A general model for the marketing of seasonal products *

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Abstract We discuss a general model for the marketing of seasonal products, namely products for which the time intervals devoted to production and sales are distinct. The firm can advertise the product, thus affecting the sales in two different ways, namely directly (customers effect) and indirectly (retailers effect). A two-dimensional *goodwill* variable represents the (retailers, customers)-advertising capital stock. The dynamics of the system in the sales interval only is analysed, in order to determine the levels of goodwill and inventory at the beginning of such time interval and the subsequent advertising policy, so as to maximize the discounted net profit.

Key words Optimal control, nonlinear programming, advertising, seasonal products, goodwill.

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Contents

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1 Introduction

The use of quantitative models for the firm marketing has received a substantial momentum since the end of the fifties, when the variety of admissible marketing actions has made the decision processes more complex. From the beginning the research has followed two main streams, as reported in [15]:

i) the econometric models which are built on a given set of observations and reflect heavily the features of the special observation context;
ii) the *a priori* models, which are more based on a study of the general features of the phenomenon, rather than on special data sets.

The present paper belongs to the latter approach and is particularly related with the models which involve the theories of Optimal Control (see for instance [20]) and Non-linear Programming (see [16]).

We address some management issues which concern the advertising effects on sales and lead to the problems of determining optimal policies both for the production and the advertising processes. The relevant scientific literature (see for instance [13], [15], [17], [21], [24]) testifies that the marketing applications of the optimal control theory are fascinating and promising, albeit not developed enough.

In this context, the marketing models for *seasonal products* represent an original topic. We call *seasonal* those products whose production and sale processes are active in two distinct and consecutive time intervals, which constitute the whole production-sale cycle of the product. Moreover, the production and the sale intervals are exogenously defined; for instance, they depend on the seasons sequence in the year for certain agricultural products, outdoor sports equipment, fashion clothing, ... or on special holidays for traditional foods, certain gift goods, ... In these examples the seasonality period is one year. Therefore, in particular, a seasonal product does not admit a "just-in-time" production organization (see for instance [14]), but, on the contrary, it needs that a production plan is determined well before demand is revealed, because of a fixed and short demand and sale interval and because of a production cycle which is both comparatively long and exogenously given. In the papers [7], [8], [9], [10], [11] and [12], the study of special production and advertising problems of seasonal products is presented on a theoretical level.

Here we propose a general model for the marketing of seasonal products in which

- no further restrictive assumption is taken for the production process;
- the firm advertising policy is assumed to affect the sale process indirectly, through the *goodwill*, a two-dimensional variable which represents how favorably the sellers keep that product for sale (sellers' goodwill) and how easily the consumers choose to buy it (consumers' goodwill);

– the firm is assumed to be able at differentiating the advertising policy according to the target being the sellers or the consumers, respectively.

We formulate an optimization problem over a single period, namely the time interval for one production-sale cycle. Some of the decision variables are real variables and some others are time functions (controls). We prove that the original problem is equivalent to a nonlinear programming problem, whose discussion depends on the solution of a parametric optimal control problem. Some of the problems studied in [7], [8], [9], [10] and [12] are special cases of such optimization problem.

We study the conditions for the existence of an optimal solution to the general problem and propose a method to determine it.

2 Literature on optimal control and marketing

The first significant contributions to the dynamic modelling of the advertising processes and to determining an optimal advertising policy are constituted by the models in [24] and [17]. They are still an important reference of the research on the subject.

First in [24] a mathematical model of sales response to advertising is provided, based on three parameters: Response Constant, Saturation Level and Sales Decay Constant. The Response Constant is the sales generated per advertising dollar when the rate of sales is zero. The Saturation Level is a practical limit of sales that can be generated and it depends both on the product being promoted and on the advertising medium used. The Sales Decay Constant is the sales decay rate in absence of promotion and it accounts for the product obsolescence and competing advertising. A knowledge of sales response to advertising for each product permits one to evaluate the return that can be expected from capital invested in advertising for each product. The Authors say that with this information it is possible to select profitable advertising programs and to estimate the optimum total size of the advertising budget.

Then, in [17] the advertising is considered as a kind of investment: the advertising expenditure affects the present and future demand for the product and, hence, the net revenue of the firm which advertises. The Authors introduce for the first time the concept of *goodwill*, which they define as a variable which "summarizes the effects of current and past advertising outlays on demand". A linear differential equation is chosen to describe the goodwill evolution. The Authors want to plan the advertising activity in the time interval $[0, +\infty)$, with the objective of maximizing the discounted profit obtained from selling the product, net of production and advertising costs. For the first time the problem of determining the advertising expenditure over time is formulated as an optimal control problem. It is an infinite horizon problem with two decision variables: the advertising expenditure rate and the price.

