Does symmetry of the operator of a dynamical system help stability?

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Abstract

In this article, we address the question of relating the stability properties of an operator with the stability properties of its associate symmetric operator. The linear-algebra results of Bendixson and Hirsch indicate that the symmetric part of a matrix is always less stable than the matrix itself. We show that in a variety of cases, including infinite dimensional cases associated to systems of PDEs, the same result is valid. We also discuss the applicability to non-autonomous systems, and we show that, in general, this result is not valid. We also review some of the the literature that in these years has appeared on the subject.

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1 Introduction

Stability of equilibria, may they be points in ODEs or stationary solutions in PDEs, plays a central role in dynamical systems and their applications. Firstly because dynamical systems, in spite of their complexity, allow stationary solutions or equilibria which are typically simple to determine. Secondly, since in application of dynamical systems one cannot fix a state exactly, but only approximately, a stationary solution or an equilibrium point must be stable to be physically meaningful.

Let a dynamical system be described by the differential equation in a Hilbert space $\dot{u} = \mathcal{F}(u, t)$, where $\mathcal{F}(\cdot, t)$ is an operator of the Hilbert space onto itself. Let u_0 be an equilibrium, that is a point u_0 of the Hilbert space such that $\mathcal{F}(u_0, t) = 0$. The stability of u_0, as is well known, can be investigated through the associated linear system $\dot{x} = \mathcal{L}x$, where $\mathcal{L}$ is the linear part of $\mathcal{F}$ at u_0, and more precisely through the spectral properties of $\mathcal{L}$ (at least in the autonomous case).

Many equilibria of physical systems are stabilized by effects that can be modeled as skew-symmetric contributions in the equations (and hence in their associated linear operator $\mathcal{L}$). Notable examples are the effect of rotation and of a solute field on the Bénard system [6, 11, 37]. It is hence reasonable to state that a physical system which can be modelled by a symmetric operator, is in some sense the least stable. When, as typically happens, the problem depends on parameters, the equilibria and their stability also depend on such parameters. Systems associated to symmetric operators will hence be those with smallest stability region (in parameter space).

The aim of this paper is to give a broader meaning to the observations above. In other words we investigate the relationship between the symmetry of the linear operator and the stability of an equilibrium in a general context (both ODE/PDEs, autonomous and non-autonomous systems, linear and non linear

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equations). This article is mainly a work of review, nevertheless, it contains new applications to the theory of PDEs, and to linear, non-autonomous systems.

In particular, we discuss in Section 2 the stability of ODE systems. We begin by recalling in Section 2.1 a classical result of Bendixson and Hirsch on the spectrum of the symmetric (or Hermitian) part of a matrix. This result implies that symmetric operators are those with least stability region. In Section 2.2 we illustrate the previous results by analyzing a simple dynamical system. In Section 2.3 we discuss the technique of energy stability and, by means of the construction of "optimal" Lyapunov functions, we investigate the coincidence of linear and nonlinear critical parameters.

Section 3 is devoted to PDE systems. After giving a list of important problems in which it is known that symmetry and stability are somehow related, in Section 3.1 we explicitly show that in the class of reaction-diffusion systems, those that are symmetric have the least stability properties. In Section 3.2 we analyze, under this point of view, Navier-Stokes equations.

In Section 4 we turn our attention to the much more difficult case of non-autonomous systems, underlining many open questions that still exist in this class of problems. In particular, we show that almost all results that are valid in the autonomous case *can not be easily extended* to non-autonomous systems, and that the literature contains results which cannot be applied to simple systems, and results in which the theory of autonomous systems is wrongly applied to non-autonomous ones.

2 Autonomous ODE systems

Consider the linear ODE system

$$\dot{\mathbf{x}} = A\,\mathbf{x}\,, \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, $(\cdot)^T$ denotes transposition, and A is a non-singular constant $n \times n$ matrix with real entries. Let us denote with $A^H = (A + A^T)/2$ and $A^S = (A - A^T)/2$ the symmetric and skew-symmetric parts of A. We want to study the stability of the equilibrium $\mathbf{x}_0 = (0, 0, \ldots, 0)^T$ of (1) and compare it to the stability of $\mathbf{x}_0$ for the systems $\dot{\mathbf{x}} = A^H\,\mathbf{x}$ and $\dot{\mathbf{x}} = A^S\,\mathbf{x}$.

2.1 The theorem of Bendixson-Hirsch

Let us begin by introducing the following

Definition 1 *We say that a matrix A is* stable *if all its eigenvalues have negative real part, it is* positively stable *if $-A$ is stable, it is* simply stable *if all its eigenvalues have non-positive real part.*

We denote by $\sigma_k = r_k + i\, s_k$, $k = 1, \ldots, n$, the eigenvalues of A, and order them by their decreasing real part r_k, so that $r_n \le \ldots \le r_1$. We also denote by λ_k, $k = 1, \ldots, n$, the (real) eigenvalues of A^H, ordered so that $\lambda_n \le \ldots \le \lambda_1$. We can then recall the following theorem, proved in 1900 by Bendixson [3]

Theorem 2 *The real parts of the eigenvalues of A are bounded by the minimum and maximum eigenvalues of its symmetric part A^H, i.e. for every $k = 1, \ldots, n$*

$$\lambda_n \le r_k \le \lambda_1.$$

This result was generalized to complex matrices by Hirsch [16] and then improved to more restrictive estimates on the region of the complex plane containing the eigenvalues (Bromwich [4], Browne [5], Bellman [2], Adam and Tsatsomeros [1]). Observe that stability of $\mathbf{x}_0$ is ensured by the condition $r_1 < 0$ for system (1), while stability of system associated to A^H is given by $\lambda_1 < 0$. From the above Theorem 2 it follows $r_1 \le \lambda_1$ and then the stability of A^H implies that of A (but not vice versa). A useful generalization of this result can be found in [18], pag. 96.

