An adaptive exponentially weighted moving average control chart

G. Capizzi, G. Masarotto

2000.13

Dipartimento di Scienze Statistiche
Università degli Studi
Via S. Francesco, 33
35121 Padova

Ottobre 2000

An Adaptive Exponentially Weighted Moving Average Control Chart

Giovanna Capizzi Guido Masarotto

Abstract

The Exponentially Weighted Moving Average (EWMA) charts (Roberts, 1959) has optimal properties in forecasting and control applications. Lucas and Saccucci (1990) showed that an EWMA scheme can be designed to quickly detect small or large shifts in the mean of a normally distributed process. However EWMA cannot be designed to be "optimal" for both small and large shifts. Furthermore in the worst-case situation (Woodall, Maragah, 1990) EWMA requires a few obsevations to overcome its initial inertia. In this article the main goal is to take the best features of the EWMA chart and the Shewhart while diminishing the inertia problem. We suggest an Adaptive Exponentially Weighted Moving Average (AEWMA) that depends on a suitable function of the current prediction error. We show the results for two weighting functions, the Huber's function (Huber, 1981) and the Tukey's bisquare function (Hampel et al., 1986). The resulting AEWMA is a smooth combination of the Shewhart's chart and the EWMA chart. A design procedure for the AEWMA control scheme is given. Parameter values are shown to be useful for detecting both small and large shifts in the mean of the process. Several control schemes are also considered. We compare the standard and worst-case *ARL*'s curves, of CUSUM, EWMA, CSEWMA and Shewhart's with supplementary rules charts (Champ and Woodall, 1987), over a wide range of shift values. All the results show a more 'balanced' response of the AEWMA scheme to any size shifts between two specified shift values.

KEY WORDS: Control charts; EWMA; Average Run Length; Adaptive coefficients; One-step forecast error.

1 Introduction

Let $\{y_t\}$, $t = 1, 2, \ldots$, be a sequence of independent random variables. Suppose that, before an unknown change time, the mean of the process, η, is equal to η_0 and after the change the mean is equal to $\eta_1 = \eta_0 + \mu$, $\mu \neq 0$. We want to be able to detect this single shift, in the mean of the process, as soon as possible.

Control charts, as the Shewhart's chart (Shewhart,1931), the CUSUM charts (Page, 1954, 1955) and the Exponentially Weighted Moving Average (EWMA) charts (Roberts, 1959), serve the purpose of detecting changes in the quality of a product in a process (Montgomery, 1985, Ryan, 1989, Rowlands and Whetherill, 1991, Whetherill and Brown, 1991, Yashchin, 1993).

The properties of the statistical change detection schemes can be investigated with the aid of the *Average Run Length, ARL*, that is the average number of observations required to produce a signal. The criteria are mainly the delay for the detection, which is related to the ability of the scheme to signal an alarm when a change actually occurs, and the mean time between false alarms. The overall criterion consists in minimizing the delay for the detection, (*out of control ARL*), for a fixed mean time between false alarms, (*in control ARL*).

An EWMA scheme is based on the statistic

$$x_t = \lambda x_{t-1} + (1 - \lambda)y_t, \tag{1}$$

The quantity x_0 represents the starting value and is often taken to be the target value η_0. The smoothing constant λ determines the rate of decay of the weights and hence the amount of information on the historical data. This scheme signals when $\mid x_t - \eta_0 \mid$ exceeds a specified action limit h_1.
In literature, efficiency and robustness of EWMA's charts is investigated with respect to the change magnitude μ in the mean. The main conclusion of this comparison is that, for a fixed λ, an EWMA scheme can be constructed to give good signaling performance for small or large shifts. Small values of λ are optimal for detecting large shifts, just as in the case of the Shewhart chart, while large values of λ are optimal for detecting small shifts, as in the case of the CUSUM scheme (Crowder, 1987A, Lucas and Saccucci, 1987). On the other hand, because of the dependence of (1) on the fixed value of λ, it is not possible to design an EWMA scheme that is "optimal" for small and large shifts simultaneously .

According to Hunter (1986) the EWMA chart may be also used as mechanism for dynamic process control, by estimating its 'current level' and the 'prediction' of its next position. In fact the control statistic (1) can be also written as the most recent observation minus λ times the one-step ahead forecast error $e_t = y_t - x_{t-1}$, i.e.

$$x_t = y_t - \lambda e_t. \tag{2}$$

If the forecast shows a future deviations from target, that seems too large, the process operator could control and adjust the process.
However, for some process shifts, the EWMA scheme (2) can be slow in terms of *ARL*. In particular when the EWMA statistic is at or near one of its control limits and a sudden change in the mean occurs in the opposite direction (*worst-case situation*), there is a large discrepancy, between the observation and its forecasted value, giving a large residual. Gradually the forecasted values shifts up also to reflect the fact that the observations have shifted upward and the EWMA scheme requires a few observations to overcome this initial *inertia* (Woodall, Maragah, 1990).
The EWMA's inertia can be guarded against by using the combined Shewhart EWMA, CSEWMA, (Lucas and Saccucci, 1987, 1990). This scheme is achieved by adding Shewhart limits to an EWMA control scheme so that an out-of-control signal is given if the EWMA statistic is outside the control limits or if the current observation is outside the Shewhart limits.
However the inertia also depends on the smoothing constant λ and for small values of λ and moderate to large shifts, this problem can be even more pronounced. In fact for values of λ close to zero the most recent forecast error receives a little weight and the evidence of a sudden

large shift will be weakened. On the other hand, when λ increases, a large shift could be detected sooner, since more weight is given to more recent errors which have a higher mean, but the addition of the Shewhart limits causes the CSEWMA to be sensitive to the occurrence of occasional outliers.

In this article the main goal is to take the best features of the EWMA chart and the Shewhart's chart while diminishing the inertia problem still present in the CSEWMA. The idea underlying is to take into account the response of the forecast errors to a shift in the mean process.
This question has been already considered within several frameworks, as for example within the dynamic Bayesian modelling (West and Harrison, 1989) or the Kalman filter estimation.
We propose an Adaptive Exponentially Weighted Moving Average (AEWMA) procedure that uses a simpler idea of weighting the one-step forecast error, based on its magnitude. Even under worst-case scenarios, a better performance than EWMA is attained by a modified EWMA prediction equation that it is sensitive to a specific pattern of the decay in the forecast errors.
The AEWMA chart is shown to give improved ARL properties when both small and large shifts are to be detected. To design the AEWMA scheme we use a design scheme different to that of Lucas and Saccucci (1990). They suggest to choose a combination of parameters which, for a given in control *ARL*, minimizes the out-of-control *ARL* of EWMA, for a specified shift in the mean. Thus the optimal choise of parameters depends on the magnitude of the shift. We recommend to design a scheme whose out-of-control *ARL*, for the optimal combination of parameters, also performs well in terms of detecting other magnitudes of the shift.

In this article we show the results for two weighting functions, one based on the Huber's function (Huber, 1981) and the other on the Tukey's bisquare function (Hampel et al., 1986). These two functions have the advantages to be parsimonious in terms of parameters and give a smooth combination of the Shewhart's and EWMA charts.
In §2 we describe the control scheme AEWMA. In §3 we obtain the *ARL* of the AEWMA using a Markov chain approach. In §4 we give the design procedure for the AEWMA scheme and in §5 we evaluate and compare the *ARL* performances of some AEWMA schemes. Finally in §6 the *ARL*'s profiles of the AEWMA schemes are compared to the corresponding profiles of CUSUM, EWMA, CSEWMA and Shewhart with supplementary rules charts (Champ and Woodall, 1987).

2 The Adaptive EWMA scheme

When a shift first occurs there is a large discrepancy, between the observation and its forecasted value, giving a large residual e_t. In this situation the control statistic x_t becomes small and the EWMA results slower to produce an out-of-control signal. Thus the control statistic (2) should be modified to accelerate its reaction to large and sudden shifts.
We suggest to use an AEWMA scheme, based on the statistic

$$x_t = y_t - \lambda\psi(e_t, \zeta), \quad x_0 = \eta_0, \tag{3}$$

where the $\psi(e_t, \zeta)$ is a suitable function of the current prediction error e_t and of a parameter vector ζ, the elements of which are to be chosen to satisfy the run length requirements.
The function $\psi(\cdot)$ should be such that, when e_t is large, it should have a relative little value. Thus

the predicted value x_t, for very small values of this weight, becomes larger and (3) should be able to overcome the inertia under worst case scenarios. Briefly if $|e_t|$ is small then $\psi(\cdot)$ should be $\sim e_t$, while if the current error is large, $\psi(\cdot)$ should be strictly less than e_t.
Then the weighting function $\psi(\cdot)$ should allow to combine the best properties of the Shewhart's and EWMA charts.
Actually many score functions could be used. To blend together the best features of the EWMA and the Shewhart's chart we recommend to use a class of monotone functions, within the envelope of the EWMA and Shewhart's $\psi(\cdot)$ functions.
In this article we show the results for two weighting functions, the Huber's function (Huber, 1981) and the Tukey's bisquare function (Hampel et al., 1986) given by

