Uniqueness Theorem for the Solutions of the Differential Equations of Incremental Thermoelectroelasticity

A. Montanaro

July 30, 2008

Key words Thermoelectroelasticity,

Abstract

We prove a uniqueness theorem for the solutions to the initial boundary value problems in incremental thermoelectroelasticity with nonlinear constitutive response functions. No restriction is made on the initial fields of deformation, electric potential, and temperature.

1 Introduction

Nowacki in [1] presented a uniqueness theorem for the solutions of the initial boundary value problems in linear thermopiezoelectricity referred to a natural state, i.e., without initial fields.

The equations of nonlinear thermoelectroelasticity were given in Tiersten [2]. Yang [3] then derived from [2] the equations for infinitesimal incremental fields superposed on finite biasing fields in a thermoelectroelastic body with no assumption on the biasing fields.

Here we extend the aforementioned Nowacki's uniqueness theorem to the incremental theory [3] without no restriction on the initial fields of deformation, electric potential, and temperature.

We explicitly to the theory [3], hence we rewrite from this paper formulae and results of incremental thermoelectroelasticity by using just the same notations.

2 Equations of Nonlinear Thermoelectroelasticity

Consider a thermoelectroelastic body that, in the reference configuration, occupies a region V with boundary surface S. The motion of the body is described by

$$y_i = y_i(X_L,\, t)\,,$$

where y_i denotes the present coordinates and X_L the reference coordinates of material points with respect to the same Cartesian coordinate system.

Let K_{Lj}, ρ_o, f_j, Δ_L, ρ_E, θ, η, Q_L and γ respectively denote the first Piola-Kirchoff stress tensor, the mass density in the reference configuration, the body force per unit mass, the reference electric displacement vector, the free charge density per unit undeformed volume, the absolute temperature, the entropy per unit mass, the reference heat flux vector, and the body heat source per unit mass. Then we have the following equations of motion, electrostatics, and heat conduction written in material form with respect to the reference configuration:

$$K_{Li,L} + \rho_o f_i = \rho_o \ddot{y}_i\,, \tag{1}$$

$$\Delta_{L,L} = \rho_E\,, \tag{2}$$

$$\rho_o \theta \dot{\eta} = -Q_{L,L} + \rho_o \gamma\,, \tag{3}$$

The above equations are adjoined by constitutive relations defined by the specification of the free energy ψ and heat flux Q_L:

$$\psi = \psi(E_{MN},\, W_M,\, \theta)\,, \qquad Q_L = Q_L(E_{MN},\, W_M,\, \theta,\, \Theta_M) \tag{4}$$

where

$$E_{MN} = (y_{j,\,M} y_{j,\,N} - \delta_{MN})/2\,, \qquad W_M = -\phi_M\,, \qquad \Theta_M = \theta_{,\,M} \tag{5}$$

are the finite strain tensor, the reference electric potential gradient, and the reference temperature gradient; of course, δ_{MN} is the Kronecker delta, and ϕ is the electric potential. Hence, by using ψ the constitutive relations (4) of [3] are deduced for K_{Li}, Δ_L, η:

$$\begin{aligned} K_{Li} &= y_{i,\,A}\rho_o \frac{\partial \psi}{\partial E_{AL}} + J X_{L,\,j}\, \varepsilon_o (E_j E_i - \frac{1}{2} E_i E_i \delta_{ji})\,, \\ \Delta_L &= \varepsilon_o J X_{L,\,j} E_j - \rho_o \frac{\partial \psi}{\partial W_L}\,, \\ \eta &= -\frac{\partial \psi}{\theta}\,. \end{aligned} \tag{6}$$

Moreover, Q_L in $(4)_2$ is restricted by

$$Q_L \Theta_L \leq 0\,. \tag{7}$$

In particular, $(4)_2$ includes the case in which Q_M is linear in Θ_L, that is,

$$Q_M = -\kappa_{ML}(\theta,\, W_A)\,\Theta_L\,. \tag{8}$$

2.1 The initial boundary value problem for a thermoelectroelastic body

To describe the corresponding boundary conditions to add to the field equations (1)-(3), three partitions (S_{i1}, S_{i2}), $i = 1,\, 2,\, 3$, of the boundary surface $S = \partial\mathcal{B}$ can be assigned. For mechanical boundary conditions, deformation $\overline{y}_i$ and traction $\overline{t}_i$ per unit undeformed area are prescribed, respectively, on S_{11} and S_{12}; for electric boundary conditions, electric potential $\overline{\phi}$ and surface-free charge $\overline{\Delta}$ per unit undeformed area are prescribed, respectively, on S_{21} and S_{22}; while for thermic boundary conditions, temperature $\overline{\theta}$ and normal heat flux $\overline{Q}$ per unit undeformed area are prescribed, respectively,on S_{31} and S_{32}. Hence, we can write