Theorem 3 *Let A a real matrix and Y a positive definite real matrix. Suppose A^H is (positively) stable, then YA is also (positively) stable.*

Assume now that the entries a_{ij} of A depend on a real parameter p, that is $a_{ij}(p) : S \to \mathbb{R}$, where S is a domain $S \subset \mathbb{R}$, which we assume for simplicity to be the same for all a_{ij}. We introduce then the following

Definition 4 *The* domain of stability *of $A(p)$ is the set $S_A \subset S$ such that $A(p)$ is stable for every $p \in S_A$.*

A similar definition holds for the stability domains of $A^H(p)$ and $A^S(p)$. The values of p at the boundaries of the domain of stability are the "critical values" p_c of p. From the above Theorem 2 then follows

Proposition 5 *The domain of stability of $A^H(p)$ is contained in the domain of stability of $A(p)$, that is $S_{A^H} \subseteq S_A$.*

We can state then that a matrix A is always "more stable" than its symmetric part. Conversely, if we consider a generic symmetric matrix B, we can always extend the stability domain of the zero solution of system $\dot{\mathbf{x}} = B\,\mathbf{x}$ by adding to B a purely skew-symmetric matrix. Note that the same definitions and proposition hold if we suppose that $a_{ij}(\mathbf{p}) : S \to \mathbb{R}$, where $\mathbf{p} = (p_1, ..., p_m) \in \mathbb{R}^m$ and $S \subset \mathbb{R}^m$. The boundary of S define in this case a "critical region" or *marginal region* in the parameter space.

We note also that the eigenvalues of the skew-symmetric matrix A^S can be either zero or appear in complex conjugate pairs. The zero solution of $\dot{\mathbf{x}} = A^S\,\mathbf{x}$ is then always simply stable (but not asymptotically stable) and the stability domain S_{A^S} of such matrix is the whole set S.

2.2 Example

To illustrate more explicitly the results of the previous section we consider the simple dynamical system

$$\begin{cases} \dot{x} = -2x + ay \\ \dot{y} = -x - 3y, \end{cases} \tag{2}$$

where a is a real number. In this case the matrix A and its symmetric and skew-symmetric parts are

$$A = \begin{pmatrix} -2 & a \\ -1 & -3 \end{pmatrix}, \quad A^H = \begin{pmatrix} -2 & \frac{a-1}{2} \\ \frac{a-1}{2} & -3 \end{pmatrix}, \quad A^S = \begin{pmatrix} 0 & \frac{a+1}{2} \\ -\frac{a+1}{2} & 0 \end{pmatrix}.$$

The eigenvalues σ_1, σ_2 of matrix A are $\sigma_{1,2} = (-5 \pm \sqrt{1-4a})/2$. Such eigenvalues are real numbers if $a \leq 1/4$ and complex conjugate if $a > 1/4$. Their real part is negative, and then the origin is asymptotically stable, if $a \in (-6, +\infty)$. The eigenvalues of A^H are $\sigma^H_{1,2} = (-5 \pm \sqrt{1 + (a-1)^2})/2$ and naturally, are always real (see Fig. 1 for a plot of the eigenvalues). The solution $\mathbf{x}_0$ of system $\dot{\mathbf{x}} = A^H\mathbf{x}$ is asymptotically stable if $a \in (1 - 2\sqrt{6}, 1 + 2\sqrt{6})$.

Finally, the zero solution of $\dot{\mathbf{x}} = A^S\mathbf{x}$ is stable for every a in $\mathbb{R}$. If S_{A^H}, S_A and S_{A^S} are the "stability domains" of A^H, A and A^S respectively, we have then, as expected, $S_{A^H} \subset S_A \subset S_{A^S}$.

2.3 Energy stability

In this section, we want to relate the observations of the previous section with the method of Lyapunov functions to study the stability of (1). We choose as energy function the Euclidean norm $E := \frac{1}{2}\|\mathbf{x}\|^2$. The derivative of E along the vector field (usually called orbital derivative) is then $\dot{E} = \mathbf{x}^T A^H \mathbf{x}$. Hence, only the symmetric part of A is relevant in the evaluation of $\dot{E}$. From elementary spectral theory we have $\mathbf{x}^T A^H \mathbf{x} \leq \lambda_1 \|\mathbf{x}\|^2$, where λ_1 is the maximum eigenvalue of A^H. It follows

Theorem 6 *If A is a symmetric matrix, the stability condition obtained with the eigenvalues method ($\lambda_1 < 0$) and Lyapunov stability with the Euclidean norm ($\dot{E} < 0$) coincide.*

This implies also that, if the p-parametric family of matrices A are all symmetric, the critical linear stability parameter p_c coincides with the Lyapunov critical one p_e. We introduce the following

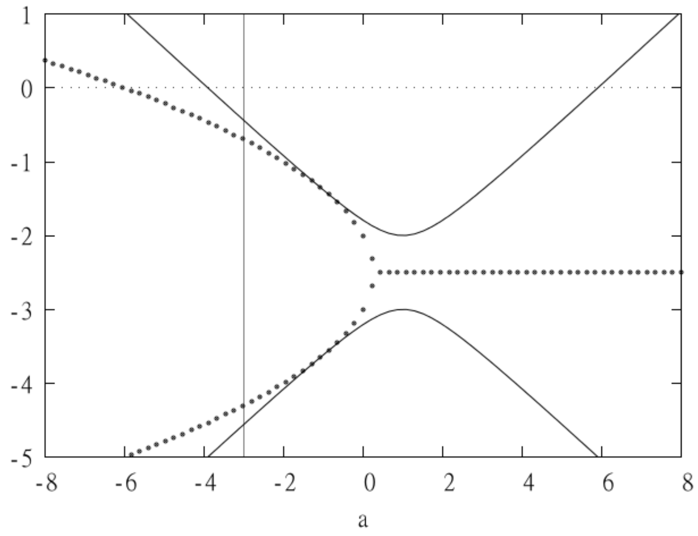

Figure 1: Eigenvalues of A^H (continuous line) and real part of eigenvalues of A (dotted line) from the sample system (2), as functions of parameter a. Note that for every choice of a (e.g. vertical line at $a = -3$ in the figure) the eigenvalues of A^H bound the eigenvalues of A both from below and above.