$$\psi_{hu}(e_t,k) = \begin{cases} e_t & \text{if } \mid e_t \mid\leq k \\ k & \text{otherwise} \end{cases} \tag{4}$$

$$\psi_{bs}(e_t,k) = \begin{cases} e_t\left[1-(\frac{e_t}{k})^2\right]^2 & \text{if } \mid e_t \mid\leq k \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Both these functions depend upon only one scalar parameter k. Moreover function (5) has the advantage to give a smooth combination of the Shewhart's and EWMA charts.
To understand the differences between these two charts and the AEWMA chart, we rewrite the control statistic (3) in the form

$$x_t = x_{t-1} + \phi_s(e_t,\zeta), \quad x_0 = \eta_0, \tag{6}$$

where $\phi_s(e_t,\zeta) = e_t - \lambda\psi_s(e_t,\zeta), \quad s = hu, bs.$
The Shewhart's chart and the EWMA chart are included in the control statistic (6), as a special case, for $\phi(e_t,\cdot) = e_t$ and $\phi(e_t,\cdot) = (1-\lambda)e_t$, respectively.
Figure 1 illustrates the ϕ-functions of the following schemes: (i) an AEWMA scheme, based on $\phi_{hu}(e_t,k) = e_t - \lambda\psi_{hu}(e_t,k)$, with $\lambda = 0.9$ and $k = 5$, (ii) an AEWMA scheme, based on $\phi_{bs}(e_t,k) = e_t - \lambda\psi_{bs}(e_t,k)$, with $\lambda = 0.9$ and $k = 14$, (iii) a Shewhart's chart, (iv) an EWMA scheme with $\lambda = 0.9$.

The figure shows that, if e_t becomes small, the $\phi_s(\cdot)$ function, with $s = hu, bs$, is close to the $\phi_{ewma}(\cdot)$, while, if e_t is large, it becomes close to the Shewhart one.

3 The ARL of the proposed scheme

Let $\gamma(\eta)$ be the *ARL*, given the AEWMA starts with $x_0 = \eta$, where

$$\gamma(\xi) = E\{N - t \mid x_t = \xi \text{ and } N > t\}$$

. The function $\gamma(\cdot)$ satisfies a Fredholm integral equation of the second kind (Crowder, 1987), given by

$$\gamma(\xi) = \begin{cases} 1 + \int_{-\infty}^{+\infty} \gamma[\xi + \phi(y-\xi,\zeta)]f(y)dy & \text{if } \mid \xi - \eta \mid\leq h_1, \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where $f(\cdot)$ is the probability density function of y_t. The solution of (7) can be obtained by replacing the equation with a system of linear algebraic equations and solve them numerically.
The scheme (6) can be also represented as a continuous state Markov process and we are able to evaluate its run length distribution, by discretizing the infinite state transition probability matrix. The interval between the upper and the lower control limits is divided in an odd number of subintervals, m, of width $\Delta = 2h/m$.
The process is assumed to be in-control if the control statistic x_t is in a 'transient' state i, at time t, that is if $v_i - \frac{\Delta}{2} < x_t \leq v_i + \frac{\Delta}{2}$, where v_i represents the midpoint of the i−th interval, I_i. The process is assumed to be out-of-control whenever x_t falls outside the control limits, that is x_t is in the 'absorbing' state a.
The transition probability matrix, represented in partitioned matrix form, is given by

$$P = \begin{pmatrix} R & (I-R)u \\ 0 & 1 \end{pmatrix},$$

where the submatrix R, $m \times m$, contains the probabilities r_{ij} of going from one transient state i to another j, in one step, I is the identity matrix and u is a column vector of ones. The elements of the vector $(I-R)u$ are the probabilities of jumping to the absorbing state, the first occurrence of which terminates the process. The $(m+1)/2$-th element of the vector $(I-R)^{-1}u$ gives the average time spent by $\{x_t\}$ to reach the absorbing state, that is the *ARL*.
The probabilities r_{ij} are approximated by assuming that the control statistic is equal to v_i whenever it is in state i. The Markov chain representation of the AEWMA procedure is

$$z_t = \begin{cases} v_i & \text{if } [z_{t-1} + \phi(y_t - z_{t-1}, \zeta)] \in I_i, \\ a & \text{otherwise} \end{cases}$$

with $z_0 = \eta_0$. Thus the elements of the R matrix are approximated by $\tilde{r}_{ij} = \Pr\{z_t \in I_j | z_{t-1} = v_i\}$. This yelds

$$\tilde{r}_{ij} = \Pr\{v_i + \phi(y_t - v_i, \zeta) \in I_j\} = \Pr\left\{v_j - v_i - \frac{\Delta}{2} < \phi(y_t - v_i, \zeta) \leq v_j - v_i + \frac{\Delta}{2}\right\} \qquad (8)$$

If $\phi(y, \cdot)$ is monotone in y the transition probabilities are given by

$$\tilde{r}_{ij} = \Pr\left\{\phi^{-1}(v_j - v_i - \Delta/2, \zeta) + v_i < y_t \leq \phi^{-1}(v_j - v_i + \Delta/2, \zeta) + v_i\right\}.$$

For the functions (4) and (5), $\phi_s^{-1}(\cdot, k)$, $s = hu, bs$ is given by

$$\phi_{hu}^{-1}(v, k) = \begin{cases} v - \lambda k & \text{if } v < -(1-\lambda)k \\ v/(1-\lambda) & \text{if } -(1-\lambda)k \leq v \leq (1-\lambda)k \\ v + \lambda k & \text{if } v > (1-\lambda)k \end{cases}.$$

$$\phi_{bs}^{-1}(v, k) = \begin{cases} \phi^*(v) & \text{if } |v| \leq k \\ v & \text{otherwise} \end{cases}$$

where $\phi^*(v)$ is the unique real root, with absolute value less than k, of the polynomial

$$[y - \lambda y\{1 - (y/k)^2\}^2 - v]$$

4 The design of the AEWMA scheme

The scheme (6), at observation t, concludes there is a lack of control, if $| x_t - \eta_0 |> h_1$. The design strategy is to choose suitable values of (λ, h, k) so that the scheme (6) has desidered properties, in terms of ARL.
The usual approach is to find the choice $\theta = (h, \lambda, \zeta)$ which, for a given in control $ARL(\mu, \theta)$, minimizes the out-of-control $ARL(0, \theta)$, for a specified shift in the mean. Rather unfortunately, as for the standard EWMA chart (Lucas and Saccucci, 1987), the values of θ obtained following this approach strongly depends on the specified magnitude of the shift. As a consequence, this design criterion is unable to produce a single AEWMA scheme which performs in a nearly optimal way for all the values of the shift. In particular, the ability of charts optimised for the detection of a small shift to signal the occurrence of a large shift is quite poor if compared to the one of a chart optimised for a large shifts, and vice versa. Thus it is desiderable to design a procedure which is more sensitive at a wider range of shifts.
Chosen two shifts, of size μ_1 and μ_2, we recommend the following strategy

1. choose the the smallest acceptable ARL, B, when the process shift is zero;

2. fix the magnitude of shift in the process, μ_2, that must be detected as soon as possible, then choose the optimal combination $\theta^* = (\lambda^*, k^*, h^*)$ which produces the minimum ARL for the size shift μ_2 and the in control ARL constraint.
   Thus θ^* is solution of the following problem

$$\begin{cases} \min_\theta ARL(\mu_2, \theta) \\ \text{subject to} \\ ARL(0, \theta) = B \end{cases} \tag{9}$$

   An AEWMA chart, with an in-control ARL of B and parameters θ^*, has a smaller out-of control ARL for the specified shift;

3. choose another magnitude of shift μ_1, $\mu_1 < \mu_2$, that must be detected quickly, and a constant $\alpha > 0$, then find the optimal $\tilde{\theta} = (\tilde{\lambda}, \tilde{k}, \tilde{h})$ as solution of

$$\begin{cases} \min_\theta ARL(\mu_1, \theta) \\ \text{subject to} \\ ARL(0, \theta) = B \\ \text{and} \\ ARL(\mu_2, \theta) \leq (1 + \alpha) ARL(\mu_2, \theta) \end{cases} \tag{10}$$

Examples will be given to demostrate that the resulting AEWMA scheme not only is optimal for two particular shifts of interest, but it also performs well in terms of detecting other shifts in the interval (μ_1, μ_2).