$$y_i = \overline{y}_i \quad \text{on} \quad S_{11}\,, \qquad K_{Li}N_L = \overline{K}_i \quad \text{on} \quad S_{12} \qquad (\text{'mechanical'})\,, \tag{9}$$

$$\phi = \overline{\phi} \quad \text{on} \quad S_{21}\,, \qquad \Delta_L N_L = \overline{\Delta} \quad \text{on} \quad S_{22}\,, \qquad (\text{'electric'}) \tag{10}$$

$$\theta = \overline{\theta} \quad \text{on} \quad S_{31}\,, \qquad Q_L N_L = \overline{Q} \quad \text{on} \quad S_{32} \qquad (\text{'thermic'})\,, \tag{11}$$

$$S_{i1} \cup S_{i2} = S\,, \quad S_{i1} \cap S_{i2} = \emptyset \quad (i = 1,\, 2,\, 3)\,, \tag{12}$$

where $\mathbf{N} = (N_L)$ is the unit exterior normal on S.

We put

$$\mathcal{A}_{body} := \Big(\, f_i,\, \rho_E,\, \gamma \,\Big)\,, \tag{13}$$

$$\mathcal{A}_{surf} := \Big(\, \overline{y}_i,\, \overline{K}_i,\, \overline{\phi},\, \overline{\Delta},\, \overline{\theta},\, \overline{Q} \,\Big)\,, \tag{14}$$

$$\mathcal{A} := (\, \mathcal{A}_{body},\, \mathcal{A}_{surf}\,) = \Big(\, f_i,\, \rho_E,\, \gamma,\, \overline{y}_i,\, \overline{K}_i,\, \overline{\phi},\, \overline{\Delta},\, \overline{\theta},\, \overline{Q} \,\Big)\,. \tag{15}$$

$\mathcal{A}_{body}$, $\mathcal{A}_{body}$, and $\mathcal{A}$ are said to be the *(external) body-action*, *surface-action*, and *action*, respectively.

The initial conditions have the form

$$\begin{gathered} y_i(\mathbf{X},\, 0) = f_i(\mathbf{X}), \quad \dot{y}_i(\mathbf{X}),\, 0) = g_i(\mathbf{X}), \\ \theta(\mathbf{X},\, 0) = h(\mathbf{X}), \qquad \phi(\mathbf{X},\, 0) = l(\mathbf{X}) \qquad (\mathbf{X} \in B,\ t = 0)\,, \end{gathered} \tag{16}$$

where

$$\mathcal{I} = \Big(f_i,\ g_i,\ h,\ l \Big)$$

are prescribed smooth functions of domain V. The initial boundary value problem is then stated as: *assigned* $\mathcal{A}_{body}$*, to find the solution* $(\phi,\, \theta,\, y_i)$ *in* $\mathcal{B}$ *to the constitutive relations (6) and field equations (1)-(3) which satisfies the boundary conditions (9)-(11) and initial conditions (16) for given* $\mathcal{A}_{surf}$ *and* $\mathcal{I}$*.*

3 Biasing and incremental fields

In incremental theories three configurations are distinguished: the reference, initial and present configuration.

3.1 The Reference Configuration

In the reference state the body is undeformed and free of all fields. A generic point at this state is denoted by $\mathbf{X}$ with rectangular coordinates X_N. The mass density in the reference configuration is denoted by ρ_o.

3.2 The Initial Configuration

We put

$$\mathcal{A}^o_{body} := \Big(f^o_i,\, \rho^o_E,\, \gamma^o \Big)\,, \tag{17}$$

$$\mathcal{A}^o_{surf} := \Big(\overline{y}^o_i,\, \overline{K}^o_i,\, \overline{\phi}^o,\, \overline{\Delta}^o,\, \overline{\theta}^o,\, \overline{Q}^o \Big)\,, \tag{18}$$

$$\mathcal{A}^o := (\, \mathcal{A}_{body},\, \mathcal{A}_{surf}\,) = \Big(f^o_i,\, \rho^o_E,\, \gamma^o,\, \overline{y}^o_i,\, \overline{K}^o_i,\, \overline{\phi}^o,\, \overline{\Delta}^o,\, \overline{\theta}^o,\, \overline{Q}^o \Big)\,. \tag{19}$$

$\mathcal{A}^o_{body}$, $\mathcal{A}^o_{body}$, and $\mathcal{A}^o$ are said to be the *(external) body-action*, *surface-action*, and *action*, respectively.

