Bivariate Lagrange interpolation at the Padua points: the generating curve approach

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Abstract

We give a simple, geometric and explicit construction of bivariate interpolation at certain points in a square (called Padua points), giving compact formulas for their fundamental Lagrange polynomials. We show that the associated norms of the interpolation operator, i.e., the Lebesgue constants, have minimal order of growth of $\mathcal{O}((\log n)^2)$. To the best of our knowledge this is the first complete, explicit example of near optimal bivariate interpolation points.

1 Introduction

Suppose that $\mathbf{K} \subset \mathbb{R}^d$ is a compact set with non-empty interior. Let $\Pi_n(\mathbb{R}^d)$ be the space of polynomials of degree $\leq n$ in d variables, and $\Pi_n(\mathbf{K})$ be the restrictions of this space to $\mathbf{K}$. Then $\Pi_n(\mathbf{K})$ is a vector space of some dimension, say, $N := \dim(\Pi_n(\mathbf{K}))$. Given N points $\mathcal{A} := \{A_k\}_{k=1}^N \subset \mathbf{K}$, the polynomial interpolation problem associated to $\Pi_n(\mathbf{K})$ and $\mathcal{A}$ is to find for each $f \in C(\mathbf{K})$ (the space of continuous functions on $\mathbf{K}$) a polynomial $p \in \Pi_n(\mathbf{K})$ such that

$$p(A_k) = f(A_k), \quad k = 1, \cdots, N.$$

If this is always possible the problem is said to be unisolvent, and if this is indeed the case we may construct the so-called fundamental Lagrange polynomials $\ell_j(x)$ with the property that

$$\ell_j(A_k) = \delta_{j,k},$$

the Kronecker delta. Further, the interpolant itself may be written as

$$(Lf)(x) = \sum_{k=1}^{N} f(x_k)\ell_k(x).$$

The mapping $f \mapsto Lf$ may be regarded as an operator from $C(\mathbf{K})$ (equipped with the uniform norm) to itself, and as such has an operator norm $\|L\|$. Classically, when $\mathbf{K} = [-1, 1]$, this norm is known as the Lebesgue constant and it is known that then $\|L\| \geq C \log n$ and that this minimal order of growth is attained, for example, by the Chebyshev points (see e.g. [4]).

In the multivariate case much less is known. Sündermann [6] has shown that for $\mathbf{K} = B^d$, the unit ball in $\mathbb{R}^d$, $d \geq 2$, the Lebesgue constant has a minimal rate of growth of at least $\mathcal{O}(n^{(d-1)/2})$.

In the tensor product case, when $\mathbf{K} = [-1, 1]^d$ and with the polynomial space $\bigotimes_{k=1}^d \Pi_n^1$, $\|L\| \geq C(\log n)^d$ and this minimal rate of growth is attained for the tensor product of the univariate Chebyshev points. However, even for the cube and the polynomials of total degree n, i.e., for $\mathbf{K} = [-1, 1]^d$, was not known until now whether the minimal rate of growth of $\mathcal{O}((\log n)^2)$ could be attained. In [2] it was shown that Xu's interpolation scheme [8, 1] in the square (which uses a space of polynomials intermediate to $\Pi_{n-1}(\mathbb{R}^2)$ and $\Pi_n(\mathbb{R}^2)$) the Lebesgue constant does grow like $\mathcal{O}((\log n)^2)$. In this paper we show that there is also such an interpolation scheme for the case $d = 2$, $\mathbf{K} = [-1,1]^2$, and the total degree polynomial space $\Pi_n(\mathbb{R}^2)$.

We give an explicit, simple, geometric construction of a point set in $\mathbf{K} = [-1,1]^2$, nicknamed "Padua points" (cf. [5]), for which we may construct compact formulas for the fundamental Lagrange polynomials. These compact formulas allow us to easily analyze the growth rate of the associated Lebesgue constants.

We point out that this establishes a remarkable dichotomy between the cases of the disk where the optimal growth would be of at least $\mathcal{O}(\sqrt{n})$ and the square where we now know that it is $\mathcal{O}((\log n)^2)$.