5 Results

The main example which is carried through this paper is concerned with the detection of a change in the mean of an independent Gaussian sequence. Without loss of generality we assume that

$\eta_0 = 0$ and $\sigma^2 = 1$.
The ARL calculations are based on a Markov chain approximation with $m = 151$ subintervals.
To illustrate the advantage of the the design procedure (10) we also consider the design scheme (9). The criterion (9) is applied to compare the ARL's performances of the AEWMA schemes, $AEW_\mu^{(s)}$, $s = hu, bs$, based on the weighting functions (4) and with (5) respectively, to that of the standard EWMA schemes, EW_μ. For an in-control ARL equal to 500 and three values of the shifts in the process that are to be detected quickly, $\mu = 0.5, 3$ and 5, Table 1 gives the EWMA's and AEWMA's parameters that will result in the minimum ARL's, for each of the specified size shifts.
Since the true value of μ is unknown, is important to evaluate whether these schemes provide sufficient protection against other shifts. Thus the ARL's of the schemes EWMA and AEWMA, optimal for $\mu = 0.5, 3$ and 5, are evaluated for several other values of μ, (Table 2).
Although, for $\mu = 0.5$, the $ARLs$, of the EWMA and the AEWMA schemes, are very similar, this table shows that as the specified shift increases, the ARL's of the AEWMA schemes become smaller than the corresponding ARL's of EWMA schemes, over the entire range of μ. This difference tends to be more pronounced for the weighting function (5).
Then the criterion (10) is applied to evaluate the AEWMA ability to detect size shifts between two specified values μ_1 and μ_2. For $B = 100, 500$, $\alpha = 0.05$ and for each combination of μ_1 and μ_2, Tables 3, 4a and 4b list the optimal parameters $\tilde{\theta} = (\tilde{\lambda}, \tilde{h}, \tilde{k})$ and the corresponding ARL's of the AEWMA schemes, $AEW_{\mu_1\mu_2}^{(s)}$, $\{(\mu_1, \mu_2)\} = \{(0.25, 3), (0.5, 3), (1, 3), \ldots, (0.25, 6), (0.5, 6), (1, 6)\}$, $s = hu, bs$.
Examination of Tables 3a and 3b illustrates that $\tilde{h}$ increases and $\tilde{\lambda}$ decreases as the specified shift μ_1 increases. On the other hand the optimal $\tilde{k}$ increases as the specified shift μ_2 increases.
We also note that both the design procedures, (9) and (10), lead to values of $\tilde{\lambda}$ that are sligthly larger than the optimal values tabled by Lucas and Saccucci (1990). Therefore, when the process is in control these optimal choices of λ are such that x_t has smoother trajectory. Note that Tables 1,3 and 5 give a threshold, h_1, equal to $h \left[\frac{1-\lambda}{1+\lambda}\right]^{1/2} \sigma_y$, where h is obtained as solution of (9) or (10).

Our results show that the ARL's of EWMA schemes, EW_3 and EW_5 are much larger than the ARL's of the six AEWMA schemes $AEW_{\mu_1 3}$ and $AEW_{\mu_1 5}$, with $\mu_1 = 0.25, 0.5$ and 1.

Table 1 Optimal parameters of EWMA and AEWMA charts

Schemes	Parameters, $B = 500$ - criterion(9)	
$EW_{0.5}$	$(\lambda^*, h_1^*) =$	$(0.952866, 1.442661)$
$AEW_{0.5}^{(hu)}$	$(\lambda^*, k^*, h_1^*) =$	$(0.95278, 4.30198, 0.49132)$
$AEW_{0.5}^{(bs)}$	$(\lambda^*, k^*, h_1^*) =$	$(0.96577, 21.03405, 0.59403)$
EW_3	$(\lambda^*, h_1^*) =$	$(0.32466, 1.69000)$
$AEW_3^{(hu)}$	$(\lambda^*, k^*, h_1^*) =$	$(0.99993, 3.05641, 1.10557)$
$AEW_3^{(bs)}$	$(\lambda^*, k^*, h_1^*) =$	$(0.66438, 7.71461, 0.69638)$
EW_5	$(\lambda^*, h_1^*) =$	$(0.02418, 1.62994)$
$AEW_5^{(hu)}$	$(\lambda^*, k^*, h_1^*) =$	$(0.96707, 1.99929, 1.67116)$
$AEW_5^{(bs)}$	$(\lambda^*, k^*, h_1^*) =$	$(0.88903, 6.09421, 1.21847)$

Table 2 ARL's of AEWMA charts and EWMA charts, $B = 500$

μ	$EW_{0.5}$	$AEW_{0.5}^{(hu)}$	$AEW_{0.5}^{(bs)}$	EW_3	$AEW_3^{(hu)}$	$AEW_3^{(bs)}$	EW_5	$AEW_5^{(hu)}$	$AEW_5^{(bs)}$
0.00	500.00	500.00	500.00	500.00	500.00	500.00	500.00	500.02	500.0
0.25	82.91	83.01	89.36	302.73	374.17	289.59	369.53	371.77	374.88
0.50	28.77	28.79	30.33	123.83	201.58	107.97	195.29	196.84	187.86
0.75	16.50	16.50	16.97	53.28	103.12	42.84	98.49	96.71	80.81
1.00	11.51	11.51	11.50	25.79	54.59	20.06	51.64	45.89	35.44
1.50	7.22	7.21	6.69	8.56	17.89	7.10	16.80	13.74	10.34
2.00	5.31	5.28	4.49	4.15	7.26	3.80	6.86	6.17	4.88
2.50	4.24	4.15	3.24	2.58	3.60	2.50	3.46	3.33	2.89
3.00	3.56	3.37	2.45	1.86	2.15	1.84	2.10	2.09	1.96
3.50	3.09	2.74	1.94	1.47	1.52	1.46	1.50	1.50	1.47
4.00	2.75	2.21	1.58	1.24	1.22	1.23	1.22	1.22	1.22
5.00	2.19	1.41	1.17	1.04	1.03	1.04	1.03	1.03	1.03
6.00	2.01	1.08	1.02	1.00	1.00	1.00	1.00	1.00	1.00

6 Comparisons

In order to discuss the properties of the AEWMA procedure respect to the more commonly used charts, we compare the ARL's profiles of the AEWMA chart to the ARL's profiles of standard EWMA charts (1), CUSUM charts, CSEWMA charts and Shewhart charts with supplementary rules.

The parameters of these schemes are chosen so that the in-control ARL would match that of the Shewhart chart with two supplementary runs rules, C_{123}, given by Champ and Woodall (1987). For an in-control ARL of 132.89, the criterion (9) is applied to design EWMA, CUSUM and CSEWMA charts optimal for shifts of size $\mu = 0.5, 3$ and 5. Using the criterion (10), AEWMA schemes are designed to detect any combination of three fixed values of μ_1 and four of μ_2. Thus we tabulate the ARL's of the following schemes

1. the AEWMA charts, $AEW_{\mu_1\mu_2}^{(s)}$, $s = hu, bs$, with weighting functions given by (4) and (5), respectively and with $\mu_1 = 0.25, 0.5, 1$, $\mu_2 = 3, 4, 5, 6$ and . The optimal parameters and the ARL's of all these AEWMA schemes are listed in Appendix (Tables 5, 6);

2. three standard EWMA charts (1): $EW_{0.5}$ with ($h_1^* = 1.2080257, \lambda^* = 0.93487$), EW_3 with ($h_1^* = 1.51642, \lambda^* = 0.23334$) and EW_5 with ($h_1^* = 1.51968, \lambda^* = 0.06356$).

3. three CUSUM charts $CUSUM_{0.5}$, $CUSUM_3$ and $CUSUM_5$.
The CUSUM chart signals if $S_t > h^*$ or $T_t < -h^*$, where $S_t = \max(0, S_{t-1} + y_t - \delta^*)$ and $T_t = \min(0, T_{t-1} + y_t + \delta^*)$. The optimal parameters of these three CUSUM schemes are ($h^* = 5.90753, \delta^* = 0.26392$), ($h^* = 1.38348, \delta^* = 1.38574$) and ($h^* = 1.159586, \delta^* = 1.56518$), respectively.

4. three CSEWMA schemes $CSEWMA_{0.5}$, $CSEWMA_3$ and $CSEWMA_5$.
The CSEWMA chart signals if (1) is greater than h_1^* or if the current observation is greater than c. Since Lucas and Saccucci (1987) suggest choosing Shewhart limits larger than

that would be used for a standard Shewhart limits to prevent a large reduction in the in-control ARL, c was chosen equal to 4. Thus the parameters of these three schemes are $(h_1^* = 1.21931, \lambda^* = 0.93338, c = 4)$, $(h_1^* = 1.523045, \lambda^* = 0.235218, c = 4)$ and $(h_1^* = 1.52620, \lambda^* = 0.043469, c = 4)$.

5. one Shewhart chart with supplementary rules, C_{123}.
   This scheme was introduced by Champ and Woodall (1987), to improve the Shewhart chart's sentivity to small shifts in the mean of the process. The procedure is given by a combination of rules of the form $T(q, r, a, b)$ which signals if q out of the last r values, of the control statistic, fall in the interval (a, b), $a < b$.
   In this article we consider the combination $C_{123} = C_1 \cup C_2 \cup C_3$ where

$$C_1 = \{T(1, 1, -\infty, 3), T(1, 1, 3, \infty)\},\ C_2 = \{T(2, 3, -3, -2), T(2, 3, 2, 3)\}$$

   and $C_3 = \{T(4, 5, -3, -1), T(4, 5, 1, 3)\}$.
   This chart signals if the present mean is larger than 3 (or less than -3), or two of the last three means are between -3 and -2 (or between 2 and 3), or four of the last five means are between -3 and -1, (or between 1 and 3).

