Linear weakly–Noetherian constants of motion are horizontal gauge momenta

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Abstract

The notion of gauge momenta is a generalization of the momentum map which is relevant for nonholonomic systems with symmetry. Weakly Noetherian functions are functions which are constants of motion of all 'natural' nonholonomic systems with a given kinetic energy and any G–invariant potential energy. We show that, when the action of the symmetry group on the configuration manifold is free and proper, a functions which is linear in the velocities is weakly–Noetherian if anf only if it is a gauge momenta which has a horizontal generator.

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1 Introduction

The relationship between constants of motion and symmetries of nonholonomic systems has been extensively studied over the last fifty years [1, 13, 14, 2, 3, 8, 9, 6, 17, 7, 18, 4, 15, 10, 11, 5, 12], but a complete comprehension of such a link—if it exists—is still missing. Investigations have mostly focussed on analogies and differences from the well understood holonomic (that is, Hamiltonian) case. There are two main differences among the holonomic and the nonholonomic cases. They are more simply explained in the simplest case of nonholonomic systems with linear constraints and natural Lagrangian (= kinetic minus potential) and of symmetry groups acting in the configuration manifold; this is in fact the only case that we consider in this article.

1. In the Hamiltonian case, a (Hamiltonian) symmetry group produces the conservation of its momentum map. In the nonholonomic case the situation is not as simple. On the one hand, only certain components of the momentum map (that we call here 'momenta') are conserved—those whose infinitesimal generators belong to a certain distribution [10]. On the other hand, there are conserved quantities which are not momenta, but which are rather conserved 'gauge momenta' [2, 11]. Conserved gauge momenta are generated by certain vector fields which are not infinitesimal generators of the group action, but are nevertheless tangent to the group orbits and have an additional invariance property (see below). Being tangent to the group orbits, they can be written as pointwise linear combinations of infinitesimal generators, and

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this is the origin of the term 'gauge'. If the generating vector field is a section of the constraint distribution as well, then one speaks of 'horizontal gauge momenta'. The case of horizontal gauge momenta is that which was originally investigated in [2], but it is not exhaustive: examples of conserved gauge momenta which are not horizontal are given in [11].

2. A second peculiarity of the link symmetries–conservation laws in the holonomic context is that the momentum map of a symplectic group action is conserved by all Hamiltonian systems with that symmetry. This property, which is called 'Noetherian condition' in [16], fails in nonholonomic mechanics [1, 14, 4, 11]. A weaker notion was introduced in [11, 12]: a *weakly Noetherian* function (or weakly Noetherian constant of motion) is a constant of motion of all nonholonomic systems with given kinetic energy and any G–invariant potential energy. It is an immediate observation that horizontal gauge momenta are weakly Noetherian [11].

The purpose of this article is to further investigate the relationship between gauge momenta and weakly Notherian functions, by proving a converse to the last statement in the realm of linear constants of motion. Specifically, we will show that, under certain conditions on the group action, see below, the weakly Noetherian functions which are linear in the velocities are exactly the horizontal gauge momenta (section 2). In addition, in section 3 we collect a few remarks on these topics.

2 Linear weakly Noetherian functions are horizontal gauge momenta

In order to properly state our result we need to recall some definitions. In the sequel, a *natural linear nonholonomic system* on a configuration manifold Q is formed by a pair (L, D), where $L : TQ \to \mathbb{R}$ is a natural Lagrangian and $\mathcal{D}$ is a (maximally) nonintegrable constant–rank distribution on Q.

Saying that the Lagrangian L is natural means that it can be written as the difference $L = T - V \circ \pi$, where the kinetic energy T is a positive definite quadratic form on TQ and $V : Q \to \mathbb{R}$ is the potential energy; $\pi : TQ \to Q$ is the canonical projection. The kinetic energy defines a Legendre transformation $TQ \to T^*Q$ and, through it, 'conjugate momenta' that we view as a fiberwise linear function $p : TQ \to T^*Q$.

The distribution $\mathcal{D}$, which is called the constraint distribution, may be viewed as a submanifold D of TQ, that we call the constraint submanifold. The constraints are assumed to fulfill d'Alembert principle, so that the nonholonomic system (L, D) defines a dynamical system on D, that is, a vector field $X_{L,D}$ on D.