2 The Padua Points and their Generating Curve

For the remainder of this paper we take $\mathbf{K} = [-1,1]^2$ for which $\Pi_n(\mathbf{K}) = \Pi_n(\mathbb{R}^2)$ and has dimension

$$\dim(\Pi_n(\mathbf{K})) = \binom{n+2}{2}.$$

For degree $n = 0, 1, 2 \cdots$, consider the parametric curve

$$\gamma_n(t) := (\cos(nt), \cos((n+1)t)), \quad 0 \le t \le \pi. \tag{1}$$

Obviously, this curve is contained in the unit square $[-1,1]^2$. We also remark that it is an algebraic curve given by $T_{n+1}(x) = T_n(y)$ where T_n denotes the classical Chebyshev polynomial of the first kind.

Now, on $\gamma_n(t)$ consider the point set of equally spaced points (in the parameter t)

$$\mathcal{A}_n := \left\{ \gamma_n \left(\frac{k}{n(n+1)} \pi \right) \; : \; k = 0, 1, \cdots, n(n+1) \right\}. \tag{2}$$

Figure 1 below shows $\gamma_n(t)$ for $n = 5$ as well as the associated point set $\mathcal{A}_5$. Note that despite the fact that we use $n(n+1)+1 = 31$ parameter values to generate $\mathcal{A}_5$ one may count that there are precisely $21 = \binom{5+2}{2} = \dim(\Pi_5(\mathbf{K}))$ different points in $\mathcal{A}_5$, i.e., exactly the correct number for polynomial interpolation.

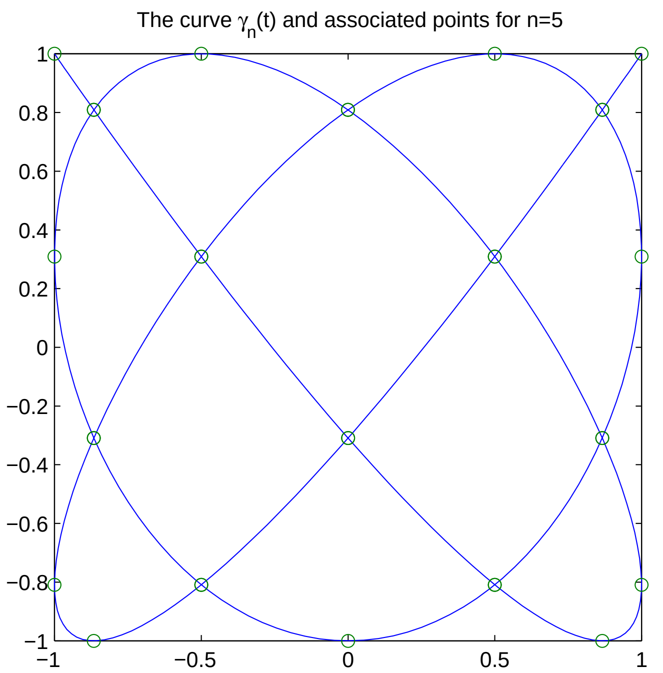

Figure 1

Notice that each of the interior points lies on a self-intersection point of the curve. Indeed, it is easy to see that

$$\gamma_n\left(\frac{jn+m(n+1)}{n(n+1)}\pi\right)=\gamma_n\left(\frac{jn-m(n+1)}{n(n+1)}\pi\right)$$

for any integers j, m. Hence we may naturally describe the set of distinct points $\mathcal{A}_n$ as

$$\mathcal{A}_n=\left\{A_{jm}:=\gamma_n\left(\frac{jn+m(n+1)}{n(n+1)}\pi\right)\ :\ j,m\geq 0 \text{ and } j+m\leq n\right\}. \quad (3)$$

More specifically, we may classify the points of $\mathcal{A}_n$ by

<u>Interior Points</u>:

$$A_{jm}=\gamma_n\left(\frac{jn+m(n+1)}{n(n+1)}\pi\right)=\left((-1)^m\cos\left(\frac{jn}{n+1}\pi\right),(-1)^j\cos\left(\frac{m(n+1)}{n}\pi\right)\right),$$

$j, m > 0$ and $j + m \leq n$.