Table 7 (see Appendix) gives the ARL's of all these procedures. Figures 2, 3 and 4 illustrate the ARL's for the EW_μ, CUSUM$_\mu$ and CSEWMA$_\mu$, optimal for $\mu = 0.5$ and 5, and the ARL of a $AEW_{\mu_1\mu_2}^{(hu)}$, optimal for $(\mu_1, \mu_2) = (0.5, 5)$.
Figure 5 illustrates the ARL of C_{123} chart and those of the $AEW_{\mu_1\mu_2}^{(hu)}$ schemes, optimal for $(\mu_1, \mu_2) = (0.5, 3), (0.5, 5)$ e (0.5,6), respectively.
Lucas and Saccucci (1990) show that in the worst-case situation the EWMA requires a few observations to overcome its initial inertia while CSEWMA shows a better performance in terms of ARL.
We compare the worst-case ARL's of AEWMA, EWMA, CSEWMA, and CUSUM charts. For the first three schemes the worst-case ARL equal to the maximum of the vector $(I - R)^{-1}u$. While for the CUSUM chart the worst-case ARL is equal to the $(m + 1)/2$-th element. Figures 6, 7 e 8 compare the worst-case ARL's of $WCAEW_{\mu_1\mu_2}$, $WCEW_\mu$, $WCSEW_\mu$ and CUSUM schemes.
All the results show a more desiderable performance of the AEWMA schemes over a wide range of shift values.

7 Summary and conclusions

In particular for small values of the shift these comparisons indicate there is a little difference between the ARL properties of AEWMA, EWMA and CUSUM, but for larger shifts the AEWMA charts is much more responsive than EWMA and CUSUM. So the AEWMA does not loose the advantage of CUSUM and AEWMA for larger shifts.
This better performance of AEWMA is shown even respect to several enhancements for EWMA's, as the CSEWMA and the Shewhart with supplementary rule, implemented when large and small shifts are to be detected. Finally our comparisons of the worst-case ARL's of the EWMA, CSEWMA and AEWMA charts (Figures 6 and 7) illustrate that the AEWMA has the advantage to combine the best features of the EWMA chart and the Shewhart chart while diminishing the

Table 3 Optimal parameters of AEWMA

	$B = 100$			$B = 500$		
	Huber function					
(μ_1, μ_2)	$\tilde{\lambda}$	$\tilde{k}$	$\tilde{h}_1$	$\tilde{\lambda}$	$\tilde{k}$	$\tilde{h}_1$
(0.25,3)	0.94591	2.42133	0.45452	0.97888	2.96697	0.53300
(0.50,3)	0.90775	2.38401	0.46996	0.93031	2.98113	0.54937
(1.00,3)	0.77329	2.44727	0.47666	0.84054	3.30842	0.55723
(0.25,4)	0.92344	2.54514	0.43336	0.98819	3.02312	0.49743
(0.50,4)	0.90725	2.51833	0.44480	0.96260	2.87963	0.56812
(1.00,4)	0.82489	2.44159	0.47163	0.87570	2.68917	0.60254
(0.25,5)	0.94189	3.15888	0.37291	0.98344	3.39274	0.43445
(0.50,5)	0.94911	3.23232	0.36141	0.94809	3.25297	0.51271
(1.00,5)	0.83037	3.13517	0.44054	0.87162	3.12009	0.55583
(0.25,6)	0.97316	3.45122	0.30570	0.98267	4.19948	0.42033
(0.50,6)	0.95417	3.20660	0.35394	0.95436	4.10324	0.48950
(1.00,6)	0.80762	2.62126	0.46066	0.86607	4.01578	0.54555
	Bisquare function					
(μ_1, μ_2)	$\tilde{\lambda}$	$\tilde{k}$	$\tilde{h}_1$	$\tilde{\lambda}$	$\tilde{k}$	$\tilde{h}_1$
(0.25,3)	0.97781	10.24508	0.81879	0.94720	10.13491	0.89432
(0.50,3)	0.95752	9.91783	0.68692	0.93376	10.21771	0.83014
(1.00,3)	0.93960	9.72418	0.63763	0.86544	10.60466	0.69487
(0.25,4)	0.99386	13.69501	0.92907	0.97060	11.42884	0.96034
(0.50,4)	0.96526	11.58546	0.62591	0.91978	10.62397	0.76645
(1.00,4)	0.91776	10.47411	0.56835	0.96524	11.30674	0.64545
(0.25,5)	0.98780	16.40169	0.58174	0.98327	16.48632	0.80171
(0.50,5)	0.97631	22.46813	0.40497	0.96243	15.05259	0.70581
(1.00,5)	0.93899	10.22665	0.61063	0.89655	13.54159	0.64394
(0.25,6)	0.99301	14.74914	0.79792	0.98773	31.06528	0.53521
(0.50,6)	0.97657	23.69023	0.39362	0.96108	25.38781	0.55254
(1.00,6)	0.93899	10.22665	0.61063	0.88145	21.56960	0.57456

inertia problem still present in the CSEWMA.

Thus the AEWMA chart not only seems optimal for the two particular shifts of interest, it also performs well in terms of detecting any size shift in the interval $(0, \max(\mu_1, \mu_2))$.

Table 4a AEWMA's ARL's, $B = 100$, criterion (10)

(μ_1, μ_2)	w.f.	shift											
		0.25	0.50	0.75	1.00	1.50	2.00	2.50	3.00	3.50	4.00	5.00	6.00
(0.25,3)	hu	48.09	21.32	12.60	8.66	4.99	3.20	2.17	1.58	1.27	1.11	1.01	1.00
	bs	49.85	21.89	12.52	8.32	4.56	2.90	2.03	1.55	1.28	1.12	1.01	1.00
(0.50,3)	hu	49.46	20.99	11.88	7.99	4.60	3.03	2.12	1.58	1.28	1.12	1.01	1.00
	bs	49.97	21.45	12.05	7.97	4.40	2.84	2.02	1.55	1.28	1.13	1.02	1.00
(1.0,3)	hu	53.77	22.39	11.72	7.42	4.08	2.75	2.03	1.58	1.31	1.14	1.02	1.00
	bs	50.47	21.40	11.85	7.79	4.31	2.81	2.01	1.55	1.28	1.13	1.02	1.00
(0.25,4)	hu	45.05	19.41	11.37	7.85	4.66	3.14	2.22	1.66	1.32	1.14	1.02	1.00
	bs	46.59	21.55	12.92	8.78	4.85	3.06	2.13	1.61	1.31	1.14	1.02	1.00
(0.50,4)	hu	46.32	19.62	11.28	7.70	4.54	3.06	2.19	1.65	1.32	1.14	1.02	1.00
	bs	46.48	20.18	11.69	7.90	4.46	2.91	2.08	1.60	1.31	1.14	1.02	1.00
(1.0,4)	hu	51.41	21.26	11.41	7.41	4.19	2.83	2.08	1.60	1.31	1.14	1.02	1.00
	bs	49.25	20.58	11.38	7.50	4.22	2.80	2.03	1.58	1.31	1.14	1.02	1.00
(0.25,5)	hu	39.73	17.59	10.78	7.71	4.88	3.51	2.66	2.05	1.61	1.31	1.05	1.00
	bs	43.32	20.02	12.20	8.45	4.84	3.14	2.22	1.69	1.37	1.18	1.03	1.00
(0.50,5)	hu	39.27	17.59	10.89	7.83	4.99	3.61	2.74	2.12	1.66	1.34	1.06	1.00
	bs	39.78	18.28	11.35	8.06	4.89	3.35	2.46	1.91	1.54	1.30	1.06	1.01
(1.0,5)	hu	46.48	19.02	10.50	7.00	4.15	2.94	2.27	1.82	1.51	1.29	1.06	1.01
	bs	49.18	20.81	11.62	7.69	4.30	2.82	2.03	1.57	1.29	1.13	1.02	1.00
(0.25,6)	hu	38.33	18.04	11.55	8.48	5.52	4.03	3.07	2.35	1.82	1.44	1.08	1.01
	bs	45.55	21.15	12.77	8.74	4.87	3.10	2.16	1.64	1.33	1.15	1.02	1.00
(0.50,6)	hu	39.12	17.68	11.01	7.95	5.07	3.65	2.76	2.11	1.64	1.33	1.05	1.00
	bs	39.56	18.23	11.35	8.09	4.93	3.39	2.50	1.94	1.57	1.32	1.07	1.01
(1.0,6)	hu	50.11	20.59	11.07	7.20	4.10	2.82	2.10	1.65	1.35	1.17	1.02	1.00
	bs	49.18	20.81	11.62	7.69	4.30	2.82	2.03	1.57	1.29	1.13	1.02	1.00

