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Keywords: SPC; Control Charts; Average Run Length; Shewhart Control Chart; Exponentially Weighted Moving Average Control Chart; Integral Equation.

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1 Introduction

Shewhart and Exponentially Weighted Moving Average (EWMA) control charts are simple and effective graphical procedures for monitoring the quality of manufactured products (see Montgomery, 2004). Since the EWMA control chart can be designed to quickly detect either small or large shifts but not both, and Shewhart control chart is efficient for detecting large shifts, it is often suggested to combine these two control schemes when both large and small shifts are to be detected. Resulting procedures, both simple and intuitive for users, permit to achieve a more balanced protection against shifts of different sizes (Lucas and Saccucci, 1990).

Charts performance is often measured in terms of run length, defined as the number of samples that are taken until an alarm is triggered. In order to better characterize the control chart performance both the zero-state and the steady-state run-length distributions are usually evaluated. Zero-state run lengths refer to the run lengths computed under the hypothesis of a mean shift occurring at the beginning

of the monitoring process while the steady-state run lengths are evaluated assuming that a shift may occur after the process has been running for a some period of time and the control statistic has reached a steady-state distribution.

Observe that, the run-length distribution of the Shewhart control chart can be computed analytically while the run-length distribution of the EWMA chart has to be approximated numerically since its closed form expression cannot be obtained. Two main approaches, both using the Markovian properties of the EWMA control statistic, have been proposed to address this issue. Following the first approach, pioneered by Brook and Evans (1972), the properties of the true continuous state Markov process are evaluated by an approximating finite-state Markov chain. This method has also been used to obtain the run-length properties of the combined Shewhart-EWMA (CSEWMA) control chart (Lucas and Saccucci, 1990). Following the second approach, a continuous characterization of the process is directly used (Crowder, 1987). Then, solutions of the corresponding integral equations and integral recurrence relations are approximated numerically. As Champ and Rigdon (1991) pointed out, this second approach is usually more efficient when an integral equation can be found; however, it has never been applied to the case of the CSEWMA chart.

In this paper, we use the second approach to evaluate the run-length distribution for the CSEWMA chart. The algorithm, based on a modified product Clenshaw-Curtis quadrature (Clenshaw and Curtis, 1960; Imhof, 1963; Sloan and Smith, 1978; Sloan, 1978; Sloan and Smith, 1980; Sloan, 1981; Sloan and Smith, 1982; Piessens, 2000), is able to overcome the problem raising from the discontinuous kernel in the integral equations related to the combined use of the Shewhart and the EWMA control chart. Indeed, when the kernel of an integral equation is not smooth, it is not possible to use the numerical approach, based on Gaussian quadrature, usually applied in the standard EWMA case. In Section 2, we describe how to compute the expected value of the CSEWMA run length. Since in many practical applications it is often useful to have a complete picture of the whole run-length distribution, we discuss in Section 3 how to evaluate the entire run length distribution of the CSEWMA chart. Section 4 is devoted to the computation of the steady-state distribution. Section 5 illustrates how to practically use the proposed procedure to design a suitable CSEWMA scheme. Results point to a good efficacy of the suggested algorithm when compared to other numerical procedures.

2 Average Run Lengths for the CSEWMA control chart

Let x_t , $t = 1, 2, \dots$, be a sequence of continuous, independent and identically distributed (i.i.d) random variables. This sequence may consist of single measurements, sample means or of any other sample statistics having common probability density function (p.d.f), $f(\cdot)$, and distribution function $F(\cdot)$. Here, the p.d.f. $f(\cdot)$ is not restricted to being normal and can represent either the in-control or the out-of-control distribution. Without loss of generality, assume that the in-control value for the process mean and variance are zero and one, respectively.

The CSEWMA control chart signals an alarm if $|x_t| > k$ or $|z_t| > h$ where

$$z_t = (1 - \lambda)z_{t-1} + \lambda x_t, \quad 0 < \lambda \leq 1, \quad z_0 = 0$$

is the EWMA control statistic. The control limit h is usually specified in terms of the asymptotic standard deviation of the EWMA control statistic, that is $h = \tilde{h}\sqrt{\lambda/(2 - \lambda)}$. The constant λ , \tilde{h} and k are design constants chosen to obtain a suitable run length performance.

Let RL be the run length of the scheme and

$$ARL(z) = \begin{cases} E(RL|z_0 = z) & \text{if } |z| \leq h, \\ 0 & \text{otherwise.} \end{cases}$$

the expected value of the CSEWMA run-length conditioned on the initial state of the EWMA statistic being z . In particular, the expected value of the first-passage time from the starting state 0 over the control limit h , i.e. $ARL(0)$, is equal to the zero-state ARL of the control chart. Since z_t is Markovian, a standard argument can be used to show that $ARL(z)$ satisfies a Fredholm integral equation of the second kind

$$ARL(z) = 1 + \int_{-k}^k ARL((1 - \lambda)z + \lambda x)f(x)dx, \quad (1)$$

with $-h \leq z \leq h$. Making the change of variable to $v = ((1 - \lambda)z + \lambda x)/h$, equation (1) can be rewritten as

$$y(u) = 1 + \int_{-1}^1 Q(u, v)y(v)dv \quad (2)$$

where $y(u) = ARL(hu)$ and $Q(u, v) = I(u, v)g(u, v)$ is the kernel of the integral equation (2). Here,

$$I(u, v) = \begin{cases} 1 & \text{if } |v - (1 - \lambda)u| \leq \frac{\lambda k}{h} \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(u, v) = \frac{h}{\lambda} f\left(\frac{h(v - (1 - \lambda)u)}{\lambda}\right).$$

For large values of k , in particular for $k \geq h(2 - \lambda)/\lambda$, the Shewhart control limits are no longer effective and $I(u, v) = 1$, for $(u, v) \in [-1, 1]^2$. In this case, since $f(\cdot)$ and the kernel $Q(\cdot, \cdot)$ are continuous, the Gauss-Legendre quadrature can be used to approximate the integral equation (2) by the system of equations

$$y(u_i) = 1 + \sum_{j=1}^n w_j g(u_i, u_j)y(u_j), \quad i = 1, \dots, n. \quad (3)$$

where u_1, \dots, u_n and w_1, \dots, w_n are the nodes and the corresponding weights of the quadrature rule. Solving the system (3) in $y(u_i)$, $i = 1, \dots, n$, and replacing u_i by u , we may obtain an approximation of $y(u)$, for any value of $u \in [-1, 1]$. This

approach, pioneered by Crowder (1987) for the EWMA control chart and known as the Nyström method in the literature regarding the numerical solution of integral equations, gives excellent results for the EWMA control chart using a limited number of nodes (see Figures 1 and 2).

However, one cannot expect a high accuracy using the Gauss-Legendre quadrature when the Shewhart and the EWMA control limits run simultaneously, i.e. when $k < h(2 - \lambda)/\lambda$ (see, for instance, Figure 3 and 4). Indeed, when the kernel $Q(\cdot, \cdot)$ of the integral equation (2) is not-smooth, the Gauss-Legendre quadrature, used by Crowder (1987) to derive the expectations of the run length of the EWMA scheme, may fail to provide a reasonable approximation to the right hand side of equation (2). Further, under this condition, we cannot be sure of getting convergence of the solution of the system of linear algebraic equations (3) to the exact result, as $n \rightarrow \infty$. This is because, when k is not large enough, a single quadrature rule cannot be used to approximate the integral on the right side of (2) for each value of u . Indeed, since singularities of the function $Q(u, \cdot)$ depend on u , different rules should be used for different values of u . On the other hand, to obtain a linear system of equations similar to (3), all the rules must be based on the same set of nodes u_i .

One feasible solution consists of using a Clenshaw-Curtis (CC) type quadrature (Clenshaw and Curtis, 1960; Imhof, 1963). This algorithm does not seem to have been previously used in the context of control charts. However, many software libraries for numerical integration are based on the CC integration rules. The QUADPACK library (Piessens et al., 1983), for instance, and also well-known commercial libraries, such as NAG (<http://www.nag.co.uk>) and IMSL (<http://www.vni.com>), make use of the CC quadrature to numerically integrate functions with singularities, oscillatory functions or functions with weights. Further, the CC quadrature has been used to solve integral equations (Sloan and Smith, 1982; Piessens, 2000; Kang et al., 2002). In particular, if f is Riemann-integrable, it is possible to show that the numerical solution, obtained using the CC quadrature, converges to the exact solution of the integral equation as $n \rightarrow \infty$ (Sloan and Smith, 1978). In addition, recent theoretical results also show that, for a several times differentiable integrand, CC and Gauss quadrature formulas have essentially the same accuracy (Trefethen, 2006). Thus, CC quadrature results is a viable alternative to the Gauss Legendre procedure even when this quadrature rule can be applied. Indeed, Figures 1 and 2 show for a standard EWMA (with no Shewhart limits) a very similar performance of the CC and Gauss-Legendre quadrature rules.