In this state the body is deformed finitely under the action of the prescribed action $\mathcal{A}^o$. The position of the material point associated with $\mathbf{X}$ is given by

$$y^o_\alpha = y^o_\alpha(\mathbf{X},\, t)\,,$$

with the Jacobian of the initial configuration denoted by

$$J_o = det(y^o_{\alpha,\,L})\,.$$

In this state the electric potential, electric field and temperature field are denoted by $\phi^o(\mathbf{X},\, t)$, $W^o_\alpha = -\phi^o_{,\alpha}$ and $\theta^o(\mathbf{X},\, t)$, respectively.

The initial fields

$$y^o_\alpha = y^o_\alpha(\mathbf{X},\, t), \quad \phi^o = \phi^o(\mathbf{X},\, t), \quad \theta^o = \theta^o(\mathbf{X},\, t) \tag{20}$$

satisfy the equations of nonlinear thermoelectroelasticity (1)-(12) under the prescribed action $\mathcal{A}^o$.

In studying the incremental fields the solution to the initial state problem is assumed known.

3.3 The Present Configuration

To the deformed body at the initial configuration, infinitesimal deformations, electric, and thermal fields are applied. The present position of the material point associated with $\mathbf{X}$ is given by $y_i(\mathbf{X},\, t)$, with electric potential $\phi(\mathbf{X},\, t)$ and temperature $\theta(\mathbf{X},\, t)$.

The fields $y_i(\mathbf{X},\, t)$, $\phi(\mathbf{X},\, t)$, $\theta(\mathbf{X},\, t)$ satisfy (1)-(3) under the action of the external action (15).

4 Equations for the incremental fields

Let ε be a small and dimensionless number. The incremental process $\varepsilon(y^1,\, \phi^1,\, \theta^1)$ for $(y,\, \phi,\, \theta)$ superposed to the initial process $(y^o,\, \phi^o,\, \theta^o)$ is assumed to be infinitesimal and, therefore, we write:

$$y_i = \delta_{i\alpha}(y^o_\alpha + \varepsilon y^1_\alpha)\,, \qquad \phi = \phi^o + \varepsilon\phi^1\,, \qquad \theta = \theta^o + \varepsilon\theta^1\,, \tag{21}$$

Corresponding to (21), other quantities of the present state can be written as:

$$\mathcal{A} \cong \mathcal{A}^o + \varepsilon\mathcal{A}^1\,, \tag{22}$$

where, due to nonlinearity, higher powers of ε may arise. For the incremental action we have

$$\mathcal{A}^1_{body} := \left(\, f^1_i,\, \rho^1_E,\, \gamma^1 \,\right) \tag{23}$$

$$\mathcal{A}^1_{surf} := \left(\, \overline{y}^1_i,\, \overline{K}^1_i,\, \overline{\phi}^1,\, \overline{\Delta}^1,\, \overline{\theta}^1,\, \overline{Q}^1 \,\right) \tag{24}$$

$$\mathcal{A}^1 := (\, \mathcal{A}_{body},\, \mathcal{A}_{surf}\,) = \left(\, f^1_i,\, \rho^1_E,\, \gamma^1,\, \overline{y}^1_i,\, \overline{K}^1_i,\, \overline{\phi}^1,\, \overline{\Delta}^1,\, \overline{\theta}^1,\, \overline{Q}^1 \,\right) \tag{25}$$

We want to derive equations governing the incremental process

$$(\mathbf{u} := \mathbf{y}^1\,, \quad \phi^1\,, \quad \theta^1\,)\,.$$

From (21) and (22), we can further write:

$$\begin{aligned} E_{KL} &\cong E^o_{KL} + \varepsilon E^1_{KL}\,, \\ W_L &\cong W^o_L + \varepsilon W^1_L\,, \\ \Theta_L &\cong \Theta^o_L + \varepsilon \Theta^1_L\,, \end{aligned} \tag{26}$$

where

$$\begin{aligned} E^o_{KL} = (y^o_{\alpha,\,K} y^o_{\alpha,\,L} - \delta_{KL})/2\,, \qquad & E^1_{KL} = (y^o_{\alpha,\,K} y^1_{\alpha,\,L} + y^o_{\alpha,\,L} y^1_{\alpha,\,K})/2\,, \\ W^o_L = -\phi^o_{,\,L}\,, \qquad & W^1_L = -\phi^1_{,\,L}\,, \\ \Theta^o_L = -\theta^o_{,\,L}\,, \qquad & \Theta^1_L = -\theta^1_{,\,L}\,. \end{aligned} \tag{27}$$