Table 4b AEWMA's ARL's, $B = 500$, criterion (10)

		shift											
(μ_1, μ_2)	w.f.	0.25	0.50	0.75	1.00	1.50	2.00	2.50	3.00	3.50	4.00	5.00	6.00
(0.25,3)	hu	100.77	37.66	22.22	15.43	8.94	5.57	3.49	2.26	1.60	1.27	1.04	1.00
	bs	177.42	49.99	21.91	12.82	6.37	3.89	2.64	1.93	1.51	1.26	1.04	1.00
(0.50,3)	hu	110.36	33.06	17.23	11.39	6.62	4.46	3.15	2.26	1.68	1.33	1.05	1.00
	bs	176.18	49.39	21.45	12.47	6.21	3.83	2.62	1.93	1.51	1.26	1.05	1.00
(1.0,3)	hu	137.48	38.07	17.23	10.39	5.64	3.85	2.90	2.26	1.80	1.46	1.11	1.01
	bs	182.32	51.45	21.36	11.90	5.78	3.63	2.55	1.93	1.54	1.29	1.05	1.00
(0.25,4)	hu	99.13	41.25	25.23	17.72	10.17	6.11	3.66	2.29	1.60	1.26	1.04	1.00
	bs	147.63	42.98	20.67	12.77	6.58	4.03	2.72	1.98	1.54	1.28	1.05	1.00
(0.50,4)	hu	113.37	36.62	20.25	13.71	7.88	5.02	3.27	2.20	1.59	1.26	1.04	1.00
	bs	149.77	43.15	20.49	12.59	6.49	3.99	2.70	1.97	1.54	1.28	1.05	1.00
(1.0,4)	hu	167.84	44.85	19.63	11.69	6.17	4.00	2.80	2.03	1.55	1.26	1.04	1.00
	bs	168.86	47.09	20.53	11.98	6.03	3.77	2.61	1.94	1.53	1.28	1.05	1.00
(0.25,5)	hu	78.20	32.60	20.23	14.57	9.10	6.22	4.28	2.90	1.99	1.47	1.08	1.01
	bs	100.32	34.44	19.09	12.69	6.97	4.39	3.00	2.19	1.70	1.39	1.08	1.01
(0.50,5)	hu	90.77	30.11	16.82	11.54	6.98	4.85	3.51	2.55	1.88	1.44	1.08	1.01
	bs	108.80	33.76	17.68	11.53	8.34	6.38	5.06	3.42	2.47	1.89	1.05	1.01
(1.0,5)	hu	128.55	35.86	16.86	10.47	5.83	3.99	2.95	2.26	1.76	1.41	1.08	1.01
	bs	139.75	38.97	17.82	10.79	5.72	3.74	2.70	2.06	1.65	1.37	1.08	1.01
(0.25,6)	hu	74.57	31.22	19.49	14.18	9.19	6.77	5.25	4.09	3.10	2.29	1.33	1.05
	bs	78.74	32.09	19.46	13.65	8.08	5.39	3.84	2.86	2.23	1.79	1.27	1.05
(0.50,6)	hu	82.48	28.81	16.58	11.59	7.26	5.30	4.13	3.30	2.61	2.05	1.32	1.05
	bs	86.27	29.49	16.63	11.38	6.77	4.65	3.44	2.66	2.13	1.75	1.27	1.05
(1.0,6)	hu	122.07	34.42	16.34	10.21	5.77	4.05	3.14	2.54	2.09	1.74	1.27	1.05
	bs	123.75	34.89	16.53	10.28	5.70	3.90	2.94	2.34	1.93	1.63	1.22	1.04

Appendix

Table 5 Optimal parameters of AEWMA, $B = 132.89$

(μ_1, μ_2)	Huber function			Bisquare function		
	$\tilde{\lambda}$	$\tilde{k}$	$\tilde{h}_1$	$\tilde{\lambda}$	$\tilde{k}$	$\tilde{h}_1$
(0.25,3)	0.90408	2.26725	0.57188	0.90669	6.88720	0.85870
(0.50,3)	0.99839	2.60778	0.51449	0.99505	10.64849	1.64918
(1.0,3)	0.99839	2.60778	0.53814	0.92247	9.78052	0.66211
(0.25,4)	0.95261	2.65061	0.44766	0.99644	13.26998	1.40053
(0.50,4)	0.93667	2.61220	0.46649	0.95737	11.13294	0.68415
(1.0,4)	0.83317	2.47293	0.51198	0.96615	11.33841	0.71588
(0.25,5)	0.97261	3.29517	0.34979	0.99124	25.39122	0.45733
(0.50,5)	0.96263	3.26627	0.37651	0.99124	16.09627	0.74723
(1.0,5)	0.86264	3.14466	0.46738	0.97658	9.54701	0.97998
(0.25,6)	0.97771	4.24204	0.32458	0.98169	25.02540	0.29567
(0.50,6)	0.93568	4.11819	0.41300	0.93223	6.59281	0.70846
(1.0,6)	0.82739	3.47754	0.47612	0.97658	9.54700	0.68368

Table 6 AEWMA's *ARL*s

(μ_1, μ_2)	w.f.	shift												
		0.00	0.25	0.50	0.75	1.00	1.50	2.00	2.50	3.00	3.50	4.00	5.00	6.00
(0.25,3)	hu	132.89	72.18	28.50	14.82	9.47	5.11	3.22	2.18	1.60	1.28	1.12	1.01	1.00
	bs	132.90	83.87	36.45	17.67	10.26	4.92	2.98	2.04	1.54	1.26	1.11	1.01	1.00
(0.50,3)	hu	132.89	91.86	51.28	30.74	19.25	8.16	3.99	2.32	1.60	1.26	1.10	1.01	1.00
	bs	132.89	62.24	26.52	14.97	9.81	5.18	3.17	2.16	1.62	1.31	1.14	1.02	1.00
(1.0,3)	hu	132.93	73.55	29.39	14.38	8.63	4.46	2.89	2.08	1.60	1.30	1.14	1.02	1.00
	bs	133.77	63.17	24.76	13.07	8.37	4.55	2.95	2.11	1.62	1.32	1.15	1.02	1.00
(0.25,4)	hu	133.07	51.40	21.88	13.06	9.11	5.41	3.56	2.45	1.76	1.36	1.16	1.02	1.00
	bs	132.89	56.98	25.11	14.71	9.85	5.31	3.28	2.25	1.68	1.35	1.16	1.02	1.00
(0.50,4)	hu	132.93	53.06	21.83	12.70	8.74	5.16	3.43	2.39	1.74	1.36	1.16	1.02	1.00
	bs	132.89	57.45	23.12	12.85	8.51	4.72	3.06	2.17	1.65	1.34	1.16	1.02	1.00
(1.0,4)	hu	132.89	63.65	24.54	12.70	8.09	4.50	3.01	2.19	1.67	1.34	1.16	1.02	1.00
	bs	133.29	57.12	23.23	13.04	8.67	4.80	3.09	2.18	1.65	1.34	1.16	1.02	1.00
(0.25,5)	hu	132.89	44.29	20.13	12.73	9.27	5.94	4.24	3.13	2.31	1.74	1.37	1.06	1.01
	bs	132.89	47.30	21.91	13.65	9.62	5.64	3.72	2.64	1.99	1.58	1.31	1.06	1.01
(0.50,5)	hu	132.89	44.72	19.74	12.28	8.87	5.65	4.04	3.00	2.25	1.71	1.36	1.06	1.01
	bs	132.89	51.94	23.10	13.85	9.48	5.30	3.37	2.34	1.76	1.41	1.20	1.03	1.00
(1.0,5)	hu	132.90	53.61	20.71	11.37	7.61	4.53	3.21	2.45	1.93	1.56	1.31	1.06	1.01
	bs	134.46	65.04	26.54	14.42	9.31	4.94	3.08	2.12	1.60	1.30	1.14	1.02	1.00
(0.25,6)	hu	133.01	43.63	20.16	12.91	9.50	6.26	4.70	3.74	3.05	2.49	2.01	1.30	1.05
	bs	132.89	45.63	20.55	12.74	9.04	5.46	3.71	2.70	2.07	1.66	1.38	1.09	1.01
(0.50,6)	hu	132.89	46.02	19.19	11.51	8.17	5.20	3.85	3.07	2.52	2.08	1.72	1.25	1.04
	bs	132.89	87.90	39.68	19.23	11.03	5.16	3.06	2.06	1.54	1.26	1.11	1.01	1.00
(1.0,6)	hu	132.89	56.26	21.50	11.48	7.52	4.39	3.12	2.43	1.98	1.66	1.42	1.11	1.01
	bs	134.46	65.04	26.54	14.42	9.31	4.94	3.08	2.12	1.60	1.30	1.14	1.02	1.00

Figure 1: Comparisons of the $\phi(\cdot)$ functions of Shewhart, EWMA and AEWMA schemes

