On the coordinate ring of spherical conjugacy classes

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Abstract

Let G be a simple algebraic group over an algebraically closed field k of characteristic zero and $\mathcal{O}$ be a spherical conjugacy class of G. We determine the decomposition of the coordinate ring $k[\mathcal{O}]$ of $\mathcal{O}$ into simple G-modules.

1 Introduction

In [9] we proved the De Concini-Kac-Procesi conjecture on the quantized enveloping algebra $\mathcal{U}_\varepsilon(\mathfrak{g})$ (introduced in [14]) for simple $\mathcal{U}_\varepsilon(\mathfrak{g})$-modules over spherical conjugacy classes of G (we recall that a conjugacy class $\mathcal{O}$ in G is called *spherical* if a Borel subgroup of G has a dense orbit in $\mathcal{O}$): our main tool was the representation theory of the quantized Borel subalgebra B_ε introduced in [15].

To fix the notation, G is a complex simple simply-connected algebraic group, $\mathfrak{g}$ its Lie algebra, B a Borel subgroup of G, T a maximal torus of B, B^- the Borel subgroup opposite to B, $\{\alpha_1, \dots, \alpha_n\}$ the set of simple roots with respect to the choice of (T, B). Let W be the Weyl group of G and let us denote by s_i the reflection corresponding to the simple root α_i: $\ell(w)$ is the length of the element $w \in W$ and $\mathrm{rk}(1-w)$ is the rank of $1-w$ in the geometric representation of W.

The representation theory of $\mathcal{U}_\varepsilon(\mathfrak{g})$ is related to the stratification of G given by conjugacy classes, while the representation theory of B_ε is related to the stratification $\{X_w \mid w \in W\}$ of B^-, where $X_w = B^- \cap BwB$ for every $w \in W$ (each X_w is an affine variety of dimension $n + \ell(w)$). We proved that for every spherical conjugacy class $\mathcal{O}$ in G, there exists $w \in W$ such that $\mathcal{O} \cap X_w \neq \varnothing$ and $\ell(w) + rk(1-w) = \dim \mathcal{O}$: this then allows to prove the De Concini-Kac-Procesi conjecture for simple $\mathcal{U}_\varepsilon(\mathfrak{g})$-modules over elements in $\mathcal{O}$. In fact we proved also a result in the opposite direction, giving therefore a characterization of spherical conjugacy classes in terms of the Weyl group ([9], Theorem 25):

let $\mathcal{O}$ be a conjugacy class of G and $w = w(\mathcal{O})$ be the unique element in W such that $\mathcal{O} \cap BwB$ is dense in $\mathcal{O}$. Then $\mathcal{O}$ is spherical if and only if $\dim \mathcal{O} = \ell(w) + rk(1-w)$.

Moreover w is always an involution (see [9], Remark 4, [10], Theorem 2.7). From this result we conjectured that, for a spherical $\mathcal{O}$, the decomposition of the ring $\mathbb{C}[\mathcal{O}]$ of regular functions on $\mathcal{O}$ (to which we refer as to *the coordinate ring* of $\mathcal{O}$) as a G-module should be strictly related to $w(\mathcal{O})$. This is the motivation for the present paper.

We recall that $\mathbb{C}[\mathcal{O}]$ is multiplicity-free, so that in order to obtain the decomposition of $\mathbb{C}[\mathcal{O}]$ into simple components one has just to determine which simple modules occur in $\mathbb{C}[\mathcal{O}]$:

$$\mathbb{C}[\mathcal{O}] \underset{G}{\cong} \bigoplus_{\lambda \in \lambda(\mathcal{O})} V(\lambda)$$

where for each dominant weight λ, $V(\lambda)$ is the simple G-module of highest weight λ (if $\lambda \in \lambda(\mathcal{O})$ we say that λ *occurs* in $\mathbb{C}[\mathcal{O}]$).

The decomposition of the coordinate ring $\mathbb{C}[X]$ for G-varieties X has been investigated by various authors. If λ is a non-zero highest weight, and $v \in V(\lambda)$ is a non-zero highest weight vector, then $\mathbb{C}[G.v]$ is isomorphic to $\underset{n \geq 0}{\oplus} V(n\lambda)^*$ ([44], Theorem 2). In particular this determines $\mathbb{C}[\mathcal{O}]$ for the minimal unipotent orbit of G. For a unipotent class in G (equivalently nilpotent orbit in $\mathfrak{g}$) McGovern ([30], Theorem 3.1) decribes $\mathbb{C}[\mathcal{O}]$ in terms of induced building blocks from a certain Levi subgroup of G (via sheaf cohomology on G/Q, Q a parabolic subgroup of G associated to $\mathcal{O}$): it is then possible to obtain multiplicities of simple G-modules in $\mathbb{C}[\mathcal{O}]$ as an alternating sum of certain partition functions. In the same paper the author gives a formula for $\mathbb{C}[\hat{\mathcal{O}}]$, where $\hat{\mathcal{O}}$ is the simply-connected cover of $\mathcal{O}$ ([30], Theorem 4.1). Then in [31] there are tables for the sets of simple modules in $\mathbb{C}[\hat{\mathcal{O}}]$ for spherical unipotent classes in the classical groups (and conjecturally in the exceptional groups). For type F_4 the monoid $\lambda(\mathcal{O})$ has been described in [7] for all spherical unipotent classes. For the maximal spherical unipotent class $\mathcal{O}$ in E_8, it has been shown in [2], Theorem 1.1, that every simple G-module occurs in $\mathbb{C}[\mathcal{O}]$ (so that $\mathcal{O}$ is a model orbit). In [36], Panyushev gives tables for the sets of simple modules for (spherical) nilpotent orbits of height 2 (and conjecturally for height 3). In [28] the author describes explicitly the structure of principal model homogeneous spaces. For semisimple spherical classes, the description of $\lambda(\mathcal{O})$ may be deduced from the tables in [26]. See also [45], Théorème 3, where symmetric varieties are considered.

The main result of this paper is the following:

Theorem. *Assume $\mathcal{O}$ is a spherical conjugacy class in G, and let $w = w(\mathcal{O})$. Then a dominant weight λ occurs in $\mathbb{C}[\mathcal{O}]$ if and only if $w(\lambda) = -\lambda$ and $\lambda(S_{\mathcal{O}}) = 1$.*

Here $S_{\mathcal{O}}$ is a certain (finite) elementary abelian 2-subgroup of T which we determine for every spherical conjugacy class, describing therefore explicitly $\lambda(\mathcal{O})$: see tables $1, \dots, 26$. In particular we completely solve the problem of determining the simple modules occurring in $\mathbb{C}[\mathcal{O}]$ for unipotent classes ([22], 8.13, Remark 2), and obtain the decomposition of $\mathbb{C}[\mathcal{O}]$ for conjugacy classes of mixed elements.

Our proof is based on the deformation result obtained by Brion in [4]. We have $\mathbb{C}[\mathcal{O}] = \mathbb{C}[G/H] = \mathbb{C}[G]^H$, where H is the centralizer of an element of $\mathcal{O}$ in G. There exists a flat deformation of G/H to a quotient G/H_0, where H_0 contains the unipotent radical U^- of B^-. We determine the decomposition of $\mathbb{C}[G/H_0]$ into simple components (i.e. we determine $\lambda(G/H_0)$), relating the group H_0 with H via the theory of elementary embeddings ([29], [5]). We then prove the crucial fact that $\lambda(\mathcal{O})$ is saturated ([34], §1.3), so that $\mathbb{C}[G/H] = \mathbb{C}[G/H_0]$ as G-modules. We also determine the decomposition of the coordinate ring $\mathbb{C}[\hat{\mathcal{O}}]$ for the simply-connected cover $\hat{\mathcal{O}}$ of $\mathcal{O}$, and of $\mathbb{C}[\overline{\mathcal{O}}]$.

The paper is structured as follows. In Section 2 we introduce the notation. In Section 3 we recall some basic facts about spherical varieties and we prove the main theorem. In Section 4 we determine the group $S_{\mathcal{O}}$ for the spherical conjugacy classes in the various groups, determining therefore the monoid $\lambda(\mathcal{O})$, and also $\lambda(\hat{O})$. In Section 5 we consider the coordinate ring $\mathbb{C}[\overline{\mathcal{O}}]$ of the closure of $\mathcal{O}$. It is well known that $\mathbb{C}[\overline{\mathcal{O}}] = \mathbb{C}[\mathcal{O}]$ if and only if $\overline{\mathcal{O}}$ is normal: we list all cases in which the spherical conjugacy class $\mathcal{O}$ has normal closure and we determine $\lambda(\overline{\mathcal{O}})$ for the classes with non-normal closure. In section 6 we consider the case when G in not necessarily simply-connected.

All the results and proofs of this article remain valid for G a simple simply-connected algebraic group over an algebraically closed field k of characteristic zero.

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2 Preliminaries

We denote by $\mathbb{C}$ the complex numbers, by $\mathbb{R}$ the reals, by $\mathbb{Z}$ the integers and by $\mathbb{N}$ the natural numbers.

Let $A = (a_{ij})$ be a finite indecomposable Cartan matrix of rank n. To A there is associated a root system Φ, a simple Lie algebra $\mathfrak{g}$ and a simple simply-connected algebraic group G over $\mathbb{C}$. We fix a maximal torus T of G, and a Borel subgroup B containing T: B^- is the Borel subgroup opposite to B, U (respectively U^-) is the unipotent radical of B (respectively of B^-). If χ is a character of T, we still denote by χ the character of B which extends χ. We denote by $\mathfrak{h}$ the Lie algebra of T. Then Φ is the set of roots relative to T, and B determines the set of positive roots Φ^+, and the simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. We fix a total ordering on Φ^+ compatible with the height function. We shall use the numbering and the description of the simple roots in terms of the canonical basis $(e_1, \ldots, e_k)$ of an appropriate $\mathbb{R}^k$ as in [3], Planches I-IX. For the exceptional groups, we shall write $\beta = (m_1, \ldots, m_n)$ for $\beta = m_1\alpha_1 + \ldots + m_n\alpha_n$.

If γ is a character of T, we shall also denote by γ the corresponding linear form $(d\gamma)_1$ on $\mathfrak{h}$. The real subspace of $\mathfrak{h}^*$ spanned by the roots is a Euclidean space E, endowed with the scalar product $(\alpha_i, \alpha_j) = d_i a_{ij}$. Here $\{d_1, \dots, d_n\}$ are relatively prime positive integers such that if D is the diagonal matrix with entries $d_1, \dots, d_n$, then DA is symmetric. P is the weight lattice, P^+ the monoid of dominant weights and W the Weyl group; s_i is the simple reflection associated to α_i, $\{\omega_1, \dots, \omega_n\}$ are the fundamental weights, w_0 is the longest element of W. In the expression $\lambda = \sum_i k_i n_i \omega_i$ we always assume k_i's and n_i's in $\mathbb{N}$. If V is a G-module, $v \in V$, $f \in V^*$, then the matrix coefficient $c_{f,v} : G \to \mathbb{C}$ is defined by $c_{f,v}(g) = f(g.v)$ for $g \in G$. We consider the action of $G \times G$ on $\mathbb{C}[G]$

$$((g, g_1).f)(c) = f(g^{-1} c g_1)$$

for c, g, $g_1 \in G$, $f \in \mathbb{C}[G]$. The algebraic version of the Peter-Weyl theorem gives the decomposition

$$\mathbb{C}[G] = \bigoplus_{\lambda \in P^+} V(-w_0\lambda)^* \otimes V(-w_0\lambda) \tag{2.1}$$

We put $\Pi = \{1, \dots, n\}$ and we fix a Chevalley basis $\{h_i, i \in \Pi; e_\alpha, \alpha \in \Phi\}$ of $\mathfrak{g}$. We shall denote by $\check{\omega}_i$, for $i = 1, \dots, n$, the elements in $\mathfrak{h}$ defined by $\alpha_j(\check{\omega}_i) = \delta_{ij}$ (recall that $\omega_j(h_i) = \delta_{ij}$) for $j = 1, \dots, n$. As usual we put $\langle x, y\rangle = \frac{2(x,y)}{(y,y)}$.

We use the notation $x_\alpha(k)$, $h_\alpha(z)$, for $\alpha \in \Phi$, $k \in \mathbb{C}$, $z \in \mathbb{C}^*$ as in [43], [11]. For $\alpha \in \Phi$ we put $X_\alpha = \{x_\alpha(k) \mid k \in \mathbb{C}\}$, the root-subgroup corresponding to α, and $H_\alpha = \{h_\alpha(z) \mid z \in \mathbb{C}^*\}$. For $h \in \mathfrak{h}$ we put $H_h = \exp \mathbb{C}h$. We identify W with N/T, where N is the normalizer of T: given an element $w \in W$ we shall denote a representative of w in N by $\dot{w}$. We choose the x_α's so that, for all $\alpha \in \Phi$, $n_\alpha = x_\alpha(1) x_{-\alpha}(-1) x_\alpha(1)$ lies in N and has image the reflection s_α in W. Then

$$x_\alpha(\xi) x_{-\alpha}(-\xi^{-1}) x_\alpha(\xi) = h_\alpha(\xi) n_\alpha \quad , \quad n_\alpha^2 = h_\alpha(-1) \tag{2.2}$$

for every $\xi \in \mathbb{C}^*$, $\alpha \in \Phi$ ([41], Proposition 11.2.1).

We put $T^w = \{t \in T \mid wtw^{-1} = t\}$, $T_2 = \{t \in T \mid t^2 = 1\}$. In particular $T^w = T_2$ if $w = w_0 = -1$.

For algebraic groups we use the notation in [19], [12]. In particular, for $J \subseteq \Pi$, $\Delta_J = \{\alpha_j \mid j \in J\}$, Φ_J is the corresponding root system, W_J the Weyl group, P_J the standard parabolic subgroup of G, $L_J = T\langle X_\alpha \mid \alpha \in \Phi_J\rangle$ the standard Levi subgroup of P_J. For $z \in W$ we put $U_z = U \cap z^{-1} U^- z$. Then the unipotent radical $R_u P_J$ of P_J is $U_{w_0 w_J}$, where w_J is the longest element of W_J. Moreover $U \cap L_J = U_{w_J}$ is a maximal unipotent subgroup of L_J.

If Ψ is a subsystem of type X_r of Φ and H is the subgroup generated by X_α, $\alpha \in \Psi$, we say that H is a X_r-subgroup of G.

If X is an algebraic variety, we denote by $\mathbb{C}[X]$ the ring of regular functions on X. If X is a multiplicity-free G-variety, then we denote by $\lambda(X)$ the set of dominant weights occurring in $\mathbb{C}[X]$, i.e. $\lambda \in P^+$ such that $\mathbb{C}[X]$ contains (a copy of) $V(\lambda)$. If $x \in X$ we denote by $G.x$

the G-orbit of x and by G_x the isotropy subgroup of x in G. If the homogeneous space G/H is spherical, we say that H is a spherical subgroup of G.

If x is an element of a group K and $H \leq K$, we shall also denote by $C(x)$ the centralizer of x in K, and by $C_H(x)$ the centralizer of x in H. If x, $y \in K$, then $x \sim y$ means that x, y are conjugate in K. For unipotent classes in exceptional groups we use the notation in [12]. We use the description of centralizers of involutions as in [21].

3 The main theorem

Let $\mathcal{O}$ be a spherical conjugacy class. Our aim is to determine $\lambda(\mathcal{O})$. For this purpose if H is the centralizer of an element in $\mathcal{O}$, we have $\mathbb{C}[\mathcal{O}] = \mathbb{C}[G/H] = \mathbb{C}[G]^H$ and, from (2.1),

$$\mathbb{C}[G]^H = \bigoplus_{\lambda \in \lambda(\mathcal{O})} V(-w_0\lambda)^* \otimes u_\lambda$$

where $0 \neq u_\lambda \in V(-w_0\lambda)^H$ ([37], Theorem 3.12). We start by considering in general a spherical homogeneous space G/H. Without loss of generality we may assume BH dense in G. By [4], Theorem 1, there exists a (flat) deformation of G/H to a homogeneuos (spherical) space G/H_0, where H_0 contains a maximal unipotent subgroup of G (such an homogeneous space is called *horospherical*, and H_0 a horospherical contraction of H). An *elementary embedding* of G/H is a pair (X, x) where X is a normal algebraic G-variety, $x \in X$ is such that $G.x$ is dense in X, $G_x = H$ and $X \setminus G.x$ is a G-orbit of codimension 1 ([6], 2.2). In [4] Brion constructs a $G \times \mathbb{C}^*$-variety and a flat $G \times \mathbb{C}^*$-morphism $p : Z \to \mathbb{C}$ (where G acts trivially on $\mathbb{C}$ and $\mathbb{C}^*$ acts via homotheties) such that $p^{-1}(\mathbb{C}^*) \cong G/H \times \mathbb{C}^*$ and $p^{-1}(0) \cong G/H_0$ ([4], Theoreme 1, [6] §3.11). One may consider Z as an elementary embedding (Z, z) of $(G \times \mathbb{C}^*)/(H \times 1)$, with closed orbit $(G \times \mathbb{C}^*)/(H_0 \times \mathbb{C}^*)$; $H \times 1$ is the isotropy subgroup of z, $H_0 \times \mathbb{C}^*$ is the isotropy subgroup of an element in the closed orbit ([6], proof of Corollaire 3.7). Let $P = P_J$ be the parabolic subgroup *associated* to H, $P = \{g \in G \mid gBH = BH\}$, and let L be a Levi subgroup (which we may assume equal to L_J, by taking an appropriate conjugate of H instead of H) of P *adapted* to H ([6], 2.9): in particular

$$P \cap H = L \cap H \quad , \quad L' \leq H \tag{3.3}$$

Then $P \times \mathbb{C}^*$ is the parabolic subgroup of $G \times \mathbb{C}^*$ associated to $H \times 1$ and $L \times \mathbb{C}^*$ is a Levi subgroup adapted to $H \times 1$ ([6], Corollaire 3.7 and its proof).

By [6], Proposition 3.10, i), we have $H_0 \times \mathbb{C}^* = (R_uQ \times 1)(L \times \mathbb{C}^* \cap H_0 \times \mathbb{C}^*)$ where Q is the opposite parabolic subgroup of P with respect to L, so that

$$H_0 = (R_uQ)(L \cap H_0) \tag{3.4}$$

We show that $L \cap H = L \cap H_0$. Let $L = CL'$, where C is the connected component of the centre of L. Then L' is contained also in H_0, by [6], Théorème 3.6.

By [6], Proposition 3.4, Z contains an open $P \times \mathbb{C}^*$-stable subset isomorphic to $R_uP \times W$ where W is $L \times \mathbb{C}^*$-stable and meets the closed orbit, and (W, z) is an elementary embedding of the torus $(C\times\mathbb{C}^*)/(C\cap H\times 1)$ ([5], proof of Lemme 4.2). Then $f = p_{|W} : W \to \mathbb{C}$ is a $(C\times\mathbb{C}^*)$-equivariant flat morphism such that $f^{-1}(\mathbb{C}^*) \cong C/C \cap H \times \mathbb{C}^*$ and $f^{-1}(0) \cong C/H_0 \cap C$. So the coordinate rings of these orbits are isomorphic C-modules and it follows that the isotropy groups of all points of W are the same. In particular

$$C \cap H = C \cap H_0 \tag{3.5}$$

With the above notation we prove

Theorem 3.1 *Let H be a spherical subgroup of G such that BH is dense in G and $L = L_J$ is a Levi subgroup adapted to H. Then $H_0 = R_uQ\,(L \cap H) = \langle U^-, U_{w_J}, C \cap H\rangle$.*

Proof. By (3.5) we have

$$L \cap H_0 = L'C \cap H_0 = L'(C \cap H_0) = L'(C \cap H) = L'C \cap H = L \cap H$$

so that by (3.4) we conclude. □

Definition 3.2 *We put $\tilde{\lambda}(G/H) = \lambda(G/H_0)$.*

Note that $\lambda(G/H) \leq \tilde{\lambda}(G/H)$ since BH is dense in G, and more generally $\mathbb{Z}\,\lambda(G/H) \cap P^+ \leq \tilde{\lambda}(G/H)$ ([34], part 2 of the proof of Proposition 1.5). Moreover

$$\lambda(G/H_0) = \{\lambda \in P^+ \mid \lambda(T \cap H) = 1\} \tag{3.6}$$

since $\prod_{j\in J} H_{\alpha_j} \leq H$ and $X_{\alpha_j}.v_{-\lambda} = v_{-\lambda}$ if $(\lambda, \alpha_j) = 0$ (here $v_{-\lambda}$ is a lowest weight vector of weight $-\lambda$ in $V(-w_0\lambda)$). Also $B \cap H \leq P \cap H = L \cap H$, so that $B \cap H = U_{w_J}(T \cap H)$. If $\lambda \in \tilde{\lambda}(G/H)$, then $F_\lambda : BH/H \to \mathbb{C}$, $b^{-1}H \mapsto \lambda(b)$ is a regular function on BH/H, and therefore a B-eigenvector of weight λ in $\mathbb{C}(G/H)$. In case G/H is quasi affine (as for conjugacy classes), then $\mathbb{Z}\,\lambda(G/H) \cap P^+ = \tilde{\lambda}(G/H)$ since $\mathbb{C}(G/H) = \text{Frac}\,\mathbb{C}[G/H]$, as in [34], Proposition 1.5. I do not know if $\mathbb{Z}\,\lambda(G/H) \cap P^+ = \tilde{\lambda}(G/H)$ holds in general.

Lemma 3.3 *Suppose F in Frac $\mathbb{C}[G/H]$ is a B-eigenvector of weight λ and $m\lambda$ lies in $\lambda(G/H)$ for a positive integer m. Then F lies in $\mathbb{C}[G/H]$.*

Proof. There exists a B-eigenvector $F_1 \in \mathbb{C}[G/H]$ of weight $m\lambda$. Then F^m/F_1 is invariant under B (as its weight is 0). So F^m/F_1 is constant, as G/H is spherical. In other words, F^m is regular on G/H. We conclude that F is in $\mathbb{C}[G/H]$, since $\mathbb{C}[G/H]$ is integrally closed ([16], Lemma 1.8). □

Let $\mathcal{O}$ be a spherical conjugacy class of G. We recall that $w = w(\mathcal{O})$ is the unique element (an involution) of W such that $BwB \cap \mathcal{O}$ is (open) dense in $\mathcal{O}$. Let v be the dense B-orbit in $\mathcal{O}$. Then BG_y is dense in G for any $y \in \mathrm{v}$. The parabolic subgroup $P = P_J$ associated to G_y coincides with $\{g \in G \mid g.\mathrm{v} = \mathrm{v}\}$. Moreover $\mathrm{v} = \mathcal{O} \cap BwB$ ([9], Corollary 26), and it is affine, as an orbit of a soluble algebraic group.

We have $w = w_0 w_J$, the subset J is invariant under ϑ, where ϑ is the symmetry of Π induced by $-w_0$, and w_0 and w_J act in the same way on Φ_J (see [10] the discussion at the end of section 3, Corollary 4.2, Remark 4.3 and Proposition 4.15).

Since all Levi subgroups of P are conjugate under $R_u P$, we may choose $y \in \mathrm{v}$ such that the standard Levi subgroup L_J is adapted to G_y. For the rest of this section we fix such a y, and we put $H = G_y$, $P = P_J$, $L = L_J$. By Theorem 3.1, we have

$$H_0 = \langle U^-, U_{w_J}, C_y \rangle = \langle U^-, U_{w_J}, T_y \rangle \tag{3.7}$$

and $\tilde{\lambda}(\mathcal{O}) = \lambda(G/H_0)$.

We shall now relate H with centralizers of elements in $\mathrm{v} \cap wB$. By the Bruhat decomposition, y is of the form $y = u\dot{w}b$, where $u \in R_u P$ and $b \in B$. We put $x_1 = u^{-1} y u = \dot{w} b u$. By [10], Corollary 4.13, $U_{w_J}(T^w)^\circ \leq C(x_1)$. Moreover, since $L' \leq C(y)$, by [10], Lemma 3.4, and commutation of y with $X_{\pm\alpha_i}$ for $i \in J$, we get $L' \leq C(x_1)$ (see also the proof of [10], Proposition 4.15).

Proposition 3.4 *Let x be in $\mathcal{O} \cap wB$. Then $T_x = T_y$ and $T \cap H^\circ = T \cap C(x)^\circ$.*

Proof. We observe that $C_{TU_w}(x) \leq T$ by the Bruhat decomposition and $C_{TU_w}(y) \leq T$, since L is adapted to $C(y)$. Now $x_1 = u^{-1} y u = y^u$ implies

$$T_{x_1} = C_T(x_1) = C_{TU_w}(x_1) \leq T \cap T^u = C_T(u)$$
$$T_y = C_T(y) = C_{TU_w}(y) \leq T \cap T^{u^{-1}} = C_T(u^{-1}) = C_T(u)$$

therefore if $t \in T_y$, then $t = t^u \in T_{x_1}$ and similarly if $t \in T_{x_1}$, then $t = t^{u^{-1}} \in T_y$. Hence $T_y = T_{x_1}$, and $T \cap C(y)^\circ = T \cap C(x_1)^\circ$. To conclude note that $\mathcal{O} \cap wB$ is the T-orbit of x_1. □

Remark 3.5 In fact $C_L(x) = C_L(y)$ for every $x \in \mathcal{O} \cap wB$, since $L' \leq C(x)$.

Remark 3.6 In general it is not true that L_J is adapted to $C(x)$ for $x \in \mathcal{O} \cap wB$. For example if $\mathcal{O}$ is the minimal unipotent class, and u is a non-identity element in $X_{-\beta}$, where β is the highest root, then $C(u) \geq U^-$, so that there is a unique Levi subgroup of P adapted to $C(u)$ ([6], Proposition 3.9), and this is L_J. Since $u \notin wB$, there is no element $x \in wB$ such that L_J is adapted to $C(x)$.

From Theorem 3.1 we get

Corollary 3.7 *Let $\mathcal{O}$ be a spherical conjugacy class, $w = w(\mathcal{O})$ and x any element in $\mathcal{O} \cap wB$. Then $H_0 = \langle U^-, U_{w_J}, T_x \rangle$, $w = w_0 w_J$.* □

By Proposition 3.4, we may put $T_{\mathcal{O}} = T_x$, for $x \in \mathcal{O} \cap wB$. Then $T_{\mathcal{O}} = T_y$ and $(T^w)^\circ \leq T_{\mathcal{O}} \leq T^w$ by [9], step 2 in the proof of Theorem 5.

We shall need the description of the monoid of weights λ such that $w(\lambda) = -\lambda$. In the next lemma we consider more generally w of the form $w = w_0 w_J$, with J ϑ-invariant.

