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**ON THE STABILITY OF THRESHOLD
ARMA MODELS**

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On the stability of threshold ARMA models

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Abstract

In the present paper we derive a sufficient condition on the coefficients of the model to ensure the stationarity and the ergodicity of the process, without imposing extra-conditions on the thresholds nor assuming the T-continuity of the related Markovian representation.

Keywords: Threshold ARMA processes, stationary processes, ergodicity.

AMS 2000 SCI: 62M10, 60G10, 60J05.

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1 Introduction

We consider a time series model with different ARMA regimes, depending on the past history of the series itself, which, in the literature, is usually called Self-Exciting Threshold ARMA (SETARMA) model. Indicating by $\underline{X}_{n-1} = (X_{n-1}, \dots, X_{n-p})'$ the vector with components the last p values of the series, we define the SETARMA($m; p, q$) model as

$$X_n = \sum_{j=1}^m \left\{ a_0^{(j)} + \sum_{i=1}^p a_i^{(j)} X_{n-i} + \sum_{i=1}^q b_i^{(j)} e_{n-i} \right\} \mathbb{1}_{R_j}(\underline{X}_{n-1}) + e_n \quad (1)$$

where $a_i^{(j)}, b_i^{(j)} \in \mathbb{R}$, for all i and j , the sets R_j , for $j = 1, \dots, m$, form a partition of the space \mathbb{R}^p , \underline{X}_0 is a suitable random variable on \mathbb{R}^p , and $\mathbb{1}_B(\cdot)$ is the indicator function of the set B . From now on, we assume that

- (H) $\{e_n\}_{n \in \mathbb{N}}$ is a sequence of independent and identically distributed (i.i.d.) random variables, independent from \underline{X}_0 . They are absolutely continuous with respect to the Lebesgue measure μ on \mathbb{R} , there exists $\eta > 0$ such that their density function $f(\cdot)$ is positive on $(-\eta, \eta)$ and $E[|e_1|] = M < +\infty$.

Various results concerning this kind of models have been obtained since the publication of the seminal work on the threshold models by Tong (1983). Among the others, we recall Brockwell, Liu and Tweedie (1992), Liu and Susko (1992) and the more recent paper by Cline and Pu (1999). Brockwell, Liu and Tweedie (1992) consider the particular case of (1) with a continuity assumption on the conditional mean $E[X_n | \underline{X}_{n-1} = \underline{x}]$, $b_i^{(j)} = b_i$, for all i and j and $R_j = \mathbb{R} \times \dots \times r_j \times \dots \times \mathbb{R} \in \mathbb{R}^p$, where the r_j 's form a partition of \mathbb{R} . For such a model they obtain a sufficient condition for the existence of a unique strictly stationary measure without using Markov chain arguments. Indeed the process $\{X_n\}$ in (1) is not a Markov chain but, as in other cases like linear and nonlinear autoregressive processes, we can build a suitable Markov chain, strictly related to the original model, so to be able to transfer the results from one process to the other. Then, one of the difficulties in obtaining stability results for this model lies on the fact that it is not straightforward to find out if the related Markov process is irreducible or T-continuous, and these are two fundamental properties in applying the 'drift'-criteria for ergodicity and geometric ergodicity (see Meyn and Tweedie (1993) for a full account on the subject). Therefore Liu and Susko (1992) develop an alternative approach based on Markov processes. They define a general nonlinear ARMA model and state sufficient conditions for the existence of an invariant probability measure for such a process. In particular, they apply their approach to a SETARMA($m; 1, q$) with the sets R_j defined as in Brockwell, Liu and Tweedie (1992). Still, they need to assume the irreducibility of the related Markov chain in order to obtain a sufficient condition for the ergodicity. Even in Cline and Pu (1999) irreducibility and T-continuity are two basic assumptions in obtaining their stability results for threshold-like ARMA models. In the present paper we show that the result on general nonlinear ARMA models by Liu and Susko (1992) applies to (1) and we prove that (1) is also irreducible and ergodic provided a condition on the coefficients of the model is satisfied. As a final note, we point out that this result is obtained without making any assumption on the thresholds.