Definition 7 *A Lyapunov function is said to be* optimal *if it ensures stability when the system is (spectrally) stable.*

For a symmetric dynamical system like (1), Theorem 6 proves that the Euclidean norm is an optimal Lyapunov function.

We investigate now how it is possible to determine optimal Lyapunov functions in more general cases. We consider the *nonlinear* ODE system $\mathcal{S}_0$

$$\dot{\mathbf{x}} = A\mathbf{x} + f(\mathbf{x}) \tag{3}$$

where $f(\mathbf{x})$ is a C^1 function with $f(\mathbf{0}) = f'(\mathbf{0}) = \mathbf{0}$. In this case, if $\mathbf{x}_0$ is a hyperbolic point, the linearization principle (Hartman-Grobman theorem [17]) holds and, at least locally, stability of (3) is equivalent to stability of A.

In the general (non-symmetric) case the linear and nonlinear critical stability thresholds are not equal if we use as energy the Euclidean norm E, so a new Lyapunov function must be introduced. For this, we recall here *the classical reduction method.* To explain this method, we find it useful to recall the following result [18]

Theorem 8 *A matrix A is stable if and only if there is a positive definite symmetric matrix G such that $GA + A^T G$ is negative definite.*

The matrix G can be though of as a quadratic bilinear form whose associated norm $E = \mathbf{x}^T G\mathbf{x}/2$ plays the role of Lyapunov function, $GA + A^T G$ turns out to be the quadratic part of the orbital derivative of E. When it is negative definite, then the norm associated to G is a good Lyapunov function.

The proof of the above theorem makes it clear that the existence of such a matrix G relies on the existence of a *generalized canonical Jordan form*, whose Jordan blocks have entries above the diagonal not 1, but small enough (smaller than half of the real part of the corresponding eigenvalues). Let Q be the non singular matrix that conjugates the matrix A in system (3) to its generalized canonical Jordan form. We can introduce the new field variables $\mathbf{y}$ so that $\mathbf{x} = Q\,\mathbf{y}$, and obtain the new (topologically equivalent) system $\mathcal{S}_1$

$$\dot{\mathbf{y}} = B\mathbf{y} + Q^{-1} f(Q\mathbf{y}),$$

where $B = Q^{-1}AQ$ is a matrix similar to A. (As it is well known, similar operators define differential equations that have the same dynamical properties, [17], pag. 39). We can hence define $F := \frac{1}{2}\|\mathbf{y}\|^2$, which turns out to be an optimal Lyapunov function. This function allows to reach the critical linear and (local) nonlinear stability thresholds, and allows to obtain a lower estimate for the basin of attraction of the equilibrium. We observe that system $\mathcal{S}_1$ is in a canonical form (generalized canonical Jordan form) and the Euclidean norm is an optimal Lyapunov function for the coincidence of the critical linear and nonlinear stability thresholds.

3 Bendixson-Hirsch theorem for PDE

The results we just recalled for ordinary differential equations, hold also in a large class of evolution partial differential equations. An evolution PDE is a differential equation of the form $u_t = \mathcal{F}(u)$, where $u : \mathbb{R}_t \times \mathbb{R}^N_x \to \mathbb{R}^n$ belongs to an appropriate Hilbert space $\mathcal{H}$, and $\mathcal{F}$ is a differential operator; an equilibrium of an evolution PDE is a function $u_0(x)$ such that $\mathcal{F}(u_0) = 0$. Denoting $\mathcal{L}$ the linear part of $\mathcal{F}$ at u_0, the adjoint of the operator $\mathcal{L}$ is the operator $\mathcal{L}^*$ such that $v \cdot \mathcal{L}(u) = \mathcal{L}^*(v) \cdot u$ for every $u, v \in \mathcal{H}$, where the dot indicates the scalar product. Consistently with the definitions above, $\mathcal{L}^H = \frac{1}{2}(\mathcal{L} + \mathcal{L}^*)$. We plan to discuss a physically relevant class of evolution PDEs for which the *equivalent* of Theorem 2 holds. The list of evolution PDE systems in which a Bendixson-Hirsch theorem holds is long. For citing a few:

- Navier-Stokes equations [20]: In the symmetric case (when u_0 is the rest state) an optimal Lyapunov function for nonlinear stability is the L^2-energy, see [37, 19, 33, 14]. This gives a lower estimate of the stability domain.
- Bénard problems: in the basic case, the associated linear operator is symmetric. To enlarge the stability region one has to add some skew-symmetric linear operator like a rotation field, a magnetic field, a solute concentration field, see [6, 15, 35, 28].
- Flows in porous media both with Darcy and Brinkman models: in the symmetrized case the optimal Lyapunov function for nonlinear stability is once again the L_2 norm in presence of an inertial term and the L_2 norm of the perturbation temperature when the inertial term can be neglected. We remark that in the symmetric case there is no need of "eccentricity" in the norm, i.e. the best coupling (Lyapunov) parameters are all equal to 1, see [37]. This is a general results (ODEs, NS, flows in porous media, RD systems).