Lemma 3.8 *Let $J \subseteq \Pi$ be ϑ-invariant and $w = w_0 w_J$. The dominant weight λ satisfies $w(\lambda) = -\lambda$ if and only if $\lambda = \sum_{i \in \Pi \setminus J} n_i \omega_i$ with $n_{\vartheta(i)} = n_i$ for all $i \in \Pi \setminus J$. Moreover $w(\lambda) = -\lambda$ implies $w_0(\lambda) = -\lambda$.*

Proof. Let $\lambda \in P^+$, $\lambda = \sum n_i \omega_i$, $n_i \in \mathbb{N}$. For $i \in \Pi \setminus J$ we have $w_J(\omega_i) = \omega_i$, so that $w(\omega_i) = -\omega_{\vartheta(i)}$.

It is clear that if $\lambda = \sum_{i \in \Pi \setminus J} n_i \omega_i$ with $n_i = n_{\vartheta(i)}$ for every $i \in \Pi \setminus J$, then $(w+1)(\lambda) = 0$. On the other hand, assume $w(\lambda) = -\lambda$. Then $w_J(\lambda) = -w_0\lambda$ and, by [20], Theorem 1.12 (a), we get $-w_0\lambda = \lambda$ and $(\lambda, \alpha_j) = 0$ for every $j \in J$. Hence $n_j = 0$ for every $j \in J$. Moreover, from $\lambda = \sum_{i \in \Pi \setminus J} n_i \omega_i$ and $-w_0\lambda = \lambda$ it follows $n_{\vartheta(i)} = n_i$ for all $i \in \Pi \setminus J$. □

Remark 3.9 If S is a ϑ-orbit in $\Pi \setminus J$, and we put $\omega_S = \sum_{i \in S} \omega_i$ then we have seen that $\{\omega_S \mid S \in (\Pi \setminus J)/\vartheta\}$ is a basis of the monoid $\{\lambda \in P^+ \mid w(\lambda) = -\lambda\}$, where $(\Pi \setminus J)/\vartheta$ is the set of ϑ-orbits in $\Pi \setminus J$. If we also assume that w acts trivially on Φ_J (as in the case of $w = w(\mathcal{O})$), then $\{\omega_S \mid S \in (\Pi \setminus J)/\vartheta\}$ is a basis of $\ker(w+1)$ in E, and so a basis of the free abelian group $\{\lambda \in P \mid w(\lambda) = -\lambda\}$.

We describe $\tilde{\lambda}(\mathcal{O})$. For this purpose we denote by $S_{\mathcal{O}}$ any supplement of $(T^w)^\circ$ in $T_{\mathcal{O}}$ (i.e. $S_{\mathcal{O}}(T^w)^\circ = T_{\mathcal{O}}$). We also put $P_w^+ = \{\lambda \in P^+ \mid w(\lambda) = -\lambda\}$. By Lemma 3.8 each element of P_w^+ satisfies $-w_0\lambda = \lambda$, so that in particular any subset X of P_w^+ is *symmetric*, i.e. $-w_0(X) = X$ ([32], 4.2, [10], Theorem 4.17)).

Theorem 3.10 *Let $\mathcal{O}$ be a spherical conjugacy class, $w = w(\mathcal{O})$ and let $S_{\mathcal{O}}$ be any supplement of $(T^w)^\circ$ in $T_{\mathcal{O}}$. Then*

$$\tilde{\lambda}(\mathcal{O}) = \{\lambda \in P_w^+ \mid \lambda(S_{\mathcal{O}}) = 1\}$$

Proof. By (3.6), $\tilde{\lambda}(\mathcal{O}) = \{\lambda \in P^+ \mid \lambda(T_{\mathcal{O}}) = 1\}$. Since $(T^w)^\circ \leq T_{\mathcal{O}}$, a necessary condition for $\lambda \in P^+$ to be in $\tilde{\lambda}(\mathcal{O})$ is that $\lambda(t\,t^w) = 1$ for every $t \in T$, as $(T^w)^\circ = \{t\,t^w \mid t \in T\}$. This condition is equivalent to $(w+1)\lambda = 0$, so that $\tilde{\lambda}(\mathcal{O}) \leq P_w^+$. Let $\lambda \in P_w^+$: then $\lambda \in \tilde{\lambda}(\mathcal{O}) \iff \lambda(S_{\mathcal{O}}) = 1$. □

We shall prove the crucial fact that $\tilde{\lambda}(\mathcal{O}) = \lambda(\mathcal{O})$, so that the monoid $\lambda(\mathcal{O})$ is *saturated* (that is $\mathbb{Z}\,\lambda(\mathcal{O}) \cap P^+ = \lambda(\mathcal{O})$, [34], Definition 1.3). In the following, x is a fixed element in $\mathcal{O} \cap wB$ and $\dot{w}$ a representative of w in N such that $x = \dot{w}u$, $u \in U$. If $u = \prod_{\alpha\in\Phi^+} x_\alpha(k_\alpha)$, and $i \in \Pi$, we say that α_i *occurs* in x if $k_{\alpha_i} \neq 0$. This is independent of the chosen total ordering on Φ^+.

For the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ in G, the monoid $\lambda(\overline{\mathcal{O}})$ of dominant weights occurring in $\mathbb{C}[\overline{\mathcal{O}}]$ is a submonoid of $\lambda(\mathcal{O})$. We start with

Proposition 3.11 *Let $\lambda \in P^+$. Then $(1-w)\lambda$ lies in $\lambda(\overline{\mathcal{O}})$.*

Proof. Let $f \in V(\lambda)^*_{-w\lambda}$, $v \in V(\lambda)_\lambda$ with $f(\dot{w}.v) = 1$. Then $c_{f,v}(t^{-1}gt) = c_{t.f,t.v}(g) = ((1-w)\lambda)(t)c_{f,v}(g)$ for every $t \in T$, $g \in G$. For every z, $z_1 \in U$ we have

$$c_{f,v}(z_1xz) = f(z_1\dot{w}\,uz.v) = f(z_1\dot{w}\,.v) = f(\dot{w}\,.v) = 1$$

since $z_1\dot{w}\,.v = \dot{w}\,.v + v_1$, where v_1 is a sum of weight vectors of weights strictly greater than $w\lambda$. Therefore for every $t \in T$, $z \in U$ we have

$$c_{f,v}(t^{-1}z^{-1}xzt) = ((1-w)\lambda)(t) \tag{3.8}$$

Since $B.x$ is dense in $\overline{\mathcal{O}}$, by (3.8) the restriction of $c_{f,v}$ to $\overline{\mathcal{O}}$ is a (non-zero) B-eigenvector of weight $(1-w)\lambda$ in $\mathbb{C}[\overline{\mathcal{O}}]$. Hence $(1-w)\lambda \in \lambda(\overline{\mathcal{O}})$. □

Corollary 3.12 *Let $\lambda \in P^+_w$. Then 2λ lies in $\lambda(\overline{\mathcal{O}})$.* □

Corollary 3.13 *Let $\lambda \in P^+$. Then $(1-w)\lambda \in \lambda(\mathcal{O})$. If moreover $\lambda \in P^+_w$, then 2λ lies in $\lambda(\mathcal{O})$.*

Proof. This follows from the fact that $\lambda(\overline{\mathcal{O}}) \leq \lambda(\mathcal{O})$. □

We have shown that

$$2P^+_w \leq (1-w)P^+ \leq \lambda(\overline{\mathcal{O}}) \leq \lambda(\mathcal{O}) \leq \tilde{\lambda}(\mathcal{O}) \leq P^+_w \tag{3.9}$$

We can prove that $\lambda(\mathcal{O})$ is saturated.

Theorem 3.14 *Let $\mathcal{O}$ be a spherical conjugacy class. Then $\lambda(\mathcal{O})$ is saturated.*

Proof. Let $\lambda \in \tilde{\lambda}(\mathcal{O})$. We put $F(b^{-1}xb) = \lambda(b)$ for $b \in B$. We observed that F is well-defined since $C_B(x) = T_xU_{w_J}$ and gives rise to a B-eigenvector of weight λ in $\mathbb{C}(\mathcal{O})$. Since $\mathcal{O}$ is quasi affine, we conclude that λ lies in $\lambda(\mathcal{O})$ by Theorem 3.10, Corollary 3.13 and Lemma 3.3. □

Theorem 3.14 in particular proves Conjecture 5.12 (and 5.10 and 5.11) in [36].

To deal with $\lambda(\overline{\mathcal{O}})$, in section 5 we shall make use of

Proposition 3.15 *Let $\lambda \in P^+$, $i \in \Pi \setminus J$ be such that α_i occurs in x and $(\lambda, \alpha_i) \neq 0$. Then $(1-w)\lambda - \alpha_i \in \lambda(\overline{\mathcal{O}})$.*

Proof. Since $\langle \lambda, \alpha_i \rangle \neq 0$, $\lambda - \alpha_i$ is a weight of $V(\lambda)$. We construct two matrix coefficients. We fix a non-zero $v \in V(\lambda)_{\lambda-\alpha_i}$. By [43], Lemma 72, there exists a (unique) $v_\lambda \in V(\lambda)_\lambda$ such that $x_{\alpha_i}(k).v = v + kv_\lambda$ for every $k \in \mathbb{C}$. Then we choose $f \in V(\lambda)^*_{-w\lambda}$ such that $f(\dot{w}.v_\lambda) = 1$.

Since α_i occurs in $x = \dot{w}\, u$, we have $u = x_{\alpha_i}(r)u'$, with $r \in \mathbb{C}^*$, $u' \in \prod_{\beta \in \Phi^+ \setminus \{\alpha_i\}} X_\beta$. Let y, $y_1 \in U$, and let $y = x_{\alpha_i}(k)y'$, $y' \in \prod_{\beta \in \Phi^+ \setminus \{\alpha_i\}} X_\beta$, then

$$y_1^{-1}xy.v = y_1^{-1}\dot{w}.v + (k+r)y_1^{-1}\dot{w}.v_\lambda$$

The vector $\dot{w}.v$ has weight $w(\lambda - \alpha_i)$, so that $y_1^{-1}\dot{w}.v$ is a sum of weight vectors of weight $w(\lambda - \alpha_i) + \beta$, where β is a sum of simple roots with non-negative coefficients. Assume $w\lambda = w(\lambda - \alpha_i) + \beta$ for a certain β. Then $w(\alpha_i) = \beta$ would be positive, a contradiction since $i \in \Pi \setminus J$. Hence $f(y_1^{-1}\dot{w}.v) = 0$. Similarly, $y_1^{-1}\dot{w}.v_\lambda = \dot{w}.v_\lambda + v'$, where v' is a sum of weight vectors of weights greater than $w\lambda$, hence $f(y_1^{-1}\dot{w}.v_\lambda) = f(\dot{w}.v_\lambda) = 1$, so that $c_{f,v}(y_1^{-1}xy) = k + r$.

The second matrix coefficient is defined dually. We fix a non-zero $f_1 \in V(-w_0\lambda)^*_{\lambda-\alpha_i}$. There exists a (unique) $f_\lambda \in V(-w_0\lambda)^*_\lambda$ such that $x_{\alpha_i}(k).f_1 = f_1 + kf_\lambda$ for every $k \in \mathbb{C}$. Then we choose $v_1 \in V(-w_0\lambda)_{-w\lambda}$ such that $f_\lambda(\dot{w}.v_1) = 1$. Let z, $z_1 \in U$, $z_1 = x_{\alpha_i}(k_1)z'$, $z' \in \prod_{\beta \in \Phi^+ \setminus \{\alpha_i\}} X_\beta$, then proceeding as before, we get $c_{f_1,v_1}(z_1^{-1}xz) = k_1$.

For $t \in T$, $z \in U$ we obtain

$$(c_{f,v} - c_{f_1,v_1})(t^{-1}z^{-1}xzt) = r\,((1-w)\lambda - \alpha_i)(t) \tag{3.10}$$

Since $B.x$ is dense in $\overline{\mathcal{O}}$, by (3.10) the restriction of $c_{f,v} - c_{f_1,v_1}$ to $\overline{\mathcal{O}}$ is a (non-zero) B-eigenvector of weight $(1-w)\lambda - \alpha_i$ in $\mathbb{C}[\overline{\mathcal{O}}]$. Hence $(1-w)\lambda - \alpha_i \in \lambda(\overline{\mathcal{O}})$. □

Corollary 3.16 *Let $i \in \Pi \setminus J$ be such that α_i occurs in x. Then $\omega_i + \omega_{\vartheta(i)} - \alpha_i$ lies in $\lambda(\overline{\mathcal{O}})$.*

Proof. This follows from Proposition 3.15 by taking $\lambda = \omega_i$. □

We can deal with other homogeneuos spaces related to $\mathcal{O}$. The simply-connected cover (or the universal covering, as in [22], p. 107) $\hat{\mathcal{O}}$ of $\mathcal{O}$ can be identified with G/H°, since G is simply-connected.

Corollary 3.17 *Let $\mathcal{O}$ be a spherical conjugacy class, and let S be a supplement of $(T^w)^\circ$ in $T \cap C(x)^\circ$. Then $\lambda(\hat{\mathcal{O}}) = \{\lambda \in P^+_w \mid \lambda(S) = 1\}$ is saturated.*

Proof. By [16], Corollary 2.2, $\hat{\mathcal{O}}$ is quasi affine and, by [6], Proposition 5.1, 5.2, L is adapted to H°, so that $\tilde{\lambda}(\hat{\mathcal{O}}) = \tilde{\lambda}(G/H^\circ) = \{\lambda \in P^+_w \mid \lambda(S) = 1\}$, since $(T^w)^\circ \leq T \cap H^\circ$. Let $\lambda \in \tilde{\lambda}(\hat{\mathcal{O}})$; then $F_\lambda : BH^\circ/H^\circ \to \mathbb{C}$, $b^{-1}H^\circ \mapsto \lambda(b)$ is a regular function on BH°/H°, and therefore a B-eigenvector of weight λ in $\mathbb{C}(G/H^\circ)$. By Corollary 3.13, $2\lambda \in \lambda(G/H) \leq \lambda(G/H^\circ)$, and we conclude by Lemma 3.3 and Proposition 3.4. □

Corollary 3.18 *Let K be a closed subgroup of G with $H^\circ \leq K \leq N(H^\circ)$. Then $\lambda(G/K) = \tilde{\lambda}(G/K)$ (and $\lambda(G/K)$ is saturated).*

Proof. Since L is adapted to H, we get $N(H) = N(H^\circ) = H(C \cap N(H))$ by [6], Corollaire 5.2, P is the parabolic subgroup corresponding to $N(H)$ and L is adapted to $N(H)$ (by the proof of [6], Proposition 5.2 a). Clearly the same holds for K, since $BH = BK$.

By Corollary 3.17, $\lambda \in \lambda(G/H^\circ) \Leftrightarrow \lambda(T \cap H^\circ) = 1$. We prove that $\lambda \in \lambda(G/K) \Leftrightarrow \lambda(T \cap K) = 1$. In one direction $\lambda \in \lambda(G/K) \Rightarrow \lambda(T \cap K) = 1$, since $\lambda(G/K) \leq \tilde{\lambda}(G/K)$. So assume $\lambda(T \cap K) = 1$. Then $\lambda(T \cap H^\circ) = 1$, so that $\lambda \in \lambda(G/H^\circ)$, and in particular $w_0\lambda = -\lambda$. Let v be a non-zero vector in $V(\lambda)^{H^0}$, and let $v = v_{-\lambda} + v'$, with $v_{-\lambda} \in V(\lambda)_{-\lambda}$, $v' \in \sum_{\mu > -\lambda} V(\lambda)_\mu$: then $v_{-\lambda} \neq 0$, since BH° is dense in G.

Since $V(\lambda)^{H^0}$ is 1-dimensional, there is a character γ of K, trivial on H°, such that $k.v = \gamma(k)v$ for $k \in K$. Since $K = H^\circ(T \cap K)$, v is K-invariant if and only if $\gamma(T \cap K) = 1$. But $v_{-\lambda} \neq 0$ implies $\gamma(k) = -\lambda(k)$ for every $k \in T \cap K$ so that v is K-invariant if and only if $\lambda(T \cap K) = 1$, and we are done. □

Remark 3.19 In general K is not quasi affine: for instance the centralizer H of $x_{-\beta}(1)$, β the highest root, contains U^-, and $T \leq N(H)$. Then $N(H)$ is epimorphic, i.e. the minimal quasi affine subgroup of G containing $N(H)$ is G ([16], p. 19, ex. 2). To our knowledge, it was known that $\lambda(G/K)$ is saturated for symmetric varieties G/K, due to the work of Vust, [45].

Proposition 3.20 *We have*

$$H/H^\circ \cong T_y/T \cap H^\circ = T_x/T \cap C(x)^\circ$$

Proof. We have $H = H^\circ(H \cap T) = H^\circ T_y$. Hence we get an epimorphism $\pi : T_y \to H/H^\circ$, inducing an isomorphism $\overline{\pi} : T_y/T \cap H^\circ \to H/H^\circ$, and we conclude by Proposition 3.4. □

Corollary 3.21 *If T^w is connected, then H is connected.*

Proof. This follows from $(T^w)^\circ \leq T \cap C(x)^\circ \leq T_x \leq T^w = (T^w)^\circ$ and Proposition 3.20. □

Due to the fact that T is 2-divisible, we have the decomposition $T = (T^w)^\circ(S^w)^\circ$ where $S^w = \{t \in T \mid t^w = t^{-1}\}$. Let $t \in T^w$, $t = s\,z$, with $s \in (T^w)^\circ$, $z \in (S^w)^\circ$. Then $z = t\,s^{-1} \in T^w \cap (S^w)^\circ \leq T^w \cap S^w \leq T_2$, the elementary abelian 2-subgroup of T of rank n. We note that $(T^w)^\circ \cap (S^w)^\circ$ is finite, even though in general not trivial. Therefore $z \in T_2$, and $T^w \leq (T^w)^\circ\, T_2$. In particular we have

$$T^w = (T^w)^\circ(T^w \cap (S^w)^\circ) = (T^w)^\circ(T^w \cap T_2)$$

and

$$T_x = (T^w)^\circ(C(x) \cap (S^w)^\circ) = (T^w)^\circ(C(x) \cap T_2)$$

Moreover every subgroup M of T_2 is a complemented group (i.e. for every subgroup X of M there exists a subgroup Y such that $XY = M$ and $X \cap Y = 1$), hence we may find a subgroup R of T_2 such that $T^w = (T^w)^\circ \times R$. Then $T_x = (T^w)^\circ \times (R \cap C(x))$ and $T \cap C(x)^\circ = (T^w)^\circ \times (R \cap C(x)^\circ)$. We put $S_{\mathcal{O}} = R \cap C(x)$, $S_{\hat{\mathcal{O}}} = R \cap C(x)^\circ$. We have therefore proved

Theorem 3.22 *Let $\mathcal{O}$ be a spherical conjugacy class, $w = w(\mathcal{O})$. Then*

$$\lambda(\mathcal{O}) = \{\lambda \in P_w^+ \mid \lambda(S_{\mathcal{O}}) = 1\} \quad , \quad \lambda(\hat{\mathcal{O}}) = \{\lambda \in P_w^+ \mid \lambda(S_{\hat{\mathcal{O}}}) = 1\}$$

□

From Proposition 3.20 it follows that H always splits over H°: if Y is a complement of $R \cap C(x)^\circ$ in $R \cap C(x)$, then Y is a complement of H° in H.

4 Description of $\lambda(\mathcal{O})$ and $\lambda(\hat{\mathcal{O}})$

From our discussion it is clear that to determine $\lambda(\mathcal{O})$ the most favourable case is when T^w is connected, so that $T_x = T^w = (T^w)^\circ$. In this case then $\lambda(\mathcal{O}) = \lambda(\hat{\mathcal{O}}) = P_w^+ = \{\sum_{i \in \Pi \setminus J} n_i \omega_i \mid n_{\vartheta(i)} = n_i\}$. We note that of course we have $Z(G) \leq T_x$, so that it is also straightforward to determine $\lambda(\mathcal{O})$ even when $T^w = (T^w)^\circ Z(G)$, so that $T_x = T^w$. In general it is quite cumbersome to determine T_x. Our strategy will be to determine T^w as $T^w = (T^w)^\circ \times R$, and then determine $R \cap C(x)$. To deal with unipotent classes, we shall usually start from the maximal one, (corresponding to w_0), and then deal with the remaining classes by an inductive procedure. In some cases we shall use an explicit form of an element x (in $\mathcal{O} \cap wB$), while in some other cases we shall determine $T \cap C(x)$ by analizing the form of eventual involutions in $T_x \setminus Z(G)(T^w)^\circ$. Note that when T^w is connected (or $T^w = (T^w)^\circ Z(G)$), it is not necessary to have an explicit description of $x \in \mathcal{O} \cap wB$ (however in certain cases it will be necessary to have such a description in section 6).

We use the fact that if $G_1 \subset G_2$ are reductive algebraic groups and u is a unipotent element in G_1 such that the conjugacy class of u in G_2 is spherical, then the conjugacy class of u in G_1 is spherical ([33], Corollary 2.3, Theorem 3.1).

The character group $X(T^w)$ is isomorphic to $P/(1-w)P$, since $P = X(T)$. Therefore T^w is connected if and only if $P/(1-w)P$ is torsion free. We are reduced to calculate elementary divisors of the endomorphism $1-w$ of P. We shall use the following results.

Lemma 4.1 *Assume the positive roots $\beta_i, \ldots, \beta_\ell$ are long and pairwise orthogonal. Then, for $\xi_1, \ldots, \xi_\ell \in \mathbb{C}^*$ and $g = x_{\beta_1}(-\xi_1^{-1}) \cdots x_{\beta_\ell}(-\xi_\ell^{-1})$ we have*

$$g x_{-\beta_1}(\xi_1) \cdots x_{-\beta_\ell}(\xi_\ell) g^{-1} = n_{\beta_1} \cdots n_{\beta_\ell} h x_{\beta_1}(2\xi_1^{-1}) \cdots x_{\beta_\ell}(2\xi_\ell^{-1})$$

for a certain $h \in T$.

Proof. By (2.2) we have $x_\alpha(-\xi^{-1})x_{-\alpha}(\xi)x_\alpha(\xi^{-1}) = n_\alpha h_\alpha(-\xi)x_\alpha(2\xi^{-1})$. Hence we get the result with $h = h_{\beta_1}(-\xi_1)\cdots h_{\beta_\ell}(-\xi_\ell)$. □

Proposition 4.2 *Let $\alpha \in \Phi$. Then T^{s_α} is connected except in the following cases:*

(i) G is of type A_1;

(ii) G is of type C_n and α is long;

(iii) G is of type B_2 and α is long.

In these cases we have $T^{s_\alpha} = (T^{s_\alpha})^\circ \times Z(G)$.

Proof. It is enough to determine in which cases the non-zero elementary divisor of $1 - s_i$ is not 1. Since $(1 - s_i)\omega_j = \delta_{ij}\alpha_i$ and $\alpha_i = \sum_k a_{ik}\omega_k$, this happens only for G of type A_1 and $i = 1$, C_n and $i = n$, or B_2 and $i = 1$ ([18], pag. 59). In these cases the non-zero elementary divisor is 2, and $T^{s_{\alpha_i}} = (T^{s_{\alpha_i}})^\circ \times Z(G)$. □

Lemma 4.3 *Let M be a connected algebraic group, S a torus of M, g a semisimple element in $C_M(S)$. Then $\langle S, g\rangle$ is contained in a torus of M.*

Proof. See [18], Corollary 22.3 B. □

Lemma 4.4 *Assume K is a connected spherical subgroup of G with no non-trivial characters. Then the monoid $\lambda(G/K)$ is free.*

Proof. We recall that we are assuming G simply-connected, so that by [16], Theorem 20.2, ${}^U\mathbb{C}[G/K]$ is a polynomial algebra. But ${}^U\mathbb{C}[G/K]$ is the monoid algebra of $\lambda(G/K)$ and the monoid algebra is factorial if and only if $\lambda(G/K)$ is free (see the proof of [32], Proposition 2). □

Lemma 4.5 *Let V be a G-module, $g \in G$, such that the image Q of the endomorphism $p(g)$ of V is 1 dimensional for a certain polynomial p. Assume $M \leq C(g)$ has no non-trivial characters. Then M acts trivially on Q.*

Proof. This is clear. □

Let $S = \{i, \vartheta(i)\}$ be a ϑ-orbit in $\Pi \backslash J$ consisting of 2 elements. We put $H_S = \{h_{\alpha_i}(z)h_{\alpha_{\vartheta(i)}}(z^{-1}) \mid z \in \mathbb{C}^*\}$. Let $\mathcal{S}_1$ be the set of ϑ-orbits in $\Pi \setminus J$ consisting of 2 elements. Then, by Remark 3.9, $\Delta_J \cup \{\alpha_i - \alpha_{\vartheta(i)}\}_{\mathcal{S}_1}$ is a basis of $\ker(1 - w)$ and

$$(T^w)^\circ = \prod_{j\in J} H_{\alpha_j} \times \prod_{S\in\mathcal{S}_1} H_S \tag{4.11}$$

We put $\Psi_J = \{\beta \in \Phi \mid w(\beta) = -\beta\}$. Then Ψ_J is a root system in $\mathrm{Im}(1-w)$ ([40], Proposition 2), and $w_{|\mathrm{Im}(1-w)}$ is -1. If $K = C((T^w)^\circ)'$, then K is semisimple with root system Ψ_J and maximal torus $T(K) := T \cap K = (S^w)^\circ$.

For each spherical (non-central) conjugacy class $\mathcal{O}$ we give the corresponding J and w as a product of commuting reflections using the tables in [9]. We give tables with corresponding $\lambda(\mathcal{O})$ and $\lambda(\hat{\mathcal{O}})$ (for semisimple classes we also give the type of the centralizer of elements in $\mathcal{O}$). In the cases when $\lambda(\hat{\mathcal{O}}) = \lambda(\mathcal{O})$, we leave a blank entry. For length reasons we shall give proofs only for some classes. In [9] for the classical groups we gave representative of semisimple conjugacy classes in $SL(n)$, $Sp(n)$ and $SO(n)$. Here we shall give an expression in terms of $\exp$. If g is in $Z(G)$, then $\mathcal{O}_g = \{g\}$, $w = 1$ and $\mathbb{C}[\mathcal{O}_g] = \mathbb{C}$.

4.1 Type A_n, $n \geq 1$.

Let $m = \left[\frac{n+1}{2}\right]$, $\beta_i = e_i - e_{n+2-i}$, for $i = 1, \ldots, m$. For $\ell = 1, \ldots, m-1$ we put $J_\ell = \{\ell+1, \ldots, n-\ell\}$, $J_m = \varnothing$. If we denote by X_i the unipotent class $(2^i, 1^{n+1-2i})$, then

$$X_\ell \longleftrightarrow J_\ell \longleftrightarrow s_{\beta_1} \cdots s_{\beta_\ell}$$

for $\ell = 1, \ldots, m$ (here $w_0 = s_{\beta_1} \cdots s_{\beta_m}$).