The paper is organized as follows. In Section 2 we define a vectorial Markovian representation of (1). In Section 3 we prove the stationarity of (1), provided a condition on the coefficients is satisfied. In Section 4 we show that under the same condition the related Markov chain is irreducible and therefore (1) is ergodic.

2 Markovian representation

If we define the vector $Y_n = (\underline{X}'_n, \underline{e}'_n)'$, where $\underline{e}_n = (e_n, \dots, e_{n-q+1})'$, we can rewrite model (1) in the following Markovian vectorial representation (without loss of generality we consider the case $a_0^{(j)} = 0, \forall j$)

$$Y_n = \sum_{j=1}^m A^{(j)} Y_{n-1} \mathbb{1}_{R_j^*}(Y_{n-1}) + \underline{c} e_n, \quad (2)$$

where $\underline{c} = (1, \dots, 0, 1, 0, \dots, 0)'$, with the 2^{nd} 1 in the $(p+1)^{th}$ position, $R_j^* = (R_j \times \mathbb{R}^q) \subset \mathbb{R}^{p+q}$ and

$$A^{(j)} = \begin{bmatrix} a_1^{(j)}, \dots, a_{p-1}^{(j)} & a_p^{(j)} & b_1^{(j)}, \dots, b_{q-1}^{(j)} & b_q^{(j)} \\ I_{(p-1) \times (p-1)} & \mathbf{0}_{p-1} & \mathbf{0}_{(p-1) \times (q-1)} & \mathbf{0}_{p-1} \\ \mathbf{0}'_{p-1} & 0 & \mathbf{0}'_{q-1} & 0 \\ \mathbf{0}_{(q-1) \times (p-1)} & \mathbf{0}_{q-1} & I_{(q-1) \times (q-1)} & \mathbf{0}_{q-1} \end{bmatrix}.$$

Note that the $(p+1)^{th}$ row of each matrix $A^{(j)}$ is a null row because it corresponds to e_n in the vector Y_n , and that $Y_{n-1} \in R_j^* = (R_j \times \mathbb{R}^q)$ if and only if $\underline{X}_{n-1} \in R_j$, since $\underline{e}_{n-1} \in \mathbb{R}^q$ by definition.

Defining the operator

$$Dy = \sum_{j=1}^m A^{(j)} \mathbb{1}_{R_j^*}(y) y,$$

we may rewrite (2) as follows

$$Y_n = DY_{n-1} + \underline{c} e_n.$$

Now let $B \in \mathcal{B}(\mathbb{R}^{p+q})$ and $y = (y_1, \dots, y_{p+q})' \in \mathbb{R}^{p+q}$ be respectively a set and a point on the state space of the Markov chain $\{Y_n\}$ and define the set

$$B_y = \{w \in \mathbb{R}^{p+q} : w \in B, w_i = y_i, i \neq 1, p+1\},$$

that is, a subset of \mathbb{R}^{p+q} with $p+q-2$ coordinates fixed and only the 1^{st} and the $(p+1)^{th}$ coordinates allowed to vary accordingly to the definition of the set B . Instead of considering the σ -algebra $\mathcal{B}(\mathbb{R}^{p+q})$, we deal with its sub- σ -algebra

$$\hat{\mathcal{B}} = \sigma \left\{ B = B_1 \times \mathbb{R}^{p-1} \times B_2 \times \mathbb{R}^{q-1}; B_1, B_2 \in \mathcal{B}(\mathbb{R}) \right\},$$

that is, the projection onto the two coordinates we mentioned above.

Let $v_{1,p+1} : \mathbb{R}^{p+q} \rightarrow \mathbb{R}^2$ be the projection map onto the 1^{st} and $(p+1)^{th}$ coordinates. Then, from the definition of B_y it follows that $v_{1,p+1}(B_y)$ does not depend on y but just on the set B .

Let us point this out by indicating, from now on, $v_{1,p+1}(B_y) = B_{1,p+1}$.