The CC quadrature rule for integral equations of the form (2) is characterized by an approximate system of linear equations of the form

$$y(u_i) = 1 + \sum_{j=1}^n w_{ij} g(u_i, u_j) y(u_j), \quad i = 1, \dots, n \quad (4)$$

Here, the nodes u_i are the zeros of the Chebyshev polynomials of degree n , $T_n(\cos \theta) = \cos(n\theta)$, i.e.

$$u_i = \cos\left(\frac{(i-1)\pi}{n-1}\right), \quad i = 1, \dots, n$$

and the weights w_{ij} are determined so that the rule is exact when the integrand is a polynomial of degree $n - 1$. Note that, this quadrature rule uses the same

integration points for every value of u in the left handside of equation (2) but adjusts the corresponding weights for handling discontinuities in $I(u_i, \cdot)$.

In practice, w_{ij} are obtained solving the following linear system

$$\sum_{j=1}^n w_{ij} T_{r-1}(u_j) = \sum_{j=1}^n w_{ij} \cos\left(\frac{(j-1)(r-1)\pi}{n-1}\right) = m_{ir} \quad \begin{array}{l} i = 1, \dots, n \\ r = 1, \dots, n \end{array} \quad (5)$$

where m_{ir} , the modified Chebyshev moments of the function $I(u_i, v)$, are defined as

$$m_{ir} = \int_{-1}^1 I(u_i, v) T_{r-1}(v) dv.$$

Reminding that $I(\cdot, \cdot)$ is an indicator function, the Chebyshev moments are easy to compute from the indefinite integral of $T_{r-1}(v)$ given by

$$\int T_{r-1}(v) dv = \begin{cases} v & \text{for } r = 1 \\ v^2/2 & \text{for } r = 2 \\ \frac{1}{2} \left[\frac{T_r(v)}{r} - \frac{T_{r-2}(v)}{r-2} \right] & \text{for } r > 2 \end{cases}$$

The solution of the system (5) can be obtained by applying the Fast Fourier Transform (FFT) or the Discrete Cosine Transform (DCT). Since the standard FFT is more widely available, we here present the solution obtained via FFT. In particular, let $(\tilde{w}_{i1}, \dots, \tilde{w}_{i(2n-2)})$ be the discrete Fourier transform of $(m_{i1}, \dots, m_{in}, m_{i(n-1)}, \dots, m_{i2})$. Then, it can be shown that

$$w_{ij} = \begin{cases} \frac{\text{Real part of } \tilde{w}_{i1}}{2(n-1)} & \text{if } i = 1 \text{ or } i = n \\ \frac{\text{Real part of } \tilde{w}_{i1}}{n-1} & \text{otherwise} \end{cases}.$$

For the CSEWMA case, we have found convenient to consider a slight modification of the weights w_{ij} given by

$$w_{ij}^+ = \frac{\max(0, w_{ij}) s_i}{\sum_i \max(0, w_{ij}) g(u_i, u_j)} \quad (6)$$

where

$$\begin{aligned} s_i &= \int_{-1}^1 Q(u_i, v) dv = \\ &= F \left\{ \min \left[k, \frac{h}{\lambda} (1 - u(1 - \lambda)) \right] \right\} - F \left\{ \max \left[-k, -\frac{h}{\lambda} (1 + u(1 - \lambda)) \right] \right\} \end{aligned} \quad (7)$$

are directly evaluable from the distribution function $F(\cdot)$ of the observations. Observe that such a modification leads to non-negative weights and makes the approximation of the integral $\int Q(u_i, v) dv$, $i = 1, \dots, n$, i.e. $\sum_j w_{ij}^+ g(u_i, u_j)$, exact. This modification is asymptotically negligible; indeed, since in the current case $g(\cdot, \cdot) \geq 0$,

the ‘‘asymptotic positivity’’ property of the weights, proved by Sloan and Smith (1982), can be used to show that

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n (w_{ij} - w_{ij}^+) g(u_i, u_j) y(u_j) = 0$$

Concerning the *ARL*, our results do not show practical differences between the modified and unmodified weights for $n > 30$. On the other hand, we will show in Sections 3 and 4 that the modified weights (6) give some advantages when the whole run-length distribution has to be evaluated.

Given the weights, the approximation to $y(u_i)$ can be obtained by solving the system of equations (4). Note that, if an odd number of nodes is used, then $u_{(n+1)/2} = 0$ and $y(u_{(n+1)/2})$ can be used to directly approximate the average run length.

For a range of values of λ , h and k , and for the case of normal and Student t density functions, Figures 1-4 show, for n ranging from 1 to 100, the approximation to the *ARL* obtained by using the following four methods:

- (i) Gauss-Legendre quadrature;
- (ii) modified CC quadrature;
- (iii) Markov chain approximation based on n states;
- (iv) ‘‘extrapolated’’ Markov chain method suggested by Lucas and Saccucci (1990).

In this case, Markov chain approximations with $n-32$, $n-24$, $n-18$, $n-8$ and n states, respectively, are computed. Then, a least square curve $a + b/n + c/n^2$ is fitted and the intercept is used to approximate the continuous state *ARL*'s. For a standard EWMA chart (Figures 1 and 2), comparisons show that Gauss-Legendre quadrature and approximations based on CC quadrature have a comparable accuracy and that these integral equation methods outperform Markov-chain approach, with or without extrapolation. When a combined Shewhart-EWMA chart is considered, solution of the integral equation using CC quadrature seems to be the most reliable and fastly convergent procedure.

3 CSEWMA Run Length Distributions and Related Summary Values

In order to gain a complete understanding of chart performance, run-length characteristics different from the average run length, such as standard deviation and quantiles of the run-length distribution or the probability of signalling alarms during a given time interval, are often evaluated. Therefore, in this section we show how to use the modified CC integration rule to approximate the whole run-length distribution.

Let

$$p_{RL}(r|u) = P(RL = r | z_0 = hu), \quad u \in [-1, 1]$$

be the probability that the run-length is r , given that the EWMA control statistic starts at time 0 with value hu . It is straightforward to show that

$$p_{RL}(1|u) = 1 - \int_{-1}^1 Q(u, v) dv \tag{8}$$

and

$$p_{RL}(r|u) = \int_{-1}^1 Q(u, v) p_{RL}(r-1|v) dv \quad (9)$$

for $r > 1$, where $Q(\cdot, \cdot)$ has been defined in Section 2.

Discretizing the recurrence integral relations (9), through a procedure similar to that used for approximating the average run length, we obtain

$$p_{RL}(r|u_i) \simeq \sum_{j=1}^n w_{ij}^+ g(u_i, u_j) p_{RL}(r-1|u_j), \quad \begin{array}{l} i = 1, \dots, n \\ j = 1, \dots, n \end{array} \quad (10)$$

where w_{ij}^+ are the modified weights (6) corresponding to the CC nodes u_i . By pretending that left-hand and right-hand sides of equation (10) are equal, we obtain the following homogeneous difference equation

$$p_{RL}(r) = A p_{RL}(r-1) \quad (11)$$

where $p_{RL}(r) = [p_{RL}(r|u_1), \dots, p_{RL}(r|u_n)]^T$ and $A = \{a_{ij}\}$ is a matrix whose (i, j) -th element is $a_{ij} = w_{ij}^+ g(u_i, u_j)$. The solution of (11) is given by

$$p_{RL}(r) = A^{r-1} p_{RL}(1)$$

where, by equations (7) and (8), $p_{RL}(1) = (1 - s_1, \dots, 1 - s_n)^T$.

Observe that, when the suggested modified weights (6) are used, then

- i) A is a sub-stochastic matrix, i.e. it is a nonnegative matrix with row sums not exceeding 1. Indeed w_{ij}^+ and $g(\cdot, \cdot)$ are non-negative and

$$\sum_j a_{ij} = \sum_j w_{ij}^+ g(u_i, u_j) = s_i \leq 1;$$

- ii) $p_{RL}(1) = (I - A)O$, where $O = (1, \dots, 1)^T$.

Thus, i) the solution to the difference equation (11) is numerically stable; ii) $p_{RL}(r|u_i) \geq 0$, $\forall r, i$; iii) since A is a sub-stochastic matrix, $(I - A)^{-1} = \lim_{r \rightarrow \infty} \sum_{j=0}^r A^j$ and consequently $\sum_r p_{RL}(r|u_i) = 1$. These properties cannot be guaranteed when the unmodified CC weights are used.