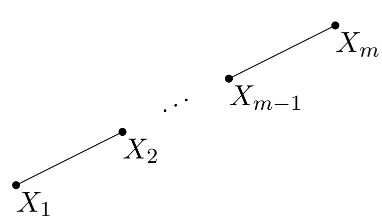
In this case T^w is almost always connected. There is only one case when it is not connected, namely when n is odd, $n+1 = 2m$, and $w = w_0$. However in this case we have $T^{w_0} = (T^{w_0})^\circ Z(G) = (T^{w_0})^\circ \times \langle h_{\alpha_m}(-1)\rangle$.

In fact we have

$$(1-w)P = \begin{cases} \mathbb{Z}\langle \omega_1 + \omega_n, \ldots, \omega_\ell + \omega_{n+1-\ell}\rangle & \text{for } \ell = 1, \ldots, m-1 \\ \mathbb{Z}\langle \omega_1 + \omega_n, \ldots, \omega_m + \omega_{m+1}\rangle & \text{for } \ell = m,\, n = 2m \\ \mathbb{Z}\langle \omega_1 + \omega_n, \ldots, \omega_{m-1} + \omega_{m+1}, 2\omega_m\rangle & \text{for } \ell = m,\, n+1 = 2m \end{cases}$$

Moreover the center $Z(G)$ of G is generated by $z = \prod_{i=1}^n h_{\alpha_i}(\xi^i)$, where ξ is a primitive $(n+1)$-th root of 1 in $\mathbb{C}$. For $n+1 = 2m$, then $z^{-1}h_{\alpha_m}(-1) \in (T^{w_0})^\circ$ since $\xi^m = -1$.

4.1.1 Unipotent classes in A_n.



Unipotent classes in A_n, $m = \left[\frac{n+1}{2}\right]$.

If n is even, or n odd with $\ell < m$, then T^w is always connected. Assume n odd, $\ell = m$. Then $T^{w_0} = (T^{w_0})^\circ Z(G)$, so that $T_x = T^{w_0}$. Moreover, the reductive part of $C(x)^\circ$ is of type A_{m-1}, so that $(T^{w_0})^\circ$ is a maximal torus of $C(x)^\circ$. Hence $Z(G) \not\leq C(x)^\circ$ and $T_x \cap C(x)^\circ = (T^{w_0})^\circ$. We get

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
X_ℓ $\ell = 1, \dots, m-1$	$\sum_{k=1}^{\ell} n_k(\omega_k + \omega_{n-k+1})$	
X_m $n = 2m$	$\sum_{k=1}^{m} n_k(\omega_k + \omega_{n-k+1})$	
X_m $n+1 = 2m$	$\sum_{k=1}^{m-1} n_k(\omega_k + \omega_{n-k+1}) + 2n_m\omega_m$	$\sum_{k=1}^{m-1} n_k(\omega_k + \omega_{n-k+1}) + n_m\omega_m$

Table 1: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in A_n.

In particular $\hat{X}_1$ is a model homogeneus space for $SL(2)$, and in fact the principal one, by [28], 3.3 (1).

4.1.2 Semisimple classes in A_n.

The centralizers of elements in spherical semisimple classes are of type $T_1A_{\ell-1}A_{n-\ell}$. Following the notation in [9], Tables 1, 5 we get

$$T_1A_{\ell-1}A_{n-\ell} \longleftrightarrow J_\ell \longleftrightarrow s_{\beta_1} \cdots s_{\beta_\ell}$$

for $\ell = 1, \dots, m$.

Type $T_1A_{\ell-1}A_{n-\ell}$. Up to a central element, the semisimple elements with centralizer of this type are conjugate to $\exp(\zeta\check{\omega}_\ell) = \text{diag}(e^{\frac{n+1-k}{n}\zeta}I_k, e^{-\frac{k}{n}\zeta}I_{n+1-k})$, $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$.

Since in all cases we have $T_x = T^w$, we get

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\zeta\check{\omega}_\ell)$ $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$ $\ell = 1, \dots, m-1$	$T_1A_{\ell-1}A_{n-\ell}$	$\sum_{k=1}^{\ell} n_k(\omega_k + \omega_{n-k+1})$
$\exp(\zeta\check{\omega}_m)$ $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$ $n = 2m$	$T_1A_{m-1}A_m$	$\sum_{k=1}^{m} n_k(\omega_k + \omega_{n-k+1})$
$\exp(\zeta\check{\omega}_m)$ $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$ $n+1 = 2m$	$T_1A_{m-1}A_{m-1}$	$\sum_{k=1}^{m-1} n_k(\omega_k + \omega_{n-k+1}) + 2n_m\omega_m$

Table 2: $\lambda(\mathcal{O})$ for semisimple classes in A_n.

4.2 Type C_n, $n \geq 2$.

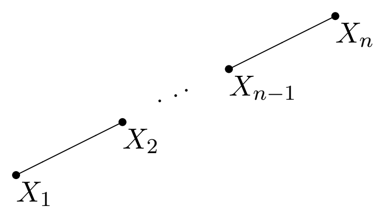
We have $\omega_\ell = e_1 + \cdots + e_\ell$ for $\ell = 1, \dots, n$ and $Z(G) = \langle z \rangle$, where $z = \prod_{i=1}^{[\frac{n+1}{2}]} h_{\alpha_{2i-1}}(-1)$. For $i = 1, \dots, n$ we denote by X_i the unipotent class $(2^i, 1^{2n-2i})$ and we put $\beta_i = 2e_i$, $J_i = \{i+1, \dots, n\}$ ($J_n = \varnothing$).

Then

$$X_\ell \longleftrightarrow J_\ell \longleftrightarrow s_{\beta_1} \cdots s_{\beta_\ell}$$

for $\ell = 1, \dots, n$ (here $w_0 = s_{\beta_1} \cdots s_{\beta_n}$).

4.2.1 Unipotent classes in C_n.



Unipotent classes in C_n

Lemma 4.6 *Let $w = s_{\beta_1} \cdots s_{\beta_\ell}$ for $\ell = 1, \dots, n$. Then*

$$T^w = (T^w)^\circ \times R \quad , \quad R = \langle h_{\alpha_1}(-1) \rangle \times \cdots \times \langle h_{\alpha_\ell}(-1) \rangle$$

Proof. For $\ell = 1, \dots, n$ we have $(1-w)P = \mathbb{Z}\langle 2\omega_1, \dots, 2\omega_\ell \rangle$. □

Proposition 4.7 *For $\ell = 1, \dots, n$ we have*

$$\lambda(X_\ell) = \{2n_1\omega_1 + \cdots + 2n_\ell\omega_\ell \mid n_k \in \mathbb{N}\}$$

Proof. In [9] we exhibit the element $x_{-\beta_1}(1) \cdots x_{-\beta_\ell}(1) \in \mathcal{O} \cap BwB \cap B^-$. By Lemma 4.1, we can choose

$$x = n_{\beta_1} \cdots n_{\beta_\ell} h \, x_{\beta_1}(2) \cdots x_{\beta_\ell}(2) \in \mathcal{O} \cap wB$$

for a certain $h \in T$. Let now $t \in R$. Then $t \in C(x) \Leftrightarrow \beta_i(t) = 1$ for $i = 1, \dots, \ell$. But $\mathbb{Z}\langle \beta_1, \dots, \beta_\ell \rangle = \mathbb{Z}\langle 2\omega_1, \dots, 2\omega_\ell \rangle$, so that $R \leq T_x$, and $T_x = T^w$. □

Proposition 4.8 *For $\ell = 1, \dots, n$ we have*

$$\lambda(\hat{X}_\ell) = \{2n_1\omega_1 + \cdots + 2n_{\ell-1}\omega_{\ell-1} + n_\ell\omega_\ell \mid n_k \in \mathbb{N}\}$$

Proof. We have $R \cap C(x)^\circ = \langle h_{\alpha_1}(-1), \dots, h_{\alpha_{\ell-1}}(-1) \rangle$. In fact, for $i = 1 \dots, \ell - 1$

$$e_{\alpha_i} - e_{-\alpha_i} \in C_{\mathfrak{g}}(\langle x_{\beta_1}(\xi) \cdots x_{\beta_\ell}(\xi) \rangle)$$

for every $\xi \in \mathbb{C}$, so that $h_{\alpha_i}(-1) = \exp(\pi(e_{\alpha_i} - e_{-\alpha_i})) \in C(x)^\circ$. On the other hand the reductive part of $C(x)$ is of type $Sp(2n-2\ell) \times O(\ell)$, so that $C(x)/C(x)^\circ$ has order 2, and we are done. □

We get

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
X_ℓ $\ell = 1, \ldots, n$	$\sum_{i=1}^{\ell} 2n_i\omega_i$	$\sum_{i=1}^{\ell-1} 2n_i\omega_i + n_\ell\omega_\ell$

Table 3: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in C_n.

4.2.2 Semisimple classes in C_n.

Let $p = [\frac{n}{2}]$. We put $\gamma_\ell = e_{2\ell-1} + e_{2\ell}$, $K_\ell = \{1, 3, \ldots, 2\ell-1, 2\ell+1, 2\ell+2, \ldots, n\}$ for $\ell = 1, \ldots, p$. Then, following the notation in [9], Tables 1, 5 we have

$$\begin{array}{llcccc} C_\ell C_{n-\ell}, & \ell = 1, \ldots, p & \longleftrightarrow & K_\ell & \longleftrightarrow & s_{\gamma_1} \cdots s_{\gamma_\ell} \\ T_1 C_{n-1} & & \longleftrightarrow & J_2 & \longleftrightarrow & s_{\beta_1} s_{\beta_2} \\ T_1 \tilde{A}_{n-1} & & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 \end{array}$$

Lemma 4.9 *Let $w = s_{\gamma_1} \cdots s_{\gamma_\ell}$ for $\ell = 1, \ldots, [\frac{n}{2}]$. Then T^w is connected.*

Proof. We have $(1-w)P = \mathbb{Z}\langle \omega_{2i} \mid i = 1, \ldots, \ell\rangle$. □

Type $T_1\tilde{A}_{n-1}$. Let $H = C(\exp(\check{\omega}_n))$. Then H is of type $T_1\tilde{A}_{n-1}$. If $\lambda = e^{\zeta/2}$, then $\exp(\zeta\check{\omega}_n) = \mathrm{diag}(\lambda I_n, \lambda^{-1} I_n)$ (in $Sp(2n)$). If $\zeta \in \mathbb{C}$, then $C(\exp(\zeta\check{\omega}_n)) = H \Leftrightarrow \zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$.

For $g = n_{\beta_1} \cdots n_{\beta_n} x_{\beta_1}(1) \cdots x_{\beta_n}(1)$, the element

$$y_\zeta = g \exp(\zeta\check{\omega}_n) g^{-1} = x_{-\beta_1}(e^\zeta - 1) \cdots x_{-\beta_n}(e^\zeta - 1) \exp(-\zeta\check{\omega}_n)$$

lies in $\mathcal{O}_{\exp(\zeta\check{\omega}_n)} \cap B s_{\beta_1} \cdots s_{\beta_n} B \cap B^-$ if $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$, and we conclude as for the class X_n.

Type T_1C_{n-1}. Let $z = \exp(\check{\omega}_1)$, $H = C(z)$. Then H is of type T_1C_{n-1}. If $\lambda = e^\zeta$, then $\exp(\zeta\check{\omega}_1) = \mathrm{diag}(\lambda, I_{n-1}, \lambda^{-1}, I_{n-1}) = h_{\beta_1}(\lambda)$. If $\lambda \in \mathbb{C}^* \setminus \{\pm 1\}$, then $C(h_{\beta_1}(\lambda)) = H$, while $C(h_{\beta_1}(-1))$ is of type C_1C_{n-1}. We assume $\lambda \in \mathbb{C}^* \setminus \{\pm 1\}$. In [9] we exhibited an element y_λ in the C_2-subgroup K of G generated by the roots $\alpha_1, \beta_2, \gamma_1, \beta_1$: $y_\lambda \in \mathcal{O}_{h_{\beta_1}(\lambda)} \cap B s_{\beta_1} s_{\beta_2} B \cap B^-$. Conjugating y_λ by an appropriate element from $B \cap K$ we get

$$x_\lambda = n_{\beta_1} n_{\beta_2} h x_{\alpha_1}(\xi_1) x_{\beta_2}(\xi_2) x_{\gamma_1}(\xi_3) x_{\beta_1}(\xi_4) \in \mathcal{O}_{h_{\beta_1}(\lambda)} \cap wB$$

for a certain $h \in T$, $\xi_i \in \mathbb{C}$, with $\xi_1 = 1 - \lambda$, $\xi_2 = -\frac{2}{\lambda}$. Since $\mathbb{Z}\langle \alpha_1, \beta_2, \gamma_1, \beta_1\rangle = \mathbb{Z}\langle \alpha_1, \beta_2\rangle = \mathbb{Z}\langle 2\omega_1, \omega_2\rangle$ we get $T_{x_\lambda} = (T^w)^\circ \times \langle h_{\alpha_1}(-1)\rangle$ and we conclude as in case $\hat{X}_2$.

Type C_kC_{n-k}, $k = 1, \ldots, p$. Let $\sigma_k = \exp(\pi i\check{\omega}_k) = \mathrm{diag}(-I_k, I_{n-k}, -I_k, I_{n-k})$, $H = C(\sigma_k)$. Then H is of type C_kC_{n-k}, $Z(H) = C(H) = \langle \sigma_k \rangle \times Z(G)$.

For type C_kC_{n-k}, T^w is connected, hence in each case we determined T_x. We get

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\zeta\check{\omega}_n)$ $\zeta\in\mathbb{C}\setminus 2\pi i\mathbb{Z}$	$T_1\tilde{A}_{n-1}$	$\sum_{k=1}^{n} 2n_k\omega_k$
$\exp(\zeta\check{\omega}_1)$ $\zeta\in\mathbb{C}\setminus \pi i\mathbb{Z}$	T_1C_{n-1}	$2n_1\omega_1+n_2\omega_2$
$\exp(\pi i\check{\omega}_\ell)$ $\ell=1,\dots,[\frac{n}{2}]$	$C_\ell C_{n-\ell}$	$\sum_{i=1}^{\ell} n_{2i}\,\omega_{2i}$

Table 4: $\lambda(\mathcal{O})$ for semisimple classes in C_n.

4.2.3 Mixed classes in C_n.

We put $p=[\frac{n}{2}]$. From [9], Table 4, we get

$$\begin{array}{llclcl} \sigma_p x_{\alpha_n}(1) & & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 \\ \sigma_k x_{\alpha_n}(1), & k=1,\dots,p-1 & \longleftrightarrow & J_{2k+1} & \longleftrightarrow & s_{\beta_1}\cdots s_{\beta_{2k+1}} \\ \sigma_k x_{\beta_1}(1), & k=1,\dots,p & \longleftrightarrow & J_{2k} & \longleftrightarrow & s_{\beta_1}\cdots s_{\beta_{2k}} \end{array}$$

Note that when n is even, then $\sigma_p x_{\beta_1}(1)\sim z\sigma_p x_{\alpha_n}(1)$.

Class of $\sigma_p x_{\alpha_n}(1)$. In [9], proof of Theorem 2.23, we exhibited an element M in $Sp(2n)$: $M\in\mathcal{O}_{\sigma_p x_{\alpha_n}(1)}\cap Bw_0B\cap B^-$. The centralizer of M in B is $Z(G)$, hence $T_x=Z(G)$.

We give also an alternative proof. Suppose for a contradiction that $T_x\neq Z(G)$, and let $\sigma\in T_x\setminus Z(G)$. Then we have $x\in K=C(\sigma)$. Since the involutions in G are conjugate (up to a central element) to σ_k, for a certain $k\in\{1,\dots,p\}$, K is of type C_kC_{n-k}.

Now x is conjugate in K to an element of the form su, with $s\in T$, $u\in U(K)$, $[s,u]=1$. We have $s=s_1s_2$, $u=u_1u_2$, with $s_1\in T(C_k)$, $s_2\in T(C_{n-k})$, $u_1\in U(C_k)$, $u_2\in T(C_{n-k})$. Note that s_1, u_1, s_2 and u_2 are uniquely determined, since $C_k\cap C_{n-k}=1$, and (u_1,u_2) must be in the class $(X_1,1)$ or $(1,X_1)$ of $C_k\times C_{n-k}$. Moreover the conjugacy classes of s_1u_1 and s_2u_2 must lie over the longest elements of the Weyl group of C_{2k} and C_{n-2k} respectively. However, at least one of u_1 and u_2 is 1, so that at least one of s_1u_1, s_2u_2 does not lie over w_0, since no involution of C_n lies over w_0. We have therefore proved that $T_x=Z(G)$.

Class of $\sigma_\ell x_{\alpha_n}(1)\sim\sigma_\ell x_{\beta_{2k\ell+1}}(1)$, $\ell=1,\cdots,p-1$. Here Ψ_J has basis $\{\alpha_1,\dots,\alpha_{2\ell},\beta_{2\ell+1}\}$, and $K=C((T^w)^\circ)'$ is of type $C_{2\ell+1}$. From the construction in [9], proof of Theorem 2.23, we can find x in K. We note that

$$R_1=\langle h_{\alpha_1}(-1)\rangle\times\cdots\times\langle h_{\alpha_{2\ell}}(-1)\rangle\times\langle h_{\beta_{2\ell+1}}(-1)\rangle$$

is another complement of $(T^w)^\circ$ in T^w, so that $T_x = (T_x \cap R_1) \times (T^w)^\circ$. By the result obtained for the mixed class of maximal dimension in $C_{2\ell+1}$ we get

$$T_x = (T^w)^\circ \times \langle (\prod_{i=1}^{\ell} h_{\alpha_{2i-1}}(-1)) h_{\beta_{2\ell+1}}(-1)\rangle = (T^w)^\circ \times \langle \prod_{i=1}^{\ell+1} h_{\alpha_{2i-1}}(-1)\rangle$$

Class of $\sigma_\ell x_{\beta_1}(1) \sim \sigma_\ell x_{\beta_\ell}(1)$, $\ell = 1, \cdots, p$. Here Ψ_J has basis $\{\alpha_1, \ldots, \alpha_{2\ell-1}, \beta_{2\ell}\}$, and K is of type $C_{2\ell}$. From the construction in [9], proof of Theorem 2.23, we can find x in K, since $\sigma_\ell x_{\beta_\ell}(1) \in C_{2\ell}$. Arguing as before, we get that

$$R_1 = \langle h_{\alpha_1}(-1)\rangle \times \cdots \times \langle h_{\alpha_{2\ell-1}}(-1)\rangle \times \langle h_{\beta_{2\ell}}(-1)\rangle = T_2(K)$$

is another complement of $(T^w)^\circ$ in T^w. Then

$$T_x \cap R_1 = T_x \cap T_2(K) = C_{T(K)}(x) = Z(K) = \langle \prod_{i=1}^{\ell} h_{\alpha_{2i-1}}(-1)\rangle$$

by the results obtained for the mixed class of maximal dimension in $C_{2\ell}$ (recall that when n is even $\sigma_p x_{\alpha_n}(-1) \sim z\sigma_p x_{\beta_1}(-1)$). Hence

$$T_x \cap R_1 = \langle \prod_{i=1}^{\ell} h_{\alpha_{2i-1}}(-1)\rangle$$

and

$$T_x = (T^w)^\circ \times (T_x \cap R_1) = (T^w)^\circ \times \langle \prod_{i=1}^{\ell} h_{\alpha_{2i-1}}(-1)\rangle$$

In order to determine $\lambda(\hat{\mathcal{O}})$, by [42], IV 2.25, in all cases the index $[C(x) : C(x)^\circ]$ is 2, hence, since in all cases $T_x/(T^w)^\circ$ has order 2, we must have $T \cap C(x)^\circ = (T^w)^\circ$. We obtain

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
$\sigma_p x_{\alpha_n}(1)$	$\sum_{i=1}^{n} n_i\omega_i$, $\sum_{i=1}^{[\frac{n+1}{2}]} n_{2i-1}$ even	$\sum_{i=1}^{n} n_i\omega_i$
$\sigma_\ell x_{\alpha_n}(1)$ $\ell = 1, \ldots, [\frac{n}{2}] - 1$	$\sum_{i=1}^{2\ell+1} n_i\omega_i$, $\sum_{i=1}^{\ell+1} n_{2i-1}$ even	$\sum_{i=1}^{2\ell+1} n_i\omega_i$
$\sigma_\ell x_{\beta_1}(1)$ $\ell = 1, \ldots, [\frac{n}{2}]$	$\sum_{i=1}^{2\ell} n_i\omega_i$, $\sum_{i=1}^{\ell} n_{2i-1}$ even	$\sum_{i=1}^{2\ell} n_i\omega_i$

Table 5: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for mixed classes in C_n.

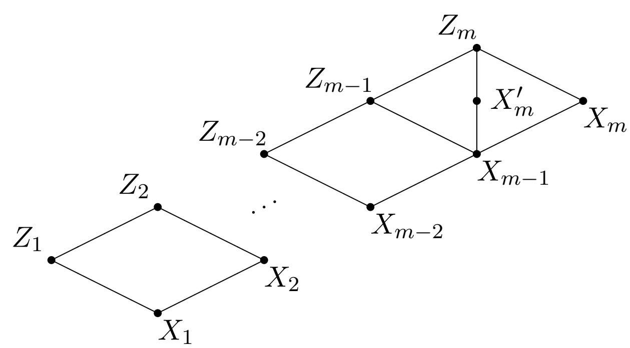
In particular $\hat{\mathcal{O}}_{\sigma_p x_{\alpha_n}(1)}$ is a model homogeneus space, and in fact the principal one, by [28], 3.3 (3).

To deal with types D_n and B_n, we denote by X_i the unipotent class which in $SO(s)$ has canonical form $(2^{2i}, 1^{s-4i})$, $i = 1, \dots, \left[\frac{s}{4}\right]$ (for $s = 4m$, $i = m$ there are 2 classes of this form: X_m and X'_m, the *very even* classes) and by Z_i the unipotent class $(3, 2^{2(i-1)}, 1^{s-4i+1})$, $i = 1, \dots, 1 + \left[\frac{s-3}{4}\right]$.

4.3 Type D_n, $n \geq 4$.

Let $m = \left[\frac{n}{2}\right]$. We have $\omega_i = e_1 + \cdots + e_i$ for $i = 1, \dots, n-2$, $\omega_{n-1} = \frac{1}{2}(e_1 + \cdots + e_{n-1}) - \frac{1}{2}e_n$, $\omega_n = \frac{1}{2}(e_1 + \cdots + e_n)$. In particular P coincides with $\mathbb{Z}\langle e_1, \dots, e_{n-1}, \frac{1}{2}(e_1 + \cdots + e_n)\rangle$. We put $\beta_i = e_{2i-1} + e_{2i}$, $\delta_i = e_{2i-1} - e_{2i}$ for $i = 1, \dots, m$. For $\ell = 1, \dots, m-1$ we put $J_\ell = \{2\ell + 1, \dots, n\}$, $J_m = \varnothing$, $K_\ell = J_\ell \cup \{1, 3, \dots, 2\ell - 1\}$ for $\ell = 1, \dots, m$.

4.3.1 Unipotent classes in D_n, n even, $n = 2m$.



Unipotent classes in D_n, $n = 2m$

The center of G is $\langle \prod_{i=1}^m h_{\alpha_{2i-1}}(-1), h_{\alpha_{n-1}}(-1)h_{\alpha_n}(-1)\rangle$. From [9] we get

$$\begin{array}{llclcl} Z_\ell, & \ell = 1, \dots, m & \longleftrightarrow & J_\ell & \longleftrightarrow & s_{\beta_1} s_{\delta_1} \cdots s_{\beta_\ell} s_{\delta_\ell} \\ X_\ell, & \ell = 1, \dots, m & \longleftrightarrow & K_\ell & \longleftrightarrow & s_{\beta_1} \cdots s_{\beta_\ell} \\ X'_m & & \longleftrightarrow & \{1, 3, \dots, n-3, n\} & \longleftrightarrow & s_{\beta_1} \cdots s_{\beta_{m-1}} s_{\alpha_{n-1}} \end{array}$$

Lemma 4.10 *Let $w = s_{\beta_1} \cdots s_{\beta_\ell}$. Then T^w is connected for $\ell = 1, \dots, m-1$, and*

$$\begin{array}{ll} T^w & = (T^w)^\circ \times \langle h_{\alpha_n}(-1)\rangle = (T^w)^\circ Z(G) \quad \textit{for } \ell = m \\ T^w & = (T^w)^\circ \times \langle h_{\alpha_{n-1}}(-1)\rangle = (T^w)^\circ Z(G) \quad \textit{for } w = s_{\beta_1} \cdots s_{\beta_{m-1}} s_{\alpha_{n-1}} \end{array}$$

Proof. We have

$$(1-w)P = \begin{cases} \mathbb{Z}\langle \omega_2, \omega_4, \dots, \omega_{2\ell}\rangle & \text{for } \ell = 1, \dots, m-1 \\ \mathbb{Z}\langle \omega_2, \omega_4, \dots, \omega_{n-2}, 2\omega_n\rangle & \text{for } \ell = m \\ \mathbb{Z}\langle \omega_2, \omega_4, \dots, \omega_{n-2}, 2\omega_{n-1}\rangle & \text{for } w = s_{\beta_1} \cdots s_{\beta_{m-1}} s_{\alpha_{n-1}} \end{cases}$$

and we conclude. □

Proposition 4.11 *For $\ell = 1, \ldots, m-1$ we have $\lambda(\hat{X}_\ell) = \lambda(X_\ell)$. Moreover*

$$\lambda(\hat{X}_m) = \left\{ \sum_{i=1}^{m-1} n_{2i}\, \omega_{2i} + n_n \omega_n \mid n_k \in \mathbb{N} \right\}$$

and

$$\lambda(\hat{X}'_m) = \left\{ \sum_{i=1}^{m-1} n_{2i}\, \omega_{2i} + n_{n-1} \omega_{n-1} \mid n_k \in \mathbb{N} \right\}$$

Proof. For $1 \leq \ell < m$ the result is clear. For $\ell = m$, $C(x)$ has rank m ([12], §13.1), so that $\prod_{j \in K_m} H_{\alpha_j}$ is a maximal torus of $C(x)$. By Lemma 4.3, $h_{\alpha_n}(-1) \notin C(x)^\circ$. Similarly for X'_m.□

Lemma 4.12 *Let $w = s_{\beta_1} \cdots s_{\beta_\ell} s_{\delta_1} \cdots s_{\delta_\ell}$ for $\ell = 1, \ldots, m$. Then*

$$T^w = (T^w)^\circ \times \langle h_{\alpha_1}(-1) \rangle \times \cdots \times \langle h_{\alpha_{2\ell-1}}(-1) \rangle$$

for $\ell = 1, \ldots, m-1$, and $T^w = T^{w_0} = T_2$ for $\ell = m$.