Some years later, in [15] the Nerlove-Arrow's model [17] is reexamined with focus on the special situation in which the advertising expenditure rate is of the "bang-bang" type:

$$a(t) = \begin{cases} a, & t \in [0, T], \\ 0, & t \in (T, +\infty). \end{cases} \tag{1}$$

The Author claims that the absence of a theory which organizes the knowledge concerning the advertising phenomenon causes a heavy waste of resources and observes, on the basis of some experimental evidence, that

- sales respond dynamically upward and downward to an increase and a decrease of advertising and frequently do so at different rates;
- sales increase occurs but its magnitude decreases with time;
- a sales saturation occurs at high advertising levels, as Vidale and Wolfe [24] had observed 20 years before.

In [23] we find the formulation of a general price-advertising model: the product price and the advertising rate at time t are the control variables and the cumulative sales at time t is the state variable; the total discounted profit is the objective function. The Authors specify assumptions which are appropriate for such a model and apply the Pontryagin's Maximum Principle (see for instance [20]) to characterize the optimal price and advertising strategies. In constrast with former papers, apart from [19], here a finite horizon problem is studied. They assume that the sales rate is equal to the production rate, so that they can represent the system state with the unique variable "sales", without considering the "inventory level".

In [22] the focus is on the price policy and the advertising expenditure of a firm faced with consumers who are only incompletely informed about both quality and price (vacation site, restaurant, ...). As quality and price cannot directly be observed before entry, advertising and reputation are utilized by prospective consumers: advertising goodwill reflects the firm's investment independently of sales and consumers' experience; reputation, on the other hand, results from sales and consumers' experience. The problem, formulated as an optimal control problem in infinite horizon, is to find a price policy and an advertising expenditure that maximize the present value of net revenues.

In [6] the finite horizon problem of determining an optimal advertising policy for new product diffusion is considered. Using standard methods of optimal control theory the Authors characterize qualitatively the structure of an optimal advertising strategy for different diffusion models.

A review on *optimal control in advertising*, updated to 1994, is the paper [13].

From the above literature it is clear that the application of Optimal Control theory to Marketing is both fascinating and open to further promising developments. In the last years, in [8], [7], [9], [10], [11], [12]) some new finite horizon optimal control problems have been studied, which concern production, advertising and sales of seasonal products. Because of the products seasonality the Authors need to distinguish the time intervals devoted to production and sales and need to consider explicitly the state variable "inventory level". On the basis of such studies, in the following sections we formulate and discuss a general model for the marketing of seasonal products.

3 The "marketing of seasonal products" problem

We formulate a general model to study the problem of a firm which produces (or purchases), sells and advertises a seasonal product. The feature of the product being seasonal amounts to assume that production and sales take place in two disjoint and consecutive time intervals. Let $[0, 1]$ be the planning interval, that is the seasonality period of the product, and let t_1, $0 \leq t_1 \leq 1$, be the final time of the production interval, $[0, t_1]$, and the starting time of the sales interval, $[t_1, 1]$. We assume that the firm can advertise the product at every time of the seasonality period.

The following parameters and variables will be employed for representing the system:

$\overline{A}_1$ sellers' goodwill level at time t_1;
$\overline{A}_2$ consumers' goodwill level at time t_1;
$\overline{m}$ quantity of product available at time t_1 (either produced in the interval $[0, t_1]$, or purchased at time t_1), to be sold in the interval $[t_1, 1]$;
$c_m(m)$ seasonal product cost function;
$A(t)$ goodwill vector function, $A(t) = (A_1(t), A_2(t))$; $A_1(t)$ is the sellers component, $A_2(t)$ is the consumers component, $t \in [0, 1]$; $A(t_1) = (\overline{A}_1, \overline{A}_2)$;
$c_A(A_1, A_2)$ goodwill cost function, the cost to have the sellers' goodwill A_1 and the consumers' goodwill A_2 at time t_1;
$a(t)$ advertising expenditure rate at time t, $t \in [0, 1]$;
$\overline{a}$ maximum advertising expenditure rate;
T sales final time;
$x(t)$ quantity of good sold in the interval $[t_1, t]$, if $t \in [t_1, T]$, and quantity of good which could be sold in the interval $[t_1, t]$, if $t \in [T, 1]$; $x(t_1) = 0$;
p product sales price in the interval $[t_1, 1]$;
ρ discount factor, $\rho \in (0, 1]$;
c_i inventory marginal cost per unit time, $c_i > 0$;
c_u marginal cost of unsold stock, $c_u > 0$.

We consider the sales interval $[t_1, 1]$ only and we want to determine the optimal advertising policy and optimal levels of goodwill, $(\overline{A}_1, \overline{A}_2)$, and inventory, $\overline{m}$, at time t_1, in order to maximize the discounted net profit.