Vertex Points: $A_{00} = (1,1)$ and $A_{0,n} = ((-1)^n, (-1)^{(n+1)})$.
Vertical Edge Points: $A_{0m} = \left((-1)^m, \cos\left(\frac{m(n+1)}{n}\pi\right)\right)$, $m = 1, 2, \cdots, n$.
Horizontal Edge Points: $A_{j0} = \left(\cos\left(\frac{jn}{n+1}\pi\right), (-1)^j\right)$, $j = 1, 2, \cdots, n$.

It follows then that the number of distinct points in $\mathcal{A}_n$ is indeed $\binom{n+2}{2}$.

3 The Fundamental Lagrange Polynomials

Consider the weighted integral

$$I(f) := \frac{1}{\pi^2}\int_{[-1,1]^2} f(x,y)\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}}dxdy$$

and the associated inner product

$$\langle f, g\rangle := \frac{1}{\pi^2}\int_{[-1,1]^2} f(x,y)g(x,y)\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}}dxdy. \tag{4}$$

Note that the polynomials $\{T_j(x)T_m(y) : j+m \le n\}$ form an orthogonal basis for $\Pi_n(\mathbb{R}^2)$ with respect to this inner product. Further, the finite dimensional space $\Pi_n(\mathbb{R}^2)$ has a reproducing kernel with respect to this inner product, which we denote by $K_n(\cdot;\cdot)$. Specifically, for each $A, B \in [-1,1]^2$, $K_n(A;\cdot) \in \Pi_n(\mathbb{R}^2)$, $K_n(\cdot;B) \in \Pi_n(\mathbb{R}^2)$ and

$$\langle p, K_n(A;\cdot)\rangle = p(A)$$

for all $p \in \Pi_n(\mathbb{R}^2)$.

We will show that there is a remarkable quadrature formula for $I(f)$ based on the Padua points (Theorem 1 below).

Because of this quadrature formula, it is perhaps not unexpected that there are formulas for the fundamental Lagrange polynomials of the Padua points in terms of K_n. We begin with the following two lemmas.

Lemma 1 *For all $p \in \Pi_{2n}(\mathbb{R}^2)$ we have*

$$\frac{1}{\pi^2}\int_{[-1,1]^2} p(x,y)\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}}dxdy = \frac{1}{\pi}\int_0^\pi p(\gamma_n(t))dt.$$

Proof. We need only show that this holds for

$$p(x,y) = T_j(x)T_m(y), \quad j+m \le 2n.$$

But, when $j=m=0$, $p(x,y)=1$, and clearly both sides are equal to 1. Otherwise, if $(j,m) \neq (0,0)$, the left side equals 0 while the right side equals

$$\begin{aligned} \frac{1}{\pi}\int_0^\pi T_j(\cos(nt))T_m(\cos((n+1)t))dt &= \frac{1}{\pi}\int_0^\pi \cos(jnt)\cos(m(n+1)t)dt \\ &= 0 \end{aligned}$$

provided $jn \neq m(n+1)$. But since n and $n+1$ are relatively prime, $jn = m(n+1)$ implies that $j=\alpha(n+1)$ and $m=\alpha n$ for some positive integer α. Hence $j+m=\alpha(n+(n+1)) > 2n$. □

Theorem 1 *Consider the following weights associated to the Padua points $A \in \mathcal{A}_n$:*

$$w_A := \frac{1}{n(n+1)}\begin{cases} 1/2 & \text{if } A \in \mathcal{A}_n \text{ is a vertex point} \\ 1 & \text{if } A \in \mathcal{A}_n \text{ is an edge point} \\ 2 & \text{if } A \in \mathcal{A}_n \text{ is an interior point} \end{cases}.$$

Then, if $p(x,y) \in \Pi_{2n}(\mathbb{R}^2)$ is such that $\langle p(x,y), T_{2n}(y)\rangle = 0$, we have

$$\frac{1}{\pi^2}\int_{[-1,1]^2} p(x,y)\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}}dxdy = \sum_{A\in\mathcal{A}_n} w_A p(A).$$

In particular, since $T_{2n}(y) = 2(T_n(y))^2 - 1$, this quadrature formula holds for $p = fg$ with $f,g \in \Pi_n(\mathbb{R}^2)$ and either $\langle f(x,y), T_n(y)\rangle = 0$ or $\langle g(x,y), T_n(y)\rangle = 0$.