Proof. We have $(1-w)P = \mathbb{Z}\langle 2\omega_1, \ldots, 2\omega_{2\ell-1}, \omega_{2\ell} \rangle$ for $\ell = 1, \ldots, m-1$. □

Let $\ell = 1$. Then $T^w = \langle h_{\alpha_1}(-1) \rangle \times (T^w)^\circ = Z(G)(T^w)^\circ$, so that $T_x = T^w$, hence

$$\lambda(Z_1) = \{2n_1\omega_1 + n_2\omega_2 \mid n_k \in \mathbb{N}\}$$

Next we consider Z_m. We claim that $T_x = Z(G)$. Suppose for a contradiction that there is an involution $\sigma \in T_x \setminus Z(G)$. Then $x \in K = C(\sigma)$, and K is the almost direct product K_1K_2, of type $D_k D_{n-k}$, for some $k = 1, \ldots, m$. We get an orthogonal decomposition $E = E_1 \oplus E_2$ and a decomposition $x = x_1x_2 \in K_1K_2$. Then $-1 = w_0 = (w_1, w_2)$, where w_i is the element of the Weyl group of K_i corresponding to x_i (the class of x_i in K_i is spherical). It follows that each $w_i = -1$, and k is even. Then x_1 is in the class $Z_{k/2}$ of K_1 and x_2 in the class $Z_{(n-k)/2}$ of K_2. However, the product x_1x_2 is then not in the class Z_m of G (since in x_1x_2 there are two rows with 3 boxes), a contradiction. Hence $T_x = Z(G)$ and

$$\lambda(Z_m) = \left\{ \sum_{i=1}^{n} n_i\omega_i \mid n_k \in \mathbb{N},\ \sum_{i=1}^{m} n_{2i-1} \text{ even, } n_{n-1} + n_n \text{ even} \right\}$$

We now deal with Z_ℓ, $\ell = 2, \ldots, m-1$. Here Ψ_J has basis $\{\alpha_1, \ldots, \alpha_{2\ell-1}, \beta_\ell\}$, and $K = C((T^w)^\circ)'$ is of type $D_{2\ell}$ (and is simply-connected). If we denote by M the $D_{n-2\ell}$-subgroup generated by $\{X_\alpha \mid \alpha \in \Phi_J\}$, then we have

$$K\,M = C(\sigma) \quad , \quad K \cap M = \langle h_{\alpha_{n-1}}(-1) h_{\alpha_n}(-1) \rangle \quad , \quad \sigma = \prod_{i=1}^{\ell} h_{\alpha_{2i-1}}(-1)$$

$$Z(K) = \langle h_{\alpha_{n-1}}(-1)h_{\alpha_n}(-1)\rangle \times \langle \sigma \rangle$$

Now $x \in K$ and

$$T^w = R \times (T^w)^\circ \quad , \quad R = \langle h_{\alpha_1}(-1)\rangle \times \cdots \times \langle h_{\alpha_{2\ell-1}}(-1)\rangle$$

with $R \leq K$, so that

$$T_x \cap R = R \cap Z(K) = \langle \sigma \rangle$$

since we have already shown that $T_y = Z(G)$ if the spherical unipotent class $\mathcal{O}_y$ lies above w_0. Hence

$$T_x = (T^w)^\circ \times \langle \sigma \rangle$$

We have proved that

$$T_x = \begin{cases} Z(G) & \text{for } x \in Z_m \cap wB \\ (T^w)^\circ \times \langle \prod_{i=1}^{\ell} h_{\alpha_{2i-1}}(-1)\rangle & \text{for } x \in Z_\ell \cap wB,\ \ell = 1, \dots m-1 \end{cases}$$

Proposition 4.13 *For $\ell = 1, \dots, m$ we have*

$$\lambda(\hat{Z}_\ell) = \left\{ \sum_{i=1}^{2\ell} n_i \omega_i \mid n_k \in \mathbb{N} \right\}$$

Proof. Let $u \in Z_\ell$, with $\ell = 1, \dots, m$. If $C(u)^\circ = RC$ with $R = R_u(C(u))$, C connected reductive, then C is of type $C_{\ell-1}B_{n-2\ell}$. In particular C is always semisimple. Then we conclude by Lemma 4.4, if $\ell \geq 2$. When $\ell = 1$, then $\operatorname{rk} C(x) = n - 2$, so that $\prod_{j \in J_1} H_{\alpha_j}$ is a maximal torus of $C(x)^\circ$. Hence $h_{\alpha_1}(-1) \notin C(x)^\circ$ by Lemma 4.3, and we are done. □

We obtained

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
X_ℓ $\ell = 1, \dots, m-1$	$\sum_{i=1}^{\ell} n_{2i}\, \omega_{2i}$	
X_m	$\sum_{i=1}^{m-1} n_{2i}\, \omega_{2i} + 2n_n\omega_n$	$\sum_{i=1}^{m-1} n_{2i}\, \omega_{2i} + n_n\omega_n$
X'_m	$\sum_{i=1}^{m-1} n_{2i}\, \omega_{2i} + 2n_{n-1}\omega_{n-1}$	$\sum_{i=1}^{m-1} n_{2i}\, \omega_{2i} + n_{n-1}\omega_{n-1}$
Z_ℓ $\ell = 1, \dots, m-1$	$\sum_{i=1}^{2\ell} n_i\omega_i,\ \sum_{i=1}^{\ell} n_{2i-1}$ even	$\sum_{i=1}^{2\ell} n_i\omega_i$
Z_m	$\sum_{i=1}^{n} n_i\omega_i,\ \sum_{i=1}^{m} n_{2i-1}$ even, $n_{n-1} + n_n$ even	$\sum_{i=1}^{n} n_i\omega_i$

Table 6: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in D_n, $n = 2m$.

In particular $\hat{Z}_m$ is a model homogeneus space, and in fact the principal one, by [28], 3.3 (4).

4.3.2 Semisimple classes in D_n, n even $n = 2m$

Following the notation in [9], Tables 1, 5 we have

$$\begin{array}{llllll} D_\ell D_{n-\ell} & \longleftrightarrow & J_\ell, & \ell = 1, \dots, m & \longleftrightarrow & s_{\beta_1} s_{\delta_1} \cdots s_{\beta_\ell} s_{\delta_\ell} \\ T_1 A_{n-1} & \longleftrightarrow & K_m, & & \longleftrightarrow & s_{\beta_1} \cdots s_{\beta_m} \\ (T_1 A_{n-1})' & \longleftrightarrow & \{1, 3, \dots, n-3, n\} & & \longleftrightarrow & s_{\beta_1} \cdots s_{\beta_{m-1}} s_{\alpha_{n-1}} \end{array}$$

There are two families of classes of semisimple elements with centralizer of type $T_1 A_{n-1}$: to distinguish them we wrote $T_1 A_{n-1}$ and $(T_1 A_{n-1})'$.

Type $D_1 D_{n-1} = T_1 D_{n-1}$. Let $\sigma_1 = \exp(\pi i \check{\omega}_1)$, $H = C(\sigma_1)$. Then H is of type $T_1 D_{n-1}$ with $Z(H) = C(H) = \exp(\mathbb{C}\check{\omega}_1) Z(G)$. If we put $\lambda = e^\zeta$, then the image of $\exp(\zeta \check{\omega}_1)$ in $SO(2n)$ is $\mathrm{diag}(\lambda, I_{n-1}, \lambda^{-1}, I_{n-1})$. We have $C(\exp(\zeta \check{\omega}_1)) = H \iff \zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$.

In this case we have

$$T^w = (T^w)^\circ Z(G)$$

so it is not necessary to give explicitly the form of an element in $wB \cap \mathcal{O}$.

Anyway for $\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$, we consider the element

$$y_\zeta = g \exp(\zeta \check{\omega}_1) g^{-1}$$

where $g = x_{-\beta_1}(1) x_{-\delta_1}(1)$. Now $\beta_1(\exp(\zeta \check{\omega}_1)) = \delta_1(\exp(\zeta \check{\omega}_1)) = e^\zeta$, so that

$$\exp(\zeta \check{\omega}_1) x_{-\delta_1}(-1) x_{-\beta_1}(-1) \exp(\zeta \check{\omega}_1)^{-1} = x_{-\delta_1}(-e^{-\zeta}) x_{-\beta_1}(-e^{-\zeta})$$

and

$$y_\zeta = x_{-\beta_1}(1 - e^{-\zeta}) x_{-\delta_1}(1 - e^{-\zeta}) \exp(\zeta \check{\omega}_1)$$

By Lemma 4.1 we may take x_ζ of the form

$$x_\zeta = n_{\beta_1} n_{\delta_1} h x_{\beta_1} \left(\frac{1 + e^{-\zeta}}{1 - e^{-\zeta}} \right) x_{\delta_1} \left(\frac{1 + e^{-\zeta}}{1 - e^{-\zeta}} \right)$$

We have $w = s_{\beta_1} s_{\delta_1}$, $T^w = \langle h_{\alpha_1}(-1) \rangle \times (T^w)^\circ = Z(G)(T^w)^\circ$, so that $T_{x_\zeta} = T^w$, hence (as for Z_1)

$$\lambda(\mathcal{O}_{\exp(\zeta \check{\omega}_1)}) = \{2n_1 \omega_1 + n_2 \omega_2 \mid n_k \in \mathbb{N}\}$$

for $\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$.

Type $D_\ell D_{n-\ell}$, $\ell = 2, \dots, m$.
Let $\sigma_\ell = \exp(\pi i \check{\omega}_\ell)$, $H = C(\sigma_\ell)$ (the image of σ_ℓ in $SO(2n)$ is $\mathrm{diag}(-I_\ell, I_{n-\ell}, -I_\ell, I_{n-\ell})$). Then H is of type $D_\ell D_{n-\ell}$. We may take

$$x_\ell = n_{\beta_1} n_{\delta_1} \cdots n_{\beta_\ell} n_{\delta_\ell} \in \mathcal{O}_{\sigma_\ell} \cap wB$$

and clearly $T_{x_\ell} = T^w$. It follows that

$$\lambda(\mathcal{O}_{\exp(\pi i\check{\omega}_\ell)}) = \left\{\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell} \mid n_i \in \mathbb{N}\right\}$$

for $\ell = 2, \dots, m-1$ and

$$\lambda(\mathcal{O}_{\exp(\pi i\check{\omega}_m)}) = \left\{\sum_{i=1}^{n} 2n_i\omega_i \mid n_i \in \mathbb{N}\right\}$$

Type T_1A_{n-1}.

Let $z = \exp(\check{\omega}_n)$, $H = C(z)$. Then H is of type T_1A_{n-1}, $Z(H) = C(H) = \exp(\mathbb{C}\check{\omega}_n)Z(G)$. If $\lambda = e^{\zeta/2}$, then the image of $\exp(\zeta\check{\omega}_n)$ in $SO(2n)$ is diag$(\lambda I_n, \lambda^{-1}I_n)$.

In this case we have

$$T^w = (T^w)^\circ Z(G)$$

so it is not necessary to give explicitly the form of an element in $wB \cap \mathcal{O}$.

Anyway if $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$, then $C(\exp(\zeta\check{\omega}_n)) = H$. Let

$$y_\zeta = g\exp(\zeta\check{\omega}_n)g^{-1}$$

where $g = n_{\beta_1}\cdots n_{\beta_m}x_{\beta_1}(1)\cdots x_{\beta_m}(1)$. Then

$$y_\zeta \in \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \cap Bs_{\beta_1}\cdots s_{\beta_m}B \cap B^-$$

By Lemma 4.1 we may take x_ζ of the form

$$x_\zeta = n_{\beta_1}\cdots n_{\beta_m}hx_{\beta_1}(\xi)\cdots x_{\beta_m}(\xi)$$

for a certain $h \in T$, $\xi = \frac{1+e^\zeta}{1-e^\zeta}$. By Lemma 4.10 we have

$$T^w = (T^w)^\circ Z(G) = (T^w)^\circ \times \langle h_{\alpha_n}(-1)\rangle$$

hence $T_{x_\zeta} = T^w$ and we conclude as for X_m.

Proposition 4.14 *Let $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$. Then*

$$\lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_n)}) = \left\{\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + 2n_n\omega_n \mid n_k \in \mathbb{N}\right\}$$

Type $(T_1A_{n-1})'$. Here we consider $z = \exp(\check{\omega}_{n-1})$, $H = C(z)$. Then H is of type $(T_1A_{n-1})'$, $Z(H) = C(H) = \exp(\mathbb{C}\check{\omega}_{n-1})Z(G)$. If $\lambda = e^{\zeta/2}$, then the image of $\exp(\zeta\check{\omega}_{n-1})$ in $SO(2n)$ is diag$(\lambda I_{n-1}, \lambda^{-1}, \lambda^{-1}I_{n-1}, \lambda)$. Applying the graph automorphism of order 2 of G interchanging α_{n-1} and α_n, from the previous result we obtain, as for X'_m,

Proposition 4.15 *Let $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$. Then*

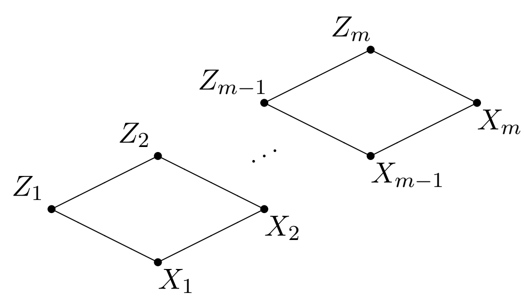
$$\lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_{n-1})}) = \left\{\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + 2n_{n-1}\omega_{n-1} \mid n_k \in \mathbb{N}\right\}$$

We got

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\zeta\check{\omega}_1)$ $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$	$T_1 D_{n-1}$	$2n_1\omega_1 + n_2\omega_2$
$\exp(\pi i\check{\omega}_\ell)$ $\ell = 2, \ldots, m-1$	$D_\ell D_{n-\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$
$\exp(\pi i\check{\omega}_m)$	$D_m D_m$	$\sum_{i=1}^{n} 2n_i\omega_i$
$\exp(\zeta\check{\omega}_n)$ $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$	$T_1 A_{n-1}$	$\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + 2n_n\omega_n$
$\exp(\zeta\check{\omega}_{n-1})$ $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$	$(T_1 A_{n-1})'$	$\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + 2n_{n-1}\omega_{n-1}$

Table 7: $\lambda(\mathcal{O})$ for semisimple classes in D_n, $n = 2m$.

4.3.3 Unipotent classes in D_n, n odd, $n = 2m + 1$.



Unipotent classes in D_n, $n = 2m + 1$

The center of G is $\langle (\prod_{j=1}^{m} h_{\alpha_{2j-1}}(-1))h_{\alpha_{n-1}}(i)h_{\alpha_n}(-i)\rangle$. From [9] we get

$$\begin{array}{lllll} Z_\ell, \quad \ell = 1, \ldots, m & \longleftrightarrow & J_\ell & \longleftrightarrow & s_{\beta_1}s_{\delta_1}\cdots s_{\beta_\ell}s_{\delta_\ell} \\ X_\ell, \quad \ell = 1, \ldots, m & \longleftrightarrow & K_\ell & \longleftrightarrow & s_{\beta_1}\cdots s_{\beta_\ell} \end{array}$$

Lemma 4.16 *Let $w = s_{\beta_1}\cdots s_{\beta_\ell}$ for $\ell = 1, \ldots, m$. Then T^w is connected.*

Proof. We have

$$(1-w)P = \begin{cases} \mathbb{Z}\langle \omega_{2i} \mid i = 1, \ldots, \ell \rangle & \text{for } \ell = 1, \ldots, m-1 \\ \mathbb{Z}\langle \omega_2, \omega_4, \ldots, \omega_{n-3}, \omega_{n-1} + \omega_n \rangle & \text{for } \ell = m \end{cases}$$

□

Therefore we have $\lambda(X_\ell) = \lambda(\hat{X}_\ell) = P_w^+$ for $\ell = 1, \ldots, m$.

Lemma 4.17 *Let $w = s_{\beta_1} \cdots s_{\beta_\ell} s_{\delta_1} \cdots s_{\delta_\ell}$ for $\ell = 1, \ldots, m$, then*

$$T^w = (T^w)^\circ \times \langle h_{\alpha_1}(-1) \rangle \cdots \times \langle h_{\alpha_{2\ell-1}}(-1) \rangle$$

Proof. We have

$$(1-w)P = \begin{cases} \mathbb{Z}\langle 2\omega_1, \ldots, 2\omega_{2\ell-1}, \omega_{2\ell} \rangle & \text{for } \ell = 1, \ldots, m-1 \\ \mathbb{Z}\langle 2\omega_1, \ldots, 2\omega_{n-2}, \omega_{n-1} + \omega_n \rangle & \text{for } \ell = m \end{cases}$$

□

For $\ell = 1$ we get $T^w = \langle h_{\alpha_1}(-1) \rangle \times (T^w)^\circ = Z(G)(T^w)^\circ$, so that $T_x = T^w$, hence

$$\lambda(Z_1) = \{2n_1\omega_1 + n_2\omega_2 \mid n_k \in \mathbb{N}\}$$

Next we consider Z_m. We claim that

$$T_x = (T^{w_0})^\circ \times \langle \sigma \rangle \quad , \quad \sigma = \prod_{i=1}^m h_{\alpha_{2i-1}}(-1)$$

(in particular $T_x = Z(G)(T^{w_0})^\circ$).

In fact, $x \in K = C((T^{w_0})^\circ)'$, and K is the D_{n-1}-subgroup of G corresponding to the subsystem Ψ_J of all roots of orthogonal to $\alpha_{n-1} - \alpha_n$: since $\alpha_{n-1} - \alpha_n = -2e_n$, $\{\alpha_1, \ldots, \alpha_{n-2}, \beta_m\}$ is a basis of Ψ_J, and K is simply-connected. We have

$$K\,(T^{w_0})^\circ = C(\sigma)\,, \quad K \cap (T^{w_0})^\circ = \langle h_{\alpha_{n-1}}(-1) h_{\alpha_n}(-1) \rangle\,, \quad \sigma = \prod_{i=1}^m h_{\alpha_{2i-1}}(-1)$$

The restriction of w_0 to $\mathbb{R}\Psi_J$ is -1 and x, as an element of K, is in the class $Z_{(n-1)/2}$ of K. Since we have already shown that $T_y = Z(K)$ if $\mathcal{O}_y$ is the spherical unipotent class of K lying over -1, and

$$T^{w_0} = R \times (T^{w_0})^\circ \quad , \quad R = \langle h_{\alpha_1}(-1) \rangle \times \cdots \times \langle h_{\alpha_{2m-1}}(-1) \rangle$$

with $R \leq K$, we get

$$T_x \cap R = R \cap Z(K) = \langle \sigma \rangle$$

hence

$$T_x = (T^w)^\circ \times \langle \sigma \rangle$$

Therefore

$$\lambda(Z_m) = \left\{ \sum_{i=1}^{n-2} n_i\omega_i + n_{n-1}(\omega_{n-1}+\omega_n) \mid n_k \in \mathbb{N},\ \sum_{i=1}^{m} n_{2i-1} \text{ even} \right\}$$

To deal with Z_ℓ, $\ell = 2, \dots, m-1$, we may use the same argument of the case D_n with even n and obtain

$$T_x = (T^w)^\circ \times \langle\sigma\rangle \quad , \quad \sigma = \prod_{i=1}^{\ell} h_{\alpha_{2i-1}}(-1)$$

Therefore

$$\lambda(Z_\ell) = \left\{ \sum_{i=1}^{2\ell} n_i\omega_i \mid n_k \in \mathbb{N},\ \sum_{i=1}^{\ell} n_{2i-1} \text{ even} \right\}$$

We summarize the results obtained in

Proposition 4.18 *For $\ell = 1, \dots, m-1$ we have*

$$\lambda(Z_\ell) = \left\{ \sum_{i=1}^{2\ell} n_i\omega_i \mid n_k \in \mathbb{N},\ \sum_{i=1}^{\ell} n_{2i-1} \text{ even} \right\}$$

Moreover

$$\lambda(Z_m) = \left\{ \sum_{i=1}^{n-2} n_i\omega_i + n_{n-1}(\omega_{n-1}+\omega_n) \mid n_k \in \mathbb{N},\ \sum_{i=1}^{m} n_{2i-1} \text{ even} \right\}$$

For the simply-connected cover we get

Proposition 4.19 *For $\ell = 1, \dots, m-1$ we have*

$$\lambda(\hat{Z}_\ell) = \left\{ \sum_{i=1}^{2\ell} n_i\omega_i \mid n_k \in \mathbb{N} \right\}$$

Moreover

$$\lambda(\hat{Z}_m) = \left\{ \sum_{i=1}^{n-2} n_i\omega_i + n_{n-1}(\omega_{n-1}+\omega_n) \mid n_k \in \mathbb{N} \right\}$$

Proof. Let $u \in Z_\ell$, with $\ell = 1, \dots, m$. If $C(u)^\circ = RC$ with $R = R_u(C(u))$, C connected reductive, then C is of type $C_{\ell-1}B_{n-2\ell}$. In particular C is always semisimple. Then we conclude by Lemma 4.4, if $\ell \geq 2$. When $\ell = 1$, then $\operatorname{rk} C(x) = n-2$, so that $\prod_{j\in J_1} H_{\alpha_j}$ is a maximal torus of $C(x)^\circ$. Hence $h_{\alpha_1}(-1) \notin C(x)^\circ$ by Lemma 4.3, and we are done. □

We got

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
X_ℓ $\ell = 1, \ldots, m-1$	$\sum_{i=1}^{\ell} n_{2i}\,\omega_{2i}$	
X_m	$\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + n_{n-1}(\omega_{n-1} + \omega_n)$	
Z_ℓ $\ell = 1, \ldots, m-1$	$\sum_{i=1}^{2\ell} n_i\omega_i,\ \sum_{i=1}^{\ell} n_{2i-1}$ even	$\sum_{i=1}^{2\ell} n_i\omega_i$
Z_m	$\sum_{i=1}^{n-2} n_i\omega_i + n_{n-1}(\omega_{n-1} + \omega_n),\ \sum_{i=1}^{m} n_{2i-1}$ even	$\sum_{i=1}^{n-2} n_i\omega_i + n_{n-1}(\omega_{n-1} + \omega_n)$

Table 8: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in D_n, $n = 2m+1$.

4.3.4 Semisimple classes in D_n, n odd, $n = 2m+1$

Following the notation in [9], Tables 1, 5 we have

$$\begin{array}{llcccc} D_\ell D_{n-\ell}, & \ell = 1, \ldots, m & \longleftrightarrow & J_\ell & \longleftrightarrow & s_{\beta_1} s_{\delta_1} \cdots s_{\beta_\ell} s_{\delta_\ell} \\ T_1 A_{n-1} & & \longleftrightarrow & K_m & \longleftrightarrow & s_{\beta_1} \cdots s_{\beta_m} \end{array}$$

Type $D_1 D_{n-1} = T_1 D_{n-1}$.

We can use the same calculations as in the case D_n, n even and obtain

$$x_\zeta = n_{\beta_1} n_{\delta_2} h_{\beta_1}(e^{-\zeta} - 1) h_{\delta_1}(e^{-\zeta} - 1) \exp(\zeta \check{\omega}_1) x_{\beta_1}\left(\frac{1 + e^{-\zeta}}{1 - e^{-\zeta}}\right) x_{\delta_1}\left(\frac{1 + e^{-\zeta}}{1 - e^{-\zeta}}\right)$$

in $\mathcal{O}_{\exp(\zeta\check{\omega}_1)} \cap wB$ for every $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$.

Since $T^w = \langle h_{\alpha_1}(-1)\rangle \times (T^w)^\circ = Z(G)(T^w)^\circ$, we get $T_{x_\zeta} = T^w$ (as for Z_1) and

$$\lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_1)}) = \{2n_1\omega_1 + n_2\omega_2 \mid n_k \in \mathbb{N}\}$$

for $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$.

Type $D_k D_{n-k}$, $k = 2, \ldots, m$. As in the case n even we may take

$$x_k = n_{\beta_1} n_{\delta_1} \cdots n_{\beta_k} n_{\delta_k} \in \mathcal{O}_{\sigma_k} \cap wB$$

where $\sigma_k = \exp(\pi i \check{\omega}_k)$. Then $T_x = T^w$, so that

$$\lambda(\mathcal{O}_{\exp(\pi i\check{\omega}_\ell)}) = \left\{\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell} \mid n_i \in \mathbb{N}\right\}$$

for $\ell = 2, \ldots, m-1$ and

$$\lambda(\mathcal{O}_{\exp(\pi i\check{\omega}_m)}) = \left\{\sum_{i=1}^{n-2} 2n_i\omega_i + n_{n-1}(\omega_{n-1} + \omega_n) \mid n_i \in \mathbb{N}\right\}$$

Type $T_1 A_{n-1}$. Here we consider elements of the form $\exp(\zeta\check{\omega}_n)$, $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$. Note that

$$w\exp(\zeta\check{\omega}_n)w^{-1} = \exp(-\zeta\check{\omega}_{n-1})$$

where $w = s_{\beta_1} \cdots s_{\beta_m}$. Hence

$$\lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_{n-1})}) = \lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_n)})$$

Proceeding as in the case n even, we may take x_ζ of the form

$$x_\zeta = n_{\beta_1} \cdots n_{\beta_m} h x_{\beta_1}(\xi) \cdots x_{\beta_m}(\xi) \in \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \cap wB$$

for a certain $h \in T$, $\xi = \frac{1+e^\zeta}{1-e^\zeta}$.

By Lemma 4.16, for $w = s_{\beta_1} \cdots s_{\beta_m}$, T^w is connected, hence

$$\lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_n)}) = \left\{\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + n_{n-1}(\omega_{n-1} + \omega_n) \mid n_k \in \mathbb{N}\right\}$$

for $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$.

We got

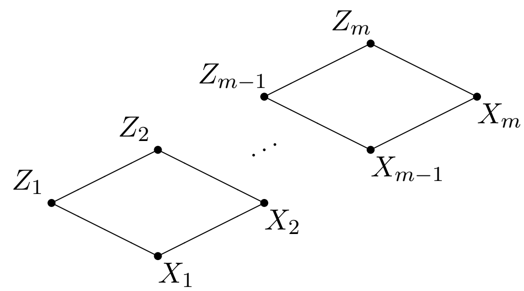
$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\zeta\check{\omega}_1)$ $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$	$T_1 D_{n-1}$	$2n_1\omega_1 + n_2\omega_2$
$\exp(\pi i\check{\omega}_\ell)$ $\ell = 2, \ldots, m-1$	$D_\ell D_{n-\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$
$\exp(\pi i\check{\omega}_m)$	$D_m D_{m+1}$	$\sum_{i=1}^{n-2} 2n_i\omega_i + n_{n-1}(\omega_{n-1} + \omega_n)$
$\exp(\zeta\check{\omega}_n)$ $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$	$T_1 A_{n-1}$	$\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + n_{n-1}(\omega_{n-1} + \omega_n)$

Table 9: $\lambda(\mathcal{O})$ for semisimple classes in D_n, $n = 2m+1$.