It is clear that if the past Y_{n-1} is given and fixed, the probabilistic behavior of Y_n is completely dependent on that of e_n . Indeed, we have that

$$P(y, B) = \Pr [(e_n, e_n)' \in B_{1,p+1} - v_{1,p+1}(Dy)].$$

It is interesting to point out that the 2^{nd} coordinate of $v_{1,p+1}(Dy)$ (that is, the $(p+1)^{th}$ coordinate of Dy) is 0. In terms of the density of e_n we have therefore

$$P(y, B) = \int_Q f(u) du \quad (3)$$

where the set $Q \in \mathbb{R}^1$ is defined by

$$Q = \{w \in \mathbb{R}^1 : (w, w)' \in B_{1,p+1} - [v_1(Dy), 0]'\},$$

with $v_1 : \mathbb{R}^{p+q} \rightarrow \mathbb{R}^1$ the projection map onto the 1st coordinate.

The sequence $\{Y_n\}$, together with the transition probabilities (3), forms a Markov chain. Moreover, since by assumption (H) the subchain $\{e_n\}$ is obviously aperiodic, so is $\{Y_n\}$.

3 Stationarity of the model

The aim of the present section is to prove that (1) induces a strictly stationary solution under a suitable condition on the coefficients. We use the following result by Liu and Susko (1992)

Theorem 3.1 *Consider the following general non-linear ARMA model, denoted by NARMA(p, q),*

$$X_n = h(X_{n-1}, \dots, X_{n-p}, e_{n-1}, \dots, e_{n-q}) + e_n \quad (4)$$

and the Markov chain $\{Y_n\}$ with $Y_n = (X_n, \dots, X_{n-p+1}, e_n, \dots, e_{n-q+1})'$. Assume that $\{e_n\}_{n \in \mathbb{N}}$ is a sequence of i.i.d. random variables, having probability density function with respect to the Lebesgue measure, and that $h(\cdot)$ is a locally bounded function.

Then $\{Y_n\}$ has an invariant probability measure if and only if there exists a non-negative measurable function $G(\cdot)$ and an increasing sequence of compact sets $K_n \uparrow \mathbb{R}^{p+q}$ with respect to the topology generated by the open sets of $\mathcal{B}(\mathbb{R}^{p+q})$ such that

$$\inf_{y \in K_n^c} G(y) \rightarrow \infty \quad \text{as } n \rightarrow \infty \quad (5)$$

and

$$\sup_{n \geq 1} E_{P_0} \{E[G(Y_n) | Y_0]\} < +\infty \quad (6)$$

for some initial probability measure P_0 of Y_0 .

Remark 3.1 *We recall that if a Markov chain has a finite positive invariant measure and the transition probabilities are time-homogeneous, then there exists a strictly stationary Markov chain, solution of the former process, with a stationary marginal probability distribution given by the normalized finite positive invariant measure.*

Remark 3.2 *As shown in Lemma 2.1 by Liu and Susko (1992), the boundedness assumption on the function $h(\cdot)$ and the assumption on the error sequence imply the uniform countable additive condition*

$$\limsup_{B \downarrow \emptyset} P(y, B) = 0,$$

$\forall B \in \hat{\mathcal{B}}$ and for any compact set K with respect to the topology generated by all open sets in $\mathcal{B}(\mathbb{R}^{p+q})$.

By hypothesis (H), the assumption on the error sequence in Theorem 3.1 is fulfilled. Moreover, in the considered model,

$$h(X_{n-1}, \dots, X_{n-p}, e_{n-1}, \dots, e_{n-q}) = \sum_{j=1}^m \{a_0^{(j)} + \sum_{i=1}^p a_i^{(j)} X_{n-i} + \sum_{i=1}^q b_i^{(j)} e_{n-i}\} \mathbb{1}_{R_j}(\underline{X}_{n-1}) \quad (7)$$

is obviously locally bounded. Thus we can prove the following result

Theorem 3.2 *The process $\{X_n\}$ defined in (1) with assumption (H) on the error sequence has a strictly stationary solution, provided that*