A *constant of motion* of the natural linear nonholonomic system (L, D) is a smooth function $F : D \longrightarrow \mathbb{R}$ which is constant along the flow of $X_{L,D}$, that is, the Lie derivative $X_{L,D}(F) = 0$ in D.

A *linear function on* D is the restriction to D of a smooth function $TQ \to \mathbb{R}$ which is linear on the fibers of TQ. Equivalently, it is a smooth function $D \to \mathbb{R}$ which is linear on the fibers of $\mathcal{D}$. A linear function on D can always be written as $\langle Z, p\rangle|_D$ for some (non–unique) smooth vector field Z on Q, which is called a *generator* of it. Any linear function on D has a unique generator which is a section of the constraint distribution [14, 10] and is therefore called the 'horizontal generator' of the linear function.

Given a smooth action $\Psi : G \times Q \to Q$ of a Lie group G on Q, we denote by $\mathcal{G}$ the distribution on Q whose fibers are the tangent spaces to the orbits of Ψ. The action Ψ defines a lifted action $\Psi^{TQ} : G \times TQ \to TQ$. For shortness, and with no danger of confusion, we speak of G–invariance, G–orbits etc to refer to either the action of G on Q or to its lift to TQ.

If $\mathcal{E}$ and $\mathcal{E}'$ are two distributions on Q, we denote by $\mathcal{E} \cap \mathcal{E}'$ the distribution on Q whose fibers are the intersection of the fibers of $\mathcal{E}$ and of $\mathcal{E}'$. We say that $\mathcal{E}$ is an overdistribution of $\mathcal{E}'$, and write $\mathcal{E} \supseteq \mathcal{E}'$, if the fibers of $\mathcal{E}$ contain—or coincide with—those of $\mathcal{E}'$. We denote by Z^{TQ} the lift to TQ of a vector field Z on Q.

We may now define the two notions of gauge momenta and of weakly Noetherian functions.

Definition 1 *Consider a natural linear nonholonomic system (L, D), an action of a Lie group G on Q and a distribution $\mathcal{E}$ on Q. An $(\mathcal{E}, G)$–*gauge momentum *of (L, D) is a linear function on D which has a generator Z which is a section of $\mathcal{G} \cap \mathcal{E}$ and which moreover satisfies $Z^{TQ}(L)|_D = 0$.*

This definition of gauge momenta is taken from [11], where it was however stated under the additional hypothesis of G–invariance of the Lagrangian. While this is of course the case of interest—and the very idea of the 'gauge method' was introduced in [2] specifically to study nonholonomic systems with symmetry—this hypothesis is logically not necessary. The reason is that, since the tangent lift Z^{TQ} of a section Z of $\mathcal{G}$ which is not an infinitesimal generator of the group action need not preserve all G–invariant functions on TQ, the definition of gauge momenta need the assumption $Z^{TQ}(L)|_D = 0$. This—not the G–invariance of L—is the relevant condition, and all the results we quote here from [11] remain true without the assumption of G–invariance of L. We will shortly comment on this fact in the Remarks.

It was proven in [10, 11] that there exists a distribution $\mathcal{R}^\circ_{T,V}$ on Q, called the reaction–annihilator distribution, which is such that an $(\mathcal{E}, G)$–gauge momentum of $(L = T - V, D)$ is a constant of motion if and only if $\mathcal{E} \subseteq \mathcal{R}^\circ_{T,V}$. This follows from the following fact, that we will use several times in the sequel:

Proposition 1. [10] *Given a natural linear nonholonomic system $(L = T - V, D)$ and a smooth vector field Z on Q, any two of the following three conditions imply the third: (C1) Z is a section of $\mathcal{R}^\circ_{T,V}$. (C2) $Z^{TQ}(L)|_D = 0$. (C3) Z is a generator of a linear constant of motion of (L, D).*

The distribution $\mathcal{R}^\circ_{T,V}$ is constructed out of the reaction forces exherted by the nonholonomic constraint and hence, given the constraint distribution D, it depends on T and V (separately, not just on the difference $T - V$); for the description of this distribution see [10, 11]. The distribution $\mathcal{R}^\circ_{T,V}$ is a (typically strict) over–distribution of $\mathcal{D}$. Therefore, the class of conserved gauge momenta include the $(\mathcal{D}, G)$–gauge momenta, but might be larger than such a family of functions. The $(\mathcal{D}, G)$–gauge momenta are called *horizontal* G–gauge momenta of (L, D).