Here, we are interested in $p_{RL}(r|0)$, i.e. in the distribution of the run length when the monitoring process starts from zero. For an odd number of nodes, this probability is given by

$$p_{RL}(r|0) = P \left[r|u \left(\frac{n+1}{2} \right) \right] = b^T A^{r-1} p_{RL}(1) = b^T A^{r-1} (I - A)O \quad (12)$$

where b is a column vector with 1 as its $(n+1)/2$ -th element and 0 otherwise.

Either quantiles of the RL distribution or the probability of signals during a given time interval can be directly computed from the probability distribution (12). By continuing to assume an odd value of n , moments of the run length distribution can be found using the following expression of the moment generating function

$$MGF(t) = \sum_{t=1}^{\infty} e^{rt} p_{RL}(r|0) = b^T e^t [I - e^t A]^{-1} (I - A)O \quad (13)$$

From (13), we can obtain directly the expected value of the approximated run-length distribution

$$E[RL] = b(I - A)^{-1}O \quad (14)$$

and the variance

$$V[RL] = b(I + A)(I - A)^{-2}O.$$

Observe that the expected value of the run-length, given by equation (14), is equal to the ARL obtained in the previous section.

An example of the algorithms performance is given in Figures 5 and 6 that show approximations, of the median and standard deviation of the CSEWMA run-length distribution, obtained using the four numerical methods described in the previous section. Even for these summary values the modified CC quadrature seems to show the best accuracy.

4 The CSEWMA Steady State Run Length Distribution

In this section we show how to compute the steady-state distribution, namely

$$p_\infty(r) = \lim_{t \rightarrow \infty} P(RL = r + t | RL > t). \quad (15)$$

that is the distribution of the run length when no false alarm is given before the signal and the shift occurs after the control statistic has reached its stationary distribution.

Given the Markovian property of the control statistic, $p_\infty(r)$ is given by

$$p_\infty(r) = \int_{-1}^1 p_{RL}(r|u)p_z(u; \infty)du \quad (16)$$

where $p_z(u; \infty)$ is the limit distribution function, as t tends to ∞ , of the density function of z_t/h , given that $RL > t$. Standard arguments can be used to show that the limit distribution $p_z(u; \infty)$ can be obtained iterating, until convergence is achieved, the following recursion

$$p_z(u; t + 1) = \frac{\int_{-1}^1 p_z(v, t)Q(v, u)dv}{\int_{-1}^1 \int_{-1}^1 p_z(v, t)Q(v, w)dv dw}, \quad (17)$$

where $p_z(v, 1) = Q(0, v) / \int_{-1}^1 Q(0, w)dw$.

Using the CC quadrature, equation (17) may be approximated by

$$p_z(u_i; t + 1) = \frac{\sum_j w_{ij}^* p_z(v_j, t)g(v_j, u_i)dv}{\sum_{j,l} w_l^{CC} w_{ij}^* p_z(v_j, t)g(v_j, u_i)} \quad (18)$$

where u_i are the CC nodes, w_{ij}^* are the CC weights modified as described in the previous section and here used for approximating integrals in the numerator of (17). Further, w_l^{CC} are the standard CC weights used to numerically integrate a continuous function on the in the interval $[-1, 1]$.

Observe that using the modified weights, the approximated recursion (18) is stable; thus, its iteration converges. Once convergence is reached, an estimate of

$p_z(u; \infty)$, denoted by $\tilde{p}_z(u; \infty)$, can be used to obtain a discretized version of (15) given by

$$p_\infty(r) \simeq \sum_i w_i^{CC} p_{RL}(r|u_i) \tilde{p}_z(u; \infty). \quad (19)$$

Figures 7 and 8 illustrate how accurate the compared quadrature rules are to approximate both the steady-state ARL and probability to give a true signal within ten observations.

5 Design of a CSEWMA control chart

In this section, we illustrate how to apply the proposed algorithms to the design of a CSEWMA chart. The CSEWMA design consists of choosing, between all combinations of constants giving a large in control *ARL*, those values λ , \tilde{h} and k for which the out-of-control *ARL* is relatively small. Since the CSEWMA control chart is desirable for its satisfactory performance for both small and large mean shifts, we here investigate the CSEWMA run length performance for a range of mean values. Results presented in the previous sections outline that, using the modified CC quadrature rule, a very high accuracy may be obtained with 65 support points. Computation of the CC weights for $n = 65$ is also particularly rapid since it requires a FFT of a vector of a highly factorizable length, i.e., 128. Thus, we show results obtained for $n = 65$ and for monitoring a Gaussian process having, in the in-control situation, mean equal to zero and unitary variance. Here, we assume that a persistent change of size μ can affect the process mean and that the desired in-control average run-length is equal to 370.4.

Figure 9 shows how the design constant k affects both the zero-state and the steady-state *ARL*'s for different values of λ . Observe that, in the case of small mean shifts, there is no practical difference between the *ARL* profiles corresponding to different values of k and λ . However, as μ increases, a better *ARL* performance can be observed, throughout the λ values, for $k = 3.25$ or $k = 3.5$. Thus, Figure 10 illustrates the comparison between the *ARL* profiles of two CSEWMA charts, based on $k = 3.25$ and 3.5 , for various λ . Note that, the choice of an optimal value of λ depends on the size of shift which should be promptly detected. However, choosing λ equal to 0.05 or 0.1 and k roughly equal to 3.25 or 3.5, seems to be a reasonable compromise for the whole range of location shifts.

In alternative to this graphical analysis, the CSEWMA design can be also based on a suitable optimization criterion. For example, given the satisfactory performance of the Shewhart chart for a relatively large value of the mean shift, the CSEWMA chart can be designed so that, between all the schemes having an in-control *ARL* equal to B , i) it is the most efficient at signalling a small mean shift μ_1 ; ii) it approximately performs as well as the Shewhart control chart, when the step shift is equal to a great value μ_2 . Thus, denoting by $\psi(\mu; \lambda, \tilde{h}, k)$ the *ARL* when the size of shift is equal to μ , a possible design procedure consists on determining λ , \tilde{h} and

k as solution of the following constrained minimization

$$\left\{ \begin{array}{l} \min_{\lambda, \tilde{h}, k} \psi(\mu_1; \lambda, \tilde{h}, k) \\ \text{subject to} \\ \psi(0; \lambda, \tilde{h}, k) = B \\ \psi(\mu_2; \lambda, \tilde{h}, k) \leq (1 + \alpha)C \end{array} \right. \quad (20)$$

where α is a small tolerance value and C is the out-of-control *ARL*, for a mean shift of μ_2 , of a Shewhart control chart having an in-control *ARL* equal to B .

Table 1 lists the solutions of (20), the corresponding CSEWMA *ARL*'s profiles for a range of shift sizes and also the 10th, 50th and 90th quantiles of the CSEWMA run length distribution. The minimization problem (20) has been solved using the Nelder-Mead algorithm (Nelder and Mead, 1965) with constraints introduced as penalties. Users should choose the design constants that guarantee a desired *ARL* profile. For example, the combination of the design constants $\lambda = 0.077$, $k = 3.201$, $\tilde{h} = 2.863$ seems able to give a satisfactory protection against both small and large shifts.

6 Conclusion

The use of a modified Clenshaw-Curtis quadrature rule is suggested to approximate integral equations and integral recurrence relations related to the run-length characteristics of the combined Shewhart-EWMA control chart. The proposed approach, as illustrated in the paper, can be used to compute a variety of performance measures (not only the average run-length). Numerical results demonstrate an increased accuracy compared to other conventional methods. In particular, this technique gives the advantage of a more rapid convergence than the Markov chain based approach previously used for this kind of monitoring schemes. In addition, since it explicitly accounts for the discontinuities in the integral kernels, it does not suffer from any drawbacks, associated to the convergence, that affect the Gauss-type quadrature methods. Further, our results outline that the performance of the suggested algorithm is not inferior to that based on the Gauss quadrature rule, even in those situations where this rule is considered optimal from the current literature, such is the case with the standard EWMA chart without Shewhart limits.

The implementation of the suggested algorithm is quite simple. A reference implementation in R (R Development Core Team, 2006) is available at the journal web page.

Finally, it should also be noted that the suggested approach can be easily adapted and seems to be promising for studying the performance of other combined control charts (e.g., combined Shewhart-CUSUM or combination of more than one EWMA [CUSUM] charts) and, more in general, of those monitoring schemes which result in integral equation or integral recurrence relations with discontinuous kernel.