3.1 Reaction diffusion systems

A reaction-diffusion system is the equation $U_t = D\Delta U + X(U)$, where $U : \mathbb{R}_t \times \mathbb{R}^N_x \to \mathbb{R}^n$ is a vector with n components, D is a real $n \times n$ matrix (the diffusion coefficients), and X is a vector field (the reaction kinetics). Assuming Dirichlet boundary conditions, the problem is well posed in the space $\mathcal{H}$ of the direct product of n Sobolev spaces $W^{2,2}_0(\Omega, \mathbb{R})$, where the zero indicates that the functions vanish at the boundary of a domain Ω. In this space, and in all function spaces we use, the scalar product will be denoted $U \cdot V = \int_\Omega U\overline{V}$, and the norm will be denoted $\|U\|^2 = U \cdot U$, where the bar indicates complex conjugation when the functions are complex valued. Assuming there exists a constant solution (a $\widetilde{U} \in \mathbb{R}^n$ such that $X(\widetilde{U}) = 0$), the linearized perturbation equations of such system are

$$u_t = (D\Delta + L)u := \mathcal{L}u, \tag{4}$$

where $L = (\partial_{x_i} X_j(\widetilde{U}))_{i,j}$ is the Jacobian of the vector field X, an $n \times n$ matrix with constant coefficients, and u is the perturbation to the equilibrium solution.

Being this system linear and autonomous, one can compute its spectrum by substituting to u all possible complex-valued functions $u_\sigma(t, x) = e^{\sigma t}u_\sigma(x)$. If one such function satisfies equations (4), the complex number $\sigma \in \mathbb{C}$ and the function u_σ are called eigenvalue and eigenfunction respectively (with a slight abuse of notation, we give the same name u_σ to two different functions).

Definition 9 *Let Σ be the set of complex numbers σ which are eigenvalues of the linear system* (4)*. The stationary solution $\widetilde{U}$ is said to be* linearly asymptotically stable *if and only if there exists $k < 0$ such that $\Re(\sigma) \leq k$ for every $\sigma \in \Sigma$; $\widetilde{U}$ is* linearly unstable *if and only if there exists $\sigma^\star \in \Sigma$ such that $\Re(\sigma^\star) > 0$.*

The stationary solution $\widetilde{U}$ is stable *if and only if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that every time $U_0 \in \mathcal{H}$ has $\|U_0\| < \delta(\varepsilon)$ it follows that $\|\Phi(t, U_0) - \widetilde{U}\| < \varepsilon$ for every $t \in (0, +\infty)$, where $\Phi(t, U_0)$ indicates the solution to equation* (4) *with initial condition $U_0(x)$. It is* unstable *if and only if it is not stable.*

The stationary solution $\widetilde{U}$ is asymptotically stable *if and only if it is stable and there exists a $\gamma \in (0, +\infty]$ such that whenever $\|U_0\| < \gamma$ then $\lim_{t\to\infty} \|\Phi(t, U_0) - \widetilde{U}\| = 0$. If $\gamma < +\infty$, then $\widetilde{U}$ is said to be* conditionally asymptotically stable*, if $\gamma = \infty$, then $\widetilde{U}$ is said to be* unconditionally (or globally) asymptotically stable*.*

When the matrices D and L depend on parameters, a natural question to investigate is to determine the *stability domain*, that is the subspace of parameters space in which the equilibrium solution is asymptotically stable.

Theorem 10 *The stability domain of the symmetric system $u_t = \mathcal{L}^H u$ is contained in the stability domain of system $u_t = \mathcal{L}\, u$.*

The proof of this theorem relies on the following lemma

Lemma 11 *Assume the matrices D, L depend on parameters and are symmetric, then the Euclidean energy $E = \|u\|^2/2$ is an optimal Lyapunov function for system (4).*

Proof The function E is of course positive definite. Its orbital derivative is

$$\dot{E} = \dot{u} \cdot u = (D\Delta u) \cdot u + (Lu) \cdot u = -(D\nabla u) \cdot \nabla u + (Lu) \cdot u. \tag{5}$$

The derivative of E is a quadratic form on the space $\mathcal{H}$, and it possibly changes signature only when the parameters are chosen so that there exists a non-zero function u_0 such that $\dot{E}(u_0) = 0$ (and hence $\dot{E}(\lambda u_0) = 0$ for every $\lambda \in \mathbb{R}$).

Consider now the spectral problem. Denoting by $u_\sigma(t, x)$ an eigenfunction, and taking the L^2 scalar product of $\mathcal{L}u_\sigma$ with $\bar{u}_\sigma$, we obtain that $\sigma\|u_\sigma\|^2 = -(D\nabla u_\sigma) \cdot \nabla \bar{u}_\sigma + (Lu_\sigma) \cdot \bar{u}_\sigma$.

The system is at marginality, i.e. the parameters are such that the spectrum of the system contains purely imaginary eigenvalues, when $\sigma = i\tau$ is purely imaginary. In this case, letting $u_\sigma = v + iw$, the equation above can be split into real and imaginary parts, and gives the system of two equations

$$\begin{cases} 0 = -(D\nabla v) \cdot \nabla v + (Lv) \cdot v - (D\nabla w) \cdot \nabla w + (Lw) \cdot w \\ \tau = (D\nabla v) \cdot \nabla w - (D\nabla w) \cdot \nabla v - (Lv) \cdot w + (Lw) \cdot v. \end{cases}$$

The symmetry of D and L imply that $\tau = 0$, which implies that $\mathcal{L}(v + iw) = 0$. By linearity of the operator, this is equivalent to $\mathcal{L}v = \mathcal{L}w = 0$. It follows that there is a non-zero real-valued function u_0 such that $0 = u_0 \cdot (\mathcal{L}u_0) = -(D\nabla u_0) \cdot \nabla u_0 + (Lu_0) \cdot u_0 = \dot{E}(u_0)$.

These facts together imply that the system is at marginality precisely when the function $\dot{E}$ changes signature. This proves that, under the hypotheses that D, L are symmetic, E is an optimal Lyapunov function for system (4). □

To prove Theorem 10, it is enough to observe that the orbital derivative of the Lyapunov function E, when the evolution equation (4) has a generic operator $\mathcal{L}$, is the same of that with operator $\mathcal{L}^H$. It follows that the stability domain of the generic system contains the stability domain of the associated symmetric system.