Definition 2 *Consider a configuration manifold Q, a constraint distribution $\mathcal{D}$ on Q, a smooth action of a Lie group G on Q, and a kinetic energy $T : TQ \to \mathbb{R}$. Then, a (T, D, G)–*weakly Noetherian function *is a smooth function $F : D \to \mathbb{R}$ which is a constant of motion of all natural linear nonholonomic systems $(T - V \circ \pi, D)$ with any G–invariant function $V : Q \to \mathbb{R}$.*

This definition is taken from [12], where it was however stated under the additional hypothesis of G–invariance of the Lagrangian. Also, the expression 'weakly Noetherian constant of motion' was used there.

Theorem 1. *Consider a configuration manifold Q, a constraint distribution $\mathcal{D}$ on Q, a kinetic energy $T : TQ \to \mathbb{R}$ and a smooth action of a Lie group G on Q. Then:*

(i) Any horizontal G–gauge momentum of (T, D) is (T, D, G)–weakly Noetherian.

(ii) Assume furthermore that the action of G on Q is free and proper. Then any linear (T, D, G)–weakly Noetherian function is a horizontal G–gauge momentum of (T, D).

Proof. Statement (i) was already proven in [11]; we give here the proof for completeness. If F is a horizontal G–gauge momentum of (T, D) then it has a generator Z which satisfies $Z^{TQ}(T)|_D = 0$ and is a section of $\mathcal{G} \cap \mathcal{D}$. Consider a G–invariant function on Q. Since Z is a section of $\mathcal{G}$, $Z(V) = 0$ and hence $Z^{TQ}(T - V \circ \pi)|_D = Z^{TQ}(T)|_D - Z(V) \circ \pi|_D = 0$. Thus, Z satisfies the two conditions C1 and C2 for $L = T - V \circ \pi$ of Proposition 1 and therefore satisfies C3 as well.

We now prove statement (ii). Denote by $\mathcal{I}_G$ the set of smooth G–invariant real functions on Q. Assume F is a linear (T, D, G)–weakly Noetherian function and consider a function $V \in \mathcal{I}_G$. Thus,

F is a constant of motion of $(T - V \circ \pi, D)$ and the unique generator Z of F which is a section of $\mathcal{D}$ satisfies conditions C1 and C3 of Proposition 1. Hence, Z satisfies condition C2 as well, namely $0 = Z^{TQ}(T - V \circ \pi)|_D = Z^{TQ}(T)|_D - Z(V) \circ \pi|_D$. By the arbitrariness of $V \in \mathcal{I}_G$ this implies the two conditions $Z^{TQ}(T)|_D = 0$ and $Z(V) = 0$ for any $V \in \mathcal{I}_G$. If the action is free and proper, the latter condition is equivalent to the fact that Z is a section of $\mathcal{G}$ (see [16], Theorem 2.5.10). Therefore, F has a generator which is a section of $\mathcal{D} \cap \mathcal{G}$ and satisfies $Z^{TQ}(T)|_D = 0$. This shows that F is a horizontal G–gauge momentum for (T, D). ■

3 Remarks

1. Definition 2 of weakly Noetherian functions is taken from [12]. A definition of weak Noetherianity for linear functions had been previously given in [11]. We thus show here that, at least if the action is free and proper, the two definitions agree, in that the definition given in [11] for linear functions coincides with the specialization of Definition 2 to linear functions. (As mentioned, all definitions in [11, 12] require the G–invariance of the Lagrangian, that we disregard here). Using the terminology of the present article, the definition given in [11] can be rephrased as follows: given T, D and G, a linear function on D is weakly Noetherian if (1) it has a generator which is a section of $\mathcal{G}$ and (2) it is a constant of motion of all natural linear nonholonomic systems $(T - V \circ \pi, D)$ with G–invariant potential energy V. This definition appears stronger than the specialization of Definition 2 to linear functions because of the first requirement. That it is not so follows from the fact that statement (ii) of Theorem 1 can, clearly, be restated as follows: *Fix T, D, G and assume that the action of G on Q is free and proper; then the horizontal generator of a linear function which is (T, D, G)–weakly Noetherian (in the sense of Definition 2) is a section of $\mathcal{G}$.*

