)$, where $\omega = h(v_0)(v_0^2 - \beta^4)/\{\beta^4[h(0) - h(v_0)] + v_0^2 h(v_0)\}$.

for estimating β tends to zero and the efficiency for estimating μ tends to one. Thus ξ_{MV} tends to the 1–point design at μ that actually is the optimum design for estimating μ. In this sense a MV–optimum design may become inefficient for the estimation of β.

On the other hand, for model (1), a SMV–optimum design guarantees equal efficiencies for individual parameter estimation, $\text{eff}_\mu(\xi_{SMV}) = \text{eff}_\beta(\xi_{SMV})$. This is a consequence of Lemma 2.3 in Dette and Sham (1998). Table 2 gives the three different candidates for SMV–optimality and their efficiencies for estimating each parameter.

s^2	ξ_{SMV}	$\text{eff}_\mu(\xi_{SMV}) = \text{eff}_\beta(\xi_{SMV})$
$(0, v_0)$	$\xi_{SMV}^{(3)} = \begin{Bmatrix} \mu - \frac{v_0}{\beta} & \mu & \mu + \frac{v_0}{\beta} \\ \frac{1-\tilde{\omega}_1}{2} & \tilde{\omega}_1 & \frac{1-\tilde{\omega}_1}{2} \end{Bmatrix}$	$(1-\tilde{\omega}_1)\, v_0^2\, h(v_0)/k^2$
$[v_0, z_2]$	$\xi_{SMV}^{(2)} = \begin{Bmatrix} \mu - \beta & \mu + \beta \\ \frac{1}{2} & \frac{1}{2} \end{Bmatrix}$	$h(s^2)/h(0)$
(z_2, z_1)	$\xi_{SMV}^{(4)} = \begin{Bmatrix} \mu - \frac{z_1}{\beta} & \mu - \frac{z_2}{\beta} & \mu - \frac{z_2}{\beta} & \mu - \frac{z_1}{\beta} \\ \frac{\tilde{\omega}_2}{2} & \frac{1-\tilde{\omega}_2}{2} & \frac{1-\tilde{\omega}_2}{2} & \frac{\tilde{\omega}_2}{2} \end{Bmatrix}$	$\frac{h(z_1)h(z_2)(z_1^2 - z_2^2)}{h(0)[h(z_2)(s^4 - z_2^2) + h(z_1)(z_1^2 - s^4)]}$
$[z_1, c)$	$\xi_{SMV}^{(2)} = \begin{Bmatrix} \mu - \beta & \mu + \beta \\ \frac{1}{2} & \frac{1}{2} \end{Bmatrix}$	$h(s^2)/h(0)$

Table 2: Efficiencies of SMV–optimum designs for a general function $h(z)$, where $\tilde{\omega}_1 = h(v_0)(v_0^2 - s^4)/\{s^4[h(0) - h(v_0)] + v_0^2 h(v_0)\}$ and $\tilde{\omega}_2 = h(z_2)(s^4 - z_2^2)/\{h(z_2)(s^4 - z_2^2) + h(z_1)(z_2^2 - s^4)\}$.

5 Some particular models

Before giving some examples where the previous results are used, we stress that the assumptions made in Theorem 3.1 and Theorem 3.2 are not too restrictive. Many cdf's satisfy these conditions. Some of the most used in practice are shown in Table 3.