References

- Brook, D. and Evans, D. (1972). An approach to the probability distribution of CUSUM run length. *Biometrika*, 59:539–549.
- Champ, C. and Rigdon, S. (1991). A comparison of the Markov chain and integral equation approaches for evaluating the run length distribution of quality control charts. *Communications in Statistics- Simulation and Computation*, 20:191–204.
- Clenshaw, C. and Curtis, A. (1960). A method for numerical integration on an automatic computer. *Numerical Mathematics*, 2:197–205.
- Crowder, S. (1987). A simple method for studying run-length distributions of Exponentially Weighted Moving Average charts. *Technometrics*, 29:401–407.
- Imhof, J. (1963). On the method for numerical integration of Clenshaw and Curtis. *Numerische Mathematik*, 5:138–141.
- Kang, S., Koltracht, I., and Rawitscher, G. (2002). Nystrom-Clenshaw-Curtis quadrature for integral equations with discontinuous kernels. *Mathematics of Computation*, 72:729–756.
- Lucas, J. and Saccucci, M. (1990). Exponentially Weighted Moving Average control schemes: Properties and enhancements quality. *Technometrics*, 32:1–29.
- Montgomery, D. (2004). *Introduction to Statistical Quality Control*. Wiley, New York, 5th edition.
- Nelder, J. and Mead, R. (1965). A simplex algorithm for function minimization. *Computer Journal*, 7:308–313.
- Piessens, R. (2000). Computing integral transforms and solving integral equations using Chebyshev polynomial approximations. *Journal of Computation and Applied Mathematics*, 121:113–124.
- Piessens, R., de Doncker-Kapenga, E., Uberhuber, C., and Kahaner, D. (1983). *QUADPACK. A subroutine package for automatic integration*. Springer Verlag.
- R Development Core Team (2006). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.
- Sloan, I. (1978). Quadrature methods for integral equations of the second kind over infinite intervals. *Mathematics of Computation*, 36:511–523.
- Sloan, I. (1981). Analysis of general quadrature methods for integral equations of the second kind. *Numerische Mathematik*, 38:263–278.
- Sloan, I. and Smith, W. (1978). Product-integration with the Clenshaw and Curtis and related points. *Numerische Mathematik*, 30:415–428.
- Sloan, I. and Smith, W. (1980). Product integration with the Clenshaw and Curtis points: implementation and error points. *Numerische Mathematik*, 34:387–401.

Sloan, I. and Smith, W. (1982). Properties of interpolatory product integration rules. *SIAM Journal on Numerical Analysis*, 19:427–442.

Trefethen, L. (2006). Is Gaussian quadrature better than Clenshaw- Curtis? *to appear in SIAM Review*.

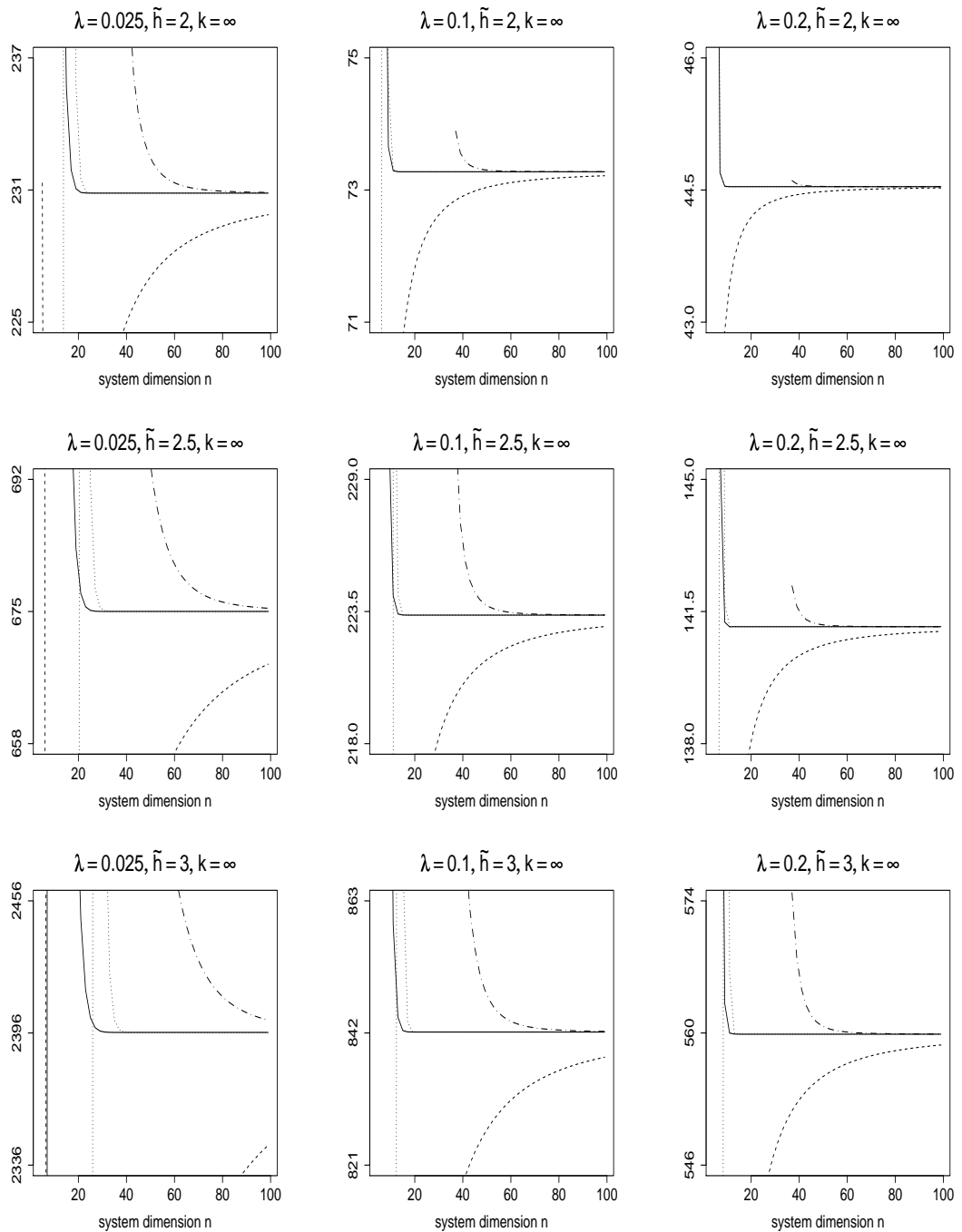


Figure 1: Approximations of the EWMA in-control average run-length (control chart without Shewhart limits) as function of the dimension of the discretized system. The underlying distribution is a standard normal. Approximations have been computed using the following methods: (i) integral equation approach using Gauss-Legendre quadrature (dot line); (ii) integral equation using Clenshaw-Curtis quadrature (solid line); (iii) Markov chain (dash line); (iv) extrapolated Markov chain (dash-dot line).