4.4 Type B_n, $n \geq 2$.

We put $m = [\frac{n}{2}]$. The center of G is $\langle h_{\alpha_n}(-1)\rangle$. We have $\omega_i = e_1 + \cdots + e_i$ for $i = 1, \ldots, n-1$, $\omega_n = \frac{1}{2}(e_1 + \cdots + e_n)$. We put $\beta_i = e_{2i-1} + e_{2i}$, $\delta_i = e_{2i-1} - e_{2i}$ for $i = 1, \ldots, m$. We put $\gamma_\ell = e_\ell$, $M_\ell = \{\ell+1, \ldots, n\}$ for $\ell = 1, \ldots, n$ and $J_\ell = \{2\ell+1, \ldots, n\}$, $K_\ell = J_\ell \cup \{1, 3, \ldots, 2\ell-1\}$ for $\ell = 1, \ldots, m$.

4.4.1 Unipotent classes in B_n, n even, $n = 2m$.



Unipotent classes in B_n, $n = 2m$.

Then

$$\begin{array}{llclcl} Z_\ell, & \ell = 1, \ldots, m & \longleftrightarrow & J_\ell & \longleftrightarrow & s_{\beta_1} s_{\delta_1} \cdots s_{\beta_\ell} s_{\delta_\ell} \\ X_\ell, & \ell = 1, \ldots, m & \longleftrightarrow & K_\ell & \longleftrightarrow & s_{\beta_1} \cdots s_{\beta_\ell} \end{array}$$

Lemma 4.20 *Let $w = s_{\beta_1} \cdots s_{\beta_\ell}$. Then T^w is connected for $\ell = 1, \ldots, m-1$ and, for $\ell = m$, $T^w = (T^w)^\circ \times \langle h_{\alpha_n}(-1) \rangle = (T^w)^\circ \times Z(G)$.*

Proof. We have

$$(1-w)P = \begin{cases} \mathbb{Z}\langle \omega_2, \omega_4, \ldots, \omega_{2\ell} \rangle & \text{for } \ell = 1, \ldots, m-1 \\ \mathbb{Z}\langle \omega_2, \omega_4, \ldots, \omega_{n-2}, 2\omega_n \rangle & \text{for } \ell = m \end{cases}$$

and we conclude. □

Proposition 4.21 *For $\ell = 1, \ldots, m-1$ we have*

$$\lambda(X_\ell) = \left\{ \sum_{i=1}^{\ell} n_{2i}\, \omega_{2i} \mid n_k \in \mathbb{N} \right\}$$

Moreover

$$\lambda(X_m) = \left\{ \sum_{i=1}^{m-1} n_{2i}\, \omega_{2i} + 2n_n \omega_n \mid n_k \in \mathbb{N} \right\}$$

Proof. This follows from Lemma 4.20, since in all cases $T_x = T^w$ (since $T^w = (T^w)^\circ Z(G)$). □

Proposition 4.22 *For $\ell = 1, \ldots, m-1$ we have $\lambda(\hat{X}_\ell) = \lambda(X_\ell)$. Moreover*

$$\lambda(\hat{X}_m) = \left\{ \sum_{i=1}^{m} n_{2i}\, \omega_{2i} \mid n_k \in \mathbb{N} \right\}$$

Proof. For $\ell = 1, \ldots, m-1$ the group T^w is connected by Lemma 4.20, and $\lambda(\hat{X}_\ell) = \lambda(X_\ell)$.

For $\ell = m$ the reductive part of $C(x)^\circ$ is of type C_m and so $\prod_{j \in K_m} H_{\alpha_j}$ is a maximal torus of $C(x)^\circ$. Hence $h_{\alpha_n}(-1) \notin C(x)^\circ$ by Lemma 4.3, and we are done. □

Lemma 4.23 *Let* $w = s_{\beta_1} \cdots s_{\beta_\ell} s_{\delta_1} \cdots s_{\delta_\ell}$. *Then*

$$T^w = \begin{cases} (T^w)^\circ \times \langle h_{\alpha_1}(-1)\rangle \times \cdots \times \langle h_{\alpha_{2\ell-1}}(-1)\rangle & \text{for } \ell = 1, \ldots, m-1 \\ T^{w_0} = T_2 & \text{for } \ell = m \end{cases}$$

Proof. We have $(1-w)P = \mathbb{Z}\langle 2\omega_1, \ldots, 2\omega_{2\ell-1}, \omega_{2\ell}\rangle$ for $\ell = 1, \ldots, m-1$. □

For $\ell = 1$ and $m \geq 2$, we get $T^w = \langle h_{\alpha_1}(-1)\rangle \times (T^w)^\circ$. In [9] we exhibit the element $x_{-\beta_1}(1)x_{-\delta_1}(1) \in \mathcal{O} \cap BwB \cap B^-$. We may therefore choose

$$x = n_{\beta_1} n_{\delta_1} h \, x_{\beta_1}(2) x_{\delta_1}(2)$$

for a certain $h \in T$. Then $h_{\alpha_1}(-1) \in C(x)$, so that $T_x = T^w$. Therefore, if $m \geq 2$,

$$\lambda(Z_1) = \{2n_1\omega_1 + n_2\omega_2 \mid n_k \in \mathbb{N}\}$$

Next we consider Z_m, $m \geq 1$. Let K be the subgroup generated by the long roots of G: K is of type D_n and it is simply-connected ([42], §II 5, 5.4 (a)). In fact $K = C(\sigma)$, where $\sigma = \prod_{i=1}^m h_{\alpha_{2i-1}}(-1)$, and $Z(K) = C(K) = Z(G) \times \langle\sigma\rangle$. Following [9], proof of Theorem 2.11, we have $x \in K$. But then we must have $T_x = Z(K)$ by the results obtained for D_n (and for $D_2 = A_1 \times A_1$ if $m = 1$), so that

$$\lambda(Z_m) = \left\{ \sum_{i=1}^n n_i\omega_i \mid n_k \in \mathbb{N}, \ \sum_{i=1}^m n_{2i-1} \text{ even}, n_n \text{ even} \right\}$$

We now deal with Z_ℓ, $\ell = 2, \ldots, m-1$. Here Ψ_J has basis $\{\alpha_1, \ldots, \alpha_{2\ell-1}, \gamma_{2\ell}\}$, and $C((T^w)^\circ)'$ is of type $B_{2\ell}$ (and is simply-connected).

From the construction in [9], proof of Theorem 2.11, we can find x in the $D_{2\ell}$-subgroup K of $C((T^w)^\circ)'$ generated by the long roots, that is the $D_{2\ell}$-subgroup with basis $\{\alpha_1, \ldots, \alpha_{2\ell-1}, \beta_\ell\}$ (which is simply-connected). We have

$$Z(K) = Z(G) \times \langle\sigma\rangle \quad , \quad \sigma = \prod_{i=1}^{\ell} h_{\alpha_{2i-1}}(-1)$$

By Lemma 4.23 we have

$$T^w = R \times (T^w)^\circ \quad , \quad R = \langle h_{\alpha_1}(-1)\rangle \times \cdots \times \langle h_{\alpha_{2\ell-1}}(-1)\rangle$$

and

$$T_x \cap R = R \cap Z(K) = \prod_{i=1}^{\ell} h_{\alpha_{2i-1}}(-1)$$

since we have already shown that $T_y = Z(D_{2\ell})$ if the spherical unipotent class $\mathcal{O}_y$ lies above w_0 in $D_{2\ell}$. Hence

$$T_x = (T^w)^\circ \times \langle\sigma\rangle$$

Therefore

$$\lambda(Z_\ell) = \left\{ \sum_{i=1}^{2\ell} n_i\omega_i \mid n_k \in \mathbb{N},\ \sum_{i=1}^{\ell} n_{2i-1} \text{ even} \right\}$$

We summarize the results obtained in

Proposition 4.24 *Let G be of type B_n, $n = 2m$, $m \geq 1$. For $\ell = 1, \ldots, m-1$ we have*

$$\lambda(Z_\ell) = \left\{ \sum_{i=1}^{2\ell} n_i\omega_i \mid n_k \in \mathbb{N},\ \sum_{i=1}^{\ell} n_{2i-1} \text{ even} \right\}$$

Moreover

$$\lambda(Z_m) = \left\{ \sum_{i=1}^{n} n_i\omega_i \mid n_k \in \mathbb{N},\ \sum_{i=1}^{m} n_{2i-1} \text{ even, } n_n \text{ even} \right\}$$

□

For the simply-connected cover we have

Proposition 4.25 *For $\ell < m$ we have*

$$\lambda(\hat{Z}_\ell) = \left\{ \sum_{i=1}^{2\ell} n_i\omega_i \mid n_k \in \mathbb{N} \right\}$$

Moreover

$$\lambda(\hat{Z}_m) = \left\{ \sum_{i=1}^{n} n_i\omega_i \mid n_k \in \mathbb{N},\ \ n_n \text{ even} \right\}$$

Proof. Let $u \in Z_\ell$, with $\ell = 1, \ldots, m$. If $C(u)^\circ = RC$ with $R = R_u(C(u))$, C connected reductive, then C is of type $C_{\ell-1}D_{n-2\ell+1}$. In particular C is semisimple except when $n-2\ell+1 = 1$, i.e. $\ell = m$. Therefore we obtain $T \cap C(x)^\circ = (T^w)^\circ$ for $\ell = 2, \ldots, m-1$ by Lemma 4.4, since in these cases $T_x = (T^w)^\circ \times \langle\sigma\rangle$.

We claim that $\omega_1 \in \lambda(\hat{Z}_\ell)$ for $\ell = 1, \ldots, m$. Let $u = x_{\alpha_{n-2\ell+2}}(1)x_{\alpha_{n-2\ell+4}}(1)\cdots x_{\alpha_n}(1)$ which is in Z_ℓ. The image Q of $(u-1)^2$ in $V(\omega_1)$ (which is the natural module for B_n) has dimension 1 and coincides with $V(\omega_1)_{\alpha_n}$. Let v be a generator of Q. Then there is a character $\gamma : C(u) \to \mathbb{C}^*$ such that $g.v = \gamma(g)v$ for every $g \in C(u)$.

Now $C(u)$ has rank $n-\ell$, so that $S = \{t \in T \mid \alpha_{n-2\ell+2}(t) = a_{n-2\ell+4}(t) = \cdots = \alpha_n(t) = 1\}$ (which is connected) is a maximal torus of $C(u)^\circ$. If $t \in S$, then $t.v = \alpha_n(t)v = v$, so that even in the case when the reductive part of $C(u)^\circ$ is not semisimple, γ is the trivial character on $C(u)^\circ$. Hence $C(u)^\circ.v = v$.

In particular, if $\ell = 1$ and $m \geq 2$, then $T \cap C(x)^\circ = (T^w)^\circ$ and

$$\lambda(\hat{Z}_1) = \{n_1\omega_1 + n_2\omega_2 \mid n_k \in \mathbb{N}\}$$

We are left to deal with Z_m. In this case we observe that taking again $u = x_{\alpha_2}(1)x_{\alpha_4}(1)\cdots x_{\alpha_n}(1)$ in Z_m, then $H_{\gamma_1} \leq C(u)$, where $\gamma_1 = e_1$. Since γ_1 is short, we have $Z(G) \leq H_{\gamma_1}$, so that $h_{\alpha_n}(-1) \in C(x)^\circ$. Therefore

$$\lambda(\hat{Z}_m) = \left\{ \sum_{i=1}^{n} n_i\omega_i \mid n_k \in \mathbb{N},\ \ n_n \text{ even} \right\}$$

since we know that $\omega_1 \in \lambda(\hat{Z}_m)$. □

We obtained

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
X_ℓ $\ell = 1, \dots, m-1$	$\sum_{i=1}^{\ell} n_{2i}\,\omega_{2i}$	
X_m	$\sum_{i=1}^{m-1} n_{2i}\,\omega_{2i} + 2n_n\omega_n$	$\sum_{i=1}^{m} n_{2i}\,\omega_{2i}$
Z_ℓ $\ell = 1, \dots, m-1$	$\sum_{i=1}^{2\ell} n_i\omega_i,\ \sum_{i=1}^{\ell} n_{2i-1}$ even	$\sum_{i=1}^{2\ell} n_i\omega_i$
Z_m	$\sum_{i=1}^{n} n_i\omega_i,\ \sum_{i=1}^{m} n_{2i-1}$ even, n_n even	$\sum_{i=1}^{n} n_i\omega_i,\ \ n_n$ even

Table 10: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in B_n, $n = 2m$.

4.4.2 Semisimple classes in B_n, n even $n = 2m$

Following the notation in [9], Tables 1, 5 we have

$$\begin{array}{llll} D_\ell B_{n-\ell}, & \ell = 1, \dots, m & \longleftrightarrow\ J_\ell & \longleftrightarrow\ s_{\beta_1}s_{\delta_1}\cdots s_{\beta_\ell}s_{\delta_\ell} \\ D_\ell B_{n-\ell}, & \ell = m+1, \dots, n & \longleftrightarrow\ M_{2(n-\ell)+1} & \longleftrightarrow\ s_{\gamma_1}s_{\gamma_2}\cdots s_{\gamma_{2(n-\ell)+1}} \\ T_1A_{n-1} & & \longleftrightarrow\ \varnothing & \longleftrightarrow\ w_0 \end{array}$$

Type $D_1B_{n-1} = T_1B_{n-1}$. Consider the element $\sigma_1 = \exp(\pi i\check{\omega}_1)$, $H = C(\sigma_1)$. Then H is of type T_1B_{n-1}. If we put $\lambda = e^\zeta$, then the image of $\exp(\zeta\check{\omega}_1)$ in $SO(2n+1)$ is $\mathrm{diag}(1, \lambda, I_{n-1}, \lambda^{-1}, I_{n-1})$. We have $C(\exp(\zeta\check{\omega}_1)) = H \iff \zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$.

For $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$, we consider the element

$$y_\zeta = g\exp(\zeta\check{\omega}_1)g^{-1}$$

where $g = x_{-\beta_1}(1)x_{-\delta_1}(1)$. Now $\beta_1(\exp(\zeta\check{\omega}_1)) = \delta_1(\exp(\zeta\check{\omega}_1)) = e^\zeta$, so that

$$\exp(\zeta\check{\omega}_1)x_{-\delta_1}(-1)x_{-\beta_1}(-1)\exp(\zeta\check{\omega}_1)^{-1} = x_{-\delta_1}(-e^{-\zeta})x_{-\beta_1}(-e^{-\zeta})$$

and

$$y_\zeta = x_{-\beta_1}(1-e^{-\zeta})x_{-\delta_1}(1-e^{-\zeta})\exp(\zeta\check{\omega}_1)$$

By Lemma 4.1 we may take x_ζ of the form

$$x_\zeta = n_{\beta_1}n_{\delta_1}hx_{\beta_1}(\xi_1)x_{\delta_1}(\xi_2)$$

for certain $h \in T$, $\xi_1, \xi_2 \in \mathbb{C}$: more precisely,

$$x_\zeta = n_{\beta_1}n_{\delta_1}h_{\beta_1}(e^{-\zeta}-1)h_{\delta_1}(e^{-\zeta}-1)\exp(\zeta\check{\omega}_1)x_{\beta_1}\left(\frac{1+e^{-\zeta}}{1-e^{-\zeta}}\right)x_{\delta_1}\left(\frac{1+e^{-\zeta}}{1-e^{-\zeta}}\right)$$

We have $w = s_{\beta_1}s_{\delta_1}$, and

$$T^w = \begin{cases} \langle h_{\alpha_1}(-1)\rangle \times (T^w)^\circ & \text{for } m \geq 2 \\ T_2 = \langle h_{\alpha_1}(-1)\rangle \times Z(G) & \text{for } m = 1 \end{cases}$$

moreover $h_{\alpha_1}(-1) \in C(x_\zeta)$, since $\beta_1(h_{\alpha_1}(-1)) = \delta_1(h_{\alpha_1}(-1)) = 1$, so that $T_x = T^w$. Therefore

$$\lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_1)}) = \begin{cases} \{2n_1\omega_1 + n_2\omega_2 \mid n_k \in \mathbb{N}\} & \text{for } m \geq 2 \\ \{2n_1\omega_1 + 2n_2\omega_2 \mid n_k \in \mathbb{N}\} & \text{for } m = 1 \end{cases}$$

for $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$ (as for Z_1).

Type D_kB_{n-k}, $k = 2, \dots, n$. Consider the element $\sigma_k = \exp(\pi i\check{\omega}_k)$, $H = C(\sigma_k)$ (the image of σ_k in $SO(2n+1)$ is diag$(1, -I_k, I_{n-k}, -I_k, I_{n-k})$). Then H is of type D_kB_{n-k}, $Z(H) = C(H) = \langle\sigma_k\rangle Z(G)$ (in fact if k is even we have $\sigma_k^2 = 1$ and $Z(H) = \langle\sigma_k\rangle \times Z(G)$, if k is odd we have $\sigma_k^2 = h_{a_n}(-1)$ and $Z(H) = \langle\sigma_k\rangle$).

Let us first assume $k = 2, \dots, m$, and let

$$x = n_{\beta_1}n_{\delta_1}\cdots n_{\beta_k}n_{\delta_k}$$

Then $x \sim h_{\beta_1}(i)h_{\delta_1}(i)\cdots h_{\beta_k}(i)h_{\delta_k}(i) \sim \sigma_k$. Now

$$T^w = \begin{cases} \langle h_{\alpha_1}(-1)\rangle \times \cdots \times \langle h_{\alpha_{2\ell-1}}(-1)\rangle \times (T^w)^\circ & \text{for } \ell = 1, \dots, m-1 \\ T_2 & \text{for } \ell = m \end{cases}$$

and clearly $T_x = T^w$. It follows that

$$\lambda(\mathcal{O}_{\exp(\pi i\check{\omega}_\ell)}) = \left\{\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell} \mid n_i \in \mathbb{N}\right\}$$

for $\ell = 2, \dots, m-1$. Moreover

$$\lambda(\mathcal{O}_{\exp(\pi i\check{\omega}_m)}) = \left\{\sum_{i=1}^{n} 2n_i\omega_i \mid n_i \in \mathbb{N}\right\}$$

Let $k = m+1, \dots, n$. In [9], proof of Theorem 2.15, we introduced a certain conjugate (in $SO(2n+1)$) $\dot{Z}_{n-k}$ of the image of σ_k in $SO(2n+1)$: $\dot{Z}_{n-k}$ is a representative of the element $Z_{n-k} = s_{\gamma_1} \cdots s_{\gamma_{2(n-k)+1}}$. Therefore the element

$$x = n_{\gamma_1} \cdots n_{\gamma_{2(n-k)+1}} t$$

is conjugate to σ_k for a certain $t \in T$. Now we have the following generalization of Lemma 4.23

Lemma 4.26 *Let $w = s_{\gamma_1} \cdots s_{\gamma_\ell}$ for $\ell = 1, \dots, n$. Then*

$$T^w = \begin{cases} (T^w)^\circ \times \langle h_{\alpha_1}(-1)\rangle \times \cdots \times \langle h_{\alpha_{\ell-1}}(-1)\rangle & \textit{for } \ell = 1, \dots, n-1 \\ T^{w_0} = T_2 & \textit{for } \ell = n \end{cases}$$

Proof. We have $(1-w)P = \mathbb{Z}\langle 2\omega_1, \dots, 2\omega_{\ell-1}, \omega_\ell\rangle$ for $\ell < n$. □

Since clearly $T_x = T^w$, we get

Proposition 4.27 *For $\ell = m+1, \dots, n$ we have*

$$\lambda(\mathcal{O}_{\exp(\pi i \check{\omega}_\ell)}) = \left\{ \sum_{i=1}^{2(n-\ell)} 2n_i\omega_i + n_{2(n-\ell)+1}\omega_{2(n-\ell)+1} \mid n_\ell \in \mathbb{N} \right\}$$

□

Type T_1A_{n-1}. Consider the element $z = \exp(\check{\omega}_n)$, $H = C(z)$. Then H is of type T_1A_{n-1}, $Z(H) = C(H) = \exp(\mathbb{C}\check{\omega}_n) \times Z(G)$. If we put $\lambda = e^\zeta$, then the image of $\exp(\zeta\check{\omega}_n)$ in $SO(2n+1)$ is $b_\lambda = \mathrm{diag}(1, \lambda I_n, \lambda^{-1} I_n)$. We have $C(\exp(\zeta\check{\omega}_n)) = H \iff \zeta \in \mathbb{C} \setminus \pi i\mathbb{Z}$.

Let $\overline{B}$ be the image of B in $SO(2n+1)$. In [9], proof of Theorem 15, we exhibited an element y_λ in $SO(2n+1)$: $y_\lambda \in \mathcal{O}_{b_\lambda} \cap \overline{B} w_0 \overline{B}$. The centralizer of y_λ in $\overline{B}$ is trivial, therefore $C_B(\tilde{y}_\lambda) = Z(G)$, where $\tilde{y}_\lambda$ is any representative of y_λ in G. Hence $T_{x_\zeta} = Z(G) = \langle h_{\alpha_n}(-1)\rangle$ for any $x_\zeta \in \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \cap w_0 B$, so that

$$\lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_n)}) = \left\{ \sum_{i=1}^{n} n_i\omega_i \mid n_k \in \mathbb{N},\ \ n_n \text{ even} \right\}$$

for $\zeta \in \mathbb{C} \setminus \pi i\mathbb{Z}$.

We obtained

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\zeta\check{\omega}_1)$ $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}, m \geq 2$	$T_1 B_{n-1}$	$2n_1\omega_1 + n_2\omega_2$
$\exp(\zeta\check{\omega}_1)$ $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}, m = 1$	$T_1 B_1$	$2n_1\omega_1 + 2n_2\omega_2$
$\exp(\pi i\check{\omega}_\ell)$ $\ell = 2, \ldots, m-1$	$D_\ell B_{n-\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$
$\exp(\pi i\check{\omega}_m)$	$D_m B_m$	$\sum_{i=1}^{n} 2n_i\omega_i$
$\exp(\pi i\check{\omega}_\ell)$ $\ell = m+1, \ldots, n$	$D_\ell B_{n-\ell}$	$\sum_{i=1}^{2(n-\ell)} 2n_i\omega_i + n_{2(n-\ell)+1}\omega_{2(n-\ell)+1}$
$\exp(\zeta\check{\omega}_n)$ $\zeta \in \mathbb{C} \setminus \pi i\mathbb{Z}$	$T_1 A_{n-1}$	$\sum_{i=1}^{n} n_i\omega_i$, n_n even

Table 11: $\lambda(\mathcal{O})$ for semisimple classes in B_n, $n = 2m$.

4.4.3 Mixed classes in B_n, n even, $n = 2m$

From [9], Table 4, we get

$$\begin{array}{lcccc} \sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1) & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 \\ \sigma_n x_{\beta_1}(1)\cdots x_{\beta_\ell}(1), \quad \ell = 1,\ldots,m-1 & \longleftrightarrow & M_{2\ell+1} & \longleftrightarrow & s_{\gamma_1}\cdots s_{\gamma_{2\ell+1}} \end{array}$$

Class of $\sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1)$. We claim that $T_x = Z(G)$ for $x \in \mathcal{O} \cap w_0 B$.

Suppose for a contradiction that $T_x \neq Z(G)$, and let $\sigma \in T_x \setminus Z(G)$. Then we have $x \in K = C(\sigma)$. Since the involutions in G are conjugate (up to a central element) to σ_{2k}, for a certain $k \in \{1,\ldots,m\}$, K is of type $D_{2k}B_{n-2k}$.

Now x is conjugate in K to an element of the form su, with $s \in T$, $u \in U(K)$, $[s,u] = 1$. We have $s = s_1 s_2$, $u = u_1 u_2$, with $s_1 \in T(D_{2k})$, $s_2 \in T(B_{n-2k})$, $u_1 \in U(D_{2k})$, $u_2 \in T(B_{n-2k})$ (note that u_1 and u_2 are uniquely determined, and u_1 must be in the classes X_k or X'_k of D_{2k}, u_2 in the class X_{m-k} of B_{n-2k}). Moreover $s_1 u_1$ and $s_2 u_2$ must lie over the longest elements of the Weyl group of D_{2k} and B_{n-2k} respectively. We want to show that $s_1 \in Z(D_{2k})$: this will lead to the contradiction that $s_1 u_1$ lies over the same element of the Weyl group of D_{2k} over which lies u_1, and this is not the longest element of the Weyl group of D_{2k}. To show that $s_1 \in Z(D_{2k})$ we may assume, up to the action of W, that $K = C(\sigma)$, where $\sigma = \prod_{i=1}^{k} h_{\alpha_{2i-1}}(-1)$.

In T there is a W-orbit $\{\sigma_n, z\sigma_n\}$, where $z = h_{\alpha_n}(-1)$, due to the fact that the long roots of B_n form a D_n-subgroup of B_n: its center is $\langle\sigma_n\rangle \times Z(G)$. Since $D_{2k} \cap B_{n-2k} = Z(G)$ and $s_1 s_2 \sim \sigma_n$ we have only the following possibilities for (s_1, s_2): $(\sigma, \sigma\sigma_n)$, $(\sigma z, \sigma\sigma_n z)$, $(\sigma z, \sigma\sigma_n)$,

$(\sigma, \sigma\sigma_n z)$. In each case we have $s_1 \in Z(D_{2k}) = \langle z, \sigma\rangle$. We have therefore proved that $T_x = Z(G)$, so that

$$\lambda(\mathcal{O}_{\sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1)}) = \left\{\sum_{i=1}^{n} n_i\omega_i \mid n_k \in \mathbb{N},\ \ n_n \text{ even}\right\}$$

Moreover, by the results for the class X_m in D_n, $n = 2m$, it follows that the centralizer of $\sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1)$ in G is not connected, hence $C(x) = C(x)^\circ \times Z(G)$ and $C(x)^\circ \cap T = 1$,

$$\lambda(\hat{\mathcal{O}}_{\sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1)}) = \left\{\sum_{i=1}^{n} n_i\omega_i \mid n_k \in \mathbb{N}\right\}$$

Class of $\sigma_n x_{\beta_1}(1)\cdots x_{\beta_\ell}(1)$, $\ell = 1, \cdots, m-1$.