We assume that the cost functions $c_A(A_1, A_2)$ and $c_m(m)$ are continuously differentiable and componentwise monotonically increasing, so that

$$\frac{\partial c_A(A_1, A_2)}{\partial A_1} \geq 0, \quad \frac{\partial c_A(A_1, A_2)}{\partial A_2} \geq 0, \quad c'_m(m) \geq 0. \tag{2}$$

The total cost for having the sellers' goodwill $\overline{A}_1$, the consumers' goodwill $\overline{A}_2$ and the good quantity $\overline{m}$ at time t_1, evaluated at t_1, is

$$C(\overline{A}_1, \overline{A}_2, \overline{m}) \;=\; c_A(\overline{A}_1, \overline{A}_2) + c_m(\overline{m}). \tag{3}$$

The objective functional is the discounted net profit at the time t_1 (but the choice might equivalently have been for any other time, for example the time 0):

$$\begin{aligned} J(\overline{A}_1, \overline{A}_2, \overline{m}, a(t), T) \;&=\; -\; C(\overline{A}_1, \overline{A}_2, \overline{m}) \\ &+ p \int_{t_1}^{T} \dot{x}(t) e^{-\rho(t-t_1)}\, dt \;-\; c_u[\overline{m} - x(1)]^+ e^{-\rho(1-t_1)} \\ &- \int_{t_1}^{1} \{a(t) + c_i[\overline{m} - x(t)]^+\} e^{-\rho(t-t_1)}\, dt. \end{aligned} \tag{4}$$

The objective functional is the difference between the discounted revenue from sales in $[t_1, T]$ and the discounted total cost. The latter is constituted by the discounted total advertising, inventory and recovering the unsold stock costs and by the cost for having the sellers' goodwill $\overline{A}_1$, the consumers' goodwill $\overline{A}_2$ and the good quantity $\overline{m}$ at time t_1. The problem MSP is stated as follows:

$$\begin{aligned} \text{maximize} \quad & J(\overline{A}_1, \overline{A}_2, \overline{m}, a(t), T), && (5)\\ \text{subject to} \quad & \dot{x}(t) = g(A(t)), && (6)\\ & x(t_1) = 0, && (7)\\ & \dot{A}(t) = f(A(t), a(t)), && (8)\\ & A(t_1) = (\overline{A}_1, \overline{A}_2), \qquad A(1) \geq (\tilde{A}_1, \tilde{A}_2), && (9)\\ & \overline{m} \in [0,\, m_{\max}], \qquad (\overline{A}_1, \overline{A}_2) \in \overline{\mathcal{A}}, && (10)\\ & a(t) \in [0,\, \overline{a}], && (11)\\ & T \in [t_1, 1], && (12)\\ & (T-1)(x(T) - \overline{m}) = 0,\; x(T) \leq \overline{m}. && (13) \end{aligned}$$

The parameters $\tilde{A}_1$ and $\tilde{A}_2$ are fixed and represent the minimum goodwill levels that have to be reached at the final time of the season, $t = 1$.

The parameter $m_{\max}$ is fixed and non-negative: it is the maximum quantity of good produceable in the interval $[0, t_1]$. The set $\overline{\mathcal{A}}$ is a compact subset of $[0, +\infty[^2$: if $A_1^{\min}$ and $A_1^{\max}$ ($A_2^{\min}$ and $A_2^{\max}$) are the minimum and maximum values of the consumers' (sellers') goodwill in t_1, then $\overline{\mathcal{A}} \subseteq [A_1^{\min},\, A_1^{\max}] \times [A_2^{\min},\, A_2^{\max}]$.

The function $g(A)$ expresses the dependence of the sales rate on the goodwill level and the function $f(A, a)$ expresses the dependence of the goodwill level increment rate on the goodwill level itself and on the advertising expenditure rate.

a) $g(A_1,\, A_2)$ is a non-negative concave function and it is componentwise monotonically increasing in (A_1, A_2), that is

$$(A_1',\, A_2') \leq (A_1'',\, A_2'') \quad \Rightarrow \quad g(A_1',\, A_2') \leq g(A_1'',\, A_2'');$$

b) $f_i(A_1,\, A_2,\, a)$ is a concave function; it is monotonically decreasing in $(A_1,\, A_2)$ and monotonically increasing in a; moreover it may not depend explicitly on A_j, $j \neq i$, and on a.

The problem MSP is a dynamic optimization problem with five decision variables: a dynamic variable, that is the advertising expenditure rate $a(t)$, and four static variables, that are the components of the goodwill level $(\overline{A}_1, \overline{A}_2)$, the inventory level $\overline{m}$ at time t_1 and the sales final time T. If we fix the values of the three static decision variables $\overline{A}_1, \overline{A}_2$ and $\overline{m}$, we remain with a dynamic optimization problem in the interval $[t_1, 1]$, with the dynamic variable $a(t)$ and the static variable T.

The final time T is the first time the equality $(T-1)(x(T)-\overline{m}) = 0$ is satisfied, in the constraint (13). Therefore we observe that, as $x(t)$ is an increasing function, because $g(A) \geq 0$, the inequality $x(T) \leq \overline{m}$, in (13), is certainly satisfied by a solution with final time T.

The state of the system at time t, $t \in [t_1, 1]$, is represented by the goodwill level, $A(t) = (A_1(t), A_2(t))$, and by the quantity of good potentially sold in $[t_1, t]$, $x(t)$.

Let us denote by $\mathcal{U}$ the set of all piecewise continuous functions $a : [t_1,\, 1] \to [0,\, \overline{a}]$, then the set of the admissible solutions of MSP is a subset of $\overline{\mathcal{A}} \times [0,\, m_{\max}] \times \mathcal{U} \times [t_1, 1]$.