When $\mathcal{L}$ is not linear nor symmetric, then an optimal Lyapunov function that proves the coincidence of linear and nonlinear critical numbers (i.e. marginal parameters) can be defined following the ideas that were laid in Theorem 8 for the ODE case. In [27, 26, 23] is given an operative method to build an optimal Lyapunov function for specific problems (an alternative method for a nonlinear binary reaction-diffusion system is given in [34]). The idea in those papers is the following: assume to be given the reaction-diffusion system

$$U_t = D\Delta U + LU + N_i(U). \tag{6}$$

In the appropriate Hilbert space one can decompose the problem in the eigenspaces of the Laplacian, with eigenvalues ξ, and reduce to a problem of the form $A_\xi = -\xi D + L$. Once identified $\tilde{\xi}$ the principal eigenvalue of the matrices (i.e. the eigenvalue corresponding to the critical linearized instability parameter), one can use the transformation Q that puts matrix $A_{\tilde{\xi}}$ in its generalized canonical Jordan form of Theorem 8, and define new field variables $V = Q^{-1}U$ and an equivalent reaction-diffusion system (see [29], pag. 53). In these coordinates, the Euclidean norm $E_1 = \frac{1}{2}\|V\|^2$ can be used to investigate the linear stability of the zero solution.

3.2 Navier-Stokes equations

The equations of Navier-Stokes are a particular case of the reaction-diffusion equations described above. Let $m_0 = (\mathbf{w}(\mathbf{x}), p(\mathbf{x}))$ be a stationary flow of a viscous incompressible fluid, solution of Navier-Stokes equations, and let $(\mathbf{u}(\mathbf{x}, t), \pi(\mathbf{x}, t))$ be a perturbation to m_0. Such perturbation must satisfy the following initial boundary value problem (IBVP)

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{w}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{w} = -\nabla\pi + \nu\Delta\mathbf{u} & \text{in } \Omega\times(0,\infty) \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,\infty) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) & \text{on } \Omega \\ \mathbf{u}(\mathbf{x},t) = 0 & \text{on } \partial\Omega\times[0,\infty), \end{cases} \tag{7}$$

where Ω is a bounded domain of $\mathbb{R}^3$. Assume that $\|\mathbf{u}\|$ and $\|\nabla\mathbf{u}\|$ are so small that we can neglect the nonlinear term $\mathbf{u}\cdot\nabla\mathbf{u}$ in (7). The resulting system is linear and autonomous, therefore we may look for solutions of the form $\mathbf{u}(\mathbf{x}, t) = e^{-\sigma t}\mathbf{q}(\mathbf{x}), \pi(\mathbf{x}, t) = e^{-\sigma t}\pi_0(\mathbf{x})$, with σ a complex number. Substituting in the linearization of (7), we have that

$$\begin{cases} -\nu\Delta\mathbf{q} + \mathbf{w}\cdot\nabla\mathbf{q} + \mathbf{q}\cdot\nabla\mathbf{w} + \nabla\pi_0 = \sigma\mathbf{q} & \text{in } \Omega \\ \nabla\cdot\mathbf{q} = 0 & \text{in } \Omega \\ \mathbf{q}(\mathbf{x}) = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

This problem is an eigenvalue problem just as the one described above. From the results [32, 36] it immediately follows

Theorem 12 *Let Ω be a bounded domain of $\mathbb{R}^3$, $\partial\Omega \in C^2$, and let $\mathbf{w} \in C^1(\bar{\Omega})$. The eigenvalue problem (8), set up in a suitable subspace $\mathcal{H}$ of a space of Sobolev functions, admits a discrete set of eigenvalues $\Sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ in the complex plane, each of finite multiplicity, which can cluster only at infinity. The eigenvalues lie in the parabolic region $c_1[\Im(\sigma)]^2 = \Re(\sigma) + c_2$, where c_1 and c_2 are some fixed positive constants. Moreover, the eigenvalues may be ordered so that*

$$re(\sigma_1) \le re(\sigma_2) \le re(\sigma_3) \le \ldots \le re(\sigma_n) \le \ldots.$$

The corresponding set of eigenfunctions $\{\mathbf{q}_{\sigma_n}\}_{n\in\mathbb{N}}$ is complete in $\mathcal{H}$, that is, the set of finite linear combinations of the eigenfunctions is dense in $\mathcal{H}$.

Let us denote $\mathcal{L}_1(\mathbf{q}) = -\nu\Delta\mathbf{q} + \mathbf{q}\cdot\mathbf{D}$, $\mathcal{L}_2(\mathbf{q}) = \mathbf{q}\cdot\mathbf{\Omega} + \mathbf{w}\cdot\nabla\mathbf{q}$, and $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ the operators on $\mathcal{H}$ into itself (cf. [32, 36, 13]), where $\mathbf{D}$ is the stretching tensor and $\mathbf{\Omega}$ is the spin tensor. Since $\nabla\mathbf{w} = \mathbf{D} + \mathbf{\Omega}$, it follows that the first line of equations (8) can be written in the form $\mathcal{L}(\mathbf{q}) + \nabla\pi_0 = \sigma\mathbf{q}$. It is also easy to prove that $\mathcal{L}_1$ is the symmetric and $\mathcal{L}_2$ is the skew-symmetric part of $\mathcal{L}$. We can hence consider the symmetric linear problem associated to (8)

$$\begin{cases} \mathcal{L}_1(\mathbf{q}) + \nabla\pi_0 = \mu\mathbf{q} & \text{in } \Omega \\ \nabla\cdot\mathbf{q} = 0 & \text{in } \Omega \\ \mathbf{q}(\mathbf{x}) = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Theorem 13 *Under the hypotheses of the previous theorem, and under the assumption that the function* $\mathbf{q}$ *belongs to a suitable subset of* $\mathcal{H}$ *of a space of Sobolev function in* Ω*, system (9) admits a countable number of real eigenvalues* $\{\mu_n\}_{n\in\mathbb{N}}$ *of finite multiplicity which can cluster only at infinity.*

Naturally, we can order the eigenvalues $\{\mu_n\}_{n\in\mathbb{N}}$ as usual so that $\mu_1 \le \mu_2 \le \mu_3 \le \ldots \le \mu_n \le \ldots$ and obtain the following

Theorem 14 (Bendixson-Hirsch) *Given the Navier-Stokes equations* (7)*, and with the notations above, one has that* $\Re(\sigma_1) \ge \mu_1$*.*

This proves that the symmetrized system has stability domain contained in that of the original system.