Here Ψ_J has basis $\{\alpha_1, \ldots, \alpha_{2\ell}, \gamma_{2\ell+1}\}$, and $K = C((T^w)^\circ)'$ is of type $B_{2\ell+1}$ (and is simply-connected). From the construction in [9], proof of Theorem 2.23, we can find x of the form $x = x_1 h$, with $h \in T$, $x_1 \in K$, x_1 in the class of $\sigma_{2\ell+1}x_{\beta_1}(1)\cdots x_{\beta_\ell}(1)$ (which is the mixed class of maximal dimension in $B_{2\ell+1}$). By Lemma 4.33 we have

$$T^w = R \times (T^w)^\circ \quad , \quad R = \langle h_{\alpha_1}(-1)\rangle \times \cdots \times \langle h_{\alpha_{2\ell}}(-1)\rangle \leq T(K)$$

$$(T^w)^\circ = H_{\alpha_{2\ell+2}} \times \cdots \times H_{\alpha_n} \quad , \quad T_x = (T_x \cap R) \times (T^w)^\circ$$

and

$$T_x \cap R \leq T_x \cap T(K) = C_{T(K)}(x) = C_{T(K)}(x_1)$$

and by the results for the mixed class of maximal dimension in $B_{2\ell+1}$ (see next subsection), we have $C_{T(K)}(x_1) = Z(K) = \langle h_{\gamma_{2\ell+1}}(-1)\rangle = \langle h_{\alpha_n}(-1)\rangle$. Hence

$$T_x \cap R \leq \langle h_{\alpha_n}(-1)\rangle \cap R = 1$$

and $T_x = (T^w)^\circ$. Therefore

$$\lambda(\hat{\mathcal{O}}_{\rho_n x_{\beta_1}(1)\cdots x_{\beta_\ell}(1)}) = \lambda(\mathcal{O}_{\rho_n x_{\beta_1}(1)\cdots x_{\beta_\ell}(1)}) = \left\{\sum_{i=1}^{2\ell+1} n_i\omega_i \mid n_k \in \mathbb{N}\right\}$$

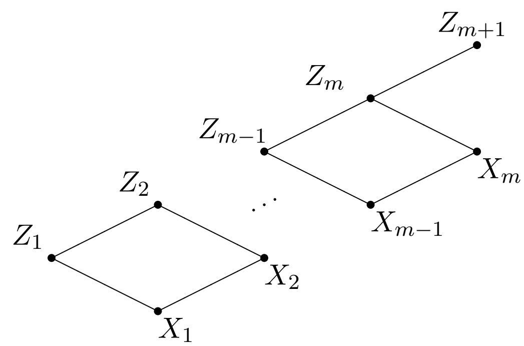
We obtained

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
$\sigma_n x_{\beta_1}(1)\cdots x_{\beta_\ell}(1)$ $\ell = 1, \cdots, m-1$	$\sum_{i=1}^{2\ell+1} n_i\omega_i$	
$\sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1)$	$\sum_{i=1}^{n} n_i\omega_i$, n_n even	$\sum_{i=1}^{n} n_i\omega_i$

Table 12: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for mixed classes in B_n, $n = 2m$.

In particular $\hat{\mathcal{O}}_{\sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1)}$ is a model homogeneus space, and in fact the principal one, by [28], 3.3 (2).

4.4.4 Unipotent classes in B_n, n odd, $n = 2m+1$.



Unipotent classes in B_n, $n = 2m+1$

Then

$$\begin{array}{llll} Z_\ell & \longleftrightarrow & J_\ell, \quad \ell = 1,\dots,m & \longleftrightarrow & s_{\beta_1}s_{\delta_1}\cdots s_{\beta_\ell}s_{\delta_\ell} \\ Z_{m+1} & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 = s_{\beta_1}s_{\delta_1}\cdots s_{\beta_m}s_{\delta_m}s_{\alpha_n} \\ X_\ell & \longleftrightarrow & K_\ell, \quad \ell = 1,\dots,m & \longleftrightarrow & s_{\beta_1}\cdots s_{\beta_\ell} \end{array}$$

Lemma 4.28 *Let $w = s_{\beta_1}\cdots s_{\beta_\ell}$ for $\ell = 1,\dots,m$. Then T^w is connected.*

Proof. For $\ell = 1,\dots,m$ we have $(1-w)P = \mathbb{Z}\langle\beta_1,\dots,\beta_\ell\rangle = \mathbb{Z}\langle\omega_{2i} \mid i = 1,\dots,\ell\rangle$. □

Proposition 4.29 *For $\ell = 1,\dots,m$ we have*

$$\lambda(\hat{X}_\ell) = \lambda(X_\ell) = \left\{\sum_{i=1}^{\ell} n_{2i}\,\omega_{2i} \mid n_k \in \mathbb{N}\right\}$$

Proof. This follows from Lemma 4.28. □

Lemma 4.30 *Let $w = s_{\beta_1}\cdots s_{\beta_\ell}s_{\delta_1}\cdots s_{\delta_\ell}$ for $\ell = 1,\dots,m$. Then*

$$T^w = (T^w)^\circ \times \langle h_{\alpha_1}(-1)\rangle \times \cdots \times \langle h_{\alpha_{2\ell-1}}(-1)\rangle$$

Proof. For $\ell = 1,\dots,m$ we have $(1-w)P = \mathbb{Z}\langle 2\omega_1,\dots,2\omega_{2\ell-1},\omega_{2\ell}\rangle$. □

For $\ell = 1$ we get $T^w = \langle h_{\alpha_1}(-1)\rangle \times (T^w)^\circ$. In [9] we exhibit the element $x_{-\beta_1}(1)x_{-\delta_1}(1) \in \mathcal{O} \cap BwB \cap B^-$. We may therefore choose $x = n_{\beta_1}n_{\delta_1}h\,x_{\beta_1}(2)x_{\delta_1}(2)$ for a certain $h \in T$. Then $h_{\alpha_1}(-1) \in C(x)$, so that $T_x = T^w$.

Next we consider Z_{m+1}. We claim that $T_x = Z(G)$. Suppose for a contradiction that there is an involution $\sigma \in T_x \setminus Z(G)$. Then $x \in K = C(\sigma)$, and K is the almost direct product K_1K_2, of type D_kB_{n-k}, for some $k = 1,\dots,n$. We get an orthogonal decomposition $E = E_1 \oplus E_2$ and a decomposition $x = x_1x_2 \in K_1K_2$. Then $-1 = w_0 = (w_1,w_2)$, where w_i is the element of the Weyl group of K_i corresponding to x_i (the class of x_i in K_i is spherical). It follows that each

$w_i = -1$, and k is even. Then x_1 is in the class $Z_{k/2}$ of K_1 and x_2 in the class $Z_{m+1-k/2}$ of K_2. However, the product x_1x_2 is not in the class Z_{m+1} of G (since in x_1x_2 there are two rows with 3 boxes), a contradiction. Hence $T_x = Z(G)$.

We now deal with Z_ℓ, $\ell = 2, \ldots, m$. Here Ψ_J has basis $\{\alpha_1, \ldots, \alpha_{2\ell-1}, \gamma_{2\ell}\}$, and $C((T^w)^\circ)'$ is of type $B_{2\ell}$. From the construction in [9], proof of Theorem 2.11, we can find x in the $D_{2\ell}$-subgroup K of $C((T^w)^\circ)'$ generated by the long roots, that is the $D_{2\ell}$-subgroup with basis $\{\alpha_1, \ldots, \alpha_{2\ell-1}, \beta_\ell\}$. We have

$$Z(K) = Z(G) \times \langle\sigma\rangle \quad , \quad \sigma = \prod_{i=1}^{\ell} h_{\alpha_{2i-1}}(-1)$$

By Lemma 4.30, $T_x = (T^w)^\circ \times (T_x \cap R)$, where $R = \langle h_{\alpha_1}(-1)\rangle \times \cdots \times \langle h_{\alpha_{2\ell-1}}(-1)\rangle \leq K$. Since x lies in the maximal spherical unipotent class of $D_{2\ell}$, from the result obtained for this class, we have $T_x \cap R = R \cap Z(K) = \langle\sigma\rangle$, hence $T_x = (T^w)^\circ \times \langle\sigma\rangle$. We have proved

Proposition 4.31 *For $\ell = 1, \ldots, m$ we have*

$$\lambda(Z_\ell) = \left\{\sum_{i=1}^{2\ell} n_i\omega_i \mid n_k \in \mathbb{N},\ \sum_{i=1}^{\ell} n_{2i-1} \text{ even}\right\}$$

Moreover

$$\lambda(Z_{m+1}) = \left\{\sum_{i=1}^{n} n_i\omega_i \mid n_k \in \mathbb{N},\ \ n_n \text{ even}\right\}$$

For the simply-connected cover we obtain

Proposition 4.32 *For $\ell = 1, \ldots, m$ we have*

$$\lambda(\hat{Z}_\ell) = \left\{\sum_{i=1}^{2\ell} n_i\omega_i \mid n_k \in \mathbb{N}\right\}$$

Moreover

$$\lambda(\hat{Z}_{m+1}) = \left\{\sum_{i=1}^{n} n_i\omega_i \mid n_k \in \mathbb{N}\right\}$$

Proof. Let $u \in Z_\ell$, with $\ell = 1, \ldots, m+1$. If $C(u)^\circ = RC$ with $R = R_u(C(u))$, C connected reductive, then C is of type $C_{\ell-1}D_{n-2\ell+1}$ ([12], §13.1). In particular C is semisimple since $n - 2\ell + 1$ is even. Hence $\lambda(\hat{Z}_\ell)$ is free by Lemma 4.4.

For $\ell = m+1$, we have $Z(G) \not\leq C(x)^\circ$. In fact, we can take $u = x_{\alpha_1}(1)x_{\alpha_3}(1)\cdots x_{\alpha_n}(1)$ in Z_{m+1}. Then $S = H_{\check{\omega}_2}H_{\check{\omega}_4}\cdots H_{\check{\omega}_{n-1}}$ is a maximal torus of $C(u)^\circ$, and since $Z(G) \cap S = \{1\}$, we get $C(u) = C(u)^\circ \times Z(G)$ by Lemma 4.3. We are left to deal with $\ell = 1$. However for each ℓ, the image Q of $(u-1)^2$ in $V(\omega_1)$ (which is the natural module for B_n) has dimension 1, so $C(u)^\circ$ acts trivially on Q by Lemma 4.5, and $\omega_1 \in \lambda(\hat{Z}_\ell)$. □

We summarize the results obtained in

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
X_ℓ $\ell=1,\ldots,m$	$\sum_{i=1}^{\ell} n_{2i}\,\omega_{2i}$	
Z_ℓ $\ell=1,\ldots,m$	$\sum_{i=1}^{2\ell} n_i\omega_i,\ \sum_{i=1}^{\ell} n_{2i-1}$ even	$\sum_{i=1}^{2\ell} n_i\omega_i$
Z_{m+1}	$\sum_{i=1}^{n} n_i\omega_i,\ \ n_n$ even	$\sum_{i=1}^{n} n_i\omega_i$

Table 13: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in B_n, $n = 2m+1$.

In particular $\hat{Z}_{m+1}$ is a model homogeneus space, and in fact the principal one, by [28], 3.3 (2).

In section 5, we shall determine the decomposition of the coordinate ring of the closure $\overline{\mathcal{O}}$ of $\mathcal{O} = Z_{m+1}$. For this purpose we shall use the fact that if $x \in \mathcal{O} \cap w_0B$, then α_{n-1} occurs in x (see the discussion before Proposition 3.11). In [9], proof of Theorem 12, we exhibit an element v in the corresponding class in $SO(2n+1)$. Working in $SO(2n+1)$, we find that $v = u'x_{\alpha_{n-1}}(-1)\dot{w}_0x_{\alpha_{n-1}}(-1)u$ for a certain representative $\dot{w}_0$ of w_0, u, $u' \in \prod_{\beta\in\Phi^+\setminus\{\alpha_{n-1}\}} X_\beta$. Then

$$x = (u'x_{\alpha_{n-1}}(-1))^{-1}vu'x_{\alpha_{n-1}}(-1) = \dot{w}_0x_{\alpha_{n-1}}(-2)u''$$

for a certain $u'' \in \prod_{\beta\in\Phi^+\setminus\{\alpha_{n-1}\}} X_\beta$. The calculation is reduced to determining the first upper off-diagonal of upper unipotent $n\times n$ matrices X, Y such that ${}^tX^{-1}Y = -\Sigma$, where Σ is the $n\times n$ matrix with diagonal equal to $(-1,0,\ldots,0)$, first upper off-diagonal equal to $(1,1,\ldots,1)$, first lower off-diagonal equal to $(-1,-1,\ldots,-1)$ and zero elsewhere.

4.4.5 Semisimple classes in B_n, n odd $n = 2m+1$

Following the notation in [9], Tables 1, 5 we get

$$\begin{array}{lllll} D_\ell B_{n-\ell}, \quad \ell = 1,\ldots,m & \longleftrightarrow & J_\ell & \longleftrightarrow & s_{\beta_1}s_{\delta_1}\cdots s_{\beta_\ell}s_{\delta_\ell} \\ D_\ell B_{n-\ell}, \quad \ell = m+1,\ldots,n & \longleftrightarrow & M_{2(n-\ell)+1} & \longleftrightarrow & s_{\gamma_1}s_{\gamma_2}\cdots s_{\gamma_{2(n-\ell)+1}} \\ T_1A_{n-1} & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 \end{array}$$

Type $D_1B_{n-1} = T_1B_{n-1}$. Consider the element $\sigma_1 = \exp(\pi i\check{\omega}_1)$, $H = C(\sigma_1)$. Then H is of type T_1B_{n-1}. If we put $\lambda = e^\zeta$, then the image of $\exp(\zeta\check{\omega}_1)$ in $SO(2n+1)$ is $\mathrm{diag}(1,\lambda,I_{n-1},\lambda^{-1},I_{n-1})$. We have $C(\exp(\zeta\check{\omega}_1)) = H \iff \zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}$.

For $\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}$, we consider the element

$$y_\zeta = g\exp(\zeta\check{\omega}_1)g^{-1}$$

where $g = x_{-\beta_1}(1)x_{-\delta_1}(1)$. Now $\beta_1(\exp(\zeta\check{\omega}_1)) = \delta_1(\exp(\zeta\check{\omega}_1)) = e^{\zeta}$, and we may take x_ζ of the form

$$x_\zeta = n_{\beta_1}n_{\delta_1}h_{\beta_1}(e^{-\zeta}-1)h_{\delta_1}(e^{-\zeta}-1)\exp(\zeta\check{\omega}_1)x_{\beta_1}\left(\frac{1+e^{-\zeta}}{1-e^{-\zeta}}\right)x_{\delta_1}\left(\frac{1+e^{-\zeta}}{1-e^{-\zeta}}\right)$$

We have $w = s_{\beta_1}s_{\delta_1}$, $T^w = \langle h_{\alpha_1}(-1)\rangle \times (T^w)^\circ$. Then $h_{\alpha_1}(-1) \in C(x_\zeta)$, since $\beta_1(h_{\alpha_1}(-1)) = \delta_1(h_{\alpha_1}(-1)) = 1$, so that $T_x = T^w$. Therefore

$$\lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_1)}) = \{2n_1\omega_1 + n_2\omega_2 \mid n_k \in \mathbb{N}\}$$

for $\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}$ (as for Z_1).

Type D_kB_{n-k}, $k = 2,\dots,n$.

Consider the element $\sigma_k = \exp(\pi i\check{\omega}_k)$, $H = C(\sigma_k)$ (the image of σ_k in $SO(2n+1)$ is $\text{diag}(1,-I_k, I_{n-k},-I_k, I_{n-k})$). Then H is of type D_kB_{n-k}, $Z(H) = C(H) = \langle\sigma_k\rangle Z(G)$ (in fact if k is even we have $\sigma_k^2 = 1$ and $Z(H) = \langle\sigma_k\rangle \times Z(G)$, if k is odd we have $\sigma_k^2 = h_{a_n}(-1)$ and $Z(H) = \langle\sigma_k\rangle$). For our purposes it is enough to deal with the elements σ_k.

Assume $k = 2,\dots,m$, and let

$$x = n_{\beta_1}n_{\delta_1}\cdots n_{\beta_k}n_{\delta_k}$$

Then $x \sim h_{\beta_1}(i)h_{\delta_1}(i)\cdots h_{\beta_k}(i)h_{\delta_k}(i) \sim \sigma_k$. Now

$$T^w = \langle h_{\alpha_1}(-1)\rangle \times \cdots \times \langle h_{\alpha_{2\ell-1}}(-1)\rangle \times (T^w)^\circ$$

and clearly $T_x = T^w$. It follows that

$$\lambda(\mathcal{O}_{\exp(\pi i\check{\omega}_\ell)}) = \left\{\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell} \mid n_i \in \mathbb{N}\right\}$$

for $\ell = 2,\dots,m$.

Assume $k = m+1,\dots,n$.

In [9], proof of Theorem 2.15, we considered a certain conjugate (in $SO(2n+1)$) $\dot{Z}_{n-k}$ of the image of σ_k in $SO(2n+1)$: $\dot{Z}_{n-k}$ is a representative of the element $Z_{n-k} = s_{\gamma_1}\cdots s_{\gamma_{2(n-k)+1}}$. Therefore the element

$$x = n_{\gamma_1}\cdots n_{\gamma_{2(n-k)+1}}t$$

is conjugate to σ_k for a certain $t \in T$. Now we have the following generalization of Lemma 4.30

Lemma 4.33 *Let* $w = s_{\gamma_1}\cdots s_{\gamma_\ell}$ *for* $\ell = 1,\dots,n$. *Then*

$$T^w = \begin{cases} (T^w)^\circ \times \langle h_{\alpha_1}(-1)\rangle \times \cdots \times \langle h_{\alpha_{\ell-1}}(-1)\rangle & \text{for } \ell = 1,\dots,n-1 \\ T^{w_0} = T_2 & \text{for } \ell = n \end{cases}$$

Proof. We have $(1-w)P = \mathbb{Z}\langle 2\omega_1, \dots, 2\omega_{\ell-1}, \omega_\ell\rangle$ For $\ell < n$. □

Since clearly $T_x = T^w$, we get

$$\lambda(\mathcal{O}_{\exp(\pi i\check{\omega}_\ell)}) = \left\{ \sum_{i=1}^{2(n-\ell)} 2n_i\omega_i + n_{2(n-\ell)+1}\omega_{2(n-\ell)+1} \mid n_k \in \mathbb{N} \right\}$$

for $\ell = m+2, \dots, n$, and

$$\lambda(\mathcal{O}_{\exp(\pi i\check{\omega}_{m+1})}) = \left\{ \sum_{i=1}^{n} 2n_i\omega_i \mid n_k \in \mathbb{N} \right\}$$

Type T_1A_{n-1}.

Consider the element $\exp(\check{\omega}_n)$, $H = C(\exp(\check{\omega}_n))$. Then H is of type T_1A_{n-1}, $Z(H) = C(H) = \exp(\mathbb{C}\check{\omega}_n)$. If we put $\lambda = e^\zeta$, then the image of $\exp(\zeta\check{\omega}_n)$ in $SO(2n+1)$ is $b_\lambda = \mathrm{diag}(1, \lambda I_n, \lambda^{-1}I_n)$. We have $C(\exp(\zeta\check{\omega}_n)) = H \iff \zeta \in \mathbb{C}\setminus \pi i\mathbb{Z}$.

With the same argument used for even n we conclude that $T_{x_\zeta} = Z(G) = \langle h_{\alpha_n}(-1)\rangle$ for any $x_\zeta \in \mathcal{O}_{\exp(\zeta\check{\omega}_n)} \cap w_0B$, so that

$$\lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_n)}) = \left\{ \sum_{i=1}^{n} n_i\omega_i \mid n_k \in \mathbb{N},\ \ n_n \text{ even} \right\}$$

for $\zeta \in \mathbb{C}\setminus \pi i\mathbb{Z}$.

We got

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\zeta\check{\omega}_1)$ $\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}$	T_1B_{n-1}	$2n_1\omega_1 + n_2\omega_2$
$\exp(\pi i\check{\omega}_\ell)$ $\ell = 2, \dots, m$	$D_\ell B_{n-\ell}$	$\sum_{i=1}^{2\ell-1} 2n_i\omega_i + n_{2\ell}\omega_{2\ell}$
$\exp(\pi i\check{\omega}_\ell)$ $\ell = m+2, \dots, n$	$D_\ell B_{n-\ell}$	$\sum_{i=1}^{2(n-\ell)} 2n_i\omega_i + n_{2(n-\ell)+1}\omega_{2(n-\ell)+1}$
$\exp(\pi i\check{\omega}_{m+1})$	$D_{m+1}B_m$	$\sum_{i=1}^{n} 2n_i\omega_i$
$\exp(\zeta\check{\omega}_n)$ $\zeta \in \mathbb{C}\setminus \pi i\mathbb{Z}$	T_1A_{n-1}	$\sum_{i=1}^{n} n_i\omega_i$, n_n even

Table 14: $\lambda(\mathcal{O})$ for semisimple classes in B_n, $n = 2m+1$.

4.4.6 Mixed classes in B_n, n odd, $n = 2m+1$

From [9], Table 4, we get

$$\sigma_n x_{\beta_1}(1)\cdots x_{\beta_\ell}(1),\ \ell = 1,\dots,m \quad\longleftrightarrow\quad M_{2\ell+1} \quad\longleftrightarrow\quad s_{\gamma_1}\cdots s_{\gamma_{2\ell+1}}$$

Class of $\sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1)$. Arguing in the same way as for the case of even n, we get $T_x = Z(G)$. In fact here the only difference is that σ_n has order 4, $\sigma_n^2 = z$, where $z = h_{\alpha_n}(-1)$. Then $\{\sigma_n, z\sigma_n = \sigma_n^{-1}\}$ is still a W-orbit.

Hence

$$\lambda(\mathcal{O}_{\sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1)}) = \left\{\sum_{i=1}^{n} n_i\omega_i \mid n_k \in \mathbb{N},\ \ n_n \text{ even}\right\}$$

Moreover we know that the centralizer of $x_{\beta_1}(1)\cdots x_{\beta_m}(1)$ in D_n is connected (since n is odd, see Table 8), therefore $C(x)$ is connected, and

$$\lambda(\hat{\mathcal{O}}_{\sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1)}) = \lambda(\mathcal{O}_{\sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1)})$$

Class of $\sigma_n x_{\beta_1}(1)\cdots x_{\beta_\ell}(1)$, $\ell = 1,\cdots,m-1$. Arguing as in the case of even n, we obtain $T_x = (T^w)^\circ$, so that

$$\lambda(\hat{\mathcal{O}}_{\sigma_n x_{\beta_1}(1)\cdots x_{\beta_\ell}(1)}) = \lambda(\mathcal{O}_{\sigma_n x_{\beta_1}(1)\cdots x_{\beta_\ell}(1)}) = \left\{\sum_{i=1}^{2\ell+1} n_i\omega_i \mid n_k \in \mathbb{N}\right\}$$

We got

$\mathcal{O}$	$\lambda(\mathcal{O}) = \lambda(\hat{\mathcal{O}})$
$\sigma_n x_{\beta_1}(1)\cdots x_{\beta_\ell}(1)$ $\ell = 1,\cdots,m-1$	$\sum_{i=1}^{2\ell+1} n_i\omega_i$
$\sigma_n x_{\beta_1}(1)\cdots x_{\beta_m}(1)$	$\sum_{i=1}^{n} n_i\omega_i$, n_n even

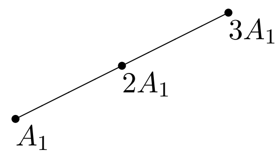
Table 15: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for mixed classes in B_n, $n = 2m+1$.

4.5 Type E_6.

We put

$$\beta_1 = (1,2,2,3,2,1),\quad \beta_2 = (1,0,1,1,1,1)$$
$$\beta_3 = (0,0,1,1,1,0),\quad \beta_4 = (0,0,0,1,0,0)$$

4.5.1 Unipotent classes in E_6.



Unipotent classes in E_6

Then

$$\begin{array}{lllll} A_1 & \longleftrightarrow & \{1,3,4,5,6\} & \longleftrightarrow & s_{\beta_1} \\ 2A_1 & \longleftrightarrow & \{3,4,5\} & \longleftrightarrow & s_{\beta_1}s_{\beta_2} \\ 3A_1 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 = s_{\beta_1}\cdots s_{\beta_4} \end{array}$$

We have

$$(1-w)P = \begin{cases} \mathbb{Z}\langle \omega_2 \rangle & \text{for } w = s_{\beta_1} \\ \mathbb{Z}\langle \omega_1+\omega_6, \omega_2 \rangle & \text{for } w = s_{\beta_1}s_{\beta_2} \\ \mathbb{Z}\langle \omega_1+\omega_6, 2\omega_2, \omega_3+\omega_5, 2\omega_4 \rangle & \text{for } w = w_0 \end{cases}$$

Here $Z(G) = \langle h_{\alpha_1}(\xi)h_{\alpha_6}(\xi^{-1})h_{\alpha_3}(\xi^{-1})h_{\alpha_5}(\xi)\rangle$, where ξ is a primitive 3rd-root of 1.

Class A_1. By Proposition 4.2, T^w is connected (in fact $(1-w)P = \mathbb{Z}\langle\omega_2\rangle$).
Class $2A_1$. Here T^w is connected since$(1-w)P = \mathbb{Z}\langle\omega_1+\omega_6,\omega_2\rangle$.
Class $3A_1$. Since $(1-w)P = \mathbb{Z}\langle\omega_1+\omega_6, 2\omega_2, \omega_3+\omega_5, 2\omega_4\rangle$, we get

$$T^{w_0} = (T^{w_0})^\circ \times R \quad , \quad R = \langle h_{\alpha_2}(-1)\rangle \times \langle h_{\alpha_4}(-1)\rangle$$

and, by 4.11, $(T^{w_0})^\circ = \{h_{\alpha_1}(t_1)h_{\alpha_6}(t_1^{-1})h_{\alpha_3}(t_3)h_{\alpha_5}(t_3^{-1}) \mid t_1, t_3 \in \mathbb{C}^*\}$.

Here Ψ_J has basis $\{\beta_2,\beta_3,\alpha_4,\alpha_2\}$, $K = C((T^w)^\circ)'$ is of type D_4 (and is simply-connected) and $Z(K) = \langle h_{\alpha_1}(-1)h_{\alpha_6}(-1), h_{\alpha_3}(-1)h_{\alpha_5}(-1)\rangle$. Since $x \in K$ and lies over the longest element of the Weyl group of K, from the result for the maximal spherical unipotent class in D_4 we get $T_x \cap K = Z(K)$. But $Z(K) \le (T^{w_0})^\circ$, so that $R \cap Z(K) = 1$, and $T_x = (T^{w_0})^\circ$.