3.1 Linear dynamics

In a simple special case the functions that describe the motion equations for the state variables of the problem are linear:

$$g(A) \;=\; G_1 A_1 \;+\; G_2 A_2, \tag{14}$$
$$f(A, a) \;=\; (-\delta_1 A_1 \;+\; \gamma_1 a, -\delta_2 A_2 \;+\; \gamma_2 a), \tag{15}$$

with $G_1 \geq 0,\; G_2 > 0,\; \delta_1 > 0,\; \delta_2 > 0,\; \gamma_1 \geq 0,\; \gamma_2 > 0$. We have some interesting special cases if

- $G_1 = 0$: the sellers' goodwill level does not affect the sales;
- $\gamma_1 = 0$: the advertising is addressed to consumers only;
- $\delta_1 = \delta_2 = \delta$ e $\gamma_1 = \gamma_2 = \gamma$: the sellers' and consumers' goodwill motion equations are the same (the initial and/or final conditions may be different).

Some linear cases in which the goodwill state variable is one dimensional, with the meaning of consumers' goodwill only, $A(t) = A_2(t)$, have been studied in [8], [9] and [12].

3.1.1 An advertising and sale problem with linear production

The problem of a firm which produces and sells a seasonal product is considered in [8], [12] and the production, advertising and sales policies in the planning interval, namely the whole seasonality period of the product, $[0, 1]$, are determined in order to maximize the firm profit. Such problems, with a one dimensional goodwill, correspond to the special cases of the problem MSP in which $G_1 = 0$, $G_2 = 1$, the single variable function $c_A(\overline{A}_1, \overline{A}_2) = c_A(\overline{A}_2)$ is the optimal value function of a linear advertising problem in the production interval,

$$c_A(\overline{A}_2) \;=\; \frac{\overline{a}}{\delta_2} \ln \left[\frac{\delta}{\gamma_2 \overline{a}} \left(A_0 e^{-\delta t_1} - \overline{A}_2 \right) + 1 \right], \tag{16}$$

and $c_m(\overline{m})$ is the optimal value function of a linear production problem with inventory costs

$$c_m(\overline{m}) \;=\; a_1 \overline{m} + a_2 \overline{m}^2, \tag{17}$$

where the parameters $a_1, a_2 > 0$ account for different production and inventory characteristics. In particular, in the problem studied in [8], the goodwill level at the final time of the interval considered is free, that is $\tilde{A}_2 = 0$. On the contrary, in the problem studied in [12], the goodwill level at the final time of the interval considered must not be less than a fixed lower bound. Such a final goodwill constraint leads to interesting considerations both from the economic and the mathematical point of view. In both cases treated in [8] and [12], the costs and the returns are not discounted ($\rho = 0$) and the inventory cost in the sales interval is not considered at all, that is $c_i = 0$, because of the assumption that the lifecycle (seasonal interval) of the product is short enough.

3.1.2 A purchase, sale and advertising problem

In [8] the Authors study the problem of maximizing the discounted net profit of a firm which purchases a quantity of some product at a given time, t_1, and afterwards, in the interval $[t_1, T]$, $t_1 \leq T \leq 1$, advertises and sells the product progressively. The firm keeps the product in store until it is sold and the advertising activity, addressed to the consumers, may start at time t_1. The purchase, advertising and sales policies in the planning interval, namely the sales interval, $[t_1, 1]$, are determined in order to maximize the firm profit. The problem is a special case of the problem MSP with the hypothesis that $G_1 = 0$, $G_2 = 1$, $\tilde{A}_2 = 0$ and that the set $\overline{\mathcal{A}}$ is a singleton (hence the function $c_A(\overline{A}_1, \overline{A}_2)$ is not considered explicitly).

3.2 Non-differentiable dynamics

A piecewise linear case is defined by using a linear function, $f(A(t), a(t))$, to describe the motion equation for the goodwill, whereas using a piecewise linear function, $g(A)$, to describe the motion equation for the quantity of good potentially sold. In particular we may represent in such a way the situation where the two components of the goodwill function affect the sales only if the goodwill level is not less than a fixed threshold. A possible form for the function $g(A)$ is the following:

$$g(A) = G_1 \max\{0, A_1 - \hat{A}_1\} + G_2 \max\{0, A_2 - \hat{A}_2\}, \tag{18}$$

with $G_i > 0,\ i = 1, 2$.

A special case of this problem has been studied in [7], [9], [10], considering a unique type of goodwill and assuming that the function $f(A, a)$ and all the other hypothesis are the same as the Section 3.1.1, with $\hat{A}_2 = 0$.

3.3 Student related problem

A natural variation of the linear goodwill dynamics case is given by the following function which is increasing and concave in the advertising expenditure rate a.

$$f(A, a) = (-\delta_1 A_1 + \gamma_1 a^\alpha, -\delta_2 A_2 + \gamma_2 a^\alpha), \tag{19}$$

with $0 < \alpha \leq 1$, $\gamma_i > 0$ and $\delta_i > 0,\ i = 1, 2$.