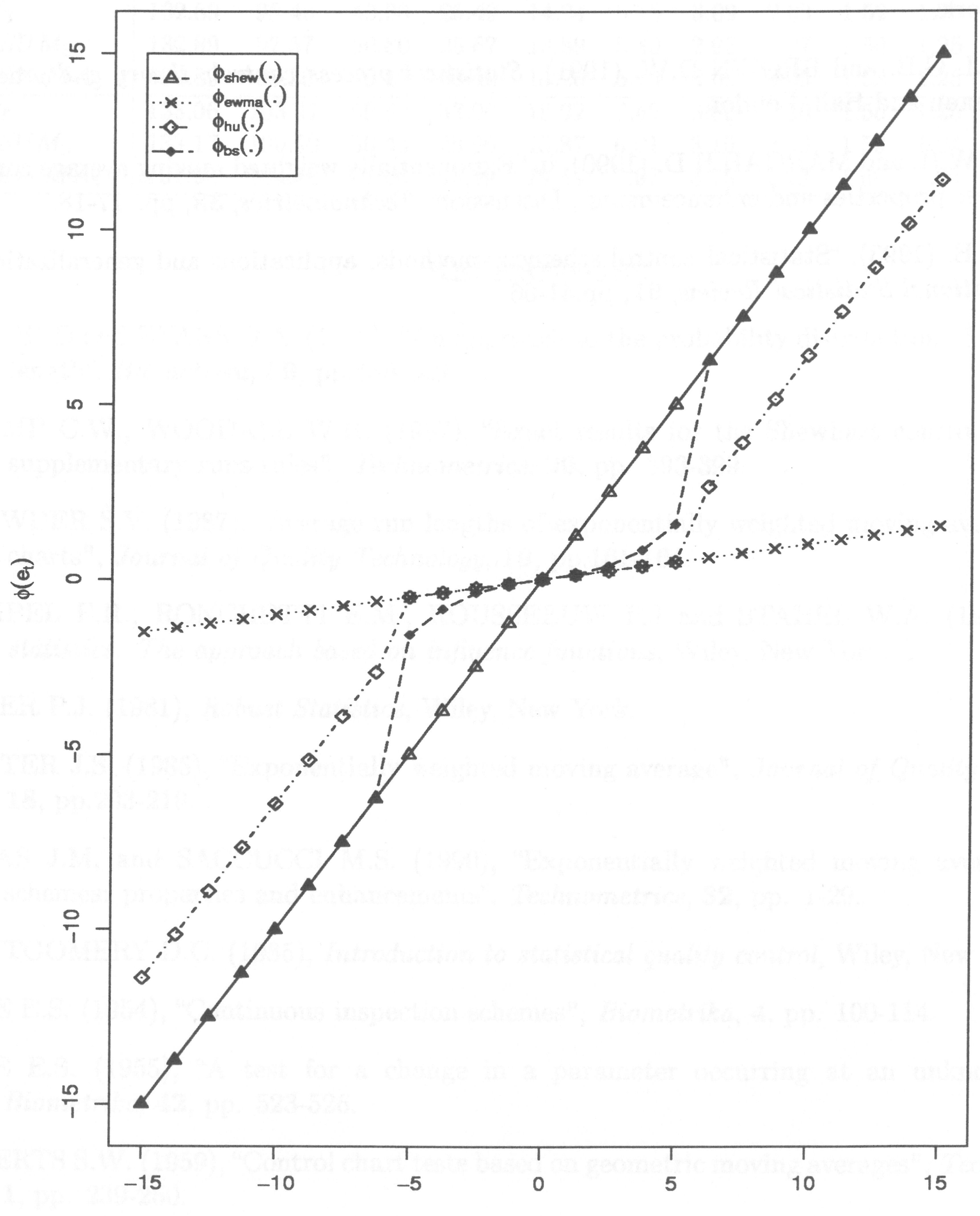

Figure 2: ARLs of the $AEW_{0.55}^{(hu)}$, $EW_{0.5}$ and EW_{5} schemes

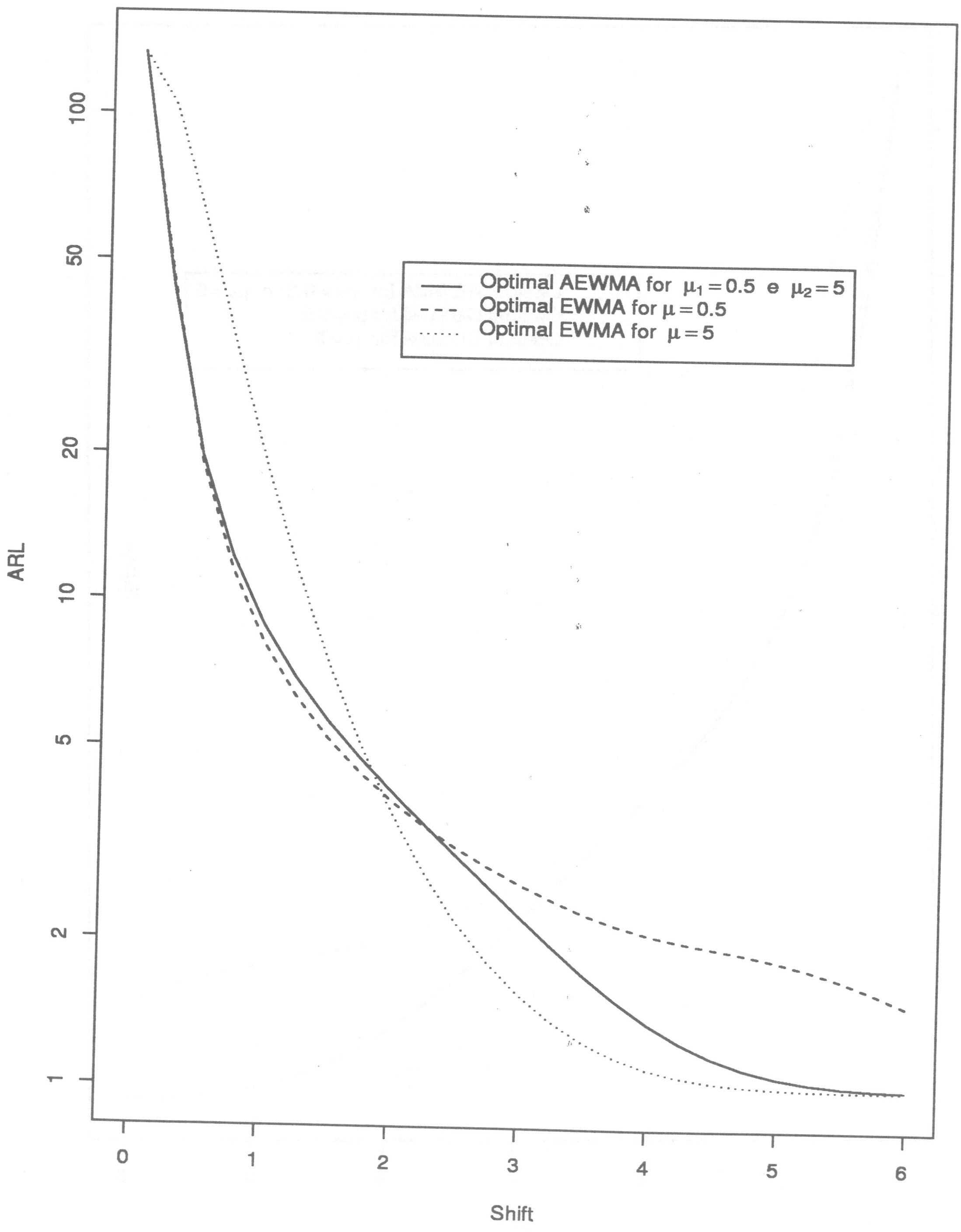

Figure 3: ARLs of the $AEW^{(hu)}_{0.55}$, $CUSUM_{0.5}$ and $CUSUM_5$ schemes

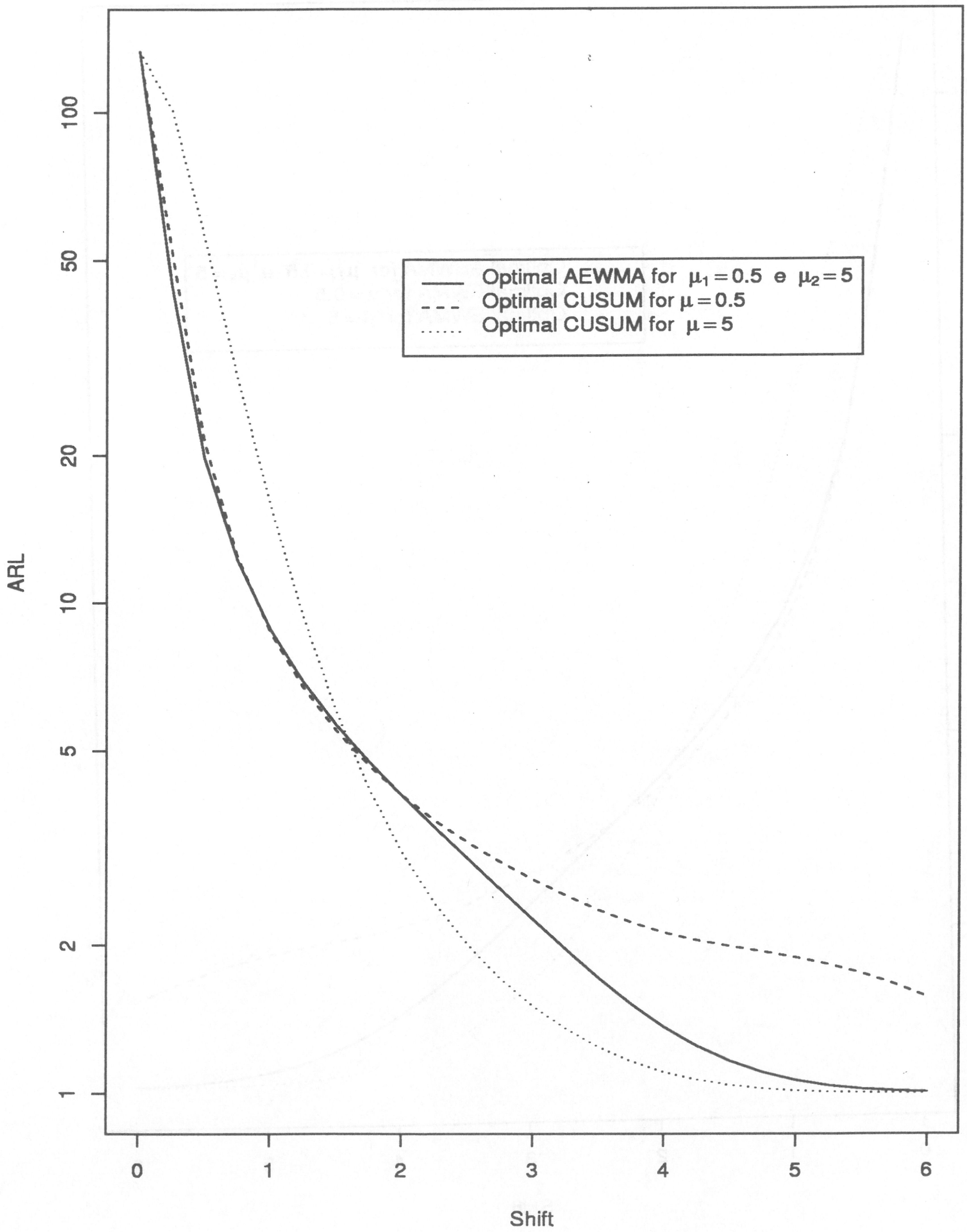