2. The previous argument shows that the unique horizontal generator of a linear weakly Noetherian function is a section of $\mathcal{G}$. However, generators of linear functions on D are not unique and not all of them need to be sections of $\mathcal{G}$. We cannot characterize in geometrical terms the class of these generators which are sections of $\mathcal{G}$. However, it is possible to characterize the class of generators which preserve the kinetic energy on D *and* are sections of $\mathcal{G}$.

In order to do this, we need to introduce some notations. Fix the kinetic energy T and let $\mathcal{D}^\perp$ be the distribution on Q whose fibers are the orthogonal complements of the fibers of $\mathcal{D}$ with respect to the Riemannian metrics g_T on Q induced by T. If Z is a vector field Z on Q denote by $Z_\mathcal{D}$ and $Z_{\mathcal{D}^\perp}$ its g_T–orthogonal projections on $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. Then, two vector fields Z and Y generate the same linear function on D if and only if $Z_\mathcal{D} = Y_\mathcal{D}$ [14].

Proposition 2. *Fix T, D and G. Assume that G acts freely and properly on Q and denote by $\mathcal{I}_G$ the class of all G–invariant functions on Q. Consider a (T, D, G)–weakly Noetherian linear function F on D and let Z be a generator of F which satisfies $Z^{TQ}(T)|_D = 0$. Then, Z is a section of $\mathcal{R}^\circ_{T,V}$ for all $V \in \mathcal{I}_G$ if and only if any of the following two conditions is verified:*

(i) Z is a section of $\mathcal{G}$.

(ii) The g_T–orthogonal projection $Z_{\mathcal{D}^\perp}$ of Z is a section of $\mathcal{G}$.

Proof. (i) Fix $V \in \mathcal{I}_G$. The vector field Z generates the first integral F of $(T - V, D)$ and is a section of $\mathcal{R}^\circ_{T,V}$. Therefore, by Proposition 1, $Z^{TQ}(T - V)|_D = 0$. By the arbitrariness of $V \in \mathcal{I}_G$ this implies $Z(V) = 0$ for all $V \in \mathcal{I}_G$, that is, for a free and proper action, Z is a section of $\mathcal{G}$.

Consider any $V \in \mathcal{I}_G$. If Z is a section of $\mathcal{G}$, then $Z(V) = 0$. Hence $Z^{TQ}(T - V)|_D = 0$ and, by Proposition 1, Z is a section of $\mathcal{R}^\circ_{T,V}$. (That the action is free and proper is not not necessary for this implication).

(ii) We resort to a coordinate description. Let $(q, \dot{q}) \in U \times \mathbb{R}^n$, $U \subset \mathbb{R}^n$, be local bundle coordinates on TQ. The fiber $\mathcal{D}_q$ of $\mathcal{D}$ over a point $q \in U$ can be written as the kernel of a $k \times n$

matrix $S(q)$ which has rank k and smoothly depends on q, where $k = \operatorname{rank}\mathcal{D}$. We denote by $A(q)$ the kinetic matrix, so that $T(q,\dot q) = \frac{1}{2}\dot q^T A(q)\dot q$. As shown e.g. in [1, 10], at each point $q \in U$ the reaction force exerted by the nonholonomic constraint is a function $R : T_qU = \mathbb{R}^n \to \mathcal{D}^\circ_q$ which in coordinates can be written as

$$R(q,\dot q) \;=\; (\Pi V')(q) + r(q,\dot q)$$

where $V' = \frac{\partial V}{\partial q}$, $\Pi = S^T(SA^{-1}S^T)^{-1}SA^{-1}$ is the A^{-1}–orthogonal projector onto $\mathcal{D}^\circ$ and $r(q,\dot q)$ is a V–independent term whose expression can be found in the quoted references but is not important here. What is important here is the fact that the V–independency of $r(q,\dot q)$ implies that it equals the reaction force exerted by the constraints in the natural linear nonholonomic system (T,D).