$F(z)$		$0 < \beta^2 < v_0$	$v_0 \leq \beta^2 < c$	$\beta^2 \geq c$	SMV
Error distribution:	ξ	$\xi_{MV}^{(3)}$	$\xi_{MV}^{(2)}$	ξ_β	$\xi_{SMV}^{(3)}$
$\frac{1+\text{sg}(z)}{2} - \frac{\text{sg}(z)}{2}\, e^{-\lvert z\rvert/2}$	eff$_\mu$	$\frac{.287}{287+.222\beta^4}$	$\frac{1}{2e^{\beta^2/2}-1}$	.0861	.526
$(c = 3.6828,\ v_0 = 3.1873)$	eff$_\beta$	$\frac{.246\beta^4}{.287+.222\beta^4}$	$\frac{.856\beta^4}{2e^{\beta^2/2}-1}$	1	.526
Normal:	ξ	$\xi_{MV}^{(2)}$		ξ_β	$\xi_{SMV}^{(2)}$
$\Phi(z)$	eff$_\mu$	$1/4\, e^{\beta^4}\Phi(\beta^2)[1-\Phi(\beta^2)]$		.385	.701
$(c = 1.575,\ v_0 = 0)$	eff$_\beta$	$1.046\beta^4/4\, e^{\beta^4}\Phi(\beta^2)[1-\Phi(\beta^2)]$		1	.701
Student-t, 4 df:	ξ	$\xi_{MV}^{(2)}$		ξ_β	$\xi_{SMV}^{(2)}$
$\frac{\Gamma(5/2)}{\sqrt{4\pi}} \int_{-\infty}^{z} \frac{dy}{(1+y^2/4)^{5/2}}$	eff$_\mu$	$256/(4+\beta^4)^2(16+3\beta^4)$		.295	.660
$(c = 1.4813,\ v_0 = 0)$	eff$_\beta$	$394.9\beta^4/(4+\beta^4)^2(16+3\beta^4)$		1	.660
Logistic:	ξ	$\xi_{MV}^{(2)}$		ξ_β	$\xi_{SMV}^{(2)}$
$e^z/(1+e^z)$	eff$_\mu$	$4e^{\beta^2}/(1+\beta^2)^2$		.305	.663
$(c = 2.3994,\ v_0 = 0)$	eff$_\beta$	$2.28e^{\beta^2}\beta^4/(1+\beta^2)^2$		1	.663
Cauchy:	ξ	$\xi_{MV}^{(2)}$		ξ_β	$\xi_{SMV}^{(2)}$
$1/2 + 1/\pi \arctan z$	eff$_\mu$	$9.87/(1+\beta^4)^2[\pi^2-(2\arctan\beta^4)^2]$		.201	.618
$(c = 1.3274,\ v_0 = 0)$	eff$_\beta$	$2.277e^{\beta^2}\beta^4/(1+\beta^4)^2[\pi^2-(2\arctan\beta^4)^2]$		1	.618
Double exponential:	ξ	$\xi_{MV}^{(3)}$	$\xi_{MV}^{(2)}$	ξ_β	$\xi_{SMV}^{(3)}$
$\frac{1+\text{sg}(z)}{2} - \frac{\text{sg}(z)}{2}\, e^{-\lvert z\rvert}$	eff$_\mu$	$\frac{.287}{287+.887\beta^4}$	$\frac{1}{2e^{\beta^2}-1}$	.0861	.526
$(c = 1.8414,\ v_0 = 1.5936)$	eff$_\beta$	$\frac{.983\beta^4}{.287+.887\beta^4}$	$\frac{.342\beta^4}{2e^{\beta^2}-1}$	1	.526

Table 3: Locally MV– and SMV–optimum designs and their efficiencies for different models. Here eff$_\mu$ and eff$_\beta$ are the efficiencies with respect to the optimum designs for estimating μ and β, respectively.

All these models satisfy one of the sufficient conditions given in Theorem 3.2, thus for none of them a 4–point symmetric MV–optimum design exists.