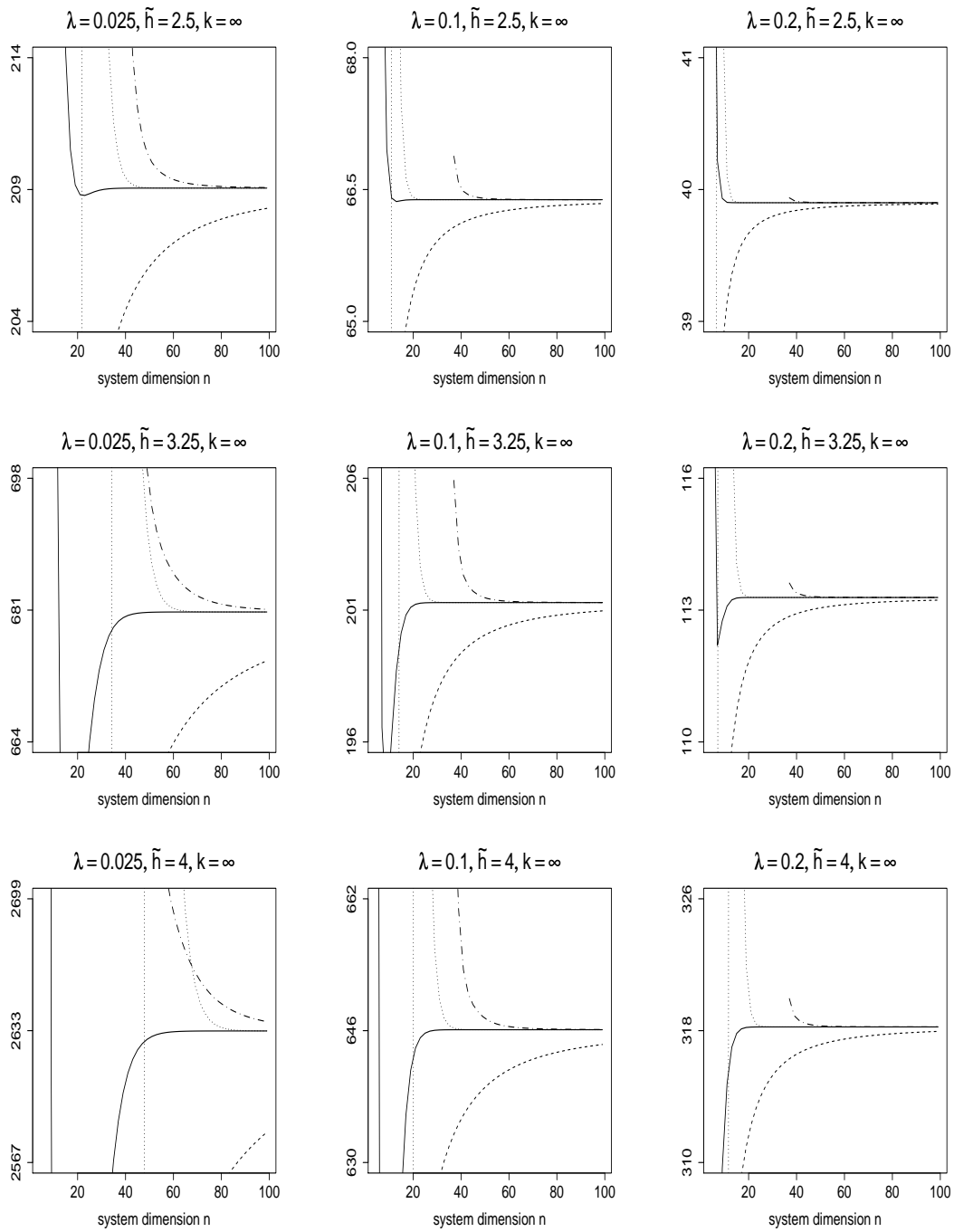


Figure 2: Approximations of the EWMA in-control average run-length (control chart without Shewhart limits) as function of the dimension of the discretized system. The underlying distribution is a Student's t with 5 degree of freedom. Legend: see Figure 1.

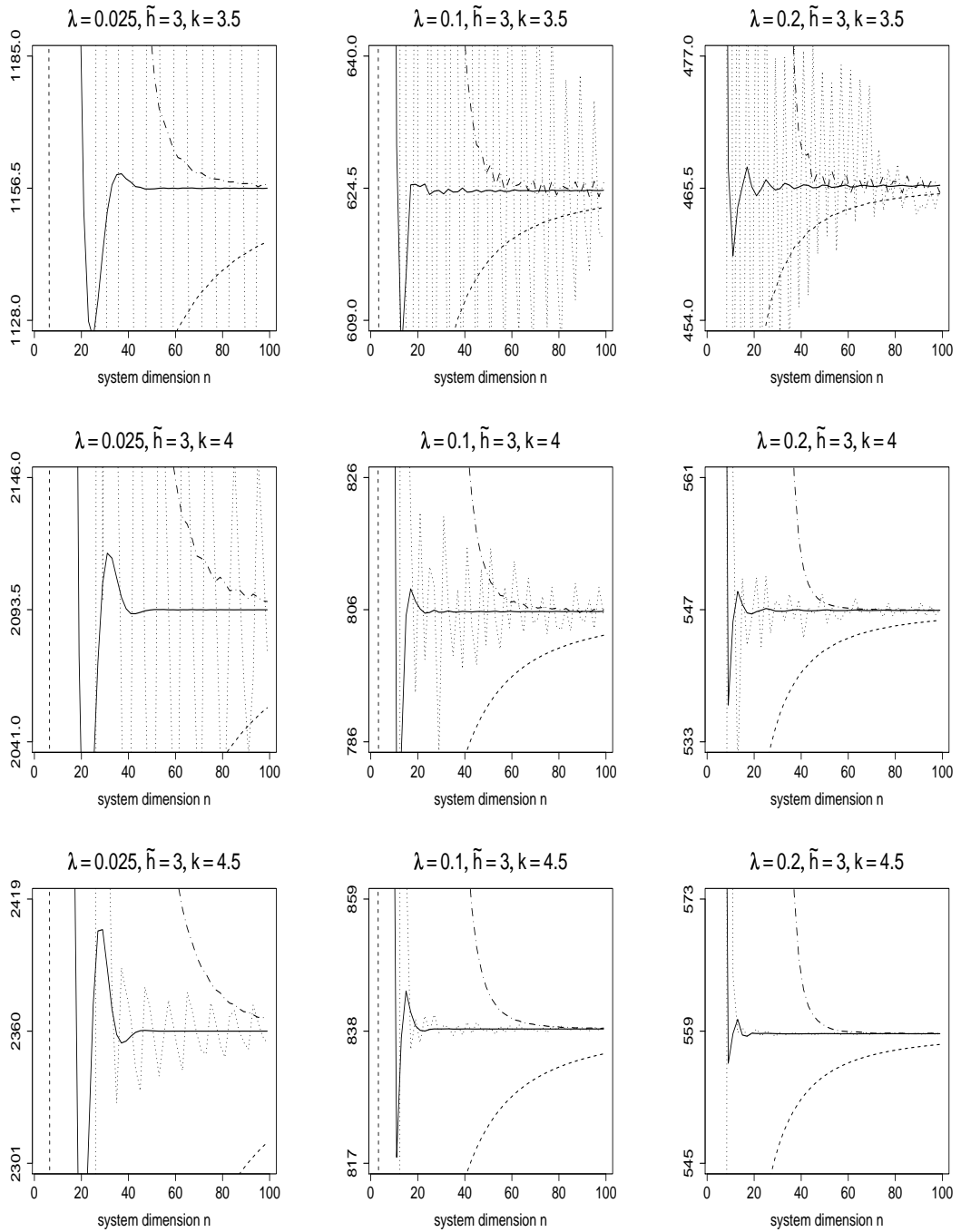


Figure 3: Approximations of the CSEWMA in-control average run-length as function of the dimension of the discretized system. The underlying distribution is a standard normal. Legend: see Figure 1.

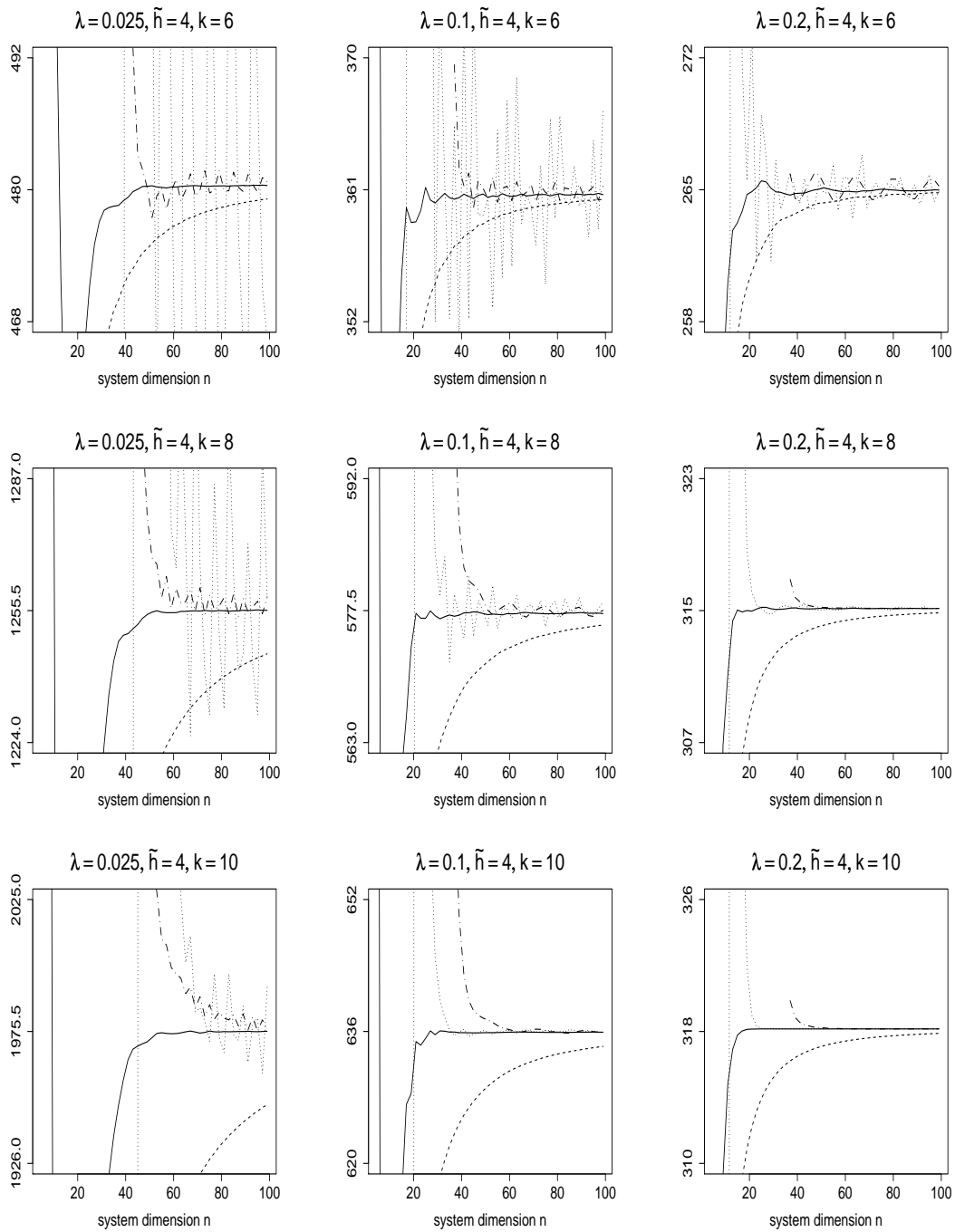


Figure 4: Approximations of the CSEWMA in-control average run-length as function of the dimension of the discretized system. The underlying distribution is a Student's t with 5 degree of freedom. Legend: see Figure 1.