Proof. First note that the quadrature formula

$$\frac{1}{\pi}\int_0^\pi q(t)dt = \frac{1}{N}\left\{\frac{1}{2}q(0) + \sum_{k=1}^{N-1} q(\frac{k}{N}\pi) + \frac{1}{2}q(\pi)\right\} \tag{5}$$

holds for all *even* trigonometric polynomials $q(t)$ (i.e. has an expansion in cosines only) for which the expansion of $q(t)$ does not contain terms with frequency an even multiple of N, i.e., such that $\int_0^\pi q(t)\cos(2\alpha N t)\,dt = 0$, $\alpha = 1,2,\cdots$.

Now, by Lemma 1, we have

$$\frac{1}{\pi^2}\int_{[-1,1]^2} p(x,y)\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}}dxdy = \frac{1}{\pi}\int_0^\pi p(\gamma_n(t))dt$$

where

$$p(\gamma_n(t)) = p(\cos(nt), \cos((n+1)t)) =: q(t)$$

is an *even* trigonometric polynomial (of degree at most $n(n+1)$) to which we may apply the quadrature formula (5) with $N = n(n+1)$ provided, under the assumptions placed on p, we have $\int_0^\pi q(t)\cos(2Nt)dt = 0$. However, if we write

$$p(x,y) =: \sum_{j=0}^{2n}\sum_{m=0}^{2n-j} a_{jm}T_j(x)T_m(y)$$

then

$$\begin{aligned} p(\gamma_n(t)) &= \sum_{j=0}^{2n}\sum_{m=0}^{2n-j} a_{jm}\cos(jnt)\cos(m(n+1)t) \\ &= \sum_{j=0}^{2n}\sum_{m=0}^{2n-j} \frac{a_{jm}}{2}\{\cos((jn+m(n+1))t) + \cos((jn-m(n+1))t)\} \end{aligned}$$

so that $a_{0,2n}$, the coefficient of $\cos(2n(n+1)t)$ in the expansion of $p(\gamma_n(t))$, is exactly the coefficient of $T_{2n}(y)$ in the expansion of $p(x,y)$. By assumption this is zero. Hence we apply Lemma 1 and (5) (with $N = n(n+1)$) to obtain

$$\begin{aligned} & \frac{1}{\pi^2}\int_{[-1,1]^2} p(x,y)\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}}dxdy \\ = & \frac{1}{\pi}\int_0^\pi p(\gamma_n(t))dt \\ = & \frac{1}{n(n+1)}\left\{\frac{1}{2}p(\gamma_n(0)) + \sum_{k=1}^{N-1} p(\gamma_n(\frac{k}{N}\pi)) + \frac{1}{2}p(\gamma_n(\pi))\right\} \\ = & \sum_{A\in\mathcal{A}_n} w_A p(A) \end{aligned}$$

since $\gamma_n(0)$ and $\gamma_n(\pi)$ are the two Vertex points and the Interior points double up. □

The quadrature just proved has precision $< 2n$ as it is exact for all polynomials of degree $2n-1$ or less. From the general theory on quadrature formulas and polynomial ideals, it follows that the set of Padua points is a variety of a polynomial ideal generated by quasi-orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle$, from which a compact formula for the Lagrange interpolation polynomial can be derived. This approach based on ideal theory will be developed fully in [3]. Here we give a proof of the compact interpolation formula.

For $A \in \mathcal{A}_n$ we will let y_A denote the y-coordinate of A.

Theorem 2 *For $B \in \mathcal{A}_n$, set*

$$L_B(x,y) := w_B[K_n(B;(x,y)) - T_n(y)T_n(y_B)], \quad B \in \mathcal{A}_n. \tag{6}$$

Then we have the interpolation formula

$$\sum_{B\in\mathcal{A}_n} L_B(A)p(B) = p(A), \quad \forall p \in \Pi_n(\mathbb{R}^2),\ \forall A \in \mathcal{A}_n. \tag{7}$$

Proof. Since, as it is easy to see, $\langle T_n(y), T_n(y)\rangle = 1/2$ and $\langle K_n(\cdot;A), T_n(y)\rangle = T_n(y_A)$, we have

$$\begin{aligned}\langle K_n(\cdot;A) - 2T_n(y_A)T_n(y), T_n(y)\rangle &= T_n(y_A) - 2T_n(y_A)\langle T_n(y), T_n(y)\rangle \\ &= T_n(y_A) - 2T_n(y_A)\frac{1}{2} \\ &= 0.\end{aligned}$$