Substituting (21)-(27) into the constitutive relations (1)-(3), with some very lengthy algebra, the following expression can be obtained:

$$\begin{aligned} K_{Mi} \cong \delta_{i\alpha}(K^o_{M\alpha} + \varepsilon K^1_{M\alpha})\,, \quad \Delta_M \cong \Delta^o_M + \varepsilon \Delta^1_M\,, \\ \eta \cong \eta^o + \varepsilon \eta^1\,, \quad Q_M \cong Q^o_M + \varepsilon Q^1_M\,. \end{aligned} \tag{28}$$

where

$$K^1_{M\alpha} = G_{M\alpha L\gamma} u_{\gamma,\,L} + R_{LM\alpha}\phi^1_{,\,L} - \rho_o \Lambda_{M\alpha}\theta^1\,, \tag{29}$$

$$\Delta^1_M = R_{MN\gamma} u_{\gamma,\,N} - L_{MN}\phi^1_{,\,N} + \rho_o P_M \theta^1\,, \tag{30}$$

$$\eta^1 = \Lambda_{M\gamma} u_{\gamma,\,M} - P_M \phi^1_{,\,M} + \alpha\theta^1\,, \tag{31}$$

$$Q^1_M = A_{MN\alpha} u_{\alpha,\,N} - B_{MN}\phi^1_{,\,N} + C_M \theta^1 + F_{MN}\theta^1_{,\,N}\,. \tag{32}$$

In (29)-(32),

- $G_{M\alpha L\gamma}$ are the effective elastic constants,
- $R_{LM\alpha}$ are the effective piezoelectric constants,
- $\Lambda_{M\alpha}$ are the effective thermoelatic constants,
- L_{MN} are the effective dielectric constants,
- P_M are the effective pyrolectric constants,
- α is related with the specific heat.

Their expressions are

$$
\begin{aligned}
G_{K\alpha L\gamma} \,=\, y^o_{\alpha,\,M}\rho_o\frac{\partial^2\psi}{\partial E_{KM}\partial E_{LN}}(\theta^o,\,E^o_{AB},\,W^o_A)\,y^o_{\alpha,\,L}+\rho_o\frac{\partial\psi}{\partial E_{KL}}(\theta^o,\,E^o_{AB},\,W^o_A)\,\delta_{\alpha\gamma}+g_{K\alpha L\gamma}\,,\\
R_{LM\gamma}=-\rho_o\frac{\partial^2\psi}{\partial W_K\partial E_{ML}}(\theta^o,\,E^o_{AB},\,W^o_A)\,y^o_{\gamma,\,M}+r_{KL\gamma}\,,\\
\Lambda_{M\gamma} \,=\, -\frac{\partial^2\psi}{\partial E_{LM}\partial\theta}(\theta^o,\,E^o_{AB},\,W^o_A)\,y^o_{\gamma,\,L}\,,\\
L_{MN} \,=\, -\rho_o\frac{\partial^2\psi}{\partial W_M\partial W_N}(\theta^o,\,E^o_{AB},\,W^o_A)+l_{MN}\,,\\
P_M \,=\, -\frac{\partial^2\psi}{\partial W_M\partial\theta}(\theta^o,\,E^o_{AB},\,W^o_A)\,,\\
\alpha \,=\, -\frac{\partial^2\psi}{\partial\theta^2}(\theta^o,\,E^o_{AB},\,W^o_A)\,,\\
A_{MN\gamma} \,=\, \frac{\partial Q_M}{\partial E_{LN}}(\theta^o,\,E^o_{AB},\,W^o_A)\,y^o_{\gamma,\,L} \,=:\, -\kappa_{MN\gamma}\,,\\
B_{MN} \,=\, \frac{\partial Q_M}{\partial W_N}(\theta^o,\,E^o_{AB},\,W^o_A) \,=:\, \kappa^E_{MN}\,,\\
C_{_M} \,=\, \frac{\partial Q_M}{\partial\theta}(\theta^o,\,E^o_{AB},\,W^o_A) \,=:\, -\kappa_M\,,\\
F_{MN} \,=\, \frac{\partial Q_M}{\partial\Theta_N}(\theta^o,\,E^o_{AB},\,W^o_A) \,=:\, -\kappa^o_{MN}\,,
\end{aligned}
\tag{33}
$$

where

$$
\begin{aligned}
g_{K\alpha L\gamma} \,=\, \varepsilon_o J_o\Big[W^o_\alpha W^o_\beta(X_{K,\,\beta}X_{L,\,\gamma}-X_{K,\,\gamma}X_{L,\,\beta})+W^o_\beta W^o_\gamma(X_{K,\,\alpha}X_{L,\,\beta}-X_{K,\,\beta}X_{L,\,\alpha})\\
+W^o_\beta W^o_\beta(X_{K,\,\gamma}X_{L,\,\alpha}-X_{K,\,\alpha}X_{L,\,\gamma})/2-W^o_\alpha W^o_\gamma X_{K,\,\beta}X_{L,\,\beta}\Big]\,,\\
r_{KL\gamma} \,=\, \varepsilon_o J_o\Big(W^o_\alpha X_{K,\,\alpha}X_{L,\,\gamma}-W^o_\alpha X_{K,\,\gamma}X_{L,\,\alpha}-W^o_\gamma X_{K,\,\alpha}X_{L,\,\alpha}\Big)\,,\quad l_{MN} \,=\, \varepsilon_o J_o X_{M,\,\alpha}X_{N,\,\alpha}\,.
\end{aligned}
\tag{34}
$$