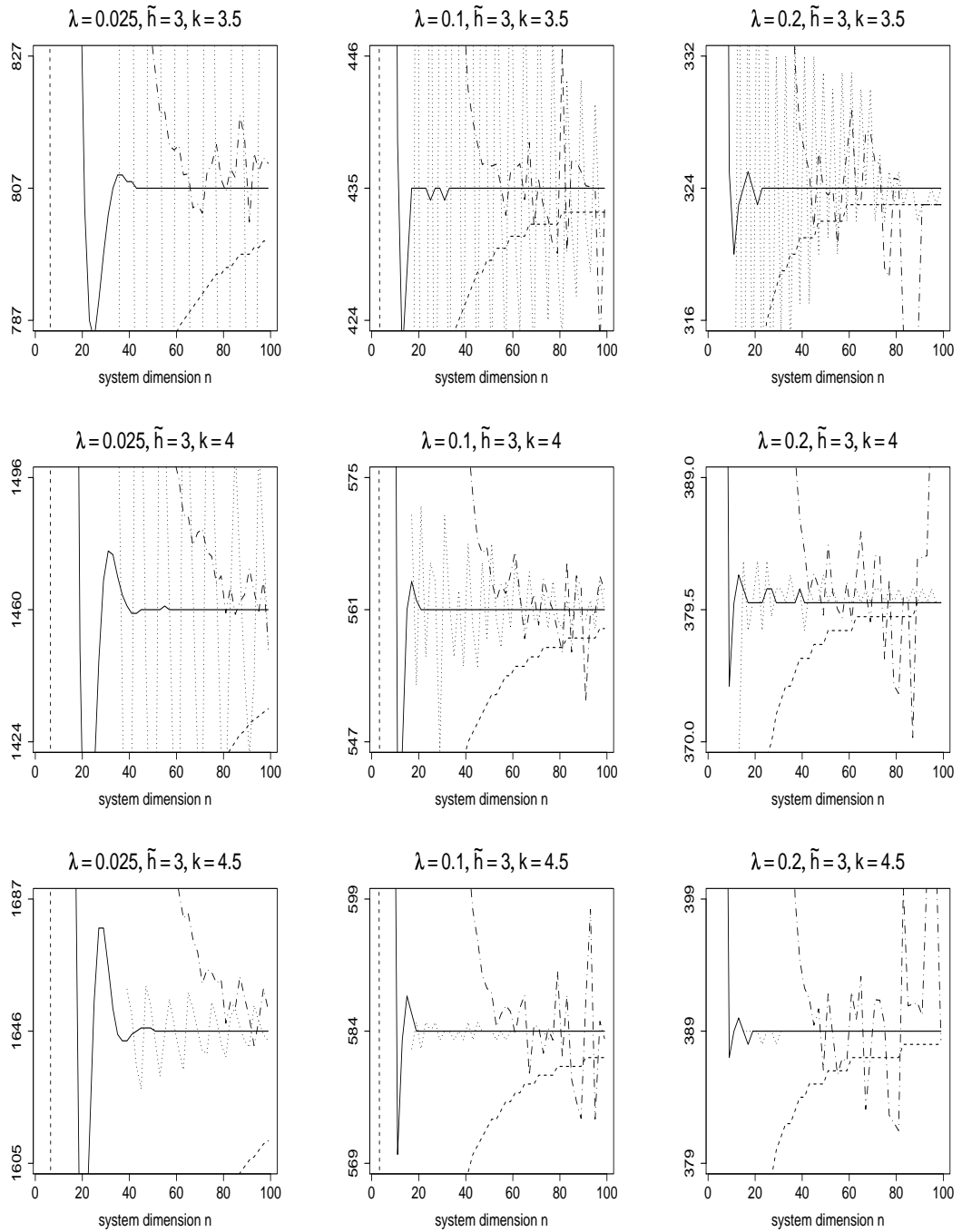


Figure 5: Approximations of the median of the CSEWMA in-control run-length as function of the dimension of the discretized system. The underlying distribution is a standard normal. Legend: see Figure 1.

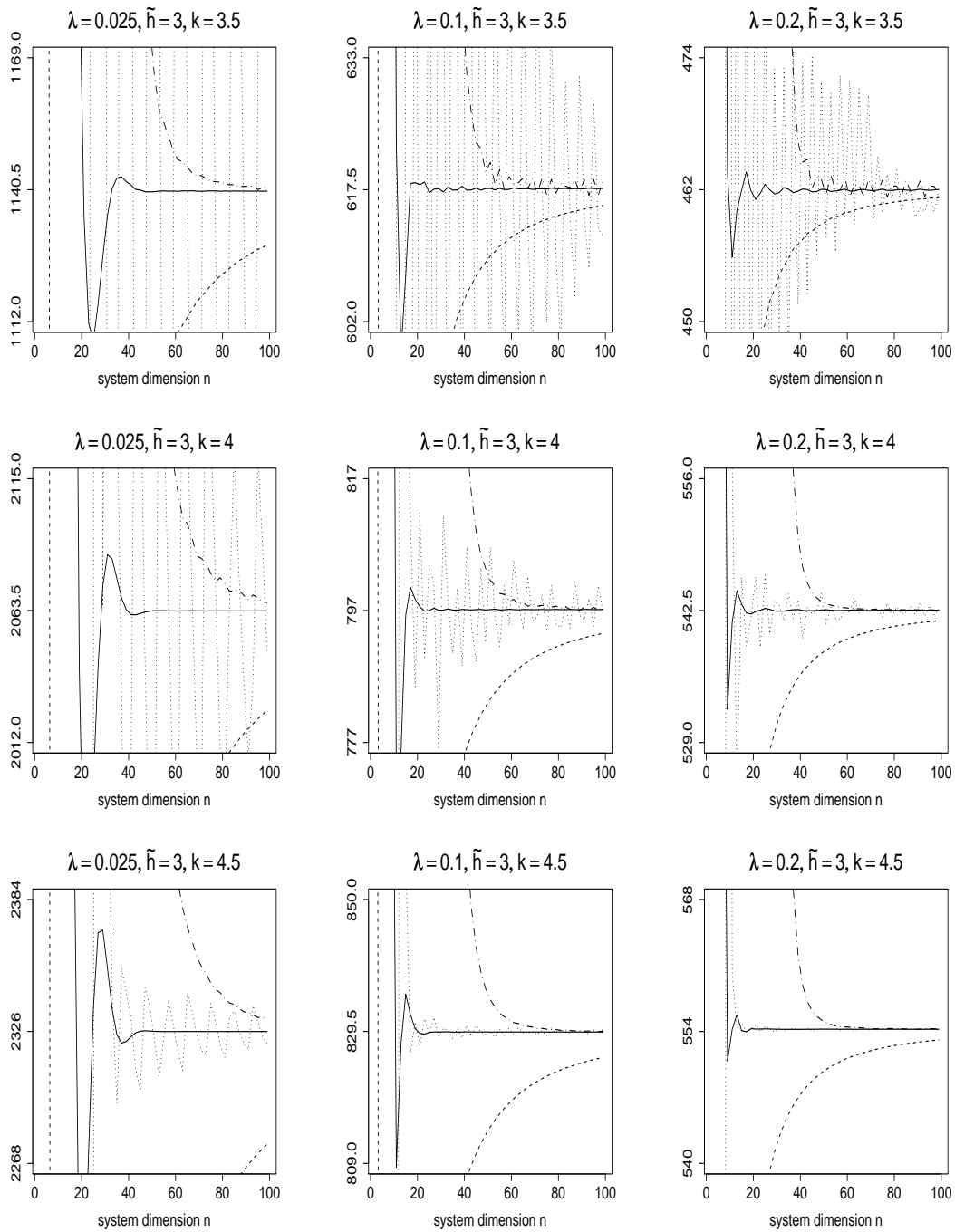


Figure 6: Approximations of the standard deviation of the CSEWMA in-control run length as function of the dimension of the discretized system. The underlying distribution is a standard normal. Legend: see Figure 1.