Figure 4: ARLs of the $AEW^{(hu)}_{0.55}$, $CSEWMA_{0.5}$ and $CSEWMA_5$ schemes

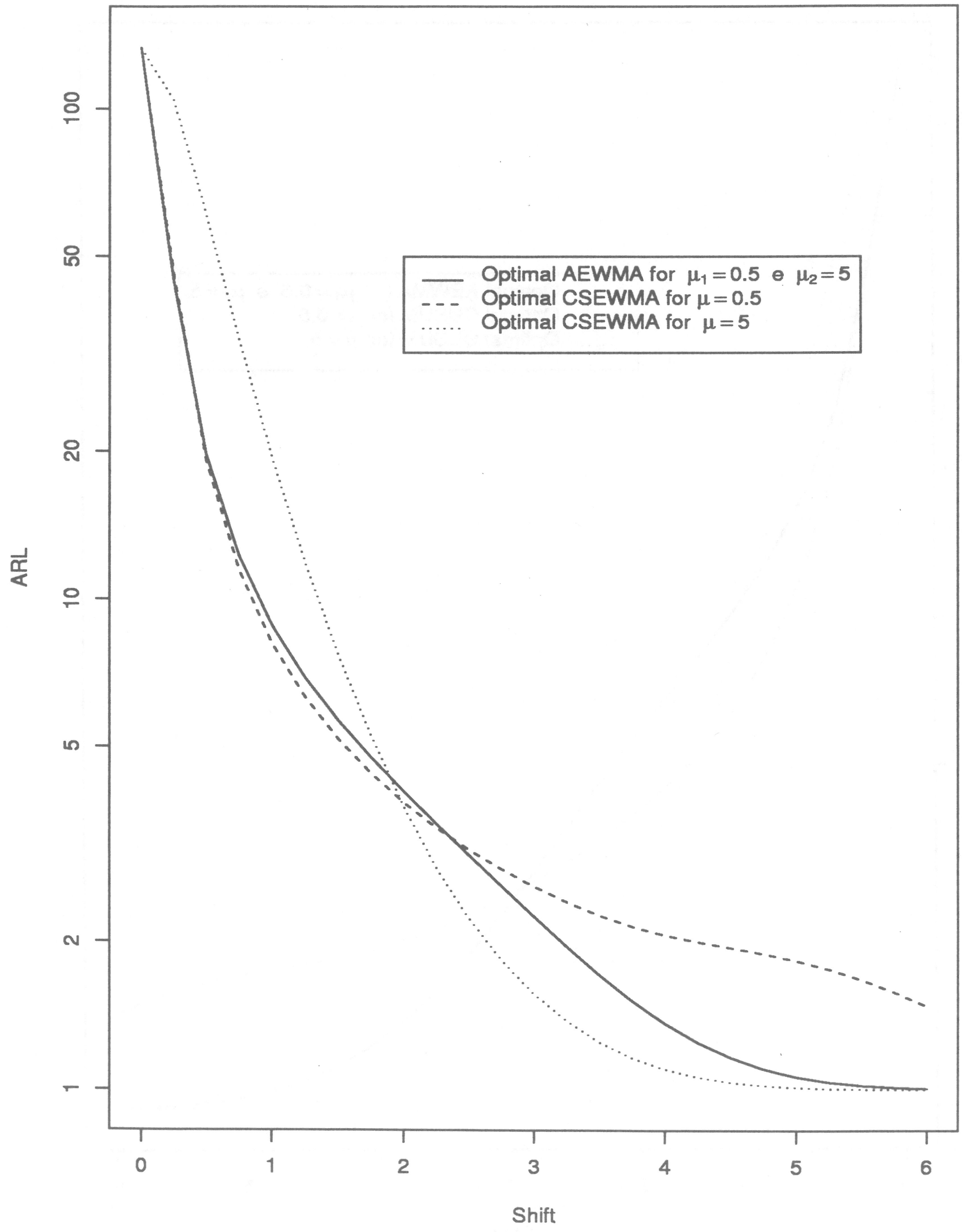

Figure 5: ARLs of the $AEW_{0.53}^{(hu)}$, $AEW_{0.56}^{(hu)}$ and C_{123} schemes

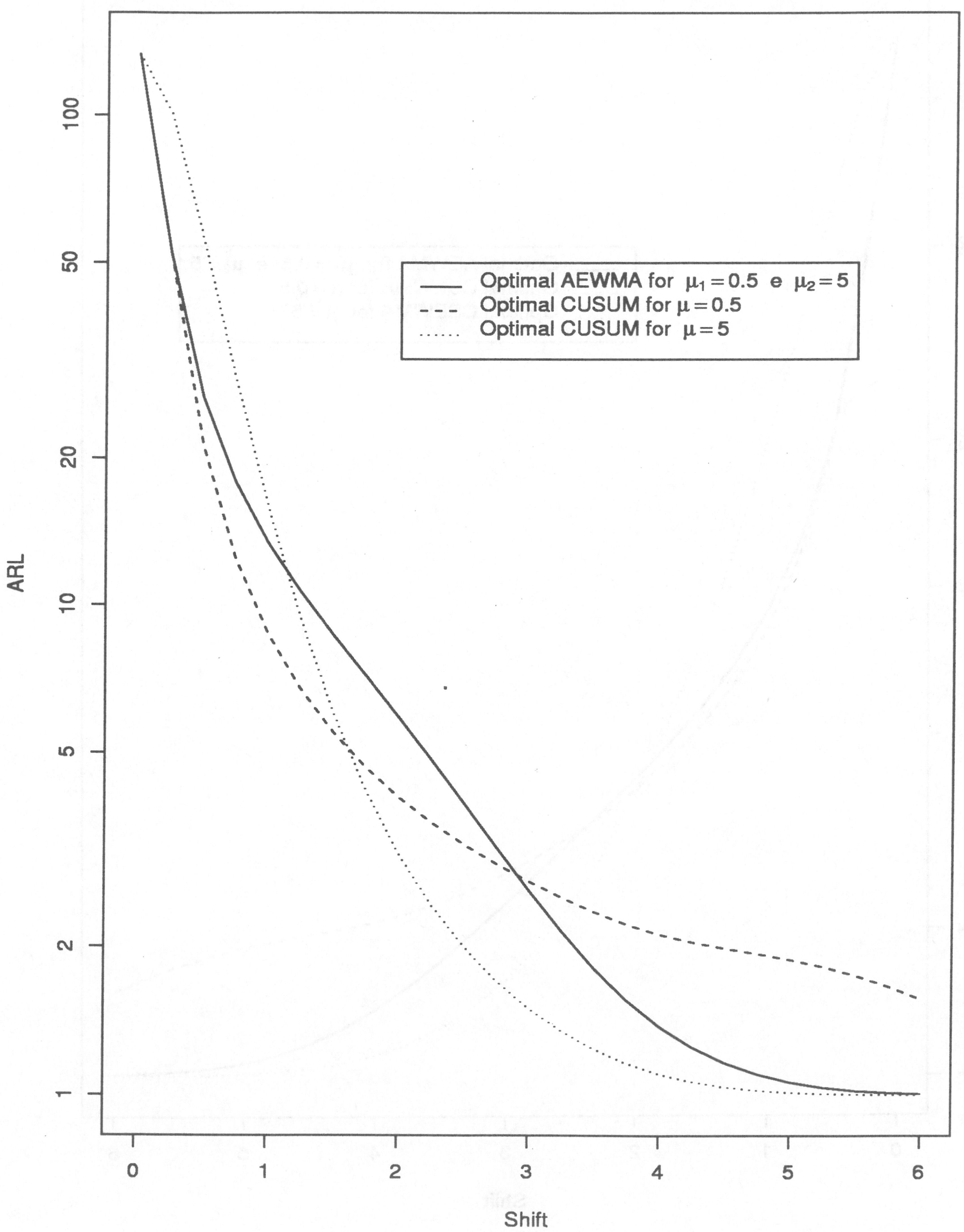

Figure 6: Worst-case ARLs of the $WCAEW_{0.55}^{(hu)}$, $WCEW_{0.5}$ and $WCEW_5$ schemes