4 Non-autonomous systems

A general non-autonomous system is a system of the type $\dot{x} = F(x,t)$ with F at least C^2. The system has a stationary equilibrium solution if there exists a point x_0 such that $F(x_0,t) = 0$. Under this hypothesis, after a translation one can assume that $x_0 = 0$, and that $F(x,t) = A(t)x + N(x,t)$ with $\|N(x,t)\| = O(\|x\|^2)$. In the autonomous case, under the hypothesis that the matrix A has no purely imaginary eigenvalues, it can be proved that the system is topologically equivalent to the system $\dot{x} = Ax$ (theorem of Grobman-Hartman). The same result is true in the non-autonomous case under strict hypotheses on $A(t)$ (existence of an exponential dichotomy) and of $N(x,t)$ (Lipschitzianity and boundedness) [30] or under less restrictive hypotheses on N but much stronger on $A(t)$ (uniform asymptotic stability) [21]. Under such assumptions, it becomes reasonable to deal with the linear non-autonomous case, that is the system of equations

$$\dot{x} = A(t)\,x \tag{10}$$

with $x \in \mathbb{R}^n$ and A a time-dependent $n \times n$ real matrix.

Even in such simplified system, the investigation of the stability of the origin is a complicate problem, and it can be shown that the symmetric system associated to A has no relation with the stability of the original system. Example (13) below, provides a non-autonomous system depending on two parameters which is unstable for some values of the parameters. On the other hand, the associated symmetric system is stable for every value of the parameters. This contradicts the results described in Proposition 5 for autonomous systems.

When dealing with non-autonomous systems, a reasonable class of changes of coordinates is the *Lyapunov transformation*, introduced by Lyapunov in [22]. Lyapunov transformations consist in all the changes of coordinates of the form $y = Q(t)\,x$, with $Q(t)$ a non-degenerate linear transformation of $\mathbb{R}^n$ in itself, smoothly depending on $t \in \mathbb{R}$, that satisfies $\sup_t \Big(||Q|| + ||Q^{-1}|| + ||\dot{Q}||\Big) < \infty$ (sometimes, condition on $||\dot{Q}||$ is discarded). In these new coordinates the system becomes

$$\dot{y} = \left(\dot{Q}(t)Q(t)^{-1} + Q(t)A(t)Q(t)^{-1}\right) y = B(t)\,y \tag{11}$$

Definition 15 *Two time-dependent matrices* $A(t), B(t)$ *are called* related *if there exists a differentiable time-dependent matrix* $Q(t)$ *bounded, invertible, with bounded inverse such that (11) holds.*

Naturally, Lyapunov transformations do not change the stability character (simple, asymptotic, uniform or not) of the origin. With such changes of coordinates, and even with the much stronger assumption that $Q(t) \in SO(n)$ (i.e. $Q(t)Q(t)^T = \mathbb{I}$, the identity matrix) one can prove that

Proposition 16 *Every matrix* $A(t)$ *can be related to a symmetric one.*

Proof In dimension 2 the result is achieved by imposing that $\dot{\gamma}J + R_\gamma A R_{-\gamma}$ be symmetric, with $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $R_\gamma = \left(\begin{smallmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{smallmatrix}\right)$. Denoting $a(t), ..., d(t)$ the four entries of the matrix A, a solution to the problem is any $\gamma(t)$ solution of the equation $\gamma'(t) = \frac{1}{2}(b(t) - c(t))$ (difference of the antidiagonal terms).

In general dimension such result can be achieved solving the differential Lie-algebra equation of Euler type $\dot{Q} = -Q\, A^S(t)$. By existence and uniqueness of solutions to Cauchy problems (in $M(n)_Q \times \mathbb{R}_t = \mathbb{R}^{n\times n} \times \mathbb{R}$), this equation admits a unique solution such that $Q(0) = \mathbb{I}$. Since at every time $-Q\, A^S(t)$ is tangent to the Lie group $SO(n) \subset M(n)$, it follows that the solution belongs to $SO(n)$. □

A particular family of time-dependent linear systems are those called *reducible.* They are the systems of type (10) that are related to a system with constant coefficients $\dot{y} = By$ by a change of variables $y = Q(t)x$. Floquet theory indicates that a large class of non-autonomous systems are reducible.

Theorem 17 (Lyapunov's theorem) *If the mapping $A(t)$ is continuous and periodically depends on t, then the system is a reducible system.*

This theorem is highly ineffective, it does not imply that the conjugation $Q(t)$ and the constant matrix B can be easily deduced starting from the matrix $A(t)$. As a matter of fact, one of the first example in the literature can be found in Markus and Yamabe [24], who provide a 2-degrees of freedom system that at every given time t is stable (both eigenvalues have real part $-1/4$) but it turns out to be unstable. The system of Markus and Yamabe is

$$\begin{cases} \dot{x} = (-1 + \frac{3}{2}\cos^2 t)x + (1 - \frac{3}{4}\sin 2t)y \\ \dot{y} = -(1 + \frac{3}{4}\sin 2t)x + (-1 + \frac{3}{2}\sin^2 t)y. \end{cases} \tag{12}$$

It does satisfy uniform Routh-Hurwitz conditions for every given time t, but this does not imply stability of the origin. This system has in fact $\mathrm{tr}(A) = -1/2$ and $\det(A) = 1/2$ for all t but, after a simple analysis, it can be shown to be related, using the one-parameter family of counterclockwise uniform rotations, to the constant matrix $\mathrm{diag}(1/2, -1)$. This implies the divergence to infinity of almost every solution (only initial data in the stable manifold will not diverge).