This kind of function for the one dimensional goodwill motion equation has been considered in [2], [3], [4] and [5], with $\alpha = 1/2$ in the first three papers and with $0 < \alpha < 1$ in the last one, although for different problems and considering a unique type of goodwill. The case $\alpha = 1/2$ was originally derived from the "student related problem" (see [18] and [1]).

3.4 Goodwill cost with common advertising

It may occur that the sellers' goodwill at time t_1 is a function of the consumers' one, $\overline{A}_1 = \alpha(\overline{A}_2)$, so that the goodwill reachable set at time t_1 is a subset of the function $\alpha(\cdot)$ graph, $\overline{\mathcal{A}} \subseteq \{(\alpha(z), z) \mid z \geq 0\}$. Such a case may occur, for instance, because the sellers' and consumers' goodwill levels are obtained by a common advertising effort. We have

$$c_A(\overline{A}_1, \overline{A}_2) \;=\; c_2(\overline{A}_2), \qquad \overline{A}_1 = \alpha(\overline{A}_2), \tag{20}$$

where $c_2(A_2)$ is the cost to drive the consumers' goodwill to the value A_2 and is a continuous, monotonically increasing and convex function.

4 Method of analysis

We observe that if the problem MSP has $(\overline{A}_1^*, \overline{A}_2^*, \overline{m}^*, a^*(t), T^*)$ as an optimal solution, with the associated state function $(A_1^*(t), A_2^*(t), x^*(t))$, $t \in [t_1, 1]$, then the relations $x^*(1) \geq x^*(T^*)$ and $x^*(T^*) = \overline{m}^*$ hold, whereas the equality $x^*(1) = \overline{m}^*$ holds if and only if $T^* = 1$. As a consequence of such observation and in view of (6), the objective functional (4) may be expanded as follows

$$J(\overline{A}_1, \overline{A}_2, \overline{m}, a(t), T) = \\ = J_1(a(t), T) + J_2(a(t), T) - C(\overline{A}_1, \overline{A}_2, \overline{m}), \tag{21}$$

$$J_1(a(t), T) = \int_{t_1}^{T} [pg(A(t)) - c_i(\overline{m} - x(t)) - a(t)]\, e^{-\rho(t-t_1)} dt, \tag{22}$$

$$J_2(a(t), T) = - \int_{T}^{1} a(t) e^{-\rho(t-t_1)}\, dt. \tag{23}$$

Now, it is interesting to notice that two special optimal control problems have $J_1(a(t), T)$ and $J_2(a(t), T)$ as their objective functionals and we will prove that an optimal solution of problem MSP may be built up from the optimal solutions of the two problems just mentioned.
For all $(\overline{A}_1, \overline{A}_2, \overline{m}, T, A_1^T, A_2^T) \in \overline{\mathcal{A}} \times [0, m_{\max}] \times [t_1, 1] \times [0, +\infty[^2$, we call *sale and goodwill* problem, denoted SG, the problem of determining an advertising policy for the interval $[t_1, T]$, given the goodwill $(\overline{A}_1, \overline{A}_2)$ and the inventory level $\overline{m}$ at time t_1, in order to maximize the discounted profit net of past goodwill and good production costs, under the requirement of the fixed time T and final goodwill (A_1^T, A_2^T). The problem SG is represented as follows:

$$\begin{array}{rll} \text{maximize} & J_1(a(t), T), & \\ \text{subject to} & (6), \quad (8), \quad (11) & \\ \text{and to} & x(t_1) = 0, & x(T) = \overline{m}, \qquad (24) \\ & A(t_1) = (\overline{A}_1, \overline{A}_2), & A(T) = (A_1^T, A_2^T). \qquad (25) \end{array}$$

For all $(T, A_1^T, A_2^T) \in [t_1, 1] \times [0, +\infty[^2$, we call *goodwill* problem, denoted G, the problem of determining an advertising policy for the interval $[T, 1]$, given the goodwill (A_1^T, A_2^T) at time T, in order to minimize the discounted advertising cost, under the requirement of the fixed initial time T and of the MSP problem final goodwill condition. The problem G is represented as follows:

$$\begin{array}{rll} \text{maximize} & J_2(a(t), T), & \\ \text{subject to} & (8), \quad (11) & \\ \text{and to} & A(T) = (A_1^T, A_2^T), & A(1) \geq (\tilde{A}_1, \tilde{A}_2). \qquad (26) \end{array}$$

If $g(A)$, $g'(A)$, $f(A, a)$ and $\partial f(A, a)/\partial A$ are continuous functions, then the Pontryagin Maximum Principle (see for instance [20, p. 85]) provides the optimality necessary conditions for the problems SG and G.

Let X_{SG} be the set of 6-tuples

$$(\overline{A}_1, \overline{A}_2, \overline{m}, T, A_1^T, A_2^T) \in \overline{\mathcal{A}} \times [0, m_{\max}] \times [t_1, 1] \times [0, +\infty[^2,$$

such that there exists an admissible solution $(a(t), A_1(t), A_2(t), x(t))$ for problem SG. Furthermore, let X_G be the set of triplets $(T, A_1^T, A_2^T) \in [t_1, T] \times [0, +\infty[^2$, such that there exists a solution $(a(t), A_1(t), A_2(t))$ admissible for problem G. The set X_G depends on the couple $(\tilde{A}_1, \tilde{A}_2)$ too, but we leave such dependence implicit, for the sake of notation simplicity.