We have shown that in all cases $T_x = (T^w)^\circ$, hence

$\mathcal{O}$	$\lambda(\mathcal{O}) = \lambda(\hat{\mathcal{O}})$
A_1	$n_2\omega_2$
$2A_1$	$n_1(\omega_1+\omega_6)+n_2\omega_2$
$3A_1$	$n_1(\omega_1+\omega_6)+n_3(\omega_3+\omega_5)+n_2\omega_2+n_4\omega_4$

Table 16: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in E_6.

4.5.2 Semisimple classes in E_6

Following the notation in [9], Table 2, we have

$$\begin{array}{lclcl} A_1A_5 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 \\ D_5\,T_1 & \longleftrightarrow & \{3,4,5\} & \longleftrightarrow & s_{\beta_1}s_{\beta_2} \end{array}$$

Type A_1A_5.

The elements of G whose centralizer is of type A_1A_5 are conjugate, up to a central element, to $\exp(\pi i\check{\omega}_2) = h_{\alpha_1}(-1)h_{\alpha_4}(-1)h_{\alpha_6}(-1)$. Let $x = n_{\beta_1}\cdots n_{\beta_4}$. Then $x^2 = h_{\beta_1}(-1)\cdots h_{\beta_4}(-1) = 1$, and $x \sim \exp(\pi i\check{\omega}_2)$. Then clearly $T_x = T^{w_0}$, so that

$$\lambda(\mathcal{O}_{\exp(\pi i\check{\omega}_2)}) = \{n_1(\omega_1+\omega_6) + n_3(\omega_3+\omega_5) + 2n_2\omega_2 + 2n_4\omega_4 \mid n_k \in \mathbb{N}\}$$

Type D_5T_1.

Let $K = C(\exp(\pi i\check{\omega}_1))$. Then $C(K) = Z(K) = \exp(\mathbb{C}\,\check{\omega}_1)$ and $C(\exp(\zeta\check{\omega}_1)) = K \Leftrightarrow \zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}$. Since T^w is connected we get

$$\lambda(\mathcal{O}_{\exp(\zeta\check{\omega}_1)}) = \{n_1(\omega_1+\omega_6) + n_2\omega_2 \mid n_k \in \mathbb{N}\}$$

if $\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}$.

We obtained

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\pi i\check{\omega}_2)$	A_1A_5	$n_1(\omega_1+\omega_6) + n_3(\omega_3+\omega_5) + 2n_2\omega_2 + 2n_4\omega_4$
$\exp(\zeta\check{\omega}_1)$ $\zeta \in \mathbb{C}\setminus 2\pi i\mathbb{Z}$	D_5T_1	$n_1(\omega_1+\omega_6) + n_2\omega_2$

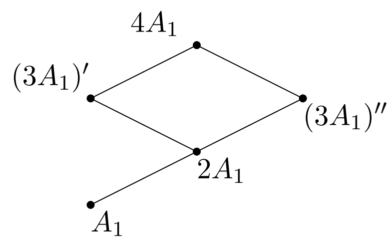
Table 17: $\lambda(\mathcal{O})$ for semisimple classes in E_6.

4.6 Type E_7.

Here $Z(G) = \langle h_{\alpha_2}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1)\rangle$. We put

$$\begin{array}{l} \beta_1 = (2,2,3,4,3,2,1),\ \beta_2 = (0,1,1,2,2,2,1),\ \beta_3 = (0,1,1,2,1,0,0), \\ \beta_4 = \alpha_7, \quad \beta_5 = \alpha_5, \quad \beta_6 = \alpha_3, \quad \beta_7 = \alpha_2 \end{array}$$

4.6.1 Unipotent classes in E_7.



Unipotent classes in E_7

Then

$$\begin{array}{ccccc} A_1 & \longleftrightarrow & \{2,3,4,5,6,7\} & \longleftrightarrow & s_{\beta_1} \\ 2A_1 & \longleftrightarrow & \{2,3,4,5,7\} & \longleftrightarrow & s_{\beta_1}s_{\beta_2} \\ (3A_1)'' & \longleftrightarrow & \{2,3,4,5\} & \longleftrightarrow & s_{\beta_1}s_{\beta_2}s_{\beta_4} \\ (3A_1)' & \longleftrightarrow & \{2,5,7\} & \longleftrightarrow & s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_6} \\ 4A_1 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 = s_{\beta_1}\cdots s_{\beta_7} \end{array}$$

We have

$$(1-w)P = \begin{cases} \mathbb{Z}\langle\omega_1\rangle & \text{for } w = s_{\beta_1} \\ \mathbb{Z}\langle\omega_1,\omega_6\rangle & \text{for } w = s_{\beta_1}s_{\beta_2} \\ \mathbb{Z}\langle\omega_1,\omega_6,2\omega_7\rangle & \text{for } w = s_{\beta_1}s_{\beta_2}s_{\beta_4} \\ \mathbb{Z}\langle 2\omega_1,2\omega_3,\omega_4,\omega_6\rangle & \text{for } w = s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_6} \end{cases}$$

Class A_1. By Proposition 4.2, T^w is connected.

Class $2A_1$. Since $(1-w)P = \mathbb{Z}\langle\omega_1,\omega_6\rangle$, T^w is connected.

Class $(3A_1)'$. Note that $Z(G) \leq (T^w)^\circ$. Since $(1-w)P = \mathbb{Z}\langle 2\omega_1, 2\omega_3, \omega_4, \omega_6\rangle$, we get

$$T^w = (T^w)^\circ \times \langle h_{\alpha_1}(-1)\rangle \times \langle h_{\alpha_3}(-1)\rangle$$

Here Ψ_J has basis $\{\alpha_1,\alpha_3,\beta_2,\beta_3\}$, $K = C((T^w)^\circ)'$ is of type D_4 (and is simply-connected) and $Z(K) = \langle h_{\alpha_2}(-1)h_{\alpha_7}(-1), h_{\alpha_2}(-1)h_{\alpha_5}(-1)\rangle$. Since $x \in K$ and lies over the longest element of the Weyl group of K, from the result for the maximal spherical unipotent class in D_4 we get $T_x \cap K = Z(K)$. But $Z(K) \leq (T^w)^\circ$, so that $R \cap Z(K) = 1$, and $T_x = (T^w)^\circ$.

Class $(3A_1)''$. Since $(1-w)P = \mathbb{Z}\langle\omega_1,\omega_6,2\omega_7\rangle$, we have

$$T^w = (T^w)^\circ \times \langle h_{\alpha_7}(-1)\rangle = (T^w)^\circ \times Z(G)$$

and $T_x = T^w$.

Do deal with the simply-connected cover of $(3A_1)''$, we note that the reductive part of $C(x)^\circ$ is of type F_4 ([12], p. 403), so in particular has rank 4: hence $S = \prod_{j\in J} H_{\alpha_j}$ is a maximal torus of $C(x)^\circ$. Since $Z(G) \not\leq S$, it follows from Proposition 3.20 that $T \cap C(x)^\circ = (T^w)^\circ$ (and $C(x) = C(x)^\circ \times Z(G)$).

Class $4A_1$. We claim that $T_x = Z(G)$. Suppose for a contradiction there exists an involution $\sigma \in T_x \setminus Z(G)$. Then $x \in K = C(\sigma)$ and K is of type D_6A_1 (see next subsection). By comparison of weighted Dynkin diagrams, the unipotent spherical class of K over w_0 does not correspond to the class $4A_1$ of E_7 (it corresponds to the class $A_2 + A_1$), a contradiction.

Do deal with the simply-connected cover of $4A_1$, we note that the reductive part of $C(x)^\circ$ is of type C_3 ([12], p. 403), so in particular it is semisimple: by Lemma 4.4, the monoid $\lambda(4\hat{A}_1)$ is free, and from

$$\lambda(4A_1) = \left\{ \sum_{i=1}^{7} n_i\omega_i,\ n_2 + n_5 + n_7 \text{ even} \right\}$$

it follows that

$$\lambda(4\hat{A}_1) = \left\{ \sum_{i=1}^{7} n_i\omega_i \right\}$$

hence $T \cap C(x)^\circ = 1$ and $C(x) = C(x)^\circ \times Z(G)$.

We obtained

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
A_1	$n_1\omega_1$	
$2A_1$	$n_1\omega_1 + n_6\omega_6$	
$(3A_1)''$	$n_1\omega_1 + n_6\omega_6 + 2n_7\omega_7$	$n_1\omega_1 + n_6\omega_6 + n_7\omega_7$
$(3A_1)'$	$n_1\omega_1 + n_3\omega_3 + n_4\omega_4 + n_6\omega_6$	
$4A_1$	$\sum_{i=1}^{7} n_i\omega_i,\ n_2 + n_5 + n_7$ even	$\sum_{i=1}^{7} n_i\omega_i$

Table 18: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in E_7.

In particular the simply-connected cover of $4A_1$ is a model homogeneus space, and in fact the principal one, by [28], 3.3 (8).

Remark 4.34 From our description, it follows that $C(x)$ is connected for the classes A_1, $2A_1$ and $(3A_1)'$, while for $(3A_1)''$ and $4A_1$ we have $C(x) = C(x)^\circ \times Z(G)$. This also follows from the tables in [1], where all unipotent classes are considered.

4.6.2 Semisimple classes in E_7

Following the notation in [9], Table 2, we have

$$\begin{array}{lllll} E_6T_1 & \longleftrightarrow & \{2,3,4,5\} & \longleftrightarrow & s_{\beta_1}s_{\beta_2}s_{\beta_4} \\ D_6A_1 & \longleftrightarrow & \{2,5,7\} & \longleftrightarrow & s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_6} \\ A_7 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 \end{array}$$

Let Y be the set of elements y of order 4 of T such that $y^2 = z$, where $Z(G) = \langle z \rangle$. Then Y is the disjoint union of 2 conjugacy classes Y_1, Y_2, where $C(y)$ is of type A_7 if $y \in Y_1$, of type E_6T_1 if $y \in Y_2$. A representative for Y_1 is $\exp(\pi i \check{\omega}_2)$, one for Y_2 is $\exp(\pi i \check{\omega}_7)$.

Type A_7. Here we consider $K = C(\exp(\pi i \check{\omega}_2))$. Then K is of type A_7, $Z(K) = \langle \exp(\pi i \check{\omega}_2) \rangle$ is of order 4. Let $x = n_{\beta_1} \cdots n_{\beta_7}$. Then $x^2 = h_{\beta_1}(-1) \cdots h_{\beta_7}(-1) = z$, $x \in w_0B$ (and $x \sim \exp(\pi i \check{\omega}_2)$), and clearly $T_x = T_2$.

Type E_6T_1. Let $K = C(\exp(\pi i \check{\omega}_7))$. Then $C(K) = Z(K) = \langle \exp(\mathbb{C}\, \check{\omega}_7) \rangle$. Now $\exp(\zeta \check{\omega}_7) = 1 \Leftrightarrow \zeta \in 4\pi i \mathbb{Z}$, and $C(\exp(\zeta \check{\omega}_7)) = K \Leftrightarrow \zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$.

In this case we have

$$T^w = (T^w)^\circ Z(G)$$

so it is not necessary to give explicitly the form of an element in $wB \cap \mathcal{O}$.

Anyway, we consider the element

$$y_\zeta = g \exp(\zeta \check{\omega}_7) g^{-1}$$

where $g = n_{\beta_1} n_{\beta_2} n_{\alpha_7} x_{\beta_1}(-1) x_{\beta_2}(-1) x_{\alpha_7}(-1)$. Now $\beta_1(\exp(\zeta \check{\omega}_7)) = \beta_2(\exp(\zeta \check{\omega}_7)) =$
$= \alpha_7(\exp(\zeta \check{\omega}_7)) = e^\zeta$, and $w(\omega_7) = -\omega_7$ so that

$$x_\zeta = n_{\beta_1} n_{\beta_2} n_{\alpha_7} h x_{\beta_1}(\xi) x_{\beta_2}(\xi) x_{\alpha_7}(\xi) \in \mathcal{O}_{\exp(\zeta \check{\omega}_7)} \cap n_{\beta_1} n_{\beta_2} n_{\alpha_7} B$$

for a certain $h \in T$, with $\xi = \frac{1+e^\zeta}{1-e^\zeta}$.

Since $T^w = (T^w)^\circ \times Z(G)$, we conclude that $T_{x_\zeta} = T^w$, as for the class $(3A_1)''$.

Type D_6A_1. The group E_7 has 2 classes of non-central involutions: $\mathcal{O}_\sigma$ and $\mathcal{O}_{\sigma z}$, where $\sigma = \exp(\pi i \check{\omega}_1) = h_{\beta_1}(-1)$. In fact there are 127 involutions in T, and z is central. The W-orbit of σ, $\{h_\alpha(-1) \mid \alpha \in \Phi^+\}$, consists of $|\Phi^+| = 63$ elements, since if the roots α and β are congruent modulo $2\mathbb{Z}\Phi$, then $\beta = \pm\alpha$ ([3], ex. 1, p. 242). Since σz is not of the form $h_\alpha(-1)$, the set $\{h_\alpha(-1)z \mid \alpha \in \Phi^+\}$ is another W-orbit (the fact that σz is not conjugate to σ also follows from the discussion in section 6).

Let $x = n_{\beta_1} n_{\beta_2} n_{\beta_3} n_{\alpha_3}$. Then $x^2 = h_{\beta_1}(-1) h_{\beta_2}(-1) h_{\beta_3}(-1) h_{\alpha_3}(-1) = 1$, so that x is an involution, and clearly $T_x = T^w$.

We obtained

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\zeta \check{\omega}_7)$ $\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$	E_6T_1	$n_1\omega_1 + n_6\omega_6 + 2n_7\omega_7$
$\exp(\pi i \check{\omega}_1)$	D_6A_1	$2n_1\omega_1 + 2n_3\omega_3 + n_4\omega_4 + n_6\omega_6$
$\exp(\pi i \check{\omega}_2)$	A_7	$\sum_{i=1}^{7} 2n_i\omega_i$

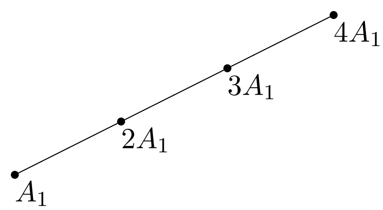
Table 19: $\lambda(\mathcal{O})$ for semisimple classes in E_7.

4.7 Type E_8.

We put

$$\beta_1 = (2,3,4,6,5,4,3,2),\ \beta_2 = (2,2,3,4,3,2,1,0),\ \beta_3 = (0,1,1,2,2,2,1,0),$$
$$\beta_4 = (0,1,1,2,1,0,0,0),\ \beta_5 = \alpha_7,\ \beta_6 = \alpha_5,\ \beta_7 = \alpha_3,\ \beta_8 = \alpha_2$$

4.7.1 Unipotent classes in E_8.



Unipotent classes in E_8

Then

$$\begin{array}{lclcl} A_1 & \longleftrightarrow & \{1,2,3,4,5,6,7\} & \longleftrightarrow & s_{\beta_1} \\ 2A_1 & \longleftrightarrow & \{2,3,4,5,6,7\} & \longleftrightarrow & s_{\beta_1}s_{\beta_2} \\ 3A_1 & \longleftrightarrow & \{2,3,4,5\} & \longleftrightarrow & s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_5} \\ 4A_1 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 = s_{\beta_1}\cdots s_{\beta_8} \end{array}$$

We have

$$(1-w)P = \begin{cases} \mathbb{Z}\langle \omega_8 \rangle & \text{for } w = s_{\beta_1} \\ \mathbb{Z}\langle \omega_1, \omega_8 \rangle & \text{for } w = s_{\beta_1}s_{\beta_2} \\ \mathbb{Z}\langle \omega_1, \omega_6, 2\omega_7, 2\omega_8 \rangle & \text{for } w = s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_5} \end{cases}$$

Class A_1. By Proposition 4.2, T^w is connected.

Class $2A_1$. Since $(1-w)P = \mathbb{Z}\langle \omega_1, \omega_8 \rangle$, T^w is connected.

Class $3A_1$. Here Ψ_J has basis $\{\alpha_7, \alpha_8, \beta_2, \beta_3\}$, $K = C((T^w)^\circ)'$ is of type D_4 and has center $\langle h_{\alpha_3}(-1)h_{\alpha_5}(-1), h_{\alpha_2}(-1)h_{\alpha_3}(-1) \rangle$ which is contained in $(T^w)^\circ$. Hence $T_x = (T^w)^\circ$.

Class $4A_1$. We claim that $T_x = 1$. Suppose for a contradiction there exists an involution $\sigma \in T_x$. Then $x \in K = C(\sigma)$. From the classification of involutions of E_8, it follows that K is of type D_8 or E_7A_1. The class of x in K is spherical, and by the uniqueness of Bruhat decomposition, x lies over the longest element of the Weyl group of K, which is w_0. By comparison of weighted Dynkin diagrams, the unipotent spherical class of K over w_0 does not correspond to the class $4A_1$ of E_8 (in both cases it corresponds to the class $A_2 + A_1$), a contradiction.

We have shown that in all cases $T_x = (T^w)^\circ$, so that $C(x)$ is connected, as also follows from [12], p. 405. We have

$\mathcal{O}$	$\lambda(\mathcal{O}) = \lambda(\hat{\mathcal{O}})$
A_1	$n_8\omega_8$
$2A_1$	$n_1\omega_1 + n_8\omega_8$
$3A_1$	$n_1\omega_1 + n_6\omega_6 + n_7\omega_7 + n_8\omega_8$
$4A_1$	$\sum_{i=1}^{8} n_i\omega_i$

Table 20: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in E_8.

In particular $4A_1$ is a model homogeneus space (see [2], Theorem 1.1), and in fact the principal one, by [28], 3.3 (9).

4.7.2 Semisimple classes in E_8.

Following the notation in [9], Table 2, we have

$$\begin{array}{ccccc} A_1E_7 & \longleftrightarrow & \{2,3,4,5\} & \longleftrightarrow & s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_5} \\ D_8 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 \end{array}$$

Type D_8. The elements of G whose centralizer is of type D_8 are conjugate to $\exp(\pi i\check{\omega}_1)$. Let $x = n_{\beta_1}\cdots n_{\beta_8}$. Then $x^2 = h_{\beta_1}(-1)\cdots h_{\beta_8}(-1) = 1$. Moreover, $x \in w_0B$ implies $x \sim \exp(\pi i\check{\omega}_1)$. Clearly $T_x = T^{w_0} = T_2$.

Type A_1E_7. The elements of G whose centralizer is of type A_1E_7 are conjugate to $\exp(\pi i\check{\omega}_8)$. Let $x = n_{\beta_1}n_{\beta_2}n_{\beta_3}n_{\alpha_7}$. Then x is conjugate to $h_{\beta_1}(i)h_{\beta_2}(i)h_{\beta_3}(i)h_{\alpha_7}(i) =$
$= h_{\alpha_2}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1)h_{\alpha_8}(-1)$ whose centralizer is of type A_1E_7, hence $x \sim \exp(\pi i\check{\omega}_8)$. Then $T_x = T^w$.

We obtained

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\pi i\check{\omega}_8)$	A_1E_7	$n_1\omega_1 + n_6\omega_6 + 2n_7\omega_7 + 2n_8\omega_8$
$\exp(\pi i\check{\omega}_1)$	D_8	$\sum_{i=1}^{8} 2n_i\omega_i$

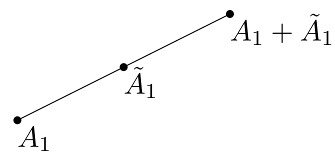
Table 21: $\lambda(\mathcal{O})$ for semisimple classes in E_8.

4.8 Type F_4.

We put

$$\beta_1 = (2,3,4,2), \quad \beta_2 = (0,1,2,2),$$
$$\beta_3 = (0,1,2,0), \quad \beta_4 = (0,1,0,0)$$

4.8.1 Unipotent classes in F_4.



Unipotent classes in F_4

Then

$$\begin{array}{ccccc} A_1 & \longleftrightarrow & \{2,3,4\} & \longleftrightarrow & s_{\beta_1} \\ \tilde{A}_1 & \longleftrightarrow & \{2,3\} & \longleftrightarrow & s_{\beta_1}s_{\beta_2} \\ A_1+\tilde{A}_1 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 = s_{\beta_1}\cdots s_{\beta_4} \end{array}$$

We have

$$(1-w)P = \begin{cases} \mathbb{Z}\langle\omega_1\rangle & \text{for } w = s_{\beta_1} \\ \mathbb{Z}\langle\omega_1, 2\omega_4\rangle & \text{for } w = s_{\beta_1}s_{\beta_2} \end{cases}$$

Class A_1. By Proposition 4.2, T^w is connected.

Class $\tilde{A}_1$. Since $(1-w)P = \mathbb{Z}\langle\omega_1, 2\omega_4\rangle$, we have $T^w = (T^w)^\circ \times \langle h_{\alpha_4}(-1)\rangle$. From [9], proof of Theorem 2.12, we get

$$x_{-\beta_1}(1)x_{-\beta_2}(1) \in \mathcal{O} \cap BwB \cap B^-$$

hence we may choose

$$x = n_{\beta_1}n_{\beta_2}hx_{\beta_1}(2)x_{\beta_2}(2)$$

for a certain $h \in T$. Since $\mathbb{Z}\langle\beta_1, \beta_2\rangle = \mathbb{Z}\langle\omega_1, 2\omega_4\rangle$, we get $\langle h_{\alpha_4}(-1)\rangle \leq T_x$, and $T_x = T^w$.

Since $[C(x) : C(x)^\circ] = 2$ ([12], p. 401), we must have $C(x) = C(x)^\circ : \langle h_{\alpha_4}(-1)\rangle$ and $C(x)^\circ \cap T = (T^w)^\circ$.

Class $A_1 + \tilde{A}_1$. Here $T^{w_0} = T_2$. We consider the subgroup K generated by the long roots of G: K is of type D_4 and it is simply-connected ([42], §II 5, 5.4 (a)). In fact $K = C(\langle h_{\alpha_3}(-1), h_{\alpha_4}(-1)\rangle)$, and $Z(K) = C(K) = \langle h_{\alpha_3}(-1), h_{\alpha_4}(-1)\rangle$. Following [9], proof of Theorems 2.12 and 2.11, we have $x \in K$ (equivalently one can show, by using weighted Dynkin diagrams, that the class in G of a unipotent element in the class Z_2 of K is precisely $A_1 + \tilde{A}_1$). But then we must have $T_x = Z(K)$ by the results obtained for D_4, so that

$$\lambda(A_1 + \tilde{A}_1) = \{n_1\omega_1 + n_2\omega_2 + 2n_3\omega_3 + 2n_4\omega_4 \mid n_k \in \mathbb{N}\}$$

By [12], p. 401, $C(x)$ is connected. We obtained

$\mathcal{O}$	$\lambda(\mathcal{O})$	$\lambda(\hat{\mathcal{O}})$
A_1	$n_1\omega_1$	
$\tilde{A}_1$	$n_1\omega_1 + 2n_4\omega_4$	$n_1\omega_1 + n_4\omega_4$
$A_1 + \tilde{A}_1$	$n_1\omega_1 + n_2\omega_2 + 2n_3\omega_3 + 2n_4\omega_4$	

Table 22: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in F_4.

4.8.2 Semisimple classes in F_4.

Following the notation in [9], Table 2, we have

$$\begin{array}{cccll} A_1C_3 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 \\ B_4 & \longleftrightarrow & \{1,2,3\} & \longleftrightarrow & s_{\gamma_1} \end{array}$$

where γ_1 is the highest short root $(1,2,3,2)$.
Type A_1C_3. The elements of G whose centralizer is of type A_1C_3 are conjugate to $\exp(\pi i\check{\omega}_1)$. Let

$$x = n_{\beta_1} \cdots n_{\beta_4}$$

Then $x^2 = h_{\beta_1}(-1) \cdots h_{\beta_4}(-1) = 1$, and $x \in w_0B$ implies $x \sim \exp(\pi i\check{\omega}_1)$. Clearly $T_x = T_2$.

Type B_4. The elements of G whose centralizer is of type B_4 are conjugate to $\exp(\pi i\check{\omega}_4)$. By Proposition 4.2, T^w is connected, hence $T_x = T^w$. Then

$$\lambda(\mathcal{O}_{\exp(\pi i\check{\omega}_4)}) = \{n_4\omega_4 \mid n_k \in \mathbb{N}\}$$

We obtained

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\pi i\check{\omega}_1)$	A_1C_3	$\sum_{i=1}^{4} 2n_i\omega_i$
$\exp(\pi i\check{\omega}_4)$	B_4	$n_4\omega_4$

Table 23: $\lambda(\mathcal{O})$ for semisimple classes in F_4.

4.8.3 Mixed class in F_4.

We put $f_2 = \exp(\pi i\check{\omega}_4) = h_{\alpha_3}(-1)$. Then following [9], Table 4

$$\mathcal{O}_{f_2x_{\beta_1}(1)} \quad \longleftrightarrow \quad \varnothing \quad \longleftrightarrow \quad w_0$$

As we already recalled, G has 2 classes of involutions. More precisely, in T there are 15 involutions, and under the action of W they fall in the 2 classes

$$\{h_\alpha(-1) \mid \alpha \in \Phi^+ \text{ is long}\} \quad , \quad \{h_\alpha(-1) \mid \alpha \in \Phi^+ \text{ is short}\}$$

where $\{h_\alpha(-1) \mid \alpha \in \Phi^+ \text{ is long}\}$, consists of 12 elements, since if the long roots α and β are congruent modulo $2\mathbb{Z}\Phi$, then $\beta = \pm\alpha$, while $\{h_\alpha(-1) \mid \alpha \in \Phi^+ \text{ is short}\}$ consists of 3 elements: $\{h_{\alpha_4}(-1), h_{\alpha_3}(-1), h_{\alpha_3}(-1)h_{\alpha_4}(-1)\}$ which are the involutions in the center of the D_4-subgroup D of G generated by the long roots.

Suppose H is a B_4-subgroup of G. Then H has 4 (non-trivial) unipotent spherical classes, and by comparison of weighted Dynkin diagrams, the class X_1 corresponds to the class A_1 of G, the classes X_2 and Z_1 to $\tilde{A}_1$, and the class Z_2 to $A_1 + \tilde{A}_1$.

Suppose H is a C_3A_1-subgroup of G. Then H has 7 (non-trivial) unipotent spherical unipotent classes, and by comparison of weighted Dynkin diagrams, the classes $(X_1, 1)$ and $(1, X_1)$ correspond to the class A_1 of G, the classes (X_1, X_1) and $(X_2, 1)$ to $\tilde{A}_1$, the classes (X_2, X_1) and $(X_3, 1)$ to $A_1 + \tilde{A}_1$ and the class (X_3, X_1) to A_2.

Now let $x \sim f_2 x_{\beta_1}(1)$, $x \in w_0 B$. We claim that $T_x = 1$. Let $x = x_s x_u$ be the Jordan-Chevalley decomposition of x. In particular $x_s \sim f_2$ and $x_u \sim x_{\beta_1}(1)$.