Spectral properties of the matrix $A(t)$ have very indecisive relations with the stability of the origin. The only simple fact that can be proven is

Proposition 18 *If $\sup_{t\geq 0} \int_0^t tr(A(s))ds = +\infty$ then the origin is unstable. If the origin is stable then $\int_0^t \mathrm{tr}(A(s))ds$ is bounded.*

Proof This fact follows straightforwardly from the fact that the Wronskian of the system, that is the matrix $W(t)$ satisfying $\dot{W} = AW$ and $W(0) = 1$, has determinant that evolves according to the rule $(\det W)^{\cdot} = e^{\int \mathrm{tr}\, A}$, and hence if $\int \mathrm{tr}\, A$ is not bounded, then some entries of $W(t)$ must be unbounded. □

It is simple to generate families depending on parameters which are spectrally stable for every t, but that are stable or unstable in some range of the parameter.

Observation 19 *Every constant matrix with negative trace can, with a uniform rotation, be related to a time-dependent matrix which is spectrally stable for every given time but that has an unstable equilibrium at the origin.*

This is a straightforward computation in dimension 2, where in fact given a diagonal matrix $A = \mathrm{diag}(a, b)$, and changing coordinates using a uniform, counterclockwise rotation $R_{\omega t}$, one obtains the matrix $\omega J + R_{\omega t} A R_{-\omega t}$, which has the same trace, but determinant $ab + \omega^2$. It follows that, when the matrix A has negative trace but is spectrally unstable (i.e. when $\det A = ab$ is negative), then changing coordinates to a rotating frame will relate the system to one which is spectrally stable (a stable node if ω is big enough and a stable spiral point if ω is even bigger).

Let us now consider the system depending on two parameters a, ε

$$\begin{cases} \dot{x} = (-1 + a\cos t)x + \epsilon y \\ \dot{y} = -\epsilon x - (1 + a\cos t)y. \end{cases} \tag{13}$$

This system has a peculiar dependence on parameters: it has trace -2 and determinant $1+\varepsilon^2 - a^2\cos^2 t$, it has two negative eigenvalues when $a < \sqrt{1+\varepsilon^2}$ and one positive and one negative eigenvalues in some interval of time in all other cases. Using Floquet theory [12] one can numerically show that the origin is stable when ε belongs to some intervals increasing in width and in number with a.

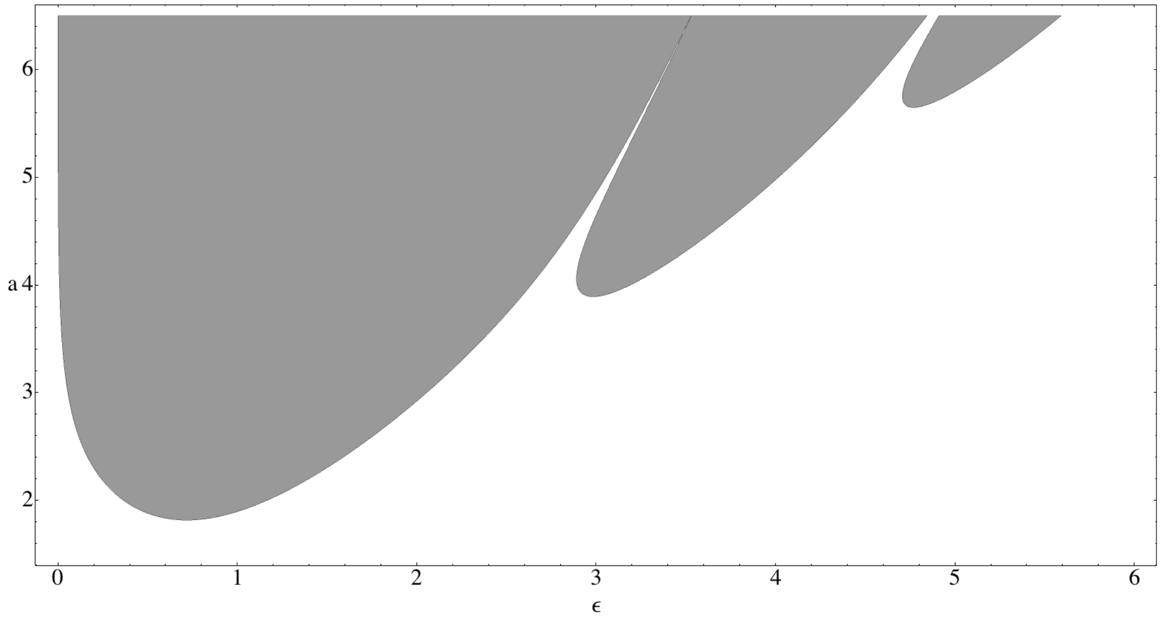

Figure 2: The region in parameter space ε, a in which systems (13) has a stable equilibrium at the origin (white) and in which the origin is unstable (gray).

The shaded region in Fig. 2 is the region in a, ε parameter space where the corresponding system has the origin as asymptotically stable equilibrium. When a is fixed and big enough, the system depends on the parameter ε and the origin is stable in a finite number of intervals, and is unstable elsewhere. In this particular system, the associated symmetric system is $\dot{x} = (-1 + a\cos t)x, \dot{y} = -(1 + a\cos t)y$, its solutions are $x = c_1 \exp(-t + a\sin t)$, $y = c_2 \exp(-t - a\sin t)$, and $(0, 0)$ is a stable equilibrium. In this case the stability region has not shrunk.