Lemma 1 *If the set $\{f(A_1, A_2, a) \mid a \in [0, \overline{a}]\}$ is convex for all (A_1, A_2) and the functions g and f are continuous in (A_1, A_2) and are uniformly Lipschitz with respect to a in (A_1, A_2), then*

a) for all $(\overline{A}_1, \overline{A}_2, \overline{m}, T, A_1^T, A_2^T) \in X_{SG}$, there exists a measurable advertising cost intensity, $a_1(t)$, $t \in [t_1, T]$, which is optimal for problem SG;
b) for all $(T, A_1^T, A_2^T) \in X_G$, there exists a measurable advertising cost intensity, $a_2(t)$, $t \in [T, 1]$, which is optimal for problem G.

Proof The existence of admissible state-control pairs for both problems is a consequence of the definitions of the sets X_{SG} and X_G. The compactness of the control range $[0, \overline{a}]$ is trivial. From the Lipschitz assumption we obtain the uniform boundedness of the state functions and from the convexity assumption for the set $\{f(A_1, A_2, a) \mid a \in [0, \overline{a}]\}$ we obtain that the convexity requirement of the Filippov-Cesari theorem [20, p. 132] holds too. □

We will denote by $w_1(\overline{A}_1, \overline{A}_2, \overline{m}, T, A_1^T, A_2^T)$ and $w_2(T, A_1^T, A_2^T)$ the optimal values of the functionals $J_1(a(t), T)$ and $J_2(a(t), T)$ respectively, whenever they exist for the problems SG and G.

Now we can prove the following equivalence result.

Theorem 1 *The original optimal control problem MSP has an optimal solution if and only if an optimal solution exists for the nonlinear programming problem MSP^*:*

$$\begin{aligned} \textit{maximize} \quad & w_1(\overline{A}_1, \overline{A}_2, \overline{m}, T, A_1^T, A_2^T) + w_2(T, A_1^T, A_2^T) \\ & \qquad - C(\overline{A}_1, \overline{A}_2, \overline{m}), \qquad (27) \\ \textit{subject to} \quad & (\overline{A}_1, \overline{A}_2, \overline{m}, T, A_1^T, A_2^T) \in X_{SG}, \qquad (28) \\ & (T, A_1^T, A_2^T) \in X_G\,. \qquad (29) \end{aligned}$$

In particular,

a) if $(\overline{A}_1^*, \overline{A}_2^*, \overline{m}^*, a^*(t), T^*)$ is an optimal solution to MSP, then there exists a couple $(A_1^{T^*}, A_2^{T^*})$, such that the associated 6–tuple $(\overline{A}_1^*, \overline{A}_2^*, \overline{m}^*, T^*, A_1^{T^*}, A_2^{T^*})$ is an optimal solution to MSP^*;
b) if $(\overline{A}_1^*, \overline{A}_2^*, \overline{m}^*, T^*, A_1^{T^*}, A_2^{T^*})$ is an optimal solution to MSP^*, then there exists a measurable function $a^*(t)$, such that the 5–tuple $(\overline{A}_1^*, \overline{A}_2^*, \overline{m}^*, a^*(t), T^*)$ is an optimal solution to MSP.

Proof a) If the original problem MSP has $(\overline{A}_1^*, \overline{A}_2^*, \overline{m}^*, a^*(t), T^*)$ as an optimal solution, then $a^*(t)$, $t \in [t_1, 1]$, is an MSP–admissible control which determines an MSP–admissible state function $A^*(t)$. Let $(A_1^{T^*}, A_2^{T^*}) = A^*(T^*)$, then

- $(\overline{A}_1^*, \overline{A}_2^*, \overline{m}^*, T^*, A_1^{T^*}, A_2^{T^*}) \in X_{SG}$ and inequality $J_1(a^*(t), T^*) \geq J_1(a(t), T^*)$ holds, for all control $a(t)$ which is admissible for the problem SG, under the requirement of fixed final time T^* and final goodwill $(A_1^{T^*}, A_2^{T^*})$, that is the equality $J_1(a^*(t), T^*) = w_1(\overline{A}_1^*, \overline{A}_2^*, \overline{m}^*, T^*, A_1^{T^*}, A_2^{T^*})$ holds;
- $(T^*, A_1^{T^*}, A_2^{T^*}) \in X_G$ and $J_2(a^*(t), T^*) \geq J_2(a(t), T^*)$, for all control $a(t)$, which is admissible for the problem G under the requirement of fixed initial time T^* and initial goodwill (A_1^*, A_2^*) and of the MSP problem final goodwill condition, that is the equality $J_2(a^*(t), T^*) = w_2(T^*, A_1^{T^*}, A_2^{T^*})$ holds.