$$\lambda = \max_{j=1, \dots, m} \sum_{i=1}^p |a_i^{(j)}| < 1. \quad (8)$$

PROOF By Theorem 3.1, we need a non-negative and measurable function $G(\cdot) : \mathbb{R}^{p+q} \rightarrow \mathbb{R}$ which satisfies conditions (5) and (6). To achieve our goal we define, as in Cline and Pu (1999),

$$G(Y_n) = g(\underline{X}_n) + \sum_{i=0}^{q-1} |e_{n-i}|$$

with

$$g(\underline{X}_n) = \max_{i=0, \dots, p-1} (\rho^i |X_{n-i}|)$$

and, by (8),

$$\rho = \left(\max_{j=1, \dots, m} \sum_{i=1}^p |a_i^{(j)}| \right)^{\frac{1}{p}} < 1. \quad (9)$$

First of all, if we define $K_n = [-n, n]^{p+q}$, then it is straightforward to conclude that $K_n \uparrow \mathbb{R}^{p+q}$ as $n \rightarrow +\infty$. Now consider $y \in \mathbb{R}^{p+q}$, then

$$\inf_{y \in K_n^c} G(y) = \inf_{y \in K_n^c} \left(\max_{i=1, \dots, p} (\rho^{i-1} |y_i|) + \sum_{i=p+1}^{p+q} |y_i| \right) \geq n (\rho^{p-1} + q) \rightarrow +\infty$$

as $n \rightarrow +\infty$ and (5) is satisfied.

Now, considering first the case $n > \max\{p, q\}$, we have

$$\begin{aligned} \mathbb{E}\{G(Y_n)|Y_0\} &= \mathbb{E}\{g(\underline{X}_n)|Y_0\} + \mathbb{E}\left\{\sum_{i=0}^{q-1} |e_{n-i}| \mid Y_0\right\} \\ &\leq \mathbb{E}\{g(\underline{X}_n)|Y_0\} + \sum_{i=0}^{q-1} \mathbb{E}\{|e_{n-i}|\} \\ &= \mathbb{E}\{\mathbb{E}\{\dots \mathbb{E}\{g(\underline{X}_n)|Y_{n-1}\}|Y_{n-2}\} \dots |Y_0\} + qM, \end{aligned} \quad (10)$$

because the error sequence is composed by i.i.d. random variables, also independent of \underline{X}_0 . Since the number of different regimes m in (1) is finite and for each of those the q coefficients of the MA part are real constants, we have $\max_{i,j} |b_i^{(j)}| < +\infty$. This implies that $\exists b > 0$ such that, for any $j \in \{1, \dots, m\}$,

$$\begin{aligned} |X_n| &= \left| \sum_{i=1}^p a_i^{(j)} X_{n-i} + \sum_{i=1}^q b_i^{(j)} e_{n-i} + e_n \right| \\ &\leq \sum_{i=1}^p |a_i^{(j)}| |X_{n-i}| + b \sum_{i=1}^q |e_{n-i}| + |e_n| \\ &\leq \left(\sum_{i=1}^p |a_i^{(j)}| \right) \max_{i=1, \dots, p} |X_{n-i}| + |e_n| + b \sum_{i=1}^q |e_{n-i}| \end{aligned}$$

$$\begin{aligned}
&\leq \max_{j=1,\dots,m} \left(\sum_{i=1}^p |a_i^{(j)}| \right) \max_{i=1,\dots,p} |X_{n-i}| + |e_n| + b \sum_{i=1}^q |e_{n-i}| \\
&= \rho^p \max_{i=1,\dots,p} |X_{n-i}| + |e_n| + b \sum_{i=1}^q |e_{n-i}| \\
&= \rho \max_{i=1,\dots,p} (\rho^{p-1} |X_{n-i}|) + |e_n| + b \sum_{i=1}^q |e_{n-i}| \\
&\leq \rho \max_{i=1,\dots,p} (\rho^{i-1} |X_{n-i}|) + |e_n| + b \sum_{i=1}^q |e_{n-i}| \\
&= \rho g(\underline{X}_{n-1}) + |e_n| + b \sum_{i=1}^q |e_{n-i}|.
\end{aligned}$$