We now compute the intersection $\bigcap_{V\in\mathcal{I}_G}\mathcal{R}^\circ_{T,V}$. The fibers of the reaction–annihilator distribution $\mathcal{R}^\circ_{T,V}$ are the annihilators of the fibers of the distribution $\mathcal{R}_{T,V}$ whose fibers are the images $\bigcup_{\dot q\in\mathcal{D}_q} R(q,\dot q)$ of the fibers of $\mathcal{D}$ under the map R. Hence $\mathcal{R}_{T,V} = \Pi V' + \mathcal{R}_{T,0}$. Therefore

$$\bigcap_{V\in\mathcal{I}_G}\mathcal{R}^\circ_{T,V} = \Big(\sum_{V\in\mathcal{I}_G}\mathcal{R}_{T,V}\Big)^\circ = \Big(\sum_{V\in\mathcal{I}_G}\Pi V' + \mathcal{R}_{T,0}\Big)^\circ .$$

Since the action of G is free and proper we have $\sum_{V\in\mathcal{I}_G} V' = \mathcal{G}^\circ$ and hence

$$\bigcap_{V\in\mathcal{I}_G}\mathcal{R}^\circ_{T,V} = (\Pi\mathcal{G}^\circ + \mathcal{R}_{T,0})^\circ = (\Pi\mathcal{G}^\circ)^\circ \cap \mathcal{R}^\circ_{T,0}$$

By definition of A^{-1}–orthogonal projection we have $\Pi\mathcal{G}^\circ = \big(\mathcal{G}^\circ + \mathcal{D}^{\circ\dagger}\big)\cap\mathcal{D}^\circ$, where the apex $\dagger$ denotes the A^{-1}–orthogonal complement, and hence

$$\bigcap_{V\in\mathcal{I}_G}\mathcal{R}^\circ_{T,V} \;=\; \big[\big(\mathcal{G}\cap\mathcal{D}^{\circ\dagger\circ}\big) + \mathcal{D}\big]\cap\mathcal{R}^\circ_{T,0}\,.$$

We now note that $\mathcal{D}^{\circ\dagger\circ} = \mathcal{D}^\perp$. In fact from $\mathcal{D} = \ker S$ it follows that $\mathcal{D}^\perp = \mathrm{Im}A^{-1}S^T$ (given that $\langle A^{-1}S^T|A|\ker S\rangle = SA^{-1}A\ker S = S\ker S = 0$) and that $\mathcal{D}^\circ = \mathrm{Im}\,S^T$, $\mathcal{D}^{\circ\dagger} = \ker SA^{-1}$, $\mathcal{D}^{\circ\dagger\circ} = \mathrm{Im}\,A^{-1}S^T$. Hence

$$\bigcap_{V\in\mathcal{I}_G}\mathcal{R}^\circ_{T,V} = \big[\big(\mathcal{G}\cap\mathcal{D}^\perp\big) + \mathcal{D}\big]\cap\mathcal{R}^\circ_{T,0}\,.$$

We can now proof statement (ii). By hypothesis, Z generates a linear constant of motion of (T,D) and its tangent lift preserves T on D. Hence, by Proposition 1, Z is a section of $\mathcal{R}_{T,0}$. Thus, the previous computation shows that Z is a section of $\bigcap_{V\in\mathcal{I}_G}\mathcal{R}^\circ_{T,V}$ if and only if it is a section of $\big(\mathcal{G}\cap\mathcal{D}^\perp\big)+\mathcal{D}$, that is, if and only if its component $Z_{\mathcal{D}^\perp}$ is a section of $\mathcal{G}$. ■

3. It has been pointed out several times that the conservation of a component of the momentum map—and more generally of a gauge momentum—does not require any G–invariance property of the constraint distribution [4, 15, 11]. The fact that, likewise, the notion of gauge momenta does not need the G–invariance of the Lagrangian indicates that, in the nonholonomic context, the conservation of a gauge momentum does not in principle require any G–invariance property of the system. Nevertheless, in all the examples we know, gauge momenta come out of a natural symmetry group of the Lagrangian.

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