If the sufficient conditions are not satisfied, a 4–point symmetric MV–optimum design is possible. For instance, let us consider a function $H(x)$ like the one showed in Figure 1 (b). As proved in part (b.3) of Theorem 3.1, for any $\beta^2 \in (v_1, v_2)$ the 3–point symmetric design (4) cannot be MV–optimum. On the other hand, following the same argument as in part (a) of the same theorem it is easy to prove that $\xi_{MV}^{(2)}$ cannot be MV–optimum as well. Therefore the only possible symmetric MV–optimum design is $\xi_{MV}^{(4)}$. It is not easy to find a cdf needing a 4–point symmetric design. Looking for a simple polynomial $H(x)$ with a shape like in Figure 1(b) and taking into account that $h(z) = [z^2\, H(z) + 1]^{-1}$ we offer the example:

$$h(z) = \frac{0.037}{1 + 1.2z^2 - 4.9\ 10^{-5}z^4 + 3.5\ 10^{-5}z^6 - 7.6\ 10^{-6}z^8 + 5.3\ 10^{-7}z^{10}}.$$

López–Fidalgo, Torsney and Ardanuy (1998) showed that any function $h(z)$ can be expressed as in (2) if and only if $h(z) \geq 0$ and $\int_{-\infty}^{\infty} \sqrt{h(z)}dz = \pi$, as it is here the case. Then $F(x) = 0.5\Big[1 - \cos\Big(\int_{-\infty}^{x} \sqrt{h(z)}dz\Big)\Big]$. Here $c = 4$, $v_0 = 1$, $z_1 = 2.4141$ and $z_2 = 1.3197$, and a 4–point symmetric MV-optimal design is needed when $1.3197 < \beta^2 < 2.4141$.

References

Abdelbasit, R. and R.L. Placket, Experimental Design for Binary Data, *J. Amer. Statist. Assoc.*, **78** (1983) 90–98.

Atkinson, A.C. and A.N. Donev, *Optimum Experimental Designs* (Oxford University Press, Oxford, 1992).

Chaloner, K. and K. Larntz, Optimal Bayesian Design Applied to Logistic Regression Experiments, *J. Statist. Planning and Inference*, **21** (1989) 191–208.

Dette, H., Designing Experiments with respect to Standardize Optimality Criteria, *J. Roy. Statist. Soc.*, **B, 59**(1) (1997) 97–110.

Dette, H. and M. Sahm, Minimax Optimal Designs in Nonlinear Regression Models, *Statistica Sinica*, **8**(4) (1998) 1249–1264.

Elfving, G., Optimum allocation in linear regression theory, *Ann. Math. Statist.*, **23** (1952) 255–262.

Elfving, G., Designs of Linear Experiments, in: Ulf Grenander (Ed.), *Prob. and Stats.*, The Harold Cramér Volume (Wiley, New York, 1959) 58–74.

Fedorov, V.V. and P. Hackl, *Model-Oriented Design of Experiments* (Springer, New York, 1997).

Ford, I., Torsney, B. and C.F.J. Wu, The Use of a Canonical Form in the Construction of Locally Optimal Designs for Nonlinear Problems, *J. Roy. Statist. Soc.*, **B, 54**(2) (1992) 569–583.

López–Fidalgo, J., Torsney, B. and R. Ardanuy, MV–Optimization in Weighted Linear Regression, in: *Model-Oriented Data Analysis*, (Springer Verlag, New York, 1998) 39–50.

López–Fidalgo, J. and W.K. Wong, A Comparative Study of MV– and SMV–Optimal Designs for Binary Response Models, in: *Advances in Stochastic Simulation Methods*, 386 (Birkhauser, Boston, 2000) 135–152.

Sitter R.R. and C.F.J. Wu, Optimal Designs for Binary Response Experiments: Fieller–, D– and A–criteria, *Scand, J. Statist.*, **20** (1993) 329–341.

Torsney, B. and J. López–Fidalgo, Minimax designs for logistic regression in a compact interval, in: Advances in Model-Oriented Design and Analysis (Springer Verlag, New York, 2001) 243-250.

Wu C.F.J., Efficient Sequential Designs with Binary Data, *J. Amer. Statist. Assoc.*, **80** (1985) 974–984.

Wu C.F.J., Optimal designs for Percentile Estimation of a Quantal Response Curve, in: J. Dodge, V.V. Fedorov, H.P. Wynn (Eds), *Optimal Design and Analysis of Experiments* (North Holland, Amsterdam, 1988) 213–233.