Since $\rho < 1$,

$$\begin{aligned}
g(\underline{X}_n) &= \max\{|X_n|, \rho|X_{n-1}|, \dots, \rho^{p-1}|X_{n-p+1}|\} \\
&\leq \max\left\{\rho g(\underline{X}_{n-1}) + |e_n| + b \sum_{i=1}^q |e_{n-i}|, \rho|X_{n-1}|, \dots, \rho^{p-1}|X_{n-p+1}|\right\} \\
&= \max\left\{\rho \max\{|X_{n-1}|, \dots, \rho^{p-1}|X_{n-p}|\} + |e_n| + b \sum_{i=1}^q |e_{n-i}|, \dots, \rho^{p-1}|X_{n-p+1}|\right\} \\
&\leq \rho g(\underline{X}_{n-1}) + |e_n| + b \sum_{i=1}^q |e_{n-i}|. \tag{11}
\end{aligned}$$

and, by (11),

$$\begin{aligned}
\mathbb{E}\{g(\underline{X}_n)|Y_{n-1}\} &\leq \rho g(\underline{X}_{n-1}) + \mathbb{E}\{|e_n|\} + b \sum_{i=1}^q |e_{n-i}| \\
&= \rho g(\underline{X}_{n-1}) + M + b \sum_{i=1}^q |e_{n-i}|. \tag{12}
\end{aligned}$$

By (12), we obtain

$$\begin{aligned}
\mathbb{E}\{\mathbb{E}\{g(\underline{X}_n)|Y_{n-1}\}|Y_{n-2}\} &\leq \mathbb{E}\left\{\rho g(\underline{X}_{n-1}) + M + b \sum_{i=1}^q |e_{n-i}| \mid Y_{n-2}\right\} \\
&= \rho \mathbb{E}\{g(\underline{X}_{n-1})|Y_{n-2}\} + M + bM + b \sum_{i=2}^q |e_{n-i}| \\
&\leq \rho \left[\rho g(\underline{X}_{n-2}) + M + b \sum_{i=1}^q |e_{n-1-i}| \right] + M + bM + b \sum_{i=2}^q |e_{n-i}| \\
&\leq \rho^2 g(\underline{X}_{n-2}) + \rho M + \rho b \sum_{i=1}^q |e_{n-1-i}| + M + bM + b \sum_{i=2}^q |e_{n-i}|
\end{aligned}$$

and, iterating n times, we have

$$\begin{aligned}
\mathbb{E}\{g(\underline{X}_n)|Y_0\} &= \mathbb{E}\{\mathbb{E}\{\dots \mathbb{E}\{g(\underline{X}_n)|Y_{n-1}\}|Y_{n-2}\}\dots|Y_0\} \\
&\leq \rho^n g(\underline{X}_0) + \rho^{n-1} M + \rho^{n-1} b \sum_{i=0}^{q-1} |e_{-i}| + \rho^{n-2} M + \rho^{n-2} bM + \rho^{n-2} b \sum_{i=0}^{q-2} |e_{-i}| +
\end{aligned}$$

$$\begin{aligned}
& + \cdots + \rho^{n-q}M + \rho^{n-q}bM + \cdots + \rho^{n-q}b|e_0| + (M + qbM) \sum_{i=0}^{n-q-1} \rho^i \\
= & \rho^n g(\underline{X}_0) + M \sum_{i=0}^{n-1} \rho^i + bM \sum_{i=0}^{n-2} \rho^i + \cdots + bM \sum_{i=0}^{n-q} \rho^i + bM \sum_{i=0}^{n-q-1} \rho^i + \\
& + b|e_0| \sum_{i=n-q}^{n-1} \rho^i + b|e_{-1}| \sum_{i=n-q+1}^{n-1} \rho^i + \cdots + b|e_{-q+1}| \rho^{n-1} \\
\leq & \rho^n g(\underline{X}_0) + (M + qbM) \frac{1 - \rho^n}{1 - \rho} + b \frac{1 - \rho^n}{1 - \rho} \sum_{i=0}^{q-1} |e_{-i}|. \tag{13}
\end{aligned}$$

We define now the initial distribution of \underline{X}_0 as

$$P_0(\cdot) = F_{X_0}(\cdot) \cdots F_{X_{-p+1}}(\cdot) F(\cdot) \cdots F(\cdot),$$

where $F(\cdot)$ is the distribution function of e_1 and $F_{X_i}(\cdot)$, $i = 0, \dots, -p + 1$, is an arbitrary distribution function with finite mean. With this choice of P_0 the case of $n = 1, \dots, \max\{p, q\}$ follows in a straightforward way from that of $n > \max\{p, q\}$.