By the last four above relations in (34) we thus have

$$
Q^1_M=-\kappa_{MN\alpha}u_{\alpha,\,N}-\kappa^E_{MN}\phi^1_{,\,N}-\kappa_M\theta^1-\kappa^o_{MN}\theta^1_{,\,N}\,. \tag{35}
$$

We have introduced the κ-notation to allow comparison between the proof written here and that written in [1]. The following symmetries hold:

$$
G_{K\alpha L\gamma} \,=\, G_{L\gamma K\alpha}\,,\qquad L_{MN} \,=\, L_{NM}\,. \tag{36}
$$

4.1 Restriction on the incremental heat flux

Now we show that the restriction (7) on the heat flux $(4)_2$, together with the condition

$$Q_L = 0 \quad \text{for} \quad \Theta_L^o = 0, \qquad \text{thus} \quad Q_L^o = 0\,, \tag{37}$$

implies an analogous restriction on the incremental heat flux (32), that is,

$$Q_L^1 \Theta_L^1 \leq 0\,. \tag{38}$$

Indeed, substituting

$$Q_L = Q_L^o + \varepsilon Q_L^1\,, \qquad \Theta_L = \Theta_L^o + \varepsilon \Theta_L^1$$

in (7), we obtain

$$\Big(Q_L^o + \varepsilon Q_L^1\Big)\Big(\Theta_L^o + \varepsilon \Theta_L^1\Big) \leq 0\,, \tag{39}$$

which for $\Theta_L^o = 0$, by (37), yields (38).

Note that the choice (8) for the heat flux response function satisfies (37).

5 Incremental field equations

We refer to a static initial state, that is, independent of time. By substituting (21)-(28) into (1)-(3) and (9)-(11), we find the governing equations for the incremental fields

$$K^1_{M\alpha,\,M} \,+\, \rho_o\, f^1_\alpha \,=\, \rho_o\, \ddot{u}_\alpha\,, \tag{40}$$

$$\Delta^1_{M,\,M} \,=\, \rho^1_E\,, \tag{41}$$

$$\rho_o\,(\theta^o \dot{\eta}^1 \,+\, \theta^1 \dot{\eta}^o) \,=\, -Q^1_{M,\,M} \,+\, \rho_o\, \gamma^1\,. \tag{42}$$

Introducing the constitutive relations (29)-(32) into the incremental equations of motion (40), the equation of the electric field (41), and the heat equation (42), for $f^1_\alpha = 0$ we have

$$G_{M\alpha L\gamma} u_{\gamma,\,LM} + R_{LM\alpha} \phi^1_{,\,LM} - \rho_o \Lambda_{M\alpha} \theta^1_{,\,M} \,=\, \rho_o\, \ddot{u}_\alpha\,, \tag{43}$$

$$R_{MN\gamma} u_{\gamma,\,NM} - L_{MN} \phi^1_{,\,NM} + \rho_o P_M \theta^1_M \,=\, \rho^1_E\,, \tag{44}$$

$$\begin{aligned} &\kappa^E_{MN} \phi^1_{,\,NM} + \kappa_M \theta^1_{,\,M} + \kappa^o_{MN} \theta^1_{,\,NM} + \kappa_{MN\alpha} u_{\alpha,\,NM} \\ &-\rho_o \theta^o \Big(\Lambda_{M\gamma} \dot{u}_{\gamma,\,M} - P_M \dot{\phi}^1_{,\,M} + \alpha \dot{\theta}^1 \Big) \,=\, -\rho_o\, \gamma^1\,. \end{aligned} \tag{45}$$

6 Uniqueness of the solution of the incremental differential equations

We follow step by step the proof of Nowacki [1] and put in evidence any difference when it will appear.