Hence, we may apply the quadrature formula of Lemma 1 to $(K_n(\cdot;A) - 2T_n(y_A)T_n(y)) \times p$ for any $p \in \Pi_n(\mathbb{R}^2)$ and obtain

$$\begin{aligned}& p(A) - 2T_n(y_A)\langle T_n(y), p\rangle \\ =\ & \langle K_n(\cdot;A) - 2T_n(y_A)T_n(y), p\rangle \\ =\ & \sum_{B\in\mathcal{A}_n} w_B(K_n(B;A) - 2T_n(y_A)T_n(y_B))p(B) \\ =\ & \sum_{B\in\mathcal{A}_n} w_B(K_n(B;A) - T_n(y_A)T_n(y_B))p(B) - T_n(y_A)\sum_{B\in\mathcal{A}_n} w_B T_n(y_B)p(B).\end{aligned}$$

It follows that

$$\begin{aligned}& \sum_{B\in\mathcal{A}_n} w_B(K_n(B;A) - T_n(y_A)T_n(y_B))p(B) \\ =\ & p(A) - T_n(y_A)\left\{2\langle T_n(y), p\rangle - \sum_{B\in\mathcal{A}_n} w_B T_n(y_B)p(B)\right\}.\end{aligned} \tag{8}$$

But, writing $p = \tilde{p} + \alpha T_n(y)$ where $\alpha := 2\langle p, T_n(y)\rangle$ and $\langle \tilde{p}, T_n(y)\rangle = 0$, we have

$$\begin{aligned}
\sum_{B\in\mathcal{A}_n} w_B T_n(y_B)p(B) &= \sum_{B\in\mathcal{A}_n} w_B T_n(y_B)[\tilde{p}(B) + \alpha T_n(y_B)] \\
&= \langle T_n(y), \tilde{p}\rangle + \alpha \sum_{B\in\mathcal{A}_n} w_B T_n^2(y_B) \\
&= 0 + \alpha \sum_{B\in\mathcal{A}_n} w_B \times 1 \\
&= \alpha \times 1 \\
&= 2\langle p, T_n(y)\rangle
\end{aligned}$$

where we have used the fact that $T_n^2(y_B) = 1$ for all $B \in \mathcal{A}_n$ and that the sum of the weights w_B is one.

Therefore, by (8), we have

$$\sum_{B\in\mathcal{A}_n} w_B(K_n(B; A) - T_n(y_A)T_n(y_B))p(B) = p(A)$$

for all $p \in \Pi_n(\mathbb{R}^2)$ and all points $A \in \mathcal{A}_n$, which completes the proof. $\square$

The polynomials L_B are indeed the fundamental Lagrange polynomials, i.e., they satisfy

$$L_B(A) = \delta_{A,B}, \qquad A, B \in \mathcal{A}_n. \tag{9}$$

This can be easily verified using the following compact formula for the reproducing kernel K_n proved in [7].

Lemma 2 *For $A, B \in [-1, 1]^2$ write $A = (\cos(\theta_1), \cos(\theta_2))$, $B = (\cos(\phi_1), \cos(\phi_2))$. Then*

$$\begin{aligned}
K_n(A; B) &= D_n(\theta_1 + \phi_1, \theta_2 + \phi_2) + D_n(\theta_1 + \phi_1, \theta_2 - \phi_2) \\
&+ D_n(\theta_1 - \phi_1, \theta_2 + \phi_2) + D_n(\theta_1 - \phi_1, \theta_2 - \phi_2)
\end{aligned}$$

where

$$D_n(\alpha, \beta) = \frac{1}{2}\frac{\cos((n + 1/2)\alpha)\cos(\alpha/2) - \cos((n + 1/2)\beta)\cos(\beta/2)}{\cos(\alpha) - \cos(\beta)}.$$

We remark that (9) does not follow from the interpolation formula (7) (as a simple example, consider the formula $(f(A) + f(B))/2 = f(C)$ valid for all $f \in \Pi_0$ and arbitrary A, B, C). A direct derivation of the compact formulas in (6) and (7) will be given in [3].