A very promising canonical form under time-dependent conjugation is that proposed in [10], where it is shown that every non-autonomous system is related to a system with upper triangular matrix. This conjugation is particularly important because when the matrix is upper triangular, the system can be solved using the "principle of variation of constants". It is simple to observe the following fact

Observation 20 *Since every initial condition can be conjugate by a Lyapunov transformation into an initial condition in the x-axis, the asymptotic behavior of the solution is determined by the asymptotic behavior of the $1,1$ entry of the upper-diagonal matrix.*

The time-dependent orthogonal matrices that relate a matrix to an upper triangular one not only are difficult to determine, but finding them is equivalent to solving Riccati equations whose solution are not known. Consider in fact the 2-dimensional case, one needs to determine a matrix $R_{\gamma(t)}$ such that $\dot{\gamma}J + RAR^{-1}$ is upper triangular. Posing $u = \tan\gamma$, this is equivalent to the Riccati equation $\dot{u} = b(t)u^2 - (a(t) - d(t))u - c(t)$. In view of the observation above, the origin is stable if and only if the integral of

$$\frac{d(t)u(t)^2 - (b(t) + c(t))u(t) + a(t)}{1 + u(t)^2}$$

is bounded for every solution $u(t)$ of the above Riccati equation; it is asymptotically stable if and only if such integral tends to $-\infty$ when t tends to infinity.

To our knowledge, there are no exact solutions to the Riccati equation associated, for example, to system (13) [31]. On the other hand, the Riccati equation associated to Markus-Yamabe system is $u' = (1 - \frac{3}{4}\sin t)u^2 - \frac{3}{2}\cos(2\,t)u + (1 + \frac{3}{4}\sin(2\,t))$. A solution to this equation is $u = -\cot t$, from which it follows that $\gamma(t) = -\arctan(\cot t) = -t + \pi/2$. Hence, rotation $R_{-t+\pi/2}$ conjugates the system to upper triangular form (actually a diagonal constant matrix, as observed above).

4.1 The literature

Also in the engineering community the theory of non-autonomous systems is relevant, for example in problems of stability of structures subject to external time-dependent forces. The literature abounds of necessary and/or sufficient conditions for stability for particular classes of non-autonomous systems. In most cases such conditions are of very narrow applicability, in some other cases they are incorrect.

Christensen [7, 8, 9] for example, plans to use Theorem 3 in the following way: given a non-degenerate A, he defines $B_\sigma = 1 + \sigma A^{-1}$, and then defines $D_\sigma = A^T A B_\sigma$. He then makes the hypothesis that $D_\sigma + D_\sigma^T = 2A^T A + \sigma(A + A^T)$ is positive definite to deduce that $B_\sigma = A^{-1}A^{-T}D_\sigma$ is positively stable. He then claims that if B_σ is positively stable for a positive σ then A is stable.

This fact is false. In fact, the eigenvalues of B_σ are the complex numbers λ such that $\det(\mathbb{I} + \sigma A^{-1} - \lambda\mathbb{I}) = 0$, that is $\det((1-\lambda)A + \sigma\mathbb{I}) = 0$. It follows that $\sigma = (\lambda - 1)\mu$ with λ eigenvalue of B_σ and μ eigenvalue of A. Denoting λ_i the eigenvalues of B_σ, the eigenvalues of A are the complex numbers $\sigma/(1-\lambda_i)$. Viceversa, denoting μ_i the eigenvalues of A, the eigenvalues of B_σ are the complex numbers $1 - \sigma/\mu_i$. It is true that if all the μ_i are negative and σ is positive then all the λ_i are positive, but it is not true what we need, i.e. if B_σ is positively stable then A is stable. It is extremely simple to build counterexamples, with 2×2 randomly generated matrices.

All further deductions in [7, 8, 9] are hence false and, apart from the integral condition on the trace, finding conditions on symmetric polynomials associated to $A(t)$ that imply stability of the origin appears of very dubious success.

Other interesting results [25], which use a measure called Lozinskiì measure, are not applicable even to the easiest example we discussed, such as system (13). The main theorem of [25] (3.1, pag. 105) states that stability follows from the convergence to zero of $\int_0^T \mu(A(t))dt$, where $\mu(A(t))$ can be either the maximum eigenvalue, the maximum among the functions $m_{ii} + \sum_{j\neq i}|m_{ij}|$ or the maximum among the functions $m_{jj} + \sum_{i\neq j}|m_{ij}|$. In all cases the dependence on ε is lost.

5 Conclusions

Very often, even in the basic theory of Lyapunov methods, one is bound to observe that the symmetric part of a differential operator plays a substantial role in the stability of an equilibrium. In this work we tried to present this phenomenon in a variety of cases, starting from ODE, through PDE, to non-autonomous ODE. We tried to shed some light on the role that self-adjoint operators play in stability theory in all these environments.

When approaching the question of stability, we have shown that evolution problems with self-adjoint reaction kinetics are those with smaller stability region but easier to treat from the viewpoint of Lyapunov functions (the norm always provides an optimal Lyapunov function). The condition of being self-adjoined depends on the choice of a scalar product, different scalar products will provide different Lyapunov "norms", and a different class of self-adjoint operators. This observation is equivalent to the fact that optimal Lyapunov functions can be obtained both in ODEs and PDEs using the technique of canonical reduction. One choses a reference frame with respect to which the linear part of the operator is almost a self-adjoint operator, one can assume such frame to be orthonormal. The associated norm can be proven to be an optimal Lyapunov function for the system.

In the non-autonomous case, things are much more difficult. It is relatively simple to create examples in which the uniform Routh-Hurwitz conditions are satisfied but the origin is unstable (see Observation 19). It is also simple to give examples of systems for which, at every given time, some eigenvalues have positive real part, but whose origin is stable. To obtain results one needs to resort to stronger principles, such as that of dichotomies [30], or to define a computable class of "normal forms" (such as Diliberto normal form) on which one can impose conditions equivalent to stability of the origin.

A natural extension of our discussion on non-autonomous ODEs should include non-autonomous PDEs, in particular reaction-diffusion equations with time-dependent coefficients, or fluid-dynamics equations with non-constant boundary conditions. This is a vast subject of research, very partially explored, which we plan to investigate in the future.

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