A modified version of energy balance is needed. It follows by substituting the virtual increments by the real increments

$$\delta u_\alpha = \frac{\partial u_\alpha}{\partial t}\, dt = v_\alpha\, dt\,, \qquad \delta u_{\alpha,\,M} = \dot{u}_{\alpha,\,M}\, dt\,, \quad \ldots$$

in the principle of virtual work

$$\int_{V^o} \big(f^1_\alpha - \rho_o \ddot{u}_\alpha\big)\delta u_\alpha\, dV + \int_{S^o} \overline{K}_\alpha\, \delta u_\alpha\, dS \;=\; \int_{V^o} K^1_{M\alpha}\, \delta u_{\alpha,\,M}\, dV\,. \tag{46}$$

Thus the fundamental energy equation

$$\int_{V^o} \big(f^1_\alpha - \rho_o \dot{v}_\alpha\big)v_\alpha\, dV + \int_{S^o} \overline{K}_\alpha\, v_\alpha\, dS \;=\; \int_{V^o} K^1_{M\alpha}\, \dot{u}_{\alpha,\,M}\, dV \tag{47}$$

is obtained, where we substitute the constitutive relations (29). Hence

$$\int_{V^o} \big(f^1_\alpha - \rho_o \dot{v}_\alpha\big)v_\alpha\, dV + \int_{S^o} \overline{K}_\alpha\, v_\alpha\, dS \;=\; \int_{V^o} \Big(G_{M\alpha L\gamma} u_{\gamma,\,L} + R_{LM\alpha}\phi^1_{,\,L} - \rho_o \Lambda_{M\alpha}\theta^1\Big)\, \dot{u}_{\alpha,\,M}\, dV\,, \tag{48}$$

thus

$$\frac{d}{dt}\big(\mathcal{W}+\mathcal{K}\big) \;=\; \int_{V^o} f^1_\alpha\, v_\alpha\, dV + \int_{S^o} \overline{K}_\alpha\, v_\alpha\, dS + \int_{V^o} \Big(\rho_o \Lambda_{M\alpha}\theta^1 - R_{LM\alpha}\phi^1_{,\,L}\Big)\, \dot{u}_{\alpha,\,M}\, dV\,, \tag{49}$$

where $\mathcal{W}$ is the work of deformation and $\mathcal{K}$ is the kinetic energy:

$$\mathcal{W} \;=\; \frac{1}{2}\int_{V^o} G_{M\alpha L\gamma}\, u_{\alpha,\,M}\, u_{\gamma,\,L}\, dV\,, \qquad \mathcal{K} \;=\; \frac{1}{2}\int_{V^o} \rho_o\, v_\alpha v_\alpha\, dV\,. \tag{50}$$

Now, to eliminate the term $\int_{V^o} \rho_o \Lambda_{M\alpha}\theta^1\, \dot{u}_{\alpha,\,M}\, dV$, we multiply the heat-conduction equation (49) by θ^1 and integrate over V^o; after simple transformations we obtain

$$\begin{aligned}
\int_{V^o} \rho_o\, \theta^1\, \Lambda_{M\alpha}\, \dot{u}_{\alpha,\,M}\, dV \;&=\; \frac{\kappa^E_{ML}}{\theta^o}\int_{S^o} \theta^1 \phi^1_{,\,L} N_M\, dS \;+ \\
&+ \frac{\kappa^L}{\theta^o}\int_{S^o} \theta^1 N_L\, dS \;+\; \frac{\kappa^o_{ML}}{\theta^o}\int_{S^o} \theta^1 \theta^1_{,\,L} N_M\, dS \;+\; \frac{\kappa_{ML\alpha}}{\theta^o}\int_{S^o} \theta^1 u_{\alpha,\,L} N_M\, dS \\
+ P_M \int_{V^o} \rho_o\, \theta^1 \dot{\phi}^1_{,\,M}\, dV \;&+\; \frac{1}{\theta^o}\int_{V^o} \rho_o\, \theta^1 \gamma^1\, dV \;-\; \frac{d}{dt}\mathcal{P} \;-\; (\,\chi \;+\; \chi_\theta \;+\; \chi_\phi \;+\; \chi_u)\,,
\end{aligned} \tag{51}$$

where

$$\mathcal{P} \,=\, \frac{\alpha}{2\theta^o}\int_{V^o} \rho_o \theta^1\,\theta^1\,dV\,, \tag{52}$$

$$\chi_\phi = \frac{\kappa^E_{ML}}{\theta^o}\int_{V^o}\theta^1_{,\,M}\phi^1_{,\,L}dV\,,\quad \chi = \frac{\kappa_L}{\theta^o}\int_{V^o}\theta^1_{,\,L}\theta^1\,dV\,,$$
$$\chi_\theta = \frac{\kappa^o_{ML}}{\theta^o}\int_{V^o}\theta^1_{,\,M}\theta^1_{,\,L}dV\,,\quad \chi_u = \frac{\kappa^o_{ML\alpha}}{\theta^o}\int_{V^o}\theta^1_{,\,M}u_{\alpha,\,L}dV\,. \tag{53}$$