In fact, if either inequality did not hold, then $J(\overline{A}_1^*, \overline{A}_2^*, \overline{m}^*, a^*(t), T^*)$ would not be maximum, because of the expansion (21). For instance, if there exists an SG–admissible control function $\hat{a}(t)$, $t \in [t_1, T^*]$, such that $J_1(\hat{a}(t), T^*) > J_1(a^*(t), T^*)$, then, after defining

$$\tilde{a}(t) = \begin{cases} \hat{a}(t), & t \in [t_1, T^*], \\ a^*(t), & t \in]T^*, 1], \end{cases}$$

we obtain that

$$\begin{aligned} J(\overline{A}_1^*, \overline{A}_2^*, \overline{m}^*, \tilde{a}(t), T^*) \ - \ J(\overline{A}_1^*, \overline{A}_2^*, \overline{m}^*, a^*(t), T^*) \ = \\ = \ J_1(\hat{a}(t), T^*) \ - \ J_1(a^*(t), T^*) \ > \ 0. \end{aligned}$$

Hence the thesis a) follows.

b) If the problem MSP^* has $(\overline{A}_1^*, \overline{A}_2^*, \overline{m}^*, T^*, A_1^{T^*}, A_2^{T^*})$ as an optimal solution, then there exists $a^*(t)$, $t \in [t_1, 1]$, such that

- $a^*(t)$, $t \in [t_1, T^*]$, is an SG–optimal measurable control which determines an SG–optimal state function $A^*(t)$, $t \in [t_1, T^*]$: therefore $J_1(a^*(t), T^*) \geq J_1(a(t), T^*)$, for all $a(t)$ admissible control of problem SG, under the requirement of the fixed final time T^* and final goodwill $(A_1^{T^*}, A_2^{T^*})$, that is the equality $w_1(\overline{A}_1^*, \overline{A}_2^*, \overline{m}^*, T^*, A_1^{T^*}, A_2^{T^*}) = J_1(a^*(t), T^*)$ holds;
- $a^*(t)$, $t \in [T^*, 1]$, is a G–optimal measurable control which determines a G–optimal state function $A^*(t)$, $t \in [T^*, 1]$: therefore $J_2(a^*(t), T^*) \geq J_2(a(t), T^*)$, for all $a(t)$ admissible control of problem G, under the requirement of the fixed initial time T^* and initial goodwill $(\overline{A}_1^*, \overline{A}_2^*)$ and of the MSP problem final goodwill condition, that is the equality $w_2(T^*, A_1^{T^*}, \overline{A}_2^{T^*}) = J_2(a^*(t), T^*)$ holds.

  The above statement can be proved in a similar way as in a). $\square$

The theorem just proved allows us to obtain the features of an MSP–optimal solution, and possibly determine it, from the knowledge of (the features of) the SG– and G–optimal solutions.

5 Analysis of special cases

If we reconsider the instances of functions $f(A, a)$ and $g(A)$, introduced in the Sections 3.1, 3.2 and 3.3, we can verify that they fulfil the requirements of the Lemma 1, which states the existence of optimal solutions of the problems SG and G. Here we review some results concerning the optimal solutions of particular problems from the literature already quoted.

5.1 Optimal solutions with linear dynamics

In this Section we review some results concerning the optimal solutions of the linear problems described in the Section 3.1, in which the goodwill state function is one dimensional.

5.1.1 Optimal solutions of an advertising and sale problem with linear production The linear problem described in Section 3.1.1 and studied in [8], which has a free final goodwill condition, has a unique optimal solution that requires that

- either there is no production, or there is a maximum rate production process, starting at a suitable time, without any break and ending at the selling starting time;
- either there is no advertising, or there is a maximum rate advertising process, without any break, which occurs in a neighborhood of the selling starting time, t_1, and with final time strictly less than the selling final time, T;
- if the advertising expenditure is non-zero, then the selling final time T is maximum, that is $T = 1$, the final time of the seasonality period.

The linear problem described in Section 3.1.1 and studied in [12], which has a bounded final goodwill condition, has an optimal solution which belongs to one of four types, called *no advertising, early advertising, late advertising, alternate advertising.* In particular, if the alternate advertising policies are optimal solutions, then their convex linear combinations are optimal solutions too. Therefore, in such a case there are infinitely many optimal solutions. Such results have been used to study a production, sale and advertising problem for a seasonal product over several seasons in [11].

5.1.2 Optimal solution af a purchase, sale and advertising problem The linear problem described in Section 3.1.2 has been studied in [9], by distinguishing among the three possibilities that the final time is either fixed, or bounded, or free. In all cases, there exists an optimal solution, which again requires that either there is no advertising, or there is a maximum rate advertising process, occuring in a neighborhood of the selling starting time, t_1, and with final time strictly less than the selling final time, T. Furthermore, the convexity of the purchase cost function c_m is recognized to be a sufficient condition for the uniqueness of the optimal solution.

5.2 Optimal solutions with non-differentiable dynamics

The piecewise linear problem described in Section 3.2 and studied in [7], [9], [10], has an optimal solution, possibly not unique. The optimal production-advertising controls exhibit the same general features as those of Section 5.1.1, but it may happen that the firm cannot start selling the product at the beginning of the interval in which the sales are allowed. Moreover, it is impossible to determine explicitly the optimal solutions and hence it is necessary to resort to an algorithm to determine a numerical solution to the problem.