4 The Growth of the Lebesgue Function

We first give a more convenient expression for D_n, which is obtained by simple trigonometric manipulations.

Lemma 3 (cf. Lemma 1 of [2]) *The function* $D_n(\alpha, \beta)$ *can be written as*

$$D_n(\alpha, \beta) = \frac{1}{4}\left\{\frac{\sin n\left(\frac{\alpha+\beta}{2}\right)\sin n\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)} + \frac{\sin(n+1)\left(\frac{\alpha+\beta}{2}\right)\sin(n+1)\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}\right\}. \tag{10}$$

Now, note that the norm of the polynomial projection

$$Lf := \sum_{A\in\mathcal{A}_n} f(A)L_A$$

considered as a map from $C(\mathbf{K})$ to $C(\mathbf{K})$, equipped with the uniform norm, is given by the maximum of the so-called Lebesgue function

$$\Lambda_n(x, y) := \sum_{A\in\mathcal{A}_n} |L_A(x, y)|. \tag{11}$$

Specifically,

$$||L|| = \max_{(x,y)\in[-1,1]^2} \Lambda_n(x, y).$$

We now proceed to give a bound on $\Lambda_n(x, y)$.

Theorem 3 *There is a constant* $C > 0$ *such that the Lebesgue function is bounded,*

$$\Lambda_n(x, y) \le C(\log n)^2, \quad n \ge 2,\ (x, y) \in [-1, 1]^2.$$

Proof. First recall that the Padua points are given by

$$A_{jm} = \gamma_n\left(\frac{jn + m(n+1)}{n(n+1)}\pi\right), \quad j, m \ge 0,\ j + m \le n.$$

Some simple manipulations show that, in fact,

$$A_{jm} = (-1)^{j+m}(\cos(\frac{j}{n+1}\pi), \cos(\frac{m}{n}\pi)). \tag{12}$$

Now,

$$\begin{aligned}
\Lambda_n(x,y) &= \sum_{A\in\mathcal{A}_n} |L_A(x,y)| \\
&= \sum_{A\in\mathcal{A}_n} w_A\,|K_n(A;(x,y)) - T_n(y)T_n(y_A)| \\
&\le \sum_{A\in\mathcal{A}_n} w_A|K_n(A;(x,y))| + \sum_{A\in\mathcal{A}_n} w_A|T_n(y)|\,|T_n(y_A)| \\
&\le \sum_{A\in\mathcal{A}_n} w_A|K_n(A;(x,y)| + 1 \qquad (13)
\end{aligned}$$

since $|T_n(y)| \le 1$ and the sum of the weights w_A is 1.

To bound this latter sum (13) we use the compact formula for K_n given in Lemma 2 and the alternate expression for D_n given in Lemma 3. Indeed it is clear that our result will follow if we find an upper bound for the sum

$$\sum_{A\in\mathcal{A}_n} w_A \left| \frac{\sin n\left(\frac{\alpha_A+\beta_A}{2}\right)}{\sin\left(\frac{\alpha_A+\beta_A}{2}\right)} \frac{\sin n\left(\frac{\alpha_A-\beta_A}{2}\right)}{\sin\left(\frac{\alpha_A-\beta_A}{2}\right)} \right| \qquad (14)$$

where we write $(x,y) =: (\cos(\phi_1), \cos(\phi_2))$ and

$$A = A_{jm}, \quad \alpha_A := \frac{j}{n+1}\pi + \phi_1, \quad \beta_A := \frac{m}{n}\pi + \phi_2.$$

The analysis of the other terms that result from expanding K_n by means of Lemmas 2 and 3 is entirely analogous. We should note, however, that the $(-1)^{j+m}$ factor in (12) has no effect on the sum (14) due to the fact that $|\sin(k(\theta+\pi))| = |\sin(k\theta)|$ for any integer k.