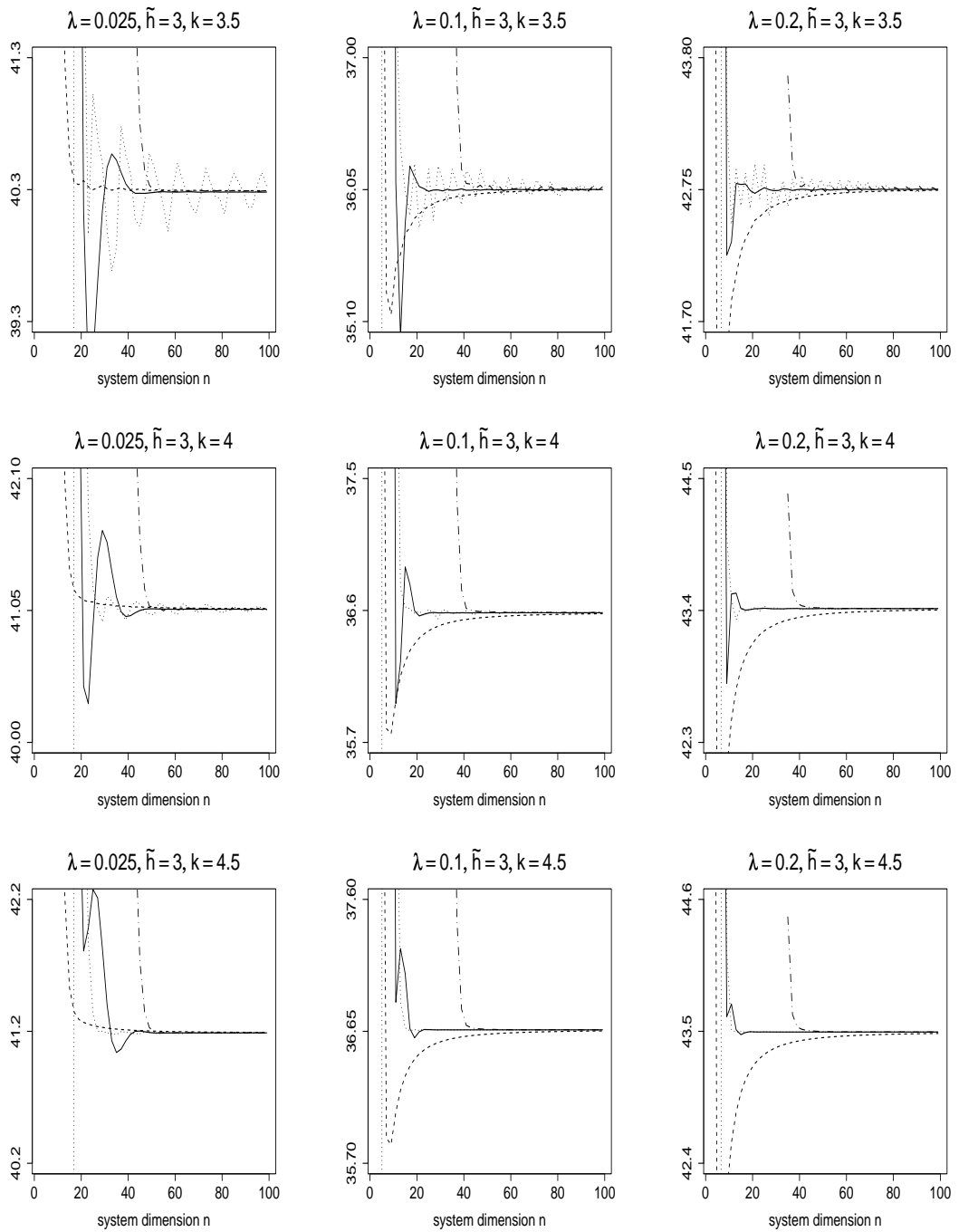


Figure 7: Approximations of the CSEWMA out-of-control steady state average run-length as function of the dimension of the discretized system. The underlying distribution is normal with mean 0.5 and unit variance. The steady state distribution has been computed assuming that the in-control distribution is a standard normal. Legend: see Figure 1.

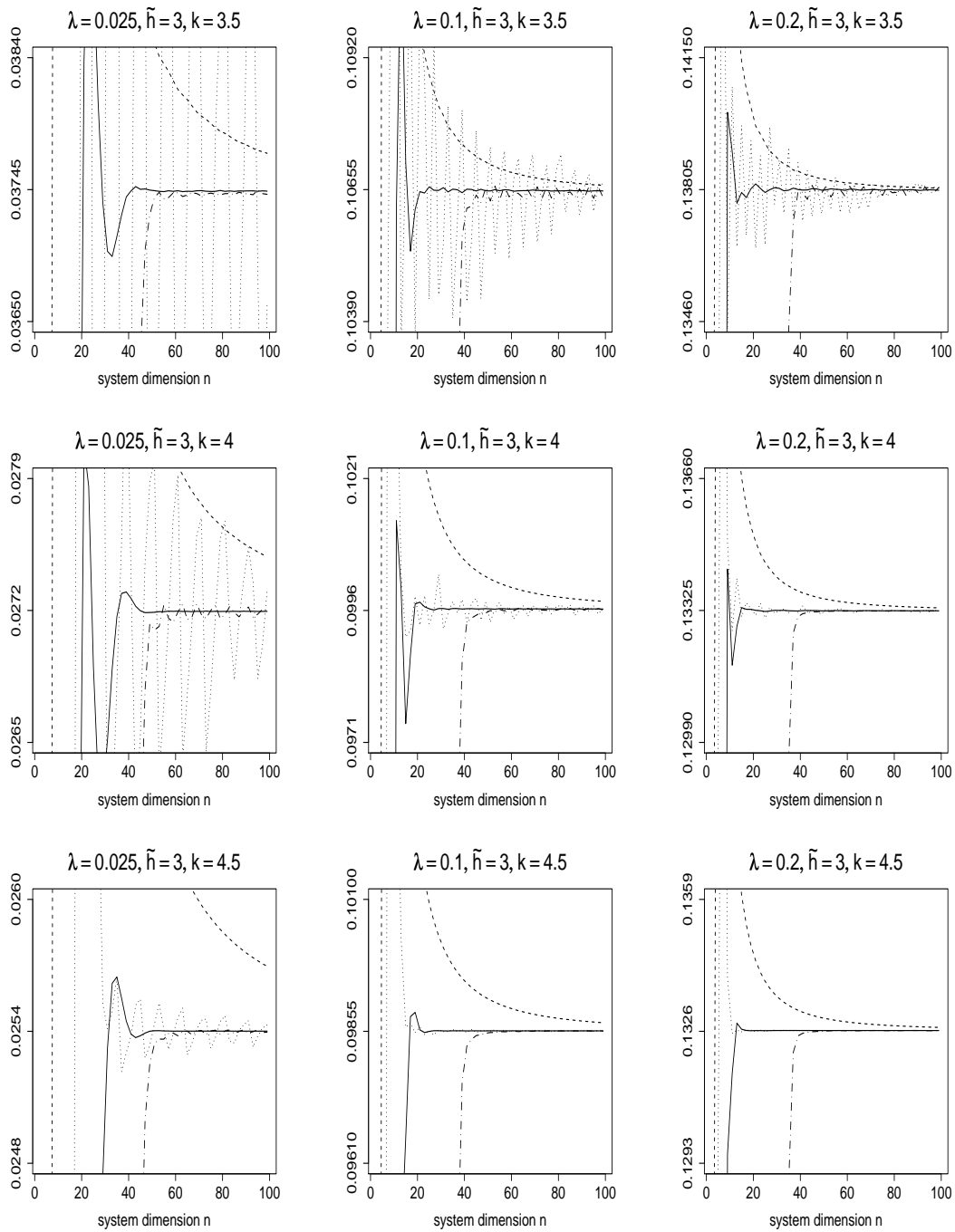


Figure 8: Approximations of the CSEWMA steady state probability to give a signal within 10 observation as a function of the dimension of the discretized system. The underlying distribution is normal with mean 0.5 and unit variance. The steady state distribution has been computed assuming that the in-control distribution is a standard normal. Legend: see Figure 1.