By (10) and (13) we can finally conclude that

$$\begin{aligned}
\sup_{n \geq 1} \mathbf{E}_{P_0}[\mathbf{E}\{G(Y_n)|Y_0\}] & \leq \sup_{n \geq 1} \mathbf{E}_{P_0}[\mathbf{E}\{\mathbf{E}\{\dots \mathbf{E}\{g(\underline{X}_n)|Y_{n-1}\}|Y_{n-2}\} \dots |Y_0\} + qM] \\
& \leq \sup_{n \geq 1} \mathbf{E}_{P_0} \left[\rho^n g(\underline{X}_0) + (M + qbM) \frac{1 - \rho^n}{1 - \rho} + b \frac{1 - \rho^n}{1 - \rho} \sum_{i=0}^{q-1} |e_{-i}| + qM \right] \\
& = \sup_{n \geq 1} \left(\rho^n \mathbf{E}_{P_0}[g(\underline{X}_0)] + (M + qbM) \frac{1 - \rho^n}{1 - \rho} + b \frac{1 - \rho^n}{1 - \rho} \sum_{i=0}^{q-1} \mathbf{E}_{P_0}[|e_{-i}|] + qM \right) \\
& = \sup_{n \geq 1} \left(\rho^n \mathbf{E}_{P_0}[g(\underline{X}_0)] + (M + 2qbM) \frac{1 - \rho^n}{1 - \rho} + qM \right) \\
& \leq \mathbf{E}_{P_0}[g(\underline{X}_0)] + (M + 2qbM) \frac{1}{1 - \rho} + qM < +\infty,
\end{aligned}$$

since $\rho < 1$ and $M = \mathbf{E}[|e_1|] < \infty$. Therefore (6) is satisfied and the proof is complete. \blacksquare

Remark 3.3 Above we have used the function described in the proof of Theorem 3.1 in Cline and Pu (1999) to obtain the suitable one-step inequality (12) and then, using this, we have showed that (6) holds, provided (8) is satisfied. Some very general results concerning the relation between the Foster-type one-step drift condition and (6) are introduced in Fonseca and Tweedie (2000), yet they are not valid in the present case. Indeed, we are not able to obtain a drift condition of the form

$$\mathbf{E}[V(Y_n)|Y_{n-1} = y] \leq \tau_1 V(y) + \tau_2,$$

where $\tau_1 \in (0, 1)$, $\tau_2 > 0$ and $V(\cdot)$ is a norm-like function.

4 Ergodicity of the model

It is well known that a stationary Markov chain is also ergodic provided it is aperiodic and irreducible. We have already proved the aperiodicity of the process $\{Y_n\}$ in section 2 and now we show that it is irreducible with respect to a suitable measure φ .

$$\begin{aligned}
&\leq \theta \max \left\{ \left[\theta \max \{ |X_{n-1}|, \dots, |X_{n-p+1}|, \theta |X_{n-p}| \} + \Delta_1(1 + \theta) \right], \theta \max_{i=1, \dots, p} |X_{n-i}| + \Delta_1, \right. \\
&\quad \left. |X_{n-1}|, \dots, |X_{n-p+2}| \right\} + \Delta_1 \\
&\leq \theta \left(\max \left\{ \theta \max \{ |X_{n-1}|, \dots, |X_{n-p+1}|, \theta |X_{n-p}| \}, \theta \max_{i=1, \dots, p} |X_{n-i}|, |X_{n-1}|, \dots, |X_{n-p+2}| \right\} \right. \\
&\quad \left. + \Delta_1(1 + \theta) \right) + \Delta_1 \\
&\leq \theta \max \{ |X_{n-1}|, \dots, |X_{n-p+2}|, \theta |X_{n-p+1}|, \theta |X_{n-p}| \} + \Delta_1 \sum_{i=0}^2 \theta^i
\end{aligned}$$