6 Conclusion

We have provided a review of some recent research concerning the marketing of seasonal products and have embedded the different contributions into a unique and general framework. The review illustrates a problem area which is largely unexplored. In fact, on one hand, some natural pieces of the sketched mosaic have still to be developed, both for the problems with a two-dimensional goodwill, which have not been studied in detail so far, and for a variety of problems with a one-dimensional goodwill and either non-linear motion equations (for example *the student related* dynamics), or special goodwill-product cost functions, or special boundary conditions. On the other hand, the issue of the numerical solution of the problems, which has been outlined for some of the problems studied, needs a particular attention for the more complex problems. Moreover, some generalizations of the proposed model are both mathematically attractive and motivated by a sound economic-management reasoning. For example, in order to address the issue of the market segmentation, we are led to consider the sale and goodwill variables as vectors with dimensions greater than 1 and 2, respectively. The model and the method of analysis proposed here should represent a general enough reference point for developing new models, suitable for special application cases, and the computation algorithms, necessary to determine the relevant solutions.

References

1. H. Bondi, Note on a student related optimal control problem, IMA Bulletin, 18 (1982), pp.10–11.
2. A. Buratto and B. Viscolani, An optimal control student problem and a marketing counterpart, Mathematical and Computer Modelling, 20 (1994), pp.19–33.
3. A. Buratto and B. Viscolani, Advertising for the launch of a product with random success, Ricerca Operativa, 26 (1996), pp.45-62.
4. A. Buratto and B. Viscolani, Optimal study/advertising plan for a knowledge/goodwill evolution process with rank memory, Journal of Information and Optimization Sciences, 18 (1997), pp.211-228.

5. A. Buratto and B. Viscolani, New product introduction: goodwill, time and advertising cost, Mathematical Methods of Operations Research, 55 (2002), pp.55–68.
6. E. Dockner and S. Jørgensen, Optimal advertising policies for diffusion models of new product innovation in monopolistic situations, Management Science, 34 (1988), pp.119–130.
7. D. Favaretto, P. Mazzega, A. Mezzaroba and B. Viscolani, Optimal control of advertising and production of a seasonal product with nondifferentiable demand, Dynamics and control, 6 (1996), pp.91–104.
8. D. Favaretto and B. Viscolani, Controllo ottimo di produzione e comunicazione per un prodotto stagionale, Rendiconti del Comitato per gli Studi Economici, XXX–XXXI (1993), pp.209–225.
9. D. Favaretto and B. Viscolani, Optimal purchase and advertising for a product with immediate sales start, Top, 4 (1996), pp.301–318.
10. D. Favaretto and B. Viscolani, Determining optimal production and advertising policies for a seasonal product, Optimization, 42 (1997), pp.359–378.
11. D. Favaretto and B. Viscolani, A multiperiod production and advertising problem for a seasonal product, Annals of Operations Research, 88 (1999), pp.31–45.
12. D. Favaretto and B. Viscolani, A single season production and advertising control problem with bounded final goodwill, Journal of Information and Optimization Sciences, 21 (2000), pp.337–357.
13. G. Feichtinger, R. F. Hartl and S. P. Sethi, Dynamic optimal control models in advertising: recent developments, Management Science, 40 (1994), pp.195–226.
14. H. Groenevelt, The Just-in-Time System, in S. C. Graves et al, eds., Logistics of Production and Inventory, North–Holland, Amsterdam, 1993, pp. 629–670.
15. J. D. C. Little, Aggregate advertising models: the state of the art, Operations Research, 27 (1979), pp.629–667.
16. D. G. Luenberger, Linear and Nonlinear Programming, Addison-Wesley, Reading, 1984.
17. M. Nerlove and K. J. Arrow, Optimal advertising policy under dynamic conditions, Economica, 29 (1962), pp.129–142.
18. G. F. Raggett, P. W. Hempson and K. A. Jukes, A student related optimal control problem, IMA Bulletin, 17 (1981), pp.133–136.
19. M. W. Sasieni, Optimal advertising expenditure, Management Science, 18 (1971), pp.64–72.
20. A. Seierstad and K. Sydsaeter, Optimal Control Theory with Economic Applications, North-Holland, Amsterdam, 1987.
21. S. P.Sethi, Optimal advertising for the Nerlove-Arrow model under a budget constraint, Operation Research Quarterly, 28 (1977), pp.683–693.
22. K. Spremann, The signaling of quality by reputation, in G. Feichtinger, ed., Optimal Control Theory and Economic Analysis 2, North-Holland, Amsterdam, 1985, pp.235–252.
23. J. T. Teng and G. L. Thompson, Optimal strategies for general price–advertising models, in G. Feichtinger, ed., Optimal Control Theory and Economic Analysis 2, North-Holland, Amsterdam, 1985, pp.183–195.
24. M. L. Vidale and H. B. Wolfe, An operations research study for sales response to advertising, Operations Research, 5 (1957), pp.370–381.