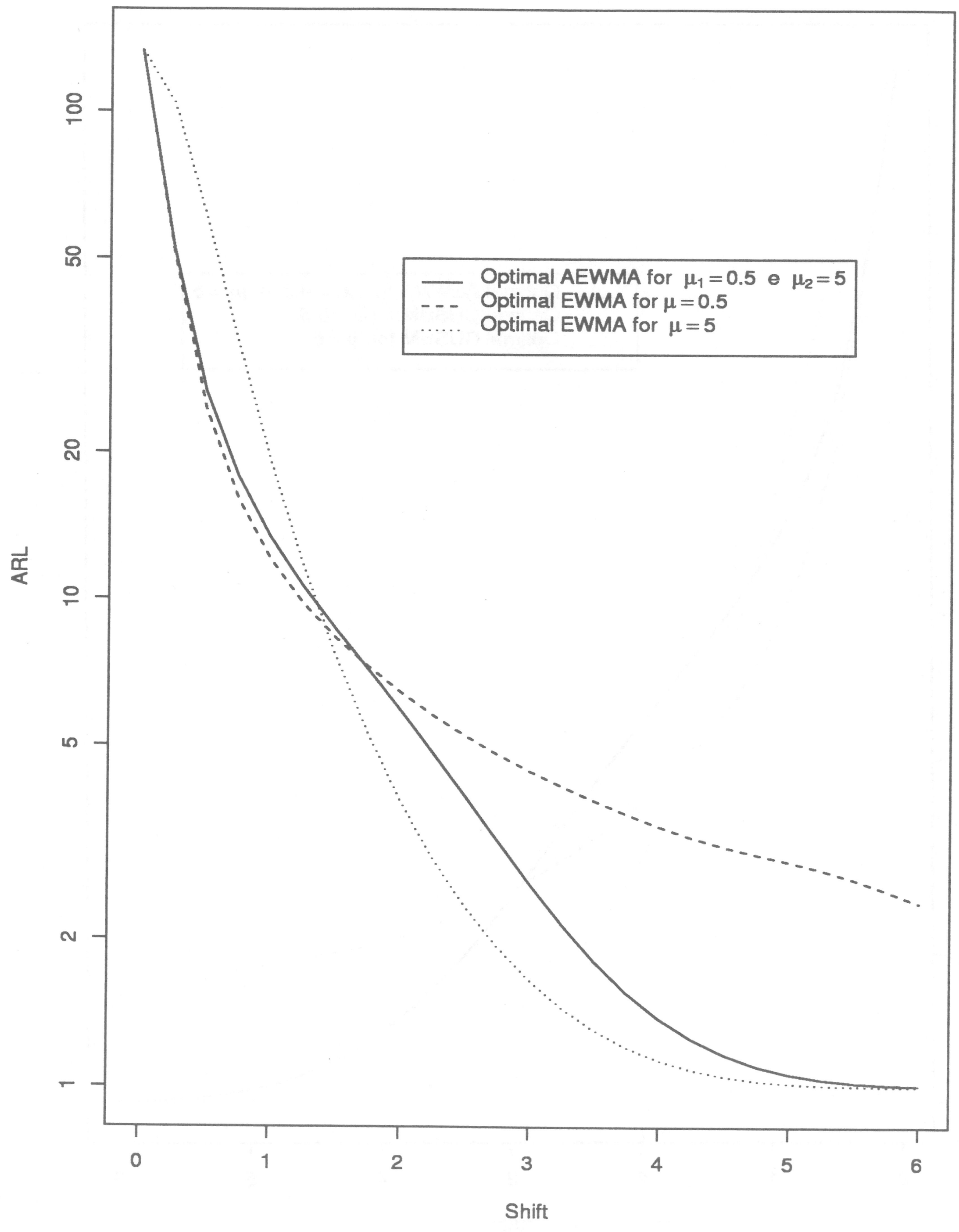

Figure 7: Worst-case ARLs of the $WCAEW_{0.55}^{(hu)}$, $CUSUM_{0.5}$ and $CUSUM_5$ schemes

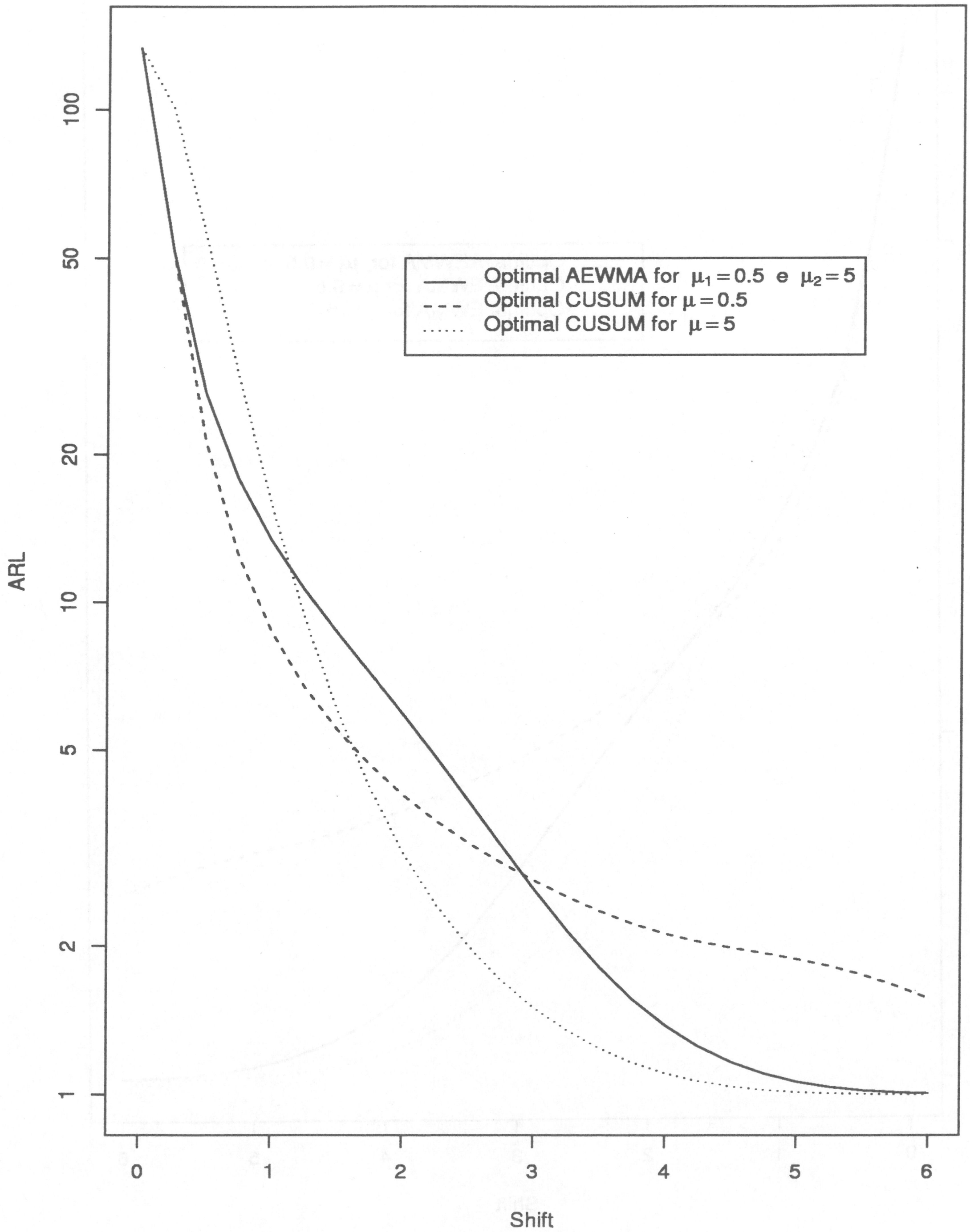

Figure 8: Worst-case ARLs of the $WCAEW_{0.55}^{(hu)}$, $WCCSEW_{0.5}$ and $WCCSEW_5$ schemes