Note that this equation differs from the corresponding Eq. (25) in [1] by the terms χ_ϕ, χ and χ_u. Now, substituting (51) into (49), we are lead to the equation

$$\begin{aligned}\frac{d}{dt}\big(\mathcal{W}+\mathcal{K}+\mathcal{P}\big) \,+\, (\,\chi \,+\, \chi_\theta \,+\, \chi_\phi \,+\, \chi_u\,) \,=\, \int_{V^o} f^1_\alpha\, v_\alpha\, dV \,+\, \int_{S^o}\overline{K}_\alpha\, v_\alpha\, dS \,+\\ +\,\frac{\kappa^E_{ML}}{\theta^o}\int_{S^o}\theta^1\phi^1_{,\,L}N_M\,dS \,+\, \frac{\kappa^L}{\theta^o}\int_{S^o}\theta^1 N_L\,dS \,+\, \frac{\kappa^o_{ML}}{\theta^o}\int_{S^o}\theta^1\theta^1_{,\,L}N_M\,dS \,+\\ +\,\frac{1}{\theta^o}\int_{V^o}\rho_o\,\theta^1\gamma^1\,dV \,-\, \int_{V^o}\Big(\,R_{LM\alpha}\phi^1_{,\,L}\,\dot{u}_{\alpha,\,M} \,-\, \rho_o P_M\,\theta^1\dot{\phi}^1_{,\,M}\,\Big)\,dV\,.\end{aligned} \tag{54}$$

To eliminate the last term $\int_{V^o}\Big(\,R_{LM\alpha}\phi^1_{,\,L}\,\dot{u}_{\alpha,\,M} \,-\, \rho_o P_M\,\theta^1\dot{\phi}^1_{,\,M}\,\Big)dV$ in Eq. (54) we use the constitutive relations (30).

Finally, we use the equation of the electric field (41) with $\rho_E = 0$. Multiplying the equation by ϕ^1 and integrating over the region of the body, we obtain

$$\int_{V^o}\dot{\Delta}_M\phi^1 N_M\,dV \,+\, \int_{SV^o}\dot{\Delta}_M W^1_M\,dV \,=\, 0\,. \tag{55}$$

Using the relations (30) and (55), after simple transformations we obtain

$$\begin{aligned}\int_{V^o}\dot{\Delta}_K W W^1_K\,dV \,=\\ =\,\int_{V^o}\Big(R_{LM\alpha}\dot{u}_{\alpha,\,N}W^1_K \,+\, L_{KN}\dot{W}^1_N W^1_K \,+\, \rho_o P_K\frac{d}{dt}(\theta^1 W^1_K) \,-\, \rho_o P_K\theta^1\dot{W}^1_K\Big)\,dV \,=\\ =\,-\int_{S^o}\dot{\Delta}^1_K N_K\phi^1\,dS\,,\end{aligned}$$

from which

$$\begin{aligned}\int_{V^o}\Big(R_{KM\alpha}\dot{u}_{\alpha,\,N}W^1_K \,-\, \rho_o P_K\theta^1\dot{W}^1_K\Big)\,dV \,=\\ =\,-\int_{S^o}\dot{\Delta}^1_K N_K\phi^1\,dS \,-\, \frac{d}{dt}\,\mathcal{E} \,-\, \frac{d}{dt}\Big(\rho_o P_K\int_{V^o}\theta^1 W^1_K\,dV\Big)\end{aligned} \tag{56}$$

where

$$\mathcal{E} \,=\, \frac{1}{2}\, L_{KN} \int_{V^o} W^1_N W^1_K \, dV \,. \tag{57}$$

In view of Eqs. (54) and (56), we arrive at the modified energy balance

$$\begin{aligned} \frac{d}{dt}\Big(\mathcal{W} + \mathcal{K} + \mathcal{P} + \mathcal{E} + \rho_o P_K \int_{V^o} \theta^1 W^1_K \, dV\Big) \,+\, (\,\chi \,+\, \chi_\theta \,+\, \chi_\phi\,) \,= \\ = \int_{V^o} f^1_\alpha \, v_\alpha \, dV \,+\, \int_{S^o} \overline{K}_\alpha \, v_\alpha \, dS \,+ \\ + \frac{\kappa^E_{ML}}{\theta^o} \int_{S^o} \theta^1 \phi^1_{,L} N_M \, dS \,+\, \frac{\kappa^L}{\theta^o} \int_{S^o} \theta^1 N_L \, dS \,+\, \frac{\kappa^o_{ML}}{\theta^o} \int_{S^o} \theta^1 \theta^1_{,L} N_M \, dS \,+ \\ + \frac{1}{\theta^o} \int_{V^o} \rho_o \, \theta^1 \gamma^1 \, dV \,-\, \int_{S^o} \dot{\Delta}^1_K N_K \phi^1 \, dS \,. \end{aligned} \tag{58}$$

The energy balance (58) makes possible the proof of the uniqueness of the solution.