Suppose for a contradiction there exists an involution $\sigma \in T_x$. Then $x \in K = C(\sigma)$, with K of type either B_4 or C_3A_1. In both cases we have $Z(K) = \langle \sigma \rangle$. Since the class (in G) of x_u is spherical, the class of x_u in K is spherical, and by the uniqueness of Bruhat decomposition, x lies over the longest element of the Weyl group of K, which is w_0.

Now x is conjugate in K to an element of the form su, with $s \in T$, $u \in U \cap K$, $[s, u] = 1$. Since $s \sim f_2$, we have $s \in \{h_{\alpha_4}(-1), h_{\alpha_3}(-1), h_{\alpha_3}(-1)h_{\alpha_4}(-1)\}$, and so s lies in $Z(D)$.

Let us assume K is of type B_4. Then u lies in the class X_1 of K, so that the class of x in K, up to a central element of K, is the class X_1 or the mixed class $\mathcal{O}_{\sigma_4 x_{\beta_1}(1)}$ (standard notation for B_4). In both cases x does not lie over w_0 (see the tables 10, 12 for $m = 2$).

Let us finally assume K is of type C_3A_1. It follows that u must be either in $(X_1, 1)$ or in $(1, X_1)$, and $s = s_1 s_2$, with $s_1 \in T(C_3)$, $s_2 \in T(A_1)$. We observe that $T(C_3) \cap T(A_1) = Z(K) = \langle \sigma \rangle$. We claim that s_2 lies in the center of A_1 (i.e. $s_2 = 1$ or σ). Up to the W-action, we may assume $\sigma = \exp(\pi i \check{\omega}_1)$. Then from the fact that $s \in \{h_{\alpha_4}(-1), h_{\alpha_3}(-1), h_{\alpha_3}(-1)h_{\alpha_4}(-1)\}$, it follows that either $s_2 = 1$, or $s_2 = \sigma$, and we are done. If we write $u = u_1 u_2$, with $u_1 \in C_3$, $u_2 \in A_1$, we must have that $s_1 u_1$ lies over w_0 in C_3, and $s_2 u_2$ lies over w_0 in A_1. But s_2 is central in A_1, therefore we must have $u_2 \neq 1$, so that u is in the class $(1, X_1)$. But then the involution s_1 does not lie over w_0 (in C_3), by the results on semisimple conjugacy classes of C_3, see table 4: only the classes $\mathcal{O}_{\exp(\zeta \check{\omega}_3)}$ for $\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$ are over w_0, but there are no involutions in these classes, since $\exp(2\pi i \check{\omega}_3)$ has order 2 (and is central).

We have therefore proved that $T_x = 1$. Hence

$\mathcal{O}$	$\lambda(\mathcal{O}) = \lambda(\hat{\mathcal{O}})$
$f_2 x_{\beta_1}(1)$	$\sum_{i=1}^{4} n_i \omega_i$

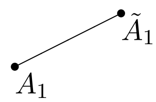
Table 24: $\lambda(\mathcal{O})$ for the mixed class in F_4.

In particular $\mathcal{O}_{f_2 x_{\beta_1}(1)}$ is a model homogeneus space, and in fact the principal one, by [28], 3.3 (6), see also [28] p. 300.

4.9 Type G_2.

We put $\beta_1 = (3,2),\ \beta_2 = \alpha_1$.

4.9.1 Unipotent classes in G_2.



Unipotent classes in G_2

Then

$$\begin{array}{ccccc} A_1 & \longleftrightarrow & \{1\} & \longleftrightarrow & s_{\beta_1} \\ \tilde{A}_1 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 = s_{\beta_1} s_{\beta_2} \end{array}$$

Class $\mathbf{A_1}$, $w = s_{\beta_1}$. By Proposition 4.2, T^w is connected, so

$$\lambda(A_1) = \{n_2\omega_2 \mid n_2 \in \mathbb{N}\}$$

Class $\tilde{\mathbf{A}}_\mathbf{1}$. We have $T^{w_0} = T_2$. We claim that $T_x = 1$. Suppose for a contradiction there exists an involution $\sigma \in T_x$. Then $x \in K = C(\sigma)$. From the classification of involutions of G_2, it follows that K is of type $A_1\tilde{A}_1$. The class of x in K is spherical, and by the uniqueness of Bruhat decomposition, x lies over the longest element of the Weyl group of K, which is w_0. By comparison of weighted Dynkin diagrams, a unipotent element of K over w_0 does not correspond to the element $\tilde{A}_1$ of G_2 (it corresponds to the subregular class $G_2(a_1)$, [12], p.401), a contradiction.

We got

$\mathcal{O}$	$\lambda(\mathcal{O}) = \lambda(\hat{\mathcal{O}})$
A_1	$n_2\omega_2$
$\tilde{A}_1$	$n_1\omega_1 + n_2\omega_2$

Table 25: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in G_2.

In particular $\tilde{A}_1$ is a model homogeneus space, and in fact the principal one, by [28], 3.3 (5).

Using the embedding of G into $SO(7)$, one can determine explicitly an $x \in \mathcal{O} \cap w_0 B$, where $\mathcal{O} = \tilde{A}_1$. Then one can check that both α_1 and α_2 occur in x (see the discussion before Proposition 3.11). This fact will be used in section 5 to determine $\mathbb{C}[\overline{\mathcal{O}}]$.

4.9.2 Semisimple classes in G_2.

Following the notation in [9], Table 2, we have

$$\begin{array}{ccccc} A_1\tilde{A}_1 & \longleftrightarrow & \varnothing & \longleftrightarrow & w_0 \\ A_2 & \longleftrightarrow & \{2\} & \longleftrightarrow & s_{\gamma_1} \end{array}$$

where γ_1 is the highest short root $(2,1)$.

The group G_2 has 1 class of involutions. However there is also a class of elements of order 3 which is spherical.

Type $A_1\tilde{A}_1$.

The elements of G whose centralizer is of type $A_1\tilde{A}_1$ are conjugate to $\exp(\pi i\check{\omega}_2)$. Let

$$x = n_{\beta_1}n_{\beta_2}$$

Then $x^2 = h_{\beta_1}(-1)h_{\beta_2}(-1) = 1$ and $x \in w_0B$. Clearly $T_x = T_2$.

Type A_2. The elements of G whose centralizer is of type A_2 are conjugate to $\exp(\frac{2\pi i}{3}\check{\omega}_1)$. By Proposition 4.2, T^w is connected, hence $T_x = T^w$.

We obtained

$\mathcal{O}$	H	$\lambda(\mathcal{O})$
$\exp(\pi i\check{\omega}_2)$	$A_1\tilde{A}_1$	$\sum_{i=1}^{2} 2n_i\omega_i$
$\exp(\frac{2\pi i}{3}\check{\omega}_1)$	A_2	$n_1\omega_1$

Table 26: $\lambda(\mathcal{O})$ for semisimple classes in G_2.

5 The coordinate ring of $\overline{\mathcal{O}}$

In this section we determine the decomposition of $\mathbb{C}[\overline{\mathcal{O}}]$ into simple G-modules, where $\overline{\mathcal{O}}$ is the closure of a spherical conjugacy class. Normality of conjugacy classes' closures has been deeply investigated. For a survey on this topic, see [23], §8, [8], 7.9, Remark (iii). The first observation is that the problem is reduced to unipotent conjugacy classes in G ([23], 8.1). In the following we are interested only in spherical conjugacy classes, and I recall the facts in this context. It is known that the closure of the minimal nilpotent orbit is always normal ([44], Theorem 2). Hesselink ([17]) proved normality for several small orbits in the classical cases and certain orbits for the exceptional cases: namely, following the notation in [12], A_1 and $2A_1$ in E_6, A_1, $2A_1$ and $(3A_1)''$ in E_7, A_1 and $2A_1$ in E_8, A_1 and $\tilde{A}_1$ in F_4, A_1 in G_2.

The classical groups have been considered in [24], [25]: for the special linear groups the closure of every conjugacy class is normal. For the symplectic and orthogonal groups there exist conjugacy classes with non-normal closure. However every spherical conjugacy class in the symplectic group has normal closure, since from the classification we know that the unipotent spherical conjugacy classes have only 2 columns (see also [17], §5, Criterion 2). For special orhogonal groups the results in [25] left open the cases of the very even unipotent classes. E. Sommers proved that these have normal closure in [39]. Taking into account the results in [25] and [39] it follows that every unipotent spherical conjugacy class in type D_n and B_n has normal closure except for the maximal class Z_{m+1} in B_n, when $n = 2m + 1$, $m \geq 1$. From this and the classification of spherical conjugacy classes, it follows that every spherical conjugacy class has normal closure, except for the above mentioned class in B_{2m+1}.

For the exceptional groups, besides the results on the minimal orbit and Hesselink's results, in [27] it is shown that the orbit $\tilde{A}_1$ in G_2 has a non-normal closure (see also [23]): here there is bijective normalization, contrary to the case of Z_{m+1} in B_{2m+1} where the closure is branched in codimension 2. In [7] the case of type F_4 is completely handled, and it follows that every spherical conjugacy class has normal closure. The same holds for E_6, as follows from [38] where every nilpotent orbit is considered. For the remaining nilpotent orbits in E_7 and E_8, in [8], 7.9, Remark (iii), A. Broer gives a list of orbits with normal closure. Among these there are all spherical nilpotent orbits in E_7 and E_8. We may therefore state

Theorem 5.1 *Let $\mathcal{O}$ be a spherical conjugacy class. Then $\overline{\mathcal{O}}$ is normal except for the class Z_{m+1} in B_{2m+1} ($m \geq 1$) and the class $\tilde{A}_1$ in G_2.* □

Remark 5.2 In [13], Example 4.4, Proposition 4.5, the authors prove normal closure for nilpotent orbits of height 2.

Remark 5.3 In [35], 6.1, normality of $\mathcal{N}^{\mathrm{sph}}$ (the union of all spherical nilpotent orbits, which is in fact the closure of the unique maximal spherical nilpotent orbit) is discussed.

Remark 5.4 From (3.9) and Corollary 3.16 it is possible to prove normality of $\overline{\mathcal{O}}$ in certain cases. For instance in type C_n from Table 3 we get $\lambda(X_\ell) = 2P_w^+$ for every unipotent class X_ℓ. From (3.9) it follows that $\lambda(\overline{\mathcal{O}}) = \lambda(\mathcal{O})$, so that $\overline{\mathcal{O}}$ is normal.

We recall that in general $\mathbb{C}[\mathcal{O}]$ is the integral closure of $\mathbb{C}[\overline{\mathcal{O}}]$ in its field of fractions and that $\mathbb{C}[\overline{\mathcal{O}}] = \mathbb{C}[\mathcal{O}]$ if and only if $\overline{\mathcal{O}}$ is normal ([22], Proposition and Corollary in 8.3). By Theorem 5.1, to describe the decomposition of $\mathbb{C}[\overline{\mathcal{O}}]$ we are left to deal with Z_{m+1} in B_{2m+1} and with $\tilde{A}_1$ in G_2. We use the notation and the tables from section 4 for the cases B_{2m+1} and G_2.

Theorem 5.5 *Let $\mathcal{O} = Z_{m+1}$ in B_n, $n = 2m+1$, $m \geq 1$. Then*

$$\lambda(\overline{\mathcal{O}}) = \left\{\sum_{i=1}^{2m} n_i\omega_i \mid \sum_{i=1}^{m} n_{2i-1} \text{ even}\right\} \cup \left\{\sum_{i=1}^{n} n_i\omega_i \mid \; n_n \text{ even}, n_n \geq 2\right\}$$

Proof. Considering the (G-equivariant) restriction $r : \mathbb{C}[\overline{\mathcal{O}}] \to \mathbb{C}[\overline{Z_m}] = \mathbb{C}[Z_m]$, we get $\left\{\sum_{i=1}^{2m} n_i\omega_i \mid \sum_{i=1}^{m} n_{2i-1} \text{ even}\right\} \leq \lambda(\overline{\mathcal{O}})$. In particular for every even j, $\omega_j \in \lambda(\overline{\mathcal{O}})$, and for every pair of odd j, k, with $1 \leq j \leq k < n$, $\omega_j + \omega_k \in \lambda(\overline{\mathcal{O}})$. By Corollary 3.12, we have $2\omega_n \in \lambda(\overline{\mathcal{O}})$. We show that $\omega_j + 2\omega_n \in \lambda(\overline{\mathcal{O}})$ for every odd j, $j < n$. We have $2\omega_{n-1} - \alpha_{n-1} = \omega_{n-2} + 2\omega_n$ and since α_{n-1} occurs in $x \in w_0B \cap \mathcal{O}$, by Corollary 3.16, we get $\omega_{n-2} + 2\omega_n \in \lambda(\overline{\mathcal{O}})$. Let j be odd, $j < n-2$. Then $\omega_j + 2\omega_n + 2\omega_{n-2} \in \lambda(\overline{\mathcal{O}})$ since $\omega_{n-2} + 2\omega_n$ and $\omega_j + \omega_{n-2}$ are in $\lambda(\overline{\mathcal{O}})$.

There exists B-eigenvectors F, H in $\mathbb{C}[\overline{\mathcal{O}}]$ of weights $\omega_j + 2\omega_n + 2\omega_{n-2}$, $2\omega_{n-2}$ respectively. Then F/H is a rational function on $\overline{\mathcal{O}}$ of weight $\omega_j + 2\omega_n$ defined at least on $\mathcal{O}$. However $2\omega_{n-2}$ is also a weight in $\lambda(Z_m)$, so that H is non-zero on the dense B-orbit v in Z_m. Hence F/H is defined on v, and it is zero on v, since F is zero on Z_m, $\omega_j + 2\omega_n + 2\omega_{n-2}$ not being in $\lambda(Z_m)$. It follows that F/H is defined on Z_m, so that it is a regular function on $\mathcal{O} \cup Z_m$. By [25], Theorem 16.2, (iii), F/H extends to $\overline{\mathcal{O}}$, and $\omega_j + 2\omega_n$ lies in $\lambda(\overline{\mathcal{O}})$. We have shown that

$$\lambda(\overline{\mathcal{O}}) \geq \left\{\sum_{i=1}^{2m} n_i\omega_i \mid \sum_{i=1}^{m} n_{2i-1} \text{ even}\right\} \cup \left\{\sum_{i=1}^{n} n_i\omega_i \mid \; n_n \text{ even}, n_n \geq 2\right\}$$

We prove that also the opposite inclusion holds. Assume $\lambda = \sum_{i=1}^{n} n_i\omega_i \in \lambda(\overline{\mathcal{O}})$. Since $\lambda(\overline{\mathcal{O}}) \leq \lambda(\mathcal{O})$, we have n_n even. If $n_n \neq 0$ we are done. So assume $n_n = 0$. Let $y \in Z_{m+1} \cap U^- \cap Bw_0B$. We observe that $y_1 := \lim_{z\to 0} h_{\alpha_n}(z)^{-1} y h_{\alpha_n}(z)$ exists, and lies in $Z_m \cap U^- \cap BwB$, where $w = w(Z_m)$ (in [9] we give representatives for both classes in $SO(2n+1)$, so that this may be checked directly). Now let $F : \overline{\mathcal{O}} \to \mathbb{C}$ be a highest weight vector of weight λ, with $F(y) = 1$. Then $F(y_1) = 1$, since $\lambda(h_{\alpha_n}(z)) = 1$ for every $z \in \mathbb{C}^*$. Since $x_1 \in Z_m \cap wB$ lies in the B-orbit of y_1, we have $F(x_1) \neq 0$. But $\sigma = \prod_{i=1}^{m} h_{\alpha_{2i-1}}(-1) \in C(x_1)$, so that $F(x_1) = F(\sigma x_1 \sigma) = \lambda(\sigma)F(x_1)$ implies $\lambda(\sigma) = 1$, and we are done. □

Theorem 5.6 *Let $\mathcal{O} = \tilde{A}_1$ in G_2. Then $\lambda(\overline{\mathcal{O}})$ is the submonoid of $\lambda(\mathcal{O})$ generated by $2\omega_1, 3\omega_1, \omega_2$.*

Proof. We know that $\omega_1 \in \lambda(\mathcal{O})$ and it follows from the proof of [27], Theorem 3.13, that $\omega_1 \notin \lambda(\overline{\mathcal{O}})$. We have

$$2\omega_1 - \alpha_1 = \omega_2 \quad , \quad 2\omega_2 - \alpha_2 = 3\omega_1$$

hence, by Corollary 3.12 and 3.16, we get $2\omega_1$, $3\omega_1$, $\omega_2 \in \lambda(\overline{\mathcal{O}})$, since both α_1, α_2 occur in $x \in w_0B \cap \mathcal{O}$. Suppose for a contradiction that $\omega_1 + n\omega_2 \in \lambda(\overline{\mathcal{O}})$ for a certain $n \in \mathbb{N}$. There exists B-eigenvectors F, H in $\mathbb{C}[\overline{\mathcal{O}}]$ of weights $\omega_1 + n\omega_2$, $n\omega_2$ respectively. Then F/H is a rational function on $\overline{\mathcal{O}}$ of weight ω_1 defined at least on $\mathcal{O}$. However $n\omega_2$ is also a weight in $\lambda(A_1)$, so that H is non-zero on the dense B-orbit v in A_1. Hence F/H is defined on v, and it is zero on v, since F is zero on A_1, because $\omega_1 + n\omega_2$ is not in $\lambda(A_1)$. It follows that F/H is defined on A_1. But A_1 has normal closure, so that F/H is defined on the closure of A_1, and then on $\overline{\mathcal{O}}$, so that there is in $\mathbb{C}[\overline{\mathcal{O}}]$ a B-eigenvector of weight ω_1, a contradiction. □

6 The general case

Let G be as usual simply-connected, $D \leq Z(G)$, $\overline{G} = G/D$, $\pi : G \to \overline{G}$ the canonical projection. For $g \in G$ we put $\overline{g} = \pi(g)$. We give a procedure to describe the coordinate ring of $\mathcal{O}_{\overline{p}}$, where $\mathcal{O}_{\overline{p}}$ is a spherical conjugacy class of $\overline{G}$. Passing to G, we have to consider the quotient $G/\pi^{-1}(C_{\overline{G}}(\overline{p}))$. Let $p = sv$ be the Jordan-Chevalley decomposition of p, $w = w(\mathcal{O}_p)$. We may assume $s \in T$. Let $W_{s,D} = \{w \in W \mid wsw^{-1} = zs, z \in D\}$, and $N_{s,D} \leq N$ such that $N_{s,D}/T = W_{s,D}$. Then $\pi^{-1}(C_{\overline{G}}(\overline{p})) = C(v) \cap N_{s,D}C(s)$. Reasoning as in [42], Corollary II, 4.4, we have a homomorphism $\pi^{-1}(C_{\overline{G}}(\overline{p})) \to D$, $g \mapsto [g,p]$ with kernel $C(p)$.

Let $y \in \mathcal{O}_p \cap BwB$ be such that $L = L_J$ is adapted to $C(y)$. If $H = \pi^{-1}(C_{\overline{G}}(\overline{y}))$, then $\lambda(\mathcal{O}_{\overline{p}}) = \lambda(G/H) = \{\lambda \in P^+_w \mid \lambda(T \cap H) = 1\}$ by Corollary 3.18. Let $x \in \mathcal{O}_p \cap wB$, $x = \dot{w}u$, with $u \in U$ and let $T_{x,D} = T \cap \pi^{-1}(C_{\overline{G}}(\overline{x}))$. By Proposition 3.4, we get $T \cap H = T_{x,D}$, hence

$$\lambda(\mathcal{O}_{\overline{x}}) = \{\lambda \in P^+_w \mid \lambda(T_{x,D}) = 1\} \tag{6.12}$$

Let $T^w_D = \{t \in T \mid wtw^{-1} = zt, z \in D\}$. From the Bruhat decomposition, we get $T_{x,D} \leq T^w_D$. Moreover since w is an involution, for $t \in T^w_D$ we have $t = w^2tw^{-2} = z^2t$, so that $z^2 = 1$. In particular $\pi^{-1}(C_{\overline{G}}(\overline{s})) = N_{s,D_2}C(s)$, $T^w_D = T^w_{D_2}$, where $D_2 = D \cap T_2$.

Let $t \in T$ and write $t = ab$, with $a \in (T^w)^\circ$, $b \in (S^w)^\circ$. Then $wtw^{-1} = tz$ with $z \in D_2$ if and only if $z = b^2$. Since $(S^w)^\circ$ is connected, we get $T^w_D = T^w_{D_2 \cap (S^w)^\circ}$ and

$$\frac{\pi^{-1}(C_{\overline{G}}(\overline{x}))}{C(x)} \cong \frac{T_{x,D}}{T_x} \hookrightarrow \frac{T^w_D}{T^w} \cong D_2 \cap (S^w)^\circ$$

with $T_x = T^w \cap C(u)$, $T_{x,D} = T^w_D \cap C(u)$. In particular, if $D_2 \cap (S^w)^\circ = 1$, then $\lambda(\mathcal{O}_{\overline{x}}) = \lambda(\mathcal{O}_x)$. This equality means that x is not conjugate to zx for any $z \in D_2$, $z \neq 1$, and this may be directly checked in many cases, for instance in type A_n or C_n (and of course always holds for x unipotent). However, to deal with orthogonal groups and E_7, we determined explicitly the cases when $D_2 \cap (S^w)^\circ$ is non-trivial, and in each case we determined $T_{x,D}$ and therefore $\lambda(\mathcal{O}_{\overline{x}})$.

Here we just observe that if $D_2 \cap (S^w)^\circ \neq 1$, then $D_2 \cap (S^w)^\circ \cong \mathbb{Z}/2\mathbb{Z}$, except possibly for $D = Z(G)$ in type D_n, $n = 2m$. It turns out that in this case for $\exp(\pi i \check{\omega}_m)$, we have $T_x = T_2$ and $T_{x,Z(G)}/T_x \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. More precisely

$$T_{x,Z(G)} = T^{w_0}_{Z(G)} = T_2 \langle h_{\alpha_{n-1}}(i)h_{\alpha_n}(i), \prod_{i=1}^{m} h_{\alpha_{2i-1}}(i) \rangle$$

so that in $G/Z(G) = PSO(2n)$, $n = 2m$,

$$\lambda(\mathcal{O}_{\overline{\exp(\pi i\check{\omega}_m)}}) = \left\{ \sum_{k=1}^{n} 2m_k\omega_k \mid m_k \in \mathbb{N},\ \ m_{n-1} + m_n \text{ and } \sum_{i=1}^{m} m_{2i-1} \text{ even} \right\}$$

We add that for $SO(2n+1)$, $n \geq 1$ and $b_\lambda = \mathrm{diag}(1, \lambda I_n, \lambda^{-1}I_n)$, $\lambda \neq \pm 1$, $\mathcal{O}_{b_\lambda}$ is a model orbit, and in fact the principal one by [28], 3.3 ($2'$).

We conclude by presenting the results for E_7.

6.1 Type E_7, $D = Z(G)$

In this case $Z(G) = \langle z \rangle$, where $z = h_{\alpha_2}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1) = \exp(2\pi i\check{\omega}_2) = \exp(2\pi i\check{\omega}_7)$.

There are 3 elements of the Weyl group to be considered and only for $w = s_{\beta_1}s_{\beta_2}s_{\beta_4}$ and $w = w_0$ we have $z \in (S^w)^\circ$.

Class of type A_7, $w = w_0$. Here $x = n_{\beta_1} \cdots n_{\beta_7}$,

$$T^{w_0}_{Z(G)} = T_2 \langle \exp(\pi i\check{\omega}_2) \rangle = T_2 \langle h_{\alpha_2}(i)h_{\alpha_5}(i)h_{\alpha_7}(i) \rangle$$

since $\exp(\pi i\check{\omega}_2) \in (S^{w_0})^\circ = T$ and $\exp(\pi i\check{\omega}_2)^2 = z$.

Proposition 6.1 *Let G be of type E_7, $D = Z(G)$, then*

$$\lambda(\mathcal{O}_{\overline{\exp(\pi i\check{\omega}_2)}}) = \left\{ \sum_{i=1}^{7} 2n_i\omega_i \mid n_2 + n_5 + n_7 \textit{ even} \right\}$$

Proof. This follows from the fact that $T_{x,Z(G)} = T^{w_0}_{Z(G)}$. □

Classes of type E_6T_1, $w = s_{\beta_1}s_{\beta_2}s_{\beta_4}$, $T^w = (T^w)^\circ \times \langle h_{\alpha_7}(-1) \rangle = (T^w)^\circ \times Z(G)$.

We have $T^w_{Z(G)} = T^w \langle \exp(\pi i\check{\omega}_7) \rangle = T^w \langle h_{\alpha_1}(-1)h_{\alpha_7}(i) \rangle$. If $\zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$, then

$$x_\zeta = n_{\beta_1}n_{\beta_2}n_{\alpha_7}hx_{\beta_1}(\xi)x_{\beta_2}(\xi)x_{\alpha_7}(\xi) \in \mathcal{O}_{\exp(\zeta\check{\omega}_7)} \cap n_{\beta_1}n_{\beta_2}n_{\alpha_7}B$$

for a certain $h \in T$, with $\xi = \frac{1+e^\zeta}{1-e^\zeta}$, so that

$$T_{x_\zeta,Z(G)} = \begin{cases} T^w_{Z(G)} & \text{if } \zeta \in \pi i\mathbb{Z} \setminus 2\pi i\mathbb{Z} \\ T^w & \text{if } \zeta \in \mathbb{C} \setminus \pi i\mathbb{Z} \end{cases}$$

since $\alpha_7(\exp(\pi i\check{\omega}_7)) = -1$.

Proposition 6.2 *Let G be of type E_7, $D = Z(G)$, then*

$$\lambda(\mathcal{O}_{\overline{\exp(\zeta\check{\omega}_7)}}) = \begin{cases} \{n_1\omega_1 + n_6\omega_6 + 2n_7\omega_7 \mid n_1 + n_7 \textit{ even}\} & \textit{if } \zeta \in \pi i\mathbb{Z} \setminus 2\pi i\mathbb{Z} \\ \{n_1\omega_1 + n_6\omega_6 + 2n_7\omega_7\} & \textit{if } \zeta \in \mathbb{C} \setminus \pi i\mathbb{Z} \end{cases}$$

□

Addendum In [9], Remark 5, we stated that if $\pi_1 : G \to G/U$ is the canonical projection, and $\mathcal{O}$ is a spherical conjugacy class, then $\pi_{1|\mathcal{O}} : \mathcal{O} \to G/U$ has finite fibers. This is not correct, and one can only say that $\pi_{1|\mathcal{O}}$ has generically finite fibers (if $w = w(\mathcal{O})$, and $g \in \mathcal{O} \cap BwB$, then $\pi_1^{-1}(gU)$ has $|T^w/T_x|$ elements, where $x \in \mathcal{O} \cap wB$).

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