and

$$\begin{aligned}
|X_{n+p}| &\leq \theta \max_{i=1, \dots, p} |X_{n+p-i}| + \Delta_1 \\
&\leq \theta \max \left\{ \left[\theta \max \{ |X_{n-1}|, \theta |X_{n-2}|, \dots, \theta |X_{n-p}| \} + \Delta_1 \sum_{i=0}^{p-1} \theta^i \right], \right. \\
&\quad \left[\theta \max \{ |X_{n-1}|, |X_{n-2}|, \theta |X_{n-3}|, \dots, \theta |X_{n-p}| \} + \Delta_1 \sum_{i=0}^{p-2} \theta^i \right], \\
&\quad \left. \dots, \theta \max_{i=1, \dots, p} |X_{n-i}| + \Delta_1 \right\} + \Delta_1 \\
&\leq \theta \max \{ \theta |X_{n-1}|, \dots, \theta |X_{n-p}| \} + \Delta_1 \sum_{i=0}^p \theta^i \\
&= \theta^2 \max_{i=1, \dots, p} |X_{n-i}| + \Delta_1 \frac{1 - \theta^{p+1}}{1 - \theta}.
\end{aligned}$$

In the same way we have

$$|X_{n+2p}| \leq \theta^3 \max_{i=1, \dots, p} |X_{n-i}| + \Delta_1 \frac{1 - \theta^{2p+1}}{1 - \theta}$$

and, finally, for $k = 0, 1, \dots$,

$$|X_{n+kp}| \leq \theta^{k+1} \max_{i=1, \dots, p} |X_{n-i}| + \Delta_1 \frac{1 - \theta^{kp+1}}{1 - \theta}. \quad (19)$$

Summarizing, by (19), we obtain that

$$|X_{n+kp}| \leq \theta^{k+1} \max_{i=1, \dots, p} |X_{n-i}| + \delta \frac{\varepsilon(bq + 1)}{2bq(1 - \theta)} \quad (20)$$

and, for $i = 1, \dots, p$,

$$\begin{aligned}
|X_{n+kp+i}| &\leq \theta \max \{ \theta^k |X_{n-1}|, \dots, \theta^k |X_{n-p+i+1}|, \theta^{k+1} |X_{n-p+i}|, \dots, \theta^{k+1} |X_{n-p}| \} \\
&\quad + \Delta_1 \frac{1 - \theta^{kp+i+1}}{1 - \theta} \\
&\leq \theta \max \{ \theta^k |X_{n-1}|, \dots, \theta^k |X_{n-p+i+1}|, \theta^{k+1} |X_{n-p+i}|, \dots, \theta^{k+1} |X_{n-p}| \} \\
&\quad + \delta \frac{\varepsilon(bq + 1)}{2bq(1 - \theta)} \\
&\leq \theta^{k+1} \max_{i=1, \dots, p} |X_{n-i}| + \delta \frac{\varepsilon(bq + 1)}{2bq(1 - \theta)}. \quad (21)
\end{aligned}$$

Therefore, by (20) and (21),

$$\exists \delta < \frac{(1-\theta)bq}{bq+1} : \forall Y_0 = y \in \mathbb{R}^{p+q} \quad \exists k < \infty : |X_{q+1+kp+i}| < \frac{\varepsilon}{2} \quad \forall i = 1, \dots, p.$$

and hence $Y_{q+1+(k+1)p}$ satisfies (18) and the proof is complete. ■

Since we have proved the irreducibility of the stationary Markov chain $\{Y_n\}$, which is then ergodic too, we can now state the following theorem

Theorem 4.1 *The process $\{X_n\}$ defined in (1) with the assumption (H) on the errors is ergodic provided that (8) is satisfied.*

This last result is in accordance with Chan and Tong (1985) who state the same sufficient condition on the coefficients for the ergodicity of a SETAR($m; p$) process.

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