We assume that two distinct solutions $(u'_i,\ \phi'^1,\ \theta'^1)$ and $(u''_i,\ \phi''^1,\ \theta''^1)$ satisfy Eqs. (40)-(42) and the appropriate boundary and initial conditions. Their difference $(\hat{u}_i = u'_i - u''_i,\ \hat{\phi} = \phi'^1 - \phi''^1,\ \hat{\theta} = \theta'^1 - \theta''^1)$ therefore satisfies the homogeneous equations (40)-(42) and the homogeneous boundary and initial conditions. Equation (58) holds for $(\hat{u}_i,\ \hat{\phi},\ \hat{\theta})$.

In view of the homogeneity of the equations and the boundary conditions, the right-hand side of Eq. (58) vanishes. Hence

$$\frac{d}{dt}\Big(\mathcal{W} + \mathcal{K} + \mathcal{P} + \mathcal{E} + \rho_o P_K \int_{V^o} \theta^1 W^1_K \, dV\Big) \,=\, -\,(\,\chi \,+\, \chi_\theta \,+\, \chi_\phi \,+\, \chi_u\,) \,\le\, 0\,, \tag{59}$$

where the last inequality is true since by (35) and (38) we have

$$-(\,\chi \,+\, \chi_\theta \,+\, \chi_\phi \,+\, \chi_u\,) \,=\, \int_{V^o} Q^1_M \Theta^1_M \, dV \,. \tag{60}$$

The integral in the left-hand side of Eq. (59) vanishes at the initial instant, since the functions $\hat{u}_i,\ \hat{\phi},\ \hat{\theta}$ satisfy the homogeneous initial conditions. On the other hand, by the inequality in (59) the left-hand side is either negative or zero.

Now we assume

(1) that the initial deformation y^o realizes that the tensor $G_{M\alpha L\gamma}$ is positive-definite, so that $\mathcal{W} \ge 0$ by (50);

(2) that the tensor L_{KN} is positive-definite, so that $\mathcal{E} \ge 0$ by (57);

$$\mathcal{E} \,=\, \frac{1}{2}\, L_{KN} \int_{V^o} W^1_N W^1_K \, dV \,; \tag{61}$$

(3) the sufficient condition of J. Ignaczak written in [1] on pages 176-177: assume that L_{IJ} is a known positive-definite symmetric tensor, $g_I = \rho_o P_I$ is a vector, and $c = \frac{\rho_o \alpha}{2\theta^o} > 0$; consider the function

$$A(\theta^1, W_L) = (\theta^1)^2 + 2\theta^1 g_I W_I^1 + L_{IJ} W_I^1 W_J^1$$

A is nonnegative for every real pair (θ^1, W_k^1), provided

$$|g_I| \leq c\lambda_m$$

where λ_m is the smallest positive eigenvalue of the tensor L_{IJ}.

Under the three above assumptions, Equations (59) imply

$$\hat{u}_{i,\,L} = 0, \qquad \hat{\theta}^1 = 0, \qquad W_L^1 = 0\,,$$

which imply the uniqueness of the solutions of the incremental thermoelectroelastic equations, i.e.,

$$u_i' = u_i'', \qquad \theta^{1\prime} = \theta^{1\prime\prime}, \qquad W_I^{1\prime} = W_I^{1\prime\prime}\,.$$

Moreover, from the constitutive relations we have that

$$K^{1\prime}{}_{I\alpha} = K^{1\prime\prime}{}_{I\alpha}, \qquad \Delta_L^{1\,\prime} = \Delta_L^{1\,\prime\prime}, \qquad \eta^{1\prime} = \eta^{1\prime\prime}\,.$$

References

[1] Nowacki, W. *Some general Theorems of Thermopiezoelectricity.* J. of Thermal Stresses, 1:171-182, 1978.

[2] Tiersten, H.F. *On the Nonlinear Equations of Thermoelectroelasticity.* Int. J. Engng Sci. Vol. 9, pp. 587-604. Pergamon Press 1971.

[3] Yang, J.S. *Equations for Small Fields Superposed on Finite Biasing Fields in a Thermoelectroelastic Body.* IEEE Transactions on Ultrasonics, Ferroelectricts, and Frequency Control, Vol. 50, 187-192, no. 2, February 2003.