We now proceed to bound (14). Using the definitions of w_A, α_A and β_A, we have

$$\begin{aligned}
&\sum_{A\in\mathcal{A}_n} w_A \left| \frac{\sin n\left(\frac{\alpha_A+\beta_A}{2}\right)}{\sin\left(\frac{\alpha_A+\beta_A}{2}\right)} \frac{\sin n\left(\frac{\alpha_A-\beta_A}{2}\right)}{\sin\left(\frac{\alpha_A-\beta_A}{2}\right)} \right| \\
&\le \frac{2}{n(n+1)} \sum_{j=0}^{n}\sum_{m=0}^{n-j} \left| \frac{\sin n\left(\frac{\phi_1+\phi_2}{2} + \frac{j}{n+1}\frac{\pi}{2} + \frac{m}{n}\frac{\pi}{2}\right)}{\sin\left(\frac{\phi_1+\phi_2}{2} + \frac{j}{n+1}\frac{\pi}{2} + \frac{m}{n}\frac{\pi}{2}\right)} \right| \left| \frac{\sin n\left(\frac{\phi_1-\phi_2}{2} + \frac{j}{n+1}\frac{\pi}{2} - \frac{m}{n}\frac{\pi}{2}\right)}{\sin\left(\frac{\phi_1-\phi_2}{2} + \frac{j}{n+1}\frac{\pi}{2} - \frac{m}{n}\frac{\pi}{2}\right)} \right| \\
&\le \frac{2}{n(n+1)} \sum_{j=0}^{n}\sum_{m=0}^{n} \left| \frac{\sin n\left(\frac{\phi_1+\phi_2}{2} + \frac{j}{n+1}\frac{\pi}{2} + \frac{m}{n}\frac{\pi}{2}\right)}{\sin\left(\frac{\phi_1+\phi_2}{2} + \frac{j}{n+1}\frac{\pi}{2} + \frac{m}{n}\frac{\pi}{2}\right)} \right| \left| \frac{\sin n\left(\frac{\phi_1-\phi_2}{2} + \frac{j}{n+1}\frac{\pi}{2} - \frac{m}{n}\frac{\pi}{2}\right)}{\sin\left(\frac{\phi_1-\phi_2}{2} + \frac{j}{n+1}\frac{\pi}{2} - \frac{m}{n}\frac{\pi}{2}\right)} \right|. \qquad (15)
\end{aligned}$$

Now this last expression (15) is a Riemann sum for a multiple of the (improper) integral

$$\int_0^{\pi/2}\int_0^{\pi/2}\left|\frac{\sin n\left(\frac{\phi_1+\phi_2}{2}+\alpha+\beta\right)}{\sin\left(\frac{\phi_1+\phi_2}{2}+\alpha+\beta\right)}\right|\left|\frac{\sin n\left(\frac{\phi_1-\phi_2}{2}+\alpha-\beta\right)}{\sin\left(\frac{\phi_1-\phi_2}{2}+\alpha-\beta\right)}\right| d\alpha d\beta. \tag{16}$$

The integrand of (16) is bounded by

$$\frac{1}{|\sin\left(\frac{\phi_1+\phi_2}{2}+\alpha+\beta\right)|}\,\frac{1}{|\sin\left(\frac{\phi_1-\phi_2}{2}+\alpha-\beta\right)|}.$$

Note that the denominator of this upper bound is zero at most along two lines in the domain $(\alpha,\beta)\in[0,\pi/2]^2$, which corresponds to at most Cn points in the corresponding sum (15), for some constant $C>0$. At each of these "singularities", we may use the fact that $|\sin(n\theta)/\sin(\theta)|\le n$ to reduce that portion of the sum (15) that involves the singular points to a univariate sum similar to

$$\frac{2}{n+1}\sum_{k=0}^{n}\left|\frac{\sin n(\theta+\frac{k}{n}\pi)}{\sin(\theta+\frac{k}{n}\pi)}\right|$$

which a classical analysis (cf. Lemma 4 of [2]) shows to be bounded by $C\log n$ for some constant $C>0$.

Thus, by the monotonicity and periodicity of $\csc(\theta)$, it follows that the sum (15) is bounded by a constant multiple of

$$\int_{\pi/(n+1)}^{\pi/2}\int_{\pi/(n+1)}^{\pi/2}|\csc(\alpha)\csc(\beta)|\,d\alpha d\beta. \tag{17}$$

This in turn is easily evaluated and seen to be bounded by $C(\log n)^2$. The result follows. □

Acknowledgments. Work supported by the ex-60% funds of the Universities of Padua and Verona, and by the GNCS-INdAM.

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