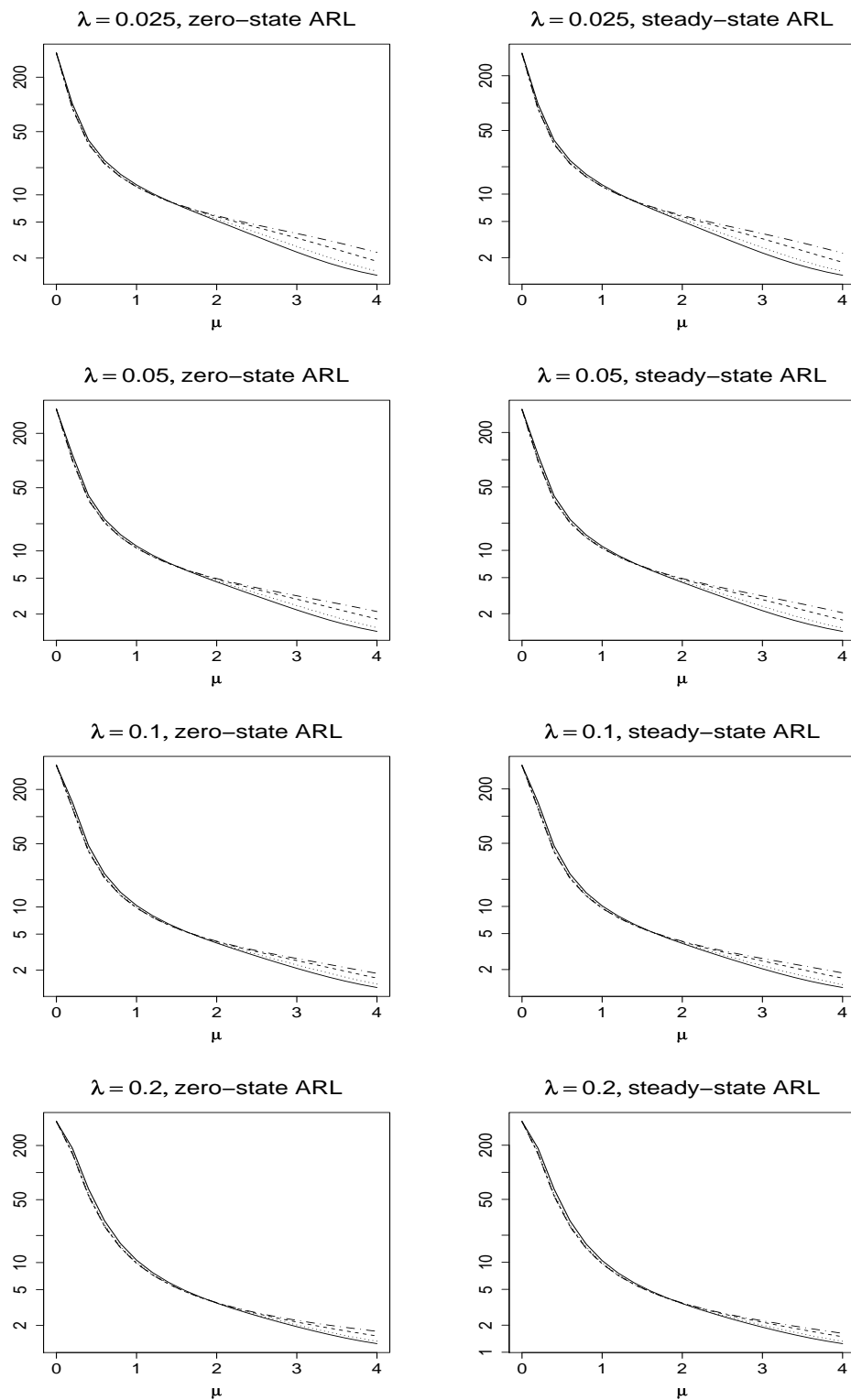


Figure 9: Profiles of the CSEWMA average run-length: $k = 3.25$ (solid line); $k = 3.5$ (dot line); $k = 4$ (dash line); $k = 4$ (dash-dot line).

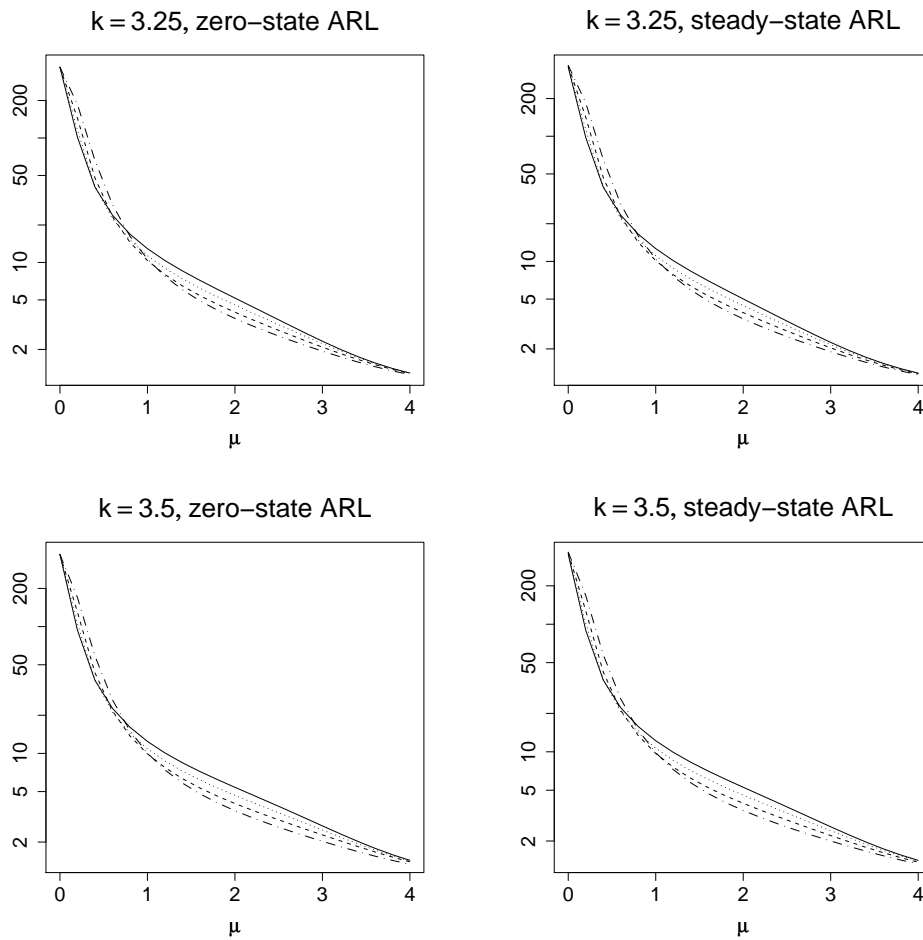


Figure 10: Profiles of the CSEWMA average run-length: $\lambda = 0.025$ (solid line); $\lambda = 0.05$ (dot line); $\lambda = 0.1$ (dash line); $\lambda = 0.2$ (dash-dot line).

Table 1: Some Combined Shewhart-EWMA control charts optimal according to criterion (20) ($B = 370.4$, $\alpha = 0.05$). ARL and Q_p denote the average and the p -quantile of the run-length distribution of the resulting control scheme.

Design shifts		Optimal parameters			Mean shifts						
μ_1	μ_2	λ	\tilde{h}	k		0	0.5	1	2	3	4
0.5	3	0.077	2.863	3.201	ARL	370.4	31.4	10.8	4.2	2.1	1.3
					$Q_{0.1}$	44.0	11.0	5.0	1.0	1.0	1.0
					$Q_{0.5}$	259.0	26.0	10.0	4.0	2.0	1.0
					$Q_{0.9}$	845.0	59.0	17.0	6.0	4.0	2.0
0.5	4	0.043	2.763	3.158	ARL	370.4	31.1	12.1	4.7	2.1	1.3
					$Q_{0.1}$	46.0	13.0	6.0	1.0	1.0	1.0
					$Q_{0.5}$	259.0	27.0	12.0	5.0	2.0	1.0
					$Q_{0.9}$	843.0	54.0	18.0	7.0	4.0	2.0
1.0	3	0.146	2.874	3.410	ARL	370.4	33.8	10.0	3.7	2.1	1.3
					$Q_{0.1}$	43.0	9.0	5.0	2.0	1.0	1.0
					$Q_{0.5}$	258.0	26.0	9.0	4.0	2.0	1.0
					$Q_{0.9}$	847.0	69.0	17.0	6.0	3.0	2.0
1.0	4	0.126	3.00	3.178	ARL	370.4	36.7	10.6	3.8	2.0	1.3
					$Q_{0.1}$	42.0	10.0	5.0	1.0	1.0	1.0
					$Q_{0.5}$	258.0	28.0	9.0	4.0	2.0	1.0
					$Q_{0.9}$	848.0	75.0	18.0	6.0	3.0	2.0

Note. \tilde{h} and k are computed assuming $\sigma = 1$ where σ denotes the standard deviation of the process. In general, this values should be multiplied by σ .

Acknowledgements

This research was partially funded by Italian MIUR-Cofin 2006 grants.

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